# VERSAL DEFORMATIONS OF LEIBNIZ ALGEBRAS

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*To My Parents*

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# Contents

# Chapter 0

# Introduction

The main objective of this thesis is to develop an algebraic deformation theory, over commutative local algebra base, for Leibniz algebras and its homomorphisms and to give a concrete construction of a formal deformation which induces all other formal deformations of a given Leibniz algebra satisfying some cohomological condition, which is unique at the infinitesimal level − the so called "Versal deformation".

Deformation theory dates back at least to Riemann's 1857 memoir on abelian functions in which he studied manifolds of complex dimension one and calculated the number of parameters (called moduli) upon which a deformation depends. The modern theory of deformations of structures on manifolds was developed extensively by Frölicher-Nijenhuis [FN57], Kodaira-Spencer [KS58b], [KS58a], Kodaira-Nirenberg-Spencer [KNS58], and Spencer [Spe62a], [Spe62b], [Spe65].

The study of deformations of algebraic structures was initiated by M. Gerstenhaber through his monumental works [Ger63], [Ger64], [Ger66], [Ger68], [Ger74]. He introduced deformation theory for associative algebras and remarked that his methods would extend to any equationally defined algebraic structure. The basic theorems and features of algebraic deformation theory are all due to him. For a comparative study of algebraic and analytic deformation theory see [Pip67].

The theory of Gerstenhaber was extended to Lie algebras by A. Nijenhuis and R. W. Richardson, Jr. [NR66], [NR67a], [NR67b]. The deformation theory of Hopf algebras, which relates to quantum groups, was studied by M. Gerstenhaber and S. D. Schack in [GS90]. An algebraic deformation theory for associative algebra homomorphisms was developed by Gerstenhaber and Schack [GS83, GS85]. For more recent results on deformation theory following Gerstenhaber see [MM02], [Yau06], [Yau07], [Man07], [Yau08].

Gerstenhaber's theory was generalized in [Bal97] by D. Balavoine to develop formal one parameter deformation theory for algebras over any quadratic operad, which includes all the classical cases. He also deduced formal one parameter deformation theory of Leibniz algebras from his theory.

Although formal deformation theory was developed in various categories following Gerstenhaber and computations were made, but the question of obtaining all non-equivalent deformations of a given object was not properly discussed for a long time. The right approach to this is the notion of versal deformation − a deformation which includes all non-equivalent ones. The existence of such a versal deformation for algebraic categories follows from the work of M. Schlessinger [Sch68].

For Lie algebras it was worked out in [Fia88] and one can deduce it in other categories as well. It turns out that (under some minor cohomology restrictions) there exists a versal element, which is universal at the infinitesimal level. For Lie algebras an explicit construction of versal deformations was given in [FF99]. The construction is parallel to the general construction in deformation theory as in [Ill71, Pal76, Lau79, GM88, Kon94].

In this thesis we give a concrete construction of versal deformation for Leibniz algebras [FMM08]. Following is a chapter-wise break-up of the thesis.

The notion of Leibniz algebras was introduced by J.-L. Loday [Lod93, Lod97, Lod01] in connection with the study of periodicity phenomenon in algebraic $K$-theory, as a non-antisymmetric analogue of Lie algebras. We recall that for a Lie algebra $\mathfrak{g}$ the Chevalley-Eilenberg complex is given by the exterior power module $\Lambda\mathfrak{g}$. The non-commutative analogue of the exterior module is the tensor module $T\mathfrak{g}$. If we replace $\wedge$ by $\otimes$ in the formula for the boundary map $d$ of the Chevalley-Eilenberg complex and put $[x_i, x_j]$ at the $i^{\text{th}}$ slot when $i < j$, we get a new complex for $\mathfrak{g}$, as the modified $d$ satisfies $d^2 = 0$. It turns out that this new complex is valid for more general objects than Lie algebras, as the only property of the Lie bracket, which is needed to prove $d^2 = 0$ is the Leibniz relation. This generalization of Lie algebras are called Leibniz algebras. Leibniz algebras turns out to be the algebras over the quadratic operad Leib [Lod01]. A (co)homology theory associated to Leibniz algebras has been developed by J.-L. Loday and T. Pirashvili [LP93]. Throughout this thesis, $\mathbb{K}$ will denote the ground field and the tensor product over $\mathbb{K}$ will be denoted by $\otimes$. In Chapter 1, we recall the definition of Leibniz algebras, discuss some examples. We also recall the definition of Leibniz algebra cohomology with coefficients in itself. Low dimensional Leibniz algebra cohomologies will be extensively used in this thesis to develop the deformation theory in question. We introduce cohomology modules associated to a Leibniz algebra homomorphism, which will be relevant in the discussion of deformation of Leibniz algebra homomorphisms.

Chapter 2, is a review of results about Harrison cohomology of a commutative algebra and its relation to extensions of the algebra. These results will be used subsequently. The basic references for this chapter are [Hoc45, Har62, Bar68]. We recall the definition of the Harrison complex of a commutative algebra $A$ and the Harrison cohomology with coefficients in an $A$-module $M$. Next, we define extension of a commutative algebra $A$ by an $A$-module $M$, describe its relation to Harrison cohomology of $A$. We also recall few basic properties that will be used later in this thesis.

In Chapter 3, we introduce the notion of deformations of Leibniz algebras and Leibniz algebra homomorphisms over a commutative local algebra base with multiplicative identity, and introduce infinitesimal deformation and other basic definitions related to deformations of a Leibniz algebra. We give a construction of an infinitesimal deformation $\eta_1$ of a Leibniz algebra $L$ for which $dim(HL^2(L;L))$ is finite. We show that this infinitesimal deformation is universal among the infinitesimal deformations of $L$ with finite dimensional local algebra base. We also prove a necessary and sufficient criterion for equivalence of two infinitesimal deformations of a Leibniz algebra. At the end we introduce the notion of infinitesimal deformations of Leibniz algebra homomorphisms and obtain a necessary and sufficient condition for equivalence of two infinitesimal deformations in this case.

In Chapter 4, we address the question of extending a given deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with a given base to a larger base. This extension problem can be described as follows. Suppose $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ is a given deformation of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with local base $A$. Let

$$0 \longrightarrow M_0 \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be a given finite dimensional extension of $A$ by $M_0$. The problem is to obtain condition for existence of a deformation $\tilde{\mathfrak{D}}$ of $f$ with base $B$ which extends the given deformation, that is, $p_*\tilde{\mathfrak{D}} = \mathfrak{D}$. We shall measure the possible obstructions that one might encounter in the above extension process as certain 3-dimensional cohomology classes, vanishing of which is a necessary and sufficient condition for an extension to exist. The set of equivalence classes of possible extensions of a given deformation $\lambda$ of $L$ with base $A$, admits certain natural actions and we shall investigate their relationship. We first take up the case of extending deformations of Leibniz algebras and then consider the relative problem of extending deformations of Leibniz algebra homomorphisms. In the last section of this chapter, we study formal deformations and obtain a necessary condition for non-triviality of a formal deformation. The results of this chapter will also enable us to obtain a sufficient criterion for existence of a formal deformation with a given differential and infinitesimal part. We end this chapter with the definition of a versal deformation.

Chapter 5 is devoted to give a construction of versal deformation of a given Leibniz algebra $L$ with $dim(HL^2(L;L)) < \infty$. We begin with the universal infinitesimal deformation $\eta_1$ of $L$ with base $C_1$ as constructed in Chapter 3, and apply the tools developed in Chapter 4 to get a finite dimensional extension $\eta_2$ with base $C_2$. We kill off the possible obstruction associated to the extension problem for a specific extension of $C_1$ to obtain $\eta_2$ with base $C_2$. We repeat this procedure successively to get a sequence of finite dimensional extensions $\eta_k$ with base $C_k$. The projective limit $C = \varprojlim_{k\to\infty} C_k$ is a complete local algebra and $\eta = \varprojlim_{k\to\infty} \eta_k$ is a formal deformation of $L$ with base $C$. We show that the algebra base $C$ can be described as a quotient of the formal power series ring over $\mathbb{K}$ in finitely many variables. Finally we prove that the formal deformation $\eta$ is a versal deformation of $L$ with base $C$.

It is well known that the construction of one parameter deformations of various algebraic structures, like associative algebras or Lie algebras, involves certain conditions on cohomology classes arising as obstructions. These condition are expressed in terms of Massey brackets [Ret77, Ret93], which are, in turn, the Lie counterpart of classical Massey products [Mas54]. The connection between obstructions in extending a given deformation and Massey products was first noticed in [Dou61]. The aim of Chapter 6, is to study this relationship in our context. More precisely, we use Massey $n$-operations as defined in [Ret77] to establish this connection in the case of one parameter deformation of a Leibniz algebra. Next, we use a general treatment of Massey brackets as introduced in [FW01] to express the obstructions arising at different steps in the inductive construction of a versal deformation as described in Chapter 5, in terms of these general Massey brackets.

Finally, in Chapter 7, we discuss two examples to illustrate the theory developed in this thesis. The first example is a three dimensional nilpotent Leibniz algebra for which we compute a versal deformation. The next example is the three dimensional Heisenberg Lie algebra. We deform this example viewing it as a Leibniz algebra to show that not only we recover all the usual Lie algebra deformations, but we get some new deformations which are Leibniz algebras and not Lie algebras. This example also illustrate the fact that versal deformation of a Lie algebra $L$ and that of $L$ when viewed as a Leibniz algebra may differ.

# Chapter 1

# Leibniz algebras and Leibniz algebra cohomology

## 1.1 Introduction

The notion of Leibniz algebras was introduced by J.-L. Loday [Lod93, Lod97, Lod01] in connection with the study of periodicity phenomenon in algebraic $K$-theory as a non-antisymmetric analogue of Lie algebras. We recall that for a Lie algebra $\mathfrak{g}$ the Chevalley-Eilenberg complex is given by the exterior power module $\Lambda\mathfrak{g}$. The non-commutative analogue of the exterior module is the tensor module $T\mathfrak{g}$. If we replace $\wedge$ by $\otimes$ in the formula for the boundary map $d$ of the Chevalley-Eilenberg complex and put $[x_i, x_j]$ at the $i^{\text{th}}$ slot when $i < j$, we get a new complex for $\mathfrak{g}$, as the modified $d$ satisfies $d^2 = 0$. It turns out that this new complex is valid for more general objects than Lie algebras, as the only property of the Lie bracket which is needed to prove $d^2 = 0$ is the Leibniz relation. This generalization of Lie algebras are called Leibniz algebras. Leibniz algebras turns out to be the algebras over the quadratic operad Leib [Lod01]. A (co)homology theory associated to Leibniz algebras has been developed by Loday and Pirashvili [LP93].

In this chapter we recall the definition of Leibniz algebras, discuss some examples. We also recall the definition of Leibniz algebra cohomology. In the last section of this chapter we introduce cohomology modules associated to a Leibniz algebra homomorphism. This will be used in subsequent chapters.

## 1.2 Leibniz algebras

Throughout this thesis, $\mathbb{K}$ will denote the ground field, and the tensor product over $\mathbb{K}$ will be denoted by $\otimes$.

**Definition 1.2.1.** *A Leibniz algebra $L$ is a $\mathbb{K}$-module, equipped with a bracket operation $[-,-]$, which is $\mathbb{K}$-bilinear and satisfies the Leibniz identity:*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] \quad \textit{for } x,\ y,\ z \in L.$$

Any Lie algebra is automatically a Leibniz algebra, as in the presence of antisymmetry, the Jacobi identity is equivalent to the Leibniz identity. Here are some more examples.

**Example 1.2.2.** *Let $(L,d)$ be a differential Lie algebra with the Lie bracket $[-,-]$. Then $L$ is a Leibniz algebra with the bracket operation $[x,y]_d := [x,dy]$. The new bracket on $L$ is called the derived bracket.*

**Example 1.2.3.** *On $\bar{T}(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$ there is a unique bracket that makes it into a Leibniz algebra and verifies*

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n = [\cdots[[v_1,v_2],v_3],\cdots,v_n] \textit{ for } v_i \in V \textit{ and } i = 1,\cdots,n.$$

*This is the free Leibniz algebra over a $\mathbb{K}$-module $V$.*

**Example 1.2.4.** *Let $A$ be any associative $\mathbb{K}$-algebra equipped with a $\mathbb{K}$-module map $D : A \longrightarrow A$ satisfying $D(x(Dy)) = (Dx)(Dy) = D((Dx)y)$ for $x,y \in A$. Then*

$$[x,y] := x(Dy) - (Dy)x$$

*is a Leibniz bracket on $A$. Some examples of $D$ satisfying the above identity are as follows:*

1. *$D$ is an algebra map satisfying $D^2 = D$,*
2. *$A$ is a superalgebra (that is, any $x \in A$ can be written uniquely as $x = x_+ + x_-$) and $D(x) = x_+$,*
3. *$D$ is a square-zero derivation, that is,*

$$D(xy) = (Dx)y + x(Dy) \textit{ and } D^2x = 0 \quad \textit{for } x,\ y \in A.$$

**Definition 1.2.5.** *For a Leibniz algebra $L$, set $L^1 = L, L^{k+1} = [L^k, L]$ is the submodule of $L$ generated by all elements of the form $[x,y]$ where $x \in L^k$ and $y \in L$, for $k \in \mathbb{N}$. Then $L$ is said to be nilpotent if there exists an integer $n \in \mathbb{N}$ such that*

$$L^1 \supset L^2 \supset \ldots \supset L^n = 0.$$

*The smallest integer $n$ for which $L^n = 0$ is called the nilindex of $L$.*

**Example 1.2.6.** *Complex nilpotent Leibniz algebras have been classified up to isomorphism for dimension* 2 *and* 3 *in [Lod93] and [AO01]. In dimension* 2 *there are two non-isomorphic nilpotent Leibniz algebras. One of them is abelian, the other is given by the non-zero Leibniz bracket,* $[e_1, e_1] = e_2$*, where* $\{e_1, e_2\}$ *is a basis of the Leibniz algebra.*

*In dimension* 3 *there are five non-isomorphic nilpotent Leibniz algebras and one infinite family of pairwise non-isomorphic Leibniz algebras. They can be described as follows. Let* $\{e_1, e_2, e_3\}$ *be a basis. In the following list we only mention the non-zero brackets of basis elements.*

$\lambda_1$ : *abelian.*

$\lambda_2$ : $[e_1, e_1] = e_2$.

$\lambda_3$ : $[e_2, e_3] = e_1, [e_3, e_2] = -e_1$.

$\lambda_4$ : $[e_2, e_2] = e_1, [e_3, e_3] = \alpha e_1, [e_2, e_3] = e_1; \alpha \in \mathbb{C}$.

$\lambda_5$ : $[e_2, e_2] = e_1, [e_3, e_2] = e_1, [e_2, e_3] = e_1$.

$\lambda_6$ : $[e_3, e_3] = e_1, [e_1, e_3] = e_2$.

Here is one geometric example [IdLMP99].
Recall that a Nambu-Poisson manifold $M$ of order $n$ is a differential manifold endowed with skew-symmetric $n$-bracket of functions $\{-, \cdots, -\}$ satisfying the Leibniz rule

$$\{f_1 g_1, f_2, \cdots, f_n\} = f_1\{g_1, f_2, \cdots, f_n\} + g_1\{f_1, f_2, \cdots, f_n\}$$

for $f_1, \cdots, f_n, g_1 \in C^\infty(M, \mathbb{R})$, and the fundamental identity

$$\{f_1, \cdots, f_{n-1}, \{g_1, \cdots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \cdots, \{f_1, \cdots, f_{n-1}, g_i\}, \cdots, g_n\}.$$

**Example 1.2.7.** *Let* $(M, \Lambda)$ *be a Nambu-Poisson manifold of order* $n$ *with Nambu-Poisson bracket* $\{-, \cdots, -\}$ *(where* $\Lambda$ *is the associated skew-symmetric tensor of type* $(n, 0)$*,*

$$\Lambda(df_1, \cdots, df_n) = \{f_1, \cdots, f_n\} \text{ for } f_1, \cdots, f_n \in C^\infty(M; \mathbb{R})).$$

*Then* $(\wedge^{n-1}(C^\infty(M; \mathbb{R})), \{-, -\}')$ *is a Leibniz algebra where*

$$\{f_1 \wedge \cdots \wedge f_{n-1}, g_1 \wedge \cdots \wedge g_{n-1}\}' = \sum_{i=1}^{n-1} g_1 \wedge \cdots \wedge \{f_1, \cdots, f_{n-1}, g_i\} \wedge \cdots \wedge g_{n-1}.$$

## 1.3 Cohomology of Leibniz algebras

In this section we recall the Leibniz algebra cohomology with coefficients in a representation as introduced by J.-L. Loday and T. Pirashvili in [LP93].

**Definition 1.3.1.** *Let $L$ be a Leibniz algebra over $\mathbb{K}$, a representation $M$ of the Leibniz algebra $L$ is a $\mathbb{K}$-module equipped with two actions (left and right) of $L$,*

$$[-,-] : L \times M \longrightarrow M \quad \text{and } [-,-] : M \times L \longrightarrow M \quad \text{such that}$$

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

*holds, whenever one of the variable is from $M$ and the others from $L$.*

In particular, $L$ is a representation of itself with the action given by the bracket in $L$. Often, we will represent an element $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in L^{\otimes n}$ by $(x_1, \cdots, x_n)$.

**Definition 1.3.2.** *Let $L$ be a Leibniz algebra and $M$ be a representation of $L$. Let $CL^n(L;M) := Hom_{\mathbb{K}}(L^{\otimes n}, M),\ n \geq 0$, and*

$$\delta^n : CL^n(L;M) \longrightarrow CL^{n+1}(L;M)$$

*be the $\mathbb{K}$-homomorphism given by*

$$\begin{aligned} &\delta^n f(x_1, \cdots, x_{n+1}) \\ &:= [x_1, f(x_2, \cdots, x_{n+1})] + \sum_{i=2}^{n+1}(-1)^i [f(x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}), x_i] \\ &\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(x_1, \cdots, x_{i-1}, [x_i, x_j], x_{i+1}, \cdots, x_{j-1}, \hat{x}_j, x_{j+1}, \cdots, x_{n+1}). \end{aligned} \tag{1.3.1}$$

The linear maps $\delta^n$, $n \geq 0$ satisfy $\delta^{n+1} \circ \delta^n = 0$.

Let $f \in CL^n(L;M)$ and $x_1, \cdots, x_{n+2} \in L$. Fix $i,j$ with $1 \leq i < j \leq n+2$ and consider the element $(x_1, \cdots, x_{i-1}, [x_i, x_j], x_{i+1}, \cdots, \hat{x}_j, \cdots, x_{n+2}) \in L^{\otimes(n+1)}$. We may denote this element as $(y_1, \cdots, y_{n+1})$ where $y_k = x_k$ for $k = 1, \cdots, i-1, i+1, \cdots, j-1$, $y_i = [x_i, x_j]$ and $y_k = x_{k+1}$ for $k = j, \cdots, n+1$.

Define

$$F_{ij}(y_1, \cdots, y_{n+1}) = \sum_{1 \leq u < v \leq n+1} (-1)^{v+1} f(y_1, \cdots, y_{u-1}, [y_u, y_v], \ldots, \hat{y}_v, \cdots, y_{n+1}).$$

Using this notations we get the following Lemma.

**Lemma 1.3.3.**

$$X = \sum_{1\le i<j\le n+2} (-1)^{j+1} F_{ij}(y_1, \cdots, y_{n+1}) = 0.$$

*Proof.* By expanding for all possible values of $1 \le i, j, u, v \le n+2$ we get,

$$\begin{aligned}
X &= \sum_{1\le i<j\le n+2} (-1)^{j+1} F_{ij}(y_1, \cdots, y_{n+1}) \\
&= \sum_{1\le i<j\le n+2} (-1)^{j+1} F_{ij}(x_1, \cdots, x_{i-1}, [x_i, x_j], x_{i+1}, \cdots, \hat{x}_j, \cdots, x_{n+2}) \\
&= \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{1\le u<v\le i-1} (-1)^{v+1} f(x_1, \cdots, x_{u-1}, [x_u, x_v], x_{u+1}, \cdots, \hat{x}_v, \cdots, [x_i, x_j], \\
&\qquad \cdots, \hat{x}_j, \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{u<v; 1\le u\le i-1, v=i} (-1)^{v+1} f(x_1, \cdots, x_{u-1}, [x_u, [x_i, x_j]], x_{u+1}, \cdots, \\
&\qquad x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, \hat{x}_j, \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{u<v; 1\le u\le i-1, i<v<j} (-1)^{v+1} f(x_1, \cdots, x_{u-1}, [x_u, x_v], x_{u+1}, \cdots, \\
&\qquad [x_i, x_j], \cdots, \hat{x}_v, \cdots, \hat{x}_j, \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{u<v; 1\le u\le i-1, j<v} (-1)^{v} f(x_1, \cdots, x_{u-1}, [x_u, x_v], x_{u+1}, \cdots, [x_i, x_j], \cdots, \\
&\qquad \hat{x}_j, \cdots, \hat{x}_v, \cdots, x_{n+2}) \cdots, \hat{x}_j, \cdots, \hat{x}_v, \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{u<v;\ u=i, v<j} (-1)^{v+1} f(x_1, \cdots, x_{i-1}, [[x_i, x_j], x_v], x_{i+1}, \cdots, \hat{x}_v, \cdots, \\
&\qquad \hat{x}_j, \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{u<v;\ u=i, j<v} (-1)^{v} f(x_1, \cdots, x_{i-1}, [[x_i, x_j], x_v], x_{i+1}, \cdots, \hat{x}_j, \cdots, \hat{x}_v, \\
&\qquad \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{i<u<v<j} (-1)^{v+1} f(x_1, \cdots, [x_i, x_j], \cdots, x_{u-1}, [x_u, x_v], x_{u+1}, \cdots, \hat{x}_v, \\
&\qquad \cdots, \hat{x}_j, \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{i<u<j<v} (-1)^{v} f(x_1, \cdots, [x_i, x_j], \cdots, x_{u-1}, [x_u, x_v], x_{u+1}, \cdots, \hat{x}_j, \cdots, \\
&\qquad \hat{x}_v, \cdots, x_{n+2}) \\
&\quad + \sum_{1\le i<j\le n+2} (-1)^{j+1} \sum_{j<u<v} (-1)^{v} f(x_1, \cdots, [x_i, x_j], \cdots, \hat{x}_j, \cdots, x_{u-1}, [x_u, x_v], x_{u+1}, \cdots, \\
&\qquad \hat{x}_v, \cdots, x_{n+2}).
\end{aligned}$$

Notice that each term in the 1st sum on the right-hand side of the above expression appears in the 9th sum with the opposite sign. Similarly the 3rd sum and the 4th sum on the right-hand side of the above expression appear in the 8th and 7th sum respectively with opposite sign. Thus by cancelling out these terms we get

$$\begin{aligned} X = &\sum_{1\leq i<j\leq n+2}(-1)^{j+1}\sum_{u<v;1\leq u\leq i-1,v=i}(-1)^{v+1}f(x_1,\cdots,x_{u-1},[x_u,[x_v,x_j]],x_{u+1},\cdots,x_{i-1},\hat{x}_v,\\ &\qquad x_{v+1},\cdots,\hat{x}_j,\cdots,x_{n+2})\\ &+\sum_{1\leq i<j\leq n+2}(-1)^{j+1}\sum_{u<v;\ u=i,v<j}(-1)^{v+1}f(x_1,\cdots,x_{u-1},[[x_u,x_j],x_v],x_{u+1},\cdots,\hat{x}_v,\cdots,\\ &\qquad \hat{x}_j,\cdots,x_{n+2})\\ &+\sum_{1\leq i<j\leq n+2}(-1)^{j+1}\sum_{u<v;\ u=i,j<v}(-1)^{v}f(x_1,\cdots,x_{u-1},[[x_u,x_j],x_v],x_{u+1},\cdots,\hat{x}_j,\cdots,\\ &\qquad \hat{x}_v,\cdots,x_{n+2}). \end{aligned}$$

If we interchange $j$ and $v$ in the 3rd sum of the right-hand side of the above expression of $X$, then using the identity $[x_u[x_v,x_j]]-[[x_u,x_v],x_j]+[[x_u,x_j],x_v]=0$ for $x_u,x_v,x_j\in L$; $1\leq u,v,j\leq n+2$ we get $X=0$. □

**Proposition 1.3.4.** *Let $L$ be a Leibniz algebra over $\mathbb{K}$ and $M$ be a representation of the Leibniz algebra $L$. Then $(CL^*(L;M),\delta)$ is a cochain complex.*

*Proof.* It is enough to show that $\delta^{n+1}\circ\delta^n f=0$ for $f\in CL^n(L;M)$ and $n\geq 0$. Let $x_1,\cdots,x_{n+2}\in L$, then by the definition of $\delta^n$ in (1.3.1) we get,

$$\begin{aligned} &\delta^{n+1}\circ\delta^n f(x_1,x_2,\cdots,x_{n+2})\\ =&[x_1,\delta^n f(x_2,\cdots,x_{n+2})]+\sum_{i=2}^{n+2}(-1)^i[\delta^n f(x_1,\cdots,\hat{x}_i,\cdots,x_{n+2}),x_i]\\ &+\sum_{1\leq i<j\leq n+2}(-1)^{j+1}\delta^n f(x_1,\cdots,x_{i-1},[x_i,x_j],x_{i+1},\cdots,x_{j-1},\hat{x}_j,x_{j+1},\cdots,x_{n+2}). \end{aligned} \tag{1.3.2}$$

Now the first term on the right-hand side of (1.3.2) can be expanded by substituting the expression of $\delta^n f(x_2,\cdots,x_{n+2})$ as follows.

$$\begin{aligned} &[x_1,\delta^n f(x_2,\cdots,x_{n+2})]\\ =&[x_1,[x_2,f(x_3,\cdots,x_{n+2})]]+\sum_{i=3}^{n+2}(-1)^{i-1}[x_1,[f(x_2,\cdots,\hat{x}_i,\cdots,x_{n+2}),x_i]] \qquad (1.3.3)\\ &+[x_1,\sum_{2\leq i<j\leq n+2}(-1)^j f(x_2,\cdots,x_{i-1},[x_i,x_j],x_{i+1},\cdots,\hat{x}_j,\cdots,x_{n+2})]. \end{aligned}$$

Similarly, the second term on the right-hand side of (1.3.2) can be expressed as

$$\begin{aligned}
&\sum_{i=2}^{n+2}(-1)^i[\delta^n f(x_1,\cdots,\hat{x}_i,\cdots,x_{n+2}),x_i]\\
=&\sum_{i=2}^{n+2}(-1)^i[[x_1,f(x_2,\cdots,\hat{x}_i,\cdots,x_{n+2})],x_i]\\
+&\sum_{i=2}^{n+2}(-1)^i\sum_{2\le u<i}(-1)^u[[f(x_1,\cdots,\hat{x}_u,\cdots,\hat{x}_i,\cdots,x_{n+2}),x_u],x_i]\\
+&\sum_{i=2}^{n+2}(-1)^i\sum_{u>i}(-1)^{u-1}[[f(x_1,\cdots,\hat{x}_i,\cdots,\hat{x}_u,\cdots,x_{n+2}),x_u],x_i]\\
+&\sum_{i=2}^{n+2}(-1)^i\sum_{u<v<i}(-1)^{v+1}[f(x_1,\cdots,x_{u-1},[x_u,x_v],\cdots,\hat{x}_v,\cdots,\hat{x}_i,\cdots,x_{n+2}),x_i]\\
+&\sum_{i=2}^{n+2}(-1)^i\sum_{u<i<v}(-1)^v[f(x_1,\cdots,x_{u-1},[x_u,x_v],x_{u+1},\cdots,\hat{x}_i,\cdots,\hat{x}_v,\cdots,x_{n+2}),x_i]\\
+&\sum_{i=2}^{n+2}(-1)^i\sum_{i<u<v}(-1)^v[f(x_1,\cdots,\hat{x}_i,\cdots,x_{u-1},[x_u,x_v],x_{u+1},\cdots,\hat{x}_v,\cdots,x_{n+2}),x_i].
\end{aligned} \tag{1.3.4}$$

Also the third term on the right-hand side of (1.3.2) can be expressed as

$$\begin{aligned}
&\sum_{1\le i<j\le n+2}(-1)^{j+1}\delta^n f(x_1,\cdots,x_{i-1},[x_i,x_j],x_{i+1},\cdots,x_{j-1},\hat{x}_j,x_{j+1},\cdots,x_{n+2})\\
=&\sum_{i=1,\ 2\le j\le n+2}(-1)^{j+1}\delta^n f([x_1,x_j],x_2,\cdots,x_{j-1},\hat{x}_j,x_{j+1},\cdots,x_{n+2})\\
&\ +\sum_{2\le i<j\le n+2}(-1)^{j+1}\delta^n f(x_1,\cdots,x_{i-1},[x_i,x_j],x_{i+1},\cdots,x_{j-1},\hat{x}_j,x_{j+1},\cdots,x_{n+2})\\
=&\sum_{j=2}^{n+2}(-1)^{j+1}[[x_1,x_j],f(x_2,x_3,\cdots,\hat{x}_j,\cdots,x_{n+2})]\\
+&\sum_{2\le i<j\le n+2}(-1)^{j+1}[x_1,f(x_2,\cdots,x_{i-1},[x_i,x_j],x_{i+1},\cdots,\hat{x}_j,\cdots,x_{n+2})]\\
+&\sum_{1\le i<j\le n+2}(-1)^{j+1}\sum_{2\le u\le i-1}(-1)^u[f(x_1,\cdots,\hat{x}_u,\cdots,x_{i-1},[x_i,x_j],\cdots,\hat{x}_j,\cdots,x_{n+2}),x_u]\\
+&\sum_{2\le i<j\le n+2}(-1)^{i+j+1}[f(x_1,\cdots,x_{i-1},\widehat{[x_i,x_j]},x_{i+1},\cdots,\hat{x}_j,\cdots,x_{n+2}),[x_i,x_j]]\\
+&\sum_{1\le i<j\le n+2}(-1)^{j+1}\sum_{i<u<j}(-1)^u[f(x_1,\cdots,x_{i-1},[x_i,x_j],\cdots,\hat{x}_u,\cdots,\hat{x}_j,\cdots,x_{n+2}),x_u]
\end{aligned}$$

$$+\sum_{1\leq i<j\leq n+2}(-1)^{j+1}\sum_{j<u\leq n+2}(-1)^{u-1}[f(x_1,\cdots,x_{i-1},[x_i,x_j],\cdots,\hat{x}_j,\cdots,\hat{x}_u,\cdots,x_{n+2}),x_u]$$
$$+X. \tag{1.3.5}$$

By the above Lemma 1.3.3, we get $X=0$. Notice that since $M$ is a representation of the Leibniz algebra $L$, if we use the identity

$$[x,[y,z]]-[[x,y],z]+[[x,z],y]=0,$$

whenever one of the variable is from $M$ and the others from $L$, we get the following.

1st term of (1.3.2) + 1st term of (1.3.4) (for $i=2$) + 1st term of (1.3.5) (for $j=2$) $=0$. Similarly,

2nd term of (1.3.2) + 1st term of (1.3.4)(for $i>2$) + 1st term of (1.3.5) (for $j>2$) $=0$,

and 2nd term of (1.3.4) + 3rd term of (1.3.4) + 4th term of (1.3.5) $=0$.

On the other hand any element in 3rd term of (1.3.2) appears with opposite sign in the 2nd term of (1.3.5), so they cancel out. Similarly any element in 4th term, 5th term or 6th term of (1.3.4) appears with opposite sign in the expression obtained from 3rd term, 5th term or 6th term of (1.3.5).

Now the result will follow from this observation, if we substitute all the three terms on the right-hand side of (1.3.2) from (1.3.3)-(1.3.5). □

**Definition 1.3.5.** *Let $ZL^n(L;M)=ker(\delta^n)$ and $BL^n(L;M)=im(\delta^{n-1})$ be submodules of $CL^n(L;M)$ consisting of cocycles and coboundaries respectively. The cohomology of the Leibniz algebra L with coefficients in the representation M is defined by*

$$HL^n(L;M):=H^n(CL^*(L;M))=\frac{ZL^n(L;M)}{BL^n(L;M)}.$$

When $M=L$ with the action given by the bracket in $L$, we denote the cohomology by $HL^*(L;L)$.

A graded module $V$ is a $\mathbb{K}$-module together with a family $\{V_i\}_{i\in\mathbb{Z}}$ of submodules of $V$, such that $V=\bigoplus_i V_i$. The elements in $V_i$ are called homogeneous of degree $i$.

**Definition 1.3.6.** *A graded Lie algebra $\mathfrak{L}$ is a graded module $\mathfrak{L}=\{L_i\}_{i\in\mathbb{Z}}$ together with a linear map of degree zero, $[-,-]:\mathfrak{L}\otimes\mathfrak{L}\longrightarrow\mathfrak{L}$, $x\otimes y\mapsto[x,y]$ satisfying*

$$(i)[x,y]=-(-1)^{|x||y|}[y,x]$$
$$(ii)(-1)^{|x||z|}[x,[y,z]]+(-1)^{|y||x|}[y,[z,x]]+(-1)^{|z||y|}[z,[x,y]]=0$$

*for $x, y, z \in \mathfrak{L}$, where $|x|$ denotes the degree of $x$.*

*A differential graded Lie algebra is a graded Lie algebra equipped with a differential $d$ satisfying*

$$d[x,y] = [dx, y] + (-1)^{|x|}[x, dy].$$

**Definition 1.3.7.** *Let $S_n$ be the symmetric group of $n$ symbols. Recall that a permutation $\sigma \in S_{p+q}$ is called a $(p,q)$-shuffle, if $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$, and, $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$. We denote the set of all $(p,q)$-shuffles in $S_{p+q}$ by $Sh(p,q)$.*

*For $\alpha \in CL^{p+1}(L;L)$ and $\beta \in CL^{q+1}(L;L)$, define $\alpha \circ \beta \in CL^{p+q+1}(L;L)$ by*

$$\begin{aligned} &\alpha \circ \beta(x_1, \dots, x_{p+q+1}) \\ =&\sum_{k=1}^{p+1}(-1)^{q(k-1)}\{\sum_{\sigma \in Sh(q,\ p-k+1)} sgn(\sigma)\alpha(x_1, \dots, x_{k-1}, \beta(x_k, x_{\sigma(k+1)}, \dots, x_{\sigma(k+q)}), \\ &\qquad x_{\sigma(k+q+1)}, \dots, x_{\sigma(p+q+1)})\}. \end{aligned}$$

Then the direct sum $CL^*(L;L) = \bigoplus_p CL^p(L;L)$ equipped with the bracket $[-,-]$ defined by,

$$[\alpha, \beta] = \alpha \circ \beta + (-1)^{pq+1}\beta \circ \alpha \tag{1.3.6}$$

for $\alpha \in CL^{p+1}(L;L)$ and $\beta \in CL^{q+1}(L;L)$, is a graded Lie algebra ( [Bal97]). The grading being reduced by one from the usual grading. Moreover the coboundary can be expressed by the Lie bracket in the graded Lie algebra as follows.

**Lemma 1.3.8.** *Suppose $\alpha \in CL^{p+1}(L;L)$, then $\delta\alpha = -[\alpha, \mu_0]$, where $\mu_0$ is the 2-cochain given by the Leibniz bracket in $L$.*

*Proof.* Here $\delta\alpha \in CL^{p+2}(L;L)$. Let $x_1, \dots, x_{p+2} \in L$. Then from the definition of coboundary we have,

$$\begin{aligned} &\delta\alpha(x_1, \dots, x_{p+2}) \\ &= \mu_0(x_1, \alpha(x_2, \dots, x_{p+2})) + \sum_{j=2}^{p+2}(-1)^j \mu_0(\alpha(x_1, \dots, x_{j-1}, \hat{x}_j, x_{j+1}, \dots, x_{p+2}), x_j) \\ &+ \sum_{1 \le k < j \le p+2} (-1)^{j+1}\alpha(x_1, \dots, x_{k-1}, \mu_0(x_k, x_j), x_{k+1}, \dots, x_{j-1}, \hat{x}_j, x_{j+1}, \dots, x_{p+2}). \end{aligned} \tag{1.3.7}$$

Now $[\alpha, \mu_0] = \alpha \circ \mu_0 + (-1)^{p+1}\mu_0 \circ \alpha$, where

$$\begin{aligned}
&\alpha \circ \mu_0(x_1, \ldots, x_{p+2})\\
=&\sum_{k=1}^{p+1}(-1)^{(k-1)} \sum_{\sigma \in Sh(1,p-k+1)} sgn(\sigma)\alpha(x_1, \ldots, x_{k-1}, \mu_0(x_k, x_{\sigma(k+1)}), x_{\sigma(k+2)}, \ldots, x_{\sigma(p+2)})\\
=&\sum_{1\leq k<j\leq p+2} (-1)^j \alpha(x_1, \ldots, x_{k-1}, \mu_0(x_k, x_j), x_{k+1}, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots, x_{p+2}).
\end{aligned}$$

And

$$\begin{aligned}
&\mu_0 \circ \alpha(x_1, \ldots, x_{p+2})\\
=&\sum_{\sigma\in Sh(p,1)} sgn(\sigma)\mu_0(\alpha(x_1, x_{\sigma(2)}, \ldots, x_{\sigma(p+1)}), x_{\sigma(p+2)}) + (-1)^p\mu_0(x_1, \alpha(x_2, \ldots, x_{p+2}))\\
=&\sum_{2\leq j\leq p+2}(-1)^{p+2-j}\mu_0(\alpha(x_1, x_2, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots, x_{p+2}), x_j)\\
&+ (-1)^p\mu_0(x_1, \alpha(x_2, \ldots, x_{p+2})).
\end{aligned}$$

Therefore

$$\begin{aligned}
&[\alpha, \mu_0](x_1, \ldots, x_{p+2})\\
=& \{\alpha \circ \mu_0 + (-1)^{p+1}\mu_0 \circ \alpha\}(x_1, \ldots, x_{p+2})\\
=& \sum_{1\leq k<j\leq p+2} (-1)^j\alpha(x_1, \ldots, x_{k-1}, \mu_0(x_k, x_j), x_{k+1}, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots, x_{p+2})\\
+& \sum_{2\leq j\leq p+2}(-1)^{j+1}\mu_0(\alpha(x_1, x_2, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots, x_{p+2}), x_j) - \mu_0(x_1, \alpha(x_2, \ldots, x_{p+2})).
\end{aligned}$$

Thus $\delta\alpha = \ - [\alpha, \mu_0]$. □

Now let us consider the linear map $d : CL^*(L; L) \longrightarrow CL^*(L; L)$ defined by $d\alpha = (-1)^{|\alpha|}\delta\alpha$ for $\alpha \in CL^*(L; L)$. Then we have

**Lemma 1.3.9.** *The differential $d$ of the graded Lie algebra $CL^*(L; L)$ is a derivation of degree 1. In otherwords, $d[\alpha, \beta] = [d\alpha, \beta] + (-1)^{|\alpha|}[\alpha, d\beta]$ for $\alpha, \beta \in CL^*(L; L)$.*

*Proof.* Let $\alpha \in CL^{p+1}(L; L)$ and $\beta \in CL^{q+1}(L; L)$ then from definition we get

$$\begin{aligned}
&d[\alpha, \beta]\\
=& (-1)^{p+q+1}\delta[\alpha, \beta]\\
=& \ - (-1)^{p+q+1}[[\alpha, \beta], \mu_0]\\
=& [\mu_0, [\alpha, \beta]] \ \text{(by antisymmetry of the graded Lie bracket } [-,-])\\
=& \ - (-1)^{q(p+1)}[\beta, [\mu_0, \alpha]] - (-1)^{p+q}[\alpha, [\beta, \mu_0]] \ \text{(by the Jacobi identity of } [-,-])
\end{aligned}$$

$$\begin{aligned}
&= [[\mu_0, \alpha], \beta] + (-1)^p[\alpha, (-1)^q \delta\beta] \\
&= (-1)^{p+1}[[\alpha, \mu_0], \beta] + (-1)^p[\alpha, d\beta] \\
&= [(-1)^p \delta\alpha, \beta] + (-1)^p[\alpha, d\beta] \\
&= [d\alpha, \beta] + (-1)^p[\alpha, d\beta].
\end{aligned}$$

□

Thus we get the following result.

**Proposition 1.3.10.** *The graded module $CL^*(L;L) = \bigoplus_p CL^p(L;L)$ equipped with the bracket defined by*

$$[\alpha, \beta] = \alpha \circ \beta + (-1)^{pq+1}\beta \circ \alpha \quad \text{for } \alpha \in CL^{p+1}(L;L) \quad \text{and } \beta \in CL^{q+1}(L;L)$$

*and the differential map $d$ by $d\alpha = (-1)^{|\alpha|}\delta\alpha$ for $\alpha \in CL^*(L;L)$ is a differential graded Lie algebra.*

From the Lemma 1.3.9, it follows that if $\alpha, \beta$ are cocycles then $[\alpha, \beta]$ is also a cocycle, and the cohomology class of $[\alpha, \beta]$ depends only on the class of $\alpha$ and $\beta$. Thus the bracket $[-,-]$ at the cochain level induces

$$[-,-] : HL^{p+1}(L;L) \otimes HL^{q+1}(L;L) \longrightarrow HL^{p+q+1}(L;L)$$

and we get the following corollary.

**Corollary 1.3.11.** *The graded module $HL^*(L;L)$ is a graded Lie algebra.*

## 1.4 Cohomology for Leibniz algebra homomorphisms

The purpose of this last section is to define cohomology modules associated to a Leibniz algebra homomorphism [MM]. We shall need them later in this thesis.

Let $L$ and $M$ be Leibniz algebras over $\mathbb{K}$. For simplicity, we use the same notation $[-,-]$ for the brackets of $L$ and $M$.

**Definition 1.4.1.** *A $\mathbb{K}$-linear map $f : L \longrightarrow M$ is said to be a Leibniz algebra homomorphism if it preserves the brackets. In other words, $f([x,y]) = [f(x), f(y)]$ for $x, y \in L$.*

Let $f : L \longrightarrow M$ be a Leibniz algebra homomorphism. We consider $M$ as a representation of $L$ via $f$ as follows.

The actions

$$[-,-] : L \times M \longrightarrow M, \; [-,-] : M \times L \longrightarrow M$$

of $L$ on $M$ are respectively

$$[l, m] := [f(l), m], \ [m, l] := [m, f(l)]$$

for $l \in L$ and $m \in M$.

Define a cochain complex $(CL^*(f; f), d)$ as follows. Set $CL^0(f; f) := 0$. For $n \geq 1$, the module of $n$-cochains is

$$CL^n(f; f) := CL^n(L; L) \times CL^n(M; M) \times CL^{n-1}(L; M).$$

The coboundary $d^n : CL^n(f; f) \longrightarrow CL^{n+1}(f; f)$ is defined by the formula

$$d^n(u, v; w) := (\delta^n u, \delta^n v; fu - vf - \delta^{n-1} w) \text{ for } (u, v; w) \in CL^n(f; f). \tag{1.4.1}$$

Here $\delta^n$ on the right-hand side are the coboundaries of the complexes defining Leibniz cohomology groups with appropriate coefficients. The map $vf : L^{\otimes n} \longrightarrow M$ is the linear map defined by $vf(x_1, \cdots, x_n) = v(f(x_1), \cdots, f(x_n))$, and $fu$ is the composition of maps.

By the Proposition 1.3.4, we have $\delta^{n+1} \circ \delta^n u = \delta^{n+1} \circ \delta^n v = \delta^n \circ \delta^{n-1} w = 0$. Moreover $\delta^n(fu) = f\delta^n u$; $(\delta^n v)f = \delta^n(vf)$ for $u \in CL^n(L; L)$, $v \in CL^n(M; M)$, and $w \in CL^{n-1}(L; M)$. So

$$f\delta^n u - (\delta^n v)f - \delta^n(fu - vf - \delta^{n-1} w) = f\delta^n u - (\delta^n v)f - \delta^n fu + \delta^n vf = 0.$$

Thus $d^{n+1} \circ d^n(u, v; w) = 0$ for $(u, v, w) \in CL^n(f; f), n \geq 0$. Hence we obtain

**Proposition 1.4.2.** *$(CL^*(f; f), d)$ is a cochain complex.* □

The corresponding cohomology modules are denoted by

$$HL^n(f; f) := H^n((CL^*(f; f), d)).$$

Next proposition relates $HL^*(f; f)$ to $HL^*(L; L)$, $HL^*(M; M)$, and $HL^*(L; M)$.

**Proposition 1.4.3.** *If $HL^n(L; L) = 0 = HL^n(M; M)$, and $HL^{n-1}(L; M) = 0$, then so is $HL^n(f; f)$.*

*Proof.* Let $(u, v; w)$ represents a cocycle in $HL^n(f; f)$. Since $HL^n(L; L) = 0 = HL^n(M; M)$, we get $u = \delta^{n-1} u_1$ and $v = \delta^{n-1} v_1$ for some $(n-1)$-cochains $u_1 \in CL^{n-1}(L; L)$ and $v_1 \in CL^{n-1}(M; M)$. Now $d^n(u, v; w) = 0$, so $fu - vf - \delta^{n-1} w = 0 = (f\delta^{n-1} u_1 - \delta^{n-1} v_1 f - \delta^{n-1} w) = \delta^{n-1}(fu_1 - v_1 f - w)$. Therefore $(fu_1 - v_1 f - w)$ is a cocycle in $CL^{n-1}(L; M)$. Since $HL^{n-1}(L; M) = 0$, we get an element $w_1 \in CL^{n-2}(L; M)$ such that $\delta^{n-2} w_1 = (fu_1 - v_1 f - w)$.

Thus $(u_1, v_1; w_1) \in CL^{n-1}(f;f)$ and $d^{n-1}(u_1, v_1; w_1) = (u, v; w)$. Consequently every cocycle is a coboundary in $CL^n(f;f)$. □

We shall need a version of the above cohomology with coefficients which is described as follows.

Let $M_0$ be a finite dimensional $\mathbb{K}$-module. For $n \geq 1$ we have the isomorphisms:

$CL^n(L; M_0 \otimes L) \cong M_0 \otimes CL^n(L;L)$, $CL^n(M; M_0 \otimes M) \cong M_0 \otimes CL^n(M;M)$ and $CL^n(L; M_0 \otimes M) \cong M_0 \otimes CL^n(L;M)$.

Set $M_0 \otimes CL^0(f;f) = 0$ and for $n \geq 1$

$M_0 \otimes CL^n(f;f) := CL^n(L; M_0 \otimes L) \times CL^n(M; M_0 \otimes M) \times CL^{n-1}(L; M_0 \otimes M)$.

Define $d^n : M_0 \otimes CL^n(f;f) \longrightarrow M_0 \otimes CL^{n+1}(f;f)$ by

$$d^n(m_1 \otimes u, m_2 \otimes v; m_3 \otimes w) := (m_1 \otimes \delta^n u, m_2 \otimes \delta^n v;\ (m_1 \otimes fu - m_2 \otimes vf - m_3 \otimes \delta^{n-1} w)\ )$$

for $(m_1 \otimes u, m_2 \otimes v; m_3 \otimes w) \in M_0 \otimes CL^n(f;f)$.

Now

$$\begin{aligned} & d^{n+1} \circ d^n(m_1 \otimes u, m_2 \otimes v; m_3 \otimes w) \\ = \ & d^{n+1}(m_1 \otimes \delta^n u, m_2 \otimes \delta^n v;\ (m_1 \otimes fu - m_2 \otimes vf - m_3 \otimes \delta^{n-1} w)\ ) \\ = \ & (0, 0;\ m_1 \otimes f(\delta^n u) - m_2 \otimes (\delta^n v) f - m_1 \otimes \delta^n(fu) + m_2 \otimes \delta^n(vf)\ ) \\ = \ & (0, 0; 0). \end{aligned} \tag{1.4.2}$$

Thus $(M_0 \otimes CL^*(f;f), d)$ is a cochain complex. We shall denote the corresponding $n$th cohomology module by $M_0 \otimes HL^n(f;f)$.

# Chapter 2

# Harrison cohomology and related results

## 2.1 Introduction

In this chapter we recall some results about Harrison cohomology of a commutative algebra and its relation to extensions of the algebra. These results will be used subsequently. The basic references for this chapter are [Hoc45, Har62, Bar68]. We first recall the definition of the Harrison complex of a commutative algebra $A$ and the Harrison cohomology with coefficients in an $A$-module $M$. Next, we define extension of a commutative algebra $A$ by an $A$-module $M$, describe its relation to Harrison cohomology of $A$. We also recall few basic properties that will be used later in this thesis.

## 2.2 Harrison complex of a commutative algebra

**Definition 2.2.1.** *By a $\mathbb{K}$-algebra we mean an associative ring $A$, which is also a $\mathbb{K}$-module satisfying the conditions*

$$k(xy) = (kx)y = x(ky), \text{ for } k \in \mathbb{K} \text{ and } x, y \in A.$$

*If the underlying ring is commutative then $A$ is called a commutative algebra.*

A local $\mathbb{K}$-algebra is the one having a unique maximal ideal or equivalently if the set of all the nonunits in $A$ forms a maximal ideal.

**Definition 2.2.2.** *Let $A$ be a $\mathbb{K}$-algebra, a bimodule $M$ over $A$ or, an $A$-bimodule is a $\mathbb{K}$-module equipped with two actions (left and right) of $A$*

$$A \times M \longrightarrow M \quad \text{and } M \times A \longrightarrow M$$

*such that* $(am)a' = a(ma')$ *for* $a, a' \in A$ *and* $m \in M$.

*The actions of* $A$ *and* $\mathbb{K}$ *on* $M$ *are compatible, that is* $(\lambda a)m = \lambda(am) = a(\lambda m)$ *for* $\lambda \in \mathbb{K}$, $a \in A$ *and* $m \in M$. *When* $A$ *has the identity* 1 *we always assume that* $1m = m1 = m$ *for* $m \in M$.

Given a bimodule $M$ over $A$, the Hochschild complex of $A$ with coefficients in $M$ is defined as follows.

Set $C_n(A, M) = M \otimes A^{\otimes n}$ ; $n \geq 0$ where $A^{\otimes n} = A \otimes \cdots \otimes A$ ($n$-copies). Let

$$\delta : C_n(A, M) \longrightarrow C_{n-1}(A, M)$$

be the $\mathbb{K}$-linear map given by

$$\begin{aligned}\delta(m, a_1, a_2, \ldots, a_n) = (ma_1, a_2, \ldots, a_n) + \sum_{i=1}^{n-1}(-1)^i(m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ + (-1)^n(a_n m, a_1, \ldots, a_{n-1}).\end{aligned} \tag{2.2.1}$$

Then $\delta^2 = 0$ and the complex $(C_*(A, M), \delta)$ is called the Hochschild complex of $A$ with coefficients in the $A$-bimodule $M$. When $M = A$, where the actions are given by algebra operation in $A$ we denote the complex $C_*(A, A)$ by $C_*(A)$.

Let $A$ be a commutative $\mathbb{K}$-algebra with 1. The Harrison complex of $A$ induced from the Hochschild complex is defined as follows.

Consider $C_n(A) = A \otimes A^{\otimes n}$ where $A$ operates on the first factor. If we denote the element $(a_0, a_1, \cdots, a_n) \in C_n(A)$ by $a_0(a_1, \cdots, a_n)$ and write $1(a_1, \cdots, a_n)$ simply by $(a_1, \cdots, a_n)$, then the boundary

$$\delta : C_n(A) \longrightarrow C_{n-1}(A)$$

is the $A$-linear map such that

$$\begin{aligned}\delta(a_1, a_2, \ldots, a_n) = a_1(a_2, \ldots, a_n) + \sum_{i=1}^{n-1}(-1)^i(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ + (-1)^n a_n(a_1, \ldots, a_{n-1}).\end{aligned} \tag{2.2.2}$$

For $a_1, a_2, \ldots, a_n \in A$ and $0 < p < n$, set

$$s_p(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in Sh(p,n-p)} sgn(\sigma)(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) \in C_n(A),$$

where $Sh(p, q)$ is the set all $(p, q)$-shuffles as defined in Chapter 1. Let $Sh_n(A)$ be the $A$-submodule of $C_n(A)$ generated by the chains $s_p(a_1, a_2, \ldots, a_n)$ for $a_1, a_2, \ldots, a_n \in$

$A$ and $0 < p < n$.

It follows from Proposition 2.2 in [Bar68] that $\delta(Sh_n(A)) \subset Sh_{n-1}(A)$. Thus we get a chain complex $Ch(A) = \{Ch_n(A), \delta\}$ where $Ch_n(A) = C_n(A)/Sh_n(A)$. This is known as the Harrision chain complex for the algebra $A$ with trivial coefficients.

Let $M$ be a left $A$-module. Since $A$ is commutative, $M$ can be considered as an $A$-bimodule. Consider the following complexes by taking $- \otimes M$ and $Hom(-; M)$ respectively.

$$\cdots \xrightarrow{\delta_{n+2}\otimes id} Ch_{n+1}(A) \otimes M \xrightarrow{\delta_{n+1}\otimes id} Ch_n(A) \otimes M \xrightarrow{\delta_n\otimes id} Ch_{n-1}(A) \otimes M \cdots$$

$$\cdots \xrightarrow{\delta^{n-2}} Hom(Ch_{n-1}(A); M) \xrightarrow{\delta^{n-1}} Hom(Ch_n(A); M) \xrightarrow{\delta^n} Hom(Ch_{n+1}(A); M) \cdots$$

**Definition 2.2.3.** *For an $A$-module $M$ we set,*

$$H_n^{Harr}(A; M) = H_n(Ch(A) \otimes M) \quad \textit{and}$$

$$H^n_{Harr}(A; M) = H^n(Hom(Ch(A), M)).$$

These are respectively the Harrision homology and cohomology modules of $A$ with coefficients in $M$.

