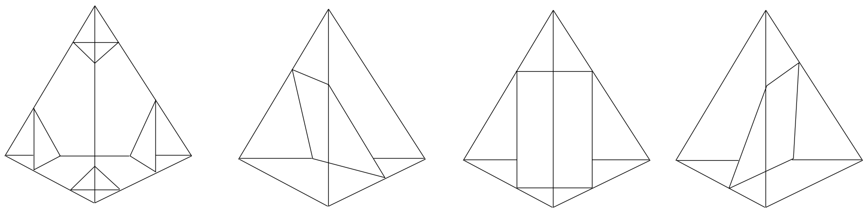
Normal surfaces and Heegaard splittings of 3-manifolds



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“To realise that you do not understand is a virtue. Not to realize that you do not understand is a defect ”

- Lao Tze

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CONTENTS

0. PREFACE

This thesis deals with various questions regarding normal surfaces and Heegaard splittings of 3-manifolds.

<u>Chapter 1</u> The first chapter is divided into two parts. In the first, we give an outline of normal surface theory and mention some of its important applications. The second part gives an overview of the theory of Heegaard splitting surfaces and a few of its applications. None of the material covered in this chapter is original and it is meant solely as an exposition of known results.

<u>Chapter 2</u> In this chapter, we give a lower bound on the Euler characteristic of a normal surface, a topological invariant, in terms of the number of normal quadrilaterals in its embedding, obtained from its combinatorial description. A closed connected normal surface having no normal quadrilaterals is a vertex-linking sphere. We make the observation that a 'strongly-connected' normal surface (with boundary) having no normal quadrilaterals, is a planar surface. Using this fact, we obtain the desired relation. In the smooth category, we expect normal triangles to correspond to positive curvature pieces. Hence by Gauss-Bonnet, such a relation is to be expected when the curvature of the quadrilateral pieces is bounded below.

Another result in a similar spirit is an upper bound on the number of normal triangles in terms of the number of normal quadrilaterals of a normal surface (having no vertex-linking spheres). Strongly-connected triangle components are shown to be subsets of vertex-linking spheres, so that the number of triangles in one such component is bounded above by the maximum number of triangles in a vertex-linking sphere. We think of quadrilaterals as 'bridges' linking the various strongly-connected triangle components and by a combinatorial argument we obtain the desired relation. Both these results pertain to original work published in the paper [43].

<u>Chapter 3</u> Here we interpret a normal surface in a (singular) three-manifold in terms of the homology of a chain complex. This allows us to study the relation between normal surfaces and their quadrilateral co-ordinates. Specifically, we give a proof of an (unpublished) result of Casson-Rubinstein saying that quadrilaterals determine a normal surface up to vertex linking spheres. We also characterise the quadrilateral coordinates that correspond to a normal surface in a (possibly ideal) triangulation. The results in this chapter are the outcome of joint work with my adviser, Siddhartha Gadgil. They have been submitted as paper [45].

CHAPTER 4 We describe a procedure for refining the given triangulation of a 3-manifold that scales the PL-metric according to a given weight function while creating no new normal surfaces.

It is known that an incompressible surface F in a triangulated irreducible 3-manifold M is isotopic to a normal surface that is of minimal PL-area in the isotopy class of F. Using the above scaling refinement we prove the converse. If F is a surface in a closed 3-manifold M such that for any triangulation τ of M, F is isotopic to a τ-normal surface $F(\tau)$ that is of minimal PL-area in its isotopy class, then we show that F is incompressible. This is the result of original work and has been published as paper [44].

CHAPTER 5 In the fifth chapter we define a space of projective maximal laminations fully carried by a branched surface. Analogous to the space of geodesic laminations on surfaces, we shall show that this space with the 'quotient Hausdorff metric' is compact, Hausdorff and that each point can be written as the intersection of a sequence of open sets. The main idea used here is to look at isotopy classes of 'splitter surfaces' in the complement of laminations (in neighbourhood of branch surface), rather than isotopy classes of the laminations themselves.

EPILOGUE In the concluding chapter we give the motivation for introducing the space of projective maximal laminations and give some conjectures regarding the structure of irreducible Heegaard surfaces in Haken manifolds. We also outline our attempts to prove them.

Incompressible surfaces and (strongly irreducible) Heegaard surfaces are in some sense opposite ends of the spectrum so it is surprising that both can be represented as 'normal' surfaces. It was shown by Li in [21], that when an almost normal branched surface carries infinitely many strongly irreducible Heegaard surfaces, their limit point in the projective measured lamination space is an essential lamination, which after a perturbation becomes a closed incompressible surface. As a result, it was shown that a non-Haken 3-manifold has only finitely many strongly irreducible Heegaard splittings. This representation of strongly irreducible Heegaard surfaces and incompressible surfaces as 'normal' surfaces is used in both our possible approaches to proving our conjecture on the structure of Heegaard splittings of Haken manifolds.

1. THEORY OF NORMAL SURFACES AND HEEGAARD SPLITTINGS

This chapter is divided into two sections. In the first, we shall discuss the theory of normal surfaces and mention some of its important applications. For a more detailed treatment of this material we refer to [35] and [22] from which most of the definitions and theorems of this chapter are sourced. In the second section, we shall give an outline of the theory of Heegaard splittings. For an exhaustive study of this material we refer to the survey paper [31] from which most of the theorems are taken. All theorems covered in this chapter are known results, and no claim is made to the originality of any of them.

1.1 Normal Surfaces

Normal surfaces are surfaces that are embedded particularly 'nicely' with respect to the given pseudo-triangulation of a 3-manifold. Normal surface theory was first introduced by Knesser in 1929 and later developed by Haken for use in an algorithm to detect the unknot. Much later, Jaco and Oertel [16] and Hemion [14] used normal surfaces to develop an algorithm for solving the homeomorphism problem for the class of manifolds that contain an incompressible two-sided surface. Since then, normal surfaces have been used for a variety of tasks such as the construction of algorithms for recognising the 3-sphere [26, 27], Seifert fibered spaces and handlebodies [17]. It has also been used for the decomposition of a closed 3-manifold into irreducible pieces and for its JSJ decomposition [17].

Many interesting classes of embedded surfaces such as incompressible surfaces and strongly-irreducible Heegaard splitting surfaces are isotopic to normal or almost-normal surfaces. The reason normal surfaces are well suited for an algorithmic approach is because in many interesting cases, if an embedded surface has a certain property, then one of a finite set of 'fundamental' surfaces must also have this property. Thus it is sufficient to check a finite set of surfaces for the required property.

A brief description of a normal surface is an embedded surface that intersects each 3-simplex of the triangulation in one of the seven types of disks, shown in Figure 1.1. We make this more precise below.

Definition 1.1.1. An arc on a 2-dimensional face F of a 3-simplex is a *normal arc* if its endpoints lie on distinct edges of F and its interior lies in the interior of F. A *normal curve* on $\Delta^{(2)}$, the

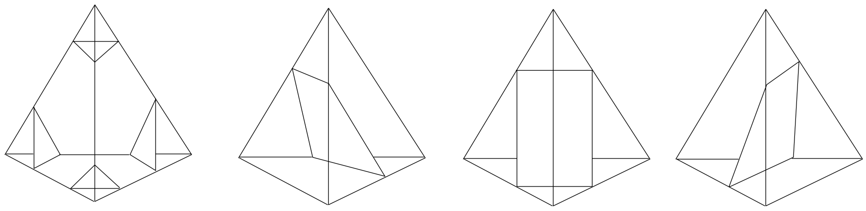

Fig. 1.1: The 4 Normal Triangle and 3 Normal Quadrilateral Types

2-dimensional faces of a 3-simplex Δ, is a closed curve c in general position with respect to $\Delta^{(2)}$ such that any component of intersection of c with a 2-dimensional face F of Δ, is a normal arc.

Definition 1.1.2. The length of a normal curve c on the 2-dimensional faces of a 3-simplex Δ is the number of points in $c \cap \Delta^{(1)}$, i.e. the number of intersections of c with the 1-skeleton of Δ.

Lemma 1.1.3 (Lemma 3.31 [35])**.** *A normal curve on the 2-dimensional faces of Δ that meets each edge at most once either has length 3 or 4.*

Proof. Here $\Delta^{(2)}$ is homeomorphic to a sphere. The closed curve c is therefore a Jordan curve on the sphere. Thus it separates $\Delta^{(2)}$ into an 'inside' disk and an 'outside' disk. The set of vertices $\Delta^{(0)}$ consists of four points. There are essentially two possibilities, the inside of c contains either one or two points of $\Delta^{(0)}$.

Suppose the inside of c contains only the vertex v. Let e_1, e_2, e_3 be the edges incident to v. Then c must intersect e_1, e_2, e_3. Furthermore, since the other vertices to which e_1, e_2, e_3 are incident lie outside of c, c must intersect e_1, e_2, e_3 an odd number of times. Similarly, it must meet the remaining three edges an even number of times. Thus if c meets no edge more than once, then c has length 3.

Suppose now that the inside of c contains the vertices v_1, v_2. Then there are four edges that are met an odd number of times and two edges that are met an even number of times. So, the former edges each meet c once and the latter are disjoint from c. Hence, the length of c is 4. □

Definition 1.1.4. A *normal disk* in a 3-simplex Δ is a disk D properly embedded in Δ, with ∂D a normal curve in $\Delta^{(2)}$. A normal disk D with ∂D a normal curve of length 3 is called a *normal triangle*, while a normal disk D with ∂D a normal curve of length 4 is called a *normal quadrilateral.*

Definition 1.1.5. Two normal disks D and D' in Δ are said to be *normally isotopic* if there is an isotopy of Δ that leaves each vertex, edge and face of Δ invariant and takes D to D' (through normal disks). We consider normal disks to be equivalent if they are normally isotopic.

Definition 1.1.6. Let M be a compact 3-manifold with a pseudo-triangulation τ. A *normal surface* in M is a surface $S \subset M$ such that for any 3-simplex Δ in τ, $S \cap \Delta$ consists of disjoint normal disks in Δ.

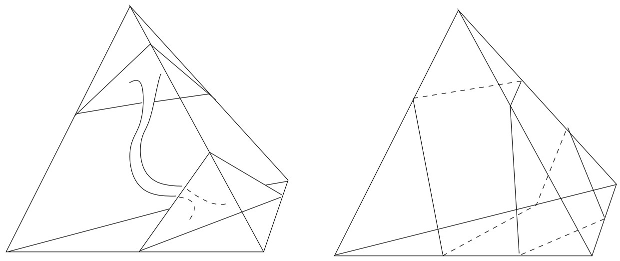

Fig. 1.2: The Almost Normal Annulus and Octagon

Definition 1.1.7. An *almost normal surface*, is an embedded surface that intersects all except one 3-simplex (say Δ) in disjoint normal disks and intersects Δ in normal triangles and exactly one octagon or annulus, as shown in Fig 1.2. The annulus must be an unknotted tube, i.e, its core must be parallel to the boundary of the tetrahedron.

Definition 1.1.8. Let v be a vertex of the triangulation and let $B(v)$ be a small ball neighbourhood of v. The boundary $\partial B(v)$ is called the *vertex-linking sphere* linking vertex v.

As normal triangles are parallel to a face of the 3-simplex, and normal quadrilaterals separate opposite edges of the 3-simplex, there are 4 types of normal triangles and 3 types of normal quadrilaterals (up to normal isotopy) as shown in Figure 1.1. Therefore, if there are t 3-simplices, then corresponding to a normal surface we get a vector in $\mathbb{Z}^{7t}$ which records the number of normal triangles and quadrilaterals of a normal surface in each of the t 3-simplices. An (unpublished) result of Casson-Rubinstein states that quadrilaterals determine a normal surface up to vertex linking spheres. This was proved by Tollefson [40] using geometric means. We prove this by different means, associating a chain complex to normal surfaces, in Chapter 3.

As two normal quadrilaterals of different types within a 3-simplex intersect, an embedded normal surface can have only one type of quadrilateral in each 3-simplex. This is called the *quadrilateral constraint.*

Let Δ_1 and Δ_2 be adjacent 3-simplices in M that meet at a common face F. Let S be a closed normal surface embedded in M, that intersects F. Then, $S \cap F$ is a union of normal arcs in F. For a vertex v of F, there is exactly one type of normal triangle T_i (and one type of normal quadrilateral Q_i) in Δ_i, $i = 1, 2$, such that $T_i \cap F$ (respectively $Q_i \cap F$) is a normal arc in F linking v. Let x_i and y_i be respectively the number of triangles T_i and number of quadrilaterals Q_i in Δ_i, $i = 1, 2$. Then, as S is a closed surface $x_1 + y_1 = x_2 + y_2$. We get such a homogenous linear equation for each vertex in each face of the triangulation. This system of homogenous linear equations are called the *matching equations.* The vector represented by S is a non-negative integer solution to the matching equations that satisfies the quadrilateral constraint. Conversely, given a non-negative integer solution to the matching equations that satisfies the quadrilateral constraints we obtain a (not necessarily connected) normal surface whose coordinates give the solution vector. Thus, properly embedded normal surfaces (up to normal isotopy) are in bijection with the non-negative integer solutions of the matching equations that satisfy the quadrilateral constraint. We refer to Theorem 3.3.27 of [22] for a proof of this theorem. Such solutions are called *admissible solutions* of the matching equations.

A non-trivial non-negative integer solution x is called a *fundamental solution* if it cannot be presented in the form $x = y + z$ where y and z are non-trivial non-negative integer solutions of the matching equations. A well known result of Haken is the following. We refer to [22], Theorem 3.2.8 for a proof.

Theorem 1.1.9. *The set of fundamental solutions to any system of linear homogenous equations with integer coefficients is finite and can be constructed algorithmically. Any non-negative integer solution to the system can be presented as a linear combination of the fundamental solutions with non-negative integer coefficients.*

One of the reasons that normal surfaces are useful is that the operation of addition of the admissible solutions has a geometric analogue called the Haken sum. In order to describe it, we first look at addition of normal arcs.

Let λ_1 and λ_2 be normal arcs on a 2-dimensional simplex F. In normal coordinates on F (which counts the number of normal arcs linking each vertex of F), they correspond to (possibly identical) unit vectors e_i, e_j in $\mathbb{Z}^3$. After a normal isotopy fixing the endpoints of λ_i, we can assume that λ_1 and λ_2 intersect in at most one point. Let $[\lambda]$ denote the vector in $\mathbb{Z}^3$ corresponding to a normal arc λ.

If the two arcs are disjoint, then we define $\lambda_1 + \lambda_2$ to be the union of the two arcs, and we observe that $[\lambda_1 + \lambda_2] = [\lambda_1] + [\lambda_2]$. If λ_1 and λ_2 intersect at a point A in F, then there are two possible cut-and-paste operations, i.e., we cut both segments at the crossing points and glue the ends together in one of the two possible ways. We will call these operations *switches*.

Definition 1.1.10. A switch at A is called a *regular switch* if it produces two normal arcs, i.e., two arcs such that the endpoints of each arc lie in different edges of the face F. Note that a regular switch is unique.

We observe again, that $[\lambda_1 + \lambda_2] = [\lambda_1] + [\lambda_2]$ for a regular switch.

We can now define a similar operation of geometric addition for normal surfaces. Let S_1 and S_2 be normal surfaces in (M, τ) and let $[S_1]$ and $[S_2]$ denote their normal coordinates. For every face F of τ we perform regular switches of the normal arcs $S_1 \cap F$ and $S_2 \cap F$. Now, for each 3-simplex Δ, there exists a union of normal disks with the boundary given by normal curves on $\partial\Delta$ obtained after these regular switches. The coordinates of the union of these normal disks corresponds to the vector $[S_1] + [S_2]$. As $[S_1]$ and $[S_2]$ are both admissible solutions to a system of linear homogenous equations, $[S_1] + [S_2]$ is a non-negative integer solution to the matching equations as well. If this solution satisfies the quadrilateral constraint, that is S_1 and S_2 do not intersect a 3-simplex in different types of quadrilaterals, then we define $S_1 + S_2$ to be the normal surface corresponding to the admissible solution $[S_1] + [S_2]$. This is called the *Haken sum* of S_1 and S_2, and by definition we have $[S_1 + S_2] = [S_1] + [S_2]$.

Euler characteristic is additive under Haken sums. The normal surfaces S_1 and S_2 intersect in a disjoint union of circles and the operation of Haken summing performs a cut-and-paste operation in a neighbourhood of these circles. As the Euler characteristic of the circles is zero, it is easy to see that $\chi(S_1 + S_2) = \chi(S_1) + \chi(S_2)$.

In Chapter 2, we shall show that the Euler characteristic of a normal surface is bounded below, by a bound that depends only on the number of normal quadrilaterals in the surface.

Definition 1.1.11. The weight $w(S)$ of a surface S in a compact triangulated 3-manifold (M, τ), is the number of intersections of S with the 1-skeleton of the triangulation, i.e., $|S \cap \tau^{(1)}|$. Similarly, let $m(S) = |S \cap (\tau^{(2)} - \tau^{(1)})|$ be the number of components in the intersection of S with the interior of the faces in τ. The pair $(w(s), m(S))$ is a measure of the complexity of the surface S.

The theory of normal surfaces is used extensively in algorithmic topology. Algorithms based on it most often follow the General Scheme described below, following [22] Chapter 4, Section 4.1.

1. Reduce the problem at hand to one of the existence in M of a surface with some specific characteristic property.

2. Choose a triangulation of M and show that if M contains at least one characteristic surface, then there exists a normal characteristic surface.

3. Show that if there is a normal characteristic surface, then there is a fundamental characteristic surface. One possible way to do that is to prove that if a characteristic surface F is not fundamental, then M contains a less complicated characteristic surface. The weight of the normal surface is one such candidate to measure the complexity of the surface.

4. Construct an algorithm to decide whether or not a given surface is characteristic.

Assume all four steps of the General Scheme are carried out. Then the algorithm that solves the problem works as follows:

1. Choose a triangulation τ of M.
2. Write down the corresponding matching system of linear equations.
3. Find the finite set of fundamental solutions by normal surfaces.
4. Realize the fundamental solutions by normal surfaces.
5. Test each of the obtained fundamental surfaces for being characteristic.

It follows that M contains a characteristic surface (i.e, that the problem in question has a positive answer) if and only if at least one of the fundamental surfaces is characteristic.

We conclude this section with an illustration of the use of this General Scheme, via the following theorem:

Theorem 1.1.12. *There is an algorithm to detect whether a 3-manifold has an incompressible surface.*

We give a complete proof of the following lemma as it shows some standard techniques used in proving the second step of the General Scheme.

Lemma 1.1.13. *Let M be an irreducible 3-manifold containing an incompressible surface S. Then for any triangulation τ of M there is an isotopy that takes S to a normal surface in (M, τ).*

Proof. Let (M, τ) be a triangulation of M and let S be an incompressible surface in M. Isotope S so that $(w(S), m(S))$ is minimal (in lexicographic ordering). The minimality of $w(S)$ implies that for each 3-simplex Δ in τ, S meets the 2-dimensional faces of Δ in a finite number of disjoint normal arcs along with simple closed curves entirely contained in the open 2-dimensional faces (i.e., S is in general position).

Let F be a 2-dimensional face of Δ and suppose that $S \cap F$ contains a simple closed curve s. Further assume that s is an innermost such curve in F. Then, s bounds a disk D in F that meets S only in its boundary. Since S is incompressible it follows that s also bounds a disk D' in S. Since D is disjoint from S away from $s = \partial D$, $D \cup D'$ is a 2-sphere. Since M is irreducible, $D \cup D'$ bounds a 3-ball. It follows that D' can be isotoped to coincide with D. A further isotopy then eliminates the component s of $S \cap \tau^{(2)}$. This contradicts the minimality of $(w(S), m(S))$. Thus for each 3-simplex Δ in τ, S meets each face F of Δ in normal arcs. Hence $S \cap \partial\Delta$ is a finite number of disjoint normal arcs.

Let c be a normal curve in $S \cap \partial\Delta$. Let S' be the component of $S \cap \Delta$ such that $c \subset \partial S'$. Since c bounds a disk E in the 3-ball Δ, it must in fact bound a disk E' in S. Here $E \cup E'$ is a 2-sphere in an irreducible 3-manifold and hence bounds a 3-ball B. A priori E may not be disjoint from S but the procedure in the above paragraph shows how to eliminate curves of intersection in $S \cap E$.

If E' does not lies entirely in Δ, then B describes an isotopy lowering $(w(S), m(S))$, a contradiction. Thus, $S' = E'$. In particular, S' is a disk.

Suppose that $\partial S'$ meets an edge e of Δ more than once. Since S' is a disk in the 3-ball Δ, it is isotopic to one of the disks bounded by $\partial S'$ in $\partial\Delta$. In particular, there is a disk E' such that $\partial E' = \alpha \cup \beta$ with $\alpha \subset S'$ and $\beta \subset \Delta^{(1)}$ and such that E' is disjoint from $S' - \alpha$. The disk E' now describes an isotopy lowering $w(S)$, a contradiction.

It now follows from Lemma 1.1.3 that $\partial S'$ has length 3 or 4. Thus S' is a normal disk. Hence S has been isotoped to a normal surface. □

In Chapter 4 we shall prove a weak converse of the above lemma. We shall show that if F is a closed surface in a closed 3-manifold M such that for any triangulation τ of M, F is isotopic to a τ-normal surface $F(\tau)$ that is of minimal PL-area in its isotopy class, then F is incompressible.

We complete the steps of the algorithm by stating the following lemmas. For proofs we refer to Theorem 4.1.15 and related theorems of [22].

Lemma 1.1.14. *If there is an incompressible normal surface, then there exists an incompressible fundamental normal surface.*

Lemma 1.1.15. *There exists an algorithm to check whether a given normal surface is incompressible.*

Following the General Scheme, using the above lemmas, we will have proved Theorem 1.1.12.

1.2 Heegaard Splittings

In this section we introduce Heegaard splittings and detail some of its properties. For an exhaustive treatment of this theory we refer to [31].

A Heegaard splitting is a splitting of a 3-manifold into simpler pieces called handlebodies. A *handlebody* is the regular neighbourhood of a finite graph in $\mathbb{R}^3$. Its *genus* is its first Betti number.

Definition 1.2.1. Let H_1 and H_2 be handlebodies of the same genus. Let $f : \partial H_1 \to \partial H_2$ be a homeomorphism of their boundaries. Gluing H_1 and H_2 along their boundaries via f, we get a closed 3-manifold M. This is said to be a *Heegaard splitting* of M and is denoted by $M = H_1 \cup_S H_2$, where S is the common boundary of the handlebodies in M. Every closed 3-manifold admits a Heegaard splitting. The common boundary of the handlebodies in M, $S = \partial H_1 = \partial H_2$ is said to be the *splitting surface* or the *Heegaard surface* of the Heegaard splitting. Two Heegaard splittings of the same manifold M with splitting surfaces S and S', are said to be *isotopic* if S and S' are isotopic in M. They are said to be *homeomorphic* if there is a homomorphism of M that takes S to S'.

Natural questions regarding Heegaard splittings are of one of two types. Either we wish to determine what Heegaard splittings a given 3-manifold admits or conversely, given the Heegaard splittings of a manifold, we wish to say something about the manifold. We shall mention a few results answering some facets of both these questions.

