

Optimal Universal Disentangling Machine for Two Qubit Quantum States

Sibasish Ghosh^{a,1}, Somshubhro Bandyopadhyay^{b,2}, Anirban Roy^{a,3},
Debasis Sarkar^{c,4} and Guruprasad Kar^a

^a *Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B.T. Road, Calcutta -700035, India.*

^b *Department of Physics, Bose Institute, 93/1 A.P.C. Road, Calcutta -700009, India.*

^c *Department of Mathematics, Barrackpore Rastraguru Surendranath College, 85 Middle Road, Barrackpore, North 24 Parganas, West Bengal, India.*

Abstract

We derive the optimal curve satisfied by the reduction factors, in the case of universal disentangling machine which uses only local operations. Impossibility of constructing a better disentangling machine, by using non-local operations, is discussed.

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¹e-mail: res9603@isical.ac.in

²e-mail: dhom@bosemain.boseinst.ernet.in

³e-mail: res9708@isical.ac.in

⁴Present address: Department of Mathematics, University of Burdwan, Burdwan, West Bengal, India.

1 Introduction

Disentanglement is the process that transforms a state of two (or more) subsystems into an unentangled state (in general, a mixture of product states) such that the reduced density matrices of each of the subsystems are unaffected.

Let ρ^{ent} be any entangled state of two qubits 1 and 2; and let ρ_1, ρ_2 be the reduced density matrices of 1 and 2 respectively. Then the operation of any disentangling machine (DM) is defined as

$$\rho^{ent} \xrightarrow{\text{DM}} \rho^{disent}$$

together with

$$\rho_i = \text{Tr}_j(\rho^{ent}) = \text{Tr}_j(\rho^{disent}), \quad i \neq j; \quad i, j = 1, 2$$

for all ρ^{ent} . This kind of ideal universal disentangling machine does not exist [1], [2].

So the next question is whether there exists a disentangling machine which disentangles entangled state, and for which

$$\text{Tr}_j(\rho^{disent}) = \eta_i \rho_i + \left(\frac{1 - \eta_i}{2}\right) I, \quad i \neq j; \quad i, j = 1, 2$$

where η_i ($0 < \eta_i < 1$ for $i = 1, 2$) is independent of ρ^{ent} [3]. Recently it has been shown that [4] this kind of machine exists, by using local cloning operations, where the input states are all pure entangled states. Reference [4] considered two cases, (1) $\eta_1 = 1$ (or $\eta_2 = 1$), *i.e.*, using only one local cloning machine, (2) $\eta_1 = \eta_2 (= \eta, \text{ say})$, *i.e.*, which uses two local cloning machines with same fidelity. For the case (1), the maximum value of η_2 (or η_1) is $1/3$. In the case (2), the maximum attainable value of η is $1/\sqrt{3}$. In the present paper, we want to find out the optimum values of η_1 and η_2 , or, in other words, the optimal curve (if it exists) satisfied by η_1 and η_2 (*i.e.*, reduction factors corresponding to the optimal disentangling machine), by using most general (asymmetric) local operations. Surprisingly, we got the same upper bounds on η as has been found in [4], in the corresponding cases (1) and (2). We have also obtained the optimal curve in the most general case, when asymmetric local operations are used.

In section 2, for simplicity, we first consider disentanglement process by applying our disentangling machine on one of the subsystems. In section 3, we deal with the symmetric case where the same disentangling machine is used locally on the two subsystems. Next, in section 4, the most general disentangling machine is considered using asymmetric local

operations, where we discuss the disentanglement of mixed states. In section 5 we sum up our results and put some arguments regarding nonlocal operations.

2 Totally asymmetric optimal universal disentangling machine

In this section, we shall consider how we can disentangle a two qubit pure entangled state by local operation on any one qubit. Suppose we have two parties x and y sharing an entangled state of two qubits given by

$$|\psi\rangle = \alpha |00\rangle_{xy} + \beta |11\rangle_{xy}, \quad (1)$$

where α and β are non-negative numbers with $\alpha^2 + \beta^2 = 1$. The first qubit belongs to x and the second belongs to y as usual. Now a universal transformation (a unitary operation) is applied to the qubit belonging to any one (say, y) of the two parties. This gives rise to a composite system ρ_{xyM} consisting of the two qubits and a (disentangling) machine M . Tracing out on the machine states we get a two qubit composite system which is disentangled (*i.e.*, separable) under certain conditions.

Consider the following unitary transformation U_j (associated with a machine state $|M\rangle_j$) applied on one subsystem j (where $j = x$ or y), defined by

$$U_j |0\rangle_j |M\rangle_j = m_{0j} |0\rangle_j |M_0\rangle_j + m_{1j} |1\rangle_j |M_1\rangle_j, \quad (2)$$

$$U_j |1\rangle_j |M\rangle_j = \widetilde{m}_{0j} |0\rangle_j |\widetilde{M}_0\rangle_j + \widetilde{m}_{1j} |1\rangle_j |\widetilde{M}_1\rangle_j, \quad (3)$$

where $\{|M_0\rangle_j, |M_1\rangle_j, |\widetilde{M}_0\rangle_j, |\widetilde{M}_1\rangle_j\}$ are four normalized machine states, and (using unitarity)

$$\left. \begin{aligned} |m_{0j}|^2 + |m_{1j}|^2 &= 1, \\ |\widetilde{m}_{0j}|^2 + |\widetilde{m}_{1j}|^2 &= 1. \end{aligned} \right\} \quad (4)$$