In our discussion we mainly require first and second Harrison cohomologies. In low dimension the Harrison cochain complex is given by

$$\begin{aligned} &0 \longrightarrow Hom(A; M) \xrightarrow{\delta^1} Hom(S^2A; M) \xrightarrow{\delta^2} Hom(A^{\otimes 3}; M), \text{ where} \\ &\delta^1\psi(a, b) = \ a\psi(b) - \psi(ab) + b\psi(a) \quad \text{for } \psi \in Hom(A; M) \\ &\delta^2\phi(a, b, c) = \ a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - c\phi(a, b) \\ &\ \text{for } \phi \in Hom(S^2A; M) \text{ and } a, b, c \in A. \end{aligned} \tag{2.2.3}$$

Note that $Ch_2(A)$ is by definition $S^2A$, the symmetric product module of $A$.

**Proposition 2.2.4.** *Let $A$ be a commutative algebra with a maximal ideal $\mathfrak{M}$ and $M$ be an $A$-module with $\mathfrak{M}M = 0$. Then we have the canonical isomorphisms,*

$$H_n^{Harr}(A; M) \cong H_n^{Harr}(A; \mathbb{K}) \otimes M \ \textit{and}$$

$$H^n_{Harr}(A; M) \cong H^n_{Harr}(A; \mathbb{K}) \otimes M.$$

*Proof.* Define

$$\mu : H_n^{Harr}(A; \mathbb{K}) \otimes M \longrightarrow H_n^{Harr}(A; M) \ \text{ by } \mu([c] \otimes m) = [c \otimes m].$$

If $c = \delta_{n+1}\alpha$ for some $(n+1)$-chain $\alpha \in Ch_{n+1}(A)$ then $[c \otimes m] = [\delta_{n+1}\alpha \otimes m] = 0$. So $\mu$ is well-defined. Suppose $\mu([c] \otimes m) = [c \otimes m]$ where $c \otimes m$ is a boundary in $Ch_n(A) \otimes M$, then $c \otimes m = \delta_{n+1}\alpha \otimes m$ for some $(n+1)$-chain $\alpha \in Ch_{n+1}(A)$. Thus $[c] \otimes m = 0$ in $H_n^{Harr}(A; \mathbb{K}) \otimes M$. So, $\mu$ is injective. From definition it is clear that $\mu$ is surjective. Next define

$$F : H^n_{Harr}(A; M) \longrightarrow Hom(H_n^{Harr}(A; \mathbb{K}); M) \ \text{ by } F([\phi])([c]) = \phi(c).$$

$$G : H^n_{Harr}(A; \mathbb{K}) \otimes M \longrightarrow Hom(H_n^{Harr}(A; \mathbb{K}); M) \ \text{ by } G([\phi] \otimes a)(c) = \phi(c)a.$$

Here $F(\phi + \delta^{n-1}(\alpha))([c]) = \phi(c) + \delta^{n-1}\alpha(c) = \phi(c) + \alpha(\delta_n c) = \phi(c)$, so $F$ is well-defined. From definition of $F$, it follows that if $\phi(c) = 0$ for all cycles $c \in Ch_n(A)$ then $\phi$ is a coboundary. Thus $F$ is injective.

To show that $F$ is surjective, let us consider $\rho \in Hom(H_n^{Harr}(A; \mathbb{K}); M)$. Then $\rho$ can be extended to a linear map $\tilde{f}$ in $H^n_{Harr}(A; M)$ such that $F(\tilde{f}) = \rho$.

Similarly $G$ is an isomorphism. □

The next proposition gives an alternative description of the first Harrison cohomology module of $A$ with coefficients in $M$.

**Definition 2.2.5.** *Let $A$ be a $\mathbb{K}$-algebra and $M$ an $A$-module. By a derivation $D : A \longrightarrow M$ (over $\mathbb{K}$) we mean a $\mathbb{K}$-linear map satisfying*

$$D(ab) = aD(b) + bD(a).$$

**Proposition 2.2.6.** *The module of derivations from $A$ to $M$ is $H^1_{Harr}(A; M)$.*

*Proof.* By definition, $H^1_{Harr}(A; M) = \frac{ker(\delta^1)}{\{0\}} = ker(\delta^1)$.
Suppose $\psi \in ker(\delta^1)$ then $\delta^1\psi(a, b) = 0$ for $a, b \in A$.
So $\psi(a, b) = a\psi(b) + b\psi(a)$. Thus $\psi : A \longrightarrow M$ is a derivation. On the other hand a derivation $f : A \longrightarrow M$ is in $ker(\delta^1)$. Therefore $H^1_{Harr}(A; M)$ is the module of derivations $A \longrightarrow M$. □

Let $A$ be a local algebra with the maximal ideal $\mathfrak{M}$, and $\varepsilon : A \to \mathbb{K}$ be the canonical augmentation. Thus $\varepsilon$ is the algebra homomorphism with $\varepsilon(1) = 1$ and $ker(\varepsilon) = \mathfrak{M}$. We consider $\mathbb{K}$ as an $A$-module, where the action of $A$ on $\mathbb{K}$, is given by the augmentation $\varepsilon$. Explicitly, $ak = \varepsilon(a)k$, for $a \in A$ and $k \in \mathbb{K}$.

**Corollary 2.2.7.** *The dual module of $(\frac{\mathfrak{M}}{\mathfrak{M}^2})$ corresponds bijectively to $H^1_{Harr}(A; \mathbb{K})$.*

*Proof.* Let $\psi \in H^1_{Harr}(A; \mathbb{K})$. By Proposition 2.2.6, $\psi$ can be viewed as a derivation $\psi : A \longrightarrow \mathbb{K}$.

Suppose $a \in \mathfrak{M}^2$ so that $a = \sum_{i,j} m_i m_j$ for some $m_i, m_j \in \mathfrak{M}$ where $i, j$ varies over a finite set. Then

$$\begin{aligned} \psi(a) &= \psi\left(\sum_{i,j} m_i m_j\right) \\ &= \sum_{i,j} \psi(m_i m_j) \\ &= \sum_{i,j} (m_i \psi(m_j) + m_j \psi(m_i)) \\ &= \sum_{i,j} (\varepsilon(m_i)\psi(m_j) + \varepsilon(m_j)\psi(m_i)) \\ &= 0. \end{aligned} \tag{2.2.4}$$

Moreover $\psi(1) = 0$, so $\psi$ vanishes on $\mathbb{K}$. Thus $\psi : \mathfrak{M} \longrightarrow \mathbb{K}$ is a linear map vanishing on $\mathfrak{M}^2$. In other words $\psi \in (\mathfrak{M}/\mathfrak{M}^2)'$.
Conversely let $\phi \in (\mathfrak{M}/\mathfrak{M}^2)'$. Then $\phi$ defines a linear map $\bar{\phi} : A \longrightarrow \mathbb{K}$ such that $\bar{\phi}(\mathfrak{M}^2) = 0$ and $\bar{\phi}(1) = 0$. For $a_1, a_2 \in A = \mathbb{K} \oplus \mathfrak{M}$ let $a_1 = k_1 + m_1$ and $a_2 = k_2 + m_2$. Now

$$\begin{aligned} \bar{\phi}(a_1 a_2) &= \bar{\phi}((k_1 + m_1)(k_2 + m_2)) \\ &= \bar{\phi}(k_1 k_2 + k_1 m_2 + k_2 m_1 + m_1 m_2) \\ &= \bar{\phi}(k_1 k_2) + k_1 \bar{\phi}(m_2) + k_2 \bar{\phi}(m_1) + \bar{\phi}(m_1 m_2) \\ &= k_1 \bar{\phi}(m_2) + k_2 \bar{\phi}(m_1) \\ &= (k_1 + m_1)\bar{\phi}(k_2 + m_2) + (k_2 + m_2)\bar{\phi}(k_1 + m_1) \\ &= a_1 \bar{\phi}(a_2) + a_2 \bar{\phi}(a_1). \end{aligned}$$

So $\bar{\phi}$ is a derivation. Therefore by Proposition 2.2.6, $\bar{\phi}$ can be viewed as an element in $H^1_{Harr}(A; \mathbb{K})$. □

We will see later that first Harrison cohomology module has another significant interpretation as the group of automorphisms of extensions of $A$ by $M$.

## 2.3 Extension of algebras and relation to Harrison cohomology

In this section we first recall the definition of extension of a commutative algebra $A$ by an $A$-module $M$. Then we recall some important characterizations of first and second Harrison cohomology modules of $A$ in terms of extension of $A$ by $M$.

**Definition 2.3.1.** *An extension of a commutative algebra $A$ with maximal ideal $\mathfrak{M}$ by an $A$-module $M$ satisfying $\mathfrak{M}M = 0$ is a commutative algebra $B$ with an algebra homomorphism $p : B \longrightarrow A$, a $\mathbb{K}$-linear map $i : M \longrightarrow B$ such that the following is an exact sequence of $\mathbb{K}$-modules*

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0.$$

*Moreover, if the image $i(M)$, which is an ideal in $B$ is denoted by $N$, then $B$-module structure on $N$ is induced by the $A$-module structure on $M$ as follows.*

$$nb = i(m)b = i(mp(b)).$$

In particular, $N$ is an ideal in $B$ with $N^2 = 0$. For if $n, n' \in N$ with $n = i(m)$ and $n' = i(m')$ where $m, m' \in M$ then $nn' = ni(m') = i(p(n)m') = 0$, because $p(n) = 0$. Therefore $N^2 = 0$.

Given $A$ and $M$ as above we can always construct an extension by considering $B = A \oplus M$ and $i(m) = (0, m)$, $p(a, m) = a$. The algebra multiplication in $B$ is given by $(a_1, m_1)(a_2, m_2) = (a_1a_2,\ a_1m_2 + a_2m_1)$ for $a_1, a_2 \in A$ and $m_1, m_2 \in M$. This extension is called the trivial extension of $A$ by $M$.

**Definition 2.3.2.** *Two extensions $B$ and $B'$ of $A$ by $M$ are said to be isomorphic if there is a $\mathbb{K}$-algebra isomorphism $f : B \longrightarrow B'$ such that the diagram below commutes.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & B & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{i'} & B' & \xrightarrow{p'} & A & \longrightarrow & 0 \end{array}$$

Next result shows that the isomorphism classes of extension of $A$ by $M$ determine and are determined by the second Harrison cohomology of $A$ with coefficients in $M$.

**Proposition 2.3.3.** *Elements of $H^2_{Harr}(A; M)$ corresponds bijectively to isomorphism classes of extensions*

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

*of the algebra $A$ by $A$-module $M$.*

*Proof.* Let $f$ be a representative class of $[f] \in H^2_{Harr}(A; M)$. So $f : A^{\otimes 2} \longrightarrow M$ is a symmetric linear map. Consider $B = A \oplus M$ and define

$$(a_1, m_1)(a_2, m_2) = (a_1a_2,\ a_1m_2 + a_2m_1 + f(a_1, a_2))$$

for $(a_1, m_1), (a_2, m_2) \in B$. Then $0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$ is an extension of $A$ by

$M$, where $i(m) = (0, m)$ and $p(a, m) = a$ for $m \in M$ and $a \in A$.

Suppose we take another representative $f'$ of the class $[f] \in H^2_{Harr}(A; M)$ and get an extension $B'$ as above. Now $f - f' = \delta^1 g$ for some $g \in Hom(A; M)$. Then $\phi : B \longrightarrow B'$ defined by $\phi(a, m) = (a, m + g(a))$ gives an isomorphism of the extension $B$ and $B'$. Thus $[f] \in H^2_{Harr}(A; M)$ corresponds to an isomorphism class of extension of $A$ by $M$.

Conversely, we consider an extension

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

of $A$ by $M$. Fix a section $q : A \longrightarrow B$ of the projection $p$.
Define

$$\alpha : B \longrightarrow A \oplus M \ \ by \ \ \alpha(b) = (p(b), i^{-1}(b - q \circ p(b))).$$

From definition $\alpha$ is a $\mathbb{K}$-linear map. Now if $\alpha(b) = (0, 0)$ then $p(b) = 0$ and $i^{-1}(b) = 0$, hence $b = 0$. So $\alpha$ is an injective map. Suppose $(a, m) \in A \oplus M$, take $b = q(a) + i(m)$, then $p(b) = a$ and $b - q \circ p(b) = q(a) + i(m) - q(a) = i(m)$. Therefore $\alpha(b) = \alpha(q(a) + i(m)) = (a, m)$, showing $\alpha$ is an onto map. Thus $\alpha$ is a $\mathbb{K}$-module isomorphism. Let $(a, m)_q$ denotes the inverse image $q(a) + i(m)$ of $(a, m) \in A \oplus M$ under the above isomorphism $\alpha$. Define $\phi_q : A^{\otimes 2} \longrightarrow M$ by

$$\phi_q(a_1, a_2) = i^{-1}((a_1, 0)_q(a_2, 0)_q - (a_1a_2, 0)_q) = i^{-1}(q(a_1)q(a_2) - q(a_1a_2)) \qquad (2.3.1)$$

for $a_1, a_2 \in A$. Now for $b_1 = (a_1, m_1)_q$ and $b_2 = (a_2, m_2)_q$ in $B$ we can describe the element $b_1b_2$ as follows.

$$\begin{aligned}
& b_1b_2 \\
&= (a_1, m_1)_q(a_2, m_2)_q \\
&= (q(a_1) + i(m_1))(q(a_2) + i(m_2)) \\
&= q(a_1)q(a_2) + q(a_1)i(m_2) + i(m_1)q(a_2) + i(m_1)i(m_2) \\
&= q(a_1)q(a_2) + i(p \circ q(a_1)m_2) + i(p \circ q(a_2)m_1) \ \text{(by the } B\text{-module structure on } i(M)) \\
&= q(a_1a_2) + \{q(a_1)q(a_2) - q(a_1a_2)\} + i(a_1m_2 + a_2m_1) \\
&= q(a_1a_2) + i(a_1m_2 + a_2m_1 + \phi_q(a_1, a_2)) \\
&= (a_1a_2 \ , \ a_1m_2 + a_2m_1 + \phi_q(a_1, a_2))_q.
\end{aligned}$$

Thus the multiplication in B can be written as,

$$(a_1, m_1)_q(a_2, m_2)_q = (a_1a_2 \ , \ a_1m_2 + a_2m_1 + \phi_q(a_1, a_2))_q.$$

Suppose $(a_1, m_1)_q, (a_2, m_2)_q$ and $(a_3, m_3)_q \in B$, using the associativity of the algebra multiplication in $B$, we get,

$$a_1\phi_q(a_2,\ a_3) - \phi_q(a_1a_2,\ a_3) + \phi_q(a_1,\ a_2a_3) - a_3.\phi_q(a_1,\ a_2) = 0.$$

or, $\delta^2\phi_q(a_1,\ a_2,\ a_3) = 0$. So $\phi_q \in Ch^2(A; M) = Hom(S^2A; M)$ is a cocycle (cf. (2.2.3)).

Let $q' : A \longrightarrow B$ be another section of $p$. Replacing $q$ by $q'$ in the above argument, will give rise to a cocycle $\phi_{q'} \in Hom(S^2A; M)$. Set $\beta = i^{-1} \circ (q' - q) \in Hom(A; M)$. Then $\phi_{q'} - \phi_q = \delta^1\beta$. Thus for a given extension of $A$ by $M$ there is a unique cohomology class $[\phi_q]$ in $H^2_{Harr}(A; M)$.

Let

$$0 \longrightarrow M \xrightarrow{i'} B' \xrightarrow{p'} A \longrightarrow 0$$

be another extension of $A$ by $M$ which is isomorphic to the extension $B$ we took before. Let $\phi : B \longrightarrow B'$ be an isomorphism of extensions $B$ and $B'$. Thus the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & B & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \| & & \phi\downarrow & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{i'} & B' & \xrightarrow{p'} & A & \longrightarrow & 0 \end{array}$$

Then $q' = \phi \circ q : A \longrightarrow B'$ is a section for $p'$. We have $\phi_{q'} : S^2A \longrightarrow M$ defined by

$$\phi_{q'}(a_1, a_2) = (i')^{-1}(q'(a_1)q'(a_2) - q'(a_1a_2)) \text{ (cf. (2.3.1))}$$

Thus $\phi_{q'}(a_1, a_2) = (i')^{-1}(q'(a_1)q'(a_2) - q'(a_1a_2)) = i^{-1}(q(a_1)q(a_2) - q(a_1a_2)) = \phi_q(a_1, a_2)$. Consequently $\phi_q$ and $\phi_{q'}$ represents the same class in $H^2_{Harr}(A; M)$.

□

We shall later need the following specific extension of a finite dimensional commutative algebra $A$.

Observe that the action of $A$ on $\mathbb{K}$ has an induced action on the module $H^2_{Harr}(A; \mathbb{K})'$ as follows. For $\alpha \in H^2_{Harr}(A; \mathbb{K})'$ and $a \in A$ we get, $a\alpha \in H^2_{Harr}(A; \mathbb{K})'$ given by $a\alpha([\psi]) = \varepsilon(a)\alpha([\psi])$ where $[\psi] \in H^2_{Harr}(A; \mathbb{K})$.

Consider a linear map

$$\mu : H^2_{Harr}(A; \mathbb{K}) \longrightarrow Hom(S^2A;\ \mathbb{K}) = (S^2A)' \tag{2.3.2}$$

where $\mu([\phi])$ is a cocycle representing the class $[\phi]$. We define this map by fixing its values on a basis of $H^2_{Harr}(A; \mathbb{K})$ and then extend it linearly. Take the dual map of $\mu$,

$$\phi_A : S^2A \longrightarrow H^2_{Harr}(A; \mathbb{K})'. \tag{2.3.3}$$

So, $\phi_A(a_1,a_2)([\alpha]) = \mu([\alpha])(a_1,a_2)$ for $a_1,a_2 \in A$ and $[\alpha] \in H^2_{Harr}(A;\mathbb{K})$. Here

$$\begin{aligned} \delta^2\phi_A(a,b,c)([\alpha]) &= \{a\phi_A(b,c) - \phi_A(ab,c) + \phi_A(a,bc) - c\phi_A(a,b)\}([\alpha]) \\ &= a\mu([\alpha])(b,c) - \mu([\alpha])(ab,c) + \mu([\alpha])(a,bc) - c\mu([\alpha])(a,b) \\ &= 0 \text{ (cf. (2.2.3))}. \end{aligned}$$

Thus $\phi_A$ is a cocycle and represent a cohomology class in the second Harrison cohomology of $A$ with coefficients in $H^2_{Harr}(A;\mathbb{K})'$, where the action of $A$ on $H^2_{Harr}(A;\mathbb{K})'$ is induced by the action of $A$ on $\mathbb{K}$. By Proposition 2.3.3, the cohomology class of $\phi_A$ determines an isomorphism class of extension represented by

$$0 \longrightarrow H^2_{Harr}(A;\mathbb{K})' \longrightarrow C \longrightarrow A \longrightarrow 0, \tag{2.3.4}$$

where the algebra structure on $C = A \oplus H^2_{Harr}(A;\mathbb{K})'$ is determined by $\phi_A$ as in the proof of Proposition 2.3.3. This extension does not depend, up to isomorphism, on the choice of $\mu$ and possesses the following partial universal property.

**Proposition 2.3.4.** *Let $M$ be an $A$-module with $\mathfrak{M}M = 0$. Then the above extension admits a unique homomorphism into an arbitrary extension*

$$0 \longrightarrow M \xrightarrow{i'} B \xrightarrow{p'} A \longrightarrow 0$$

*of $A$.*

*Proof.* Let $[f] \in H^2_{Harr}(A;M) = H^2_{Harr}(A;\mathbb{K}) \otimes M$ be the cohomology class determined by the isomorphism class of the extension

$$0 \longrightarrow M \xrightarrow{i'} B \xrightarrow{p'} A \longrightarrow 0.$$

This $[f] \in H^2_{Harr}(A;\mathbb{K}) \otimes M$ defines a linear map

$$\tilde{f} : H^2_{Harr}(A;\mathbb{K})' \longrightarrow M \quad \text{(see Proposition 3.4.1 in Chapter 3).}$$

Using this we get the following homomorphism of extensions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2_{Harr}(A;\mathbb{K})' & \xrightarrow{i} & C & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \tilde{f}\big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & M & \xrightarrow{i'} & B & \xrightarrow{p'} & A & \longrightarrow & 0 \end{array}$$

The resulting homomorphism is unique as it is determined by $[f]$. □

The following proposition shows that an extension $B$ of a commutative local algebra

$A$ with identity by an $A$-module $M$ is also local and has identity.

**Proposition 2.3.5.** *Let*

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

*be an extension of a commutative local algebra $A$ with 1 by an $A$-module $M$ satisfying $\mathfrak{M}M = 0$, where $\mathfrak{M}$ is the maximal ideal in $A$. Then $B$ is local and has multiplicative identity.*

*Proof.* Let

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be an extension of a commutative local algebra $A$ by an $A$-module $M$ satisfying $\mathfrak{M}M = 0$, where $\mathfrak{M}$ is the maximal ideal in $A$. Then by Proposition 2.3.3, this extension determines a cocycle $\phi_q$ representing a cohomology class $[\phi_q] \in H^2_{Harr}(A;\mathbb{K})$ where $q$ is a section of the projection $p : B \longrightarrow A$.

Now $\delta^2\phi_q(1,1,a) = 0$, so $\phi_q(1,a) - \phi_q(1,a) + \phi_q(1,a) - a\phi_q(1,1) = 0$. Therefore $\phi_q(1,a) = a\phi_q(1,1)$. Consider the element $(1, -\phi_q(1,1))_q \in B$. Then for any arbitrary element $(a,m)_q \in B$ we have

$$(1, -\phi_q(1,1))_q(a,m)_q = (a, m - a\phi_q(1,1) + \phi_q(1,a))_q = (a,m)_q.$$

So, $(1, -\phi_q(1,1))_q$ is the identity in $B$.

Observe that if we set $\phi' = \phi_q - \delta^1\psi$, where $\psi \in Hom(A;M)$ is arbitrary map satisfying $\psi(1) = \phi_q(1,1)$, then the cocycle $\phi'$ satisfies $\phi'(1,a) = 0$ for $a \in A$ and $\phi' = \phi_{q'}$ for the section $q' = q - (i \circ \psi)$ (see the proof of Proposition 2.3.3). With respect to the identification $B \cong A \oplus M$ defined by $q'$, the multiplicative identity in $B$ can be expressed as $(1,0)_{q'}$.

Next, we show that $B$ is local.

Suppose $\mathfrak{M}$ is the maximal ideal in $A$. Consider the ideal $p^{-1}(\mathfrak{M}) \subset B$. Let $\mathfrak{N}$ be an ideal of $B$ containing $p^{-1}(\mathfrak{M})$. If possible, suppose there is an element $x \in \mathfrak{N}$ such that $p(x)$ is not in $\mathfrak{M}$. But $\mathfrak{M}$ is the unique maximal ideal in the local algebra $A$ so $p(x)$ must be a unit in $A$, that is $p(x)a_1 = 1$, for some $a_1 \in A$. Since $p$ is onto there is a $y \in B$ such that $p(y) = a_1$. This implies $p(xy) = p(x)p(y) = 1$. Thus $b = xy$ is an element in $\mathfrak{N}$ such that $p(b) = 1$. Now choose a section $q : A \longrightarrow B$ of $p$ with $q(1) = b$. Then according to our earlier notations $b = (1,0)_q$. Now for any element $(a,m)_q \in B$ we can write

$$(a,m)_q = (1,0)_q(a, m - \phi_q(1,a))_q.$$

Since $b = (1,0)_q \in \mathfrak{N}$ and $\mathfrak{N}$ is an ideal of $B$ we conclude that $(a,m)_q \in \mathfrak{N}$.

Thus our assumption implies that $\mathfrak{N} = B$. Hence $p^{-1}(\mathfrak{M})$ is maximal in $B$. □

Let us denote the maximal ideal of $B$ by $\mathfrak{M}_B$.

**Corollary 2.3.6.** *For $i(M) = N$, we have $\mathfrak{M}_B N = 0$.*

*Proof.* Take an element $x$ from $\mathfrak{M}_B$ and an element $n \in N$ such that $n = i(m)$, $m \in M$. Then $xi(m) = i(p(x)m) = i(\varepsilon(p(x))m) = 0$, where $\varepsilon$ is the augmentation in $A$ (cf. Definition 2.3.1). □

Let $\mathcal{A}$ denote the group of all automorphisms of any given extension.

**Proposition 2.3.7.** *There is a one-to-one correspondence between the group $\mathcal{A}$ of automorphisms of any given extension,*

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

*of $A$ and $H^1_{Harr}(A; M)$.*

*Proof.* Suppose $f : B \longrightarrow B$ is a $\mathbb{K}$-algebra isomorphism giving an automorphism of the given extension,

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0.$$

Thus we get the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & B & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & B & \xrightarrow{p} & A & \longrightarrow & 0 \end{array}$$

Now fix a section $q : A \longrightarrow B$ of $p$. As in the proof of Proposition 2.3.3, we have a linear isomorphism $\alpha : B \cong A \oplus M$ and let $(a, m)_q$ denote the inverse image $q(a) + i(m)$ of $(a, m)$ under this isomorphism.

Suppose $f((a, m)_q) = (f_1((a, m)_q), f_2((a, m)_q))_q$ for $(a, m)_q \in B$, where $f_1, f_2$ are maps obtained from $\alpha \circ f$ by taking projection into first and second components. By the above diagram we have $p \circ f = p$ and $f \circ i = i$. These in turn give $f_1((a, m)_q) = a$ and $f_2((0, m)_q) = m$ respectively. Therefore for $(a, m)_q \in B$, we get

$$\begin{aligned} & f((a, m)_q) = (f_1((a, m)_q), f_2((a, m)_q))_q \\ \text{or, } & f((a, 0)_q + (0, m)_q) = (a, f_2((a, m)_q))_q \\ \text{or, } & f((a, 0)_q) + f((0, m)_q) = (a, f_2((a, m)_q))_q \\ \text{or, } & (a, f_2(a, 0)_q)_q + (0, m)_q = (a, f_2((a, m)_q))_q \\ \text{or, } & (a, f_2((a, 0)_q) + m)_q = (a, f_2((a, m)_q))_q. \end{aligned}$$

So $f_2((a,m)_q) = \psi(a) + m$ where the map $\psi : A \longrightarrow M$ is given by $\psi(a) = f_2((a,0)_q)$. Let $a_1, a_2 \in A$. Then $(a_1,0)_q, (a_2,0)_q \in B$. Since $f$ is a $\mathbb{K}$-algebra homomorphism, we have,

$$\begin{aligned}
&f((a_1,\ 0)_q(a_2,\ 0)_q) = f((a_1,\ 0)_q)f((a_2,\ 0)_q)\\
&\text{or, } f((a_1a_2,\ \phi_q(a_1,\ a_2))_q) = (a_1,\ f_2((a_1,\ 0)_q))_q(a_2,\ f_2((a_2,\ 0)_q))_q\\
&\text{or, } (a_1a_2,\ \phi_q(a_1,\ a_2) + \psi(a_1,\ a_2))_q = (a_1,\ \psi(a_1))_q(a_2,\ \psi(a_2))_q\\
&\text{or, } (a_1a_2,\ \phi_q(a_1,\ a_2) + \psi(a_1,\ a_2))_q = (a_1a_2,\ a_1\psi(a_2) + a_2\psi(a_1) + \phi_q(a_1,a_2))_q\\
&\text{or, } \psi(a_1a_2) = a_1\psi(a_2) + a_2\psi(a_1)\\
&\text{or, } \delta^1\psi(a_1,a_2) = 0.
\end{aligned} \tag{2.3.5}$$

Therefore $\psi$ represents a cohomology class in $H^1_{Harr}(A;M)$.
Conversely, suppose $\psi : A \longrightarrow M$ is a linear map with $\delta^1\psi = 0$. So $\delta^1\psi(1,1) = 0$, which gives $\psi(1) = 0$ (cf. (2.2.3)).

Define, $f : B \longrightarrow B$ by $f((a,m)_q) = (a, m+\psi(a))_q$. Then $f(0,1)_q = (0, 1+\psi(1))_q = (0,1)_q$ and $f$ is an automorphism of the given extension.

The assignment $f \mapsto [\psi]$ is the required bijection. □

We shall use the following results ( [Har62]) in Chapter 5.

**Proposition 2.3.8.** *Let $A = \mathbb{K}[x_1, x_2, \cdots, x_n]$ be polynomial algebra, and let $\mathfrak{M}$ be the ideal of polynomials without constant terms. If an ideal $I$ of $A$ is contained in $\mathfrak{M}^2$, then $H^2_{Harr}(A/I;\mathbb{K}) \cong (I/\mathfrak{M}I)'$.*

**Proposition 2.3.9.** *If $A, \mathfrak{M}$ and $I$ are as in Proposition 2.3.8, then the extension*

$$0 \longrightarrow H^2_{Harr}(A/I;\mathbb{K})' \longrightarrow C \longrightarrow A/I \longrightarrow 0$$

*as given in (2.3.4) for $A/I$ is*

$$0 \longrightarrow I/\mathfrak{M}I \xrightarrow{i} A/\mathfrak{M}I \xrightarrow{p} A/I \longrightarrow 0,$$

*where $i$ and $p$ are induced by the inclusions $I \hookrightarrow A$ and $\mathfrak{M}I \hookrightarrow I$.*

To simplify notations, we henceforth omit superscripts for coboundaries, it should be clear from the context which coboundary is being used.

# Chapter 3

# Deformations of Leibniz algebras and homomorphisms of Leibniz algebras

## 3.1 Introduction

In this chapter we introduce the notion of deformations of Leibniz algebras and Leibniz algebra homomorphisms over a commutative local algebra base with multiplicative identity, and introduce infinitesimal deformation and other basic definitions related to deformations of a Leibniz algebra. We give a construction of an infinitesimal deformation $\eta_1$ of a Leibniz algebra $L$ for which $dim(HL^2(L;L))$ is finite. Next, we show that infinitesimal deformation $\eta_1$ is universal among the infinitesimal deformations of $L$ with finite dimensional local algebra base. We also prove a necessary and sufficient criterion for equivalence of two infinitesimal deformations of a Leibniz algebra. At the end we introduce the notion of infinitesimal deformations of Leibniz algebra homomorphisms and obtain a necessary and sufficient condition for equivalence of two infinitesimal deformations in this case. From now on we assume that $\mathbb{K}$ is a field of characteristic zero.

## 3.2 Deformations

Let $L$ be a Leibniz algebra and $A$ be a commutative local algebra with identity 1 over $\mathbb{K}$. Let $\mathfrak{M}$ be the maximal ideal of $A$ and $\varepsilon : A \longrightarrow A/\mathfrak{M} \cong \mathbb{K}$ be the canonical augmentation. Note that $\varepsilon$ is an algebra homomorphism with $\varepsilon(1) = 1$ and $ker(\varepsilon) = \mathfrak{M}$. By $(A, \mathfrak{M})$ we will mean that $A$ is a commutative local algebra with 1 and $\mathfrak{M}$ is the maximal ideal in $A$.

**Definition 3.2.1.** *A deformation $\lambda$ of $L$ with base $(A, \mathfrak{M})$, or simply with base $A$, is an $A$-Leibniz algebra structure on the tensor product $A \otimes L$ with the bracket $[-,-]_\lambda$ such that*

$$(\varepsilon \otimes id) : A \otimes L \to \mathbb{K} \otimes L$$

*is an $A$-Leibniz algebra homomorphism, where the $A$-Leibniz algebra structure on $\mathbb{K} \otimes L \cong L$ is given via $\varepsilon$, that is, $a(k \otimes l) = \varepsilon(a)k \otimes l$.*

**Remark 3.2.2.** *More generally, one has the notion of deformation with a commutative algebra base (not necessarily local) by fixing an augmentation. In this case, if the base is also local, then any deformation over such base is called a local deformation. Throughout, we will be concerned with deformations over local algebra base and we omit the adjective local and simply use the term deformation.*

**Definition 3.2.3.** *A deformation $\lambda$ over a commutative local algebra base $A$ is called infinitesimal (of first order) if $\mathfrak{M}^2 = 0$. In general it is called a $k$th order deformation when $\mathfrak{M}^{k+1} = 0$.*

Observe that if $\lambda$ is a deformation as in Definition 3.2.1, then for $l_1, l_2 \in L$ and $a, b \in A$ we have

$$[a \otimes l_1, b \otimes l_2]_\lambda = ab[1 \otimes l_1, 1 \otimes l_2]_\lambda,$$

by $A$- linearity of the bracket $[-,-]_\lambda$. Thus to define a deformation $\lambda$ of the Leibniz algebra $L$, it is enough to specify the brackets $[1 \otimes l_1, 1 \otimes l_2]_\lambda$ for $l_1, l_2 \in L$. Moreover, since $(\varepsilon \otimes id) : A \otimes L \to \mathbb{K} \otimes L$ is an $A$-Leibniz algebra homomorphism,

$$(\varepsilon \otimes id)([1 \otimes l_1, 1 \otimes l_2]_\lambda) = [l_1, l_2] = (\varepsilon \otimes id)(1 \otimes [l_1, l_2])$$

which implies

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda - 1 \otimes [l_1, l_2] \in ker(\varepsilon \otimes id).$$

Hence we can write

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_j c_j \otimes y_j,$$

where $\sum_j c_j \otimes y_j$ is a finite sum with $c_j \in ker(\varepsilon) = \mathfrak{M}$ and $y_j \in L$.

Next we define the notion of deformation of a homomorphism of Leibniz algebras.

**Definition 3.2.4.** *Let $f : L \longrightarrow M$ be a Leibniz algebra homomorphism from a Leibniz algebra $L$ to a Leibniz algebra $M$. A deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of $f$ with base $(A, \mathfrak{M})$ (or simply with base $A$) consists of deformations $\lambda$ and $\mu$ (with base $A$) of the Leibniz* *algebras $L$ and $M$ respectively along with an $A$-Leibniz algebra homomorphism*

$$f_{\lambda\mu} : (A \otimes L, \lambda) \longrightarrow (A \otimes M, \mu)$$

*such that the following diagram commutes.*

$$\begin{array}{ccc} A \otimes L & \xrightarrow{f_{\lambda\mu}} & A \otimes M \\ {\scriptstyle \varepsilon\otimes id}\downarrow & & {\scriptstyle \varepsilon\otimes id}\downarrow \\ \mathbb{K} \otimes L \cong L & \xrightarrow{f} & M \cong \mathbb{K} \otimes M \end{array}$$

Often we shall use the simpler notation $f_{\lambda\mu}$ to denote a deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of $f$. By $A$-linearity, $f_{\lambda\mu}$ is determined by its value $f_{\lambda\mu}(1 \otimes l) \in A \otimes M$ for $l \in L$. The commutativity of the above diagram implies

$$f_{\lambda\mu}(1 \otimes l) = 1 \otimes f(l) + \sum_j m_j \otimes x_j \text{ for } m_j \in \mathfrak{M} \text{ and } x_j \in M.$$

**Remark 3.2.5.** *If the algebra $A$ is a finite dimensional $\mathbb{K}$-module and $\{m_i\}_{1\leq i\leq r}$ is a basis of $\mathfrak{M}$ then a deformation $\lambda$ of $L$ can be written as*

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes l_i' \textit{ for } l_1, l_2, l_i' \in L.$$

*Similarly, a deformation $f_{\lambda\mu}$ of a Leibniz algebra homomorphism $f : L \longrightarrow M$ can be written as*

$$f_{\lambda\mu}(1 \otimes l_1) = 1 \otimes f(l_1) + \sum_{j=1}^{r} m_j \otimes x_j \textit{ for } l_1 \textit{ and } x_j \in M.$$

**Definition 3.2.6.** *Suppose $\lambda_1$ and $\lambda_2$ are two deformations of a Leibniz algebra $L$ with base $A$. We call them equivalent if there exists a Leibniz algebra isomorphism*

$$\phi : (A \otimes L, [-,-]_{\lambda_1}) \to (A \otimes L, [-,-]_{\lambda_2})$$

*such that $(\varepsilon \otimes id) \circ \phi = (\varepsilon \otimes id)$.*

We write $\lambda_1 \cong \lambda_2$ if $\lambda_1$ is equivalent to $\lambda_2$. The equivalence class of a deformation $\lambda$ will be denoted by $< \lambda >$.

Equivalence of two deformations $(\lambda, \mu; f_{\lambda\mu})$ and $(\lambda', \mu'; f_{\lambda'\mu'})$ of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with base $A$ is defined as follows.

**Definition 3.2.7.** *Any two deformations $(\lambda, \mu; f_{\lambda\mu})$ and $(\lambda', \mu'; f_{\lambda'\mu'})$ of $f : L \longrightarrow M$ with base $A$ are said to be equivalent, written as $(\lambda, \mu; f_{\lambda\mu}) \cong (\lambda', \mu'; f_{\lambda'\mu'})$, if there exist* *equivalences*

$$\Phi_{\lambda\lambda'} : (A \otimes L, [-,-]_\lambda) \longrightarrow (A \otimes L, [-,-]_{\lambda'})$$

$$\text{and } \Psi_{\mu\mu'} : (A \otimes M, [-,-]_\mu) \longrightarrow (A \otimes M, [-,-]_{\mu'})$$

*such that* $\Psi_{\mu\mu'} \circ f_{\lambda\mu} = f_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}$.

The equivalence class of a deformation $(\lambda, \mu; f_{\lambda\mu})$ will be denoted by $< \lambda, \mu; f_{\lambda\mu} >$. We denote by $\lambda_0$ the Leibniz algebra structure on $A \otimes L$ given by

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda_0} = 1 \otimes [l_1, l_2] \text{ for } l_1, l_2 \in L.$$

This is a deformation of $L$ with base $A$. Any deformation of $L$ with base $A$ which is equivalent to $\lambda_0$ is called a trivial deformation. Similarly for a Leibniz algebra homomorphism $f : L \longrightarrow M$, we have a deformation $(\lambda_0, \mu_0; f_{\lambda_0\mu_0})$ of $f$ where

$$f_{\lambda_0\mu_0}(1 \otimes l) = 1 \otimes f(l) \in A \otimes M \text{ for } l \in L,$$

and any deformation of $f$ equivalent to $(\lambda_0, \mu_0; f_{\lambda_0\mu_0})$ is said to be a trivial deformation of $f$.

## 3.3 Push-out of a deformation

Push-out is a method to produce new deformation from a given one by changing base by a given homomorphism from the given base to a new base.

Suppose $\lambda$ is a given deformation of a Leibniz algebra $L$ with commutative local algebra base with 1 and augmentation $\varepsilon : A \to \mathbb{K}$. Let $A'$ be another commutative local algebra with identity. Denote the augmentation of $A'$ by $\varepsilon'$. Suppose $\phi : A \to A'$ is an algebra homomorphism with $\phi(1) = 1$ so that $(\varepsilon' \circ \phi) = \varepsilon$. Let $ker(\varepsilon') = \mathfrak{M}'$ be the maximal ideal in $A'$.

**Definition 3.3.1.** *The push-out $\phi_*\lambda$ is a deformation of $L$ with base $(A', \mathfrak{M}')$ with bracket $[-,-]_{\phi_*\lambda}$ as given below. Consider $A'$ as an $A$-module by $a'a = a'\phi(a)$ so that*

$$A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L).$$

*Then the deformation $\phi_*\lambda$ is given by the bracket*

$$[{a_1}' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]_{\phi_*\lambda} = a_1' a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda$$

*where $a_1', a_2' \in A'$, $a_1, a_2 \in A$ and $l_1, l_2 \in L$.*

**Remark 3.3.2.** *If the bracket* $[-,-]_\lambda$ *is given by*

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_j c_j \otimes y_j \text{ for } c_j \in \mathfrak{M} \text{ and } y_j \in L, \tag{3.3.1}$$

*then the bracket* $[-,-]_{\phi_*\lambda}$ *can be written as*

$$[1 \otimes l_1, 1 \otimes l_2]_{\phi_*\lambda} = 1 \otimes [l_1, l_2] + \sum_j \phi(c_j) \otimes y_j. \tag{3.3.2}$$

**Proposition 3.3.3.** *With* $\phi$ *as above,* $(\phi \otimes id) : (A \otimes L, \lambda) \longrightarrow (A' \otimes L, \phi_*\lambda)$ *is an* $A$*-Leibniz algebra homomorphism.*

*Proof.* For $a \otimes l \in A \otimes L$, we have $(\phi \otimes id)(a \otimes l) = \phi(a) \otimes l = a\phi(1) \otimes l = a(\phi \otimes id)(1 \otimes l)$. So $\phi \otimes id$ is an $A$-linear map. We now show that $(\phi \otimes id)$ preserves the brackets. Suppose $1 \otimes l_1, 1 \otimes l_2 \in A \otimes L$ , then

$$\begin{aligned}
&[(\phi \otimes id)(1 \otimes l_1), (\phi \otimes id)(1 \otimes l_2)]_{\phi_*\lambda} \\
&= [\phi(1) \otimes l_1, \phi(1) \otimes l_2]_{\phi_*\lambda} \\
&= [1 \otimes_A (1 \otimes l_1), 1 \otimes_A (1 \otimes l_2)]_{\phi_*\lambda} \\
&= 1 \otimes_A [1 \otimes l_1, 1 \otimes l_2]_\lambda \\
&= 1 \otimes_A \{1 \otimes [l_1, l_2] + \sum_j c_j \otimes y_j\} \text{ ( by (3.3.1))} \\
&= 1 \otimes_A (1 \otimes [l_1, l_2]) + \sum_j 1 \otimes_A (c_j \otimes y_j) \\
&= \phi(1) \otimes [l_1, l_2] + \sum_j \phi(c_j) \otimes y_j \\
&= (\phi \otimes id)\{1 \otimes [l_1, l_2] + \sum_j c_j \otimes y_j\} \\
&= (\phi \otimes id)[1 \otimes l_1, 1 \otimes l_2]_\lambda.
\end{aligned} \tag{3.3.3}$$

□

**Remark 3.3.4.** *Observe that if* $\lambda$ *is a deformation of a Leibniz algebra* $L$ *with base* $A$ *then the push-out* $\varepsilon_*\lambda$ *via* $\varepsilon$ *is the original Leibniz bracket in* $L$*. To see this, note that the Leibniz bracket* $\varepsilon_*\lambda$ *on* $(\mathbb{K} \otimes L) = \mathbb{K} \otimes_A (A \otimes L)$ *where the* $A$*-module structure on* $\mathbb{K}$ *is given via* $\varepsilon$*, is obtained as follows.*

*For* $k_1 \otimes_A (a_1 \otimes l_1), k_2 \otimes_A (a_2 \otimes l_2) \in \mathbb{K} \otimes L \cong L$,

$$\begin{aligned}
&[k_1 \otimes_A (a_1 \otimes l_1), k_2 \otimes_A (a_2 \otimes l_2)]_{\varepsilon_*\lambda} \\
&= k_1 k_2 \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda
\end{aligned}$$

$$
\begin{aligned}
&= k_1k_2 \otimes_A a_1a_2[1 \otimes l_1, 1 \otimes l_2]_\lambda \\
&= k_1k_2 \otimes_A a_1a_2 \left(1 \otimes [l_1, l_2] + \sum_j c_j \otimes y_j\right), \ c_j \in \mathfrak{M}, y_j \in L, (\ \textit{by (3.3.1)}\ ) \\
&= k_1k_2 \otimes_A (a_1a_2 \otimes [l_1, l_2]) + k_1k_2 \otimes_A a_1a_2 \left(\sum_j c_j \otimes y_j\right) \\
&= k_1k_2 \otimes_A (a_1a_2 \otimes [l_1, l_2]) + \sum_j k_1k_2 \otimes_A (a_1a_2c_j \otimes y_j) \\
&= k_1k_2\varepsilon(a_1a_2) \otimes [l_1, l_2] + \sum_j k_1k_2\varepsilon(a_1a_2c_j) \otimes y_j \\
&= k_1k_2\varepsilon(a_1)\varepsilon(a_2) \otimes [l_1, l_2] + \sum_j k_1k_2\varepsilon(a_1)\varepsilon(a_2)\varepsilon(c_j) \otimes y_j \\
&= [k_1\varepsilon(a_1)l_1, k_2\varepsilon(a_2)l_2] \quad (\textit{since } \varepsilon(c_j) = 0, \ \textit{for } c_j \in \textit{ker}(\varepsilon) = \mathfrak{M}) \\
&= [k_1 \otimes_A (a_1 \otimes l_1), k_2 \otimes_A (a_2 \otimes l_2)].
\end{aligned}
$$

*Therefore $\varepsilon_*\lambda$ is the given bracket in $L$.*

**Proposition 3.3.5.** *Push-out is preserved under composition, in other words, if $\phi : A \longrightarrow A'$ and $\psi : A' \longrightarrow A''$ are homomorphisms of commutative local algebras with* 1 *and $\phi(1) = 1$, $\psi(1) = 1$ and $\lambda$ is a given deformation of $L$ with base $A$ then $(\psi \circ \phi)_*\lambda = \psi_*(\phi_*\lambda)$.*

*Proof.* $\phi : A \longrightarrow A'$ and $\psi : A' \longrightarrow A''$. So we have two brackets $[-,-]_{(\psi\circ\phi)_*\lambda}$ and $[-,-]_{\psi_*(\phi_*\lambda)}$ on $A'' \otimes L$. We need to show that they are equal.

As before, we write $A'' \otimes L = A'' \otimes_A (A \otimes L)$, where $A''$ is considered as an $A$-module via $\psi \circ \phi$. Let $a''_1 \otimes_A (a_1 \otimes l_1)$ and $a''_2 \otimes_A (a_2 \otimes l_2)$ be any two elements in $A'' \otimes L$. Then by definition of push-out

$$
\begin{aligned}
&[a''_1 \otimes_A (a_1 \otimes l_1), a''_2 \otimes_A (a_2 \otimes l_2)]_{(\psi\circ\phi)_*\lambda} \\
&= a''_1 a'' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda \\
&= (a''_1 a''_2)(a_1a_2) \otimes_A [1 \otimes l_1, 1 \otimes l_2]_\lambda \\
&= a''_1 a''_2 \psi(\phi(a_1))\psi(\phi(a_2)) \otimes_A [1 \otimes l_1, 1 \otimes l_2]_\lambda \\
&= (a''_1 \otimes_{A'} \phi(a_1))(a''_2 \otimes_{A'} \phi(a_2)) \otimes_A [1 \otimes l_1, 1 \otimes l_2]_\lambda \\
&\quad (\text{writing } A'' = A'' \otimes_{A'} A', \text{ an element } a''\psi(a') \text{ corresponds to } a'' \otimes_{A'} a') \\
&= (a''_1 a''_2 \phi(a_1)\phi(a_2)) \otimes_A [1 \otimes l_1, 1 \otimes l_2]_\lambda \\
&= a''_1 a''_2 \otimes_{A'} (\phi(a_1)\phi(a_2) \otimes_A [1 \otimes l_1, 1 \otimes l_2]_\lambda) \\
&= a''_1 a''_2 \otimes_{A'} (1 \otimes_A a_1a_2[1 \otimes l_1, 1 \otimes l_2]_\lambda) \\
&= a''_1 a''_2 \otimes_{A'} \otimes_{A'} (1 \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda)
\end{aligned}
$$

$= a_1''a_2'' \otimes_{A'} \otimes_{A'}([1 \otimes_A a_1 \otimes l_1, 1 \otimes_A a_2 \otimes l_2]_{\phi_*\lambda})$ ( 1 on the right-hand side is $1 \in A'$)
$= [a_1'' \otimes_{A'} ((1 \otimes_A a_1) \otimes l_1), a_2'' \otimes_{A'} ((1 \otimes_A a_2) \otimes l_2)]_{\psi_*(\phi_*\lambda)}$
$= [(a_1'' \otimes_{A'} 1) \otimes_A (a_1 \otimes l_1), (a_2'' \otimes_{A'} 1) \otimes_A (a_2 \otimes l_2)]_{\psi_*(\phi_*\lambda)}$
$= [a_1'' \otimes_A (a_1 \otimes l_1), a_2'' \otimes_A (a_2 \otimes l_2)]_{\psi_*(\phi_*\lambda)}$.

Note that under the isomorphism $A'' \cong A'' \otimes_{A'} A'$, an element $a''$ corresponds to $a'' \otimes_{A'} 1$. □

Let $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ be a given deformation of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with base $A$. Then the push-out of $\mathfrak{D}$ by $\phi : A \longrightarrow A'$ is defined as follows.

**Definition 3.3.6.** *The push-out $\phi_*\mathfrak{D} = (\phi_*\lambda, \phi_*\mu; \phi_* f_{\lambda\mu})$ is a deformation of $f$ with base $A'$ where $\phi_*\lambda$ and $\phi_*\mu$ are as in Definition 3.3.1, and*

$$\phi_* f_{\lambda\mu} : (A' \otimes L, \phi_*\lambda) \longrightarrow (A' \otimes M, \phi_*\mu)$$

$$\text{is given by } \phi_* f_{\lambda\mu}(a_1' \otimes_A (a_1 \otimes l_1)) = a_1' \otimes_A f_{\lambda\mu}(a_1 \otimes l_1)$$

*for $a_1', a_2' \in A', a_1, a_2 \in A$ and $l_1, l_2 \in L$.*

**Proposition 3.3.7.** *Suppose $\lambda$ and $\lambda'$ are two deformations of the Leibniz algebra $L$ with base $A$. If $\lambda$ and $\lambda'$ are equivalent deformations of $L$ with base $A$ then $\phi_*\lambda$ and $\phi_*\lambda'$ are also equivalent.*

*Proof.* Let $U : (A \otimes L, \lambda) \longrightarrow (A \otimes L, \lambda')$ be an isomorphism of the Leibniz algebras $\lambda$ and $\lambda'$. Now consider the Leibniz algebras $\phi_*\lambda$ and $\phi_*\lambda'$ on

$$A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L),$$

which are given by

$$[{a_1}' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]_{\phi_*\lambda} = a_1'a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda$$

$$\text{and } [{a_1}' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]_{\phi_*\lambda'} = a_1'a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_{\lambda'}$$

where $a_1', a_2' \in A',\ a_1, a_2 \in A$ and $l_1, l_2 \in L$.

Define an $A'$-linear map $U' : (A' \otimes L, \phi_*\lambda) \longrightarrow (A' \otimes L, \phi_*\lambda')$ by

$$U'(a' \otimes_A (a \otimes l)) = a' \otimes_A U(a \otimes l).$$

Now

$$\begin{aligned}
U'[a_1{}' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]_{\phi_*\lambda} &= U'(a_1'a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda) \\
&= a_1'a_2' \otimes_A U([a_1 \otimes l_1, a_2 \otimes l_2]_\lambda) \\
&= a_1'a_2' \otimes_A [U(a_1 \otimes l_1), U(a_2 \otimes l_2)]_{\lambda'} \\
&= [a_1{}' \otimes_A U(a_1 \otimes l_1), a_2' \otimes_A U(a_2 \otimes l_2)]_{\phi_*\lambda'} \\
&= [U'(a_1{}' \otimes_A (a_1 \otimes l_1)), U'(a_2' \otimes_A (a_2 \otimes l_2))]_{\phi_*\lambda'}.
\end{aligned}$$

So $U'$ is a homomorphism of Leibniz algebra. It is clear that $U'$ is bijective since $U$ is bijective and

$$\begin{aligned}
(\varepsilon' \otimes id) \circ U'(a' \otimes_A (a \otimes l)) &= (\varepsilon' \otimes id)(a' \otimes_A U(a \otimes l)) \\
&= (\varepsilon' \otimes id)(a' \otimes_A (\phi \otimes id) \circ U(a \otimes l)) \\
&= \varepsilon'(a')(\varepsilon' \circ \phi \otimes id) \circ U(a \otimes l) \\
&= \varepsilon'(a')(\varepsilon \otimes id) \circ U(a \otimes l) \\
&= \varepsilon'(a)(\varepsilon \otimes id)(a \otimes l) \quad (\text{since } (\varepsilon \otimes id) \circ U = \varepsilon \otimes id) \\
&= \varepsilon'(a)\varepsilon(a) \otimes l \\
&= \varepsilon'(a)\varepsilon' \circ \phi(a) \otimes l \\
&= (\varepsilon' \otimes id)(a'\phi(a) \otimes l) \\
&= (\varepsilon' \otimes id)(a' \otimes_A (a \otimes l)).
\end{aligned}$$

Consequently, $U'$ is an equivalence of the Leibniz algebras $\phi_*\lambda$ and $\phi_*\lambda'$. □

**Proposition 3.3.8.** *Suppose $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda', \mu'; f_{\lambda'\mu'})$ are two deformations of a Leibniz algebra homomorphisms $f : L \longrightarrow M$ with base $A$ then $\phi_*\mathfrak{D}$ and $\phi_*\mathfrak{D}'$ are also equivalent.*

*Proof.* Let $U : (A \otimes L, \lambda) \longrightarrow (A \otimes L, \lambda')$ and $V : (A \otimes M, \mu) \longrightarrow (A \otimes M, \mu')$ gives equivalences of $\mathfrak{D}$ and $\mathfrak{D}'$.

So, $U$ is an isomorphism of Leibniz algebras $\lambda$ and $\lambda'$, $V$ is an isomorphism of Leibniz algebras $\mu$ and $\mu'$ such that $V \circ f_{\lambda\mu} = f_{\lambda'\mu'} \circ U$.