Firstly, we observe that every closed 3-manifold has a Heegaard splitting. Moise and Bing [23, 2] showed that every compact 3-manifold can be triangulated. The regular neighbourhood of the 1-skeleton of a triangulation is a handlebody. Its complement is the regular neighbourhood of the graph dual to the 1-skeleton. As a result, we get a splitting of the manifold into two handlebodies, which is precisely a Heegaard splitting.

The simplest example of a Heegaard splitting is the splitting of S^3 by any embedded S^2 within it. By the Schoenflies Theorem, both the complementary components of S^2 are balls. This gives a splitting of S^3 into balls, which can be thought of as genus zero handlebodies. Manifolds which admit a Heegaard splitting of genus one are called *lens spaces*.

Next we wish to address the question of uniqueness of this splitting. Heegaard splittings of a 3-manifold are far from unique. One way of increasing the genus of a splitting is by adding trivial handles. This process, similar in nature to taking a connected sum with S^3, is called stabilisation.

Definition 1.2.2. Let $M = H_1 \cup_S H_2$ be a Heegaard splitting of a 3-manifold M. Let α be a properly embedded arc in H_2 parallel to an arc in S. Add a neighbourhood of α to H_1 and delete it from H_2. This adds a 1-handle to both H_1 and H_2. The result is a Heegaard splitting $H_1' \cup_{S'} H_2'$, where the genus of each H_i' is one more than the genus of H_i. This process is called a *stabilisation* of S.

It was first shown by Reidemeister and Singer (see [1] for details) that any two splittings of a 3-manifold are isotopic after finitely many stabilisations. Until recently, it was conjectured that precisely one stabilisation of the higher genus splitting (and suitably many of the lower genus splitting) would suffice to make any two Heegaard splittings equivalent. This has been disproved by Hass, Thompson, Thurston [12] where for any $n > 0$ they construct a manifold $M(n)$ which has a pair of splittings needing n stabilisations to become equivalent. There is however a bound on the number of stabilisation required to make any two Heegaard splittings equivalent [28, 29, 30]. This bound depends polynomially on the genus of the two splittings.

As every splitting can be stabilised, we consider the question of uniqueness only for destabilised splittings. Waldhausen [41] showed that any positive genus splitting of S^3 is a stabilisation of the standard genus zero splitting. Bonahaon and Ottal [4] obtained a similar result for lens spaces. Scharlemann and Thompson [33] proved this uniqueness for $(surface) \times I$, and Schultens [36] generalised this proof, showing that Heegaard splittings of $(surface) \times S^1$ are standard. Heegaard splittings of Seifert Fibered Spaces though not unique, are also well understood [24, 37].

Analogous to the concept of reducibility of a manifold, we have the following definition for reducibility of a Heegaard splitting.

Definition 1.2.3. A Heegaard splitting $H_1 \cup_S H_2$ is *reducible* if there is a 2-sphere which intersects S in a single essential curve (of S).

A stabilised splitting, that is not the standard genus one splitting of S^3, is reducible. Haken showed that every splitting of a reducible manifold is a reducible splitting [11]. As a weak converse, every reducible splitting of an irreducible 3-manifold is stabilised. As we shall henceforth consider only destabilised splittings in irreducible closed 3-manifolds, we need consider only irreducible splittings.

Heegaard splittings can be thought of as a handle-decomposition of the 3-manifold. A standard trick in handle theory is that the order of handle addition can be rearranged. An $(r+1)$-handle can be added before a r-handle precisely when the attaching r-sphere of the $(r+1)$-handle is disjoint from the belt $(n-r-1)$-sphere of the r handle. In the case of Heegaard splittings, the natural order of handle-addition can be rearranged exactly when there is an essential disk in H_1 and an essential disk in H_2 with disjoint boundaries. With this in mind, Casson and Gordon [6] defined a notion of weak reducibility of Heegaard splittings.

Definition 1.2.4. A Heegaard splitting $H_1 \cup_S H_2$ is *weakly reducible* if there are essential disks $D_i \subset H_i$, such that ∂D_1 and ∂D_2 are disjoint in S.

Any reducible Heegaard splitting $H_1 \cup_S H_2$ in a closed orientable 3-manifold is weakly reducible, by the following argument. Let D_i be properly embedded disks in H_i with $\partial D_1 = \partial D_2 = c$ a simple closed curve in S. As S is an orientable surface in an orientable 3-manifold, a regular neighbourhood of c is homeomorphic to $c \times [-1,1]$, with $\partial D_1 = \partial D_2 = c \times \{0\}$. Let $D_1' = D_1 \cup (c \times [0,1])$ and $D_2' = D_2 \cup (c \times [-1,0])$. After a slight perturbation we can assume D_1' and D_2' are properly embedded in H_1 and H_2 respectively, and are disjoint, so that S is a weakly reducible splitting. A splitting that is not weakly reducible is called *strongly irreducible.*

Weak reducibility has many topological consequences, primarily, a manifold with an irreducible, weakly reducible splitting contains an incompressible surface [6]. Thus in particular for non-Haken manifolds, every irreducible splitting is strongly irreducible.

There are manifolds with infinitely many non-isotopic strongly irreducible splittings. The first such examples were discovered by Casson, Gordon and Parris. Kobayashi [19] generalised their examples and found a class of manifolds with $p(g)$ (non-isotopic) strongly irreducible Heegaard splittings of genus g, where $p(g)$ increases polynomially fast in g.

The generalised Waldhausen conjecture states that orientable, atoroidal, irreducible 3-manifolds have are only finitely many Heegaard splittings of each genus up to isotopy. This was recently proved by Tao Li [20]. He also showed that for non-Haken 3-manifolds there are, in fact, only finitely many irreducible Heegaard splittings up to isotopy [21].

Moriah, Schleimer and Sedgwick [25] have shown that for all known examples of manifolds with infinitely many irreducible splittings, there exists a splitting surface H and a surface K, such that each of the splittings is given by the Haken sum $H + nK$, where n is some non-negative integer. They also show that such a surface K is incompressible.

These results lead us to state the following conjecture.

Conjecture 1.2.5. *Let M be a closed, orientable, irreducible and atoroidal 3-manifold with infinitely many strongly irreducible Heegaard splittings. Then, there exists an incompressible surface*

K and a strongly irreducible Heegaard splitting H such that there are infinitely many strongly irreducible Heegaard splittings given by the Haken sum $H + nK$, for $n \in \mathbb{N}$.

As a first step in this direction we aim to prove the following weaker result.

Conjecture 1.2.6. *Let M be a closed, orientable, irreducible and atoroidal 3-manifold, with infinitely many strongly irreducible Heegaard splittings. Then, there exists a strongly irreducible Heegaard splitting surface H and a sequence of (possibly disconnected) incompressible surfaces K_n such that $H_n = H + K_n$ is a sequence of strongly irreducible Heegaard splittings.*

Strongly irreducible Heegaard surfaces are almost normal surfaces. There exists an almost normal branch surface that carries all except finitely many of these surfaces. Tao Li [21] has shown that the limit of strongly irreducible Heegaard surfaces in the projective measured lamination space of this branch surface is an essential lamination.

We have defined a space of projective maximal laminations, and shown that it is compact and Hausdorff (in Chapter 5). Our attempt then is to show that an 'unbounded limit' of strongly irreducible Heegaard surfaces in this space is an essential lamination, using the same methods employed in [21].

2. EULER CHARACTERISTIC AND QUADRILATERALS OF NORMAL SURFACES

2.1 Introduction

The goal of this chapter is to give a relation between the Euler characteristic of a normal surface, a topological invariant, and the number of normal quadrilaterals in its embedding, obtained from its combinatorial description. Secondly, we get a relation between the number of normal triangles and normal quadrilaterals.

Theorem 2.1.1. *Let M be a 3-manifold with a pseudo-triangulation τ. Let F be a normal surface in (M, τ). Let Q be the number of normal quadrilaterals in F. Then,*

$$\chi(F) \geq 2 - 7Q$$

In particular, if F is an oriented, closed and connected normal surface of genus g,

$$g \leq \frac{7}{2}Q$$

Definition 2.1.2. Let F be a normal surface in M. Let t be a normal triangle of F that lies in a tetrahedron Δ. The triangle t is said to link a vertex v of Δ if t separates $\partial\Delta$ into two disks such that the disk containing v has no other vertices of Δ. Similarly, a normal arc α in a face $\mathcal{F}$ is said to link a vertex v of $\mathcal{F}$ if the segment containing v in $\partial\mathcal{F} - \alpha$ has no other vertices of $\mathcal{F}$.

Definition 2.1.3. Let $S(v)$ be the boundary of a small ball neighbourhood of v. The sphere $S(v)$ is a normal surface composed of normal triangles linking v, each from a distinct normal isotopy class. This is defined to be the vertex linking sphere linking vertex v.

Remark 2.1.4. Any closed connected normal surface S in M composed of normal triangles is normally isotopic to a vertex-linking sphere. This is due to the following reasons.

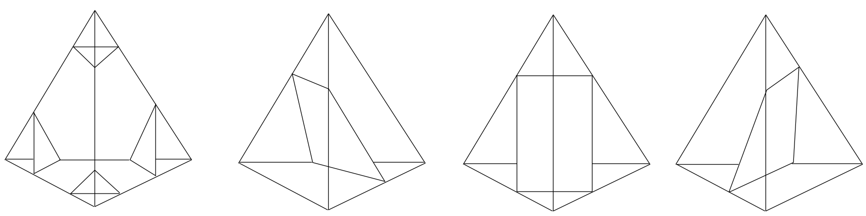

Fig. 2.1: The 4 Normal Triangle and 3 Normal Quadrilateral Types

Firstly, all normal triangles in S link the same vertex. If this were not true, there would be normal triangles t_1 and t_2 in S, linking distinct vertices of τ, that intersect in normal arcs. This is not possible as normal arcs linking different vertices of a face are not normally isotopic. So, let v be the common vertex linked by triangles of S. Then S is a cover of $S(v)$, as $S(v)$ is composed of one triangle from each of the normal isotopy classes of triangles linking vertex v. As $S(v)$ is a sphere and S is a closed connected cover of $S(v)$, the covering projection map is a homeomorphism. Therefore, there is only one triangle of S in each normal isotopy class of triangles linking v and so S is normally isotopic to $S(v)$.

This remark is the motivation for Theorem 2.1.1. The remark is not true for closed connected normal surfaces embedded in 3-complexes. For example, take a triangulation of a surface F and construct a 3-complex by taking the join with a common point of all triangles in F. Push F into the interior of this 3-complex so that F is embedded as a closed normal surface composed entirely of normal triangles.

We expect that in Riemannian 3-manifolds, triangles correspond to positive curvature pieces. This would imply that when quadrilaterals have a curvature that is bounded below, the smaller the Euler characteristic, the greater would be the number of quadrilaterals.

Remark 2.1.5. An inequality relating the Euler characteristic with the total number of normal disks is easy to obtain. This is because the number of normal discs gives a bound on the number of disjoint closed curves on the surface, up to isotopy. This, in turn, gives a lower bound on the Euler characteristic.

Our second theorem, Theorem 2.1.7, gives a relation between the number of triangles and the number of quadrilaterals of an embedded normal surface.

Definition 2.1.6. The degree of a vertex v with respect to a tetrahedron Δ, denoted by $N_v(\Delta)$, is the number of vertices of Δ that are equal to v. The degree of the vertex v, denoted by N_v, is the sum $\Sigma_{\Delta \in \tau} N_v(\Delta)$. This is equal to the number of triangles in a vertex-linking sphere linking the vertex v.

The degree N of the pseudo-triangulation τ is then defined as the maximum of N_v over all vertices v. This is the maximum number of triangles in a vertex-linking sphere in τ.

Let A be the union of triangles of an embedded normal surface F. Let v be a vertex of the pseudo-triangulation τ. Let σ be a *strongly connected* (defined in Definition 2.2.3) component of the triangles of A linking vertex v. By lemma 2.2.4, σ has at most N_v triangles. Any two such components are 'connected' by quadrilaterals. So we get an upper bound on the number of triangles in terms of the number of quadrilaterals and the degree of the pseudo-triangulation N.

Theorem 2.1.7. *Let F be a normal surface in (M, τ), no component of which is a vertex-linking sphere. Let T and Q be the number of normal triangles and normal quadrilaterals of F respectively.*

Let N be the degree of the pseudo-triangulation τ of M. Then,

$$T \leq 4NQ$$

2.2 An inequality relating the Euler characteristic and the number of quadrilaterals of normal surfaces

In this section we give a proof of Theorem 2.1.1. Let F be a normal surface in M. Let Q be the number of normal quadrilaterals in F. Let A (respectively B) be the union of normal triangles (respectively normal quadrilaterals) of F. Note that A and B are 2-complexes which are subsets of the surface F. Define the boundary of a 2-complex σ to be the union of edges of σ that do not intersect $int(\sigma)$.

An outline of the proof is as follows. We suitably modify the 2-complexes A and B, at singular points, to get surfaces A' and B' that intersect in a disjoint union of circles with $A' \cup B' = F$. We then show that the connected components of A' are planar surfaces. We, therefore, get a lower bound on the Euler characteristic of A' in terms of the number of components $|\partial A'|$ of $\partial A'$. The boundary of A' is the same as the boundary of B'. We introduce weights and get an inequality relating $|\partial B'|$ with the weight $w(\partial B)$ of ∂B. Now as $w(\partial B) \leq w(B) = 4Q$ we obtain an upper bound on $|\partial A'|$ in terms of the number of quadrilaterals. Thus, we get a lower bound on the Euler characteristic of A' in terms of the number of quadrilaterals.

For a bound on the Euler characteristic of B', we take an increasing union of quadrilaterals and use the fact that each new quadrilateral intersects the union of the preceding stage in a circle or at most 4 disjoint segments. This again gives us a lower bound in terms of the number of quadrilaterals.

As the Euler characteristic of F is the sum of Euler characteristics of A' and B', we get a lower bound for the Euler characteristic of F in terms of Q.

Definition 2.2.1. Let D_n be the collection of normal discs of F. Let $\{D''_m\}$ be the collection of triangles obtained by taking the second barycentric subdivision of all D_n. Then the regular neighbourhood of a point w in F is defined as $N(w) = int(\cup\{D''_m : w \in D''_m\})$.

Definition 2.2.2. A point w in A is said to be a singular point if $N(w) \cap A$ is not homeomorphic to $\mathbb{R}^2$ nor to the upper-half plane. Let ω be the set of singular points and $N(\omega) = \cup\{N(w) : w \in \omega\}$.

Let $A' = A - N(\omega)$, $B' = B \cup \overline{N(\omega)}$. Therefore, A' and B' are compact surfaces with $F = A' \cup B'$ and $A' \cap B'$ a disjoint union of circles. The surface A' is the closure of the interior of A, while B' is the closure of a neighbourhood of B in F.

Definition 2.2.3. A subcomplex σ of A is said to be strongly connected if for any two triangles t and t' in σ, there exists a sequence of triangles $\{t_i\}_{i=0}^{n}$ with $t = t_0$ and $t' = t_n$, such that for all i, t_i and t_{i+1} intersect in at least one normal arc. A subcomplex consisting of a single triangle is taken to be strongly connected. It is easy to see that A can be decomposed as a union of maximal strongly connected components.

Lemma 2.2.4. *The interior of a strongly connected component σ of A is a planar surface and there are at most N triangles in σ.*

Proof. Away from the singular vertices on the boundary, σ is locally $\mathbb{R}^2$ or the upper half plane. So in particular, the interior of σ is a surface. We shall show that it is planar.

All triangles in σ link the same vertex. Otherwise, as σ is strongly connected, there would be a common normal arc shared by normal triangles in σ linking distinct vertices. This is a contradiction as normal arcs in a face, linking different vertices are not normally isotopic. So we may assume that all triangles of σ link the same vertex v.

Let $S(v)$ be a vertex-linking sphere linking the vertex v. Let $\pi : \sigma \to S(v)$ be the projection map that takes each triangle in σ to the corresponding triangle in $S(v)$ (in the same tetrahedron). We claim that $\pi|_{int(\sigma)}$ is an embedding, so that $int(\sigma)$ is a planar surface and as there are at most N triangles in $S(v)$ there are at most N triangles in σ as well.

If $\pi|_{int(\sigma)}$ is not an embedding, then there exist triangles t and t' in σ such that $\pi(t) = \pi(t')$. We say triangle t is below triangle t' if t and t' both link the same vertex v and lie in the same tetrahedron with t in between v and t'. We may assume that t is below t'.

As σ is strongly-connected, there exists a sequence of triangles $\{t_i\}_{i=0}^k$ such that $t_0 = t'$, $t_k = t$ and t_i intersects t_{i+1} in a normal arc. Note that t_k is below t_0. We now extend this sequence to a maximal sequence of triangles in σ with the property that t_i intersects t_{i+1} in a normal arc and t_i is below t_{i-k} for all $i \geq k$. As σ is compact, this sequence is finite.

Suppose this sequence extended till t_N. Let Δ be the tetrahedron containing t_{N-k+1}. Let α be the normal arc $t_N \cap \Delta$ and let α' be the normal arc $t_{N-k} \cap t_{N-k+1}$. Then as t_N is below t_{N-k}, α is below α'.

As the surface F has no boundary, there exists a normal disk D in Δ that contains α. If D were a triangle then we could extend the sequence $\{t_i\}_{i=0}^N$ by appending D to it, contradicting the maximality of this sequence. So D must be a quadrilateral. But as α is below α', D would necessarily intersect the triangle t_{N-k+1}, contradicting the fact that F is embedded in M. Hence, $\pi|_{int(\sigma)}$ must be an embedding. □

Lemma 2.2.5. *Let $|A'|$ denote the number of components of A'. Then,*

$$\chi(A') \geq 2|A'| - 4Q$$

Proof.

Claim. $\chi(A') = 2|A'| - |\partial A'|$.

Let $\{A_n\}$ be the strongly connected components of A. Then, $int(A') = int(A) = \cup_n int(A_n)$. By Lemma 2.2.4, the interior of A_n are planar surfaces. So, as A' is a surface (not just a 2-complex), A' is itself a disjoint union of connected planar surfaces.

A connected planar surface S with b boundary components has $\chi(S) = 2 - b$. As A' is a disjoint union of connected planar surfaces $\chi(A') = 2|A'| - |\partial A'|$ as claimed.

Definition 2.2.6. Let σ be a 2-complex. Define $w(e)$, weight of an edge $e \in \sigma$, to be

$$w(e) = \begin{cases} 1 & \text{if } e \subset \partial\sigma \\ 2 & \text{otherwise} \end{cases}$$

Define $w(\sigma)$, the weight of σ, to be the total weight of all edges of σ.
For ψ a 1-complex. Define $w(\psi)$ to be the total number of edges of ψ.
From the definition, it is clear that $w(\sigma) \geq w(\partial\sigma)$.

Claim. $4Q \geq |\partial A'|$

As $\partial B'$ is obtained from ∂B by smoothening around singular points, each component of $\partial B'$ contains the midpoint of some edge of ∂B. As $\partial B'$ is a disjoint union of circles, the midpoint of an edge of ∂B can not lie in two components of $\partial B'$. As a result, the number of edges in ∂B is at least the number of components of $\partial B'$, i.e $w(\partial B) \geq |\partial B'|$. The weight of B is $4Q$, so that $4Q = w(B) \geq w(\partial B) \geq |\partial B'|$. As $\partial B' = \partial A'$, we obtain the claimed inequality.

Using the above two Claims, we get the desired relation

$$\chi(A') \geq 2|A'| - 4Q$$

□

Lemma 2.2.7. *For $B \neq \phi$,* $\chi(B') \geq \begin{cases} |\omega| - 3Q & \text{when } \omega \neq \phi \\ 4 - 3Q & \text{when } \omega = \phi \end{cases}$

Proof. Let $B'_0 = \overline{N(\omega)}$. Let B'_n be a union of n quadrilaterals of B' such that $B'_{n+1} \supset B'_n$. For a quadrilateral q such that $B'_{n+1} = B'_n \cup q$, $B'_n \cap q$ is either empty, a circle or is homotopically equivalent to a set with at most four points. So, $\chi(B'_n \cap q) \leq 4$, and we get

$$\begin{aligned} \chi(B'_{n+1}) &= \chi(B'_n) + \chi(q) - \chi(B'_n \cap q) \\ &\geq (\chi(B'_n) + 1) - 4 \end{aligned}$$

As $\overline{N(\omega)}$ is a disjoint union of discs,

$$\chi(B'_0) = |\omega|$$

Therefore by induction on n, we have $\chi(B'_n) \geq |\omega| - 3n$ when $\omega \neq \phi$, and if $\omega = \phi$ then $\chi(B'_1) = 1$ and we get $\chi(B'_n) \geq 1 - 3(n-1)$. Putting $n = Q$, we have $B'_Q = B'$ giving us the desired result. □

Proof of Theorem 2.1.1. When $B = \phi$, the surface F is composed of normal triangles and so is a union of vertex linking spheres. Therefore $\chi(F) \geq 2$.

When $B \neq \phi$, as $A' \cap B'$ is a disjoint union of circles, so using Lemmas 2.2.5 and 2.2.7 we get,

$$\begin{aligned} \chi(F) &= \chi(A' \cup B') \\ &= \chi(A') + \chi(B') \\ &\geq \begin{cases} (2|A'| - 4Q) + (|\omega| - 3Q) & \text{when } \omega \neq \phi \\ (2|A'| - 4Q) + 4 - 3Q & \text{when } \omega = \phi \end{cases} \end{aligned}$$

When $|A'| = 0$ the surface is composed entirely of quadrilaterals, so that $B \neq \phi$ and $\omega = \phi$. So, we get the desired relation

$$\chi(F) \geq 2 - 7Q$$

□

2.3 An inequality relating the number of triangles and quadrilaterals of normal surfaces

In this section we prove an inequality relating the number of triangles and quadrilaterals of a normal surface.

Proof of Theorem 2.1.7. Let $F(v)$ be the union of normal triangles of F linking the vertex v. Let $\{F(v)_n\}$ be the strongly connected components of $F(v)$.

Consider a graph Γ whose vertices are the normal quadrilaterals of F and the non-empty $F(v)_n$. Call the former vertices Q-vertices and the latter as S-vertices. Let there be an edge of Γ joining an S-vertex and a Q-vertex for every edge shared by the corresponding $F(v)_n$ and the corresponding quadrilateral. Let there also be edges of Γ between Q-vertices corresponding to every edge shared by the corresponding quadrilaterals. For a quadrilateral in the pseudo cell decomposition of the surface we could have simple loops corresponding to quadrilaterals that are not injective on its edges.

As each quadrilateral has 4 sides, the degree of each Q-vertex is 4 and as there are no vertex-linking spheres, there are no isolated S-vertices. As each edge of Γ has at least one vertex incident on a Q-vertex (no edges between S-vertices) therefore, $4Q =$ total degree of Q-vertices $\geq$ total degree of S-vertices $\geq S$, where Q and S are the number of Q-vertices and S-vertices respectively.

