SPECTRAL ANALYSIS AND SYNTHESIS FOR RADIAL SECTIONS OF HOMOGENOUS VECTOR BUNDLES ON CERTAIN NONCOMPACT RIEMANNIAN SYMMETRIC SPACES

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Contents

Chapter 0

Introduction

We consider two classical theorems of real analysis which deals with translation invariant subspaces of integrable and smooth functions on $\mathbb{R}$ respectively. The first one is a theorem of Norbert Wiener [63] which states that if the Fourier transform of a function $f \in L^1(\mathbb{R})$ has no real zeros then the finite linear combinations of translations $f(x-a)$ of f with complex coefficients form a dense subspace in $L^1(\mathbb{R})$, equivalently, $\mathsf{span}\{g * f \mid g \in L^1(\mathbb{R})\}$ is dense in $L^1(\mathbb{R})$. This theorem is well known as the Wiener-Tauberian Theorem (WTT). The second theorem on spectral analysis on $\mathbb{R}$, due to Laurent Schwartz [56] states that a closed nonzero translation invariant subspace of $C^\infty(\mathbb{R})$ with its usual Fréchet topology contains the map $x \mapsto e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. This is equivalent to the statement that if $f \in C^\infty(\mathbb{R})$ then the closure of the set $\{W * f \mid W \in C^\infty(\mathbb{R})'\}$ in $C^\infty(\mathbb{R})$ contains the map $x \mapsto e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$, where $C^\infty(\mathbb{R})'$ denotes the set of compactly supported distributions on $\mathbb{R}$. We shall call this Schwartz's theorem. It is well known that the statement above is false for $\mathbb{R}^n$ if $n > 1$ (see [31]).

We use the terms *spectral analysis* and *spectral synthesis* in the sense of Schwartz [56]. We endeavour to study these theorems in the context of homogenous vector bundles on a noncompact rank one Riemannian symmetric space X. We recall that such a space X can be identified with G/K where G is a connected noncompact semisimple Lie group with finite centre having real rank one and K is a maximal compact subgroup of G. This makes X a G-space with canonical G-action. Any function on X can be identified with a right K-invariant function on G and in particular left K-invariant functions on X are K-biinvariant (also called radial) functions on G. In this setup we shall consider the two theorems mentioned above. We shall discuss them one after the other.

Wiener-Tauberian Theorem was extended to abelian locally compact groups where the hypothesis is on a Haar integrable function which has nonvanishing

Fourier transform on all unitary characters (see [51]). Analogues of this result hold also for many nonabelian Lie groups (see e.g. [27, 43]). On the other hand back in 1955 failure of WTT even for the commutative Banach algebra of integrable radial functions on $\mathrm{SL}(2,\mathbb{R})$ was noticed by Ehrenpreis and Mautner in [22]. A simple proof due to M. Duflo of the fact that the WTT based on unitary dual is false for any noncompact semisimple Lie group appears in [43]. This failure can be attributed to the existence of the nonunitary uniformly bounded representations in groups of this class (see [23, 41]).

However a modified version of the theorem was established in [22] for radial functions in $L^1(\mathrm{SL}(2,\mathbb{R}))$. There were a few attempts to generalize this result to more general semisimple Lie groups and with lesser restriction on functions (see [57, 58, 7, 5, 6, 54, 55, 45, 18]). Research remains incomplete as almost all of these papers deal only with radial functions. Apart from the group $\mathrm{SL}(2,\mathbb{R})$, where we have the advantage of one dimensional K-types (see [54]), going beyond the K-biinvariant setup is difficult, perhaps insuperably so.

Our departure in this thesis is in two directions. Firstly we come out of the setup of the radial functions and deal with the radial sections of certain homogenous vector bundle on the noncompact Riemannian symmetric spaces. For a unitary representation (τ, V_τ) of K we consider the vector bundle E_τ over G/K which is defined as follows: The equivalence relation ρ_τ on $G\times V_\tau$ is defined by $(g,v)\ \rho_\tau\ (g',v')$ if and only if there exists $k\in K$ such that $g'=gk$ and $v'=\tau(k^{-1})v$. Then the quotient space $E_\tau = G\times V_\tau/\rho_\tau$ with the projection $p: G\times V_\tau/\rho_\tau \to G/K$ defined by $[(g,v)]\mapsto gK$ is a vector bundle over G/K. There is a one-to-one correspondence between the sections of E_τ and functions on G in the class, $\Gamma(G,\tau)=\{f: G\to V_\tau \mid f(gk)=\tau(k^{-1})f(g), \text{ for all } g\in G, k\in K\}$. A τ-radial section of this bundle is associated with an $\mathrm{End}V_\tau$-valued τ-radial function on G defined by

$$f(k_1 g k_2) = \tau(k_2^{-1})\circ f(g)\circ \tau(k_1^{-1})$$

or with its scalar version $f: G\longrightarrow \mathbb{C}$ defined by $f(x) = d_\tau\overline{\chi_\tau} * f * d_\tau\overline{\chi_\tau}(x), f(kxk^{-1}) = f(x)$ for $x\in G$ and $k\in K$. Here χ_τ and d_τ are respectively the character and dimension of τ. We restrict our attention to the vector bundle associated with a K-type τ for which (G,K,τ) is a Gelfand triple, i.e. when compactly supported (or integrable) τ radial functions form a commutative algebra under convolution. This is perhaps the natural step after dealing with radial functions. Here the role of the elementary spherical function ϕ_λ is taken up by the τ-spherical function $\Phi^\tau_{\sigma,\lambda}$ which is an eigensection (of the Laplace-Beltrami operator of X) of the bundle E_τ. The τ-spherical transform $\widehat{f}$ of a τ-radial function is defined using $\Phi^\tau_{\sigma,\lambda}$ and is the object corresponding to the spherical Fourier transform of a radial function. We denote $\mathrm{Tr}\,\Phi^\tau_{\sigma,\lambda}$ by $\phi^\tau_{\sigma,\lambda}$. For a function space $\mathcal{L}(G)$ on G, $\mathcal{L}(G//K)$ and $\mathcal{L}_\tau(G)$ denote respectively the set of radial and τ-radial functions in $\mathcal{L}(G)$.

For the sake of being explicit, we will be working with the example of the spinor bundle on the real hyperbolic spaces for which a well-developed L^2 theory is available (see [11, 13]). All the results we obtain here will go through for many other examples of Gelfand triple (see towards the end of the section for details of such Gelfand triples). In particular all the theorems are valid for K-biinvariant functions of any noncompact semisimple Lie group with finite centre which has real rank one. Our results however improve on the existing results for K-biinvariant functions (cf. [7, 55]) and also add new results in that context. We identify the real hyperbolic space as $\mathrm{Spin}_0(n,1)/\mathrm{Spin}(n)$, where $\mathrm{Spin}_0(n,1)$ is the identity component of the group $\mathrm{Spin}(n,1)$. Camporesi and Pedon have used this identification in [13]. Let τ_n be the classical complex spin representation of K. The spinor bundle is the homogenous vector bundle $\sum H^n(\mathbb{R}) = G \times V_{\tau_n}/\rho_{\tau_n}$ and the sections of this bundle are the *spinors*. It is known that τ_n is irreducible when n is odd and splits into two inequivalent irreducible components when n is even. We will work with the vector bundle corresponding to the irreducible components of the representations τ_n.

As our second point of departure we view WTT as a problem associated to a space of functions $\mathcal{F}_1$ acting on another say $\mathcal{F}_2$ by convolution. One tries to put *sufficient condition* on a family of functions $\mathcal{G} \subset \mathcal{F}_2$ so that $\mathcal{G}$ generates $\mathcal{F}_2$ under $\mathcal{F}_1$ action. We point out that the Banach algebras like $L^1(G//K)$ or their counter part $L^1_\tau(G)$ (which forms the usual setup for WTT) can be considered as particular cases of two different families of Banach algebras or modules. The first family consists of analogues of Beurling algebras with *analytic* weights (see [14]) while the second consists of Lorentz spaces and algebras. The first family remains close to the classical in behavior, but that of the latter family which in particular includes the L^p as well as the weak L^p spaces is rooted in the Kunze-Stein phenomenon ([15]) and hence has no euclidean analogue. We can formulate WTT for all these Banach algebras and modules and by a more or less uniform approach we can prove the theorem in all the cases (see Theorem 6.1.1, Theorem 6.1.2, Remark 6.1.8), except for a degenerate case which we shall treat separately (see (B) below).

We also see that two Wiener-Tauberian type theorems arise naturally in our context which are based on the unitary dual.

(A) Unlike the classical WTT which considers $L^1(\mathbb{R})$ action on $L^1(\mathbb{R})$, we view WTT as a theorem involving naturally arising pairs of spaces $(\mathcal{F}_1, \mathcal{F}_2)$ with $\mathcal{F}_1$ acting on $\mathcal{F}_2$ by convolution. For the same space $\mathcal{F}_2$ we may find several spaces $\{\mathcal{F}_1^\alpha\}$ so that WTT can be formulated for $(\mathcal{F}_1^\alpha, \mathcal{F}_2)$ for each α. WTT finds sufficient conditions on a collection of functions in $\mathcal{F}_2$ so that under $\mathcal{F}_1^\alpha$ action it generates a dense space in $\mathcal{F}_2$. The core of the sufficient condition is the *nonvanishing condition* of the Fourier transform on its natural domain of definition for the functions in $\mathcal{F}_2$ and thus depends solely on the function space $\mathcal{F}_2$. More precisely this condition remains unaltered if we change the first space of the pair say from $\mathcal{F}_1^\alpha$ to $\mathcal{F}_1^\beta$.

However one can ask: Given a collection of functions $\mathcal{G}$ in $\mathcal{F}_2$ which satisfies a weaker nonvanishing condition, can we bring in the action of some additional convolutors on $\mathcal{G}$ which enables $\mathcal{G}$ to generate $\mathcal{F}_2$? In particular we are interested in finding a WTT where the nonvanishing condition is only on the unitary dual. Our next theorem (Theorem 6.2.1) is an attempt in this direction where (for instance) we see that before the usual $L^1_\tau(G)$-action if we are allowed to convolve the generator $f \in L^1_\tau(G)$ with a few other measurable τ-radial functions, then f can generate a dense space in $L^1_\tau(G)$ if (apart from satisfying the estimate at infinity of the usual WTT) its τ-spherical transform $\widehat{f}$ is nonvanishing only on the unitary dual. That is, the condition of nonvanishing Fourier transform here is much weaker than what is necessary for L^1-action: $\widehat{f}$ is nonvanishing on the Gelfand-Spectrum of the Banach algebra $L^1_\tau(G)$.

(B) A reason why many theorems of harmonic analysis on X or on G are unlike their euclidean analogue or have no analogue at all lies in the fact that the elementary spherical function ϕ_λ (in particular ϕ_0) satisfies certain decay estimate. This is in deep contrast with the euclidean case where the modulus of the unitary characters are constants and the nonunitary characters are unbounded functions. The degenerate case of the weighted algebra we mentioned above is given by the set of τ-radial functions which are integrable with weight $\phi_0(x)$. This is a commutative Banach algebra and is the largest space of measurable τ-radial functions for which the τ-spherical transform exists as absolutely convergent integral. We observe that unlike in other Banach algebras and modules mentioned above the domain of the τ-spherical transform of the elements of this Banach algebra shrinks from the strip to the line $\mathbb{R}$. We consider this space as the *test case* where we have deactivated the role of the decay of ϕ_λ. We show that indeed in this case the algebra loses its semisimple flavor so far the WTT is concerned and we obtain a WTT which resembles the theorem on $\mathbb{R}$ (see Theorem 6.2.2).

Our treatment relies on the method developed in [7] which substantially modified the theorem for the radial functions in $L^1(\mathrm{SL}(2,\mathbb{R}))$ and in [11, 13, 46, 47, 48] which extensively studied radial functions of a K-type τ. We may add here that a first systematic study of this subject appeared simultaneously in [11] and in [46]. (See also [47, 48] and the references in p.165 of [46].) Some of these results will appear in [49].

Next we take the Schwartz's theorem in the same setup as above. Here also we work on the τ-radial sections of spinor bundle (see Theorem 7.1.3) though as in the case of WTT the results are valid for some other Gelfand triples (see below). Like WTT in the context of Riemannian symmetric spaces or of the semisimple Lie groups the first account of Schwartz's theorem is again in the celebrated work of Ehrenpreis and Mautner [24] where it was proved for $\mathrm{SL}(2,\mathbb{R})$. For radial functions in a real rank one noncompact semisimple Lie group with finite centre the result is obtained by a different method in Bagchi and Sitaram [3].

As a consequence of the Schwartz's theorem we obtain a Wiener Tauberian type theorem for compactly supported distributions (see Theorem 7.2.1). Recalling that the elementary spherical functions ϕ_λ and its τ-radial version $\phi^\tau_{\sigma,\lambda}$ are in $L^{2+\varepsilon}$ for any $\varepsilon > 0$, we also observe how failure of the classical WTT for $L^p, 1 < p < 2$ functions can be related to the failure of Schwartz type theorem for $L^{p'}$ functions where $1/p + 1/p' = 1$.

As mentioned earlier Schwartz's theorem was extended for the group $\mathrm{SL}(2,\mathbb{R})$ in [24]. We shall try to improve the result. We recall that $\mathrm{SO}(2) \cong S^1$ is a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. We parametrize elements of $K = S^1$ as $\{k_\theta \mid \theta \in [0, 2\pi)\}$. The one dimensional K-types e_n are parametrized by integers n where $e_n(k_\theta) = e^{in\theta}$. For every pair of integers (m, n) of the same parity we have a spherical function $\Phi^{m,n}_\lambda$. In this setup the elementary spherical function $\phi_\lambda = \Phi^{0,0}_\lambda$. Theorem in [24] states that if V is a nonzero closed translation invariant subspace of $C^\infty(\mathrm{SL}(2,\mathbb{R}))$, then either for every even m, n or for every odd m, n, V contains $\Phi^{m,n}_\lambda$ for some $\lambda \in \mathbb{C}$ which depends on m, n. We consider the bundle E_n over $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ (see definition of E_τ above). Then the C^∞-sections of this bundle can be identified with $C^\infty(\mathrm{SL}(2,\mathbb{R}))_n$ which are the right n-type C^∞-functions on $\mathrm{SL}(2,\mathbb{R})$. The object which corresponds to $x \mapsto e^{i\lambda x}$ here is $e^n_{\lambda,k} : x \mapsto e^{\lambda(H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1})), \lambda \in \mathbb{C}, k \in K$ where $H(x)$ and $K(x)$ are the A-part and the K-part of the Iwasawa decomposition $G = KAN$ of the element x. We show that every left translation invariant nonzero closed subspace of $C^\infty(\mathrm{SL}(2,\mathbb{R}))_n$ contains $e^n_{\lambda,k}$ for some $\lambda \in \mathbb{C}$ and all $k \in K$. Since $\int_K e^n_{\lambda,k}(x)\overline{e_m(k)}\,dk = \Phi^{n,m}_{\lambda-\rho}(x)$ it follows from this result that for every m of the parity of n, V contains $\Phi^{n,m}_\nu(x)$ for some $\nu \in \mathbb{C}$ which depends on m. Using this step we shall finally prove that any nonzero closed (both-sided) translation invariant subspace V of $C^\infty(\mathrm{SL}(2,\mathbb{R}))$ contains $e^n_{\lambda,k}$ either for every even n or every odd n for some $\lambda \in \mathbb{C}$ which depends on n and for all $k \in K$ (see Theorem 8.1.2).

We indicate at the end how our method applies, *mutatis mutandis*, to obtain similar versions of WTT as well as Schwartz's theorem in some other Gelfand triples; e.g.

1. $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$, $\tau \in \widehat{K}$;
2. $G = \mathrm{SU}(n,1)$, $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ and τ is some irreducible component of Spin representation;
3. $G = \mathrm{Sp}(1,n)$, $K = \mathrm{Sp}(1) \times \mathrm{Sp}(n)$ and $\tau|_{\mathrm{Sp}(n)} \equiv 1$;
4. G connected, noncompact real rank one semisimple Lie group with finite centre and $\tau \in \widehat{K}$ with $\tau|_M$ is irreducible.

Actually our method relies on an explicit understanding of the images of certain spaces of functions and distributions under τ-spherical transform. The proofs work readily when as function spaces these images become identical with that of our working example namely the spinor bundle.

Crucial ingredients for the proofs of our main results are: (a) L^p-Schwartz space isomorphism theorems ($0 < p \leq 2$) for τ-radial functions, (b) Paley-Wiener theorem and (c) slice-projection property of the Abel transform; the latter two results for compactly supported τ-radial distributions. We prove these intermediate results. Our proof of the Schwartz space isomorphism theorems is an adaptation of the Anker's proof ([2]) of the corresponding theorem for the K-biinvariant case.

The thesis is organized as follows:

In Chapter 1 we establish the required properties of the elementary spherical functions and spherical transform, part of which is not so standard.

In Chapter 2 we extend some of the properties obtained in Chapter 1 to τ-spherical functions and τ-spherical transform. We also define Abel transform and its adjoint for τ-radial functions and distributions, obtain the *slice-projection* theorem.

In Chapter 3 we obtain the Banach algebras and modules, on which we consider the Wiener-Tauberian theorems in Chapter 6.

Chapter 4 contains preliminaries for the Spin group, Spin representations.

Chapter 5 has the L^p-Schwartz space isomorphism theorem for τ-radial functions and Paley-Wiener theorem for τ-radial distributions. These are intermediate steps for the proofs of our main results.

In Chapter 6 we prove analogue of Wiener-Tauberian theorems for τ-radial functions.

In Chapter 7 we prove an analogue of Schwartz's theorem on spectral analysis for τ-radial functions, a Wiener-Tauberian theorem for compactly supported τ-radial distributions and some related results.

In Chapter 8 we revisit Schwartz's theorem on $\mathrm{SL}(2,\mathbb{R})$ obtained in [24] and establish a stronger version of it.

In Chapter 9 we provide some other examples of Gelfand triple for which all the theorems proved in this thesis will hold. We indicate the reasons.

In Chapter $5, 6, 7$ (G, K, τ) are as defined in Chapter 4.

0.1 Notation

The following table summarizes some of the notation we shall use frequently. $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{N}$ are respectively set of real numbers, complex numbers, integers and natural numbers.

For $z \in \mathbb{C}$			
	$\Re z$	:	real part of z
	$\Im z$	:	imaginary part of z
	$\overline{z}$	:	complex conjugate of z
For a set S in a topological space			
	$\overline{S}$	:	closure of S
	S°	:	interior of S
	∂S	:	boundary of S
For any $p \in \mathbb{R}$,	p'	$=$	$\frac{p}{p-1}$
$\Sigma = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$		:	set of restricted roots
$m_\alpha, m_{2\alpha}$		:	dimensions of root spaces $\mathfrak{g}_\alpha, \mathfrak{g}_{2\alpha}$ respectively
ρ		:	the half sum of positive roots
For $0 < p \leq 2, \delta > 0$			
γ_p		$=$	$\frac{2}{p} - 1$
S_p		$=$	$\{z \in \mathbb{C} \mid \lvert\Im z\rvert \leq \gamma_p \rho\}$
$S_{p,\delta}$		$=$	$\{z \in \mathbb{C} \mid \lvert\Im z\rvert \leq \gamma_p \rho + \delta\}$
$\widehat{G}$		:	unitary dual of a group G
$C^\infty(G)$		:	infinitely differentiable functions on G
$C_c^\infty(G)$		:	compactly supported functions in $C^\infty(G)$
$C^p(G)$		:	L^p-Schwartz space on G
$L^{p,q}(G)$		:	Lorentz space on G with norm $\lVert \cdot \rVert_{p,q}^*$
For a function space $\mathcal{L}(G)$ of G			
$\mathcal{L}(G//K)$		:	K-biinvariant functions in $\mathcal{L}(G)$
$\mathcal{L}_\tau(G)$		:	τ-radial functions in $\mathcal{L}(G)$ where $(\tau, V_\tau) \in \widehat{K}$.

For a topological vector space V

$\mathrm{End}V$	:	set of endomorphisms on V
V'	:	set of continuous linear functionals on V
$\Phi^{\tau}_{\sigma,\lambda}$	:	$\mathrm{End}V_{\tau}$ valued τ-spherical function
$\phi^{\tau}_{\sigma,\lambda}(x)$	:	$\mathrm{Tr}\Phi^{\tau}_{\sigma,\lambda}(x)$
ϕ_{λ}	:	elementary spherical function

For a function space $\mathcal{F}$ on a symmetric domain in $\mathbb{C}$ or $\mathbb{R}$:

$\mathcal{F}_e$	:	set of even functions in $\mathcal{F}$
$\mathcal{F}_o$	:	set of odd functions in $\mathcal{F}$
$\sigma(x)$	:	distance of the point $x \in G/K$ from origin in the metric induced from the Killing form

$$L^1(w_{p,r}) = \{f \text{ measurable on } G \mid \int_G |f(x)|\phi_{i\gamma_p\rho}(x)(1+\sigma(x))^r\,dx < \infty\}$$

Chapter 1

Elementary Spherical Functions and Spherical Transform

We begin this chapter recalling some notation and establishing preliminaries which will be used throughout this thesis. Most of our notation related to the semisimple Lie groups and the associated symmetric spaces is standard and can be found for example in [33, 28]. Here we shall recall a few of them which are required to describe the results. We shall follow the standard practice of using the letter C, C_1, C_2 etc. for constants, whose value may change from one line to another. Occasionally the constants will be suffixed to show their dependency on important parameters. Everywhere in this thesis the symbol $f_1 \asymp f_2$ for two positive expressions f_1 and f_2 means that there are positive constants C_1, C_2 such that $C_1 f_1 \leq f_2 \leq C_2 f_1$. For a complex valued function f, $\overline{f}$ will denote its complex conjugation and for a set S in a topological space $\overline{S}$ will denote its closure. For a complex number z, we will use $\Re z$ and $\Im z$ to denote respectively the real and imaginary parts of z.

Let G be a connected noncompact semisimple Lie group with finite centre and $\mathfrak{g}$ its Lie algebra. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We assume that G is of real rank one, i.e. $\dim \mathfrak{a} = 1$. We denote the real dual of $\mathfrak{a}$ by $\mathfrak{a}^*$. Let $\Sigma \subset \mathfrak{a}^*$ be the subset of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$. We recall that either $\Sigma = \{-\alpha, \alpha\}$ or $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$ where α is a positive root and the Weyl group W associated to Σ is $\{\mathrm{Id}, -\mathrm{Id}\}$ where Id is the identity operator. Let $m_\alpha = \dim \mathfrak{g}_\alpha$ and $m_{2\alpha} = \dim \mathfrak{g}_{2\alpha}$ where $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ are the root spaces corresponding to α and 2α. As usual then $\rho = \frac{1}{2}(m_\alpha + 2m_{2\alpha})\alpha$ denotes the half sum of the positive roots. Let H_0 be the unique element in $\mathfrak{a}$ such that $\alpha(H_0) = 1$ and through this we identify $\mathfrak{a}$ with $\mathbb{R}$ as $t \leftrightarrow tH_0$. Then $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0\}$ is identified with the set of positive real numbers.

We also identify $\mathfrak{a}^*$ and its complexification $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{R}$ and $\mathbb{C}$ respectively by $t \leftrightarrow t\alpha$ and $z \leftrightarrow z\alpha$, $t \in \mathbb{R}$, $z \in \mathbb{C}$. By abuse of notation we will denote $\rho(H_0) = \frac{1}{2}(m_\alpha + 2m_{2\alpha})$ by ρ. Let $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$, $N = \exp \mathfrak{n}$, $K = \exp \mathfrak{k}$, $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}_+$ and $\overline{A^+} = \exp \overline{\mathfrak{a}_+}$. Then K is a maximal compact subgroup of G, N is a nilpotent Lie group and A is a one dimensional vector subgroup identified with $\mathbb{R}$. More precisely, A is parametrized by $a_s = \exp(sH_0)$. The Lebesgue measure on $\mathbb{R}$ induces the Haar measure on A as $da_s = ds$. Let M be the centralizer of A in K. Let $X = G/K$ be the Riemannian symmetric space of noncompact type associated with the pair (G, K). Let $\sigma(x) = d(xK, eK)$ where d is the distance function of X induced by the Killing form on $\mathfrak{g}$. The sets of (equivalence classes of) irreducible unitary representations of G, K, M are denoted respectively by $\widehat{G}, \widehat{K}, \widehat{M}$.

The group G has the Iwasawa decomposition $G = KAN$ and the polar decomposition $G = K\overline{A^+}K$. Using the Iwasawa decomposition we write an element $x \in G$ uniquely as $K(x) \exp H(x) N(x)$ where $K(x), H(x)$ and $N(x)$ are respectively the K-part, A-part and N-part of x in this decomposition. Let dg, dn, dk and dm be the Haar measures of G, N, K and M respectively where $\int_K dk = 1$ and $\int_M dm = 1$. We have the following integral formulae corresponding to the two decompositions above which hold for any integrable function:

$$\int_G f(g)dg = C_1 \int_K \int_{\mathbb{R}} \int_N f(ka_t n) e^{2\rho t}\, dn\, dt\, dk, \tag{1.0.1}$$

and

$$\int_G f(g)dg = C_2 \int_K \int_{\mathbb{R}^+} \int_K f(k_1 a_t k_2)(\sinh t)^{m_\alpha} (\sinh 2t)^{m_{2\alpha}}\, dk_1\, dt\, dk_2. \tag{1.0.2}$$

The constants C_1, C_2 depend on the normalizations of the Haar measures involved. We also use the Iwasawa decomposition $G = NAK$ which has the same Jacobian as the decomposition $G = KAN$, and the decompositions $G = KNA$ and $G = ANK$ each of which has Jacobian 1. The following identities will be useful in our computations [34] :

$$H(ghk) = H(hk) + H(gK(hk)) \text{ and } K(ghk) = K(gK(hk)). \tag{1.0.3}$$

We also note, using the well known estimate $\sinh t \asymp te^t/(1+t)$, in (1.0.2) above that

$$\begin{aligned}\int_G |f(g)|dg &\asymp C_3 \int_K \int_0^1 \int_K |f(k_1 a_t k_2)| t^{d-1}\, dk_1\, dt\, dk_2 \\ &+ C_4 \int_K \int_1^\infty \int_K |f(k_1 a_t k_2)| e^{2\rho t}\, dk_1\, dt\, dk_2 \qquad (1.0.4)\end{aligned}$$

where $d = m_\alpha + m_{2\alpha} + 1$.

A function is called K-biinvariant if $f(k_1 x k_2) = f(x)$ for all $x \in G, k_1, k_2 \in K$. For any function space $\mathcal{L}(G)$ on G we denote the set of K-biinvariant functions in $\mathcal{L}(G)$ by $\mathcal{L}(G//K)$. For any $\lambda \in \mathbb{C}$ we define the elementary spherical function ϕ_λ by

$$\phi_\lambda(x) = \int_K e^{-(i\lambda+\rho)H(xk)}\, dk \text{ for all } x \in G.$$

Then ϕ_λ is a K-biinvariant function and $\phi_\lambda = \phi_{-\lambda}$, $\phi_\lambda(x) = \phi_\lambda(x^{-1})$. It is clear that $|\phi_\lambda(x)| \leq \phi_{i\Im\lambda}(x)$ for any $\lambda \in \mathbb{C}$ and $x \in G$. The spherical transform $\widehat{f}$ of a function $f \in L^1(G//K)$ is defined by the formula

$$\widehat{f}(\lambda) = \int_G f(x)\phi_\lambda(x^{-1})\, dx \text{ for all } \lambda \in \mathbb{R}.$$

We have following Plancherel Theorem for spherical transform: For $f \in L^2(G//K)$

$$\int_G |f(x)|^2\, dx = \int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda$$

where $c(\lambda)$ is the (suitably normalized) Harish-Chandra c-function, $|c(\lambda)|^{-2}$ is the Plancherel density and $d\lambda$ is the Lebesgue measure on $\mathbb{R}$ (see [28]).

For $p \in (0, 2]$ we define $\gamma_p = (2/p - 1)$. We consider the strip

$$S_p = \{z \in \mathbb{C} \mid |\Im z| \leq \gamma_p \rho\}$$

and note that when $p = 2$ then the strip becomes the line $\mathbb{R}$. For $0 < p < 2$ let S_p° and ∂S_p respectively be the interior and the boundary of the strip.

We have the following asymptotic estimate of ϕ_λ ([33, p. 447]). For $\Im\lambda < 0$, $t > 0$

$$\lim_{t\to\infty} e^{(-i\lambda+\rho)(tH)} \phi_\lambda(a_t) = c(\lambda). \qquad (1.0.5)$$

As the c-function has neither zero nor pole in the region $\Im\lambda < 0$ (see [33, Theorem 6.4, Ch. IV]) it follows that for every $\varepsilon > 0$ there is a $M_\varepsilon > 0$ such that for all t

with $|t| > M_\varepsilon$

$$(1-\varepsilon)e^{-(\Im\lambda+\rho)|t|}|c(\lambda)| \leq |\phi_\lambda(a_t)| \leq (1+\varepsilon)e^{-(\Im\lambda+\rho)|t|}|c(\lambda)|. \tag{1.0.6}$$

Using continuity of ϕ_λ we get that for any fixed $\lambda \in \mathbb{C}$ with $\Im\lambda < 0$:

$$|\phi_\lambda(a_t)| \asymp e^{-(\Im\lambda+\rho)|t|} \tag{1.0.7}$$

and in particular for $\lambda = -i\gamma_p\rho$, $0 < p < 2$, we have

$$\phi_{i\gamma_p\rho}(a_t) = \phi_{-i\gamma_p\rho}(a_t) \asymp e^{-2/p'\rho|t|}. \tag{1.0.8}$$

This estimate becomes degenerate when $p = 2$, i.e. when $\gamma_p = 0$. However we have the following estimate for $\lambda = 0$: $\phi_0(a_t) \asymp (1+|t|)e^{-\rho|t|}$ (see [1]). Apart from these pointwise or uniform estimates of ϕ_λ there are L^p estimates, which leads to the celebrated *Kunze-Stein phenomenon* (see [41], [15]). It is clear from the estimate of ϕ_0 and the fact that $|\phi_\lambda| \leq \phi_0$ if $\lambda \in \mathbb{R}$ that for $\lambda \in \mathbb{R}$, $\phi_\lambda \in L^{2+\varepsilon}(G//K)$ for any $\varepsilon > 0$. From this it follows that for any function f in $L^p(G//K)$ with $1 \leq p < 2$, $|\widehat{f}(\lambda)| \leq C\|f\|_p$ when $\lambda \in \mathbb{R}$. Using Plancherel theorem we immediately get that $L^p(G//K) * L^2(G//K) \subset L^2(G//K)$ with the corresponding norm inequality. This can be considered as a starting point of the Kunze-Stein phenomenon or at least the "convolution-inequality version" of it (see [16] for comprehensive survey). Using an interpolation with the known fact that $L^1(G//K) * L^1(G//K) \subset L^1(G//K)$ we obtain $L^p(G//K) * L^q(G//K) \subset L^q(G//K)$ where $1 \leq p < q \leq 2$ with the corresponding norm inequality.

More recently a sharper version of the Kunze-Stein phenomenon is obtained for the groups of real rank one which involves Lorentz space estimates of ϕ_λ (see [16, 39]). Before we embark upon further studies of the behavior of ϕ_λ along this line we need the following definitions and results for the Lorentz spaces (see [30, 59] for details). Let (M, m) be a σ-finite measure space, $f : M \longrightarrow \mathbb{C}$ be a measurable function and $p \in [1, \infty)$, $q \in [1, \infty]$. We define

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [f^*(t)t^{1/p}]^q \frac{dt}{t}\right)^{1/q} & \text{when } q < \infty \\ \\ \sup_{t>0} t d_f(t)^{1/p} & \text{when } q = \infty. \end{cases}$$

Here d_f is the distribution function of f, i.e. for $\alpha > 0$, $d_f(\alpha)$ is the Haar measure of the set $\{x \in G \mid |f(x)| > \alpha\}$ and $f^*(t) = \inf\{s \mid d_f(s) \leq t\}$ is the *nonincreasing rearrangement* of f ([30, p. 45]). We take $L^{p,q}(M)$ to be the set of all measurable functions $f : M \longrightarrow \mathbb{C}$ such that $\|f\|^*_{p,q} < \infty$. By $L^{\infty,\infty}(M)$ and $\|\cdot\|_{\infty,\infty}$ we mean respectively the space $L^\infty(M)$ and the norm $\|\cdot\|_\infty$. The space $L^{p,\infty}(M)$ is also called weak L^p-space on M.

For $p, q \in [1, \infty)$ the following identity gives an alternative expression of $\|\cdot\|^*_{p,q}$ which we will use:

$$\frac{q}{p}\int_0^\infty (t^{1/p}f^*(t))^q \frac{dt}{t} = q\int_0^\infty (td_f(t)^{1/p})^q \frac{dt}{t}.$$

Though this is well known and used in many places (see e.g. [8]) we give here a sketch of the proof as we could not locate one.

Proof. We use the substitution $t = s^\alpha$ where $\alpha = \frac{p}{q}$ in the left hand side integral. Then $\frac{dt}{t} = \alpha\frac{ds}{s}$ and we get,

$$\begin{aligned}
\frac{q}{p}\int_0^\infty t^{q/p} f^*(t)^q \frac{dt}{t} &= \frac{q}{p}\alpha \int_0^\infty (s^\alpha)^{q/p} f^*(s^\alpha)^q \frac{ds}{s} \\
&= \int_0^\infty f^*(s^\alpha)^q ds \\
&= \int_0^\infty \left(q\int_0^{f^*(s^\alpha)} \lambda^{q-1} d\lambda \right) ds \\
&= q\int_0^\infty \lambda^{q-1} \int_{f^*(s^\alpha)\geq\lambda} ds\, d\lambda.
\end{aligned}$$

To prove the assertion, it is now enough to show that $\int_{f^*(s^\alpha)\geq\lambda} ds = d_f(\lambda)^{q/p}$.

For a set A let $|A|$ be its Lebesgue measure. Then $\int_{f^*(s^\alpha)\geq\lambda} ds = |\{s \mid f^*(s^\alpha) \geq \lambda\}|$. The set

$$\begin{aligned}
\{s \mid f^*(s^\alpha) \geq \lambda\} &= \{s \mid \inf\{u > 0 \mid d_f(u) \leq s^\alpha\} \geq \lambda\} \\
&= \{s \mid d_f(u) > s^\alpha \text{ for all } u \in (0, \lambda)\} \\
&= \{s \mid d_f(\lambda - \varepsilon) > s^\alpha \text{ for all } \varepsilon > 0\} \\
&= \{s \mid d_f(\lambda) \geq s^\alpha\} \text{ for almost every } \lambda,
\end{aligned}$$

as d_f is monotone function. Thus $\int_{f^*(s^\alpha)\geq\lambda} ds = d_f(\lambda)^{1/\alpha} = d_f(\lambda)^{q/p}$ for almost every λ as $\alpha = \frac{p}{q}$. This completes the proof. □

For p, q in the range above, $L^{p,p}(M) = L^p(M)$ and if $q_1 \leq q_2$ then $\|f\|^*_{p,q_2} \leq \|f\|^*_{p,q_1}$ and consequently $L^{p,q_1}(M) \subset L^{p,q_2}(M)$. We recall that for $1 < p < \infty$ and $1 \leq q < \infty$, the dual of $L^{p,q}(M)$ is $L^{p',q'}(M)$ where $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$ and the dual space of $L^{1,q}(M)$ is $\{0\}$ for $1 < q < \infty$ (see [30, p. 52]). Everywhere in this thesis any $p \in [1, \infty)$ is related to p' as above.

Proposition 1.0.1. *The elementary spherical function ϕ_λ satisfies the following properties.*

(1) *For $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\Im\lambda_1| > |\Im\lambda_2| > 0$ and $r \geq 0$, $|\phi_{\lambda_2}(x)|(1+\sigma(x))^r \leq C|\phi_{\lambda_1}(x)|$ for all $x \in G$ for some constant C which depends on λ_1, λ_2.*

(2) *For $1 \leq p < 2$, $\phi_\lambda \in L^{p',\infty}(G//K)$ if and only if $\lambda \in S_p$.*

(3) *For $1 < p < 2$ and $1 \leq r \leq \infty$, $\phi_\lambda \in L^{p',r}(G//K)$ if and only if $\lambda \in S_p^\circ$.*

(4) $\frac{\phi_0(a_r)}{(1+r)} \in L^{2,\infty}(G//K)$.

Proof. The assertion (1) follows from (1.0.7), noting that $\phi_\lambda = \phi_{-\lambda}$.

For proving (2) and (3) we first note that when $\lambda = \xi + i\gamma_p\rho$ where $\xi \in \mathbb{R}$ and $\gamma_p = 2/p - 1$ then for $t > 0$, $\phi_\lambda(a_t) \asymp e^{-2\rho t/p'}$ (see (1.0.8)).

Let $f(a_t) = e^{-2\rho t/p'}$. Then

$$d_f(\alpha) = m(\{t \mid e^{-2\rho t/p'} > \alpha\}),$$

where m is the Haar measure on G in polar decomposition. Thus $d_f(\alpha) = 0$ if $\alpha > 1$ and hence we need to consider $\alpha \in (0,1)$. We have $d_f(\alpha) = m(\{t \mid t < p'/2\rho \log 1/\alpha\})$.

