Isomorphism of Schwartz spaces under Fourier transform

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To the holy trinity

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Contents

Chapter 1

Introduction

Classical Fourier analysis derives much of its power from the fact that there are three function spaces whose images under the Fourier transform can be exactly determined. They are the Schwartz space, the L^2 space and the space of all $\mathcal{C}^\infty$ functions of compact support. The determination of the image is obtained from the definition in the case of Schwartz space, through the Plancherel theorem for the L^2 space and through the Paley-Wiener theorem for the other space.

In harmonic analysis of semisimple Lie groups, function spaces on various restricted set-ups are of interest. Among the multitude of these spaces it is again the spaces analogous to the three spaces above for which characterization of images under Fourier transform has been possible. Having neither the advantage of the duality nor the well behaved characters as the Euclidean set-up, the determination of images has been hard work in all the situations here- leading to the Schwartz space isomorphism theorems, the Paley-Wiener theorems and the Plancherel theorems. Some of these results have been reworked in recent years resulting in simpler approaches and redefining the interrelationships of these results. This the context of the present thesis.

Our set-up is a connected, non-compact, semisimple real Lie group G having finite center and K a maximal compact subgroup of G. A main inspiration for our work is J-P. Anker's [2] proof of Schwartz space isomorphism under the Fourier transform for bi-K-invariant functions on G. Unlike the earlier proofs of this result, this beautiful proof relies on the Paley-Wiener theorem and takes no recourse to the asymptotics of elementary spherical functions due to Harish-Chandra except, indirectly, for what is involved in the Paley-Wiener theorem. Since a proof of the Paley-Wiener theorem had already been found that did not use the Schwartz space isomorphism theorem as well, Anker's proof thus scripted an 'elementary' development of Harmonic Analysis of bi-K-invariant functions.

It is in the above spirit that we take up our first function space, the L^p-Schwartz space $\mathcal{S}^p_\delta(X)$ $(0 < p \leq 2)$ of a given (left) K-type δ on the symmetric space $X = G/K$ under the assumption that G/K is of real rank-1. The relevant Fourier transform here is the δ-spherical transform. In characterizing the image of the δ-spherical transform, we do not attempt to adopt the arguments of Anker as suggested in [2]. Instead we exploit the Kostant polynomials to reduce the problem to the bi-K-invariant case and use Anker's result thereafter. Again this provides arguments relying on the Paley-Wiener theorem to prove our result which is a part of the Eguchi-Kowata theorem [9] (where they established the isomorphism for $\mathcal{S}^p(X)$ without the restriction of the left type).

The second function space that we consider is in connection with the theory of spectral projection advocated by Stricharz [41,42,44]. Bray [8] worked on spectral projections in the semisimple context to obtain the Paley-Wiener theorem. We work with the L^p-Schwartz space $\mathcal{S}^p(X)_K$ $(0 < p \leq 2)$ of K-finite functions on $X = G/K$. With the assumption of real rank-1, like in Bray's [8] result, we are able to obtain a characterization of the image of this space under spectral projection; we also have partial results removing the rank restriction. Our result looks akin to what Stricharz obtains for Euclidean spaces.

In the third function space we go out of the bi-K-invariant or right-K-invariant situation. As is well-known, harmonic analysis on the full group has not yet gone very far. Indeed, it is only for the group $SL_2(\mathbb{R})$ that the characterization problem for the L^p-Schwartz space $\mathcal{S}^p(G)$ have so far been solved (Barker [7]). On the same group we take up the case of L^p-Schwartz spaces $(1 < p \leq 2)$ of functions having given left and right-K-types. We obtain again a (somewhat) elementary proof of Barker's result in this case.

The thesis is organized as follows. In Chapter 2 we set down our notation and collect useful results and formulae. Chapter 3 is devoted to the Schwartz space isomorphism of $\mathcal{S}^p_\delta(X)$. In Chapter 4 we give our results on spectral projection. This chapter can also be viewed as an application of the isomorphism theorem obtained in Chapter 3. In the last chapter, Chapter 5, we come back to Schwartz space isomorphism under Fourier transform, this time on the group $SL_2(\mathbb{R})$, for the space of functions with fixed left and right K-types.

Chapter 2

Notation and preliminaries

In this chapter we shall briefly recall some facts and results about noncompact Riemannian symmetric space realized as $X = G/K$, where G be a connected noncompact semisimple Lie group with finite center and K a maximal compact subgroup of G. In our discussion we shall concentrate on the 'rank-1' Riemannian symmetric spaces, that is, the semisimple Lie group G will be of 'real rank-1'. Many of the basic notions, and results will be stated without proof. We refer to the standard textbooks [14,20,26,27,32] for more details and proofs.

We denote $\mathfrak{g}$ and $\mathfrak{k}$ for the Lie algebras of G and K respectively. As K is a maximal compact subgroup of G, there exists an involutive automorphism θ, called the *Cartan involution*, of G whose set of fixed points is precisely K. The Lie algebra $\mathfrak{g}$ has the Cartan decomposition into the eigenspaces of the Cartan involution:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{2.0.1}$$

where, $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\}$ and $\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta X = -X\}$. The corresponding decomposition $G = K \exp \mathfrak{p}$ is called the *Cartan decompositions* of G respectively. Let us denote $\mathfrak{g}_{\mathbb{C}}$ for the *complexification* [28, p-180] of the Lie algebra $\mathfrak{g}$. The *Killing form* $\mathfrak{B}$ on $\mathfrak{g}$ has the properties

(i) $\mathfrak{B}$ is invariant under the action of G and θ;

(ii) it is real valued on $\mathfrak{g} \times \mathfrak{g}$, positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

Then $\mathfrak{B}$ induces an inner product on $\mathfrak{g}$ by $\langle X, Y\rangle = -\mathfrak{B}(X, \theta Y)$ which extends to $\mathfrak{g}_{\mathbb{C}}$ as a Hermitian inner product-called the *Cartan-Killing form*.

The corresponding norm is denoted by $\|X\|^2 = -\mathfrak{B}(X, \theta X)$. The Cartan-Killing form also induces the Riemannian structure on $X = G/K$, whose tangent space at eK is identified with $\mathfrak{p}$.
Let us choose and fix a one dimensional subspace $\mathfrak{a} \subset \mathfrak{p}$. We denote by $\mathfrak{a}^*$ its real dual and $\mathfrak{a}^*_{\mathbb{C}}$ its complex dual vector space. The Cartan-Killing form induces an inner product on $\mathfrak{a}$ and hence on $\mathfrak{a}^*$. We denote $\langle \cdot, \cdot \rangle_1$ for the extension of the inner product to $\mathfrak{a}^*_{\mathbb{C}}$. For any $\alpha \in \mathfrak{a}^*$ we set

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \ H \in \mathfrak{a}\}. \tag{2.0.2}$$

We denote $\mathfrak{g}_0 = \mathfrak{m}$ which is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We shall call α a *root* of the system $(\mathfrak{g}, \mathfrak{a})$ (called restricted root of $\mathfrak{g}$) if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. We denote by Σ the set of all roots of $(\mathfrak{g}, \mathfrak{a})$. For each root α, $m_\alpha = dim \mathfrak{g}_\alpha$ is the *multiplicity* of the root α. Selecting a non-zero element $X \in \mathfrak{a}$, we call a root α positive if $\alpha(X) > 0$; the set of positive roots is denoted by Σ^+ in Σ and $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ is a nilpotent subalgebra of the Lie algebra $\mathfrak{g}$. Let $N = \exp \mathfrak{n}$ be the analytic subgroup of G defined by $\mathfrak{n}$. N is a closed subgroup of G and the exp map is a diffeomorphism from $\mathfrak{n}$ onto N. The sub group $A = \exp \mathfrak{a}$ normalizes N. We denote $M = \exp \mathfrak{m}$; then clearly M is the centralizer of A in K. The subgroup M of K also normalizes N. Let M' be the normalizer of A in K. The Weyl group $W = M'/M$ is the group $\{1, -1\}$ which acts on $\mathfrak{a}$; identifying $\mathfrak{a}$ and $\mathbb{R}$ with the help of X above, W acts on $\mathbb{R}$ by multiplication. The cone $\mathfrak{a}^+$ in $\mathfrak{a}$ corresponds to the set of all positive numbers; $\mathfrak{a}^{*+}$ will be the dual cone of $\mathfrak{a}^*$. Let $\rho = \frac{1}{2} \sum_{\gamma \in \Sigma^+} m_\gamma \gamma \in \mathfrak{a}^{*+}$. We now change our choice of X so that $\rho(X) = 1$. This normalization identifies A, $\mathfrak{a}$ and $\mathfrak{a}^*$ all with $\mathbb{R}$ and in particular ρ is identified with 1. The complexifiction $\mathfrak{a}^*_{\mathbb{C}}$ is identified with $\mathbb{C}$. The group elements of A will now be denoted by a_t where $t \in \mathbb{R}$ and $\exp t = a_t$. With the normalization the positive chambers A^+, $\mathfrak{a}^+$ and $\mathfrak{a}^{*+}$ are all identified with $\mathbb{R}^+$.

We shall be using the Iwasawa and the Cartan decomposition of G and the corresponding expressions of the Haar measure on G. The Iwasawa decomposition gives

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad \text{and} \quad G = KAN, \tag{2.0.3}$$

where the map $(k, a, n) \mapsto kan \in G$ is a diffeomorphism from $K \times A \times N$ onto G. The group G can also be expressed as $G = NAK$, the map being again a diffeomorphism. Let $H : g = ka_t n \mapsto H(g) = t$ and $A : g = na_{t_1}k \mapsto A(g) = t_1$ are Iwasawa-$\mathfrak{a}$-projections of $g \in G$ in $\mathfrak{a}$ for KAN and NAK decomposition of the group respectively. These two projections are related by $A(g) = -H(g^{-1})$ for all $g \in G$.

The *Cartan decomposition* gives $G = K\overline{A^+}K$. It induces a diffeomorphism from $K/M \times A^+ \times K$ (or $K \times A^+ \times M/K$) onto an open dense subset of G. Let x^+ be the $\overline{\mathfrak{a}^+}$ projection of $x \in G$ for the Cartan decomposition $x = k_1(\exp x^+)k_2$ and we denote $|x| = \|x^+\|$. For all $x \in G$ the Iwasawa-$\mathfrak{a}$-projection $H(x)$ and the quantity $|x|$ are related by the inequality:

$$\|H(x)\| \leq c|x|, \quad x \in G, \text{ where } c > 0 \text{ is a fixed constant.} \tag{2.0.4}$$

We also note that in the symmetric space $X = G/K$, $|x|$ is the Riemannian distance of xK from the coset eK, e being the identity element of G.

The Haar measure corresponding to the Iwasawa-KAN decomposition is given by

$$\int_G f(x)dx = const. \int_K dk \int_{\mathfrak{a}^+} e^{2t}dt \int_N dnf(ka_t n), \tag{2.0.5}$$

where, the *const* stands for a normalizing constant. For Iwasawa-NAK decomposition the expression for the Haar measure is even simpler

$$\int_G f(x)dx = const. \int_N dk \int_{\mathfrak{a}^+} dt \int_K dnf(ka_t n). \tag{2.0.6}$$

In the case of the Cartan decomposition the Haar measure on G is given by

$$\int_G f(x)dx = const. \int_K dk_1 \int_{\mathfrak{a}^+} \Delta(t)dt \int_K dk_2 f(k_1 a_t k_2), \tag{2.0.7}$$

where the weight function $\Delta(t) = \prod_{\alpha \in \Sigma^+} \sinh^{m_\alpha} \alpha(t)$. We shall use the estimate $\Delta(t) = O(e^{2t})$ of the density function. The maximal compact subgroup K acts on the group G from left as well as from right. A function f on G is said to be *bi-K-invariant* if it satisfies the relation

$$f(k_1 x k_2) = f(x), \text{ for all } x \in G \text{ and } k_1, k_2 \in K, \tag{2.0.8}$$

and it may also be regarded as a function on the double cosets $K \setminus G/K \equiv G//K$. A function f will be called *right-K-invariant* if for all $x \in G$ and

$k \in K$ it satisfies

$$f(xk) = f(x). \tag{2.0.9}$$

Althrough in this thesis we shall consider a function on the symmetric space $X = G/K$ as a right-K-invariant function on the group G. For any function space $\mathfrak{F}(G)$ on G or $\mathfrak{F}(G/K)$ on X, we shall denote $\mathfrak{F}(G//K)$ for the corresponding subspace of bi-K-invariant functions.

We denote $\mathcal{C}^\infty(G)$ for the set of all smooth functions on G. We fix a basis $\{X_j\}$ for the Lie algebra $\mathfrak{g}$. Let $\mathcal{U}(\mathfrak{g})$ be the *universal enveloping algebra* over $\mathfrak{g}$. Let $D_1 \cdots D_m, E_1 \cdots E_n \in \mathcal{U}(\mathfrak{g})$, then the action of $\mathcal{U}(\mathfrak{g})$ on a function $f \in \mathcal{C}^\infty(G)$ is defined as follows:

$$f(D_1 \cdots D_m; x; E_1 \cdots E_n) =$$
$$\frac{d}{dt_1}\,|_{t_1=0} \cdots \frac{d}{dt_m}|_{t_m=0}\, \frac{d}{ds_1}\,|_{s_1=0} \cdots \frac{d}{ds_n}\,|_{s_n=0}\, f(e^{t_1 D_1} \cdots e^{t_m D_m} x e^{s_1 E_1} \cdots e^{s_n E_n}). \tag{2.0.10}$$

Let $b_{ij} = \mathfrak{B}(X_i, X_j)$ and (b^{ij}) be the inverse of the matrix (b_{ij}). We now define a distinguished element, called the *Casimir element*, of $\mathcal{U}(\mathfrak{g})$ by the following:

$$\Omega = \sum_{i,j} b^{ij} X_i X_j. \tag{2.0.11}$$

The differential operator Ω lies in the center of $\mathcal{U}(\mathfrak{g})$. The action of the *Laplace-Beltrami operator* $\mathbf{L}$ on X is defined by the action of Ω:

$$\mathbf{L} f(xK) = f(x; \Omega), \qquad x \in G. \tag{2.0.12}$$

Let $P = MAN$ be a *minimal parabolic* subgroup of G. We describe the spherical representations of interest for analysis of right-K-invariant functions. The one dimensional representation $\pi^P_\lambda : ma_t n \mapsto e^{i\lambda t}$ $(\lambda \in \mathbb{C})$ of P induces the *principal series* representations π_λ $(\lambda \in \mathbb{C})$ of G realized on the Hilbert space $L^2(K/M)$, given by the formula:

$$\{\pi_\lambda(x)\zeta\}(kM) = e^{-(i\lambda+1)H(x,kM)}\zeta(K(x^{-1}k)M), \qquad \zeta \in L^2(K/M), \tag{2.0.13}$$

where, $(x, kM) \mapsto H(x, kM)$ is the function from $G \times K/M$ into $\mathfrak{a}$ defined by $H(x, kM) = H(x^{-1}k)$ and $K(y)$ denotes the K projection of $y \in G$ in Iwasawa-KAN decomposition. For each $x \in G$ and $\lambda \in \mathbb{C}$, the adjoint of the operator $\pi_\lambda(x)$ is given by

$$\{\pi_\lambda(x)\}^* = \pi_{\overline{\lambda}}(x^{-1}). \qquad (2.0.14)$$

The representation π_λ is unitary if and only if $\lambda \in \mathbb{R}$ [14, Proposition 3.1.1]. It follows from the definition that $\pi_\lambda|_K$ is the left regular representation of K in $L^2(K/M)$. Clearly, the constants $\mathbb{C} \cdot 1$ are in $L^2(K/M)$ and they are precisely the K-invariant vectors for each π_λ in $L^2(K/M)$. The *elementary spherical functions* are the following matrix coefficients of the principal series representations corresponding to the function 1:

$$\varphi_\lambda(x) = \langle \pi_\lambda(x)1, 1\rangle_{L^2(K/M)} = \int_K e^{-(i\lambda+1)H(x^{-1}k)}dk. \qquad (2.0.15)$$

Using (2.0.14) one can get an alternative integral expression for the elementary spherical functions as follows:

$$\begin{aligned}\varphi_\lambda(x) = \langle 1, \pi_\lambda(x)^* 1\rangle_{L^2(K/M)} &= \langle 1, \pi_{\overline{\lambda}}(x^{-1})\rangle_{L^2(K/M)}, \\ &= \int_K e^{(i\lambda-1)H(xk)}dk. \end{aligned} \qquad (2.0.16)$$

We collect some of the very basic properties of the elementary spherical functions, which will be used throughout.

Proposition 2.0.1. *(i) The expression $\varphi_\lambda(x)$ is a bi-K-invariant $\mathcal{C}^\infty$ function in the x variable and it is a W-invariant holomorphic function in $\lambda \in \mathbb{C}$.*

(ii) For each $\lambda \in \mathbb{C}$, $x \mapsto \varphi_\lambda(x)$ is a joint eigenfunction of all the G-invariant differential operators on G/K; in particular for the Laplace-Beltrami operator we have:

$$\boldsymbol{L}\varphi_\lambda(\cdot) = -(\langle \lambda, \lambda\rangle_1 + 1)\varphi_\lambda(\cdot), \qquad \lambda \in \mathbb{C}. \qquad (2.0.17)$$

(iii) For each $\lambda \in \mathbb{C}$ and $x, y \in G$, the following property is referred to as the 'symmetric property of the elementary spherical functions'

$$\varphi_\lambda(x^{-1}y) = \int_K e^{-(i\lambda+1)H(y^{-1}k^{-1})}e^{(i\lambda-1)H(x^{-1}k^{-1})}dk. \qquad (2.0.18)$$

(iv) *For any given* $\boldsymbol{D}, \boldsymbol{E} \in \mathcal{U}(\mathfrak{g})$, *there exists a constant* $c > 0$ *such that*

$$|\varphi_\lambda(\boldsymbol{D}; x; \boldsymbol{E})| \leq c(|\lambda| + 1)^{deg\boldsymbol{E}+deg\boldsymbol{D}} \varphi_{i\Im\lambda}(x) \quad \text{for all } x \in G,\ \lambda \in \mathbb{C}. \tag{2.0.19}$$

(v) *Given any polynomial* P *in the algebra* $\boldsymbol{S}(\mathfrak{a}^*)$ *of symmetric polynomials over* $\mathfrak{a}^*$, *there exists a positive constant* c *such that:*

$$\left| P\left(\frac{\partial}{\partial\lambda}\right) \varphi_\lambda(x) \right| \leq c(1 + |x|)^{degP} \varphi_{i\Im\lambda}(x), \qquad x \in G. \tag{2.0.20}$$

(vi) *For all* t *and* λ *in* $\overline{\mathbb{R}^+}$ *we have:*

$$0 < \varphi_{-i\lambda}(a_t) \leq e^{\lambda t} \varphi_0(a_t). \tag{2.0.21}$$

(vii) *For all* $x \in G$, *we have* $0 < \varphi_0(x) = \varphi_0(x^{-1}) \leq 1$;

(viii) *For all* $t \in \overline{\mathbb{R}^+}$, *we have the following two-side estimate of* φ_0:

$$e^{-t} \leq \varphi_0(a_t) \leq c(1 + t)^a e^{-t}, \tag{2.0.22}$$

where $c, a > 0$ *are group dependent constants;*

Property (i) is a very basic fact which follows from the definition. For a proof one can see [14, Ch. 4]. Property (ii) was proved by Helgason [27]. For (iii) we refer to [26, Ch. III, Theorem 1.1]. The estimates (iv), (v), and (vi) follows from the results in [14, Sec. 4.6]. For a direct and a simple proof of (iv) and (v) one can see [2, Proposition 3]. The estimate (vii) of φ_0 is due to Harish-Chandra. A proof of this can be found in [14, Theorem 4.6.4, Theorem 4.6.5]. We should note that a sharper two-sided estimate of φ_0 is given by Anker [1].

Let δ be a unitary irreducible representation of K realized on a finite dimensional vector space V_δ with an inner product $\langle \cdot, \cdot \rangle$. Let us denote $dimV_\delta = d_\delta$. We denote by $\widehat{K}$ the set of equivalence classes of unitary irreducible representations of K and by customary abuse of notation regard each element of $\widehat{K}$ as a representation from its equivalence class. For each $\delta \in \widehat{K}$, let χ_δ stand for the character of the representation δ and $V_\delta^M = \{v \in V_\delta \mid \delta(m)v = v \text{ for all } m \in M\}$ is the subspace of V_δ fixed under $\delta|_M$. For a group with real rank-1, V_δ^M can be of dimension either zero or 1 (see [34]). Let $\widehat{K}_M$ stands for the subset of $\widehat{K}$ consisting of δ for which $V_\delta^M \neq \{0\}$ and we will mostly be interested in representations $\delta \in \widehat{K}_M$. We set an orthogonal basis $\{v_j\}_{1\leq j\leq d_\delta}$ of V_δ and we assume that v_1 generates V_δ^M. We also define a norm for each unitary irreducible representation of K. Let Θ be the restriction of the Cartan-Killing form $\mathfrak{B}$ to $\mathfrak{k}\times\mathfrak{k}$. Let $\mathcal{K}_1, ..., \mathcal{K}_r$ be a basis for $\mathfrak{k}$ over $\mathbb{R}$ orthonormal with respect to Θ. Let

$$\omega_{\mathfrak{k}} = -(\mathcal{K}_1^2 + ... + \mathcal{K}_r^2)$$

be the Casimir element of K. Clearly $\omega_{\mathfrak{k}}$ is a differential operator which commutes with both left and right translations of K. Thus $\delta(\omega_{\mathfrak{k}})$ commutes with $\delta(k)$ for all $k \in K$. Hence by *Schur's lemma* [46, Ch.I, Theorem 2.1]:

$$\delta(\omega_{\mathfrak{k}}) = c(\delta)\delta(e), \quad \text{where} \quad c(\delta) \in \mathbb{C}.$$

As $\delta(\mathcal{K}_i)$ $(1 \leq i \leq r)$ are skew-adjoint operators, $c(\delta)$ is real and $c(\delta) \geq 0$. We define $|\delta|^2 = c(\delta)$, for $\delta \in \widehat{K}_M$. As, $\delta \in \widehat{K}_M$, $\delta(k)$ is a unitary matrix of order $d_\delta \times d_\delta$. So $\|\delta(k)\|_{\mathbf{2}} = \sqrt{d_\delta}$ where $\|\cdot\|_{\mathbf{2}}$ denotes the Hilbert Schmidt norm. Also, from Weyl's dimension formula we can choose an $r \in \mathbb{Z}^+$ and a positive constant c independent of δ such that

$$\|\delta(k)\|_{\mathbf{2}} \quad \leq \quad c(1 + |\delta|)^r \tag{2.0.23}$$

for all $k \in K$. Thus, $d_\delta \leq c'(1 + |\delta|)^{2r}$ with $c' > 0$ independent of δ.
For any $f \in \mathcal{C}^\infty(X)$ we put:

$$f^\delta(x) = d_\delta \int_K f(kx)\delta(k^{-1})dk. \tag{2.0.24}$$

Clearly, f^δ is a $\mathcal{C}^\infty$ map from X to $Hom(V_\delta, V_\delta)$ satisfying

$$f^\delta(kx) = \delta(k)f^\delta(x), \quad \text{for all } x \in X, k \in K. \tag{2.0.25}$$

Any function satisfying the property (2.0.25) will be referred to as (a $d_\delta \times d_\delta$ matrix valued) left δ-type function. For any function space $\mathcal{E}(X) \subseteq \mathcal{C}^\infty(X)$, we write $\mathcal{E}^\delta(X) = \{f^\delta \mid f \in \mathcal{E}(X)\}$. We shall denote by $\check{\delta}$ the *contragradient* representation of the representation $\delta \in \widehat{K}_M$. and a function f will be called a scalar valued left $\check{\delta}$-type function if $f \equiv d_\delta \chi_\delta * f$, where the operation $*$

is the convolution over K. For any class of scalar valued functions $\mathcal{G}(X)$ we shall denote

$$\mathcal{G}(\check{\delta}, X) = \{g \in \mathcal{G}(X) \mid g \equiv d_\delta \chi_\delta * g\}.$$

Throughout our discussion we fix the notation $\mathcal{D}(X)$ for the subclass of functions in $\mathcal{C}^\infty(X)$ which are of compact support. The following theorem, due to Helgason, identifies the two classes $\mathcal{D}^\delta(X)$ and $\mathcal{D}(\check{\delta}, X)$ corresponding to each $\delta \in \widehat{K}_M$.

Theorem 2.0.2. [Helgason [26, Ch.III, Proposition 5.10]]
The map $\mathcal{Q} : f \mapsto g$, $g(x) = tr\,(f(x))$ $(x \in X)$ is a homeomorphism from $\mathcal{D}^\delta(X)$ onto $\mathcal{D}(\check{\delta}, X)$ and its inverse is given by $g \mapsto f = g^\delta$.

Remark 2.0.3. *For each $\delta \in \widehat{K}_M$, the space $\mathcal{D}(X, Hom(V_\delta, V_\delta))$ of $\mathcal{C}^\infty$ functions on X taking values in $Hom(V_\delta, V_\delta)$, carries the inductive limit topology of the Fréchet spaces*

$$\mathcal{D}_R(X, Hom(V_\delta, V_\delta)) = \{F \in \mathcal{D}(X, Hom(V_\delta, V_\delta)) \mid suppF \subseteq \overline{B^R(0)}\},$$

for $R = 0, 1, 2, \cdots$. As $\mathcal{D}(\check{\delta}, X) \subset \mathcal{D}(X)$, so the natural topology of $\mathcal{D}(\check{\delta}, X)$ is the inherited subspace topology.

A consequence of the *Peter-Weyl theorem* can be stated [27, Ch.IV, Corollary 3.4] in the form that any $f \in \mathcal{C}^\infty(X)$ has the decomposition

$$f(x) = \sum_{\delta \in \widehat{K}_M} tr(f^\delta(x)). \tag{2.0.26}$$

A function $f \in \mathcal{C}^\infty(X)$ is said to be *left K finite* if there exists a finite subset $\Gamma(f) \subset \widehat{K}_M$ (depending on the function f) such that $tr(f^\gamma(\cdot)) \equiv 0$ for all $\gamma \in \widehat{K}_M \setminus \Gamma(f)$. For any class $\mathfrak{H}(X) \subseteq \mathcal{C}^\infty(X)$ of function we shall denote $\mathfrak{H}(X)_K$ for its left K finite subclass. Let Γ be a fixed subset (finite or infinite) of $\widehat{K}_M$. Then we shall use the notation $\mathfrak{H}(X; \Gamma)$ for the subclass of $\mathfrak{H}(X)$

$$\mathfrak{H}(X; \Gamma) = \{g \in \mathfrak{H}(X) \mid g^\delta(\cdot) \equiv 0, \text{ for all } \delta \in \widehat{K}_M \setminus \Gamma\}. \tag{2.0.27}$$

2.1 Generalized spherical functions

Definition 2.1.1. *For each $\delta \in \widehat{K}_M$ and $\lambda \in \mathbb{C}$, the function*

$$\Phi_{\lambda,\delta}(x) = \int_K e^{-(i\lambda+1)H(x^{-1}k)}\delta(k)dk, \quad x \in G, \tag{2.1.1}$$

is called the 'generalized spherical function' of class δ. For each $x \in G$, $\Phi_{\lambda,\delta}(x)$ is an operator in $Hom(V_\delta, V_\delta)$. Taking point-wise adjoints leads to the expression

$$\Phi_{\overline{\lambda},\delta}(x)^* = \int_K e^{(i\lambda-1)H(x^{-1}k)}\delta(k^{-1})dk, \quad x \in G. \tag{2.1.2}$$

Remark 2.1.2. *From the Iwasawa decomposition, if $x \in G$ and $\tau \in K$, $H(\tau x) = H(x)$. Hence, the expressions (2.1.1) and (2.1.2) show that both $\Phi_{\lambda,\delta}(\cdot)$ and $\Phi_{\overline{\lambda},\delta}(\cdot)^*$ can be considered as functions on the space $X = G/K$.*

In the following proposition we list out some basic properties of the generalized spherical functions that we will be using.