By Lemma 2.2.4 each $F(v)_n$ has at most N triangles, where N is the degree of the pseudo-triangulation τ. Therefore, $NS \geq T$, where T is the number of triangles in F. Therefore, we get the required relation,

$$T \leq 4NQ$$

□

Remark 2.3.1. This proof also holds for a normal surface F embedded in a 3-complex, if we assumed that no component of F was composed solely of normal triangles.

3. A CHAIN COMPLEX AND QUADRILATERALS FOR NORMAL SURFACES

3.1 Introduction

Casson-Rubinstein independently observed that a normal surface is essentially determined by its normal quadrilaterals. Their (unpublished) observation was that normal surfaces are determined up to vertex linking spheres by quadrilateral coordinates. This allows a considerable increase in efficiency of algorithms based on normal surfaces.

The purpose of this chapter is to clarify this observation, as well as the complementary question of when a given set of quadrilateral coordinates corresponds to a normal surface, by interpreting normal surfaces in terms of the homology of a chain complex associated to M. Our methods also allow us to address the analogous questions for *ideal triangulations*. A criterion for quadrilateral coordinates determining a normal surface and a proof of Casson-Rubinstein's observation was earlier given by Tollefson[40] for compact manifolds, using geometric constructions. Tillmann [39] proves a similar result for ideal triangulations in the context of spun-normal surfaces. Spun-normal surfaces, introduced by Thurston, are the analogue of normal surfaces in ideal triangulations. For a detailed treatment of spun-normal surfaces we refer to [39].

As we wish to consider ideal triangulations, we consider a context more general than triangulated 3-manifolds. Namely, let M be an orientable three-dimensional simplicial complex that is a manifold away from vertices, and so that the link of each vertex v is a closed, connected, orientable surface (not necessarily a sphere). We can define normal surfaces in this situation exactly as in the case of 3-manifolds.

Henceforth, we assume M is as above. We can associate to a vertex v the vertex linking normal surface $S(v)$, which is a closed orientable surface (but not in general a sphere). The space $\hat{M}$ obtained from M by deleting those vertices v for which $S(v)$ is not a sphere is a (non-compact) 3-manifold with an ideal triangulation.

3.2 The chain complex

In this section, we associate a chain complex $(\mathcal{C}, \partial_*)$ to M, such that normal surfaces are in bijection with cycles of $\mathcal{C}_2$ (Lemma 3.2.2).

Fix an orientation of M. For each vertex v, assume that $S(v)$ is oriented so that its co-orientation at each point is along a vector pointing away from v. As $S(v)$ is a union of normal triangles (linking v), we get a triangulation of $S(v)$. Let $(C_*(v), \partial_*(v))$ be the simplicial chain complex associated to this triangulation. Then, we shall show that $C_2(v)$ embeds in $\mathcal{C}_2$, $C_1(v)$ embeds in $\mathcal{C}_1$ and the restriction of the boundary map ∂_2 (of $\mathcal{C}_2$) to $C_2(v)$ agrees with $\partial_2(v)$ (Proposition 3.2.1).

We now define the chain complex $(\mathcal{C}_*, \partial_*)$ as follows.

Firstly, observe that a normal arc is uniquely determined up to normal isotopy by the face in which it lies and the vertex that it links. Let v be a vertex of a face F, then we denote by $\alpha(F, v)$ the normal arc that lies in F and links v.

We give an arbitrary orientation to the edges of the triangulation of M and let $e(F, v)$ denote the edge in F opposite to v. We orient the normal arc $\alpha(F, v)$ so that it is in the same direction as $e(F, v)$. Let $\mathcal{C}_1$ be the free abelian group generated by these oriented normal arcs up to normal isotopy.

Let $\mathcal{C}_2^t$ be the free abelian group generated by normal triangles (up to normal isotopy) and $\mathcal{C}_2^q$ be the free abelian group generated by normal quadrilaterals (up to normal isotopy). Define $\mathcal{C}_2 = \mathcal{C}_2^t \oplus \mathcal{C}_2^q$ to be the free abelian group generated by normal disks (up to normal isotopy). For all $k < 1$ and $k > 2$, let $\mathcal{C}_k$ be zero.

Next we define the boundary maps of $(\mathcal{C}_*, \partial_*)$. Take ∂_k to be zero for all $k \neq 2$. To define the boundary map ∂_2, we proceed as follows.

Let v be a vertex of a face F of a tetrahedron Δ. Let $e(F, v)$ denote the edge in F opposite to v, and let $\hat{e}(F, v)$ denote its midpoint. Let $a(\Delta, F, v)$ denote a unit vector based at $\hat{e}(F, v)$, perpendicular to F pointing out of Δ (when translated along $\partial\Delta$ so that its base point lies in the interior of F). Let $F(\Delta, v)$ be the face in Δ opposite to v. Let $b(\Delta, F, v)$ denote the unit vector based at $\hat{e}(F, v)$ perpendicular to $F(\Delta, v)$ pointing out of Δ (when translated along $\partial\Delta$ so that its base point lies in the interior of $F(\Delta, v)$). Let $u(\Delta, F, v) = a(\Delta, F, v) \times b(\Delta, F, v)$ be their cross product with respect to the orientation of M. Observe that $u(\Delta, F, v)$ either points along $e(F, v)$ or in the direction opposite to it.

Given a normal disk D in Δ, suppose that ∂D is the union of normal arcs $\{\alpha(F, v)\}_{(F,v)\in A}$. Recall that $\alpha(F, v)$ is oriented in the direction of $e(F, v)$. The boundary map $\partial_2(D)$ is defined to be $\sum_{(F,v)\in A} \varepsilon(F, v)\alpha(F, v)$ where $\varepsilon(F, v)$ is 1 if $\alpha(F, v)$ (and consequently $e(F, v)$) is oriented along $u(\Delta, F, v)$, and is (-1) otherwise. This extends uniquely to a homomorphism $\partial_2 : \mathcal{C}_2 \to \mathcal{C}_1$.

It is easy to see the following proposition.

Proposition 3.2.1. *There is an embedding of $C_2(v)$ in $\mathcal{C}_2^t \subset \mathcal{C}_2$, and an embedding of $C_1(v)$ in $\mathcal{C}_1$. When D is a triangle linking a vertex v, $\partial_2 D$ has edges oriented cyclically. Also, the co-orientation of D coming from the cyclically oriented edges is the normal vector pointing away from v. So the boundary map ∂_2 restricted to $C_2(v)$ is $\partial_2(v)$, the (usual) boundary map of $C_2(v)$.*

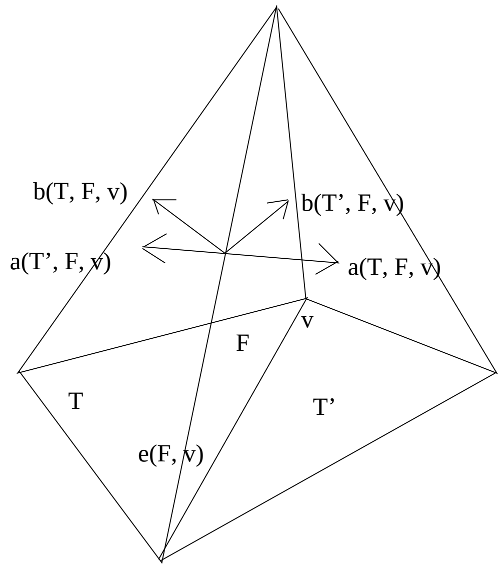

Fig. 3.1: The vectors determining the boundary map

We can interpret normal surfaces in terms of the chain complex $(\mathcal{C}_*, \partial_*)$ as follows.

Lemma 3.2.2. *There is a bijective correspondence between normal coordinates and* 2*-chains of the chain complex. Further, normal coordinates corresponding to a* 2*-chain* ξ *satisfy the matching equations if and only if* $\partial_2\xi = 0$*.*

Proof. The first statement follows as $\mathcal{C}_2$ is the free abelian group generated by normal isotopy classes of normal discs.

Let $F = T \cap T'$ be a common face of the tetrahedra T and T'. Let $D \subset T$ and $D' \subset T'$ be normal disks such that $D \cap F = D' \cap F = \alpha(F, v)$ for some vertex v of F. Then clearly $a(T, F, v) = -a(T', F, v)$. Moreover, we can observe that there exists $\lambda \in \mathbb{R}$ such that $b(T, F, v) = b(T', F, v) + \lambda a(T, F, v)$ (see Fig 3.1). Therefore, $u(T, F, v) = -u(T', F, v)$ and so $\varepsilon(T, F, v) = -\varepsilon(T', F, v)$.

So, the boundary of a 2-chain ξ is zero if and only if for each normal arc α in face $F = \Delta_1 \cap \Delta_2$, the number of normal disks of ξ in Δ_1 that have α in their boundary equals the number of normal disks of ξ in Δ_2 having α in their boundary. This is precisely when ξ is a solution of the matching equations. Therefore, $\partial_2(\xi) = 0$ if and only if its normal coordinates satisfy the matching equations. □

Thus, as there are no three-chains, normal surfaces are in bijective correspondence with the homology $H_2(\mathcal{C})$.

3.3 Quadrilateral co-ordinates

We now turn to the question regarding quadrilateral co-ordinates determining normal surfaces. Quadrilateral co-ordinates are in bijective correspondence with chains $\zeta \in \mathcal{C}_2^q$. We shall henceforth consider such 2-chains.

Note that admissibility is a condition determined by the quadrilateral coordinates. We shall assume that ζ corresponds to non-negative, admissible quadrilateral coordinates.

Corresponding to the decomposition $\mathcal{C}_1 = \bigoplus_{v \in V} C_1(v)$, we define homomorphisms $\bar{\partial}_v : \mathcal{C}_2 \to C_1(v)$ as the composition $\pi(v) \circ \partial_2$ of the boundary map with the projection onto $C_1(v)$. As $\mathcal{C}_1 = \bigoplus_{v \in V} C_1(v)$, for $\xi \in \mathcal{C}_2$, $\partial_2(\xi) = 0$ if and only if $\bar{\partial}_v(\xi) = 0$ for all $v \in V$.

As $\mathcal{C}_2 = \mathcal{C}_2^t \oplus \mathcal{C}_2^q$, by Lemma 3.2.2 the 2-chain ζ corresponds to quadrilateral co-ordinates of a normal surface F with normal coordinates ξ if and only if there is a 2-chain $\zeta' \in \mathcal{C}_2^t$ with $\partial_2(\zeta + \zeta') = 0$. In this case, the normal co-ordinates of F are $\xi = \zeta + \zeta'$.

We first give a necessary condition for ζ to correspond to the quadrilateral coordinates of a normal surface.

Theorem 3.3.1. *There is a normal surface F with quadrilateral coordinates corresponding to ζ if and only if $\bar{\partial}_v \zeta \in C_1(v)$ is a boundary in $C_*(v)$ for all $v \in V$.*

Proof. First, assume that ζ corresponds to the quadrilateral co-ordinates of a surface F. Then there is a 2-chain $\zeta' \in \mathcal{C}_2^t$ with $\partial(\zeta + \zeta') = \partial\zeta + \partial\zeta' = 0$. Hence for each vertex $v \in V$, $\bar{\partial}_v \zeta + \bar{\partial}_v \zeta' = 0$

As $\mathcal{C}_2^t = \bigoplus C_2(v)$, we can write $\zeta' = \bigoplus_{v \in V} \zeta'(v)$, $\zeta'(v) \in C_2(v)$. For each $v \in V$, $\bar{\partial}_v \zeta' = \partial_2(v)\zeta'(v)$ is a boundary in the complex $C_*(v)$. Hence $\bar{\partial}_v \zeta = -\bar{\partial}_v \zeta'$ is also a boundary.

Conversely, if $\bar{\partial}_v \zeta$ is a boundary for each $v \in V$, then there are 2-chains $\zeta'(v) \in C_2(v)$ with $\partial_2(v)\zeta'(v) = -\bar{\partial}_v \zeta$. We claim that we can choose $\zeta'(v)$ so that all the corresponding (triangle) coordinates are non-negative. Namely, as $S(v)$ is closed and oriented, the sum of the triangles in $S(v)$ is a cycle $[S(v)]$. By replacing $\zeta'(v)$ by $\zeta'(v) + k[S(v)]$, for k sufficiently large, we can ensure that all the co-ordinates are non-negative.

Let $\zeta' = \Sigma_{v \in V} \zeta'(v) \in \mathcal{C}_2^t$. By construction $\bar{\partial}_v(\zeta + \zeta') = 0$ for all $v \in V$, and hence $\partial(\zeta + \zeta') = 0$.

Let $\xi = \zeta + \zeta'$. By Lemma 3.2.2, ξ satisfies the matching equations. Further, as ζ is assumed to correspond to admissible, non-negative quadrilateral coordinates, and the coordinates of $\zeta'(v)$ are non-negative triangular coordinates, ξ is an admissible, non-negative solution. □

Remark 3.3.2. When $\bar{\partial}_v \zeta$ is a cycle in $C_1(v)$ for all $v \in V$, then ζ is the quadrilateral coordinates of a normal or spun-normal surface. The above theorem says that when $\bar{\partial}_v \zeta$ is in fact a boundary the normal or spun-normal surface is compact, so it is a normal surface.

In the important case where M is a manifold, Theorem 3.3.1 takes a particularly useful form.

Corollary 3.3.3. *If M is a manifold, ζ corresponds to quadrilateral coordinates of a normal surface if and only if $\bar{\partial}_v(\zeta) \in C_1(v)$ is a cycle in $C_*(v)$ for all $v \in V$.*

Proof. This follows from Theorem 3.3.1 as $H_1(S(v), \mathbb{Z}) = 0$. □

The class $\bar{\partial}_v(\zeta)$ is a cycle if and only if its boundary is zero. This is a condition that is simple to check and also conceptually very simple.

In the general case, we need to check whether $\bar{\partial}_v(\zeta)$ is a cycle and represents the trivial homology class. The latter can be checked, for instance, by evaluating on a basis of cohomology.

We now turn to Casson-Rubinstein-Tollefson's observation on uniqueness. The following is a useful way to state the result.

Theorem 3.3.4 (Casson-Rubinstein-Tollefson)**.** *Let ζ be an admissible, non-negative set of quadrilateral coordinates that can be represented by a normal surface. Then there is a set of admissible, non-negative normal surface coordinates ξ corresponding to ζ such that if ξ' is another set of such coordinates, then $\xi' = \xi + \sum_{v \in V} m_v[S(v)]$, with $m_v \geq 0$.*

Proof. By Theorem 3.3.1, $\bar{\partial}_v\zeta$ is the boundary of a 2-chain $\zeta'(v) \in C_2(v)$. If $\zeta''(v)$ is another such 2-chain, then $\zeta'(v) - \zeta''(v)$ is a 2-cycle, hence represents an element of the homology $H_2(S(v), \mathbb{Z})$. As $H_2(S(v), \mathbb{Z}) = \mathbb{Z}$ and is generated by $S(v)$, $\zeta''(v) = \zeta'(v) + m[S(v)]$.

Consider the coefficients of the triangles of $S(v)$ in $\zeta'(v)$ and let m be the smallest such coefficient. The chain $\zeta'(v) - m[S(v)]$ then has all coefficients non-negative and at least one coefficient zero. Further, if we replace $\zeta'(v)$ by $\zeta'(v) - m[S(v)]$, we see that for any non-negative chain $\zeta''(v)$ with $\partial(v)\zeta''(v) = \partial(v)\zeta'(v)$, $\zeta''(v) = \zeta'(v) + m'[S(v)]$ with $m' \geq 0$.

Now let $\zeta' = \sum_{v \in V} \zeta'(v)$ and let $\xi = \zeta + \zeta'$. It is easy to see that ξ is as claimed. □

Let S be a normal surface, and let (S) denote its quadrilateral coordinates. Then the above theorem says that there exists a normal surface F with $(F) = (S)$ such that if F' is any other normal surface with $(F') = (S)$ then F' is the union of F with some vertex-linking surfaces.

4. INCOMPRESSIBILITY AND NORMAL MINIMAL SURFACES

4.1 Introduction

Given a Riemannian manifold (M, g), we can scale the metric by multiplying g with a smooth positive real-valued function. Such a rescaling may, however, introduce new minimal surfaces. Given a triangulated 3-manifold (M, τ), we can scale the PL-metric by taking a refinement of τ, by repeatedly subdividing the tetrahedra in τ according to a positive integer-valued scaling function. In general, such a scaling may introduce new minimal normal surfaces. We describe here a procedure for scaling the PL-metric that introduces no new normal surfaces.

Definition 4.1.1. Let Δ be a tetrahedron with vertices labeled $\{a, b, c, d\}$. Let e be a point in the interior of Δ. Take a simplicial triangulation of Δ using the tetrahedra $\Delta_A = [b, c, d, e], \Delta_B = [a, c, d, e], \Delta_C = [a, b, d, e]$ and $\Delta_D = [a, b, c, e]$. Define ϕ on a triangulation τ to be the function that gives a refinement of τ by dividing each tetrahedron Δ of τ into 4 tetrahedra, as described above. We call this the refinement function. This is shown in Figure 4.1 where the additional edges in the refinement of Δ are shown as dotted-lines.

Let $f : \{\Delta : \Delta \in \tau\} \to \mathbb{Z}$ be a function that associates a non-negative integer to each tetrahedron Δ of τ. We call such a function a scaling function. Define ϕ_f on τ to be the function that gives to each $\Delta \in \tau$ the triangulation $\phi^{f(\Delta)}(\Delta)$, obtained by taking $f(\Delta)$ iterates of ϕ on Δ. As the faces of Δ are also faces of $\phi(\Delta)$, $\phi_f(\tau) = \tau'$ is a refined triangulation of τ.

The main theorem in this chapter is Theorem 4.1.2.

Theorem 4.1.2. *Let F be a closed surface embedded in a 3-manifold M no component of which*

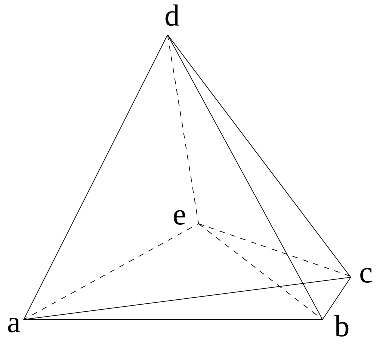

Fig. 4.1: A tetrahedron in τ, partitioned by ϕ.

is a 2-sphere. Let τ be a triangulation of M. Let $f : \{\Delta : \Delta \in \tau\} \to \mathbb{Z}$ be a scaling function and let $\tau' = \phi_f(\tau)$ be the corresponding refinement of τ. Then, F is τ-normal $\Leftrightarrow$ F is τ'-normal.

Every τ-normal surface is τ'-normal by observing that each τ-normal disk is a union of τ'-normal disks, as shown in lemma 4.2.5. For the converse, the proof depends on a simple examination of the possible τ'-normal proper embeddings of a surface in a tetrahedron Δ of the triangulation τ. We show that every τ'-normal surface within Δ is in fact a τ-normal disk, hence every τ'-normal surface is also τ-normal.

In the second part of this chapter we use such a refinement of the triangulation to obtain a PL-analogue of the theorem proved in [8]. It is known that if F is a smooth incompressible surface in an irreducible Riemannian 3-manifold M, then the isotopy class of F has a least area surface. The theorem proved by Gadgil in [8] proves the converse, that is, if F is a smooth surface in a closed, irreducible 3-manifold M such that for each Riemannian metric g of M, F is isotopic to a least-area surface $F(g)$, then F is incompressible.

Similarly, in the PL case, it is known that an incompressible surface F in a triangulated 3-manifold M is isotopic to a normal surface that is of minimal PL-area in the isotopy class of F. We prove here the converse.

Theorem 4.1.3. *Let F be a closed orientable surface in an irreducible orientable closed 3-manifold M. Then, F is incompressible if and only if for any triangulation τ of M, there exists a τ-normal surface $F(\tau)$ isotopic to F that is of minimal PL-area in the isotopy class of F.*

If F is an incompressible surface that is not normal in a triangulation τ of M, then it is known that a PL-area decreasing isotopy exists. To prove the converse, we show that given a compressible surface F, there exists a triangulation τ' for which the isotopy class of F has no normal minimal surface. An outline of the proof is as follows. Let $\hat{F}$ be the surface obtained by compressing F along a compressing disc. Therefore, F is obtained from $\hat{F}$ by attaching a 1-handle to $\hat{F}$. We start with a certain 'prism' triangulation of a regular neighbourhood $N(\hat{F})$ of $\hat{F}$ which is such that any connected normal surface lying in $N(\hat{F})$ is isotopic to a component of $\hat{F}$. We extend this triangulation to a triangulation τ of M. Let $Ar(F)$ be the PL-Area of F in τ, then as the 1-handle can be chosen to avoid all edges of τ, we always get a representative of F (in its isotopy class) such that $Ar(F) = Ar(\hat{F})$.

We define a scaling function $f : \{\Delta : \Delta \in \tau\} \to \mathbb{Z}$ that takes the value 0 on $\Delta \subset N(\hat{F})$ and a value greater than $Ar(\hat{F})$ for Δ not in $N(\hat{F})$. We now take the refinement τ' of τ given by ϕ_f. As every τ'-normal surface is also τ-normal by Theorem 4.1.2, a τ'-normal surface that does not lie in $N(\hat{F})$ has a τ-normal disk outside $N(\hat{F})$. This disk has τ' PL-area more than the τ' PL-area of $\hat{F}$. As $\hat{F}$ is not homeomorphic to F, a normal surface that lies entirely in $N(\hat{F})$ is not isotopic to F, while a normal surface that does not lie in $N(\hat{F})$ has τ' PL-area more than that of $\hat{F}$. As there is always a surface isotopic to F that has τ' PL-area equal to that of $\hat{F}$ (and which is not normal) so, the isotopy class of F has no normal minimal-area surfaces in τ'.

4.2 Proof of Theorem 4.1.2

In this section, we first show lemma 4.2.4 which determines normal disks using normal arcs in the boundary of the disk. Then, using Fig 4.2 we prove lemma 4.2.5 which says that a τ-normal disk is a τ'-normal surface. We then prove Theorem 4.1.2. We introduce the following notation:

Definition 4.2.1. A τ'-normal triangle T in $\Delta_X \subset \Delta, X \in \{A, B, C, D\}$, is said to link a vertex w in Δ_X if $\partial\Delta_X - \partial T$ has a component that contains the vertex w and no other vertices of Δ_X. We say the coordinates of T are $[T] = (X, T_w)$. When the context is clear we shall denote the triangle T itself by its coordinates (X, T_w).

Similarly, a τ'-normal quadrilateral Q in $\Delta_X \subset \Delta, X \in \{A, B, C, D\}$, is said to link an edge yz in Δ_X if $\partial\Delta_X - \partial Q$ has a component that contains the vertices y and z, and no other vertices of Δ_X. We say the coordinates of Q are $[Q] = (X, Q_{yz})$. When the context is clear we shall denote the quadrilateral Q itself by its coordinates (X, Q_{yz}).