Using orthogonality, we have (from (2) and (3)),

$$m_{0j}^* \widetilde{m}_{0j} \langle M_0 | \widetilde{M}_0 \rangle_j + m_{1j}^* \widetilde{m}_{1j} \langle M_1 | \widetilde{M}_1 \rangle_j = 0. \quad (5)$$

Now applying this operation U_j on an arbitrary one qubit state $|\phi\rangle_j = a|0\rangle_j + b|1\rangle_j$ (where $|a|^2 + |b|^2 = 1$), we get the following composite state,

$$U_j |\phi\rangle_j |M\rangle_j =$$

$$\begin{aligned}
& a \left[m_{0j} |0\rangle_j |M_0\rangle_j + m_{1j} |1\rangle_j |M_1\rangle_j \right] + b \left[\widetilde{m}_{0j} |0\rangle_j |\widetilde{M}_0\rangle_j + \widetilde{m}_{1j} |1\rangle_j |\widetilde{M}_1\rangle_j \right] \\
& = am_{0j} |0\rangle_j |M_0\rangle_j + am_{1j} |1\rangle_j |M_1\rangle_j + b\widetilde{m}_{0j} |0\rangle_j |\widetilde{M}_0\rangle_j + b\widetilde{m}_{1j} |1\rangle_j |\widetilde{M}_1\rangle_j. \quad (6)
\end{aligned}$$

We got (6) by applying the above unitary operation on any one qubit (x or y) pure state $|\phi\rangle_j$ ($j = x, y$), where $\rho_j = |\phi\rangle_j\langle\phi| = \frac{1}{2}(\mathbf{1} + \vec{s}\cdot\vec{\sigma})$ (with $|\vec{s}| = 1$). And now we demand that the reduced density matrix, after tracing out the machine states in the equation (6), is of the form $\frac{1}{2}(\mathbf{1} + \eta_j\vec{s}\cdot\vec{\sigma})$ (where $0 < \eta_j \leq 1$) for all \vec{s} (isotropy) [3]. Then the machine has to satisfy the following equations :

$$\left. \begin{aligned}
m_{0j}m_{1j}^* \langle M_1 | M_0 \rangle_j &= 0, \\
\widetilde{m}_{0j}\widetilde{m}_{1j}^* \langle \widetilde{M}_1 | \widetilde{M}_0 \rangle_j &= 0, \\
m_{1j}^*\widetilde{m}_{0j} \langle M_1 | \widetilde{M}_0 \rangle_j &= 0.
\end{aligned} \right\} \quad (7)$$

$$\left. \begin{aligned}
\text{Re} \left\{ m_{0j}^*\widetilde{m}_{0j} \langle M_0 | \widetilde{M}_0 \rangle_j \right\} &= 0, \\
\text{Re} \left\{ m_{1j}^*\widetilde{m}_{1j} \langle M_1 | \widetilde{M}_1 \rangle_j \right\} &= 0.
\end{aligned} \right\} \quad (8)$$

$$\eta_j = m_{0j}\widetilde{m}_{1j}^* \langle \widetilde{M}_1 | M_0 \rangle_j. \quad (9)$$

$$\left. \begin{aligned}
|m_{0j}| &= \left(\frac{1 + \eta_j}{2} \right)^{1/2}, \\
|m_{1j}| &= \left(\frac{1 - \eta_j}{2} \right)^{1/2}, \\
|\widetilde{m}_{0j}| &= \left(\frac{1 - \eta_j}{2} \right)^{1/2}, \\
|\widetilde{m}_{1j}| &= \left(\frac{1 + \eta_j}{2} \right)^{1/2}.
\end{aligned} \right\} \quad (10)$$

Now we apply the above-mentioned operation on one of the two qubits (say on y in the state $|\psi\rangle$, in equation (1)). Then the state $|\psi\rangle = \alpha |00\rangle_{xy} + \beta |11\rangle_{xy}$ is transformed (after tracing out the machine states, and applying all the above conditions *i.e.*, (4), (5), (7) – (10)) to the following density matrix :

$$\begin{aligned}
D_{xy}^{TA} &= \text{Tr}_M [\rho_{xyM}] = \\
& \begin{bmatrix} \frac{\alpha^2(1+\eta_y)}{2} & 0 & -i\alpha\beta\Lambda_y & \alpha\beta\eta_y \\ 0 & \frac{\alpha^2(1-\eta_y)}{2} & 0 & i\alpha\beta\Lambda_y \\ i\alpha\beta\Lambda_y & 0 & \frac{\beta^2(1-\eta_y)}{2} & 0 \\ \alpha\beta\eta_y & -i\alpha\beta\Lambda_y & 0 & \frac{\beta^2(1+\eta_y)}{2} \end{bmatrix} \quad (11)
\end{aligned}$$

⁵As the system x is unchanged, therefore, $\eta_x = 1$

where $\Lambda_j = \text{Im} \left\{ m_{0j}^* \widetilde{m}_{0j} \langle M_0 | \widetilde{M}_0 \rangle_j \right\}$ (for $j = x, y$), and the entries of this matrix are arranged in accordance with the ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of the two qubits.