We now consider the push-out deformations $\phi_*\mathfrak{D} = (\phi_*\lambda, \phi_*\mu; \phi_* f_{\lambda\mu})$ and $\phi_*\mathfrak{D}' = (\phi_*\lambda', \phi_*\mu'\phi_* f_{\lambda'\mu'})$ of $f$ with base $A'$.

Take $U' : (A' \otimes L, \phi_*\lambda) \longrightarrow (A' \otimes L, \phi_*\lambda')$ defined by $U'(a' \otimes_A (a \otimes l)) = a' \otimes_A U(a \otimes l)$ and $V' : (A' \otimes M, \phi_*\mu) \longrightarrow (A' \otimes M, \phi_*\mu')$ defined by $V'(a' \otimes_A (a \otimes x)) = a' \otimes_A U(a \otimes x)$ for $a' \otimes_A (a \otimes l) \in A' \otimes_A (A \otimes L) = A' \otimes L$ and $a' \otimes_A (a \otimes x) \in A' \otimes_A (A \otimes M) = A' \otimes M$.

From Proposition 3.3.7 it follows that $U'$ and $V'$, respectively give equivalences $\phi_*\lambda \cong \phi_*\lambda'$ and $\phi_*\mu \cong \phi_*\mu'$. The homomorphisms $\phi_* f_{\lambda\mu}$ and $\phi_* f_{\lambda'\mu'}$ are given by

$\phi_* f_{\lambda\mu}(a' \otimes_A (a \otimes l)) = a' \otimes_A f_{\lambda\mu}(a \otimes l)$ and $\phi_* f_{\lambda'\mu'}(a' \otimes_A (a \otimes l)) = a' \otimes_A f_{\lambda'\mu'}(a \otimes l)$ for $a' \otimes_A (a \otimes l) \in A' \otimes L$. Then we get

$$\begin{aligned}\phi_* f_{\lambda'\mu'} \circ U'(a' \otimes_A (a \otimes l)) =& \phi_* f_{\lambda'\mu'}(a' \otimes_A U(a \otimes l))\\ =& a' \otimes_A f_{\lambda'\mu'} \circ U(a \otimes l)\\ =& a' \otimes_A V \circ f_{\lambda\mu}(a \otimes l)\\ =& V'(a' \otimes_A f_{\lambda\mu}(a \otimes l))\\ =& V' \circ \phi_* f_{\lambda\mu}(a' \otimes_A (a \otimes l)).\end{aligned}$$

Therefore it follows that $\phi_*\mathfrak{D} = (\phi_*\lambda, \phi_*\mu; \phi_* f_{\lambda\mu})$ and $\phi_*\mathfrak{D}' = (\phi_*\lambda', \phi_*\mu'; \phi_* f_{\lambda'\mu'})$ are equivalent deformations of $f$ with base $A'$. □

**Corollary 3.3.9.** *Push-out of a trivial deformation is trivial.*

## 3.4 Construction of an infinitesimal deformation

In this section we construct a specific infinitesimal deformation of a Leibniz algebra $L$ with $dim(HL^2(L; L)) < \infty$, which turns out to be universal in the class of all infinitesimal deformations of $L$.

Let $L$ be a Leibniz algebra which satisfies the condition that the second cohomology module $HL^2(L; L)$ is a finite dimensional module over $\mathbb{K}$. This is true for example, if $L$ is a finite dimensional module over $\mathbb{K}$. Thoughout this chapter, we denote $HL^2(L; L)$ by $\mathbb{H}$.

Consider the algebra $C_1 = \mathbb{K} \oplus \mathbb{H}'$ by setting $(k_1, h_1)(k_2, h_2) = (k_1k_2, k_1h_2 + k_2h_1)$ where $\mathbb{H}'$ is the dual module of $\mathbb{H}$. Observe that the second summand is an ideal of $C_1$ with zero multiplication and we get the following trivial extension of $\mathbb{K}$ by $\mathbb{H}'$.

$$0 \longrightarrow \mathbb{H}' \xrightarrow{i_1} C_1 \xrightarrow{p_1} \mathbb{K} \longrightarrow 0, \ i_1(\alpha) = (0, \alpha) \text{ and } p_1(k, \alpha) = k.$$

Fix a $\mathbb{K}$-linear map

$$\mu : \mathbb{H} \longrightarrow CL^2(L; L) = Hom(L^{\otimes 2}; L)$$

which takes a cohomology class into a cocycle representing it. Such a linear map $\mu$ can be obtained by fixing its values on a basis of $\mathbb{H}$ and then extending it linearly. We need the following isomorphism of $\mathbb{K}$-modules.

**Proposition 3.4.1.** $\mathbb{H}' \otimes L \cong Hom(\mathbb{H}\ ; L)$.

*Proof.* Let $dim(\mathbb{H}) = n$. Suppose $\{h_i\}_{1 \leq i \leq n}$ is a basis of $\mathbb{H}$ and $\{g_i\}_{1 \leq i \leq n}$ is the dual basis. Let $\xi = \sum_{i=1}^{n} \alpha_i g_i \in H'$. Then for $l \in L$, $\xi \otimes l = \sum_{i=1}^{n} \alpha_i (g_i \otimes l) \in \mathbb{H}' \otimes L$.

Define a linear map $F : \mathbb{H}' \otimes L \longrightarrow Hom(\mathbb{H}; L)$ by $F(g_i \otimes l) = \phi_i \in Hom(\mathbb{H}; L)$ where $\phi_i(h_j) = \delta_{ij} l$ for $0 \leq i, j \leq n$. Then

$$F(\xi \otimes l) = F\left(\left(\sum_{i=1}^{n} \alpha_i g_i\right) \otimes l\right) = \sum_{i=1}^{n} F(\alpha_i g_i \otimes l) = \sum_{i=1}^{n} \alpha_i \phi_i.$$

Clearly $F$ is linear and having $kerF = \{0\}$. Now let $\phi \in Hom(\mathbb{H}; L)$ where $\phi(h_i) = \beta_i \in L$. Consider $\sum_{i=1}^{n} g_i \otimes \beta_i \in \mathbb{H}' \otimes L$. Then

$$F\left(\sum_{i=1}^{n} g_i \otimes \beta_i\right)(h_j) = \sum_{i=1}^{n} F(g_i \otimes \beta_i)(h_j) = \sum_{i=1}^{n} \delta_{ij} \beta_i = \beta_j = \phi(h_j).$$

This shows that $\sum_{i=1}^{n} g_i \otimes \beta_i \in \mathbb{H}' \otimes L$ is the preimage of $\phi \in Hom(H; L)$, so $F$ is onto. Consequently $F$ is an isomorphism. □

By Proposition 3.4.1, we have

$$C_1 \otimes L = (\mathbb{K} \oplus \mathbb{H}') \otimes L = (\mathbb{K} \otimes L) \oplus (\mathbb{H}' \otimes L) = L \oplus Hom(\mathbb{H}\ ; L).$$

We use the above identification to define a bilinear bracket $[-,-]$ on $C_1 \otimes L$ as follows. For $(l_1, \phi_1), (l_2, \phi_2) \in L \oplus Hom(\mathbb{H}\ ; L)$,

$$[(l_1, \phi_1), (l_2, \phi_2)] = ([l_1, l_2], \psi)$$

where the map $\psi : \mathbb{H} \longrightarrow L$ is given by

$$\psi(\alpha) = \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)] \text{ for } \alpha \in \mathbb{H}\ .$$

**Proposition 3.4.2.** *The $\mathbb{K}$-module $C_1 \otimes L$ equipped with the bracket defined above is a Leibniz algebra.*

*Proof.* The bracket is clearly additive in each variable.
Let $(l_1, \phi_1), (l_2, \phi_2) \in L \oplus Hom(\mathbb{H}; L)$ with $[(l_1, \phi_1), (l_2, \phi_2)] = ([l_1, l_2], \psi)$. So

$$\psi(\alpha) = \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)] \quad \text{for } \alpha \in \mathbb{H}.$$

Now for $a \in C_1$,

$$\begin{aligned} a[(l_1, \phi_1), (l_2, \phi_2)] &= a([l_1, l_2], \psi) \\ &= (a[l_1, l_2], a\psi) \\ &= (\varepsilon(a)[l_1, l_2], \varepsilon(a)\psi) \\ &= (\ [\varepsilon(a)l_1, l_2],\ \varepsilon(a)\psi\ ) \end{aligned}$$

$$
\begin{aligned}
&= [(\varepsilon(a)l_1, \varepsilon(a)\phi_1), (l_2, \phi_2)] \\
&= [(al_1, a\phi_1), (l_2, \phi_2)] \\
&= [a(l_1, \phi_1), (l_2, \phi_2)].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
a[(l_1, \phi_1), (l_2, \phi_2)] &= a([l_1, l_2], \psi) \\
&= (a[l_1, l_2], a\psi) \\
&= (\varepsilon(a)[l_1, l_2], \varepsilon(a)\psi) \\
&= ([l_1, \varepsilon(a)l_2], \varepsilon(a)\psi) \\
&= [(l_1, \phi_1), (\varepsilon(a)l_2, \varepsilon(a)\phi_2)] \\
&= [(l_1, \phi_1), (al_2, a\phi_2)] \\
&= [(l_1, \phi_1), a(l_2, \phi_2)].
\end{aligned}
$$

This shows that the bracket $[-,-]$ defined above on the module $C_1 \otimes L$ is $C_1$-bilinear. It remains to check the Leibniz relation. Let $(l_1, \phi_1), (l_2, \phi_2), (l_3, \phi_3) \in L \oplus Hom(\mathbb{H}; L)$. Suppose

$$
\begin{aligned}
&[(l_1, \phi_1), (l_2, \phi_2)] = ([l_1, l_2], \psi_{12}), \quad [(l_2, \phi_2), (l_3, \phi_3)] = ([l_2, l_3], \psi_{23}) \\
\text{and } &[(l_1, \phi_1), (l_3, \phi_3)] = ([l_1, l_3], \psi_{13}),
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_{12}(\alpha) &= \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)], \\
\psi_{23}(\alpha) &= \mu(\alpha)(l_2, l_3) + [\phi_2(\alpha), l_3] + [l_2, \phi_3(\alpha)] \\
\text{and } \psi_{13}(\alpha) &= \mu(\alpha)(l_1, l_3) + [\phi_1(\alpha), l_3] + [l_1, \phi_3(\alpha)].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&[(l_1, \phi_1), [(l_2, \phi_2), (l_3, \phi_3)]] - [[(l_1, \phi_1), (l_2, \phi_2)], (l_3, \phi_3)][[(l_1, \phi_1), (l_3, \phi_3)], (l_2, \phi_2)] \\
&= [(l_1, \phi_1), ([l_2, l_3], \psi_{23})] - [([l_1, l_2], \psi_{12}), (l_3, \phi_3)] + [([l_1, l_3], \psi_{13}), (l_2, \phi_2)] \\
&= ([l_1, [l_2, l_3]], \psi_{1(23)}) - ([[l_1, l_2], l_3], \psi_{(12)3}) + ([[l_1, l_3], l_2], \psi_{(13)2}) \\
&= ([l_1, [l_2, l_3]] - [[l_1, l_2], l_3] + [[l_1, l_3], l_2], \psi_{1(23)} - \psi_{(12)3} + \psi_{(13)2}).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&(\psi_{1(23)} - \psi_{(12)3} + \psi_{(13)2})(\alpha) \\
&= \psi_{1(23)}(\alpha) - \psi_{(12)3}(\alpha) + \psi_{(13)2})(\alpha) \\
&= \mu(\alpha)(l_1, [l_2, l_3]) + [\phi_1(\alpha), [l_2, l_3]] + [l_1, \psi_{23}(\alpha)] - \mu(\alpha)([l_1, l_2], l_3) \\
&\quad - [\psi_{12}(\alpha), l_3] - [[l_1, l_2], \phi_3(\alpha)] + \mu(\alpha)([l_1, l_3], l_2) + [\psi_{13}(\alpha), l_2] + [[l_1, l_3], \phi_2(\alpha)] \\
&= \delta\mu(\alpha)(l_1, l_2, l_3) \text{ (by (1.3.1) in Chapter1)} \\
&= 0 \quad \text{(since } \mu(\alpha) \text{ is a cochain representing } \alpha,\ \delta\mu(\alpha) = 0).
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
&[(l_1, \phi_1), [(l_2, \phi_2), (l_3, \phi_3)]] - [[(l_1, \phi_1), (l_2, \phi_2)], (l_3, \phi_3)] + [[(l_1, \phi_1), (l_3, \phi_3)], (l_2, \phi_2)] \\
&= 0 \in C_1 \otimes L.
\end{aligned}
$$

So

$$
[(l_1, \phi_1), [(l_2, \phi_2), (l_3, \phi_3)]] = [[(l_1, \phi_1), (l_2, \phi_2)], (l_3, \phi_3)] - [[(l_1, \phi_1), (l_3, \phi_3)], (l_2, \phi_2)].
$$

□

Since $C_1$ is local with maximal ideal $\mathbb{H}'$ such that $\mathbb{H}'^2 = 0$, we get an infinitesimal deformation of $L$ with base $C_1 = \mathbb{K} \oplus \mathbb{H}'$.

**Proposition 3.4.3.** *Up to an isomorphism, this deformation does not depend on the choice of $\mu$.*

*Proof.* Let

$$\mu' : \mathbb{H} \longrightarrow CL^2(L; L)$$

be another choice for $\mu$. Then for $\alpha \in \mathbb{H}$ , the cochains $\mu(\alpha)$ and $\mu'(\alpha)$ in $CL^2(L; L)$ represent the same cohomology class $\alpha$. So $\mu(\alpha) - \mu'(\alpha)$ is a coboundary. Hence we can define a $\mathbb{K}$-linear map

$$\gamma : \mathbb{H} \longrightarrow CL^1(L; L)$$

on a basis $\{h_i\}_{1 \leq i \leq n}$ of $\mathbb{H}$ by $\gamma(h_i) = \gamma_i$ with $\delta\gamma_i = \mu(h_i) - \mu'(h_i)$. Clearly, $\mu - \mu' = \delta\gamma$.

Using the identification $C_1 \otimes L \cong L \oplus Hom(\mathbb{H}\ ; L)$, define

$$\rho : C_1 \otimes L \longrightarrow C_1 \otimes L \ \text{ by } \rho((l, \phi)) = (l, \psi),$$

where $\psi(\alpha) = \phi(\alpha) + \gamma(\alpha)(l)$, $l \in L$ and $\phi \in Hom(\mathbb{H}\ ; L)$. Then $\rho$ is a $C_1$-linear automorphism of $C_1 \otimes L$ with $\rho^{-1}(l, \psi) = (l, \phi)$ with $\phi(\alpha) = \psi(\alpha) - \gamma(\alpha)(l)$ for $\alpha \in \mathbb{H}$ .

It remains to show that $\rho$ preserves the bracket.

Let $(l_1,\phi_1),(l_2,\phi_2)\in C_1\otimes L$ with $\rho(l_1,\phi_1)=(l_1,\psi_1)$ and $\rho(l_2,\phi_2)=(l_2,\psi_2)$. Suppose $[(l_1,\phi_1),(l_2,\phi_2)]=([l_1,l_2],\phi_3)$ where $\phi_3(\alpha)=\mu(\alpha)(l_1,l_2)+[\phi_1(\alpha),l_2]+[l_1,\phi_2(\alpha)]$, and $[(l_1,\psi_1),(l_2,\psi_2)]=([l_1,l_2],\psi_3)$ where $\psi_3(\alpha)=\mu'(\alpha)(l_1,l_2)+[\psi_1(\alpha),l_2]+[l_1,\psi_2(\alpha)]$. Then

$$\begin{aligned}\psi_3(\alpha)&=\mu'(\alpha)(l_1,l_2)+[\psi_1(\alpha),l_2]+[l_1,\psi_2(\alpha)]\\&=\mu(\alpha)(l_1,l_2)-\delta\gamma(\alpha)(l_1,l_2)+[\phi_1(\alpha)+\gamma(\alpha)(l_1),l_2]+[l_1,\phi_2(\alpha)+\gamma(\alpha)(l_2)]\\&=\mu(\alpha)(l_1,l_2)-[l_1,\gamma(\alpha)(l_2)]-[\gamma(\alpha)(l_1),l_2]+\gamma(\alpha)([l_1,l_2])+[\phi_1(\alpha),l_2]\\&\qquad\qquad+[\gamma(\alpha)(l_1),l_2]+[\phi_1(\alpha),l_2]+[l_1,\phi_2(\alpha)]+[l_1,\gamma(\alpha)(l_2)]\\&=\mu(\alpha)(l_1,l_2)+[\phi_1(\alpha),l_2]+[l_1,\phi_2(\alpha)]+\gamma(\alpha)([l_1,l_2])\\&=\phi_3(\alpha)+\gamma(\alpha)([l_1,l_2]).\end{aligned}$$

Hence $\rho([l_1,l_2],\phi_3)=([l_1,l_2],\psi_3)=[(l_1,\psi_1),(l_2,\psi_2)]=[\rho(l_1,\phi_1),\rho(l_2,\phi_2)]$.

Therefore, up to an isomorphism, the infinitesimal deformation obtained above is independent of the choice of $\mu$. □

We shall denote this deformation of $L$ by $\eta_1$.

**Remark 3.4.4.** *Suppose $\{h_i\}_{1\le i\le n}$ is a basis of $\mathbb{H}$ and $\{g_i\}_{1\le i\le n}$ is the dual basis. Let $\mu(h_i)=\mu_i\in CL^2(L;L)$. Under the identification $C_1\otimes L\cong L\oplus Hom(\mathbb{H};L)$, an element $(l,\phi)\in L\oplus Hom(\mathbb{H};L)$ corresponds to $1\otimes l+\sum_{i=1}^n g_i\otimes\phi(h_i)$. Then for $(l_1,\phi_1),(l_2,\phi_2)\in L\oplus Hom(\mathbb{H};L)$ their bracket $([l_1,l_2],\psi)$ corresponds to*

$$1\otimes[l_1,l_2]+\sum_{i=1}^n g_i\otimes(\mu_i(l_1,l_2)+[\phi_1(h_i),l_2]+[l_1,\phi_2(h_i)]).$$

*In particular, for $l_1,l_2\in L$ we have*

$$[1\otimes l_1,1\otimes l_2]_{\eta_1}=1\otimes[l_1,l_2]+\sum_{i=1}^n g_i\otimes\mu_i(l_1,l_2).$$

The main property of $\eta_1$ is that it is universal in the class of infinitesimal deformations with a finite dimensional local algebra base (Theorem 3.4.11).

Let $\mathcal{C}$ be the category of finite dimensional commutative local algebras with 1. Let $\lambda$ be any infinitesimal deformation of $L$ with base $(A,\mathfrak{M})\in\mathcal{C}$. Let $\{m_i\}_{1\le i\le r}$ be a basis of $\mathfrak{M}$ and $\{\xi_i\}_{1\le i\le r}$ be the corresponding dual basis. Any element $\xi\in\mathfrak{M}'$ can be extended as an element $\xi\in A'$ with $\xi(1)=0$.

Define a cochain $\alpha_{\lambda,\xi}\in CL^2(L;L)$ by

$$\alpha_{\lambda,\xi}(l_1,l_2)=(\xi\otimes id)([1\otimes l_1,1\otimes l_2]_\lambda),\ \text{for } l_1,l_2\in L. \tag{3.4.1}$$

Setting $\psi_i = \alpha_{\lambda,\xi_i}$ for $1 \leq i \leq r$, the Leibniz bracket (3.3.1) in terms of the basis of $\mathfrak{M}$ takes the form

$$\begin{aligned}[1 \otimes l_1, 1 \otimes l_2]_\lambda &= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes x_i \\ &= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes \psi_i(l_1, l_2).\end{aligned} \tag{3.4.2}$$

**Proposition 3.4.5.** *With $\lambda$ as above, the cochain $\alpha_{\lambda,\xi} \in CL^2(L; L)$ is a cocycle.*

*Proof.* By definition of coboundary in (1.3.1), if we take $l_1, l_2, l_3 \in L$ then

$$\begin{aligned}\delta\alpha_{\lambda,\xi}(l_1, l_2, l_3) =& [l_1, \alpha_{\lambda,\xi}(l_2, l_3)] + [\alpha_{\lambda,\xi}(l_1, l_3), l_2] - [\alpha_{\lambda,\xi}(l_1, l_2), l_3] \\ &- \alpha_{\lambda,\xi}([l_1, l_2], l_3) + \alpha_{\lambda,\xi}([l_1, l_3], l_2) + \alpha_{\lambda,\xi}(l_1, [l_2, l_3]).\end{aligned} \tag{3.4.3}$$

For $1 \otimes l_1, 1 \otimes l_2, 1 \otimes l_3 \in A \otimes L$,

$$\begin{aligned}&(\xi \otimes id)([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda) \\ &= (\xi \otimes id)([1 \otimes l_1, 1 \otimes [l_2, l_3]]_\lambda + [1 \otimes l_1, \sum_{j=1}^{r} m_j \otimes \psi_j(l_2, l_3)]_\lambda) \text{ (using (3.4.2))} \\ &= \alpha_{\lambda,\xi}(l_1, [l_2, l_3]) + \sum_{j=1}^{r} (\xi \otimes id)[1 \otimes l_1, m_j \otimes \psi_j(l_2, l_3)]_\lambda.\end{aligned}$$

Observe that,

$$\begin{aligned}&(\xi \otimes id)[1 \otimes l_1, m_j \otimes \psi_j(l_2, l_3)]_\lambda \\ &= (\xi \otimes id)m_j[1 \otimes l_1, 1 \otimes \psi_j(l_2, l_3)]_\lambda \\ &= (\xi \otimes id)m_j \left(1 \otimes [l_1, \psi_j(l_2, l_3)] + \sum_{k=1}^{r} m_k \otimes \psi_k(l_1, \psi_j(l_2, l_3))\right) \\ &= (\xi \otimes id)(m_j \otimes [l_1, \psi_j(l_2, l_3)]) \ \ (\lambda \text{ being an infinitesimal deformation, } \mathfrak{M}^2 = 0) \\ &= [l_1, (\xi \otimes id)(m_j \otimes \psi_j(l_2, l_3))].\end{aligned}$$

Therefore $(\xi \otimes id)([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda)$ can be written as

$$\begin{aligned}&(\xi \otimes id)([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda) \\ &= \alpha_{\lambda,\xi}(l_1, [l_2, l_3]) + [l_1, (\xi \otimes id) \sum_{j=1}^{r} m_j \otimes \psi_j(l_2, l_3)]\end{aligned} \tag{3.4.4}$$

by using (3.4.2)

$$
\begin{aligned}
&= \alpha_{\lambda,\xi}(l_1,[l_2,l_3]) + [l_1,(\xi\otimes id)([1\otimes l_2, 1\otimes l_3]_\lambda - 1\otimes [l_2,l_3])] \\
&= \alpha_{\lambda,\xi}(l_1,[l_2,l_3]) + [l_1,\alpha_{\lambda,\xi}(l_2,l_3)] \quad (\text{since } \xi(1)=0).
\end{aligned}
$$

Similarly

$$(\xi\otimes id)([[1\otimes l_1, 1\otimes l_2]_\lambda, 1\otimes l_3]_\lambda) = \alpha_{\lambda,\xi}([l_1,l_2],l_3) + [\alpha_{\lambda,\xi}(l_1,l_2),l_3] \tag{3.4.5}$$

$$\text{and } (\xi\otimes id)([[1\otimes l_1, 1\otimes l_3]_\lambda, 1\otimes l_2]_\lambda) = \alpha_{\lambda,\xi}([l_1,l_3],l_2) + [\alpha_{\lambda,\xi}(l_1,l_3),l_2]. \tag{3.4.6}$$

Thus using (3.4.4)-(3.4.6) in (3.4.3) we get,

$$
\begin{aligned}
&\delta\alpha_{\lambda,\xi}(l_1,l_2,l_3) \\
&= \{[l_1,\alpha_{\lambda,\xi}(l_2,l_3)] + \alpha_{\lambda,\xi}(l_1,[l_2,l_3])\} - \{[\alpha_{\lambda,\xi}(l_1,l_2),l_3] + \alpha_{\lambda,\xi}([l_1,l_2],l_3)\} \\
&\qquad + \{[\alpha_{\lambda,\xi}(l_1,l_3),l_2] + \alpha_{\lambda,\xi}([l_1,l_3],l_2)\} \\
&= (\xi\otimes id)([1\otimes l_1,[1\otimes l_2,1\otimes l_3]_\lambda]_\lambda - [[1\otimes l_1,1\otimes l_2]_\lambda,1\otimes l_3]_\lambda \\
&\qquad + [[1\otimes l_1,1\otimes l_3]_\lambda,1\otimes l_2]_\lambda) \\
&= 0 \qquad (\text{since } [-,-]_\lambda \text{ satisfies the Leibniz relation on } A\otimes L).
\end{aligned}
$$

□

Let $\lambda$ be a deformation of a Leibniz algebra $L$ with base $(A,\mathfrak{M})\in\mathcal{C}$. Let $\mathfrak{M}'$ be the dual of $\mathfrak{M}$. We shall use the following standard identifications.

**Lemma 3.4.6.** *$Hom(L;\mathfrak{M}\otimes L)\cong \mathfrak{M}\otimes Hom(L;L)\cong Hom(\mathfrak{M}';Hom(L;L))$*

*Proof.* Suppose $\{m_i\}_{1\le i\le r}$ is a basis of $\mathfrak{M}$ and $\{\xi_i\}_{1\le i\le r}$ is the corresponding dual basis of $\mathfrak{M}'$. An arbitrary element in $\mathfrak{M}\otimes Hom(L;L)$ is of the form $\sum_j a_j\otimes f_j$ where $j$ varies over a finite sum and $a_j\in\mathfrak{M}$, $f_j\in Hom(L;L)$. Let $a_j=\sum_{i=1}^r c_{ij}m_i$ for $c_{ij}\in\mathbb{K}$, then

$$\sum_j a_j\otimes f_j = \sum_j\left(\sum_{i=1}^r c_{ij}m_i\right)\otimes f_j = \sum_{i=1}^r m_i\otimes\left(\sum_j c_{ij}f_j\right) = \sum_{i=1}^r m_i\otimes\phi_i,$$

where $\phi_i=\sum_j c_{ij}f_j\in Hom(L;L)$.

Define a $\mathbb{K}$-linear map, $G:\mathfrak{M}\otimes Hom(L;L)\longrightarrow Hom(L;\mathfrak{M}\otimes L)$ by

$$G\left(\sum_{i=1}^r m_i\otimes\phi_i\right)(l) = \sum_{i=1}^r m_i\otimes\phi_i(l) \quad \text{for } l\in L.$$

Observe that $G$ is an injective map. Suppose

$$G\left(\sum_{i=1}^{r} m_i \otimes \phi_i\right) = 0 \in Hom(L; \mathfrak{M} \otimes L).$$

So we get, $\sum_{i=1}^{r} m_i \otimes \phi_i(l) = 0$ for $l \in L$. Equivalently,

$$(\xi_i \otimes id)\left(\sum_{i=1}^{r} m_i \otimes \phi_i(l)\right) = 0.$$

This gives $\phi_i(l) = 0$ for $l \in L$. So, $\phi_i = 0$.

Now let $\rho_2$ be a linear map in $Hom(L; \mathfrak{M} \otimes L)$ then we have $\rho_2(l) \in \mathfrak{M} \otimes L$ for $l \in L$. Suppose $\rho_2(l) = \sum_{i=1}^{r} m_i \otimes l_i$ for some $l_i \in L$. We now define $\phi_i \in Hom(L; L),\ 1 \le i \le r$ by

$$\phi_i(l) = \xi_i \otimes id(\rho_2(l)) \text{ for } l \in L.$$

Therefore

$$\rho_2(l) = \sum_{i=1}^{r} m_i \otimes \phi_i(l) = G\left(\sum_{i=1}^{r} m_i \otimes \phi_i\right)(l) \text{ for } l \in L.$$

So

$$\sum_{i=1}^{r} G(m_i \otimes \phi_i) = \rho_2.$$

Therefore $G$ is an onto map and consequently $G$ is an isomorphism. The last isomorphism has already been proved in Proposition 3.4.1 and is given by

$$F : \mathfrak{M} \otimes Hom(L; L) \longrightarrow Hom(\mathfrak{M}'; Hom(L; L))$$

where

$$F(m_i \otimes \phi_i)(\xi_j) = \delta_{i,j}\phi_i \in Hom(L; L).$$

□

**Proposition 3.4.7.** *Suppose $\lambda_1$ and $\lambda_2$ are infinitesimal deformations of a Leibniz algebra $L$ with base $(A, \mathfrak{M}) \in \mathcal{C}$. Then $\lambda_1$ and $\lambda_2$ are equivalent if and only if $\alpha_{\lambda_1,\xi}$ and $\alpha_{\lambda_2,\xi}$ represent the same cohomology class for $\xi \in \mathfrak{M}'$.*

*Proof.* Let $dim(\mathfrak{M}) = r$. Suppose $\{m_i\}_{1 \le i \le r}$ is a basis of $\mathfrak{M}$ and $\{\xi_i\}_{1 \le i \le r}$ be the corresponding dual basis of $\mathfrak{M}'$. For $\xi_i \in \mathfrak{M}'$, let $a_{\lambda,\xi_i} \in \mathbb{H}$ be the cohomology class of the cocycle $\alpha_{\lambda,\xi_i}$ for any infinitesimal deformation $\lambda$ of $L$ with base $A$. The correspondences

$$\xi_i \longmapsto \alpha_{\lambda,\xi_i} \text{ and } \xi_i \longmapsto a_{\lambda,\xi_i}$$

for $1 \le i \le r$ define homomorphisms

$$\alpha_\lambda : \mathfrak{M}' \longrightarrow CL^2(L;L) \text{ with } \delta \circ \alpha_\lambda = 0 \text{ and } a_\lambda : \mathfrak{M}' \longrightarrow \mathbb{H}\ .$$

Let $\lambda_1$ and $\lambda_2$ be two equivalent deformations of the Leibniz algebra $L$ with base $A$. Then there exists an $A$-Leibniz algebra isomorphism

$$\rho : (A \otimes L, [-,-]_{\lambda_1}) \longrightarrow (A \otimes L, [-,-]_{\lambda_2}) \text{ with } (\varepsilon \otimes id) \circ \rho = (\varepsilon \otimes id).$$

Now $A \otimes L = (\mathbb{K} \oplus \mathfrak{M}) \otimes L = (\mathbb{K} \otimes L) \oplus (\mathfrak{M} \otimes L) = L \oplus (\mathfrak{M} \otimes L)$. Thus any element of $A \otimes L$ is of the form $(l, \sum_{i=1}^r m_i \otimes l_i)$ where $l_i \in L$ for $1 \le i \le r$. By $A$-linearity, $\rho$ is determined by the values $\rho(1 \otimes l)$ for $l \in L$ and hence $\rho$ is of the form $\rho = \rho_1 + \rho_2$ where $\rho_1 : L \longrightarrow L$ and $\rho_2 : L \longrightarrow \mathfrak{M} \otimes L$. The map $\rho_1$ must be the identity map $id : L \longrightarrow L$ by the compatibility $(\varepsilon \otimes id) \circ \rho = (\varepsilon \otimes id)$.

Under the isomorphisms in Lemma 3.4.6, we have

$$\rho_2 \longmapsto \sum_{i=1}^r m_i \otimes \phi_i \longmapsto b_\rho \tag{3.4.7}$$

where $\phi_i = (\xi_i \otimes id) \circ \rho_2$ and $b_\rho(\xi_i) = \phi_i$. Thus we may write,

$$\rho(1 \otimes l) = \rho_1(1 \otimes l) + \rho_2(1 \otimes l) = 1 \otimes l + \sum_{i=1}^r m_i \otimes \phi_i(l) \text{ for } l \in L.$$

The map $\rho$ is a Leibniz algebra homomorphism if and only if

$$\rho([1 \otimes l_1, 1 \otimes l_2]_{\lambda_1}) = [\rho(1 \otimes l_1), \rho(1 \otimes l_2)]_{\lambda_2} \text{ for } l_1, l_2 \in L. \tag{3.4.8}$$

Set ${\psi_i}^k = \alpha_{\lambda_k, \xi_i}$ $1 \le i \le r$ for $k = 1$ and 2. Then

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda_k} = 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes {\psi_i}^k(l_1, l_2).$$

Therefore

$$\begin{aligned} &\rho([1 \otimes l_1, 1 \otimes l_2]_{\lambda_1}) \\ =&\rho(1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes {\psi_i}^1(l_1, l_2)) \\ =&\rho(1 \otimes [l_1, l_2]) + \sum_{i=1}^r m_i \rho(1 \otimes {\psi_i}^1(l_1, l_2)) \end{aligned}$$

$$= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes \phi_i([l_1, l_2]) + \sum_{i=1}^{r} m_i\{1 \otimes \psi_i^1(l_1, l_2) + \sum_{j=1}^{r} m_j \otimes \phi_j({\psi_i}^1(l_1, l_2))\}$$

$$= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes \phi_i([l_1, l_2]) + \sum_{i=1}^{r} m_i(1 \otimes {\psi_i}^1(l_1, l_2))$$

( as $m_i m_j = 0$, $\lambda_1$ being infinitesimal ).

Similarly

$$[\rho(1 \otimes l_1), \rho(1 \otimes l_2)]_{\lambda_2}$$
$$= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes \psi_i^2(l_1, l_2) + \sum_{i=1}^{r} m_i \otimes [l_1, \phi_i(l_2)] + \sum_{i=1}^{r} m_i \otimes [\phi_i(l_1), l_2].$$

Thus (3.4.8) holds if and only if,

$$[\rho(1 \otimes l_1), \rho(1 \otimes l_2)]_{\lambda_2} - \rho([1 \otimes l_1, 1 \otimes l_2]_{\lambda_1}) = 0.$$

Equivalently, $\sum_{i=1}^{r} m_i \otimes (\psi_i^2(l_1, l_2) - \psi_i^1(l_1, l_2)) + \sum_{i=1}^{r} m_i \otimes \delta\phi_i(l_1, l_2) = 0$ holds for $l_1, l_2 \in L$.

Equivalently, $\psi_i^1(l_1, l_2) - \psi_i^2(l_1, l_2) = \delta\phi_i(l_1, l_2)$, for $l_1, l_2 \in L$ holds.

Equivalently, $\alpha_{\lambda_1, \xi_i} - \alpha_{\lambda_2, \xi_i} = \delta\phi_i$ for $1 \leq i \leq r$.

Thus $\lambda_1$ and $\lambda_2$ are equivalent if and only if $a_{\lambda_1} = a_{\lambda_2}$.

□

Suppose $(A, \mathfrak{M}) \in \mathcal{C}$. The algebra $A/\mathfrak{M}^2$ is obviously local with maximal ideal $\mathfrak{M}/\mathfrak{M}^2$ and having the additional property $(\mathfrak{M}/\mathfrak{M}^2)^2 = 0$. Let $p_2 : A \longrightarrow A/\mathfrak{M}^2$ be the obvious quotient map. If $\lambda$ is any deformation of $L$ with base $A$ then we get the induced deformation $p_{2*}\lambda$ with base $A/\mathfrak{M}^2$, which is obviously infinitesimal. As a consequence, $\alpha_{p_{2*}\lambda}$ takes values in cocycles and hence we have a map

$$a_{p_{2*}\lambda} : (\mathfrak{M}/\mathfrak{M}^2)' \to HL^2(L; L) \text{ defined by } \; a_{p_{2*}\mathfrak{D}}(\xi) = [\alpha_{p_{2*}\lambda}]$$

where $[\alpha_{p_{2*}\lambda}]$ denotes the cohomology class represented by $\alpha_{p_{2*}\lambda}$.

**Definition 3.4.8.** *The linear maps $\alpha_{p_{2*}\lambda}$ and $a_{p_{2*}\lambda}$ are respectively called the infinitesimal and the differential of $\mathfrak{D}$. The differential $a_{p_{2*}\lambda} : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(L; L)$ is also denoted by $d\lambda$. The deformation $p_{2*}\lambda$ may be called the infinitesimal part of $\lambda$.*

**Corollary 3.4.9.** *Two infinitesimal deformations $\lambda$ and $\lambda'$ with base $(A, \mathfrak{M}) \in \mathcal{C}$ are equivalent if and only if they have the same differential.*

**Corollary 3.4.10.** *Suppose $\lambda$ and $\lambda'$ are two equivalent deformations of $L$ with base $A$ then they have the same differential.*

The main property of $\eta_1$ is given by the following theorem.

**Theorem 3.4.11.** *For any infinitesimal deformation $\lambda$ of a Leibniz algebra $L$ with a finite dimensional base $(A, \mathfrak{M}) \in \mathcal{C}$ there exists a unique homomorphism $\phi : C_1 = (\mathbb{K} \oplus \mathbb{H}') \longrightarrow A$ such that $\lambda$ is equivalent to the push-out $\phi_* \eta_1$.*

*Proof.* Let $\lambda$ be an infinitesimal deformation of a Leibniz algebra $L$ with base $(A, \mathfrak{M}) \in \mathcal{C}$. Let $\mathfrak{M}$ be the maximal ideal in $A$ and $dim(\mathfrak{M}) = r$. As before let $\{\xi\}_{1 \leq i \leq r}$ be the dual basis in $\mathfrak{M}'$ corresponding to a basis $\{m_i\}_{1 \leq i \leq r}$ of $\mathfrak{M}$. Let $\alpha_\lambda : \mathfrak{M}' \longrightarrow CL^2(L; L)$ and $a_\lambda : \mathfrak{M}' \longrightarrow \mathbb{H}$ be the homomorphisms as defined in Proposition 3.4.7.

Now consider the map $\phi = (id \oplus {a_\lambda}') : C_1 \longrightarrow \mathbb{K} \oplus \mathfrak{M} = A$. By Proposition 3.4.7, it is enough to show that $\alpha_{\phi_* \eta_1} = \mu \circ a_\lambda$. Let $\{h_i\}_{1 \leq i \leq n}$ be a basis of $\mathbb{H}$ and $\{g_i\}_{1 \leq i \leq n}$ be the corresponding dual basis of $\mathbb{H}'$. By Remarks 3.3.2 and 3.4.4 we have

$$[1 \otimes l_1, 1 \otimes l_2]_{\phi_* \eta_1} = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} \phi(g_i) \otimes \mu(h_i)(l_1, l_2).$$

Let ${a_\lambda}' : \mathbb{H}' \longrightarrow \mathfrak{M}$ be the dual of $a_\lambda$. Then

$${a_\lambda}'(g_j) = \sum_{i=1}^{r} \xi_i({a_\lambda}'(g_j)) m_i \text{ and } a_\lambda(\xi_i) = \sum_{j=1}^{n} g_j(a_\lambda(\xi_i)) h_j.$$

Thus

$$\begin{aligned}
\alpha_{\phi_* \eta_1}(\xi_i)(l_1, l_2) &= (\xi_i \otimes id)[1 \otimes l_1, 1 \otimes l_2]_{\phi_* \eta_1} \\
&= (\xi_i \otimes id)(1 \otimes [l_1, l_2] + \sum_{j=1}^{n} \phi(g_j) \otimes \mu(h_j)(l_1, l_2)) \\
&= (\xi_i \otimes id) \left( \sum_{j=1}^{n} {a_\lambda}'(g_j) \otimes \mu(h_j)(l_1, l_2) \right) \\
&= \sum_{j=1}^{n} \xi_i({a_\lambda}'(g_j)) \otimes \mu(h_j)(l_1, l_2)) \\
&= \sum_{j=1}^{n} g_j(a_\lambda(\xi_i)) \otimes \mu(h_i)(l_1, l_2)) \\
&= \mu \left( \sum_{j=1}^{n} g_j(a_\lambda(\xi_i)) h_j \right) (l_1, l_2) \\
&= \mu \circ a_\lambda(\xi_i)(l_1, l_2).
\end{aligned}$$

Now we show that the above map $\phi$ is unique.

Let $\psi : C_1 = (\mathbb{K} \oplus \mathbb{H}') \longrightarrow A = \mathbb{K} \oplus \mathfrak{M}$ be an arbitrary $\mathbb{K}$-module homomorphism such that $\psi(1) = 1$ and $(\varepsilon \circ \psi)$ is the canonical augmentation in $C_1$.

Suppose $\psi_*\eta_1$ is equivalent to the deformation $\lambda$ of $L$ with base $A$. From Proposition 3.4.7, it follows that $a_{\psi_*\eta_1} = a_\lambda$. So, $a_{\psi_*\eta_1,\xi_i} = a_{\lambda,\xi_i}$ for $1 \leq i \leq r$. We know

$$\begin{aligned}
[1 \otimes l_1, 1 \otimes l_2]_{\psi_*\eta_1} &= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} \psi(g_i) \otimes \mu(h_i)(l_1, l_2) \\
&= 1 \otimes [l_1, l_2] + \sum_{i=1}^{n} \{\sum_{j=1}^{r} \xi_j(\psi(g_i))m_j\} \otimes \mu(h_i)(l_1, l_2) \qquad (3.4.9)\\
&= 1 \otimes [l_1, l_2] + \sum_{j=1}^{r} m_j \otimes \{\sum_{i=1}^{n} \xi_j(\psi(g_j))\mu(h_i)\}(l_1, l_2).
\end{aligned}$$

Thus it follows that

$$\alpha_{\psi_*\eta_1}(\xi_j) = \sum_{i=1}^{n} \xi_j(\psi(g_i))\mu(h_i) = \mu\left(\sum_{i=1}^{n} \xi_j(\psi(g_i))h_i\right).$$

So,

$$a_{\psi_*\eta_1}(\xi_j) = \left[\mu\left(\sum_{i=1}^{n} \xi_j(\psi(g_i))h_i\right)\right] = \sum_{i=1}^{n} \xi_j(\psi(g_i))h_i,$$

the cohomology class represented by $\alpha_{\psi_*\eta_1}(\xi_j)$.

On the other hand,

$$a_\lambda(\xi_j) = \sum_{i=1}^{n} g_i(a_\lambda(\xi_j))h_i = \sum_{i=1}^{n} \xi_j(a'_\lambda(g_i))h_i.$$

Hence by comparing expression of $\alpha_{\psi_*\eta_1}(\xi_j)$ and $a_\lambda(\xi_j)$ we get $\psi(g_i) = a'_\lambda(g_i)$ for $1 \leq i \leq n$.

Therefore $\psi = \phi = (id \oplus a'_\lambda) : C_1 \longrightarrow A$. □

## 3.5 Infinitesimal deformations of Leibniz algebra homomorphisms

Next we focus our attention to infinitesimal deformations of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with finite dimensional base. Let $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ be any deformation of $f$ with base $(A, \mathfrak{M}) \in \mathcal{C}$. Let $\{m_i\}_{1 \le i \le r}$ be a basis of $\mathfrak{M}$ and $\{\xi_i\}_{1 \le i \le r}$ be the corresponding dual basis. As before, we regard any $\xi \in \mathfrak{M}'$ as an element of $A'$ with $\xi(1) = 0$. Define a cochain $f_{\lambda\mu,\xi} \in CL^1(L; M)$ by

$$f_{\lambda\mu,\xi}(l) = (\xi \otimes id) \circ f_{\lambda\mu}(1 \otimes l) \text{ for } l \in L. \tag{3.5.1}$$

In particular, for the basis elements $\xi_j,\ 1 \le j \le r$, if we set $f_j = f_{\lambda\mu,\xi_j}$, then by Remark 3.2.5 the deformation $f_{\lambda\mu}$ of $f$ can be written as

$$f_{\lambda\mu}(1 \otimes l) = 1 \otimes f(l) + \sum_{j=1}^{r} m_j \otimes f_j(l), l \in L. \tag{3.5.2}$$

Thus we have a linear map $\alpha_{\mathfrak{D}} : \mathfrak{M}' \longrightarrow CL^2(f; f)$ given by

$$\alpha_{\mathfrak{D}}(\xi) = (\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_{\lambda\mu,\xi}) \text{ for } \xi \in \mathfrak{M}',$$

where $\alpha_{\lambda,\xi} \in CL^2(L; L), \alpha_{\mu,\xi} \in CL^2(M; M)$ are the cochains as defined in (3.4.1).

**Proposition 3.5.1.** *For any infinitesimal deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of $f$ with base $(A, \mathfrak{M}) \in \mathcal{C}$, $\alpha_{\mathfrak{D}}$ takes values in cocycles.*

*Proof.* By the definition of the coboundary in $CL^*(f; f)$, we have to show that $\delta\alpha_{\lambda,\xi} = 0 = \delta\alpha_{\mu,\xi}$ and $f\alpha_{\lambda,\xi} - \alpha_{\mu,\xi} f = \delta f_{\lambda\mu,\xi}$ for any $\xi \in \mathfrak{M}'$. Since $\lambda, \mu$ are infinitesimal deformations of the Leibniz algebras $L$ and $M$ respectively, by Proposition 3.4.5, $\delta\alpha_{\lambda,\xi} = 0 = \delta\alpha_{\mu\xi}$ for any $\xi \in \mathfrak{M}'$.

To complete the proof it is enough to show that

$$f\psi_i^{\lambda} - \psi_i^{\mu} f - \delta f_i = f\alpha_{\lambda,\xi_i} - \alpha_{\mu,\xi_i} f - \delta f_{\lambda\mu,\xi_i} = 0, \quad 1 \le i \le r.$$

We know that $f_{\lambda\mu} : A \otimes L \longrightarrow A \otimes M$ is a Leibniz algebra homomorphism, that is,

$$f_{\lambda\mu}[1 \otimes l_1, 1 \otimes l_2]_{\lambda} - [f_{\lambda\mu}(1 \otimes l_1), f_{\lambda\mu}(1 \otimes l_2)]_{\mu} = 0$$

for $l_1, l_2 \in L$.

We have from (3.5.2)

$$
\begin{aligned}
& f_{\lambda\mu}[1\otimes l_1, 1\otimes l_2]_\lambda \\
& = f_{\lambda\mu}(1\otimes [l_1,l_2] + \sum_{i=1}^{r} m_i\otimes \psi_i^\lambda(l_1,l_2)) \\
& = 1\otimes f([l_1,l_2]) + \sum_{i=1}^{r} m_i\otimes f_i([l_1,l_2]) + f_{\lambda\mu}\left(\sum_{i=1}^{r} m_i\otimes \psi_i^\lambda(l_1,l_2)\right) \\
& = 1\otimes f([l_1,l_2]) + \sum_{i=1}^{r} m_i\otimes f_i([l_1,l_2]) + \sum_{i=1}^{r} m_i\otimes f\psi_i^\lambda(l_1,l_2) \text{ (since } \mathfrak{M}^2=0).
\end{aligned}
$$

Also,

$$
\begin{aligned}
& [f_{\lambda\mu}(1\otimes l_1), f_{\lambda\mu}(1\otimes l_2)]_\mu \\
& = [1\otimes f(l_1) + \sum_{j=1}^{r} m_j\otimes f_j(l_1), 1\otimes f(l_2) + \sum_{j=1}^{r} m_j\otimes f_j(l_2)]_\mu \\
& = [1\otimes f(l_1), 1\otimes f(l_2)]_\mu + \sum_{j=1}^{r} m_j[1\otimes f(l_1), 1\otimes f_j(l_2)]_\mu \\
& + \sum_{j=1}^{r} m_j[1\otimes f_j(l_1), 1\otimes f(l_2)]_\mu + \sum_{i,j=1}^{r} m_j m_i[1\otimes f_i(l_1), 1\otimes f_j(l_2)]_\mu \\
& = 1\otimes [f(l_1), f(l_2)] + \sum_{j=1}^{r} m_j\otimes \psi_j^\mu(f(l_1), f(l_2)) + \sum_{j=1}^{r} m_i\otimes [f(l_1), f_j(l_2)] \\
& + \sum_{j=1}^{r} m_j\otimes [f_j(l_1), f(l_2)] \quad \text{( by using the fact that } \mathfrak{M}^2=0).
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
& (\xi_i\otimes id)(f_{\lambda\mu}[1\otimes l_1, 1\otimes l_2]_\lambda - [f_{\lambda\mu}(1\otimes l_1), f_{\lambda\mu}(1\otimes l_2)]_\mu) \\
& = (f\alpha_{\lambda,\xi_i} - \alpha_{\mu,\xi_i} f - \delta f_{\lambda\mu,\xi_i})(l_1,l_2).
\end{aligned}
$$

Since $f_{\lambda\mu}$ preserves the brackets we get $(f\alpha_{\lambda,\xi_i} - \alpha_{\mu,\xi_i} f - \delta f_{\lambda\mu,\xi_i}) = 0$ for $1\leq i\leq r$. □

**Proposition 3.5.2.** *Let $\mathfrak{D} = (\lambda,\mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda',\mu'; f'_{\lambda'\mu'})$ be two infinitesimal deformations of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with base $(A,\mathfrak{M}) \in \mathcal{C}$. Then $\alpha_{\mathfrak{D}}(\xi)$ and $\alpha_{\mathfrak{D}'}(\xi)$ represent the same cohomology class for $\xi\in\mathfrak{M}'$, if and only if $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent deformations.*

*Proof.* Suppose $\mathfrak{D} = (\lambda,\mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda',\mu'; f'_{\lambda'\mu'})$ are two equivalent infinitesimal deformations of $f$ with base $A$. Let $(\alpha_{\lambda,\xi},\alpha_{\mu,\xi}; f_{\lambda\mu,\xi})$ and $(\alpha_{\lambda',\xi},\alpha_{\mu',\xi}; f'_{\lambda'\mu',\xi})$ be the associated 2-cocycles in $CL^2(f;f)$ determined by $\mathfrak{D}$ and $\mathfrak{D}'$ respectively.

Let $\Phi_{\lambda\lambda'} : (A \otimes L, \lambda) \longrightarrow (A \otimes L, \lambda')$ and $\Psi_{\mu\mu'} : (A \otimes M, \mu) \longrightarrow (A \otimes M, \mu')$ be as in Definition 3.2.7 so that

$$\Psi_{\mu\mu'} \circ f_{\lambda\mu} = f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}. \tag{3.5.3}$$

Since $\lambda$ and $\mu$ are equivalent to $\lambda'$ and $\mu'$ respectively, it follows from the Proposition 3.4.7 that $\alpha_{\lambda,\xi}$ and $\alpha_{\mu,\xi}$ determine the same cohomology class as $\alpha_{\lambda',\xi}$ and $\alpha_{\mu',\xi}$ respectively. In fact, as shown in the proof of Proposition 3.4.7, (cf. (3.4.7)) the $A$-Leibniz algebra isomorphisms $\Phi_{\lambda\lambda'}$ and $\Psi_{\mu\mu'}$ are determined by some linear maps $b_\Phi : \mathfrak{M}' \longrightarrow Hom(L; L)$ and $b_\Psi : \mathfrak{M}' \longrightarrow Hom(M; M)$ respectively so that for $\xi \in \mathfrak{M}'$ and $l \in L, x \in M$ we have,

$$\Phi_{\lambda\lambda'}(1 \otimes l) = 1 \otimes l + \sum_{i=1}^{r} m_i \otimes b_\Phi(\xi_i)(l)$$
$$\Psi_{\mu\mu'}(1 \otimes x) = 1 \otimes x + \sum_{i=1}^{r} m_i \otimes b_\Psi(\xi_i)(x),$$

where $\alpha_{\lambda,\xi} - \alpha_{\lambda',\xi} = \delta b_\Phi(\xi)$ and $\alpha_{\mu,\xi} - \alpha_{\mu',\xi} = \delta b_\Psi(\xi)$. Now if we denote $f_j = f_{\lambda\mu,\xi_j}$ and $f'_j = f_{\lambda'\mu',\xi_j}$ we get,

$$\begin{aligned}
&\Psi_{\mu\mu'} \circ f_{\lambda\mu}(1 \otimes l)\\
=&\Psi_{\mu\mu'}(1 \otimes f(l) + \sum_{j=1}^{r} m_j \otimes f_j(l))\\
=&1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes b_\Psi(\xi_i)(f(l)) + \sum_{j=1}^{r} m_j \otimes f_j(l) + \sum_{1\le i,j\le r} m_j m_i \otimes b_\Psi(\xi_i)(f_j(l))\\
=&1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes b_\Psi(\xi_i)(f(l)) + \sum_{i=1}^{r} m_i \otimes f_i(l) \quad (\text{since } \mathfrak{M}^2 = 0),
\end{aligned}$$

and $f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}(1 \otimes l)$

$$\begin{aligned}
&= f'_{\lambda'\mu'}(1 \otimes l + \sum_{i=1}^{r} m_i \otimes b_\Phi(\xi_i)(l))\\
&= 1 \otimes f(l) + \sum_{j=1}^{r} m_j \otimes f'_j(l) + \sum_{i=1}^{r} m_i \otimes f b_\Phi(\xi_i)(l)\\
&\quad + \sum_{1\le i,j\le r} m_i m_j \otimes f'_j(b_\Phi(\xi_i)(l))\\
&= 1 \otimes f(l) + \sum_{i=1}^{r} m_i \otimes f'_i(l) + \sum_{i=1}^{r} m_i \otimes f b_\Phi(\xi_i)(l) \quad (\text{since } \mathfrak{M}^2 = 0).
\end{aligned}$$

It follows from the above expressions that

$$(\xi_i \otimes id) \circ \Psi_{\mu\mu'} \circ f_{\lambda\mu}(1 \otimes l) = b_\Psi(\xi_i)(f(l)) + f_i(l)$$

$$\text{and } (\xi_i \otimes id) \circ f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}(1 \otimes l) = f b_\Phi(\xi_i)(l) + f'_i(l).$$

Hence by (3.5.3) we get

$$(\xi_i \otimes id)(\Psi_{\mu\mu'} \circ f_{\lambda\mu} - f'_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'})(1 \otimes l) = 0$$
$$\text{or, } b_\Psi(\xi_i)f - f b_\phi(\xi_i) + (f_i - f'_i) = 0 \text{ for } 1 \leq i \leq r.$$

Thus it follows that $(\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_\xi) - (\alpha_{\lambda',\xi}, \alpha_{\mu',\xi}; f'_\xi) = d(b_\Phi(\xi), b_\Psi(\xi); 0)$ for $\xi \in \mathfrak{M}'$. Conversely, suppose $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda', \mu'; f_{\lambda'\mu'})$ are two infinitesimal deformations of $f$ with base $A$ such that for $\xi \in \mathfrak{M}'$, $\alpha_{\mathfrak{D}}(\xi)$ and $\alpha_{\mathfrak{D}'}(\xi)$ represent the same cohomology class.