Definition 4.2.2. A τ'-normal arc λ is said to link a vertex x (respectively an edge yz) in a face F of τ' if $F - \lambda$ has a component that contains the vertex x and no other vertices of F (respectively contains the vertices y and z and no other vertices of F). Denote the set of τ'-normal arcs in faces of tetrahedra of τ', linking vertex x (respectively edge yz) by Λ_x (respectively Λ_{yz}).

Definition 4.2.3. We define $\Lambda_x * \Lambda_y$ (respectively $\Lambda_x * \Lambda_{xy}$) to be the set of τ'-normal paths that are not contained in a single face, and are given by the concatenation of an arc in Λ_x with an arc in Λ_y (respectively Λ_{xy}).

We now state the following lemma which says that given a pair of contiguous normal arcs in the boundary of a normal disc, we can determine whether the disc is a triangle or a quadrilateral and we can determine which vertex (respectively which edge) it links. Also, if we are given that the normal disk is a triangle (respectively a quadrilateral) and we are given one normal arc in its boundary, then the vertex linked by the normal triangle (respectively the edge linked by the normal quadrilateral) can be determined.

Lemma 4.2.4. *For a τ'-normal disk D with a normal path $\lambda \subset \partial D$,*
(i) If D is a triangle with $\lambda \in \Lambda_x$ then D is a triangle linking the vertex x.
(ii) If D is a quadrilateral with $\lambda \in \Lambda_{xy}$ then D is a quadrilateral linking the edge xy.
(iii) If D is a quadrilateral in the tetrahedron $[w, x, y, z]$ with λ in the face $[x, y, z]$ and $\lambda \in \Lambda_z$, then D is a quadrilateral linking the edge wz.
(iv) If $\lambda \in \Lambda_x * \Lambda_x$ then D is a triangle linking the vertex x.
(v) If $\lambda \in \Lambda_x * \Lambda_{xy}$ then D is a quadrilateral linking the edge xy.

Proof. We make the following simple observations:
a. A normal disk D is a quadrilateral if and only if each normal arc in ∂D links a distinct vertex, which is the same as saying normal arcs in ∂D link more than one vertex.
b. A normal triangle T in a tetrahedron $[w, x, y, z]$ links the vertex x if and only if any normal arc in ∂T links x.
c. A normal quadrilateral Q in a tetrahedron $[w, x, y, z]$ links edge xy if and only if $\partial Q \cap xy = \phi$.

The statement (i) follows from observation b.

Let D be a quadrilateral in a tetrahedron $[w, x, y, z]$, with $\lambda \in \Lambda_{xy}$ and λ contained in the face $F = [x, y, z]$. Then $\partial\lambda$ is contained in the arcs xz and yz. Therefore, if $\partial D \cap xy \neq \phi$ then as ∂D is transverse to edges, ∂D is a circle in $\partial\Delta$ transversely intersecting each edge of the triangle $[x, y, z]$. Therefore, ∂D must intersect some edge of this triangle more than once. This is a contradiction as D is a normal disc, so that ∂D intersects each edge at most once. Therefore, $\partial D \cap xy = \phi$ and statement (ii) follows from observation c.

Statement (iii) follows from a similar argument replacing Λ_{xy} with Λ_z and observing that quadrilaterals that link xy are precisely the quadrilaterals that link wz.

Statement (iv) follows from observations a and b.

The disk D in statement (v) is a quadrilateral from observation a. As there exists an arc $\lambda \subset \partial D$ with $\lambda \in \Lambda_{xy}$, from statement (ii) we can see that D links the edge xy. □

We now state the lemma which shows that every τ-normal disk is a union of τ'-normal disks (see Figure 4.2).

Lemma 4.2.5. *Let S be a properly embedded surface in Δ. Then,*
(i) S is a τ'-normal surface with $S = (A, T_d) \cup (B, T_d) \cup (C, T_d)$ or $S = (A, Q_{de}) \cup (B, Q_{de}) \cup (C, Q_{de}) \cup (D, T_e) \Leftrightarrow S$ is τ-isotopic to a τ-normal triangle linking vertex d.
(ii) S is a τ'-normal surface with $S = (D, T_a) \cup (B, Q_{ad}) \cup (C, Q_{ad}) \cup (A, T_d)$ or $S = (B, T_c) \cup (D, Q_{bc}) \cup (A, Q_{bc}) \cup (C, T_b) \Leftrightarrow S$ is τ-isotopic to a τ-normal quadrilateral linking edge ad.
(iii) S is a τ'-normal surface with $S = (A, T_e) \cup (B, T_e) \cup (C, T_e) \cup (D, T_e) \Leftrightarrow S$ is a τ' vertex-linking sphere linking the vertex e.

Proof. If S is a τ-normal triangle in the tetrahedron Δ, linking vertex d, then after a τ-normal isotopy we may assume that $S = \partial B(d) \cap \Delta$, where $B(d)$ is a small ball neighbourhood of d in M. This is shown in Figure 4.2 (i) a. Then, S intersects the faces of τ' as shown in Figure 4.2 (i) b, so that $S = (A, T_d) \cup (B, T_d) \cup (C, T_d)$. Conversely, if $S = (A, T_d) \cup (B, T_d) \cup (C, T_d)$ (Figure 4.2 (i) b) or $S = (A, Q_{de}) \cup (B, Q_{de}) \cup (C, Q_{de}) \cup (D, T_e)$ (Figure 4.2 (ii) b.), then S is a properly embedded disk in Δ with ∂S a circle in $\partial\Delta$ linking vertex d. So, S is a τ-normal triangle linking vertex d.

If S is a τ-normal quadrilateral in the tetrahedron Δ, linking edge ad, then after a τ-normal

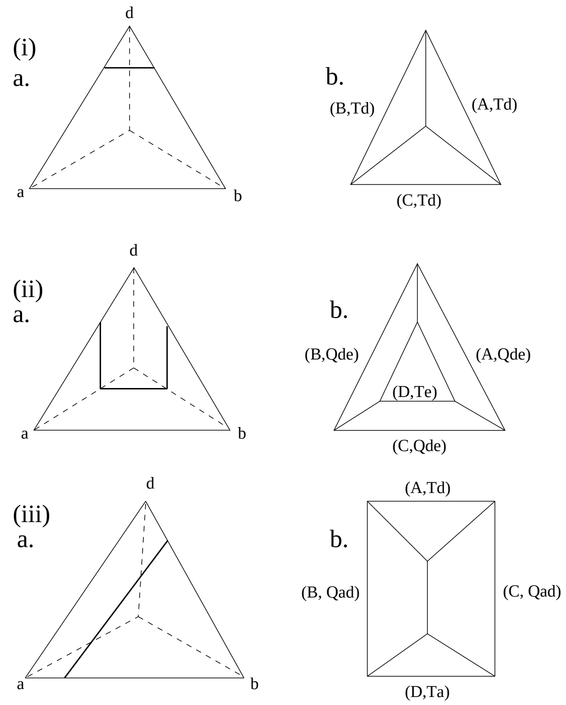

Fig. 4.2: Diagrams (i) and (ii) represent a normal triangle linking vertex d. Diagram (iii) represents a normal quadrilateral linking edge ad.

isotopy we may assume that $S = \partial B(ad) \cap \Delta$, where $B(ad)$ is a small ball neighbourhood of ad in M. This is shown in Figure 4.2 (iii) a. Then, S intersects the faces of τ' as shown in Figure 4.2 (iii) b, so that $S = (D, T_a) \cup (B, Q_{ad}) \cup (C, Q_{ad}) \cup (A, T_d)$. Conversely if $S = (D, T_a) \cup (B, Q_{ad}) \cup (C, Q_{ad}) \cup (A, T_d)$ (Figure 4.2 (iii) b) or $S = (B, T_c) \cup (D, Q_{bc}) \cup (A, Q_{bc}) \cup (C, T_b)$(corresponding to a quadrilateral linking the edge bc), then S is a properly embedded disk in Δ with ∂S a circle in $\partial\Delta$ linking the edge ad. So, S is a τ-normal quadrilateral linking edge ad.

If S is a vertex linking sphere linking vertex e, then $S = \partial B(e)$ where $B(e)$ is a small ball-neighbourhood of e in M. So that $S = (A, T_e) \cup (B, T_e) \cup (C, T_e) \cup (D, T_e)$. Conversely, if $S = (A, T_e) \cup (B, T_e) \cup (C, T_e) \cup (D, T_e)$ then it is easy to see that $S = \partial B(e)$ and therefore S is a vertex-linking sphere in τ' linking vertex e. □

We now give a proof of Theorem 4.1.2.

Proof. If F is τ-normal then by lemma 4.2.5 it follows that F is a union of τ'-normal disks and hence is τ'-normal as well.

To prove the converse, let S be a connected component of $F \cap \Delta$. We shall show in Claims 1 and 2 that if S contains a τ'-normal triangle, then S must either be a τ-normal disk or a vertex-linking sphere. In Claim 3 we shall show that S is not the union of τ'-normal quadrilaterals. So every component of $F \cap \Delta$ is either a τ-normal disk or a τ' vertex-linking sphere. Thus we would have shown that any τ'-normal surface in M is either a τ-normal surface or it has a component which is a τ' vertex-linking sphere.

Claim. *If $S' \subset S$ is a τ'-normal triangle with coordinates (X, T_e) for some $X \in \{A, B, C, D\}$ then S is either a vertex linking sphere in τ' linking vertex e or S is a τ-normal triangle.*

Without loss of generality, assume $X = D$. The boundary $\partial S'$ is composed of normal arcs linking vertex e, i.e., for $Y \in \{A, B, C\}$, $S' \cap \Delta_Y$ gives a normal arc in Λ_e.

If $S' \cap \Delta_A$ meets a τ'-normal triangle T in Δ_A, then by lemma 4.2.4 (i), $[T] = (A, T_e)$. Let $S'' = S' \cup T$. Now for $Y \in \{B, C\}$; $S'' \cap \Delta_Y \in \Lambda_e * \Lambda_e$, therefore by lemma 4.2.4 (iv) S'' meets normal triangles with coordinates (B, T_e) and (C, T_e). So as S is connected, $S = (A, T_e) \cup (B, T_e) \cup (C, T_e) \cup (D, T_e)$, therefore by lemma 4.2.5, S is a vertex-linking sphere linking vertex e.

If $S' \cap \Delta_A$ meets a τ'-normal quadrilateral Q in Δ_A , then by lemma 4.2.4 (iii), $[Q] = (A, Q_{de})$. Let $S'' = S' \cup Q$. Now for $Y \in \{B, C\}$; $S'' \cap \Delta_Y \in \Lambda_e * \Lambda_{de}$, therefore by lemma 4.2.4 (v), S'' meets normal quadrilaterals with coordinates (B, Q_{de}) and (C, Q_{de}). So as S is connected, $S = (A, Q_{de}) \cup (B, Q_{de}) \cup (C, Q_{de}) \cup (D, T_e)$, therefore by lemma 4.2.5, S is a τ-normal triangle (linking vertex d).

Claim. *If $S' \subset S$ is a τ'-normal triangle with coordinates (X, T_w), for $X \in \{A, B, C, D\}$, where w is a vertex in Δ_X other than e, then S is either a τ-normal triangle or a τ-normal quadrilateral.*

Without loss of generality, assume $X = D$ and $w = a$. Then $S' \cap \Delta_B$ and $S' \cap \Delta_C$ are in Λ_a, while $S' \cap \Delta_A = \phi$.

If $S' \cap \Delta_B$ meets a τ'-normal triangle T, then by lemma 4.2.4 (i), $[T] = (B, T_a)$. Let $S'' = S' \cup T$. Then $S'' \cap \Delta_C \in \Lambda_a * \Lambda_a$ therefore by lemma 4.2.4 (iv), S'' meets a normal triangle with coordinates (C, T_a) in Δ_C. So we have $S = (D, T_a) \cup (B, T_a) \cup (C, T_a)$, therefore by Lemma 4.2.5, S is a τ-normal triangle (linking vertex a).

If $S' \cap \Delta_B$ meets a τ'-normal quadrilateral Q, then by lemma 4.2.4 (iii), $[Q] = (B, Q_{ad})$. We have $(S' \cup Q) \cap \Delta_C \in \Lambda_a * \Lambda_{ad}$ therefore by lemma 4.2.4 (v), $S' \cup Q$ meets Δ_C in a quadrilateral Q' with $[Q'] = (C, Q_{ad})$. Let $S'' = S \cup Q \cup Q'$. Then $S'' \cap \Delta_A \in \Lambda_d * \Lambda_d$. So that by lemma 4.2.4 (iv), S'' meets Δ_A in a normal triangle with coordinates (A, T_d). Therefore $S = (D, T_a) \cup (B, Q_{ad}) \cup (C, Q_{ad}) \cup (A, T_d)$ and by Lemma 4.2.5, S is a τ-normal quadrilateral (linking edge ad).

Claim. *S is not a union of τ'-normal quadrilaterals.*

Without loss of generality we assume $S \cap \Delta_D \neq \phi$. Let Q be a normal quadrilateral in $S \cap \Delta_D$. Then as the normal arcs in ∂Q link distinct vertices, there exists an arc $\lambda \subset \partial Q$ that belongs to Λ_e. Assume, without loss of generality, that $Q \cap \Delta_A = \lambda \in \Lambda_e$. By lemma 4.2.4 (iii), Q meets Δ_A in a normal quadrilateral Q_A with $[Q_A] = (A, Q_{de})$. As $Q_A \cap \Delta_B \in \Lambda_{de}$, by lemma 4.2.4 (ii), Q_A

meets Δ_B in quadrilateral Q_B with $Q_B = (B, Q_{de})$. Let $S' = Q_A \cup Q_B$. Then $S' \cap \Delta_D \in \Lambda_e * \Lambda_e$, so by lemma 4.2.4 (iv), S' must meet Δ_D in a triangle with coordinates (D, T_e) contradicting our assumption that S is composed solely of τ'-normal quadrilaterals.

□

4.3 The prism triangulation of N(F)

Let F be a closed oriented connected surface lying in an oriented 3-manifold M. Denote by I the closed interval $[-1, 1]$. Let $N(F) \cong F \times I$ be a regular neighbourhood of F. In this section we define a triangulation of $N(F)$, which is such that any closed connected normal surface lying in $N(F)$ is normally isotopic to $F \times \{0\}$.

Take a triangular disc T with oriented edges. Assume the edges are not cyclically oriented. Label the vertices $\{v_0, v_1, v_2\}$ of T in such a way that the edges are oriented as $\{v_0v_1, v_1v_2, v_0v_2\}$. In $T \times I$, let $T \times \{-1\}$ be identified with $[v_0, v_1, v_2]$ labeled as above and $T \times \{1\} = [w_0, w_1, w_2]$, where v_i and w_i have the same image under the projection $T \times I \to T$. Then we get a triangulation of $T \times I$, using the tetrahedra $\Delta_0 = [v_0, w_0, w_1, w_2]$, $\Delta_1 = [v_0, v_1, w_1, w_2]$ and $\Delta_2 = [v_0, v_1, v_2, w_2]$. We call this the prism triangulation of $T \times I$. (See proof of Theorem 2.10 [13] for details.)

Lemma 4.3.1. *Let T_1 and T_2 be triangles with non-cyclic oriented edges that intersect in an edge $e = T_1 \cap T_2$. Assume the orientation on the edge e coming from T_1 is the same as that coming from T_2. Let τ_1 and τ_2 be the prism triangulations of $T_1 \times I$ and $T_2 \times I$ respectively. Then $\tau = \tau_1 \cup \tau_2$ is a triangulation of $(T_1 \cup T_2) \times I$.*

Proof. In the prism triangulation of T, we note that the 1-skeleton lies in $\partial T \times I$ and is the union of $\partial T \times \partial I$ and the edges $\{v_0w_0, v_1w_1, v_2w_2, v_0w_1, v_0w_2, v_1w_2\}$. Recall that the edges of T were oriented as $\{v_0v_1, v_1v_2, v_0v_2\}$. So given an oriented edge $e = [-1, 1]$ of T, with e oriented in the direction from -1 to 1, $e \times I$ is the union of two triangles given by the join of $e \times \{-1\}$ with the point $(1, 1)$ and the join of $e \times \{1\}$ with the point $(-1, -1)$. In particular, the triangles divide the square $e \times I$ along the diagonal from $(-1, -1)$ to $(1, 1)$.

Therefore if two triangles T_1 and T_2 with oriented edges intersect in an edge $e = T_1 \cap T_2$, where the orientation of e coming from T_1 is the same as that from T_2, then the prism triangulation of $T_1 \times I$ and $T_2 \times I$ agree on the intersection $e \times I$. So by taking the union $\tau_1 \cup \tau_2$ we get a triangulation on $(T_1 \cup T_2) \times I$. □

We can now define the prism triangulation on $F \times I$. Firstly, we claim that given a triangulation τ of F there exists a refinement τ' of τ and an orientation of the edges of τ' such that no triangle has edges oriented cyclically.

Give any orientation to the edges of the 1-skeleton of τ. Let N_τ be the number of triangles of

τ with edges oriented cyclically. If $N_\tau > 0$, then take a triangle $T = [a, b, c]$ in τ with cyclically oriented edges $\{ab, bc, ca\}$. Let d be a point in the interior of T. Define the triangulation τ' as a refinement of τ given by subdividing T into the triangles $[a, b, d]$, $[b, c, d]$, $[c, a, d]$. Orient the newly introduced edges of the 1-skeleton as da, db and dc. Then none of the triangles in the subdivision of T has cyclically oriented edges. Therefore the number of triangles with cyclically oriented edges in τ', $N_{\tau'} = N_\tau - 1$. So after N_τ such refinements we obtain a triangulation of F with no triangles having edges oriented cyclically.

Now by lemma 4.3.1, we can patch up the prism triangulations of triangles of F to get a triangulation of $F \times I$. We call this the prism triangulation of $F \times I$, relative to the triangulation τ' of F.

We now show that any properly embedded normal surface in the prism triangulation of $T \times I$ with boundary in $\partial T \times I$ is normally isotopic to $T \times \{0\}$.

Lemma 4.3.2. *Let $T = [v_0, v_1, v_2]$ be a triangle with non-cyclic edges. Let τ be the prism triangulation on $T \times I$. Let S be a properly embedded normal surface with $\partial S \subset \partial T \times I$. Then S is normally isotopic to $T \times \{0\}$.*

Proof. In the prism $T \times I$, let $T \times \{-1\} = [v_0, v_1, v_2]$ and $T \times \{1\} = [w_0, w_1, w_2]$ with v_i and w_i projecting to the same point on T. Then, the prism triangulation of $T \times I$ is composed of the tetrahedra $\Delta_0 = [v_0, w_0, w_1, w_2]$, $\Delta_1 = [v_0, v_1, w_1, w_2]$ and $\Delta_2 = [v_0, v_1, v_2, w_2]$.

Observe that Δ_0 contains the face $[w_0 w_1 w_2] = T \times \{1\}$, while Δ_2 contains the face $[v_0 v_1 v_2] = T \times \{-1\}$. As S does not intersect $T \times \partial I$, $S \cap \Delta_0$ is parallel to $[w_0 w_1 w_2]$ and is therefore a union of triangles linking v_0. Similarly, $S \cap \Delta_2$ is a union of triangles linking w_2. The tetrahedron Δ_1 has a pair of opposing edges $v_0 v_1$ and $w_1 w_2$ that lie in $T \times \partial I$. Therefore $S \cap \Delta_1$ is a union of normal disks that separates these pair of edges and is therefore a union of normal quadrilaterals linking edge $v_0 v_1$.

Note that $\Delta_0 \cap \Delta_1 = [v_0 w_1 w_2]$ and $\Delta_1 \cap \Delta_2 = [v_0 v_1 w_2]$. So by the matching equations, the number of triangles in $\Delta_0 \cap S$ equals the number of quadrilaterals in $\Delta_1 \cap S$ which is the same as the number of triangles in $\Delta_2 \cap S$.

Let T_0 be a triangle in $\Delta_0 \cap S$, Q_1 a quadrilateral in $\Delta_1 \cap S$ and T_2 a triangle in $\Delta_2 \cap S$ such that T_0 meets Q_1 in $\Delta_0 \cap \Delta_1$ and Q_1 meets T_2 in $\Delta_1 \cap \Delta_2$. Then $T_0 \cup Q_1 \cup T_2 = S'$ is a connected properly embedded normal surface in $T \times I$ that projects homeomorphically onto T and so S' is normally isotopic to $T \times \{0\}$. Any normal surface in $T \times I$ that does not intersect $T \times \partial I$ is therefore, a disjoint union of discs parallel to $T \times \{0\}$. As S is a normal connected surface, $S = S'$ as required. □

Theorem 4.3.3. *Let F be a closed oriented connected surface. Let τ be a prism triangulation of $N(F) \simeq F \times I$. Then any normal closed connected surface $F' \subset N(F)$ is normally isotopic to*

$F \times \{0\}$.

Proof. Let τ' be a triangulation of F, and let τ be the prism triangulation of $N(F)$ relative to τ'. Let T be a triangle in τ' then $\tau|_T$ is the prism triangulation on $T \times I$. As F' is normal in τ and is closed, $F' \cap (T \times I)$ is a $\tau|_T$-normal properly embedded surface S with $\partial S \subset \partial T \times I$ so by lemma 4.3.2, S is normally isotopic to $T \times \{0\}$. As this is true for every triangle T in the triangulation τ' of F and the surface F' is connected, F' is normally isotopic to $F \times \{0\}$. □

4.4 Proof of Theorem 4.1.3

As before, let F be a closed oriented surface in a compact oriented 3-manifold M. Let τ_1 be a triangulation of $N(F)$. In this section we firstly show that given any integer W, there exists an extension of τ_1 to a triangulation τ of M such that any τ-normal surface F' that does not lie in $N(F)$ has PL-area more than W. This is shown in lemma 4.4.4, using which we prove Theorem 4.1.3.

Definition 4.4.1. Let Γ be a simplicial complex of dimension n. Then $|\Gamma|$ denotes the number of n-cells in Γ.

Definition 4.4.2. Let τ be a triangulation of M. Let the i-weight of F be defined as $w^{(i)}(F) = |F \cap \tau^{(i)}|$, where $\tau^{(i)}$ is the i-the skeleton of the triangulation τ. Then, the PL-area of F is given by the ordered pair $w(F) = (w^{(1)}(F), w^{(2)}(F))$.

Lemma 4.4.3. *Let ϕ be the refinement function (Definition 4.1.1) that gives the refinement of a tetrahedron Δ into 4 tetrahedra. Let τ be a triangulation of Δ consisting of the single tetrahedron Δ. Let $\tau^n = \phi^n(\tau)$ be a triangulation of Δ obtained by taking n iterates of ϕ. Let D be a τ-normal disk. Then the 1-weight of D in τ^n is greater than n.*

Proof. As D is τ-normal, by Theorem 4.1.2, D is τ^n - normal. Let d_n be the number of τ^n-normal disks in D. Let $w_n(D) = w^{(1)}(D)$ in τ^n, be the 1-weight of D in τ^n. As D is a τ-normal disk, its weight in $\tau^0 = \tau$ is greater than equal to 3, therefore $d_0 = 1$ and $w_0(D) \geq 3$. Now we claim that for $n > 0$, $d_n \geq 3d_{n-1}$ and $w_n(D) \geq w_{n-1}(D) + d_{n-1}$.