Proposition 2.1.3. *(i) Let $\delta \in \widehat{K}_M$ and $\lambda \in \mathbb{C}$. Then for each $x \in X$ and $k \in K$ we have*

$$\Phi_{\lambda,\delta}(kx) = \delta(k)\Phi_{\lambda,\delta}(x) \quad \text{and} \quad \Phi_{\overline{\lambda},\delta}(kx)^* = \Phi_{\overline{\lambda},\delta}(x)^*\delta(k^{-1}). \tag{2.1.3}$$

Let $v \in V_\delta$ and $m \in M$ then

$$\delta(m)\left(\Phi_{\overline{\lambda},\delta}(x)^* v\right) = \Phi_{\overline{\lambda},\delta}(x)^* v. \tag{2.1.4}$$

(ii) For each fixed λ and δ, the function $\Phi_{\lambda,\delta}(x)$ and its adjoint are both joint eigenfunction of all G-invariant differential operators of X. Particularly, for the Laplace-Beltrami operator $\boldsymbol{L}$, the eigenvalues are as follows:

$$\left(\boldsymbol{L}\Phi_{\lambda,\delta}\right)(x) = -\left(\langle\lambda,\lambda\rangle_1 + 1\right)\Phi_{\lambda,\delta}(x), \quad x \in X. \tag{2.1.5}$$

(iii) For each fixed $x \in X$, the function $\lambda \mapsto \Phi_{\lambda,\delta}(x)$ is holomorphic.

(iv) For any $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ there exist constants $c = c(\boldsymbol{g}_1, \boldsymbol{g}_2)$, $b = b(\boldsymbol{g}_1, \boldsymbol{g}_2)$

and $c_0 > 0$ *so that*

$$\|\Phi_{\lambda,\delta}(\boldsymbol{g}_1; x; \boldsymbol{g}_2)\|_{\boldsymbol{2}} \leq c(1+|\delta|)^b(1+|\lambda|)^b\varphi_0(x)e^{|\Im\lambda|(1+|x|)}, \qquad (2.1.6)$$

for all $x \in X$ *and* $\lambda \in \mathbb{C}$.

Proof. Property (2.1.3) follows trivially from the definition of the generalized spherical function. (2.1.4) also follows from (2.1.1) as below:

$$\delta(m)\left(\Phi_{\overline{\lambda},\delta}(x)^* v\right) = \left\{\int_K e^{(i\lambda-1)H(x^{-1}k)}\delta(mk^{-1})dk\right\} v$$

by a simple change of variable mk^{-1} to k'^{-1} we get the right side

$$= \left\{\int_K e^{(i\lambda-1)H(x^{-1}k'm)}\delta(k'^{-1})dk'\right\} v.$$

In the above, let $x^{-1}k' = K(x^{-1}k')(\exp H(x^{-1}k'))n'$ for some $n' \in N$. As M normalizes N and centralizes A we have,

$$x^{-1}k'm = K(x^{-1}k')m(\exp H(x^{-1}k'))N(x^{-1}k')$$

this shows that $H(x^{-1}k') = H(x^{-1}k'm)$. Thus

$$\delta(m)\left(\Phi_{\overline{\lambda},\delta}(x)^* v\right) = \left\{\int_K e^{(i\lambda-1)H(x^{-1}k')}\delta(k'^{-1})dk'\right\} v = \Phi_{\overline{\lambda},\delta}(x)^* v. \qquad (2.1.7)$$

A proof of property (ii) may be found in [26, §1 (6)] and [27, Ch.II, Corollary 5.20]. The estimate (2.1.6) is a work of Arthur [6]. ∎

Remark 2.1.4. *The property (2.1.4) clearly shows that for each* $x \in X$ *the operator* $\Phi_{\overline{\lambda},\delta}(x)^*$ *maps* V_δ *to* V_δ^M. *Hence* $\Phi_{\overline{\lambda},\delta}(x)^*$ *is a* $d_\delta \times d_\delta$ *matrix whose only the first row can nonzero. Consequently, for each* $x \in X$, $\Phi_{\lambda,\delta}(x)$ *is a* $d_\delta \times d_\delta$ *matrix of which only the first column can be nonzero. In other words, the operator* $\Phi_{\lambda,\delta}(x)$ *vanishes identically on the orthogonal complement of the subspace* V_δ^M.

Unlike the elementary spherical functions, the generalized spherical functions $\Phi_{\lambda,\delta}(\cdot)$ and $\Phi_{\overline{\lambda},\delta}(\cdot)^*$ are not even in the λ variable. The following theorem, due to Helgason, determines the effect of Weyl group action on the λ variable the generalized spherical function.

Theorem 2.1.5. [Helgason [26, Ch.III, Theorem 5.15]]
For each $\delta \in \widehat{K}_M$ and for all $\lambda \in \mathbb{C}$, the restrictions $\Phi_{\lambda,\delta}|_A$ and $\Phi_{\overline{\lambda},\delta}(\cdot)^*|_A$ satisfy the relations

$$\Phi_{\lambda,\delta}|_A Q_\delta(1-i\lambda) = \Phi_{-\lambda,\delta}|_A Q_\delta(1+i\lambda); \tag{2.1.8}$$

$$Q_\delta(1-i\lambda)^{-1}\Phi_{\overline{\lambda},\delta}|_A(\cdot)^* = Q_\delta(1+i\lambda)\Phi_{-\overline{\lambda},\delta}|_A(\cdot)^*, \tag{2.1.9}$$

where, $Q_\delta(1+i\lambda)$ is a polynomial on $\mathbb{C}$, called the Kostant polynomial. Furthermore, both sides of (2.1.9) are holomorphic for all $\lambda \in \mathbb{C}$, implying that $\Phi_{\overline{\lambda},\delta}|_A(\cdot)^*$ is divisible by $Q_\delta(1-i\lambda)$ in the ring of entire functions.

A description of the polynomial $Q_\delta(1-i\lambda)$ can be found in [26, p.-238]. The polynomial $Q_\delta(1-i\lambda)$ has the representation in terms of the Gamma functions [26, Theorem 11.2, Ch. III, §11]

$$Q_\delta(1-i\lambda) = \left(\frac{1}{2}(\alpha+\beta+1-i\lambda)\right)_{\frac{r+s}{2}} \left(\frac{1}{2}(\alpha-\beta+1-i\lambda)\right)_{\frac{r-s}{2}} \tag{2.1.10}$$

where $(z)_m = \frac{\Gamma(z+m)}{\Gamma(z)}$ and r, s are integers. Two group dependent constants α and β are given by $\alpha = \frac{1}{2}(m_\gamma + m_{2\gamma} - 1)$, $\beta = \frac{1}{2}(m_{2\gamma} - 1)$. The pair of integers (r, s) gives a certain parameterization of the representation $\delta \in \widehat{K}_M$ (this parameterization was originally done by Kostant [34]; here we use a related parameterization due to Johnson and Wallach [31]). Clearly $Q_\delta(1-i\lambda)$ is a polynomial in λ of degree r. Helgason [26, Ch. III, § 11] further showed that all the zeros of the polynomial $Q_\delta(1-i\lambda)$ lie on the imaginary axis and, for all $\delta \in \widehat{K}_M$, none lies in the interior of the strip $\mathfrak{a}_1^* := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \,:\, |Im\lambda| \le 1\}$. The following Lemma is an immediate corollary of the expression (2.1.10) of the Kostant Polynomial.

Lemma 2.1.6. *For each $\delta \in \widehat{K}_M$, $Q_\delta(1-i\lambda) \neq 0$ for all $\lambda \in \mathfrak{a}^* + i\overline{\mathfrak{a}^{*+}} \subset \mathbb{C}$.*

The following theorem, due to Helgason, establishes an interrelation between the generalized spherical function corresponding to δ and the elementary spherical function. This theorem will be the main pathway for extending certain results from the bi-K-invariant class to the left-δ type class of functions on X.

Theorem 2.1.7. *For each nontrivial $\delta \in \widehat{K}_M$ and for all $\lambda \in \mathbb{C}$, we have*

$$\Phi_{\lambda,\delta}|_{V_\delta^M}(x) = \left(\boldsymbol{D}^\delta \varphi_\lambda\right)(x) Q_\delta(1-i\lambda)^{-1}, \quad x \in X, \tag{2.1.11}$$

where, $\boldsymbol{D}^\delta$ is a left invariant differential operator on X.

For a proof of the theorem and a description of the differential operator $\mathbf{D}^\delta$ we refer to [26, Ch.III, §5].

2.2 Fourier transforms

In this section we shall recall some basic definitions and results of Fourier transforms defined on function classes with different left-K-types. We shall confine our discussion here mostly to the compactly supported functions. We begin our discussion with the class $\mathcal{D}(G//K)$.

Definition 2.2.1. *For each $f \in \mathcal{D}(G//K)$, its* spherical transform *or* Harish-Chandra transform *is a function $\mathcal{S}f$ on $\mathbb{C}$ defined by*

$$\mathcal{S}f(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx. \tag{2.2.1}$$

From *Morera's theorem* $\mathcal{S}f$ is holomorphic for all $\lambda \in \mathbb{C}$. As the elementary spherical function is even in the λ variable, it immediately follows that

$$\mathcal{S}f(\lambda) = \mathcal{S}f(-\lambda), \quad \lambda \in \mathbb{C}. \tag{2.2.2}$$

Before we give a topological characterization of the image of $\mathcal{D}(G//K)$ under the spherical transformation we define a function space on $\mathbb{C}$.

Definition 2.2.2. *A holomorphic function $\psi(\lambda)$ on $\mathfrak{a}_{\mathbb{C}}^*$ is called a* holomorphic function of exponential type-R *if there exists a positive constant R such that for each $N \in \mathbb{Z}^+$*

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*} |\psi(\lambda)|(1+|\lambda|)^N e^{-R|\Im\lambda|} \leq C_N < +\infty, \tag{2.2.3}$$

where C_N is a positive constant depending on N. We denote $\mathcal{H}^R(\mathbb{C})$ for the class of all holomorphic exponential type-R functions on $\mathbb{C}$. Let $\mathcal{H}^R(\mathbb{C})_W \subset \mathcal{H}(\mathbb{C})$ be the subclass of even functions.

We denote

$$\mathcal{H}(\mathbb{C}) = \bigcup_{R>0} \mathcal{H}^R(\mathbb{C}), \quad \text{and} \quad \mathcal{H}(\mathbb{C})_W = \bigcup_{R>0} \mathcal{H}^R(\mathbb{C})_W. \tag{2.2.4}$$

The space $\mathcal{H}^R(\mathbb{C})$ has the topology of *uniform convergence on compacta* and $\mathcal{H}(\mathbb{C})$ is given the inductive limit topology. The subspace $\mathcal{H}(\mathbb{C})_W$ inherits its topology from $\mathcal{H}(\mathbb{C})$. In our discussion we shall refer to the following theorem as the *Paley-Wiener theorem.*

Theorem 2.2.3. *The spherical transform $f \mapsto \mathcal{S}f$ is a topological isomorphism from $\mathcal{D}(G//K)$ onto $\mathcal{H}(\mathbb{C})_W$, with the inverse transform given by*

$$f(x) = \frac{1}{2}\int_{\mathfrak{a}^+} \psi(\lambda)\varphi_\lambda(x)|\boldsymbol{c}(\lambda)|^{-2}d\lambda, \quad x \in G, \tag{2.2.5}$$

here, $|\boldsymbol{c}(\lambda)|^{-2}d\lambda$ is the Plancherel measure. More precisely, $f \in \mathcal{D}_R(G//K) = \{f \in \mathcal{D}(G//K) \mid suppf \subseteq \overline{B^R(0)}\}$ if and only if $\mathcal{S}f \in \mathcal{H}(\mathbb{C})_W$.

This theorem was originally proved by Helgason [22] and Gangolli [13]. Rosenberg [38] removed the dependence of the proof on Harish-Chandra's Schwartz space isomorphism theorem.

The Plancherel measure is a fundamental contribution of Harish-Chandra. The function $\mathbf{c}(\lambda)$ in the measure is also called the Harish-Chandra **c**-function. A complete expression of $\mathbf{c}(\lambda)$ can be found in [27, Chap. IV] or [14, Sect. 4.7]. For our purpose we shall only need the simple estimate [3] : for constants $c, b > 0$

$$|\mathbf{c}(\lambda)|^{-2} \leq \ c(|\lambda| + 1)^b \quad \text{for all } \lambda \in \mathfrak{a}^* = \mathbb{R}. \tag{2.2.6}$$

Remark 2.2.4. *For any $f \in \mathcal{D}(G//K)$, let $\mathcal{A}f$ be the function on $\mathfrak{a}$ defined by*

$$(\mathcal{A}f)(t) = e^t \int_N f(a_t n)dn. \tag{2.2.7}$$

This map $f \mapsto \mathcal{A}f$ is called the Abel transform. It can be shown that

$$\mathcal{S}f(\lambda) = \int_{\mathfrak{a}=\mathbb{R}} (\mathcal{A}f)(t)e^{\lambda t}dH, \qquad \lambda \in \mathbb{C}. \tag{2.2.8}$$

From the above we get a commutative diagram involving the operators $\mathcal{A}$, $\mathcal{S}$ and Euclidean Fourier transform. The Paley-Wiener theorem (Theorem 2.2.3) now shows that the Abel transform is a topological isomorphism between the spaces $\mathcal{D}(G//K)$ and $\mathcal{D}(\mathbb{R})_W$. The relation (2.2.8) plays crucial roles in proving several results for the spherical transform.

The Fourier transform of functions on the symmetric space X, that we shall be considering here, was introduced by Helgason. Geometrically it is the analogue of the Euclidian Fourier transform of functions F on $\mathbb{R}^n$ in polar coordinate form

$$\widehat{F}(\lambda\omega) = \int_{\mathbb{R}^n} F(x)e^{-(x,\omega)}dx, \qquad |\omega| = 1, \lambda \in \mathbb{R}. \tag{2.2.9}$$

In this formula, the inner product (x, ω) stands for the signed distance from the origin to the hyperplane passing through the point x with an unit normal ω.

We make the formal definition of the Fourier transform now.

Definition 2.2.5. *Let $f \in \mathcal{D}(X)$, then its Helgason Fourier transform (HFT) [26, Ch.III, §1] denoted by $\mathcal{F}f$ is a function defined on $\mathbb{C} \times K/M$ given by the integral*

$$\mathcal{F}f(\lambda, kM) = \int_G f(x)e^{(i\lambda-1)H(x^{-1}k)}dx. \tag{2.2.10}$$

For the sake of simplicity we fix the notational convention $\mathcal{F}f(\lambda, kM) = \mathcal{F}f(\lambda, k)$. We should note that, for a bi-K-invariant function (that is a left K-invariant function) g on X, the HFT reduces to the spherical transform: $\mathcal{F}g(\lambda, k) = \mathcal{F}g(\lambda, e) = \mathcal{S}g(\lambda)$.

A $\mathcal{C}^\infty$ function $\psi(\lambda, k)$ on $\mathbb{C} \times K/M$ is said to be of *uniformly exponential type-R* if there exists a constant $R > 0$ and for each $N \in \mathbb{Z}^+$ there is a constant $C_N > 0$ such that

$$\sup_{\lambda\in\mathbb{C}, k\in K/M} e^{-R|\Im\lambda|}(1 + |\lambda|)^N |\psi(\lambda, k)| \le C_N < +\infty. \tag{2.2.11}$$

We denote the class of such function by $\mathcal{H}^R(\mathbb{C} \times K/M)$. Let $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times K/M) = \cup_{R>0}\mathcal{H}^R(\mathbb{C} \times K/M)$. Let $\mathcal{H}(\mathbb{C} \times K/M)_W \subset \mathcal{H}(\mathbb{C} \times K/M)$ be the subclass of functions ψ with the property: $\lambda \mapsto \int_K e^{-(i\lambda+1)H(x^{-1}k)}\psi(\lambda, k)dk$ is an even entire function. We now state the analogue of the Paly-Wiener theorem for the Helgason Fourier transform.

Theorem 2.2.6. Helgason [26, Ch.III, §5]
The HFT is a bijection of $\mathcal{D}(X)$ onto $\mathcal{H}(\mathbb{C} \times K/M)_W$ with the inversion

formula

$$f(x) = (\mathcal{F}^{-1}(\mathcal{F}f))(x) = \frac{1}{2}\int_{\mathfrak{a}^*=\mathbb{R}}\int_K e^{-(i\lambda+1)H(x^{-1}k)}\mathcal{F}f(\lambda,k)|\boldsymbol{c}(\lambda)|^{-2}d\lambda dk. \tag{2.2.12}$$

In particular, if for some $R>0$, $\psi\in\mathcal{H}^R(\mathbb{C}\times K/M)_W$ *then* $\mathcal{F}^{-1}\psi\in\mathcal{D}^R(X)$.

Remark 2.2.7. *A commutative diagram similar to (2.2.8) can also be obtained for the HFT. In this case the role of the Abel transform is played by the Radon transform. For a symmetric space* $X=G/K$ *the space of all horocycles is identified with the quotient* $\Xi = G/MN$. *Each horocycle* $\xi\in\Xi$ *is a submanifold, hence it inherits a Riemannian structure. The* N *orbit* ξ_0, *is the horocycle passing through the origin. Left-*G*-action on* ξ_0 *generates all the members of* Ξ. *The 'Radon transform' or the 'horocycle transform' of any suitable function* f *on* X, *denoted by* $\mathcal{R}f$, *is a function on the horocycle space* Ξ *and is given by the following integral.*

$$(\mathcal{R}f)(g\cdot\xi_0) = \int_N f(gn\cdot 0)dn. \tag{2.2.13}$$

For any $f\in\mathcal{D}(X)$, *the HFT* $\mathcal{F}f$ *is sliced through the Radon transform* $\mathcal{R}f$ *as follows:*

$$\mathcal{F}f(\lambda,k) = \int_{\mathbb{R}}(\mathcal{R}f)(ka_t\cdot\xi_0)e^{(-i\lambda+1)(t)}dt. \tag{2.2.14}$$

For each $f\in\mathcal{D}(X)$ and $\delta\in\widehat{K}_M$, we define the δ projection of its HFT $\mathcal{F}f$ as follows:

$$(\mathcal{F}f)^{\delta}(\lambda,k) = d_\delta\int_K \mathcal{F}f(\lambda,k_1k)\delta(k_1)dk_1, \quad \text{where } \lambda\in\mathbb{C} \text{ and } k\in K. \tag{2.2.15}$$

The HFT $\mathcal{F}(f^\delta)$ of f^δ is also defined by the formula (2.2.10), in this case the integral of the matrix function being the matrix of the entry-wise integrals.

Lemma 2.2.8. *For each* $f\in\mathcal{D}(X)$ *and* $\delta\in\widehat{K}_M$ *the following are true*

(i) $(\mathcal{F}f)^{\delta}(\lambda,k) = \delta(k)\,(\mathcal{F}f)^{\delta}(\lambda,e)$,

(ii) $\mathcal{F}(f^\delta)(\lambda,k) = (\mathcal{F}f)^{\delta}(\lambda,k)$, *for all* $\lambda\in\mathbb{C}$ *and* $k\in K$.

Proof. Part (i) of the Lemma follows trivially from (2.2.15). Part (ii) can be deduced from the following calculation

$$\begin{aligned}\mathcal{F}(f^\delta)(\lambda, kM) &= \int_X f^\delta(x) e^{(i\lambda-1)H(x^{-1}k)} dx \\ &= d_\delta \int_G \left\{ \int_K f(k_1 x)\delta(k_1^{-1}) dk_1 \right\} e^{(i\lambda-1)H(x^{-1}k)} dx. \qquad (2.2.16)\end{aligned}$$

Now the desired result follows from (2.2.16) by Fubini's theorem.

■

If $f \in \mathcal{D}(X)$ then $f^\delta \in \mathcal{D}^\delta(X)$. The quantity $(\mathcal{F}f)^\delta(\lambda, e)$ (for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$), which is same as $\mathcal{F}(f^\delta)(\lambda, e)$, is called the *$\delta$-spherical transform* of the function $f^\delta \in \mathcal{D}^\delta(X)$. Let us give an alternative integral representation of this matrix valued Fourier transform using the generalized spherical functions, that is, the δ-spherical transform on $\mathcal{D}^\delta(X)$. Most of the basic analysis was done by Helgason [26] on $\mathcal{D}(\check{\delta}, X)$, we shall follow those results closely and prove them on $\mathcal{D}^\delta(X)$ using the homeomorphism $\mathcal{Q}$, defined in Theorem 2.0.2.

Definition 2.2.9. *For $f \in \mathcal{D}^\delta(X)$ the δ-spherical transform $\widetilde{f}$ is a matrix valued function on $\mathfrak{a}^*_{\mathbb{C}}$ and is given by*

$$\widetilde{f}(\lambda) = d_\delta \int_G trf(x)\ \Phi_{\overline{\lambda},\delta}(x)^* dx, \qquad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \qquad (2.2.17)$$

Clearly, for each λ, $\widetilde{f}(\lambda) \in Hom(V_\delta, V_\delta^M)$ and $\lambda \mapsto \widetilde{f}(\lambda)$ is an entire function. The following Lemma identifies the two definitions of the δ-spherical transform mentioned above.

Lemma 2.2.10. *If $f \in \mathcal{D}^\delta(X)$, where $\delta \in \widehat{K}_M$, then $\mathcal{F}f(\lambda, e) = \widetilde{f}(\lambda)$ for all $\lambda \in \mathbb{C}$.*

Proof. For any $f \in \mathcal{D}^\delta(X)$, using the topological isomorphism $\mathcal{Q}$ as described in Theorem 2.0.2, we get $trf(\cdot) \in \mathcal{D}(\check{\delta}, X)$ and also $f(x) = d_\delta \int_K trf(kx)\delta(k^{-1})dk$. Now from the definition of HFT (2.2.10) we get:

$$\begin{aligned}\mathcal{F}f(\lambda, e) &= \int_G f(x) e^{(i\lambda-1)(H(x^{-1}e))} dx \\ &= d_\delta \int_G \int_K trf(kx)\delta(k^{-1}) dk e^{(i\lambda-1)(H(x^{-1}e))} dx.\end{aligned}$$

By Fubini's theorem and on substituting $kx = y$ the last expression

$$
\begin{aligned}
&= d_\delta \int_G trf(y) \left\{ \int_K e^{(i\lambda-1)(H(y^{-1}k))} \delta(k^{-1}) dk \right\} dy \\
&= d_\delta \int_G trf(y) \Phi_{\overline{\lambda},\delta}(y)^* dy, \quad \text{from} \ \ (2.1.2) \\
&= \widetilde{f}(\lambda), \quad \text{by (2.2.17).}
\end{aligned}
$$

■

Let us now determine the behavior of the δ-spherical transform under the action of the Weyl group.

Lemma 2.2.11. *Let $\delta \in \widehat{K}_M$ and $f \in \mathcal{D}(X)$, then the map*

$$\lambda \mapsto Q_\delta(1 - i\lambda)^{-1} \widetilde{f}(\lambda), \tag{2.2.18}$$

where Q_δ is the polynomial defined in Theorem 2.1.5, is an even holomorphic function on $\mathbb{C}$.

Proof. This result is an easy consequence of the Definition 2.2.9 and Theorem 2.1.5. ■

Lemma 2.2.12. *Let $f \in \mathcal{D}^\delta(X)$, then the inversion formula for the δ-spherical transform (Definition 2.2.9) is given by:*

$$f(x) = \frac{1}{2} \int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x) \widetilde{f}(\lambda) |\boldsymbol{c}(\lambda)|^{-2} d\lambda. \tag{2.2.19}$$

Proof. The formula (2.2.19) comes from the inversion formula (2.2.12) of the HFT.

$$
\begin{aligned}
f(x) &= \frac{1}{2} \int_{\mathfrak{a}^*} \int_K \mathcal{F}f(\lambda, k) e^{-(i\lambda+1)(H(x^{-1}k))} |\mathbf{c}(\lambda)|^{-2} dk \ d\lambda \text{ from (2.2.10)} \\
&= \frac{1}{2} \int_{\mathfrak{a}^*} \int_K \delta(k) \mathcal{F}f(\lambda, e) e^{-(i\lambda+1)(H(x^{-1}k))} |\mathbf{c}(\lambda)|^{-2} dk \ d\lambda \text{ by Lemma 2.2.8} \\
&= \frac{1}{2} \int_{\mathfrak{a}^*} \left\{ \int_K e^{-(i\lambda+1)(H(x^{-1}k))} \delta(k) dk \right\} \widetilde{f}(\lambda) |\mathbf{c}(\lambda)|^{-2} d\lambda \text{ by Lemma 2.2.10} \\
&= \frac{1}{2} \int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x) \widetilde{f}(\lambda) |\mathbf{c}(\lambda)|^{-2} d\lambda \text{ by (2.1.1).}
\end{aligned}
$$

■

A topological Paley-Wiener theorem can be deduced for the δ-spherical transform, which we shall present in the next chapter.
The above transforms and most of the results mentioned above can be extended to larger classes of functions containing the compactly supported functions. In our discussion we shall consider the Schwartz class of functions.

Chapter 3

Schwartz space isomorphism theorem

3.1 Introduction

In this chapter we give a simple proof of the L^p-Schwartz space isomorphism ($0 < p \leq 2$) under the Fourier transform for the class of functions of left δ-type on a *rank*-1 Riemaniann symmetric space X realized as G/K, where G is a connected, noncompact semisimple Lie group with finite center and with real rank-1, and K a maximal compact subgroup of G.

The L^p-Schwartz space isomorphism theorem ($0 < p \leq 2$) for the bi-K-invariant class of functions has a long history. This theorem was first proved for $p = 2$ by Harish-Chandra [18–20]. Later, it was extended to $0 < p < 2$ by Trombi and Varadarajan [49]. Particular cases were considered in [10, 11, 23]. Rouvière [39] proved this theorem for real rank-1 groups by using an explicit form of the inverse of the Abel transform. The book by Gangolli and Varadarajan [14] contains a detailed and complete proof of the Schwartz spaces isomorphism theorem. Our point of departure is the work of Anker [2] who gave a remarkably short and elegant proof of the L^p-Schwartz space isomorphism theorem ($0 < p \leq 2$) for K-bi-invariant functions on the group G under the spherical transform. In his proof, Anker obtained the Schwartz space isomorphism theorem as a consequence of the Paley-Wiener theorem for the bi-K-invariant class of functions. He avoids use of accurate estimate of the behavior of the elementary spherical functions, which played the crucial role in all the earlier works. In this chapter we have used Anker's result to obtain the isomorphism of the L^p-Schwartz space $(0 < p \leq 2)$ of the functions on X with a fixed left-K-type, under the *Helgason Fourier transform*. Our technique is to first reduce the problem to the bi-K-invariant situation by the use of the Kostant's polynomial, so that we are able to invoke Anker's result. Apart from giving a simple proof our treatment has the same advantage as Anker's of not using higher asymptotics of the $\varphi_\lambda(x)$.