Now we apply the Peres–Horodecki theorem to test the inseparability of D_{xy}^{TA} , which states that a density matrix ρ of a bipartite system in $\mathcal{C}^2 \otimes \mathcal{C}^2$ is separable if and only if the partial transposition of ρ is positive semidefinite, *i.e.*, each of the eigen values of the partial transposition of ρ is non-negative [5], [6], [7]. It turns out that the state $|\psi\rangle$ will be disentangled (*i.e.*, D_{xy}^{TA} is separable) if the following conditions are satisfied:

$$\left. \begin{aligned} 1 - \eta_y^2 + 2\alpha^2\beta^2 (1 - \eta_y^2 - 4\Lambda_y^2) &\geq 0, \\ \alpha^2\beta^2 \left\{ (1 + \eta_y)^2(1 - 2\eta_y) - 4\Lambda_y^2 \right\} &\geq 0, \\ \alpha^4\beta^4 \left\{ (1 - 3\eta_y)(1 + \eta_y)^3 + 8\Lambda_y^2 (2\Lambda_y^2 - 1 + \eta_y^2) \right\} &\geq 0. \end{aligned} \right\} \quad (12)$$

All the three conditions in (12) will be satisfied for all $\alpha\beta \in [0, 1/2]$ (as we are seeking for universal disentangling machine), if $\eta_y \leq 1/3$.

Thus we see that *all pure states of two qubits x and y will be disentangled by applying a disentangling machine locally on y , provided $\eta_y \leq 1/3$.*

Now that our requirement is also to have reduced density matrices $D_{ad}^{(x)} = \text{Tr}_y[D_{xy}^{TA}]$, $D_{ad}^{(y)} = \text{Tr}_x[D_{xy}^{TA}]$ of the disentangled state D_{xy} as close as possible to those of the entangled one (*i.e.*, $\rho_{bd}^{(x)} = \text{Tr}_y[|\psi\rangle\langle\psi|]$ and $\rho_{bd}^{(y)} = \text{Tr}_x[|\psi\rangle\langle\psi|]$ respectively), we note that the reduced density matrix of the subsystem x is unaltered whereas that of the subsystem y is changed. Let us summarise these results.

1. It is possible to disentangle any arbitrary bipartite entangled state by applying local disentangling machine on one of its qubits provided the reduction factor (η) of the isotropic machine is less than or equal to $1/3$.

2. After disentanglement the reduced density matrices of the subsystems are given by

$$\begin{aligned} D_{ad}^{(x)} &= \rho_{bd}^{(x)} \\ D_{ad}^{(y)} &= \eta\rho_{bd}^{(y)} + \left(\frac{1-\eta}{2}\right) I \end{aligned}$$

where $\eta_{max} = 1/3$.

3 Symmetric optimal universal disentangling machine

In the previous section we have shown how to disentangle any pure state of two qubits by applying local operation on one of the qubits. In this section we apply the local unitary

operation $U = U_x = U_y$ ^[1], defined by equations (2) and (3), on both the parties x and y (in the state $|\psi\rangle$, given in equation (1)) separately.

Each of the two parties now performs the same local unitary operation U on their own qubit, as described in the previous section. After this local operation, the reduced density matrix (tracing out the machine states) of the two parties x and y (applying all the constraints on the machine states, *i.e.*, conditions (4), (5), (7) – (10)) is given by,

$$\rho_{xy} = \begin{bmatrix} \frac{(1-\eta)^2}{4} + \alpha^2\eta - 2\alpha\beta\Lambda^2 & -i\alpha\beta\Lambda\eta & -i\alpha\beta\Lambda\eta & \alpha\beta\eta^2 \\ i\alpha\beta\Lambda\eta & \frac{1-\eta^2}{4} + 2\alpha\beta\Lambda^2 & 0 & i\alpha\beta\Lambda\eta \\ i\alpha\beta\Lambda\eta & 0 & \frac{1-\eta^2}{4} + 2\alpha\beta\Lambda^2 & i\alpha\beta\Lambda\eta \\ \alpha\beta\eta^2 & -i\alpha\beta\Lambda\eta & -i\alpha\beta\Lambda\eta & \frac{(1-\eta)^2}{4} + \beta^2\eta - 2\alpha\beta\Lambda^2 \end{bmatrix} \quad (13)$$

It follows from the Peres-Horodecki theorem [5], [6], [7], that ρ_{xy} is separable (*i.e.*, the state $|\psi\rangle$ is disentangled) if

$$a_1(2a_2 + a_3) + a_2(a_2 + 2a_3) - (\alpha\beta\eta)^2(4\Lambda^2 + \eta^2) \geq 0, \quad (14)$$

$$\begin{aligned} a_1a_2(a_2 + 2a_3) + a_2^2a_3 - (a_1 + a_3)(\alpha\beta\eta)^2(2\Lambda^2 + \eta^2) \\ - 4a_2(\alpha\beta\eta\Lambda)^2 - 4(\alpha\beta)^3\eta^4\Lambda^2 \geq 0, \end{aligned} \quad (15)$$

⁶so here $\eta_x = \eta_y \equiv \eta$ (say), $m_{ix} = m_{iy}$ ($i = 0, 1$), $\widetilde{m}_{ix} = \widetilde{m}_{iy}$ ($i = 0, 1$), $|M_i\rangle_x = |M_i\rangle_y$ ($i = 0, 1$), $|\widetilde{M}_i\rangle_x = |\widetilde{M}_i\rangle_y$ ($i = 0, 1$), $\Lambda_x = \Lambda_y \equiv \Lambda$ (say)

$$a_1 a_2^2 a_3 - 2a_2 (a_1 + a_3) (\alpha\beta\eta\Lambda)^2 - a_1 a_3 (\alpha\beta)^2 \eta^4 - 2(a_1 + a_3) (\alpha\beta)^3 \eta^4 \Lambda^2 \geq 0, \quad (16)$$

where

$$a_1 = \frac{(1-\eta)^2}{4} + \alpha^2 \eta - 2\alpha\beta\Lambda^2,$$

$$a_2 = \frac{1-\eta^2}{4} + 2\alpha\beta\Lambda^2,$$

$$a_3 = \frac{(1-\eta)^2}{4} + \beta^2 \eta - 2\alpha\beta\Lambda^2.$$

Now we shall consider the following two special cases, where in the first case, we put constraint on the machine, and in the second, we take the original state as a maximally entangled state.