By lemma 4.2.5, D is divided into at least 3 τ^1-normal disks on taking the refinement along ϕ, and its weight is increased by at least one. Therefore $d_1 \geq 3d_0$ and $w_1 \geq w_0 + d_0$.

Similarly now, if D is the union of d_{n-1} τ^{n-1}-normal disks then each such disk is divided into at least 3 τ^n-normal disks by taking the refinement along ϕ, while its weight is incremented by at least one for each of the τ^{n-1}-normal disks. Therefore $d_n \geq 3d_{n-1}$, while $w_n \geq w_{n-1} + d_{n-1}$. So by induction, $w_n \geq w_0 + \Sigma_{i=0}^{n-1} d_i \geq w_0 + (\Sigma_{i=0}^{n-1} 3^i) d_0$

Therefore the weight $w_n(D) \geq 3 + 1 + 3 + 3^2 + 3^3 ... + 3^{n-1} > n$ for all $n > 0$. □

Lemma 4.4.4. *Let $\hat{F}$ be a closed surface in M and let W be a positive integer. Let τ_1 be a triangulation of a regular neighbourhood $N(\hat{F})$ of $\hat{F}$ in M. Then, there exists an extension of τ_1 to a triangulation τ of M such that for any τ-normal surface S that is not contained in $N(\hat{F})$, $w^{(1)}(S) > W$.*

Proof. We extend the triangulation of $\partial N(\hat{F})$ given by τ_1, to a triangulation τ_2 of $M - int(N(\hat{F}))$. Then, $\tau' = \tau_1 \cup \tau_2$ is a triangulation of M. Let f be the scaling function that takes the value W on tetrahedra of τ_2 and the value 0 on tetrahedra of τ_1. Let $\tau = \phi_f(\tau')$ be the corresponding refined triangulation. We claim that τ is the required triangulation.

Let S be a τ-normal surface in M that is not contained in $N(S)$. By Theorem 4.1.2 then, S is τ'-normal as well. As S is not contained in $N(\hat{F})$ there exists a τ_2-normal disk D in $S - int(N(\hat{F}))$. By lemma 4.4.3 now, the 1-weight of D in τ is greater than W, therefore $w^{(1)}(S) > W$ in τ. □

We are now in a position to prove Theorem 4.1.3.

Proof. Assume F is incompressible. Let τ be any triangulation of M. Let F' be a surface isotopic to F of minimal PL-area in the isotopy class of F. If F' is not τ-normal then it is known that there exists a weight minimising isotopy of F', which is a contradiction. So every minimal PL-area surface in the isotopy class of F is normal.

Conversely, suppose F is compressible. Let $\hat{F}$ be the surface obtained by compressing F along a compressing disk. The surface F is obtained from $\hat{F}$ by attaching a 1-handle γ.

Let τ' be a prism triangulation of $N(\hat{F}) \cong \hat{F} \times I$. Let the 1-weight of the normal surface $\hat{F} \times \{0\}$ be denoted by W. By applying lemma 4.4.4, we obtain an extension of τ' to a triangulation τ of M such that any normal surface that does not lie in $N(\hat{F})$ has 1-weight greater than W.

We can assume the 1-handle γ is disjoint from the 1-skeleton of τ. As F is obtained from $\hat{F} \times \{0\}$ by attaching this 1-handle, the 1-weight $W = w^{(1)}(\hat{F} \times \{0\}) = w^{(1)}(F)$.

Assume there exists a normal minimal surface F' isotopic to F. By construction of τ, any normal surface that does not lie in $N(\hat{F})$ has 1-weight more than $W = w(F)$. So, F' lies in $N(\hat{F})$.

By Theorem 4.3.3 then, F' is isotopic to a connected component of $\hat{F}$. As F' is isotopic to F, we have F isotopic to a connected component of $\hat{F}$. This is a contradiction as F is compressible and hence every component of $\hat{F}$ has genus strictly lower than the genus of F. □

5. SPACE OF MAXIMAL LAMINATIONS

5.1 Introduction

Geodesic laminations on surfaces were first introduced by Thurston, and have since been an important tool in hyperbolic geometry, low-dimensional topology and dynamical systems [3]. We wish to study laminations in 3-manifolds, along the lines of geodesic laminations on surfaces [7]. The concept of a lamination whose leaves are totally geodesic surfaces is, however, not that interesting. A theorem of Zeghib [42] says that in a compact Riemannian manifold of dimension $n \geq 3$ and of negative curvature, if λ is a codimension 1 lamination whose leaves are totally geodesic, then all leaves of λ are compact.

Definition 5.1.1. A branch surface B is a compact subset of M locally modeled on Fig 5.1 (a) (Figure 1.1 of [7]). We denote by $N(B)$ a fibered regular neighbourhood of B in M locally modeled on Fig 5.1 (b). Fig 5.1 (b) also shows the horizontal boundary components $\partial_h N(B)$ and the vertical boundary components $\partial_v N(B)$ of $N(B)$.

The branch locus L of B is the union of those points of B that are not locally R^2. The closure (under the path metric) of components of $B - L$ are called the branch sectors of B.

Every lamination in a closed 3-manifold is fully carried by a branch surface (after 'splitting' finitely many leaves) (Remark 4.4 [7]). In this chapter, we restrict our attention to laminations fully carried by a fixed branch surface B, in a closed 3-manifold M. We can now use the additional structure of the transverse I-fibers of the regular fibered neighbourhood of the branch surface, to study the space of laminations.

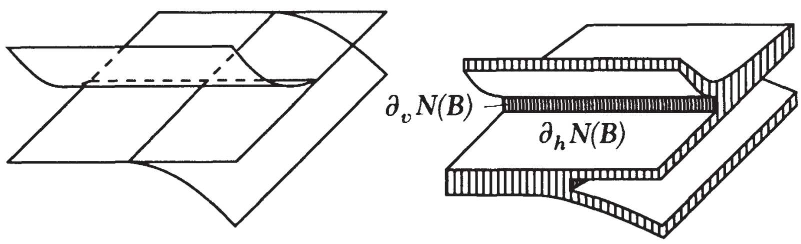

Fig. 5.1: Branched surface

Hyperbolic surfaces have the special property that the homotopy class of a closed curve has a unique geodesic. As a result, geodesic laminations are 'rigid' under isotopy via geodesic laminations. Laminations carried by a branch surface have no such property, so we consider laminations only up to isotopy.

A geodesic lamination λ has no interior, and therefore the closed subset of $N(B)$ representing λ is uniquely foliated. Laminations in neighbourhood of branched surfaces, on the other hand, may have interior. So we identify laminations with the closed subset of $N(B)$ representing λ. That is, two laminations that have different foliations but are represented by the same closed subset of $N(B)$ are said to be equivalent. In particular, if l is a surface (without boundary) carried by B, then the subset $l \times I \subset N(B)$ may have different foliations, but we consider all such laminations the same. If a lamination λ fully carried by B has a point $p \in int(\lambda)$ then, p lies in a component of λ homeomorphic to $l \times I$, for l a surface (without boundary and possibly non-compact) carried by B. If λ has no interior, then it is uniquely foliated by the closed set representing λ, so the only ambiguity we have allowed is different foliations of $l \times I$ components of λ. Let $\mathcal{F}'(B)$ be the space of closed subsets of $N(B)$ with the Hausdorff metric. Then the space of laminations is a sub-space of $\mathcal{F}'(B)$.

A maximal geodesic lamination is defined as a geodesic lamination that is maximal with respect to inclusion of geodesic laminations on the surface. This is equivalent to saying that its a geodesic lamination all of whose complementary components (in the surface) are ideal triangles. We wish to similarly define the notion of maximality for laminations carried by B.

We shall only consider laminations λ for which $\partial_h N(B) \subset \lambda$. We say a lamination λ is a maximal lamination if each component of $N(B) - \lambda$ contains a component of $\partial_v N(B)$. This implies, in particular, that no component of $N(B) - \lambda$ is of the form $l \times I$ for a surface l without boundary (carried by B). We denote by $\mathcal{L}'(B)$ the space of maximal laminations. As $\mathcal{L}'(B)$ is a subspace of the metric space of closed subsets of $N(B)$, $\mathcal{L}'(B)$ has a metric topology.

The projectivisation of two measured laminations carried by B are considered equivalent if the ratio of their weights (on all branch sectors) is a constant. So if each leaf l of a measured lamination λ is replaced by $l \times I$ the projectivisation of λ does not change.

A B-isotopy is an isotopy that fixes each I-fiber of $N(B)$ (as a set). We define the space of projective maximal laminations $\mathcal{PL}(B)$ as the space of B-isotopy classes of maximal laminations that have no interior. We say a lamination $\bar{\lambda}$ is the projectivisation of a lamination λ, if $\bar{\lambda}$ has no interior and λ can be obtained from $\bar{\lambda}$ by 'thickening' some of the leaves, i.e., by replacing some of the leaves l of λ by $l \times I$.

As mentioned above, the space of laminations with no interior is exactly the space of those laminations which do not have a component of the form $l \times I$ (for l a possibly non-compact surface without boundary, carried by B). Therefore, there is a surjective map from $\mathcal{L}'(B)$ to $\mathcal{PL}(B)$ which collapses every $l \times I$ component (of a lamination in $\mathcal{L}'(B)$) to the leaf l. As laminations are taken up to isotopy, this collapsing is well-defined. This map gives us an equivalence relation on $\mathcal{L}'(B)$. Maximal laminations λ and λ' in $\mathcal{L}'(B)$ are considered equivalent if after collapsing all $l \times I$ components of λ and of λ', there is an isotopy that takes one to the other. We shall henceforth consider $\mathcal{PL}(B)$ as equivalence classes of laminations in $\mathcal{L}'(B)$, and give it the quotient topology.

The natural topology on the space of geodesic laminations is the one induced by the Hausdorff metric. With this topology, the space of geodesic laminations becomes sequentially compact. One way of showing this is that the space of closed subsets of a surface is compact under the Hausdorff metric. The space of geodesic laminations is a closed subset of this space, and is therefore compact.

For an attempt along similar lines, to show that $\mathcal{PL}(B)$ is compact under the quotient topology, a naive approach would be to show that $\mathcal{L}'(B)$ is a closed subset of $\mathcal{F}'(B)$. As $\mathcal{F}'(B)$ is compact, $\mathcal{L}'(B)$ would be as well. So $\mathcal{PL}(B)$, being the quotient of a compact set, would be compact. Such an approach fails however as $\mathcal{L}'(B)$ is not closed in $\mathcal{F}'(B)$.

As a counter example, consider a closed connected surface S fully carried by B, and assume $\partial_h N(B) \subset S$. Then S is a maximal lamination. Assume that J is an I-fiber of $N(B)$ that intersects S more than once. Let $\hat{S}$ be a closed set formed by squeezing S along J so that $S \cap J$ is a single point. Then, $\hat{S}$ equals S outside a neighbourhood of J while $\hat{S} \cap J$ is a single point. Any neighbourhood of $\hat{S}$ contains a surface parallel to S, but $\hat{S}$ is not a lamination. Therefore, the complement of $\mathcal{L}'(B)$ is not open in $\mathcal{F}'(B)$.

Instead, we shall show that $\mathcal{PL}(B)$ is sequentially compact by the following arguments. Given a sequence $[\lambda_n]$ in $\mathcal{PL}(B)$, we shall obtain a subsequence λ_{n_k} and by thickening its leaves get a sequence $\mu_{n_k} \sim \lambda_{n_k}$ in $\mathcal{L}'(B)$ which is such that $\mu_{n_k} \to \mu$ in $\mathcal{L}'(B)$. As a result, $[\lambda_{n_k}]$ converges to $[\mu]$ in $\mathcal{PL}(B)$ as required.

The subspace $\mathcal{L}'(B)$ of $\mathcal{F}'(B)$ has an induced Hausdorff metric and the topology is, therefore, normal. In order to show that $\mathcal{PL}(B)$ is a Hausdorff space, it is enough to show that the equivalence class $[\lambda]$ of a given lamination $\lambda \in \mathcal{L}'(B)$ is a closed subset of $\mathcal{L}'(B)$. This will also show that $[\lambda]$ is the intersection of a decreasing sequence of open sets of $\mathcal{PL}(B)$.

Our aim in this chapter is to show these three properties for the space $\mathcal{PL}(B)$, that is, we wish to show that $\mathcal{PL}(B)$ is sequentially compact (Theorem 5.3.20), Hausdorff (Corollary 5.3.25) and each point of $\mathcal{PL}(B)$ is the intersection of a decreasing sequence of open sets (Corollary 5.3.25).

The splitting of a branch surface B is defined as follows:

Definition 5.1.2. Let K be a compact surface (possibly with boundary and possibly disconnected) carried by B such that $N(B) - int(N(K)) = N(B')$ is the fibered regular neighbourhood of a branch surface B'. We say B' is the splitting of B along K.

We characterise such compact surfaces K whose complements are branch surfaces. We call such surfaces branch-splitters.

Similarly, we show that the complement of a lamination is the neighbourhood of a lamination-splitter surface. We characterise laminations by the existence of such a surface in its complement (in $N(B)$).

An advantage of such an approach is that both branch splittings and laminations can be dealt with in one stroke, by looking at the splitters in their complements. Another (more genuine) advantage is that isotopy classes of projective maximal laminations are difficult to deal with, however the isotopy classes of a minimal lamination-splitter surface is easier to handle. This is because it has only finitely many components each of which has a boundary in $\partial_v N(B)$.

Firstly, to go back-and-forth between laminations and the splitter surfaces in their complement, we show that an inclusion isotopy of laminations or 'normal' splitter surfaces can be extended to an ambient isotopy of $N(B)$ (in Theorem 5.2.19).

Next, we get a convenient basis of opens sets in $\mathcal{PL}(B)$. Let V'_{B_1} be the set of maximal laminations fully carried by B_1. Let V_{B_1} be the set of isotopy classes of maximal laminations with a representative that is fully carried by B_1. We shall show (in lemma 5.3.4 and theorem 5.3.21) that the set of V_{B_1}, for B_1 any 'maximal' branch splitting of B, forms a basis of the topology on $\mathcal{PL}(B)$.

An outline of the proof is as follows. For B_1 a maximal branch splitting of B, the neighbourhood $int(N(B_1))$ is an open subset of $N(B)$. We shall show that the set V'_{B_1} is an open subset of $\mathcal{PL}'(B)$. Consequently, the set V_{B_1} is an open subset of $\mathcal{PL}(B)$.

Let $[\lambda] \in \mathcal{PL}(B)$ and let V be an open set containing $[\lambda]$. Then we shall show that there is an open set W of $N(B)$ that contains λ such that the if μ is a maximal lamination in W then $[\mu] \in V$. Lemma 4.2 [7] shows that a maximal lamination λ is precisely the inverse limit of a 'strictly decreasing' sequence of 'maximal' branch-splittings along λ (stated in these terms in Theorem 5.3.17). Therefore, there exists a maximal branch splitting B_1 of B along λ with a neighbourhood $N(B_1)$ such that $N(B_1) \subset W$. We can then conclude that $V_{B_1} \subset V$ as required.

If B_1 is a maximal branch splitting of B, then we can speak of the space of laminations of B_1. We shall later show in this chapter, that $\mathcal{PL}(B_1) = V_{B_1}$. In other words, two laminations carried by B_1 are isotopic in $N(B_1)$ if and only if they are isotopic in $N(B)$ (Theorem 5.3.8).

We divide this chapter into two parts. In the first we shall show (in Theorems 5.2.6 and 5.2.7) that the complements (in $N(B)$) of laminations (and similarly neighbourhood of branch surfaces) are exactly the neighbourhoods of splitter surfaces in $N(B)$. Then, the main effort shall be to prove Theorem 5.2.19 which says that inclusion isotopies of a normal splitter surface, lamination or the neighbourhood of a branch splitting in $N(B)$ can be extended to an identity isotopy of $N(B)$.

In section 5.3, we shall show that $\mathcal{PL}(B)$ is sequentially compact (Theorem 5.3.20), Hausdorff (Corollary 5.3.25) and that each point in $\mathcal{PL}(B)$ can be obtained as the intersection of a sequence of open sets (Corollary 5.3.25) . We also show, in Theorem 5.3.8, that if B_1 is a maximal branch splitting of B, then $\mathcal{PL}(B_1) = V_{B_1}$. That is, laminations carried by B_1 are isotopic in $N(B_1)$ if and only if they are isotopic in $N(B)$.

5.2 Splitters and Isotopies

We divide this section into two parts. In the first part, we shall characterise laminations (and branch splittings) in terms of the lamination-splitter surfaces (and similarly branch-splitter surfaces) in their complements. In the second, we shall show that an isotopy of laminations or 'normal' splitter surfaces in $N(B)$ can be extended to an an ambient isotopy of $N(B)$.

5.2.1 Splitter Surfaces

Our goal in this subsection is to prove Theorem 5.2.6 and Theorem 5.2.7.

Let $\pi : N(B) \to B$ be the map that collapses each I-fiber of $N(B)$ to a point. A lamination is a closed disjoint union of connected surfaces in $N(B)$ (called leaves of λ), transverse to the I-fibers of $N(B)$. We say a lamination λ is fully carried by B if $\pi^{-1}(x) \cap \lambda \neq \phi$ for all $x \in B$. Similarly, we say a branch surface B_1 is fully carried by B if the neighbourhood $N(B_1) \subset N(B)$ is fibered by sub-fibers of I-fibers of $N(B)$, and the set $N(B_1) \cap \pi^{-1}(x) \neq \phi$ for all $x \in B$.

Remark 5.2.1. Unless stated explicitly, surfaces carried by B may not be properly embedded in $N(B)$, i.e., they may have a boundary in the interior of $N(B)$. They may also be non-compact or disconnected, but they are always required to be closed subsets of $N(B)$.

Definition 5.2.2. Let K be a surface in $N(B)$ transverse to the I-fibers of $N(B)$ that intersects $\partial_v N(B)$ in circles. Let $K \cap \partial_h N(B) = \phi$ and assume K intersects no component of $\partial_v N(B)$ in more than one circle. Then, K is called a splitter surface.

If K intersects each component of $\partial_v N(B)$ and $\partial K \subset \partial_v N(B)$ then we say K is a lamination-splitter surface. If K is a compact surface we call it a branch-splitter surface. A splitter surface may be simultaneously both a branch and a lamination-splitter.

Definition 5.2.3. For a splitter surface K, we denote by $N(K)$ the fibered closed neighbourhood of K in $N(B)$ and let $\partial_h N(K)$ and $\partial_v N(K)$ denote respectively the horizontal and vertical boundary components of $\partial N(K)$. We define $int(N(K))$ to be $N(K) - \partial_h N(K)$. Furthermore, if K intersects the annulus V of $\partial_v N(B)$, then we require V to be a component of $\partial_v N(K)$.

Similarly for a branch surface B_1 obtained by splitting B, we denote by $N(B_1)$ the fibered closed neighbourhood of B_1 in $N(B)$. We insist that $\partial_n N(B) \subset \partial_h N(B_1)$.

The following lemma shows the existence of splitter surfaces.

Lemma 5.2.4. *Let $\lambda \subset N(B)$ be a lamination that is fully carried by B. Assume $\partial_h N(B) \subset \lambda$. Then there exists a lamination-splitter surface K and a fibered neighbourhood $N(K)$ of K such that $N(B) - \lambda = int(N(K))$.*

Proof. Let σ be a component of $N(B) - \lambda$. Let $J_p = \pi^{-1}(\pi(p))$ be the I-fiber containing the point $p \in N(B)$. For each $p \in N(B)$, $\sigma \cap J_p$ is an open subset of J_p. As $\partial J_p \subset \partial_h N(B) \subset \lambda$, so $\sigma \cap \partial J_p = \phi$. Therefore, $\sigma \cap J_p$ is a disjoint union of open intervals. As $N(B) = \cup\{J_p : p \in N(B)\}$, so $\sigma = \cup\{J_p \cap \sigma : p \in N(B)\}$. Therefore, σ is an open sub-manifold of $N(B)$ that is fibered by open sub-intervals of the I-fibers of $N(B)$. We shall show that σ is the total space of a $(-1,1)$-bundle over a surface with boundary in $\partial_v N(B)$.

Let J'_p be the component of $J_p - \partial_h N(B)$ that contains p (so if $p \in int(\partial_v N(B))$ then J'_p is a fiber of $int(\partial_v N(B))$, otherwise J'_p is a component of $J_p - \partial N(B)$). For any point $p \in int(N(B))$, there exists an open disk D_p in $N(B)$ transverse to the I-fibers such that $p \in W(p) = D_p \times J'_p$. For any point $p \in int(\partial_v N(B))$ there exists a half-open disk D_p (i.e., a closed disk minus an arc in the boundary) whose boundary lies in $int(\partial_v N(B))$, such that $p \in W(p) = D_p \times J'_p$.

The set $\mathcal{W} = \{W(p) : p \in N(B) - \partial_h N(B)\}$ then becomes a cover of $N(B) - \partial_h N(B)$. As $\sigma \subset N(B) - \partial_h N(B)$, the set $\{W(p) \cap \sigma : p \in \sigma\}$ forms a cover of σ. As $\partial_h N(B) \subset \lambda$, so $\partial_h N(B) \cap \partial_v N(B) = \partial(\partial_v N(B)) \subset \lambda$. Therefore, for each fiber J' of σ, either $J' \subset int(N(B))$ or J' is a fiber of $int(\partial_v N(B))$. So, σ with cover $\mathcal{W}$, becomes the total space of a $(-1,1)$-bundle over a surface (with boundary in $\partial_v N(B)$) as required.

We denote this surface, obtained by collapsing each fiber of σ to a point, by K_σ. We can represent K_σ by a surface properly embedded in σ, transverse to the fibers of σ, with $\partial K_\sigma \subset \partial_v \sigma \subset \partial_v N(B)$. Also we observe that σ is a fibered neighbourhood of K_σ.

Furthermore, as $\sigma \cap \partial_h N(B) = \phi$ and $K_\sigma \subset \sigma$, $K_\sigma \cap \partial_h N(B) = \phi$. Also as $K_\sigma \cap \partial(\partial_v N(B)) = \phi$, so K_σ intersects the annuli of $\partial_v N(B)$ in circles. As K_σ intersects each fiber of σ in exactly one point and each fiber of σ is either a fiber of $int(\partial_v N(B))$ or is disjoint from $\partial_v N(B)$, so K_σ

intersects each fiber of $\partial_v N(B)$ in at most one point. Therefore, K_σ intersects each component of $\partial_v N(B)$ in at most one circle. Hence, K_σ is a splitter surface.

Let $K = \cup\{K_\sigma : \sigma \text{ is a component of } N(B) - \lambda\}$. As $\partial K \subset \partial_v N(B)$, K intersects each component of $\partial_v N(B)$ and by construction there exists a fibered neighbourhood $N(K)$ of K such that $int(N(K)) = N(B) - \lambda$, so K is in fact the required lamination-splitter surface. □

We now prove that the complement of a fibered neighbourhood of a lamination-splitter surface is a lamination fully carried by B.