In the last section of this chapter we further extend the isomorphism (Theorem 3.3.3) to the Schwartz class $\mathcal{S}^p(F;X) \subset \mathcal{C}^\infty(F : X)$, where F is a finite subset of $\widehat{K}_M$. The main content of this chapter is a joint work [30] with Rudra P. Sarkar. Below, our plan is to give the explicit statement of the isomorphism theorem for the bi-K-invariant class of functions, before we take up our work on the function spaces on X.

3.2 Bi-K-invariant results

We begin this section with the definition of the L^p-Schwartz space $\mathcal{S}^p(G)$ where G is a semisimple Lie group as mentioned earlier.

Definition 3.2.1. *For every $0 < p \leq 2$, the L^p-Schwartz space $\mathcal{S}^p(G)$ is the space of functions $f \in \mathcal{C}^\infty(G)$ with the following decay: for each $\boldsymbol{D}, \boldsymbol{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $m \in \mathbb{Z}^+ \cup \{0\}$*

$$\mu_{\boldsymbol{D},\boldsymbol{E},m}(f) = \sup_{x \in G} |f(\boldsymbol{D}; x; \boldsymbol{E})|(1 + |x|)^m \varphi_0^{-\frac{2}{p}}(x) < +\infty. \tag{3.2.1}$$

We denote by $\mathcal{S}^p(G//K)$ the subspaces of $\mathcal{S}^p(G)$ consisting of bi-K-invariant functions. The space $\mathcal{S}^p(G//K)$ is a Fréchet space with the topology induced by the seminorms $\{\mu_{\mathbf{D},\mathbf{E},m}\}$ defined in (3.2.1).

The space $\mathcal{D}(G//K)$ is a dense subspace of $\mathcal{S}^p(G//K)$ with respect to the topology of the Schwartz space. The image of $\mathcal{D}(G//K)$ under the spherical Fourier transform is completely characterized in the Paley-Wiener theorem. It can be shown that for each $f \in \mathcal{S}^p(G//K)$, the spherical transform $\mathcal{S}f$ given by (2.2.1) exists for all λ in a strip $\mathfrak{a}^*_\varepsilon \subset \mathbb{C}$ where $\varepsilon = \left(\frac{2}{p} - 1\right)$ and

$$\mathfrak{a}^*_\varepsilon = \{\lambda \in \mathbb{C} \mid |\Im\lambda| \leq \varepsilon\}. \tag{3.2.2}$$

On the λ variable domain we have.

Definition 3.2.2. *Let $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W$ be the space of all complex valued functions defined on $\mathfrak{a}_\varepsilon^*$ satisfying the following properties.*

(i) Each $h \in \mathcal{S}(\mathfrak{a}_\varepsilon^*)_W$ is holomorphic in the interior of the strip $\mathfrak{a}_\varepsilon^*$ and extends as a continuous function to the closed strip.

(ii) For each $\lambda \in \mathfrak{a}_\varepsilon^*$, $h(\lambda) = h(-\lambda)$ (This is the Weyl group invariance in our case).

(iii) For each polynomial $P \in \boldsymbol{S}(\mathfrak{a}^*)$ and $t \in \mathbb{Z}^+ \cup \{0\}$ we have,

$$\tau_{P,t}(h) = \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \left| P\left(\frac{d}{d\lambda}\right) h(\lambda) \right| (1+|\lambda|)^t < +\infty, \tag{3.2.3}$$

where $\boldsymbol{S}(\mathfrak{a}^*)$ is the symmetric algebra of constant coefficient polynomials on $\mathfrak{a}^*$ and $P\left(\frac{d}{d\lambda}\right)$ is the differential operator obtained by replacing the variable λ in $P(\lambda)$ with $\frac{d}{d\lambda}$.

It can be shown that, with the topology induced by the countable family $\{\tau_{P,t}\}$ of seminorms, $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W$ becomes a Fréchet space. Moreover, $\mathcal{H}(\mathbb{C})_W|_{\mathfrak{a}_\varepsilon^*}$ (see (2.2.4)) is a dense subset of $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W$. The Schwartz space isomorphism theorem states the following.

Theorem 3.2.3. *The spherical transform (2.2.1) is a topological isomorphism from $\mathcal{S}^p(G//K)$, for $0 < p \leq 2$, onto the space $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W$; the inverse transform is given by the integral (2.2.5).*

We observe that the topology of the space $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$ can also be determined by two other families of seminorms, both of them equivalent to the one given in (3.2.3). For simplicity, we use the same notation for these seminorms. The first one is

$$\tau_{P,t}(h) = \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \left| P\left(\frac{d}{d\lambda}\right) \left\{ (\langle \lambda, \lambda \rangle_1 + 1)^t h(\lambda) \right\} \right| < +\infty, \tag{3.2.4}$$

where $P \in \mathbf{S}(\mathfrak{a}^*)$ and $t \geq 0$ is an integer. The equivalence of (3.2.3) and (3.2.4) is trivial. As the members of $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W$ are all even, so the seminorms on this space can also be defined alternatively as follows:

$$\tau_{P,t}(h) = \sup_{\lambda \in Int\left(\mathfrak{a}_\varepsilon^* \cap (\mathbb{R} + i\overline{\mathbb{R}^+})\right)} \left| P\left(\frac{d}{d\lambda}\right) h(\lambda) \right| (|\lambda|+1)^t < +\infty. \tag{3.2.5}$$

These, alternative forms of the seminorms will be useful for proving certain results in the course of our discussion.

3.3 Left-δ-type case

We now come to the function spaces of our interest, where we find it more convenient to work with matrix valued functions. Let us choose one $\delta \in \widehat{K}$ with the representation space V_δ . Let d_δ be the dimension of the space V_δ. The basic L^p-Schwartz space $\mathcal{S}^p_\delta(X)$ is the space of $Hom(V_\delta, V_\delta)$ valued $\mathcal{C}^\infty$ functions f on the symmetric space X satisfying the properties:

(i) for each $x \in X$ and $k \in K$, $f(kx) = \delta(k) f(x)$;

(ii) for each $\mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and for each integer $n \geq 0$ one has

$$\mu_{\mathbf{D},\mathbf{E},n}(f) = \sup_{x \in G} \|f(\mathbf{D}; x; \mathbf{E})\|_{\mathbf{2}} (1 + |x|)^n \varphi_0^{-\frac{2}{p}}(x) < +\infty. \qquad (3.3.1)$$

Remark 3.3.1. *(i) In fact it can be shown that: $\mathcal{S}^p_\delta(X) = \left\{ f^\delta \mid f \in \mathcal{S}^p(X) \right\}$, where $\mathcal{S}^p(X)$ is the subspace of $\mathcal{S}^p(G)$ consisting of right K-invariant functions. [We note that any function on X can also be regarded as a right K-invariant function on G.] The projection $f \mapsto f^\delta$ is as defined in (2.0.24).*

(ii) Let $\mathcal{S}^p(\check{\delta}; X) = \{ f \in \mathcal{S}^p(X) \mid f \equiv d_\delta \chi_{\check{\delta}} * f \}$, where $\check{\delta}$ is the contragradient representation of δ. Being a subspace of $\mathcal{S}^p(X)$, $\mathcal{S}^p(\check{\delta}; X)$ has the subspace topology. Theorem 2.0.2 can be easily extended to the Schwartz space level and the map $f(x) \mapsto tr f(x)$ $(x \in X)$ is a homeomorphism from $\mathcal{S}^p_\delta(X)$ onto $\mathcal{S}^p(\check{\delta}; X)$ with the inverse given by (2.0.24).

We now define a function space in the Fourier domain which is a prospective candidate for the image of $\mathcal{S}^p_\delta(X)$ under the δ-spherical transform.

Definition 3.3.2. *We denote $\mathcal{S}_\delta(\mathfrak{a}^*_\varepsilon)$ for the space of all $Hom(V_\delta, V_\delta)$ valued functions ψ on the complex strip $\mathfrak{a}^*_\varepsilon$ with the properties:*

(i) For each $\lambda \in \mathfrak{a}^*_\varepsilon$, $\psi(\lambda)$ maps V_δ to V_δ^M. We have already mentioned that $dim V_\delta^M = 1$, so with a convenient choice of basis $\psi(\lambda)$ is a $d_\delta \times d_\delta$ matrix with all the rows except the first one being identically zero.

(ii) *Each ψ is holomorphic in the interior of the strip $\mathfrak{a}_\varepsilon^*$ and extends as a continuous function to the closed strip.*

(iii) *ψ satisfies the identity*

$$Q_\delta(\lambda)^{-1}\psi(\lambda) = Q_\delta(-\lambda)^{-1}\psi(-\lambda), \qquad \lambda \in \mathfrak{a}_\varepsilon^*, \tag{3.3.2}$$

where $Q_\delta(\lambda) \equiv Q_\delta(1 - i\lambda)$ is the Kostant polynomial (2.1.10).

(iv) *For each $P \in \boldsymbol{S}(\mathfrak{a}^*)$ and for each integer $t \geq 0$ we have:*

$$\tau_{P,t}(\psi) = \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \left\| P\left(\frac{d}{d\lambda}\right)\psi(\lambda)\right\|_{\boldsymbol{2}} (1 + |\lambda|)^t < +\infty. \tag{3.3.3}$$

It can be shown that the space $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ is a Fréchet space with the topology induced by the countable family of seminorms $\{\tau_{P,t}\}$. Let us now state the main theorem of this chapter.

Theorem 3.3.3. *For $0 < p \leq 2$ and $\varepsilon = (2/p - 1)$ the δ-spherical transform $f \mapsto \widetilde{f}$ is a topological isomorphism between the spaces $\mathcal{S}_\delta^p(X)$ and $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$.*

This theorem is a part of the result of Eguchi and Kowata [9].
In the following discussion we shall actually show that the δ-spherical transform is a continuous bijection from $\mathcal{S}_\delta^p(X)$ onto $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$. Hence Theorem 3.3.3 will follow from the *open mapping theorem*. Before that, let us state the topological Paley-Wiener theorem, due to Helgason, for the δ-spherical transform, which will be crucially used in our proof.

3.3.1 A topological Paley-Wiener theorem

A holomorphic function $\psi : \mathfrak{a}_\mathbb{C}^* \simeq \mathbb{C} \longrightarrow Hom(V_\delta, V_\delta^M)$ is said to be of *exponential type* R if for each $N \in \mathbb{Z}^+$

$$\sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} e^{-R|\Im\lambda|}(1 + |\lambda|)^N \|\psi(\lambda)\|_{\boldsymbol{2}} < +\infty. \tag{3.3.4}$$

We denote the class of such functions by $\mathcal{H}_\delta^R(\mathbb{C})$. Let $\mathcal{H}_\delta(\mathbb{C}) = \bigcup_{R>0} \mathcal{H}_\delta^R(\mathbb{C})$.

Theorem 3.3.4. [Topological Paley-Wiener Theorem for K-types]
For each fixed $\delta \in \widehat{K}_M$, the δ-spherical transform given by(2.2.17) is a home-

omorphism between the spaces $\mathcal{D}^\delta(X)$ *and* $\mathcal{P}^\delta(\mathbb{C})$*, where*

$$\mathcal{P}^\delta(\mathbb{C}) = \left\{\xi \in \mathcal{H}_\delta(\mathbb{C}) \mid \lambda \mapsto Q_\delta(\lambda)^{-1}\xi(\lambda) \text{ is an even entire function}\right\}. \tag{3.3.5}$$

Here $Q_\delta(\lambda)$ *is the Kostant polynomial. The inverse transform is given by (2.2.19).*

Proof. We rely on the proof of the topological Paley-Wiener theorem given by Helgason ([26, Ch.III, Theorem 5.11]), where he characterized the image of the space $\mathcal{D}(\check{\delta}, X)$ under the transform $f \mapsto \widehat{\widehat{f}}$, where

$$\widehat{\widehat{f}}(\lambda) = d_\delta \int_G f(x)\Phi_{\overline{\lambda},\delta}(x)^* dx, \quad (\lambda \in \mathfrak{a}^*_{\mathbb{C}}). \tag{3.3.6}$$

Helgason showed that, the above transform is a topological isomorphism between the spaces $\mathcal{D}(\check{\delta}, X)$ and $\mathcal{P}^\delta(\mathbb{C})$. From Theorem 2.0.2, and the definition of the δ-spherical transform given in (2.2.17) the following diagram commutes: for each $f \in \mathcal{D}_\delta(X)$, $\widehat{\widehat{(\mathcal{Q}f)}}(\lambda) = \widetilde{f}(\lambda)$, for all $\lambda \in \mathbb{C}$.

The theorem follows from the facts that the maps $\mathcal{Q}$ and $f \mapsto \widehat{\widehat{f}}$ are homeomorphisms. ∎

Let us consider the function space $\mathcal{P}^\delta_0(\mathbb{C}) = \{h \in \mathcal{H}_\delta(\mathbb{C}) \mid h \text{ is even}\}$ with the relative topology. Any $h \in \mathcal{P}^\delta_0(\mathfrak{a}^*_{\mathbb{C}})$ can be written as a row $h \equiv (h_j)_{1\leq j\leq d_\delta}$. Each of the scalar valued component function h_j is entire, W-invariant and of exponential type. Let $\mathcal{D}(G//K, Hom(V_\delta, V_\delta^M))$ be the spaces of all $Hom(V_\delta, V_\delta^M)$ valued bi-K-invariant, compactly supported, $\mathcal{C}^\infty$ functions on G. From the Paley-Wiener theorem (Theorem 2.2.3) for the spherical transform, we get an unique $f_j \in \mathcal{D}(G//K)$ such that $\mathcal{S}f_j = h_j$. We set $f \equiv (f_j)_{1\leq j\leq d_\delta}$. We denote $\mathcal{S}f := (\mathcal{S}f_j)_{1\leq j\leq d_\delta}$, and so $\mathcal{S}f = h$. Moreover the Paley-Wiener theorem concludes that, $\mathcal{S}$ is a homeomorphism between $\mathcal{D}(G//K, Hom(V_\delta, V_\delta^M))$ and $\mathcal{P}^\delta_0(\mathbb{C})$. Furthermore the following Lemma shows that the two Paley-Wiener spaces $\mathcal{P}^\delta(\mathbb{C})$ and $\mathcal{P}^\delta_0(\mathbb{C})$ are homeomorphic.

Lemma 3.3.5. [Helgason]
The mapping $\psi(\lambda) \mapsto Q_\delta(\lambda)\psi(\lambda)$ *(*$\lambda \in \mathbb{C}$*) is a homeomorphism from* $\mathcal{P}^\delta_0(\mathbb{C})$ *onto* $\mathcal{P}^\delta(\mathbb{C})$.

Proof. For a proof of the above Lemma see [26, Ch.-III, §5, Lemma 5.12]. ∎

The following is a key lemma for the proof of Theorem 3.3.3.

Lemma 3.3.6. *Any function $f \in \mathcal{D}^\delta(X)$ can be written as $f(x) = \boldsymbol{D}^\delta \phi(x)$ ($\forall x \in G$), where $\phi \in \mathcal{D}(G//K, Hom(V_\delta, V_\delta^M))$ and $\boldsymbol{D}^\delta$ is the constant coefficient differential operator corresponding to Q_δ as introduced in (2.1.11).*

Proof. Let $f \in \mathcal{D}^\delta(X)$, then by the Paley-Wiener theorem (Theorem 3.3.4), its δ-spherical transform $\widetilde{f} \in \mathcal{P}^\delta(\mathbb{C})$. Using the homeomorphism given in Lemma 3.3.5, we get an unique function $\lambda \mapsto \Phi(\lambda) = Q_\delta(\lambda)^{-1}\widetilde{f}(\lambda)$ in $\mathcal{P}_0^\delta(\mathbb{C})$. The Paley-Wiener theorem for the spherical transform gives a function $\phi \in \mathcal{D}(G//K, Hom(V_\delta, V_\delta^M))$ such that:

$$\phi(x) = \frac{1}{2}\int_{\mathfrak{a}^*} \varphi_\lambda(x)\Phi(\lambda)|\mathbf{c}(\lambda)|^{-2}d\lambda. \tag{3.3.7}$$

Now we apply the differential operator $\mathbf{D}^\delta$, introduced in Theorem 2.1.7, on the both sides of (3.3.7). As the integral in the above expression converges absolutely, so we can write:

$$\begin{aligned}
\left(\mathbf{D}^\delta \phi\right)(x) &= \frac{1}{2}\int_{\mathfrak{a}^*} \left(\mathbf{D}^\delta \varphi_\lambda(x)\right)\Phi(\lambda)|\mathbf{c}(\lambda)|^{-2}d\lambda \\
&\text{now we use (2.1.11) to get ,} \\
&= \frac{1}{2}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)Q_\delta(\lambda)\Phi(\lambda)|\mathbf{c}(\lambda)|^{-2}d\lambda \\
&= \frac{1}{2}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)\widetilde{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}d\lambda \\
&= f(x), \quad \text{by the inversion formula (2.2.19).}
\end{aligned}$$

■

Looking at the bi-K-invariant result that, $\mathcal{H}(\mathbb{C})_W|_{\mathfrak{a}_\varepsilon^*}$ is a dense subspace of $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$, the following result is expected.

Lemma 3.3.7. *The Paley-Wiener space $\mathcal{P}^\delta(\mathbb{C})|_{\mathfrak{a}_\varepsilon^*}$ is a dense subspace of the Fréchet space $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$.*

To prove this Lemma, we again need to go back and forth between the Fréchet space $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ and its symmetric counterpart.
Let $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ denotes the space of all $Hom(V_\delta, V_\delta^M)$ valued $\mathcal{C}^\infty$ functions h on $\mathfrak{a}_\varepsilon^*$ such that

(i) h is holomorphic on $Int\mathfrak{a}_\varepsilon^*$ and it extends as a continuous function to the closed strip $\mathfrak{a}_\varepsilon^*$;

(ii) h is an even function on $\mathfrak{a}_\varepsilon^*$;

(iii) for each polynomial $P \in \mathbf{S}(\mathfrak{a}^*)$ and integer $n \geq 0$

$$\tau^+_{P,n}(h) = \sup_{\lambda \in Int\left(\mathfrak{a}_\varepsilon^* \cap \left(\mathbb{R}+i\overline{\mathbb{R}^+}\right)\right)} (1+|\lambda|)^n \left\| P\left(\frac{d}{d\lambda}\right) h(\lambda) \right\|_{\mathbf{2}} < +\infty. \quad (3.3.8)$$

The seminorms $\tau^+_{P,n}(\cdot)$ makes $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ a Fréchet space. Note that the space $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ can also be viewed as the space of all $Hom(V_\delta, V_\delta^M)$ valued functions whose matrix entries are in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W$. For our purpose we shall be using another equivalent (inducing the same topology) family of seminorms on $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ given by

$$\tau^{+*}_{P,r}(h) = \sup_{\lambda \in Int\left(\mathfrak{a}_\varepsilon^* \cap (\mathbb{R}+i\overline{\mathbb{R}^+})\right)} \left\| \left\{ P\left(\frac{d}{d\lambda}\right) h(\lambda) \right\} (1+\langle\lambda,\lambda\rangle_1)^r \right\|_{\mathbf{2}}. \quad (3.3.9)$$

As the spherical transform $\mathcal{S}$ can be extended to the class of operator valued functions, we can extend the isomorphism given in Theorem 3.2.3 for this class as follows:

Lemma 3.3.8. *The spherical transform is a topological isomorphism between the spaces $\mathcal{S}^p(G//K, Hom(V_\delta, V_\delta^M))$ and $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$.*

Proof. This Lemma can be proved easily by using the conclusion of the Theorem 3.2.3 for each matrix entry of the functions of $\mathcal{S}^p(G//K, Hom(V_\delta, V_\delta^M))$. ∎

The next Lemma proves that the two spaces $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ and $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ are homeomorphic. Infact the following Lemma extends the homeomorphism given in Lemma 3.3.5 between the Paley-Wiener spaces to the corresponding Schwartz classes.

Lemma 3.3.9. *The map*

$$g(\lambda) \mapsto Q_\delta(\lambda) g(\lambda), \qquad \text{for all} \;\; \lambda \in \mathfrak{a}_\varepsilon^*, \quad (3.3.10)$$

is a homeomorphism from the space $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ onto $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$.

Proof. Let us first take $g \in \mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$. We denote $h(\lambda) = Q_\delta(\lambda)g(\lambda)$ ($\lambda \in \mathfrak{a}_\varepsilon^*$). We shall show that $h \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$. It is clear that h is holomorphic in the interior of the tube $\mathfrak{a}_\varepsilon^*$.

It trivially follows from the definition that the function $\lambda \mapsto Q_\delta(\lambda)^{-1}h(\lambda)$ is W-invariant. We shall now show that h also satisfies the Schwartz space decay. Let us take a polynomial $P \in \mathbf{S}(\mathfrak{a}^*)$ and $m \in \mathbb{Z}^+ \cup \{0\}$. Then

$$\begin{aligned}
&\sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \left\| P\left(\frac{d}{d\lambda}\right) h(\lambda) \right\|_{\mathbf{2}} (1+|\lambda|)^m \\
&\qquad \leq \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \sum_\kappa c_\kappa \left\| \left\{ P'_\kappa\left(\frac{d}{d\lambda}\right) Q_\delta(\lambda) \right\} \left\{ P_\kappa\left(\frac{d}{d\lambda}\right) g \right\}(\lambda) \right\|_{\mathbf{2}} (1+|\lambda|)^m
\end{aligned}$$

by the simple application of Leibniz rule, where P_κ, P'_κ are polynomials and the sum over a finite set,

$$\begin{aligned}
&\leq \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \sum_\kappa c_\kappa \left| \left\{ P'_\kappa\left(\frac{d}{d\lambda}\right) Q_\delta(\lambda) \right\} \right| \left\| \left\{ P_\kappa\left(\frac{d}{d\lambda}\right) g \right\}(\lambda) \right\|_{\mathbf{2}} (1+|\lambda|)^m \\
&\leq \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \sum_\kappa c_\kappa^\delta \left\| \left\{ P_\kappa\left(\frac{d}{d\lambda}\right) g \right\}(\lambda) \right\|_{\mathbf{2}} (1+|\lambda|)^{m_\kappa^\delta} \qquad (3.3.11)
\end{aligned}$$

where m_κ^δ are nonnegative integers and c_κ^δ are positive constants both depending on $\delta \in \widehat{K}_M$. As $g \in \mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$, the right hand side of (3.3.11) is clearly finite. The inequality (3.3.11) shows that the map (3.3.10) is continuous from $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ into $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$.

Now let $\psi \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ and define $g(\lambda) := Q_\delta(\lambda)^{-1}\psi(\lambda)$ ($\lambda \in \mathfrak{a}_\varepsilon^*$). As, $\psi \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$, by Definition 3.3.2 the function g is even and it is holomorphic in the interior of the tube $\mathfrak{a}_\varepsilon^*$. To conclude $g \in \mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ all we have to show is that the function g has a certain decay. At this point we use the alternative form (3.3.9) of the seminorms on $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$.

Let $P \in \mathbf{S}(\mathfrak{a}^*)$ and t be any nonnegative integer, then

$$\begin{aligned}
&\left\{ P\left(\frac{d}{d\lambda}\right) g(\lambda) \right\} (1+\langle\lambda,\lambda\rangle_1)^r \\
&\qquad = \left\{ P\left(\frac{d}{d\lambda}\right) Q_\delta(\lambda)^{-1}\psi(\lambda) \right\} (1+\langle\lambda,\lambda\rangle_1)^r \\
&\qquad = \frac{P_1(\lambda)\ \Box\psi(\lambda)}{P_2(\lambda)}, \qquad (3.3.12)
\end{aligned}$$

where $P_1(\lambda), P_2(\lambda)$ are a polynomials whose degrees are clearly dependent on d_δ and $\square$ is a certain constant coefficient polynomial in $\left(\frac{d}{d\lambda}\right)$. We also note that the polynomial P_2 is precisely of the form $(Q_\delta(\lambda))^m$ $(m \in \mathbb{Z}^+)$ hence it does not vanish on $(\mathfrak{a}_\varepsilon^* \cap (\mathbb{R} + i\overline{\mathbb{R}^+}))$ (see, Lemma 2.1.6) . Hence there exists some constant $\theta_\delta > 0$, so that, $\inf_{\lambda \in Int\mathfrak{a}_\varepsilon^* \cap \left(\mathbb{R}+i\overline{\mathbb{R}^+}\right)} |P_2(\lambda)| \geq \theta_\delta$. Thus the following inequality follows easily from (3.3.12).

$$\sup_{\lambda \in Int\left(\mathfrak{a}_\varepsilon^* \cap (\mathbb{R}+i\overline{\mathbb{R}^+})\right)} \left\| \left\{ P\left(\frac{\partial}{\partial\lambda}\right) g(\lambda) \right\} (1 + \langle \lambda, \lambda \rangle_1)^r \right\|_{\mathbf{2}}$$
$$\leq C_\delta/\theta_\delta \sup_{\mu \in Int\mathfrak{a}_\varepsilon^*} \|P_1(\lambda)\ \square\psi(\lambda)\|_{\mathbf{2}} \qquad (3.3.13)$$

As $\psi \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$, so the right-hand side of (3.3.13) is clearly finite which proves that the function g has the required decay of the Schwartz class $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$. This also proves that the map (3.3.10) is a bijection. As both the spaces $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ and $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ are Fréchet spaces, so an application of the open mapping theorem completes the proof of the Lemma. ■

We shall now give a proof of the Lemma 3.3.7.

Proof. of Lemma 3.3.7

Let us take any $H \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$, it is enough to show that, there is a sequence $\{G_n\}$ ($G_n \in \mathcal{P}^\delta(\mathbb{C})$) converging to H in the topology of the space $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$. Let $H(\lambda) = (H_j(\lambda))_{1\leq j\leq d_\delta}$. By the isomorphism obtained in Lemma 3.3.9, we get one unique $G \in \mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ such that

$$H(\lambda) = (H_j(\lambda))_{1\leq j\leq d_\delta} = Q_\delta(\lambda)G(\lambda) = (Q_\delta(\lambda)G_j(\lambda))_{1\leq j\leq d_\delta}. \qquad (3.3.14)$$

As $G \in \mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$, so from the definition of the Schwartz space $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ it follows that the entry functions $G_j \in \mathcal{S}(\mathfrak{a}_\varepsilon^*)$ for each j, $1 \leq j \leq d_\delta$. We know that the Paley-Wiener space $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$ under the spherical transform is dense in the Schwartz class $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$ [14]. Therefore we can get a sequence $\left\{g_{j_n}\right\}_n \subset \mathcal{P}^\delta(\mathbb{C})$ converging to G_j in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$. As, $Q_\delta(\lambda)$ is only a polynomial, hence for each $\lambda \in \mathfrak{a}_\varepsilon^*$, $\left\{Q_\delta(\lambda)g_{j_n}(\lambda)\right\}$ converges to $Q_\delta(\lambda)G_j(\lambda)$ in $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$.