Let $(\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_\xi) - (\alpha_{\lambda',\xi}, \alpha_{\mu',\xi}; f'_\xi) = d(u, v; w)$ for some 1-cochain $(u, v; w) \in CL^1(f; f)$.

In particular we can take $(\alpha_{\lambda,\xi}, \alpha_{\mu,\xi}; f_\xi) - (\alpha_{\lambda',\xi}, \alpha_{\mu',\xi}; f'_\xi) = d(u, v; 0)$ as $d(u, v; w) = d(u, v + \delta w; 0)$. For $\xi = \xi_i$ let $(u_i, v_i; 0) \in CL^1(f; f)$ be such that

$$(\alpha_{\lambda,\xi_i} - \alpha_{\lambda',\xi_i}, \alpha_{\mu,\xi_i} - \alpha_{\mu',\xi_i}; f_i - f'_i) = d(u_i, v_i; 0) = (\delta u_i, \delta v_i; f u_i - v_i f) \tag{3.5.4}$$

for $1 \leq i \leq r$.

Define $A$-linear maps

$$\Phi_{\lambda\lambda'} : (A \otimes L, \lambda) \longrightarrow (A \otimes L, \lambda')$$
$$\text{by } \Phi_{\lambda\lambda'}(1 \otimes l) = 1 \otimes l + \sum_{i=1}^{r} m_i \otimes u_i(l),$$
$$\text{and } \Psi_{\mu\mu'} : (A \otimes M, \mu) \longrightarrow (A \otimes M, \mu')$$
$$\text{by } \Psi_{\mu\mu'}(1 \otimes x) = 1 \otimes x + \sum_{i=1}^{r} m_i \otimes v_i(x), \text{ for } l \in L \text{ and } x \in M.$$

Then Proposition 3.4.7 and (3.5.4) together imply that $\Phi_{\lambda\lambda'}$ and $\Psi_{\mu\mu'}$ are equivalences $\lambda \cong \lambda'$ and $\mu \cong \mu'$ respectively. To show that $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ and $\mathfrak{D}' = (\lambda', \mu'; f_{\lambda'\mu'})$ are equivalent deformations of $f$, it remains to check the relation

$$\Psi_{\mu\mu'} \circ f_{\lambda\mu} = f_{\lambda'\mu'} \circ \Phi_{\lambda\lambda'}.$$

Suppose $f_{\lambda\mu}$ and $f_{\lambda'\mu'}$ are given by

$$\begin{aligned} f_{\lambda\mu}(1\otimes l) =& 1\otimes f(l)+\sum_{i=1}^{r} m_i\otimes f_i(l) \\ \text{and } f_{\lambda'\mu'}(1\otimes l) =& 1\otimes f(l)+\sum_{i=1}^{r} m_i\otimes f'_i(l) \text{ for } l\in L. \end{aligned}$$

For $l \in L$ we get

$$\begin{aligned} \Psi_{\mu\mu'}\circ f_{\lambda\mu} =& \Psi_{\mu\mu'}\{1\otimes f(l)+\sum_{i=1}^{r} m_i\otimes f_i(l)\} \\ =& \Psi_{\mu\mu'}(1\otimes f(l))+\sum_{i=1}^{r} m_i\otimes \Psi_{\mu\mu'}(1\otimes f_i(l)) \\ =& 1\otimes f(l)+\sum_{i=1}^{r} m_i\otimes v_i(f(l))+\sum_{i=1}^{r} m_i\{1\otimes f_i(l)+\sum_{j=1}^{r} m_j\otimes v_j(f_i(l))\} \\ =& 1\otimes f(l)+\sum_{i=1}^{r} m_i\otimes v_i f(l)+\sum_{i=1}^{r} m_i\otimes f_i(l) \quad \text{(by using } \mathfrak{M}^2=0\text{)}, \end{aligned}$$

$$\begin{aligned} &\text{and } f_{\lambda'\mu'}\circ\Phi_{\lambda\lambda'}(1\otimes l) \\ =& f_{\lambda'\mu'}\{1\otimes l+\sum_{i=1}^{r} m_i\otimes u_i(l)\} \\ =& f_{\lambda'\mu'}(1\otimes l)\sum_{i=1}^{r} m_i\otimes f_{\lambda'\mu'}(1\otimes u_i(l)) \\ =& 1\otimes f(l)+\sum_{i=1}^{r} m_i\otimes f'_i(l)+\sum_{i=1}^{r} m_i\{1\otimes f(l)+\sum_{j=1}^{r} m_i\otimes f'_j(l)\} \\ =& 1\otimes f(l)+\sum_{i=1}^{r} m_i\otimes f'_i(l)+\sum_{i=1}^{r} m_i\otimes fu_i(l) \quad \text{(by using } \mathfrak{M}^2=0\text{)}. \end{aligned}$$

Thus we have

$$\begin{aligned} (\Psi_{\mu\mu'}\circ f_{\lambda\mu})(1\otimes l)-(f_{\lambda'\mu'}\circ\Phi_{\lambda\lambda'})(1\otimes l) =& \sum_{i=1}^{r} m_i\otimes (v_i f(l)+f_i(l)-f'_i(l)-fu_i(l)) \\ =& \sum_{i=1}^{r} m_i\otimes\{(f_i-f'_i)(l)-(fu_i-v_i f)(l)\} \end{aligned}$$

From (3.5.4) we get $f_i - f'_i = fu_i - v_i f$.

So it follows that

$$\Psi_{\mu\mu'}\circ f_{\lambda\mu} = f'_{\lambda'\mu'}\circ\Phi_{\lambda\lambda'}.$$

□

Suppose $(A, \mathfrak{M}) \in \mathcal{C}$. Let $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ be a given deformation of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with base $A$. As observed before, the algebra $A/\mathfrak{M}^2$ is local with maximal ideal $\mathfrak{M}/\mathfrak{M}^2$ and $p_{2*}\mathfrak{D} = (p_{2*}\lambda, p_{2*}\mu; f_{p_{2*}\lambda\ p_{2*}\mu})$ with base $A/\mathfrak{M}^2$ is an infinitesimal deformation of $f$ where $p_2 : A \longrightarrow A/\mathfrak{M}^2$ is the quotient map. As a consequence, $\alpha_{p_{2*}\mathfrak{D}}$ takes values in cocycles and hence we have a map

$$a_{p_{2*}\mathfrak{D}} : (\mathfrak{M}/\mathfrak{M}^2)' \to HL^2(f; f) \text{ defined by } \ a_{p_{2*}\mathfrak{D}}(\xi) = [\alpha_{p_{2*}\mathfrak{D}}]$$

where $[\alpha_{p_{2*}\mathfrak{D}}]$ denotes the cohomology class represented by $\alpha_{p_{2*}\mathfrak{D}}$.

**Definition 3.5.3.** *The linear maps $\alpha_{p_{2*}\mathfrak{D}}$ and $a_{p_{2*}\mathfrak{D}}$ are respectively called the infinitesimal and the differential of $\mathfrak{D}$. The deformation $p_{2*}\mathfrak{D}$ may be called the infinitesimal part of $\mathfrak{D}$.*

**Corollary 3.5.4.** *Two infinitesimal deformations $\mathfrak{D}$ and $\mathfrak{D}'$ with base $(A, \mathfrak{M}) \in \mathcal{C}$ are equivalent if and only if they have the same differential.*

**Corollary 3.5.5.** *Suppose $\mathfrak{D}$ and $\mathfrak{D}'$ are two equivalent deformations of $f$ with base $A$ then they have the same differential.*

# Chapter 4

# Extension of deformations

## 4.1 Introduction

The aim of this chapter is to address the question of extending a given deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with a given base to a larger base. This extension problem can be described as follows. Suppose $\mathfrak{D}$ is a given deformation of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with local base $A$. Let

$$0 \longrightarrow M_0 \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} A \longrightarrow 0$$

be a given finite dimensional extension of $A$ by $M_0$. The problem is to obtain condition for existence of a deformation $\tilde{\mathfrak{D}}$ of $f$ with base $B$ which extends the given deformation, that is, $p_*\tilde{\mathfrak{D}} = \mathfrak{D}$. We shall measure the possible obstructions that one might encounter in the above extension process as certain 3-dimensional cohomology classes, vanishing of which is a necessary and sufficient condition for an extension to exist. The set of equivalence classes of possible extensions of a given deformation $\lambda$ of $L$ with base $A$, admits certain natural actions and we shall investigate their relationship.

We first take up the case of extending deformations of Leibniz algebras and then consider the relative problem of extending deformations of Leibniz algebra homomorphisms. In the last section, we study formal deformations and obtain a necessary condition for non-triviality of a formal deformation. The results of this chapter will also enable us to obtain a sufficient criterion for existence of a formal deformation with a given differential and infinitesimal part. We end with the definition of a versal deformation.

## 4.2 Extension of a deformation of Leibniz algebras

Let $\lambda$ be a deformation of a Leibniz algebra $L$ with a finite dimensional local algebra base $(A, \mathfrak{M}) \in \mathcal{C}$. The primary aim of this section is to derive a necessary and sufficient condition for the existence of a deformation of $L$ with an algebra base $B$ extending $\lambda$. The extension process naturally leads to an obstruction cochain which turns out to be a cocycle. We use this cocycle to formulate the desired criterion.

We recall (Proposition 2.3.3) that the set of equivalence classes of 1-dimensional extensions of $A$ corresponds bijectively to $H^2_{Harr}(A;\mathbb{K})$. Consider $[\psi] \in H^2_{Harr}(A;\mathbb{K})$ and suppose

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0 \tag{4.2.1}$$

is a representative of the class of 1- dimensional extensions of $A$, which corresponds to $[\psi]$. Set the following $\mathbb{K}$-linear maps.

$$I = (i \otimes id) : L \cong \mathbb{K} \otimes L \longrightarrow B \otimes L, P = (p \otimes id) : B \otimes L \longrightarrow A \otimes L$$
$$\text{and } E = (\hat{\varepsilon} \otimes id) : B \otimes L \longrightarrow \mathbb{K} \otimes L \cong L,$$

where $\hat{\varepsilon} = \varepsilon \circ p$, with $\varepsilon$ is the augmentation of $A$. Fix a section $q : A \longrightarrow B$ of $p$ in the above extension, then

$$b \longmapsto (p(b), i^{-1}(b - q \circ p(b))) \tag{4.2.2}$$

is a $\mathbb{K}$-module isomorphism $B \longrightarrow (A \oplus \mathbb{K})$. Following the notations in Chapter 2, let $(a,k)_q \in B$ be the inverse of $(a,k) \in (A \oplus \mathbb{K})$ under the above isomorphism. Then the algebra structure of $B$ is determined by $\psi$ and is given by

$$(a_1,k_1)_q(a_2,k_2)_q = (a_1a_2\ ,\ \varepsilon(a_1)k_2 + \varepsilon(a_2)k_1 + \psi(a_1,a_2))_q. \tag{4.2.3}$$

Suppose $dim(A) = r+1$ and $\{m_j\}_{1\le j\le r}$ is a basis of the maximal ideal $\mathfrak{M}_A$ of A. Let $\{\xi_j\}_{1\le j\le r}$ denote the dual basis of $\mathfrak{M}'_A$. Then the Leibniz bracket $[-,-]_\lambda$ on $A \otimes L$ can be written as (cf. (3.4.2))

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_{j=1}^{r} m_j \otimes \psi_j(l_1,l_2) \text{ for } l_1, l_2 \in L.$$

Here $\psi_j = \alpha_{\lambda,\xi_j} \in CL^2(L;L)$ is given by $\alpha_{\lambda,\xi_j}(l_1,l_2) = (\xi_j \otimes id)([1 \otimes l_1, 1 \otimes l_2 l]_\lambda)$ for $l_1, l_2 \in L$ and $1 \le i \le r$. By Proposition 2.3.5, $B$ is a local algebra with the maximal ideal $\mathfrak{M}_B = p^{-1}(\mathfrak{M}_A)$ and $\hat{\varepsilon}$ is the augmentation of $B$. If $n_j = (m_j, 0)_q$, for $1 \le j \le r$ and $n_{r+1} = (0,1)_q$ then $\{n_j\}_{1\le j\le r+1}$ is a basis of $\mathfrak{M}_B$.

Let $\psi_{r+1} \in CL^2(L;L) = Hom(L^{\otimes 2}; L)$ be an arbitrary element. We define a $B$-bilinear operation $\{-,-\} : (B \otimes L)^{\otimes 2} \longrightarrow B \otimes L$ by

$$\{b_1 \otimes l_1, b_2 \otimes l_2\} = b_1b_2 \otimes [l_1,l_2] + \sum_{j=1}^{r+1} b_1b_2n_j \otimes \psi_j(l_1,l_2) \text{ for } b_1 \otimes l_1, b_2 \otimes l_2 \in B \otimes L.$$

**Lemma 4.2.1.** *The $B$-bilinear map $\{-,-\}$ as defined above satisfies the following conditions.*

$$\begin{aligned} &(i)\ P\{l_1,l_2\} = [P(l_1),P(l_2)]_\lambda \quad \text{for } l_1,l_2 \in B\otimes L \\ &(ii)\ \{I(l),l_1\} = I[l,E(l_1)] \quad \text{for } l\in L \text{ and } l_1 \in B\otimes L. \end{aligned} \tag{4.2.4}$$

*Proof.* **(i)** By linearity it is enough to prove the statement for elements $\bar{l_1} = b_1 \otimes l_1$ and $\bar{l_2} = b_2\otimes l_2$ in $B\otimes L$. By definition of $\{-,-\}$,

$$\begin{aligned} & P\{\bar{l_1},\bar{l_2}\} \\ &= P(b_1b_2\otimes[l_1,l_2] + \sum_{j=1}^{r+1} b_1b_2n_j\otimes\psi_j(l_1,l_2)) \\ &= p(b_1b_2)\otimes[l_1,l_2] + \sum_{j=1}^{r+1} p(b_1b_2n_j)\otimes\psi_j(l_1,l_2) \\ &= p(b_1)p(b_2)(1\otimes[l_1,l_2] + \sum_{j=1}^{r+1} p(n_j)\otimes\psi_j(l_1,l_2)) \\ &= p(b_1)p(b_2)(1\otimes[l_1,l_2] + \sum_{j=1}^{r} m_j\otimes\psi_j(l_1,l_2)) \\ &\quad (\text{ since } p(n_j)=m_j \text{ for } 1\le j\le r \text{ and } p(n_{r+1})=0\ ) \\ &= p(b_1)p(b_2)[1\otimes l_1, 1\otimes l_2]_\lambda \\ &= [p(b_1)\otimes l_1, p(b_2)\otimes l_2]_\lambda \quad (\text{ since } [-,-]_\lambda \text{ is a } A\text{-bilinear operation on } A\otimes L) \\ &= [P(b_1\otimes l_1), P(b_2\otimes l_2)]_\lambda \\ &= [P(\bar{l_1}),P(\bar{l_2})]_\lambda. \end{aligned} \tag{4.2.5}$$

Therefore $P\{\bar{l_1},\bar{l_2}\} = [P(\bar{l_1}),P(\bar{l_2})]_\lambda$ for $\bar{l_1},\bar{l_2}\in B\otimes L$.

**(ii)** Let $l\in L$ and $\bar{l_1} = b_1\otimes l_1 \in B\otimes L$ with $b_1 = (a_1,k_1)_q$. Then,

$$\begin{aligned} & \{I(l),\bar{l_1}\} \\ &= \{(i\otimes id)(1\otimes l), b_1\otimes l_1\} \\ &= \{i(1)\otimes l, b_1\otimes l_1\} \\ &= \{n_{r+1}\otimes l, b_1\otimes l_1\} \\ &= n_{r+1}b_1\otimes[l,l_1] + \sum_{j=1}^{r+1} n_{r+1}b_1n_j\otimes\psi_j(l,l_1) \end{aligned} \tag{4.2.6}$$

$$= ((0,1)_q(a_1,k_1)_q)\otimes[l,l_1] + \sum_{j=1}^{r}(0,1)_q(a_1,k_1)_q(m_j,0)_q\otimes\psi_j(l,l_1)$$
$$+ ((a_1,k_1)_q(0,1)_q^2)\otimes\psi_{r+1}(l,l_1)$$
$$= (0,\varepsilon(a_1))_q\otimes[l,l_1] \qquad \text{(by (4.2.3))}$$
( since $(0,1)_q(m_j,0)_q = (0,0)_q, (0,1)_q^2 = (0,0)_q$ and $(0,1)_q(a_1,k_1)_q = (0,\varepsilon(a_1))_q$ )
$$= i(\varepsilon(a_1))\otimes[l,l_1]$$
$$= i(1)\otimes\varepsilon(a_1)[l,l_1]$$
$$= i(1)\otimes[l,\hat{\varepsilon}(b_1)l_1]$$
$$= I(1\otimes[l,E(b_1\otimes l_1)])$$
$$= I[l,E(\bar{l_1})].$$

Since $E$ is linear the statement in $(ii)$ holds. □

Thus the Leibniz algebra structure $\lambda$ on $A\otimes L$ can be lifted to a $B$-bilinear operation $\{-,-\}:(B\otimes L)^{\otimes 2}\longrightarrow B\otimes L$ satisfying (4.2.4). In addition to this if $\{-,-\}$ satisfies the Leibniz relation on $B\otimes L$ then it is indeed a deformation of $L$ with base $B$ extending the deformation $\lambda$ with base $A$.

In our next step we show that a bilinear operation as obtained above gives rise to a 3-cochain, which we call an obstruction cochain.

Suppose $\{-,-\}$ is a $B$-bilinear operation on $B\otimes L$ satisfying the conditions in (4.2.4).

Let us define a linear map $\phi:(B\otimes L)^{\otimes 3}\longrightarrow B\otimes L$ by

$$\phi(l_1,l_2,l_3) = \{l_1,\{l_2,l_3\}\} - \{\{l_1,l_2\},l_3\} + \{\{l_1,l_3\},l_2\} \quad \text{for } l_1,l_2,l_3\in B\otimes L. \qquad (4.2.7)$$

It is clear that $\{-,-\}$ satisfies the Leibniz relation if and only if $\phi = 0$. Now from property $(i)$ in (4.2.4) and the definition of $\phi$ it follows that

$$P\circ\phi(l_1,l_2,l_3)$$
$$= P(\{l_1,\{l_2,l_3\}\} - \{\{l_1,l_2\},l_3\} + \{\{l_1,l_3\},l_2\})$$
$$= [P(l_1),[P(l_2),P(l_3)]_\lambda]_\lambda - [[P(l_1),P(l_2)]_\lambda,P(l_3)]_\lambda + [P(l_1),P(l_3)]_\lambda,P(l_2)]_\lambda$$
$$= 0 \text{ (since } P(l_1),P(l_2),P(l_3)\in A\otimes L \text{ and } [-,-]_\lambda \text{ satisfy the Leibniz relation).} \qquad (4.2.8)$$

Therefore $\phi$ takes values in $ker(P)$.

Observe that $\phi(l_1,l_2,l_3) = 0$, whenever one of the arguments belongs to $ker(E)$. To see this, suppose $l_1 = (b\otimes l)\in ker(E)\subseteq B\otimes L$. Since $ker(E) = ker(\hat{\varepsilon})\otimes L = p^{-1}(ker(\varepsilon))\otimes L = \mathfrak{M}_B\otimes L$, we can write $l_1 = \sum_{j=1}^{r+1}(n_j\otimes l'_j)$ with $l'_j\in L$, $1\le j\le r+1$.

Then for $l_2, l_3 \in B \otimes L$, we get

$$\phi(l_1, l_2, l_3) = \phi\left(\sum_{j=1}^{r+1}(n_j \otimes l'_j), l_2, l_3\right) = \sum_{j=1}^{r+1} n_j \phi(1 \otimes l'_j, l_2, l_3) = 0.$$

This is because $\phi(1 \otimes l'_j, l_2, l_3) \in ker(P) = im(I) = im(i) \otimes L = i(\mathbb{K}) \otimes L$ and for any element $k \in \mathbb{K}$ and $l \in L$,

$$\begin{aligned}&n_j\ i(k) \otimes l = i(p(n_j)k) \otimes l = i(m_j k) \otimes l = i(\varepsilon(m_j)k) \otimes l = 0 \ \text{ for } 1 \leq j \leq r\\ &\text{and } n_{r+1} i(k) \otimes l = k n_{r+1}^2 \otimes l = 0 \ \ (m_j \in \mathfrak{M} \subset A \text{ and } m_j k = \varepsilon(m_j)k).\end{aligned}$$

The other two cases are similar. Thus $\phi$ defines a linear map

$$\tilde{\phi} : \left(\frac{B \otimes L}{ker(E)}\right)^{\otimes 3} \longrightarrow \ ker(P),$$

$\tilde{\phi}(b_1 \otimes l_1 + ker(E), b_2 \otimes l_2 + ker(E), b_3 \otimes l_3 + ker(E)) = \phi(b_1 \otimes l_1, b_2 \otimes l_2, b_3 \otimes l_3)$. Moreover, the surjective map $E : B \otimes L \longrightarrow \mathbb{K} \otimes L \cong L$, defined by $b \otimes l \longmapsto \hat{\varepsilon}(b) \otimes l$, induces an isomorphism $\frac{B \otimes L}{ker(E)} \overset{\alpha}{\cong} L$, where

$$\alpha : L \longrightarrow \frac{B \otimes L}{ker(E)} \ \ ; \ \ \alpha(l) = (1 \otimes l) + ker(E).$$

Also, $ker(P) = im(I) = i(\mathbb{K}) \otimes L = \mathbb{K}\ i(1) \otimes L \overset{\beta}{\cong} L$ where the isomorphism $\beta$ is given by $\beta(k n_{r+1} \otimes l) = kl$ with inverse $\beta^{-1}(l) = n_{r+1} \otimes l$. Thus we get a linear map $\bar{\phi} : L^{\otimes 3} \longrightarrow L$, such that $\bar{\phi} = \beta \circ \tilde{\phi} \circ \alpha^{\otimes 3}$. The cochains $\bar{\phi} \in CL^3(L; L)$ and $\phi$ are related by

$$n_{r+1} \otimes \bar{\phi}(l_1, l_2, l_3) = \phi(1 \otimes l_1, 1 \otimes l_2, 1 \otimes l_3). \tag{4.2.9}$$

**Proposition 4.2.2.** *The 3-cochain $\bar{\phi} \in CL^3(L; L)$ is a cocycle.*

*Proof.* Let $l_1, l_2, l_3 \in L$. Then from the coboundary formula (1.3.1) we get

$$\begin{aligned}&\delta\bar{\phi}(l_1, l_2, l_3, l_4)\\ &= [l_1, \bar{\phi}(l_2, l_3, l_4)] + [\bar{\phi}(l_1, l_3, l_4), l_2] - [\bar{\phi}(l_1, l_2, l_4), l_3] + [\bar{\phi}(l_1, l_2, l_3), l_4]\\ &\quad - \bar{\phi}([l_1, l_2], l_3, l_4) + \bar{\phi}([l_1, l_3], l_2, l_4) - \bar{\phi}([l_1, l_4], l_2, l_3)\\ &\quad + \bar{\phi}(l_1, [l_2, l_3], l_4) - \bar{\phi}(l_1, [l_2, l_4], l_3) - \bar{\phi}(l_1, l_2, [l_3, l_4]).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\beta^{-1} \circ \delta\bar{\phi}(l_1, l_2, l_3, l_4) \\
&= \beta^{-1}([l_1, \bar{\phi}(l_2, l_3, l_4)]) + \beta^{-1}([\bar{\phi}(l_1, l_3, l_4), l_2]) - \beta^{-1}([\bar{\phi}(l_1, l_2, l_4), l_3]) \\
&\quad + \beta^{-1}([\bar{\phi}(l_1, l_2, l_3), l_4]) - \beta^{-1}(\bar{\phi}([l_1, l_2], l_3, l_4)) + \beta^{-1}(\bar{\phi}([l_1, l_3], l_2, l_4)) \\
&\quad - \beta^{-1}(\bar{\phi}([l_1, l_4], l_2, l_3)) + \beta^{-1}(\bar{\phi}(l_1, [l_2, l_3], l_4)) - \beta^{-1}(\bar{\phi}(l_1, [l_2, l_4], l_3)) \\
&\quad - \beta^{-1}(\bar{\phi}(l_1, l_2, [l_3, l_4])).
\end{aligned} \tag{4.2.10}$$

Now,

$$\begin{aligned}
&\beta^{-1}([l_1, \bar{\phi}(l_2, l_3, l_4)]) \\
&= n_{r+1} \otimes [l_1, \bar{\phi}(l_2, l_3, l_4)] \\
&= I([l_1, \bar{\phi}(l_2, l_3, l_4)]) \quad (i(1) = n_{r+1}) \\
&= I([l_1, E(1 \otimes \bar{\phi}(l_2, l_3, l_4))]) \\
&= \{I(l_1), 1 \otimes \bar{\phi}(l_2, l_3, l_4)\} \quad (\text{by } (ii) \text{ of } (4.2.4)) \\
&= \{n_{r+1} \otimes l_1, 1 \otimes \bar{\phi}(l_2, l_3, l_4)\} \\
&= \{1 \otimes l_1, n_{r+1} \otimes \bar{\phi}(l_2, l_3, l_4)\} \\
&= \{1 \otimes l_1, \phi(1 \otimes l_2, 1 \otimes l_3, 1 \otimes l_4)\} \quad (\text{by } (4.2.9)) \\
&= \{1 \otimes l_1, \{1 \otimes l_2, \{1 \otimes l_3, 1 \otimes l_4\}\}\} - \{1 \otimes l_1, \{\{1 \otimes l_2, 1 \otimes l_3\}, 1 \otimes l_4\}\} \\
&\quad + \{1 \otimes l_1, \{\{1 \otimes l_2, 1 \otimes l_4\}, 1 \otimes l_3\}\}.
\end{aligned} \tag{4.2.11}$$

Also,

$$\begin{aligned}
&\beta^{-1}([\bar{\phi}(l_1, l_3, l_4), l_2]) \\
&= n_{r+1} \otimes [\bar{\phi}(l_1, l_3, l_4), l_2] \\
&= I([\bar{\phi}(l_1, l_3, l_4), E(1 \otimes l_2)]) \quad (i(1) = n_{r+1}) \\
&= \{I(\bar{\phi}(l_1, l_3, l_4)), 1 \otimes l_2\} \quad (\text{by } (ii) \text{ of } (4.2.4)) \\
&= \{n_{r+1} \otimes \bar{\phi}(l_1, l_3, l_4), 1 \otimes l_2\} \\
&= \{\phi(1 \otimes l_1, 1 \otimes l_3, 1 \otimes l_4), 1 \otimes l_2\} \quad (\text{by } (4.2.9)) \\
&= \{\{1 \otimes l_1, \{1 \otimes l_3, 1 \otimes l_4\}\} - \{\{1 \otimes l_1, 1 \otimes l_3\}, 1 \otimes l_4\} \\
&\quad + \{\{1 \otimes l_1, 1 \otimes l_4\}, 1 \otimes l_3\}, 1 \otimes l_2\} \\
&= \{\{1 \otimes l_1, \{1 \otimes l_3, 1 \otimes l_4\}\}, 1 \otimes l_2\} - \{\{\{1 \otimes l_1, 1 \otimes l_3\}, 1 \otimes l_4\}, 1 \otimes l_2\} \\
&\quad + \{\{\{1 \otimes l_1, 1 \otimes l_4\}, 1 \otimes l_3\}, 1 \otimes l_2\}.
\end{aligned} \tag{4.2.12}$$

Similarly,

$$\begin{aligned} &\beta^{-1}([\bar{\phi}(l_1,l_2,l_4),l_3]) \\ =\ &\{\{1\otimes l_1,\{1\otimes l_2,1\otimes l_4\}\},1\otimes l_3\}-\{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_4\},1\otimes l_3\} \\ &+\{\{\{1\otimes l_1,1\otimes l_4\},1\otimes l_2\},1\otimes l_3\} \end{aligned} \tag{4.2.13}$$

and

$$\begin{aligned} &\beta^{-1}([\bar{\phi}(l_1,l_2,l_3),l_4]) \\ =\ &\{\{1\otimes l_1,\{1\otimes l_2,1\otimes l_3\}\},1\otimes l_4\}-\{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_3\},1\otimes l_4\} \\ &+\{\{\{1\otimes l_1,1\otimes l_3\},1\otimes l_2\},1\otimes l_4\}. \end{aligned} \tag{4.2.14}$$

Again,

$$\begin{aligned} &\beta^{-1}(\bar{\phi}([l_1,l_2],l_3,l_4)) \\ &=n_{r+1}\otimes\bar{\phi}([l_1,l_2],l_3,l_4) \\ &=\phi(1\otimes[l_1,l_2],1\otimes l_3,1\otimes l_4) \\ &=\phi(\{1\otimes l_1,1\otimes l_2\}-X,1\otimes l_3,1\otimes l_4) \\ &\quad(\text{since } E(\{1\otimes l_1,1\otimes l_2\})=1\otimes[l_1,l_2] \text{ by } (i) \text{ of } (4.2.4)\text{, we may write} \\ &\quad\{1\otimes l_1,1\otimes l_2\}=1\otimes[l_1,l_2]+X,\ \text{where } X \text{ denotes an element in } ker(E)) \\ &=\phi(\{1\otimes l_1,1\otimes l_2\},1\otimes l_3,1\otimes l_4) \\ &\quad(\text{since } \phi(l_1,l_2,l_3)=0 \text{ for at least one } l_i\in ker(E) \text{ for } i=1,2 \text{ and } 3) \\ &=\{\{1\otimes l_1,1\otimes l_2\},\{1\otimes l_3,1\otimes l_4\}\}-\{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_3\},1\otimes l_4\} \\ &\quad+\{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_4\},1\otimes l_3\}. \end{aligned} \tag{4.2.15}$$

Similarly we have,

$$\begin{aligned} &\beta^{-1}(\bar{\phi}([l_1,l_3],l_2,l_4)) \\ &=n_{r+1}\otimes\bar{\phi}([l_1,l_2],l_3,l_4) \\ &=\{\{1\otimes l_1,1\otimes l_3\},\{1\otimes l_2,1\otimes l_4\}\}-\{\{\{1\otimes l_1,1\otimes l_3\},1\otimes l_2\},1\otimes l_4\} \\ &\quad+\{\{\{1\otimes l_1,1\otimes l_3\},1\otimes l_4\},1\otimes l_2\}, \end{aligned} \tag{4.2.16}$$

$$\begin{aligned} &\beta^{-1}(\bar{\phi}([l_1,l_4],l_2,l_3)) \\ &=n_{r+1}\otimes\bar{\phi}([l_1,l_2],l_4,l_3) \\ &=\{\{1\otimes l_1,1\otimes l_4\},\{1\otimes l_2,1\otimes l_3\}\}-\{\{\{1\otimes l_1,1\otimes l_4\},1\otimes l_2\},1\otimes l_3\} \\ &\quad+\{\{\{1\otimes l_1,1\otimes l_4\},1\otimes l_3\},1\otimes l_2\}, \end{aligned} \tag{4.2.17}$$

$$\begin{aligned}
&\beta^{-1}(\bar{\phi}(l_1,[l_2,l_3],l_4)) \\
&= n_{r+1}\otimes\bar{\phi}(l_1,[l_2,l_3],l_4) \\
&= \phi(1\otimes l_1,\{1\otimes l_2,1\otimes l_3\},1\otimes l_4) \\
&= \{1\otimes l_1,\{\{1\otimes l_2,1\otimes l_3\},1\otimes l_4\}\} - \{\{1\otimes l_1,\{1\otimes l_2,1\otimes l_3\}\},1\otimes l_4\} \\
&\qquad + \{\{1\otimes l_1,1\otimes l_4\},\{1\otimes l_2,1\otimes l_3\}\}.
\end{aligned} \tag{4.2.18}$$

$$\begin{aligned}
&\beta^{-1}(\bar{\phi}(l_1,[l_2,l_4],l_3)) \\
&= n_{r+1}\otimes\bar{\phi}(l_1,[l_2,l_4],l_3) \\
&= \phi(1\otimes l_1,\{1\otimes l_2,1\otimes l_4\},1\otimes l_3) \\
&= \{1\otimes l_1,\{\{1\otimes l_2,1\otimes l_4\},1\otimes l_3\}\} - \{\{1\otimes l_1,\{1\otimes l_2,1\otimes l_4\}\},1\otimes l_3\} \\
&\qquad + \{\{1\otimes l_1,1\otimes l_3\},\{1\otimes l_2,1\otimes l_4\}\},
\end{aligned} \tag{4.2.19}$$

and

$$\begin{aligned}
&\beta^{-1}(\bar{\phi}(l_1,l_2,[l_3,l_4])) \\
&= n_{r+1}\otimes\bar{\phi}(l_1,l_2,[l_3,l_4]) \\
&= \phi(1\otimes l_1,1\otimes l_2,\{1\otimes l_3,1\otimes l_4\}) \\
&= \{1\otimes l_1,\{1\otimes l_2,\{1\otimes l_3,1\otimes l_4\}\} - \{\{1\otimes l_1,1\otimes l_2\},\{1\otimes l_3,1\otimes l_4\}\} \\
&\qquad + \{\{1\otimes l_1,\{1\otimes l_3,1\otimes l_4\}\},1\otimes l_2\}.
\end{aligned} \tag{4.2.20}$$

Substituting each term on the right-hand side of (4.2.10) from (4.2.11) - (4.2.20) we get,

$$\begin{aligned}
&\beta^{-1}\circ\delta\bar{\phi}(l_1,l_2,l_3,l_4) \\
=&\{1\otimes l_1,\{1\otimes l_2,\{1\otimes l_3,1\otimes l_4\}\}\} - \{1\otimes l_1,\{\{1\otimes l_2,1\otimes l_3\},1\otimes l_4\}\} \\
&+\{1\otimes l_1,\{\{1\otimes l_2,1\otimes l_4\},1\otimes l_3\}\} + \{\{1\otimes l_1,\{1\otimes l_3,1\otimes l_4\}\},1\otimes l_2\} \\
&-\{\{\{1\otimes l_1,1\otimes l_3\},1\otimes l_4\},1\otimes l_2\} + \{\{\{1\otimes l_1,1\otimes l_4\},1\otimes l_3\},1\otimes l_2\} \\
&-\{\{1\otimes l_1,\{1\otimes l_2,1\otimes l_4\}\},1\otimes l_3\} + \{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_4\},1\otimes l_3\} \\
&-\{\{\{1\otimes l_1,1\otimes l_4\},1\otimes l_2\},1\otimes l_3\} + \{\{1\otimes l_1,\{1\otimes l_2,1\otimes l_3\}\},1\otimes l_4\} \\
&-\{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_3\},1\otimes l_4\} + \{\{\{1\otimes l_1,1\otimes l_3\},1\otimes l_2\},1\otimes l_4\} \\
&-\{\{1\otimes l_1,1\otimes l_2\},\{1\otimes l_3,1\otimes l_4\}\} + \{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_3\},1\otimes l_4\} \\
&-\{\{\{1\otimes l_1,1\otimes l_2\},1\otimes l_4\},1\otimes l_3\} + \{\{1\otimes l_1,1\otimes l_3\},\{1\otimes l_2,1\otimes l_4\}\} \\
&-\{\{\{1\otimes l_1,1\otimes l_3\},1\otimes l_2\},1\otimes l_4\} + \{\{\{1\otimes l_1,1\otimes l_3\},1\otimes l_4\},1\otimes l_2\} \\
&-\{\{1\otimes l_1,1\otimes l_4\},\{1\otimes l_2,1\otimes l_3\}\} + \{\{\{1\otimes l_1,1\otimes l_4\},1\otimes l_2\},1\otimes l_3\} \\
&-\{\{\{1\otimes l_1,1\otimes l_4\},1\otimes l_3\},1\otimes l_2\} + \{1\otimes l_1,\{\{1\otimes l_2,1\otimes l_3\},1\otimes l_4\}\} \\
&-\{\{1\otimes l_1,\{1\otimes l_2,1\otimes l_3\}\},1\otimes l_4\} + \{\{1\otimes l_1,1\otimes l_4\},\{1\otimes l_2,1\otimes l_3\}\}
\end{aligned}$$

$$
\begin{aligned}
&- \{1\otimes l_1, \{\{1\otimes l_2, 1\otimes l_4\}, 1\otimes l_3\}\} + \{\{1\otimes l_1, \{1\otimes l_2, 1\otimes l_4\}\}, 1\otimes l_3\}\\
&- \{\{1\otimes l_1, 1\otimes l_3\}, \{1\otimes l_2, 1\otimes l_4\}\} - \{1\otimes l_1, \{1\otimes l_2, \{1\otimes l_3, 1\otimes l_4\}\}\\
&+ \{\{1\otimes l_1, 1\otimes l_2\}, \{1\otimes l_3, 1\otimes l_4\}\} - \{\{1\otimes l_1, \{1\otimes l_3, 1\otimes l_4\}\}, 1\otimes l_2\}\\
= {} & 0.
\end{aligned}
$$

Thus $\beta^{-1}\circ\delta\bar{\phi}(l_1,l_2,l_3,l_4) = 0$ for $l_1, l_2, l_3$ and $l_4 \in L$. Since $\beta^{-1}$ is an isomorphism, it follows that $\delta\bar{\phi} = 0$. □

Let us show now that the cohomology class of $\bar{\phi}$ is independent of the choice of the lifting $\{-,-\}$.

**Proposition 4.2.3.** *Suppose $\{-,-\}$ and $\{-,-\}'$ are any two $B$-bilinear operations on $B\otimes L$, satisfying (4.2.4). Let $\bar{\phi}$ and $\bar{\phi}'$ be the corresponding cocycles determined by $\{-,-\}$ and $\{-,-\}'$ respectively. Then $\bar{\phi}$ and $\bar{\phi}'$ represent the same cohomology class in $HL^3(L;L)$.*

*Proof.* Set $\rho = \{-,-\}' - \{-,-\}$. Then $\rho : (B\otimes L)^{\otimes 2} \longrightarrow B\otimes L$ is a $B$-linear map. Now $\rho$ takes values in $ker(P)$ because for $l_1, l_2 \in B\otimes L$,

$$
\begin{aligned}
& P\circ\rho\ (l_1,l_2)\\
= {} & P\{l_1,l_2\} - P\{l_1,l_2\}'\\
= {} & [P(l_1),P(l_2)]_\lambda - [P(l_1),P(l_2)]_\lambda \text{ (by } (i) \text{ in (4.2.4))}\\
= {} & 0.
\end{aligned}
$$

Since $\rho$ takes values in $ker(P)$, $\rho(l_1,l_2) = 0$, whenever one of the arguments is in $ker(E)$. This is similar to the argument given for $\phi$. Thus $\rho$ induces a linear map

$$\tilde{\rho} : \left(\frac{B\otimes L}{ker(E)}\right)^{\otimes 2} \longrightarrow ker(P),$$

$$\tilde{\rho}(l_1 + ker(E), l_2 + ker(E)) = \rho(l_1,l_2) \text{ for } l_1, l_2 \in B\otimes L.$$

Hence we get a 2-cochain $\bar{\rho} : L^{\otimes 2} \longrightarrow L$ such that $\bar{\rho} = \beta\circ\tilde{\rho}\circ\alpha^{\otimes 2} \in CL^2(L;L)$. The map $\rho$ and $\bar{\rho}$ are related by $n_{r+1}\otimes\bar{\rho}(l_1,l_2) = \rho(1\otimes l_1, 1\otimes l_2)$ for $l_1, l_2 \in L$.

Let us denote by $\phi'$ respectively, $\bar{\phi}'$ the corresponding maps determined using $\{-,-\}'$ as in (4.2.9). Next we will show that

$$(\bar{\phi}' - \bar{\phi}) = \delta\bar{\rho}.$$

Suppose $l_1, l_2, l_3 \in L$. From the coboundary formula (1.3.1), we have

$$\begin{aligned}\delta\bar{\rho}(l_1, l_2, l_3) = [l_1, \bar{\rho}(l_2, l_3)] + [\bar{\rho}(l_1, l_3), l_2] - [\bar{\rho}(l_1, l_2), l_3] - \bar{\rho}([l_1, l_2], l_3)\\ + \bar{\rho}([l_1, l_3], l_2) + \bar{\rho}(l_1, [l_2, l_3]).\end{aligned}$$

Let us compute the terms appearing on the right-hand side of $\beta^{-1} \circ \delta\bar{\rho}(l_1, l_2, l_3)$

$$\begin{aligned}\beta^{-1}([l_1, \bar{\rho}(l_2, l_3)]) &= n_{r+1} \otimes [l_1, \bar{\rho}(l_2, l_3)]\\ &= I[l_1, \bar{\rho}(l_2, l_3)]\\ &= I[l_1, E(1 \otimes \bar{\rho}(l_2, l_3))]\\ &= \{I(l_1), 1 \otimes \bar{\rho}(l_2, l_3)\}'\\ &= \{n_{r+1} \otimes l_1, 1 \otimes \bar{\rho}(l_2, l_3)\}'\\ &= \{1 \otimes l_1, n_{r+1} \otimes \bar{\rho}(l_2, l_3)\}'\\ &= \{1 \otimes l_1, \rho(l_2, l_3)\}'\\ &= \{1 \otimes l_1, \{1 \otimes l_2, 1 \otimes l_3\}' - \{1 \otimes l_2, 1 \otimes l_3\}\}'\\ &= \{1 \otimes l_1, \{1 \otimes l_2, 1 \otimes l_3\}'\}' - \{1 \otimes l_1, \{1 \otimes l_2, 1 \otimes l_3\}\}'.\end{aligned}$$

$$\begin{aligned}\beta^{-1}[\bar{\rho}(l_1, l_3), l_2] &= n_{r+1} \otimes [\bar{\rho}(l_1, l_3), l_2]\\ &= I(1 \otimes [\bar{\rho}(l_1, l_3), E(1 \otimes l_2)])\\ &= \{I(\bar{\rho}(l_1, l_3)), 1 \otimes l_2\}'\\ &= \{n_{r+1} \otimes \bar{\rho}(l_1, l_3), 1 \otimes l_2\}'\\ &= \{\rho(l_1, l_3), 1 \otimes l_2\}'\\ &= \{\{1 \otimes l_1, 1 \otimes l_3\}' - \{1 \otimes l_1, 1 \otimes l_3\}, 1 \otimes l_2\}'\\ &= \{\{1 \otimes l_1, 1 \otimes l_3\}', 1 \otimes l_2\}' - \{\{1 \otimes l_1, 1 \otimes l_3\}, 1 \otimes l_2\}'.\end{aligned}$$

Similarly,

$$\beta^{-1}[\bar{\rho}(l_1, l_2), l_3] = \{\{1 \otimes l_1, 1 \otimes l_2\}', 1 \otimes l_3\}' - \{\{1 \otimes l_1, 1 \otimes l_2\}, 1 \otimes l_3\}'.$$

Also

$$\begin{aligned}\beta^{-1} \circ \bar{\rho}([l_1, l_2], l_3) &= n_{r+1} \otimes \bar{\rho}([l_1, l_2], l_3)\\ &= \rho(1 \otimes [l_1, l_2], 1 \otimes l_3)\\ &= \{1 \otimes [l_1, l_2], 1 \otimes l_3\}' - \{1 \otimes [l_1, l_2], 1 \otimes l_3\}.\end{aligned}$$

Since $E(\{1\otimes l_1, 1\otimes l_2\}) = 1\otimes[l_1,l_2]$ so we get $\{1\otimes l_1, 1\otimes l_2\} = 1\otimes[l_1,l_2]+X$, where $X$ denotes an element in $ker(E)$, using this the last expression is given by

$$\begin{aligned}
&= \{\{1\otimes l_1, 1\otimes l_2\} - X, 1\otimes l_3\}' - \{\{1\otimes l_1, 1\otimes l_2\} - X, 1\otimes l_3\}\\
&= \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\}' - \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\} - \rho(X, 1\otimes l_3)\\
&= \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\}' - \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\}\\
&\qquad (\text{since } X\in ker(E),\ \rho(X, 1\otimes l_3)=0).
\end{aligned}$$

Similarly,

$$\beta^{-1}\circ\bar{\rho}([l_1,l_3],l_2) = \{\{1\otimes l_1, 1\otimes l_3\}, 1\otimes l_2\}' - \{\{1\otimes l_1, 1\otimes l_3\}, 1\otimes l_2\}$$

$$\begin{aligned}
\text{and } \beta^{-1}\circ\bar{\rho}(l_1,[l_2,l_3]) &= n_{r+1}\otimes\bar{\rho}(l_1,[l_2,l_3])\\
&= \rho(1\otimes l_1, 1\otimes[l_2,l_3])\\
&= \{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}\}' - \{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}\}.
\end{aligned}$$

Substituting all the above terms on the right-hand side of $\beta^{-1}\circ\delta\bar{\rho}(l_1,l_2,l_3)$ we get,

$$\begin{aligned}
\beta^{-1}\circ\delta\bar{\rho}(l_1,l_2,l_3) &= \beta^{-1}[l_1,\bar{\rho}(l_2,l_3)] + \beta^{-1}[\bar{\rho}(l_1,l_3),l_2] - \beta^{-1}[\bar{\rho}(l_1,l_2),l_3]\\
&\quad - \beta^{-1}\bar{\rho}([l_1,l_2],l_3) + \beta^{-1}\bar{\rho}([l_1,l_3],l_2) + \beta^{-1}\bar{\rho}(l_1,[l_2,l_3])\\
&= \{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}'\}' - \{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}\}'\\
&\quad + \{\{1\otimes l_1, 1\otimes l_3\}', 1\otimes l_2\}' - \{\{1\otimes l_1, 1\otimes l_3\}, 1\otimes l_2\}'\\
&\quad - \{\{1\otimes l_1, 1\otimes l_2\}', 1\otimes l_3\}' + \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\}'\\
&\quad - \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\}' + \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\}\\
&\quad + \{\{1\otimes l_1, 1\otimes l_3\}, 1\otimes l_2\}' - \{\{1\otimes l_1, 1\otimes l_3\}, 1\otimes l_2\}\\
&\quad + \{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}\}' - \{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}\}\\
&= \phi'(1\otimes l_1, 1\otimes l_2, 1\otimes l_3) - \phi(1\otimes l_1, 1\otimes l_2, 1\otimes l_3)\\
&= n_{r+1}\otimes\bar{\phi}'(l_1,l_2,l_3) - n_{r+1}\otimes\bar{\phi}(l_1,l_2,l_3)\\
&= n_{r+1}\otimes(\bar{\phi}'-\bar{\phi})(l_1,l_2,l_3)\\
&= \beta^{-1}\circ(\bar{\phi}'-\bar{\phi})(l_1,l_2,l_3),
\end{aligned}$$

where $\phi'$ and $\bar{\phi}'$ are the maps corresponding to $\phi$ and $\bar{\phi}$ respectively for the pairing $\{-,-\}'$. Therefore $\beta^{-1}\circ\delta\bar{\rho}(l_1,l_2,l_3) = \beta^{-1}\circ(\bar{\phi}'-\bar{\phi})(l_1,l_2,l_3)$ for $l_1,l_2,l_3\in L$. Since $\beta^{-1}$ is an isomorphism we get $\delta\bar{\rho} = (\bar{\phi}'-\bar{\phi})$. □

We note that the 3-cocycle $\bar{\phi}$ as determined by the extension (4.2.1) depends only on its isomorphism class.

The above considerations together with Proposition 2.3.3 define a map

$$\theta_\lambda : H^2_{Harr}(A;\mathbb{K}) \longrightarrow HL^3(L;L) \ \text{ by } \theta_\lambda([\psi]) = [\bar{\phi}], \tag{4.2.21}$$

where $[\bar{\phi}]$ is the cohomology class of $\bar{\phi}$. We call the map $\theta_\lambda$ the *obstruction map*.

We are now in a position to formulate a necessary and sufficient condition for existence of an extension of the Leibniz algebra structure $\lambda$ on $A \otimes L$.

**Theorem 4.2.4.** *Let $\lambda$ be a deformation of the Leibniz algebra $L$ with base $A$ and let $B$ be a 1-dimensional extension of $A$ corresponding to the cohomology class $[\psi] \in H^2_{Harr}(A;\mathbb{K})$. Then $\lambda$ can be extended to a deformation of $L$ with base $B$ if and only if the obstruction $\theta_\lambda([\psi]) = 0$.*

*Proof.* Suppose $\theta_\lambda([\psi]) = 0$. Let

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be a 1-dimensional extension representing the cohomology class $[\psi]$. Let $\{-,-\}$ be a $B$-bilinear operation on $B \otimes L$ satisfying (4.2.4), lifting the Leibniz algebra structure $\lambda$ on $A \otimes L$. Let $\bar{\phi}$ be the associated cocycle in $CL^3(L;L)$ as described above in (4.2.9). Then $\theta_\lambda([\psi]) = [\bar{\phi}] = 0$, implies $\bar{\phi} = \delta\rho$ for some $\rho \in CL^2(L;L)$. Now take $\rho' = -\rho$, and define a new linear map

$$\{-,-\}' : (B \otimes L)^{\otimes 2} \longrightarrow B \otimes L \ \text{ by } \{l_1, l_2\}' = \{l_1, l_2\} + I \circ \rho'(E(l_1), E(l_2)).$$

Let $\phi'$ and $\bar{\phi}'$ be the maps as defined in (4.2.9) corresponding to $\{-,-\}'$. Then we have $(\bar{\phi}' - \bar{\phi}) = \delta\bar{\rho}' = -\bar{\phi}$. Hence $\bar{\phi}' = 0$, which implies $\phi' = 0$. Therefore, $\{-,-\}'$ is a Leibniz algebra structure on $B \otimes L$ extending $\lambda$.

The converse part is clear from the fact that if we have an extension $\{-,-\}$ of the deformation $\lambda$ of $L$ with base $A$, the map $\phi$ as defined in (4.2.7) is the zero map. So the induced cochain $\bar{\phi}$ is the zero cochain. ☐

**Example 4.2.5.** *Let $\lambda_t$ be a finite order 1-parameter deformation of a Leibniz algebra $L$ with base $A = \mathbb{K}[[t]]/(t^{N+1})$. Explicitly, $\lambda_t(x,y) = \sum_{i\geq 0} \lambda_i(x,y)t^i$ (modulo $t^{N+1}$), where $x, y \in L$, $\lambda_i \in CL^2(L;L)$ with $\lambda_0$ the original Leibniz bracket in $L$ and $N$ is the order of the deformation. By Leibniz relation we have*

$$\lambda_t(x, \lambda_t(y,z)) - \lambda_t(\lambda_t(x,y),z) + \lambda_t(\lambda_t(x,z),y) = 0$$

*modulo $t^{N+1}$, for $x, y, z \in L$.*

*If we try to extend it to a deformation of order $N+1$, starting with the extension*

$$0 \longrightarrow (t^{N+1})/(t^{N+2}) \longrightarrow \mathbb{K}[[t]]/(t^{N+2}) \longrightarrow \mathbb{K}[[t]]/(t^{N+1}) \longrightarrow 0,$$

*the obstruction cocycle in this case can be written as ( [Bal97])*

$$\bar{\phi}(x,y,z) = \sum_{\substack{i+j=N+1\\ i,j>0}} \{\lambda_i(\lambda_j(x,y),z) - \lambda_i(\lambda_j(x,z),y) - \lambda_i(x,\lambda_j(y,z))\}.$$

*The given deformation extends to a deformation of order $N+2$, if the cohomology class of the above obstruction cocycle is zero.*

Suppose now that $M_0$ is a finite dimensional $A$-module satisfying the condition $\mathfrak{M}M_0 = 0$, where $\mathfrak{M}$ is the maximal ideal in $A$. The previous results can be generalized from the 1-dimensional extension (4.2.1) to a more general extension

$$0 \longrightarrow M_0 \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} A \longrightarrow 0,$$

representing an isomorphism class of extensions corresponding to a cohomology class $[\psi] \in H^2_{Harr}(A;M_0)$ (Proposition 2.3.3).

If we try to extend a deformation with base $A$ to a deformation with base $B$, as in the beginning of this section, then an analogous computation yields

$$\tilde{\phi} : \left(\frac{B\otimes L}{ker\ (E)}\right)^{\otimes 3} \longrightarrow ker\ (P) = im(I) \cong M_0 \otimes L.$$

It will give rise to a cocycle $\bar{\phi} \in CL^3(L; M_0 \otimes L)$ with the cohomology class

$$[\bar{\phi}] \in HL^3(L; M_0 \otimes L) = M_0 \otimes HL^3(L;L).$$

The obstruction map in this case is

$$\theta_\lambda : H^2_{Harr}(A;M_0) \longrightarrow M_0 \otimes HL^3(L;L) \text{ defined by } \theta_\lambda([\psi]) = [\bar{\phi}].$$

Then, as in the case of 1-dimensional extension, we have the following.