Lemma 5.2.5. *Let K be a lamination-splitter surface. Then there exists a fibered neighbourhood $N(K)$ of K and a lamination λ fully carried by B such that $\lambda = N(B) - int(N(K))$.*

Proof. As K intersects each annulus of $\partial_v N(B)$ in exactly one essential circle and $\partial K = K \cap \partial_v N(B)$, so there exists a fibered neighbourhood $N(K)$ of K such that $\partial_v N(B) = \partial_v N(K)$. Let $\lambda = N(B) - int(N(K))$. Observe that λ is disjoint from $int(\partial_v N(B)) = int(\partial_v N(K))$. We shall show that λ is a lamination.

Along the lines of lemma 5.2.4, let J'_p be the component of $J_p - int(\partial_v N(B))$ that contains the point $p \in \lambda$. Then, for each such point, there exists a disk $D(p)$ transverse to the I-fibers such that for $W(p) = D(p) \times J'_p$. The set $\{W(p) \cap \lambda : p \in \lambda\}$ forms a cover of λ.

Let $F_p \subset J_p$ be the closed set $J'_p - int(N(K))$. Observe that $int(\partial_v N(B))$ equals $int(\partial_v N(K))$ and $\partial_h N(K)$ is transverse to the J-fibers. So for any point $s \in F_p$, we can choose the open disk $D(p)$ so that $D(p) \times \{s\} \subset \lambda$. Then, $W(p) \cap \lambda = D(p) \times F_p$. Therefore λ is covered by open sets $D(p) \times F_p$ where $D(p)$ is homeomorphic to $\mathbb{R}^2$, F_p is a closed subset of $\mathbb{R}$ and these open sets match-up nicely. So λ is a lamination. □

Combining the above two lemmas, Lemma 5.2.4 and Lemma 5.2.5, we get the following theorem.

Theorem 5.2.6. *Let $\lambda \subset N(B)$ be a lamination fully carried by B. Assume $\partial_h N(B) \subset \lambda$. Then, there exists a lamination-splitter surface $K \subset N(B)$ and a fibered neighbourhood $N(K)$ of K, such that $int(N(K)) = N(B) - \lambda$.*

Conversely, let K be a lamination-splitter surface. Then there exists a fibered neighbourhood $N(K)$ of K and a lamination λ fully carried by B such that $\lambda = N(B) - int(N(K))$.

We say λ is a splitting of B and denote it by $\lambda = B - K$.

Working in exactly the same way, we show the similar result for branch splittings.

Theorem 5.2.7. *Let B_1 be a branch surface fully carried by B and let $N(B_1) \subset N(B)$ be a fibered neighbourhood of B_1. Assume $\partial_h N(B) \subset N(B_1)$. Then, there exists a branch-splitter surface $K \subset N(B)$ and a fibered neighbourhood $N(K)$ of K, such that $int(N(K)) = N(B) - N(B_1)$.*

Conversely, let K be a branch-splitter surface. Then there exists a fibered neighbourhood $N(K)$ of K and a branch surface B_1 fully carried by B such that $N(B_1) = N(B) - int(N(K))$.

We say B_1 is a splitting of B and denote it by $B_1 = B - K$.

Proof. Similar in nature to above Theorem 5.2.6. □

Remark 5.2.8. When K is both a lamination and a branch splitter, then $\beta = B - K$ is the neighbourhood of a closed surface. We may then refer to β as either a lamination or the neighbourhood of a branch surface fully carried by B.

Lemma 5.2.9. *Let $B_1 = B - K_1$ be a branch-splitting of B and let $\lambda = B - K_\lambda$ be a lamination fully carried by B_1. Then, after a B-isotopy of K_1 (in $N(K_1)$) we may assume that $K_1 \subset K_\lambda$.*

Proof. By definition of branch splitting, there exists a fibered neighbourhood $N(B_1)$ of B_1 and a neighbourhood $N(K_1)$ of K_1 such that $N(B_1) = N(B) - int(N(K))$. So in particular, as $N(K) \cap \partial_h N(B) = \phi$ so $\partial_h N(B) \subset \partial_h N(B_1)$.

Let λ be a lamination fully carried by B_1, then after an isotopy of λ (in $N(B_1)$) we may assume that $\partial_h N(B_1) \subset \lambda$ (after possibly replacing some leaf l of λ by $l \times I$).

So, we have $\partial_h N(B) \subset \partial_h N(B_1) \subset \lambda$. As $\lambda \subset N(B_1)$, so $N(K_1) \subset N(K_\lambda)$. Secondly as $\partial_h N(B_1) \subset \lambda$, so $N(K_1) \cap \lambda = \partial_h N(K_1)$. So if a fiber of $int(N(K_\lambda))$ intersects $N(K_1)$ then it is in fact a fiber of $int(N(K_1))$. Therefore, $N(K_1)$ is a union of fibers of $N(K_\lambda)$, so K_λ intersects each fiber of $N(K_1)$ exactly once. Hence there is an isotopy of K_1 (in $N(K_1)$) that takes it to a subsurface of K_λ. □

5.2.2 *Isotopies*

Our aim in this subsection is to prove Theorem 5.2.19.

We firstly, define normal surfaces carried by B.

Definition 5.2.10. Let Λ_b be a union of transversally intersecting paths or circles properly embedded in a branch sector b of B, such that $b - \Lambda_b$ is a disk. Let $\Lambda = \cup\{\Lambda_b : b \in B\}$. Let L be the branch locus of B. Then $L' = L \cup \Lambda$ is called a normal branch locus of B and the closure of components of $B - L'$ is called a system of normal branch sectors of B.

A system of normal branch sectors of B partitions B into disks $\{b' : b \in B\}$ such that $L \subset \cup\{\partial b' : b \in B\}$.

Henceforth we fix a system of normal branch sectors of B.

Definition 5.2.11. Let $K \subset N(B)$ be a surface transverse to the I-fibers of $N(B)$ such that $\pi(\partial K) \subset L'$, where L' is the normal branch locus of B. For each component δ of $\partial_h N(B)$ we assume that either $\delta \subset K$ or $\delta \cap K = \phi$. We call such a surface a normal surface.

Let Δ be a normal branch sector of B, then $K \cap \pi^{-1}(\Delta)$ is a disjoint union of disks B-isotopic to Δ. We call these the normal disks of K. Let $|B'|$ be the number of normal branch sectors of B. There are at most $|B'|$ (types of) normal disks up to B-isotopy.

A curve γ in K is said to be a normal curve if it is transversal to ∂D for every normal disk D of K.

For a set $A \subset K$, the normal neighbourhood of A in K, $\overline{N}_K(A)$ is the union of all normal disks of K that intersect A.

Definition 5.2.12. Let $B_1 \subset N(B)$ be a branch surface such that $\pi(\partial_v N(B_1)) \subset L'$, where L' is the normal branch locus of B. For each component δ of $\partial_h N(B)$ we assume that either $\delta \subset \partial_h N(B_1)$ or $\delta \cap \partial_h N(B_1) = \phi$. We call such a branch surface a normal branch surface.

Definition 5.2.13. For a set $A \subset N(B)$, an injection $i : A \to N(B)$ is said to be a B-injection if $\pi(i(a)) = \pi(a)$ for all $a \in A$. We further insist that $i(a) \in \partial_v N(B)$ if and only if $a \in \partial_v N(B)$.

An isotopy $H_t : A \to N(B)$ is said to be a B-isotopy if each H_t is a B-injection of A in $N(B)$. We say an isotopy $H_t : A \to N(B)$ takes A to A' if H_0 is the inclusion map of A in $N(B)$ and $H_1(A) = A'$. We say an isotopy $H_t : N(B) \to N(B)$ takes A to A' if H_0 is the identity map of $N(B)$ and $H_1(A) = A'$.

Our aim in the rest of this sub-section is to prove Theorem 5.2.19 which says that any inclusion B-isotopy of a normal surface, lamination or the neighbourhood of a normal branch surface in $N(B)$ can be extended to an ambient B-isotopy of $N(B)$. We start with a cell-decomposition of B along the normal branch sectors. We then obtain an ambient isotopy by inductively extending along each i-skeleton of B as shown in lemmas 5.2.15, 5.2.16 and 5.2.18.

Definition 5.2.14. Let $E \subset \mathbb{R}^2$ be a closed set. Let J be a closed interval. Let $K \subset E \times J$ be a closed set such that $E \times \partial J \subset K$. We say an injection $i : K \to E \times J$ is an E-injection, if $\pi(i(x)) = \pi(x)$ for all $x \in K \subset E \times J$, where $\pi : E \times J \to E$ is the map that collapses each J-fiber to a point. We say i is a monotonic E-injection if, in addition, i restricted to each J-fiber is a monotonic map (in J) that fixes the endpoints of J (i.e., it point-wise fixes $E \times \partial J \subset K$).

Suppose for every such monotonic E-injection of K in $E \times J$, there exists an extension $\tilde{i} : E \times J \to E \times J$ that is an E-homeomorphism (i.e, it is invariant on each J-fiber of E). We then say that $\tilde{i}$ is a canonical extension of i, if for any continuous family i_t of monotonic E-injections, the corresponding family of extensions $\tilde{i}_t$ is continuous (i.e., $i \to \tilde{i}$ is a continuous map from the space of monotonic E-injections of K in $E \times J$ to the space of E-homeomorphisms of $E \times J$).

We shall now prove lemmas 5.2.15, 5.2.16 and 5.2.18 which give canonical extensions of E-injections to E-homeomorphisms, for E a single point, an edge and a disk respectively. For uniformity, we assume that if E is a single point, then $\partial E = \phi$ and $\partial E \times J = \phi$.

Lemma 5.2.15. *Let E be a single point. Let K^0 be a closed subset of $E \times J = J$ such that $E \times \partial J \subset K^0$. Let $K = (\partial E \times J) \cup K^0 = K^0$. Then for any monotonic E-injective map*

$i : K \to E \times J$ there exists a canonical extension of i to an E-homeomorphism $\tilde{i} : E \times J \to E \times J$ that fixes $E \times \partial J$.

If i is the inclusion of $K \subset E \times J$ in $E \times J$, then $\tilde{i}$ is the identity map of $E \times J$.

Proof. As $K \subset J$ is a closed set, $J - K = \cup int(J_n)$ where J_n are closed sub-intervals of J. Let $J_n = [a_n, b_n]$, for $a_n, b_n \in K$. As i is monotonic, we assume i is increasing so that $i(a_n) \leq i(b_n)$. Then there exists a unique linear map $i_n : [a_n, b_n] \to [i(a_n), i(b_n)]$ such that $i_n(a_n) = i(a_n)$ and $i_n(b_n) = i(b_n)$. Define $\tilde{i}(t) = i(t)$ if $t \in K$, $\tilde{i}(t) = i_n(t)$ if $t \in J_n$.

Then, $\tilde{i}$ is clearly a well defined and continuous map from J to J. Note that $\partial J \subset K$ and i point-wise fixes $S^1 \times \partial J$ by definition of monotonicity of i. So as $\tilde{i}(J)$ is a closed sub-interval of J that contains the end-points of J, $\tilde{i}$ is surjective. As i is monotonic, $\tilde{i}(int(J_n)) \cap \tilde{i}(int(J_m)) = \phi$ for all $n \neq m$ and $\tilde{i}(int(J_n)) \cap \tilde{i}(K) = \phi$ for all n. As $\tilde{i}(= i)$ is injective on K and $\tilde{i}$ is injective on each J_n therefore, $\tilde{i}$ is injective on J as well. Hence, $\tilde{i}$ is a homeomorphism of J. A similar argument shows the existence of such an E-homeomorphism that extends i when i is monotonically decreasing.

Observe that if i_t is a continuous family of monotonic injections of K, then $\tilde{i}_t(J_n) = [i_t(a_n), i_t(b_n)]$. As $\tilde{i}_t$ is a linear extension of i_t on J_n, $\tilde{i}_t$ restricted to J_n is continuous with respect to t, and $\tilde{i}_t = i_t$ on K therefore $\tilde{i}_t$ is a continuous family of extensions. So, $\tilde{i}$ is a canonical extension of i.

Clearly if i is the inclusion map of K in J, then each of the linear extensions on J_n are identity maps and $\tilde{i} : J \to J$ is the identity map of J. □

Lemma 5.2.16. *Let $E = [0, 1]$. Let K^1 be a closed disjoint union of arcs in $E \times J$ that are transverse to the J-fibers and such that $E \times \partial J \subset K^1$. Let $K = (\partial E \times J) \cup K^1$. Then for any monotonic E-injective map $i : K \to E \times J$ there exists a canonical extension of i to an E-homeomorphism $\tilde{i} : E \times J \to E \times J$ that fixes $E \times \partial J$.*

If i is the inclusion of $K \subset E \times J$ in $E \times J$, then $\tilde{i}$ is the identity map of $E \times J$.

Proof. We first prove the following claim.
Claim: Let σ_0 and σ_1 be homeomorphisms of J that point-wise fix ∂J. Then, there exists an isotopy of homeomorphisms $h_t : J \to J$ such that $h_0 = \sigma_0$ and $h_1 = \sigma_1$.

Define $h_t(x) = \sigma_0(x)(1 - t) + \sigma_1(x)t$ for all $x \in J$, $t \in I$. Then, h_t is a homotopy taking σ_0 to σ_1. As $\sigma_i(0) = 0$ and σ_i is a homeomorphism, σ_i is strictly increasing (for $i = 1, 2$). Therefore, for $x < y$ in J, $h_t(x) = \sigma_0(x)(1 - t) + \sigma_1(x)t < \sigma_0(y)(1 - t) + \sigma_1(y)t = h_t(y)$. So that h_t is injective for all $t \in I$. As $h_t(0) = 0$, $h_t(1) = 1$ and $h_t(J)$ is a closed sub-interval of J, for all $t \in I$, h_t is surjective as well. So, $h_t : J \to J$ is an isotopy of homeomorphisms taking σ_0 to σ_1.

Define $h : E \times J \to E \times J$ by $h(t, x) = (t, h_t(x))$. Then h is an E-homeomorphism that extends $\sigma_0 \cup \sigma_1$ (which is defined on $\partial E \times J$) to $E \times J$. We also observe that if $(\sigma_0)_s$ and $(\sigma_1)_s$ is a continuous family of homeomorphisms of J, then the corresponding family of homeomorphisms h_s is a continuous family.

Let $\cup int(A_n) = (E\times J) - K$, where A_n is a fibered neighbourhood of an arc (transverse to the J-fibers) carried by $E \times J$ (i.e, A_n is a fibered open rectangle). We denote A_n by $E \times J_n$, where $J_n = A_n \cap (\{0\}\times J)$. As in above claim, we can canonically extend $i|_{\partial E\times J_n}$ to an E-homeomorphism $i_n : E \times J_n \to E \times J_n$. We define $\tilde{i}(x) = i(x)$ if $x \in K$, and $\tilde{i}(x) = i_n(x)$ if $x \in A_n$. By arguments as in lemma 5.2.15, $\tilde{i}$ restricted to each J-fiber $\{e\}\times J$ is a homeomorphism of $\{e\}\times J$. Therefore, $\tilde{i}$ is an E-homeomorphism of $E \times J$. As the extension on each A_n is a canonical extension, $\tilde{i}$ is a canonical extension of i.

Also, if i is the inclusion of K in $E \times J$, then $\tilde{i}$ is the identity map of $E \times J$. □

Lemma 5.2.17. *Let $E = S^1 \times [0, 1]$. Let K^2 be a closed disjoint union of annuli in $E \times J$ that are transverse to the J-fibers and such that $E \times \partial J \subset K^2$. Let $K = (\partial E \times J) \cup K^2$. Then for any monotonic E-injective map $i : K \to E \times J$ there exists a canonical extension of i to an E-homeomorphism $\tilde{i} : E \times J \to E \times J$ that fixes $E \times \partial J$.*

If i is the inclusion of $K \subset E \times J$ in $E \times J$, then $\tilde{i}$ is the identity map of $E \times J$.

Proof. We first prove the following claim.
Claim: Let σ_0 and σ_1 be S^1-homeomorphisms of $S^1 \times J$ that point-wise fix ∂J. Then, there exists an isotopy of homeomorphisms $h_t : S^1 \times J \to S^1 \times J$ such that $h_0 = \sigma_0$ and $h_1 = \sigma_1$.

Let $p \in S^1$. Let $E' = cl(S^1 - p)$ then $E' = [0, 1]$. Also σ_0 (similarly σ_1) restricted to $E' \times J$, is an E'-homeomorphisms that is identical on $\{0\} \times J$ and $\{1\} \times J$. Let h_t be the extension as described in the claim in lemma 5.2.16 (taking E' as E). Then as h_t is uniquely determined by σ_0 and σ_1, so h_t on $\{0\} \times J$ is identical to h_t on $\{1\} \times J$ for all $t \in I$. Therefore, h_t is an isotopy of S^1-homeomorphisms of $S^1 \times J$ taking σ_0 to σ_1.

Let $h : [0, 1] \times (S^1 \times J) \to [0, 1] \times (S^1 \times J)$ be defined by $h(t, x) = (t, h_t(x))$. Then h is an E-homeomorphism that extends $\sigma_0 \cup \sigma_1$ (defined on $\partial E \times J$) to $E \times J$. We also observe that if $(\sigma_0)_s$ and $(\sigma_1)_s$ is a continuous family of homeomorphisms of $E \times J$, then the corresponding family of homeomorphisms h_s is a continuous family.

Let $\cup int(A_n) = (E\times J) - K$, where A_n is the fibered neighbourhood of an annulus (transverse to the J-fibers) carried by $E \times J$. We denote A_n by $E \times J_n$, where $J_n = A_n \cap (\{e\} \times J)$ for some fixed point $e \in E$. As in above claim, we can extend $i|_{\partial E\times J_n}$ to a canonical homeomorphism $i_n : E \times J_n \to E \times J_n$. We define $\tilde{i}(x) = i(x)$ if $x \in K$, and $\tilde{i}(x) = i_n(x)$ if $x \in A_n$. By arguments as in lemma 5.2.15, $\tilde{i}$ restricted to each J-fiber $\{e\}\times J$ is a homeomorphism of $\{e\}\times J$. Therefore, $\tilde{i}$ is an E-homeomorphism of $E \times J$. As the extension on each A_n is a canonical extension, $\tilde{i}$ is a canonical extension of i.

Also, if i is the inclusion of $K \subset E \times J$ in $E \times J$, then $\tilde{i}$ is the identity map of $E \times J$. □

Lemma 5.2.18. *Let E be a closed 2-disk. Let K^2 be a closed disjoint union of disks in $E \times J$ that are transverse to the J-fibers and such that $E \times \partial J \subset K^2$. Let $K = (\partial E \times J) \cup K^2$ be a closed subset of $E \times J$. Then for any monotonic E-injective map $i : K \to E \times J$ there exists a canonical extension of i to an E-homeomorphism $\tilde{i} : E \times J \to E \times J$ that fixes $E \times \partial J$.*

If i is the inclusion of $K \subset E \times J$ in $E \times J$, then $\tilde{i}$ is the identity map of $E \times J$.

Proof. Let p be a point in $int(E)$ and let J_p be the I-fiber $\pi^{-1}(\pi(p))$. Let $K_p = K \cap J_p$ and let i_p be the restriction of i to K_p. Then by lemma 5.2.15, there exists a canonical extension of i_p to a homeomorphism $\tilde{i}_p$ of J_p. We extend $i : K \to E \times J$ to $i : K \cup J_p \to E \times J$ by defining $i = \tilde{i}_p$ on J_p.

Let $E' = S^1 \times [0,1]$, let c be the boundary component $S^1 \times \{1\}$ and let $\sigma : E' \times J \to E \times J$ be a bundle map that identifies $E' - c$ with $E - p$, sends c to p and identifies the J-fiber J_e with the J-fiber $J_{\sigma(e)}$, $\forall e \in E'$. Let $K' = \sigma^{-}(K)$. Then, K' is closed disjoint union of annuli in $E' \times J$ that are transverse to the J-fibers and such that $E' \times \partial J \subset K'$. Also, for any $s, t \in c$, $K' \cap J_s = K' \cap J_t$ where J_s and J_t are the J-fibers on s and t respectively. Let $i_1(x) = \sigma^{-1}(i(\sigma(x)))$ for $x \in K' \cup \partial E \times J$, where i_1 acts identically on J-fibers of c.

By the above lemma 5.2.17 then, there is a canonical extension of i_1 to a homeomorphism $\tilde{i}_1 : E' \times J \to E' \times J$. By construction, this extension is identical on J-fibers of c (because K' and i_1 are identical on J-fibers of c). Therefore, $\tilde{i} : E \times J \to E \times J$, defined as $\tilde{i}(x,s) = i(\sigma^{-1}(x,s))$ is a well defined E-homeomorphism that canonically extends i. □

Theorem 5.2.19. *Let B be a branch surface. Let $K \subset N(B)$ be a normal surface, lamination or the neighbourhood of a normal branch surface fully carried by B. Let $h_t : K \to N(B)$ be a B-isotopy of K such that h_0 is the inclusion map of K in $N(B)$. For each component δ of $\partial_h N(B)$ assume that either $h_t(K) \supset \delta$ for all $t \in I$ or $h_t(K) \cap \delta = \phi$ for all $t \in I$. Then there exists a B-isotopy $H_t : N(B) \to N(B)$ that extends h_t, such that H_t point-wise fixes $\partial_h N(B)$ and H_0 is the identity map of $N(B)$.*

As H_t point-wise fixes $\partial_h N(B)$, in particular, H_t is invariant on $\partial_v N(B)$.

Proof. If a component δ of $\partial_h N(B)$ is not contained in K then include it in K and define h_t on δ as the inclusion map of δ in $N(B)$, for all $t \in I$. Therefore we may assume that $\partial_h N(B) \subset h_t(K)$ for all $t \in I$ and by hypothesis, h_t is still an isotopy of K.

Let J be an I-fiber of $N(B)$. Take $x, y \in J$ with $x < y$. As h_0 is the inclusion map of K, $h_0(x) = x$, $h_0(y) = y$ so that $h_0(x) < h_0(y)$. Suppose for some $t \in I$, $h_t(x) > h_t(y)$ then by continuity of h_t there exists $s \in I$ such that $h_s(x) = h_s(y)$ contradicting the injectivity of h_s.

Therefore, h_t is a monotonic B-injection for all $t \in I$.

We take a cell decomposition of B, with B^i denoting the i-th skeleton of B, as follows. Let B^0 be the set of double points in the normal locus L', let $B^1 = L'$ and $B^2 = B$. Let $F^i = \pi^{-1}(B^i)$ and let $K^i = K \cap F^i$ for $i = 0, 1, 2$, where $\pi : N(B) \to B$ is the map that collapses each I-fiber to a point.

Observe that on each edge e of L', $K^1 \cap \pi^{-1}(e)$ is a disjoint union of arcs transversal to the J-fibers of $N(B)$. And as $\partial_h N(B) \subset K$, $e \times \partial J \subset K^1$. Similarly, for each disk D in $B - L'$, $K^2 \cap \pi^{-1}(D)$ is a closed disjoint union of disks transversal to the J-fibers of $N(B)$. And as $\partial_h N(B) \subset K$, $D \times \partial J \subset K^2$.