If $0 < \alpha < e^{-2\rho/p'}$ then $p'/2\rho \log 1/\alpha > 1$. Thus in this range of α using (1.0.4) we have

$$d_f(\alpha) \asymp \left[\int_0^1 t^{d-1}dt + \int_1^{\frac{p'}{2\rho}\log\frac{1}{\alpha}} e^{2\rho t}\,dt\right] \asymp \frac{1}{\alpha^{p'}}. \tag{1.0.9}$$

If $e^{-2\rho/p'} < \alpha < 1$ then $0 < p'/2\rho \log 1/\alpha < 1$ and hence for this range of α we have

$$d_f(\alpha) \asymp \int_0^{\frac{p'}{2\rho}\log\frac{1}{\alpha}} t^{d-1}dt = \frac{1}{d}\left(\frac{p'}{2\rho}\log\frac{1}{\alpha}\right)^d. \tag{1.0.10}$$

Thus from the definition of Lorentz spaces given above, (1.0.9) and (1.0.10) it follows that $\phi_\lambda \in L^{q,\infty}(G//K)$ if and only if

$$\sup_{0<\alpha<e^{-2\rho/p'}} \frac{\alpha}{\alpha^{p'/q}} < \infty,$$

that is if and only if $p' \leq q$. Similarly it follows from (1.0.9) and (1.0.10) that

$\phi_\lambda \in L^{q,r}(G//K)$ if and only if

$$\int_0^{e^{-2\rho/p'}} \frac{d\alpha}{\alpha^{1-r+rp'/q}} < \infty,$$

that is if and only if $p' < q$.

Now we take $p > s$. Then $\gamma_p < \gamma_s$ and hence by (1.0.10) $\phi_{i\gamma_p\rho} \in L^{s',r}(G//K)$ by taking $s' = q$.

(4) As before let m be the Haar measure on G. We consider the function $f(r) = e^{-\rho r}$ for $r \geq 0$. We note that for $\alpha \geq 1$, $d_f(\alpha) = m\{r \mid e^{-\rho r} > \alpha\} = 0$. For $\alpha < 1$ we have

$$\begin{aligned} d_f(\alpha) = & \; m\{r \mid r < \frac{1}{\rho}\log\frac{1}{\alpha}\} \\ \leq & \int_0^{1/\rho \log 1/\alpha} e^{2\rho r}\, dr \quad (\text{as } \sinh r \leq e^r) \\ = & \; \frac{1}{2\rho}\left(e^{2\log 1/\alpha} - 1\right) \leq \frac{1}{2\rho\alpha^2}. \end{aligned}$$

Hence $\sup_{0<\alpha<1} \alpha d_f(\alpha)^{1/2} \leq (2\rho)^{-1/2} < \infty$. As $\phi_0(a_r)/(1+r) \asymp e^{-\rho r}$, the proof is complete. □

Remark 1.0.2. Proposition 1.0.1(2) for the case $p = 1$ is well known as the Helgason-Johnson theorem (see [35]) and holds for groups of arbitrary real rank. In the language of Lorentz space Helgason-Johnson theorem restates as $\phi_\lambda \in L^{\infty,\infty}(G//K)$ if and only if $\lambda \in S_1$. Proposition 1.0.1(2) is its expected generalization: $\phi_\lambda \in L^{p',\infty}(G//K)$ if and only if $\lambda \in S_p$. However it is known that for $p > 1$, Proposition 1.0.1(2) is false when real rank of G is more than one (see [39]).

Proposition 1.0.1 readily determines the domain where the spherical transform of a function exists as a convergent integral. For instance for $L^{p,1}$ functions with $1 \leq p < 2$ the domain is S_p and for $L^{p,q}$ functions with $1 < p < 2, 1 < q \leq \infty$ it is S_p°. We may point out that the latter includes the weak L^p spaces for $p > 1$. The phenomenon however fails for weak L^1. For example we consider the K-biinvariant function $f(k_1 a_r k_2) = r^{-(m_\alpha + m_{2\alpha}+1)}\chi_{[0,1]}(r)$ where $\chi_{[0,1]}$ is the indicator function of $[0,1]$. Then it can be verified that f is in weak L^1, but the integral $\int_{\mathbb{R}} f(r)\phi_0(r)J(r)\, dr$ does not converge. Here $J(r)$ is the Jacobian of the polar decomposition. This shows that while for $p > 1$ the pointwise existence of the spherical transform is guaranteed for weak L^p functions, the situation is different for weak L^1 functions (see [50] for details).

Chapter 2

τ-Spherical Functions and τ-Spherical Transform

2.1 τ-Radial Functions

In this section we recall the definitions of τ-radial functions and their τ-spherical transforms. We discuss both endomorphism valued and scalar valued τ-radial functions. We will follow mainly [11] for basic notation and argument.

Definition 2.1.1. For G and K as in Chapter 1 and $(\tau, V_\tau) \in \widehat{K}$ a function $F : G \to \mathrm{End}V_\tau$ is said to be τ-radial if $F(k_1 x k_2) = \tau(k_2^{-1})F(x)\tau(k_1^{-1})$ for all $k_1, k_2 \in K$, $x \in G$.

When τ is the trivial representation of K, a τ-radial function is simply a K-biinvariant function. The τ-radial functions are radial sections of the homogenous vector bundle E_τ over G/K associated with the representation $\tau \in \widehat{K}$ (see Introduction).

Let $\Gamma(G, \tau, \tau)$ be the set of all τ-radial functions. Also let $L^2(G, \tau, \tau)$ be the square integrable τ-radial functions with inner product

$$\langle F_1, F_2 \rangle = \int_G \mathrm{Tr}\left[F_1(x)F_2(x)^*\right] dx,$$

where $F_2(x)^*$ denotes adjoint of $F_2(x)$. For suitable $F_1, F_2 \in \Gamma(G, \tau, \tau)$ their convolution is defined by

$$F_1 * F_2(x) = \int_G F_1(y^{-1}x)F_2(y)\, dy.$$

Then for $F_1, F_2 \in \Gamma(G, \tau, \tau)$, we can verify that $F_1 * F_2 \in \Gamma(G, \tau, \tau)$ whenever convolution makes sense. In fact

$$\begin{aligned}
(F_1 * F_2)(k_1 x k_2) &= \int_G F_1(y^{-1}k_1 x k_2)F_2(y)\,dy \\
&= \tau(k_2^{-1}) \int_G F_1(y^{-1}k_1 x)F_2(y)\,dy \\
&= \tau(k_2^{-1}) \int_G F_1(z^{-1}x)F_2(k_1 z)\,dz \\
&= \tau(k_2^{-1}) \int_G F_1(z^{-1}x)F_2(z)\,dz\ \tau(k_1^{-1}) \\
&= \tau(k_2^{-1})(F_1 * F_2)(x)\tau(k_1^{-1}).
\end{aligned}$$

We let $I_\tau(G)$ denote the set of all scalar valued functions f on G such that $f(kxk^{-1}) = f(x)$ for $k \in K, x \in G$ and $d_\tau\overline{\chi_\tau} * f = f = f * d_\tau\overline{\chi_\tau}$ where χ_τ and d_τ are character and dimension of τ respectively. We call elements of $I_\tau(G)$ as scalar valued τ-radial functions. For $f_1, f_2 \in I_\tau(G)$, their convolution is defined by

$$f_1 * f_2(x) = \int_G f_1(xy^{-1})f_2(y)\,dy,$$

whenever the integral converges and it can be verified that $f_1 * f_2 \in I_\tau(G)$. We have the following proposition which gives an bijection between $\Gamma(G,\tau,\tau)$ and $I_\tau(G)$.

Proposition 2.1.2. *There is a one-to-one correspondence between the spaces $\Gamma(G,\tau,\tau)$ and $I_\tau(G)$.*

Proof. For given $F \in \Gamma(G,\tau,\tau)$ we define f_F by $f_F(x) = d_\tau \mathrm{Tr} F(x)$. Then $f_F(kxk^{-1}) = d_\tau \mathrm{Tr} F(kxk^{-1}) = d_\tau \mathrm{Tr}(\tau(k)F(x)\tau(k^{-1})) = d_\tau \mathrm{Tr} F(x) = f_F(x)$, that is f_F is K-central. Now

$$\begin{aligned}
\left(f_F * d_\tau\overline{\chi_\tau}\right)(x) &= d_\tau \int_K f_F(xk)\chi_\tau(k)\,dk \\
&= d_\tau^2 \int_K \mathrm{Tr} F(xk)\chi_\tau(k)\,dk \\
&= d_\tau^2 \mathrm{Tr}\left[\int_K \tau(k^{-1})\chi_\tau(k)\,dk\ F(x)\right] \\
&= d_\tau \mathrm{Tr} F(x) \\
&= f_F(x),
\end{aligned}$$

where we have used the Schur orthogonality relation for K with normalization $\int_K dk = 1$. Similarly we have $d_\tau\overline{\chi_\tau} * f_F = f_F$. Hence $f_F \in I_\tau(G)$. Conversely, suppose $f \in I_\tau(G)$. We define F_f by

$$F_f(x) = \int_K \tau(k) f(kx)\, dk.$$

Then

$$F_f(k_1 x k_2) = \int_K \tau(k) f(k k_1 x k_2)\, dk.$$

We put $kk_1 = k_3$ in the above to get

$$\begin{aligned}
F_f(k_1 x k_2) &= \textstyle\int_K \tau(k_3 k_1^{-1}) f(k_3 x k_2)\, dk_3 \\
&= \textstyle\int_K \tau(k_3) f(k_3 x k_2)\, dk_3 \ \tau(k_1^{-1}) \\
&= \textstyle\int_K \tau(k_3) f(k_2^{-1} k_2 k_3 x k_2)\, dk_3 \ \tau(k_1^{-1}) \\
&= \textstyle\int_K \tau(k_3) f(k_2 k_3 x)\, dk_3 \ \tau(k_1^{-1}), \text{ since } f \text{ is } K\text{-central.}
\end{aligned}$$

Also we put $k_2 k_3 = k_4$ in the above to get

$$\begin{aligned}
F_f(k_1 x k_2) &= \textstyle\int_K \tau(k_2^{-1} k_4) f(k_4 x)\, dk_4 \ \tau(k_1^{-1}) \\
&= \tau(k_2^{-1}) \textstyle\int_K \tau(k_4) f(k_4 x)\, dk_4 \ \tau(k_1^{-1}) \\
&= \tau(k_2^{-1}) F_f(x) \tau(k_1^{-1}).
\end{aligned}$$

Therefore $F_f \in \Gamma(G, \tau, \tau)$. Also for $F \in \Gamma(G, \tau, \tau)$ we have

$$F_{f_F}(x) = \int_K \tau(k) f_F(kx)\, dk = d_\tau \int_K \tau(k) \mathrm{Tr}\left(F(kx)\right)\, dk.$$

Hence $F_{f_F}(x) = d_\tau \int_K \tau(k) \mathrm{Tr}\left(F(x)\tau(k^{-1})\right) dk = F(x)$ by Schur orthogonality relation.

Again for $f \in I_\tau(G)$ we have

$$f_{F_f}(x) = d_\tau \mathrm{Tr}(F_f(x)) = d_\tau \mathrm{Tr}\left(\int_K \tau(k) f(kx)\, dk\right) = d_\tau \int_K \chi_\tau(k) f(kx)\, dk.$$

Therefore $f_{F_f}(x) = d_\tau \overline{\chi_\tau} * f(x) = f(x)$. This shows that $F \mapsto f_F$ is a bijection between $\Gamma(G, \tau, \tau)$ and $I_\tau(G)$ with inverse $f \mapsto f_F$. □

For $f_1, f_2 \in I_\tau$ and $F_1, F_2 \in \Gamma(G,\tau,\tau)$ we have $F_{f_1 * f_2} = F_{f_2} * F_{f_1}$ and $f_{F_1 * F_2} = f_{F_2} * f_{F_1}$ whenever the convolutions make sense (see [62, p.3]). Therefore it follows that $I_\tau(G)$ is commutative if and only if $\Gamma(G,\tau,\tau)$ is commutative.

Let $C_R^\infty(G,\tau,\tau)$ be the space of all τ-radial infinitely differentiable compactly supported functions with support contained in the ball of radius R, that is $F \in C_R^\infty(G,\tau,\tau)$, when $F(Ka_tK) = 0$, for all $|t| > R$. The set of all compactly supported τ-radial infinitely differentiable functions is denoted by $C_c^\infty(G,\tau,\tau)$. The corresponding sets for scalar valued functions are denoted by $C_{c,\tau}^\infty(G)_R$ and $C_{c,\tau}^\infty(G)$ respectively. Precisely, $C_{c,\tau}^\infty(G) = \{f \in I_\tau(G) \mid f$ is compactly supported and $C^\infty\}$ and $C_{c,\tau}^\infty(G)_R = \{f \in I_\tau(G) \mid f$ is compactly supported in a ball of radius R and $C^\infty\}$. Also the set of infinitely differentiable τ-radial functions and the set of corresponding scalar valued functions are denoted by $C^\infty(G,\tau,\tau)$ and $C_\tau^\infty(G)$ respectively. We topologize $C_{c,\tau}^\infty(G)$ and $C_\tau^\infty(G)$ as follows (see [24]): A sequence $\{f_i\}$ in $C_{c,\tau}^\infty(G)$ converges to 0 if and only if there exists a compact set C of G such that $\text{supp} f_i \subseteq C$ for all i and f_i along with all derivatives converges to 0 uniformly on C. A sequence $\{f_i\}$ in $C_\tau^\infty(G)$ converges to 0 if and only if f_i along with all derivatives converges to 0 uniformly on each compact subsets of G.

The τ-radial L^p-Schwartz spaces for $0 < p \leq 2$ are defined by

$$C^p(G,\tau,\tau) = \{F \in C^\infty(G,\tau,\tau) \mid \forall D_1, D_2 \in \mathcal{U}(\mathfrak{g}), \forall N \in \mathbb{N}, \sup_{t \geq 0} \|F(D_1; a_t; D_2\|_{\text{End}V_\tau}(1+t)^N e^{\frac{2}{p}\rho t} < \infty\},$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of G.

The corresponding space of scalar valued functions is defined by

$$C_\tau^p(G) = \{f \in C_\tau^\infty(G) \mid \forall D_1, D_2 \in \mathcal{U}(\mathfrak{g}), \forall N \in \mathbb{N}, \sup_{t \geq 0} |f(D_1; a_t; D_2)|(1+t)^N e^{\frac{2}{p}\rho t} < \infty\}.$$

Here $F(D_1; a_t; D_2)$ (respectively $f(D_1; a_t; D_2)$) is the usual left and right derivatives of F (respectively of f) by D_1 and D_2 evaluated at a_t.

The spaces $C_c^\infty(G,\tau,\tau), C^\infty(G,\tau,\tau)$ and $C^p(G,\tau,\tau)$ are topologically isomorphic with the function spaces $C_{c,\tau}^\infty(G), C_\tau^\infty(G)$ and $C_\tau^p(G)$ respectively through the map $F \mapsto f_F$ and its inverse $f \mapsto F_f$. We will mostly work with the scalar valued τ-radial functions. However it will be clear from the context whether we are considering scalar or $\text{End}V_\tau$ valued functions. For any (scalar valued) function space $\mathcal{L}(G)$ the set of τ-radial functions in $\mathcal{L}(G)$ will be denoted by $\mathcal{L}_\tau(G)$. We recall the well known facts: For $0 < p \leq q \leq 2$, $C^{\infty}_{c,\tau}(G)$ is dense in $C^p_\tau(G)$, $C^p_\tau(G)$ is dense in $C^q_\tau(G)$ and for $1 \leq p \leq 2$, $C^p_\tau(G)$ is dense in $L^p_\tau(G)$.

Let $\mathbb{D}(G,\tau)$ denotes the algebra of left-invariant differential operators acting on $C^\infty(G,\tau) = \{f : G \to V_\tau \mid f \text{ is } C^\infty \text{ and } f(xk) = \tau(k^{-1})f(x)\}$.

Definition 2.1.3. A function $\Phi \in C^\infty(G,\tau,\tau)$, with $\Phi(e) = \mathrm{Id}$ is called τ-spherical function if Φ is an eigenfunction for $\mathbb{D}(G,\tau)$, i.e., there is a character χ_Φ of $\mathbb{D}(G,\tau)$ such that

$$D\Phi(\cdot)v = \chi_\Phi(D)\Phi(\cdot)v$$

for all $D \in \mathbb{D}(G,\tau)$ and all $v \in V_\tau$ (the representation space of τ).

Then we have the following characterizations for the τ-spherical functions (see [11], [46, Theorems B.2, B.12], [47, Theorems 6, 8]).

Theorem 2.1.4. *Let $\Phi \in C^\infty(G,\tau,\tau)$, with $\Phi(e) = \mathrm{Id}$. Then the following conditions are equivalent:*

(1) *Φ is a τ-spherical function,*

(2) *The map $F \mapsto \lambda_\Phi(F) = \frac{1}{d_\tau}\int_G \mathrm{Tr}\,[F(x)\Phi(x^{-1})]\,dx$ is a character of $C^\infty_c(G,\tau,\tau)$,*

(3) *$\Phi * F = \lambda_\Phi(F)\Phi$, for all $F \in C^\infty_c(G,\tau,\tau)$,*

(4) *Φ satisfies either one of the following (equivalent) functional equations:*

   (a) $d_\tau \int_K \tau(k)\Phi(xky)\,dk = \mathrm{Tr}(\Phi(y))\Phi(x)$,

   (b) $d_\tau \int_K \Phi(xky)\chi_\tau(k)\,dk = \Phi(y)\Phi(x)$ *for all* $x, y \in G$.

Let $P = MAN$ be a minimal parabolic subgroup of G. Given $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{C}$, we have the representation $\sigma \otimes e_\lambda \otimes 1$ of P where $e_\lambda(x) = e^{i\lambda x}$ is the (not necessarily unitary) character of A and 1 is the trivial representation of N. The minimal principal series representation $\pi_{\sigma,\lambda} = \mathrm{ind}_P^G(\sigma \otimes e_\lambda \otimes 1)$ is the representation induced by $\sigma \otimes e_\lambda \otimes 1$ from P to G. In our parameterization $\pi_{\sigma,\lambda}$ is unitary if and only if $\lambda \in \mathbb{R}$ and they are also irreducible except maybe for $\lambda = 0$. The subquotient theorem of Harish-Chandra implies that each $\pi \in \widehat{G}$ is infinitesimally equivalent to a subquotient representation of a nonunitary principal series $\pi_{\sigma,\lambda}$, for suitable $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{C}$. For a detailed account on construction and parametrization of representations we refer to [40, Ch. VII].

For $\pi \in \widehat{G}, \tau \in \widehat{K}$ and $\sigma \in \widehat{M}$ we let $m(\tau,\pi)$ (respectively $m(\sigma,\tau)$) denote the multiplicity of τ in $\pi|_K$ (respectively multiplicity of σ in $\tau|_M$). Also for $\tau \in \widehat{K}$ we let $\widehat{M}(\tau) = \{\sigma \in \widehat{M} \mid m(\sigma, \tau) > 0\}$. We have the following result regarding the link between the commutativity of the algebra $I_\tau(G)$ and the multiplicity of τ in the elements of $\widehat{G}$. (See [29], [19, Theorem 3]. See also Proposition 5.1 and the Remarks following it in [46] for a relevant discussion.)

Proposition 2.1.5. *The following conditions are equivalent:*

(1) *For $f_1, f_2 \in I_\tau(G)$, $f_1 * f_2 = f_2 * f_1$ whenever convolutions on both sides make sense.*

(2) *$m(\tau, \pi) \leq 1$ for all $\pi \in \widehat{G}$.*

We digress briefly to recall that for a unimodular locally compact group G, a compact subgroup K of G and a unitary irreducible representation τ of K, if the convolution algebra of continuous compactly supported τ-radial functions on G is commutative, then (G, K, τ) is called a Gelfand triple. The term *Gelfand triple* is coined by E. Pedon which generalizes the well known concept of Gelfand pair (see [46, section 5.2, Appendix B].)

We come back to the context of G, K and τ of the previous proposition. From now on we restrict our attention to those $\tau \in \widehat{K}$ for which $m(\tau, \pi) \leq 1$ for any $\pi \in \widehat{G}$. Then in particular $C^\infty_{c,\tau}(G)$ is commutative i.e. (G, K, τ) is a Gelfand triple. We note that by Frobenius reciprocity theorem $m(\tau, \pi_{\sigma,\lambda}) \leq 1$ is equivalent to the condition that $\tau|_M$ is multiplicity free. Unless stated otherwise, by $\tau \in \widehat{K}$ we shall mean such a τ in $\widehat{K}$.

For $\tau \in \widehat{K}$, $\sigma \in \widehat{M}(\tau)$ and $\lambda \in \mathbb{C}$, let $\Phi^\tau_{\sigma,\lambda}(x)$ be the matrix block of type τ of $\pi_{\sigma,\lambda}(x)$. Precisely $\Phi^\tau_{\sigma,\lambda}(x) := P_\tau \pi_{\sigma,\lambda}(x^{-1})(P_\tau)^*$, where P_τ is the projection of $H_{\pi_{\sigma,\lambda}}$ (the representation space of $\pi_{\sigma,\lambda}$) onto V_τ given by $P_\tau = d_\tau \int_K \pi_{\sigma,\lambda}(k)\chi_\tau(k^{-1})\, dk$. (See [46, 12, 13]. See also [11] where by abuse of notation the author writes the right-side projector P_τ to mean its dual operator P^*_τ.) The subquotient theorem implies that every (nonzero) τ-spherical function on G can be written as $\Phi^\tau_{\sigma,\lambda}$ for suitable $\sigma \in \widehat{M}(\tau)$ and $\lambda \in \mathbb{C}$. Moreover this spherical function $\Phi^\tau_{\sigma,\lambda}$ admits the following integral representation

$$\Phi^\tau_{\sigma,\lambda}(x) = \frac{d_\tau}{d_\sigma} \int_K e^{-(i\lambda+\rho)H(xk)} \left[\tau(k) \circ P_\sigma \circ \tau(K(xk)^{-1})\right] dk \tag{2.1.1}$$

where

$$P_\sigma = d_\sigma \int_M \tau(m^{-1})\chi_\sigma(m)\, dm$$

is the projection of V_τ onto V_σ (representation space of σ) $\subseteq V_\tau$ and d_σ is the dimension of σ. The corresponding scalar valued τ-spherical function $\phi^\tau_{\sigma,\lambda}$ is given by:

$$\phi^\tau_{\sigma,\lambda}(x) = \mathrm{Tr}(\Phi^\tau_{\sigma,\lambda}(x)) = d_\tau \int_{K\times M} e^{-(i\lambda+\rho)H(xk)}\chi_\tau(km^{-1}K(xk)^{-1})\chi_\sigma(m)\,dm\,dk. \tag{2.1.2}$$

The following proposition indicates the relation between $\Phi^\tau_{\sigma,\lambda}(x)$ and $\Phi^\tau_{\sigma,\lambda}(x^{-1})$.

Proposition 2.1.6. *For $A \in \mathrm{End}V_\tau$, let A^* be its adjoint. Then $\Phi^\tau_{\sigma,\lambda}(x) = \left(\Phi^\tau_{\sigma,\overline{\lambda}}(x^{-1})\right)^*$ and $\phi^\tau_{\sigma,\lambda}(x) = \overline{\phi^\tau_{\sigma,\overline{\lambda}}(x^{-1})}$ for $\tau \in \widehat{K}$ as above, $x \in G, \lambda \in \mathbb{C}$ and $\sigma \in \widehat{M}(\tau)$.*

Proof. Putting $g = h^{-1}$ in the identities $H(ghk) = H(hk) + H(gK(hk))$ and $K(ghk) = K(gK(hk))$, we get

$$H(hk) + H(h^{-1}K(hk)) = 0 \text{ and } K(h^{-1}K(hk)) = k.$$

Using these we have,

$$\begin{aligned}
\Phi^\tau_{\sigma,\lambda}(x) &= \tfrac{d_\tau}{d_\sigma}\int_K e^{-(i\lambda+\rho)H(xk)}\left[\tau(k)\circ P_\sigma\circ\tau(K(xk)^{-1})\right]dk \\
&= \tfrac{d_\tau}{d_\sigma}\int_K e^{(i\lambda+\rho)H(x^{-1}K(xk))}\left[\tau(K(x^{-1}K(xk)))\circ P_\sigma\circ\tau(K(xk)^{-1})\right]dk.
\end{aligned}$$

We put $K(xk) = k_1$ in the above to get

$$\begin{aligned}
\Phi^\tau_{\sigma,\lambda}(x) &= \tfrac{d_\tau}{d_\sigma}\int_K e^{(i\lambda+\rho)H(x^{-1}k_1)}\left[\tau(K(x^{-1}k_1))\circ P_\sigma\circ\tau(k_1^{-1})\right]e^{-2\rho H(x^{-1}k_1)}\,dk_1 \\
&= \tfrac{d_\tau}{d_\sigma}\int_K e^{(i\lambda-\rho)H(x^{-1}k_1)}\left[\tau(k_1)\circ P_\sigma\circ\tau\left(K(x^{-1}k_1)^{-1}\right)\right]^*dk_1 \\
&= \tfrac{d_\tau}{d_\sigma}\int_K \overline{e^{-(i\overline{\lambda}+\rho)H(x^{-1}k_1)}}\left[\tau(k_1)\circ P_\sigma\circ\tau\left(K(x^{-1}k_1)^{-1}\right)\right]^*dk_1 \\
&= \left(\Phi^\tau_{\sigma,\overline{\lambda}}(x^{-1})\right)^*.
\end{aligned}$$

We also have $\phi^\tau_{\sigma,\lambda}(x) = \mathrm{Tr}\left(\Phi^\tau_{\sigma,\lambda}(x)\right) = \mathrm{Tr}\left(\Phi^\tau_{\sigma,\overline{\lambda}}(x^{-1})\right)^* = \overline{\mathrm{Tr}\left(\Phi^\tau_{\sigma,\overline{\lambda}}(x^{-1})\right)} = \overline{\phi^\tau_{\sigma,\overline{\lambda}}(x^{-1})}$. □

One also has the following easier proof of the proposition above. For $\tau \in \widehat{K}$, $\sigma \in \widehat{M}(\tau)$, let $\pi_{\sigma,\lambda}(x)^*$ be the adjoint of the operator $\pi_{\sigma,\lambda}(x)$. Then by unitarity it follows that $\pi_{\sigma,\lambda}(x)^* = \pi_{\sigma,\lambda}(x^{-1})$ for $\lambda \in \mathbb{R}$. Hence $\pi_{\sigma,\lambda}(x)^* = \pi_{\sigma,\bar{\lambda}}(x^{-1})$ for $\lambda \in \mathbb{C}$ by analytic continuation. This proves the assertion.

Definition 2.1.7. For a suitable τ-radial function $F \in \Gamma(G, \tau, \tau)$, its τ-spherical transform at $(\sigma, \lambda) \in \widehat{M}(\tau) \times \mathbb{C}$ is defined by

$$\widehat{F}(\sigma, \lambda) = \frac{1}{d_\tau} \int_G \mathrm{Tr}[F(x)\Phi^\tau_{\sigma,\lambda}(x^{-1})]\, dx, \tag{2.1.3}$$

whenever the integral exists.

Correspondingly, for a suitable function $f \in I_\tau(G)$, its τ-spherical transform is given by

$$\widehat{f}(\sigma, \lambda) = \frac{1}{d_\tau} \int_G f(x)\phi^\tau_{\sigma,\lambda}(x^{-1})\, dx, \tag{2.1.4}$$

whenever the integral exists. We have the following theorem which shows that a scalar valued function and its corresponding endomorphism valued function has same τ-spherical transform.

Theorem 2.1.8. *For a suitable function $f \in I_\tau(G)$ and $F \in \Gamma(G, \tau, \tau)$ we have*

$$\widehat{f}(\sigma, \lambda) = \widehat{F_f}(\sigma, \lambda) \text{ for all } \sigma \in \widehat{M}(\tau), \lambda \in \mathbb{C}$$

and

$$\widehat{F}(\sigma, \lambda) = \widehat{f_F}(\sigma, \lambda) \text{ for all } \sigma \in \widehat{M}(\tau), \lambda \in \mathbb{C}.$$

Proof. Let e be the identity element of G. We have

$$\widehat{f}(\sigma, \lambda) = \frac{1}{d_\tau} \int_G f(x)\phi^\tau_{\sigma,\lambda}(x^{-1})\, dx = \frac{1}{d_\tau}(f * \phi^\tau_{\sigma,\lambda})(e) = \frac{1}{d_\tau^2}(f * d_\tau\phi^\tau_{\sigma,\lambda})(e).$$

By Proposition 2.1.2 there is a $F \in \Gamma(G, \tau, \tau)$ such that $f(x) = d_\tau \mathrm{Tr}(F(x)) = f_F(x)$. Therefore $F = F_f$. Also we have $\phi^\tau_{\sigma,\lambda}(x) = \mathrm{Tr}(\Phi^\tau_{\sigma,\lambda}(x))$. This shows that

$$\widehat{f}(\sigma, \lambda) = \frac{1}{d_\tau^2}(f_F * f_{\Phi^\tau_{\sigma,\lambda}})(e) = \frac{1}{d_\tau^2} f_{\Phi^\tau_{\sigma,\lambda} * F}(e) = \frac{1}{d_\tau}\mathrm{Tr}\left(\Phi^\tau_{\sigma,\lambda} * F(e)\right).$$

That is

$$\widehat{f}(\sigma, \lambda) = \frac{1}{d_\tau}\mathrm{Tr}\left(\int_G \Phi^\tau_{\sigma,\lambda}(y^{-1})F(y)\, dy\right) = \frac{1}{d_\tau}\int_G \mathrm{Tr}\left(F(y)\Phi^\tau_{\sigma,\lambda}(y^{-1})\right)\, dy.$$

Therefore $\widehat{f}(\sigma, \lambda) = \widehat{F}(\sigma, \lambda) = \widehat{F_f}(\sigma, \lambda)$. The other equality will follow similarly. This completes the proof. □

The τ-spherical functions $\phi^\tau_{\sigma,\lambda}$, $\Phi^\tau_{\sigma,\lambda}$ satisfy the following functional equations (see [11, Theorem 3.6], [46, Theorem B.2], [47, Theorem 6]):

$$d_\tau \int_K \phi^\tau_{\sigma,\lambda}(xkyk^{-1})\,dk = \phi^\tau_{\sigma,\lambda}(x)\phi^\tau_{\sigma,\lambda}(y), \tag{2.1.5}$$

$$d_\tau \int_K \Phi^\tau_{\sigma,\lambda}(xkyk^{-1})\,dk = \Phi^\tau_{\sigma,\lambda}(x)\phi^\tau_{\sigma,\lambda}(y). \tag{2.1.6}$$

Using these it is easy to verify that for $f_1, f_2 \in I_\tau(G)$

$$\widehat{f_1 * f_2}(\sigma,\lambda) = \widehat{f_1}(\sigma,\lambda)\widehat{f_2}(\sigma,\lambda),$$

for $\lambda \in \mathbb{C}$ and $\sigma \in \widehat{M}(\tau)$ whenever both sides make sense.

Indeed using K-central property of f_1, f_2 and $\phi^\tau_{\sigma,\lambda}$ we have,

$$\begin{aligned}
\widehat{f_1 * f_2}(\sigma,\lambda) &= \int_G (f_1 * f_2)(x)\phi^\tau_{\sigma,\lambda}(x^{-1})\,dx \\
&= \int_G\int_G f_1(z)f_2(y)\phi^\tau_{\sigma,\lambda}(y^{-1}z^{-1})\,dz\,dy \\
&= \int_G\int_G\int_K f_1(z)f_2(kyk^{-1})\phi^\tau_{\sigma,\lambda}(y^{-1}z^{-1})\,dk\,dz\,dy \\
&= \int_G\int_G\int_K f_1(z)f_2(y_1)\phi^\tau_{\sigma,\lambda}(k^{-1}y_1^{-1}kz^{-1})\,dk\,dz\,dy_1 \\
&= \int_G\int_G\int_K f_1(z)f_2(y_1)\phi^\tau_{\sigma,\lambda}(y_1^{-1}kz^{-1}k^{-1})\,dk\,dz\,dy_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\widehat{f_1 * f_2}(\sigma,\lambda) &= \int_F f_1(z)\phi^\tau_{\sigma,\lambda}(z^{-1})\,dz \int_G f_2(y_1)\phi^\tau_{\sigma,\lambda}(y_1^{-1})\,dy_1 \\
&= \widehat{f_1}(\lambda)\widehat{f_2}(\lambda).
\end{aligned}$$

From the integral representation (2.1.1) and (2.1.2) it follows that the operator norm of $\Phi^\tau_{\sigma,\lambda}(x)$ and the absolute value of $\phi^\tau_{\sigma,\lambda}$ are bounded by a constant multiple of the elementary spherical function $\phi_{\Im\lambda}(x)$. From the estimates of ϕ_λ (see Proposition 1.0.1) we get the following:

Proposition 2.1.9. *The τ-spherical functions satisfy the following properties.*

(1) *For $1 < p < 2$ and $1 \le q \le \infty$, $\phi^\tau_{\sigma,\lambda} \in L^{p',q}_\tau(G)$ if $\lambda \in S_p^\circ$.*

(2) *$\phi^\tau_{\sigma,\lambda} \in L^{p',\infty}_\tau(G)$ if $\lambda \in S_p$ for $1 \le p < 2$.*

Corresponding statements are easy to formulate for EndV_τ-valued τ-spherical functions $\Phi^\tau_{\sigma,\lambda}$. For this we substitute the absolute value by the norm of the matrix $\Phi^{\tau}_{\sigma,\lambda}$ in the definition of L^p-norms, distribution functions etc.. The converse of these statements are not immediate from the corresponding statements for elementary spherical functions in Proposition 1.0.1. We will come back to this question (see Remark 6.1.8 (2)).

It is clear from Proposition 2.1.9 that for a function $f \in L^{p,1}_{\tau}(G), 1 \leq p < 2$ (respectively for $L^{p,q}_{\tau}(G), 1 < p < 2, 1 < q \leq \infty$) its τ-spherical transform exists as convergent integral on S_p (respectively on S_p°).

2.2 τ-Radial Distributions

In this section we will introduce τ-radial distributions, tempered distributions and compactly supported distributions and their τ-spherical transforms. We begin by recalling some basic facts about distributions on $\mathbb{R}$.

The set of compactly supported infinitely differentiable functions, Schwartz space functions and infinitely differentiable functions on $\mathbb{R}$ are denoted by $C_c^{\infty}(\mathbb{R}), \mathcal{S}(\mathbb{R})$ and $C^{\infty}(\mathbb{R})$ respectively. Any continuous linear functional on $C_c^{\infty}(\mathbb{R})$ (respectively on $\mathcal{S}(\mathbb{R})$ and $C^{\infty}(\mathbb{R})$) is called a *distribution* (respectively *tempered distribution* and *compactly supported distribution*) on $\mathbb{R}$. The set of all distributions, tempered distributions and compactly supported distributions on $\mathbb{R}$ are denoted by $C_c^{\infty}(\mathbb{R})', \mathcal{S}(\mathbb{R})', C^{\infty}(\mathbb{R})'$ respectively. Any locally integrable function h on $\mathbb{R}$ can be considered as a distribution T_h by $T_h(g) = \int_{\mathbb{R}} h(t)g(t)\,dt$ for $g \in C_c^{\infty}(\mathbb{R})$. For $T_1, T_2 \in C_c^{\infty}(\mathbb{R})', h \in C_c^{\infty}(\mathbb{R})$ we define $T_1 * h(t) = T_1(L(t)h)$, $(T_1 * T_2)(h) = T_1 * (T_2 * h)(0)$ where $(L(t)h)(s) = h(s-t)$.

Definition 2.2.1. For a compactly supported distribution T on $\mathbb{R}$, its euclidean Fourier transform is defined by $\widetilde{T}(\lambda) = T(e^{-i\lambda(\cdot)})$ for $\lambda \in \mathbb{C}$.

Then it follows that if T_h is induced by a compactly supported function h on $\mathbb{R}$ then $\widetilde{T_h}(\lambda) = T_h(e^{-i\lambda(\cdot)}) = \int_{\mathbb{R}} h(t)e^{-i\lambda t}\,dt = \widetilde{h}(\lambda)$.