**Theorem 4.2.6.** *Let $\lambda$ be a deformation of a Leibniz algebra $L$ with base $(A,\mathfrak{M})$ and let $M_0$ be a finite dimensional $A$-module with $\mathfrak{M}M_0 = 0$. Consider an extension $B$ of $A$*

$$0 \longrightarrow M_0 \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} A \longrightarrow 0$$

*corresponding to some $[\psi] \in H^2_{Harr}(A;M_0)$. A deformation $\mu$ of $L$ with base $B$ such that $p_*\mu = \lambda$ exists if and only if the obstruction $\theta_\lambda([\psi]) = 0$.*

## 4.3 Two actions on the set of extensions of the deformation $\lambda$

Suppose $\mu$ and $\mu'$ are any two deformations with base $B$ extending a given deformation $\lambda$ with base $A$ of a Leibniz algebra $L$. We would like to know how $\mu$ and $\mu'$ are related. In the present section we study this relationship.

Let $[\psi] \in H^2_{Harr}(A;\mathbb{K})$ and consider the extension problem as in the previous section. Assume $\theta_\lambda([\psi]) = 0$. Let $S$ denote the set of equivalence classes of deformations $\mu$ of $L$ with base $B$ such that $p_*\mu = \lambda$.

We define two natural actions on $S$. Let $\mathcal{A}$ denote the group of automorphisms of the extension (4.2.1). Let $u \in \mathcal{A}$, then $u : B \longrightarrow B$ is an algebra isomorphism such that following diagram commutes (see Proposition 2.3.7).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \xrightarrow{i} & B & \xrightarrow{p} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle u} & & \| & & \\
0 & \longrightarrow & \mathbb{K} & \xrightarrow{i} & B & \xrightarrow{p} & A & \longrightarrow & 0
\end{array}$$

Now $u_*\mu$ is a deformation of $L$ with base $B$ and $p_*(u_*\mu) = (p \circ u)_*\mu = p_*\mu = \lambda$. Also if $\mu \cong \mu'$, then $u_*\mu \cong u_*\mu'$ (by Proposition 3.3.7 ). Thus we get an action

$$\sigma_1 : \mathcal{A} \times S :\longrightarrow S \text{ defined by } \sigma_1(u, <\mu>) = <u_*\mu>,$$

where $<\mu>$ is the equivalence class of $\mu$ (cf. Definition 3.2.6).

On the other hand, $HL^2(L;L)$ acts on $S$ as follows.

Suppose a $B$-bilinear operation $\{-,-\}$ is given on $B \otimes L$, satisfying (4.2.4) so that $p_*\{-,-\} = \lambda$ and $\rho \in CL^2(L;L)$ a given cochain. Define

$$\{l_1, l_2\}' = \{l_1, l_2\} + I \circ \rho(E(l_1), E(l_2) \text{ for } l_1, l_2 \in B \otimes L.$$

Then $\{-,-\}'$ is a $B$-bilinear operation on $B \otimes L$ and satisfies (4.2.4). Moreover the 2-cochain $\bar{\rho}$ determined by the difference $\{-,-\}' - \{-,-\}$ as in the beginning of the proof of Proposition 4.2.3, is the given $\rho$. It follows that $\bar{\phi}' - \bar{\phi} = \delta\rho$.

In particular, if $\{-,-\}'$ and $\{-,-\}$ satisfy Leibniz relation on $B \otimes L$, $\rho$ must be a cocycle.

Define an action

$$\sigma_2 : HL^2(L;L) \times S \longrightarrow S \text{ by } \sigma_2([\rho], <\mu>) = <\mu'>,$$

where $[l_1, l_2]_{\mu'} = [l_1, l_2]_\mu + I \circ \rho(E(l_1, E(l_2)))$.

We first check that the above action is well-defined.
Suppose $\rho_1, \rho_2 \in CL^2(L;L)$ represent the same class $[\rho] \in HL^2(L;L)$, so that $\rho_1 - \rho_2 = \delta\alpha$ for some 1-cochain $\alpha \in CL^1(L;L)$. Define two deformations $\mu_1'$ and $\mu_2'$ of $L$ with base $B$, which are extensions of $\lambda$, where the brackets $[-,-]_{\mu_1'}$ and $[-,-]_{\mu_2'}$ are given by

$$\begin{aligned} &[l_1, l_2]_{\mu_1'} = [l_1, l_2]_\mu + I \circ \rho_1(E(l_1), E(l_2)) \text{ and} \\ &[l_1, l_2]_{\mu_2'} = [l_1, l_2]_\mu + I \circ \rho_2(E(l_1), E(l_2)) \text{ for } l_1, l_2 \in B \otimes L. \end{aligned} \tag{4.3.1}$$

We claim that $\mu_1' \cong \mu_2'$.
For this we define a $B$-linear isomorphism $\Phi : B \otimes L \longrightarrow B \otimes L$
by $\Phi(1 \otimes l) = 1 \otimes l + n_{r+1} \otimes \alpha(l)$ for $1 \otimes l \in B \otimes L$. Note that $E \circ \Phi = E$. From definition it follows that for $l \in L$, $\Phi \circ I(1 \otimes l) = \Phi(n_{r+1} \otimes l) = n_{r+1} \otimes l = I(1 \otimes l)$. To prove our claim it remains to check that $\Phi$ preserves the brackets. Now

$$\begin{aligned} &\Phi[1 \otimes l_1, 1 \otimes l_2]_{\mu_1'} \\ &= \Phi([1 \otimes l_1, 1 \otimes l_2]_\mu + I \circ \rho(E(1 \otimes l_1), E(1 \otimes l_2))) \\ &= \Phi[1 \otimes l_1, 1 \otimes l_2] + \Phi \circ I \circ \rho_1(l_1, l_2) \\ &= \Phi(1 \otimes [l_1, l_2] + \sum_{i=1}^{r+1} n_j \otimes \psi_j(l_1, l_2)) + \Phi \circ I \circ \rho_1(l_1, l_2) \\ &= \Phi(1 \otimes [l_1, l_2]) + \sum_{i=1}^{r+1} n_j \Phi(1 \otimes \psi_j(l_1, l_2)) + \Phi \circ I \circ \rho_1(l_1, l_2) \\ &= 1 \otimes [l_1, l_2] + n_{r+1} \otimes \alpha([l_1, l_2]) \\ &\quad + \sum_{i=1}^{r+1} n_j (1 \otimes \psi_j(l_1, l_2) + n_{r+1} \otimes \alpha \circ \psi_j(l_1, l_2)) + \Phi \circ I \circ \rho_1(l_1, l_2) \\ &= 1 \otimes [l_1, l_2] + n_{r+1} \otimes \alpha([l_1, l_2]) + \sum_{j=1}^{r+1} n_j \otimes \psi_j(l_1, l_2) \\ &\quad + \sum_{j=1}^{r+1} n_j n_{r+1} \otimes \alpha \circ \psi_j(l_1, l_2) + \Phi \circ I \circ \rho_1(l_1, l_2) \\ &= 1 \otimes [l_1, l_2] + \sum_{j=1}^{r+1} n_j \otimes \psi_j(l_1, l_2) + n_{r+1} \otimes \alpha([l_1, l_2]) + \Phi \circ I \circ \rho_1(l_1, l_2) \\ &= [1 \otimes l_1, 1 \otimes l_2]_\mu + n_{r+1} \otimes \alpha([l_1, l_2]) + \Phi \circ I \circ \rho_1(l_1, l_2). \end{aligned} \tag{4.3.2}$$

On the other hand

$$\begin{aligned}
&[\Phi(1\otimes l_1),\Phi(1\otimes l_2)]_{\mu_2'}\\
&=[1\otimes l_1+n_{r+1}\otimes\alpha(l_1),1\otimes l_2+n_{r+1}\otimes\alpha(l_2)]_{\mu_2'}\\
&=[1\otimes l_1,1\otimes l_2]_{\mu_2'}+[1\otimes l_1,n_{r+1}\otimes\alpha(l_2)]_{\mu_2'}\\
&\quad+[n_{r+1}\otimes\alpha(l_1),1\otimes l_2]_{\mu_2'}+[n_{r+1}\otimes\alpha(l_1),n_{r+1}\otimes\alpha(l_2)]_{\mu_2'}\\
&=[1\otimes l_1,1\otimes l_2]_{\mu}+I\circ\rho_2(E(1\otimes l_1),E(1\otimes l_2))\\
&\quad+n_{r+1}[1\otimes l_1,1\otimes\alpha(l_2)]_{\mu}+n_{r+1}I\circ\rho_2(E(1\otimes l_1),E(1\otimes\alpha(l_2)))\\
&\quad+n_{r+1}[1\otimes\alpha(l_1),1\otimes l_2]_{\mu}+n_{r+1}I\circ\rho_2(E(1\otimes\alpha(l_1)),E(1\otimes l_2))\\
&=[1\otimes l_1,1\otimes l_2]_{\mu}+I\circ\rho_2(l_1,l_2)\\
&\quad+n_{r+1}(1\otimes[l_1,\alpha(l_2)]+\sum_{j=1}^{r+1}n_j\otimes\psi_j(l_1,\alpha(l_2)))\\
&\quad+n_{r+1}(1\otimes[\alpha(l_1),l_2]+\sum_{j=1}^{r+1}n_j\otimes\psi_j(\alpha(l_1),l_2))\\
&=[1\otimes l_1,1\otimes l_2]_{\mu}+I\circ\rho_2(l_1,l_2)\\
&\quad+n_{r+1}\otimes[l_1,\alpha(l_2)]+n_{r+1}\otimes[\alpha(l_1),l_2].
\end{aligned}\tag{4.3.3}$$

Thus

$$\begin{aligned}
&\Phi[1\otimes l_1,1\otimes l_2]_{\mu_1'}-[\Phi(1\otimes l_1),\Phi(1\otimes l_2)]_{\mu_2'}\\
&=n_{r+1}\otimes\alpha([l_1,l_2])-n_{r+1}\otimes[l_1,\alpha(l_2)]-n_{r+1}\otimes[\alpha(l_1),l_2]\\
&\quad+\Phi\circ I\circ\rho_1(l_1,l_2)-I\circ\rho_2(l_1,l_2)\\
&=n_{r+1}\otimes(-\delta\alpha(l_1,l_2))+I\circ(\rho_1-\rho_2)(l_1,l_2)\\
&=0.
\end{aligned}\tag{4.3.4}$$

Hence $\sigma_2([\rho_1],<\mu>)=\ <\mu_1'>\ =\ <\mu_2'>\ =\sigma_2([\rho_2],<\mu_2>)$.

In order to check that $\sigma_2$ is well defined it remains to show that $\sigma_2([\rho],<\mu_1>)=\sigma_2([\rho],<\mu_2>)$ for $\mu_1\cong\mu_2$.

Suppose $\mu_1\cong\mu_2$ and $\sigma_2([\rho],<\mu_1>)=\ <\mu_1'>$, $\sigma_2([\rho],<\mu_2>)=\ <\mu_2'>$. Let

$$\Phi:(B\otimes L,[-,-]_{\mu_1})\longrightarrow(B\otimes L,[-,-]_{\mu_2})$$

be an equivalence of $\mu_1$ and $\mu_2$. By definition

$$\begin{aligned}
&[l_1,l_2]_{\mu_1'}=[l_1,l_2]_{\mu_1}+I\circ\rho(E(l_1),E(l_2))\\
&[l_1,l_2]_{\mu_2'}=[l_1,l_2]_{\mu_2}+I\circ\rho(E(l_1),E(l_2))\ \text{for}\ l_1,l_2\in B\otimes L.
\end{aligned}\tag{4.3.5}$$

We claim that

$$\Phi[l_1, l_2]_{\mu'_1} = [\Phi(l_1), \Phi(l_2)]_{\mu'_2}.$$

Now

$$\begin{aligned}\Phi[l_1, l_2]_{\mu'_1} &= \Phi[l_1, l_2]_{\mu_1} + \Phi \circ I \circ \rho(E(l_1), E(l_2)) \\ &= [\Phi(l_1), \Phi(l_2)]_{\mu_2} + \Phi \circ I \circ \rho(E(l_1), E(l_2)).\end{aligned} \tag{4.3.6}$$

On the other hand

$$\begin{aligned}[\Phi(l_1), \Phi(l_2)]_{\mu'_2} &= [\Phi(l_1), \Phi(l_2)]_{\mu_2} + I \circ \rho(E \circ \Phi(l_1), E \circ \Phi(l_2)) \\ &= [\Phi(l_1), \Phi(l_2)]_{\mu_2} + I \circ \rho(E(l_1), E(l_2)) \quad \text{(by using } E \circ \Phi = \Phi).\end{aligned} \tag{4.3.7}$$

So $\mu'_1 \cong \mu'_2$. Therefore $\sigma_2([\rho], < \mu_1 >) = \; < \mu'_1 > \; = \; < \mu'_2 > \; = \sigma_2([\rho], < \mu_2 >)$. Consequently, the action of $HL^2(L; L)$ on $S$ is well defined. The transitivity of the action follows from the definition of $\sigma_2$.

We now show that the actions $\sigma_1$ and $\sigma_2$ on $S$ are related to each other by the differential $d\lambda : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(L; L)$ (see Definition 3.4.8).

Recall that the group $\mathcal{A}$ is identified with $H^1_{Harr}(A; \mathbb{K})$ (Proposition 2.3.7) so that for any element $u \in \mathcal{A}$, the corresponding cohomology class in $H^1_{Harr}(A; \mathbb{K})$ is represented by a 1-cocycle $\psi$ such that $u((a, m)_q) = (a, m + \psi(a))_q$, where $q : A \longrightarrow B$ is a section of $p$. On the other hand $H^1_{Harr}(A; \mathbb{K})$ is identified with $(\mathfrak{M}/\mathfrak{M}^2)'$ (Corollary 2.2.7). For if $\psi : A \longrightarrow \mathbb{K}$ is a linear map with $\delta\psi = 0$, then $\psi(1) = 0$ and $\psi$ vanishes on $\mathfrak{M}^2$. As a consequence, $\psi$ can be viewed as an element in $(\mathfrak{M}/\mathfrak{M}^2)'$. Using these identifications we get the following result.

**Proposition 4.3.1.** *Let $\lambda$ be a deformation of the Leibniz algebra $L$ with base $A$ and let*

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

*be a given extension of $A$. If $u : B \longrightarrow B$ is an automorphism of this extension which corresponds to an element $\psi \in (\mathfrak{M}/\mathfrak{M}^2)'$, then for any deformation $\mu$ of $L$ with base $B$, such that $p_*\mu = \lambda$, the difference $([-,-]_{u_*\mu} - [-,-]_\mu)$ induces a cocycle representing the cohomology class $d\lambda(\psi)$.*

*Proof.* Suppose $< \mu > \in S$ and $u \in \mathcal{A}$. By definition of $\sigma_1$, $< u_*\mu > \in S$. Consider the Leibniz brackets $[-,-]_{u_*\mu}$ and $[-,-]_\mu$ on $B \otimes L$ for the deformations $u_*\mu$ and $\mu$ respectively.

Suppose $\psi \in (\mathfrak{M}/\mathfrak{M}^2)'$ corresponds to $u \in \mathcal{A}$ under the identification mentioned above. We now proceed to show that the 2-cocycle determined by the difference $([-,-]_{u_*\mu} - [-,-]_\mu)$ represents $d\lambda(\psi)$.

Choose a basis $\{m_1, m_2, \cdots, m_k, m_{k+1}, \cdots, m_r\}$ of the maximal ideal $\mathfrak{M}$ of $A$ such that $\{\bar{m}_j\}_{1\leq j\leq k}$ is a basis of $\mathfrak{M}/\mathfrak{M}^2$. Fix a section $q : A \longrightarrow B$ of $p$. Let $\{n_1, n_2, \cdots, n_k, n_{k+1}, \cdots, n_{r+1}\}$ be the corresponding basis of $\mathfrak{M}_B = p^{-1}(\mathfrak{M})$ where $p(n_j) = m_j$ for $1 \leq j \leq r$ and $i(1) = n_{r+1}$. For $l_1, l_2 \in L$ suppose

$$[1\otimes l_1, 1\otimes l_2]_\mu = 1\otimes [l_1, l_2] + \sum_{j=1}^{r+1} n_j \otimes \psi_j(l_1, l_2).$$

Now we have $u(n_j) = u(m_j, 0)_q = (m_j, \psi(m_j))_q = (m_j, \psi(\bar{m}_j))$ for $1 \leq j \leq r$ and $u(n_{r+1}) = u(0,1)_q = (0,1)_q = n_{r+1}$. Then by (3.3.2)

$$\begin{aligned}
&[1\otimes l_1, 1\otimes l_2]_{u*\mu}\\
&= 1\otimes [l_1, l_2] + \sum_{j=1}^{r} u(n_j)\otimes \psi_j(l_1,l_2) + u(n_{r+1})\otimes \psi_{r+1}(l_1,l_2)\\
&= 1\otimes [l_1, l_2] + \sum_{j=1}^{r} (m_j, \psi(\bar{m}_j))_q \otimes \psi_j(l_1,l_2) + n_{r+1}\otimes \psi_{r+1}(l_1,l_2)\\
&= 1\otimes [l_1, l_2] + \sum_{j=1}^{r} (m_j, 0)_q \otimes \psi_j(l_1,l_2) + \sum_{j=1}^{r} (0, \psi(\bar{m}_j))_q \otimes \psi_j(l_1,l_2) + n_{r+1}\otimes \psi_{r+1}(l_1,l_2)\\
&= 1\otimes [l_1, l_2] + \sum_{j=1}^{r} n_j \otimes \psi_j(l_1,l_2) + n_{r+1}\otimes \psi_{r+1}(l_1,l_2) + \sum_{j=1}^{k} \psi(\bar{m}_j) n_{r+1}\otimes \psi_j(l_1,l_2)\\
&= [1\otimes l_1, 1\otimes l_2]_\mu + \sum_{j=1}^{k} \psi(\bar{m}_j) n_{r+1}\otimes \psi_j(l_1,l_2)\\
&= [1\otimes l_1, 1\otimes l_2]_\mu + \sum_{j=1}^{k} n_{r+1}\otimes \psi(\bar{m}_j)\psi_j(l_1,l_2)\\
&= [1\otimes l_1, 1\otimes l_2]_\mu + I\left(\sum_{j=1}^{k} 1\otimes \psi(\bar{m}_j)\psi_j(l_1,l_2)\right)\\
&= [1\otimes l_1, 1\otimes l_2]_\mu + I\circ\rho(l_1,l_2) \text{ where } \rho(l_1,l_2) = \textstyle\sum_{j=1}^{k} 1\otimes \psi(\bar{m}_j)\psi_j(l_1,l_2).
\end{aligned}$$

Since $\mu$ extends $\lambda$, we have

$$\begin{aligned}
[1\otimes l_1, 1\otimes l_2]_\lambda &= [1\otimes l_1, 1\otimes l_2]_{p*\mu}\\
&= 1\otimes [l_1, l_2] + \sum_{j=1}^{r} p(n_j)\otimes \psi_j(l_1,l_2) + p(n_{r+1})\otimes \psi_{r+1}(l_1,l_2)\\
&= 1\otimes [l_1, l_2] + \sum_{j=1}^{r} m_j\otimes \psi_j(l_1,l_2).
\end{aligned}$$

Thus $\alpha_{\lambda, m_j'} = \psi_j$ for $1 \leq j \leq r$, where $\{m_j'\}_{1\leq j\leq r}$ denotes the dual basis (see (3.4.1)).

Then the push-out $p_{2*}\lambda$ via $p_2 : A \longrightarrow A/\mathfrak{M}^2$ may be written as

$$\begin{aligned}[1\otimes l_1, 1\otimes l_2]_{p_{2*}\lambda} &= 1\otimes[l_1,l_2] + \sum_{j=1}^{r} p_2(m_j)\otimes\psi_j(l_1,l_2)\\ &= 1\otimes[l_1,l_2] + \sum_{j=1}^{r} \bar{m}_j\otimes\psi_j(l_1,l_2).\end{aligned}$$

So, $\alpha_{p_{2*}\lambda,\bar{m}'_j} = \psi_j$ and $a_{p_{2*}\lambda,\bar{m}'_j} = [\psi_j]$, the cohomology class of $\psi_j$ for $1 \le j \le k$. Thus $\rho(l_1,l_2) = \sum_{j=1}^{k} 1\otimes\psi(\bar{m}_j)\psi_j(l_1,l_2) = \sum_{j=1}^{k} 1\otimes\psi(\bar{m}_j)\alpha_{p_{2*}\lambda,\bar{m}'_j}(l_1,l_2)$. Now

$$d\lambda(\psi) = d\lambda(\sum_{j=1}^{k}\psi(\bar{m}_j)m'_j) = \sum_{j=1}^{k}\psi(\bar{m}_j)d\lambda(\bar{m}'_j) = \sum_{j=1}^{k}\psi(\bar{m}_j)a_{p_{2*}\lambda,\bar{m}'_j}.$$

This shows that $d\lambda(\psi)$ is represented by the cochain $\rho$. □

**Corollary 4.3.2.** *Suppose that for a deformation $\lambda$ of the Leibniz algebra $L$ with base $A$, the differential $d\lambda : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(L;L)$ is onto. Then the group of automorphisms $\mathcal{A}$ of the extension (4.2.1) operates transitively on the set $S$ of equivalence classes of deformations $\mu$ of $L$ with base $B$ such that $p_*\mu = \lambda$.*

**Remark 4.3.3.** *If $\mu$ and $\mu'$ are two extensions of $\lambda$ then the difference $[-,-]_{\mu'} - [-,-]_{\mu}$ determines a cocycle representing an element in $HL^2(L;L)$. Now if the differential map $d\lambda : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(L;L)$ is onto then we have an element $\psi \in (\mathfrak{M}/\mathfrak{M}^2)'$ corresponding to an element $u \in \mathcal{A}$ so that*

$$[-,-]_{u_*\mu} - [-,-]_{\mu} = d\lambda(\psi) = [-,-]_{\mu'} - [-,-]_{\mu},$$

*which gives $[-,-]_{u_*\mu} = [-,-]_{\mu'}$. Thus it follows that for a deformation $\lambda$ of the Leibniz algebra $L$ with base $(A,\mathfrak{M}) \in \mathcal{C}$, if the differential $d\lambda : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(L;L)$ is onto and if $\mu$ exists, then it is unique up to an isomorphism and an automorphism of the extension (4.2.1).*

We end this section with the following naturality property of the obstruction map.

**Proposition 4.3.4.** *Suppose $(A_1,\mathfrak{M}_1)$ and $(A_2,\mathfrak{M}_2)$ are in $\mathcal{C}$ with augmentations $\varepsilon_1$ and $\varepsilon_2$ respectively. Let $\phi : A_2 \longrightarrow A_1$ be an algebra homomorphism with $\phi(1) = 1$ and $\varepsilon_1\circ\phi = \varepsilon_2$. Suppose $\lambda_2$ is a deformation of a Leibniz algebra $L$ with base $A_2$ and $\lambda_1 = \phi_*\lambda_2$ is the push-out via $\phi$. Then the diagram in Figure 4.1 commutes.*

*Proof.* Let $[\psi_{A_1}] \in H^2_{Harr}(A_1;\mathbb{K})$ and $[\psi_{A_2}] = \phi^*([\psi_{A_1}]) \in H^2_{Harr}(A_2;\mathbb{K})$ correspond to the classes of 1- dimensional extensions of $A_1$ and $A_2$, represented by

$$0 \longrightarrow \mathbb{K} \xrightarrow{i_k} B_k \xrightarrow{p_k} A_k \longrightarrow 0, \quad k = 1,2.$$

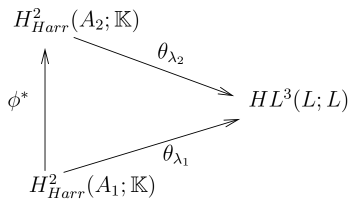


Figure 4.1:

Fix some sections $q_k : A_k \longrightarrow B_k$ of $p_k$ for $k = 1, 2$. Then, as in (4.2.2), we get $\mathbb{K}$-module isomorphisms $B_k \cong A_k \oplus \mathbb{K}$ . Let $(a, x)_{q_k}$ denote the inverse of $(a, x)$ under the above isomorphisms. The algebra structures on $B_k$ are determined as in (4.2.3).

Let $I_k = (i_k \otimes id)$, $P_k = (p_k \otimes id)$ and $E_k = (\hat{\varepsilon}_k \otimes id)$ , where $\hat{\varepsilon}_k = (\varepsilon_k \circ p_k)$ for $k = 1, 2$. Suppose $\mathfrak{M}_k$ is the unique maximal ideal in $A_k$. Then $\mathfrak{N}_k = p_k^{-1}(\mathfrak{M}_k)$ is the unique maximal ideal of $B_k$ (cf. Proposition 2.3.5). Let $\{m_{ki}\}_{1\leq i\leq r_k}$ be a basis of $\mathfrak{M}_k$ and $\{n_{ki}\}_{1\leq i\leq r_k+1}$ a basis of $\mathfrak{N}_k$ for $k = 1, 2$ (as obtained in Section 4.2).

Thus $n_{ki} = (m_{ki}, 0)_{q_k}$ for $1 \leq i \leq r_k$ and $n_{k(r_k+1)} = (0, 1)_{q_k}$. Let $\{\xi_{ki}\}_{1\leq i\leq r_k}$ be the dual basis of $\{m_{ki}\}$. As in (3.5.2), the Leibniz bracket on $A_2 \otimes L$ may be written as

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda_2} = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r_2} m_{2i} \otimes \psi_i^2(l_1, l_2) \quad \text{for } l_1, l_2 \in L,$$

where $\psi_i^2 = \alpha_{\lambda_2,\xi_{2i}}$. Let $\phi(m_{2i}) = \sum_{j=1}^{r_1} c_{i,j} m_{1j}$ where $c_{i,j} \in \mathbb{K}$ for $1 \leq i \leq r_2$. Then the push-out $\lambda_1 = \phi_*\lambda_2$ on $A_1 \otimes L$ may be written as (cf. (3.3.2))

$$\begin{aligned}[1 \otimes l_1, 1 \otimes l_2]_{\lambda_1} &= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r_2} \left( \sum_{j=1}^{r_1} c_{i,j} m_{1j} \right) \otimes \psi_i^2(l_1, l_2) \\ &= 1 \otimes [l_1, l_2] + \sum_{j=1}^{r_1} m_{1j} \otimes \psi_j^1(l_1, l_2) \quad \text{for } l_1, l_2 \in L,\end{aligned}$$

where $\psi_j^1 \in CL^2(L; L)$ is defined by $\psi_j^1(l_1, l_2) = \sum_{i=1}^{r_2} c_{i,j}\psi_i^2(l_1, l_2)$ for $l_1, l_2 \in L$.

For any 2-cochain $\chi \in CL^2(L; L)$, let $\{-, -\}_k : (B_k \otimes L)^{\otimes 2} \longrightarrow B_k \otimes L$ be the $B_k$-bilinear operation on $B_k \otimes L$ lifting $\lambda_k$, defined by

$$\{1 \otimes l_1, 1 \otimes l_2\}_k = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r_k} n_{ki} \otimes \psi_i^k(l_1, l_2) + n_{k(r_k+1)}\chi(l_1, l_2)$$

for $k = 1, 2$ and $l_1, l_2 \in L$. We know that $\{-, -\}_k$ satisfies properties $(i)$ and $(ii)$ of (4.2.4) (Lemma 4.2.1).

Define $\psi : B_2 \cong (A_2 \oplus \mathbb{K}) \longrightarrow B_1 \cong (A_1 \oplus \mathbb{K})$ by $\psi((a, x)_{q_2}) = (\phi(a), x)_{q_1}$ for

$(a,x)_{q_2} \in B_2$. It is clear that $\psi$ is a $\mathbb{K}$-algebra homomorphism. Because, for $1 \leq j,k \leq r_2$

$$\begin{aligned}
\psi(n_{2j}n_{2k}) &= \psi((m_{2j},0)_{q_2}(m_{2k},0)_{q_2}) \\
&= \psi(m_{2j}m_{2k}, \psi_{A_2}(m_{2j},m_{2k}))_{q_2} \\
&= (\phi(m_{2j}m_{2k}), \psi_{A_1}(\phi(m_{2j}),\phi(m_{2k})))_{q_1} \quad (\text{since } [\psi_{A_2}] = \phi^*[\psi_{A_1}]) \\
&= (\phi(m_{2j})\phi(m_{2k}), \psi_{A_1}(\phi(m_{2j}),\phi(m_{2k})))_{q_1} \\
&= (\phi(m_{2j}),0)_{q_1}(\phi(m_{2k}),0)_{q_1} \\
&= \psi(n_{2j})\psi(n_{2k}).
\end{aligned}$$

Moreover note that $\psi(n_{2j}n_{2(r_2+1)}) = \psi(n_{2j})\psi(n_{2(r_2+1)})$ for any $j$, $1 \leq j \leq r_k$ as $n_{kj}n_{k(r_k+1)} = 0$, $k = 1,2$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \xrightarrow{i_1} & B_1 & \xrightarrow{p_1} & A_1 & \longrightarrow & 0 \\
 & & \| & & \psi\downarrow & & \phi\downarrow & & \\
0 & \longrightarrow & \mathbb{K} & \xrightarrow{i_2} & B_2 & \xrightarrow{p_2} & A_2 & \longrightarrow & 0
\end{array}$$

We claim that $(\psi \otimes id)(\{1\otimes l_1, 1\otimes l_2\}_2) = \{\psi\otimes id\ (1\otimes l_1), \psi\otimes id\ (1\otimes l_2)\}_1$ for $l_1,l_2 \in L$. Now

$$\begin{aligned}
&(\psi\otimes id)(\{1\otimes l_1, 1\otimes l_2\}_2) \\
&= 1\otimes[l_1,l_2] + \sum_{i=1}^{r_2} \psi(n_{2i})\otimes\psi_i^2(l_1,l_2) + \psi(n_{2(r_2+1)})\otimes\chi(l_1,l_2) \\
&= 1\otimes[l_1,l_2] + \sum_{i=1}^{r_2} (\phi(m_{2i}),0)_{q_1}\otimes\psi_i^2(l_1,l_2) + \psi(n_{2(r_2+1)})\otimes\chi(l_1,l_2) \\
&= 1\otimes[l_1,l_2] + \sum_{i=1}^{r_2} \left(\sum_{j=1}^{r_1} c_{ij}m_{1j}, 0\right)_{q_1} \otimes\psi_i^2(l_1,l_2) + n_{1(r_1+1)}\otimes\chi(l_1,l_2) \\
&\quad (\phi(m_{2i}) = \sum_{j=1}^{r_1} c_{ij}m_{1j} \text{ and } \psi(n_{2(r_2+1)}) = \psi((0,1)_{q_2}) = (\phi(0),1)_{q_1} = n_{1(r_1+1)}) \\
&= 1\otimes[l_1,l_2] + \sum_{i=1}^{r_2} \left(\sum_{j=1}^{r_1} c_{ij}n_{1j}\right) \otimes\psi_i^2(l_1,l_2) + n_{1(r_1+1)}\otimes\chi(l_1,l_2) \\
&= 1\otimes[l_1,l_2] + \sum_{j=1}^{r_1} n_{1j}\otimes\psi_j^1(l_1,l_2) + n_{1(r_1+1)}\otimes\chi(l_1,l_2) \\
&= \{1\otimes l_1, 1\otimes l_2\}_1 \\
&= \{\psi\otimes id\ (1\otimes l_1), \psi\otimes id\ (1\otimes l_2)\}_1, \text{ which proves our claim.}
\end{aligned}$$

Let $\phi_k$ be defined by $\{-,-\}_k$ as in (4.2.7) and $\bar{\phi}_k$ the corresponding cocycle as in (4.2.9). Then

$$\begin{aligned}
&(\psi\otimes id)\circ\phi_2(1\otimes l_1, 1\otimes l_2, 1\otimes l_3)\\
&=(\psi\otimes id)(\{1\otimes l_1,\{1\otimes l_2,1\otimes l_3\}_2\}_2-\{\{1\otimes l_1,1\otimes l_2\}_2,1\otimes l_3\}_2\\
&\qquad\qquad+\{\{1\otimes l_1,1\otimes l_3\}_2,1\otimes l_2\}_2)\\
&=\{1\otimes l_1,\{1\otimes l_2,1\otimes l_3\}_1\}_1-\{\{1\otimes l_1,1\otimes l_2\}_1,1\otimes l_3\}_1\\
&\qquad\qquad+\{\{1\otimes l_1,1\otimes l_3\}_1,1\otimes l_2\}_1\\
&=\phi_1(1\otimes l_1,1\otimes l_2,1\otimes l_3)\\
&=n_{1(r_1+1)}\otimes\bar{\phi}_1(l_1,l_2,l_3),
\end{aligned}\tag{4.3.8}$$

$$\begin{aligned}
\text{and } (\psi\otimes id)\phi_2(1\otimes l_1,1\otimes l_2,1\otimes l_3)&=(\psi\otimes id)(n_{2(r_2+1)}\otimes\bar{\phi}_2(l_1,l_2,l_3))\\
&=n_{1(r_1+1)}\otimes\bar{\phi}_2(l_1,l_2,l_3).
\end{aligned}\tag{4.3.9}$$

From (4.3.8) and (4.3.9) we get, $\bar{\phi}_1=\bar{\phi}_2$.
Therefore, $\theta_{\lambda_1}([\psi_{A_1}])=[\bar{\phi}_1]=[\bar{\phi}_2]=\theta_{\lambda_2}([\psi_{A_2}])=\theta_{\lambda_2}\circ\phi^*([\psi_{A_1}])$.
Hence $\theta_{\lambda_1}=\ \theta_{\lambda_2}\circ\phi^*$. □

## 4.4 Extension of a deformation of Leibniz algebra homomorphisms

This section is analogous to Section 4.2. We extend the results of Section 4.2 to the case of a Leibniz algebra homomorphism.

Recall that a deformation $\mathfrak{D}=(\lambda,\mu;f_{\lambda\mu})$ of a Leibniz algebra homomorphism $f: L\longrightarrow M$ with base $A$ consists of a deformation $\lambda$ of $L$, a deformation $\mu$ of $M$ with base $A$ and a Leibniz algebra homomorphism $f_{\lambda\mu}:(A\otimes L,\lambda)\longrightarrow(A\otimes M,\mu)$.

Let $\mathfrak{D}=(\lambda,\mu;f_{\lambda\mu})$ be a deformation of a Leibniz algebra homomorphism $f:L\to M$ with base $(A,\mathfrak{M})\in\mathcal{C}$. Let $\varepsilon: A\longrightarrow\mathbb{K}$ be the augmentation of $A$. We fix a cohomology class $[\psi]\in H^2_{Harr}(A;\mathbb{K})$. Let

$$0\longrightarrow\mathbb{K}\overset{i}{\longrightarrow}B\overset{p}{\longrightarrow}A\longrightarrow 0$$

represents the isomorphism class of 1-dimensional extension of $A$ corresponding to $[\psi]$. As before, we consider the problem of extending $\mathfrak{D}$ from the base $A$ to the base $B$. We follow the notations of Section 4.2.

As in Chapter 3, (3.4.1) and (3.5.1), let $\psi_j^\lambda=\alpha_{\lambda,\xi_j}\in CL^2(L;L)$, $\psi_j^\mu=\alpha_{\mu,\xi_j}\in CL^2(M;M)$ and $f_j=f_{\lambda\mu,\xi_j}\in CL^1(L;M)$ for $1\leq j\leq r$. Then by (3.4.2) the brackets

$[-,-]_\lambda$ and $[-,-]_\mu$ can be written as

$$[1\otimes l_1, 1\otimes l_2]_\lambda = 1\otimes[l_1,l_2] + \sum_{j=1}^{r} m_j \otimes \psi_j^\lambda(l_1,l_2) \quad \text{for } l_1,l_2\in L$$

$$\text{and } [1\otimes x_1, 1\otimes x_2]_\mu = 1\otimes[x_1,x_2] + \sum_{j=1}^{r} m_j \otimes \psi_j^\mu(x_1,x_2) \quad \text{for } x_1,x_2\in M.$$

Also by (3.5.2),

$$f_{\lambda\mu}(1\otimes l) = 1\otimes f(l) + \sum_{j=1}^{r} m_i\otimes f_j(l).$$

Using an arbitrary element $(\psi_L,\psi_M)\in CL^2(L;L)\times CL^2(M;M)$ we define $B$-bilinear operations,

$$\{-,-\}_L : (B\otimes L)^{\otimes 2} \longrightarrow B\otimes L \text{ and } \{-,-\}_M : (B\otimes M)^{\otimes 2} \longrightarrow B\otimes M$$

as follows:

$$\{b_1\otimes l_1, b_2\otimes l_2\}_L = b_1b_2\otimes[l_1,l_2] + \sum_{j=1}^{r} b_1b_2n_j\otimes\alpha_{\lambda,\xi_j}(l_1,l_2) + b_1b_2n_{r+1}\otimes\psi_L(l_1,l_2)$$

$$\text{and } \{b_1\otimes x_1, b_2\otimes x_2\}_M = b_1b_2\otimes[x_1,x_2] + \sum_{j=1}^{r} b_1b_2n_j\otimes\alpha_{\mu,\xi_j}(x_1,x_2) + b_1b_2n_{r+1}\otimes\psi_M(x_1,x_2).$$

Moreover, we have a $B$-linear map $\tilde{f} : B\otimes L \longrightarrow B\otimes M$ defined by

$$\tilde{f}(b\otimes l) = b\otimes f(l) + \sum_{j=1}^{r} bn_j\otimes f_j(l).$$

Let $I, P, E$ be the linear maps as defined in Section 4.2, and $I_1, P_1$ and $E_1$ denote the corresponding maps obtained by replacing $L$ by $M$.

We claim that the triple $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ satisfy the following identities.

$$\begin{aligned}
&(i)\ P\{l_1,l_2\}_L = [P(l_1),P(l_2)]_\lambda \text{ and } P_1\{x_1,x_2\}_M = [P_1(x_1),P_1(x_2)]_\mu \\
&\qquad \text{for } l_1,l_2\in B\otimes L, \text{ and } x_1,x_2\in B\otimes M. \\
&(ii)\ \{I(l),l_1\}_L = I[l,E(l_1)] \text{ for } l\in L,\ l_1\in B\otimes L \text{ and} \\
&\qquad \{I_1(x),x_1\}_M = I_1[x,E_1(x_1)] \text{ for } x\in M,\ x_1\in B\otimes M. \\
&(iii)\ (\hat{\varepsilon}\otimes id)\circ\tilde{f} = f\circ(\hat{\varepsilon}\otimes id). \\
&(iv)\ f_{\lambda\mu}\circ P = P_1\circ\tilde{f}.
\end{aligned} \tag{4.4.1}$$

In view of (4.2.5) and (4.2.6) we need only to verify the last two identities. Let $b \otimes l \in B \otimes L$.

**(iii)** Then,

$$\begin{aligned}
(\hat{\varepsilon} \otimes id) \circ \tilde{f}(b \otimes l) &= (\hat{\varepsilon} \otimes id) \left( b \otimes f(l) + \sum_{j=1}^{r} bn_j \otimes f_j(l) \right) \\
&= \hat{\varepsilon}(b) \otimes f(l) + \sum_{j=1}^{r} \hat{\varepsilon}(bn_j) \otimes f_j(l) \\
&= \hat{\varepsilon}(b) f(l) + \sum_{j=1}^{r} \hat{\varepsilon}(b)\hat{\varepsilon}(n_j) \otimes f_j(l) \\
&= f(\hat{\varepsilon}(b) l) \quad (\text{since } n_j \in ker(\hat{\varepsilon}),\ \hat{\varepsilon}(n_j) = 0) \\
&= f \circ (\hat{\varepsilon} \otimes id)(b \otimes l).
\end{aligned} \tag{4.4.2}$$

**(iv)** We have,

$$\begin{aligned}
f_{\lambda\mu} \circ P(b \otimes l) &= f_{\lambda\mu}(p(b) \otimes l) \\
&= p(b) f_{\lambda\mu}(1 \otimes l) \\
&= p(b) \left( 1 \otimes f(l) + \sum_{j=1}^{r} m_j \otimes f_j(l) \right) \\
&= p(b) \otimes f(l) + \sum_{j=1}^{r} p(b) m_j \otimes f_j(l) \\
&= (p \otimes id)(b \otimes f(l)) + \sum_{j=1}^{r} p(b)p(n_j) \otimes f_j(l) \\
&= P_1(b \otimes f(l)) + \sum_{j=1}^{r} P_1(bn_j \otimes f_j(l)) \\
&= P_1 \circ \tilde{f}(b \otimes l).
\end{aligned} \tag{4.4.3}$$

Therefore the triple $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ satisfies the conditions in (4.4.1) as claimed.

If in addition the brackets $\{-,-\}_L$ and $\{,-\}_M$ satisfy the Leibniz relation and the map $\tilde{f} : B \otimes L \longrightarrow B \otimes M$ preserves the respective brackets, then the triple $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ is a deformation of $f$ with base $B$ extending $\mathfrak{D}$. Just like the case of Leibniz algebra, this extension process leads to an obstruction map as described below. Consider the $\mathbb{K}$-linear maps

$$\phi_L : (B \otimes L)^{\otimes 3} \longrightarrow B \otimes L, \quad \phi_M : (B \otimes M)^{\otimes 3} \longrightarrow B \otimes M,$$
$$\text{and } \phi_{f_{\lambda\mu}} : (B \otimes L)^{\otimes 2} \longrightarrow B \otimes M \quad \text{given by}$$

$$\begin{aligned}\phi_L(l_1,l_2,l_3) &= \{l_1,\{l_2,l_3\}_L\}_L - \{\{l_1,l_2\}_L,l_3\}_L + \{\{l_1,l_3\}_L,l_2\}_L\\ \phi_M(x_1,x_2,x_3) &= \{x_1,\{x_2,x_3\}_M\}_M - \{\{x_1,x_2\}_M,x_3\}_M + \{\{x_1,x_3\}_M,x_2\}_M \quad (4.4.4)\\ \text{and } \phi_{f_{\lambda\mu}}(l_1,l_2) &= \tilde{f}\{l_1,l_2\}_L - \{\tilde{f}(l_1),\tilde{f}(l_2)\}_M,\end{aligned}$$

where $l_1,l_2,l_3 \in B\otimes L$; $x_1,x_2,x_3 \in B\otimes M$. Here $\phi_L$ and $\phi_M$ are the map $\phi$ for $L$ and $M$ respectively, as defined in (4.2.7) in Section 4.2. As observed before, $\phi_L = 0$ if and only if $\{-,-\}_L$ is a Leibniz bracket on $B\otimes L$,and similarly $\phi_M = 0$ if and only if $\{-,-\}_M$ is a Leibniz bracket on $B\otimes M$. Moreover $\phi_{f_{\lambda\mu}} = 0$ if and only if $\tilde{f}$ preserves the $B$-bilinear operations $\{-,-\}_L$ and $\{-,-\}_M$.

By (4.2.8) we get, $P\circ\phi_L(l_1,l_2,l_3) = 0$ and $P_1\circ\phi_M(x_1,x_2,x_3) = 0$ for $l_1,l_2,l_3 \in B\otimes L$ and $x_1,x_2,x_3 \in B\otimes M$ .

By the properties $(i)$ and $(iv)$ in (4.4.1) we also have for $l_1,l_2 \in B\otimes L$,

$$\begin{aligned}P_1\circ\phi_{f_{\lambda\mu}}(l_1,l_2) &= P_1(\tilde{f}\{l_1,l_2\}_L - \{\tilde{f}(l_1),\tilde{f}(l_2)\}_M)\\ &= P_1\circ\tilde{f}\{l_1,l_2\}_L - P_1\circ\{\tilde{f}(l_1),\tilde{f}(l_2)\}_M\\ &= f_{\lambda\mu}\circ P\{l_1,l_2\}_L - [P_1\circ\tilde{f}(l_1),P_1\circ\tilde{f}(l_2)]_\mu\\ &= f_{\lambda\mu}[P(l_1),P(l_2)]_\lambda - [f_{\lambda\mu}\circ P(l_1), f_{\lambda\mu}\circ P(l_2)]_\mu\\ &= 0 \;\; (\text{ as } f_{\lambda\mu} \text{ is a Leibniz algebra homomorphism}).\end{aligned}$$

Therefore $\phi_L$ and $\phi_M$ take values in $ker(P)$ and $ker(P_1)$ respectively, and $\phi_{f_{\lambda\mu}}$ takes values in $ker(P_1)$. As observed in Section 4.2, we have $\phi_L(l_1,l_2,l_3) = 0$ whenever one of the arguments is in $ker(E)$ and similarly $\phi_M(x_1,x_2,x_3) = 0$ whenever one of the arguments is in $ker(E_1)$. This is because, $ker(P) = im(i)\otimes L = \mathbb{K}n_{r+1}\otimes L$, $ker(P_1) = im(i)\otimes M = \mathbb{K}n_{r+1}\otimes M$, and $n_j n_{r+1} = 0$ for $1\le j\le r+1$. Moreover $\phi_{f_{\lambda\mu}} = 0$ whenever one of the arguments is in $ker(E)$. For suppose $l_1 = (b\otimes l) \in ker(E) \subseteq B\otimes L$. Since $ker(E) = ker(\hat{\varepsilon})\otimes L = p^{-1}(ker(\varepsilon))\otimes L = \mathfrak{M}_B\otimes L$, we can write $l_1 = \sum_{j=1}^{r+1} n_j\otimes l'_j$ with $l'_j \in L$; $1\le j\le r+1$. Then for $l_2 \in B\otimes L$, we get

$$\phi_{f_{\lambda\mu}}(l_1,l_2) = \phi_{f_{\lambda\mu}}\left(\sum_{j=1}^{r+1} n_j\otimes l'_j, l_2\right) = \sum_{j=1}^{r+1} n_j\phi_{f_{\lambda\mu}}(l'_j,l_2) = 0.$$

Note that $\phi_{f_{\lambda\mu}}(l'_j,l_2) \in ker(P_1) = i(\mathbb{K})\otimes M$ and for any element $k\in\mathbb{K}$ and $x\in M$,

$$\begin{aligned}&n_j\; i(k)\otimes x = i(p(n_j)k)\otimes x = i(m_jk)\otimes x = i(\varepsilon(m_j)k)\otimes x = 0 \;\text{ for } 1\le j\le r\\ &\text{and } n_{r+1}i(k)\otimes x = kn_{r+1}^2\otimes x = 0 \;\; (m_j\in\mathfrak{M}\subset A \text{ and } m_jk = \varepsilon(m_j)k).\end{aligned}$$

A similar argument shows that $\phi_{f_{\lambda\mu}} = 0$ whenever $l_2 \in ker(E)$.

Thus we have the following induced linear maps

$$\begin{aligned} &\tilde{\phi}_L : \left(\frac{B \otimes L}{ker(E)}\right)^{\otimes 3} \longrightarrow ker(P), \quad \tilde{\phi}_M : \left(\frac{B \otimes M}{ker(E_1)}\right)^{\otimes 3} \longrightarrow ker(P_1), \\ \text{and } &\tilde{\phi}_{f_{\lambda\mu}} : \left(\frac{B \otimes L}{ker(E)}\right)^{\otimes 2} \longrightarrow ker(P_1), \end{aligned} \tag{4.4.5}$$

determined by the values of $\phi_L, \phi_M$ and $\phi_{f_{\lambda\mu}}$ on the coset representatives respectively. Let $\alpha$ denote the isomorphism $L \cong \frac{B\otimes L}{ker(E)}$ as defined in Section 4.2. Similarly we denote by $\alpha_1$ the isomorphism $M \cong \frac{B\otimes M}{ker(E_1)}$ induced by the linear maps $E$ and $E_1$ respectively. Also recall that we denoted the isomorphism $ker(P) \cong L$ by $\beta$; $\beta(kn_{r+1} \otimes l) = kl$, for $k \in \mathbb{K}$ and $l \in L$. Similarly, let $\beta_1$ denote the isomorphism $ker(P_1) \cong M$.

We use these isomorphisms and the linear maps $\tilde{\phi}_L, \tilde{\phi}_M$ to get cochains $\bar{\phi}_L \in CL^3(L; L)$, $\bar{\phi}_M \in CL^3(M; M)$ as in (4.2.9). Moreover, the linear map $\tilde{\phi}_{f_{\lambda\mu}}$ defines a cochain $\bar{\phi}_{f_{\lambda\mu}} \in CL^2(L; M)$ by $\beta_1 \circ \bar{\phi}_{f_{\lambda\mu}} \circ \alpha^{\otimes 2}$.

Thus for $l_1, l_2, l_3 \in L$ and $x_1, x_2, x_3 \in M$, we have

$$\begin{aligned} n_{r+1} \otimes \bar{\phi}_L(l_1, l_2, l_3) &= \phi_L(1 \otimes l_1, 1 \otimes l_2, 1 \otimes l_3) \\ n_{r+1} \otimes \bar{\phi}_M(x_1, x_2, x_3) &= \phi_M(1 \otimes x_1, 1 \otimes x_2, 1 \otimes x_3), \\ \text{and } n_{r+1} \otimes \bar{\phi}_{f_{\lambda\mu}}(l_1, l_2) &= \phi_{f_{\lambda\mu}}(1 \otimes l_1, 1 \otimes l_2). \end{aligned} \tag{4.4.6}$$

The resulting 3-cochain $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) \in CL^3(f; f)$ is called the obstruction cochain for the triple $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ in extending the deformation $\mathfrak{D}$ of $f$ with base $A$ to a deformation with base $B$.

Next we show that the obstruction cochain $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) \in CL^3(f; f)$ is a 3-cocycle. For this we shall need the following observation.

**Lemma 4.4.1.** *For the triple $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ satisfying (4.4.1) we have*

(*i*) $\tilde{f}(1 \otimes l) - 1 \otimes f(l) \in ker(E_1)$

(*ii*) $\{1 \otimes l, 1 \otimes l'\}_L - 1 \otimes [l, l'] \in ker(E)$

(*iii*) $\tilde{f}\{1 \otimes l, 1 \otimes l'\}_L - \{\tilde{f}(1 \otimes l), \tilde{f}(1 \otimes l')\}_M \in ker(P_1)$,

*for $l, l' \in L$.*

*Proof.* For $l \in L$, we have

$$\begin{aligned} E_1 \circ \tilde{f}(1 \otimes l) =& f \circ E(1 \otimes l) \quad (\text{ by } (iii) \text{ of } (4.4.1)\ ) \\ =& f(\hat{\varepsilon}(1) \otimes l) \\ =& f(l) \\ =& E_1(1 \otimes f(l)). \quad \text{This proves } (i). \end{aligned}$$

For $l, l' \in L$,

$E(\{1 \otimes l, 1 \otimes l'\}_L)$
$= (\varepsilon \otimes id) \circ P\{1 \otimes l, 1 \otimes l'\}_L$
$= (\varepsilon \otimes id)[P(1 \otimes l), P(1 \otimes l')]_\lambda$ (by $(i)$ of (4.4.1))
$= (\varepsilon \otimes id)[1 \otimes l, 1 \otimes l']_\lambda$
$= 1 \otimes [l, l']$ ($E = (\varepsilon \otimes id) : A \otimes L \longrightarrow \mathbb{K} \otimes L \cong L$ is a Leibniz algebra homomorphism)
$= E(1 \otimes [l, l'])$, proving $(ii)$.

Finally for $l, l' \in L$,

$$\begin{aligned} & P_1 \circ \tilde{f}\{1 \otimes l, 1 \otimes l'\}_L \\ &= f_{\lambda\mu} \circ P\{1 \otimes l, 1 \otimes l'\}_L \text{ ( by } (iv) \text{ in (4.4.1) )} \\ &= f_{\lambda\mu}([P(1 \otimes l), P(1 \otimes l')]_\lambda) \text{ ( by } (i) \text{ in (4.4.1) )} \\ &= f_{\lambda\mu}([1 \otimes l, 1 \otimes l']_\lambda) \\ &= [f_{\lambda\mu}(1 \otimes l), f_{\lambda\mu}(1 \otimes l')]_\mu \\ &= [f_{\lambda\mu} \circ P(1 \otimes l), f_{\lambda\mu} \circ P(1 \otimes l')]_\mu \\ &= [P_1 \circ \tilde{f}(1 \otimes l), P_1 \circ \tilde{f}(1 \otimes l')]_\mu \\ &= P_1\{\tilde{f}(1 \otimes l), \tilde{f}(1 \otimes l')\}_M. \end{aligned}$$

This proves $(iii)$. □

**Proposition 4.4.2.** *The obstruction cochain* $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ *is a* 3*-cocycle in* $CL^3(f; f)$.