Then taking E as B^0, by lemma 5.2.15 we get a canonical extension of $h_t : K^0 \to F^0$ to a B-homeomorphism $H_t : F^0 \to F^0$ for all $t \in I$ (As $K \supset \partial_h N(B)$, and $\partial J_p \subset \partial_h N(B)$ so $K \cap J_p \neq \phi$ for all $p \in B^0$). Next, taking E as the edges of B^1, by lemma 5.2.16 we get a canonical extension of $H_t \cup h_t : F^0 \cup K^1 \to F^1$ to a B-homeomorphism $H_t : F^1 \to F^1$ for all $t \in I$. And lastly, taking E as the disks of B^2, by lemma 5.2.18 we get a canonical extension of $H_t \cup h_t : F^1 \cup K^2 \to F^2$ to a B-homeomorphism $H_t : F^2 \to F^2$.

As h_0 is the inclusion map of K in $N(B)$, so H_0 is the identity on $F^2 = N(B)$ as required. As H_t is a canonical extension of h_t for all $t \in I$, H_t is a continuous family of homeomorphisms of $N(B)$ as required. Also, as h_t restricted to $\partial_h N(B)$ is identity, therefore H_t point-wise fixes $\partial_h N(B)$. □

This theorem implies the following proposition.

Proposition 5.2.20. *Let $\lambda_1 = B - K_1$ and $\lambda_2 = B - K_2$ be maximal lamination with, K_1 and K_2 their lamination-splitter surfaces. Then $\lambda_1 \sim \lambda_2$ if and only if K_1 is B-isotopic to K_2.*

Proof. If there exists an isotopy $h_t : \lambda_1 \to N(B)$ taking λ_1 to λ_2 then by Theorem 5.2.19, there is an extension of h_t to an ambient isotopy $H_t : N(B) \to N(B)$ such that H_0 is the identity map. Then H_t takes $N(K_1)$ to $N(K_2)$, so $H_1(K_1) \subset N(K_2)$ intersects each I-fiber of $N(K_2)$ exactly once, therefore K_1 is isotopic to K_2.

Conversely, if K_1 is inclusion-isotopic to K_2, we can extend the isotopy to an ambient isotopy H_t of $N(B)$ that takes K_1 to K_2 and therefore the complement of $N(K_1)$ in $N(B)$ is taken to the complement of $H_1(N(K_1)) = N'(K_2)$ in $N(B)$. Let $\lambda_2' = N(B) - int(N'(K_2))$. Then $\lambda_2' \sim \lambda_2 = N(B) - int(N(K_2))$ and so, $\lambda_1 \sim \lambda_2$ as required. □

5.3 Space of laminations

In this section we shall show that $\mathcal{PL}(B)$ is sequentially compact (Theorem 5.3.20), Hausdorff (Corollary 5.3.25) and that each point of $\mathcal{PL}(B)$ is the intersection of a sequence of open sets (Corollary 5.3.25). We shall also show, in Theorem 5.3.8, that laminations carried by a maximal splitting B_1 of B are isotopic in $N(B_1)$ if and only if they are isotopic in $N(B)$, i.e, $\mathcal{PL}(B_1) = V_{B_1}$.

Definition 5.3.1. Let $\lambda = B - K$ be a lamination (with K its lamination-splitter surface). If each component of K intersects $\partial_v N(B)$, then we say K is a minimal lamination-splitter surface and we say λ is a maximal lamination. In other words, we do not want $(l \times I)$ components in the complement of λ, where l is a surface without boundary (carried by B).

Similarly let $B_1 = B - K_1$ be a branch surface (with K_1 its branch-splitter surface). If each component of K_1 intersects $\partial_v N(B)$, we say K_1 is a minimal branch-splitter surface and we say B_1 is a maximal branch surface.

Given a splitter K we can discard components of K that do not intersect $\partial_v N(B)$ to obtain a minimal splitter surface.

Remark 5.3.2. Let K be a minimal lamination-splitter surface and let K' be a sub-(lamination-splitter) surface of K. Then each component of K and K' intersects $\partial_v N(B)$. Also K and K' intersect each annulus of $\partial_v N(B)$ in exactly one circle. So K' is a subsurface and a closed subset of K, with $\partial K' = \partial K$. Also, K' intersects each component of K and therefore, $K' = K$.

Definition 5.3.3. Let $\beta_1 = B - K_1$ and $\beta_2 = B - K_2$ be laminations or neighbourhoods of normal branch surfaces fully carried by B. We say $\beta_2 \leq \beta_1$, if after a B-isotopy of K_2 in $N(B)$, $K_1 \subset K_2$. If $\beta_1 \leq \beta_2$ and $\beta_2 \leq \beta_1$ then we say β_1 is isotopic to β_2, and denote it by $\beta_1 \simeq \beta_2$.

By this notation, $\lambda \leq B_1$ if after a B-isotopy of λ in $N(B)$, λ is fully carried by $N(B_1)$ (see Lemma 5.2.9). Laminations $\lambda_2 \leq \lambda_1$ if after a B-isotopy of λ_2 in $N(B)$, λ_2 is a sub-lamination of λ_1. Similarly, $B_2 \leq B_1$ if after a B-isotopy of $N(B_2)$, $N(B_2) \subset N(B_1)$. This happens precisely when B_2 is a splitting of B_1.

For any maximal branch splitting B_1, V'_{B_1} was defined to be the set of maximal laminations fully carried by B_1 while V_{B_1} was defined to be the equivalence classes of laminations $[\lambda]$ such that $\lambda \leq B_1$. Let $\mathcal{B} = \{V_{B_1} : B_1 \text{ is a maximal splitting of } B\}$.

Lemma 5.3.4. *The set $\mathcal{B}$ is a basis for a topology on $\mathcal{PL}(B)$.*

Proof. Let $B_1 = B - K_1$ and $B_2 = B - K_2$ be splittings of B and let $B - K_\lambda = \lambda$ for some $[\lambda] \in V_{B_1} \cap V_{B_2}$. Then we shall show that there exists a maximal branch surface $B' = B - K'$ obtained by splitting B such that $[\lambda] \in V_{B'}$ and $V_{B'} \subset V_{B_1} \cap V_{B_2}$.

As $\lambda \leq B_1$, we may assume after a B-isotopy of K_λ in $N(B)$ that $K_1 \subset K_\lambda$ (Lemma 5.2.9). Let $H_t : N(B) \to N(B)$ be an isotopy that takes K_λ to $K_{\lambda'}$, where $K_2 \subset K_{\lambda'}$. As K_λ is a minimal splitter surface each component of K_λ intersects $\partial_v N(B)$. So there exists a minimal branch-splitter $K' \subset K_\lambda$ that contains $H_1^{-1}(K_2) \cup K_1$. By construction, as $K' \subset K_\lambda$ therefore λ is fully carried by $B' = B - K'$.

Let $\mu = B - K_\mu$ be a lamination fully carried by B', then $K_1 \subset K' \subset K_\mu$, that is μ is fully carried by B_1. As $K_2 \subset H_1(K') \subset H_1(K_\mu)$, so H isotopes μ to a lamination that is fully carried by B_2. Hence, $[\lambda] \in V_{B_1} \cap V_{B_2}$, so we have $V_{B'} \subset V_{B_1} \cap V_{B_2}$. □

Let X be the set $\mathcal{PL}(B)$, with the topology generated by $\mathcal{B}$. From now on, we shall prove results about X and then later show that X is homeomorphic to the quotient Hausdorff metric topology of $\mathcal{PL}(B)$ (Theorem 5.3.21).

Our aim now is to prove Theorem 5.3.8, which says that laminations carried by a maximal splitting B_1 of B are isotopic in $N(B_1)$ if and only if they are isotopic in $N(B)$.

Let K_1 and K_2 be splitter surfaces, let $F \subset K_1$ be a connected subsurface and let $h : F \to N(B)$ be a B-injection such that $h(F) \subset K_2$. Then the following lemma says that image of a single point of F determines h uniquely.

Lemma 5.3.5. *Let K_1 and K_2 be splitter surfaces. Let F be a connected subsurface of K_1. Let $h_1 : F \to N(B)$ and $h_2 : F \to N(B)$ be B-injections such that $h_1(F) \subset K_2$ and $h_2(F) \subset K_2$. Let $p \in F$ be such that $h_1(p) = h_2(p)$ then $h_1(x) = h_2(x)$ for all $x \in F$.*

Proof. Let $A = \{x \in F : h_1(x) = h_2(x)\}$. By continuity of h_1 and h_2, A is closed in F. As F and K_2 are transverse to the I-fibers of $N(B)$, for all $a \in A$ there exists a neighbourhood $N(a) \subset F$ such that $\pi^{-1}(N(a)) \cap K_2$ is a disjoint union of disks $\{D_j\}_{j\in J}$ such that for all $x \in N(a)$, for all $j \in J$, $\pi^{-1}(x) \cap D_j$ is a single point.

As $h_1(F) \subset K_2$ and $h_2(F) \subset K_2$ and h_1, h_2 are invariant on I-fibers, therefore $h_1(N(a)) \subset D_j$ and $h_2(N(a)) \subset D_k$ for some $j, k \in J$. As $h_1(a) = h_2(a)$, $D_j = D_k$. As $\pi^{-1}(x) \cap D_j$ is a single point for each $x \in N(a)$, $h_1(x) = h_2(x) = \pi^{-1}(x) \cap D_j$ for all $x \in N(a)$. Therefore $N(a) \subset A$ and A is open in F.

By hypothesis $A \neq \phi$ and F is connected, therefore $A = F$ as required. □

The following technical lemma will be used repeatedly, and is one of the reasons why we deal only with maximal branch surfaces obtained by splitting B and not just any branch surface obtained by splitting B. Let F be a sub-surface (and a closed subset) of a lamination-splitter K and let K' any other lamination-splitter. Then, in general, there may exist B-injections $g : F \to N(B)$ and $h : F \to N(B)$ such that $g(F) \subset K'$ and $h(F) \subset K'$ however $g \neq h$. For F a minimal lamination-splitter, this doesn't happen.

Lemma 5.3.6. *Let K be a lamination-splitter surface and let $F_1 \subset F_2 \subset K$ be minimal branch-splitter surfaces. Let K' be another lamination splitter surface. Let $g : F_1 \to N(B)$ and $h : F_2 \to N(B)$ be B-injections such that $g(F_1) \subset K'$ and $h(F_2) \subset K'$. Then, $h|_{F_1} = g$.*

Proof. By previous lemma 5.3.5, all we need to show is that for each connected component σ of F_1 there exists a point $p \in \sigma$ such that $h(p) = g(p)$.

As F_1 is a minimal splitter surface, σ intersects an annulus V of $\partial_v N(B)$. Let $\sigma \cap V = \gamma$ be an essential circle in V. As K' is a lamination splitter surface, $K' \cap V = \gamma'$ is another essential circle in V. As g and h are invariant on V (by definition of B-isotopy) and as $g(F_1) \subset K'$, $h(F_1) \subset K'$ therefore $g(\gamma) = h(\gamma) = \gamma'$. For any point $p \in \gamma$, $\pi^{-1}(p)$ intersects γ' in a single point, say p'. As g and h are both invariant on I-fibers of $N(B)$, therefore $g(p) = h(p) = p'$. □

Lemma 5.3.7. *Let K_0 and K_1 be lamination-splitter surfaces and let $F \subset K_0 \cap K_1$ be a minimal branch-splitter surface. Then K_0 is isotopic to K_1 in $N(B)$ if and only if K_0 is isotopic to K_1 (in $N(B)$) point-wise fixing F.*

Proof. Let $h_t : N(B) \to N(B)$ be an isotopy that takes K_0 to K_1. Let $i : F \to N(B)$ be the inclusion of F in $N(B)$ and let $i_t : F \to N(B)$ be $i_t = h_t|_F$.

As h_0 is the identity map of $N(B)$, $i_0 = i$. Taking $F_1 = F_2 = F$, $K = K_0$, $K' = K_1$, $g = i$ and $h = i_1$ in lemma 5.3.6, $i_1 = i$ as well. That is, both i_0 and i_1 are just the inclusion of F in $N(B)$.

Let $\tilde{i}_t : N(B) \to N(B)$ be the canonical extension of i_t. Then as i_0 and i_1 are inclusion maps of F in $N(B)$, $\tilde{i}_0$ and $\tilde{i}_1$ are identity maps of $N(B)$. Let $H_t = \tilde{i}_t^{-1} \circ h_t : N(B) \to N(B)$. Then H_0 is the identity map of $N(B)$ while $H_1 = h_1$. Also for all $t \in I$, $H_t|_F = \tilde{i}_t^{-1}|_{h_t(F)} \circ h_t|_F = h_t^{-1} \circ h_t|_F = i$. Therefore, $H_t : N(B) \to N(B)$ is an isotopy that point-wise fixes F and takes K_0 to K_1. □

Looking at the complement in $N(B)$, we get the following theorem.

Theorem 5.3.8. *Let λ_0 and λ_1 be laminations fully carried by a maximal branch surface B_1 inside $N(B)$. Then $\lambda_0 \sim \lambda_1$ in $\mathcal{PL}(B)$ if and only if $\lambda_0 \sim \lambda_1$ in $\mathcal{PL}(B_1)$.*

Proof. Let $\lambda_0 = B - K_0$, $\lambda_1 = B - K_1$ and let $B_1 = B - F$ then $F \subset K_0 \cap K_1$. By lemma 5.3.7, there exists an isotopy $H_t : N(B) \to N(B)$ taking K_0 to K_1, while point-wise fixing F. It therefore takes the complements, λ_0 to λ_1 while point-wise fixing F. After a slight modification of H_t we can obtain an isotopy $H'_t : N(B) \to N(B)$ taking λ_0 to λ_1 while point-wise fixing $N(F)$, where $N(F)$ is a fibered neighbourhood of F with $N(B) = N(B_1) \cup N(F)$ and $N(B_1) \cap N(F) \subset \partial N(F)$. Therefore, restricting H'_t to $N(B_1)$ we get an isotopy of $N(B_1)$ taking λ_0 to λ_1.

Conversely, if λ_0 is isotopic to λ_1 in $N(B_1)$, then there is an extension to an isotopy in $N(B)$, taking λ_0 to λ_1. □

Remark 5.3.9. This theorem is certainly not true if the branch splitting B_1 is not minimal. For example, let F be a closed disk in $N(B)$, transverse to the fibers of $N(B)$, with $D \cap \partial N(B) = \phi$.

Let $N(F) = F \times [0,1] \subset int(N(B))$ be a fibered neighbourhood of F and let $B_1 = B - F$. Let λ be a lamination fully carried by B_1 such that $\lambda \cap \pi^{-1}(F)$ is a disjoint union of finitely many disks. Suppose λ_1 is a lamination obtained from λ by an isotopy across $N(F)$. Then, λ_1 is not isotopic to λ in $N(B_1)$.

Our goal now is to prove Theorem 5.3.20, which says that X is sequentially compact. Along the way we shall prove Theorem 5.3.17, which says that every maximal lamination is the inverse limit of a sequence of maximal normal branch surfaces.

Definition 5.3.10. Let $|B'|$ be the number of normal branch sectors in the branch surface B. Let K be a compact normal surface carried by B. Let $[K]$ be the 'coordinates' of K corresponding to the weight of K on each normal branch sector of B. Let $|K|$ be the sum of the coordinates of $[K]$. Let $|\partial K|$ be the sum of the coordinates of $[\partial K]$ on the normal branch locus L'.

Remark 5.3.11. Given a compact normal surface K, there are up to B-isotopy only finitely many normal surface K_i, such that $[K_i] = [K]$.

Lemma 5.3.12. *For some $N \in \mathbb{N}$, let $K_n \subset N(B)$ be normal surfaces such that $|K_n| \leq N$. Then there exists an infinite subsequence K_{n_i} of K_n and a normal surface $K \subset N(B)$ such that K_{n_i} are B-isotopic to K.*

Proof. Let $d = |B|$. There are only finitely many points $(x_i) \in \mathbb{Z}^d$ which satisfy the system of equations $x_i \geq 0$, $\Sigma_{i=1}^{d} x_i \leq N$. Therefore there exists a subsequence K_m of K_n such that $[K_m] = v$ for some vector $v \in \mathbb{Z}^d$. There are up to B-isotopy only finitely many normal surfaces with coordinate v. Therefore, there exists a normal surface K and a subsequence K_{n_i} of K_m such that $K_{n_i} \simeq K$ for all $i \in \mathbb{N}$. ☐

Definition 5.3.13. Assume B has a Riemannian metric, which induces a metric on $N(B)$. Let K be a splitter surface in $N(B)$ and F_1, F_2 subsets of K. Then $d_K(F_1, F_2)$ denotes the distance in K between F_1 and F_2. By convention, if $F_1 = \phi$ or $F_2 = \phi$, then $d_K(F_1, F_2) = \infty$.

We now show that given a sequence of lamination-splitter surfaces K_n, there is an increasing sequence of normal minimal branch-splitter surfaces in each K_n, that 'uniformly' increases on the family K_n.

Lemma 5.3.14. *Let K_n be a sequence of lamination-splitter surfaces. Then there exists a real number $d > 0$ and for each $n \in \mathbb{N}$, a sequence of normal minimal branch-splitter surfaces $\{F_n(i)\}_{i=1}^{\infty}$ such that*
(i) $\partial K_n \subset \partial F_n(1)$, $F_n(i) \subset F_n(i+1)$ and $F_n(i) \subset K_n$ for all $n, i \in \mathbb{N}$.
(ii) $|F_n(1)| \leq |\partial K_n|$ and $|F_n(i+1)| \leq |F_n(i)| + |\partial F_n(i)|$ for all $n, i \in \mathbb{N}$.
(iii) Let $\partial F_n(i) = \partial_1 F_n(i) \cup \partial_2 F_n(i)$, where $\partial_1 F_n(i) = \partial K_n$. Then $d_{F_n(i+1)}(\partial_2 F_n(i+1), \partial_2 F_n(i)) \geq d$ for all $n, i \in \mathbb{N}$.

Proof. Let $F_n(1) = \overline{N}(\partial K_n)$, the normal neighbourhood of ∂K_n in K_n, for all $n \in \mathbb{N}$. Let $F_n(i+1) = \overline{N}(F_n(i)) = F_n(i) \cup \overline{N}(\partial_2 F_n(i))$ for all $i, n \in \mathbb{N}$. Then, by construction $\partial K_n \subset F_n(1)$ and $F_n(i) \subset F_n(i+1) \subset K_n$.

Let d be the minimum distance between non-intersecting edges of L'. If $\partial_2 F_n(i) = \phi$ (i.e., $F_n(i) = F_n(i+1)$) then $d_{F_n(i+1)}(\partial_2 F_n(i+1), \partial_2 F_n(i)) = \infty$. If $\partial_2 F_n(i) \neq \phi$, then as $\partial_2 F_n(i) \cap \partial_2 F_n(i+1) = \phi$ so $d_{F_n(i+1)}(\partial_2 F_n(i+1), \partial_2 F_n(i)) \geq d$.

As $F_n(i+1)$ is obtained from $F_n(i)$ by attaching at most $|\partial F_n(i)|$ normal disks to $\partial F_n(i)$, $|F_n(i+1)| \leq |F_n(i)| + |\partial F_n(i)|$.

As each component of $F_n(i)$ contains a component of $F_n(1) = \overline{N}(\partial K_n)$, each component of $F_n(i)$ intersects ∂K_n. As K_n is a lamination-splitter surface and $\partial K_n = F_n(i) \cap \partial_v N(B)$, $F_n(i)$ intersects $\partial_v N(B)$ in circles, with at most one circle in each annulus of $\partial_v N(B)$. Also, $F_n(i)$ is closed and bounded, hence compact. Therefore, $F_n(i)$ is a normal minimal branch-splitter surface. □

Lemma 5.3.15. *Let $F(n)$ be a sequence of minimal normal branch-splitter surfaces and $d > 0$, such that $F(n) \subset F(n+1)$ and $d_{F(n+1)}(\partial_2 F(n), \partial_2 F(n+1)) > d$ and $F(1)$ intersects each component of $\partial_v N(B)$. Then $K = \cup F(n)$ is a minimal lamination-splitter surface.*

Proof. Suppose there exists $p \in cl(K) - K$. As $K = \cup_{n=1}^{\infty} F(n)$, there exists a sequence $p_n \to p$ in K, such that $p_n \in \partial_2 F(n)$ for all $n \in \mathbb{N}$. This contradicts the assumption that $d_{F(n+1)}(\partial_2 F(n), \partial_2 F(n+1)) > d$ for all $n \in \mathbb{N}$. Therefore K is a closed set.

Let $\partial K = \partial_1 K \cup \partial_2 K$ where $\partial_1 K = K \cap \partial_v N(B)$. We shall show that $\partial_2 K = \phi$. Let c be a component of $\partial_2 K$. Then there exists $m \in \mathbb{N}$ such that $c \subset \partial_2 F(m)$. So $c \subset int(\overline{N}(F(m))) = int(F(m+1))$. Therefore, $c \subset int(K)$ which is a contradiction as $c \subset \partial K$. Therefore, $\partial K = K \cap \partial_v N(B)$.

As $F(n)$ are splitter surfaces, $K \cap \partial_h N(B) = \phi$ and K intersects $\partial_v N(B)$ in circles. As each $F(n)$ intersects annuli of $\partial_v N(B)$ in not more than one circle and K is an increasing union of $F(n)$, K intersects no component of $\partial_v N(B)$ in more than one circle. By assumption, $K \supset F(1)$ intersects each component of $\partial_v N(B)$. Therefore, K is a minimal lamination-splitter surface. □

We now give the converse of the above lemma 5.3.15

Lemma 5.3.16. *Let K be a minimal lamination-splitter surface. Then there exists a sequence $F(i)$ of normal minimal branch-splitter surfaces such that $\partial K \subset \partial F(1)$, $F(i) \subset F(i+1)$, $d_{F(i+1)}(\partial_2 F(i), \partial_2 F(i+1)) > d$ and $K = \cup_{i=1}^{\infty} F(i)$.*

Proof. Taking $K_n = K$ for all $n \in \mathbb{N}$ in Lemma 5.3.14, we get $F(i)$ as the required normal minimal branch-splitter surfaces. Let $F = \cup_{i=1}^{\infty} F(i)$, we need to show that $K = F$. By previous lemma 5.3.15, F is a lamination-splitter surface. By remark 5.3.2 then, as K is a minimal splitter surface, $F = K$. □

The above two lemmas imply the following theorem, which is a re-statement of Lemma 4.2 of [7].

Theorem 5.3.17. *Let λ be a maximal lamination. Then, there exists a sequence of maximal branch surfaces B_n obtained by splitting B and a real number $d > 0$ such that B_{n+1} is a splitting of B_n, $d(B_n, B_{n+1}) > d$ and $\lambda = \cap N(B_n)$.*

Conversely given a sequence of branch surfaces B_n obtained by splitting B such B_{n+1} is a splitting of B_n and $d(B_n, B_{n+1}) > d$, $\cap N(B_n)$ is a maximal lamination.