Case I: $\Lambda = 0$, i.e., $\text{Im} \left\{ \langle M_0 | \widetilde{M}_0 \rangle \right\} = 0$ [\[8\]](#).

In this case, conditions [\(14\)](#) – [\(16\)](#) will be reduced to the following three conditions respectively:

$$\frac{1-\eta^2}{8} \{3 + \eta^2 + 8(\alpha\beta\eta)^2\} \geq 0, \quad (17)$$

$$(1-\eta^2)^2 + 8(\alpha\beta\eta)^2(1-2\eta^2-\eta^4) \geq 0, \quad (18)$$

$$\left\{ \frac{(1-\eta^2)^2}{16} + (\alpha\beta\eta)^2 \right\} \left\{ \frac{(1-\eta^2)^2}{16} - (\alpha\beta\eta^2)^2 \right\} \geq 0. \quad (19)$$

It is clear from the above conditions [\(17\)](#) – [\(19\)](#) that all bipartite pure entangled states (i.e., for all $\alpha\beta \in [0, 1/2]$), will be disentangled, if the reduction factor (η) is less than or equal to $1/\sqrt{3}$, and so the maximum value η_{max} of η is equal to $1/\sqrt{3}$.[\[7\]](#)

Case II: $\alpha\beta = 1/2$ [\[9\]](#).

Here, above conditions [\(14\)](#) – [\(16\)](#) will be reduced to the following three conditions respectively:

$$\Lambda^4 \leq \frac{3}{16}(1-\eta^4), \quad (20)$$

$$\Lambda^4 \leq \frac{1-3\eta^4-2\eta^6}{16}, \quad (21)$$

$$(1+\eta^2+4\Lambda^2)(1+\eta^2-4\Lambda^2)(1-2\eta^2-3\eta^4+8\Lambda^2\eta^2-16\Lambda^4) \geq 0. \quad (22)$$

As we have to maximize η , we have to check [\(20\)](#) – [\(22\)](#) with all possible values of Λ . For that reason, we take, in the above three conditions [\(20\)](#) – [\(22\)](#), the values of Λ^2 ,

⁷Note the error in equation no. (17) of [\[10\]](#), and also in [\[4\]](#).

starting from 0. Keeping all these in mind, it can be shown that all the conditions (20) – (22) will be satisfied for all $\alpha\beta \in [0, 1/2]$, if the maximum value of η is $1/\sqrt{3}$.

As mentioned in footnote 6, $\Lambda \equiv \text{Im} \left\{ m_0^* \widetilde{m}_0 \langle M_0 | \widetilde{M}_0 \rangle \right\} = \sqrt{(1 - \eta^2)/4} \times \text{Im} \left\{ \langle M_0 | \widetilde{M}_0 \rangle \right\} = \sqrt{(1 - \eta^2)/4} \lambda$, where $\lambda = \text{Im} \left\{ \langle M_0 | \widetilde{M}_0 \rangle \right\}$, so $\lambda \in [-1, 1]$. In the general situation, for an arbitrary (universal) disentangling machine (*i.e.*, for arbitrary value of $\lambda \in [-1, 1]$), we have to test whether the maximum value η_{max} of the reduction factor can be made greater than $1/\sqrt{3}$. Now we note that the conditions given in (14) to (16) are non-linear in λ^2 , $\alpha\beta$ and η , making it very difficult to get η_{max} analytically from these conditions. And so we proceed numerically. As we are concerned with universal disentangling machines (different machines correspond to different values of λ), therefore, we have to find out the maximum value $\eta_{max}(\lambda^2)$ among all possible values $\eta(\lambda^2)$ of the reduction factor for which all the states (*i.e.*, for all the values of $\alpha\beta \in [0, 1/2]$) will be disentangled by a disentangling machine corresponding to the given value of λ , so that all the conditions (14) – (16) are satisfied. Note that, our required η_{max} is the maximum value of all these $\eta_{max}(\lambda^2)$'s. From our numerical results, we have plotted $\eta_{max}(\lambda^2)$ against λ^2 , in figure 1, which shows that the maximum value η_{max} of all $\eta_{max}(\lambda^2)$'s occurs at $\lambda = 0$ (*i.e.*, at $\Lambda = 0$), the maximum value being $1/\sqrt{3}$.

4 Asymmetric optimal universal disentangling machine

Here we apply the operations U_x and U_y (given in equations (2) and (3)) separately on the qubits x and y of the state $|\psi\rangle$ (given by equation (1)) respectively. After taking trace over the machine states, and using the unitarity, orthogonality and isotropy conditions (*i.e.*, equations (4), (5), (7) – (10)) for both the parties x and y , the reduced density matrix (of the two parties x, y) becomes:

$$D_{xy} =$$

$$\begin{bmatrix} b_1 - c & -i\alpha\beta\Lambda_x\eta_y & -i\alpha\beta\Lambda_y\eta_x & \alpha\beta\eta_x\eta_y \\ i\alpha\beta\Lambda_x\eta_y & b_2 + c & 0 & i\alpha\beta\Lambda_y\eta_x \\ i\alpha\beta\Lambda_y\eta_x & 0 & b_3 + c & i\alpha\beta\Lambda_x\eta_y \\ \alpha\beta\eta_x\eta_y & -i\alpha\beta\Lambda_y\eta_x & -i\alpha\beta\Lambda_x\eta_y & b_4 - c \end{bmatrix}, \quad (23)$$

where

$$\left. \begin{aligned} b_1 &= \frac{(1-\eta_x)(1-\eta_y)}{4} + \frac{\alpha^2(\eta_x+\eta_y)}{2}, \\ b_2 &= \frac{(1-\eta_x)(1+\eta_y)}{4} + \frac{\alpha^2(\eta_x-\eta_y)}{2}, \\ b_3 &= \frac{(1-\eta_x)(1+\eta_y)}{4} + \frac{\beta^2(\eta_x-\eta_y)}{2}, \\ b_4 &= \frac{(1-\eta_x)(1-\eta_y)}{4} + \frac{\beta^2(\eta_x+\eta_y)}{2}, \\ c &= 2\alpha\beta\Lambda_x\Lambda_y. \end{aligned} \right\} \quad (24)$$