*Proof.* By the definition of the coboundary $d$ in (1.4.1) we have

$$d(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) = (\delta\bar{\phi}_L, \delta\bar{\phi}_M; f\bar{\phi}_L - \bar{\phi}_M f - \delta\bar{\phi}_{f_{\lambda\mu}}).$$

Thus it is enough to show that $\delta\bar{\phi}_L = 0 = \delta\bar{\phi}_M$, and $\delta\bar{\phi}_{f_{\lambda\mu}} = f\bar{\phi}_L - \bar{\phi}_M f$. The result will follow if we show that

$$\beta^{-1} \circ \delta\bar{\phi}_L = 0 = {\beta_1}^{-1} \circ \delta\bar{\phi}_M \text{ and } {\beta_1}^{-1} \circ \delta\bar{\phi}_{f_{\lambda\mu}} = {\beta_1}^{-1} \circ (f\bar{\phi}_L - \bar{\phi}_M f).$$

First two equalities follows from Proposition 4.2.2. Thus we only need to verify the last equality. For $l_1, l_2, l_3 \in L$,

$$\begin{aligned} \beta_1^{-1} \circ \delta\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2, l_3) = {} & \beta_1^{-1}[f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)] + \beta_1^{-1}[\bar{\phi}_{f_{\lambda\mu}}(l_1, l_3), f(l_2)] \\ & - \beta_1^{-1}[\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2), f(l_3)] - \beta_1^{-1}(\bar{\phi}_{f_{\lambda\mu}}([l_1, l_2], l_3)) \\ & + \beta_1^{-1}(\bar{\phi}_{f_{\lambda\mu}}([l_1, l_3], l_2)) + \beta_1^{-1}(\bar{\phi}_{f_{\lambda\mu}}(l_1, [l_2, l_3])). \end{aligned} \tag{4.4.7}$$

Let us compute the terms on the right-hand side of (4.4.7). The first term is

$$
\begin{aligned}
&\beta_1^{-1}[f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)] \\
&= n_{r+1} \otimes [f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)] \\
&= I_1[f(l_1), \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)] \quad (i(1) = n_{r+1}) \\
&= \{I_1 f(l_1), 1 \otimes \bar{\phi}_{f_{\lambda\mu}}(l_2, l_3)\}_M \quad (\text{by } (ii) \text{ of } (4.4.1)) \\
&= \{1 \otimes f(l_1), \phi_{f_{\lambda\mu}}(1 \otimes l_2, 1 \otimes l_3)\}_M \\
&\qquad (\text{using B-bilinearity of } \{-,-\}_M \text{ and by } (4.4.6)) \\
&= \{1 \otimes f(l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L - \{\tilde{f}(1 \otimes l_2), \tilde{f}(1 \otimes l_3)\}_M\}_M \quad (\text{by } (4.4.4)) \\
&= \{\tilde{f}(1 \otimes l_1) - X_1, \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\}_M \\
&\quad - \{1 \otimes f(l_1), \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\}_M \\
&\quad (\text{ by Lemma 4.4.1 } (i) \text{ where } X_1, X_2, X_3 \in ker(E_1)\ ) \\
&= \{\tilde{f}(1 \otimes l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\}_M - \{X_1, \{\tilde{f}(1 \otimes l_2), \tilde{f}(1 \otimes l_3)\}_M + Z_{2,3}\}_M \\
&\quad - \{1 \otimes f(l_1), \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\}_M \\
&\quad (\text{ by Lemma 4.4.1 } (iii) \text{ where } Z_{2,3} \in ker(P_1)\ ) \\
&= \{\tilde{f}(1 \otimes l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\}_M - \{X_1, \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\}_M \\
&\quad - \{X_1, Z_{2,3}\}_M - \{1 \otimes f(l_1), \{1 \otimes f(l_2) + X_2, 1 \otimes f(l_3) + X_3\}_M\}_M \\
&\quad (\text{ by Lemma 4.4.1 } (i) \text{ where } X_2, X_3 \in ker(E_1)\ ) \\
&= \{\tilde{f}(1 \otimes l_1), \tilde{f}\{1 \otimes l_2, 1 \otimes l_3\}_L\}_M - \{X_1, \{1 \otimes f(l_2), 1 \otimes f(l_3)\}_M\}_M \\
&\quad - \{X_1, \{1 \otimes f(l_2), X_3\}_M\}_M - \{X_1, \{X_2, 1 \otimes f(l_3)\}_M\}_M - \{X_1, \{X_2, X_3\}_M\}_M \\
&\quad - \{X_1, Z_{2,3}\}_M - \{1 \otimes f(l_1), \{1 \otimes f(l_2), 1 \otimes f(l_3)\}_M\}_M \\
&\quad - \{1 \otimes f(l_1), \{X_2, 1 \otimes f(l_3)\}_M\}_M - \{1 \otimes f(l_1), \{1 \otimes f(l_2), X_3\}_M\}_M \\
&\quad - \{1 \otimes f(l_1), \{X_2, X_3\}_M\}_M.
\end{aligned} \tag{4.4.8}
$$

Similarly,

$$
\begin{aligned}
&\beta_1^{-1}[\bar{\phi}_{f_{\lambda\mu}}(l_1, l_3), f(l_2)] \\
&= \{\tilde{f}\{1 \otimes l_1, 1 \otimes l_3\}_L, \tilde{f}(1 \otimes l_2)\}_M - \{\{1 \otimes f(l_1), 1 \otimes f(l_3)\}_M, X_2\}_M \\
&\quad - \{\{1 \otimes f(l_1), X_3\}_M, X_2\}_M - \{\{X_1, 1 \otimes f(l_3)\}_M, X_2\}_M - \{\{X_1, X_3\}_M, X_2\}_M \\
&\quad - \{Z_{1,3}, X_2\}_M - \{\{1 \otimes f(l_1), 1 \otimes f(l_3)\}_M, 1 \otimes f(l_2)\}_M \\
&\quad - \{\{X_1, 1 \otimes f(l_3)\}_M, 1 \otimes f(l_2)\}_M - \{\{1 \otimes f(l_1), X_3\}_M, 1 \otimes f(l_2)\}_M \\
&\quad - \{\{X_1, X_3\}_M, 1 \otimes f(l_2)\}_M
\end{aligned} \tag{4.4.9}
$$

and

$$\begin{aligned}
&\beta_1^{-1}[\bar{\phi}_{f_{\lambda\mu}}(l_1,l_2),f(l_3)]\\
&=\{\tilde{f}\{1\otimes l_1,1\otimes l_2\}_L,\tilde{f}(1\otimes l_3)\}_M-\{\{1\otimes f(l_1),1\otimes f(l_2)\}_M,X_3\}_M\\
&\quad-\{\{1\otimes f(l_1),X_2\}_M,X_3\}_M-\{\{X_1,1\otimes f(l_2)\}_M,X_3\}_M-\{\{X_1,X_2\}_M,X_3\}_M\\
&\quad-\{Z_{1,2},X_3\}_M-\{\{1\otimes f(l_1),1\otimes f(l_2)\}_M,1\otimes f(l_3)\}_M\\
&\quad-\{\{X_1,1\otimes f(l_2)\}_M,1\otimes f(l_3)\}_M-\{\{1\otimes f(l_1),X_2\}_M,1\otimes f(l_3)\}_M\\
&\quad-\{\{X_1,X_2\}_M,1\otimes f(l_3)\}_M.
\end{aligned}\tag{4.4.10}$$

Also

$$\begin{aligned}
&\beta_1^{-1}(\bar{\phi}_{f_{\lambda\mu}}([l_1,l_2],l_3))\\
&=n_{r+1}\otimes\bar{\phi}_{f_{\lambda\mu}}([l_1,l_2],l_3)\\
&=\phi_{\lambda\mu}(1\otimes[l_1,l_2],1\otimes l_3)\ (\text{ by the 3rd expression in (4.4.6) })\\
&=\tilde{f}\{1\otimes[l_1,l_2],1\otimes l_3\}_L-\{\tilde{f}(1\otimes[l_1,l_2]),\tilde{f}(1\otimes l_3)\}_M\\
&\qquad\qquad(\text{ by the 3rd expression in (4.4.4) })\\
&=\tilde{f}\{\{1\otimes l_1,1\otimes l_2\}_L-Y_{1,2},1\otimes l_3\}_L-\{\tilde{f}(\{1\otimes l_1,1\otimes l_2\}_L-Y_{1,2}),\tilde{f}(1\otimes l_3)\}_M\\
&\qquad\qquad(\text{ by Lemma 4.4.1 } (ii) \text{ where } Y_{1,2}\in ker(E)\ )\\
&=\tilde{f}\{\{1\otimes l_1,1\otimes l_2\}_L,1\otimes l_3\}_L-\tilde{f}\{Y_{1,2},1\otimes l_3\}_L-\{\tilde{f}\{1\otimes l_1,1\otimes l_2\}_L,\tilde{f}(1\otimes l_3)\}_M\\
&\qquad\qquad+\{\tilde{f}(Y_{1,2}),\tilde{f}(1\otimes l_3)\}_M.
\end{aligned}\tag{4.4.11}$$

Similarly,

$$\begin{aligned}
&\beta_1^{-1}(\bar{\phi}_{f_{\lambda\mu}}([l_1,l_3],l_2))\\
=&\tilde{f}\{\{1\otimes l_1,1\otimes l_3\}_L,1\otimes l_2\}_L-\tilde{f}\{Y_{1,3},1\otimes l_2\}_L\\
&-\{\tilde{f}\{1\otimes l_1,1\otimes l_3\}_L,\tilde{f}(1\otimes l_2)\}_M+\{\tilde{f}(Y_{1,3}),\tilde{f}(1\otimes l_2)\}_M
\end{aligned}\tag{4.4.12}$$

and

$$\begin{aligned}
&\beta_1^{-1}(\bar{\phi}_{f_{\lambda\mu}}(l_1,[l_2,l_3]))\\
=&\tilde{f}\{1\otimes l_1,\{1\otimes l_2,1\otimes l_3\}_L\}_L-\tilde{f}\{1\otimes l_1,Y_{2,3}\}_L\\
&-\{\tilde{f}(1\otimes l_1),\tilde{f}\{1\otimes l_2,1\otimes l_3\}_L\}_M+\{\tilde{f}(1\otimes l_1),\tilde{f}(Y_{2,3})\}_M.
\end{aligned}\tag{4.4.13}$$

Thus using (4.4.8)-(4.4.13) in (4.4.7) we get

$$\begin{aligned}
&\beta_1^{-1} \circ \delta\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2, l_3)\\
=& - \phi_M(X_1, 1\otimes f(l_2), 1\otimes f(l_3)) - \phi_M(X_1, 1\otimes f(l_2), X_3) - \phi_M(X_1, X_2, 1\otimes f(l_3))\\
&- \phi_M(X_1, X_2, X_3) - \phi_M(1\otimes f(l_1), X_2, 1\otimes f(l_3)) - \phi_M(1\otimes f(l_1), 1\otimes f(l_2), X_3)\\
&- \phi_M(1\otimes f(l_1), X_2, X_3) + \phi_{f_{\lambda\mu}}(Y_{1,2}, 1\otimes l_3) - \phi_{f_{\lambda\mu}}(Y_{1,3}, 1\otimes l_2)\\
&- \phi_{f_{\lambda\mu}}(1\otimes l_1, Y_{2,3}) - \{X_1, Z_{2,3}\}_M - \{Z_{1,3}, X_2\}_M + \{Z_{1,2}, X_3\}_M\\
&+ \tilde{f}\phi_L(1\otimes l_1, 1\otimes l_2, 1\otimes l_3) - \phi_M(1\otimes f(l_1), 1\otimes f(l_2), 1\otimes f(l_3)).
\end{aligned} \tag{4.4.14}$$

Now recall that $\phi_M(l_1, l_2, l_3) = 0$, whenever one of the arguments is in $ker(E_1)$ and $\phi_{f_{\lambda\mu}}(l_1, l_2) = 0$, whenever one of the arguments is in $ker(E)$. Moreover note that $\{X_1, Z_{2,3}\}_M = 0$ as $\{-,-\}_M$ is $B$-bilinear and $n_j n_{r+1} = 0$ for $1 \leq j \leq r$. Similarly, $\{Z_{1,3}, X_2\}_M = 0 = \{Z_{1,2}, X_3\}_M$.

Therefore from (4.4.14) we get,

$$\begin{aligned}
&\beta_1^{-1} \circ \delta\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2, l_3)\\
=&\tilde{f}\phi_L(1\otimes l_1, 1\otimes l_2, 1\otimes l_3) - \phi_M(1\otimes f(l_1), 1\otimes f(l_2), 1\otimes f(l_3))\\
=&\beta_1^{-1}(f\bar{\phi}_L - \bar{\phi}_M f)(l_1, l_2, l_3).
\end{aligned}$$

This completes the proof. □

**Proposition 4.4.3.** *Suppose* $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ *and* $(\{-,-\}'_L, \{-,-\}'_M; \tilde{f}')$ *are any two triples satisfying conditions (4.4.1). Let* $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ *and* $(\bar{\phi}'_L, \bar{\phi}'_M; \bar{\phi}'_{f_{\lambda\mu}})$ *be the corresponding cocycles determined by* $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ *and* $(\{-,-\}'_L, \{-,-\}'_M; \tilde{f}')$ *respectively. Then* $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ *and* $(\bar{\phi}'_L, \bar{\phi}'_M; \bar{\phi}'_{f_{\lambda\mu}})$ *represent the same cohomology class in* $HL^3(f; f)$.

*Proof.* Suppose $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ and $(\{-,-\}'_L, \{-,-\}'_M; \tilde{f}')$ are two liftings satisfying (4.4.1). Set $\rho_L = \{-,-\}'_L - \{-,-\}_L$, $\rho_M = \{-,-\}'_M - \{-,-\}_M$ and $\rho_f = \tilde{f}' - \tilde{f}$. Then $\rho_L : (B\otimes L)^{\otimes 2} \longrightarrow B\otimes L$, $\rho_M : (B\otimes M)^{\otimes 2} \longrightarrow B\otimes M$ and $\rho_f : B\otimes L \longrightarrow B\otimes M$ are $B$-linear maps. It follows from Proposition 4.2.3 that $\rho_L$ and $\rho_M$ induce two 2-cochains $\bar{\rho}_L$ and $\bar{\rho}_M$ respectively such that

$$(\bar{\phi}'_L - \bar{\phi}_L) = \delta\bar{\rho}_L \text{ and } (\bar{\phi}'_M - \bar{\phi}_M) = \delta\bar{\rho}_M. \tag{4.4.15}$$

Now $\rho_f$ takes values in $ker(P_1)$ because for $l \in B \otimes L$

$$\begin{aligned} P_1 \circ \rho_f(l) &= P_1(\tilde{f}'(l) - \tilde{f}(l)) \\ &= P_1 \circ \tilde{f}'(l) - P_1 \circ \tilde{f}(l) \\ &= f_{\lambda\mu} \circ P(l) - f_{\lambda\mu} \circ P(l) \quad \text{(by } (iv) \text{ in (4.4.1))} \\ &= 0. \end{aligned}$$

Also from $(iii)$ in (4.4.1) it follows that $E_1 \circ \tilde{f}' = f \circ E = E_1 \circ \tilde{f}$, so $\rho_f$ vanishes on $ker(E_1)$. Thus $\rho_f$ induces a linear map $\tilde{\rho}_f : (B \otimes L/kerE_1) \longrightarrow ker(P_1)$, which defines a 1-cochain $\bar{\rho}_f \in CL^1(L;M)$ such that $n_{r+1} \otimes \bar{\rho}_f(l) = \rho_f(1 \otimes l)$ for $l \in L$.

We claim that

$$(\bar{\phi\prime}_L, \bar{\phi}'_M; \bar{\phi}'_{f_{\lambda\mu}}) - (\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) = d(\bar{\rho}_L, \bar{\rho}_M; \bar{\rho}_f) = (\delta\bar{\rho}_L, \delta\bar{\rho}_M; f\bar{\rho}_L - \bar{\rho}_M f - \delta\bar{\rho}_f).$$

In view of (4.4.15) it is enough to show that $(\bar{\phi}'_{f_{\lambda\mu}} - \bar{\phi}_{f_{\lambda\mu}}) = (f\bar{\rho}_L - \bar{\rho}_M f - \delta\bar{\rho}_f)$. For $l_1, l_2 \in L$,

$$\begin{aligned} &\beta^{-1} \circ \delta\bar{\rho}_f(l_1, l_2) \\ &= n_{r+1} \otimes \delta\bar{\rho}_f(l_1, l_2) \\ &= n_{r+1} \otimes ([f(l_1), \bar{\rho}_f(l_2)] + [\bar{\rho}_f(l_1), f(l_2)] - \bar{\rho}_f[l_1, l_2]) \\ &= n_{r+1} \otimes [f(l_1), \bar{\rho}_f(l_2)] + n_{r+1} \otimes [\bar{\rho}_f(l_1), f(l_2)] - n_{r+1} \otimes \bar{\rho}_f[l_1, l_2] \\ &= I_1[f(l_1), \bar{\rho}_f(l_2)] + I_1[\bar{\rho}_f(l_1), f(l_2)] - \rho_f(1 \otimes [l_1, l_2]) \\ &= \{I_1 f(l_1), 1 \otimes \bar{\rho}_f(l_2)\}'_M + \{I_1\bar{\rho}_f, 1 \otimes f(l_2)\}'_M - (\tilde{f}' - \tilde{f})(1 \otimes [l_1, l_2]) \\ &= \{n_{r+1} \otimes f(l_1), 1 \otimes \bar{\rho}_f(l_2)\}'_M + \{n_{r+1} \otimes \bar{\rho}_f, 1 \otimes f(l_2)\}'_M - \tilde{f}'(1 \otimes [l_1, l_2]) \\ &\qquad + \tilde{f}(1 \otimes [l_1, l_2]) \\ &= \{1 \otimes f(l_1), n_{r+1} \otimes \bar{\rho}_f(l_2)\}'_M + \{n_{r+1} \otimes \bar{\rho}_f, 1 \otimes f(l_2)\}'_M - \tilde{f}'(1 \otimes [l_1, l_2]) \\ &\qquad + \tilde{f}(1 \otimes [l_1, l_2]) \\ &= \{\tilde{f}'(1 \otimes l_1) - X'_1, (\tilde{f}' - \tilde{f})(1 \otimes l_2)\}'_M + \{(\tilde{f}' - \tilde{f})(1 \otimes l_1), \tilde{f}(1 \otimes l_2) - X_2\}'_M \\ &\qquad - \tilde{f}'(\{1 \otimes l_1, 1 \otimes l_2\}'_L - Y'_{1,2}) + \tilde{f}(\{1 \otimes l_1, 1 \otimes l_2\}'_L - Y'_{1,2}) \\ &\qquad \text{( by Lemma 4.4.1 } (i) \text{ and } (ii) \text{ )} \\ &= \{\tilde{f}'(1 \otimes l_1), \tilde{f}'(1 \otimes l_2)\}'_M - \{\tilde{f}'(1 \otimes l_1), \tilde{f}(1 \otimes l_2)\}'_M - \{X'_1, (\tilde{f}' - \tilde{f})(1 \otimes l_2)\}'_M \\ &\quad + \{\tilde{f}'(1 \otimes l_1), \tilde{f}(1 \otimes l_2)\}'_M - \{\tilde{f}(1 \otimes l_1), \tilde{f}(1 \otimes l_2)\}'_M - \{(\tilde{f}' - \tilde{f})(1 \otimes l_1), X_2\}'_M \\ &\quad - \tilde{f}'\{1 \otimes l_1, 1 \otimes l_2\}'_L + \tilde{f}\{1 \otimes l_1, 1 \otimes l_2\}'_L + (\tilde{f}' - \tilde{f})(Y'_{1,2}) \\ &= \{\tilde{f}'(1 \otimes l_1), \tilde{f}'(1 \otimes l_2)\}'_M - \{\tilde{f}(1 \otimes l_1), \tilde{f}(1 \otimes l_2)\}'_M - \tilde{f}'\{1 \otimes l_1, 1 \otimes l_2\}'_L \\ &\qquad + \tilde{f}\{1 \otimes l_1, 1 \otimes l_2\}'_L. \end{aligned} \tag{4.4.16}$$

On the other hand

$$\beta^{-1}\circ(f\bar{\rho}_L)(l_1,l_2) = n_{r+1}\otimes f\bar{\rho}_L(l_1,l_2) = \tilde{f}\{1\otimes l_1,1\otimes l_2\}'_L - \tilde{f}\{1\otimes l_1,1\otimes l_2\}_L, \quad (4.4.17)$$

$$\begin{aligned}\beta^{-1}\circ\bar{\phi}'_{f_{\lambda\mu}}(l_1,l_2) &= n_{r+1}\otimes\bar{\phi}'_{f_{\lambda\mu}}(l_1,l_2)\\ &= \phi'_{f_{\lambda\mu}}(1\otimes l_1,1\otimes l_2)\\ &= \tilde{f}'\{1\otimes l_1,1\otimes l_2\}'_L - \{\tilde{f}'(1\otimes l_1),\tilde{f}'(1\otimes l_2)\}'_M,\end{aligned} \quad (4.4.18)$$

$$\begin{aligned}\beta^{-1}\circ\bar{\phi}_{f_{\lambda\mu}}(l_1,l_2) &= n_{r+1}\otimes\bar{\phi}_{f_{\lambda\mu}}(l_1,l_2)\\ &= \phi_{f_{\lambda\mu}}(1\otimes l_1,1\otimes l_2)\\ &= \tilde{f}\{1\otimes l_1,1\otimes l_2\}_L - \{\tilde{f}(1\otimes l_1),\tilde{f}(1\otimes l_2)\}_M,\end{aligned} \quad (4.4.19)$$

and

$$\begin{aligned}&\beta^{-1}\circ\bar{\rho}_M f(l_1,l_2)\\ &= n_{r+1}\otimes\bar{\rho}_M f(l_1,l_2)\\ &= n_{r+1}\otimes\bar{\rho}_M(f(l_1),f(l_2))\\ &= \rho_M(1\otimes f(l_1),1\otimes f(l_2))\\ &= \{1\otimes f(l_1),1\otimes f(l_2)\}'_M - \{1\otimes f(l_1),1\otimes f(l_2)\}_M\\ &= \{\tilde{f}(1\otimes l_1)-X_1,\tilde{f}(1\otimes l_2)-X_2\}'_M - \{\tilde{f}(1\otimes l_1)-X_1,\tilde{f}(1\otimes l_2)-X_2\}_M\\ &\qquad\qquad (\text{ by Lemma 4.4.1 } (i), \text{ where } X_1,X_2\in ker(E_1)\ )\\ &= \{\tilde{f}(1\otimes l_1),\tilde{f}(1\otimes l_2)\}'_M - \{\tilde{f}(1\otimes l_1),\tilde{f}(1\otimes l_2)\}_M\\ &\quad -\rho_M(\tilde{f}(1\otimes l_1),X_2) - \rho_M(X_1,\tilde{f}(1\otimes l_2)) + \rho_M(X_1,X_2)\\ &= \{\tilde{f}(1\otimes l_1),\tilde{f}(1\otimes l_2)\}'_M - \{\tilde{f}(1\otimes l_1),\tilde{f}(1\otimes l_2)\}_M\\ &\quad (\text{ since } \rho_M(x_1,x_2)=0, \text{ whenever one of the arguments is in } ker(E_1)\ ).\end{aligned} \quad (4.4.20)$$

Using (4.4.17)-(4.4.20) we get

$$\begin{aligned}&\beta^{-1}\circ(f\bar{\rho}_L - \bar{\rho}_M f - \bar{\phi}'_{f_{\lambda\mu}} + \bar{\phi}_{f_{\lambda\mu}})(l_1,l_2)\\ =&\{\tilde{f}'(1\otimes l_1),\tilde{f}'(1\otimes l_2)\}'_M - \{\tilde{f}(1\otimes l_1),\tilde{f}(1\otimes l_2)\}'_M\\ &-\tilde{f}'\{1\otimes l_1,1\otimes l_2\}'_L + \tilde{f}\{1\otimes l_1,1\otimes l_2\}'_L.\end{aligned} \quad (4.4.21)$$

Therefore from (4.4.16) and (4.4.21) it follows that

$$\beta^{-1}\circ\delta\bar{\rho}_f(l_1,l_2) = \beta^{-1}\circ(f\bar{\rho}_L - \bar{\rho}_M f - \bar{\phi}'_{f_{\lambda\mu}} + \bar{\phi}_{f_{\lambda\mu}})(l_1,l_2)$$

□

Thus we have a map

$$\Theta_{\mathfrak{D}} : H^2_{Harr}(A;\mathbb{K}) \longrightarrow HL^3(f;f) \text{ given by } \Theta_{\mathfrak{D}}([\psi]) = [(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})],$$

where $[(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})]$ denotes the cohomology class of $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$, which is the *obstruction map* in the present context.

**Theorem 4.4.4.** *Given an* 1*-dimensional extension*

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

*of $A$ representing $[\psi] \in H^2_{Harr}(A;\mathbb{K})$, a deformation $\mathfrak{D}$ of $f$ with base $A$ can be extended to a deformation of $f$ with base $B$ if and only if $\Theta_{\mathfrak{D}}([\psi]) = 0$.*

*Proof.* Suppose $\Theta_{\mathfrak{D}}([\psi]) = 0$. Let $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ be a triple satisfying conditions in (4.4.1). Let $\phi_L, \phi_M$ and $\phi_{f_{\lambda\mu}}$ be maps as defined in (4.4.4). Let $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ be the associated cocycle.

Since $\Theta_{\mathfrak{D}}([\psi]) = [(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})] = 0$, there is a 2-cochain $(u, v; w) \in CL^2(f;f)$ such that $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}}) = d(u, v; w)$. Therefore $\delta u = \bar{\phi}_L$, $\delta v = \bar{\phi}_M$, and $(fu - vf - \delta w) = \bar{\phi}_{f_{\lambda\mu}}$. Take $\rho'_L = -u, \rho'_M = -v$, and $\rho'_f = -w$. Define the linear maps

$$\{-,-\}'_L : (B \otimes L)^{\otimes 2} \longrightarrow B \otimes L \text{ by } \{l_1, l_2\}'_L = \{l_1, l_2\} - I \circ u(E(l_1), E(l_2)),$$
$$\{-,-\}'_M : (B \otimes M)^{\otimes 2} \longrightarrow B \otimes M \text{ by } \{x_1, x_2\}'_M = \{x_1, x_2\} - I_1 \circ v(E_1(x_1), E_1(x_2)),$$
$$\text{and } \tilde{f}' : B \otimes L \longrightarrow B \otimes M \text{ by } \tilde{f}'(l) = \tilde{f}(l) - I_1 \circ w(E(l))$$

for $l, l_1, l_2 \in B \otimes L$ and $x_1, x_2 \in B \otimes M$.

We claim that $(\{-,-\}'_L, \{-,-\}'_M; \tilde{f}')$ is a deformation of $f$ with base $B$ extending the deformation $\mathfrak{D}$. Let $\phi'_L, \phi'_M$, and $\phi'_{f_{\lambda\mu}}$ be the associated maps as defined in (4.4.4). Let $(\bar{\phi}'_L, \bar{\phi}'_M; \bar{\phi}'_{f_{\lambda\mu}})$ be corresponding 3-cocycle. Then it is easy to see that $(\bar{\phi}_L - \bar{\phi}'_L) = \delta u$, $(\bar{\phi}_M - \bar{\phi}'_M) = \delta v$, and $(\bar{\phi}_{f_{\lambda\mu}} - \bar{\phi}'_{f_{\lambda\mu}}) = fu - vf - \delta w$. Thus $\bar{\phi}'_L = 0$, $\bar{\phi}'_M = 0$, and $\bar{\phi}'_{f_{\lambda\mu}} = 0$. It follows from (4.4.6) that $\phi'_L = 0 = \phi'_M$ and $\phi'_{f_{\lambda\mu}} = 0$. Consequently $(\{-,-\}'_L, \{-,-\}'_M; \tilde{f}')$ is a deformation of $f$ with base $B$ extending $\mathfrak{D}$.

The converse part follows easily, since if we have an extension $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ of the deformation $\mathfrak{D}$ then $\phi_L = \phi_M = \phi f_{\lambda\mu} = 0$, by definition (4.4.4). So $\Theta_{\mathfrak{D}}([\psi])$ is represented by the zero cochain. ☐

**Example 4.4.5.** *Let $\mathfrak{D} = (\lambda_t, \mu_t; f_t)$ be a finite order 1-parameter deformation of Leibniz algebra homomorphism $f : L \longrightarrow M$ with base $A = \mathbb{K}[[t]]/(t^{N+1})$. Explicitly,*

$$\lambda_t(l_1, l_2) = \sum_{i\geq 0} \lambda_i(l_1, l_2)t^i \text{ (modulo } t^{N+1})$$
$$\mu_t(x_1, x_2) = \sum_{i\geq 0} \mu_i(x_1, x_2)t^i \text{ (modulo } t^{N+1})$$
$$f_t(l) = \sum_{i\geq 0} f_i(l)t^i \text{ (modulo } t^{N+1})$$

*where $l, l_i \in L$, $x_i \in M$, $\lambda_i \in CL^2(L; L)$, $\mu_i \in CL^2(M; M)$ and $f_i \in CL^1(L; M)$, $i = 1, 2$ with $\lambda_0, \mu_0$ are the original brackets in $L$ and $M$ respectively, and $f_0$ is the Leibniz algebra homomorphism $f$.*

*By Leibniz relations satisfied by $\lambda_t$ and $\mu_t$ we have*

$$\lambda_t(l_1, \lambda_t(l_2, l_3)) - \lambda_t(\lambda_t(l_1, l_2), l_3) + \lambda_t(\lambda_t(l_1, l_3), l_2) = 0$$
$$\mu_t(x_1, \mu_t(x_2, x_3)) - \mu_t(\mu_t(x_1, x_2), x_3) + \mu_t(\mu_t(x_1, x_3), x_2) = 0$$
$$\text{for } l_i \in L \text{ and } x_i \in M,\ i = 1, 2, 3.$$

*Also $f_t$ is a Leibniz algebra homomorphism, so we have*

$$f_t(\lambda_t(l_1, l_2)) = \mu_t(f_t(l_1), f_t(l_2)) \text{ for } l_1, l_2 \in L.$$

*If we try to extend $\mathfrak{D}$ to a deformation of order $N + 1$, starting with the extension*

$$0 \longrightarrow (t^{N+1})/(t^{N+2}) \longrightarrow \mathbb{K}[[t]]/(t^{N+2}) \longrightarrow \mathbb{K}[[t]]/(t^{N+1}) \longrightarrow 0,$$

*the obstruction cocycle $(\bar{\phi}_L, \bar{\phi}_M; \bar{\phi}_{f_{\lambda\mu}})$ in this case can be written as ( [Man07])*

$$\bar{\phi}_L(l_1, l_2, l_3) = \sum_{\substack{i+j=N+1\\ i,j>0}} \{\lambda_i(\lambda_j(x, y), z) - \lambda_i(\lambda_j(x, z), y) - \lambda_i(x, \lambda_j(y, z))\}$$
$$\bar{\phi}_M(x_1, x_2, x_3) = \sum_{\substack{i+j=N+1\\ i,j>0}} \{\mu_i(\mu_j(x_1, x_2), x_3) - \mu_i(\mu_j(x_1, x_3), x_2) - \mu_i(x_1, \mu_j(x_2, x_3))\}$$
$$\bar{\phi}_{f_{\lambda\mu}}(l_1, l_2) = \sum{}' \mu_i(f_j(l_1), f_k(l_2)) - \sum_{\substack{i+j=N+1\\ i,j>0}} f_i(\lambda_j(l_1, l_2))$$

*for $l_i \in L$ and $x_i \in M$, $i = 1, 2, 3$.*

*Here,*

$$\sum{}' = \sum_{\substack{i+j=N+1\\ i,j>0;k=0}} + \sum_{\substack{i+k=N+1\\ i,k>0;j=0}} + \sum_{\substack{j+k=N+1\\ j,k>0;i=0}} + \sum_{\substack{i+j+k=N+1\\ i,j,k>0}} .$$

*The given deformation extends to a deformation of order $N+2$ if the cohomology class of the obstruction cocycle is zero.*

So far we were concerned with the lifting problem for 1-dimensional extension of the base $(A, \mathfrak{M}) \in \mathcal{C}$ of a deformation. An analogous consideration holds for any finite dimensional extension of the algebra $A$ by an $A$-module.

Let $M_0$ be a finite dimensional $A$-module satisfying $\mathfrak{M}M_0 = 0$. From Proposition 2.3.3 , it follows that $H^2_{Harr}(A; M_0)$ is in one to one correspondence with the isomorphism classes of extensions

$$0 \longrightarrow M_0 \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0. \tag{4.4.22}$$

Let $[\psi] \in H^2_{Harr}(A; M_0)$ correspond to the class of extensions represented by the extension in (4.4.22). If we proceed with the above extension as in the case of 1-dimensional extension, we obtain a triple $(\{-,-\}_L, \{-,-\}_M; \tilde{f})$ and the maps $\phi_L, \phi_M$ and $\phi_{f_{\lambda\mu}}$ as determined in (4.4.4). We define $\tilde{\phi}_L, \tilde{\phi}_M$ and $\tilde{\phi}_{f_{\lambda\mu}}$ using $\phi_L, \phi_M$ and $\phi_{f_{\lambda\mu}}$ respectively as in (4.4.5). As before we have isomorphisms $(B \otimes L/ker(E)) \cong L$ and $(B \otimes M/ker(E_1)) \cong M$. Moreover in this general case, we have isomorphisms $ker(P) \cong M_0 \otimes L$ and $ker(P_1) \cong M_0 \otimes M$. We use these isomorphisms to obtain cochains

$$\begin{aligned} \bar{\phi}_L &\in CL^3(L; M_0 \otimes L) \cong M_0 \otimes CL^3(L; L), \\ \bar{\phi}_M &\in CL^3(M; M_0 \otimes M) \cong M_0 \otimes CL^3(M; M) \\ \text{and } \bar{\phi}_{f_{\lambda\mu}} &\in CL^2(L; M_0 \otimes M) \cong M_0 \otimes CL^2(L; M). \end{aligned}$$

An argument analogous to the proof of Proposition 4.4.2 shows that $(\bar{\phi}_L, \bar{\phi}_M, \bar{\phi}_{f_{\lambda\mu}})$ is a cocycle in $M_0 \otimes CL^3(f; f)$.

As a consequence we have the *obstruction map*

$$\Theta_{\mathfrak{D}} : H^2_{Harr}(A; M_0) \longrightarrow M_0 \otimes HL^3(f; f).$$

As in Theorem 4.4.4, we have

**Theorem 4.4.6.** *Given an extension (4.4.22) of $A$ by the $A$-module $M_0$ representing $[\psi]$, a deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of $f$ with base $(A, \mathfrak{M}) \in \mathcal{C}$ can be lifted to a deformation of $f$ with base $B$ if and only if $\Theta_{\mathfrak{D}}([\psi]) = 0$.*

□

## 4.5 Formal deformations

An $\mathfrak{M}$-adic filtration of a commutative algebra $A$ is a filtration

$$A = \mathfrak{M}^0 \supset \mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots .$$

by ideals of $A$. Given an $\mathfrak{M}$-adic filtration, the completion $\hat{A}_{\mathfrak{M}}$ of $A$ with respect to the $\mathfrak{M}$-adic filtration is the projective limit of the quotient algebras $A/\mathfrak{M}^i$ equipped with the family of homomorphisms $p_i^j : A/\mathfrak{M}^j \longrightarrow A/\mathfrak{M}^i$ for $j \geq i$ defined by $p_i^j(a+\mathfrak{M}^j) = a+\mathfrak{M}^i$. It is a subalgebra of the direct product

$$\begin{aligned}\hat{A}_{\mathfrak{M}} &:= \varprojlim_{n\to\infty} (A/\mathfrak{M}^n) \\ &:= \{a = (a_1, a_2, \cdots) \in \Pi_i A/\mathfrak{M}^i \ ; \ p_i^j(a_j + \mathfrak{M}^j) = a_i + \mathfrak{M}^i \text{ for all } j > i\}.\end{aligned}$$

The completed ring $\hat{A}_{\mathfrak{M}}$ is equipped with maps $p_j : \hat{A}_{\mathfrak{M}} \longrightarrow A/\mathfrak{M}^j$ for all $j$ such that for $j \geq i$, $p_i^j \circ p_j = p_i$.

If $A$ is local with maximal ideal $\mathfrak{M}$ then $\hat{A}_{\mathfrak{M}}$ is a local ring with the maximal ideal $\hat{\mathfrak{M}}$, where $\hat{\mathfrak{M}} = \{a = (a_1, a_2, \cdots) \in \Pi_i A/\mathfrak{M}^i; \ a_1 = 0\}$.

In case the natural map $A \longrightarrow \hat{A}_{\mathfrak{M}}$ is an isomorphism, then we say that $A$ is complete with respect to $\mathfrak{M}$. In this case we shall always assume that $dim(\mathfrak{M}^k/\mathfrak{M}^{k+1})$ is finite for all $k$.

**Example 4.5.1.** *If $A = R[x_1, \cdots, x_n]$ is a polynomial ring over the ring $R$, and $\mathfrak{M} =< x_1, \cdots, x_n >$, the ideal generated by $x_1, \cdots, x_n$, then the completion $\hat{A}_{\mathfrak{M}}$ of $A$ with respect to $\mathfrak{M}$ is the formal power series ring*

$$\hat{A}_{\mathfrak{M}} \cong R[[x_1, \cdots, x_n]].$$

**Definition 4.5.2.** *Let $A$ be a complete local algebra, where $\mathfrak{M}$ is the maximal ideal in $A$. A deformation $\lambda$ of $L$ with base $A$ is called formal if the $A$-Leibniz algebra structure $\lambda$ on*

$$A\hat{\otimes}L = \varprojlim_{n\to\infty} ((A/\mathfrak{M}^n) \otimes L),$$

*is the projective limit of deformations $\lambda_n$ with base $A/\mathfrak{M}^n$.*

**Example 4.5.3.** *If $A = \mathbb{K}[[t]]$ then a formal deformation of a type of algebra $L$ (associative, Lie, Leibniz etc.) over $A$ is a formal one parameter deformation of $L$ [Ger64, Ger66, Ger68, Ger74]. For example, the deformation $\lambda$ of a Leibniz algebra $L$ with base $A = \mathbb{K}[[t]]$ is given by the bracket*

$$\lambda_t = [-,-] + \lambda_1 t + \lambda_2 t^2 + \cdots , \quad \text{where } \lambda_i \in CL^2(L; L)$$

*with $[-,-]$ being the original Leibniz bracket on $L$. This is studied in [Bal97].*

**Definition 4.5.4.** *A formal deformation of a Leibniz algebra homomorphism $f : L \longrightarrow M$ with base $A$, where $A$ is complete, is a deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ which is obtained as a projective limit of deformations $\mathfrak{D}_n = (\lambda_n, \mu_n; f_{\lambda_n\mu_n})$ with base $A/\mathfrak{M}^n$.*

**Example 4.5.5.** *If $A = \mathbb{K}[[t]]$ then a formal deformation $(\lambda_t, \mu_t; f_t)$ of $f : L \longrightarrow M$ is a Leibniz algebra homomorphism $f_t : L_t \longrightarrow M_t$ of the form*

$$f_t = f + f_1 t + f_2 t^2 + \cdots .$$

*Here each $f_i : L \longrightarrow M$ is a $\mathbb{K}$-linear map for $i \geq 1$, and $L_t = L \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ , $M_t = M \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ are formal one parameter family of deformations of $L$ and $M$ given by brackets $\lambda_t$ and $\mu_t$, respectively. This is studied in [Man07].*

**Definition 4.5.6.** *For a formal deformation $\mathfrak{D}$ of $f$ with base $A$, $p_{2*}\mathfrak{D}$ is called the infinitesimal part of $\mathfrak{D}$ and $a_{p_{2*}\mathfrak{D}}$ is the differential of $\mathfrak{D}$, where $p_2 : A \longrightarrow A/\mathfrak{M}^2$ is the map introduced above.*

More generally, let $\mathfrak{D}_k$ denote the push-out $p_{k*}\mathfrak{D}$, where $p_k : A \longrightarrow A/\mathfrak{M}^k$. Then $\mathfrak{D}_k$ is a deformation with base $A/\mathfrak{M}^k$.

**Definition 4.5.7.** *For a formal deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of $f$ with base $A$, the linear map*

$$\alpha_{\mathfrak{D}_2} : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow CL^2(f; f)$$

*defined by $\alpha_{\mathfrak{D}_2}(\xi) = (\alpha_{\lambda_2,\xi}, \alpha_{\mu_2,\xi}; f_{\lambda_2\mu_2,\xi})$ is called the infinitesimal of the deformation $\mathfrak{D}$. More generally, if $\alpha_{\mathfrak{D}_k} = 0$, the zero map for $1 \leq k \leq n$, and $\alpha_{\mathfrak{D}_{n+1}}$ is a non-zero linear map, then $\alpha_{\mathfrak{D}_{n+1}}$ is called the $n$-infinitesimal of the deformation $\mathfrak{D}$, where $\mathfrak{D}_k = p_{k*}\mathfrak{D}$.*

**Remark 4.5.8.** *Note that for every $n$,*

$$0 \longrightarrow \mathfrak{M}^n/\mathfrak{M}^{n+1} \longrightarrow A/\mathfrak{M}^{n+1} \longrightarrow A/\mathfrak{M}^n \longrightarrow 0$$

*is an extension of the algebra $A/\mathfrak{M}^n$ and $\mathfrak{D}_{n+1}$ is an extension of the deformation $\mathfrak{D}_n$ from the base $A/\mathfrak{M}^n$ to the base $A/\mathfrak{M}^{n+1}$.*

**Proposition 4.5.9.** *The infinitesimal $\alpha_{\mathfrak{D}_2}$ of the deformation $\mathfrak{D}$ takes values in cocycles in $CL^2(f; f)$. More generally, the $n$-infinitesimal $\alpha_{\mathfrak{D}_{n+1}}$ takes values values in cocycles in $CL^2(f; f)$.*

*Proof.* By Theorem 3.5.1, $\alpha_{\mathfrak{D}_2}(\xi)$ is a 2-cocycle in $CL^2(f; f)$. In general let

$$\alpha_{\mathfrak{D}_k} : (\mathfrak{M}/\mathfrak{M}^{k+1})' \longrightarrow CL^2(f; f)$$

be the $k$-infinitesimal of the deformation $\mathfrak{D}$. Consider the extension

$$0 \longrightarrow \mathfrak{M}^k/\mathfrak{M}^{k+1} \xrightarrow{i_k^{k+1}} A/\mathfrak{M}^{k+1} \xrightarrow{p_k^{k+1}} A/\mathfrak{M}^k \longrightarrow 0$$

of $A/\mathfrak{M}^k$. The maximal ideal of $A/\mathfrak{M}^k$ is $\mathfrak{M}/\mathfrak{M}^k$ and the maximal ideal $\mathfrak{M}/\mathfrak{M}^{k+1}$ of $A/\mathfrak{M}^{k+1}$ is given by $\mathfrak{M}/\mathfrak{M}^{k+1} = \mathfrak{M}/\mathfrak{M}^k \oplus \mathfrak{M}^k/\mathfrak{M}^{k+1}$. Let $dim(\mathfrak{M}/\mathfrak{M}^k) = r$ and $dim(\mathfrak{M}^k/\mathfrak{M}^{k+1}) = s$, so $dim(\mathfrak{M}/\mathfrak{M}^{k+1}) = r + s$. Let $\{m_i + \mathfrak{M}^k\}_{1\leq i\leq r}$ be a basis of $\mathfrak{M}/\mathfrak{M}^k$ and $\{m_{r+i} + \mathfrak{M}^{k+1}\}_{1\leq i\leq s}$ be a basis of $\mathfrak{M}^k/\mathfrak{M}^{k+1}$. Thus we get a basis of $\mathfrak{M}/\mathfrak{M}^{k+1}$, which we write as $\{\bar{m}_i\}_{1\leq i\leq r+s}$.

From definition, $\alpha_{\mathfrak{D}_k}(\bar{m}_i') = (\alpha_{\lambda_k,\bar{m}_i'}, \alpha_{\mu_k,\bar{m}_i'}; f_{\lambda_k\mu_k,\bar{m}_i'})$. It is sufficient to show that

$$d\alpha_{\mathfrak{D}_k}(\bar{m}_i') = (\delta\alpha_{\lambda_k,\bar{m}_i'}, \delta\alpha_{\mu_k,\bar{m}_i'}; f\alpha_{\lambda_k,\bar{m}_i'} - \alpha_{\mu_k,\bar{m}_i'}f - \delta f_{\lambda_k\mu_k,\bar{m}_i'}) = 0$$

for $1 \leq i \leq r+s$. Notice that $\mathfrak{D}_k$ is an extension of $\mathfrak{D}_{k-1}$ and $\alpha_{\mathfrak{D}_k}$ is the $k$-infinitesimal, so $\alpha_{\mathfrak{D}_k}(\bar{m}_i') = \alpha_{\mathfrak{D}_{k-1}}(\bar{m}_i') = 0$ for $1 \leq i \leq r$. Thus $\mathfrak{D}_k = (\lambda_k, \mu_k; f_{\lambda_k\mu_k})$ is given by the following.

$$[1\otimes l_1, 1\otimes l_2]_{\lambda_k} = 1\otimes[l_1,l_2] + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes \alpha_{\lambda_k,\bar{m}_i'}(l_1,l_2), \text{ for } l_1,l_2 \in L$$

$$[1\otimes x_1, 1\otimes x_2]_{\mu_k} = 1\otimes[x_1,x_2] + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes \alpha_{\mu_k,\bar{m}_i'}(x_1,x_2) \; x_1,x_2 \in M \text{ and}$$

$$f_{\lambda_k\mu_k}(1\otimes l) = 1\otimes f(l) + \sum_{i=r+1}^{r+s} \bar{m}_j \otimes f_{\lambda_k\mu_k,\bar{m}_i'}(l) \text{ for } l \in L.$$

Now $\{\bar{m}_i\}_{r+1\leq i\leq r+s}$ is a basis of $\mathfrak{M}^k/\mathfrak{M}^{k+1}$ and $(\mathfrak{M}^k/\mathfrak{M}^{k+1})^2 = 0$. Thus by the analogous computations as in Theorem 3.5.1 we get $\delta\alpha_{\lambda_k,\bar{m}_i'} = 0 = \delta\alpha_{\mu_k,\bar{m}_i'}$ and $f\alpha_{\lambda_k,\bar{m}_i'} - \alpha_{\mu_k,\bar{m}_i'}f - \delta f_{\lambda_k\mu_k,\bar{m}_i'} = 0$ for $r+1 \leq i \leq r+s$. Consequently, $\alpha_{\mathfrak{D}_k}(\bar{m}_i')$ is a 2-cocycle in $CL^2(f;f)$ for $1 \leq i \leq r+s$.

□

**Theorem 4.5.10.** *A non-trivial formal deformation of a Leibniz algebra homomorphism is equivalent to a deformation whose $n$-infinitesimal is not a coboundary (in the sense that the image of the $n$-infinitesimal is not contained in 2-coboundaries) for some $n \geq 1$.*

*Proof.* Let $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ be a formal deformation of $f$ with complete local algebra base $(A, \mathfrak{M})$. Suppose $\mathfrak{D}$ has $n$-infinitesimal $\alpha_{\mathfrak{D}_{n+1}}$ for some $n \geq 1$. Note that for every $k$, $\mathfrak{D}_k = p_{k*}\mathfrak{D} = (\lambda_k, \mu_k; f_{\lambda_k\mu_k})$ is a deformation with base $A/\mathfrak{M}^k$ such that $\mathfrak{D} = \varprojlim_{k\to\infty} \mathfrak{D}_k$ where $p_k : A \longrightarrow A/\mathfrak{M}^k$.

Assume that $\alpha_{\mathfrak{D}_{n+1}}(\xi)$ is a coboundary in $CL^2(f;f)$ for all $\xi \in (\mathfrak{M}/\mathfrak{M}^{n+1})'$.

We claim that $\mathfrak{D}_{n+1}$ is equivalent to the trivial deformation $\mathfrak{D}_0 = (\lambda_0, \mu_0; f_0)$ with base $A/\mathfrak{M}^{n+1}$, where $\lambda_0$ and $\mu_0$ are given by $[1\otimes l_1, 1\otimes l_2]_{\lambda_0} = 1\otimes [l_1, l_2]$ and $[1\otimes x_1, 1\otimes x_2]_{\mu_0} = 1\otimes [x_1, x_2]$ for $l_1, l_2 \in L$, $x_1, x_2 \in M$, and $f_0(1\otimes l) = 1\otimes f(l)$ for $l \in L$.

Let $\{\bar{m}_i = m_i + \mathfrak{M}^n\}_{1\le i\le r}$ be a basis of $\mathfrak{M}/\mathfrak{M}^n$. We extend this to basis $\{\bar{m}_i\}_{1\le i\le r+s}$ of $\mathfrak{M}/\mathfrak{M}^{n+1} = \mathfrak{M}/\mathfrak{M}^n \oplus \mathfrak{M}^n/\mathfrak{M}^{n+1}$ by adding a basis $\{\bar{m}_{r+i} = m_{r+i} + \mathfrak{M}^{n+1}\}_{1\le i\le s}$ of $\mathfrak{M}^n/\mathfrak{M}^{n+1}$. By our assumption $\alpha_{\mathfrak{D}_n} = 0$, the zero map so that $\alpha_{\mathfrak{D}_{n+1}}(\bar{m}_i') = \alpha_{\mathfrak{D}_n}(\bar{m}_i') = 0$ for $1\le i\le r$ as $\mathfrak{D}_{n+1}$ is an extension of $\mathfrak{D}_n$ (cf. Remark 4.5.8 ) and $\alpha_{\mathfrak{D}_{n+1}}(\bar{m}_i') = d(u_i, v_i; w_i), r+1\le i\le r+s$ for some 1-cochains $(u_i, v_i; w_i) \in CL^1(f; f)$.

We may assume that $\alpha_{\mathfrak{D}_{n+1}}(\bar{m}_i') = d(u_i, v_i; 0)$ as $d(u_i, v_i; w_i) = d(u_i, v_i + \delta w_i; 0)$ for $r+1\le i\le r+s$. Thus we have following relations

$$(\alpha_{\lambda_n, \bar{m}_i'}, \alpha_{\mu_n, \bar{m}_i'}; f_{\lambda_n\mu_n, \bar{m}_i'}) = (\delta u_i, \delta v_i, f u_i - v_i f) \text{ for } r+1\le i\le r+s. \tag{4.5.1}$$

Define $A/\mathfrak{M}^{n+1}$-linear maps

$$\begin{aligned}
&\Phi : ((A/\mathfrak{M}^{n+1})\otimes L, \lambda_{n+1}) \longrightarrow ((A/\mathfrak{M}^{n+1})\otimes L, \lambda_0) \text{ by}\\
&\Phi(1\otimes l) = 1\otimes l + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes u_i(l) \text{ and}\\
&\Psi : ((A/\mathfrak{M}^{n+1})\otimes M, \mu_{n+1}) \longrightarrow ((A/\mathfrak{M}^{n+1})\otimes M, \mu_0) \text{ by}\\
&\Psi(1\otimes x) = 1\otimes l + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes v_i(x) \text{ for } l\in L \text{ and } x\in M.
\end{aligned}$$

Observe that

$$\begin{aligned}
&[\Phi(1\otimes l_1), \Phi(1\otimes l_2)]_{\lambda_0}\\
&= [1\otimes l_1 + \sum_{i=r+1}^{r+s} \bar{m}_i\otimes u_i(l_1), 1\otimes l_2 + \sum_{i=r+1}^{r+s} \bar{m}_i\otimes u_i(l_2)]_{\lambda_0}\\
&= [1\otimes l_1, 1\otimes l_2]_{\lambda_0} + \sum_{i=r+1}^{r+s} \bar{m}_i[1\otimes u_i(l_1), 1\otimes l_2]_{\lambda_0}\\
&\quad + \sum_{i=r+1}^{r+s} \bar{m}_i[1\otimes l_1, 1\otimes u_i(l_2)]_{\lambda_0} + \sum_{i,j=r+1}^{r+s} \bar{m}_i\bar{m}_j[1\otimes u_i(l_1), 1\otimes u_j(l_2)]_{\lambda_0}\\
&= 1\otimes [l_1, l_2] + \sum_{i=r+1}^{r+s} \bar{m}_i\otimes [u_i(l_1), l_2] + \sum_{i=r+1}^{r+s} \bar{m}_i[l_1, u_i(l_2)]\\
&\qquad\qquad (\text{ using } \bar{m}_i\bar{m}_j = 0, r+1\le i, j\le r+s\ ).
\end{aligned} \tag{4.5.2}$$

On the other hand

$$\begin{aligned}
&\Phi[1\otimes l_1, 1\otimes l_2]_{\lambda_{n+1}}\\
&=\Phi(1\otimes[l_1,l_2]+\sum_{i=r+1}^{r+s}\bar{m}_i\otimes\alpha_{\lambda_{n+1},\bar{m}_i'}(l_1,l_2))\\
&=\Phi(1\otimes[l_1,l_2]+\sum_{i=r+1}^{r+s}\bar{m}_i\Phi(1\otimes\alpha_{\lambda_{n+1},\bar{m}_i'}(l_1,l_2))\\
&=1\otimes[l_1,l_2]+\sum_{i=r+1}^{r+s}\bar{m}_i\otimes u_i([l_1,l_2])\\
&\quad+\sum_{i=r+1}^{r+s}\bar{m}_i(1\otimes\alpha_{\lambda_{n+1},\bar{m}_i'}(l_1,l_2)+\sum_{j=r+1}^{r+s}\bar{m}_j\otimes u_j(\alpha_{\lambda_{n+1},\bar{m}_i'}(l_1,l_2)))\\
&=1\otimes[l_1,l_2]+\sum_{i=r+1}^{r+s}\bar{m}_i\otimes u_i([l_1,l_2])+\sum_{i=r+1}^{r+s}\bar{m}_i\otimes\alpha_{\lambda_{n+1},\bar{m}_i'}(l_1,l_2).
\end{aligned}\tag{4.5.3}$$

The $(A/\mathfrak{M}^{n+1})$- linear map $\Phi$ defines an equivalence of $\lambda_{n+1}$ and $\lambda_0$ if and only if

$$[\Phi(1\otimes l_1),\Phi(1\otimes l_2)]_{\lambda_0}-\Phi([1\otimes l_1,1\otimes l_2]_{\lambda_{n+1}})=0.$$

Equivalently,

$$\sum_{i=r+1}^{r+s}\bar{m}_i\otimes\delta u_i(l_1,l_2)-\sum_{i=r+1}^{r+s}\bar{m}_i\otimes\alpha_{\lambda_{n+1},\bar{m}_i'}(l_1,l_2)=0.$$

Equivalently, $\sum_{i=r+1}^{r+s}\bar{m}_i\otimes(\delta u_i-\alpha_{\lambda_{n+1},\bar{m}_i'})(l_1,l_2)=0$.