Let $g_n(\lambda) = \left(g_{j_n}(\lambda)\right)_{1\leq j\leq d_\delta}$. As each $g_{j_n} \in \mathcal{P}(\mathbb{C})$, so from the definition it follows that, the matrix valued function $g_n \in \mathcal{P}_0^\delta(\mathbb{C})$. Clearly by Lemma 3.3.5 for each natural number n, $Q_\delta g_n \in \mathcal{P}^\delta(\mathbb{C})$. Let P be any polynomial in

$\mathbf{S}(\mathfrak{a}^*)$ and t be any nonnegative integer then:

$$\begin{aligned}
&\tau_{P,t}\left(Q_\delta(\cdot)g_n(\cdot) - Q_\delta(\cdot)G(\cdot)\right) \\
&\quad = \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \left\| P\left(\frac{d}{d\lambda}\right)\{Q_\delta(\lambda)g_n(\lambda) - Q_\delta(\lambda)G(\lambda)\}\right\|_{\mathbf{2}} (1+|\lambda|)^t \\
&\quad = c_\delta \sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*} \left\| P\left(\frac{d}{d\lambda}\right)(g_n - G)(\lambda)\right\|_{\mathbf{2}} (1+|\lambda|)^{t_\delta}. \qquad (3.3.15)
\end{aligned}$$

Suitably large n makes the right hand side of (3.3.15) arbitrarily small. Hence we get the sequence $\{Q_\delta(\cdot)g_n\}_n$ in $\mathcal{P}^\delta(\mathbb{C})$ converging to H in the topology of $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$. This completes the proof of the Lemma. ■

3.4 Proof of Theorem 3.3.3

We shall first extend the definition of the δ-spherical transform (2.2.17) to the Schwartz class $\mathcal{S}_\delta^p(X)$ where $0 < p \leq 2$.

Lemma 3.4.1. *For each $f \in \mathcal{S}_\delta^p(X)$, the function $\lambda \mapsto \widetilde{f}(\lambda)$, where $\widetilde{f}$ is given by (2.2.17), is a holomorphic function in the interior of the complex strip $\mathfrak{a}_\varepsilon^*$ and it extends as a continuous function to the boundary of the strip. If $p = 2$, $\widetilde{f}$ is continuous on $\mathfrak{a}^*$.*

Proof. For each $f \in \mathcal{S}_\delta^p(X)$, it is easy to observe that $|trf(x)| \leq \|f(x)\|_{\mathbf{2}}$ $(x \in X)$. The function $x \mapsto trf(x)$ has the following decay: for each $\mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and integer $n \geq 0$

$$\sup_{x \in G} |trf(\mathbf{D}; x; \mathbf{E})|(1+|x|)^n \varphi_0^{-\frac{2}{p}}(x) < +\infty, \qquad (3.4.1)$$

which follows easily from (3.3.3). We also notice that

$$\begin{aligned}
\|\Phi_{\lambda,\delta}(x)\|_{\mathbf{2}} &\leq \int_K \left| e^{-(i\lambda+1)H(x^{-1}k)}\right| \|\delta(k)\|_{\mathbf{2}} dk \\
&= c_\delta \int_K e^{-(-\Im\lambda+1)H(x^{-1}k)} dk \\
&= c_\delta \varphi_{i\Im\lambda}(x), \quad x \in G, \lambda \in \mathbb{C}. \qquad (3.4.2)
\end{aligned}$$

Observe now that the integrand in the definition of $\widetilde{f}(\lambda)$ satisfies, for $\lambda \in \mathfrak{a}_\varepsilon^*$

and $x \in G$

$$\begin{aligned}
\|trf(x)\Phi_{\overline{\lambda},\delta}(x)^*\|_{\mathbf{2}} &\leq c\varphi_0^{\frac{2}{p}}(x)(1+|x|)^{-n}e^{|\Im\lambda||x|}\varphi_0(|x|), \text{ using (3.4.2)} \\
&\leq c(1+|x|)^{-n}e^{|\Im\lambda||x|}\varphi_0^{\frac{2}{p}+1}(|x|) \\
&\leq c(1+|x|)^{-n+d_p}e^{(\varepsilon-\frac{2}{p}-1)|x|} \\
&= c(1+|x|)^{-n+d_p}e^{-2|x|}, \qquad (3.4.3)
\end{aligned}$$

where the constant c (for a given n) comes from (3.4.1) and d_p from the estimate (2.0.22). The function $(1+|x|)^{-n+d_p}e^{-2|x|}$ is integrable for large n and dominates the integrand for all $\lambda \in \mathfrak{a}_\varepsilon^*$. The continuity of $\widetilde{f}(\lambda)$ for $\lambda \in \mathfrak{a}_\varepsilon^*$ now follows from the dominated convergence theorem.

Let γ be a closed curve in the interior of the tube $\mathfrak{a}_\varepsilon^*$. From the definition of the δ-spherical transform we get: for $f \in \mathcal{S}_\delta^p(X)$

$$\int_\gamma \widetilde{f}(\lambda)d\lambda = d_\delta \int_\gamma \left\{ \int_G trf(x)\Phi_{\overline{\lambda},\delta}(x)^* dx \right\} d\lambda,$$

as we have already noticed that the integral within braces exists absolutely for $\lambda \in \mathfrak{a}_\varepsilon^*$, so a simple application of Fubini's theorem gives:

$$= d_\delta \int_G trf(x) \left\{ \int_\gamma \Phi_{\overline{\lambda},\delta}(x)^* d\lambda \right\} dx,$$

as $\lambda \mapsto \Phi_{\overline{\lambda},\delta}(x)^*$ is an entire function, so by the Morera's Theorem the inner integral vanishes and hence we get:

$$= 0.$$

Hence by Morera's theorem again it follows that, for each $f \in \mathcal{S}_\delta^p(X)$ the transform $\widetilde{f}$ in a holomorphic function on $Int\mathfrak{a}_\varepsilon^*$. ∎

Lemma 3.4.2. *The δ-spherical transform $f \mapsto \widetilde{f}$ is a continuous map from the Schwartz space $\mathcal{S}_\delta^p(X)$ into $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$.*

Proof. Lemma 3.4.1 shows that the integral representation (2.2.17) is well-defined for the Schwartz class on $\mathfrak{a}_\varepsilon^*$ furthermore, the transform $\widetilde{f}$ is holomorphic in the interior of the tube $\mathfrak{a}_\varepsilon^*$. Lemma 2.2.11 has an extension for all $f \in \mathcal{S}_\delta^p(X)$ and $\widetilde{f}$ satisfies (3.3.2) for all $\lambda \in \mathfrak{a}_\varepsilon^*$. It is now enough to show that given any seminorm τ on $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ one can find a seminorm μ on $\mathcal{S}_\delta^p(X)$ such that

$$\tau(\widetilde{f}) \leq c\mu(f) \quad \text{for each } f \in \mathcal{S}_\delta^p(X).$$

Let P be any polynomial in $\mathbf{S}(\mathfrak{a}^*)$ and n be any nonnegative integer. From the definition of the δ-spherical transform we get

$$\begin{aligned}
&P\left(\frac{d}{d\lambda}\right)\left\{(\langle\lambda,\lambda\rangle_1+1)^n\,\widetilde{f}(\lambda)\right\}\\
&\qquad= d_\delta\ P\left(\frac{d}{d\lambda}\right)\int_G trf(x)\ \ (\langle\lambda,\lambda\rangle_1+1)^n\,\Phi_{\overline{\lambda},\delta}(x)^*dx\\
&\qquad= (-1)^n d_\delta\ P\left(\frac{d}{d\lambda}\right)\int_G trf(x)\ \ \mathbf{L}^n\Phi_{\overline{\lambda},\delta}(x)^*dx\\
&\qquad= (-1)^n d_\delta\ P\left(\frac{d}{d\lambda}\right)\int_G \mathbf{L}^n trf(x)\ \ \Phi_{\overline{\lambda},\delta}(x)^*dx\\
&\qquad= (-1)^n d_\delta\ P\left(\frac{d}{d\lambda}\right)\int_G \mathbf{L}^n trf(x)\ \ \Phi_{\overline{\lambda},\delta}(x)^*dx.
\end{aligned}$$

The second equality in the chain uses (2.1.5) and the third follows from an application of integration by parts. As the differential operator $\mathbf{L}$ acts entry-wise to the operator valued function f, so it is easy to check that $\mathbf{L}trf(x) = tr\mathbf{L}f(x)$ and so the last expression becomes

$$\begin{aligned}
&= (-1)^n d_\delta\ P\left(\frac{d}{d\lambda}\right)\int_G tr\ \mathbf{L}^n f(x)\ \ \Phi_{\overline{\lambda},\delta}(x)^*dx\\
&= (-1)^n d_\delta\ P\left(\frac{d}{d\lambda}\right)\int_G tr\ \mathbf{L}^n f(x)\left\{\int_K e^{(i\lambda-1)(H(x^{-1}k))}\delta(k^{-1})dk\right\}dx,
\end{aligned}$$

using the integral expression (2.1.9) for the generalized spherical function. Here $f \in \mathcal{S}_\delta^p(X)$, so as discussed earlier, f can also be considered as a right K invariant function on the group G. The action of the Laplace Beltrami operator $\mathbf{L}$ on f is the same as the action of the *Casimir operator* on f considering it as a function on G. Hence, by the property of the Casimir operator, the action of $\mathbf{L}$ does not change the left K-type of the function f, that is, the function $\mathbf{L}^n f$ is again of left δ-type. Moreover, for each nonnegative integer n the function $\mathbf{L}^n f(x) \in \mathcal{S}_\delta^p(X)$. Hence by Lemma 3.4.1, the repeated integral in the last line exists absolutly. We now apply Fubini's theorem to interchange the integrals and then we put $x^{-1}k = y^{-1}$ to get:

$$
\begin{aligned}
&= (-1)^n d_\delta P\left(\frac{d}{d\lambda}\right)\int_K\int_G tr\mathbf{L}^n f(ky)\ \ e^{(i\lambda-1)H(y^{-1})}\delta(k^{-1})dydk \\
&= (-1)^n d_\delta P\left(\frac{d}{d\lambda}\right)\int_G\left\{\int_K tr\mathbf{L}^n f(ky^{-1})\ \ \delta(k^{-1})dk\right\}\ \ e^{(i\lambda-1)H(y)}dy \\
&= (-1)^n \int_G P\left(iH(y)\right)\mathbf{L}^n f(y^{-1})\ \ e^{(i\lambda-1)H(y)}dy,
\end{aligned}
$$

the last equality follows by using (ii) of Remark 3.3.1. Let us now break up the group G as well as the Haar measure using the Iwasawa KAN decomposition and write $y = ka_r n$, where $r \in \mathfrak{a} = \mathbb{R}$ and $\exp r = a_r$ to obtain

$$
\begin{aligned}
&= (-1)^n \int_K\int_{\mathfrak{a}}\int_N P(iH(ka_rn))\mathbf{L}^n f(n^{-1}{a_r}^{-1}k^{-1})e^{(i\lambda-1)(H(ka_rn))}dke^{2r}drdn \\
&= (-1)^n \int_{\mathfrak{a}}\int_N P(ir)\mathbf{L}^n f((a_rn)^{-1})e^{(i\lambda+1)r}drdn. \qquad (3.4.4)
\end{aligned}
$$

From (3.4.4) it follows that:

$$
\begin{aligned}
&\left\|P\left(\frac{d}{d\lambda}\right)\left\{\left(\langle\lambda,\lambda\rangle_1+1\right)^n\widetilde{f}(\lambda)\right\}\right\|_{\mathbf{2}} \\
&\qquad\qquad \leq \int_{\mathfrak{a}}\int_N \|\mathbf{L}^n f((a_rn)^{-1})\|_{\mathbf{2}}\ |P(ir)|e^{(|\Im\lambda|+1)r}drdn. \qquad (3.4.5)
\end{aligned}
$$

Using the basic estimate (2.0.4) we ge $|r| \leq c(1+|a_rn|)$ and hence one can find $d_P \in \mathbb{Z}^+$ such that

$$
|P(ir)| \leq c(1+|a_rn|)^{d_P}. \qquad (3.4.6)
$$

As $f \in \mathcal{S}^p_\delta(X)$, so for each $m \in \mathbb{Z}^+$ we have

$$
\|\mathbf{L}^n f((a_rn)^{-1})\|_{\mathbf{2}} \leq \mu_{\mathbf{L}^n,m}(f)(1+|a_rn|)^{-m}\varphi_0^{\frac{2}{p}}((a_rn)^{-1}). \qquad (3.4.7)
$$

The above inequality also uses the fact that $|g| = |g^{-1}|$ for all $g \in G$. The estimates (3.4.6) and (3.4.7) reduces the inequality (3.4.5) to the following.

$$
\begin{aligned}
&\left\|P\left(\frac{d}{d\lambda}\right)\left\{\left(\langle\lambda,\lambda\rangle_1+1\right)^n\widetilde{f}(\lambda)\right\}\right\|_{\mathbf{2}} \\
&\leq c\mu_{\mathbf{L}^n,m}(f)\int_{\mathfrak{a}}\int_N (1+|a_rn|)^{-m+d_P}\varphi_0^{\frac{2}{p}}((a_rn)^{-1})e^{(|\Im\lambda|+1)r}drdn
\end{aligned}
$$

$$= c\mu_{\mathbf{L}^n,m}(f)\int_K\int_{\mathfrak{a}}\int_N (1+|ka_rn|)^{-m+d_P}\varphi_0^{\frac{2}{p}}((ka_rn)^{-1})e^{(|\Im\lambda|-1)H(ka_rn)}dke^{2r}drdn$$
$$= c\mu_{\mathbf{L}^n,m}(f)\int_G (1+|x|)^{d_P-m}\varphi_0^{\frac{2}{p}}(x^{-1})e^{(|\Im\lambda|-1)H(x)}dx. \tag{3.4.8}$$

Now we use the Cartan decomposition i.e. $x = k_1\exp|x|k_2$ and the appropriate form of the Haar measure (2.0.7) to get

$$\begin{aligned}
(3.4.8) &= c\mu_{\mathbf{L}^n,m}(f)\int_K\int_{\mathfrak{a}^+}\int_K \varphi_0^{\frac{2}{p}}(\exp|x^{-1}|)\ (1+\big|k_1\exp|x|k_2\big|)^{-m+d_P} \\
&\qquad\qquad e^{(|\Im\lambda|-1)(H(k_1\exp|x|k_2))}\ dk_1\ \Delta(|x|)d|x|\ dk_2 \\
&= c\mu_{\mathbf{L}^n,m}(f)\int_{\mathfrak{a}^+}\int_K \varphi_0^{\frac{2}{p}}(\exp|x|)\ (1+|x|)^{-m+d_P}e^{(|\Im\lambda|-1)(H(\exp|x|k_2))} \\
&\qquad\qquad \Delta(|x|)d|x|\ dk_2 \\
&= c\mu_{\mathbf{L}^n,m}(f)\int_{\mathfrak{a}^+} \varphi_0^{\frac{2}{p}}(\exp|x|)\ (1+|x|)^{-m+d_P} \\
&\qquad\qquad \left\{\int_K e^{(|\Im\lambda|-1)(H(\exp|x|k_2))}\ dk_2\right\}\Delta(|x|)d|x| \\
&= c\mu_{\mathbf{L}^n,m}(f)\int_{\mathfrak{a}^+} \varphi_0^{\frac{2}{p}}(\exp|x|)\ (1+|x|)^{-m+d_P}\varphi_{-i|\Im\lambda|}(\exp|x^{-1}|) \\
&\qquad\qquad \Delta(|x|)d|x| \\
&\le c\mu_{\mathbf{L}^n,m}(f)\int_{\mathfrak{a}^+} \varphi_0^{\frac{2}{p}+1}(\exp|x|)\ (1+|x|)^{-m+d_P}e^{|\Im\lambda||x|}\Delta(|x|)d|x|,
\end{aligned} \tag{3.4.9}$$

where the last inequality in this chain follows by using the estimate (2.0.21) of the elementary spherical function. We take $\lambda\in Int\mathfrak{a}_\varepsilon^*$, therefore $|\Im\lambda| < \varepsilon = (\frac{2}{p}-1)$. By using the fundamental estimate (2.0.22), we reduce the inequality (3.4.9) to the following

$$\begin{aligned}
&\left\|P\left(\frac{d}{d\lambda}\right)\left\{(\langle\lambda,\lambda\rangle_1+1)^n\,\widetilde{f}(\lambda)\right\}\right\|_{\mathbf{2}} \\
&\qquad \le c\mu_{\mathbf{L}^n,m}(f)\int_{\mathfrak{a}^+}(1+|x|)^{-m+d_P+\frac{2}{p}-1}\varphi_0^2(\exp|x|)\Delta(|x|)d|x| \\
&\qquad = c\mu_{\mathbf{L}^n,m}(f)\int_G(1+|x|)^{-m+d_P+\frac{2}{p}-1}\varphi_0^2(x)dx.
\end{aligned} \tag{3.4.10}$$

We choose a suitably large $m\in\mathbb{Z}^+$ so that the integral in (3.4.10) converges ([20, Lemma 11]). This completes the proof of the Lemma. ∎

We now take up the extension of the inversion formula (2.2.19) of the δ-spherical transform.

Lemma 3.4.3. *For each $h \in \mathcal{S}_\delta(\mathfrak{a}^*_\varepsilon)$ the integral*

$$\frac{1}{2}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)\ h(\lambda)\ |\boldsymbol{c}(\lambda)|^{-2} d\lambda, \tag{3.4.11}$$

gives a $Hom(V_\delta, V_\delta)$ valued, left δ-type C^∞ function of $x \in X$. [From now on we shall denote this function by $\mathcal{I}h$].

Proof. Let us take any $\mathbf{D} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. Since $h \in \mathcal{S}_\delta(\mathfrak{a}^*_\varepsilon)$, for all $x \in X$

$$\begin{aligned}\frac{1}{2}\int_{\mathfrak{a}^*} &\|\Phi_{\lambda,\delta}(\mathbf{D}; x)\|_{\mathbf{2}}\|h(\lambda)\|_{\mathbf{2}}|\mathbf{c}(\lambda)|^{-2} d\lambda \\ &\leq c_\delta \varphi_0(x)\int_{\mathfrak{a}^*}(1+|\lambda|)^{b_{\mathbf{D}}+b-n} d\lambda,\end{aligned} \tag{3.4.12}$$

by using the decay (3.2.3) and the estimates (2.2.6) and (2.1.6) one can choose a suitably large n so that the integral in (3.4.12) exists finitely. This proves $\mathcal{I}h$ is a function on X and $\mathbf{D}\mathcal{I}h$ exists for all $\mathbf{D} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. Hence $\mathcal{I}h \in \mathcal{C}^\infty(X, Hom(V_\delta, V_\delta))$. As, $\Phi_{\lambda,\delta}(\cdot)$ is of left δ-type (Proposition 2.1.3), so is $\mathcal{I}h$. ∎

Lemma 3.4.4. *If $h \in \mathcal{S}_\delta(\mathfrak{a}^*_\varepsilon)$ then the inverse $\mathcal{I}h \in \mathcal{S}^p_\delta(X)$.*

Proof. To prove this Lemma we shall first consider the spaces $\mathcal{P}^\delta(\mathbb{C})$ and $\mathcal{D}^\delta(X)$ equipped with the topologies of the respective Schwartz spaces containing them. We have seen that $\mathcal{P}^\delta(\mathbb{C})$ and $\mathcal{D}^\delta(X)$ are dense subspaces of $\mathcal{S}_\delta(\mathfrak{a}^*_\varepsilon)$ and $\mathcal{S}^p_\delta(X)$ respectively.
We shall show that the map $\mathcal{I}$ is a continuous map from $\mathcal{P}^\delta(\mathbb{C})$ onto (by Theorem 3.3.4) $\mathcal{D}^\delta(X)$ with respect to the Schwartz space topologies. That is for each seminorm μ on $\mathcal{D}^\delta(X)$, there exists a seminorm τ on $P^\delta(\mathbb{C})$ such that $\mu(f) \leq c\tau(h)$, where $f = \mathcal{I}h \in \mathcal{D}^\delta(X)$ and c is a positive constant depending on $\delta \in \widehat{K}_M$.
As, $f \in \mathcal{D}^\delta(X)$, by Lemma 3.3.6, we get a function $\phi \in \mathcal{D}(G//K, Hom(V_\delta, V_\delta))$ such that $f \equiv \mathbf{D}^\delta\phi$. If Φ be the image of ϕ under the spherical transform then it follows easily that $h(\cdot) = Q_\delta(\cdot)\Phi(\cdot)$. Let $\mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and n be any

nonnegative integer, then

$$\begin{aligned}\mu_{\mathbf{D},\mathbf{E},n}(f) &= \sup_{x\in G}\|f(\mathbf{D};x;\mathbf{E})\|_{\mathbf{2}}(1+|x|)^n\varphi_0^{-\frac{2}{p}}(x)\\ &= \sup_{x\in G}\|\mathbf{D}^{\delta}\phi(\mathbf{D};x;\mathbf{E})\|_{\mathbf{2}}(1+|x|)^n\varphi_0^{-\frac{2}{p}}(x)\\ &= \zeta_{\mathbf{D}^{\delta}\mathbf{D},\mathbf{E},n}(\phi). \end{aligned}\tag{3.4.13}$$

[Here ζ denote the seminorms on the Fréchet space $\mathcal{S}^p(G//K, Hom(V_\delta, V_\delta))$.] At this point we use Anker's [2] proof of the Schwartz space isomorphism theorem for K-bi invariant functions. For each $\mathbf{D},\mathbf{E}\in\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ and $n\in\mathbb{Z}^+$ one can find a polynomial $P\in\mathbf{S}(\mathfrak{a}^*)$ and $m_\delta\in\mathbb{Z}^+$ (depending on d_δ) such that,

$$\zeta_{\mathbf{D}^{\delta}\mathbf{D},\mathbf{E},n}(\phi)\leq c\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}\left\|P\left(\frac{d}{d\lambda}\right)\Phi(\lambda)\right\|_{\mathbf{2}}(1+|\lambda|)^{m_\delta}.$$

Using the isomorphism, proved in Lemma 3.3.9, between the Schwartz spaces $\mathcal{S}_0(\mathfrak{a}_\varepsilon^*)$ and $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ we get that

$$\mu_{\mathbf{D},\mathbf{E},n}(f)\leq c\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}\left\|P_1\left(\frac{d}{d\lambda}\right)h(\lambda)\right\|_{\mathbf{2}}(1+|\lambda|)^{m'_\delta}=\tau_{P_1,m_\delta}(h),\tag{3.4.14}$$

where $\tau_{P_1,m_\delta}(h)$ is one of $\mathcal{S}(\mathfrak{a}_\varepsilon^*)$ seminorms and $h\in\mathcal{P}^\delta(\mathbb{C})\subset\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$.
Now we apply the density argument to conclude the Lemma. Let us now take $h\in\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$. As, $\mathcal{P}_\delta(\mathbb{C})$ is dense in $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$, there exists a Cauchy sequence $\{h_n\}\subset\mathcal{P}_\delta(\mathfrak{a}_{\mathbb{C}}^*)$ converging to h. Then, by what we have proved above, we can get a Cauchy sequence $\{f_n\}\subset\mathcal{D}^\delta(X)$ such that $\widetilde{f_n}=h_n$. As $\mathcal{S}_\delta^p(X)$ is a Fréchet space the sequence must converge to some $f\in\mathcal{S}_\delta^p(X)$. Now $f=\mathcal{I}h$ by a pointwise convergence argument from Lemma 3.4.3. This completes the proof of the Lemma 3.4.11. ∎

We note that the Lemma 3.4.4 also implies the fact that the δ- spherical transform is an injection in the corresponding Schwartz space level.
Finally, Lemma 3.4.2 and Lemma 3.4.4 together show that the δ-spherical transform is a continuous surjection of $\mathcal{S}_\delta^p(X)$ onto $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ for $(0<p\leq 2)$. A simple application of the open mapping theorem concludes that the δ-spherical transform is a topological isomorphism between the corresponding Schwartz spaces. This proves the Theorem 3.3.3. In the next section we shall extend this result to a slightly larger class of functions.

3.5 Finite K-type functions

Let X be a rank-1 Riemannian symmetric space, as before. Let $\mathcal{D}(F;X) \subset \mathcal{C}^{\infty}(F;X)$ denote the subspace of compactly supported scalar valued functions f on X such that f is left-K-finite with the left types lying in the fixed subset $F \subseteq \widehat{K}_M$.

$$\mathcal{S}^p(F;X) = \{f \in \mathcal{S}^p(X) \mid f(x) = \sum_{\delta \in F} tr f^{\delta}(x) \quad \text{for all} \;\; x \in X\}, \qquad (3.5.1)$$

the Schwartz space containing $\mathcal{D}(F;X)$. It also follows easily from the definition that, if $f \in \mathcal{S}^p(F;X)$ then for each $\delta \in F$ the projection f^{δ} (defined in (2.0.24)) is a member of $\mathcal{S}^p_{\delta}(X)$. For these classes of functions the transform we shall mainly consider is the Helgason Fourier transform.

Let us now define the Schwartz class of functions on the domain $\mathfrak{a}^*_{\varepsilon} \times K/M$.

Definition 3.5.1. *Let $\mathcal{S}(F;\mathfrak{a}^*_{\varepsilon} \times K/M)$ denotes the class of functions h on $\mathfrak{a}^*_{\varepsilon} \times K/M$ satisfying the following properties:*

(i) For each $kM \in K/M$, the function $\lambda \mapsto h(\lambda, kM)$ is holomorphic on $Int\mathfrak{a}^*_{\varepsilon}$, and it extends as a continuous function on the closed strip $\mathfrak{a}^*_{\varepsilon}$. The function h is a smooth function in the $k \in K/M$ variable.

(ii) For all $\lambda \in \mathfrak{a}^*_{\varepsilon}$ and $x \in G$

$$\check{h}(\lambda, x) = \check{h}(-\lambda, x), \qquad (3.5.2)$$

where, $\check{h}(\lambda, x) = \int_K h(\lambda, k) e^{-(i\lambda+1)H(x^{-1}k)} dk$.

(iii) For each $P \in \boldsymbol{S}(\mathfrak{a}^*)$ and for integers $n, m > 0$ the function h satisfies the following decay condition

$$\sup_{(\lambda,k) \in Int\mathfrak{a}^*_{\varepsilon} \times K/M} \left| P\left(\frac{d}{d\lambda}\right) h(\lambda, k; \omega^m_{\mathfrak{k}}) \right| \; (1 + |\lambda|)^n < \; +\infty. \qquad (3.5.3)$$

(iv) For each $\delta \in \widehat{K}_M \setminus F$ the left-δ-projection h^{δ} defined by

$$h^{\delta}(\lambda, k) = d_{\delta} \int_K h(\lambda, k_1 k) \delta(k_1^{-1}) dk_1, \qquad (3.5.4)$$

is identically a zero function on $\mathfrak{a}^*_{\varepsilon} \times K/M$.

The space $\mathcal{S}(F;\mathfrak{a}_\varepsilon^* \times K/M)$ is a Fréchet space with the topology induced by the seminorms (3.5.3). By the theory of smooth functions on compact groups [45], the topology of the space $\mathcal{S}(F;\mathfrak{a}_\varepsilon^* \times K/M)$ can be given by the following equivalent family of seminorms; for each $P \in \mathbf{S}(\mathfrak{a}^*)$ and $m \in \mathbb{Z}^+$ we have

$$\sup_{\lambda \in Int\mathfrak{a}_\varepsilon^*, \delta \in F} \left\| P\left(\frac{d}{d\lambda}\right) h^\delta(\lambda, eM) \right\|_{\mathbf{2}} (1+|\lambda|)^m < +\infty, \text{ for } h \in \mathcal{S}(F;\mathfrak{a}_\varepsilon^* \times K/M). \tag{3.5.5}$$

We denote by $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times K/M)$ the Fréchet space satisfying all the conditions of the Definition 3.5.1 except condition (iv). The space $\mathcal{S}(F;\mathfrak{a}_\varepsilon^* \times K/M)$ is a closed subspace of $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times K/M)$. We know that HFT can extended to the Schwartz class $\mathcal{S}^p(X)$ [9], further more HFT is a continuous map from $\mathcal{S}^p(X)$ into $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times K/M)$. Hence HFT is a continuous map from $\mathcal{S}^p(F;X)$ into $\mathcal{S}(F;\mathfrak{a}_\varepsilon^* \times K/M)$.