Using Peres – Horodecki theorem [5], [6], [7], we see that D_{xy} will be separable (*i.e.*, $|\psi\rangle$ will be disentangled) if the following three conditions are satisfied:

$$F_1 + 2\alpha\beta \left\{ \eta_x\eta_y\Lambda_x\Lambda_y - \alpha\beta \left(4\Lambda_x^2\Lambda_y^2 + \Lambda_x^2\eta_y^2 + \Lambda_y^2\eta_x^2 \right) \right\} \geq 0, \quad (25)$$

$$F_2 + (\alpha\beta)^2 \left\{ 4\alpha\beta\eta_x\eta_y\Lambda_x\Lambda_y - \left(\Lambda_x^2\eta_y^2 + \Lambda_y^2\eta_x^2 \right) \right\} \geq 0, \quad (26)$$

$$\begin{aligned} & F_3 + \\ & \frac{\alpha^4\beta^4}{2} \left\{ 4\Lambda_x^2\Lambda_y^2 \left(\eta_x^2 + \eta_y^2 + 8\Lambda_x^2\Lambda_y^2 + 2\eta_x^2\eta_y^2 \right) + 16\Lambda_x^2\Lambda_y^2 \left(\Lambda_x^2\eta_y^2 + \Lambda_y^2\eta_x^2 \right) \right\} \\ & + \frac{\alpha^4\beta^4}{2} \left(\Lambda_x^2\eta_y^2 - \Lambda_y^2\eta_x^2 \right) \left\{ 2 \left(\Lambda_x^2\eta_y^2 - \Lambda_y^2\eta_x^2 \right) + \eta_x^2 - \eta_y^2 \right\} \\ & + \frac{\alpha^3\beta^3\Lambda_x\Lambda_y\eta_x\eta_y}{2} \left\{ \left(2 - \eta_x^2 - \eta_y^2 - 16\Lambda_x^2\Lambda_y^2 \right) - 4 \left(\Lambda_x^2\eta_y^2 + \Lambda_y^2\eta_x^2 \right) \right\} \\ & - \frac{\alpha^2\beta^2}{8} \left\{ 4\Lambda_x^2\Lambda_y^2 \left(1 + \eta_x^2 + \eta_y^2 - 3\eta_x^2\eta_y^2 \right) + \left(\Lambda_x^2\eta_y^2 + \Lambda_y^2\eta_x^2 \right) \left(1 - \eta_x^2\eta_y^2 \right) + \left(\Lambda_x^2\eta_y^2 - \Lambda_y^2\eta_x^2 \right) \left(\eta_x^2 - \eta_y^2 \right) \right\} \\ & - \frac{1}{8} \alpha\beta\Lambda_x\Lambda_y\eta_x\eta_y \left(1 - \eta_x^2 \right) \left(1 - \eta_y^2 \right) \geq 0, \quad (27) \end{aligned}$$

where

$$\left. \begin{aligned} F_1 &= b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - \alpha^2\beta^2\eta_x^2\eta_y^2, \\ F_2 &= b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 - \alpha^2\beta^2\eta_x^2\eta_y^2(b_1 + b_4), \\ F_3 &= b_1b_2b_3b_4 - b_1b_4\alpha^2\beta^2\eta_x^2\eta_y^2, \end{aligned} \right\} \quad (28)$$

b_1, b_2, b_3, b_4 being given by equations (24). Here $\Lambda_j = \lambda_j\sqrt{(1-\eta_j^2)/4}$, where $\lambda_j = \text{Im} \left\{ {}_j \langle M_0 | \widetilde{M}_0 \rangle_j \right\}$ for $j = x, y$.

Let us first consider the case when $\Lambda_x = 0$, and $\Lambda_y = 0$. In this case, above three conditions (25) – (27) will be reduced to the following three conditions respectively.

$$F_1 \geq 0, \quad (29)$$

$$F_2 \geq 0, \quad (30)$$

$$F_3 \geq 0, \quad (31)$$

where F_i 's are given in (28). Conditions (29) – (31) will be satisfied for all $\alpha\beta \in [0, 1/2]$ if the reduction factors η_x and η_y satisfy the following relation:

$$\eta_x \eta_y \leq \frac{1}{3}. \quad (32)$$

Thus for given any η_x (η_y) in $(0, 1]$, the maximum value $\eta_{y(max)}$ ($\eta_{x(max)}$) of η_y (η_x) is $1/3\eta_x$ ($1/3\eta_y$).