From the relations in (4.5.1) it follows that $\Phi$ defines an equivalence of $\lambda_{n+1}$ and $\lambda_0$. Similarly $\Psi$ defines an equivalence of $\mu_{n+1}$ and $\mu_0$.

We now check that

$$\Psi\circ f_{\lambda_{n+1}\mu_{n+1}}=f\circ\Phi.$$

Now

$$\begin{aligned}
&\Psi\circ f_{\lambda_{n+1}\mu_{n+1}}\\
&=\Psi\{1\otimes f(l)+\sum_{i=r+1}^{r+s}\bar{m}_i\otimes f_{\lambda_{n+1}\mu_{n+1},\bar{m}_i'}(l)\}\\
&=\Psi(1\otimes f(l))+\sum_{i=r+1}^{r+s}\bar{m}_i\Psi(1\otimes f_{\lambda_{n+1}\mu_{n+1},\bar{m}_i'}(l))
\end{aligned}$$

$$= 1 \otimes f(l) + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes v_i(f(l)) + \sum_{i=r+1}^{r+s} \bar{m}_i \{1 \otimes f_{\lambda_{n+1}\mu_{n+1},\bar{m}'_i}(l)$$

$$+ \sum_{j=r+1}^{r+s} \bar{m}_j \otimes v_j(f_{\lambda_{n+1}\mu_{n+1},\bar{m}'_i}(l))\}$$

$$= 1 \otimes f(l) + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes v_i(f(l)) + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes f_{\lambda_{n+1}\mu_{n+1},\bar{m}'_i}(l)$$

$$\text{( since } (\mathfrak{M}^n/\mathfrak{M}^{n+1})^2 = 0 \text{ ).}$$

On the other hand

$$f_0 \circ \Phi(1 \otimes l) = f_0(1 \otimes l + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes u_i(l))$$

$$= f_0(1 \otimes l) + \sum_{i=r+1}^{r+s} \bar{m}_i f_0(1 \otimes u_i(l))$$

$$= 1 \otimes f(l) + \sum_{i=r+1}^{r+s} \bar{m}_i \otimes f u_i(l).$$

Using the expression $fu_i - v_i f = f_{\lambda_{n+1}\mu_{n+1},\bar{m}'_i}$ in (4.5.1) we get

$$\Psi \circ f_{\lambda_{n+1}\mu_{n+1}} = f \circ \Phi.$$

Therefore $\mathfrak{D}_{n+1}$ is equivalent to the trivial deformation of $f$ with base $A/\mathfrak{M}^{n+1}$.

Thus we may assume that $\mathfrak{D}$ has $(n+1)$-infinitesimal. If $\alpha_{\mathfrak{D}_{n+2}}$ takes values in coboundaries, we can repeat the argument and the process must stop as $\mathfrak{D} = \varprojlim_{n\to\infty} \mathfrak{D}_n$ is given to be non-trivial. □

**Definition 4.5.11.** *A Leibniz algebra homomorphism $f : L \longrightarrow M$ is said to be rigid if any formal deformation of $f$ is trivial.*

**Corollary 4.5.12.** *If $HL^2(f; f) = 0$, then $f : L \longrightarrow M$ is rigid.*

*Proof.* Since $HL^2(f; f) = 0$, any 2-cocycle in $CL^2(f; f)$ is a 2-coboundary. Suppose $\mathfrak{D}$ is a formal deformation of $f$ with base $A$. By Theorem 3.5.1, $\alpha_{\mathfrak{D}_2}(\xi)$ is a 2-cocycle, hence a coboundary in $CL^2(f; f)$ for any $\xi \in (\mathfrak{M}/\mathfrak{M}^2)'$. Then as shown in the proof of Theorem 4.5.10, $\mathfrak{D}_2$ is a trivial deformation. But then by Proposition 4.5.9, the 2-infinitesimal takes values in cocycles and hence in coboundaries. By repeating the argument, we see that $\mathfrak{D}_n$ is trivial for every $n$. Hence $\mathfrak{D} = \varprojlim_{n\to\infty} \mathfrak{D}_n$ must be trivial. □

Suppose $A$ is a complete local algebra with the maximal ideal $\mathfrak{M}$. Let

$$a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f;f)$$

be any linear map and $\mathfrak{D}$ be an infinitesimal deformation with base $A/\mathfrak{M}^2$ and $a_{\mathfrak{D}} = a$. For instance, let $\{\bar{m}_i\}_{1\leq i\leq r}$ be a basis of $\mathfrak{M}/\mathfrak{M}^2$ with $\{\bar{\xi}_i\}_{1\leq i\leq r}$ be the corresponding dual basis of $(\mathfrak{M}/\mathfrak{M}^2)'$. Consider a linear map $\alpha : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow CL^2(f;f)$ such that $a(\bar{\xi}_i) = [\alpha(\bar{\xi}_i)]$, $1 \leq i \leq r$. Suppose $\alpha(\bar{\xi}_i) = (\psi_i^{\lambda}, \psi_i^{\mu}; f_i)$. We define a deformation $\mathfrak{D} = (\lambda, \mu; f_{\lambda\mu})$ of $f$ with base $A/\mathfrak{M}^2$ as follows.

$$[1\otimes l_1, 1\otimes l_2]_\lambda = 1\otimes [l_1,l_2] + \sum_{i=1}^{r} m_i \otimes \psi_i^{\lambda}(l_1,l_2),$$

$$[1\otimes x_1, 1\otimes x_2]_\mu = 1\otimes [x_1,x_2] + \sum_{i=1}^{r} m_i \otimes \psi_i^{\mu}(x_1,x_2),$$

$$f_{\lambda\mu}(1\otimes l_1) = 1\otimes f(l_1) + \sum_{i=1}^{r} m_i \otimes f_i(l_1) \quad \text{for } l_1,l_2 \in L \quad \text{and } x_1,x_2 \in M.$$

Then $a_{\mathfrak{D}} = a$.

Suppose the deformation $\mathfrak{D}$ can be lifted to a deformation $\mathfrak{D}_k$ with base $A/\mathfrak{M}^k$ for $k \geq 2$. Consider the extension

$$0 \longrightarrow \mathfrak{M}^k/\mathfrak{M}^{k+1} \xrightarrow{i_k^{k+1}} A/\mathfrak{M}^{k+1} \xrightarrow{p_k^{k+1}} A/\mathfrak{M}^k \longrightarrow 0,$$

representing a cohomology class $[\psi_k] \in H^2_{Harr}(A/\mathfrak{M}^k; M_k)$, where $M_k = \mathfrak{M}^k/\mathfrak{M}^{k+1}$.

Let $\theta_k = \Theta_{\mathfrak{D}_k}([\psi_k]) \in M_k \otimes HL^3(f;f)$. Then by Proposition 4.4.6 we obtain

**Proposition 4.5.13.** *Let $A$ be a complete local algebra with the maximal ideal $\mathfrak{M}$. Let $a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f;f)$ be a given linear map. Let $\mathfrak{D}$ be any infinitesimal deformation with base $A/\mathfrak{M}^2$ and $a_{\mathfrak{D}} = a$. Then there exists a formal deformation of $f$ with base $A$ and with the given map $a$ as its differential if and only if $\theta_k = 0$ for all $k \geq 2$.*

□

**Corollary 4.5.14.** *If $HL^3(f;f) = 0$, then every linear map*

$$a : (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow HL^2(f;f)$$

*is the differential of some formal deformation $\mathfrak{D}$ of $f$ with base $A$.*

□

We end this chapter with the definition of a versal deformation of a Leibniz algebra $L$.

**Definition 4.5.15.** *A formal deformation $\eta$ of a Leibniz algebra $L$ with base $C$ is called versal, if*

**(i)** *for any formal deformation $\lambda$ of the Leibniz algebra $L$ with base $A$ there exists a homomorphism $f : C \to A$ such that the deformation $\lambda$ is equivalent to $f_*\eta$;*

**(ii)** *$f$ is unique whenever the the maximal ideal $\mathfrak{M}$ of $A$ satisfies the condition $\mathfrak{M}^2 = 0$.*

An explicit construction of a versal deformation will be given in the next chapter.

# Chapter 5

# Construction of a Versal deformation of Leibniz algebras

## 5.1 Introduction

The present chapter is devoted to give a construction of versal deformation of a given Leibniz algebra $L$ with $dim(HL^2(L;L)) < \infty$. We begin with the universal infinitesimal deformation $\eta_1$ of $L$ with base $C_1$ as constructed in Chapter 3, and apply the tools developed in the last chapter to get a finite dimensional extension $\eta_2$ with base $C_2$. We kill off the possible obstruction associated to the extension problem for the specific extension (2.3.4) of $C_1$ to obtain $\eta_2$ with base $C_2$. We repeat this procedure successively to get a sequence of finite dimensional extensions $\eta_k$ with base $C_k$. The projective limit $C = \varprojlim_{k\to\infty} C_k$ is a complete local algebra and $\eta = \varprojlim_{k\to\infty} \eta_k$ is a formal deformation of $L$ with base $C$. We show that the algebra base $C$ can be described as a quotient of the formal power series ring over $\mathbb{K}$ in finitely many variables. Finally we prove that the formal deformation $\eta$ is a versal deformation of $L$ with base $C$.

## 5.2 Construction of a formal deformation $\eta$

Let $L$ be a Leibniz algebra satisfying $dim(HL^2(L;L)) < \infty$. As in Chapter 3, we denote $HL^2(L;L)$ by $\mathbb{H}$.

Set $C_0 = \mathbb{K}$ and $C_1 = \mathbb{K} \oplus \mathbb{H}'$. As in Section 3.4, we consider the algebra $C_1$ as a finite dimensional extension of $\mathbb{K}$ given by the trivial extension

$$0 \longrightarrow \mathbb{H}' \xrightarrow{i_1} C_1 \xrightarrow{p_1} C_0 \longrightarrow 0,$$

where the multiplication in $C_1$ is defined by

$$(k_1, h_1)(k_2, h_2) = (k_1 k_2\ ,\ k_1 h_2 + k_2 h_1) \ \text{ for } (k_1, h_1), (k_2, h_2) \in C_1.$$

Let $\eta_1$ be the universal infinitesimal deformation of $L$ with base $C_1$ as constructed in Section 3.4. We proceed by induction. Suppose for some $k \geq 1$ we have constructed a finite dimensional local algebra $C_k$ in $\mathcal{C}$ and a deformation $\eta_k$ of $L$ with base $C_k$ such that $p_{k*}\eta_k = \eta_{k-1}$.

Consider a linear map

$$\mu_k : H^2_{Harr}(C_k; \mathbb{K}) \longrightarrow Hom(S^2 C_k; \mathbb{K}) = (S^2 C_k)'$$

where $\mu_k$ takes a cohomology class $[\psi]$ to a cocycle representing it. Such a linear map can be obtained by fixing its values on a basis of $H^2_{Harr}(C_k; \mathbb{K})$ and then extending it linearly (as in (2.3.2) ). Let the dual map of $\mu_k$ be

$$f_k : S^2 C_k \longrightarrow H^2_{Harr}(C_k; \mathbb{K})'.$$

We have seen that $f_k$ is a cocycle and represents a cohomology class in the second Harrison cohomology of $C_k$ with coefficients in $H^2_{Harr}(C_k; \mathbb{K})'$. Therefore by Proposition 2.3.3, the cohomology class of $f_k$ corresponds to an isomorphism class of extensions of $C_k$ represented by an extension

$$0 \longrightarrow H^2_{Harr}(C_k; \mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}_{k+1}} C_k \longrightarrow 0. \tag{5.2.1}$$

For this extension we now consider the problem of extending the deformation $\eta_k$ of $L$ with base $C_k$ as discussed in Chapter 4. As in (4.2.21) the associated obstruction is $\theta_k([f_k]) \in H^2_{Harr}(C_k; \mathbb{K})' \otimes HL^3(L; L)$. By Proposition 3.4.1, $\theta_k([f_k])$ gives a linear map

$$\omega_k : H^2_{Harr}(C_k; \mathbb{K}) \longrightarrow HL^3(L; L) \tag{5.2.2}$$

with the dual map

$$\omega_k{}' : HL^3(L; L)' \longrightarrow H^2_{Harr}(C_k; \mathbb{K})'.$$

Thus to get an extension of $\eta_k$ we modify the extension (5.2.1) to a new extension of $C_k$ for which obstruction vanishes.

We take the quotient module $coker(\omega'_k)$ of $H^2_{Harr}(C_k; \mathbb{K})'$ and obtain an induced extension of the algebra $C_k$ by $coker(\omega'_k)$ as in the commutative diagram in Figure 5.1, where the vertical arrows are projection maps.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2_{Harr}(C_k;\mathbb{K})' & \xrightarrow{\bar{i}_{k+1}} & \bar{C}_{k+1} & \xrightarrow{\bar{p}_{k+1}} & C_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \frac{H^2_{Harr}(C_k;\mathbb{K})'}{\omega_k'(HL^3(L;L)')} & \longrightarrow & \frac{\bar{C}_{k+1}}{\bar{i}_{k+1}\circ\omega_k'(HL^3(L;L)')} & \longrightarrow & C_k & \longrightarrow & 0
\end{array}$$

Figure 5.1:

This yields a new extension

$$0 \longrightarrow coker(\omega_k') \longrightarrow \bar{C}_{k+1}/\bar{i}_{k+1} \circ \omega_k'(HL^3(L;L)') \longrightarrow C_k \longrightarrow 0.$$

Observe that $coker(\omega_k') \cong (ker(\omega_k))'$, so it induces an extension

$$0 \longrightarrow (ker(\omega_k))' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p_{k+1}} C_k \longrightarrow 0 \tag{5.2.3}$$

where $C_{k+1} = \bar{C}_{k+1}/\bar{i}_{k+1} \circ \omega_k'(HL^3(L;L)')$ and $i_{k+1}$, $p_{k+1}$ are the mappings induced by $\bar{i}_{k+1}$ and $\bar{p}_{k+1}$, respectively. Since $C_k$ is finite dimensional, the cohomology module $H^2_{Harr}(C_k;\mathbb{K})$ is also finite dimensional and hence by Proposition 2.3.5, $C_{k+1}$ is in $\mathcal{C}$ as well.

**Remark 5.2.1.** *It follows from Proposition 2.3.4 that the specific extension (5.2.1) has the following "universal property". For any $C_k$-module $M$ with $\mathfrak{M}M = 0$, (5.2.1) admits a unique homomorphism into an arbitrary extension of $C_k$:*

$$0 \longrightarrow M \longrightarrow B \longrightarrow C_k \longrightarrow 0.$$

As a consequence we get the following result.

**Proposition 5.2.2.** *The deformation $\eta_k$ with base $C_k$ of the Leibniz algebra $L$ admits an extension to a deformation with base $C_{k+1}$, which is unique up to an isomorphism and an automorphism of the extension*

$$0 \longrightarrow (ker(\omega_k))' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p_{k+1}} C_k \longrightarrow 0.$$

*Proof.* From the above construction of the extension (5.2.3) it is clear that the corresponding obstruction map is the restriction of $\omega_k$ to the submodule $ker(\omega_k)$ and is given by

$$\omega_k|_{ker(\omega_k)} : ker(\omega_k) \longrightarrow HL^3(L;L).$$

Hence, it is the zero map. Thus the result follows from Theorem 4.2.6. $\square$

Therefore we get a new deformation $\eta_{k+1}$ of $L$ with base $C_{k+1}$ extending the deformation $\eta_k$ endowed with the projection map $p_{k+1} : C_{k+1} \longrightarrow C_k$ such that $p_{k+1*}\eta_{k+1} = \eta_k$. Using induction on $k$, the above process yields a sequence of finite dimensional local algebras $C_k$ and deformations $\eta_k$ of the Leibniz algebra $L$ with base $C_k$

$$\mathbb{K} \xleftarrow{p_1} C_1 \xleftarrow{p_2} C_2 \xleftarrow{p_3} \ldots\ldots \xleftarrow{p_k} C_k \xleftarrow{p_{k+1}} C_{k+1} \ldots$$

such that $p_{k+1*}\eta_{k+1} = \eta_k$. Thus by taking the projective limit we obtain a formal deformation $\eta$ of $L$ with base $C = \varprojlim_{k\to\infty} C_k$.

## 5.3 Versality property of $\eta$

Suppose $dim(\mathbb{H}) = n$. Let $\{h_i\}_{1\leq i\leq n}$ be a basis of $\mathbb{H}$ and $\{g_i\}_{1\leq i\leq n}$ be the corresponding dual basis. Let $\mathbb{K}[[\mathbb{H}']]$ denote the formal power series ring $\mathbb{K}[[g_1, \ldots, g_n]]$ over $\mathbb{K}$ in $n$ variables $g_1, \ldots, g_n$. A typical element in $\mathbb{K}[[\mathbb{H}']]$ is of the form

$$\sum_{i=0}^{\infty} a_i\alpha_i(g_1, \ldots, g_n) = a_0 + a_1\alpha_1(g_1, \ldots, g_n) + a_2\alpha_2(g_1, \ldots, g_n) + \cdots ,$$

where $a_i \in \mathbb{K}$ and $\alpha_i$ is a monomial of degree $i$ in $n$-variables $g_1, \ldots, g_n$ for $i = 0, 1, 2, \ldots$. Let $\mathfrak{M}$ denote the unique maximal ideal in $\mathbb{K}[[\mathbb{H}']]$, consisting of all elements in $\mathbb{K}[[\mathbb{H}']]$ with constant term being equal to zero.

The next result gives a description of the finite dimensional local algebras $C_k$, for $k \geq 1$ constructed above.

**Proposition 5.3.1.** *For $k \geq 1$, $C_k \cong \mathbb{K}[[\mathbb{H}']]/I_k$ for some ideal $I_k$, satisfying $\mathfrak{M}^2 = I_1 \supset I_2 \supset \ldots \supset I_k \supset \mathfrak{M}^{k+1}$ and the maximal ideal of $C_k$ is $\mathfrak{M}/I_k$.*

*Proof.* By construction of the universal infinitesimal deformation $\eta_1$, the algebra base $C_1 = \mathbb{K} \oplus \mathbb{H}' \cong \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^2$. Take $I_1 = \mathfrak{M}^2 \supset \mathfrak{M}^3$. Clearly, the maximal ideal of $C_1$ is $\mathfrak{M}/\mathfrak{M}^2 = \mathfrak{M}/I_1$.

We use induction on $k$ to prove the result. Suppose we already know that $C_k \cong \mathbb{K}[[\mathbb{H}']]/I_k$ where $\mathfrak{M}^2 \supset I_k \supset \mathfrak{M}^{k+1}$.

Then taking $A = \mathbb{K}[[\mathbb{H}']]$ and $I = I_k$ in Proposition 2.3.8 , we get

$$H^2_{Harr}(C_k; \mathbb{K}) \cong (I_k/\mathfrak{M}I_k)'.$$

Then the algebra $\bar{C}_{k+1}$ which is the extension of $C_k$ by $H^2_{Harr}(C_k; \mathbb{K})'$ can be written as

$$\bar{C}_{k+1} \cong \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}I_k, \text{ by Proposition 2.3.9.}$$

By construction in the previous section, $C_{k+1}$ is the quotient of $\bar{C}_{k+1}$ by an ideal contained in $H^2_{Harr}(C_k;\mathbb{K})' \cong I_k/\mathfrak{M}I_k \subset \mathfrak{M}^2/\mathfrak{M}I_k$. Let $I_{k+1}/\mathfrak{M}I_k$ be the ideal by which we take quotient of $\bar{C}_{k+1}$ to get $C_{k+1}$. So $I_{k+1}/\mathfrak{M}I_k \subset I_k/\mathfrak{M}I_k \subset \mathfrak{M}^2/\mathfrak{M}I_k$.

Therefore $C_{k+1} = \bar{C}_{k+1}/(I_{k+1}/\mathfrak{M}I_k) \cong \mathbb{K}[[\mathbb{H}']]/I_{k+1}$ where $\mathfrak{M}^2 \supset I_{k+1} \supset \mathfrak{M}I_k \supset \mathfrak{M}^{k+2}$. Then the maximal ideal of $C_{k+1}$ is $\mathfrak{M}/I_{k+1}$. The proof is now complete by induction. □

**Remark 5.3.2.** *In Proposition 2.3.9, the projection map* $p : A/\mathfrak{M}I \longrightarrow A/I$ *is induced by the inclusion* $\mathfrak{M}I \hookrightarrow I$. *So the projection map* $p_{k+1} : C_{k+1} \longrightarrow C_k$ *for* $k \geq 1$ *discussed in the Proposition 5.3.1 is given by the natural map* $p_{k+1}(f + I_{k+1}) = f + I_k$.

**Corollary 5.3.3.** *For* $k \geq 2$ *the projection* $p_k : C_k \longrightarrow C_{k-1}$ *induces an isomorphism* $(\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k})' \longrightarrow (\frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}})'$. *Moreover, the differential* $d\eta_k : (\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k})' \longrightarrow \mathbb{H}$ *is the identity map.*

*Proof.* By Proposition 5.3.1, we have $C_0 = \mathbb{K}$ ; $C_1 = (\mathbb{K} \oplus \mathbb{H}') \cong \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^2$ and for $k \geq 2$, $C_k \cong \mathbb{K}[[\mathbb{H}']]/I_k$ where $\mathfrak{M}^2 = I_1 \supset I_2 \supset \ldots \supset I_k \supset \mathfrak{M}^{k+1}$.

By Remark 5.3.2, the projection $p_k : C_k \longrightarrow C_{k-1}$ is given by $p_k(f + I_k) = f + I_{k-1}$ for $f \in C_k$ and $k \geq 1$. The map $p_k$ gives rise to a surjective linear map $p_k|_{\mathfrak{M}/I_k} : \mathfrak{M}/I_k \longrightarrow \mathfrak{M}/I_{k-1}$ by restriction of $p_k$ on the maximal ideal $\mathfrak{M}/I_k$ of $C_k$. Consider the quotient map $q_k : \mathfrak{M}/I_{k-1} \longrightarrow \frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}$. Thus $q_k \circ (p_k|_{\mathfrak{M}/I_k}) : \mathfrak{M}/I_k \longrightarrow \frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}$ is a surjective linear map with kernel $\mathfrak{M}^2/I_k$. Consequently we get an isomorphism

$$\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k} \longrightarrow \frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}.$$

As a result we get the desired isomorphism

$$\left(\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k}\right)' \longrightarrow \left(\frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}\right)'.$$

In particular any $k \geq 1$ we get $\left(\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k}\right)' = \left(\frac{\mathfrak{M}}{\mathfrak{M}^2}\right)'$. Observe that $C_1 = \mathbb{K} \oplus \mathbb{H}'$ has the maximal ideal $\mathbb{H}'$ with $(\mathbb{H}')^2 = 0$. Hence $(\frac{\mathfrak{M}}{\mathfrak{M}^2})' = (\mathbb{H}')' = \mathbb{H}$.

By definition, the differential $d\eta_k : \left(\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k}\right)' = \left(\frac{\mathfrak{M}}{\mathfrak{M}^2}\right)' \longrightarrow \mathbb{H}$ is given by

$$d\eta_k(h_i) = a_{\pi_{k*}\eta_k}(h_i) = a_{\pi_{k*}\eta_k,h_i} = [\alpha_{\pi_{k*}\eta_k,h_i}], \text{ the cohomology class of } \alpha_{\pi_{k*}\eta_k,h_i},$$

where $\pi_k : C_k \longrightarrow C_k/(\mathfrak{M}^2/I_k)$ is the canonical projection. Now $p_{k*}\eta_k = \eta_{k-1}$ for $k \geq 1$ and $p(h_i + I_k) = h_i + I_{k-1}$. Thus from (3.3.2) and (3.4.1) it follows that $\alpha_{\eta_k,h_i} = \alpha_{\eta_{k-1},h_i} = \cdots = \alpha_{\eta_1,h_i} = \mu(h_i)$ for $1 \leq i \leq n$. Also we have $\alpha_{\pi_{k*}\eta_k,h_i} = \alpha_{\eta_k,h_i} = \mu(h_i)$ for $i \leq i \leq n$.

Therefore $d\eta_k(h_i) = [\alpha_{\pi_{k*}\eta_k,h_i}] = [\mu(h_i)] = h_i$. □

**Proposition 5.3.4.** *The complete local algebra* $C = \varprojlim_{k\to\infty} C_k$ *can be described as* $C \cong \mathbb{K}[[\mathbb{H}']]/I$*, where* $I$ *is an ideal contained in* $\mathfrak{M}^2$ *and the maximal ideal in* $C$ *is* $\mathfrak{M}/I$.

*Proof.* Consider the map

$$\phi : \mathbb{K}[[\mathbb{H}']] \longrightarrow C_k = \mathbb{K}[[\mathbb{H}']]/I_k \ \text{ defined by } \phi(f) = f + I_k \ \text{ for } \ f \in \mathbb{K}[[\mathbb{H}']].$$

Since $I_k \supset \mathfrak{M}^{k+1}$, the map $\phi$ induces a surjective linear map

$$\phi_k : \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^{k+1} \longrightarrow C_k = \mathbb{K}[[\mathbb{H}']]/I_k \ \text{ for each } k \geq 1.$$

In the limit we get a surjective linear map

$$\hat{\phi} : \mathbb{K}[[\mathbb{H}']] = \varprojlim_{k\to\infty} \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^{k+1} \longrightarrow \varprojlim_{k\to\infty} C_k = C.$$

given by $\hat{\phi}(f_1+\mathfrak{M}^2, f_2+\mathfrak{M}^3, \cdots) = (\phi_1(f_1+\mathfrak{M}^2), \phi_2(f_2+\mathfrak{M}^3), \cdots) = (f_1+I_1, f_2+I_2, \cdots)$. Therefore we get an isomorphism $C \cong \mathbb{K}[[\mathbb{H}']]/ker(\hat{\phi})$.

Let $I$ denote the ideal $ker(\hat{\phi}) = \bigcap_k I_k \subset \mathfrak{M}^2$. So $C \cong \mathbb{K}[[\mathbb{H}']]/I$. Clearly the maximal ideal of $C$ is $\mathfrak{M}/I$. □

Finally we prove the versality property of the formal deformation $\eta$ with base $C$. We need the following standard lemma.

**Lemma 5.3.5.** *Suppose* $0 \longrightarrow M_r \xrightarrow{i} B_r \xrightarrow{p} A \longrightarrow 0$ *is an* $r$*-dimensional extension of* $A$*. Then this extension yields an* $(r-1)$*-dimensional extension*

$$0 \longrightarrow M_{r-1} \xrightarrow{\bar{i}} B_{r-1} \xrightarrow{\bar{p}} A \longrightarrow 0$$

*of* $A$ *and a* 1*-dimensional extension*

$$0 \longrightarrow \mathbb{K} \longrightarrow B_r \longrightarrow B_{r-1} \longrightarrow 0.$$

*Proof.* Let $\{x_i\}_{1\leq i\leq r}$ be a basis of $M_r$. We take $M_{r-1} = M_r/ < x_r >$ and $B_{r-1} = B_r/i(< x_r >)$, where $< x_r >$ denotes the 1-dimensional submodule over $\mathbb{K}$ spanned by $x_r$. Then we get an $(r-1)$-dimensional extension $0 \longrightarrow M_{r-1} \xrightarrow{\bar{i}} B_{r-1} \xrightarrow{\bar{p}} A \longrightarrow 0$, where $\bar{i}$ and $\bar{p}$ are maps induced by $i$ and $p$ respectively. If $p' : B_r \longrightarrow B_{r-1}$ is the quotient map then the 1-dimensional extension in question is

$$0 \longrightarrow < x_r > \longrightarrow B_r \xrightarrow{p'} B_{r-1} \longrightarrow 0.$$

□

**Theorem 5.3.6.** *Let $L$ be a Leibniz algebra with $dim(\mathbb{H}) < \infty$. Then the formal deformation $\eta$ with base $C$ constructed in Section 5.2, is a versal deformation of $L$.*

*Proof.* Suppose $dim(\mathbb{H}) = n$. Let $\{h_i\}_{1\leq i\leq n}$ be a basis of $\mathbb{H}$ and $\{g_i\}_{1\leq i\leq n}$ the corresponding dual basis of $\mathbb{H}'$.

Let $A$ be a complete local algebra with maximal ideal $\mathfrak{M}$ and $\lambda$ be a formal deformation of $L$ with base $A$. We want to find a $\mathbb{K}$-algebra homomorphism $\phi : C \longrightarrow A$ such that the deformation $\lambda$ is equivalent to the push-out $\phi_*\eta$ of the deformation $\eta$ via the map $\phi$.

Denote $A_0 = A/\mathfrak{M} \cong \mathbb{K}$ and $A_1 = A/\mathfrak{M}^2 \cong \mathbb{K} \oplus (\mathfrak{M}/\mathfrak{M}^2)$. Since $A$ is a complete local algebra, we have $A = \varprojlim_{k\to\infty} A/\mathfrak{M}^k$. Moreover, for each $k \geq 1$ we have the following finite dimensional extension

$$0 \longrightarrow \frac{\mathfrak{M}^k}{\mathfrak{M}^{k+1}} \stackrel{i}{\longrightarrow} \frac{A}{\mathfrak{M}^{k+1}} \stackrel{p}{\longrightarrow} \frac{A}{\mathfrak{M}^k} \longrightarrow 0 \tag{5.3.1}$$

because $dim(\frac{\mathfrak{M}^k}{\mathfrak{M}^{k+1}}) < \infty$.

Let $n_k = dim(\frac{\mathfrak{M}^k}{\mathfrak{M}^{k+1}})$. A repeated application of Lemma 5.3.5 to the extension

$$0 \longrightarrow \frac{\mathfrak{M}^2}{\mathfrak{M}^3} \longrightarrow \frac{A}{\mathfrak{M}^3} \longrightarrow \frac{A}{\mathfrak{M}^2} = A_1 \longrightarrow 0$$

yields $n_1$ number of 1-dimensional extensions as follows.

$$\begin{gathered} 0 \longrightarrow \mathbb{K} \longrightarrow A_2 \longrightarrow A_1 \longrightarrow 0 \\ 0 \longrightarrow \mathbb{K} \longrightarrow A_3 \longrightarrow A_2 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow \mathbb{K} \longrightarrow A_{n_1+1} = \frac{A}{\mathfrak{M}^3} \longrightarrow A_{n_1} \longrightarrow 0. \end{gathered}$$

Similarly, the extension

$$0 \longrightarrow \frac{\mathfrak{M}^3}{\mathfrak{M}^4} \longrightarrow \frac{A}{\mathfrak{M}^4} \longrightarrow \frac{A}{\mathfrak{M}^3} = A_{n_1+1} \longrightarrow 0$$

splits into $n_2$ number of 1-dimensional extensions and so on. Consequently, we get a sequence of 1- dimensional extensions

$$0 \longrightarrow \mathbb{K} \stackrel{j_{k+1}}{\longrightarrow} A_{k+1} \stackrel{q_{k+1}}{\longrightarrow} A_k \longrightarrow 0 \quad ; \ k \geq 1.$$

Since $A = \varprojlim_{k\to\infty} A/\mathfrak{M}^k$, it follows that $A = \varprojlim_{k\to\infty} A_k$. Let $Q_k : A \longrightarrow A_k$ be the projection map for the inverse system $\{A_k, q_k\}_{k\geq 1}$ with the limit $A$, where $Q_1 : A \longrightarrow A_1 = A/\mathfrak{M}^2$

is the natural projection. Let $Q_{k*}\lambda = \lambda_k$, then $\lambda_k$ is a deformation of $L$ with base $A_k$. Thus $\lambda_k = Q_{k*}\lambda = (q_{k+1} \circ Q_{k+1})_*\lambda = q_{k+1*}\lambda_{k+1}$. Now we will construct inductively homomorphisms $\phi_j : C_j \longrightarrow A_j$ for $j = 1, 2 \ldots$, compatible with the corresponding projections $C_{j+1} \longrightarrow C_j$ and $A_{j+1} \longrightarrow A_j$, along with the conditions $\phi_{j*}\eta_j \cong \lambda_j$. Define $\phi_1 : C_1 \longrightarrow A_1$ as

$$(id \oplus (d\lambda)') : \mathbb{K} \oplus \mathbb{H}' \longrightarrow \mathbb{K} \oplus (\mathfrak{M}/\mathfrak{M}^2).$$

From Theorem 3.4.11, we have $\phi_{1*}\eta_1 \cong \lambda_1$.

Suppose we have constructed a $\mathbb{K}$-algebra homomorphism $\phi_k : C_k \longrightarrow A_k$ with $\phi_{k*}\eta_k \cong \lambda_k$. Consider the homomorphism $\phi_k^* : H^2_{Harr}(A_k; \mathbb{K}) \longrightarrow H^2_{Harr}(C_k; \mathbb{K})$ induced by $\phi_k$. Let

$$0 \longrightarrow \mathbb{K} \stackrel{i_{k+1}}{\longrightarrow} B \stackrel{p_{k+1}}{\longrightarrow} C_k \longrightarrow 0$$

represent the image under $\phi_k^*$ of the isomorphism class of the extension (Proposition 2.3.3)

$$0 \longrightarrow \mathbb{K} \stackrel{j_{k+1}}{\longrightarrow} A_{k+1} \stackrel{q_{k+1}}{\longrightarrow} A_k \longrightarrow 0.$$

Then we have a commutative diagram

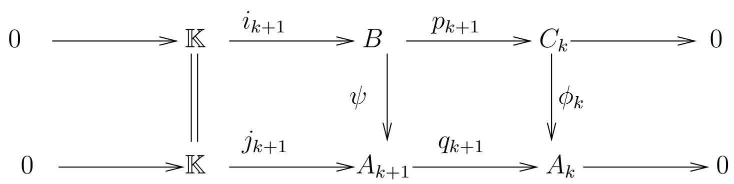


Figure 5.2:

where $\psi$ is given by $\psi((x,k)_q) = (\phi_k(x), k)_{q'}$ for some fixed sections $q$ and $q'$ of $p_{k+1}$ and $q_{k+1}$ respectively. Observe that by Proposition 4.3.4, the obstructions in extending $\lambda_k$ to the base $A_{k+1}$ and that of $\eta_k$ to the base $B$ coincide. Since $\lambda_k$ has an extension $\lambda_{k+1}$, the corresponding obstruction is zero. Hence there exists a deformation $\xi$ of $L$ with base $B$ which extends $\eta_k$ with base $C_k$ such that $\psi_*\xi = \lambda_{k+1}$. By Remark 5.2.1 we get a unique homomorphism of extensions given by the commutative diagram in Figure 5.3. Since the deformation $\eta_k$ has been extended to $B$, the obstruction map

$$\omega_k : H^2_{Harr}(C_k; \mathbb{K}) \longrightarrow HL^3(L; L)$$

is zero and hence $\omega'_k$ is also the zero map.

Therefore the composition $\tau' \circ \omega'_k : HL^3(L; L)' \longrightarrow \mathbb{K}$ is zero. Consequently, $\tau'$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2_{Harr}(C_k;\mathbb{K})' & \xrightarrow{\bar{i}_{k+1}} & \bar{C}_{k+1} & \xrightarrow{\bar{p}_{k+1}} & C_k & \longrightarrow & 0 \\ & & \downarrow{\tau'} & & \downarrow{\bar{\chi}} & & \| & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0 \end{array}$$

Figure 5.3:

induces a linear map

$$\tau : H^2_{Harr}(C_k;\mathbb{K})'/\omega'_k(HL^3(L;L)') \longrightarrow \mathbb{K}.$$

Moreover the map $\bar{\chi} : \bar{C}_{k+1} \longrightarrow B$ induces a linear map

$$\chi : C_{k+1} = \bar{C}_{k+1}/\bar{i}_{k+1} \circ \omega'_k(HL^3(L;L)') \longrightarrow B.$$

Since $coker(\omega'_k) \cong (ker(\omega_k))'$, the last diagram yields the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (ker(\omega_k))' & \longrightarrow & C_{k+1} & \longrightarrow & C_k & \longrightarrow & 0 \\ & & \downarrow{\tau} & & \downarrow{\chi} & & \| & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0 \end{array}$$

Figure 5.4:

By Corollary 5.3.3, the differential

$$d\eta_k : \left(\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k}\right)' \longrightarrow \mathbb{H}$$

is onto, so by Corollary 4.3.2, the deformations $\chi_*\eta_{k+1}$ and $\xi$ are related by some automorphism $u : B \longrightarrow B$ of the extension

$$0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow C_k \longrightarrow 0$$

with $u_*(\chi_*\eta_{k+1}) = \xi$. Now set $\phi_{k+1} = (\psi \circ u \circ \chi) : C_{k+1} \longrightarrow A_{k+1}$, where $\psi$ is as in Figure 5.2. As push-out is preserved under composition of maps (Proposition 3.3.5) we get

$$\phi_{k+1*}\eta_{k+1} = \psi_* \circ u_* \circ \chi_*\eta_{k+1} = \psi_*\xi = \lambda_{k+1}.$$

Thus by induction we get a sequence of homomorphisms $\phi_k : C_k \longrightarrow A_k$ with $\phi_{k*}\eta_k = \lambda_k$. Consequently, in the limit, we find a homomorphism $\phi : C \longrightarrow A$ such that $\phi_*\eta = \lambda$. If $\mathfrak{M}^2 = 0$, then the uniqueness of $\phi$ follows from the corresponding property of $\phi_1$ in Theorem 3.4.11. $\square$

# Chapter 6

# Massey Brackets, relation with Obstructions

## 6.1 Introduction

It is well known that the construction of one parameter deformations of various algebraic structures, like associative algebras or Lie algebras, involves certain conditions on cohomology classes, arising as obstructions. This is also mentioned in this thesis for Leibniz algebras in Examples 4.2.5 and 4.4.5. These conditions are expressed in terms of Massey brackets [Ret77, Ret93], which are, in turn, the Lie counterpart of classical Massey products [Mas54]. The connection between obstructions in extending a given deformation and Massey products was first noticed in [Dou61]. The aim of this chapter is to study this relationship in our context.

First, we relate obstructions in extending a finite order one parameter deformation of a Leibniz algebra to Massey brackets of 2-cocycles using the notion of Massey $n$-operations as defined by V.S. Retakh [Ret77]. Recall (Definition 3.4.8) that for $(A, \mathfrak{M}) \in \mathcal{C}$, the differential of a deformation $\lambda$ with base $A$ of a Leibniz algebra $L$ is a linear map, $\big(\mathfrak{M}/\mathfrak{M}^2\big)' \longrightarrow HL^2(L; L)$. A natural question is whether any such linear map is realized as a differential. We prove a necessary and sufficient condition for an arbitrary linear map $a : \big(\mathfrak{M}/\mathfrak{M}^2\big)' \longrightarrow HL^2(L; L)$ to be the differential of a deformation with base $A$. This is done by using a general approach to Massey brackets as introduced in [FW01]. Finally, we express the obstructions arising at different steps in the inductive construction of a versal deformation $\eta$ as discussed in Chapter 5, in terms of these general Massey brackets.

## 6.2 Massey Brackets

Let $L$ be a Leibniz algebra over $\mathbb{K}$ with bracket $[-,-]$. Recall that a formal deformation of $L$ (see Example 4.5.3) is defined as a formal power series

$$[l_1,l_2]_{\lambda_t} = [l_1,l_2] + \sum_{i=1}^{\infty} t^i \lambda_i(l_1,l_2) \text{ for } \lambda_i \in CL^2(L;L) \text{ and } l_1,l_2 \in L,$$

which makes $L[[t]] = L \otimes \mathbb{K}[[t]]$ a Leibniz algebra. The bracket $[-,-]_{\lambda_t}$ satisfies the Leibniz relation is equivalent to the fact that ( [Bal97])

$$\delta\lambda_1 = 0 \text{ and } \delta\lambda_i = \frac{1}{2}\sum_{k=1}^{i-1}[\lambda_k,\lambda_{i-k}] \text{ for } i \geq 2. \tag{6.2.1}$$

Here $[\lambda_k,\lambda_{i-k}]$ denotes the product in the differential graded Lie algebra structure in $(CL^*(L;L),[-,-],d)$ (cf. Proposition 1.3.10).

The first condition $\delta\lambda_1 = 0$ in (6.2.1) means that $\lambda_1$ is a 2-cocycle. Then $[-,-]_{\lambda_t} = [-,-]+t\lambda_1$ is an infinitesimal deformation of $L$ with base $\mathbb{K}[[t]]/(t^2)$. This infinitesimal deformation can be extended to a formal deformation of $L$ with base $\mathbb{K}[[t]]$ if and only if there exist cochains $\lambda_i \in CL^2(L;L)$ such that each $\lambda_i$ satisfies (6.2.1). These conditions can be conveniently expressed by Massey brackets [Ret77] defined on the graded module $HL^*(L;L)$ (cf. Corollary 1.3.11).

Let $M$ be an ordered set of homogeneous elements of $CL^*(L;L)$, and, $P$ and $Q$ be non intersecting ordered set of elements of $M$. Denote by $\varepsilon(P,Q)$ the sum of numbers of form $(|x|+1)(|y|+1)$ such that $x \in P, y \in Q$ and $y$ precedes $x$ in $M$, where $|x|$ denotes the homogeneous degree of $x \in CL^*(L;L)$. The pair $P,Q$ is called proper if the minor element of $P$ precedes the minor element of $Q$ and $P \cup Q = M$.

**Definition 6.2.1.** *Let $y_i = [x_i] \in HL^*(L;L)$, $1 \leq i \leq n$. We say that the Massey operation $< y_1,\cdots,y_n >$ is defined for the elements $y_i$, if for $m < n$ and for any set $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ there exist elements $x_{i_1\cdots i_m}$ such that for each ordered set $I = (x_{i_1},\cdots,x_{i_m})$ we have $dx_{i_1\cdots i_m} = \sum(-1)^{\varepsilon(J,K)}[\bar{x}_{j_1,\cdots j_s}, x_{k_1,\cdots,k_t}]$, where the sum is taken over all proper pairs of the sets $J = (x_{j_1},\cdots,x_{j_s})$, $K = (x_{k_1},\cdots,x_{k_t})$ and $\bar{x} = (-1)^{|x|+1}x$ for $x \in CL^*(L;L)$. The set $\{x_{i_1\cdots i_m}\}$ is called the defining system of $< y_1,\cdots,y_n >$.*

**Proposition 6.2.2.** *[Ret77], The element $\bar{x}_{1\cdots n} = \sum(-1)^{\varepsilon(J,K)}[\bar{x}_{j_1\cdots j_s}, x_{k_1\cdots k_t}]$, where the sum is taken over all proper decompositions of the set $\{x_1,\cdots,x_n\}$, is a cocycle. The cohomology class represented by the cocycle $\bar{x}_{1\cdots n}$ is independent of the choice of different representatives of $\{y_i\}$.*

□

The class in $HL^*(L;L)$ corresponding to $\bar{x}_{1\cdots n}$ is called the value of the operation $< y_1, \cdots, y_n >$. Denote by $[y_1, \cdots, y_n]$ the set of classes constructed for all defining systems.

Mostly we need to consider 2-cocycles $y_i \in HL^2(L;L)$, so that the representative cochains $x_i \in CL^2(L;L)$. Then $|x_i| = 1$ and $\bar{x}_i = x_i$. For $y_i = [x_i] \in HL^2(L;L)$, $i = 1, 2$ it follows from Definition 6.2.1 that $< y_1, y_2 >$ is represented by the 2-cochain $x_{12}$ obtained from graded Lie bracket $[x_1, x_2]$.

Suppose that $y_i \in HL^2(L;L)$, $1 \leq i \leq 3$ such that $< y_i, y_j >= 0$ for every $i$ and $j$. This means that for a cocycle $x_i$ representing $y_i$ we have $[x_i, x_j] = dx_{ij}$ for some 2- cochain $x_{ij}$. Then the third order Massey operation $< y_1, y_2, y_3 >$ is defined and is represented by

$$[x_{12}, x_3] + [x_1, x_{23}] + [x_{13}, x_2].$$

The cohomology class is independent of the choice of $x_{ij}$. The higher order Massey brackets are defined inductively.

Observe that the obstruction cocycle $\bar{\phi}$ in Example 4.2.5 can be written as

$$\begin{aligned} \bar{\phi}(x,y,z) &= \sum_{\substack{i+j=N+1 \\ i,j>0}} \{\lambda_i(\lambda_j(x,y),z) - \lambda_i(\lambda_j(x,z),y) - \lambda_i(x,\lambda_j(y,z))\} \\ &= \frac{1}{2} \sum_{\substack{i+j=N+1 \\ i,j>0}} [\lambda_i, \lambda_j](x,y,z). \end{aligned} \tag{6.2.2}$$

It follows from Definition 6.2.1 that the cohomology class of

$$\sum_{\substack{i+j=N+1 \\ i,j>0}} [\lambda_i, \lambda_j]$$

denotes the $N$th Massey bracket $< [\lambda_1], \cdots, [\lambda_1] >$. Now this obstruction $\bar{\phi}$ represents the 0 class if and only if the $N$th Massey bracket is defined and $[\ [\lambda_1], \cdots, [\lambda_1]\ ]$ ($N$-many) contains the class 0. If $\bar{\phi} = \delta\lambda_{N+1}$ then $\lambda_t = \sum_{i=1}^{N+1} \lambda_i t^i$ is a $(N+1)$th order deformation of $L$ with base $\mathbb{K}[[t]]/(t^{N+2})$.

Using Definition 6.2.1, the conditions in (6.2.1) for the bracket $[-,-]_{\lambda_t}$ is given by the fact that the set $[\ [\lambda_1], \cdots, [\lambda_1]\ ]$ (the bracket contain $i$-many $\lambda_1$) contains 0 for all $i \geq 2$.

**Remark 6.2.3.** *One can establish a similar relationship between Massey brackets and obstructions that arise in extending deformations not merely with one parameter base but with more general base. This connection will be used in the next chapter.*

Next we recall a more general definition of Massey brackets [FW01] to relate it to the obstruction cocycles obtained in the construction in 5.2.

Suppose $(\mathcal{L}, \nu, d)$ is a differential graded Lie algebra, $\nu$ being the bracket and $d$ is the differential on the graded module $\mathcal{L}$ over $\mathbb{K}$. We denote by $\mathcal{H} = \bigoplus_i \mathcal{H}^i$, the cohomology of $\mathcal{L}$ with respect to the differential $d$. For our purpose we consider the differential graded Lie algebra $(CL^*(L; L), \nu, d)$ (cf. Proposition 1.3.10).

Let $F$ be a graded cocommutative coassociative coalgebra, that is a graded module with a degree 0 mapping (comultiplication) $\Delta : F \longrightarrow F \otimes F$ satisfying the conditions $S \circ \Delta = \Delta$ and $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$, where

$$S : F \otimes F \longrightarrow F \otimes F$$

is defined as

$$S(\phi \otimes \psi) = (-1)^{|\phi||\psi|}(\psi \otimes \phi).$$

Suppose also that a filtration $F_0 \subset F_1 \subset F$ is given in $F$, such that $F_0 \subset ker(\Delta)$ and $Im(\Delta) \subset F_1 \otimes F_1$. We need the following result (see [FW01]).

**Proposition 6.2.4.** *Suppose a linear mapping $\alpha : F_1 \longrightarrow \mathcal{L}$ of degree 1 satisfies the condition*

$$d\alpha = \nu \circ (\alpha \otimes \alpha) \circ \Delta. \tag{6.2.3}$$

*Then $\nu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset ker(d)$.*

□

**Definition 6.2.5.** *Let $a : F_0 \longrightarrow \mathcal{H}$, $b : F/F_1 \longrightarrow \mathcal{H}$ be two linear maps of degree 1 and 2 respectively. We say that $b$ is contained in the Massey $F$-bracket of $a$, and write $b \in [a]_F$, or $b \in [a]$, if there exists a degree 1 linear mapping $\alpha : F_1 \longrightarrow \mathcal{L}$ satisfying condition (6.2.3) and such that the diagrams in Figure 6.1 are commutative, where the vertical maps labeled by $\pi$ denote the projections of each module onto the quotient module.*

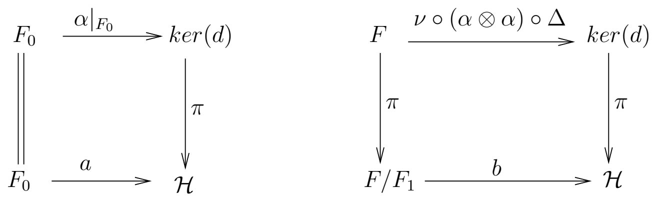


Figure 6.1:

Note that the upper horizontal maps of the above diagrams are well defined, since $\alpha(F_0) \subset \alpha(ker\Delta) \subset ker(d)$ by virtue of (6.2.3), and $\nu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset ker(d)$ by Proposition 6.2.4.

The definition makes sense even if $F_1 = F$. In that case $Hom(F/F_1,\ \mathbb{K}) = 0$, and $[a]_F$ may either be empty or contain 0. In the last case we say that *a satisfies the condition of triviality of Massey F-brackets.*

Let $(A, \mathfrak{M}) \in \mathcal{C}$ be a finite dimensional commutative local algebra with 1. Let $\varepsilon : A \longrightarrow A/\mathfrak{M} \cong \mathbb{K}$ be the canonical augmentation.

Suppose $\rho : (A \otimes L) \times (A \otimes L) \longrightarrow (A \otimes L)$ is an $A$-bilinear operation on $A \otimes L$ ($\rho$ need not satisfy the Leibniz identity) such that $(\varepsilon \otimes id) : A \otimes L \longrightarrow \mathbb{K} \otimes L \cong L$ is a homomorphism with respect to the operation $\rho$ on $A \otimes L$ and the Leibniz bracket $[-,-]$ on $L$. In other words,

$$(\varepsilon \otimes id) \circ \rho(a_1 \otimes l_1, a_2 \otimes l_2) = \varepsilon(a_1 a_2)[l_1, l_2] \text{ for } a_1 \otimes l_1, a_2 \otimes l_2 \in A \otimes L.$$

We prove a necessary and sufficient condition for $\rho$ to be a Leibniz bracket on $A \otimes L$.

Suppose $dim(A) = r + 1$. Choose a basis $\{m_1, m_2, \cdots, m_k, m_{k+1}, \cdots, m_r\}$ of $\mathfrak{M}$ such that $\{\bar{m}_i = m_i + \mathfrak{M}^2\}_{1 \le i \le k}$ is a basis of $\mathfrak{M}/\mathfrak{M}^2$.

Note that for $1 \otimes l_1,\ 1 \otimes l_2 \in A \otimes L$ we have

$$(\varepsilon \otimes id) \circ \rho(1 \otimes l_1, 1 \otimes l_2) = [(\varepsilon \otimes id)(1 \otimes l_1), (\varepsilon \otimes id)(1 \otimes l_2)] = \varepsilon \otimes id(1 \otimes [l_1, l_2]).$$

Therefore

$$\rho(1 \otimes l_1, 1 \otimes l_2) - 1 \otimes [l_1, l_2] \in ker(\varepsilon \otimes id) = ker(\varepsilon) \otimes L = \mathfrak{M} \otimes L. \tag{6.2.4}$$

Hence we can write

$$\rho(1 \otimes l_1, 1 \otimes l_2) = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r} m_i \otimes l_i' \text{ for } l_1, l_2, l_i' \in L. \tag{6.2.5}$$

Any linear map $\phi \in (\mathfrak{M}/\mathfrak{M}^2)'$ can be extended to a linear map $\tilde{\phi} \in \mathfrak{M}'$ by defining $\tilde{\phi}(m_i) = 0$ for $k + 1 \le i \le r$. Denote $F = F_1 = \mathfrak{M}'$ and $F_0 = (\mathfrak{M}/\mathfrak{M}^2)'$. Then we get a filtration $F_0 \subset F_1 = F$. The maximal ideal $\mathfrak{M}$ of $A$ is a commutative associative algebra. So the dual module $\mathfrak{M}' = F = F_1$ is a cocommutative coassociative coalgebra with the comultiplication $\Delta : F \longrightarrow F \otimes F$ being the dual of the multiplication in $\mathfrak{M}$.

Suppose $\phi \in \mathfrak{M}'$ is given, define $\alpha_\phi \in CL^2(L; L)$ by

$$\alpha_\phi(l_1, l_2) = (\phi \otimes id)(\rho(1 \otimes l_1, 1 \otimes l_2) - 1 \otimes [l_1, l_2]) \tag{6.2.6}$$

for $l_1, l_2 \in L$ (cf. (6.2.4)). This gives a linear map

$$\alpha : \mathfrak{M}' \longrightarrow CL^2(L; L) \text{ defined by } \phi \mapsto \alpha_\phi.$$

Suppose $\{m'_i\}_{1\leq i\leq r}$ is the basis of $\mathfrak{M}'$ dual to the basis $\{m_i\}_{1\leq i\leq r}$ of $\mathfrak{M}$. From (6.2.5) it follows that $\alpha_{m'_i}(l_1,l_2) = (m'_i\otimes id)(\rho(1\otimes l_1, 1\otimes l_2) - 1\otimes [l_1,l_2]) = l'_i$.