We define the Paley-Wiener space PW^D for distributions on $\mathbb{R}$ as the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfying $|f(\lambda)| \leq C(1+|\lambda|)^M e^{R|\Im\lambda|}$ for all $\lambda \in \mathbb{C}$, for some $M \in \mathbb{N} \cup \{0\}$ and $R > 0$. Let PW_e^D be the set of even functions in PW^D. We endow PW^D with the topology of *"analytic uniform structure"* (see [21, p. 9], [38, p. 414]). The topology is defined as follows: Let P denote the set of all continuous positive functions $a(z) = a_1(\Re z)a_2(\Im z)$ $(z \in \mathbb{C})$, where a_1 dominates all polynomials and a_2 dominates all linear exponentials. For an $a \in P$ we let U_a denotes the set of all functions $F \in PW^D$ such that $|F(z)| \leq a(z)$ for all $z \in \mathbb{C}$. We topologize PW^D so that $\{U_a\}_{a \in P}$ is a fundamental system of neighborhoods of 0. Then a sequence $\{F_j\}$ converges to 0 in PW^D if and only if there exists $\alpha > 0$ such that

$$\sup_{z\in\mathbb{C}} |F_j(z)| \exp(-\alpha|\Im z|)(1+|z|)^{-\alpha} \to 0 \text{ as } j \to \infty.$$

Let PW_e^D be topologized by the subspace topology of PW^D. We have the following topological Paley-Wiener theorem for distributions on $\mathbb{R}$ [21, Theorem 5.19]:

Theorem 2.2.2. *The euclidean Fourier transform for distribution is a topological isomorphism between $C^\infty(\mathbb{R})'$ and PW^D (respectively between $C^\infty(\mathbb{R})'_e$ and PW_e^D).*

For a function $f \in C^\infty(G)$ we define its τ-radial projection f_τ as

$$f_\tau(x) = d_\tau^2 \int_K (\overline{\chi_\tau} * f * \overline{\chi_\tau})(kxk^{-1})\, dk.$$

A distribution W on G is called τ-radial if $W(f) = W(f_\tau)$. Similarly we define τ-radial tempered distribution and τ-radial compactly supported distribution. Let $C_{c,\tau}^\infty(G)'$, $C_\tau^2(G)'$ and $C_\tau^\infty(G)'$ be the dual spaces of $C_{c,\tau}^\infty(G)$, $C_\tau^2(G)$ and $C_\tau^\infty(G)$ respectively. Then their elements are τ-radial distributions, tempered distributions and compactly supported distributions respectively. We use on $C_{c,\tau}^\infty(G)'$ the topology of the uniform convergence on bounded subsets of $C_{c,\tau}^\infty(G)$.

We will be using the following notation:

$$f^\vee(x) = f(x^{-1}), f^*(x) = \overline{f(x^{-1})},\ L(x)f(y) = f(x^{-1}y),\ \text{and } R(x)f(y) = f(yx)$$

where f is a function on G. Any locally integrable function $f \in I_\tau(G)$ can be considered as a τ-radial distribution W_f by $W_f(g) = \int_G f(x)g^\vee(x)\, dx$ for $g \in C_{c,\tau}^\infty(G)$. For $W, W_1, W_2 \in C_{c,\tau}^\infty(G)'$ we define the following:

$$\overline{W}(f) = \overline{W(\overline{f})},\ W^\vee(f) = W(f^\vee),\ W^*(f) = \overline{W(f^*)}, (W * f)(x) = W^\vee(L(x)f^\vee),$$

$$(f * W)(x) = W^\vee(R(x^{-1})f^\vee) \text{ and } (W_1 * W_2)(f) = W_1^\vee * [W_2^\vee * f^\vee](e)$$

where $f \in C_{c,\tau}^\infty(G)$ and e is the identity element of G.

For suitable functions $f, g \in I_\tau(G)$, following results are easy to verify:

$$\overline{W}_f = W_{\overline{f}}, W_f^\vee = W_{f^\vee}, W_f^* = W_{f^*}, W_f * g = f * g,$$

$$g * W_f = g * f \text{ and } W_f * W_g = W_{f*g}.$$

Indeed,

1. $\overline{W_f}(g) = \overline{W_f(\overline{g})} = \overline{\int_G f(x)\overline{g}^\vee(x)\,dx} = \int_G \overline{f}(x)g^\vee(x)\,dx = W_{\overline{f}}(g)$. This proves $\overline{W}_f = W_{\overline{f}}$.

2. $W_f^\vee(g) = W_f(g^\vee) = \int_G f(x)g(x)\,dx = W_{f^\vee}(g)$ which proves $W_f^\vee = W_{f^\vee}$.

3. $W_f^*(g) = \overline{W_f(g^*)} = \overline{\int_G f(x)(g^*)^\vee(x)\,dx} = \int_G f^*(x)g^\vee(x)\,dx = W_{f^*}(g)$. This proves $W_f^* = W_{f^*}$.

4. $(W_f * g)(x) = (W_f)^\vee(L(x)g^\vee) = \int_G f(y)(L(x)(g)^\vee)(y)\,dy = (f*g)(x)$. This proves $W_f * g = f * g$. Similarly we can prove $g * W_f = g * f$.

5. $W_f * W_g(h) = W_f^\vee * [W_g^\vee * h^\vee](e) = W_f^\vee * [g^\vee * h^\vee](e) = [f^\vee * g^\vee * h^\vee](e) = \int_G h(x)(f*g)^\vee(x)\,dx = W_{f*g}(h)$.

Definition 2.2.3. For $W \in C_\tau^\infty(G)'$, its τ-spherical transform is defined pointwise at $(\sigma,\lambda) \in \widehat{M}(\tau) \times \mathbb{C}$ by $\widehat{W}(\sigma,\lambda) = \frac{1}{d_\tau}W(\phi_{\sigma,\lambda}^\tau)$.

It is clear that when a distribution W is induced by a function $w \in C_{c,\tau}^\infty(G)$, then $\widehat{W}(\sigma,\lambda) = \widehat{w}(\sigma,\lambda)$. Using deneness of $C_{c,\tau}^\infty(G)$ in $C_\tau^\infty(G)'$ and the continuity of the τ-spherical transform it can be verified that for $W_1, W_2 \in C_\tau^\infty(G)'$ and $h \in C_{c,\tau}^\infty(G)$, $\widehat{W_1 * h}(\sigma,\lambda) = \widehat{W_1}(\sigma,\lambda)\widehat{h}(\sigma,\lambda)$. Also $(W_1 * W_2)\widehat{\ }(\sigma,\lambda) = \widehat{W_1}(\sigma,\lambda).\widehat{W_2}(\sigma,\lambda)$.

2.3 Abel Transform and its Adjoint for τ-Radial Functions

This section is devoted to introduce Abel transform and the adjoint of Abel transforms. We establish a relation between the τ-spherical transform and Abel transform of suitable τ-radial functions and τ-radial compactly supported distributions.

Definition 2.3.1. For a function $F \in \Gamma(G,\tau,\tau)$ and $\sigma \in \widehat{M}(\tau)$ the Abel transform of F is defined as:

$$\mathcal{A}_\sigma F(t) = \frac{1}{d_\tau} e^{\rho t} \int_N \mathrm{Tr}(F(a_t n) \circ P_\sigma)\,dn \text{ for each } \sigma \in \widehat{M}(\tau) \tag{2.3.1}$$

whenever the integral make sense.

Correspondingly for a function $f \in I_\tau(G)$ and $\sigma \in \widehat{M}(\tau)$ Abel transform of f is defined as:

$$\mathcal{A}_\sigma(f)(t) = \frac{1}{d_\tau} e^{\rho t} \int_{M \times N} f(ma_t n) \chi_\sigma(m)\, dm\, dn \text{ for each } \sigma \in \widehat{M}(\tau) \tag{2.3.2}$$

whenever the integral make sense.

Note that when τ is trivial (i.e. the function f is K-biinvariant) then σ is also trivial and $\mathcal{A}_\sigma(f)(t)$ coincides with the well known Abel transform for K-biinvariant functions. We have the following *slice projection* theorem for Abel transform (cf. [46, eq. 7.1]). For a suitable function h on $\mathbb{R}$ let $\widetilde{h}$ be its euclidean Fourier transform that is, $\widetilde{h}(\lambda) = \int_{\mathbb{R}} h(x) e^{-i\lambda x}\, dx$.

Theorem 2.3.2. *For $\sigma \in \widehat{M}(\tau)$ and $f \in I_\tau(G)$, $\widehat{f}(\sigma, \lambda) = \widetilde{\mathcal{A}_\sigma(f)}(-\lambda)$ whenever both sides exist.*

Proof. Let f be a suitable function in $I_\tau(G)$. Then

$$\begin{aligned} \widehat{f}(\sigma,\lambda) &= \int_G f(x) \int_{K \times M} e^{-(i\lambda+\rho)H(x^{-1}k)} \chi_\tau(km^{-1}(K(x^{-1}k)^{-1}) \chi_\sigma(m)\, dm\, dk\, dx \\ &= \int_{K\times M} \int_G f(ky^{-1}) e^{-(i\lambda+\rho)H(y)} \chi_\tau(km^{-1}(K(y)^{-1}) \chi_\sigma(m)\, dy\, dm\, dk. \end{aligned}$$

We use the Iwasawa decomposition $G = KNA$ and write $y = k_1 n^{-1} a_{-t}$ to get

$$\begin{aligned} \widehat{f}(\sigma,\lambda) &= \int_{K\times M} \int_{K\times A\times N} f(ka_t n k_1^{-1}) e^{(i\lambda+\rho)t} \chi_\tau(km^{-1}k_1^{-1}) \chi_\sigma(m)\, dk_1\, da_t\, dn\, dm\, dk \\ &= \int_K \int_{K\times A\times N} \int_M f(ka_t n k_1^{-1}) e^{(i\lambda+\rho)t} \chi_\tau(km^{-1}k_1^{-1}) \chi_\sigma(m)\, dm\, dk_1\, da_t\, dn\, dk. \end{aligned}$$

Substituting first $km^{-1} = k_3$ and then $k_1 k_3^{-1} = k_2$ and using $f(k_3 x k_3^{-1}) = f(x)$ we get

$$\begin{aligned} \widehat{f}(\sigma,\lambda) &= \int_M \int_{K\times A\times N} \int_K f(k_3 m a_t n k_1^{-1}) \chi_\tau(k_3 k_1^{-1})\, dk_3\, dk_1 e^{(i\lambda+\rho)t}\, da_t\, dn \chi_\sigma(m)\, dm \\ &= \int_M \int_{A\times N} \int_K \int_K f(k_3 m a_t n k_3^{-1} k_2^{-1}) \overline{\chi_\tau(k_2)}\, dk_2\, dk_3 e^{(i\lambda+\rho)t}\, da_t\, dn \chi_\sigma(m)\, dm \\ &= \frac{1}{d_\tau} \int_M \chi_\sigma(m) \int_{A\times N} \int_K f(k_3 m a_t n k_3^{-1}) dk_3 e^{(i\lambda+\rho)t}\, da_t\, dn\, dm \\ &= \frac{1}{d_\tau} \int_A \left(e^{\rho t} \int_{M\times N} f(ma_t n) \chi_\sigma(m)\, dm\, dn \right) e^{i\lambda t}\, dt \\ &= \int_{\mathbb{R}} \mathcal{A}_\sigma f(t) e^{i\lambda t}\, dt. \end{aligned}$$

□

From the theorem above and the injectivity of the Fourier transform it follows that for two functions $f_1, f_2 \in I_\tau$ for which the τ-spherical transform and Abel transform exist,

$$\mathcal{A}_\sigma(f_1 * f_2) = \mathcal{A}_\sigma f_1 * \mathcal{A}_\sigma f_2 \quad \text{for all } \sigma \in \widehat{M}(\tau).$$

From the domain of existence of the τ-spherical transform (see Section 2.1), Theorem 2.3.2 and Fubini's theorem we have the following *mapping properties* of the Abel transform:

Proposition 2.3.3. *Let f be a measurable scalar valued τ-radial function on G. Then for all $\sigma \in \widehat{M}(\tau)$:*

(1) *If $f \in L^{p,1}_\tau(G), 1 \leq p < 2$ then $\int_{\mathbb{R}} |\mathcal{A}_\sigma f(t)|\, e^{\gamma_p \rho |t|}\, dt \leq C\|f\|^*_{p,1}$.*

(2) *If $f \in L^{p,q}_\tau(G), 1 < p < 2, 1 < q \leq \infty$ then $\int_{\mathbb{R}} |\mathcal{A}_\sigma f(t)|\, e^{\alpha|t|}\, dt \leq C\|f\|^*_{p,q}$ for any $0 < \alpha < \gamma_p\rho$.*

Proof. We note that $|\mathcal{A}_\sigma f| \leq \mathcal{A}_\sigma |f|$ and

$$\int_{\mathbb{R}} \mathcal{A}_\sigma |f|(t) e^{\pm\gamma_p\rho t}\, dt = \widehat{|f|}(\sigma, \mp i\gamma_p\rho) = \int |f(x)| \phi^\tau_{\sigma, \mp i\gamma_p\rho}(x^{-1})\, dx \leq \|\phi^\tau_{\sigma,\mp i\gamma_p\rho}\|^*_{p',\infty}\|f\|^*_{p,1}.$$

This proves (1). Similar argument proves (2). □

Definition 2.3.4. For a measurable function f on $\mathbb{R}$ adjoint of Abel transform $\mathcal{A}^*_\sigma$ for $\sigma \in \widehat{M}(\tau)$ is defined by

$$A^*_\sigma f(y) \quad = \quad \frac{1}{d_\sigma} \int_K f(e^{H(yk)}) e^{-\rho(H(yk))} \left(\tau(k) \circ P_\sigma \circ \tau(K(yk)^{-1})\right)\, dk.$$

It is clear that if f is a bounded function then $\mathcal{A}^*_\sigma f$ exists. As Iwasawa decomposition is a diffeomorphism $\mathcal{A}^*_\sigma f$ is infinitely differentiable whenever $f \in C^\infty(\mathbb{R})$. Also we observe that $\mathcal{A}^*_\sigma f$ is a $\mathrm{End}V_\tau$ valued τ-radial function. That is $\mathcal{A}^*_\sigma f \in C^\infty(G, \tau, \tau)$. The following theorem justifies the definition of adjoint of Abel transform.

Theorem 2.3.5. *For a measurable function $F \in \Gamma(G, \tau, \tau)$ on G and a measurable function f on $\mathbb{R}$ the following holds*

$$\langle \mathcal{A}_\sigma F, f\rangle = \langle F, \mathcal{A}^*_\sigma f\rangle$$

whenever the inner products on both sides make sense.

Proof. We have

$$\begin{aligned}
\langle \mathcal{A}_\sigma F, f\rangle &= \textstyle\int_{\mathbb{R}} \mathcal{A}_\sigma F(a_t)\overline{f(a_t)}\, dt \\
&= \tfrac{1}{d_\sigma} \textstyle\int_{\mathbb{R}} e^{\rho t} \int_N \operatorname{Tr}\left(F(a_t n)\circ P_\sigma\right)\, dn\overline{f(a_t)}\, dt \\
&= \tfrac{1}{d_\sigma} \textstyle\int_{K\times A\times N} e^{\rho t}\operatorname{Tr}\left(F(ka_t n)\circ \tau(k)\circ P_\sigma\right)\overline{f(a_t)}\, dn\, dt\, dk.
\end{aligned}$$

We put $ka_tn = x$ in the above integral to get,

$$\begin{aligned}
\langle \mathcal{A}_\sigma F, f\rangle &= \tfrac{1}{d_\sigma} \textstyle\int_G \operatorname{Tr}\left(F(x)\circ \tau(K(x))\circ P_\sigma\right) e^{-\rho H(x)}\overline{f(e^{H(x)})}\, dx \\
&= \tfrac{1}{d_\sigma} \textstyle\int_G \operatorname{Tr}\left(F(x)\circ \overline{f(e^{H(x)})} e^{-\rho H(x)}\tau(K(x))\circ P_\sigma\right)\, dx.
\end{aligned}$$

Again we put $x = yk$ in the above and get,

$$\begin{aligned}
\langle \mathcal{A}_\sigma F, f\rangle &= \tfrac{1}{d_\sigma} \textstyle\int_G\int_K \operatorname{Tr}\left(\tau(k^{-1})\circ F(y)\circ \tau(K(yk))\circ P_\sigma\overline{f(e^{H(yk)})}e^{-\rho H(yk)}\right)\, dk\, dy \\
&= \textstyle\int_G \operatorname{Tr}\left(F(y)\circ \tfrac{1}{d_\sigma}\int_K \tau(K(yk))\circ P_\sigma\circ\tau(k^{-1})\overline{f(e^{H(yk)})}e^{-\rho H(yk)}\, dk\right)\, dy \\
&= \textstyle\int_G \operatorname{Tr}\left(F(y)\circ (\mathcal{A}_\sigma^* f(y))^*\right)\, dy \\
&= \langle F, \mathcal{A}_\sigma^* f\rangle.
\end{aligned}$$

□

Remark 2.3.6. It is clear from the definition that if $f(a_t) = e^{-i\lambda t}$, then $\mathcal{A}_\sigma^* f(y) = \frac{1}{d_\tau}\Phi^\tau_{\sigma,\lambda}(y)$. Therefore by the theorem above and Proposition 2.1.6 we have the following slice projection theorem for endomorphism valued functions. For a suitable τ-radial function F and $\sigma\in\widehat{M}(\tau)$

$$\widehat{F}(\sigma,\lambda) = \widetilde{\mathcal{A}_\sigma F}(-\lambda), \tag{2.3.3}$$

where $\widetilde{\mathcal{A}_\sigma F}(-\lambda) = \int_{\mathbb{R}} \mathcal{A}_\sigma F(t) e^{i\lambda t} dt$ denotes the euclidean Fourier transform of $\mathcal{A}_\sigma F$ at $-\lambda$. In fact

$$\widehat{F}(\sigma,\lambda) = \frac{1}{d_\tau}\int_G \operatorname{Tr}\left(F(x)\Phi^\tau_{\sigma,\lambda}(x^{-1})\right)\, dx = \frac{1}{d_\tau}\int_G \operatorname{Tr}\left(F(x)(\Phi^\tau_{\sigma,\overline{\lambda}}(x))^*\right)\, dx.$$

Therefore,

$$\widehat{F}(\sigma,\lambda) = \frac{1}{d_\tau}\langle F, \Phi^\tau_{\sigma,\overline{\lambda}}\rangle = \langle F, \mathcal{A}^*_\sigma(e^{-i\overline{\lambda}(\cdot)})\rangle = \langle \mathcal{A}_\sigma F, e^{-i\overline{\lambda}(\cdot)}\rangle = \int_{\mathbb{R}} \mathcal{A}_\sigma F(t) e^{i\lambda t} dt.$$

We note that this result is equivalent to Theorem 2.3.2. We can extend the definition of Abel transform to compactly supported τ-radial distributions in the following way.

Definition 2.3.7. For $W \in C^\infty_\tau(G)'$, its Abel transform $\mathcal{A}_\sigma W$ is defined by:

$$\mathcal{A}_\sigma W(f) = W(\mathrm{Tr}\mathcal{A}^*_\sigma f) \text{ for } f \in C^\infty(A), \text{ for each } \sigma \in \widehat{M}(\tau). \tag{2.3.4}$$

We note that $\mathcal{A}_\sigma W$ is a compactly supported distribution on $\mathbb{R}$ and the *slice projection property* is in-built in the definition above of the Abel transform for compactly supported distributions. That is

$$\widehat{W}(\sigma,\lambda) = \widetilde{\mathcal{A}_\sigma W}(\lambda). \tag{2.3.5}$$

In fact

$$\widehat{W}(\sigma,\lambda) = \frac{1}{d_\tau} W(\phi^\tau_{\sigma,\lambda}) = W\left(\mathrm{Tr}\mathcal{A}^*_\sigma e^{-i\lambda(\cdot)}\right) = \mathcal{A}_\sigma W(e^{-i\lambda(\cdot)}) = \widetilde{\mathcal{A}_\sigma W}(\lambda).$$

Use of the slice projection property also yields the following for $W_1, W_2 \in C^\infty_\tau(G)'$:

$$(\mathcal{A}_\sigma W_1 * \mathcal{A}_\sigma W_2)\tilde{}(\sigma,\lambda) \quad = \quad \widetilde{\mathcal{A}_\sigma W_1}(\sigma,\lambda)\widetilde{\mathcal{A}_\sigma W_2}(\sigma,\lambda) = \widehat{W_1}(\sigma,\lambda)\widehat{W_2}(\sigma,\lambda).$$

Therefore

$$(\mathcal{A}_\sigma W_1 * \mathcal{A}_\sigma W_2)\tilde{}(\sigma,\lambda) = (W_1 * W_2)\hat{}(\sigma,\lambda) = \mathcal{A}_\sigma(W_1 * W_2)\tilde{}(\sigma,\lambda).$$

By the injectivity of the euclidean Fourier transform from the relation above we get

$$\mathcal{A}_\sigma(W_1 * W_2) = \mathcal{A}_\sigma(W_1) * \mathcal{A}_\sigma(W_2).$$

Chapter 3

Some Banach Algebras and Modules

In this chapter we shall continue to work with a Gelfand triple (G, K, τ). We shall set our basic objects for which the Wiener-Tauberian type theorems will be proved in Chapter 6. We shall consider two different sets of Banach spaces of scalar valued τ-radial functions. Members of these sets can be viewed as generalizations of the group algebra $L^1_\tau(G)$. We shall investigate some properties of these spaces and find the domains of the τ-spherical transforms of the functions in these spaces. We shall also identify the Banach algebras and modules among these spaces.

For normed linear spaces $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ by $\mathcal{V} \subseteq \mathcal{W}$ (respectively by $\mathcal{U} * \mathcal{V} \subseteq \mathcal{W}$) we mean both set inclusions and the associated norm inequalities:

$$\|v\|_{\mathcal{W}} \leq C\|v\|_{\mathcal{V}} \text{ (respectively } \|u * v\|_{\mathcal{W}} \leq C\|u\|_{\mathcal{U}}\|v\|_{\mathcal{V}}) \text{ for all } u \in \mathcal{U}, v \in \mathcal{V}.$$

3.1 Weighted Spaces

We shall consider the weights $w_{p,r}(x) = \phi_{i\gamma_p\rho}(x)(1+\sigma(x))^r$, for $r \geq 0$ $0 < p \leq 2$, which are naturally associated with the group. For r and p as above we define the weighted L^1-spaces:

$$L^1_\tau(G, w_{p,r}) = \left\{f : G \to \mathbb{C} \mid f \text{ is measurable and } \tau\text{-radial with } \|f\|_{w_{p,r}} < \infty\right\}$$

where

$$\|f\|_{w_{p,r}} = d_\tau^2 \int_G |f(x)| w_{p,r}(x)\, dx.$$

Similar spaces have appeared already for instance in [17, 18]. It is clear that the τ-spherical transform of any function in these weighted spaces has analytic extension (except for the degenerate case $p = 2$, which we shall treat separately). For convenience henceforth we will write $L^1_\tau(w_{p,r})$ for $L^1_\tau(G, w_{p,r})$.

We note that $L^1_\tau(w_{1,0}) = L^1_\tau(G)$ and if $r \geq s \geq 0$ then $L^1_\tau(w_{p,r}) \subseteq L^1_\tau(w_{p,s})$. A few more such observations we write in the form of a proposition.

Proposition 3.1.1. *We have the following set inclusions and corresponding norm inequalities:*

(*a*) *If* $1 \leq q < p \leq 2$ *then* $L^1_\tau(w_{q,s}) \subseteq L^1_\tau(w_{p,r})$ *for any* $r > 0$, $s > 0$. *If* $1 \leq q \leq p \leq 2$ *then* $L^1_\tau(w_{q,s}) \subseteq L^1_\tau(w_{p,r})$ *for* $s > r > 0$. *In particular* $L^1_\tau(G) \subseteq L^1_\tau(w_{p,r})$ *if* $1 \leq p \leq 2$ *and* $r \geq 0$.

(*b*) *If* $0 < p < 1$ *and* $r \geq 0$ *then* $L^1_\tau(w_{p,r}) \subseteq L^1_\tau(G)$.

(*c*) *If* $1 < p < 2$ *then* $L^{p,1}_\tau(G) \subsetneq L^1_\tau(w_{p,0})$ *and if* $r > 1$ *then* $L^{p,r}_\tau(G) \not\subset L^1_\tau(w_{p,0})$.

(*d*) *For* $1 \leq q < p$, $L^{q,s}_\tau(G) \subseteq L^1_\tau(w_{p,r})$ *if* $1 \leq s \leq \infty$, $r \geq 0$.

Proof. Assertion (a) and (b) follow comparing the weights (see Proposition 1.0.1) and noting that $L^1_\tau(G) = L^1_\tau(w_{1,0})$.

For (c) we note that $\phi_{i\gamma_p\rho} \in L^{p',\infty}(G)$ by Proposition 1.0.1 and hence $L^{p,1}_\tau(G) \subset L^1_\tau(w_{p,0})$.

Next we shall show that $L^1_\tau(w_{p,0}) \setminus L^{p,1}_\tau(G)$ is nonempty for $1 < p \leq 2$. For p in this range, we take a $g \in L^1_\tau(G) \setminus L^p_\tau(G)$ and consider $h = g\phi^{-1}_{i\gamma_p\rho}$. Then $h \in L^1_\tau(w_{p,0})$, but as $\phi^{-1}_{i\gamma_p\rho} \geq 1$, $|h(x)| \geq |g(x)|$ and hence $h \notin L^p_\tau(G)$. As $L^{p,1}_\tau(G) \subseteq L^p_\tau(G)$, the assertion follows.

Next we show $L^{p,r}_\tau(G) \setminus L^1_\tau(w_{p,0})$ is nonempty when $r > 1$.

As by Proposition 1.0.1, $\phi_{i\gamma_p\rho} \notin L^{p',s}(G//K)$ unless $s = \infty$ we have that there exists a nonnegative function $\psi \in L^{p,r}(G//K) \setminus L^1(G//K, w_{p,0})$. It is now easy to construct a function $f \in L^{p,r}_\tau(G) \setminus L^1_\tau(w_{p,0})$ in the following way: We define $\widetilde{f} : G \to \mathrm{End}V_\tau$ by

$$\widetilde{f}(x)v = \widetilde{f}(k_1 a_t k_2)v = \psi(a_t)\tau(k_2^{-1}k_1^{-1})v, \; v \in V_\tau.$$

Let $f(x) = \mathrm{Tr}(\widetilde{f}(x)) = \psi(a_t)\mathrm{Tr}(\tau(k_2^{-1}k_1^{-1}))$. Then f is a τ-radial function and $|f(x)| \leq C_\tau \psi(x)$. Hence $f \in L^{p,r}_\tau(G)$. Also

$$\int_G |f(x)|\phi_{i\gamma_p\rho}(x)dx = C'_\tau \int_G \psi(x)\phi_{i\gamma_p\rho}(x)dx$$

which is infinite by the choice of ψ.

For (d) we use that for any $r \geq 0$ and $q < p_1 < p$, $\phi_{i\gamma_p \rho}(x)(1+\sigma(x))^r \leq \phi_{i\gamma_{p_1}\rho}(x)$ and that $\phi_{i\gamma_{p_1}\rho} \in L^{q',1}(G)$ by Proposition 1.0.1. The result now follows from the properties of the Lorentz spaces. $\square$

We need now the following lemma.

Lemma 3.1.2. *Let w_1 and w_2 be two radial measurable positive functions on G such that $w_2(x) \leq w_1(x)$ for all $x \in G$ and*

$$\int_K w_2(xky)\, dk \leq w_2(x).w_2(y).$$

Let for $i = 1, 2$

$$L^1_\tau(G, w_i) = \{f \in I_\tau(G) \mid f \text{ is measurable with } \|f\|_{w_i} = d_\tau^2 \int_G |f(x)| w_i(x)\, dx < \infty\}.$$

Then $L^1_\tau(G, w_1) \subset L^1_\tau(G, w_2)$ and $\|f_1 * f_2\|_{w_2} \leq \|f_1\|_{w_1} \|f_2\|_{w_2}$. In particular $L^1(G, w_2)$ is a Banach algebra.

Proof. We suppose $f_1 \in L^1_\tau(G, w_1)$, $f_2 \in L^1_\tau(G, w_2)$. Then

$$\begin{aligned}
\int_G |f_1 * f_2(x)| w_2(x)\, dx &= \int_G \left| \int_G f_1(xy^{-1}) f_2(y)\, dy \right| w_2(x)\, dx \\
&\leq \int_G \int_G |f_1(xy^{-1})|\; |f_2(y)|\, dy w_2(x)\, dx \\
&= \int_G \int_G |f_1(z)|\; |f_2(y)| w_2(zy)\, dy\, dz \quad (\text{putting } xy^{-1} = z) \\
&= \int_G \int_G \left| \int_K d_\tau f_1(zk^{-1}) \overline{\chi_\tau(k)}\, dk \right|\; |f_2(y)| w_2(zy)\, dy\, dz \\
&\leq d_\tau \int_G \int_G \int_K |\overline{\chi_\tau(k)}|\; |f_1(z_1)| |f_2(y)| w_2(z_1 k y)\, dk\, dy\, dz_1 \\
&= d_\tau^2 \int_G \int_G \int_K |f_1(z_1)| w_1(z_1) |f_2(y)| w_2(z_1 k y) w_1^{-1}(z_1)\, dk\, dy\, dz_1 \\
&\leq d_\tau^2 \int_G \int_G |f_1(z_1)| w_1(z_1) |f_2(y)| w_2(y) dy.
\end{aligned}$$

Therefore $\|f_1 * f_2\|_{w_2} \leq \|f_1\|_{w_1} \|f_2\|_{w_2}$. $\square$

From the lemma above and Proposition 3.1.1 the following corollary is immediate.

Corollary 3.1.3. *For $p, q \in (0, 2]$ and positive real numbers r, s*

(a) *$L^1_\tau(w_{p,r})$ is a Banach algebra,*

(b) $L^1_\tau(w_{p,s})$ *is a* $L^1_\tau(w_{p,r})$ *module if* $r > s$,

(c) $L^1_\tau(w_{p,r})$ *is a* $L^1_\tau(w_{q,s})$ *module if* $q < p$. *In particular if* $p > 1$ *then* $L^1_\tau(w_{p,r})$ *is a* $L^1_\tau(G)$ *module and if* $p \le 1$ *then* $L^1_\tau(G)$ *is a* $L^1_\tau(w_{p,r})$ *module.*

Proposition 3.1.4. *For any* $r \ge 0$ *and* $0 < p \le 2$, $C^p_\tau(G)$ *is dense in* $L^1_\tau(w_{p,r})$ *and the inclusion map* $i : C^p_\tau(G) \to L^1_\tau(w_{p,r})$ *is continuous.*

Proof. We suppose that $f \in C^p_\tau(G)$ and $\rho_1(f) = \sup_{x\in G} |f(x)|(1+\sigma(x))^{r+s} e^{\frac{2}{p}\rho\sigma(x)}$ for some $s > 1$. Then

$$\begin{aligned} \|f\|_{w_{p,r}} &= d_\tau^2 \textstyle\int_G |f(x)|(1+\sigma(x))^r \phi_{i\gamma_p\rho}(x)\, dx \\ &\le d_\tau^2 \rho_1(f) \textstyle\int_G e^{-\frac{2}{p}\rho\sigma(x)} \phi_{i\gamma_p\rho}(x) \frac{1}{(1+\sigma(x))^s}\, dx. \end{aligned}$$

Therefore

$$\|f\|_{w_{p,r}} \le d_\tau^2 \rho_1(f) \textstyle\int_G \frac{e^{-2\rho\sigma(x)}}{(1+\sigma(x))^s}\, dx.$$

In the last step we have used the estimate (1.0.8) of ϕ_λ. Now by using polar decomposition it is easy to check that the integral is finite and thus $\|f\|_{w_{p,r}} \le C\rho_1(f)$. Therefore $C^p_\tau(G) \subseteq L^1_\tau(w_{p,r})$ and the inclusion map $i : C^p_\tau(G) \to L^1_\tau(w_{p,r})$ is continuous. Also we have $C^\infty_{c,\tau}(G) \subseteq C^p_\tau(G) \subseteq L^1_\tau(w_{p,r})$ and $C^\infty_{c,\tau}(G)$ is dense in $L^1_\tau(w_{p,r})$. Therefore $C^p_\tau(G)$ is dense in $L^1_\tau(w_{p,r})$. ☐

Proposition 3.1.5. *The* τ*-spherical transform of the functions in* $L^1_\tau(w_{p,r})$ *have the following properties:*
(1) *If* $f \in L^1_\tau(w_{p,r}), 0 < p < 2$, $r \ge 0$, *then for all* $\sigma \in \widehat{M}(\tau)$, $\widehat{f}(\sigma, \cdot)$ *is analytic on* S_p° *and continuous on the boundary. If* $f \in L^1_\tau(w_{2,r}), r \ge 0$, *then* $\widehat{f}(\sigma,\cdot), \sigma \in \widehat{M}(\tau)$ *exists as a convergent integral on* $\mathbb{R}$.
(2) *Let* $f \in L^1_\tau(w_{p,r})$ *with* $0 < p < 2$ *and* $r \ge 0$. *Then for all* $\sigma \in \widehat{M}(\tau)$, $\mathcal{A}_\sigma f \in L^1(\mathbb{R}, w)$, *where* $w(t) = e^{\gamma_p\rho|t|}$. *Moreover*

$$\lim_{|\xi|\to\infty} \widehat{f}(\sigma, \xi + i\eta) = 0$$

uniformly in $\eta \in [-\gamma_p\rho, \gamma_p\rho]$.

Proof. (1) We take a function $f \in L^1_\tau(w_{p,r})$ with $0 < p \le 2$ and $r \ge 0$ and $\lambda \in S_p$. Then

$$\int_G |f(x)|\ |\phi^\tau_{\sigma,\lambda}(x)|\, dx = \int_G |f(x)|(1+\sigma(x))^r \phi_{i\gamma_p\rho}(x) \frac{|\phi^\tau_{\sigma,\lambda}(x)|}{\phi_{i\gamma_p\rho}(x)(1+\sigma(x))^r}\, dx < \infty$$

as $|\phi^\tau_{\sigma,\lambda}(x)| \le C\ \phi_{i\Im\lambda}(x) \le C\ \phi_{i\gamma_p\rho}(x)$. For $0 < p < 2$, a standard use of Fubini's

theorem, Morera's theorem and dominated convergence theorem shows that $\widehat{f}(\sigma,\cdot)$ is analytic on S_p° and continuous on the boundary.
(2) From (1) above we know that $\widehat{f}(\sigma,\cdot)$ exists on S_p. We recall that for a suitable function h on $\mathbb{R}$, $\widetilde{h}$ is its Euclidean Fourier transform i.e., $\widetilde{h}(\lambda)=\int_{\mathbb{R}} h(x)e^{-i\lambda x}\,dx$. As $f\in L^1_\tau(w_{p,r})$ with $0<p<2, r\geq 0$, $|\mathcal{A}_\sigma f|\leq \mathcal{A}_\sigma|f|$ and $\widetilde{\mathcal{A}_\sigma f}(-\lambda)=\widehat{f}(\sigma,\lambda)$ for $\lambda\in S_p$, we have:

$$\int_{\mathbb{R}} |\mathcal{A}_\sigma f(t)|e^{\gamma_p\rho|t|}\,dt<\infty.$$

Thus $g=\mathcal{A}_\sigma f$ is in the weighted space $L^1(\mathbb{R},w)$ with weight $w(t)=e^{\gamma_p\rho|t|}$. This reduces the assertion to the Riemann-Lebesgue lemma for functions on $\mathbb{R}$ which are integrable with an exponential weight. It is also clear that for η as in the hypothesis

$$|\widetilde{g}(\xi+i\eta)|\leq \int_{\mathbb{R}} |g(t)|e^{\eta|t|}\,dt\leq \|g\|_{w,1},$$

where $\|g\|_{w,1}=\int_{\mathbb{R}}|g(x)|e^{\gamma_p\rho|x|}\,dx$ the weighted L^1-norm of g. To complete the proof of the assertion we now approximate g in $L^1(\mathbb{R},w)$ by finite sums h of step functions and use $\widetilde{h}(\xi+i\eta)\to 0$ as $|\xi|\to\infty$ uniformly in $\eta\in[-\gamma_p\rho,\gamma_p\rho]$ and note that

$$\begin{aligned}|\widetilde{g}(\xi+i\eta)| &\leq |\widetilde{g}(\xi+i\eta)-\widetilde{h}(\xi+i\eta)|+|\widetilde{h}(\xi+i\eta)|\\ &\leq \|g-h\|_{w,1}+|\widetilde{h}(\xi+i\eta)|.\end{aligned}$$

□

Remark 3.1.6. The following remarks are in order.

(a) The spaces $L^1_\tau(w_{2,r}), r\geq 0$ stand apart as the domains of the τ-spherical transforms of the functions in these spaces are no longer strips in the complex plane. We shall call this the degenerate case. The Wiener-Tauberian theorem for this case will be proved separately.

(b) One can also consider weighted spaces $L^1_\tau(w_{p,r})$ for $1\leq p\leq 2$ and $r<0$. Using the inequality $(1+\sigma(y))/(1+\sigma(x))\leq(1+\sigma(xy))\leq(1+\sigma(x))(1+\sigma(y))$ ([28, Prop. 4.6.11]) it can be shown that $L^1_\tau(w_{p,r})$ is an $L^1_\tau(w_{p,s})$ module when p,r are as above and $|r|<s$. Argument similar to what we have used in the previous proposition shows that the τ-spherical transform of functions in $L^1_\tau(w_{p,r})$ extends analytically on S_p° and a Riemann-Lebesgue lemma holds on S_p°.

3.2 Lorentz Spaces

Our second set consists of Lorentz spaces $L^{p,q}_\tau(G)$ of scalar valued τ-radial functions.