Lemma 3.5.2. *Let $h \in \mathcal{S}(F;\mathfrak{a}_\varepsilon^* \times K/M)$, then for each $\delta \in F$, the left-δ-projection $h^\delta \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$.*

Proof. The function $\lambda \mapsto h^\delta(\lambda, eM)$ trivially satisfies condition (i) and (ii) of Definition 3.3.2. The required decay (3.3.3) is also an easy consequence of (3.5.3). It can be shown that the δ-projection h^δ also satisfies the condition

$$(\check{h^\delta})(\lambda, x) = (h^\delta)(\check{-\lambda}, x), \quad x \in X.$$

It is easy to check that $h^\delta(\lambda, kM) = \delta(k)h^\delta(\lambda, eM)$, hence for each $(\lambda, a) \in \mathfrak{a}_\varepsilon^* \times A \cdot 0$ we write $(\check{h^\delta})(\lambda, a)$ as follows

$$(\check{h^\delta})(\lambda, a) = \Phi_{\lambda,\delta}(a)h^\delta(\lambda, eM).$$

By the property (2.1.8) of the generalized spherical functions it follows that $\lambda \mapsto Q_\delta(\lambda)^{-1}h^\delta(\lambda, eM)$ is an even function. Hence we conclude that $h^\delta(\cdot, eM) \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$. ∎

By using Theorem 3.3.3, for each $h \in \mathcal{S}(F;\mathfrak{a}_\varepsilon^* \times K/M)$ we get an unique finite sequence $\{f^\delta\}_{\delta \in F}$ of $\mathcal{C}^\infty$ functions on X such that each member $f^\delta \in \mathcal{S}_\delta^p(X)$. We consider the following scalar valued function

$$f(x) = \sum_{\delta \in F} tr f^\delta(x), \quad x \in X. \tag{3.5.6}$$

For each $\delta \in F$, $\left(\mathcal{F}^{-1}h\right)^{\delta}(x) = \mathcal{J}h^{\delta}(x) = f^{\delta}(x)$. Hence, we get $\mathcal{F}^{-1}h(x) = f(x)$ for all $x \in X$. The function $f \in \mathcal{S}^p(F;X)$, furthermore, for each $\mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ and $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}
&\sup_{x\in G} |f(\mathbf{D};x;\mathbf{E})|(1+|x|)^n \varphi_0^{-\frac{2}{p}}(x) \\
&\qquad \leq c \sup_{x\in G,\delta\in F} \|f^{\delta}(\mathbf{D};x;\mathbf{E})\|_{\mathbf{2}}(1+|x|)^n \varphi_0^{-\frac{2}{p}}(x) \\
&\qquad \leq c_1 \sup_{\lambda\in Int\mathfrak{a}_{\varepsilon}^*,\delta\in F} \left\| P\left(\frac{d}{d\lambda}\right) h^{\delta}(\lambda, eM)\right\|_{\mathbf{2}} (1+|\lambda|)^m \\
&\qquad \leq c_2 \sup_{\lambda\in Int\mathfrak{a}_{\varepsilon}^*,k\in K} \left| P_1\left(\frac{d}{d\lambda}\right) h(\lambda, k; \omega_{\mathfrak{k}}^r)\right| (1+|\lambda|)^{m_1},
\end{aligned}$$

for some $P_1 \in \mathbf{S}(\mathfrak{a}^*)$ and $r, m_1 \in \mathbb{Z}^+$. Thus, the HFT is a bijective map from $\mathcal{S}(F;\mathfrak{a}_{\varepsilon}^* \times K/M)$ to $\mathcal{S}^p(F;X)$. Once again, by the open mapping theorem we conclude the following.

Theorem 3.5.3. *Let F be a finite subset of $\widehat{K}_M$, then the Helgason Fourier transform is topological isomorphism of the space $\mathcal{S}^p(F;X)$ onto the Fréchet space $\mathcal{S}(F;\mathfrak{a}_{\varepsilon}^* \times K/M)$.*

Chapter 4

Image of Schwartz Space Under Spectral Projection

4.1 Introduction

This chapter can be viewed as an application of the Schwartz space isomorphism for the rank-1 Riemannian symmetric space obtained in the previous chapter. Let X be the Riemannian symmetric space realized as G/K where, G is a connected, noncompact, real rank-1 semi-simple Lie group with finite center. Let φ_λ ($\lambda \in \mathfrak{a}^*_{\mathbb{C}}$) be the elementary spherical functions of G. For $f \in \mathcal{S}^p(X)$ ($0 < p \leq 2$) (for the case $1 < p \leq 2$, $f \in L^p(X)$) we consider the transform $f \mapsto P_\lambda f(x) = f * \varphi_\lambda(x)$ for each λ in a suitable domain. The function $P_\lambda f$ is an eigenfunction of the Laplacian $\mathbf{L}$, satisfying $\mathbf{L}P_\lambda f(x) = -(1+\lambda^2)P_\lambda f(x)$ and the transform $f \mapsto P_\lambda f$ is called the generalized spectral projection. Strichartz in his series of papers [41], [42], [43], [44] initiated the project of reviewing Harmonic Analysis in terms of the generalized spectral projection. Continuing this project Bray [8] proved a spectral Paley-Wiener theorem for the symmetric space $X = G/K$. Ionescu [29] characterized the image $P_\lambda\left(L^2(X)\right)$. Strichartz [42] determined the image of Euclidean Schwartz class functions under spectral projection. The aim of this chapter is to characterize the image of the L^p-Schwartz space $\mathcal{S}^p(X)$ ($0 < p \leq 2$) under the transform $f \mapsto P_\lambda f$. This chapter is mainly divided in three parts. In Section 4.2 we obtain properties of the functions $P_\lambda f$ for $f \in \mathcal{S}^p(X)$. Sufficient conditions for a left K finite function $f(\lambda, x)$ on $\mathfrak{a}^*_\varepsilon \times X$, to be of the form $P_\lambda g(x) = f(\lambda, x)$, for some $g \in \mathcal{S}^p(X)$, are

taken up in Section 4.3. Finally, in Section 4.4, we shall characterize the image of certain subspace of $L^2(X)$ under the spectral projection in the light of the inverse Paley-Wiener theorem [47]. Main references for this chapter are [26, 27] and [8]. For Section 4.4 we refer [15–17, 35, 37] and [47].
Most of the basic notation used here have been defined in Chapter 2 and Chapter 3. Throughout this chapter we shall denote $\mathcal{E}_\lambda(X) = \{g \in \mathcal{C}^\infty(X) \mid \mathbf{L}g = -(1+\lambda^2)g\}$ $(\lambda \in \mathbb{C})$. The space $\mathcal{E}^\delta_\lambda(X)$ is the space of all matrix valued left δ-projection of the members of $\mathcal{E}_\lambda(X)$. We denote $\mathcal{E}_\lambda(\delta, X) = \mathcal{C}^\infty(\delta, X) \cap \mathcal{E}_\lambda(X)$.

4.2 Necessary Conditions

In this section we start with the L^p-Schwartz space $(0 < p \leq 2)$ $\mathcal{S}^p(X)$. For $f \in \mathcal{S}^p(X)$ we define $P_\lambda f(x) = (f * \varphi_\lambda)(x)$, for $\lambda \in \mathfrak{a}^*$. We get an alternative expression for the spectral projection $P_\lambda f(\cdot)$ in terms of the Helgason Fourier transform $\mathcal{F}f$ of the function $f \in \mathcal{S}^p(X)$. Beginning with $P_\lambda f(x) = \int_G f(y)\varphi_\lambda(y^{-1}x)dy$, we use the following standard property

$$\varphi_\lambda(y^{-1}x) = \int_K e^{-(i\lambda+1)H(x^{-1}k^{-1})} e^{(i\lambda-1)H(y^{-1}k^{-1})} dk$$

of the elementary spherical functions and use Fubini's theorem to write:

$$\begin{aligned} P_\lambda f(x) &= \int_K \left\{ \int_G f(y) e^{(i\lambda-1)H(y^{-1}k^{-1})} dy \right\} e^{-(i\lambda+1)H(x^{-1}k^{-1})} dk \\ &= \int_K \mathcal{F}f(\lambda, k^{-1}) e^{-(i\lambda+1)H(x^{-1}k^{-1})} dk \\ &= \int_K \mathcal{F}f(\lambda, k) e^{-(i\lambda+1)H(x^{-1}k)} dk. \end{aligned} \tag{4.2.1}$$

We have already noticed in Chapter 3 that for any $f \in \mathcal{S}^p(X)$ the Helgason Fourier transform $\mathcal{F}f$ is defined on the domain $\mathfrak{a}^*_\varepsilon \times K/M$, where $\mathfrak{a}^*_\varepsilon$ is the closed strip $\left\{\lambda \in \mathbb{C} \mid |\Im\lambda| \leq \varepsilon = \left(\frac{2}{p} - 1\right)\right\}$. Hence, (4.2.1) implies that for each $f \in \mathcal{S}^p(X)$ the function $(\lambda, x) \mapsto P_\lambda f(x)$ is defined on $\mathfrak{a}^*_\varepsilon \times X$. We could also obtain a direct proof of the existence of $P_\lambda f(x)$ and its continuity in both the variables by closely following the arguments in Lemma 3.4.1.

We use the notation $\mathfrak{e}_{\lambda,k}(x)$ for $e^{-(i\lambda+1)H(x^{-1}k)}$, which is the kernel of the integral in the definition (4.2.1). As we have already mentioned that the

Iwasawa decomposition KAN is diffeomorphic to G, so, for each $k \in K$, the Iwasawa-A-projection $x \mapsto H(x^{-1}k)$ is a $\mathcal{C}^\infty$ map on G. Thus $\mathfrak{e}_{\lambda,k} \in \mathcal{C}^\infty(X)$ for each $\lambda \in \mathbb{C}$ and $k \in K$. Hence, from (4.2.1) one can conclude that for each $\lambda \in \mathfrak{a}_\varepsilon^*$ and for each $f \in \mathcal{S}^p(X)$, $P_\lambda f \in \mathcal{C}^\infty(X)$. Furthermore the kernel $\mathfrak{e}_{\lambda,k}$ is a joint eigenfunction of the algebra $D(X)$ of all G-invariant differential operators on X. In particular for the Laplace-Beltrami operator $\mathbf{L}$ the eigenvalue for $\mathfrak{e}_{\lambda,k}$ we have:

$$\mathbf{L}\mathfrak{e}_{\lambda,k}(x) = -(1+\lambda^2)\mathfrak{e}_{\lambda,k}(x), \tag{4.2.2}$$

for each $\lambda \in \mathbb{C}$ and $k \in K$ [14]. According to our notation $\mathfrak{e}_{\lambda,k} \in \mathcal{E}_\lambda(X)$ for each $\lambda \in \mathbb{C}$. Note that the integral in the definition (4.2.1) of $P_\lambda f$ is over a compact set. Therefore, it follows easily that for each $\lambda \in \mathfrak{a}_\varepsilon^*$ and $f \in \mathcal{S}^p(X)$, $P_\lambda f \in \mathcal{E}_\lambda(X)$.

Before we prove other characteristic properties of the function space $P_\lambda(\mathcal{S}^p(X))$ $(\lambda \in \mathfrak{a}_\varepsilon^*)$, we need to recapitulate some of the results proved in the previous chapters.

If $f \in \mathcal{S}^p(X)$ then f^δ belongs to the space $\mathcal{S}^p_\delta(X)$ which is the operator valued left δ-projection of the space $\mathcal{S}^p(X)$. The image of $\mathcal{S}^p_\delta(X)$ under the δ-spherical transform (2.2.17) is characterized in the previous chapter (see Theorem 3.3.3). In this section we shall mainly use the continuity (see Lemma 3.4.2) of the transform from $\mathcal{S}^p_\delta(X)$ onto the space $\mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ (as defined in Definition 3.3.2).

For each $f \in \mathcal{S}^p(X)$ and for each $\lambda \in \mathfrak{a}_\varepsilon^*$, $P_\lambda f \in \mathcal{C}^\infty(X)$, we define its matrix valued left δ-projection $(P_\lambda f)^\delta$ by

$$(P_\lambda f)^\delta(x) = d_\delta \int_K P_\lambda f(kx)\delta(k^{-1})dk. \tag{4.2.3}$$

It is clear that for each $\delta \in \widehat{K}_M$, $(P_\lambda f)^\delta \in \mathcal{E}^\delta_\lambda(X)$. Now $(P_\lambda f)^\delta$ satisfies $(P_\lambda f)^\delta(kx) = \delta(k)(P_\lambda f)^\delta(x)$ $(k \in K, x \in X)$. Hence $tr(P_\lambda f)^\delta$ is a left $\check{\delta}$-type scalar valued function and hence $tr(P_\lambda f)^\delta \in \mathcal{E}(\check{\delta}, X)$.
The following proposition relates the projection $(P_\lambda f)^\delta$ with the generalized spherical function (2.1.1). This structure will be very useful for estimating the decay of the function $P_\lambda f$ for each $f \in \mathcal{S}^p(X)$ $(0 < p \leq 2)$.

Proposition 4.2.1. *Let $f \in L^p(X)$ if $1 < p \leq 2$ and $f \in \mathcal{S}^p(X)$ if $0 < p \leq 1$. Then* $(P_\lambda f)^\delta(x) = P_\lambda(f^\delta)(x) = \Phi_{\lambda,\delta}(x)\ \widetilde{f^\delta}(\lambda)$, *for $x \in X$ and $\lambda \in Int\mathfrak{a}_\varepsilon^*$,* *where $\widetilde{f^\delta}$ is the δ-spherical transform of f^δ as defined in (2.0.24).*

Proof. The existence of $P_\lambda f$ needs a proof in case of $f \in L^p(X)$, $1 < p \leq 2$. It is a consequence of the estimates (2.0.19) and (2.0.21) of the function φ_λ, that for $\lambda \in \mathfrak{a}_\varepsilon^*$, $\varphi_\lambda(\cdot) \in L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Further, it can be shown that, for each compact set $U \subset Int\mathfrak{a}_\varepsilon^*$, there exists a $g \in L^q(X)$ with $g \geq 0$ such that $|\varphi_\lambda(x)| < g(x)$, for $\lambda \in U$ and $x \in X$. Thus by Hölder's inequality $f * \varphi_\lambda(x)$ exists for all $\lambda \in U$. Moreover, the uniform domination of the φ_λ means that $\lambda \mapsto \varphi_\lambda$ is a continuous map of U to $L^q(X)$. Hölder's inequality will then make $f * \varphi_\lambda(x)$ continuous in $\lambda \in U$. The compact set U being arbitrary we get the existence and continuity in both the variables on $Int\mathfrak{a}_\varepsilon^* \times X$. From (4.2.3) we now have:

$$\begin{aligned}
(P_\lambda f)^\delta(x) &= d_\delta \int_K P_\lambda f(kx)\ \delta(k^{-1})\ dk \\
&= d_\delta \int_K \int_G f(y)\varphi_\lambda(y^{-1}kx)\ dy\ \delta(k^{-1})\ dk \\
&= d_\delta \int_K \int_G f(kz)\varphi_\lambda(z^{-1}x)\ dz\ \delta(k^{-1})\ dk \\
&= d_\delta \int_G \varphi_\lambda(z^{-1}x) \int_K f(kz)\ \delta(k^{-1})\ dk\ dz \\
&= \int_G f^\delta(z)\varphi_\lambda(z^{-1}x)\ dz \qquad (4.2.4) \\
&= P_\lambda(f^\delta)(x).
\end{aligned}$$

Using the *symmetry of the spherical function* (2.0.18) the expression (4.2.4) can be written as

$$(P_\lambda f)^\delta(x) = \int_G f^\delta(z) \int_K e^{-(i\lambda+1)H(x^{-1}k)} e^{(i\lambda-1)H(z^{-1}k)} dk\ dz. \qquad (4.2.5)$$

The repeated integral on the right-hand side converging absolutely, we interchange the integrals to obtain

$$\begin{aligned}
(P_\lambda f)^\delta(x) &= \int_K e^{-(i\lambda+1)H(x^{-1}k)} \int_G f^\delta(z) e^{(i\lambda-1)H(z^{-1}k)} dz\ dk \\
&= \int_K e^{-(i\lambda+1)H(x^{-1}k)} \mathfrak{F} f^\delta(\lambda, k) dk.
\end{aligned}$$

We have already noticed that $\mathcal{F}f^\delta(\lambda,k) = \delta(k)\mathcal{F}f^\delta(\lambda,e)$ and also we have observed that $\mathcal{F}f^\delta(\lambda,e) = \widetilde{f^\delta}(\lambda)$. Then

$$(P_\lambda f)^\delta(x) = \left\{ \int_K e^{-(i\lambda+1)H(x^{-1}k)}\delta(k)dk \right\} \widetilde{f^\delta}(\lambda) = \Phi_{\lambda,\delta}(x)\widetilde{f^\delta}(\lambda). \qquad (4.2.6)$$

■

From the above Proposition it follows easily that, for all $f \in \mathcal{S}^p(G//K)$ $(0 < p \leq 2)$, $P_\lambda f(x) = \varphi_\lambda(x)\mathcal{S}f(\lambda)$, where $\lambda \in \mathfrak{a}_\varepsilon^*$ and $\mathcal{S}f$ is the spherical transform of f.

Remark 4.2.2. *As $\varphi_\lambda(x) = \varphi_{-\lambda}(x)$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$, so both the functions $P_\lambda f(\cdot)$, $(P_\lambda f)^\delta(\cdot)$ and $tr(P_\lambda f)^\delta$ are even in the λ variable.*

To characterize $P_\lambda f$ for $f \in \mathcal{S}^p(X)$, we shall first concentrate on each of its δ-projections $(P_\lambda f)^\delta$. The following proposition summarizes the properties of $(P_\lambda f)^\delta$ which will be useful to characterize $P_\lambda(\mathcal{S}^p(X))$ $(\lambda \in \mathfrak{a}_\varepsilon^*)$.

Proposition 4.2.3. *For $f \in \mathcal{S}^p(X)$, where $0 < p \leq 2$ and for each fixed $\delta \in \widehat{K}_M$ the operator valued left δ-projection $(P_\lambda f)^\delta$ of $P_\lambda f$ $(\lambda \in \mathfrak{a}_\varepsilon^*)$ has the properties:*

(i) For each $\lambda \in \mathfrak{a}_\varepsilon^*$, the function $(P_\lambda f)^\delta \in \mathcal{E}_\lambda^\delta(X)$.

(ii) For each $x \in X$, $\lambda \mapsto (P_\lambda f)^\delta(x)$ is an even holomorphic function in the interior of the strip $\mathfrak{a}_\varepsilon^*$ and it extends as an even continuous function on the closed strip. The map $\lambda \mapsto Q_\delta(1-i\lambda)^{-1}(P_\lambda f)^\delta(x)$ is a holomorphic function on the open strip $Int\mathfrak{a}_\varepsilon^*$. For $p = 2$, $(P_\lambda f)^\delta(x)$ is a $\mathcal{C}^\infty$-function of $\lambda \in \mathbb{R}$.

(iii) For each $\boldsymbol{D}, \boldsymbol{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$, $m, n, s \in \mathbb{Z}^+ \cup \{0\}$ and for any real number $r_p < \frac{2p-2}{p}$ we can find positive constants c_i and positive integers l, t such that

$$\sup_{x\in G, \lambda \in Int\mathfrak{a}_\varepsilon^*} \left\| \left(\frac{d}{d\lambda}\right)^m (P_\lambda f^\delta)(\boldsymbol{D};x;\boldsymbol{E}) \right\|_2 (1+|x|)^n(1+|\lambda|)^s \varphi_0^{-r_p}(x)$$
$$\leq \sum_{i=0}^m c_i(1+|\delta|)^q \sup_{x\in X} \|\boldsymbol{L}^l f^\delta(x)\|_2 (1+|x|)^t \varphi_0^{-\frac{2}{p}}(x). \qquad (4.2.7)$$

Proof. Property (i) has already been discussed. The property (ii) is a consequence of the expression (4.2.6) and Lemma 3.4.1.
(iii) Using the result of the Proposition 4.2.1 we get

$$\left\| \left(\frac{d}{d\lambda}\right)^m (P_\lambda f)^\delta(\mathbf{D};x;\mathbf{E}) \right\|_{\mathbf{2}} = \left\| \left(\frac{d}{d\lambda}\right)^m \{\Phi_{\lambda,\delta}(\mathbf{D};x;\mathbf{E})\widetilde{f^\delta}(\lambda)\} \right\|_{\mathbf{2}}$$
$$\leq \sum_{\ell=0}^{t} \left\| \left(\frac{d}{d\lambda}\right)^\ell \Phi_{\lambda,\delta}(\mathbf{D};x;\mathbf{E}) \right\|_{\mathbf{2}} \left\| \left(\frac{d}{d\lambda}\right)^{m-\ell} \widetilde{f^\delta}(\lambda) \right\|_{\mathbf{2}} \tag{4.2.8}$$

We shall use the following estimates for the various derivatives of the matrix coefficients of the principal series representation [21, §17, Lemma 1]:

$$\left\| \left(\frac{d}{d\lambda}\right)^\ell \Phi_{\lambda,\delta}(\mathbf{D};x;\mathbf{E}) \right\|_{\mathbf{2}} \leq c(1+|\delta|)^q(1+|\lambda|)^q(1+|x|)^u\varphi_0(x)e^{|\Im\lambda||x|} \tag{4.2.9}$$

where, $c>0$ is a constant (may depend on the derivatives chosen but independent of $\delta \in \widehat{K}_M$), $q \in \mathbb{Z}^+$ depends on $\mathbf{D},\mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $u \in \mathbb{Z}^+$ depends on the integer ℓ. As $\lambda \in \mathfrak{a}_\varepsilon^*$ we can replace $|\Im\lambda|$ by $\varepsilon = (\frac{2}{p}-1)$ in (4.2.9). Now from (4.2.8) and (4.2.9) we get:

$$\left\| \left(\frac{d}{d\lambda}\right)^m (P_\lambda f)^\delta(\mathbf{D};x;\mathbf{E}) \right\|_{\mathbf{2}} (1+|x|)^n(1+|\lambda|)^s\varphi_0^{-r_p}(x)$$
$$\leq \sum_{\ell=0}^{m} c_\ell(1+|x|)^{n+u}(1+|\lambda|)^{s+q}(1+|\delta|)^q\varphi_0^{1-r_p}(x)e^{\varepsilon|x|} \left\| \left(\frac{d}{d\lambda}\right)^{m-\ell} \widetilde{f^\delta}(\lambda) \right\|_{\mathbf{2}}.$$

We notice that for all $0<p\leq 2$, $1-r_p>0$. Now we make use of the estimate $\varphi_0^{1-r_p}(x) \leq a(1+|x|)^{b_p}\, e^{(r_p-1)|x|}$, $x \in X$ (*where, b_p is a positive real number, to be precise, it is exactly* $1-r_p$). This is an easy consequence of the two-sided estimate (2.0.22) of the elementary spherical function $\varphi_0(x)$. Hence we can continue the above chain of inequalities by

$$\leq \sum_{\ell=0}^{m} c_\ell(1+|x|)^{n+u+b_p}(1+|\lambda|)^{s+q}(1+|\delta|)^q e^{-\gamma|x|} \left\| \left(\frac{d}{d\lambda}\right)^{m-\ell} \widetilde{f^\delta}(\lambda) \right\|_{\mathbf{2}}, \tag{4.2.10}$$

where, $\gamma = 2 - \frac{2}{p} - r_p$. Clearly, $\gamma > 0$ as $r_p < \frac{2p-2}{p}$. Hence from (4.2.10)

$$\begin{aligned}
&\sup_{x\in G,\lambda\in Int\mathfrak{a}_\varepsilon^*,}\left\| \left(\frac{d}{d\lambda}\right)^m (P_\lambda f)^\delta(\mathbf{D};x;\mathbf{E})\right\|_{\mathbf{2}} (1+|x|)^n(1+|\lambda|)^s\varphi_0^{-r_p}(x)\\
&\leq \sum_{\ell=0}^{t} c_\ell(1+|\delta|)^q\left\{\sup_{x\in X}(1+|x|)^{n+u+b_p}e^{-\gamma|x|}\right\}\\
&\qquad\left\{\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}(1+|\lambda|)^{s+q}\left\|\left(\frac{d}{d\lambda}\right)^{m-\ell}\widetilde{f^\delta}(\lambda)\right\|_{\mathbf{2}}\right\}\\
&\leq \sum_{\ell=0}^{m}\overline{c_\ell}\,(1+|\delta|)^q\left\{\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}(1+|\lambda|)^{s+q}\left\|\left(\frac{d}{d\lambda}\right)^{m-\ell}\widetilde{f^\delta}(\lambda)\right\|_{\mathbf{2}}\right\}.
\end{aligned} \tag{4.2.11}$$

Now the expression within braces is the norm $\tau_{s+q,m-\ell}(\widetilde{f^\delta})$. Using the continuity of the δ-spherical transform (Lemma 3.4.2), we write: there exists positive integers l, t such that the last expression (4.2.11) is dominated by

$$c(1+|\delta|)^q\sup_{x\in G}\|\mathbf{L}^l f^\delta(x)\|_{\mathbf{2}}(1+|x|)^t\varphi_0^{-\frac{2}{p}}(x). \tag{4.2.12}$$

■

Remark 4.2.4. *The fact that $\lambda \mapsto Q_\delta(1-i\lambda)^{-1}(P_\lambda f)^\delta$ (for all $f \in \mathcal{S}^p(X)$) is holomorphic on $Int\mathfrak{a}_\varepsilon^*$ can be given a separate proof by using the structural form (4.2.6) of $(P_\lambda f)^\delta(\cdot)$. It can be shown that (in fact we shall discuss about this in detail in the next section), for each $x = ka_t\cdot 0$, $\Phi_{\lambda,\delta}(ka_t\cdot 0) = \delta(k)Q_\delta(1+i\lambda)\Phi(\lambda,t)$, where $\Phi(\lambda,t)$ is a scalar valued function on $\mathbb{C}\times\overline{\mathbb{R}^+}$ such that for each value of λ it is a nonzero function in the t variable. Hence, by (4.2.6), $Q_\delta(1+i\lambda)^{-1}$ $(P_\lambda f)^\delta$ is holomorphic on $Int\mathfrak{a}_\varepsilon^*$. Now $\lambda\mapsto(P_\lambda f)^\delta$ being an even function, it is easy to notice that, actually, $[Q_\delta(1-i\lambda)Q_\delta(1+i\lambda)]^{-1}(P_\lambda f)^\delta$ is holomorphic on $Int\mathfrak{a}_\varepsilon^*$.*

The above proposition helps us to get the decay/growth of $P_\lambda f$ when $f\in\mathcal{S}^p(X)$ and $\lambda\in\mathfrak{a}_\varepsilon^*$. The following is the main theorem of this section.