Thus

$$\rho(1\otimes l_1, 1\otimes l_2) = 1\otimes [l_1,l_2] + \sum_{i=1}^{r} m_i\otimes \alpha_{m'_i}(l_1,l_2) \ \text{ for } l_1,l_2\in L.$$

So, $\rho$ and $\alpha$ determine each other.

We now consider the differential graded Lie algebra $(CL^*(L;L),\nu,d)$ (see Proposition 1.3.10) and get the following result.

**Proposition 6.2.6.** *The operation $\rho$ satisfies the Leibniz relation on $A\otimes L$ if and only if the map $\alpha$ satisfies the equation $d\alpha - \frac{1}{2}\nu\circ(\alpha\otimes\alpha)\circ\Delta = 0$.*

*Proof.* Let $1\otimes l_1, 1\otimes l_2\in A\otimes L$, then from (6.2.5) we write

$$\rho(1\otimes l_1, 1\otimes l_2) = 1\otimes [l_1,l_2] + \sum_{i=1}^{r} m_i\otimes \psi_i(l_1,l_2) \qquad (6.2.7)$$

where $\psi_i\in CL^2(L;L)$ is given by $\psi_i = \alpha_{m'_i}$. For $1\otimes l_1, 1\otimes l_2, 1\otimes l_3\in A\otimes L$ we have

$$\begin{aligned}
&\rho(1\otimes l_1, \rho(1\otimes l_2, 1\otimes l_3))\\
&= \rho(1\otimes l_1, 1\otimes [l_2,l_3] + \sum_{i=1}^{r} m_i\otimes\psi_i(l_2,l_3)) \text{ (by (6.2.7))}\\
&= \rho(1\otimes l_1, 1\otimes [l_2,l_3]) + \sum_{i=1}^{r} m_i\rho(1\otimes l_1, 1\otimes \psi_i(l_2,l_3))\\
&= 1\otimes [l_1,[l_2,l_3]] + \sum_{i=1}^{r} m_i\otimes\psi_i(l_1,[l_2,l_3]) + \sum_{i=1}^{r} m_i\otimes [l_1,\psi_i(l_2,l_3)]\\
&\qquad + \sum_{1\leq i,j\leq r} m_im_j\otimes\psi_j(l_1,\psi_i(l_2,l_3)) \text{ (by (6.2.7))}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\rho(\rho(1\otimes l_1, 1\otimes l_2), 1\otimes l_3)\\
&= 1\otimes [[l_1,l_2],l_3] + \sum_{i=1}^{r} m_i\otimes\psi_i([l_1,l_2],l_3) + \sum_{i=1}^{r} m_i\otimes [\psi_i(l_1,l_2),l_3]\\
&\qquad + \sum_{1\leq i,j\leq r} m_im_j\otimes\psi_j(\psi_i(l_1,l_2),l_3)
\end{aligned}$$

and

$$\begin{aligned}&\rho(\rho(1\otimes l_1,1\otimes l_3),1\otimes l_2)\\&=1\otimes[[l_1,l_3],l_2]+\sum_{i=1}^{r}m_i\otimes\psi_i([l_1,l_3],l_2)+\sum_{i=1}^{r}m_i\otimes[\psi_i(l_1,l_3),l_2]\\&\qquad+\sum_{1\le i,j\le r}m_im_j\otimes\psi_j(\psi_i(l_1,l_3),l_2).\end{aligned}$$

For a linear map $\phi:\mathfrak{M}\longrightarrow\mathbb{K}$, let $\phi(m_i)=x_i\in\mathbb{K}$. Then by (6.2.6) and (6.2.7) we get

$$\alpha_\phi(l_1,l_2)=(\phi\otimes id)\left(\sum_i m_i\otimes\psi_i(l_1,l_2)\right)=\sum_i x_i\otimes\psi_i(l_1,l_2)=1\otimes\left(\sum_i x_i\psi_i\right)(l_1,l_2). \tag{6.2.8}$$

This shows that $\alpha_\phi$ can be expressed as $\sum_i x_i\psi_i$.

Let $\Delta(\phi)=\sum_p\xi_p\otimes\eta_p$ for some $\xi_p,\eta_p\in\mathfrak{M}'$. We set $\xi_p(m_i)=\xi_{p,i}$ and $\eta_p(m_i)=\eta_{p,i}$. Thus

$$\phi(m_i\ m_j)=\Delta(\phi)(m_i\otimes m_j)=\left(\sum_p\xi_p\otimes\eta_p\right)(m_i\otimes m_j)=\sum_p\xi_{p,i}\ \eta_{p,j}.$$

Now

$$\begin{aligned}&(\phi\otimes id)\left(\sum_{1\le i,j\le r}m_im_j\otimes\psi_j(l_1,\psi_i(l_2,l_3))\right)\\&=\sum_{1\le i,j\le r}\phi(m_im_j)\otimes\psi_j(l_1,\psi_i(l_2,l_3)\\&=\sum_{1\le i,j\le r}\left(\sum_p\xi_{p,i}\ \eta_{p,j}\right)\ \psi_j(l_1,\psi_i(l_2,l_3))\\&=\sum_p\left(\sum_{i=1}^{r}\xi_{p,i}\left(\sum_{j=1}^{r}\eta_{p,j}\psi_j(l_1,\psi_i(l_2,l_3))\right)\right)\\&=\sum_p\left(\sum_{i=1}^{r}\xi_{p,i}\alpha_{\eta_p}(l_1,\psi_i(l_2,l_3))\right)\\&=\sum_p\alpha_{\eta_p}\left(l_1,\sum_{i=1}^{r}\xi_{p,i}\psi_i(l_2,l_3)\right)\\&=\sum_p\alpha_{\eta_p}(l_1,\alpha_{\xi_p}(l_2,l_3)).\end{aligned}$$

Therefore

$$\begin{aligned}
&(\phi\otimes id)(\rho(1\otimes l_1,\rho(1\otimes l_2,1\otimes l_3)))\\
&=\sum_{i=1}^{r}\phi(m_i)\otimes\psi_i(l_1,[l_2,l_3])+\sum_{i=1}^{r}\phi(m_i)\otimes[l_1,\psi_i(l_2,l_3)]\\
&\qquad\qquad+\sum_p\alpha_{\eta_p}(l_1,\alpha_{\xi_p}(l_2,l_3))\\
&=\alpha_\phi(l_1,[l_2,l_3])+[l_1,\alpha_\phi(l_2,l_3)]+\sum_p\alpha_{\eta_p}(l_1,\alpha_{\xi_p}(l_2,l_3)).
\end{aligned} \tag{6.2.9}$$

Similarly

$$\begin{aligned}
&(\phi\otimes id)(\rho(\rho(1\otimes l_1,1\otimes l_2),1\otimes l_3))\\
&=\alpha_\phi([l_1,l_2],l_3)+[\alpha_\phi(l_1,l_2),l_3]+\sum_p\alpha_{\eta_p}(\alpha_{\xi_p}(l_1,l_2),l_3)
\end{aligned} \tag{6.2.10}$$

and

$$\begin{aligned}
&(\phi\otimes id)(\rho(\rho(1\otimes l_1,1\otimes l_3),1\otimes l_2))\\
&=\alpha_\phi([l_1,l_3],l_2)+[\alpha_\phi(l_1,l_3),l_2]+\sum_p\alpha_{\eta_p}(\alpha_{\xi_p}(l_1,l_3),l_2).
\end{aligned} \tag{6.2.11}$$

Hence by substituting (6.2.9)-(6.2.11), we get

$$\begin{aligned}
&(\phi\otimes id)(\rho(1\otimes l_1,\rho(1\otimes l_2,1\otimes l_3))-\rho(\rho(1\otimes l_1,1\otimes l_2),1\otimes l_3)\\
&\qquad\qquad+\rho(\rho(1\otimes l_1,1\otimes l_3),1\otimes l_2))\\
&=\alpha_\phi(l_1,[l_2,l_3])+[l_1,\alpha_\phi(l_2,l_3)]-\alpha_\phi([l_1,l_2],l_3)-[\alpha_\phi(l_1,l_2),l_3]\\
&\qquad+\alpha_\phi([l_1,l_3],l_2)+[\alpha_\phi(l_1,l_3),l_2]+\sum_p\{\alpha_{\eta_p}(l_1,\alpha_{\xi_p}(l_2,l_3))\\
&\qquad-\alpha_{\eta_p}(\alpha_{\xi_p}(l_1,l_2),l_3)+\alpha_{\eta_p}(\alpha_{\xi_p}(l_1,l_3),l_2)\}\\
&=\delta\alpha_\phi(l_1,l_2,l_3)+\frac{1}{2}\sum_p[\alpha_{\eta_p},\alpha_{\xi_p}](l_1,l_2,l_3)\\
&=(-d\alpha+\frac{1}{2}\nu\circ(\alpha\otimes\alpha)\circ\Delta)\phi(l_1,l_2,l_3).
\end{aligned}$$

Thus the operation $\rho$ on $A\otimes L$ satisfies the Leibniz relation if and only if the linear map $\alpha$ (determined by $\rho$ ) satisfies the equation $d\alpha-\frac{1}{2}\nu\circ(\alpha\otimes\alpha)\circ\Delta=0$. □

Now let $\rho$ be a deformation of $L$ with base $(A,\mathfrak{M})\in\mathcal{C}$. Then $\rho$ satisfies the Leibniz relation on $A\otimes L$. From Proposition 6.2.6, the linear map $\alpha$ determined by $\rho$ satisfies the equation

$$d\alpha - \frac{1}{2}\nu \circ (\alpha \otimes \alpha) \circ \Delta = 0. \tag{6.2.12}$$

Here $\Delta(F_0) = 0$, since for $\xi \in F_0 = (\mathfrak{M}/\mathfrak{M}^2)'$, $\xi : \mathfrak{M} \longrightarrow \mathbb{K}$ is a linear map vanishing on $\mathfrak{M}^2$, which gives $\Delta(\xi)(m_i \otimes m_j) = \xi(m_i m_j) = 0$ for $1 \leq i, j \leq r$. So, $F_0 \subset ker(\Delta)$. Therefore by (6.2.12) we get $\alpha(F_0) \subset ker(d)$.

Let $a$ denote the composition

$$a : F_0 \xrightarrow{\alpha} ker(d) \xrightarrow{\pi} HL^2(L; L), \text{ where } \pi \text{ is the quotient map.}$$

Then from Definition 3.4.8 it follows that $a$ is the differential of the deformation $\rho$ of $L$ with base $A$.

**Corollary 6.2.7.** *A linear map $a : F_0 \longrightarrow HL^2(L; L)$ is a differential of some deformation of the Leibniz algebra $L$ with base $A$ if and only if $\frac{1}{2}a$ satisfies the condition of triviality of Massey $F$-brackets.*

*Proof.* Suppose $a : F_0 \longrightarrow HL^2(L; L)$ is a linear map so that $\frac{1}{2}a$ satisfies the condition of triviality of Massey $F$-brackets. From Definition 6.2.5 we get, $\frac{1}{2}a : F_0 \longrightarrow HL^2(L; L)$ is a linear map such that there exists a linear map $\frac{1}{2}\alpha : F_1 = \mathfrak{M}' \longrightarrow CL^2(L; L)$ satisfying $d(\frac{1}{2}\alpha) = \nu \circ (\frac{1}{2}\alpha \otimes \frac{1}{2}\alpha) \circ \Delta$ or, $d\alpha - \frac{1}{2}\nu \circ (\alpha \otimes \alpha) \circ \Delta = 0$.

By Proposition 6.2.6, the map $\alpha$ determines a $A$-bilinear operation $\rho$ on $A \otimes L$ such that $\rho$ is a deformation of $L$ with base $A$. Also from Definition 3.5.3 it follows that the linear map $a : F_0 \longrightarrow HL^2(L; L)$ is the differential $d\rho$ of the deformation $\rho$.

Conversely, suppose $\rho$ is a deformation of $L$ with base $A$ such that the differential $d\rho$ is the linear map $a : F_0 \longrightarrow HL^2(L; L)$. Now the operation $\rho$ on $A \otimes L$ determines a linear map $\alpha : F \longrightarrow CL^2(L; L)$ satisfying the equation $d\alpha - \frac{1}{2}\nu \circ (\alpha \otimes \alpha) \circ \Delta = 0$ (by Proposition 6.2.6). This means $\nu \circ (\alpha \otimes \alpha) \circ \Delta$ takes values in coboundaries. Hence for $b$ if we take the zero map, by Definition 6.2.5, $\frac{1}{2}a$ satisfies the condition of triviality of Massey $F$-brackets. □

## 6.3 Computation of Obstructions

In this section we relate the obstruction $\omega_k$ appeared in Section 5.2 at the $k$th stage of the construction of a versal deformation $\eta$ to Massey brackets.

Recall that a versal deformation $\eta$ is obtained by constructing a sequence of finite dimensional local algebras $C_k$ with maximal ideals $\mathfrak{M}_k$, and deformations $\eta_k$ of the Leibniz algebra $L$ with base $C_k$ yielding an inverse system

$$\mathbb{K} \xleftarrow{p_1} C_1 \xleftarrow{p_2} C_2 \xleftarrow{p_3} \ldots\ldots \xleftarrow{p_k} C_k \xleftarrow{p_{k+1}} C_{k+1} \ldots$$

with $p_{k+1*}\eta_{k+1} = \eta_k$, so that $\eta$ is the projective limit of $\eta_k$. Taking the dual of the above system we get a direct system

$$\mathbb{K} \xrightarrow{p_1'} C_1' \xrightarrow{p_2'} C_2' \xrightarrow{p_3'} \ldots\ldots \xrightarrow{p_k'} C_k' \xrightarrow{p_{k+1}'} C_{k+1}' \cdots .$$

Moreover, the corresponding maximal ideals $\mathfrak{M}_k$ give another system

$$\mathbb{K} \xrightarrow{p_1'} \mathfrak{M}_1' \xrightarrow{p_2'} \mathfrak{M}_2' \xrightarrow{p_3'} \ldots\ldots \xrightarrow{p_k'} \mathfrak{M}_k' \xrightarrow{p_{k+1}'} \mathfrak{M}_{k+1}' \cdots ,$$

where each $p_k'$ is an injective linear map. In the induction process, for any $k$ we get an extension of $C_k$ given by

$$0 \longrightarrow H^2_{Harr}(C_k;\mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}_{k+1}} C_k \longrightarrow 0. \tag{6.3.1}$$

The cohomology class represented by the obstruction cocycle in the extension process gives a linear map (cf. (5.2.2))

$$\omega_k : H^2_{Harr}(C_k;\mathbb{K}) \longrightarrow HL^3(L;L).$$

To kill this obstruction, we consider the new base

$$C_{k+1} = \bar{C}_{k+1}/\bar{i}_{k+1} \circ \ \omega_k'(HL^3(L;L)).$$

Take $F = (\bar{\mathfrak{M}}_{k+1})'$, $F_1 = \mathfrak{M}_k'$ and $F_0 = \mathfrak{M}_1' = HL^2(L;L)$.

From the extension (6.3.1) we have $F/F_1 = (\bar{\mathfrak{M}}_{k+1})'/\mathfrak{M}_k' = H^2_{Harr}(C_k;\mathbb{K})$. Thus $\omega_k$ can be viewed as a linear map

$$\omega_k : F/F_1 \longrightarrow HL^3(L;L).$$

**Theorem 6.3.1.** *The map $\omega_k$ obtained from the cohomology class represented by the obstruction cochain, has the property, $2\omega_k \in [id]_F$. Moreover, an arbitrary element of $[id]_F$ is equal to $2\omega_k$ for an appropriate extension of the deformation $\eta_1$ of $L$ with base $C_1$ to a deformation $\eta_k$ of $L$ with base $C_k$.*

*Proof.* By Definition 6.2.5, in order to show that $2\omega_k \in [id]_F$, we need to find a linear map $\alpha : F_1 \longrightarrow CL^2(L;L)$ satisfying the condition (6.2.3) such that the diagrams in Figure 6.1 are commutative. Here the projection map in the first diagram in Figure 6.1 is the map $CL^2(L;L) \longrightarrow HL^2(L;L)$, the left vertical projection in the second diagram is $F = (\bar{\mathfrak{M}}_{k+1})' \longrightarrow F/F_1 = H^2_{Harr}(C_k;\mathbb{K})$ whereas the right vertical projection is the map $CL^3(L:L) \longrightarrow HL^3(L;L)$.

Consider $\eta_k$, a deformation of $L$ with base $C_k$ extending the deformation $\eta_1$ of $L$

with base $C_1$. Now $[-,-]_{\eta_k}$ is a Leibniz algebra structure on $C_k \otimes L$. As in (6.2.6) define

$$\alpha : \mathfrak{M}'_k \longrightarrow CL^2(L;L)$$

by $\alpha_\phi(l_1,l_2) = (\phi \otimes id)([1\otimes l_1, 1\otimes l_2]_{\eta_k} - 1\otimes[l_1,l_2])$ for $\phi \in \mathfrak{M}'_k$ and $l_1, l_2 \in L$. Since $\eta_k$ is a Leibniz algebra structure on $C_k \otimes L$, Proposition 6.2.6 implies that $d\alpha = \frac{1}{2}\nu \circ (\alpha \otimes \alpha) \circ \Delta$.

Observe that $\alpha|_{F_0} : F_0 \longrightarrow CL^2(L;L)$ is given by $\alpha|_{F_0}(h_i) = \mu(h_i)$, a representative of the cohomology class $h_i$. So the composition

$$a = \pi \circ \alpha|_{F_0} : F_0 \longrightarrow \mathbb{H}$$

is the identity map.

Now consider the extension $\bar{C}_{k+1}$ (given in (6.3.1)) of $C_k$ by $H^2_{Harr}(C_k;\mathbb{K})'$ and recall that $\omega_k$ is a linear map determined by the cohomology class represented by the associated obstruction cocycle.

Let $\{m_i\}_{1\le i\le r}$ be a basis of $\mathfrak{M}_k$. We extend this to a basis $\{\bar{m}_i\}_{1\le i\le r+s}$ of the maximal ideal $\bar{\mathfrak{M}}_{k+1}$ of $\bar{C}_{k+1}$. Let the multiplication in $\bar{\mathfrak{M}}_{k+1}$ be defined (on the basis) as

$$\bar{m}_i\ \bar{m}_j = \sum_{p=1}^{r+s} c_{i\ j}^{p} \bar{m}_p \text{ for } 1 \le i,j \le r+s.$$

Then the comultiplication obtained by taking the dual of the multiplication in $\bar{\mathfrak{M}}_{k+1}$,

$$\Delta : (\bar{\mathfrak{M}}_{k+1})' \longrightarrow \mathfrak{M}'_k \otimes \mathfrak{M}'_k$$

is given by $\Delta(\bar{m}'_p) = \sum_{i,j=1}^{r} c_{ij}^{p} m'_i \otimes m'_j$.

As in (3.4.2) we write the Leibniz bracket $[-,-]_{\eta_k}$ on $C_k \otimes L$ as

$$[1\otimes l_1, 1\otimes l_2]_{\eta_k} = 1 \otimes [l_1,l_2] + \sum_{i=1}^{r} m_i \otimes \psi_i(l_1,l_2).$$

From the definition of $\alpha$ we have $\alpha(m'_i)(l_1,l_2) = \psi_i(l_1,l_2)$ for $i \le r$. For arbitrary cochains $\psi_i \in CL^2(L;L)$, $r+1 \le i \le s$, a $\bar{C}_{k+1}$-bilinear map $\{-,-\}$ on $\bar{C}_{k+1} \otimes L$ is given by

$$\{1\otimes l_1, 1\otimes l_2\} = 1 \otimes [l_1,l_2] + \sum_{i=1}^{r+s} \bar{m}_i \otimes \psi_i(l_1,l_2).$$

( Compare the construction of the obstruction map in Chapter 4 .) To prove that $\alpha$ and $2\omega_k$ satisfy required conditions we proceed as follows.

For $1\otimes l_1, 1\otimes l_2, 1\otimes l_3 \in \bar{\mathfrak{M}}_{k+1}\otimes L$,

$$\begin{aligned}
&\{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\}\\
&= \{1\otimes [l_1,l_2] + \sum_{i=1}^{r+s} \bar{m}_i \otimes \psi_i(l_1,l_2), 1\otimes l_3\}\\
&= \{1\otimes [l_1,l_2], 1\otimes l_3\} + \sum_{i=1}^{r+s} \bar{m}_i\{1\otimes \psi_i(l_1,l_2), 1\otimes l_3\}\\
&= 1\otimes [[l_1,l_2],l_3] + \sum_{i=1}^{r+s} \bar{m}_i\otimes \psi_i([l_1,l_2],l_3) + \sum_{i=1}^{r+s} \bar{m}_i \otimes [\psi_i(l_1,l_2),l_3]\\
&\qquad + \sum_{i,j=1}^{r} \bar{m}_j\bar{m}_i \otimes \psi_j(\psi_i(l_1,l_2),l_3)\\
&= 1\otimes [[l_1,l_2],l_3] + \sum_{i=1}^{r+s} \bar{m}_i\otimes \psi_i([l_1,l_2],l_3) + \sum_{i=1}^{r+s} \bar{m}_i \otimes [\psi_i(l_1,l_2),l_3]\\
&\qquad + \sum_{i,j=1}^{r}\sum_{p=1}^{r+s} c_{ij}^p \bar{m}_p \otimes \psi_j(\psi_i(l_1,l_2),l_3).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\{\{1\otimes l_1, 1\otimes l_3\}, 1\otimes l_2\}\\
&= 1\otimes [[l_1,l_3],l_2] + \sum_{i=1}^{r+s} \bar{m}_i\otimes \psi_i([l_1,l_3],l_2) + \sum_{i=1}^{r+s} \bar{m}_i \otimes [\psi_i(l_1,l_3),l_2]\\
&\qquad + \sum_{i,j=1}^{r}\sum_{p=1}^{r+s} c_{ij}^p \bar{m}_p \otimes \psi_j(\psi_i(l_1,l_3),l_2)
\end{aligned}$$

and

$$\begin{aligned}
&\{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}\}\\
&= 1\otimes [l_1,[l_2,l_3]] + \sum_{i=1}^{r+s} \bar{m}_i\otimes \psi_i(l_1,[l_2,l_3]) + \sum_{i=1}^{r+s} \bar{m}_i \otimes [l_1,\psi_i(l_2,l_3)]\\
&\qquad + \sum_{i,j=1}^{r}\sum_{p=1}^{r+s} c_{ij}^p \bar{m}_p \otimes \psi_j(l_1,\psi_i(l_2,l_3)).
\end{aligned}$$

Therefore

$$
\begin{aligned}
&(\bar{m}'_p \otimes id)(\{1\otimes l_1, \{1\otimes l_2, 1\otimes l_3\}\} - \{\{1\otimes l_1, 1\otimes l_2\}, 1\otimes l_3\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \{\{1\otimes l_1, 1\otimes l_3\}, 1\otimes l_2\}) \\
&= \psi_i([l_1,l_2],l_3) + [\psi_i(l_1,l_2),l_3] - \psi_i([l_1,l_3],l_2) - [\psi_i(l_1,l_3),l_2] + \psi_i(l_1,[l_2,l_3]) \\
&\quad + [l_1,\psi_i(l_2,l_3)] + \sum_{i,j=1}^{r} c_{ij}^p(\ \psi_j(\psi_i(l_1,l_2),l_3) - \psi_j(\psi_i(l_1,l_3),l_2) + \psi_j(l_1,\psi_i(l_2,l_3))\ ) \\
&= \delta\psi_p(l_1,l_2,l_3) + \frac{1}{2}\sum_{i,j=1}^{r} c_{ij}^p[\psi_j,\psi_i](l_1,l_2,l_3) \\
&= \delta\psi_p(l_1,l_2,l_3) + \frac{1}{2}\ \nu\circ(\alpha\otimes\alpha)\circ\Delta(\bar{m}'_p)(l_1,l_2,l_3) \\
&= -\ d\alpha(m'_p)(l_1,l_2,l_3) + \frac{1}{2}\ \nu\circ(\alpha\otimes\alpha)\circ\Delta(\bar{m}'_p)(l_1,l_2,l_3).
\end{aligned}
$$

Thus with $b = 2\omega_k$ and $a = [\alpha] = id|_{F_0=HL^2(L;L)}$, the conditions in Definition 6.2.5 are satisfied. This completes the proof. □

# Chapter 7

# Computations and examples

## 7.1 Introduction

The aim of this final chapter is to illustrate the theory developed in this thesis by two examples. First example is a three dimensional nilpotent Leibniz algebra over $\mathbb{C}$. We compute cohomologies necessary for our purpose, Massey brackets and construct a versal deformation of this example, [Man08].

Since any Lie algebra $L$ is a Leibniz algebra it is natural to investigate whether one recovers the same deformation picture of $L$ if it is seen as a Leibniz algebra. Our next example is a three dimensional Lie algebra over $\mathbb{C}$. This example illustrates that a Lie algebra $L$ when viewed as a Leibniz algebra may admit new deformations which are Leibniz algebras but not Lie algebras. Moreover, versal deformation of $L$ as Lie algebra and that of $L$ when viewed as a Leibniz algebra may differ.

## 7.2 Computation of second and third Leibniz cohomology of a nilpotent Leibniz algebra

Consider a three dimensional module $L$ spanned by $\{e_1, e_2, e_3\}$ over $\mathbb{C}$. Define a bilinear map $[-,-] : L \times L \longrightarrow L$ by $[e_1, e_3] = e_2$ and $[e_3, e_3] = e_1$, all other products of basis elements being 0. Then $(L, [-,-])$ is a three dimensional Leibniz algebra over $\mathbb{C}$. The Leibniz algebra $L$ is nilpotent and is denoted by $\lambda_6$ in the classification of three dimensional nilpotent Leibniz algebras (see Example 1.2.6).

To construct a versal deformation of $\lambda_6$, we need to compute the second and third cohomology modules of $\lambda_6 = L$. First consider $HL^2(L; L)$. Our computation consists of the following steps:

(i) To determine a basis of the module of cocycles $ZL^2(L; L)$,

(ii) to find out a basis of the coboundary module $BL^2(L;L)$,

(iii) to determine the quotient module $HL^2(L;L)$.

(i) Let $\psi \in ZL^2(L;L)$. Then $\psi : L \otimes L \longrightarrow L$ is a linear map and $\delta\psi = 0$, where

$$\delta\psi(e_i, e_j, e_k) = [e_i, \psi(e_j, e_k)] + [\psi(e_i, e_k), e_j] - [\psi(e_i, e_j), e_k] - \psi([e_i, e_j], e_k) + \psi(e_i, [e_j, e_k]) + \psi([e_i, e_k], e_j) \text{ for } 0 \leq i, j, k \leq 3.$$

Suppose the linear map $\psi$ is defined by $\psi(e_i, e_j) = \sum_{k=1}^{3} a_{i,j}^k e_k$ where $a_{i,j}^k \in \mathbb{C}$ for $1 \leq i, j, k \leq 3$. Since $\delta\psi = 0$, equating the coefficients of $e_1, e_2$ and $e_3$ in the expression of $\delta\psi(e_i, e_j, e_k)$ we get the following relations:

$$\begin{array}{ccc} a_{1,1}^1 = a_{1,1}^3 = 0; & a_{1,2}^1 = a_{1,2}^3 = 0; & a_{2,1}^1 = a_{2,1}^2 = a_{2,1}^3 = 0 \\ a_{2,2}^1 = a_{2,2}^2 = a_{2,2}^3 = 0; & a_{3,1}^2 = a_{3,1}^3 = 0; & a_{3,2}^2 = a_{3,2}^3 = 0 \\ a_{2,3}^3 = 0; & a_{1,1}^2 = a_{3,1}^1 = -a_{3,3}^3; & a_{1,2}^2 = -a_{1,3}^3 = a_{3,2}^1. \end{array}$$

Observe that there is no relation among the coefficients $a_{1,3}^1$, $a_{1,3}^2$, $a_{2,3}^1$, $a_{2,3}^2$, $a_{3,3}^1$ and $a_{3,3}^2$. Therefore, in terms of the ordered basis $\{e_i \otimes e_j\}_{1\leq i,j\leq 3}$ of $L \otimes L$ and $\{e_i\}_{1\leq i\leq 3}$ of $L$, the matrix corresponding to $\psi$ is of the form

$$M = \begin{pmatrix} 0 & 0 & x_3 & 0 & 0 & x_5 & x_1 & x_2 & x_7 \\ x_1 & x_2 & x_4 & 0 & 0 & x_6 & 0 & 0 & x_8 \\ 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & -x_1 \end{pmatrix}$$

where $x_1 = a_{1,1}^2, x_2 = a_{1,2}^2, x_3 = a_{1,3}^1, x_4 = a_{1,3}^2, x_5 = a_{2,3}^1, x_6 = a_{2,3}^2, x_7 = a_{3,3}^1$ and $x_8 = a_{3,3}^2$ are in $\mathbb{C}$.

We define the cocycles $\phi_i \in ZL^2(L;L)$ for $1 \leq i \leq 8$, by taking $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of $\psi$. Then $\{\phi_i\}_{1\leq i\leq 8}$ forms a basis of $ZL^2(L;L)$. Next we compute the coboundary module $BL^2(L;L)$.

(ii) Let $\psi_0 \in BL^2(L;L)$. We have $\psi_0 = \delta g$ for some 1-cochain $g \in CL^1(L;L) = Hom(L;L)$. Let $g(e_i) = g_i^1 e_1 + g_i^2 e_2 + g_i^3 e_3$ for $1 \leq i \leq 3$. Then the matrix associated to $g$ is given by

$$\begin{pmatrix} g_1^1 & g_2^1 & g_3^1 \\ g_1^2 & g_2^2 & g_3^2 \\ g_1^3 & g_2^3 & g_3^3 \end{pmatrix}.$$

From the definition of coboundary we get

$$\delta g(e_i, e_j) = [e_i, g(e_j)] + [g(e_i), e_j] - g([e_i, e_j]) \;\text{ for } 0 \leq i, j \leq 3.$$

The matrix $\delta g$ can be written as

$$\begin{pmatrix} 0 & 0 & (g_1^3 - g_2^1) & 0 & 0 & g_2^3 & g_1^3 & g_2^3 & (2g_3^3 - g_1^1) \\ g_1^3 & g_2^3 & (g_3^3 + g_1^1 - g_2^2) & 0 & 0 & g_2^1 & 0 & 0 & (g_3^1 - g_1^2) \\ 0 & 0 & -g_2^3 & 0 & 0 & 0 & 0 & 0 & -g_1^3 \end{pmatrix}.$$

Since $\psi_0 = \delta g$ is also a cocycle in $CL^2(L;L)$, comparing matrices $\delta g$ and $M$ we get $x_2 = x_5$ and $x_6 = x_1 - x_3$. Thus we conclude that the matrix of $\psi_0$ is of the form

$$\begin{pmatrix} 0 & 0 & x_3 & 0 & 0 & x_2 & x_1 & x_2 & x_7 \\ x_1 & x_2 & x_4 & 0 & 0 & (x_1 - x_3) & 0 & 0 & x_8 \\ 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & -x_1 \end{pmatrix}.$$

Let $\phi_i' \in BL^2(L;L)$ for $i = 1,2,3,4,7,8$ be the coboundary with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of $\psi_0$. It follows that $\{\phi_1', \phi_2', \phi_3', \phi_4', \phi_7', \phi_8'\}$ forms a basis of the coboundary module $BL^2(L;L)$. Notice that

$$\begin{aligned} \phi_1' = \phi_1 + \phi_3 - \phi_3' \qquad & \phi_3' = \phi_3 - \phi_6 \\ \phi_2' = \phi_2 + \phi_5 \qquad & \phi_i' = \phi_i \text{ for } i = 4,7,8 \end{aligned} \tag{7.2.1}$$

(iii) For $[\psi] \in HL^2(L;L) = \frac{ZL^2(L;L)}{BL^2(L;L)}$, we can express the cocycle $\psi$ as

$$\begin{aligned} \psi =& \sum_{i=1}^{8} x_i \phi_i \quad \text{for } x_i \in \mathbb{C} \\ =& x_1(\phi_1' - \phi_3 + \phi_3') + x_2\phi_2 + x_3\phi_3 + x_4\phi_4' + x_5(\phi_2' - \phi_2) + x_6(\phi_3 - \phi_3') \\ & \qquad + x_7\phi_7' + x_8\phi_8' \quad (\text{using } (7.2.1)) \\ =& (x_2 - x_5)\phi_2 - (x_1 - x_3 - x_6)\phi_3 + \phi, \end{aligned}$$

where $\phi = x_1\phi_1' + x_5\phi_2' + (x_1 - x_6)\phi_3' + x_4\phi_4' + x_7\phi_7' + x_8\phi_8' \in BL^2(L;L)$. Thus an arbitrary element $[\psi] \in HL^2(L;L)$ is in the submodule generated by $\{[\phi_2], [\phi_3]\}$. Also the set $\{[\phi_2], [\phi_3]\}$ is linearly independent. Therefore $dim(HL^2(L;L)) = 2$.

Next let us consider the module $HL^3(L;L)$.

If $\psi \in ZL^3(L;L)$, then a computation similar to 2-cocycles shows that the transpose of the matrix of $\psi$ is

$$\begin{pmatrix}
0 & x_1 & 0 \\
0 & x_2 & 0 \\
x_3 & x_4 & (x_2+x_5) \\
0 & x_5 & 0 \\
0 & 0 & 0 \\
x_6 & x_{17} & 0 \\
x_7 & x_8 & -x_5 \\
\frac{1}{5}(2x_2-3x_6+2x_{11}) & (x_{13}-x_{10}+2x_7+x_3-2x_1) & 0 \\
(2x_{16}-x_{14}) & x_9 & x_1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{1}{5}(3x_2+3x_6-2x_{11})-x_5 & x_{10} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & x_{11} & 0 \\
x_5 & (x_1-x_7) & 0 \\
0 & \frac{1}{5}(3x_2+3x_6-2x_{11}) & 0 \\
(x_1-x_7) & (3x_{16}-x_{14}-x_8) & x_5 \\
x_1 & 0 & 0 \\
x_2 & 0 & 0 \\
x_{12} & x_{18} & x_{13} \\
x_5 & 0 & 0 \\
0 & 0 & 0 \\
(x_{17}-x_{13}-x_{10}+3x_7+2x_3) & x_{19} & \frac{1}{5}(6x_2+x_6+x_{11}) \\
x_{14} & x_{15} & -x_1 \\
(2x_{13}-2x_1-x_3-x_7) & (x_{14}+x_{12}-x_8-x_4) & -x_2 \\
(x_9+x_{15}) & x_{20} & x_{16}
\end{pmatrix}.$$

Let $\tau_i \in ZL^3(L;L)$ for $1 \leq i \leq 20$ be the cocycle with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix. Then $\{\tau_i\}_{1\leq i\leq 20}$ forms a basis of $ZL^3(L;L)$. So $dim(ZL^3(L;L)) = 20$.

On the other hand suppose $\psi \in CL^3(L;L)$ is a coboundary with $\psi = \delta g$. Let $g(e_i, e_j) = g^1_{i,j}e_1 + g^2_{i,j}e_2 + g^3_{i,j}e_3$; for $1 \leq i, j \leq 3$. Then the transpose of the matrix of $\psi = \delta g$ is

$$\begin{pmatrix}
0 & g^3_{1,1} & 0 \\
0 & g^3_{1,2} & 0 \\
(g^1_{2,1}+g^1_{1,2}-g^3_{1,1}) & (g^2_{2,1}+g^2_{1,2}-g^1_{1,1}+g^3_{1,3}) & (g^3_{2,1}+g^3_{1,2}) \\
0 & g^3_{2,1} & 0 \\
0 & g^3_{2,2} & 0 \\
(g^1_{2,2}-g^3_{1,2}) & (g^2_{2,2}+g^3_{2,3}-g^1_{1,2}) & g^3_{2,2} \\
(g^3_{1,1}-g^1_{2,1}) & (g^1_{1,1}+g^3_{3,1}-g^2_{2,1}) & -g^3_{2,1} \\
(g^3_{1,2}-g^1_{2,2}) & (g^1_{1,2}+g^3_{3,2}-g^2_{2,2}) & -g^3_{2,2} \\
g^1_{1,1} & (g^3_{3,3}+g^2_{1,1}) & g^3_{1,1} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
(g^1_{2,2}-g^3_{2,1}) & (g^2_{2,2}-g^1_{2,1}) & g^3_{2,2} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-g^3_{2,2} & -g^1_{2,2} & 0 \\
g^3_{2,1} & g^1_{2,1} & 0 \\
g^3_{2,2} & g^1_{2,2} & 0 \\
g^1_{2,1} & g^2_{2,1} & g^3_{2,1} \\
g^3_{1,1} & 0 & 0 \\
g^3_{1,2} & 0 & 0 \\
(g^1_{1,1}+g^1_{3,2}-g^3_{3,1}+g^3_{1,3}) & (g^2_{1,1}+g^2_{3,2}-g^1_{3,1}) & (g^3_{1,1}+g^3_{3,2}) \\
g^3_{2,1} & 0 & 0 \\
g^3_{2,2} & 0 & 0 \\
(g^3_{2,3}-g^3_{3,2}+g^1_{1,2}) & (g^2_{1,2}-g^1_{3,2}) & g^3_{1,2} \\
(2g^3_{3,1}-g^1_{1,1}) & (g^1_{3,1}-g^2_{1,1}) & -g^3_{1,1} \\
(2g^3_{3,2}-g^1_{1,2}) & (g^1_{3,2}-g^2_{1,2}) & -g^3_{1,2} \\
(g^1_{3,1}+g^3_{3,3}) & g^2_{3,1} & g^3_{3,1}
\end{pmatrix}.$$

Since $\delta\psi$ is also zero, the transpose of the matrix of $\psi$ is of the previous form as well. Comparing these two matrices we get $x_6 = -(x_2 + x_{11})$ and $x_{19} = x_4 - x_8 - x_{12} - x_{14}$. Thus a coboundary $\psi$ has the following transpose matrix.

$$\begin{pmatrix}
0 & x_1 & 0 \\
0 & x_2 & 0 \\
x_3 & x_4 & (x_2+x_5) \\
0 & x_5 & 0 \\
0 & 0 & 0 \\
-(x_2+x_{11}) & x_{17} & 0 \\
x_7 & x_8 & -x_5 \\
(x_2+x_{11}) & (x_{13}-x_{10}+2x_7+x_3-2x_1) & 0 \\
(2x_{16}-x_{14}) & x_9 & x_1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-(x_{11}+x_5) & x_{10} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & x_{11} & 0 \\
x_5 & (x_1-x_7) & 0 \\
0 & -x_{11} & 0 \\
(x_1-x_7) & (3x_{16}-x_{14}-x_8) & x_5 \\
x_1 & 0 & 0 \\
x_2 & 0 & 0 \\
x_{12} & x_{18} & x_{13} \\
x_5 & 0 & 0 \\
0 & 0 & 0 \\
(x_{17}-x_{10}+3x_7+2x_3-x_{13}) & (x_4+x_8-x_{12}-x_{14}) & x_2 \\
x_{14} & x_{15} & -x_1 \\
(2x_{13}-2x_1-x_3-x_7) & (x_{14}+x_{12}-x_8-x_4) & -x_2 \\
(x_9+x_{15}) & x_{20} & x_{16}
\end{pmatrix}.$$

Let $\tau_i{}' \in BL^3(L;L)$ for $1 \leq i \leq 20$ and $i \neq 6, 19$ be the coboundary with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of $\psi$. It follows that $\{\tau'\}_{1\leq i\leq 20, i\neq 6,19}$ forms a basis of the coboundary module $BL^3(L;L)$. Consequently by considering the quotient module $\frac{ZL^3(L;L)}{BL^3(L;L)} = HL^3(L;L)$ we get, $dim(HL^3(L;L)) = 2$.

## 7.3 Computation of a versal deformation of $\lambda_6$

Let $L = \lambda_6$. Since $HL^3(L;L)$ is nontrivial, it is necessary to compute possible obstructions in order to extend an infinitesimal deformation to a higher order one.

First we describe the universal infinitesimal deformation $\eta_1$ for our Leibniz algebra. To make our computation simpler, we choose the representative cocycles $\mu_1, \mu_2$ where $\mu_1 = (\phi_2 - \phi_2')$ and $\mu_2 = \phi_3$. Let us denote a basis of $HL^2(L; L)'$ by $\{t, s\}$. By Remark 3.4.4, the universal infinitesimal deformation $\eta_1$ of $L$ can be written as

$$[1 \otimes e_i, 1 \otimes e_j]_{\eta_1} = 1 \otimes [e_i, e_j] + t \otimes \mu_1(e_i, e_j) + s \otimes \mu_2(e_i, e_j).$$

with base $C_1 = \mathbb{C} \oplus \mathbb{C}\, t \oplus \mathbb{C}\, s$.

In order to extend $\eta_1$ to a deformation of $L$ with larger base we need to compute possible obstructions which arise in the extension process. We shall compute these obstructions using the inductive definition of Massey brackets by [Ret77] (see Definition 6.2.1), which is a particular case of more general definition, Definition 6.2.5 ( [FW01]).

Recall that if $y_1 = [x_1], y_2 = [x_2]$ are 2- cohomology classes, then the second order bracket $< y_1, y_2 >$ is represented by the graded Lie bracket $[x_1, x_2]$. Assume that for $y_i \in HL^2(L; L)$, $< y_i, y_j >= 0$ for $1 \leq i \leq 3$. This means that for a cocycle $x_i$ representing $y_i$ we have $[x_i, x_j] = dx_{ij}$. Then the third order Massey bracket $< y_1, y_2, y_3 >$ is defined and is represented by

$$[x_{12}, x_3] + [x_1, x_{23}] + [x_{13}, x_2].$$

The cohomology class is independent of the choice of $x_{ij}$. The higher order Massey brackets are defined inductively.

By Definition 6.2.1, we have.

**(i)** $< [\mu_1], [\mu_1] >$ is represented by $[\mu_1, \mu_1] = 2(\mu_1 \circ \mu_1)$.
Now $(\mu_1 \circ \mu_1)(e_i, e_j, e_k)$
$= \mu_1(\mu_1(e_i, e_j), e_k) - \mu_1(\mu_1(e_i, e_k), e_j) - \mu_1(e_i, \mu_1(e_j, e_k))$ for $1 \leq i, j, k \leq 3$.

Since $\mu_1(e_2, e_3) = -e_1$ and takes value zero on all other basis element of $L \otimes L$, it follows that $\mu_1 \circ \mu_1 = 0$.

**(ii)** Similarly $< [\mu_1], [\mu_2] >$ is represented by $[\mu_1, \mu_2] = \mu_1 \circ \mu_2 + \mu_2 \circ \mu_1$. Since $\mu_2(e_1, e_3) = e_1$ and takes value zero on all other basis elements of $L \otimes L$ it follows that $< [\mu_1], [\mu_2] >= 0$.

**(iii)** The bracket $< [\mu_2], [\mu_2] >$ is represented by $[\mu_2, \mu_2] = 2(\mu_2 \circ \mu_2) = 0$.

Since $\{[\mu_1], [\mu_2]\}$ forms a basis for $HL^2(L; L)$, it follows that all the Massey 2- brackets are trivial. So all the Massey 3- brackets are defined.

From the definition of Massey 3- bracket it follows that all the Massey 3- brackets $< [\mu_i], [\mu_j], [\mu_k] >$ are trivial and represented by the 0-cocycle. By induction it follows that any $< [\mu_1], [\mu_2], \cdots, [\mu_k] >= 0$ for $[\mu_i] \in HL^2(L; L)$ and moreover, they are represented by the 0 cocycle.

By Theorem 6.3.1 and Remark 6.2.3, it follows that the possible obstruction at each stage in extending $\eta_1$ to a versal deformation with base $\mathbb{C}[[t,s]]$ can be realized as the Massey brackets of $[\mu_1]$ and $[\mu_2]$. So the possible obstructions are zero.

As there are no obstructions to extending the universal infinitesimal deformation $\eta_1$, it means that $\eta_1$ extends to a versal deformation with base $\mathbb{C}[[t,s]]$. Moreover, observe that by our choice of $\mu_1$ and $\mu_2$ every Massey bracket is represented by the 0- cocycle, and so $\eta_1$ is itself a Leibniz bracket with base $\mathbb{C}[[t,s]]$. It follows by the construction in 5.2 that $\eta_1$ is a versal deformation.

Explicitly, the versal deformation that we have constructed can be written as

$$[e_1, e_3]_{t,s} = e_2 + e_1 s, \quad [e_3, e_3]_{t,s} = e_1, \quad [e_2, e_3]_{t,s} = -e_1 t$$

with all the other brackets of basis elements are zero.

Thus we obtain the following two non-equivalent one parameter deformations for the Leibniz algebra $\lambda_6$.

$$(i)\ [e_1, e_3]_t = e_2, \quad [e_2, e_3]_t = -e_1 t, \quad [e_3, e_3]_t = e_1,$$

all the other brackets of basis elements are zero.

$$(ii)\ [e_1, e_3]_s = e_2 + e_1 s, \quad [e_3, e_3]_s = e_1,$$

all the other brackets of basis elements are zero.

## 7.4 The three dimensional Heisenberg Lie algebra

Since any Lie algebra $L$ is a Leibniz algebra it is natural to investigate whether one recovers the same deformation picture of $L$ if it is seen as a Leibniz algebra. The following example illustrates that a Lie algebra $L$ when viewed as a Leibniz algebra may admit new deformations which are Leibniz algebras but not Lie algebras. Moreover, the versal deformation of $L$ as Lie algebra and that of $L$ when viewed as a Leibniz algebra may differ.

Let $L$ be a module over $\mathbb{C}$ with basis $\{e_1,\ e_2,\ e_3\}$. Define a bilinear map $[-,-]$ : $L \times L \longrightarrow L$, by $[e_1, e_3] = e_2$, $[e_3, e_1] = -e_2$, and, all other products of basis elements being zero. Then $(L,\ [-,-])$ is the complex three-dimensional Heisenberg Lie algebra.

Let us first determine the universal infinitesimal Leibniz deformation $\eta_1$ of $L$. For this, we need to compute $HL^2(L; L)$. This computation is similar to that of the first example in Section 7.2.

Let $\psi : L^{\otimes 2} \longrightarrow L$ be a 2-cocycle. Suppose $\psi(e_i, e_j) = \sum_{k=1}^{3} a_{i,j}^k e_k$ where $a_{i,j}^k \in \mathbb{C}$; for $1 \leq i, j, k \leq 3$. Since $\psi$ is a cocycle, we have

$$\begin{aligned}\delta\psi(e_i, e_j, e_k) &= [e_i, \psi(e_j, e_k)] + [\psi(e_i, e_k), e_j] - [\psi(e_i, e_j), e_k] - \psi([e_i, e_j], e_k) \\ &+ \psi(e_i, [e_j, e_k]) + \psi([e_i, e_k], e_j) = 0 \text{ for } 0 \leq i, j, k \leq 3.\end{aligned}$$

Using the expression of $\psi(e_i, e_j)$ above we get some relations between the coefficients $a_{i,j}^k$. If we use the resulting relations then the matrix of $\psi$ with respect to the ordered basis $\{e_i \otimes e_j\}_{1 \leq i,j \leq 3}$ of $L^{\otimes 2}$ and $\{e_i\}_{1 \leq i \leq 3}$ of $L$, takes the form

$$\begin{pmatrix} 0 & x_2 & x_5 & -x_2 & 0 & x_8 & -x_5 & -x_8 & 0 \\ x_1 & x_3 & x_6 & -x_3 & 0 & x_9 & x_{10} & -x_9 & x_{11} \\ 0 & x_4 & x_7 & -x_4 & 0 & x_2 & -x_7 & 0 & 0 \end{pmatrix}.$$

where $x_1 = a_{1,1}^2$; $x_2 = a_{1,2}^1$; $x_3 = a_{1,2}^2$; $x_4 = a_{1,2}^3$; $x_5 = a_{1,3}^1$; $x_6 = a_{1,3}^2$; $x_7 = a_{1,3}^3$; $x_8 = a_{2,3}^1$; $x_9 = a_{2,3}^2$; $x_{10} = a_{3,1}^2$; and $x_{11} = a_{3,3}^2$ are in $\mathbb{C}$ .

Let $\phi_i$ for $1 \leq i \leq 11$, be the cocycle with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of $\psi$. Then $\{\phi_i\}_{1 \leq i \leq 11}$ forms a basis of the submodule of 2-cocycles in $CL^2(L; L)$.

On the other hand, let $\psi_0$ be a 2-coboundary so that $\psi_0 = \delta g$ for some 1-cochain $g$. Let $g(e_i) = g_i^1 e_1 + g_i^2 e_2 + g_i^3 e_3$ for $i = 1, 2, 3$. The coboundary formula gives

$$\delta g(e_i, e_j) = [e_i, g(e_j)] + [g(e_i), e_j] - g([e_i, e_j])$$

for $1 \leq i, j \leq 3$. From this we write down the matrix of $\delta g$. But $\delta g = \psi_0$ is also a cocycle and we know the form of the matrix for a 2-cocycle as given above. Comparing these two matrices we get $x_1 = x_2 = x_4 = x_8 = x_{11} = 0$, $x_3 = -x_7$, $x_5 = -x_9$ and $x_6 = -x_{10}$. Thus matrix of $\psi_0$ takes the form

$$\begin{pmatrix} 0 & 0 & x_5 & 0 & 0 & 0 & -x_5 & 0 & 0 \\ 0 & x_3 & x_6 & -x_3 & 0 & -x_5 & -x_6 & x_5 & 0 \\ 0 & 0 & -x_3 & 0 & 0 & 0 & x_3 & 0 & 0 \end{pmatrix}.$$

Let $\phi'_i \in BL^2(L; L)$ for $i = 3, 5, 6$ be the coboundary with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of $\psi_0$. It follows that $\{\phi'_3, \phi'_5, \phi'_6\}$ forms a basis for the submodule of 2-coboundaries in $CL^2(L; L)$. Observe that

$$\phi'_3 + \phi_7 = \phi_3,\ \phi'_5 + \phi_9 = \phi_5 \text{ and } \phi'_6 + \phi_{10} = \phi_6.$$

Using this it follows that

$$\{[\phi_1], [\phi_2], [\phi_4], [\phi_7], [\phi_8], [\phi_9], [\phi_{10}], [\phi_{11}]\}$$

forms a basis of $HL^2(L; L)$ where $[\phi_i]$ denotes the cohomology class represented by the cocycle $\phi_i$. Thus $dim(HL^2(L; L)) = 8$.

Let $\{t_i\}_{1\leq i\leq 8}$ denote the dual basis. Then by Remark 3.4.4, the universal infinitesimal deformation $\eta_1$ of $L$ with base $\mathbb{C} \oplus HL^2(L; L)'$ is given by

$$\begin{aligned}[e_i, e_j]_{\eta_1} =& 1 \otimes [e_i, e_j] + t_1 \otimes \phi_1(e_i, e_j) + t_2 \otimes \phi_2(e_i, e_j) + t_3 \otimes \phi_4(e_i, e_j)\\ &+ t_4 \otimes \phi_7(e_i, e_j) + t_5 \otimes \phi_8(e_i, e_j) + t_6 \otimes \phi_9(e_i, e_j)\\ &+ t_7 \otimes \phi_{10}(e_i, e_j) + t_8 \otimes \phi_{11}(e_i, e_j).\end{aligned}$$

In particular, we get eight non-equivalent infinitesimal deformations of $L$ given by

$$\mu_i = \mu_0 + t\phi_i \quad \text{for } i = 1, 2, 4, 7, 8, 9, 10, 11,$$

where $\mu_0$ denotes the original bracket in $L$. Observe that $\phi_j$ is skew-symmetric for $j = 2, 4, 7, 8, 9$ and hence the infinitesimal deformations $\mu_j$ for $j = 2, 4, 7, 8, 9$ are Lie algebras.

A similar computation yields that two dimensional Chevalley-Eilenberg cohomology module is of dimension 5, and spanned by the cohomology classes of $\phi_j$ for $j = 2, 4, 7, 8, 9$.

Hence as before we can write down the universal infinitesimal deformation $\eta_1'$ of $L$ as a Lie algebra as follows.

$$\begin{aligned}[e_i, e_j]_{\eta_1'} =& 1 \otimes [e_i, e_j] + t_1 \otimes \phi_2(e_i, e_j) + t_2 \otimes \phi_4(e_i, e_j) + t_3 \otimes \phi_7(e_i, e_j)\\ &+ t_4 \otimes \phi_8(e_i, e_j) + t_5 \otimes \phi_9(e_i, e_j).\end{aligned}$$

The universal infinitesimal deformation of $L$ as Lie algebra is not the same as the one when we view it as Leibniz algebra. Thus we see that even at the infinitesimal level the universal deformation of a Lie algebra differs from that when the Lie algebra is deformed as a Leibniz algebra. This example shows that by deforming a Lie algebra $L$ in the category of Leibniz algebras not only one recovers its Lie algebra deformations but can get new deformations of $L$ which are only Leibniz algebras as one might expect. Moreover, versal deformation of $L$ as Lie algebra and that of $L$ when viewed as a Leibniz algebra may differ.

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