Next we consider the general situation. Since we are looking for universal disentangling machine(s), thus, for given any value of the pair (λ_x, λ_y) in $[0, 1] \times [0, 1]$, and for given any value of $\eta_x \in (0, 1]$, each of the three conditions (25) – (27) has to be satisfied for all $\alpha\beta \in [0, 1/2]$, and for all values of $\eta_y \in (0, \eta_{y(max)}]$, where $0 \leq \eta_{y(max)} \leq 1$. Now we see that (i) satisfaction of the condition (25) (universally) implies satisfaction of condition (29) (universally) (as the term other than F_1 on the left hand side of (25) becomes non-positive for $\alpha\beta = 1/2$), (ii) satisfaction of the condition (26) (universally) implies satisfaction of condition (30) (universally) (as the term other than F_2 on the left hand side of (26) becomes non-positive for $\alpha\beta = 1/2$). But it can be shown that one cannot find even a single value of $\alpha\beta \in (0, 1/2]$ for which the term on the left hand side of (27), other than F_3 , always remains non-positive for every choice of $\lambda_x, \lambda_y \in [0, 1]$, and for every choice of $\eta_x, \eta_y \in [0, 1]$.⁸ So we have to take into account the three conditions (25) – (27) all together. We have obtained numerically that for given any value of the pair (λ_x, λ_y) in $[0, 1] \times [0, 1]$, and for given any value of $\eta_x \in (0, 1]$, the maximum value $\eta_{y(max)}$ (say) of $\eta_y \in (0, 1]$, for which all the three conditions (25) – (27) are satisfied for all $\alpha\beta \in [0, 1/2]$, satisfies the relation $\eta_{y(max)} \leq 1/3\eta_x$ (see figure 2 for a comparison). So the optimal disentangling machine, in the asymmetric case, will be obtained from the

⁸We have verified this numerically.

case where $\Lambda_x = 0$ and $\Lambda_y = 0$, and so the optimal curve to be satisfied by the reduction factors η_x and η_y , is given by the following rectangular hyperbola.

$$\eta_x \eta_y = \frac{1}{3}. \quad (33)$$

It is clear from condition (32) that (i) the maximum value of η_y , in the totally asymmetric case (*i.e.*, when $\eta_x = 1$) is $1/3$ (as shown in section 2), and (ii) the maximum value of η , in the symmetric case (*i.e.*, when $\eta_x = \eta_y \equiv \eta$) is $1/\sqrt{3}$ (as shown in section 3).

Now we come to the question of disentanglement (by using local operations only) of an arbitrary mixed state ρ of the two qubits x and y . In its spectral representation, ρ takes the following form:

$$\rho = \sum_i \mu_i P[|\psi_i\rangle], \quad (34)$$

where $\mu_i \geq 0$, $\sum_i \mu_i = 1$, and $P[|\psi_i\rangle]$ is the projection operator on the (normalized) pure state $|\psi_i\rangle$ of the two qubits x and y . One can express $|\psi_i\rangle$, in its Schmidt form, as

$$|\psi_i\rangle = a_i |0_i 0_i\rangle_{xy} + b_i |1_i 1_i\rangle_{xy}, \quad (35)$$

where a_i, b_i are non-negative numbers with $a_i^2 + b_i^2 = 1$, and $|0_i\rangle_j, |1_i\rangle_j$ are two orthonormal states in the Hilbert space of the qubit j (for $j = x, y$). As discussed earlier, each of the states $|\psi_i\rangle$ will be disentangled by using local operations, with some common values of the reduction factors η_x and η_y . Let $\rho_{j(\psi_i)}^{bd}$ and ρ_j^{bd} be the single particle reduced density matrices of $|\psi_i\rangle$ and ρ respectively, corresponding to the qubit j ($j = x, y$), before disentanglement; and let $\rho_{j(\psi_i)}^{ad}$ and ρ_j^{ad} be the single particle reduced density matrices of $|\psi_i\rangle$ and ρ respectively, corresponding to the qubit j ($j = x, y$), after disentanglement. If η'_j is the reduction factor associated with the disentanglement of ρ , corresponding to the qubit j ($j = x, y$), we then have

$$\rho_j^{ad} \equiv \eta'_j \rho_j^{bd} + \frac{1 - \eta'_j}{2} I = \eta'_j \sum_i \mu_i \rho_{j(\psi_i)}^{bd} + \frac{1 - \eta'_j}{2} I, \quad (36)$$

or

$$\sum_i \mu_i \rho_{j(\psi_i)}^{ad} \equiv \sum_i \mu_i \left\{ \eta_j \rho_{j(\psi_i)}^{bd} + \frac{1 - \eta_j}{2} I \right\} = \eta'_j \sum_i \mu_i \rho_{j(\psi_i)}^{bd} + \frac{1 - \eta'_j}{2} I. \quad (37)$$

Thus we get from equation (37) that $\eta'_j = \eta_j$ for $j = x, y$.

5 Discussion

Since an ideal universal disentangling machine does not exist, we have explored how well a universal disentangling machine can be, using most general type of local operations. We have shown that in the case of optimal universal disentangling machines, reduction factors lie on a rectangular hyperbola. Here one should mention that in the case of optimal universal cloning machines, the relation between the reduction factors follows directly from no-signalling phenomenon [11]. So one may look for the physical phenomenon (or phenomena), from which above-mentioned rectangular hyperbola would follow directly. This is still an unresolved problem.

We now address the issue of obtaining a better disentangling machine, if possible, using non-local operations.

For all (non-unitary) local operations, entanglement decreases, whereas, for non-local operations, it may increase, decrease, or remain same. In the disentanglement process described here, a separable state remains separable without any further constraints (other than isotropy condition) on the machine, but even in order to keep a separable state separable under non-local operations, some constraints other than isotropy condition have to be imposed on the machine, which seems to decrease the reduction factor. But if one redefine the notion of disentanglement as a process which also keeps every pure bipartite product state, a product of two (single particle) density matrices of the two particles, then the allowed class of disentangling machines will comprise of only local operations.