Theorem 4.2.5. *For $f\in\mathcal{S}^p(X)$ $(0<p\leq 2)$, the complex valued function $P_\lambda f$ defined on $\mathfrak{a}_\varepsilon^*\times X$ has the properties:*

(i) For each $\lambda\in\mathfrak{a}_\varepsilon^*$ $\quad P_\lambda f\in\mathcal{E}_\lambda(X)$;

(ii) *For each $x \in X$ the function $\lambda \mapsto P_\lambda f(x)$ is an even holomorphic function on $Int\mathfrak{a}_\varepsilon^*$ and it extends as an even continuous function to the closed strip $\mathfrak{a}_\varepsilon^*$. For each $\delta \in \widehat{K}_M$, the δ-projection $(P_\lambda f)^\delta$ is identically zero function on X at all the zeros of the Kostant polynomial $Q_\delta(1-i\lambda)$ lying in $Int\mathfrak{a}_\varepsilon^*$;*

(iii) *For each $\boldsymbol{D}, \boldsymbol{E} \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, $m, n, s \in \mathbb{Z}^+ \cup \{0\}$ and for all real number $r_p < \frac{2p-2}{p}$, one can find integers $\ell, t \in \mathbb{Z}^+$ and a positive constant c depending on m, n, s and r_p such that:*

$$\begin{aligned}\sup_{x\in G, \lambda \in Int\mathfrak{a}_\varepsilon^*} \left| \left(\frac{d}{d\lambda}\right)^m P_\lambda f(\boldsymbol{D}; x; \boldsymbol{E}) \right| (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_p}(x) \\ \leq c \sup_{x\in G} |\boldsymbol{L}^\ell f(x)| (1+|x|)^t \varphi_0^{-\frac{2}{p}}(x) < +\infty. \end{aligned} \tag{4.2.13}$$

Proof. (i) This property has already been discussed.
(ii) The Peter-Weyl decomposition of the function $P_\lambda f(x)$ is given by:

$$P_\lambda f(x) = \sum_{\delta \in \widehat{K}_M} tr (P_\lambda f)^\delta (x), \tag{4.2.14}$$

where the convergence is in the sense of uniform convergence on compact sets. Condition (ii) is an easy consequence of the above decomposition and Proposition 4.2.3.
(iii) By using (4.2.14) we get:

$$\begin{aligned} &\left| \left(\frac{d}{d\lambda}\right)^m P_\lambda f(\mathbf{D}; x; \mathbf{E}) \right| (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_p}(x) \\ &\quad \leq \sum_{\delta \in \widehat{K}_M} \left| \left(\frac{d}{d\lambda}\right)^m tr (P_\lambda f)^\delta (\mathbf{D}; x; \mathbf{E}) \right| (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_p}(x) \\ &\quad = \sum_{\delta \in \widehat{K}_M} \left| tr \left[\left(\frac{d}{d\lambda}\right)^m (P_\lambda f)^\delta \right] (\mathbf{D}; x; \mathbf{E}) \right| (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_p}(x) \\ &\quad \leq \sum_{\delta \in \widehat{K}_M} \left\| \left(\frac{d}{d\lambda}\right)^m (P_\lambda f)^\delta (\mathbf{D}; x; \mathbf{E}) \right\|_{\mathbf{2}} (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_p}(x). \end{aligned} \tag{4.2.15}$$

The next inequality follows easily from (4.2.15) by applying (4.2.11).

$$\sup_{x\in G,\lambda\in Int\mathfrak{a}_\varepsilon^*}\left|\left(\frac{d}{d\lambda}\right)^m P_\lambda f(\mathbf{D};x;\mathbf{E})\right|(1+|x|)^n(1+|\lambda|)^s\varphi_0^{-r_p}(x)$$
$$\leq\ c\sum_{\delta\in\widehat{K}_M}\sum_{j=0}^{m}\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}\left\{(1+|\lambda|)^{s+q}(1+|\delta|)^q\left\|\left(\frac{d}{d\lambda}\right)^{m-j}\widetilde{f}^\delta(\lambda)\right\|_{\mathbf{2}}\right\}$$
$$\leq\sum_{j=1}^{m}\sum_{\delta\in\widehat{K}_M}(1+|\delta|)^{-2}\left\{\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}(1+|\lambda|)^{s+q}(1+|\delta|)^{q+2}\left\|\left(\frac{d}{d\lambda}\right)^{m-j}\widetilde{f}^\delta(\lambda)\right\|_{\mathbf{2}}\right\}. \tag{4.2.16}$$

As, $f\in\mathcal{S}^p(X)$, so its Helgason Fourier transform $\mathcal{F}f\in\mathcal{S}(\mathfrak{a}_\varepsilon^*\times K/M)$ [9]. Thus, by the definition of the Schwartz space $\mathcal{S}(\mathfrak{a}_\varepsilon^*\times K/M)$ (see Definition 3.5.1), one gets: for each $\mathtt{N}_1,\mathtt{N}_2\in\mathbb{Z}^+$ and $P\in\mathbf{S}(\mathfrak{a}^*)$,

$$\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*,k\in K/M}\left|P\left(\frac{d}{d\lambda}\right)\mathcal{F}f(\lambda,k;\omega_{\mathfrak{k}}^{\mathtt{N}_1})\right|(1+|\lambda|)^{\mathtt{N}_2}<+\infty.$$

The above countable family of seminorms induces a Fréchet topology on $\mathcal{S}(\mathfrak{a}_\varepsilon^*\times K/M)$. By the theory of smooth functions on the compact group [45, Theorem 4], it follows that the topology of $\mathcal{S}(\mathfrak{a}_\varepsilon^*\times K/M)$ can also be obtained from the equivalent family of seminorms, given by

$$\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*,\delta\in\widehat{K}_M}\left\|P\left(\frac{d}{d\lambda}\right)\widetilde{f}^\delta(\lambda)\right\|_{\mathbf{2}}(1+|\lambda|)^{\mathtt{n}}(1+|\delta|)^{\mathtt{m}}<+\infty.$$

Hence, we can state that the expression within braces of each of the summands of (4.2.16) is dominated by the single finite quantity :

$$\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*;\delta\in\widehat{K}_M}\left\|\left(\frac{d}{d\lambda}\right)^{m-j}\widetilde{f}^\delta(\lambda)\right\|_{\mathbf{2}}(1+|\lambda|)^{s+q}(1+|\delta|)^{q+2}. \tag{4.2.17}$$

This coupled with the summability of $\sum(1+|\delta|)^{-2}$ reduces the inequality (4.2.16) to:

$$\sup_{x\in X,\lambda\in Int\mathfrak{a}_\varepsilon^*}\left|\left(\frac{d}{d\lambda}\right)^m P_\lambda f(\mathbf{D};x;\mathbf{E})\right|(1+|x|)^n(1+|\lambda|)^s\varphi_0^{-r_p}(x)$$
$$\leq c\sum_{j=0}^{m}\left\{\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*;\delta\in\widehat{K}_M}\left\|\left(\frac{d}{d\lambda}\right)^{m-j}\widetilde{f}^\delta(\lambda)\right\|_{\mathbf{2}}(1+|\lambda|)^{s+q}(1+|\delta|)^{q+2}\right\}.$$

Again by the equivalence of the seminorms on $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times K/M)$, we can find positive integers m_1, m_2, m_3 such that the last expression is

$$\leq c \sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*; k\in K/M} \left| \left(\frac{d}{d\lambda}\right)^{m_1} \mathcal{F}f(\lambda, k; \omega_{\mathfrak{k}}^{m_2}) \right| (1+|\lambda|)^{m_3},$$

and by the continuity of the Helgason Fourier transform on the Schwartz space $\mathcal{S}^p(X)$ [9], we get nonnegative integers $\ell, t \in \mathbb{Z}^+$ such that the above expression is

$$\leq c \sup_{x\in G} |\mathbf{L}^\ell f(x)|(1+|x|)^t \varphi_0^{-\frac{2}{p}}(x). \tag{4.2.18}$$

This completes the proof of theorem. ■

Remark 4.2.6. *This part of the characterization does not really use the fact that G is of real rank-1. Thus it can also be obtained for any Riemannian symmetric space realized as G/K with G a non-compact, connected semisimple Lie group with finite center.*

For each $\varepsilon > 0$, let us now define a function space $\mathcal{P}_\varepsilon(X)$.

Definition 4.2.7. *For $\varepsilon > 0$, then $\mathcal{P}_\varepsilon(X)$ denotes the class of functions $(\lambda, x) \mapsto f_\lambda(x)$ defined on $\mathfrak{a}_\varepsilon^* \times X$ and satisfying the following conditions:*

(i) For each $x \in X$ the function $\lambda \mapsto f_\lambda(x)$ is an even $\mathcal{C}^\infty$ function on $\mathfrak{a}^*$ and is analytic on the interior of the strip $\mathfrak{a}_\varepsilon^* = \{\lambda \mid |\Im\lambda| \leq \varepsilon\}$. On the boundary it extends as a continuous function.

(ii) For each $\lambda \in \mathfrak{a}_\varepsilon^*$ the map $x \mapsto f_\lambda(x)$ is a $\mathcal{C}^\infty$ function on X, an eigenfunction of $\boldsymbol{L}$, $f_\lambda \in \mathcal{E}_\lambda(X)$.
Moreover, for each $\delta \in \widehat{K}_M$ and $x \in G$, the function $\lambda \mapsto Q_\delta(1 - i\lambda)^{-1} f_\lambda^\delta(x)$ is holomorphic on $Int\mathfrak{a}_\varepsilon^*$.

(iii) For each $\boldsymbol{D}, \boldsymbol{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $m, n, s \in \mathbb{Z}^+ \cup \{0\}$

$$\sup_{x\in G, \lambda\in Int\mathfrak{a}_\varepsilon^*} \left| \left(\frac{d}{d\lambda}\right)^m f_\lambda(\boldsymbol{D}; x; \boldsymbol{E}) \right| (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_\varepsilon}(x) < +\infty, \tag{4.2.19}$$

where, $r_\varepsilon < \frac{1-\varepsilon}{1+\varepsilon}$.

It is easy to verify that $\mathcal{P}_\varepsilon(X)$ is a Fréchet space with the topology induced by the countable family of seminorms (4.2.19).
We shall conclude this section by restating the Theorem 4.2.5 in the light of Definition 4.2.7.

Theorem 4.2.8. *The spectral projection $f \mapsto P_\lambda f$ is a continuous map from the Schwartz space $\mathcal{S}^p(X)$ $(0 < p \leq 2)$ into $\mathcal{P}_\varepsilon(X)$, where $\varepsilon = \left(\frac{2}{p} - 1\right)$.*

In the next section we shall obtain sufficient conditions for the image of $\mathcal{S}^p(X)$ under the transform $f \mapsto P_\lambda f$. The fact that G is of real rank-1 plays a crucial role there.

4.3 Sufficient Conditions

We begin this section with the definition of a specific subspace of the function space $\mathcal{P}_\varepsilon(X)$ for each $\varepsilon \geq 0$.

Definition 4.3.1. *we denote by $\overline{\mathcal{P}}_\varepsilon(X)$, for each $\varepsilon > 0$, the class of functions $f_\lambda(x)$ in $\mathcal{P}_\varepsilon(X)$ which are of left K-finite type in the x variable, where the finite set of $\delta \in \widehat{K}_M$ involved can be chosen independently of λ.*

In this section we shall try to establish a sufficient condition for a measurable function $(\lambda, x) \mapsto f_\lambda(x) \in \overline{\mathcal{P}}_\varepsilon(X)$ to be of the form $P_\lambda f(x)$ with some $f \in \mathcal{S}^p(X)$ for suitable $0 < p \leq 2$. Let us fix one $\varepsilon \geq 0$ and a function $f_\lambda(x) \in \overline{\mathcal{P}}_\varepsilon(X)$.

Because of the decay (4.2.19) the integral

$$f_n(x) := (-1)^n \int_{\mathfrak{a}^{*+}} (1 + \lambda^2)^n f_\lambda(x) \; |\mathbf{c}(\lambda)|^{-2} \, d\lambda, \quad (\, n \in \mathbb{Z}^+ \,) \tag{4.3.1}$$

converges absolutely, where $\mathbf{c}(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function. Let us set $f_0 = f$, i.e

$$f(x) = \int_{\mathfrak{a}^{*+}} f_\lambda(x) |\mathbf{c}(\lambda)|^{-2} d\lambda. \tag{4.3.2}$$

It also follows from the specified decay (4.2.19) of the function $(\lambda, x) \mapsto f_\lambda(x)$: that for $n = 0, 1, \cdots,$ the function $f_n \in \mathcal{C}^\infty(X)$. As, $f_\lambda(\cdot) \in \mathcal{E}_\lambda(X)$, it can be shown that $\mathbf{L}^n f = f_n$.

For each $\lambda \in \mathfrak{a}_\varepsilon^*$ and $\delta \in \widehat{K}_M$ we define the operator valued left δ-projection by

$$f_\lambda^\delta(x) = \int_K f_\lambda(kx)\delta(k^{-1})dk. \tag{4.3.3}$$

It is clear from the definition of the function space $\overline{\mathcal{P}}_\varepsilon(X)$ that for each $\delta \in \widehat{K}_M$ and each $\lambda \in \mathfrak{a}_\varepsilon^*$, $f_\lambda^\delta \in \mathcal{E}_\lambda^\delta(X)$. The Peter-Weyl decomposition of the function $f_\lambda(\cdot)$ is as follows

$$f_\lambda(x) = \sum_{\delta \in \widehat{K}_M} tr f_\lambda^\delta(x), \qquad \lambda \in \mathfrak{a}_\varepsilon^*. \tag{4.3.4}$$

As f_λ is assumed to be left K-finite, so in the above decomposition (4.3.4) all but finitely many terms are identically zero functions. Let us denote F for the finite subset of $\widehat{K}_M$ corresponding to the function f_λ for which the summands are non zero functions. It follows from the earlier discussion that for each $\delta \in F$ and $\lambda \in \mathfrak{a}_\varepsilon^*$, $x \mapsto tr f_\lambda^\delta(x)$ is of left $\check{\delta}$-type. Hence $tr f_\lambda^\delta(\cdot) \in \mathcal{E}_\lambda(\check{\delta}, X)$.

Lemma 4.3.2. *For each $\delta \in F$ the map $(\lambda, x) \mapsto f_\lambda^\delta(x)$ satisfies the decay*

$$\sup_{x \in X, \lambda \in Int\,\mathfrak{a}_\varepsilon^*} \left\| \left(\frac{d}{d\lambda}\right)^m f_\lambda^\delta(x) \right\|_{\boldsymbol{2}} (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_\varepsilon}(x) < \mathtt{K} < +\infty \tag{4.3.5}$$

where, $\mathtt{K} = d_\delta^{3/2} \cdot c$, the constant c being independent of δ.

Proof. The assertion is true because

$$\begin{aligned}
&\left\{ \left(\frac{d}{d\lambda}\right)^m f_\lambda^\delta(x) \right\} (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_\varepsilon}(x) \\
&\quad = d_\delta \left\{ \int_K \left(\frac{d}{d\lambda}\right)^m f_\lambda(kx)\delta(k^{-1})dk \right\} (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_\varepsilon}(x) \\
&\quad = d_\delta \int_K \left\{ \left(\frac{d}{d\lambda}\right)^m f_\lambda(kx) \right\} \delta(k^{-1})(1+|kx|)^n (1+|\lambda|)^s \varphi_0^{-r_\varepsilon}(kx) dk.
\end{aligned}$$

Now taking Hilbert Schmidt norms on both sides and using the fact $\|\delta(k)\|_{\boldsymbol{2}} = \sqrt{d_\delta}$ we get an inequality from which we get the required conclusion. (*It is easy to check that the δ dependent part in the dominating constant is precisely* $d_\delta^{\frac{3}{2}}$). An immediate corollary is ■

Corollary 4.3.3. *For each $m, n, s \in \mathbb{Z}^+ \cup \{0\}$ and for each fixed $\delta \in F$*

$$\sup_{x\in X, \lambda \in Int\mathfrak{a}^*_\varepsilon} \left| \left(\frac{d}{d\lambda}\right)^m tr f^\delta_\lambda(x) \right| (1+|x|)^n (1+|\lambda|)^s \varphi_0^{-r_\varepsilon}(x) < +\infty. \qquad (4.3.6)$$

Lemma 4.3.4. *The function f obtained in (4.3.2) is also left K-finite and moreover,*

$$tr f^\delta(x) = \int_{\mathfrak{a}^{*+}} tr f^\delta_\lambda(x)\ |\mathbf{c}(\lambda)|^{-2} d\lambda\ . \qquad (4.3.7)$$

Proof. Using (4.3.2) and (4.3.4) we get the following:

$$f(x) = \int_{\mathfrak{a}^{*+}} \sum_{\delta\in F} tr f^\delta_\lambda(x)\ |\mathbf{c}(\lambda)|^{-2} d\lambda,$$

as the above sum is over a finite set F, so

$$= \sum_{\delta\in F} \int_{\mathfrak{a}^{*+}} tr f^\delta_\lambda(x)\ |\mathbf{c}(\lambda)|^{-2} d\lambda. \qquad (4.3.8)$$

Let us denote $\psi_\delta(x) = \int_{\mathfrak{a}^{*+}} tr f^\delta_\lambda(x)\ |\mathbf{c}(\lambda)|^{-2} d\lambda$. The integral converges absolutely because of the decay (4.3.6). We have already noticed that, $tr f^\delta_\lambda(\cdot)$ are of left $\check{\delta}$-type and we only need the routine checking

$$\begin{aligned}
\psi_\delta(x) &= d_\delta \int_{\mathfrak{a}^{*+}} \{\chi_\delta * tr f^\delta_\lambda\ \}(x)\ |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&= d_\delta \int_{\mathfrak{a}^{*+}} \left\{ \int_K \chi_\delta(k^{-1}) tr f^\delta_\lambda(kx) dk \right\}\ |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&= d_\delta \int_K \chi_\delta(k^{-1}) \left\{ \int_{\mathfrak{a}^{*+}} tr f^\delta_\lambda(kx) dk\ |\mathbf{c}(\lambda)|^{-2} d\lambda \right\} dk \\
&= d_\delta \{\chi_\delta * \psi_\delta\}(x) \qquad (4.3.9)
\end{aligned}$$

to conclude that each ψ_δ is a scalar valued left $\check{\delta}$-type. The rest follows from the Peter-Weyl decomposition. ■

So far we have noted that the function f obtained in (4.3.2) is in $\mathcal{C}^\infty(X)$ and it is of left K-finite type. Now we shall try to show that $f \in \mathcal{S}^p(X)$ for some $0 < p \leq 2$. Towards that we shall first try to obtain a structural form of f^δ_λ analogous to the one given in Proposition 4.2.6. The assumption that $f^\delta_\lambda \in \mathcal{E}^\delta_\lambda(X)$ will now play a crucial role. The following theorem is the key to the desired form of f^δ_λ. Let $\delta \in \widehat{K}_M$ and V_δ ($d_\delta = dimV_\delta$)be the representation space for δ with the orthonormal basis $v_1, v_2, \cdots, v_{d_\delta}$ where,

$V_\delta^M = \mathbb{C}v_1$.

Theorem 4.3.5. [Helgason, [23, Theorem 1.4, p-133]]
Let $\lambda \in \mathbb{C}$ be such that $\Re\langle i\lambda, \alpha\rangle \geq 0$, where α is positive restricted root. Then the functions

$$\Psi_{\lambda,\check{\delta}j}(x) = \sqrt{d_\delta}\int_K e^{-(i\lambda+1)H(x^{-1}k)}\,\langle\delta(k)v_1\ ,\ v_j\rangle\,dk, \qquad 1 \leq j \leq d_\delta, \tag{4.3.10}$$

form a basis of the eigenspace $\mathcal{E}_\lambda(\check{\delta}, X)$.

We note that, by using the definition (2.1.1) of the generalized spherical functions we can write the basis vectors as follows

$$\Psi_{\lambda,\check{\delta}j}(x) = \sqrt{d_\delta}\,\langle\Phi_{\lambda,\delta}(x)v_1\ ,\ v_j\rangle\,. \tag{4.3.11}$$

Remark 4.3.6. *For a real rank-1 group G, we have identified the Iwasawa A subgroup with $\mathbb{R}$. With this normalization $\mathfrak{a}$, $\mathfrak{a}^*$ are identified with $\mathbb{R}$ and $\mathfrak{a}^+$, $\mathfrak{a}^{*+}$ with $\mathbb{R}^+$. As we are only considering the real rank-1 group, so there will be a smallest positive restricted root α and at most one more which will be 2α. Clearly, $\alpha \in \mathbb{R}^+$. This immediately suggests that for all λ with $\Im\lambda \leq 0$, $\Re\langle i\lambda, \alpha\rangle \geq 0$.*
Hence, for all $\lambda \in \mathfrak{a}_\varepsilon^{*-} = \{\lambda \in \mathfrak{a}_\varepsilon^* \mid \Im\lambda \leq 0\}$, the vectors $\Psi_{\lambda,\check{\delta}j}(x)$ forms a basis of $\mathcal{E}_\lambda(\check{\delta}, X)$.

Lemma 4.3.7. *For each $\delta \in \widehat{K}_M$ the matrix valued projection f_λ^δ of f_λ for each $\lambda \in \mathfrak{a}_\varepsilon^{*-}$ has the following structural form*

$$f_\lambda^\delta(ka_t.0) = \sqrt{d_\delta}\ \Phi_{\lambda,\delta}(ka_t.0)h^\delta(\lambda) \tag{4.3.12}$$

where, $\Phi_{\lambda,\delta}(ka_t.0)$ is a $(d_\delta \times 1)$ matrix and $h^\delta(\lambda)$ is a $(1 \times d_\delta)$ matrix

Proof. We have noticed that $tr f_\lambda^\delta \in \mathcal{E}_\lambda(\check{\delta}, X)$ for all $\lambda \in \mathfrak{a}_\varepsilon^*$ and $\delta \in F$. Now we shall write $tr f_\lambda^\delta(x)$ in terms of the basis vectors given in (4.3.11).

$$tr f_\lambda^\delta(x) = \sqrt{d_\delta}\ \sum_{j=1}^{d_\delta} h_j^\delta(\lambda)\,\langle\Phi_{\lambda,\delta}(x)v_1, v_j\rangle\,, \tag{4.3.13}$$

where, $h_j^\delta(\lambda)$ are coefficients depending on λ. Let us denote the $(1\times d_\delta)$ matrix $h^\delta(\lambda) = \big(h_1^\delta(\lambda), \cdots, h_{d_\delta}^\delta(\lambda)\big)$. We also recall the fact that the generalized spherical function $\Phi_{\lambda,\delta}(x)$ vanishes on the orthogonal complement of V_δ^M, so we can regard $\Phi_{\lambda,\delta}(x)$ as the $(d_\delta \times 1)$ column vector with the entries $\langle \Phi_{\lambda,\delta}(x)v_1, v_j\rangle$. Then it is clear from (4.3.13) that

$$tr f_\lambda^\delta(x) = \sqrt{d_\delta}\; tr[\Phi_{\lambda,\delta}(x)h^\delta(\lambda)]. \tag{4.3.14}$$

Next we shall show that the matrices $f_\lambda^\delta(x)$ and $\Phi_{\lambda,\delta}(x)h^\delta(\lambda)$ have identical entries.

$$\begin{aligned} f_\lambda^\delta(x)_{\imath\ \ell} &= \left\langle f_\lambda^\delta(x)v_\ell, v_\imath \right\rangle \\ &= d_\delta \left\langle \int_K tr f_\lambda^\delta(kx)\delta(k^{-1})dk v_\ell, v_\imath \right\rangle \\ &= d_\delta \int_K tr f_\lambda^\delta(kx) \left\langle \delta(k^{-1})v_\ell, v_\imath \right\rangle dk. \end{aligned} \tag{4.3.15}$$

Now we use (4.3.14) to replace $tr f_\lambda^\delta(kx)$ to get

$$\begin{aligned} f_\lambda^\delta(x)_{\imath\ \ell} &= d_\delta^{\frac{3}{2}} \sum_{j=1}^{d_\delta} \int_K \langle \Phi_{\lambda,\delta}(kx)v_1\ ,\ v_j\rangle\ \ \left\langle \delta(k^{-1})v_\ell, v_\imath \right\rangle h_j^\delta(\lambda)\ dk \\ &= d_\delta^{\frac{3}{2}} \sum_{j=1}^{d_\delta} \int_K \langle \delta(k)\Phi_{\lambda,\delta}(x)v_1, v_j\rangle\ \ \left\langle \delta(k^{-1})v_\ell, v_\imath \right\rangle h_j^\delta(\lambda)\ dk \\ &= d_\delta^{\frac{3}{2}} \sum_{j=1}^{d_\delta} \int_K \overline{\langle \delta(k^{-1})v_j, \Phi_{\lambda,\delta}(x)v_1\rangle}\ \ \left\langle \delta(k^{-1})v_\ell, v_\imath \right\rangle h_j^\delta(\lambda)\ dk. \end{aligned} \tag{4.3.16}$$

The representation coefficients $k \mapsto \langle \delta(k)v, u\rangle$ $(u, v \in V_\delta)$ satisfies the following consequences of the *Schur's Orthogonality Relations* : If $u, v, u', v' \in V_\delta$, then

$$\int_K \langle \delta(k)u, v\rangle\, \overline{\langle \delta(k)u', v'\rangle} = d_\delta^{-1}\, \langle u, u'\rangle\, \overline{\langle v, v'\rangle}. \tag{4.3.17}$$

Using (4.3.17) in (4.3.16) as also the fact that $\{v_i\}(1 \le i \le d_\delta)$ forms an orthonormal basis of the representation space V_δ we write

$$f_\lambda^\delta(x)_{\imath\ \ell} = \sqrt{d_\delta}\, \langle \Phi_{\lambda,\delta}(x)v_1, v_\imath\rangle\, h_\ell^\delta(\lambda). \tag{4.3.18}$$

The right hand side of (4.3.18) is precisely the $(\imath, \ell)$ entry of the matrix $\sqrt{d_\delta}\ \Phi_{\lambda,\delta}(x)h^\delta(\lambda)$.

Hence the Lemma follows. ∎

Remark 4.3.8. *By the assumption for each $x \in X$ the function $\lambda \mapsto f_\lambda^\delta(x)$ is even, so the structural form given in (4.3.12) is valid for all $\lambda \in \mathfrak{a}_\varepsilon^*$.*

Remark 4.3.9. *(i) Writing*

$$\begin{aligned} f_\lambda^\delta(x) &= \sqrt{d_\delta}\Phi_{\lambda,\delta}(x)h^\delta(\lambda) \\ &= \sqrt{d_\delta}\{Q_\delta(1-i\lambda)\Phi_{\lambda,\delta}(x)\}\{Q_\delta(1-i\lambda)^{-1}h^\delta(\lambda)\}. \end{aligned} \tag{4.3.19}$$

We notice that f_λ^δ is even in the λ variable and the function $\lambda \mapsto Q_\delta(1-i\lambda)\Phi_{\lambda,\delta}(x)$ is even by (Theorem 2.1.5). Hence for all $\lambda \in \mathfrak{a}_\varepsilon^*$, the function $\lambda \mapsto Q_\delta(1-i\lambda)^{-1}h^\delta(\lambda)$ is an even function.