We have seen in Chapter 2 that like the elementary spherical function ϕ_λ, the τ-spherical function $\phi^\tau_{\sigma,\lambda}$ also satisfies some uniform estimates, L^p estimates as well as the Lorentz space estimates. The L^p estimates of ϕ_λ leads to the Kunze-Stein phenomenon for K-biinvariant functions (see Chapter 1). From Similar argument it follows that $L^p_\tau(G) * L^q_\tau(G) \subset L^q_\tau(G)$ with the associated norm inequality: $\|f * g\|_q \leq C\|f\|_p\|g\|_q$ when $1 \leq p < q \leq 2$ for τ-radial functions f, g. This is the Kunze-Stein phenomenon in its classical form for τ-radial functions.

Through the works of Herz, Lohoué, Lohoué and Rychner, Cowling and Ionescu sharper version of Kunze-Stein phenomenon is obtained for groups of real rank one which involves Lorentz spaces (see [16] for a comprehensive survey and for the references, see also [44, 17, 39]). This gives rise to new modules and algebras, which we shall see now.

Proposition 3.2.1. *The spaces $L^{p,q}_\tau(G)$ satisfy the following properties:*

(1) *For $1 \leq p < 2$, $L^{p,1}_\tau(G)$ is a Banach algebra.*

(2) *$L^{\alpha,r}_\tau(G)$ is an $L^{q,1}_\tau(G)$ module for $1 < q \leq \alpha < 2$ and $1 \leq r \leq \infty$.*

(3) *If $1 \leq q < p \leq 2$ then $C^q_\tau(G)$ is dense in $L^{p,r}_\tau(G)$ for $0 \leq r < \infty$ and the inclusion map $i : C^q_\tau(G) \to L^{p,r}_\tau(G)$ is continuous.*

We need the following theorem. [64, 16].

Theorem 3.2.2 (Zafran)**.** *Suppose T is a bilinear operator which is bounded from $L^{a_i,r_i}(A) \times L^{b_i,s_i}(B)$ to $L^{c_i,t_i}(C)$ for $i = 0, 1$ and $a_0 < a_1, b_0 < b_1, c_0 < c_1$. Suppose for $\theta \in (0, 1)$,*

$$\frac{1}{a_\theta} = \frac{1-\theta}{a_0} + \frac{\theta}{a_1}; \quad \frac{1}{b_\theta} = \frac{1-\theta}{b_0} + \frac{\theta}{b_1}; \quad \frac{1}{c_\theta} = \frac{1-\theta}{c_0} + \frac{\theta}{c_1}$$

and $(r, s, t) \in [1, \infty] \times [1, \infty] \times [1, \infty]$ satisfies $\frac{1}{r} + \frac{1}{s} \geq 1 + \frac{1}{t}$. Then T is bounded operator from $L^{a_\theta,r}(A) \times L^{b_\theta,s}(B)$ to $L^{c_\theta,t}(C)$.

Proof of Proposition 3.2.1. Assertion (1) is proved in [44, 16]. However it is possible to give a unified proof of (1) and (2). In this regard we need the following fundamental end point estimate of Ionescu ([39]):

$$L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G).$$

Specializing to τ-radial functions we get,

$$L^{2,1}_\tau(G) * L^{2,1}_\tau(G) \subseteq L^{2,\infty}_\tau(G).$$

We also have

$$L^p_\tau(G) * L^1_\tau(G) \subseteq L^p_\tau(G)$$

for $p \geq 1$. This can be restated as:

$$L^{p,p}_\tau(G) * L^{1,1}_\tau(G) \subseteq L^{p,p}_\tau(G), \text{ for } p \geq 1, \text{ in particular for } 1 \leq p < 2.$$

Using Theorem 3.2.2 we get

$$L^{\alpha,r}_\tau(G) * L^{q,s}_\tau(G) \subseteq L^{\alpha,t}_\tau(G)$$

where $\frac{1}{r} + \frac{1}{s} \geq 1 + \frac{1}{t}$ and $q \leq \alpha$ (varying $1 \leq p < 2$ we get all q such that $q \leq \alpha$). We note that $(r, 1, r)$ satisfies the relation above. Therefore

$$L^{\alpha,r}_\tau(G) * L^{q,1}_\tau(G) \subseteq L^{\alpha,r}_\tau(G) \text{ when } q \leq \alpha.$$

This proves (2). In particular taking $r = 1$ and $\alpha = q = p$ we get $L^{p,1}_\tau(G)$ is a Banach algebra. This proves (1).

(3) We recall that $C^q_\tau(G) \subseteq L^q_\tau(G) \subseteq L^{q,\infty}_\tau(G)$. Again $C^q_\tau(G) \subseteq L^2_\tau(G) \subseteq L^{2,\infty}_\tau(G)$. Therefore by interpolation ([30, p. 64]) $C^q_\tau(G) \subseteq L^{p,r}_\tau(G)$ when $q < p < 2$ and $0 < r \leq \infty$.

For a function $g \in C^q_\tau(G)$ we consider the seminorm

$$\rho_1(g) = \sup_{x \in G} |g(x)| e^{\frac{2}{q}\rho\sigma(x)}.$$

A function $g \in C^q_\tau(G)$ implies $\rho_1(g) < \infty$, that is for all $x \in G$,

$$|g(x)| \leq B e^{-\frac{2}{q}\rho\sigma(x)}, \text{ where } B = \rho_1(g) > 0.$$

Let $f = \frac{1}{B} g$. We have, $|f(x)| \leq e^{-\frac{2}{q}\rho\sigma(x)}$.

We recall that

$$\|f\|^*_{p,1} = \int_0^\infty d_f(t)^{1/p}\, dt$$

where $d_f(t) = m\{x \mid |f(x)| > t\}$ for $t > 0$, m being the Haar measure of G.

Then we have,

$$\begin{aligned} d_f(t) &= \quad 0 \text{ if } t > 1 \\ &= m\{x \mid \sigma(x) < \tfrac{q}{2\rho}\log\tfrac{1}{t}\} \text{ if } t \leq 1. \end{aligned}$$

Let us write β for $m_\alpha + m_{2\alpha}$. A direct calculation using the expression of the measure m yields the estimates

$$d_f(t) \leq C\left[\frac{1}{\beta+1} + \frac{1}{2\rho}(e^{q\log\frac{1}{t}} - e^{2\rho})\right] \leq C + \frac{1}{2\rho t^q} \text{ if } 0 < t < e^{-\frac{2\rho}{q}}$$

and

$$d_f(t) \leq C\int_0^{\frac{q}{2\rho}\log\frac{1}{t}} r^\beta\, dr = \frac{C}{\beta+1}(\frac{q}{2\rho}\log\frac{1}{t})^{\beta+1} \text{ if } e^{-\frac{2\rho}{q}} \leq t \leq 1.$$

Now

$$\int_0^\infty d_f(t)^{1/p}\, dt \leq \int_{e^{-\frac{2\rho}{q}}}^1 \left[\frac{C}{\beta+1}(\frac{q}{2\rho}\log\frac{1}{t})^{\beta+1}\right]^{1/p} dt + \int_0^{e^{-\frac{2\rho}{q}}} \left[C + \frac{1}{2\rho t^q}\right]^{1/p} dt.$$

The first integral converges as the integrand is continuous in t and the second integral converges as $q < p$. Thus $\|f\|^*_{p,1} \leq C$, where C is independent of f. Therefore $\|g\|^*_{p,1} \leq C\rho_1(g)$ for all $g \in C^q_\tau(G)$. Also since $\|g\|^*_{p,r} \leq \|g\|^*_{p,1}$ for $r \geq 1$ the second part of the assertion follows. As $C^\infty_{c,\tau}(G)$ is dense in $L^{p,r}_\tau(G)$ and $C^\infty_{c,\tau}(G) \subset C^q_\tau(G)$ it follows that $C^q_\tau(G)$ is dense in $L^{p,r}_\tau(G)$. $\square$

Remark 3.2.3. Note that it follows as a special case of a more general result proved in [9] that the spaces $L^{p,q}_\tau(G)$ are $L^1_\tau(G)$ modules for $1 < p < \infty$ and for $1 \leq q \leq \infty$. Using a result of Saeki modified by Cowling ([53, 16]) one can show that $L^{\alpha,r}_\tau(G), r > 1$ is not an algebra and $L^{\alpha,r}_\tau(G)$ is not an $L^{\alpha,s}_\tau(G)$ module for $s > 1$.

Since by Proposition 3.1.1, $L^{p,1}_\tau(G) \subseteq L^1_\tau(w_{p,0}), 1 \leq p < 2$ and $L^{p,r}_\tau(G) \subseteq L^1_\tau(w_{q,s}), 1 < p < q < 2, r > 1$, the following proposition is immediate from Proposition 3.1.5.

Proposition 3.2.4. *The τ-spherical transform of the functions in $L^{p,q}_\tau(G)$ have the following properties:*

(1) *Let $f \in L^{p,1}_\tau(G)$, $1 \leq p < 2$. Then for all $\sigma \in \widehat{M}(\tau)$, $\widehat{f}(\sigma,\cdot)$ is analytic on*

S_p°, continuous on ∂S_p and

$$\lim_{|\xi|\to\infty} \widehat{f}(\sigma, \xi + i\eta) = 0$$

uniformly in $\eta \in [-\gamma_p\rho, \gamma_p\rho]$.

(2) *Let $f \in L_\tau^{p,r}(G)$ with $1 < p < 2$, $r > 1$. Then for all $\sigma \in \widehat{M}(\tau)$, $\widehat{f}(\sigma, \cdot)$ is analytic on S_p° and*

$$\lim_{|\xi|\to\infty} \widehat{f}(\sigma, \xi + i\eta) = 0$$

uniformly in $\eta \in [-(\gamma_p\rho - \delta), (\gamma_p\rho - \delta)]$ for any $0 < \delta < \gamma_p\rho$.

Chapter 4

Spin Group, Spin Representations and Spherical Functions

4.1 Spin Group and Spin Representations

In this section we establish the required preliminaries for our main working example namely: $G = \mathrm{Spin}_0(n,1)$, the identity component of $\mathrm{Spin}(n,1)$ and the spin representations. First we recall the algebraic definition of $\mathrm{Spin}(n,1)$ (see [42] for details).

Let V be a vector space over a field k, q a quadratic form on V and $\mathcal{F}(V) = \sum_{r=0}^{\infty} \otimes^r V$ the tensor algebra of V. We suppose that $I_q(V)$ is the ideal in $\mathcal{F}(V)$ generated by all elements of the form $v \otimes v + q(v)1$, $v \in V$. Then the Clifford algebra is defined by $\mathrm{Cl}(V,q) = \mathcal{F}(V)/I_q(V)$. The vector space V is naturally embedded in $\mathrm{Cl}(V,q)$ as the image of $V = \otimes^1 V$ under canonical projection $\pi_q : \mathcal{F}(V) \to \mathrm{Cl}(V,q)$.

Proposition 4.1.1. *Let $\mathcal{A}$ be an associative k-algebra with unit and $f : V \to \mathcal{A}$, be a linear map such that $f(v).f(v) = -q(v).1$ for all $v \in V$. Then f extends uniquely to a k-algebra homomorphism $\widetilde{f} : \mathrm{Cl}(V,q) \to \mathcal{A}$. Moreover $\mathrm{Cl}(V,q)$ is the unique associative k-algebra with this property.*

We consider the automorphism $\alpha : \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$ which extends the map $\alpha(v) = -v$ on V. Since α^2 is the identity, there is a decomposition $\mathrm{Cl}(V,q) = \mathrm{Cl}^0(V,q) \oplus \mathrm{Cl}^1(V,q)$ where $\mathrm{Cl}^i(V,q) = \{\phi \in \mathrm{Cl}(V,q) : \alpha(\phi) = (-1)^i \phi\}$ for $i = 0,1$ are the eigenspaces of α.

An element $\phi \in \mathrm{Cl}(V,q)$ is invertible if there is a $\psi \in \mathrm{Cl}(V,q)$ such that $\phi\psi = \psi\phi = \mathrm{Id}$, where Id is the identity in $\mathrm{Cl}(V,q)$. Then the set $\mathrm{Cl}^{\times}(V,q)$ of all invertible elements in $\mathrm{Cl}(V,q)$ forms a group containing all elements $v \in V$ such that $q(v) \neq 0$. Let $P(V,q)$ be the subgroup of $\mathrm{Cl}^{\times}(V,q)$ generated by all elements $v \in V$ with $q(v) \neq 0$.

Definition 4.1.2. The subgroup of $P(V,q)$ generated by all $v \in V$ with $q(v) = \pm 1$ is called the Pin group of (V,q) and is denoted by $\mathrm{Pin}(V,q)$. The Spin group of (V,q) is $\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap \mathrm{Cl}^0(V,q)$.

Let V be an n dimensional vector space over $\mathbb{R}$ which we identify with $\mathbb{R}^n$ by choosing a basis. Let $q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$, $r+s = n, 0 \leq r \leq n$ be a non-degenerate quadratic form on V. Then we denote $\mathrm{Spin}(V,q)$ by $\mathrm{Spin}(r,s)$. In particular when $s = 0$, i.e., when $q(x) = x_1^2 + \cdots + x_n^2$ then $\mathrm{Spin}(n,0)$ is denoted by $\mathrm{Spin}(n)$. We also denote $\mathrm{Cl}(V,q)$ by Cl_n and $\mathrm{Cl}^0(V,q)$ by Cl_n^0 for V and q as above and $s = 0$.

Definition 4.1.3. The complex spin representation of $\mathrm{Spin}(n)$ is the homomorphism $\tau_n : \mathrm{Spin}(n) \to \mathrm{GL}_{\mathbb{C}}(S)$, given by restricting an irreducible complex representation $\mathrm{Cl}_n \to \mathrm{Hom}_{\mathbb{C}}(S,S)$ to $\mathrm{Spin}(n) \subseteq \mathrm{Cl}_n^0 \subseteq \mathrm{Cl}_n$.

Note that the definition of τ_n does not depend on the choice of the (equivalence class of the) irreducible representation of the Clifford algebra Cl_n which is restricted.

Let G be $\mathrm{Spin}_0(n,1)$, the identity component of $\mathrm{Spin}(n,1)$. Then in the notation of Chapter 1 $K = \mathrm{Spin}(n)$ and $M = \mathrm{Spin}(n-1)$. We recall that G and K are respectively the universal two-fold coverings of $\mathrm{SO}_0(n,1)$ and $\mathrm{SO}(n)$. The rest of the chapter is a reproduction of the relevant part from [13].

If τ_n is the classical complex spin representation of K then $\dim \tau_n = \mathbb{C}^{2^{[n/2]}}$ where $[\cdot]$ denotes the integral part and:

(a) If n is even then τ_n splits into two irreducible components positive and negative half-spin representations $\tau_n = \tau_n^+ \oplus \tau_n^-$ and $\tau_n^{\pm}|_M = \sigma_{n-1}$, where σ_{n-1} is the spin representation of M.

(b) If n is odd then τ_n is irreducible and $\tau_n|_M = \sigma_{n-1}^+ \oplus \sigma_{n-1}^-$, where $\sigma_{n-1}^{\pm}$ are irreducible components of the spin representation σ_{n-1} of M.

We note that $\sigma_{n-1} = \tau_{n-1}$. Following [13] we use different notation to emphasize the group M or K of which this is a representation.

The Lie algebras of G and K are $\mathfrak{g} = \mathfrak{spin}(n,1) \cong \mathfrak{so}(n,1)$ and $\mathfrak{k} = \mathfrak{spin}(n) \cong \mathfrak{so}(n)$ respectively. We consider the element

$$H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{p},$$

where 0_{n-1} is the $(n-1)\times(n-1)$ zero matrix. Then $\mathfrak{a} := \{tH_0 \mid t \in \mathbb{R}\}$ is a Cartan subspace in $\mathfrak{p}$, and the corresponding analytic Lie subgroup $A = \{a_t \mid t \in \mathbb{R}\}$ where

$$a_t := \exp(tH_0) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix},$$

I_{n-1} being the $(n-1)\times(n-1)$ identity matrix. As usual we define $\alpha \in \mathfrak{a}^*$ by $\alpha(H_0) = 1$. Then $\Sigma = \{-\alpha, \alpha\}$, and the corresponding Weyl group $W = \{\pm \mathrm{Id}\}$. We shall use the identification of $\mathfrak{a}^*$ and $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{R}$ and $\mathbb{C}$ respectively as in Chapter 1. We have here $\mathfrak{n} = \mathfrak{g}_\alpha$, the unique positive root subspace and N the corresponding (abelian) analytic subgroup of G. Then the half-sum of positive roots ρ reduces simply to $\rho = \frac{n-1}{2}\alpha$, and will always be considered as a scalar $\rho = \frac{n-1}{2}$ by the above identification.

4.2 τ-Spherical Functions

Instead of an arbitrary element $\tau \in \widehat{K}$, from now on we will confine ourselves to the irreducible components of the complex spin representations τ_n. Precisely when n is even, then τ will denote one of $\{\tau_n^+, \tau_n^-\}$ and when n is odd then τ will denote τ_n. Also for n odd σ^+ and σ^- will denote respectively the representations σ_{n-1}^+ and σ_{n-1}^- and for n even σ will denote the representation σ_{n-1}. Henceforth in this chapter and in Chapter 5, 6, 7 we shall use the expression "n is even" and "n is odd" to distinguish between these two cases. For n even as $\widehat{M}(\tau)$ contains only one σ sometimes we will suppress σ and write $\phi^\tau_\lambda, \Phi^\tau_\lambda, \widehat{f}, \mathcal{A}f$ instead of $\phi^\tau_{\sigma,\lambda}, \Phi^\tau_{\sigma,\lambda}, \widehat{f}(\sigma,\cdot), \mathcal{A}_\sigma f$ respectively.

We recall that for an integer $n \geq 2$, the real hyperbolic space of dimension n is given by

$$\mathbb{H}^n(\mathbb{R}) = \{x \in \mathbb{R}^{n+1} \mid L(x,x) = -1, x_{n+1} > 0\}$$

where $x = (x_1, x_2, \ldots, x_{n+1})$ and

$$L(x, y) = y_1 x_1 + \cdots + y_n x_n - y_{n+1} x_{n+1}.$$

We realize $\mathbb{H}^n(\mathbb{R})$ as G/K instead of more well known $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$. Then the τ-radial functions for τ as above are radial sections of the spinor bundles.

It is clear from the above that (G, K, τ) is a Gelfand triple by virtue of the multiplicity free criterion for $\tau|_M$ when $\tau \in \widehat{K}$ is either τ_n^+ or τ_n^- if n is even and is τ_n if n is odd (see Chapter 2, Proposition 2.15).

The τ-spherical functions reduces to,

$$\Phi_\lambda^\tau(x) = \int_K e^{-(i\lambda+\rho)H(xk)} \tau(kK(xk)^{-1})\, dk \text{ when } n \text{ is even and}$$

$$\Phi_{\sigma^\pm,\lambda}^\tau(x) = 2\int_K e^{-(i\lambda+\rho)H(xk)} [\tau(k) \circ P_{\sigma^\pm} \circ \tau(K(xk)^{-1})]\, dk, \text{ when } n \text{ is odd.}$$

In the above $P_{\sigma^\pm}$ is the orthogonal projection of V_τ onto its $\sigma^\pm$-isotypical component of $V_\tau(\sigma^\pm) \cong V_{\sigma^\pm}$.

It follows from the action of the nontrivial Weyl group element on the principal series representations (see [40, Chapter VII]) that $\phi_\lambda^\tau(x) = \phi_{-\lambda}^\tau(x)$ when n is even and $\phi_{\sigma^+,\lambda}^\tau(x) = \phi_{\sigma^-,-\lambda}^\tau(x)$, $\phi_{\sigma^-,\lambda}^\tau(x) = \phi_{\sigma^+,-\lambda}^\tau(x)$ when n is odd. From this we get the corresponding properties of the τ-spherical transform: For n even, $\widehat{f}(\lambda) = \widehat{f}(-\lambda)$ and for n odd $\widehat{f}(\sigma^+, \lambda) = \widehat{f}(\sigma^-, -\lambda)$, $\widehat{f}(\sigma^-, \lambda) = \widehat{f}(\sigma^+, -\lambda)$.

If $\Omega_{\mathfrak{g}}$ denotes the *Casimir operator* in the enveloping algebra of $\mathfrak{g}$ then (see [13])

$$(-\Omega_{\mathfrak{g}})\Phi_{\sigma,\lambda}^\tau(\cdot) = \{\lambda^2 + \rho^2 - \frac{(n-1)(n-2)}{8}\}\Phi_{\sigma,\lambda}^\tau(\cdot) \text{ for each } \sigma \in \widehat{M}(\tau). \quad (4.2.0)$$

Also if D is the Dirac operator then we have (see [13])

$$\mathrm{D}^2 = -\Omega_{\mathfrak{g}} - \frac{n(n-1)}{8}\mathrm{Id}.$$

This shows that

$$\mathrm{D}^2\Phi_{\sigma,\lambda}^\tau(\cdot) = \lambda^2\Phi_{\sigma,\lambda}^\tau(\cdot) \text{ for each } \sigma \in \widehat{M}(\tau) \text{ and } \lambda \in \mathbb{C}.$$

Let $d\nu(\lambda) = \nu(\lambda)d\lambda$ denote the Plancherel measure on $\mathfrak{a}^*$ which is identified as $\mathbb{R}$. Here $d\lambda$ is the Lebesgue measure on $\mathbb{R}$ and the density $\nu(\lambda)$ is given by,

$$\nu(\lambda) = 2^{3-2n}[(\frac{n}{2}-1)!]^{-2}\lambda\coth(\pi\lambda)\prod_{j=1}^{\frac{n}{2}-1}[\lambda^2+j^2] \text{ for } n \text{ even and}$$

$$\nu(\lambda) = 2^{-2}\pi^{-1}[\frac{n-1}{2}(\frac{n-1}{2}+1)\cdots(n-2)]^{-2}\prod_{j=1}^{\frac{n-1}{2}}(\lambda^2+(j-\frac{1}{2})^2) \text{ for } n \text{ odd.}$$

We have the following inversion formula for τ-radial functions. We note that the discrete series representations of G does not appear in the formula (see [13]).

Theorem 4.2.1. *Let $F \in C^2(G,\tau,\tau)$. Then for n even and n odd respectively we have*

$$F(x) = \int_0^\infty \widehat{F}(\lambda)\Phi^\tau_\lambda(x)\,d\nu(\lambda) \text{ and} \tag{4.2.1}$$

$$F(x) = \sum_{\sigma\in\widehat{M}(\tau)}\int_0^\infty \widehat{F}(\sigma,\lambda)\Phi^\tau_{\sigma,\lambda}(x)\,d\nu(\lambda) = \int_{\mathbb{R}}\widehat{F}(\sigma^+,\lambda)\Phi^\tau_{\sigma^+,\lambda}(x)\,d\nu(\lambda). \tag{4.2.2}$$

Consequently for $f \in C^2_\tau(G)$ and n even and n odd respectively

$$f(x) = d_\tau\int_0^\infty \widehat{f}(\lambda)\phi^\tau_\lambda(x)\,d\nu(\lambda) \text{ and} \tag{4.2.3}$$

$$f(x) = d_\tau\sum_{\sigma\in\widehat{M}(\tau)}\int_0^\infty \widehat{f}(\sigma,\lambda)\phi^\tau_{\sigma,\lambda}(x)\,d\nu(\lambda) = d_\tau\int_{\mathbb{R}}\widehat{f}(\sigma^+,\lambda)\phi^\tau_{\sigma^+,\lambda}(x)\,d\nu(\lambda). \tag{4.2.4}$$

We also have the following Plancherel formula (see [13]):

Theorem 4.2.2. *Let $F \in C^2(G,\tau,\tau)$. Then for n even and n odd respectively we have*

$$\|F\|^2_{L^2(G,\tau,\tau)} = 2^{\frac{n}{2}-1}\int_0^\infty |\widehat{F}(\lambda)|^2 d\nu(\lambda) \text{ and}$$

$$\|F\|^2_{L^2(G,\tau,\tau)} = 2^{\frac{n-1}{2}}\int_0^\infty \left[|\widehat{F}(\sigma^+,\lambda)|^2+|\widehat{F}(\sigma^-,\lambda)|^2\right]d\nu(\lambda) = 2^{\frac{n-1}{2}}\int_{\mathbb{R}}|\widehat{F}(\sigma^+,\lambda)|^2 d\nu(\lambda).$$

Now we have

$$\|F\|^2_{L^2(G,\tau,\tau)} = \langle F,F\rangle = \int_G \mathrm{Tr}\left(F(x)F(x)^*\right)\,dx = \mathrm{Tr}\left(\int_G F(x)F(x)^*\,dx\right).$$

Therefore

$$\|F\|^2_{L^2(G,\tau,\tau)} = \operatorname{Tr}\left(\int_G F(x)G(x^{-1})\,dx\right) \text{ where } G(x) = F(x^{-1})^*.$$

This shows that (see Chapter 2)

$$\|F\|^2_{L^2(G,\tau,\tau)} = \operatorname{Tr}\left((F * G)(e)\right) = \frac{1}{d_\tau} f_{F*G}(e) = \frac{1}{d_\tau}(f_G * f_F)(e).$$

Hence $\|F\|^2_{L^2(G,\tau,\tau)} = \frac{1}{d_\tau}\int_G f_G(x) f_F(x^{-1})\,dx$. Since $f_G(x) = \overline{f_F(x^{-1})}$ it follows that

$$\|F\|^2_{L^2(G,\tau,\tau)} = \frac{1}{d_\tau}\|f_F\|^2_{L^2_\tau(G)}. \tag{4.2.5}$$

As a consequence of Plancherel Theorem 4.2.2 and equation (4.2.5) we have the following Plancherel Theorem for scalar valued functions:

Theorem 4.2.3. *Let $f \in C^2_\tau(G)$. Then for n even and n odd respectively we have*

$$\|f\|^2_{L^2_\tau(G)} = 2^{\frac{n}{2}-1} d_\tau \int_0^\infty |\widehat{f}(\lambda)|^2 d\nu(\lambda) \text{ and}$$

$$\|f\|^2_{L^2_\tau(G)} = 2^{\frac{n-1}{2}} d_\tau \int_0^\infty \left[|\widehat{f}(\sigma^+,\lambda)|^2 + |\widehat{f}(\sigma^-,\lambda)|^2\right] d\nu(\lambda) = 2^{\frac{n-1}{2}} d_\tau \int_{\mathbb{R}} |\widehat{f}(\sigma^+,\lambda)|^2 d\nu(\lambda).$$

Chapter 5

Schwartz Space Isomorphism and Paley-Wiener Theorems

5.1 Introduction

In this Chapter we shall prove two theorems which characterize τ-radial elements of two spaces on G, namely the L^p-Schwartz spaces $(0 < p \leq 2)$ and the space of compactly supported distributions, where G and τ are as in the previous chapter. These characterizations will be used to prove the main theorems.

Our starting point is the following Paley-Wiener theorem for τ-radial function which is proved in [13]. For $R > 0$ we define the Paley-Wiener space $PW_R(\mathbb{C})$ as the space of all entire functions $h : \mathbb{C} \to \mathbb{C}$ satisfying for each $N \in \mathbb{N}$

$$|h(\lambda)| \leq C_N(1 + |\lambda|)^{-N} e^{R|\Im\lambda|} \text{ for all } \lambda \in \mathbb{C}$$

for some constant $C_N > 0$ depending on N. We will denote the set of all even functions in $PW_R(\mathbb{C})$ by $PW_R(\mathbb{C})_e$. Also we let $PW(\mathbb{C}) = \cup_{R>0} PW_R(\mathbb{C})$ and $PW(\mathbb{C})_e = \cup_{R>0} PW_R(\mathbb{C})_e$. We topologize $PW_R(\mathbb{C})$ in the following way: A sequence $\{F_j\}$ in $PW_R(\mathbb{C})$ converges to 0 in $PW_R(\mathbb{C})$ if and only if for any polynomial P, $P(\lambda)F_j(\lambda)$ converges to 0 uniformly in some strip about $\Re\lambda = 0$. We endow $PW(\mathbb{C})$ with the inductive limit topology (see [24]).

We quote the following Paley-Wiener theorem from [13]:

Theorem 5.1.1 (Camporesi-Pedon)**.** *The following Paley-Wiener theorem is true for τ-radial functions:*

(a) *For n even, the map $F \mapsto \widehat{F}$ is a topological isomorphism between: $C_R^\infty(G, \tau, \tau)$ and $PW_R(\mathbb{C})_e$. As a consequence the map $f \mapsto \widehat{f}$ is a topological isomorphism between $C_{c,\tau}^\infty(G)_R$ and $PW_R(\mathbb{C})_e$.*

(b) *For n odd, the maps $F \mapsto \widehat{F}(\sigma^+, \cdot)$ and $F \mapsto \widehat{F}(\sigma^-, \cdot)$ both are topological isomorphisms between: $C_R^\infty(G, \tau, \tau)$ and $PW_R(\mathbb{C})$. As a consequence the maps $f \mapsto \widehat{f}(\sigma^+, \cdot)$ and $f \mapsto \widehat{f}(\sigma^-, \cdot)$ both are topological isomorphisms between $C_{c,\tau}^\infty(G)_R$ and $PW_R(\mathbb{C})$.*

From this it follows (as in the K-biinvariant case) that if n is even the map $F \mapsto \widehat{F}$ is a topological isomorphism between: $C_c^\infty(G, \tau, \tau)$ and $PW(\mathbb{C})_e$. Also if n is odd the maps $F \mapsto \widehat{F}(\sigma^+, \cdot)$, $F \mapsto \widehat{F}(\sigma^-, \cdot)$ are topological isomorphisms between: $C_c^\infty(G, \tau, \tau)$ and $PW(\mathbb{C})$.

Let $C_R^\infty(\mathbb{R})$ denote the set of infinitely differentiable functions on $\mathbb{R}$ supported in $[-R, R]$ and let $C_R^\infty(\mathbb{R})_e$ be the set of even functions in $C_R^\infty(\mathbb{R})$. From the euclidean Paley-Wiener Theorem (see [52]) and the slice-projection property of the Abel transform (see Section 2.3), it is clear that for n even $F \mapsto \mathcal{A}F$ is a topological isomorphism between $C_R^\infty(G, \tau, \tau)$ and $C_R^\infty(\mathbb{R})_e$; also for n odd $F \mapsto \mathcal{A}_{\sigma^+}F$ and $F \mapsto \mathcal{A}_{\sigma^-}F$ are topological isomorphsims between $C_R^\infty(G, \tau, \tau)$ and $C_R^\infty(\mathbb{R})$.

We shall first take up the Schwartz space isomorphism theorem.

5.2 Schwartz Space Isomorphism Theorem

Our proof for L^p-Schwartz space isomorphism theorem for τ-radial function is an adaptation of Anker's proof of the corresponding theorem for K-biinvariant functions (see [1]) which cleverly uses the Paley-Wiener Theorem to avoid the intricacies of asymptotic expansion of spherical functions. The L^p-Schwartz spaces $C^p(G, \tau, \tau)$ and $C_\tau^p(G)$ are defined in Section 2.1.

We recall the following properties of the τ-spherical functions for both n even and odd (see Chapter 1, Chapter 4 and equation (5.3.3) below):

(1) $\|\Phi_{\sigma,\lambda}^\tau(x)\|_{\mathrm{End}V_\tau} \leq C\phi_{i\Im\lambda}(x) \leq C(1 + \sigma(x))^r e^{-(\rho - |\Im\lambda|)\sigma(x)}$,
for all $x \in G, \lambda \in \mathbb{C}$ and $\sigma \in \widehat{M}(\tau)$,

(2) $\|\Phi_{\sigma,\lambda}^\tau(E_1; x; E_2)\|_{\mathrm{End}V_\tau} \leq C(1+|\lambda|)^{r_1}\phi_{i\Im\lambda}(x) \leq C(1+|\lambda|)^{r_1}(1+\sigma(x))^r e^{(-\rho+|\Im\lambda|)\sigma(x)}$
for each $E_1, E_2 \in \mathcal{U}(\mathfrak{g})$ and for x, λ, σ as above,

(3) $(-\Omega_\mathfrak{g})\Phi_{\sigma,\lambda}^\tau(\cdot) = \left\{\lambda^2 + \rho^2 - \frac{(n-1)(n-2)}{8}\right\} \Phi_{\sigma,\lambda}^\tau(\cdot)$ for λ and σ as above.

We also recall that for both the cases n even and odd the Plancherel density ν satisfies $|\nu(\lambda)| \leq C(1+|\lambda|)^b$ for some constant $b > 0$, for all $\lambda \in \mathbb{R}$. It is well known that $PW(\mathbb{C})$ (respectively $PW(\mathbb{C})_e$) is a dense subspace of $\mathcal{S}(\mathbb{R})$ (respectively of $\mathcal{S}(\mathbb{R})_e$).

We define $\mathcal{S}(S_p)$ to be the set of all functions $h : S_p \to \mathbb{C}$ which are continuous on S_p, holomorphic on S_p° (when $p = 2$ then the function is simply C^∞ on $S_2 = \mathbb{R}$) and satisfies $\sup_{\lambda \in S_p}(1 + |\lambda|)^r |\frac{d^m}{d\lambda^m} h(\lambda)| < \infty$, for all $r, m \in \mathbb{N} \cup \{0\}$. Let $\mathcal{S}(S_p)_e$ and $\mathcal{S}(S_p)_o$ denote the subspaces of $\mathcal{S}(S_p)$ consisting of even and odd functions respectively. Topologized by the seminorms above it can be verified that $\mathcal{S}(S_p), \mathcal{S}(S_p)_e$ and $\mathcal{S}(S_p)_o$ are Fréchet spaces. With this preparation we are ready to state the theorems.

Theorem 5.2.1. *Let $0 < p \leq 2$ be fixed. Then for any $\sigma \in \widehat{M}(\tau)$ $F \mapsto \widehat{F}(\sigma, \cdot)$ is a topological isomorphism between $C^p(G, \tau, \tau)$ and $S(S_p)_e$ when n is even and between $C^p(G, \tau, \tau)$ and $S(S_p)$ when n is odd.*

Proof. Suppose n is even. Since in this case $\tau|_M$ contains unique $\sigma \in \widehat{M}$ we omit σ from the notation. Let $F \in C^p(G, \tau, \tau)$. It is clear from Proposition 3.1.4 and Proposition 3.1.5 that $\widehat{F}(\lambda)$ is analytic on S_p° and continuous on the boundary of S_p. As $\phi_\lambda^\tau = \phi_{-\lambda}^\tau$ it follows that $\widehat{F}$ is even. We shall first show that $\sup_{\lambda \in S_p}(1 + |\lambda|)^r \left| \frac{d^s}{d\lambda^s} \widehat{F}(\lambda) \right| < \infty$ for any nonnegative integers r and s, which will prove $\widehat{F} \in \mathcal{S}(S_p)_e$. For this it is sufficient to show that

$$\sup_{\lambda \in S_p} \left| P(\frac{d}{d\lambda}) \left[\left\{ \lambda^2 + \rho^2 - \frac{(n-1)(n-2)}{8} \right\}^s \widehat{F}(\lambda) \right] \right| < \infty \qquad (5.2.0)$$

for any polynomial P and any nonnegative integer s. Indeed if we assume (5.2.0), then taking P to be a constant and using the continuity of $\widehat{F}$ we get

$$\sup_{\lambda \in S_p} (1 + |\lambda|)^r |\widehat{F}(\lambda)| < \infty$$

for any $r > 0$. Using this and

$$|(\lambda^2 + c^2)^s \frac{d}{d\lambda} \widehat{F}(\lambda)| \leq |2s\lambda(\lambda^2 + c^2)^{s-1} \widehat{F}(\lambda)| + \left| \frac{d}{d\lambda} \left[(\lambda^2 + c^2)^s \widehat{F}(\lambda) \right] \right|,$$

for any constant c, we have $\sup_{\lambda \in S_p} |(\lambda^2 + c^2)^s \frac{d}{d\lambda} \widehat{F}(\lambda)| < \infty$. Estimates defining the seminorms of $\mathcal{S}(S_p)$ involving higher derivatives will follow in a similar fashion.