In this regard, for intuitive understanding, we point out that local cloning machine, with the blank copy, functions as universal disentangling machine which can be made optimal. Now if we use optimal local cloning machine, which produces three copies instead of two, then the cloning machine along with the two blank copies acts as a universal disentangling machine with reduction factor being $5/9$ [12]. A non-local cloning machine [10] along with six blank copies, which produces seven copies of the bipartite states, acts as universal disentangling machine with reduction factor being $11/35$ [12], which is much less than the former case. Recently Mor and Terno [13] obtained a set of bipartite entangled states, which can be perfectly disentangled, provided the reduced density matrices of one of the parties commute. But this is achieved using local operation, namely local broadcasting.

In conclusion, we have obtained the optimal disentangling machine exploiting the most

general local operations, and discussed that there can not be any non-local operation which may give a better one.

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8. Here we have taken the identical values $m_{0x} = m_{0y}$ as m_0 , which can be taken as the real quantity $\sqrt{(1 + \eta)/2}$ (see equation (10)); similarly we have taken the identical values $\widetilde{m}_{0x} = \widetilde{m}_{0y}$ as \widetilde{m}_0 , which can be taken as the real quantity $\sqrt{(1 - \eta)/2}$ (see equation (10)). Also we denote here the identical states $|M_0\rangle_x = |M_0\rangle_y$ as $|M_0\rangle$, and the identical states $|\widetilde{M}_0\rangle_x = |\widetilde{M}_0\rangle_y$ as $|\widetilde{M}_0\rangle$. Thus $\Lambda \equiv \text{Im} \{m_0^* \widetilde{m}_0 \langle M_0 | \widetilde{M}_0 \rangle\} = 0$ implies that $\text{Im} \{\langle M_0 | \widetilde{M}_0 \rangle\} = 0$, as our requirement is to obtain universal disentangling machines, and it is known that such machines may exist provided η is less than 1 [1].
9. In this case, the reduced density matrices ρ_j^{bd} and ρ_j^{ad} (of the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle_{xy} + |11\rangle_{xy})$, before and after disentanglement respectively) are both equal to $(1/2)I$, I being the 2×2 identity matrix. So there is no direct effect of the reduction factor η on the disentanglement of $\frac{1}{\sqrt{2}}(|00\rangle_{xy} + |11\rangle_{xy})$. But we try to see here how the universal disentangling machines work on this maximally entangled state.
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FIGURE CAPTION : Figure 1 describes the decreasing nature of $\eta_{max}(\lambda^2)$ with the increment of λ^2 . This graph shows that the maximum value η_{max} of $\eta_{max}(\lambda^2)$ occurs at $\lambda^2 = 0$, and (so) η_{max} is equal to $1/\sqrt{3}$.

FIGURE CAPTION : Figure 2 shows the optimal curve $\eta_x \eta_y = 1/3$, which corresponds to the case $\lambda_x = 0, \lambda_y = 0$ (represented in the figure by the continuous line). Also, for a comparison, the optimal curves corresponding to the cases $(\lambda_x, \lambda_y) = (0.2, -0.2)$ (represented in the figure by the broken line of the type ‘- - -’), $(\lambda_x, \lambda_y) = (0.5, -0.5)$ (represented in the figure by the broken line of the type ‘- - -’), $(\lambda_x, \lambda_y) = (0.9, 0.1)$ (represented in the figure by the broken line of the type ‘- - - - -’) are given here.