At this point we need to look in a different direction. The matrix entries of the generalized spherical functions are associated with Jacobi functions. Let $x = ka_t.0 \in X$. Then

$$\Phi_{\lambda,\delta\ j}(ka_t) = \langle \Phi_{\lambda,\delta}(ka_t)v_1, v_j\rangle = \langle \delta(k)\Phi_{\lambda,\delta}(a_t)v_1, v_j\rangle.$$

It can easily be seen that $\Phi_{\lambda,\delta}(a_t)v \in V_\delta^M$ for all $v \in V_\delta$. Hence on V_δ^M, $\Phi_{\lambda,\delta}(a_t)$ will be a multiplication operator

$$\Phi_{\lambda,\delta}(a_t)v_1 = \varphi_{\lambda,\delta}(t)v_1, \tag{4.3.20}$$

where, $\varphi_{\lambda,\delta}(t)$ is a function of t depending on λ and δ. For each $\delta \in \widehat{K}_M$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the function $t \mapsto \varphi_{\lambda,\delta}(t)$ has an expression in terms of the hypergeometric functions (Helgason [24], Koornwinder [33])

$$\varphi_{\lambda,\delta}(t) = Q_\delta(i\lambda+1)(\alpha+1)_r^{-1}(\sinh t)^r(\cosh t)^s\varphi_\lambda^{\alpha+r,\beta+s}(t), \tag{4.3.21}$$

where, $\varphi_\lambda^{\alpha+r,\beta+s}$ is the Jacobi function of the first kind with parameters $(\alpha+r, \beta+s)$. The integers (r,s) and the quantities α, β are already introduced in (2.1.10). This Jacobi function has the integral representation [33]:

$$\varphi_\lambda^{\alpha+r,\beta+s}(t) = \int_0^1\int_0^\pi |\cosh t - \mathfrak{r}e^{i\theta}\sinh t|^{-i\lambda-\varrho}dP_{\alpha+r,\beta+s}(\mathfrak{r},\theta), \tag{4.3.22}$$

where, $\varrho = \alpha+r+\beta+s+1$ and $dP_{\alpha+r,\beta+s}(\mathfrak{r},\theta)$ is a probability measure [25] on $[0,1]\times[0,\pi]$.

Lemma 4.3.10. *For all $\lambda \in \mathbb{C}$ the Jacobi function $\varphi_\lambda^{\alpha+r,\beta+s}$ satisfies the following:*

$$\varphi_\lambda^{\alpha+r,\beta+s}(0) = 1 \tag{4.3.23}$$

$$\left|\left(\frac{d}{d\lambda}\right)^k \varphi_\lambda^{\alpha+r,\beta+s}(t)\right| \leq ct^k e^{(|\Im\lambda|+\varrho)t}, \quad t \in \mathbb{R}^+,\ k \in \mathbb{Z}^+. \tag{4.3.24}$$

Proof. These two properties follows from the integral representation (4.3.22). We use the estimate

$$\log|\cosh t - re^{i\theta}\sinh t| \leq ct, \qquad t > 0,$$

to get the inequality (4.3.24). ∎

Our next Lemma concerns the domain on which the function h^δ holomorphic.

Lemma 4.3.11. *For each $\delta \in \widehat{K}_M$, the functions $\lambda \mapsto h^\delta(\lambda)$ and $\lambda \mapsto Q_\delta(1-i\lambda)^{-1}h^\delta(\lambda) = g^\delta(\lambda)$ are holomorphic in the interior of the complex strip $\mathfrak{a}_\varepsilon^*$.*

Proof. We note that the zeros of the polynomial $Q_\delta(1-i\lambda)$ are purely imaginary. We have assumed that $x \mapsto f_\lambda^\delta(x)$ is identically zero function on X for all λ which are zeros of the polynomial $Q_\delta(1-i\lambda)$. Also we have assumed that f_λ^δ is even in λ. So, $x \mapsto f_\lambda^\delta(x)$ is also zero for the zeros of $Q_\delta(1+i\lambda)$ in $Int\mathfrak{a}_\varepsilon^*$. Hence, $\lambda \mapsto Q_\delta(1+i\lambda)^{-1}f_\lambda^\delta(\cdot)$ is holomorphic on $Int\mathfrak{a}_\varepsilon^*$.
We restrict the function $f_\lambda^\delta(\cdot)$ to $(\mathfrak{a}_\varepsilon^* \times A^+.0)$. Then by the structural form obtained in Lemma 4.3.7 we write:

$$f_\lambda^\delta(a_t) = \sqrt{d_\delta}\ \Phi_{\lambda,\delta}(a_t)h^\delta(\lambda), \qquad (t > 0). \tag{4.3.25}$$

For proving $\lambda \mapsto h^\delta(\lambda)$ is holomorphic on $Int\mathfrak{a}_\varepsilon^*$ it is enough to prove that each of its matrix entries are so. By definition the $(1,j)$th $(1 \leq j \leq d_\delta)$ matrix entry of $f_\lambda^\delta(a_t)$ is given by $f_\lambda^\delta(a_t)_{1j} = \sqrt{d_\delta}\Phi_{\lambda,\delta\ 1}(a_t)h^\delta(\lambda)_j$.

$$\begin{aligned}\Phi_{\lambda,\delta\ 1}(a_t) &= \langle \Phi_{\lambda,\delta}(a_t)v_1, v_1\rangle, \\ &= \|v_1\|\varphi_{\lambda,\delta}(t).\end{aligned} \tag{4.3.26}$$

Hence by (4.3.26) and the expression (4.3.21) we get:

$$f_\lambda^\delta(a_t)_{1j} = \|v_1\|(\alpha+1)_r^{-1} Q_\delta(1+i\lambda)(\sinh t)^r(\cosh t)^s \varphi_\lambda^{\alpha+r,\beta+s}(t) h_j^\delta(\lambda). \tag{4.3.27}$$

As the first order zeros of $Q_\delta(1+i\lambda)$ are neutralized by that of $f_\lambda^\delta(a_t)_{1j}$, so we write:

$$Q_\delta(1+i\lambda)^{-1} f_\lambda^\delta(a_t)_{1j} = C_{v_1,\alpha}(\sinh t)^r(\cosh t)^s \varphi_\lambda^{\alpha+r,\beta+s}(t) h_j^\delta(\lambda). \tag{4.3.28}$$

The left hand side of (4.3.28) is holomorphic on $Int\mathfrak{a}_\varepsilon^*$. To conclude that h_j^δ is holomorphic at $\lambda \in Int\mathfrak{a}_\varepsilon^*$, we can choose $t_0 > 0$ so that $\varphi_\lambda^{\alpha+r,\beta+s}(t_0) \neq 0$ as is possible by the observation (4.3.23). Noting that $\varphi_\lambda^{\alpha+r,\beta+s}(t_0)$ is holomorphic in λ and that both $\sinh t_0$ and $\cosh t_0$ are positive we reach our conclusion. To see that $\lambda \mapsto Q_\delta(1-i\lambda)^{-1}h^\delta(\lambda)$ is holomorphic on $Int\mathfrak{a}_\varepsilon^*$, we note that $f_\lambda^\delta(a_t)_{1j}$ is symmetric in λ and so from (4.3.28) $Q_\delta(1+i\lambda)^{-1}f_\lambda^\delta(a_t)_{1j}$ as well as $Q_\delta(1-i\lambda)^{-1}f_\lambda^\delta(a_t)_{1j}$ are analytic in $Int\mathfrak{a}_\varepsilon^*$. From the exact expression of $Q_\delta(1-i\lambda)$ (2.1.10) we further notice that the polynomials $Q_\delta(1+i\lambda)$ and $Q_\delta(1-i\lambda)$ have no common zeros. We can hence conclude that $[Q_\delta(1-i\lambda)Q_\delta(1+i\lambda)]^{-1}f_\lambda^\delta(a_t)_{1j}$ is analytic on $Int\mathfrak{a}_\varepsilon^*$. Using (4.3.28) again we get the desired analyticity of $Q_\delta(1-i\lambda)^{-1}h^\delta(\lambda)$ in $Int\mathfrak{a}_\varepsilon^*$. ■

Remark 4.3.12. *For each $x \in X$ and $\delta \in \widehat{K}_M$; $\lambda \mapsto f_\lambda^\delta(x)$ extends as a continuous function to the closed strip $\mathfrak{a}_\varepsilon^*$. From (4.3.12) it follows that $\lambda \mapsto h^\delta(\lambda)$ also extends as a continuous function to $\mathfrak{a}_\varepsilon^*$.*

Our next aim is to determine the decay of the function h^δ, for that we need a lower bound of the associated Jacobi function.

Lemma 4.3.13. *(Bray [8, Lemma 2.4])*
Let $\mu, \tau \geq -\frac{1}{2}$, then for any $\Lambda > \frac{2}{\pi}$, there is a constant C depending on μ, τ, Λ such that

$$\left|\varphi_\lambda^{\mu,\tau}\left(\frac{1}{|\lambda|^2}\right)\right| \geq C_{\Lambda,\mu,\tau}, \quad \text{for } |\lambda| > \Lambda \tag{4.3.29}$$

Infact the constant $C_{\Lambda,\mu,\tau}$ has the following form:

$$C_{\Lambda,\mu,\tau} = e^{-\frac{2+\mu+\tau}{\Lambda}} \cos(1/\Lambda).$$

Remark 4.3.14. [The following observation is due to R. P. Sarkar]
We note that for $G = SU(n,1)$*, the quantities* $\alpha \geq 0$*,* $\beta = 0$ *(where* α*,* β *are as in (2.1.10)) and the parameterization* (r,s) *of* $\widehat{K}_M$ *runs over* $\mathbb{Z}^+ \times \mathbb{Z}$ *with* $r \pm s \in 2\mathbb{Z}^+$*. Suppose for some* $\delta \in \widehat{K}_M$*,* $s_\delta < 0$*. In such a case we use the relation [33, (5.75)]*

$$\varphi_\lambda^{\alpha+r_\delta, s_\delta}(t) = (\cosh t)^{2|s_\delta|}\varphi_\lambda^{\alpha+r_\delta,|s_\delta|}(t), \quad t > 0, \lambda \in \mathbb{C}, \tag{4.3.30}$$

and rewrite (4.3.21) as follows

$$\varphi_{\lambda,\delta}(t) = Q_\delta(i\lambda+1)(\alpha+1)_{r_\delta}^{-1}(\sinh t)^{r_\delta}(\cosh t)^{|s_\delta|}\varphi_\lambda^{\alpha+r_\delta,|s_\delta|}(t). \tag{4.3.31}$$

The Jacobi function $\varphi_\lambda^{\alpha+r_\delta,|s_\delta|}(t)$ *clearly satisfies the conditions of Lemma 4.3.13. For other classes of real rank-1 groups the parameters* $\alpha + r$ *and* $\beta + s$ *are positive integers. Thus, for all* G *of real rank-*1*, the condition of Lemma 4.3.13 holds for the function* $\varphi_\lambda^{\alpha+r,\beta+s}$*.*

Proposition 4.3.15. *For each* $\delta \in F$ *the function* $\lambda \mapsto h_i^\delta(\lambda)$ *for each* $i = 1, 2, \cdots, d_\delta$ *satisfies the following decay condition:*

$$\sup_{\lambda \in \mathfrak{a}_\varepsilon^*} \left| \left(\frac{d}{d\lambda}\right)^n h_i^\delta(\lambda) \right| (1+|\lambda|)^m < +\infty. \tag{4.3.32}$$

Proof. The structural form obtained in Lemma 4.3.7 and (4.3.5) gives the following decay/growth condition for each $(1,j)$th matrix entry of $f_\lambda^\delta(a_t)$: for each $m, n \in \mathbb{Z}^+ \cup \{0\}$

$$\sup_{a_t \in \mathfrak{a}^+, \lambda \in Int\mathfrak{a}_\varepsilon^*} \left| \Phi_{\lambda,\delta\ 1}(a_t)\, h_j^\delta(\lambda) \right| (1+t)^n (1+|\lambda|)^m \varphi_0^{-r_\varepsilon}(a_t) = c_{1j} < +\infty. \tag{4.3.33}$$

This immediately implies that: for $\lambda \in Int\mathfrak{a}_\varepsilon^*$ and $t > 0$

$$\begin{aligned} |\Phi_{\lambda,\delta\ 1}(a_t)||h_j^\delta(\lambda)|(1+|\lambda|)^m &\leq c_{1j}\frac{1}{(1+t)^n}\varphi_0^{r_\varepsilon}(a_t) \\ &\leq c_{1j}(r_\varepsilon, t), \end{aligned} \tag{4.3.34}$$

where,

$$c_{1j}(r_\varepsilon, t) = \begin{cases} c_{1j} & \text{if } \ r_\varepsilon \geq 0 \\ c_{1j}e^{|r_\varepsilon|t} & \text{if } \ r_\varepsilon < 0. \end{cases}$$

The last line of the inequality (4.3.34) is a consequence of the fact that

$\varphi_0(a_t) < 1$ for all $t > 0$ and the two-sided estimate (2.0.22) of $\varphi_0(a_t)$.
Now we express $\Phi_{\lambda,\delta\ 1}(a_t)$ in terms of the Jacobi function (4.3.21), which reduces (4.3.34) to the following:

$$|h_j^\delta(\lambda)||\varphi_\lambda^{\alpha+r,\beta+s}(t)||Q_\delta(i\lambda+1)(\sinh t)^r(\cosh t)^s|\ (1+|\lambda|)^m \leq \frac{1}{\|v_1\|}c_{1j}(r_\varepsilon,t). \tag{4.3.35}$$

We note that $h^\delta g^\delta(\lambda)Q_\delta(1-i\lambda)$, $\lambda \in Int\mathfrak{a}_\varepsilon^*$. Hence we get the inequality.

$$|g_j^\delta(\lambda)||\varphi_\lambda^{\alpha+r,\beta+s}(t)||Q_\delta(1-i\lambda)Q_\delta(i\lambda+1)(\sinh t)^r||(\cosh t)^s|(1+|\lambda|)^m \leq \frac{1}{\|v_1\|}c_{1j}(r_\varepsilon,t). \tag{4.3.36}$$

We now let $t = \frac{1}{|\lambda|^2}$. We choose one $\Lambda > \frac{2}{\pi}$ large enough so that the disk $B^\Lambda(0) = \{\lambda \mid |\lambda| \leq \Lambda\}$ contains all the zeros of the polynomial $Q_\delta(1+i\lambda)$ lying in $Int\mathfrak{a}_\varepsilon^*$. For $\lambda \in Int\mathfrak{a}_\varepsilon^* \setminus B^\Lambda(0)$, by Lemma 4.3.13,

$$\left|\varphi_\lambda^{\alpha+r,\beta+s}\left(\frac{1}{|\lambda|^2}\right)\right| > C_{\Lambda,r,s} > 0. \tag{4.3.37}$$

We note that the polynomial $Q_\delta(1+i\lambda)Q_\delta(1-i\lambda)$ is of degree $2r$ (see, (2.1.10)). Hence for all $\lambda \in Int\mathfrak{a}_\varepsilon^* \setminus B^\Lambda(0)$ one can find a positive constant $\mathfrak{d}$ such that $\left|Q_\delta(1+i\lambda)Q_\delta(1-i\lambda)\left(\sinh\frac{1}{|\lambda|^2}\right)^r\right| > \mathfrak{d}$.

Also, for the above choice of λ, $e^{|r_\varepsilon|\frac{1}{|\lambda|^2}} \leq e^{|r_\varepsilon|\frac{1}{|\Lambda|^2}}$. Hence from (4.3.36) we conclude that: for all $\lambda \in Int\mathfrak{a}_\varepsilon^* \setminus B^\Lambda(0)$

$$|g_j^\delta(\lambda)|(1+|\lambda|)^m \leq \frac{e^{|r_\varepsilon|\frac{1}{|\Lambda|^2}}}{C_{\Lambda,\alpha+r,\beta+s}\mathfrak{d}} = c_\delta(\text{say}) \text{ for each } m \in \mathbb{Z}^+. \tag{4.3.38}$$

As $B^\Lambda(0)$ contains all the zeros of the polynomial $Q_\delta(1-i\lambda)$, so there exists a constant $k_\delta > 0$ such that $|Q(1-i\lambda)| > k_\delta$ for all $\lambda \in B^\Lambda(0)^c$. Thus, for $\lambda \in Int\mathfrak{a}_\varepsilon^* \setminus B^\Lambda(0)$

$$|h_j^\delta(\lambda)|(1+|\lambda|)^m = \frac{1}{|Q_\delta(1-i\lambda)|}|g_j^\delta(\lambda)|(1+|\lambda|)^m \leq \frac{c_\delta}{k_\delta}. \tag{4.3.39}$$

For each $1 \leq j \leq d_\delta$ the following inequality is also obtained from (4.3.5):

for all $a_t \in A^+$ and $\lambda \in Int\mathfrak{a}_\varepsilon^*$

$$\left|\left(\frac{d}{d\lambda}\right)\{\Phi_{\lambda,\delta\ 1}(a_t)Q_\delta(1-i\lambda)g_j^\delta(\lambda)\}\right|(1+t)^n(1+|\lambda|)^m\varphi_0^{-r_\varepsilon}(a_t) \le c'_{1j}.$$

That is

$$\left|\left\{\frac{d}{d\lambda}g_j^\delta(\lambda)\right\}Q_\delta(1-i\lambda)\Phi_{\lambda,\delta\ 1}(a_t) + g_j^\delta(\lambda)\ \left(\frac{d}{d\lambda}\right)\{Q_\delta(1-i\lambda)\Phi_{\lambda,\delta\ 1}(a_t)\}\right| (1+|\lambda|)^m \le c'_{1j}(r_\varepsilon, t).$$

The last line can be written as

$$\begin{aligned}\left|\left\{\frac{d}{d\lambda}g_j^\delta(\lambda)\right\}Q_\delta(1-i\lambda)\Phi_{\lambda,\delta\ 1}(a_t)\right|&(1+|\lambda|)^m \\ &\le c'_{1j}(r_\varepsilon,t) + \left|g_j^\delta(\lambda)\left(\frac{d}{d\lambda}\right)\{Q_\delta(1-i\lambda)\Phi_{\lambda,\delta\ 1}(a_t)\}\right|(1+|\lambda|)^m,\end{aligned}$$

writing $\Phi_{\lambda,\delta\ 1}(a_t)$ in terms of the Jacobi functions as in (4.3.21) we get,

$$\begin{aligned}&\le c'_{1j}(r_\varepsilon,t) + \\ &\frac{(1+|\lambda|)^m}{(1+\alpha)_r}\left|g_j^\delta(\lambda)\left(\frac{d}{d\lambda}\right)\left\{Q_\delta(1-i\lambda)Q_\delta(1+i\lambda)(\sinh t)^r(\cosh t)^s\varphi_\lambda^{\alpha+r,\beta+s}(t)\right\}\right| \\ &\le c'_{1j}(r_\varepsilon,t) + \\ &\frac{(1+|\lambda|)^m}{(1+\alpha)_r}\left|g_j^\delta(\lambda)\left\{\left(\frac{d}{d\lambda}\right)Q_\delta(1-i\lambda)Q_\delta(1+i\lambda)\right\}(\sinh t)^r(\cosh t)^s\varphi_\lambda^{\alpha+r,\beta+s}(t)\right| \\ &+\frac{(1+|\lambda|)^m}{(1+\alpha)_r}\left|g_j^\delta(\lambda)Q_\delta(1-i\lambda)Q_\delta(1+i\lambda)(\sinh t)^r(\cosh t)^s\left\{\left(\frac{d}{d\lambda}\right)\varphi_\lambda^{\alpha+r,\beta+s}(t)\right\}\right|.\end{aligned} \tag{4.3.40}$$

We rewrite the inequality as

$$\begin{aligned}\left|\left(\frac{d}{d\lambda}\right)g_j^\delta(\lambda)\right|&(1+|\lambda|)^m \\ <c'_{1j}(r_\varepsilon,t)&\left[\left|Q_\delta(1+i\lambda)Q_\delta(1-i\lambda)\right.(\sinh t)^r\,|\,(\cosh t)^s\,|\varphi_\lambda^{\alpha+r,\beta+s}(t)|\right]^{-1} \\ &+|g_j^\delta(\lambda)|(1+|\lambda|)^m\frac{|\frac{d}{d\lambda}(Q_\delta(1-i\lambda)Q_\delta(1+i\lambda)|}{|Q_\delta(1+i\lambda)Q_\delta(1-i\lambda)|} \\ &+|g_j^\delta(\lambda)|(1+|\lambda|)^m\frac{\left|\left(\frac{d}{d\lambda}\right)\varphi_\lambda^{\alpha+r,\beta+s}(t)\right|}{\left|\varphi_\lambda^{\alpha+r,\beta+s}(t)\right|}.\end{aligned} \tag{4.3.41}$$

Again we take $t = \frac{1}{|\lambda|^2}$ and choose $\Lambda > \frac{2}{\pi}$ suitably large so that $B^\Lambda(0)$ contains all the zeros of the polynomial $Q_\delta(1-i\lambda)$. We will presently obtain bounds for the three terms on the right-hand side separately for $\lambda \in Int\mathfrak{a}_\varepsilon^* \setminus B^\Lambda(0)$. The first term is bounded by a constant, as seen earlier. We have already obtained a bound for the factor $|g_j^\delta(\lambda)|(1+|\lambda|)^m$ present in the last two terms. The quotient in the second term is clearly bounded in the region $Int\mathfrak{a}_\varepsilon^* \setminus B^\Lambda(0)$. Finally, a bound for the quotient in the last term is obtained from the following facts $|\varphi_\lambda^{\alpha+r,\beta+s}(\frac{1}{|\lambda|^2})| \geq C_{\Lambda,\alpha+r,\beta+s}$ (by Lemma 4.3.13) and $\left|\left(\frac{d}{d\lambda}\right)\varphi_\lambda^{\alpha+r,\beta+s}(\frac{1}{|\lambda|^2})\right| \leq c\frac{1}{|\Lambda|^2}e^{(\varepsilon+\varrho)(\frac{1}{|\Lambda|^2})}$ (by (4.3.23)).

As $h_j^\delta(\lambda) = Q_\delta(1-i\lambda)g_j^\delta(\lambda)$, so for each $m \in \mathbb{Z}^+$

$$\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}\left|\left(\frac{d}{d\lambda}\right)h_j^\delta(\lambda)\right|(1+|\lambda|)^m \leq C_\delta < +\infty. \tag{4.3.42}$$

For any order of the derivative on λ, we can use essentially the same argument. ■

Corollary 4.3.16. *From the decay (4.3.32) of each matrix entry of the function h^δ obtained in the above Proposition 4.3.15 we get: for each fixed $\delta \in F$, for each $m, n \in \mathbb{Z}^+ \cup \{0\}$*

$$\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}\left\|\left(\frac{d}{d\lambda}\right)^m h^\delta(\lambda)\right\|_2 (1+|\lambda|)^n < +\infty. \tag{4.3.43}$$

Here the norm $\|\cdot\|_2$ stands for the Hilbert Schmidt norm.

Let us now recollect the properties we have obtained for the function h^δ ($\delta \in \widehat{K}_M$) in the following Lemma.

Lemma 4.3.17. *The function h^δ (as obtained in (4.3.12)) is a $Hom(V_\delta, V_\delta^M)$ valued function on $\mathfrak{a}_\varepsilon^*$ which satisfies the following properties:*

(i) *h^δ is holomorphic in interior of $\mathfrak{a}_\varepsilon^*$ and it extends to the closed strip $\mathfrak{a}_\varepsilon^*$ as a continuous function.*

(ii) *$\lambda \mapsto Q_\delta(1-i\lambda)^{-1}h^\delta(\lambda)$ is an even function and also it is holomorphic on $Int\mathfrak{a}_\varepsilon^*$.*

(iii) *for each $m, n \in \mathbb{Z}^+ \cup \{0\}$*

$$\sup_{\lambda\in Int\mathfrak{a}_\varepsilon^*}\left\|\left(\frac{d}{d\lambda}\right)^m h^\delta(\lambda)\right\|_2 (1+|\lambda|)^n < +\infty \tag{4.3.44}$$

The above Lemma shows that, for each $\delta \in \widehat{K}_M$, $h^\delta \in \mathcal{S}_\delta(\mathfrak{a}_\varepsilon^*)$ (see Definition 3.3.2). Hence by Theorem 3.3.3 the inversion $\mathcal{J}h^\delta$ given by

$$\mathcal{J}h^\delta(x) = \int_{\mathfrak{a}^{*+}} \Phi_{\lambda,\delta}(x) h^\delta(\lambda) |\mathbf{c}(\lambda)|^{-2} d\lambda, \tag{4.3.45}$$

belongs to the left δ-type operator valued projection $\mathcal{S}_\delta^p(X)$ of the p th Schwartz class of functions $\mathcal{S}^p(X)$, where p is determined by the chosen ε by $p = \frac{2}{1+\varepsilon}$.
It is clear from (4.3.45), (4.3.12) and (4.3.2) that $f^\delta \equiv \sqrt{d_\delta}\ \mathcal{J}h^\delta$. Hence for each δ, trf^δ satisfies the Schwartz space decay condition: for each $\mathbf{D},\ \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $n \in \mathbb{Z}^+ \cup \{0\}$,

$$\sup_{x \in X} |trf^\delta(\mathbf{D}; x;\ \mathbf{E})| (1 + |x|)^n \varphi_0^{-\frac{2}{p}}(x) \leq +\infty. \tag{4.3.46}$$

The function f obtained in (4.3.2) was proved to be left K-finite. Hence $f \in \mathcal{S}^p(X)$ where, $p = \frac{2}{1+\varepsilon}$.

Finally we shall show that $P_\lambda f(x) = f_\lambda(x)$ ($\lambda \in \mathfrak{a}_\varepsilon^*$) where the function f is obtained in (4.3.2). As f is left K finite $P_\lambda f$ can be decomposed as follows

$$\begin{aligned} P_\lambda f(x) &= \sum_{\delta \in F} tr(P_\lambda f)^\delta(x) \\ &= \sum_{\delta \in F} trP_\lambda(f)^\delta(x), \end{aligned} \tag{4.3.47}$$

where the last line follows by using Lemma 4.2.1, and F is a finite subset of $\widehat{K}_M$. Now from the above discussion and Lemma 4.2.1 we get the following:

$$P_\lambda(f)^\delta(x) = \Phi_{\lambda,\delta}(x)\widetilde{f^\delta}(\lambda) = \Phi_{\lambda,\delta}(x)h^\delta(\lambda) = f_\lambda^\delta(x), \quad \lambda \in \mathfrak{a}_\varepsilon^*. \tag{4.3.48}$$

Thus (4.3.47) can be reformulated as follows

$$P_\lambda f(x) = \sum_{\delta \in F} trf_\lambda^\delta(x),$$

which immediately gives $P_\lambda f(x) = f_\lambda(x)$ for all $\lambda \in \mathfrak{a}_\varepsilon^*$ and $x \in X$. We give the gist of what we have shown in this section in the form of the following theorem

Theorem 4.3.18. *Any continuous function* $g : (x, \lambda) \mapsto g_\lambda(x)$ *defined on* $X \times \mathfrak{a}^*_\varepsilon$ *with certain* $\varepsilon \geq 0$ *and satisfying the conditions of Definition 4.3.1 is of the form* $g_\lambda(x) = P_\lambda f(x)$ *for some left* K *finite function* $f \in S^p(X)$ *where* $p = \frac{2}{1+\varepsilon}$.

4.4 Inverse Paley-Wiener Theorem

We begin the section with a definition.

Definition 4.4.1. *Let* $P_R(X)$ *be the class of scalar valued functions* $(\lambda, x) \mapsto f_\lambda(x)$ *on* $\mathfrak{a}^* \times X$ *satisfying:*

(i) the map $\lambda \mapsto f_\lambda(\cdot)$ *is an even, compactly supported* $\mathcal{C}^\infty$ *function on* $\mathfrak{a}^*$ *with it support lying in* $[-R, R]$ *(note that our group* G *is of real rank-1 and thus we have identified* $\mathfrak{a}^*$ *with* $\mathbb{R}$*),*

(ii) for each $\lambda \in \mathfrak{a}^*$, $x \mapsto f_\lambda(x)$ *is a* C^∞ *function on* X *and* $f_\lambda(\cdot) \in \mathcal{E}_\lambda(X)$.