Now using the fact that the Casimir operator $\Omega_\mathfrak{g}$ is formally self-adjoint with respect the L^2 inner product and (4.2.0) we have,

$$\begin{aligned}
& P(\tfrac{d}{d\lambda})\left[\left\{\lambda^2+\rho^2-\tfrac{(n-1)(n-2)}{8}\right\}^s \widehat{F}(\lambda)\right] \\
= \; & P(\tfrac{d}{d\lambda})\left[\left\{\lambda^2+\rho^2-\tfrac{(n-1)(n-2)}{8}\right\}^s \tfrac{1}{d_\tau}\int_G \operatorname{Tr}\left(F(x)\circ \Phi_\lambda^\tau(x^{-1})\right)dx\right] \\
= \; & P(\tfrac{d}{d\lambda})\left[\tfrac{1}{d_\tau}\int_G \operatorname{Tr}\left(F(x)\circ (-\Omega_{\mathfrak{g}})^s\Phi_\lambda^\tau(x^{-1})\right)dx\right] \\
= \; & P(\tfrac{d}{d\lambda})\left[\tfrac{1}{d_\tau}\int_G \operatorname{Tr}\left((-\Omega_{\mathfrak{g}})^s F(x)\circ \Phi_\lambda^\tau(x^{-1})\right)dx\right] \\
= \; & \tfrac{1}{d_\tau}\int_G \operatorname{Tr}\left((-\Omega_{\mathfrak{g}})^s F(x)\circ P(\tfrac{d}{d\lambda})\Phi_\lambda^\tau(x^{-1})\right)dx.
\end{aligned}$$

Using the polar decomposition (see Chapter 1, equation (1.0.4)) we get

$$\begin{aligned}
& \left|P(\tfrac{d}{d\lambda})\left[\left\{\lambda^2+\rho^2-\tfrac{(n-1)(n-2)}{8}\right\}^s \widehat{F}(\lambda)\right]\right| \\
\leq \; & C\int_{K\times\overline{\mathbb{R}^+}\times K}\left|\operatorname{Tr}\left((-\Omega_{\mathfrak{g}})^s F(k_1a_tk_2)\circ (P(\tfrac{d}{d\lambda})\Phi_\lambda^\tau(k_2^{-1}a_{-t}k_1^{-1}))\right)\right| e^{2\rho t}\,dk_1\,dt\,dk_2 \\
\leq \; & C\int_{K\times\overline{\mathbb{R}^+}\times K}\|(-\Omega_{\mathfrak{g}})^s F(k_1a_tk_2)\|_{\operatorname{End}V_\tau}\;\;\left\|P(\tfrac{d}{d\lambda})\Phi_\lambda^\tau(k_2^{-1}a_{-t}k_1^{-1})\right\|_{\operatorname{End}V_\tau} e^{2\rho t}\,dk_1\,dt\,dk_2.
\end{aligned}$$

We have (see [32])

$$\left\|P(\frac{d}{d\lambda})\Phi_\lambda^\tau(k_2^{-1}a_{-t}k_1^{-1})\right\|_{\operatorname{End}V_\tau} \leq C(1+|t|)^{r_1}e^{(|\Im\lambda|-\rho)|t|}\leq C(1+|t|)^{r_2}e^{(\frac{2}{p}-2)\rho|t|}.$$

Also since $F\in C^p(G,\tau,\tau)$ we have

$$\|(-\Omega_{\mathfrak{g}})^s F(k_1a_tk_2)\|_{\operatorname{End}V_\tau}\leq C(1+t)^{-(r_2+2)}e^{-\frac{2}{p}\rho t}\beta(F) \text{ for } t\geq 0$$

where β is a seminorm of the L^p-Schwartz space function. Therefore, we have

$$\sup_{\lambda\in S_p}\left|P(\frac{d}{d\lambda})\left[\left\{\lambda^2+\rho^2-\frac{(n-1)(n-2)}{8}\right\}^s \widehat{F}(\lambda)\right]\right|\leq C\beta(F).$$

This shows that $\widehat{F}\in\mathcal{S}(S_p)_e$ and the map $F\mapsto\widehat{F}$ is continuous.

Since $C^p(G,\tau,\tau)\subseteq L^2(G,\tau,\tau)$ by Plancherel theorem (Theorem 4.2.2) it follows that $F\mapsto\widehat{F}$ is injective. Now we shall show that the map $F\mapsto\widehat{F}$ from $C^p(G,\tau,\tau)$ to $\mathcal{S}(S_p)_e$ is surjective and the inverse map is continuous. We note that $C_c^\infty(G,\tau,\tau)$ and $PW(\mathbb{C})_e$ are dense in $C^p(G,\tau,\tau)$ and $\mathcal{S}(S_p)_e$ respectively. Therefore to complete the theorem it is sufficient to show that given a seminorm β of $C^p(G,\tau,\tau)$, there is a seminorm γ on $\mathcal{S}(S_p)_e$ such that

$$\beta(F) \leq C\gamma(\widehat{F}), \text{ for all } F \in C_c^\infty(G,\tau,\tau).$$

We shall break the remainder of the proof in a few steps and use the following notation. Let $\gamma_{r,s}^{(p)}(f) = \sup_{\lambda\in S_p}(1+|\lambda|)^r|\frac{d^s}{d\lambda^s}f(\lambda)|$ for $f \in \mathcal{S}(S_p)_e$ and

$$\beta(F) = \sup_{x\in G}(1+\sigma(x))^r\|F(E_1;x;E_2)\|_{\mathrm{End}V_\tau}e^{\frac{2}{p}\rho\sigma(x)} \text{ for } F \in C^p(G,\tau,\tau).$$

From now on we shall use the notation $\|\cdot\|$ to mean $\|\cdot\|_{\mathrm{End}V_\tau}$.

For each positive integer l, let $C_l = [-l\rho, l\rho]$ and $G_l = K\exp(C_lH_0)K$. Then G is disjoint union of $G_2, G_3\setminus G_2, G_4\setminus G_3, \cdots$.

Step 1: From inversion formula of C_c^∞ function (Theorem 4.2.1) we have $F(E_1;x;E_2) = \int_0^\infty \widehat{F}(\lambda)\Phi_\lambda^\tau(E_1;x;E_2)\nu(\lambda)\,d\lambda$. Therefore using the estimates of $\Phi_\lambda(E_1;x;E_2)$ and $\nu(\lambda)$ given above we have,

$$\begin{aligned}\|F(E_1;x;E_2)\| &\leq \int_0^\infty |\widehat{F}(\lambda)|\|\Phi_\lambda^\tau(E_1;x;E_2)\|\nu(\lambda)\,d\lambda\\ &\leq C\phi_0(x)\int_0^\infty(1+|\lambda|)^{b_1}|\widehat{F}(\lambda)|\,d\lambda.\end{aligned}$$

Now, $\sup_{x\in G_2}(1+\sigma(x))^r\|F(E_1;x;E_2)\|e^{\frac{2}{p}\rho\sigma(x)}$

$$\begin{aligned}&\leq C\sup_{x\in G_2}(1+\sigma(x))^r\phi_0(x)e^{\frac{2}{p}\rho\sigma(x)}\int_0^\infty(1+|\lambda|)^b|\widehat{F}(\lambda)|\,d\lambda\\ &\leq C\sup_{\lambda\in\mathbb{R}}(1+|\lambda|)^{b+2}|\widehat{F}(\lambda)| \text{ (since } G_2 \text{ is compact)}\\ &= C\gamma_{b+2,0}^{(2)}(\widehat{F}).\end{aligned}$$

Hence we have

$$\sup_{x\in G_2}(1+\sigma(x))^r\|F(D;x;E)\|\,e^{\frac{2}{p}\rho\sigma(x)} \leq C\gamma_{b+2,0}^{(2)}(\widehat{F}). \qquad (5.2.1)$$

Step 2: Let $w \in C^{\infty}(\mathbb{R})$ be such that

$$w = \begin{cases} 0 & \text{on} \quad (-\infty, 0] \\ \\ 1 & \text{on} \quad [1, \infty). \end{cases}$$

We define $w_l(t) = w(l + \frac{t}{\rho})w(l - \frac{t}{\rho})$. Then w_l is an even function and

$$w_l = \begin{cases} 1 & \text{on} \quad C_{l-1} \\ \\ 0 & \text{on} \quad C_l^c. \end{cases}$$

Here C_l^c denotes the complement of C_l in $\mathbb{R}$. We have $\mathcal{A}F = w_l\mathcal{A}F + (1 - w_l)\mathcal{A}F$. Let $g_l(t) = (1 - w_l(t))\mathcal{A}F(t)$. Since $F \in C_c^{\infty}(G, \tau, \tau)$, $\mathcal{A}F \in C_c^{\infty}(\mathbb{R})_e$ and hence $g_l \in C_c^{\infty}(\mathbb{R})_e$. Therefore, there is $F_l \in C_c^{\infty}(G, \tau, \tau)$ such that $\mathcal{A}F_l = g_l$.

Step 3: As $\mathcal{A}F = w_l\mathcal{A}F + g_l$, $\mathcal{A}F$ and g_l are equal on C_l^c. That is $\mathcal{A}F$ and g_l may differ only on C_l. This shows that F and F_l may differ only on G_l. Hence for $x \in G_{l+1} \setminus G_l$, $F(x) = F_l(x)$. Also arguing as in Step 1 we have,

$$\|F_l(E_1; x; E_2)\| \leq C\phi_0(x)\gamma^{(2)}_{b_1+2,0}(\widehat{F}).$$

Step 4: Let ∇ denotes the Laplacian on $\mathbb{R}$. Then,

$$\begin{aligned}
\gamma^{(2)}_{b_1+2,0}(\widehat{F_l}) &= \sup\nolimits_{\lambda\in\mathbb{R}}(1+|\lambda|)^{b_1+2}|\widehat{F_l}(\lambda)| \\
&= \sup\nolimits_{\lambda\in\mathbb{R}}(1+|\lambda|)^{b_1+2}|\widetilde{g_l}(\lambda)| \text{ since } \widehat{F_l}=\widetilde{g_l} \\
&= \sup\nolimits_{\lambda\in\mathbb{R}}(1+|\lambda|)^{b_1+2}\left|\int_{\mathbb{R}} g_l(t)e^{-i\lambda t}dt\right| \\
&\leq C\sup\nolimits_{\lambda\in\mathbb{R}}\sum_{k=0}^{b_1+2}\left|\int_{\mathbb{R}} g_l(t)\nabla^k e^{-i\lambda t}dt\right| \\
&= C\sup\nolimits_{\lambda\in\mathbb{R}}\sum_{k=0}^{b_1+2}\left|\int_{\mathbb{R}} \nabla^k g_l(t)e^{-i\lambda t}dt\right| \\
&\leq C\sum_{k=0}^{b_1+2}\int_{\mathbb{R}}\left|\nabla^k g_l(t)\right|dt \\
&\leq C\sum_{k=0}^{b_1+2}\sup\nolimits_{t\in\mathbb{R}}(1+|t|)^2\left|\nabla^k g_l(t)\right| \\
&= C\sum_{k=0}^{b_1+2}\sup\nolimits_{t\in\mathbb{R}}\left[(1+|t|)^2\left|\nabla^k\left\{(1-w_l(t))\mathcal{A}F(t)\right\}\right|\right] \\
&= C\sum_{k=0}^{b_1+2}\sup\nolimits_{\mathbb{R}\setminus C_{l-1}}\left[(1+|t|)^2\left|\nabla^k\left\{(1-w_l(t))\mathcal{A}F(t)\right\}\right|\right] \\
&\leq C\sum_{k=0}^{b_1+2}\sup\nolimits_{\mathbb{R}\setminus C_{l-1}}(1+|t|)^2|\nabla^k\mathcal{A}F(t)|.
\end{aligned}$$

The last inequality follows as $1-w_l$ and all its derivatives are bounded.

Step 5: Now $\sup_{x\in G_{l+1}\setminus G_l}(1+\sigma(x))^r\|F(E_1;x;E_2)\|e^{\frac{2}{p}\rho\sigma(x)}$

$$\begin{aligned}
&= \sup\nolimits_{x\in G_{l+1}\setminus G_l}(1+\sigma(x))^r\|F_l(E_1;x;E_2)\|e^{\frac{2}{p}\rho\sigma(x)} \\
&\leq \sup\nolimits_{x\in G_{l+1}\setminus G_l}(1+\sigma(x))^r e^{(\frac{2}{p}-1)\rho\sigma(x)}\gamma^{(2)}_{b_1+2,0}(\widehat{F_l}) \text{ (by Step 3)} \\
&\leq Cl^{r_1}e^{(\frac{2}{p}-1)\rho l}\gamma^{(2)}_{b_1+2,0}(\widehat{F_l}) \\
&\leq C\sum_{k=0}^{b_1+2}\sup\nolimits_{\mathbb{R}\setminus C_{l-1}}(1+|t|)^{r_1+2}e^{(\frac{2}{p}-1)\rho t}\left|\nabla^k\mathcal{A}F(t)\right| \text{ (by Step 4)} \\
&\leq C\sum_{m=0}^{r_1+2}\int_{\mathbb{R}}(1+|\lambda|)^{b_1+2}\left|\nabla^m\widehat{F}(\lambda+i(\tfrac{2}{p}-1)\rho)\right|d\lambda \\
&\leq C\sum_{m=0}^{r_1+2}\sup\nolimits_{\lambda\in S_p}(1+|\lambda|)^{b_1+4}|\nabla^m\widehat{F}(\lambda)|.
\end{aligned}$$

The last but one inequality follows from the fact that

$$P(t)e^{(\frac{2}{p}-1)\rho t}Q(\frac{\partial}{\partial t})g(t) = C\int_{\mathbb{R}} P(i\frac{\partial}{\partial\lambda})\left\{Q(i\lambda-(\frac{2}{p}-1)\rho)h(\lambda+i(\frac{2}{p}-1)\rho)\right\}e^{i\lambda t}dt$$

where $g(t) = c\int_{\mathbb{R}} h(\lambda)e^{i\lambda t}d\lambda$ and P, Q are polynomials. Hence we have

$$\sup_{x\in G_{l+1}\setminus G_l}(1+\sigma(x))^r\|F(E_1;x;E_2)\|e^{\frac{2}{p}\rho\sigma(x)} \le C\sum_{m=0}^{r_1+2}\sup_{\lambda\in S_p}(1+|\lambda|)^{b_1+4}|\nabla^m\widehat{F}(\lambda)|. \tag{5.2.2}$$

Therefore, it follows from equation (5.2.1) and equation (5.2.2) that

$$\sup_{x\in G}(1+\sigma(x))^r\|F(D;x;E)\|e^{\frac{2}{p}\rho\sigma(x)} \le C\sum_{m=0}^{r_1+2}\sup_{\lambda\in S_p}(1+|\lambda|)^{b_1+4}|\nabla^m\widehat{F}(\lambda)|$$

where C is independent of F. Hence

$$\beta(F) \le C\gamma(\widehat{F}) \text{ for all } F\in C_c^\infty(G,\tau,\tau) \text{ for some seminorm } \gamma \text{ on } \mathcal{S}(S_p)_e.$$

This completes the proof for n even.

Similarly for n odd we can prove that $F\mapsto\widehat{F}(\sigma^+,\cdot)$ and $F\mapsto\widehat{F}(\sigma^-,\cdot)$ are topological isomorphisms between $C^p_\tau(G)$ and $\mathcal{S}(S_p)$. □

As an immediate consequence we have the following corollary:

Corollary 5.2.2. *Let $0<p\le 2$ be fixed. Then for any $\sigma\in\widehat{M}(\tau)$ $f\mapsto\widehat{f}(\sigma,\cdot)$ is a topological isomorphism between $C^p_\tau(G)$ and $S(S_p)_e$ when n is even and between $C^p_\tau(G)$ and $S(S_p)$ when n is odd.*

Remark 5.2.3. The theorem can also be proved relating τ-spherical transform to Jacobi transform (see [13]) and using the corresponding theorems for Jacobi transform (see [26]). In [13] Camporesi-Pedon proved the case $p=2$ of the theorem above using Jacobi transform.

5.3 Paley-Wiener Theorem for Distributions

Let G, K, τ be as in the previous section. We associate a locally integrable function f on $\mathbb{R}$, with a distribution W_f on $\mathbb{R}$ in the following way: For $g\in C_c^\infty(\mathbb{R})$,

$$W_f(g) = d_\tau^2\int_0^\infty f(\lambda)g(\lambda)\,d\nu(\lambda) \text{ if } n \text{ is even and} \tag{5.3.1}$$

$$W_f(g) = d_\tau^2 \int_{\mathbb{R}} f(\lambda) g(\lambda)\, d\nu(\lambda) \text{ if } n \text{ is odd,} \tag{5.3.2}$$

where $d\nu(\lambda) = \nu(\lambda) d\lambda$ is the Plancherel measure for τ-radial function on G.

A distribution W on $\mathbb{R}$ is called even if $W(f) = W(f_e)$ where $f_e(x) = (f(x) + f(-x))/2$. Let $C^\infty(\mathbb{R})'$ and $C^\infty(\mathbb{R})'_e$ denote the set of compactly supported and compactly supported even distributions on $\mathbb{R}$ respectively.

Definition 5.3.1. Let $W \in C_\tau^2(G)'$. For any $\sigma \in \widehat{M}(\tau)$, we define its τ-spherical transform $\widehat{W}(\sigma, \cdot) = \widehat{W}_\sigma$ by

$$\widehat{W}_\sigma(\widehat{f}(\sigma, \cdot)) = W(f).$$

We recall that when n is even $\tau|_M$ contains a unique $\sigma \in \widehat{M}$. Therefore in this case $\widehat{W}_\sigma$ will be denoted by $\widehat{W}$ and the definition above reduces to $\widehat{W}(\widehat{f}) = W(f)$. Note that in this case (i.e. when n is even) $f \mapsto \widehat{f}$ is an isomorphism between $C_\tau^2(G)$ and $\mathcal{S}(\mathbb{R})_e$. We extend $\widehat{W}$ on $\mathcal{S}(\mathbb{R})$ as for any $g \in \mathcal{S}(\mathbb{R})$, $\widehat{W}(g) = \widehat{W}(g_e)$. This makes $\widehat{W}$ an even tempered distribution on $\mathbb{R}$.

When n is odd we note that $f \mapsto \widehat{f}(\sigma, \cdot)$ is an isomorphism between $C_\tau^2(G)$ and $\mathcal{S}(\mathbb{R})$ for any $\sigma \in \widehat{M}(\tau)$. Therefore in this case $\widehat{W}_\sigma$ defines a tempered distribution on $\mathbb{R}$. We also note that

$$\widehat{W}_{\sigma^+}\left(\widehat{f}(\sigma^-, \cdot)\right) = \widehat{W}_{\sigma^-}\left(\widehat{f}(\sigma^+, \cdot)\right)$$

which can be verified by noting that for any $f \in C_\tau^2(G)$ (with n odd) there exists $g \in C_\tau^2(G)$ such that $\widehat{g}(\sigma^+, \cdot) = \widehat{f}(\sigma^-, \cdot)$ and consequently $\widehat{g}(\sigma^-, \cdot) = \widehat{f}(\sigma^+, \cdot)$.

It is easy to verify that the Definition 5.3.1 matches with Definition 2.2.3 when $W \in C_\tau^\infty(G)'$. Here we indicate the proof. It is enough to prove that the two definitions coincide for functions in $C_{c,\tau}^\infty(G)$ as $C_{c,\tau}^\infty(G)$ is dense in $C_\tau^\infty(G)'$. When n is even, for a function w in $C_{c,\tau}^\infty(G)$ using Theorem 4.2.1 we have

$$\int_G w(x) f^\vee(x)\, dx = d_\tau^2 \int_0^\infty \widehat{w}(\lambda) \widehat{f}(\lambda)\, d\nu(\lambda),$$

for every $f \in C_\tau^2(G)$ where we recall that $f^\vee(x) = f(x^{-1})$. The left hand side of this identity is $w(f)$ when w is interpreted as a tempered distribution (see Chapter 2) and hence is the same as $\widehat{w}(\widehat{f})$ by Definition 5.3.1. Here $\widehat{w}$ is the τ-spherical transform of w interpreted as a tempered distribution. Again interpreting the function $\widehat{w}$ (which is the τ-spherical transform of the function w) as a distribution on $\mathbb{R}$, the right hand side is its action on the function $\widehat{f}$ (see equation (5.3.1)) which is in the Schwartz space on $\mathbb{R}$. Thus the two definitions of τ-spherical transform agree for functions in $C^\infty_{c,\tau}(G)$ when n is even. Similarly for n odd we can show that the definitions coincide for $W \in C^\infty_\tau(G)'$.

Lemma 5.3.2. *For $W \in C^2_\tau(G)'$, $f \in C^2_\tau(G)$ and $h \in C^\infty_{c,\tau}(G)$, $W(h * f) = (W * h)(f)$.*

Proof. We have $(h * f)(x) = \int_G h(xy)f(y^{-1})\,dy = \int_G (L(y)h^\vee)^\vee(x)f^\vee(y)\,dy$. Therefore,

$$W(h * f) = \int_G W\left((L(y)h^\vee)^\vee\right) f^\vee(y)\,dy = \int_G (W * h)(y)f^\vee(y)\,dy = (W * h)(f).$$

□

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *rapidly decreasing* if for each $N \in \mathbb{N}$ there is $C_N > 0$ such that $|f(\lambda)| \leq C_N(1 + |\lambda|)^{-N}$ for all $\lambda \in \mathbb{C}$. Also a function $f : \mathbb{C} \to \mathbb{C}$ is called *slowly increasing* if there exists a nonnegative integer M such that $|f(\lambda)| \leq C(1 + |\lambda|)^M$ for all $\lambda \in \mathbb{C}$. We recall that the spaces PW^D and PW^D_e are Paley-Wiener spaces for distributions defined in Section 2.2.

We prove the following *Paley-Wiener theorem* for τ-radial distributions.

Theorem 5.3.3. *For any $\sigma \in \widehat{M}(\tau)$, the τ-spherical transform $W \mapsto \widehat{W}_\sigma$ is a topological isomorphisms between $C^\infty_\tau(G)'$ and PW^D_e when n is even and between $C^\infty_\tau(G)'$ and PW^D when n is odd.*

Proof. We shall deal with cases n even and odd separately.

(a) Suppose n is even. As mentioned above in this case $\tau|_M$ contains a unique $\sigma \in \widehat{M}$ and we will omit σ from the notation. We take a W_1 in PW^D_e. Using W_1 we define a tempered τ-radial distribution $W : C^2_\tau(G) \to \mathbb{C}$ by $W(f) = d^2_\tau \int_0^\infty W_1(\lambda)\widehat{f}(\lambda)\,d\nu(\lambda)$. Since $\nu(\lambda)$ has polynomial growth on $\mathbb{R}$ and $W_1(\lambda)$ is slowly increasing, by dominated convergence theorem it follows that $f \mapsto W(f)$ is continuous. We shall show that W is actually compactly supported. For this it is sufficient to show that $W * h \in C^\infty_{c,\tau}(G)$, for all $h \in C^\infty_{c,\tau}(G)$ (see [20, Theorem 1]). We note that for all $h \in C^\infty_{c,\tau}(G)$, $W * h$ is a tempered distribution and $\widehat{W * h}$ is an even tempered distribution on $\mathbb{R}$ (since by Corollary 5.2.2, any element $f \in \mathcal{S}(\mathbb{R})_e$, can be written as $f(\lambda) = \widehat{g}(\lambda)$ for some $g \in C^2_\tau(G)$).

For $f \in C^2_\tau(G)$ and h as above we have,

$$W(h * f) = d^2_\tau \int_0^\infty W_1(\lambda)\widehat{h}(\lambda)\widehat{f}(\lambda)\,d\nu(\lambda).$$

Therefore by Lemma 5.3.2

$$\widehat{W * h}(\widehat{f}) = d_\tau^2 \int_0^\infty W_1(\lambda)\widehat{h}(\lambda)\widehat{f}(\lambda)\, d\nu(\lambda).$$

Let $g(\lambda) = W_1(\lambda)\widehat{h}(\lambda), \lambda \in \mathbb{C}$. Then $\widehat{W * h}(\widehat{f}) = g(\widehat{f})$ for all $f \in C_\tau^2(G)$, i.e., $\widehat{W * h} = g$ (where equality is in the sense of distribution). Hence, $\widehat{W * h}(\lambda) = g(\lambda)$. Since W_1 is slowly increasing and $\widehat{h}$ is rapidly decreasing, g is rapidly decreasing. Again as both of them are of exponential type, so is g. Moreover g is even. Therefore by Theorem 5.1.1, $W * h \in C_{c,\tau}^\infty(G)$. This proves that W is a compactly supported distribution.

We shall show that $\widehat{W}(\lambda) = W_1(\lambda)$ for all $\lambda \in \mathbb{C}$. Since W is compactly supported, for $h(\not\equiv 0) \in C_{c,\tau}^\infty(G)$, we have $\widehat{W * h}(\lambda) = \widehat{W}(\lambda)\widehat{h}(\lambda)$ and hence $\widehat{W}(\lambda)\widehat{h}(\lambda) = W_1(\lambda)\widehat{h}(\lambda)$, i.e., $[\widehat{W}(\lambda) - W_1(\lambda)]\widehat{h}(\lambda) = 0$. Since both the factors are entire and $h \neq 0$, $\widehat{W}(\lambda) = W_1(\lambda)$, for all $\lambda \in \mathbb{C}$.

For the converse we take an element $W \in C_\tau^\infty(G)'$. Then $\widehat{W}(\lambda) = \widehat{W}(-\lambda)$, as $\phi_\lambda^\tau(x) = \phi_{-\lambda}^\tau(x)$.

Since, $W : C_\tau^\infty(G) \to \mathbb{C}$ is continuous with respect to the topology of $C_\tau^\infty(G)$, there exists $X_1, X_2, \cdots X_r \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and a compact set Ω in G such that for any $f \in C_\tau^\infty(G)$,

$$|W(f)| \leq C \sum_{i=1}^r \sup_{x\in\Omega} |f(X_i; x)| \text{ for some } C > 0.$$

Now

$$\begin{aligned}
& \sup\nolimits_{x\in\Omega} |\phi_\lambda^\tau(X_i; x)| \\
= \;& \sup\nolimits_{x\in\Omega} \left| \tfrac{d}{dt}\Big|_{t=0} \phi_\lambda^\tau(\exp(tX_i)x)\right| \\
= \;& \sup\nolimits_{x\in\Omega} \left| \tfrac{d}{dt}\Big|_{t=0} \int_K e^{-(i\lambda+\rho)H(\exp(tX_i)xk)} \chi_\tau[kK(\exp(tX_i)xk)^{-1}]\, dk\right| \\
= \;& \sup\nolimits_{x\in\Omega} \Big| \int_K \tfrac{d}{dt}\left(e^{-(i\lambda+\rho)H(\exp(tX_i)xk)}\right) \chi_\tau[kK(\exp(tX_i)xk)^{-1}]\Big|_{t=0} \\
+ \;& \int_K e^{-(i\lambda+\rho)H(\exp(tX_i)xk)} \tfrac{d}{dt}\left(\chi_\tau[kK(\exp(tX_i)xk)^{-1}]\right)\Big|_{t=0} dk\Big| \\
\leq \;& C_1(\sup\nolimits_{x\in\Omega} |\phi_\lambda(X_i; x)| + \sup\nolimits_{x\in\Omega} |\phi_\lambda(x)|) \\
\leq \;& C_2((1 + |\lambda|)^d e^{R|\Im\lambda|} + e^{R|\Im\lambda|}),
\end{aligned}$$

for some positive constants C_1, C_2, R, d.

In the last step we have used the inequality (see [28, Proposition 4.6.2])

$$|\phi_\lambda(X_i;x)| \leq C(1+|\lambda|)^d \phi_{i\Im\lambda}(x) \text{ for some constants } C, d > 0. \tag{5.3.3}$$

Therefore $\sup_{x\in\Omega}|\phi^\tau_\lambda(X_i;x)| \leq a(1+|\lambda|)^d e^{R|\Im\lambda|}$. This shows that $|\widehat{W}(\lambda)| \leq C(1+|\lambda|)^M e^{R|\Im\lambda|}$, for some $C, M, R > 0, (M \in \mathbb{Z})$.

We shall show that $\lambda \mapsto \widehat{W}(\lambda)$ is entire. Firstly $\lambda \mapsto \widehat{W}(\lambda)$ is continuous (since, $\lambda_n \to \lambda_0$ implies $\phi^\tau_{\lambda_n} \to \phi^\tau_{\lambda_0}$ in the topology of $C^\infty_\tau(G)$. So $\widehat{W}(\lambda_n) \to \widehat{W}(\lambda_0)$). Let Γ be a closed rectangular path in $\mathbb{C}$. We shall show that $\int_\Gamma \widehat{W}(\lambda)\, d\lambda = 0$. Let $F = \int_\Gamma \phi^\tau_\lambda\, d\lambda$, then $F(x) = \int_\Gamma \phi^\tau_\lambda(x)\, d\lambda = 0$ (since $\lambda \mapsto \phi^\tau_\lambda(x)$ is entire). Therefore $F \equiv 0$. This shows that $0 = W(F) = \int_\Gamma W(\phi^\tau_\lambda)\, d\lambda = \int_\Gamma \widehat{W}(\lambda)\, d\lambda$. Hence $\lambda \mapsto \widehat{W}(\lambda)$ is entire.

Hence the τ-spherical transform gives a bijection between these two spaces $C^\infty_\tau(G)'$ and PW^D_e. Therefore we give the topology on PW^D_e so that the τ-spherical transform becomes a topological isomorphism. The proof of [38, Proposition 1] shows that this topology coincide with the topology given by the analytic uniform structure. This implies that the τ-spherical transform is a topological isomorphism.

(b) Suppose n is odd. Temporarily for this proof we shall use the following notation for convenience

$$\widehat{f}_+(\cdot) = \widehat{f}(\sigma^+,\cdot), \widehat{f}_-(\cdot) = \widehat{f}(\sigma^-,\cdot), \widehat{W}_+ = \widehat{W}_{\sigma^+}, \widehat{W}_- = \widehat{W}_{\sigma^-}.$$

Let W_1 be in PW^D. We define

$$W(f) = d^2_\tau \int_{\mathbb{R}} W_1(\lambda)\widehat{f}_+(\lambda)\, d\nu(\lambda), \text{ for } f \in C^\infty_{c,\tau}(G).$$

Then as in the case (a) we can show that W is a tempered distribution. Next we shall show that W is compactly supported distribution. For this we shall show $W * h \in C^\infty_{c,\tau}(G)$, for all $h \in C^\infty_{c,\tau}(G)$ (see [20, Theorem 1]). Let $f \in C^2_\tau(G)$, then

$$W(h*f) = d^2_\tau \textstyle\int_{\mathbb{R}} W_1(\lambda)\widehat{h}_+(\lambda)\widehat{f}_+(\lambda)\, d\nu(\lambda),$$

$$i.e., \ (W*h)(f) = d^2_\tau \textstyle\int_{\mathbb{R}} W_1(\lambda)\widehat{h}_+(\lambda)\widehat{f}_+(\lambda)\, d\nu(\lambda),$$

$$i.e., \ (\widehat{W*h})_+(\widehat{f}_+) = d^2_\tau \textstyle\int_{\mathbb{R}} W_1(\lambda)\widehat{h}_+(\lambda)\widehat{f}_+(\lambda)\, d\nu(\lambda).$$

Thus we have $(\widehat{W*h})_+(\widehat{f}_+) = g_+(\widehat{f}_+)$, where $g_+(\lambda) = W_1(\lambda)\widehat{h}_+(\lambda)$. This shows that $\widehat{(W*h)}_+ = g_+$ in the sense of distribution on $\mathbb{R}$. Therefore, $\widehat{(W*h)}_+(\lambda) = W_1(\lambda)\widehat{h}_+(\lambda)$. As in (a) above we can show that, $\widehat{(W*h)}_+$ is rapidly decreasing, exponential type function. So by Theorem 5.1.1 $W*h$ is a compactly supported function. Therefore, W is a compactly supported τ-radial distribution on G. Also similar to the even case it follows that $\widehat{W}_+(\lambda) = W_1(\lambda)$.

For the converse direction our argument is similar to that of (a).

Therefore, $W \mapsto \widehat{W}_+$ is a bijection between $C_\tau^\infty(G)'$ and PW^D. Since the map $W \mapsto \widehat{W}_+$ is a bijection we give the topology on PW^D so that the map is a topological isomorphism. Here also the proof of [38, Proposition 1] shows that this topology coincides with the topology given by "analytic uniform structure" on PW^D. So the τ-spherical transform $W \mapsto \widehat{W}_+$ is a topological isomorphism between $C_\tau^\infty(G)'$ and PW^D.

Exactly through the same steps as above we can also prove that $W \mapsto \widehat{W}_-$ is a topological isomorphism between $C_\tau^\infty(G)'$ and PW^D. This completes the proof. □

It follows from Theorem 5.3.3 and slice projection property (2.3.5) that if n is even then for $X \in C_\tau^\infty(G)', \mathcal{A}X$ is a unique element of $C^\infty(\mathbb{R})'_e$ such that

$$\widetilde{\mathcal{A}X}(\lambda) = \mathcal{A}X(\psi_\lambda) = X(\mathrm{Tr}\mathcal{A}^*\psi_\lambda) = \widehat{X}(\lambda) \text{ where } \psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2.$$

Similarly when n is odd then for $X \in C_\tau^\infty(G)'$, $\mathcal{A}_{\sigma^+}X$ and $\mathcal{A}_{\sigma^-}X$ are unique elements in $C^\infty(\mathbb{R})'$ such that

$$\widetilde{\mathcal{A}_{\sigma^+}}X(\lambda) = \mathcal{A}_{\sigma^+}X(e^{-i\lambda(\cdot)}) = X(\mathrm{Tr}\mathcal{A}^*_{\sigma^+}e^{-i\lambda(\cdot)}) = \widehat{X}_{\sigma^+}(\lambda) \text{ and}$$

$$\widetilde{\mathcal{A}_{\sigma^-}}X(\lambda) = \mathcal{A}_{\sigma^-}X(e^{-i\lambda(\cdot)}) = X(\mathrm{Tr}\mathcal{A}^*_{\sigma^-}e^{-i\lambda(\cdot)}) = \widehat{X}_{\sigma^-}(\lambda).$$

From the Paley-Wiener theorems for distributions (Theorem 2.2.2 and Theorem 5.3.3) it is immediate that for n even $\mathcal{A} : C_\tau^\infty(G)' \to C^\infty(\mathbb{R})'_e$ is a topological isomorphism and for n odd both $\mathcal{A}_{\sigma^+}$ and $\mathcal{A}_{\sigma^-} : C_\tau^\infty(G)' \to C^\infty(\mathbb{R})'$ are topological isomorphisms.

Chapter 6

Wiener-Tauberian Theorems

6.1 Wiener-Tauberian Theorems for Lorentz Spaces and Weighted Spaces

In this section we shall state and prove analogues of Wiener-Tauberian theorem for the Banach algebras and modules considered in Chapter 3, where the triple (G, K, τ) is as in Chapter 4.

We state here the two main theorems of this section.

Theorem 6.1.1. *For some index set* Λ, *let* $\{f_\alpha \mid \alpha \in \Lambda\}$ *be a collection of functions in* $L^{p,1}_\tau(G) \cap L^{q,1}_\tau(G)$ *where* $1 \leq q < p < 2$. *For* $\sigma \in \widehat{M}(\tau)$ *let*

$$Z_\sigma = \{z \in S_p \mid \widehat{f_\alpha}(\sigma, z) = 0 \text{ for all } \alpha \in \Lambda\}.$$

If for all $\sigma \in \widehat{M}(\tau)$, Z_σ *is empty and there exists an* $\alpha(\sigma) \in \Lambda$ *such that* $\widehat{f_{\alpha(\sigma)}}$ *satisfies for* $t \in \mathbb{R}$

$$\limsup_{|t| \to \infty} |\widehat{f_{\alpha(\sigma)}}(\sigma, t) e^{Ke^{|t|}}| > 0 \text{ for all } K > 0 \tag{6.1.1}$$

then the ideal generated by $\{f_\alpha \mid \alpha \in \Lambda\}$ *in* $L^{p,1}_\tau(G)$ *is dense in* $L^{p,1}_\tau(G)$.

Theorem 6.1.2. *Let* $0 < p < 2, r \geq 0$ *and* $\{f_\alpha : \alpha \in \Lambda\}$ *be a collection of functions in* $L^1_\tau(w_{q,r})$ *for some* $0 < q < p$ *where* Λ *is an index set. For* $\sigma \in \widehat{M}(\tau)$, *let*

$$Z_\sigma = \{z \in S_p \mid \widehat{f_\alpha}(\sigma, z) = 0 \text{ for all } \alpha \in \Lambda\}.$$

If for every $\sigma \in \widehat{M}(\tau)$, Z_σ *is empty and there exits* $\alpha(\sigma) \in \Lambda$ *such that* $f_{\alpha(\sigma)}$ *satisfies* (6.1.1) *then the ideal generated by* $\{f_\alpha : \alpha \in \Lambda\}$ *in* $L^1_\tau(w_{p,r})$ *is dense in* $L^1_\tau(w_{p,r})$.

Before entering into the proofs we shall discuss the necessity of the hypothesis. First we shall give a brief sketch of the construction of a function in $L^{p,1}_\tau(G)$ whose τ-spherical transform does not vanish anywhere in S_p, which does not satisfy condition (6.1.1) and the ideal generated by f is not dense in $L^{p,1}_\tau(G)$. This will establish the necessity of an additional condition like (6.1.1). This is an adaptation of the corresponding construction of a K-biinvariant integrable function on $\mathrm{SL}(2,\mathbb{R})$ (see [25]).