Proof. The proof is a combination of lemmas 5.3.16 and 5.3.15. Let $\lambda = B - K$, where K is a minimal lamination-splitter surface. Then, by lemma 5.3.16 we get a real number $d > 0$ and an increasing sequence of minimal branch-splitter surfaces $F(n)$, such that $K = \cup F(n)$ and $d_{F(n+1)}(\partial_2 F(n), \partial_2 F(n+1)) > d$. Let $B_n = B - F(n)$. Then B_{n+1} is a splitting of B_n, $d(B_n, B_{n+1}) > d$ and $\cap N(B_n) = \lambda$.

Conversely, given a sequence of branch surfaces B_n such that B_{n+1} is a splitting of B_n and $d(B_n, B_{n+1}) > d$, taking complements we have $F(n) \subset F(n+1)$, $d_{F(n+1)}(\partial_2 F(n), \partial_2 F(n+1)) > d$ and $F(1)$ intersects each component of $\partial_v N(B)$. Therefore, by lemma 5.3.15, $\cup F(n) = K$ is a minimal lamination-splitter surface. Taking complement again, $\cap N(B_n) = B - K$ is a maximal lamination, as required. □

Definition 5.3.18. Let $\lambda_n = B - K_n$ be a sequence of laminations. If λ_n converges to $\lambda = B - K$ in X, then we say K_n converges to K.

The sequence λ_n converges to λ, if given any maximal branch splitting B_1 that fully carries λ, there exists $m \in \mathbb{N}$ such that for all $n > m$, λ_n is isotopic to a lamination fully carried by B_1. Looking at the complements, the sequence K_n converges to K, if for any compact sub-surface F of K there exists an $m \in \mathbb{N}$ such that for all $n > m$ there exists a B-isotopy $H(n)_t : N(B) \to N(B)$ taking K_n to K'_n where $K'_n \supset F$.

In the following theorem we shall show sequential compactness for minimal lamination-splitter surfaces. Given a sequence K_n, we shall exhaust them by an increasing union of minimal branch-splitters $K_n(i)$ that increases uniformly as per lemma 5.3.14. As $K_n(1)$ is a compact normal surface, by lemma 5.3.12 we shall obtain a subsequence K_n^1 of K_n and a minimal branch-splitter $F(1)$ so that after an isotopy $K_n^1(1) = F(1)$.

Having got a subsequence K_n^m of K_n^{m-1} for which $K_n^m(m) = F(m)$ we shall proceed to obtain a subsequence K_n^{m+1} of K_n^m and a minimal branch-splitter $F(m+1) \supset F(m)$ such that after an isotopy we may assume $K_n^{m+1}(m+1) = F(m+1)$.

Finally, let $K = \cup F(m)$. We shall show that K is a minimal lamination splitter. Then we take a diagonal sequence K_{n_i} of K_n^i to get a subsequence of K_n such that given any $m \in \mathbb{N}$ for all $i > m$, there exists an isotopy after which $K_{n_i}(i) = F(i) \supset F(m)$. So we would have shown that K_{n_i} converges to K.

Lemma 5.3.19. *Let K_n be a sequence of lamination-splitter surfaces. Then, there exists a subsequence K_{n_i} of K_n and a minimal lamination-splitter surface K such that $K_{n_i} \to K$.*

Proof. For each K_n, we take an increasing union of normal minimal branch-splitter surfaces $K_n(i)$ as defined in Lemma 5.3.14. We shall now obtain an increasing sequence of branch-splitter surfaces $F(i)$ such that $\cup F(i) = K$ is the limit lamination-splitter surface.

As ∂K_n consists of an essential circle in each component $\partial_v N(B)$, $|\partial K_n|$ is a constant c for all $n \in \mathbb{N}$. As $|K_n(1)| \leq |\partial K_n| = c$ for all $n \in \mathbb{N}$, by Lemma 5.3.12 there exists a normal surface $K'(1)$ and a subsequence K_n^1 of K_n such that $K_n^1(1) \simeq K'(1)$. Let $F(1) = K_1^1(1)$. As $K_n^1(1) \simeq F(1)$ for all $n \in \mathbb{N}$, by Theorem 5.2.19, we can extend this isotopy to an ambient B-isotopy $H_t^n : N(B) \to N(B)$, taking $K_n^1(1)$ to $F(1)$, i.e., H_0^n is the identity map while $F(1) = H_1^n(K_n^1(1)) \subset H_1^n(K_n^1)$. So, after such a B-isotopy for each K_n^1 we may assume that $F(1) = K_n^1(1) \subset K_n^1$ for all $n \in \mathbb{N}$.

Suppose for $i = m - 1$, there exists $F(i)$ such that $F(i-1) \subset F(i)$ and there exists a subsequence K_n^i of K_n^{i-1} such that $K_n^i(i) = F(i)$ for all $n \in \mathbb{N}$.

By Lemma 5.3.14, $|K_n^{m-1}(m)| \leq |K_n^{m-1}(m-1)| + |\partial K_n^{m-1}(m-1)|$ for all $n \in \mathbb{N}$. As $K_n^{m-1}(m-1) = F(m-1)$ for all $n \in \mathbb{N}$, $|K_n^{m-1}(m)| \leq |F(m-1)| + |\partial F(m-1)|$. By Lemma 5.3.12, there exists a normal surface $K'(m)$ and a subsequence K_n^m of K_n^{m-1} such that $K_n^m(m) \simeq K'(m)$. Let $F(m) = K_1^m(m)$. Then as $F(m-1) = K_n^{m-1}(m-1)$ for all $n \in \mathbb{N}$, so $F(m-1) = K_1^m(m-1) \subset K_1^m(m) = F(m)$ as required. Again, by Theorem 5.2.19, after isotopies of K_n^m we may assume that $F(m) = K_n^m(m)$ for all $n \in \mathbb{N}$.

Observe that K_n^{i+1} is, up to isotopy, a subsequence of K_n^i for all $i \in \mathbb{N}$. Let K_{n_i} be a diagonal sequence of K_n^i. Then, for all $i \in \mathbb{N}$ there exists a B-isotopy $H(i)_t : N(B) \to N(B)$ such that $F(i) = H(i)_1(K_{n_i}(i))$.

Let $F = \cup_{i=1}^{\infty} F(i)$. Given a compact subsurface F' of F, there exists $m \in \mathbb{N}$ such that $F' \subset F(m)$. For all $i > m$, there exists an ambient isotopy of $N(B)$ that takes K_{n_i} to a surface that contains $F(i) \supset F(m)$. Therefore, F is the limit of K_{n_i}. We now show that F is a minimal lamination splitter surface.

For each $n \in \mathbb{N}$, $d_{F(n+1)}(\partial_2 F(n), \partial_2 F(n+1)) = d_{K_1^{n+1}}(\partial_2 K_1^{n+1}(n), \partial_2 K_1^{n+1}(n+1)) > d$ by construction of the sequence $\{K_1^{n+1}(i)\}_i$. Also, as K_1^1 is a lamination-splitter surface, $F(1) = K_1^1(1)$ intersects each component of $\partial_v N(B)$. So, by lemma 5.3.15, $K = \cup F(n)$ is a minimal lamination-splitter surface. □

Taking the complements, we get the desired sequential compactness of laminations.

Theorem 5.3.20. *The space X is sequentially compact.*

Proof. Let $\lambda_n = B - K_n$ be a sequence of maximal laminations, then by the above lemma 5.3.19 there exists a maximal lamination $\lambda = B - K$ and a subsequence K_{n_i} of K_n such that $K_{n_i} \to K$. So given any maximal branch-splitter surface $B_1 = B - F_1$ such that $\lambda \leq B_1$, there exists $m \in \mathbb{N}$ such that for all $i > m$, $\lambda_{n_i} \leq B_1$. □

We now show that the topology on X is the same as the topology on $\mathcal{PL}(B)$.

Theorem 5.3.21. *The space X is homeomorphic to $\mathcal{PL}(B)$.*

Proof. What we need to show is that the set $\mathcal{B} = \{V_{B_1} : B_1 \text{ is a maximal branch splitting of } B\}$ is a basis for $\mathcal{PL}(B)$ as well. That is, we need to show V_{B_1} is an open set in $\mathcal{PL}(B)$ and that given any point $[\lambda] \in \mathcal{PL}(B)$ and an open set V containing $[\lambda]$, there exists a maximal branch splitting B_1 such that $[\lambda] \in V_{B_1} \subset V$.

For a maximal branch splitting B_1, let B_1 be represented as a closed subset of $N(B)$ (i.e., a branched surface embedded in $N(B)$). Then for $\delta > 0$, $N'_\delta(B_1)$, a δ-neighbourhood of B_1 in $\mathcal{L}'(B)$ (i.e., a δ-neighbourhood in $\mathcal{F}'(B)$ intersected with $\mathcal{L}'(B)$), is the union of those maximal laminations that lie in a δ-neighbourhood $N_\delta(B_1)$ of B_1 (in $N(B)$) and are fully carried by B_1. So the corresponding neighbourhood $\hat{N}_\delta(B_1)$ in $\mathcal{PL}(B)$, is exactly V_{B_1} which is all those isotopy classes of maximal laminations that have a representative fully carried by B_1. Therefore, in particular, V_{B_1} is open.

Given $[\lambda] \in V$, take a δ-neighbourhood $N'_\delta(\lambda)$ in $\mathcal{L}'(B)$ such that the corresponding neighbourhood $\hat{N}_\delta([\lambda]) \subset V$. Let $N_\delta(\lambda)$ be a δ neighbourhood of λ in $N(B)$. Then if a maximal lamination μ lies in $N_\delta(\lambda)$ and intersects each I-fiber of $N_\delta(\lambda)$, then $[\mu] \in \hat{N}_\delta([\lambda])$.

By Theorem 5.3.17 there exists a sequence of maximal branch splittings B_n such that $\lambda = \cap N(B_n)$. Therefore, there exists $m \in \mathbb{N}$ such that $\lambda \subset N(B_m) \subset N_\delta(\lambda)$ and each fiber of $N_\delta(\lambda)$ intersects $N(B_m)$. Now if $[\mu] \in V_{B_m}$, then after an isotopy $\mu \subset N(B_m) \subset N_\delta(\lambda)$ and each fiber of $N_\delta(\lambda)$ intersects μ. Therefore, $[\mu] \in \hat{N}_\delta([\lambda]) \subset V$ and $V_{B_m} \subset V$ as required. □

As detailed in the introduction, to show that $\mathcal{PL}(B)$ is Hausdorff and that each point in $\mathcal{PL}(B)$ is the intersection of a sequence of open sets it is enough to show that for each $\lambda \in \mathcal{L}'(B)$ the equivalence class $[\lambda]$ of λ is a closed subset of $\mathcal{L}'(B)$.

Definition 5.3.22. Let $K \subset N(B)$ be a lamination-splitter surface. Let $h : K \times [0, n] \to N(B)$ and $i : K \times [0, m] \to N(B)$ be B-isotopies such that $h_n = i_0$. Then we define $h * i : K \times [0, n+m] \to N(B)$ by $(h * i)_t = h(t)$ if $t \in [0, n]$ and $(h * i)_t = i_{t-n}$ if $t \in [n, n+m]$. This is the concatenation of paths in the space of B-injections of K in $N(B)$.

We define $\overline{h} : K \times [0, n] \to N(B)$ as the isotopy $\overline{h}_t = h_{n-t}$.

Lemma 5.3.23. *Let K and K' be minimal lamination-splitter surfaces in $N(B)$. Let F_n be an increasing sequence of minimal normal branch-splitter surfaces such that $\cup F_n = K$. For each $n \in \mathbb{N}$, let $h(n)_t : F_n \to N(B)$ be an isotopy taking F_n to a subsurface of K'. Then there exists an isotopy $f : K \to N(B)$ taking K to K'.*

Proof. By lemma 5.2.19, there exists an extension of the inclusion isotopy $h(n) : F_n \times [0,1] \to N(B)$ to an ambient isotopy of $N(B)$. Let $\tilde{h}(n) : K \times [0,1] \to N(B)$ be the restriction of this ambient isotopy to the lamination splitter surface K.

Let us assume there exists a family of isotopies $g(n) : K \times [0,n] \to N(B)$ having the following properties.
(i) $g(1)_0$ is the inclusion map $i : K \to N(B)$
(ii) $g(n)_t = g(n-1)_t$, $\forall\, t \in [0, n-1]$, $n > 1$
(iii) $g(n)_t(x) = g(n)_{n-1}(x) = g(n-1)_{n-1}(x)$, $\forall\, t \in [n-1, n]$, $x \in F_{n-1}$, $n > 1$
(iv) $g(n)_n = \tilde{h}(n)_1$, $n \geq 1$

Then for all positive integers $k \leq n$, $\forall\, x \in F_k$, $\forall\, t \in [k, n]$, $g(n)_t(x) = g(n)_k(x) = g(k)_k(x) = \tilde{h}(k)_1(x)$. That is, $g(m)$ isotopes K so that F_m 'sticks' to $h(m)(F_m) = F'_m \subset K'$ in time $t \in [0, m]$. Then, for all time $t \geq m$, F_m remains stuck to F'_m. In particular, $F'_m \subset F'_{m+1}$.

As $g(n)_t = g(n-1)_t$ for $t \in [0, n-1]$, we can think of $g(n)$ as an increasing sequence of paths in the space of B-injections of K in $N(B)$. We now define the composite path $g(\infty) : K \times [0, \infty) \to N(B)$ by $g(\infty)_t = g(n)_t$ for $t \in [0, n]$. This is a well-defined inclusion isotopy of K in $N(B)$.

Let $\sigma : [0,1) \to [0, \infty)$ be a homeomorphism. Define $f : K \times [0,1] \to N(B)$ by $f_t(x) = g(\infty)_{\sigma(t)}(x)$ for $t \in [0,1)$ and $f_1(x) = h(k)_1(x)$ for $x \in F_k$.

For positive integers $k < n$, $F_k \subset F_n$ and by lemma 5.3.6, $h(n)_1|_{F_k} = h(k)_1$ therefore f_1 is well-defined on $\cup F_n = K$. As each $h(n)_1 : F_n \to N(B)$ is a B-injection and F_n is an increasing sequence with $\cup F_n = K$, $f_1 : K \to N(B)$ is a B-injection as well.

For all positive integers k, $\forall x \in F_k$, $t \in [k, \infty)$, $g(\infty)_t(x) = g(k)_k(x) = \tilde{h}(k)_1(x)$, i.e., for each $x \in K$, $g(\infty)_t(x)$ is eventually constant (as a function of t). Therefore for $x \in F_k$, $f_1(x) = f_t(x)$ for all $t > \sigma^{-1}(k)$ so that f is continuous with respect to t at $t = 1$.

Therefore, f is an isotopy taking K to a sublamination of K'. As K' is a minimal lamination splitter surface by Remark 5.3.2, f is the required isotopy taking K to K'.

We now give a construction of the family of isotopies $g(n)$. Let $g(1) = \tilde{h}(1)$. Then $g(1)_0 = \tilde{h}(1)_0 = i$ the inclusion of K in $N(B)$ and $g(1)_1 = \tilde{h}(1)_1$ as required.

Having defined $g(n)$ with the required properties, we now define $g(n+1) : K \times [0, n+1] \to N(B)$ by $g(n+1) = g(n) * g'(n)$ where $g'(n) : K \times [0,1] \to N(B)$ is defined as follows.

Let $K_0 = g(n)_n(K) = \tilde{h}(n)_1(K)$ and let $K_1 = \tilde{h}(n+1)_1(K)$. By lemma 5.3.6, as $\tilde{h}(n+1)_1|_{F_n} = h(n)$ therefore, $F'_n = h(n)(F_n) \subset K_0 \cap K_1$. The isotopy $\overline{g(n)} * \tilde{h}(n+1) : K \times [0, n+1] \to N(B)$ takes K_0 (via K) to K_1 therefore by lemma 5.3.7, there is an isotopy $g'(n) : K \times [0,1] \to N(B)$ taking K_0 to K_1 while point-wise fixing F'_n.

So by construction of $g(n+1)$, $g(n+1)_t = g(n)_t$ for all $t \in [0,n]$ and $g(n+1)_{n+1} = \tilde{h}(n+1)_1$. Furthermore, by the construction of $g'(n)$, as $g'(n)_t(x) = x$ for all $x \in F'_n$, $t \in [0,1]$. So $\forall x \in F_n$, $\forall t \in [n, n+1]$, $g(n+1)_t(x) = g(n)_n(x) = h(n)(x)$. Therefore, $g(n)$ is the required family of isotopies. □

Looking at the complement now, we have the required theorem.

Theorem 5.3.24. *The equivalence class of a lamination is closed in $\mathcal{L}'(B)$.*

Proof. Let λ_n be a sequence of laminations such that λ_n converges to μ in $\mathcal{L}'(B)$ and each λ_n is isotopic to a fixed lamination λ. Then, we shall show that λ is equivalent to μ.

By Theorem 5.3.17, there exists a sequence of maximal branch splittings B_n such that $\cap N(B_n) = \mu$. As $\lambda_n \to \mu$ in $\mathcal{L}'(B)$, for every $m > 0$, there exists $k > 0$ such that for all $n > k$, λ_n lies in $N(B_m)$. In terms of complementary splitter surfaces, if $B_n = B - F'_n$, $\lambda = B - K$ and $\mu = B - K'$, then there is an isotopy $h(n)$ of K such that that $h(n)_1(K) \supset F'_n$. Let $F_n \subset K$ be such that $h(n)_1(F_n) = F'_n$. Applying lemma 5.3.23, we get an isotopy $h : N(B) \to N(B)$ that takes K to K', and therefore takes the complements λ to μ as required. □

Corollary 5.3.25. *The space of maximal laminations $\mathcal{PL}(B)$ is Hausdorff and each point in it is the intersection of a sequence of open sets.*

Proof. As $\mathcal{L}'(B)$ is a metric space, the topology generated by it is normal. Therefore, for $\lambda \in \mathcal{L}'(B)$ if $[\lambda]$ is a closed set, then identifying equivalence classes, we get $\mathcal{L}(B)$ to be a Hausdorff space and each point in it as the intersection of a sequence of open sets. □

Remark 5.3.26. The topology on $\mathcal{PL}(B)$, coming from the Hausdorff metric on $\mathcal{L}'(B)$, is in many respects close to a metric topology. However the obvious topology of distance in $\mathcal{L}'(B)$ between closed sets (equivalence classes of laminations) fails. With respect to this 'metric', the distance between any two isotopy classes of laminations is zero.

To see this, take equivalence classes $[\lambda]$ and $[\mu]$ and let B be represented by an embedded branched surface in $N(B)$. Then, for any neighbourhood $N_\delta(B)$ of B in $N(B)$, there is a lamination isotopic to λ (similarly to μ) that lies in $N_\delta(B)$. So there is a sequence of laminations λ_n isotopic to λ (similarly a sequence μ_n isotopic to μ) that converges to B in $F'(B)$. Therefore, $d(\lambda_n, \mu_n) \to 0$ in $\mathcal{L}'(B)$ and the 'distance' between $[\lambda]$ and $[\mu]$ is zero.

6. EPILOGUE

It was shown by Tao Li in [21] that a non-Haken manifold has only finitely many strongly irreducible Heegaard splittings. Moriah, Schleimer and Sedgwick [25] have shown that for all existing examples of manifolds with infinitely many irreducible splittings, there exists a splitting surface H and a surface K, such that each of the splittings is given by then Haken sum $H + nK$, where n is some non-negative integer. They also show that such a surface K is incompressible.

These results lead us to state the following conjecture.

Conjecture 6.0.27. *Let M be a closed, orientable, irreducible and atoroidal 3-manifold with infinitely many strongly irreducible Heegaard splittings. Then, there exists an incompressible surface K and a strongly irreducible Heegaard splitting H such that there are infinitely many strongly irreducible Heegaard splittings given by the Haken sum $H + nK$, for $n \in \mathbb{N}$.*

As a first step in this direction we first aim to prove the weaker result.

Conjecture 6.0.28. *Let M be a closed, orientable, irreducible and atoroidal 3-manifold, with infinitely many strongly irreducible Heegaard splittings. Then, there exists a strongly irreducible Heegaard splitting surface H and a sequence of (possibly disconnected) incompressible surfaces K_n such that $S_n = H + K_n$ is a sequence of strongly irreducible Heegaard splittings.*

In our first approach at proving this conjecture, we attempt to extend the methods employed by Li in[21] to the space of projective maximal laminations carried by an almost normal branch surface. We give an outline of the proof here.

Let M be a closed orientable irreducible atoroidal 3-manifold that is not a small Seifert fiber space, that contains infinitely many strongly irreducible Heegaard splittings. By a theorem in [20], there is a finite collection of branch surfaces in M such that every strongly irreducible Heegaard surface is fully carried by a branch surface in this collection. Moreover, the branch surfaces in this collection do not carry any normal 2-sphere or normal torus. Let B be a branch surface in this collection that carries infinitely many strongly irreducible Heegaard surfaces $\{S_n\}$.

We have defined a space of projective maximal laminations $\mathcal{PL}(B)$ fully carried by B, analogous to the space of geodesic laminations on surfaces, and the space of projective measured laminations carried by branch surfaces. We have shown that this space is compact and Hausdorff, so the sequence $\{S_n\}$ has a limit point.

By passing to a subsequence we can assume that for each branch sector $b \in B$, the weight of S_n at b, $w_b(S_n)$, is either a constant with respect to n or tends to infinity as n tends to infinity.

Let B^- be the sub-branch surface of B obtained by taking the union of those branch sectors $b \in B$ where $w_b(S_n) \to \infty$. We construct a lamination μ fully carried by B^- that is the 'unbounded limit' of S_n. Our main task then is to prove that μ is an essential lamination. We use the same techniques employed by Li in [21], with the difference being that μ is not a point in the projective measured lamination space and in fact may not have a measure.

Once we show that μ is an essential lamination we show that there is a splitting B_1 of B along μ that carries infinitely many of the S_n, and is such that B_1^- (a splitting of B^-) is an essential branch surface. By passing to a subsequence, we shall show that for each branch sector $b \in B_1 - B_1^-$ the weight of S_n is a constant $c(b)$ (independent of n). As each coordinate $(w_b(S_n))_{b \in B^-}$ of S_n tends to infinity as $n \to \infty$, there exists $k, m \in \mathbb{N}$ such that for all $n > m$, $w_b(S_n) - w_b(S_k) > 0$ if $b \in B_1^-$ (while $w_b(S_n) - w_b(S_k) = 0$ for all $b \in B_1 - B_1^-$). Let $H = S_k$ and $K_n = S_n - S$. Then K_n is a closed surface fully carried by the essential branch surface B_1^-, and therefore by [7] is an incompressible surface. So we will have shown that $S_n = H + K_n$ as required.

In our second approach at proving this conjecture, we first try and prove the result of Li for non-Haken manifolds, by different means. We shall extend the ideas developed in [44] to characterise incompressible surfaces as 'canonically stable normal' surfaces and to characterise strongly irreducible Heegaard surfaces as 'canonically stable almost normal' surfaces. This is analogous, in the smooth category, to saying that incompressible surfaces are minimal area surfaces while strongly irreducible Heegaard surfaces are minimal surfaces of index 1. We shall then show that the projective limit of canonically stable almost normal surfaces is a canonically stable normal surface.

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