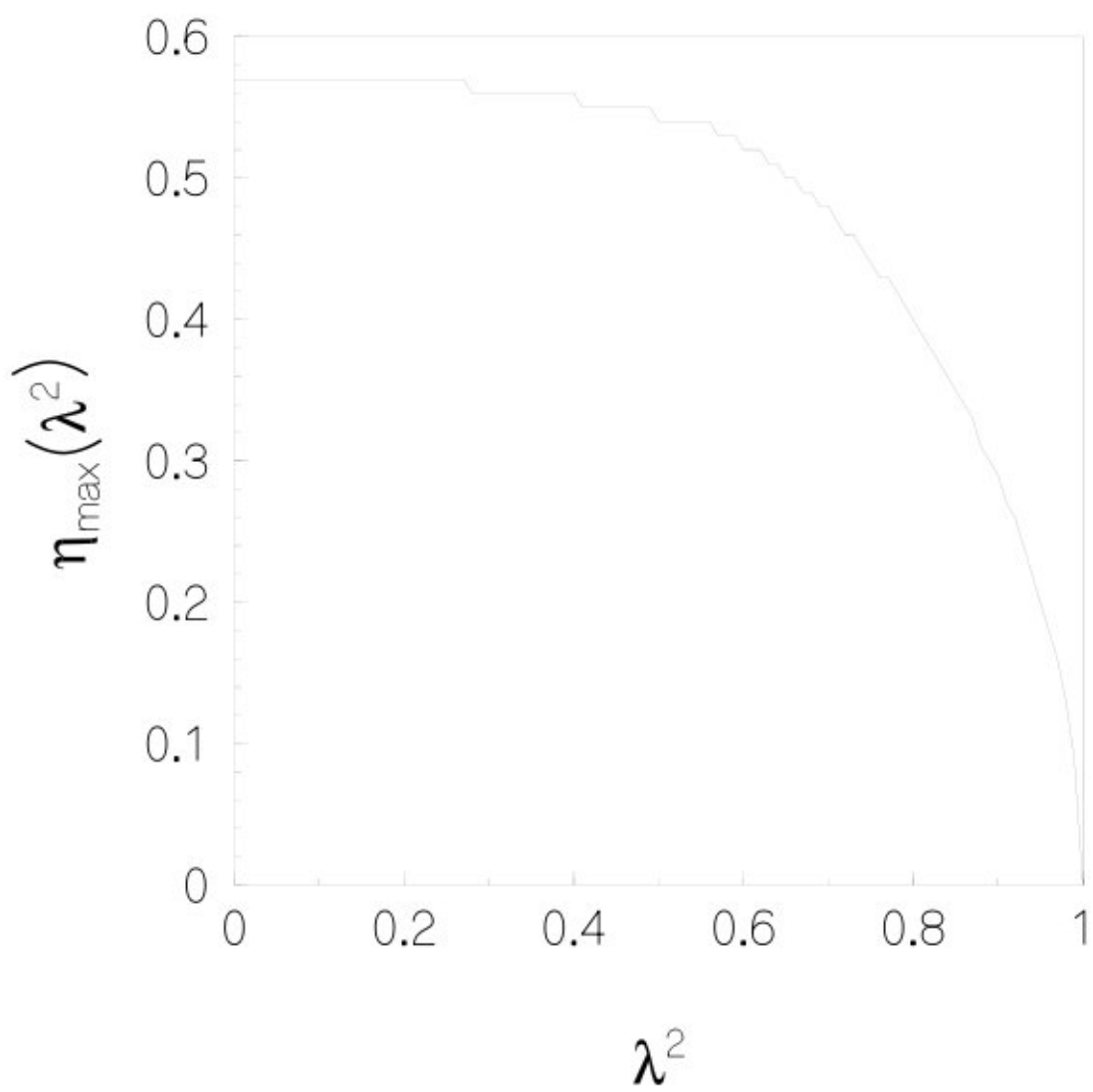


Figure 1

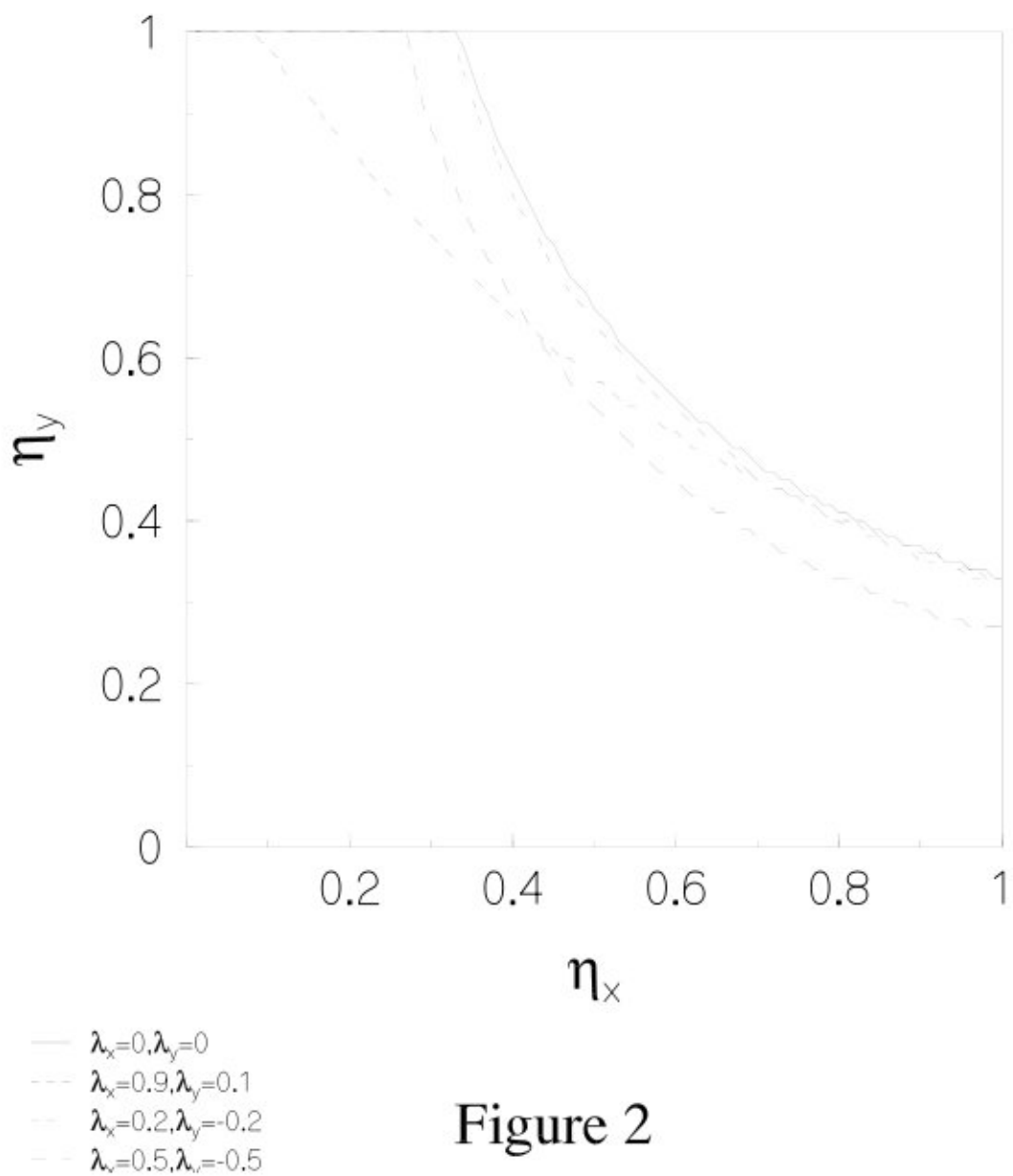


Figure 2