Let n be even. We fix p_1, p_2 such that $p_2 < p < p_1$. Let $\alpha = \pi/2\gamma_{p_1}\rho$. For $\lambda \in \mathbb{C}$, we define the functions $F(\lambda) = e^{-\cosh \lambda\alpha}$ and $G(\lambda) = 4(e^{\lambda\alpha} + e^{-\lambda\alpha} + 2)^{-1}$. It is easy to check that both $F, G \in \mathcal{S}(S_{p_2})_e$. By Corollary 5.2.2 there exist $f, g \in C^{p_2}_\tau(G)$ such that $\widehat{f} = F$ and $\widehat{g} = G$. It is clear that F does not vanish anywhere on S_p. We recall that $C^{p_2}_\tau(G)$ is a dense subspace of $L^{p,1}_\tau(G)$ (see Chapter 3). We assume that $I_f = \{\beta * f \mid \beta \in L^1_\tau(G)\}$ is dense in $L^{p,1}_\tau(G)$. Then there is a sequence $\{g_n\}$ in $L^1_\tau(G)$ such that $g_n * f$ converges to g in $L^{p,1}_\tau$. For $\lambda \in S_{p_1}$ we have,

$$|\widehat{g_n * f}(\lambda) - \widehat{g}(\lambda)| \leq \int_G |(g_n * f - g)(x)\phi^\tau_\lambda(x^{-1})|\, dx \leq C_\tau \|g_n * f - g\|^*_{p,1} \|\phi_{i\Im\lambda}\|^*_{p',\infty}.$$

Therefore

$$|\widehat{g_n * f}(\lambda) - \widehat{g}(\lambda)| \leq C_\tau \|g_n * f - g\|^*_{p,1} \|\phi_{i\gamma_{p_1}\rho}\|^*_{p',\infty}.$$

This implies that $\widehat{g_n * f} \to \widehat{g}$ uniformly on S_{p_1}. That is

$$\widehat{g_n}(\lambda)F(\lambda) \to G(\lambda) \text{ uniformly on } S_{p_1}. \tag{6.1.2}$$

Let $\mathbb{D}$ be the open unit disc and let $\overline{\mathbb{D}}$ be its closure. Let $A(\mathbb{D})$ be the disc algebra, that is the algebra of all functions $f : \mathbb{D} \to \mathbb{C}$ which are holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, endowed with the supremum norm. Let $A_0(\mathbb{D}) = \{f \in A(\mathbb{D}) \mid f(z) = f(-z)\, \forall z \in \overline{\mathbb{D}} \text{ and } f(i) = f(-i) = 0\}$. We equip $A_0(\mathbb{D})$ with the subspace topology of $A(\mathbb{D})$.

We consider the conformal map $\psi(\lambda) = i(e^{\pi\lambda/2\gamma_{p_1}\rho} - 1)(e^{\pi\lambda/2\gamma_{p_1}\rho} + 1)^{-1}$ from the strip S_{p_1} onto $\overline{\mathbb{D}}$, which maps $\mathbb{R}$ on the line segment joining i and $-i$. In particular $\psi(0) = 0$, $\psi(\infty) = i$, $\psi(-\infty) = -i$, $\psi(i\gamma_{p_1}\rho) = -1$ and $\psi(-i\gamma_{p_1}\rho) = 1$. Then from (6.1.2) we have $\widehat{g_n}(\psi^{-1}(z))F(\psi^{-1}(z)) \to G(\psi^{-1}(z))$ uniformly on $\overline{\mathbb{D}}$. Since $F(\psi^{-1}(z)) = \exp\left(\frac{z^2-1}{z^2+1}\right)$ and $G(\psi^{-1}(z)) = z^2 + 1$, it follows that $z^2 + 1$ is in the closed ideal I (with respect to topology of uniform convergence) generated by $\exp\left(\frac{z^2-1}{z^2+1}\right)$ in $A_0(\mathbb{D})$. This is not possible since every element of I decays very fast along imaginary axis. Therefore $\overline{I_f}$ is a proper subset of $L^{p,1}_\tau(G)$. We note that f does not satisfy condition (6.1.1). This construction works also for the case n odd where we have to substitute $\widehat{f}$ by $\widehat{f}(\sigma^+, \cdot)$ in the argument above.

For the nonvanishing condition of the τ-spherical transform, we suppose that there is a $\lambda_0 \in Z_\sigma$ for a fixed $\sigma \in \widehat{M}$. We define $T(f) = \widehat{f}(\sigma, \lambda_0)$. Then $T : L^{p,1}_\tau(G) \to \mathbb{C}$ is an algebra homomorphism and hence Ker$T$ is a proper maximal ideal in $L^{p,1}_\tau(G)$ where Ker$T = \{f \in L^{p,1}_\tau(G) \mid T(f) = 0\}$. Precisely Ker$T \supseteq \{f_\alpha : \alpha \in \Lambda\}$. Hence the ideal generated by $\{f_\alpha : \alpha \in \Lambda\}$ is proper, which shows the necessity of the nonvanishing condition in Theorem 6.1.1. Similar argument works for Theorem 6.1.2.

We may point out here that all the available analogues of WTT (see [22,7]) impose the nonvanishing condition of the Fourier transform in a strip slightly larger than the domain of the Fourier transform. Only exceptions are [6] (announced in [5]) and [18]. But both of these papers deal only with the case of integrable K-biinvariant functions on $\mathrm{SL}(2, \mathbb{R})$ and it appears difficult to extend the method for more general groups or for more general functions. Here for τ-radial functions we have considered the nonvanishing condition only on the appropriate domain.

We shall isolate a few steps of the proof of the theorems as the following lemmas. We recall that for any $p \in (0, 2]$, $\gamma_p = 2/p - 1$. For any $\delta > 0$ we define the augmented strip

$$S_{p,\delta} = \{z \in \mathbb{C} \mid |\Im z| \leq \gamma_p \rho + \delta\}.$$

Let $A^p(\delta)$ be the space of continuous functions $F : S_{p,\delta} \to \mathbb{C}$ which are holomorphic on $S^\circ_{p,\delta}$, and satisfy $\lim_{|\xi|\to\infty} F(\xi + i\eta) = 0$ on $S_{p,\delta}$. We endow $A^p(\delta)$ with the supremum norm topology. Let $A^p_0(\delta)$ be the subspace of even functions in $A^p(\delta)$, equipped with the subspace topology. Let $\mathbb{T} = \partial\mathbb{D}$ be the boundary of $\mathbb{D}$ and $A_1(\mathbb{D}) = \{f \in A(\mathbb{D}) \mid f(i) = f(-i) = 0\}$.

We equip both $A_1(\mathbb{D})$ and $A_0(\mathbb{D})$ with the subspace topology of $A(\mathbb{D})$. We denote the closed ideal generated by a function $f \in A(\mathbb{D})$ in $A(\mathbb{D})$ by $\langle f(z) \rangle$. Henceforth by an ideal we will mean a closed ideal unless mentioned otherwise. It is easy to show that $A_1(\mathbb{D}) = \langle (z - i)(z + i) \rangle$. We recall that every maximal ideal of $A(\mathbb{D})$ is of the form $M_\lambda = \{f \in A(\mathbb{D}) \mid f(\lambda) = 0\}$ for some point λ in the closed unit disc (see [37, p. 87]). We also need the following generalization of a theorem given in [37, p.88].

Lemma 6.1.3. *If J is a non-zero closed ideal of $A(\mathbb{D})$ contained in precisely two maximal ideals M_{λ_1} and M_{λ_2} with $K = \{\lambda_1, \lambda_2\} \subset \mathbb{T}$ then J is the closed principal* *ideal generated by*

$$f(z) = (z-\lambda_1)(z-\lambda_2)\exp\left[\rho_1\frac{z+\lambda_1}{z-\lambda_1} + \rho_2\frac{z+\lambda_2}{z-\lambda_2}\right]$$

where ρ_1, ρ_2 *are non-negative real numbers.*

Proof. Clearly $J \subseteq H^1(\mathbb{D})$ where $H^1(\mathbb{D})$ is the class of analytic functions f in the open unit disc $\mathbb{D}$ for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in L^1-norm as $r \to 1$. If the greatest common divisor of the inner parts of the nonzero functions in J is $F = B.S$ then $B = 1$ and $S(z) = \exp\left[-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\right]$ for some unique singular positive measure μ on $\mathbb{T}$ (see [37, Theorem p. 67, Lemma p. 85]). Therefore $J = \{Sg \mid g \in A(\mathbb{D})$ which vanishes on $K\}$ (see [37, Theorem on p. 85]). We know that, S is analytic on $\mathbb{D}$ and continuous on $\mathbb{T} \setminus K_1$ where K_1 is the closed support of μ (see [37, second Theorem on p. 68]). Also S is not extendable continuously to the points of K_1. Let g be any element of $A(\mathbb{D})$ which vanishes exactly on K. The continuity of Sg on $\mathbb{T}$ then implies that $K_1 \subseteq K$. Therefore $S(z) = \exp\left[\rho_1\frac{z+\lambda_1}{z-\lambda_1} + \rho_2\frac{z+\lambda_2}{z-\lambda_2}\right]$ where $\rho_1 = \mu(\{\lambda_1\})$ and $\rho_2 = \mu(\{\lambda_2\})$. Let $J_1 = \langle (z-\lambda_1)(z-\lambda_2)S(z)\rangle$. We have $J_1 \subseteq J$. Suppose $\Re\lambda_1, \Re\lambda_2 \geq 0$. Then for any $g \in A(\mathbb{D})$, which vanishes on K,

$$S(z)g(z)\frac{(z-\lambda_1)(z-\lambda_2)}{(z-\lambda_1-\frac{1}{n})(z-\lambda_2-\frac{1}{n})} \in J_1.$$

Also

$$S(z)g(z)\frac{(z-\lambda_1)(z-\lambda_2)}{(z-\lambda_1-\frac{1}{n})(z-\lambda_2-\frac{1}{n})} \to S(z)g(z)$$

uniformly in $\overline{\mathbb{D}}$ as $n \to \infty$. Therefore $J \subseteq J_1$ as J_1 is closed. Hence $J = J_1$. Similar argument works for the other cases (i.e when both $\Re\lambda_1$ and $\Re\lambda_2$ are nonpositive or one of them is positive while the other is negative). Therefore $J = \left\langle (z-\lambda_1)(z-\lambda_2)\exp\left[\rho_1\frac{z+\lambda_1}{z-\lambda_1} + \rho_2\frac{z+\lambda_2}{z-\lambda_2}\right]\right\rangle$ where ρ_1, ρ_2 are non-negative real numbers. This completes the proof. □

Lemma 6.1.4. *If* I_0 *and* I_1 *are ideals of* $A_0(\mathbb{D})$ *and* $A_1(\mathbb{D})$ *respectively then there are ideals* J_0 *and* J_1 *of* $A(\mathbb{D})$ *such that* $I_0 = J_0 \cap A_0(D)$ *and* $I_1 = J_1 \cap A_1(D)$.

The proof of this lemma is omitted as it is similar to [7, Lemma 1.2].

We say that a function f on $\overline{\mathbb{D}}$ *decays less than exponential* at μ if for all $K > 0$

$$\limsup_{z\to\mu} |f(z)e^{K\frac{\mu+z}{\mu-z}}| > 0. \tag{6.1.3}$$

Lemma 6.1.5. (a) *Let I_1 be an ideal in $A_1(\mathbb{D})$. If the functions in I_1 have no common zero other than $\pm i$ and if I_1 contains a function which satisfies (6.1.3) for $\mu = \pm i$, then $I_1 = A_1(\mathbb{D})$.*

(b) *Let I_0 be an ideal in $A_0(\mathbb{D})$. If the functions in I_0 have no common zero other than $\pm i$ and if I_0 contains a function which satisfies (6.1.3) for $\mu = \pm i$, then $I_0 = A_0(\mathbb{D})$.*

Proof. (a) We have by Lemma 6.1.4, $I_1 = J_1 \cap A_1(\mathbb{D})$ where J_1 is an ideal of $A(\mathbb{D})$.

If J_1 is contained in three or more maximal ideals, then the zero set of J_1 has at least three elements. This contradicts the fact that $I_1 = J_1 \cap A_1(\mathbb{D})$ since the zero set of I_1 has two elements. Hence J_1 can be contained in precisely one maximal ideal or in precisely two maximal ideals.

If J_1 is contained in only one maximal ideal then by Beurling-Rudin theorem (see [37, Corollary p. 88]) J_1 is $\langle (z-\alpha)^k \rangle$ where α is a point in open unit disc $\mathbb{D}$ and k is a positive integer, or it is of the form $\left\langle (z-\lambda)e^{\rho\frac{z+\lambda}{z-\lambda}} \right\rangle$ where λ is a point on the boundary $\mathbb{T}$ and ρ is a non-negative real number. But the first case is not possible as the functions in I_1 have no common zero in $\mathbb{D}$. The second case with $\lambda \in \mathbb{T} \setminus \{i, -i\}$ contradicts the hypothesis as the only common zero of the functions in $I_1 \subseteq J_1$ is $\{\pm i\}$. Also the second case with $\rho > 0$ and $\lambda = \pm i$ is not also possible as it contradicts (6.1.3). Only possibility thus is $\rho = 0$ and $\lambda = \pm i$ in the second case. Hence $J_1 = \langle z - i \rangle$ or $J_1 = \langle z + i \rangle$. Since $A_1(\mathbb{D}) = \langle (z-i)(z+i) \rangle$, $J_1 \supseteq A_1(\mathbb{D})$. Therefore $I_1 = A_1(\mathbb{D})$.

If J_1 is contained in precisely two maximal ideals, say P and Q then P and Q are of the form $P = M_i = \{f \in A(\mathbb{D}) \mid f(i) = 0\}, Q = M_{-i} = \{g \in A(\mathbb{D}) \mid g(-i) = 0\}$. Therefore by Lemma 6.1.3

$$J_1 = \left\langle (z-i)(z+i)\exp\left[\rho_1 \frac{z+i}{z-i} + \rho_2 \frac{z-i}{z+i}\right] \right\rangle$$

where ρ_1 and ρ_2 are non-negative real numbers. But this is possible only when $\rho_1 = \rho_2 = 0$ as otherwise it contradicts (6.1.3). Therefore $J_1 = \langle (z-i)(z+i) \rangle = A_1(\mathbb{D})$. Hence $I = A_1(\mathbb{D})$. This completes the proof.

(b) Slight modification of the argument above proves (b). □

Lemma 6.1.6. *Let $\delta > 0$ and $0 < p < 2$ be fixed.*

(a) *If n is odd then the set of functions*

$$\mathcal{F}_{p,\delta} = \{f \in C^p_\tau(G) : \widehat{f}(\sigma^+, \lambda) \in A^p(\delta) \text{ and } \widehat{f}(\sigma^+, \lambda)e^{K\lambda^2} \in A^p(\delta) \text{ for some } K > 0\}$$

is dense in $C^p_\tau(G)$.

(b) *If* n *is even then the set of functions*

$$\mathcal{F}_{p,\delta} = \{f \in C^p_\tau(G) : \widehat{f}(\lambda) \in A^p_0(\delta) \text{ and } \widehat{f}(\lambda)e^{K\lambda^2} \in A^p_0(\delta) \text{ for some } K > 0\}$$

is dense in $C^p_\tau(G)$.

Proof. We will prove (a). For proving (b) we only have to replace $\widehat{f}(\sigma^+, \cdot)$ by $\widehat{f}$ and $A^p(\delta)$ by $A^p_0(\delta)$. Let $g \in C^\infty_{c,\tau}(G)$. Since $f \mapsto \widehat{f}(\sigma^+, \cdot)$ is a topological isomorphism between $C^p_\tau(G)$ and $\mathcal{S}(S_p)$ (see Chapter 5), it is sufficient to show that there is a sequence $f_m \in \mathcal{F}_{p,\delta}$ such that $\widehat{f_m}(\sigma^+, \cdot)$ converges to $\widehat{g}(\sigma^+, \cdot)$ in $\mathcal{S}(S_p)$. Let $F_m(\lambda) = \widehat{g}(\sigma^+, \cdot)(\lambda)e^{-\lambda^2/m}$ for $m \in \mathbb{N}$. Clearly $F_m \in \mathcal{S}(S_p)$. By Corollary 5.2.2 there is $f_m \in C^p_\tau(G)$ such that $\widehat{f_m}(\sigma^+, \lambda) = F_m(\lambda)$. Therefore $f_m \in \mathcal{F}_{p,\delta}$. We shall show that $\widehat{f_m}(\sigma^+, \cdot)$ converges to $\widehat{g}(\sigma^+, \cdot)$ in the topology of $\mathcal{S}(S_p)$. For fixed $r \in \mathbb{N}$, we consider $|\lambda^r[\widehat{f_m}(\sigma^+, \lambda) - \widehat{g}(\sigma^+, \lambda)]| = |\lambda^r \widehat{g}_+(\lambda)||1 - e^{-\lambda^2/m}|$. The first term converges to zero as $|\lambda| \to \infty$ and the second term converges to zero uniformly on every compact subset of $S_{p,\delta}$ as $m \to \infty$. Hence $|\lambda^r[\widehat{f_m}(\sigma^+, \lambda) - \widehat{g}(\sigma^+, \lambda)]|$ converges to zero uniformly on $S_{p,\delta}$ as $m \to \infty$. Since $\widehat{f_m}(\sigma^+, \cdot)$ and $\widehat{g}(\sigma^+, \cdot)$ are analytic, by Cauchy's integral formula it follows that on the smaller strip S_p the same is true for all derivatives of $\lambda^r(\widehat{f_m}(\sigma^+, \lambda) - \widehat{g}(\sigma^+, \lambda))$. This completes the proof. □

Lemma 6.1.7. *Let* $\delta > 0$ *and* $0 < p < 2$ *be fixed. For* n *even, let* f *be a* τ*-radial measurable function and* $\{f_i\}$ *be a sequence of* τ*-radial measurable functions such that* $\widehat{f_i}, \widehat{f} \in A^p_0(\delta)$. *If there is a* $K > 0$ *such that* $\widehat{f}(\lambda)e^{K\lambda^2} \in A^p_0(\delta)$ *and if* $\widehat{f_i}(\lambda)$ *converges to* $\widehat{f}(\lambda)e^{K\lambda^2}$ *in the topology of* $A^p_0(\delta)$, *then* $\widehat{f_i}(\lambda)e^{-K\lambda^2}$ *converges to* $\widehat{f}(\lambda)$ *in the topology of* $\mathcal{S}(S_p)_e$.

For n *odd if we replace* $\widehat{f}$ *and* $\widehat{f_i}$ *respectively by* $\widehat{f}(\sigma^+, \cdot)$ *and* $\widehat{f_i}(\sigma^+, \cdot)$ *and* $A^p_0(\delta)$ *by* $A^p(\delta)$ *in the hypothesis then* $\widehat{f_i}(\sigma^+, \lambda)e^{-K\lambda^2}$ *converges to* $\widehat{f}(\sigma^+, \lambda)$ *in the topology of* $\mathcal{S}(S_p)$.

Proof. We will prove only for the case when n is even. The case of odd n can be proved through similar steps.

As $\widehat{f}(\lambda)e^{K\lambda^2} \in A^p_0(\delta)$ and $\widehat{f_i}(\lambda) \in A^p_0(\delta)$, using Cauchy's integral formula it can be shown that $\widehat{f}$ and $\widehat{f_i}(\lambda)e^{-K\lambda^2}$ are in $\mathcal{S}(S_p)_e$.

Since both $\widehat{f_i}(\lambda)e^{-K\lambda^2}$ and $\widehat{f}(\lambda)$ are holomorphic, it is sufficient to prove that, for an arbitrary $r \in \mathbb{N}$, $|\lambda^r[\widehat{f_i}(\lambda)e^{-K\lambda^2} - \widehat{f}(\lambda)]|$ converges to zero uniformly in the strip $S_{p,\delta}$ as $i \to \infty$. Now $|\lambda^r[\widehat{f_i}(\lambda)e^{-K\lambda^2} - \widehat{f}(\lambda)]| = |\lambda^r e^{-K\lambda^2}||\widehat{f}(\lambda)e^{K\lambda^2} - \widehat{f_i}(\lambda)|$. The first factor is a bounded function and the second factor converges to zero uniformly on $S_{p,\delta}$. Hence $|\lambda^r[\widehat{f_i}(\lambda)e^{K\lambda^2} - \widehat{f}(\lambda)]|$ converges to zero uniformly on $S_{p,\delta}$ as $i \to \infty$. Therefore the lemma follows. □

With this preparation we shall enter into the main part of the proof.

Proof of Theorem 6.1.1. We shall first prove the theorem for n even. In this case $\tau|_M$ contains a unique $\sigma \in \widehat{M}$. Therefore we can omit σ from the notation for this case and in particular denote the function $f_{\alpha(\sigma)}$ simply by f_{α_0}.

Since $f_\alpha \in L^{p,1}_\tau(G) \cap L^{q,1}_\tau(G)$, $\widehat{f_\alpha}$ is analytic on S_q° and continuous on S_q by Proposition 3.2.4 (1). That is $\widehat{f_\alpha}$ is analytic on $S^\circ_{p,\delta}$ and continuous on $S_{p,\delta}$, where $\delta = (\gamma_q - \gamma_p)\rho > 0$. Also by Proposition 3.2.4 $\lim_{|\xi| \longrightarrow \infty} \widehat{f_\alpha}(\xi + i\eta) = 0$ for $|\eta| \leq \gamma_q\rho$. We consider the strip

$$T = \left\{ z \mid \gamma_p\rho < |\Im z| \leq \gamma_p\rho + \frac{\delta}{2} \right\}.$$

We take a function f_β from the collection $\{f_\alpha \mid \alpha \in \Lambda\}$. Let $Z(T)$ be the set of all zeros of $\widehat{f_\beta}$ in T.

As zeros of an analytic function in a connected open set are isolated we choose countably many disjoint open rectangles R_i such that $Z(T) \subset \cup_i R_i \subset T$ and each R_i contains only one zero of $\widehat{f_\beta}$.

We suppose R_1 has a zero of $\widehat{f_\beta}$ at a_1 of order n_1. Note that a_1 is at a positive distance from S_p. As $\widehat{f_\beta}(\lambda) = \widehat{f_\beta}(-\lambda)$, we have $\widehat{f_\beta}(\lambda) = (\lambda^2 - a_1^2)^{n_1} g(\lambda)$ where $g(\lambda)$ is an even function which is analytic on $S^\circ_{p,\delta/2}$ and $g(\pm a_1) \neq 0$. We choose a $q < q_1 < p$ such that $\gamma_{q_1}\rho < |\Im a_1|$. That is $a_1 \notin S_{q_1}$.

We define a function H on S_{q_1} by

$$H(\lambda) = \frac{e^{-\lambda^2}}{(\lambda^2 - a_1^2)^{n_1}}.$$

Then $H \in \mathcal{S}(S_{q_1})_e$ and hence by Corollary 5.2.2 there exists $h \in C^{q_1}_\tau(G)$ such that $\widehat{h}(\lambda) = H(\lambda)$ for $\lambda \in S_{q_1}$. Let $f_\beta * h = f_{\beta,1}$. Then $f_{\beta,1} \in L^{p,1}_\tau(G)$ by Proposition 3.2.1 (3) and $\widehat{f_{\beta,1}}(\lambda) = \widehat{f_\beta}(\lambda)\widehat{h}(\lambda) = g(\lambda)e^{-\lambda^2}$ for $\lambda \in S_{q_1}$. But as g is analytic on $S^\circ_{p,\delta/2}$ it follows that $\widehat{f_{\beta,1}}$ extends analytically on $S^\circ_{p,\delta/2}$ and $\lim_{|\xi| \to \infty} \widehat{f_{\beta,1}}(\xi + i\eta) = 0$ on $S_{p,\delta/2}$. From this using Cauchy's integral formula and Corollary 5.2.2 we get that $f_{\beta,1} \in C^{q_1}_\tau(G) \subseteq L^{p,1}_\tau(G)$. We also note that $\widehat{f_{\beta,1}}(\lambda) \neq 0$ for all $\lambda \in R_1$ and $\widehat{f_{\beta,1}}(\lambda) \neq 0$ whenever $\widehat{f_\beta}(\lambda) \neq 0$. In this way we can construct a collection $f_{\beta,i} \in L^{p,1}_\tau(G)$ such that $\widehat{f_{\beta,i}}(\lambda) \neq 0$ on R_i and $\widehat{f_{\beta,i}}(\lambda) \neq 0$ whenever $\widehat{f_\beta}(\lambda) \neq 0$. By definition the functions $f_{\beta,i}$ is in the ideal generated by $\{f_\alpha \mid \alpha \in \Lambda\}$ in $L^{p,1}_\tau(G)$

as $C_\tau^{q_1}(G) \subseteq L_\tau^{p,1}(G)$. We consider the collection

$$\Gamma = \{f_\alpha, f_{\beta,i} \mid \alpha \in \Lambda, i = 1, 2, \cdots\}.$$

Then $Z_\Gamma = \{z \in S_{p,\delta/2} \mid \widehat{f}(z) = 0, \text{ for all } f \in \Gamma\}$ is empty.

It is clear that $\widehat{f} \in A_0^p(\delta/2)$ for all $f \in \Gamma$. We consider the conformal map

$$\psi(\lambda) = \frac{i(e^{\pi\lambda/2(\gamma_p\rho+\frac{\delta}{2})} - 1)}{e^{\pi\lambda/2(\gamma_p\rho+\frac{\delta}{2})} + 1}$$

from the strip $S_{p,\delta/2}$ onto the closure of $\mathbb{D}$, which maps $\mathbb{R}$ on the line segment joining i and $-i$. In particular $\psi(0) = 0$, $\psi(\infty) = i$, $\psi(-\infty) = -i$, $\psi(i(\gamma_p\rho+\frac{\delta}{2})) = -1$ and $\psi(-i(\gamma_p\rho+\frac{\delta}{2})) = 1$. Through ψ we can identify the functions on $A_0^p(\delta/2)$ as the functions on $A_0(\mathbb{D})$. We abuse the notation to denote both the function on $A_0^p(\delta/2)$ and its realization on $A_0(\mathbb{D})$ by $\widehat{f}$. Note that it follows from (6.1.1) that, as a function in $A_0(\mathbb{D})$, $\widehat{f_{\alpha_0}}$ satisfies (6.1.3).

Let $\widehat{I}$ be the algebraic ideal in $A_0(\mathbb{D})$ generated by the τ-spherical transforms of the functions in Γ . Then $\widehat{I}$ satisfies the hypothesis of Lemma 6.1.5(b) and hence is dense in $A_0^p(\delta/2)$ under the supremum norm. We note that $A_0^p(\delta/2) = A_0^{p_1}(\delta')$ for some $p_1 < p$ such that $\gamma_{p_1}\rho < \gamma_p\rho + \delta/2$ and $0 < \delta' < \delta/2$. We take a function $\xi \in \mathcal{F}_{p_1,\delta'} \subset C_\tau^{p_1}(G)$ (as described in Lemma 6.1.6). Then by definition the function $\lambda \mapsto \widehat{\xi}(\lambda)e^{K\lambda^2}$ is in $A_0^{p_1}(\delta')$ for some $K > 0$. Now by denseness of $\widehat{I}$ there exists $\{F_n\} \subset \widehat{I}$ such that $F_n(\lambda) \to \widehat{\xi}(\lambda)e^{K\lambda^2}$ as $n \to \infty$ uniformly in $\lambda \in S_{p_1,\delta'}$. Since $e^{-K\lambda^2} \in A_0^{p_1}(\delta')$ we have that $F_n(\lambda)e^{-K\lambda^2} \in \widehat{I}$.

Therefore by Lemma 6.1.7, $F_n(\lambda)e^{-K\lambda^2} \to \widehat{\xi}(\lambda)$ in $\mathcal{S}(S_{p_1})_e$. It follows easily from Cauchy's integral formula that the function $F_n(\lambda)e^{-K\lambda^2}$ is in $S(S_{p_1})_e$.

Since $F_n \in \widehat{I}$, we can assume without loss of generality that $F_n = \widehat{f}\ h_1$ for some $f \in \Gamma$ and $h_1 \in A_0^{p_1}(\delta')$. Thus for $K = K_1 + K_2$, $K_1, K_2 > 0$, $F_n(\lambda)e^{-K\lambda^2} = \widehat{f}(\lambda)e^{-K_1\lambda^2}h_1(\lambda)e^{-K_2\lambda^2} = (m_1 * m_2)\widehat{\ }(\lambda)$ where $\widehat{m_1}(\lambda) = \widehat{f}(\lambda)e^{-K_1\lambda^2}$ and $\widehat{m_2}(\lambda) = h_1(\lambda)e^{-K_2\lambda^2}$ and $\widehat{m_1}, \widehat{m_2} \in \mathcal{S}(S_{p_1})_e$. As the τ-spherical transform is a topological isomorphism from $C_\tau^{p_1}(G)$ to $\mathcal{S}(S_{p_1})_e$ it follows that $m_1, m_2 \in C_\tau^{p_1}(G)$. It is also clear that m_1 and hence $m_1 * m_2$ is in the ideal generated by elements of Γ in $L_\tau^{p,1}(G)$. This completes the proof for the case n even.

For the case when n is odd we have to modify the proof in the following way. First we note that as $\widehat{f}(\sigma_+, \lambda) = \widehat{f}(\sigma_-, -\lambda)$, it is enough to work with $\widehat{f}(\sigma_+, \lambda)$. In the above line of argument we take $H(\sigma_+, \lambda) = \frac{e^{-\lambda^2}}{(\lambda-a_1)^{n_1}}$, and proceed in an analogous fashion. □

For the proof of Theorem 6.1.2 one can follow the similar line of argument and use the corresponding Propositions and Lemmas. We omit the proof.

Remark 6.1.8. The following remarks are in order.
(1) The Kunze-Stein phenomenon (see Chapter 3), Corollary 3.1.3 and Proposition 3.2.1 show that we can formulate WTT also in many other setup; for instance:

(a) $L^p_\tau(G)$ under $L^{q,r}_\tau(G)$ action where $1 \leq q < p$, $1 \leq r \leq q$,

(b) $L^{p,r}_\tau(G)$ as an $L^{q,1}_\tau(G)$ module, where $1 < q \leq p < 2$ and $1 \leq r \leq \infty$, $L^{p,r}_\tau(G)$ as $L^1_\tau(G)$ module for $1 < p < 2$ and $1 \leq r \leq \infty$,

(c) $L^1_\tau(w_{p,s})$ as $L^1_\tau(w_{p,r})$ module where $r > s \geq 0$,

(d) $L^1_\tau(w_{p,s})$ as $L^1_\tau(w_{q,r})$ module where $q < p$ and $r, s \in \mathbb{R}$ with $s \geq 0$.

We notice that (b) includes the weak L^p spaces for $1 < p < 2$. Starting from similar hypothesis and with easy modifications of the method used in Theorem 6.1.1 and in Theorem 6.1.2 we can prove these theorems. We omit them for brevity.

(2) We recall that for $1 < p < 2$, the elementary spherical function $\phi_\lambda \in L^{p',\infty}(G)$ if and only if $\lambda \in S_p$ and from this it follows that if $\lambda \in S_p$ then $\phi^\tau_{\sigma,\lambda} \in L^{p',\infty}_\tau(G)$ (see Chapter 2). The following simple argument using WTT proves that if $\phi^\tau_{\sigma,\lambda} \in L^{p',\infty}_\tau(G)$ then $\lambda \in S_p$.

We suppose for some point $(\sigma, \lambda_0) \in \widehat{M}(\tau) \times (\mathbb{C} \setminus S_p)$, $\phi^\tau_{\sigma,\lambda_0} \in L^{p',\infty}_\tau(G)$. Then $\phi^\tau_{\sigma,\lambda_0}$ defines a continuous linear functional on $L^{p,1}_\tau(G)$. In other words τ-spherical transform of any function in $L^{p,1}_\tau$ exists as a convergent integral at λ_0. We choose a $\delta > 0$ such that the augmented strip $S_{p,\delta}$ does not contain λ_0. We consider a collection of τ-radial C^∞_c functions on G such that the τ-spherical transforms of this collection have no common zero in $S_{p,\delta}$, but all of them vanish at λ_0. Then by Theorem 6.1.2, this collection of functions generates a dense ideal in $L^{p,1}_\tau(G)$ because any C^∞_c function automatically satisfies condition (6.1.1) by Phragmén-Lindelöf Theorem. Using continuity of spherical transform for $L^{p,1}_\tau$-functions, thus all functions in $L^{p,1}_\tau(G)$ has zero τ-spherical transform at λ_0. This contradicts the fact that there is a τ-radial $C^\infty_c(G)$ function which has nonzero τ-spherical transform at λ_0. (Such a function can be constructed using Paley-Wiener Theorem.)

6.2 Wiener-Tauberian Theorems based on Unitary Dual

In this section we shall prove two Wiener-Tauberian type theorems to point out two different features of the theorem. In both of these theorems the nonvanishing condition imposed on the function will be only on the unitary dual of the group. The triple (G, K, τ) is as in the previous section.

In Remark 6.1.8 we have observed that it is possible to formulate and prove WTT for many function spaces acting by convolution on many others. Thus the basic formulation of the Wiener-Tauberian theorem involves two function spaces $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_1$ acts on $\mathcal{F}_2$ by convolution. We find sufficient condition which we put on a set of functions $\mathcal{G}$ in $\mathcal{F}_2$ such that under $\mathcal{F}_1$ action $\mathcal{G}$ generates a dense linear subspace of $\mathcal{F}_2$. Often the sufficient condition is also necessary, for instance for a WTT for $L^{p,1}$ functions the generator should have nonvanishing τ-spherical transform on the strip S_p. This makes the situation extremely rigid. We will see below that the rigidity of the condition on the generator can be weakened.

To keep things simple we shall consider only the Banach algebra $L^1_\tau(G)$ and we shall put condition on a single function instead of an arbitrary collection. It is not difficult to see how the theorem below generalizes for other Banach algebras and modules discussed in Chapter 3 and for arbitrary collection of functions as generators. Easy modification of the argument will prove these generalizations. We recall that

$$S_{1,\delta} = \{z \mid |\Im z| \leq 1 + \delta\}.$$

Theorem 6.2.1. *Let $f \in L^1_\tau(G)$ be such that $\widehat{f}(\sigma, \cdot)$ extends holomorphically to $S^\circ_{1,\delta}$ for some $\delta > 0$. Furthermore if*

(1) $|\widehat{f}(\sigma, \xi + i\eta)| \longrightarrow 0$ *as* $|\xi| \longrightarrow \infty$ *on* $S_{1,\delta}$,

(2) f *satisfies* (6.1.1)

(3) *For each* $\sigma \in \widehat{M}(\tau)$, $\widehat{f}(\sigma, \lambda) \neq 0$ *for all* $\lambda \in \mathbb{R}$,

then there exists a vector subspace A of $C^2_\tau(G)$ such that $h * f \in L^1_\tau(G)$ for all $h \in A$ and the ideal generated by $\{h * f \mid h \in A\}$ in $L^1_\tau(G)$ is dense in $L^1_\tau(G)$.

Proof. First we take up the case n even. Since $\widehat{f}$ is holomorphic on $S^\circ_{1,\delta}$, zeros of $\widehat{f}$ cannot have limit point in a smaller strip $S_{1,\delta/2}$ except at $\pm\infty$. We can divide the strip $S_{1,\delta/2}$ into countably many rectangles R_i with disjoint interiors such that exactly one zero of $\widehat{f}$ is in R_i°. By hypothesis this zero does not lie on the real line. Suppose R_i° has a zero of $\widehat{f}$ at a_i of order n_i. Then $\widehat{f}(\lambda) = F_i(\lambda)(\lambda^2 - a_i^2)^{n_i}$ where $F_i(\pm a_i) \neq 0$. It is clear that a_i is at a positive distance from the real line.

We consider the function $H_i(\lambda) = \frac{e^{-\lambda^2}}{(\lambda^2-a_i^2)^{n_i}}$.

One verifies that $H_i|_{\mathbb{R}}$ is in $\mathcal{S}(\mathbb{R})_e$ and hence using Schwartz space isomorphism there is a $h_i \in C^2_\tau(G)$ such that $\widehat{h_i}(\lambda) = H_i(\lambda)$ for all $\lambda \in \mathbb{R}$.

Now

$$\widehat{h_i * f}(\lambda) = \widehat{h_i}(\lambda)\widehat{f}(\lambda) = e^{-\lambda^2}F_i(\lambda)$$

has analytic extension on $S_{1,\delta/2}$, does not vanish anywhere on R_i and satisfies (1) of the hypothesis. We also note that $\widehat{h_i * f}(\lambda) \neq 0$ whenever $\widehat{f}(\lambda) \neq 0$.

From the condition $|\widehat{f}(\xi + i\eta)| \longrightarrow 0$ as $|\xi| \longrightarrow \infty$ on $S_{1,\delta}$, it follows that $|F_i(\xi + i\eta)| \longrightarrow 0$ as $|\xi| \longrightarrow \infty$ on $S_{1,\delta/2}$. Using Cauchy's integral formula it can be shown that the derivatives of F_i are bounded on S_1 and hence $F_i(\lambda).e^{-\lambda^2}$ is in the image of $C^1_\tau(G)$ under τ-spherical transform. This shows that the function $f * h_i$ is in $L^1_\tau(G)$.

In this way for each i, we can construct h_i and consider the collection of functions $\{h_i * f \mid i = 0, 1, 2, \ldots\}$. We note that τ-spherical transform of these collection of functions have no common zero in $S_{1,\delta/2}$. It is also clear that $f * h_i$ satisfies condition (2) of the hypothesis. Thus this collection satisfy the conditions of Theorem 6.1.2 for $p = 1, r = 0$. Therefore by Theorem 6.1.2 the ideal generated by $h_i * f$ in $L^1_\tau(G)$ is dense in $L^1_\tau(G)$. It is clear that $A = \{h_i\} \subseteq C^2_\tau(G)$.

For the case when n is odd a routine modification will prove the theorem. □

Next we take up the WTT for a degenerate case of the algebras described in Chapter 3. We recall that for $r \geq 0$

$$L^1_\tau(w_{2,r}) = \{f \text{ measurable } \mid \int_G |f(x)|\phi_0(x)(1+\sigma(x))^r\,dx < \infty\}$$

is an algebra and $L^p_\tau(G) \subseteq L^1_\tau(w_{2,r})$ for $1 \leq p < 2$. We shall prove the following Wiener-Tauberian theorem for this algebra.

Theorem 6.2.2. *For an index set Λ let $\{f_\alpha : \alpha \in \Lambda\}$ be a collection of functions such that for every $\alpha \in \Lambda$, $f_\alpha \in L^{p_\alpha}_\tau(G)$ for some $p_\alpha \in [1, 2)$. Then the ideal generated by the collection in $L^1_\tau(w_{2,r}), r \geq 0$ is dense in $L^1_\tau(w_{2,r})$ if and only if for any $\sigma \in \widehat{M}(\tau)$ the collection $\{\widehat{f_\alpha}(\sigma, \lambda) \mid \alpha \in \Lambda\}$ does not have common zero on $\mathbb{R}$.*

We note that unlike the Wiener-Tauberian theorems proved in Section 6.1, this theorem resembles the euclidean Wiener-Tauberian theorem.

For a tempered distribution W on $\mathbb{R}$ and $g \in \mathcal{S}(\mathbb{R})$, we define a tempered distribution $W.g$ on $\mathbb{R}$ by the rule $W.g(h) = W(gh)$, where $h \in \mathcal{S}(\mathbb{R})$.

We need the following two lemmas.

Lemma 6.2.3. *Let W be a tempered distribution on $\mathbb{R}$ and $\{g_\alpha \mid \alpha \in \Lambda\} \subseteq \mathcal{S}(\mathbb{R})$ be a collection of functions such that $W.g_\alpha = 0$ for all $\alpha \in \Lambda$. Then support of W is contained in $\cap_{\alpha\in\Lambda}\{\lambda \in \mathbb{R} \mid g_\alpha(\lambda) = 0\}$.*

Proof. We suppose that t is in the support of W but $g_{\alpha_0}(t) \neq 0$ for some $\alpha_0 \in \Lambda$. Since $g_{\alpha_0}(t) \neq 0$, there is a neighbourhood V_t of t such that $g_{\alpha_0}(s) \neq 0$ for all $s \in V_t$. Therefore there is a function $\psi \in C^\infty(\mathbb{R})$ such that $g_{\alpha_0}(s)\psi(s) = 1$ for all $s \in V_t$. Also t is in the support of W implies that there is a C_c^∞ function ϕ supported on V_t such that $W(\phi) \neq 0$. Therefore $g_{\alpha_0}\psi\phi = \phi$. Now $W.g_{\alpha_0} = 0$ implies $W.g_{\alpha_0}(\phi\psi) = 0$. This shows that $W(g_{\alpha_0}\phi\psi) = 0$. Therefore $W(\phi) = 0$, which is not possible. Hence $g_\alpha(t) = 0$ for all $\alpha \in \Lambda$. ☐

Lemma 6.2.4. *Let $\{f_\alpha \mid \alpha \in \Lambda\} \subseteq \mathcal{S}(\mathbb{R})$ (respectively $\{f_\alpha \mid \alpha \in \Lambda\} \subseteq \mathcal{S}(\mathbb{R})_e$) be a collection of functions such that $\{\widetilde{f_\alpha} \mid \alpha \in \Lambda\}$ has no common zero on $\mathbb{R}$, where $\widetilde{f_\alpha}$ is the euclidean Fourier transform of f_α. Then the ideal generated by f_α (under convolution) in $\mathcal{S}(\mathbb{R})$ (respectively in $\mathcal{S}(\mathbb{R})_e$) is dense in $\mathcal{S}(\mathbb{R})$ (respectively in $\mathcal{S}(\mathbb{R})_e$).*

Proof. Let V be the ideal generated by $\{f_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{S}(\mathbb{R})$. Let T be a tempered distribution on $\mathbb{R}$ such that $T(\phi) = 0$ for all $\phi \in V$. Then for any $g \in \mathcal{S}(\mathbb{R})$ we have $T(g * f_\alpha) = 0$ which implies that $(T * f_\alpha)(g) = 0$. Hence $T * f_\alpha \equiv 0$ as a distribution. Therefore $\widetilde{T * f_\alpha} \equiv 0$, i.e., $\widetilde{T}.\widetilde{f_\alpha} \equiv 0$. So support of $\widetilde{T}$ is contained in $\cap_{\alpha\in\Lambda}\{\lambda \mid \widetilde{f_\alpha}(\lambda) = 0\}$ by Lemma 6.2.3. By the hypothesis it follows that $\widetilde{T} \equiv 0$. Hence $T = 0$. Similar argument works when f_α's are also even. ☐

Proof of Theorem 6.2.2. Let n be even. Then $\sigma \in \widehat{M}(\tau)$ is unique and we shall omit the obvious σ from the notation. We take $f_\alpha \in L_\tau^{p_\alpha}(G)$ from the collection in the hypothesis. By Corollary 5.2.2 there exists a function $g \in C_\tau^{p_\alpha}(G)$ such that $\widehat{g}(\lambda) = e^{-\lambda^2}$ for all $\lambda \in S_{p_\alpha}$. Then for any $p \in (p_\alpha, 2)$, by Theorem 3.2.4 $\widehat{f_\alpha}$ is bounded on S_p. Using Cauchy's integral formula it follows that $\widehat{f_\alpha}.\widehat{g} \in \mathcal{S}(S_2)_e = \mathcal{S}(\mathbb{R})_e$. Using Corollary 5.2.2 again we have $f_\alpha * g \in C_\tau^2(G)$. Let $h_\alpha = f_\alpha * g$. In this way we get a collection $\{h_\alpha \mid \alpha \in \Lambda\}$ inside the ideal generated by $\{f_\alpha \mid \alpha \in \Lambda\}$ in $C_\tau^2(G)$. We notice that $\widehat{h_\alpha}(\lambda) \neq 0$ whenever $\widehat{f_\alpha}(\lambda) \neq 0$. Therefore the collection $\{\widehat{h_\alpha} \mid \alpha \in \Lambda\}$ does not have common zero on $\mathbb{R}$. It is thus sufficient to prove that the ideal generated by $\{h_\alpha \mid \alpha \in \Lambda\}$ in $L_\tau^1(w_{2,r})$ is dense in $L_\tau^1(w_{2,r})$.

As Abel transform is a topological isomorphism between $C_\tau^2(G)$ and $\mathcal{S}(\mathbb{R})_e$, we also have $\mathcal{A}h_\alpha \in \mathcal{S}(\mathbb{R})_e$ for all $\alpha \in \Lambda$. Let I and J respectively be the ideal generated by $\{h_\alpha \mid \alpha \in \Lambda\}$ in $C^2_\tau(G)$ and the ideal generated by $\{\mathcal{A}h_\alpha \mid \alpha \in \Lambda\}$ in $\mathcal{S}(\mathbb{R})_e$. Let $\overline{I}$ and $\overline{J}$ be the closure of I in $C^2_\tau(G)$ and the closure of J in $S(\mathbb{R})_e$ respectively. By Theorem 2.3.2 $\widetilde{\mathcal{A}h_\alpha}$ has no common zero. Hence by Lemma 6.2.4 $\overline{J} = \mathcal{S}(\mathbb{R})_e$.

Let $*_{\mathbb{R}}$ be the convolution on $\mathbb{R}$. For any $g_\alpha \in \mathcal{S}(\mathbb{R})_e$ as $g_\alpha *_{\mathbb{R}} \mathcal{A}h_\alpha = \mathcal{A}(\mathcal{A}^{-1}g_\alpha * h_\alpha)$ where $\mathcal{A}^{-1}g_\alpha \in C^2_\tau(G)$ we have $J \subset \mathcal{A}I$ and hence $\overline{\mathcal{A}I} = \mathcal{S}(\mathbb{R})_e$. But as $\mathcal{A}(\overline{I}) = \overline{\mathcal{A}I}$ we have $\mathcal{A}\overline{I} = \mathcal{S}(\mathbb{R})_e$. As Abel transform is a topological isomorphism between $C^2_\tau(G)$ and $\mathcal{S}(\mathbb{R})_e$ we finally have $\overline{I} = C^2_\tau(G)$. The result now follows from the facts that topology of $C^2_\tau(G)$ is stronger than the topology of $L^1_\tau(w_{2,r})$ and $C^2_\tau(G)$ is dense in $L^1_\tau(w_{2,r})$ (see Chapter 3).

A routine modification of the argument above proves the assertion for the case when n is odd. □

Remark 6.2.5. We note that this proof cannot be adopted for the Wiener-Tauberian theorems we have proved earlier. For example in the case of $L^1_\tau(G)$ we have to work with an L^1-tempered distribution W and a function $g \in \mathcal{S}(S_1)$ in Lemma 6.2.3. To make $W.g(\phi\psi)$ meaningful $\phi\psi$ has to be in $\mathcal{S}(S_1)$ which is the image of $C^1_\tau(G)$ under τ-spherical transform. Here ϕ cannot be compactly supported as ϕ has to be analytic on S_1°. So we can only assume that ϕ analytic on S_1° and very rapidly decreasing. Therefore to make $\phi\psi$ rapidly decreasing ψ cannot grow fast. But the function ψ chosen in the proof has to be analytic on S_1° and that takes away the liberty to make it decay arbitrarily outside V_t.

Chapter 7

Invariant Subspace Theorem of Schwartz

7.1 Schwartz's Theorem

We recall Schwartz's theorem on $\mathbb{R}$ ([56]).

Theorem 7.1.1. (Schwartz 1947) *Let f be a nonzero function in $C^\infty(\mathbb{R})$. Then the space $V_f = \overline{\{W * f \mid W \in C^\infty(\mathbb{R})'\}}$ contains the function $x \mapsto e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. Moreover the linear space generated by these $e^{i\lambda x}$ and their derivatives $\frac{d^j}{d\lambda^j} e^{i\lambda x}$, j=1, 2,... which are in V_f, is dense in V_f.*

For even functions on $\mathbb{R}$ we have the following version of this theorem (see [3]): For $\lambda \in \mathbb{C}$, let $\psi_\lambda(x) = \frac{e^{i\lambda x} + e^{-i\lambda x}}{2}$.

Theorem 7.1.2. *Let f be a nonzero even function in $C^\infty(\mathbb{R})$. Then the space $V_f = \overline{\{W * f \mid W \in C^\infty(\mathbb{R})'_e\}}$ contains ψ_λ for some $\lambda \in \mathbb{C}$. Moreover the linear space generated by these $\psi_\lambda(x)$ and their derivatives $\frac{d^j}{d\lambda^j}\psi_\lambda(x)$, j=1, 2,... which are in V_f, is dense in V_f.*

With G, K, τ as in Chapter 4, for τ-radial functions the object which corresponds to the plane wave $x \mapsto e^{i\lambda x}$ is the τ-spherical function $\phi^\tau_{\sigma,\lambda}$. In view of this we offer the following analogue of the theorem above.

Theorem 7.1.3. *Let f be a nonzero function in $C^\infty_\tau(G)$ and V_f is the closure of $\{W * f \mid W \in C^\infty_\tau(G)'\}$ (in $C^\infty_\tau(G)$ topology). Then for each $\sigma \in \widehat{M}(\tau)$, V_f contains the function $x \mapsto \phi^\tau_{\sigma,\lambda}(x)$ for some $\lambda \in \mathbb{C}$ (λ depends on σ). Moreover the linear space generated by these $\phi^\tau_{\sigma,\lambda}(x)$ and their derivatives $\frac{d^j}{d\lambda^j}\phi^\tau_{\sigma,\lambda}(x)$, $j = 1, 2, \ldots$ which are in V_f, is dense in V_f.*

Remark 7.1.4. We recall that for a τ-radial function f, its translations are not in general τ-radial. The appropriate operation which substitute translation in this setup is the *τ-radial translation*, by which we mean the projection of the translated function on the space of τ-radial functions. We note that the space V_f defined in the statement above is also the closure of the τ-radial translations of f. Precisely we define the τ-radial translation $\rho_x^\tau(f)$ of a function $f \in C_\tau^\infty(G)$ by an element $x \in G$ by

$$\rho_x^\tau(f)(y) = d_\tau \int_K (\overline{\chi_\tau} * L(x)f)(kyk^{-1})\,dk = d_\tau \int_K \int_K f(x^{-1}k_1^{-1}kyk^{-1})\overline{\chi_\tau(k_1)}\,dk_1\,dk.$$

Let W_f be the closure of $\{\rho_x^\tau(f) \mid x \in G\}$. Then $V_f = W_f$. Indeed using the denseness of $C_{c,\tau}^\infty(G)$ in $C_\tau^\infty(G)'$ it is not difficult to show that $V_f \subseteq W_f$. For the other side for every $x \in G$, we define $W_x \in C_\tau^\infty(G)'$ by $W_x(g) = g(x^{-1})$ for $g \in C_\tau^\infty(G)$. A straightforward computation then shows that $(W_x * f)(y) = \rho_x^\tau f(y)$. Thus Theorem 7.1.3 can be rewritten using τ-radial translations, instead of convolution with elements in $C_\tau^\infty(G)'$.

Proof of Theorem 7.1.3. (a) Suppose n is even. Then as in this case $\tau|_M$ contains a unique $\sigma \in \widehat{M}$ we will omit σ from the notation. Using the isomorphism of the Abel transform $\mathcal{A}$ between $C_\tau^\infty(G)'$ and $C^\infty(\mathbb{R})_e'$ and the reflexivity of $C^\infty(\mathbb{R})_e$ we define a linear operator $T : C_\tau^\infty(G) \longrightarrow C^\infty(\mathbb{R})_e$ by the following:

$$\mathcal{A}W(Tg) = \frac{1}{d_\tau} W(g) \text{ for } W \in C_\tau^\infty(G)', g \in C_\tau^\infty(G).$$

As $\mathcal{A}$ is a topological isomorphism between the dual spaces $C_\tau^\infty(G)'$ and $C^\infty(\mathbb{R})_e'$ (see Section 2.3 and Section 5.3), T is also a topological isomorphism.

Since $\mathcal{A}(W)(T\phi_\lambda^\tau) = \frac{1}{d_\tau} W(\phi_\lambda^\tau) = \widehat{W}(\lambda) = \mathcal{A}(W)(\psi_\lambda)$ for all $W \in C_\tau^\infty(G)'$ and as $\mathcal{A}$ is an isomorphism, we have $T\phi_\lambda^\tau = \psi_\lambda$, equivalently $T^{-1}\psi_\lambda = \phi_\lambda^\tau$.

We note that

$$\frac{\psi_{\lambda+h} - \psi_\lambda}{h} \to \frac{d}{d\lambda}\psi_\lambda \text{ as } h \to 0$$

in the topology of $C^\infty(\mathbb{R})_e$. Using the fact that T is a topological isomorphism we have

$$(T)^{-1}\left(\frac{\psi_{\lambda+h} - \psi_\lambda}{h}\right) \to (T)^{-1}\left(\frac{d}{d\lambda}\psi_\lambda\right)$$

as $h \to 0$ in the topology of $C^\infty_\tau(G)$. But $(T)^{-1}(\psi_{\lambda+h}) = \phi^\tau_{\lambda+h}$. Therefore

$$\lim_{h\to 0}\frac{\phi^\tau_{\lambda+h}-\phi^\tau_\lambda}{h} = (T)^{-1}\left(\frac{d}{d\lambda}\psi_\lambda\right).$$

In other words we have $T\left(\frac{d}{d\lambda}\phi^\tau_\lambda\right) = \frac{d}{d\lambda}(\psi_\lambda)$.

By repeated use of the argument above we get,

$$T(\frac{d^j}{d\lambda^j}\phi^\tau_\lambda) = \frac{d^j}{d\lambda^j}\psi_\lambda,\ j = 0,1,2,\ldots$$

and equivalently

$$T^{-1}(\frac{d^j}{d\lambda^j}\psi_\lambda) = \frac{d^j}{d\lambda^j}\phi^\tau_\lambda,\ j = 0,1,2,\ldots. \tag{7.1.1}$$

Now we consider Tf where f is as in the hypothesis.

Let $V_{Tf} = \overline{\{S * Tf \mid S \in C^\infty(\mathbb{R})'_e\}}$. We claim that

$$V_{Tf} = \overline{\{T(W * f) \mid W \in C^\infty_\tau(G)'\}}. \tag{7.1.2}$$

Note that every $S \in C^\infty(\mathbb{R})'_e$ can be written as $S = \mathcal{A}W$ for some $W \in C^\infty_\tau(G)'$. Thus we have $V_{Tf} = \overline{\{\mathcal{A}W * Tf \mid W \in C^\infty_\tau(G)'\}}$. It is sufficient to show that

$$\mathcal{A}W * Tf = T(W * f). \tag{7.1.3}$$

We take an arbitrary $S_1 \in C^\infty(\mathbb{R})'_e$. Then $S_1 = \mathcal{A}W_1$ for some $W_1 \in C^\infty_\tau(G)'$. Now,

$$\begin{aligned}\mathcal{A}(W_1)(\mathcal{A}W * Tf) &= \mathcal{A}W_1 * (\mathcal{A}W * Tf)(0)\\ &= (\mathcal{A}W_1 * \mathcal{A}W) * Tf(0)\\ &= \mathcal{A}(W_1 * W) * Tf(0)\\ &= \mathcal{A}(W_1 * W)(T(f))\\ &= \tfrac{1}{d_\tau}(W_1 * W)(f).\end{aligned}$$

On the other hand,

$$\mathcal{A}(W_1)(T(W * f)) = \frac{1}{d_\tau}W_1(W * f) = \frac{1}{d_\tau}W_1(f * W) = \frac{1}{d_\tau}(W_1 * W)(f).$$

As both sides of (7.1.3) are functions in $C^\infty(\mathbb{R})_e$ and $S_1 = \mathcal{A}W_1$ is an arbitrary element of the dual space, this proves (7.1.3) and establishes the claim.

Now it is clear from Theorem 7.1.2 that V_{Tf} contains ψ_λ for some λ and these ψ_λ and their derivatives $\frac{d^j}{d\lambda^j}\psi_\lambda$ which are in V_{Tf} generate a dense subspace in V_{Tf}. From (7.1.2) and (7.1.1) now the theorem follows using that T is a topological isomorphism.

(b) Suppose n is odd. As in (a) using the isomorphism $\mathcal{A}_{\sigma^+} : C_\tau^\infty(G)' \to C^\infty(\mathbb{R})'$ and the reflexivity of $C^\infty(\mathbb{R})$ we define a linear operator $T : C_\tau^\infty(G) \to C^\infty(\mathbb{R})$ by the following:

$$\mathcal{A}_{\sigma^+}W(Tg) = \frac{1}{d_\tau}W(g) \text{ for } W \in C_\tau^\infty(G)', g \in C_\tau^\infty(G).$$

Since $\mathcal{A}_{\sigma^+}$ is a topological isomorphism between $C_\tau^\infty(G)'$ and $C^\infty(\mathbb{R})'$ (see Section 2.3 and Section 5.3), T is also a topological isomorphism. Also since $\mathcal{A}_{\sigma^+}(W)(T\phi^\tau_{\sigma^+,\lambda}) = \frac{1}{d_\tau}W(\phi^\tau_{\sigma^+,\lambda}) = \widehat{W}_{\sigma^+}(\lambda) = \mathcal{A}_{\sigma^+}(W)(e^{-i\lambda(\cdot)})$ for all $W \in C_\tau^\infty(G)'$ and as $\mathcal{A}_{\sigma^+}$ is an isomorphism, we have $T\phi^\tau_{\sigma^+,\lambda} = e^{-i\lambda(\cdot)}$. Following the steps similar to (a) we get

$$T(\frac{d^j}{d\lambda^j}\phi^\tau_{\sigma^+,\lambda}) = \frac{d^j}{d\lambda^j}e^{-i\lambda(\cdot)}, j = 0, 1, 2, \cdots$$

and equivalently

$$T^{-1}(\frac{d^j}{d\lambda^j}e^{-i\lambda(\cdot)}) = \frac{d^j}{d\lambda^j}\phi^\tau_{\sigma^+,\lambda}, j = 0, 1, 2, \cdots \tag{7.1.4}$$

Let $V_{Tf} = \overline{\{S * Tf \mid S \in C^\infty(\mathbb{R})'\}}$. Then as in ($a$) we have

$$V_{Tf} = \overline{\{T(W * f) \mid W \in C_\tau^\infty(G)'\}}.$$

By Theorem 7.1.1 V_{Tf} contains $e^{-i\lambda(\cdot)}$ for some $\lambda \in \mathbb{C}$. These $e^{-i\lambda(\cdot)}$ and their derivatives $\frac{d^j}{d\lambda^j}e^{-i\lambda(\cdot)}$ which are in V_{Tf} generate a dense subspace in V_{Tf}. From (7.1.4) it follows that V_f contains $\phi^\tau_{\sigma^+,\lambda}$ and these $\phi^\tau_{\sigma^+,\lambda}$ and their derivatives $\frac{d^j}{d\lambda^j}\phi^\tau_{\sigma^+,\lambda}$ generate a dense subspace in V_f. Since $\phi^\tau_{\sigma^+,\lambda} = \phi^\tau_{\sigma^-,-\lambda}$, the theorem follows for $\sigma = \sigma^-$ also. This completes the proof. □

We shall conclude this section with the observation that if a higher derivative of a τ-spherical function belongs to V_f then so does any of its lower derivative. Apart from those used in the theorem above we shall use the notation:

$$\phi^\tau_{\lambda_0,k}(x) = \frac{d^k}{d\lambda^k}\,\phi^\tau_\lambda(x)\big|_{\lambda=\lambda_0}, \phi^\tau_{\sigma^\pm,\lambda_0,k}(x) = \frac{d^k}{d\lambda^k}\,\phi^\tau_{\sigma^\pm,\lambda}(x)\big|_{\lambda=\lambda_0} \text{ and } \psi_{\lambda_0,k} = \frac{d^k}{d\lambda^k}\psi_\lambda\,|_{\lambda=\lambda_0}.$$

We claim that if $\phi^\tau_{\lambda_0,k}(x)$ (respectively $\phi^\tau_{\sigma^\pm,\lambda_0,k}(x)$) is in V_f for n even (respectively for n odd) then so is $\phi^\tau_{\lambda_0,l}(x)$ (respectively $\phi^\tau_{\sigma^\pm,\lambda_0,l}(x)$) for any $0 \le l \le k$.

We consider the case when n is even. We suppose that $\phi^\tau_{\lambda_0,1} \in V_f$. Then

$\psi_{\lambda_0,1} = T(\phi^\tau_{\lambda_0,1}) \in V_{Tf}$. This implies that $s \mapsto \psi_{\lambda_0,1}(s+t) + \psi_{\lambda_0,1}(s-t) \in V_{Tf}$ for any $t \in \mathbb{R}$ as V_{Tf} is invariant under convolutions by even compactly supported distributions (hence invariant under *even translations*). We use the following identity

$$\psi_{\lambda_0,1}(s+t) + \psi_{\lambda_0,1}(s-t) - 2\psi_{\lambda_0}(t)\psi_{\lambda_0,1}(s) = -2t\sin(\lambda_0 t)\psi_{\lambda_0}(s)$$

to infer that the map $s \mapsto \psi_{\lambda_0}(s) = T(\phi^\tau_{\lambda_0})(s) \in V_{Tf}$. Therefore $\phi^\tau_{\lambda_0} \in V_f$, since T is a topological isomorphism. A repeated use of this argument proves the assertion.

For the case n odd the argument is similar except we have to use the identity:

$$\frac{d}{d\lambda}e^{-i\lambda(s+t)} - e^{-i\lambda t}\frac{d}{d\lambda}e^{-i\lambda s} = (-ite^{-i\lambda t})e^{-i\lambda s} \text{ for all } s, t \in \mathbb{R}.$$

7.2 Some Related Results

In this section we continue to work with G, K and τ as in the previous section. The following is a consequence of Theorem 7.1.3 which can be thought of as a Wiener-Tauberian theorem for distributions.

Corollary 7.2.1. *Let $\{T_\alpha\}_{\alpha\in\Lambda}$ be a collection of compactly supported τ-radial distributions on G such that for each $\sigma \in \widehat{M}(\tau)$, $\widehat{T_\alpha}(\sigma, \cdot)$ has no common zero in $\mathbb{C}$. Then the space generated by $\{T_\alpha\}_{\alpha\in\Lambda}$ defined by $\{T_\alpha * W \mid W \in C^\infty_\tau(G)', \alpha \in \Lambda\}$ is dense in $C^\infty_\tau(G)'$.*

Proof. We suppose that the space generated by $\{T_\alpha\}_{\alpha\in\Lambda}$ is not dense in $C^\infty_\tau(G)'$. Then there exists $f \in C^\infty_\tau(G)$ such that $(T_\alpha * W)(f) = 0$ for all $W \in C^\infty_\tau(G)', \alpha \in \Lambda$. Therefore we have $T_\alpha(W * f) = 0$ for all $W \in C^\infty_\tau(G)', \alpha \in \Lambda$. Hence $T_\alpha(V_f) = 0$ for all $\alpha \in \Lambda$. But for each $\sigma \in \widehat{M}(\tau)$ by Theorem 7.1.3 V_f contains $\phi^\tau_{\sigma,\lambda_0}$ for some $\lambda_0 \in \mathbb{C}$. Hence $\widehat{T_\alpha}(\sigma, \lambda_0) = 0$ for all $\alpha \in \Lambda$. This contradicts the hypothesis. This completes the proof. ☐

Mean periodic functions: A function in $f \in C^\infty_\tau(G)$ is *mean periodic* if there is a $W \in C^\infty_\tau(G)'$ such that $W * f = 0$ (see [24]), equivalently if V_f is proper subspace of $C^\infty_\tau(G)$, where V_f is as defined in Theorem 7.1.3. We shall see that if a nonzero τ-radial C^∞ function is in one of the following classes then it is not mean periodic:

(1) $f \in L^p_\tau(G)$, $1 \leq p \leq 2$,

(2) $f \in L^{p,\infty}_\tau(G), 1 < p < 2$, that is f is weak-L^p,

(3) $f \in L^1_\tau(G, w_{2,0}) = \{f$ measurable and τ-radial $\mid \int_G |f(x)|\phi_0(x)dx < \infty\}$,

We notice that by the properties of ϕ_λ mentioned in Chapter 1, it follows that $L^1_\tau(G, w_{2,0})$ contains the classes described in (2) and hence in particular contains $L^p_\tau(G), 1 < p < 2$. It is also clear that $L^1_\tau(G, w_{2,0}) \supset L^1_\tau(G)$ as ϕ_0 is bounded by 1. Therefore we shall check the property only for functions in $L^1_\tau(G, \phi_0) \cap C^\infty_\tau(G)$ and in $L^2_\tau(G) \cap C^\infty_\tau(G)$.

Let f be a nonzero function either in $L^1_\tau(G, w_{2,0}) \cap C^\infty_\tau(G)$ or in $L^2_\tau(G) \cap C^\infty_\tau(G)$. We suppose that for some $W \in C^\infty_\tau(G)'$, $W * f = 0$. Then $\widehat{W}(\sigma, \lambda)\widehat{f}(\sigma, \lambda) = 0$ for almost every $\lambda \in \mathbb{R}$. Since f is a nonzero function $\widehat{f}(\sigma, \cdot)$ is nonzero on a set of positive measure. Therefore $\widehat{W}(\sigma, \cdot)$ is zero on that set of positive measure. Since by Paley-Wiener Theorem (see Theorem 5.3.3) $\widehat{W}(\sigma, \cdot)$ is an entire function, it follows that $\widehat{W}(\sigma, \cdot) \equiv 0$. Therefore $W = 0$.

The range $1 \leq p \leq 2$ is sharp in the sense that for any $q > 2$, there are functions in $L^q_\tau(G) \cap C^\infty_\tau(G)$ which are mean periodic. Indeed for a fixed $q > 2$ we consider the function $f = \phi^\tau_{\sigma,\lambda}$ for some $\lambda \in S^\circ_{q'}$. Then $f \in L^q_\tau(G) \cap C^\infty_\tau(G)$. It is easy to construct a nonzero $W \in C^\infty_\tau(G)'$ such that $\widehat{W}(\sigma, \lambda) = 0$ (see Theorem 5.3.3). For this W,

$$\begin{aligned} W * \phi^\tau_{\sigma,\lambda}(x) &= \int_K (W * \phi^\tau_{\sigma,\lambda})(k^{-1}xk)\, dk = \int_K W^\vee\left(L(k^{-1}xk)(\phi^\tau_{\sigma,\lambda})^\vee(\cdot)\right) dk \\ &= \int_K W(\phi^\tau_{\sigma,\lambda}(\cdot k^{-1}xk))\, dk = \frac{1}{d_\tau} W(\phi^\tau_{\sigma,\lambda}(x)\phi^\tau_{\sigma,\lambda}(\cdot)) = \widehat{W}(\sigma, \lambda)\phi^\tau_{\sigma,\lambda}(x). \end{aligned}$$

Therefore $W * \phi^\tau_{\sigma,\lambda}(x) = 0$. This shows that $\phi^\tau_{\sigma,\lambda}$ is mean periodic.

A question connected with failure of the Wiener-Tauberian Theorem:

As the functions $\phi^\tau_{\sigma,\lambda} \in L^{q,\infty}_\tau(G)$ when $\lambda \in S_{q'}, q > 2$, (see Chapter 2) a natural question at this point is: Does an arbitrary closed L^1-invariant subspace of $L^{q,\infty}_\tau(G)$ contain $\phi^\tau_{\sigma,\lambda}$ for some λ? That is we ask if we can have an analogue of Schwartz's theorem where the space $C^\infty_\tau(G)$ is replaced by $L^{q,\infty}_\tau(G)$. We shall show that the answer is negative. Precisely for any $q > 2$ there is a function $g \in L^{q,\infty}_\tau(G)$ such that the closure of $I_g = \{\beta * g \mid \beta \in L^1_\tau(G)\}$ does not contain $\phi^\tau_{\sigma,\lambda}$ for any $\sigma \in \widehat{M}(\tau), \lambda \in \mathbb{C}$. Interestingly this is related with the failure of the Wiener-Tauberian theorem of the commutative Banach algebra $L^{p,1}_\tau(G), 1 \leq p < 2$ which we shall discuss now.

We fix a $p \in [1, 2)$. We have noted in Chapter 2 that the Gelfand spectrum of $L^{p,1}_\tau(G)$ is S_p and hence the τ-spherical transform of a function $f \in L^{p,1}_\tau(G)$ has to be necessarily nonvanishing on S_p so that f generates a dense ideal under convolution in $L^{p,1}_\tau(G)$. However we recall that this condition is not sufficient. A counter example is constructed in Chapter 6 to show that there is a function f in $L^{p,1}_\tau(G)$ whose τ-spherical transform does not vanish anywhere in S_p, but the ideal generated by f is not dense in $L^{p,1}_\tau(G)$.

We take that function $f \in L^{p,1}_\tau(G)$, whose τ-spherical transform is nonzero for all $\lambda \in S_p$ but $I_f = \{\beta * f \mid \beta \in L^1_\tau(G)\}$ is not dense in $L^{p,1}_\tau(G)$. Then there exists $g \in L^{p',\infty}_\tau(G)$ such that $\int_G g(x)\overline{h(x)}\,dx = 0$ for all $h \in \overline{I_f}$. A use of Fubini's theorem shows that $\int_G f(x)\overline{k(x)}\,dx = 0$ for all $k \in \overline{I_g}$, where $I_g = \{\beta * g \mid \beta \in L^1_\tau(G)\}$. We suppose that $\phi^\tau_{\sigma,\lambda_0} \in \overline{I_g} \subset L^{p',\infty}_\tau(G)$ for some $\lambda_0 \in \mathbb{C}$. Then $\int_G f(x)\overline{\phi^\tau_{\sigma,\lambda_0}(x)}\,dx = 0$, i.e., $\int_G f(x)\phi^\tau_{\sigma,\overline{\lambda_0}}(x^{-1})\,dx = 0$ by Proposition 2.1.6. Therefore $\widehat{f}(\sigma,\overline{\lambda_0}) = 0$. But as $\phi^\tau_{\sigma,\lambda_0} \in L^{p',\infty}_\tau(G)$, $\lambda_0 \in S_p$ (see Chapter 6). Hence $\overline{\lambda_0} \in S_p$. This contradicts the assumption.

We may point out that the argument above works also for L^1-invariant subspace of $L^{q,r}_\tau(G)$, with $q > 2, 1 \le r < \infty$. Precisely, we recall that (see Chapter 2) for $q > 2$, if $\lambda \in S^\circ_{q'}$, then $\phi^\tau_{\sigma,\lambda} \in L^{q,r}_\tau(G)$ for $1 \le r < \infty$. We can show as above that there exists a function $g \in L^{q,r}_\tau(G)$, such that the closed L^1-invariant subspace generated by g does not contain any $\phi^\tau_{\sigma,\lambda}$.

Conversely we suppose that there exists a function $g \in L^{q,r}_\tau(G)$ with $q > 2$, $1 \le r < \infty$ such that the closed L^1-invariant subspace I_g generated by g does not contain any $\phi^\tau_{\sigma,\lambda}$. Then for any $\lambda \in S^\circ_{q'}$ there exists a function $f_\lambda \in L^{q',r'}_\tau(G)$ such that $f_\lambda(I_g) = 0$ and $f_\lambda(\phi^\tau_{\sigma,\lambda}) \neq 0$. Thus we get a collection $\mathcal{F} = \{f_\lambda \mid \lambda \in S^\circ_{q'}\}$ in $L^{q',r'}_\tau(G)$ such that their τ-spherical transform do not have common zero in $S^\circ_{q'}$. But as $f_\lambda(I_g) = 0$, we have $g(I_{f_\lambda}) = 0$ for all $\lambda \in S^\circ_{q'}$ where $I_{f_\lambda} = \{\beta * f_\lambda \mid \beta \in L^1_\tau(G)\}$. Therefore $g(I) = 0$ where I is the closure of $\mathsf{span} \cup_{\lambda \in S^\circ_{q'}} I_{f_\lambda}$. Thus assuming that there is a L^1-invariant subspace of $L^{q,r}_\tau(G)$ which does not contain any $\phi^\tau_{\sigma,\lambda}$, we can show that the Wiener-Tauberian theorem for $L^{q',r'}_\tau(G)$ based on nonvanishing τ-spherical transform on its domain fails.

Chapter 8

Revisiting Schwartz's Theorem on $\mathrm{SL}(2,\mathbb{R})$

8.1 Statement of the Theorem

In this Chapter we shall prove a version of the Schwartz's theorem for the group $\mathrm{SL}(2,\mathbb{R})$ without any restriction on K-finiteness. This result will strengthen the analogue of Schwartz's theorem proved in [24]. Our method is based on a generalization of the notion of *simplicity* (see [34, p.315] for the corresponding notion on right K-invariant functions). Our theorem is inspired by a result of Helgason and Sengupta where the corresponding result is proved for rank one symmetric spaces (see [36]).

Throughout this chapter G and $\mathfrak{g}$ will denote the group $\mathrm{SL}(2,\mathbb{R})$ and its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ respectively. In the notation of Chapter 1, here $K = \mathrm{SO}(2) \cong S^1$ which we parametrize as $\{k_\theta \mid \theta \in [0, 2\pi)\}$ and equip with the normalized Haar measure $dk = d\theta/2\pi$. The unitary dual $\widehat{K}$ of K is parametrized by integers. Precisely $\widehat{K} = \{e_n \mid n \in \mathbb{Z}\}$ where $e_n(k_\theta) = e^{in\theta}$. We note that for any integer n, (G, K, e_n) is a Gelfand triple (see [4]). By abuse of notation we shall denote the K-types simply by the integers n. For each pair of integers (m, n) of the same parity (i.e. either both even or both odd integers) we have the (m, n)-th spherical function $\Phi_\lambda^{m,n}$ which is the object corresponding to the elementary spherical function in analysis of K-biinvariant functions. In particular $\Phi_\lambda^{n,n}$ is the n-spherical function in the language of Section 2.1. An explicit definition of $\Phi_\lambda^{m,n}$ and other required preliminaries are given in the next Section.

The following theorem is due to Ehrenpreis and Mautner (see [24]).

Theorem 8.1.1 (Ehrenpreis-Mautner). *Let V be a (both sided) translation in-*

variant nonzero closed subspace of $C^\infty(G)$. Then either for each $m,n \in 2\mathbb{Z}$ or for each $m,n \in 2\mathbb{Z}+1$, there exists $\lambda \in \mathbb{C}$ such that V contains $x \mapsto \Phi_\lambda^{m,n}(x)$ where λ depends on m,n.

We shall prove the following stronger version. Here $C^\infty(G)_n$ is the set of right n type functions in $C^\infty(G)$.

Theorem 8.1.2. *Let V_n be a (left) translation invariant nonzero closed subspace of $C^\infty(G)_n$. Then there exists $\lambda \in \mathbb{C}$ such that V_n contains the function*

$$e^n_{\lambda,k} : x \mapsto e^{-\lambda H(x^{-1}k^{-1})} e_n(K(x^{-1}k^{-1})^{-1}) \text{ for all } k \in K.$$

If V is a (both sided) translation invariant nonzero closed subspace of $C^\infty(G)$ then either for each $n \in 2\mathbb{Z}$ or for each $n \in 2\mathbb{Z}+1$, there exists $\lambda \in \mathbb{C}$ which depends on n such that V contains the function $x \mapsto e^n_{\lambda,k}(x)$ for all $k \in K$.

The spaces $C^\infty(G)$ and $C^\infty(G)_n$ are equipped with the usual Fréchet topology (see Chapter 2). We note that the elements of $C^\infty(G)_n$ can be considered as smooth sections of the line bundle E_n (see Introduction for definition) associated with the K-type n. Here the object which naturally corresponds with the exponential function $e^{i\lambda x}$ is the function $e^n_{\lambda,k}$. It is an eigensection (of the Laplace-Beltrami operator of SL$(2,\mathbb{R})/$SO(2)) of the bundle E_n. It is not difficult to verify that Theorem 8.1.1 follows from Theorem 8.1.2 as $\int_K e^n_{\lambda,k}(x)\overline{e_m(k)}\,dk = \Phi^{n,m}_{\lambda-1}(x)$. Theorem 8.1.2 will be proved in Section 8.3.

8.2 Preliminaries

We need the following additional preliminaries apart from those given above. Four important elements of $\mathfrak{g}$ are

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \overline{Y} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

We suppose that $\exp(\theta X) = k_\theta, \exp(2tH) = a_t, \exp(\xi Y) = n_\xi$. Then $A = \{a_t : t \in \mathbb{R}\}$ is a vector subgroup and $N = \{n_\xi : \xi \in \mathbb{R}\}$ is a nilpotent subgroup of G. We fix the Iwasawa decomposition $G = KAN$ and if for $x \in G$, $x = k_\theta a_t n_\xi$ is its corresponding decomposition then we write $H(x)$ for t and $K(x)$ for k_θ. Clearly H is left K-invariant and right N-invariant. In this group $m_\alpha = 0$ and $m_{2\alpha} = 1$ and hence $\rho = 1$. The Haar measure splits according to the Iwasawa decomposition as

$dx = e^{2t} dk\, dt\, dn$ where $dk = dk_\theta = \frac{d\theta}{2\pi}$ is the normalized Haar measure of K and $dn = dn_\xi = d\xi$ as well as $da = da_t = dt$ are both Lebesgue measures on $\mathbb{R}$.

A complex valued function f on G is said to be of *left* (respectively *right*) K-type n if $f(k_\theta x) = e^{in\theta} f(x)$ (respectively $f(xk_\theta) = f(x)e^{in\theta}$). Functions of left K-type m and right K-type n are also referred as functions of type (m, n). By $f_{m,n}$ we denote the projection of f in left type m and right type n, which is defined (whenever possible) by

$$f_{m,n}(x) = \int_K \int_K f(k_\theta x k_\phi) e^{-im\theta} e^{-in\phi} dk_\theta dk_\phi.$$

It can be verified that $f_{m,n}$ is a (m, n)-type function. It will also be called to as the (m, n)-th component of f. Let $C^\infty(G)_{m,n}$ and $C^\infty(G)_n$ respectively denote the set of (m, n) type and right n type C^∞ functions on G. We recall that (see [4]) if $f \in C^\infty(G)$ then $f = \sum_{m,n\in\mathbb{Z}} f_{m,n}$ in the $C^\infty(G)$ topology. The following results are easy to verify:

(1) If f is of type (m, n) where m is odd and n is even then $f \equiv 0$,

(2) If $n \neq r$ then $f_{m,n} * g_{r,s} \equiv 0$ and $f_{m,n} * g_{n,s}$ is of type (m, s),

(3) If either $m \neq -r$ or $n \neq -s$ then $\int_G f_{m,n}(x) g_{r,s}(x) dx = 0$.

The complexification of $\mathfrak{g}$ is denoted by $\mathfrak{g}_\mathbb{C}$ and the universal algebra of $\mathfrak{g}_\mathbb{C}$ is denoted by $\mathcal{U}$. The Casimir element Ω of $\mathcal{U}$ is defined by

$$\Omega = H^2 + H - Y\overline{Y}.$$

The centre $\mathcal{Z}$ of $\mathcal{U}$ is generated by Ω. Each $X \in \mathfrak{g}$ gives a left invariant vector field L_X and right invariant vector field R_X by the formulas

$$\begin{aligned} L_X f(x) &= f(x; X) = \tfrac{d}{dt} f(x \exp(tX))\big|_{t=0} \\ R_X f(x) &= f(X; x) = \tfrac{d}{dt} f(\exp(tX)x)\big|_{t=0}. \end{aligned}$$

These identifications give an isomorphism between $\mathcal{U}$ and the algebra of left invariant differential operators on G, and an anti-isomorphism between $\mathcal{U}$ and the algebra of right invariant operators. If $g_1, g_2 \in \mathcal{U}$ are considered as right invariant and left invariant differential operators respectively, then their action at any $x \in G$ will be denoted by $f(g_1; x; g_2)$. The elements of $\mathcal{Z}$ corresponds to the biinvariant differential operators on G.

Let $M=\{\pm \mathrm{I}\}$ where I is the identity matrix. Then M is the centre of G and $\widehat{M}=\{\sigma^+,\sigma^-\}$ where σ^- is the only nontrivial element of $\widehat{M}$. We let $\mathbb{Z}^\sigma=2\mathbb{Z}$ if $\sigma=\sigma^+$ and $\mathbb{Z}^\sigma=2\mathbb{Z}+1$ if $\sigma=\sigma^-$. Corresponding to each $\sigma\in\widehat{M}$ and $\lambda\in\mathbb{C}$ the principal series representation $\pi_{\sigma,\lambda}$ in the *compact picture* is given by

$$\pi_{\sigma,\lambda}(x)f(k)=e^{-(\lambda+1)H(x^{-1}k^{-1})}f(K(x^{-1}k^{-1})^{-1}) \text{ for } f\in L^2(K,\sigma) \tag{8.2.1}$$

where $L^2(K,\sigma)$ is the subspace of $L^2(K)$ generated by $\{e_n \mid n\in\mathbb{Z}^\sigma\}$. Precisely $L^2(K,\sigma)=\{f\in L^2(K)\mid f(km)=\sigma(m)f(k), \text{ for all } k\in K,\sigma\in M\}$. We also have the following relation:

$$\pi_{\sigma,\lambda}(x)^*=\pi_{\sigma,-\overline{\lambda}}(x^{-1}) \text{ for each } x\in G,\lambda\in\mathbb{C},$$

where $\pi_{\sigma,\lambda}(x)^*$ is the adjoint of the operator $\pi_{\sigma,\lambda}(x)$. For $\sigma\in\widehat{M},\lambda\in\mathbb{C}$ and $m,n\in\mathbb{Z}^\sigma$ we define

$$\Phi^{m,n}_{\sigma,\lambda}(x)=\langle\pi_{\sigma,\lambda}(x)e_m,e_n\rangle=\int_K e^{-(\lambda+1)H(x^{-1}k^{-1})}e_m(K(x^{-1}k^{-1})^{-1})\overline{e_n(k)}\,dk$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(K,\sigma)$. Thus $\Phi^{m,n}_{\sigma,\lambda}(x)$ is the (m,n)-th *matrix coefficient* of the operator $\pi_{\sigma,\lambda}(x)$. In particular when $m=n=0$ and $\sigma=\sigma^+$, then $\Phi^{0,0}_{\sigma,\lambda}$ is the elementary spherical function ϕ_λ. It is easy to check that $\Phi^{m,n}_{\sigma,\lambda}$ is of type (n,m). For an (m,n) type function f on G its (m,n)-th spherical transform is defined by

$$\widehat{f}(\sigma,\lambda)_{m,n}=\int_G f(x)\Phi^{m,n}_{\sigma,\lambda}(x^{-1})\,dx.$$

It also follows from (3) above that if f is of type (m,n), then

$$\widehat{f}(\sigma,\lambda)_{r,s}=\int_G f(x)\Phi^{r,s}_{\sigma,\lambda}(x^{-1})\,dx=0 \text{ if } r\neq m \text{ or if } s\neq n.$$

We note that an integer n determines a $\sigma=\sigma(n)\in\widehat{M}$ by the condition $n\in\mathbb{Z}^\sigma$. Therefore sometimes we may omit the obvious σ and write $\Phi^{m,n}_\lambda$ for $\Phi^{m,n}_{\sigma,\lambda}$ and $\widehat{f}(\lambda)_{m,n}$ for $\widehat{f}(\sigma,\lambda)_{m,n}$.

The infinitesimal representation of $\mathfrak{g}$ induced by $\pi_{\sigma,\lambda}$ is given by

$$\pi_{\sigma,\lambda}(L)v=\frac{d}{dt}\pi_{\sigma,\lambda}(\exp tL)v\Big|_{t=0} \text{ for } L\in\mathfrak{g},v\in L^2(K,\sigma).$$

We define $E=2H+i(Y-\overline{Y})$ and $F=-2H+i(Y-\overline{Y})$. Then $\{X,E,F\}$ form a basis of $\mathfrak{g}_\mathbb{C}$ and for $\sigma\in\widehat{M},\lambda\in\mathbb{C}$ and $n\in\mathbb{Z}^\sigma$, we have (see [4, 4.4 and

4.6])

$$\pi_{\sigma,\lambda}(E)e_n = (n+\lambda+1)e_{n+2}$$
$$\pi_{\sigma,\lambda}(F)e_n = (n-\lambda-1)e_{n-2}$$
$$\pi_{\sigma,\lambda}(X)e_n = ine_n.$$

Using these it is easy to verify that:

$$\Phi^{m,n}_{\sigma,\lambda}(E;x) = (n+\lambda-1)\Phi^{m,n-2}_{\sigma,\lambda}(x) \tag{8.2.2}$$
$$\Phi^{m,n}_{\sigma,\lambda}(x;E) = (m+\lambda+1)\Phi^{m+2,n}_{\sigma,\lambda}(x) \tag{8.2.3}$$
$$\Phi^{m,n}_{\sigma,\lambda}(F;x) = (n-\lambda+1)\Phi^{m,n+2}_{\sigma,\lambda}(x) \tag{8.2.4}$$
$$\Phi^{m,n}_{\sigma,\lambda}(x;F) = (m-\lambda-1)\Phi^{m-2,n}_{\sigma,\lambda}(x) \tag{8.2.5}$$

for $m,n \in \mathbb{Z}^\sigma$ and $\lambda \in \mathbb{C}$.

It follows from equation (8.2.4), (8.2.5) that $\Phi^{m+2,n-2}_{\sigma,\lambda}(F;x^{-1};F)$ is a constant multiple of $\Phi^{m,n}_{\sigma,\lambda}(x^{-1})$ where the constant is a polynomial in λ, depending on m,n. We take a nonzero function $f \in C^\infty_c(G)_{m,n}$. Then there is a $\lambda \in \mathbb{C}$ such that $\int_G f(x)\Phi^{m+2,n-2}_{\sigma,\lambda}(F;x^{-1};F)\,dx$ is nonzero. Hence by change of variable it follows that $\int_G f(F;x;F)\Phi^{m+2,n-2}_{\sigma,\lambda}(x^{-1})\,dx$ is nonzero. This implies that $f(F;\cdot;F)$ is of type $(m+2,n-2)$. Through similar argument we can show that $f(x;E), f(x;F)$ and $f(E;x)$ are of type $(m,n+2),(m,n-2)$ and $(m-2,n)$ respectively. Thus given integers (r,s) of the parity of (m,n) repeating the process above we can find D_1, D_2 in $\mathcal{U}$ such that $f(D_1;x;D_2)$ is of type (r,s).

These facts and a density argument yields the following:

Proposition 8.2.1. *For a right n type C^∞ function f on G and a fixed integer m (same parity of n), there is a $D \in \mathcal{U}$ such that $f(\cdot;D)$ is a right m type function.*

8.3 Proof of the Theorem

The following lemma gives a *symmetry property* of $\Phi^{n,n}_\lambda$.

Lemma 8.3.1. *For each $n \in \mathbb{Z}^\sigma$ we have,*

$$\begin{aligned}&\Phi^{n,n}_{\sigma,\lambda}(y^{-1}x)\\ =\ &\int_K \left[e^{-(\lambda+1)H(x^{-1}k^{-1})}e^{-(-\lambda+1)H(y^{-1}k^{-1})}e_n(K(x^{-1}k^{-1})^{-1})\overline{e_n(K(y^{-1}k^{-1})^{-1})}\right]dk.\end{aligned}$$

Proof. We have

$$\begin{aligned}\Phi^{n,n}_{\sigma,\lambda}(y^{-1}x) &= \langle \pi_{\sigma,\lambda}(y^{-1}x)e_n, e_n\rangle \\ &= \langle \pi_{\sigma,\lambda}(x)e_n, (\pi_{\sigma,\lambda}(y^{-1}))^{*}e_n\rangle \\ &= \langle \pi_{\sigma,\lambda}(x)e_n, \pi_{\sigma,-\bar{\lambda}}(y)e_n\rangle \\ &= \int_K \pi_{\sigma,\lambda}(x)e_n(k)\overline{\pi_{\sigma,-\bar{\lambda}}(y)e_n(k)}\, dk.\end{aligned}$$

The lemma now follows from the definition of $\pi_{\sigma,\lambda}$ (see equation (8.2.1)). □

Definition 8.3.2. *For a fixed $n \in \mathbb{Z}^\sigma$ and $\lambda \in \mathbb{C}$, the Poisson transform of $F \in L^2(K,\sigma)$ is defined by*

$$\mathcal{P}^n_\lambda(F)(x) = \int_K F(k)e^{-(\lambda+1)H(x^{-1}k^{-1})}e_n(K(x^{-1}k^{-1})^{-1})\, dk.$$

The map $x \mapsto \mathcal{P}^n_\lambda(F)(x)$ is of right type n. For n and λ as above and $k \in K$ the Helgason Fourier transform for a right n type function f is defined by

$$\widetilde{f}(\lambda,k,n) = \int_G f(x)e^{(\lambda-1)H(x^{-1}k^{-1})}e_n(K(x^{-1}k^{-1}))\, dx, \text{ whenever the integral converges.}$$

Since M is the centre of G, $H(x^{-1}k^{-1}m^{-1}) = H(x^{-1}k^{-1})$ and $K(x^{-1}k^{-1}m^{-1}) = K(x^{-1}k^{-1})m^{-1}$. Therefore $\widetilde{f}(\lambda,km,n) = e_{-n}(m)\widetilde{f}(\lambda,k,n) = \sigma(m)\widetilde{f}(\lambda,k,m)$, i.e., $\widetilde{f}(\lambda,\cdot,m) \in L^2(K,\sigma)$. The following lemma gives a relation between the Poisson transform and the Helgason Fourier transform.

Lemma 8.3.3. *For a right n-type function f, $f * \Phi^{n,n}_{\sigma,\lambda}(x) = \mathcal{P}^n_\lambda(\widetilde{f}(\lambda,.,n))(x)$ for $n \in \mathbb{Z}^\sigma$, whenever both sides make sense.*

Proof. Using Lemma 8.3.1 and Fubini's theorem we have

$$\begin{aligned}&f * \Phi^{n,n}_{\sigma,\lambda}(x) \\ =\ & \int_G f(y)\Phi^{n,n}_{\sigma,\lambda}(y^{-1}x)\, dy \\ =\ & \int_G f(y)\int_K e^{-(\lambda+1)H(x^{-1}k^{-1})}e^{(\lambda-1)H(y^{-1}k^{-1})}e_n(K(x^{-1}k^{-1})^{-1})e_n(K(y^{-1}k^{-1}))\, dk\, dy \\ =\ & \int_K e^{-(\lambda+1)H(x^{-1}k^{-1})}e_n(K(x^{-1}k^{-1})^{-1})\int_G f(y)e^{(\lambda-1)H(y^{-1}k^{-1})}e_n(K(y^{-1}k^{-1}))\, dy\, dk \\ =\ & \int_K e^{-(\lambda+1)H(x^{-1}k^{-1})}e_n(K(x^{-1}k^{-1})^{-1})\widetilde{f}(\lambda,k,n)\, dk.\end{aligned}$$

Hence

$$f * \Phi^{n,n}_{\sigma,\lambda}(x) = \mathcal{P}^n_\lambda(\widetilde{f}(\lambda, \cdot, n))(x).$$

□

Definition 8.3.4. For a fixed integer n, a point $\lambda \in \mathbb{C}$ is said to be n-simple if the map $F \mapsto \mathcal{P}^n_\lambda(F)$ is injective on $L^2(K,\sigma)$ where $n \in \mathbb{Z}^\sigma$.

The following lemma gives a criterion for n-simplicity of a point $\lambda \in \mathbb{C}$.

Lemma 8.3.5. *Let n be a fixed integer and $\sigma \in \widehat{M}$ such that $n \in \mathbb{Z}^\sigma$. A point λ in $\mathbb{C}$ is n-simple, if for every $m \in \mathbb{Z}^\sigma$, $\Phi^{n,m}_{\sigma,\lambda}$ is a non-zero function.*

Proof. Let f be a function in $L^2(K,\sigma)$. Let $\mathcal{P}^n_\lambda(f)(x)$ be denoted by $F(x)$.

We suppose $F \equiv 0$. Then $F_m \equiv 0$, for all $m \in \mathbb{Z}$, where

$$\begin{aligned} F_m(x) &= \int_K F(kx)e_{-m}(k)\,dk \\ &= \int_K \left(\int_K f(k_1)e^{-(\lambda+1)H(x^{-1}k^{-1}k_1^{-1})}e_n(K(x^{-1}k^{-1}k_1^{-1})^{-1})\,dk_1\right) e_{-m}(k)\,dk. \end{aligned}$$

We put $k_1k = k_2$ in the above to get,

$$\begin{aligned} F_m(x) &= \int_K \int_K f(k_1)e^{-(\lambda+1)H(x^{-1}k_2^{-1})}e_n(K(x^{-1}k_2^{-1})^{-1})e_{-m}(k_1^{-1}k_2)\,dk_2\,dk_1 \\ &= \int_K \left(\int_K f(k_1)\,e_m(k_1)\,dk_1\right)e^{-(\lambda+1)H(x^{-1}k_2^{-1})}e_n(K(x^{-1}k_2^{-1})^{-1})e_{-m}(k_2)\,dk_2 \\ &= \int_K \hat{f}(-m)e^{-(\lambda+1)H(x^{-1}k_2^{-1})}e_n(K(x^{-1}k_2^{-1})^{-1})e_{-m}(k_2)\,dk_2 \\ &= \widehat{f}(-m)\Phi^{n,m}_{\sigma,\lambda}(x). \end{aligned}$$

Therefore if λ is such that $\Phi^{n,m}_{\sigma,\lambda} \not\equiv 0$ for every $m \in \mathbb{Z}^\sigma$, then $\widehat{f}(-m) = 0$, for all $m \in \mathbb{Z}^\sigma$. As $f \in L^2(K,\sigma)$ it is clear that $\widehat{f}(r) = 0$ for all $r \in \mathbb{Z} \setminus \mathbb{Z}^\sigma$. Thus $\widehat{f}(m) = 0$ for all $m \in \mathbb{Z}$ and hence $f \equiv 0$. Then from Definition 8.3.4 it follows that λ is n-simple. □

An explicit description of the combinations of $\{\lambda, m, n\}$ for which $\Phi^{m,n}_{\sigma,\lambda} \equiv 0$ is given in [4, Proposition 7.1, 7.2]. Using that we get the following immediate Corollary.

Corollary 8.3.6. *For a fixed integer n and any $\lambda \in \mathbb{C}$, either λ or,$-\lambda$ is n-simple.*

Lemma 8.3.7. *Let* $\sigma \in \widehat{M}$. *If* $\lambda \in \mathbb{C}$ *is* n*-simple for some* $n \in \mathbb{Z}^\sigma$ *then,*

$$\mathcal{B} = \left\{ k \mapsto \sum_{j=1}^{N} a_j e^{-(\bar{\lambda}+1)H(x_j^{-1}k^{-1})} \overline{e_n(K(x_j^{-1}k^{-1})^{-1})} : a_j \in \mathbb{C}, x_j \in G, N \in \mathbb{N} \right\}$$

is dense in $L^2(K,\sigma)$.

Proof. If $\mathcal{B}$ is not dense then there is a nonzero function $F \in L^2(K,\sigma)$ such that for all $x \in G$,

$$\int_K F(k) e^{-(\lambda+1)H(x^{-1}k^{-1})} e_n(K(x^{-1}k^{-1})^{-1})\, dk = 0.$$

That is $\mathcal{P}^n_\lambda(F) \equiv 0$. As λ is n-simple we have $F \equiv 0$. This contradicts our assumption on F. □

Lemma 8.3.8. *If* $-\bar{\lambda}$ *is* n*-simple for some* $n \in \mathbb{Z}^\sigma$ *then* $\overline{\mathsf{span}\{{}^y\Phi^{n,n}_{\sigma,\lambda} : y \in G\}}$ *contains* $\mathcal{P}^n_\lambda(L^2(K,\sigma))$ *where* ${}^y\Phi^{n,n}_{\sigma,\lambda}$ *denotes the left translation of* $\Phi^{n,n}_{\sigma,\lambda}$ *by* $y \in G$.

Proof. Let $F \in L^2(K,\sigma)$, then by Lemma 8.3.7 there is a sequence $\{f_\alpha\}$ in $\mathcal{B}$ such that $f_\alpha \to F$ as $\alpha \to \infty$. As Poisson transform is continuous, $\mathcal{P}^n_\lambda(f_\alpha)(y) \to \mathcal{P}^n_\lambda(F)(y)$. Therefore

$$\sum_{j=1}^{N_\alpha} a_{j,\alpha} \int_K e^{(\lambda-1)H(x_{j,\alpha}^{-1}k^{-1})} \overline{e_n(K(x_{j,\alpha}^{-1}k^{-1})^{-1})} e^{-(\lambda+1)H(y^{-1}k^{-1})} e_n(K(y^{-1}k^{-1})^{-1})\, dk$$

converges to $\mathcal{P}^n_\lambda(F)(y)$. Therefore by Lemma 8.3.1, $\sum_{j=1}^{N_\alpha} a_{j,\alpha} \Phi^{n,n}_{\sigma,\lambda}(x_{j,\alpha}^{-1}y)$ converges to $\mathcal{P}^n_\lambda(F)(y)$. This shows that the space $\overline{\mathsf{span}\{{}^x\Phi^{n,n}_{\sigma,\lambda} \mid x \in G\}}$ contains $\mathcal{P}^n_\lambda(L^2(K,\sigma))$. □

We need the following Corollary of Theorem 8.1.1 which can also be proved using the method of Section 7.1 as (G,K,n) is a Gelfand triple.

Theorem 8.3.9. *For a nonzero function* $f \in C^\infty(G)_{n,n}$ *the closure of the set* $\{W * f \mid W \in C^\infty(G)'_{n,n}\}$ *contains the function* $x \mapsto \Phi^{n,n}_\lambda$ *for some* $\lambda \in \mathbb{C}$.

Here $C^\infty(G)'_{n,n}$ is the dual space of $C^\infty(G)_{n,n}$; in other words the set of compactly supported distributions on G of type (n,n).

Proof of theorem 8.1.2. Let V_n be a left translation invariant closed subspace of $C^\infty(G)_n$. We consider $V_{n,n}$ the closed subspace of left n type functions in V_n. That is $V_{n,n}$ contains all (n,n) type functions in V_n. We take a nonzero function $g \in V_n$. Since V_n is left translation invariant we may assume that $g(e) \neq 0$ where e is the identity element of the group G. Let $g_n(x) = \int_K g(k_\theta x)e^{-in\theta}d\theta$. Then it follows that $g_n \in V_{n,n}$ and $g_n(e) = g(e) \neq 0$ which shows that $V_{n,n}$ is a nonzero subspace. As V_n is left translation invariant and $g_n \in V_n$, for any $h \in C_c^\infty(G)_{n,n}$ $h * g_n \in V_n$. This implies that V_n contains the closure of the set $\{W * g_n \mid W \in C^\infty(G)'_{n,n}\}$, since $C_c^\infty(G)_{n,n}$ is dense in $C^\infty(G)'_{n,n}$. Hence $V_{n,n}$ contains the closure of the set $\{W * g_n \mid W \in C^\infty(G)'_{n,n}\}$ as $W * g_n$ is of type (n,n) for $W \in C^\infty(G)'_{n,n}$. By Theorem 8.3.9, closure of $\{W * g_n \mid W \in C^\infty(G)'_{n,n}\}$ contains $\Phi_\lambda^{n,n}$ for some $\lambda \in \mathbb{C}$. That is there is a $\lambda \in \mathbb{C}$ such that $\Phi_\lambda^{n,n} \in V_{n,n} \subseteq V_n$. We consider that fixed λ. Since by Corollary 8.3.6 either $\overline{\lambda}$ or $-\overline{\lambda}$ is n-simple and $\Phi_\lambda^{n,n} = \Phi_{-\lambda}^{n,n}$ without loss of generality we assume that $-\overline{\lambda}$ is n-simple. By Lemma 8.3.8, $\mathcal{P}_\lambda^n(L^2(K,\sigma)) \subseteq \overline{\mathsf{span}\{{}^y\Phi_\lambda^{n,n} \mid y \in G\}}$. But we have $\mathsf{span}\{{}^y\Phi_\lambda^{n,n} \mid y \in G\} \subseteq V_n$. Therefore $\mathcal{P}_\lambda^n(L^2(K,\sigma)) \subseteq V_n$.

For an integer $l \in \mathbb{Z}^\sigma$ let $F_l = e_{-l} \in L^2(K,\sigma)$. Then

$$P_\lambda^n(F_l)(x) = \int_K e^{-(\lambda+1)H(x^{-1}k^{-1})} e_n(K(x^{-1}k^{-1})^{-1})\overline{e_l(k)}\, dk = \Phi_\lambda^{n,l}(x).$$

Thus $\Phi_\lambda^{n,l}(x) \in V_n$ for all $l \in \mathbb{Z}^\sigma$. We note that λ is independent of l.

For a fixed $k \in K$, $e^n_{\lambda+1,k}$ is a right n-type function on G. We decompose $e^n_{\lambda+1,k}$ in left K-types as a absolutely and uniformly convergent series in the $C^\infty(G)$ topology:

$$e^n_{\lambda+1,k}(x) = \sum_{l\in\mathbb{Z}^\sigma} e^{n,l}_{\lambda+1,k}(x) \tag{8.3.1}$$

where

$$\begin{aligned} e^{n,l}_{\lambda+1,k}(x) &= \int_K e^n_{\lambda+1,k}(k_1 x)e_{-l}(k_1)dk_1 \\ &= \int_K e^{-(\lambda+1)H(x^{-1}k_1^{-1}k^{-1})} e_n(K(x^{-1}k_1^{-1}k^{-1})^{-1})e_{-l}(k_1)\, dk_1. \end{aligned}$$

Substituting $kk_1 = k_2$ in the above, we have,

$$\begin{aligned} e^{n,l}_{\lambda+1,k}(x) &= \int_K e^{-(\lambda+1)H(x^{-1}k_2^{-1})} e_n(K(x^{-1}k_2^{-1})^{-1})e_{-l}(k^{-1}k_2)\, dk_2 \\ &= e_l(k)\int_K e^{-(\lambda+1)H(x^{-1}k_2^{-1})} e_n(K(x^{-1}k_2^{-1})^{-1})e_{-l}(k_2)\, dk_2 \\ &= e_l(k)\Phi_\lambda^{n,l}(x). \end{aligned}$$

As $\Phi_\lambda^{n,l} \in V_n$ for each $l \in \mathbb{Z}^\sigma$, the function $x \mapsto e_l(k)\Phi_\lambda^{n,l}(x) = e^{n,l}_{\lambda+1,k}(x)$ is also in V_n, for each $l \in \mathbb{Z}^\sigma$ and each $k \in K$. Hence by equation (8.3.1) we have $x \mapsto e^n_{\lambda+1,k}(x) \in V_n$, for all $k \in K$. This completes the first part of the theorem.

For the second part we let V be a both sided translation invariant subspace of $C^\infty(G)$. For each integer n, we let V_n be the closed subspace of right n type functions in V. Then V_n is left translation invariant. It is also clear that there is an integer n such that V_n is nonzero. From Proposition 8.2.1 it then follows that if $n \in \mathbb{Z}^\sigma$ then V_m is nonzero for each $m \in \mathbb{Z}^\sigma$. That is, either for each even n or for each odd n, V_n is nonzero. Now it follows from the first part of the theorem that each nonzero V_n contains $x \mapsto e^n_{\lambda,k}(x)$ for some $\lambda \in \mathbb{C}$ (λ depends on n), for all $k \in K$. This completes the proof. □

Remark 8.3.10. We note that there are (both sided) translation invariant nonzero closed subspace of $C^\infty(G)$ which contain $e^n_{\lambda,k}$ only for one parity of n (even or odd). For instance we take a C^∞-function f such that $f(x) = f(-x)$ for all $x \in G$. Then all the K-types of f in its decomposition are $n \in 2\mathbb{Z}$. From this it is clear that the closed translation invariant subspace V_f generated by f does not contain $e^n_{\lambda,k}$ with $n \in 2\mathbb{Z}+1$. Similarly if our function $f \in C^\infty(G)$ is odd i.e. $f(-x) = -f(x)$ for all $x \in G$ then V_f will not contain $e^n_{\lambda,k}$ for $n \in 2\mathbb{Z}$.

Chapter 9

Some Other Examples

The readers will observe that important ingredients of the proofs of the main results are the L^p-Schwartz space isomorphism theorem and the Paley-Wiener theorem for compactly supported distributions along with an explicit understanding of the images of these spaces under the τ-spherical transform. If these are available for a Gelfand triple then one can expect that the results in this thesis can be extended to that. For the proof L^p-Schwartz space isomorphism theorem we can adapt the method of Anker (see [2]) as we have done in the case of spinor bundle on the real hyperbolic case. In doing so one needs to define the Abel transform for τ-radial functions along with its adjoint and prove the slice projection property of the Abel transform (see Chapter 2 and Chapter 5). An alternative approach would be relating the τ-spherical transform with the Jacobi transform and then use the corresponding theorems for the Jacobi transform (see [26]). For the Paley-Wiener theorem for compactly supported τ-radial distributions, the starting point is Paley-Wiener theorem for compactly supported infinitely differentiable τ-radial functions. We mention here a few examples of Gelfand triple where we can achieve these targets and to which therefore all our results in Chapter 6 and Chapter 7 can be readily extended.

(A) Let G be $\mathrm{Sp}(1,n)$ and K be a maximal compact subgroup of G. Then K can be realized as

$$\left\{ \begin{pmatrix} u & 0 \\ 0 & U \end{pmatrix} : u \in \mathrm{Sp}(1), U \in \mathrm{Sp}(n) \right\} = \mathrm{Sp}(1) \times \mathrm{Sp}(n).$$

Let $\mathbb{N}/2$ be the set of non-negative half integers $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}$. Then $\widehat{\mathrm{Sp}(1)}$ is parametrized by $\mathbb{N}/2$ (for details see [60]). Let $(\tau_l, V_l), l \in \mathbb{N}/2$ be an element of $\widehat{\mathrm{Sp}(1)}$. We extend τ_l to a representation of K by setting $\tau_l = 1$ on $\mathrm{Sp}(n)$. We shall continue to call this representation as $\tau_l \in \widehat{K}$. Then (G, K, τ_l) is a Gelfand triple. It can be shown that for a suitable τ_l-radial function f on G, its τ_l-spherical transform is given by,

$$\widehat{f}_l(\lambda) = C_l \mathcal{F}^{(2n-1,2l+1)} \left(\frac{f(a_t)}{(\cosh t)^{2l}} \right) (i\lambda)$$

where $\mathcal{F}^{(2n-1,2l+1)}$ is the Jacobi transform and that $\widehat{f}_l(\lambda) = \widehat{f}_l(-\lambda)$ (see [60]). We also note that when $2l < 2n-1$, there is no discrete series representation which contains τ_l. Therefore it is straightforward to use the L^p-Schwartz space isomorphism for Jacobi functions to prove the corresponding theorems for τ_l-radial functions. With this preparation we can prove analogues of all the theorems mentioned above for the triple (G, K, τ_l). We note that in the cases where there are discrete series representation relevant for $\tau = \tau_l$-radial functions our scheme works easily if none of the discrete series representations is embedded as a subrepresentation of a principal series representation parametrized by a λ in the domain of the τ_l-spherical transform. Because in that scenario through the inversion formula, $C^p_\tau(G)$ can be divided explicitly into the principal and discrete parts as $C^p_\tau(G) = C^p_\tau(G)_P \oplus C^p_\tau(G)_D$. For the principal part $C^p_\tau(G)_P$ we prove the isomorphism again using Jacobi transform. For the discrete part we note that there can only be finitely many discrete series representations containing τ (see [61]) and therefore any function defined on this finite set (which parametrizes the discrete series representations containing τ) will be in the image of $C^p_\tau(G)_D$. A similar argument is used in [54].

(B) Let us now consider the complex hyperbolic spaces $\mathbb{H}^n(\mathbb{C}) = G/K$ where $G = \mathrm{SU}(n,1)$ and $K = S(\mathrm{U}(n) \times \mathrm{U}(1))$. The L^2 harmonic analysis of the Dirac spinors on $\mathbb{H}^n(\mathbb{C})$ were developed in [12]. Let τ be the spin representation of K. Then τ has the decomposition into irreducible K-types as $\tau = \tau_0 \oplus \tau_1 \oplus \cdots \oplus \tau_n$. We note that for τ_j-radial functions $j = 0, \ldots, n, j \neq n/2$ there is no relevant discrete series representations. There is a discrete series representation, restriction to K of which contains $\tau_{n/2}$ when n is even. This discrete series is infinitesimally embedded in this parametrization at $i/2$. We notice that for $4n(2n+1)^{-1} < p \leq 2$, $i/2 \notin S_p = \{z \in \mathbb{C} \mid |\Im z| \leq (2/p - 1)n\}$.

Relating the τ_j-spherical functions to the Jacobi function and thus the τ_j-spherical transform to the Jacobi transform, a Paley-Wiener theorem is obtained in ([12, section 4]). It is natural to expect that in an analogous way (i.e. through Jacobi transform) the L^p-Schwartz space isomorphism theorems for $0 < p \leq 2$

and Paley-Wiener Theorem for compactly supported distributions can also be proved. We observe that the relation between the τ_j-spherical transform and Jacobi transform for the cases $j = 0$ and $j = n$ are somewhat simpler than that for the other τ_j. A straightforward adaptation of the proofs given in the thesis will yield the Schwartz space isomorphism theorems and Paley-Wiener theorem respectively for τ_j-radial functions and distributions when $j = 0$ or n. It appears to us that with a little more effort the corresponding theorems for τ_j-radial functions with $1 \leq j \leq n-1, j \neq n/2$ can also be proved. Lastly for the case of an even n and $j = n/2$, the proof of L^p-Schwartz space isomorphism theorem for $4n/2n+1 < p \leq 2$ will not face any further obstacle as for this range of p a function in $C^p_{\tau_{n/2}}(G)$ can be decomposed in principal and discrete parts through the inversion formula. We have discussed a similar situation in (A) above.

Once these results are obtained, it will not be difficult to verify that the main results proved in this thesis will hold for τ_j-radial functions. As commented in [12] there is a strong similarity between the results obtained in [12] with those obtained in [46] for the p-forms on the real hyperbolic spaces. We note that harmonic analytic aspects of these differential forms were extensively studied in [46]. It will be interesting to see if the methods of this thesis can be applied for these cases.

(C) Let G be a connected, noncompact semisimple Lie group of real rank one with finite centre and K be a maximal compact subgroup of G. We also assume $(\tau, V_\tau) \in \widehat{K}$ is such that $(\tau|_M, V_\tau)$ is irreducible. Then (G, K, τ) is a Gelfand triple. A Paley-Wiener theorem for this is proved by Campoli (see [10, Theorem 3.3.1]). We note that in this case the τ-spherical transform is even and it is possible to adapt the proof in [2] to prove the L^p-Schwartz space isomorphism theorems for τ-radial functions (see Chapter 5). Staring from the Paley-Wiener theorem for functions one can also prove the corresponding theorem for distributions. This will enable us to extend the main results to this case.
(D) We recall that if $G = \mathrm{SL}(2, \mathbb{R})$, $K = \mathrm{SO}(2)$ and $\tau \in \widehat{K}$, then (G, K, τ) is a Gelfand triple. The L^p-Schwartz space theorem and the Paley-Wiener theorem are available for this case (see for instance [4]). In general the K-type τ is contained in a finite number of discrete series representations of G. However apart from the case of Wiener-Tauberian theorem for L^1 functions the discrete series will pose no difficulty (see the argument in (A) above). The case of integrable functions can also be dealt with along the line of argument in [54] and thus all the theorems in Chapter 6 are extendable here. Schwartz's theorem for this case has been already discussed in Chapter 8.

We conclude with the following remarks.

Remark 9.0.11. All the results in this thesis are valid for the K-biinvariant functions of a connected noncompact semisimple Lie group of real rank one with finite centre. All the theorems in Chapter 6 are new in this context.

Our basic objects in this thesis were homogenous vector bundles associated with Gelfand triples. Apart from those which are mentioned above some notable examples of Gelfand triples are $G/K = H^n(\mathbb{R})$ or $H^n(\mathbb{C})$ and τ is an arbitrary representation in $\widehat{K}$ (see Remarks 2 and 3 in [46, p. 82]). There are Gelfand triples also for quaternionic hyperbolic spaces. It is natural to ask if the targets of this thesis could be achieved for these cases.

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