In this section we shall try to characterize the space $P_R(X)$ as an image of certain subspace of $L^2(X)$ under the spectral projection.
We need to recall some basic results regarding a certain G-invariant domain Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$ called the *complex crown*. Here $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ is the natural complexification (see [36]) of the symmetric space X. The domain can be explicitly written as $\Xi = G \exp i\Omega \cdot x_0$, where $x_0 = eK$ and $\Omega = \{H \in \mathfrak{a} \mid |\alpha(H)| < \frac{\pi}{2}, \alpha \in \Sigma\}$ [35]. Let $\mathcal{G}(\Xi)$ be space of all holomorphic functions on the complex crown. For $\lambda \in i\mathfrak{a}^*$, the function $H \mapsto \varphi_\lambda(\exp iH)$ $(H \in \mathfrak{a})$ can be analytically continued in the tube domain $\mathfrak{a} + 2i\Omega$ [37]. Almost all the basic analysis on the crown domain uses a fundamental tool called the *orbital integrals* developed by Gindikin et al. [15]. Let h be a function on Ξ suitably decreasing at the boundary and $Y \in 2\Omega$, then the orbital integral is defined by

$$O_h(iY) = \int_G h\left(g \exp\left(\frac{i}{2}Y\right) \cdot x_0\right) \, dg. \tag{4.4.1}$$

If a holomorphic function θ on the tube $\mathfrak{a}+2i\Omega$ has the representation $\theta(Y) = \int_{i\mathfrak{a}^*} g(\lambda)\ \varphi_\lambda(\exp Y)|\mathbf{c}(\lambda)|^{-2}d\lambda$, then we define (with certain condition on g, see [36]) the operator $D\theta(Y) = \int_{i\mathfrak{a}^*} g(\lambda)\psi_\lambda(Y)|\mathbf{c}(\lambda)|^{-2}d\lambda$, where $\psi_\lambda(Y) = e^{\langle\lambda,Y\rangle} + e^{\langle\lambda,-Y\rangle}$. The operator D is a pseudo-differential shift operator [36]. The following is an inverse Paley-Wiener theorem for the Helgason Fourier transform.

Theorem 4.4.2. [Thangavelu [47, Theorem 2.3]]
Let $f \in L^2(X)$, then the Helgason Fourier transform $\widetilde{f}(\lambda, kM)$ is supported in $|\lambda| \leq R$ if and only if the function f has a holomorphic extension $F \in \mathcal{G}(\Xi)$ which satisfies the estimate

$$DO_{|F|^2}(iY) \leq Ce^{2R|Y|}, \quad \text{where } C \text{ is independent of } Y \in 2\Omega. \tag{4.4.2}$$

Our main theorem in this section is a consequence of the above theorem.

Theorem 4.4.3. *A function $f_\lambda(x)$ $(x \in X,\ \lambda \in \mathfrak{a}^*)$ is in $P_R(X)$ if and only if $f_\lambda(x) = (f * \varphi_\lambda)(x)$ $(\forall\ x \in X,\ \lambda \in \mathfrak{a}^*)$ for some $f \in L^2(X)$ which admits a holomorphic extension $F \in \mathcal{G}(\Xi)$ satisfying the estimate (4.4.2)*

Proof. Let $f_\lambda(x) \in P_R(X)$; then we get a function

$$f(x) = \int_{\mathfrak{a}^{*+}} f_\lambda(x)\ |\mathbf{c}(\lambda)|^{-2} d\lambda. \tag{4.4.3}$$

The integral (4.4.3) is obviously convergent and $f \in \mathcal{C}^\infty(X)$. A simple application of the Peter-Weyl theorem gives

$$f(x) = \int_{\mathfrak{a}^{*+}} \left[\sum_{\delta \in \widehat{K}_M} tr\ f_\lambda^\delta(x) \right] |\mathbf{c}(\lambda)|^{-2} d\lambda. \tag{4.4.4}$$

Now for each $\delta \in \widehat{K}_M$ and $\lambda \in \mathfrak{a}^*$, $f_\lambda^\delta(\cdot) \in \mathcal{E}_\lambda^\delta(X)$, hence by Theorem 4.3.12 we can write $f_\lambda^\delta(x) = \sqrt{d_\delta}\Phi_{\lambda,\delta}(x) h^\delta(\lambda)$. As $\lambda \mapsto f_\lambda^\delta(\cdot)$ is compactly supported and $\Phi_{\lambda,\delta}(\cdot)$ is an entire function so the function h^δ must have its support in $[-R, R]$. The above structural form can further reduce (4.4.4) as follows:

$$\begin{aligned}
f(x) &= \int_{\mathfrak{a}^{*+}} \left[\sum_{\delta \in \widehat{K}_M} \sqrt{d_\delta} \sum_{i=1}^{d_\delta} \langle \Phi_{\lambda,\delta}(x) h^\delta(\lambda) v_i, v_i \rangle \right] |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&= \int_{\mathfrak{a}^{*+}} \left[\sum_{\delta \in \widehat{K}_M} \sqrt{d_\delta} \sum_{i=1}^{d_\delta} \langle \Phi_{\lambda,\delta}(x) v_1, v_i \rangle \langle h^\delta(\lambda) v_i, v_1 \rangle \right] |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&= \int_{\mathfrak{a}^{*+}} \left[\sum_{\delta \in \widehat{K}_M} \sqrt{d_\delta} \sum_{i=1}^{d_\delta} \langle \int_K e^{-(i\lambda+1)H(x^{-1}k)} \delta(k) dk\ v_1, v_i \rangle\ \langle h^\delta(\lambda) v_i, v_1 \rangle \right] \\
&\qquad |\mathbf{c}(\lambda)|^{-2} d\lambda
\end{aligned}$$

$$= \int_{\mathfrak{a}^{*+}} \int_K e^{-(i\lambda+1)H(x^{-1}k)} \left[\sum_{\delta \in \widehat{K}_M} \sqrt{d_\delta} \sum_{i=1}^{d_\delta} \langle \delta(k)\ v_1, v_i \rangle\ \langle h^\delta(\lambda) v_i, v_1 \rangle \right]$$
$$|\mathbf{c}(\lambda)|^{-2} dk\ d\lambda$$
$$= \int_{\mathfrak{a}^{*+}} \int_K e^{-(i\lambda+1)H(x^{-1}k)} \left[\sum_{\delta \in \widehat{K}_M} \sqrt{d_\delta}\ tr(\delta(k) h^\delta(\lambda)) \right] |\mathbf{c}(\lambda)|^{-2} dk d\lambda. \tag{4.4.5}$$

We denote the function $h(\lambda, k) = \sum_{\delta \in \widehat{K}_M} \sqrt{d_\delta}\ tr(\delta(k) h^\delta(\lambda))$ in (4.4.5). Then clearly it is a $\mathcal{C}^\infty$ function in the λ variable and the function $\lambda \mapsto \int_K e^{(-i\lambda+1)H(x^{-1}k)} h(\lambda, k) dk$ is even. Also $h(\lambda, \cdot)$ is a compactly supported function and f is nothing but the Helgason Fourier inversion of the function h. Hence by [47, Theorem 2.3], the function $f \in L^2(X)$ and it admits a holomorphic extension on the complex crown satisfying the estimate (4.4.2). On the other hand if $g \in L^2(X)$ then, for all $\lambda \in \mathfrak{a}^*$ and $x \in X$, $P_\lambda g(x) = g * \varphi_\lambda(x) = \int_K e^{-(i\lambda+1)H(x^{-1}k)} \widetilde{g}(\lambda, k)\ dk$. Furthermore if g can be extended holomorphically to some $\overline{g} \in \mathcal{G}(\Xi)$ with $\overline{g}$ satisfying (4.4.2) then by Theorem 4.4.2, $\widetilde{g}(\lambda, \cdot)$ is supported in $[-R, R]$. It is now easy to show that $g * \varphi_\lambda(\cdot) \in P_R(X)$. ■

Chapter 5

Characterization of Fourier transforms of rapidly decreasing functions on $SL_2(\mathbb{R})$-of given left and right K-types

5.1 Introduction

In this chapter we shall establish the L^p-Schwartz space isomorphism theorem $(1 < p \leq 2)$ for the Schwartz class functions on the group $SL_2(\mathbb{R})$ with fixed left and right K-types. Characterization of the image of the L^p-Schwartz space under the group Fourier transform started with Ehrenpreis and Mautner [12], where, they have considered the case $p = 2$ and characterized the image of $\mathcal{S}^2(G)$ where $G = PSL_2(\mathbb{R})$. Later, the complete $p = 2$ result, for any reductive group, was established by Arthur in [4, 5]. The corresponding theorem for values other than $p = 2$, to be specific $1 < p < 2$, was obtained by Trombi [48], who proved his result under the K-finite restriction for any semisimple Lie group with real rank-1. The main reference for this section is W. H. Barker [7], where, he has established the L^p-Schwartz space isomorphism theorem $(0 < p \leq 2)$ for the group $SL_2(\mathbb{R})$. In each of the works recounted above, the Harish-Chandra expansion of the matrix coefficients of the principal series as well as the discrete series representations plays an important role.

Following Anker [2], we give here a simple proof of the isomorphism between the space $\mathcal{S}^p_{m,n}(G)$ $(G = SL_2(\mathbb{R}))$, of all L^p $(1 < p < 2)$ Schwartz class functions with fixed left type m and right type n, and its image space $\mathcal{S}^p_{m,n}(\widehat{G})$. Our proof does not involve the asymptotic expansions of the matrix entries of the representations, except what goes into the Paley-Wiener theorem. Our notation in this chapter closely follows Barker [7]. In particular there is a change in the parameterization of $\mathfrak{a}^*_{\mathbb{C}}$ (by the rotation $\lambda \mapsto i\lambda$) from what we have used in the previous chapters. Whereas so far the real axis represented the unitary dual, it would be the imaginary axis in this chapter.

5.2 Notations and some basic results

In this section we concentrate on the 2×2 real *special linear* group $SL_2(\mathbb{R})$, that is

$$G = SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\}. \tag{5.2.1}$$

The Lie algebra of G denoted by $\mathfrak{sl}_2(\mathbb{R})$ is realized as

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0,\ a, b, c, d \in \mathbb{R} \right\}. \tag{5.2.2}$$

The elements

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ H = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \overline{Y} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ are of special interest. The corresponding group elements to the first three of these elements are:

$$\begin{aligned} k_\theta &= \exp(\theta X) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}, \\ a_t &= \exp(2tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad t \in \mathbb{R}, \text{ and} \\ n_\xi &= \exp(\xi Y) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \quad \xi \in \mathbb{R}. \end{aligned}$$

The subgroups of G,

$$K=\{k_\theta \mid \theta\in\mathbb{R}\}\,,\quad A=\{a_t \mid t\in\mathbb{R}\}\,,\quad N=\{n_\xi \mid \xi\in\mathbb{R}\}\,,$$

have corresponding Lie algebras denoted respectively by $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$. The compact subgroup K is maximal compact in G and the Iwasawa decomposition is $G=K\times A\times N$. Through the element $H\in\mathfrak{a}$, the Lie subalgebra $\mathfrak{a}$ gets identified with $\mathbb{R}$ and so also $\mathfrak{a}^*$ through the usual paring. The only root is then $2\in\mathbb{R}=\mathfrak{a}$ and we have the half sum of the positive roots $\rho=1\in\mathbb{R}$. The complexification $\mathfrak{a}^*_{\mathbb{C}}$ of the space $\mathfrak{a}^*$ is identified with $\mathbb{C}$, the Weyl group in this case is simply $W=\{\pm1\}$ and the positive Weyl chamber is thus $\mathbb{R}^+$. The Cartan decomposition of the group is as follows $G=K\cdot Cl(A^+)\cdot K$, where $A^+=\{a_t \mid t>0\}$. If according to this decomposition any element $x\in G$ be written as $x=k_\theta a_t k_\psi$, the element a_t is unique and we denote $|x|=t$. The Haar measure corresponding to the Iwasawa KAN decomposition is this case is given by $dx=e^{2t}dkdadn$ where $x=k_\theta a_t n_\xi$, $dk=dk_\theta=\frac{1}{2\pi}d\theta$ $(0\le\theta\le2\pi)$, $da=da_t=dt$ and $dn=dn_\xi=d\xi$.
The maximal compact subgroup K being isomorphic to $SO(2)$, the set $\widehat{K}$ of equivalence classes of all irreducible representations of K is parameterized by $\mathbb{Z}$ and the character corresponding to $n\in\mathbb{Z}$ is given by $\tau_n(k_\theta)=e^{in\theta}$.
Let $n,m\in\mathbb{Z}$. A complex valued function f on the group G is said to of *left type n and right type m* or simply type (n,m) if for each $k_1,k_2\in K$ and $x\in G$:

$$f(k_1xk_2)=\tau_n(k_1)f(x)\tau_m(k_2). \tag{5.2.3}$$

In particular, $(0,0)$-type functions are precisely the K-bi-invariant functions on the group $G=SL_2(\mathbb{R})$. For any $\mathcal{C}^\infty$ function on G, the (m,n) th component or the (m,n) projection of f denoted by $\mathcal{P}_{m,n}(f)$ or $f^{(m,n)}$ is simply given by

$$\begin{aligned}\mathcal{P}_{m,n}(f)=f^{(m,n)}(x)&=\int_K\int_K\overline{\tau_m(k_1)}\;\overline{\tau_n(k_2)}f(k_1xk_2)dk_1dk_2\\&=\int_K\int_K e^{-im\theta}e^{-in\phi}f(k_\theta xk_\phi)dk_\theta dk_\phi.\end{aligned} \tag{5.2.4}$$

Clearly, $f^{(m,n)}$ is of (m,n) type. Any $\mathcal{C}^\infty$ function f can be decomposed as $f=\sum_{m,n\in\mathbb{Z}}f^{(m,n)}$ and this sum is absolutely convergent. For any function class $\mathfrak{F}(G)$, we shall denote $\mathfrak{F}_{m,n}(G)$ for the (m,n)-type projection of the

corresponding function class.
The centralizer M of A in K is given by $M = \{\pm I\}$, where I is the 2×2 identity matrix. Let $\widehat{M} = \{\sigma_+, \sigma_-\}$ be the equivalence classes of the irreducible representations of M, where $\sigma_+(-I) = 1$ and $\sigma_-(-I) = -1$. For each $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{C}$ we get a one dimensional representation (σ, λ) of the minimal parabolic subgroup $P = MAN$: $(\sigma, \lambda)(man) = \sigma(m)\lambda(a)$, $m \in M,\ a \in A,\ n \in N$. Let $(\pi_{\sigma,\lambda}, H_\sigma)$ be the principal series representation of G, induced from (σ, λ). The representation space H_σ is a subspace of $L^2(K)$ generated by the orthonormal basis $\{\tau_n \mid n \in \mathbb{Z}^\sigma\}$, where $\mathbb{Z}^\sigma$ is the set of even integers for $\sigma = \sigma_+$ and the set of odd integers for $\sigma = \sigma_-$. The representation $\pi_{\sigma,\lambda}$ is unitary if and only if $\lambda = i\mathbb{R}$. On H_σ it has the following form [7, 4.1]

$$[\pi_{\sigma,\lambda}(x)\tau_n](k) = e^{-(\lambda+1)H(x^{-1}k^{-1})}\ \tau_{-n}(K(x^{-1}k^{-1})). \tag{5.2.5}$$

For each $k \in \mathbb{Z}^*$, the set of all non zero integers, there exists a discrete series representation π_k of the group G. These representations appear as sub representations of $\pi_{\sigma,|k|}$, where σ is chosen so that $k \in \mathbb{Z}^* \setminus \mathbb{Z}^\sigma$. For the representation $\pi_{\sigma,\lambda}$ we need the infinitesimal action of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, and $\mathfrak{g}_\mathbb{C}$. The following three elements of $\mathfrak{g}_\mathbb{C}$ are of special interest while dealing with the principal series representations:

$$X,\ E = 2H + i(Y - \overline{Y}) = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},\ F = -2H + i(Y - \overline{Y}) = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}. \tag{5.2.6}$$

The $\pi_{\sigma,\lambda}$ action of the above elements are as follows:

$$\pi_{\sigma,\lambda}(X)\tau_n = in\tau_n,\ \pi_{\sigma,\lambda}(E)\tau_n = (n+\lambda+1)\tau_{n+2},\ \pi_{\sigma,\lambda}(F)\tau_n = (n-\lambda-1)\tau_{n-2}. \tag{5.2.7}$$

For $m, n \in \mathbb{Z}^\sigma$ and $k \in \mathbb{Z}^* \setminus \mathbb{Z}^\sigma$, let $\Phi^{m,n}_{\sigma,\lambda}(x) = \langle \pi_{\sigma,\lambda}(x)\tau_m, \tau_n\rangle$ and $\Psi^{m,n}_k = \langle \pi_k(x)\tau^k_m, \tau^k_n\rangle_k$ be the matrix coefficients of the principle and the discrete series representations respectively. Here $\langle\cdot,\cdot\rangle_k$ is a fixed normalizing inner product of π_k and τ^k_n denotes the normalized version of the basis elements of the representation space of π_k (see, [7, p. 20]). For each $\sigma \in \widehat{M}$, $\lambda \in \mathbb{C}$ and $m, n \in \mathbb{Z}^\sigma$, the matrix coefficients of the principal series representation can be given an explicit integral representation as follows:

$$\Phi^{m,n}_{\sigma,\lambda}(x) = \int_K e^{-(\lambda+1)H(x^{-1}k^{-1})}\tau_{-m}(K(x^{-1}k^{-1}))\tau_n(k^{-1})\, dk. \tag{5.2.8}$$

From (5.2.8), it is clear that $\Phi^{0,0}_{\sigma_+,\lambda}(x) = \varphi_\lambda(x)$. Some very elementary properties and estimates of $\Phi^{m,n}_{\sigma,\lambda}$ are highlighted in the following remark, which will needed later.

Remark 5.2.1. *(i) Suppose $\sigma \in \widehat{M}$, $m,n \in \mathbb{Z}^\sigma$ and $k \in \mathbb{Z}^{\sigma_-}$, then $\Phi^{m,n}_{\sigma,k}$ is identically zero if and only if $m < -k < n$ or $n < k < m$.*

(ii) For all $\sigma \in \widehat{M}$, $\lambda \in \mathbb{C}$ and $m,n \in \mathbb{Z}^\sigma$, the function $x \mapsto \Phi^{m,n}_{\sigma,\lambda}(x)$ is of type (m,n). It is also an eigenfunction of the Casimir operator Ω ([7, 4.7])

$$\Omega\Phi^{m,n}_{\sigma,\lambda}(x) = \frac{\lambda^2-1}{4}\Phi^{m,n}_{\sigma,\lambda}(x), \qquad \forall x \in G. \tag{5.2.9}$$

(iii) For all $\sigma \in \widehat{M}$, $\lambda \in \mathbb{C}$ and $m,n \in \mathbb{Z}^\sigma$:

$$|\Phi^{m,n}_{\sigma,\lambda}(x)| \le \Phi^{0,0}_{\sigma,\Re\lambda}(x) = \varphi_{\Re\lambda}(x). \tag{5.2.10}$$

Further we have the following estimate for $\Phi^{m,n}_{\sigma,\lambda}$ and its derivatives due to Harish-Chandra [21, Lemma 17.1]. For each $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$, $s \in \mathbb{Z}^+$ and $\epsilon \ge 0$, there exists a positive constant c and integers $r_1, r_2 \ge 0$ such that

$$\left|\left(\frac{d}{d\lambda}\right)^s \Phi^{m,n}_{\sigma,\lambda}(\boldsymbol{g}_1; x; \boldsymbol{g}_2)\right| \le c(1+|m|)^{r_1}(1+|n|)^{r_2}(1+|\lambda|)^{r_1+r_2} (1+|x|)^{s+\epsilon}\varphi_0^{1-\epsilon}(x), \tag{5.2.11}$$

for all $x \in G$ and for all $|\Re\lambda| \le \epsilon$. The integers r_α $(\alpha = 1,2)$ may be chosen so that $r_\alpha \le degree(\boldsymbol{g}_\alpha)$.

(iv) The matrix coefficients $\Psi^{m,n}_k$ of the discrete series representations are merely multiples of the corresponding entries $\Phi^{m,n}_{\sigma,|k|}$ [7, Proposition 7.6], hence they are also of spherical type (m,n). Fixing $\ell \in \mathbb{Z}^+$ and $\boldsymbol{g}, \boldsymbol{g}' \in$

$\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, there exists constants $c, r_j (1 \le j \le 4) > 0$ such that

$$|\Psi_k^{m,n}(\boldsymbol{g}; x; \boldsymbol{g'})| \le c(1+|m|)^{r_1}(1+|n|)^{r_2}(1+|k|)^{r_3}(1+|x|)^{r_4}\varphi_0^{1+\ell}(x), \tag{5.2.12}$$

for all $k \in \mathbb{Z}^\sigma$ with $|k| \ge \ell$ and for all $m, n \in \mathbb{Z} \setminus \mathbb{Z}^\sigma$. This estimate is due to Barker [7] (see also Trombi and Varadarajan [50]).

For a suitable function f on G the Fourier transform of f relative to the principal series and the discrete series representation, denoted respectively by $\mathcal{F}_H(f)(\sigma, \lambda)$ and $\mathcal{F}_B(f)(k)$, are defined to be the operators

$$\mathcal{F}_H(f)(\sigma, \lambda) = \int_G f(x)\pi_{\sigma,\lambda}(x^{-1})dx, \quad \text{and } \mathcal{F}_B(f)(k) = \int_G f(x)\pi_k(x^{-1})dx. \tag{5.2.13}$$

Hence the group Fourier transform $\mathcal{F}$ is given by the ordered pair $\mathcal{F} = (\mathcal{F}_H, \mathcal{F}_B)$. The canonical matrix coefficients of the operators in (5.2.13) are denoted by $\mathcal{F}_H^{m,n}(f)$ and $\mathcal{F}_B^{m,n}(f)$, also they have the following integral representation:

$$\mathcal{F}_H^{m,n}(f)(\sigma, \lambda) = \int_G f(x)\Phi_{\sigma,\lambda}^{m,n}(x^{-1})dx, \tag{5.2.14}$$

$$\mathcal{F}_B^{m,n}(f)(k) = \int_G f(x)\Psi_k^{m,n}(x^{-1}). \tag{5.2.15}$$

For notational simplicity, once we fix $m, n \in \mathbb{Z}^\sigma$, we shall simply write $\mathcal{F}_H^{m,n}(f)(\lambda)$ instead of $\mathcal{F}_H^{m,n}(f)(\sigma, \lambda)$.

Remark 5.2.2. *Let g be a (m, n)-type compactly supported function on G. Then only the (m, n)th matrix coefficient of the operator valued Fourier transform $\mathcal{F}_H g$ will be non zero. Hence, in effect, for any (m, n)-type function g $\mathcal{F}_H(g) = \mathcal{F}_H^{m,n}(g)$. Similarly, for the discrete part of the Fourier transform we have: $\mathcal{F}_B(g) = \mathcal{F}_B^{m,n}(g)$.*

5.3 Paley-Wiener Theorem and its consequences

We shall denote by $\mathcal{D}^T(G)$ the space of all $\mathcal{C}^\infty$ functions on G whose supports are in $[-T, T]$ and $\mathcal{D}_{m,n}^T(G)$ be its (m, n)-type projection. Before we give a formal description of what we call the Paley-Wiener space $\mathcal{D}_{H;m,n}^T(\widehat{G})$, let us define the Kostant functions on $\mathbb{C}$, which are related to intertwining operations between principal series representation. For $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $\lambda \in \mathbb{C}$

$$\phi^{m,n}_{\sigma,\lambda} = \begin{cases} \frac{(|n|-1+\lambda)(|n|-3+\lambda)\cdots(|m|+1+\lambda)}{(|n|-1-\lambda)(|n|-3-\lambda)\cdots(|m|+1-\lambda)}, & \text{when } |n| > |m|; \\ (-1)^{\frac{(n-m)}{2}}, & \text{when } |n| = |m|; \\ \frac{(|m|-1-\lambda)(|m|-3-\lambda)\cdots(|n|+1-\lambda)}{(|m|-1+\lambda)(|m|-3+\lambda)\cdots(|n|+1+\lambda)}, & \text{when } |m| > |n|. \end{cases} \tag{5.3.1}$$

The following proposition list out some properties of this function $\phi^{m,n}_{\sigma,\lambda}$.

Proposition 5.3.1. [Barker [7, Proposition7.2]]
Let $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^\sigma$. Then

(i) the function $\lambda \mapsto \phi^{m,n}_{\sigma,\lambda}$ is meromorphic and its only singularities are the first order poles at $\lambda = k \in \mathbb{Z}^{\sigma-}$ such that $\min(|m|, |n|) < |k| < \max(|m|, |n|)$*;*

(ii) $\phi^{m,n}_{\sigma,-\lambda} = \left(\phi^{m,n}_{\sigma,\lambda}\right)^{-1} = \phi^{n,m}_{\sigma,\lambda}$;

(iii) $\Phi^{m,n}_{\sigma,\lambda}(x) = \phi^{m,n}_{\sigma,\lambda}\Phi^{m,n}_{\sigma,-\lambda}$, for all $x \in G$ and for all $\lambda \in \mathbb{C}$ which are not poles of $\phi^{m,n}_{\sigma,\lambda}$;

(iv) $\phi^{m,n}_{\sigma,\lambda} = 0$ precisely when $\Phi^{m,n}_{\sigma,\lambda}(x)$ is identically zero and $\Phi^{m,n}_{\sigma,-\lambda}(x)$ is not identically zero;

(v) $\phi^{m,n}_{\sigma,\lambda}$ has first order poles in the λ variable when $\Phi^{m,n}_{\sigma,-\lambda}$ is identically zero but $\Phi^{m,n}_{\sigma,\lambda}$ is not identically zero.

Let us now formally define the Paley-Wiener space $\mathcal{D}^T_{H;m,n}(\widehat{G})$.

Definition 5.3.2. *The space $\mathcal{D}^T_{H;m,n}(\widehat{G})$ is the space of all entire functions $\phi : \mathbb{C} \mapsto \mathbb{C}$ such that*

(i) $\phi(\lambda) = \phi^{m,n}_{\sigma,\lambda}\phi(-\lambda)$, for all λ where $\phi^{m,n}_{\sigma,\lambda}$ is defined;

(ii) for each $N \geq 0$, there exists a constant $C_N < \infty$ such that

$$|\phi(\lambda)| \leq C_N(1 + |\lambda|)^{-N} e^{T|\Re\lambda|}, \qquad \lambda \in \mathbb{C}.$$

(iii) if $m, n \in \mathbb{Z}^\sigma$ and $mn < 0$, then $\phi(k) = 0$ for all $k \in \mathbb{Z}^{\sigma-}$ such that $|k| \leq \min\{|m|, |n|\}$.

Below is the Paley-Wiener theorem for the spherical (m,n)-type functions (Barker [7, Theorem 10.5]).

Theorem 5.3.3. *The transform $\mathcal{F}_H^{m,n}$ is an isomorphism from $\mathcal{D}_{m,n}^T(G)$ onto $\mathcal{D}_{H;m,n}^T(\widehat{G})$, with the inverse $\mathcal{I}_H^{m,n}$ given by:*

$$(\mathcal{I}_H^{m,n}\phi)(x) = \left(\frac{1}{2\pi}\right)^2 \int_{\Re\lambda=0} \phi(\lambda)\Phi_{\sigma,\lambda}^{n,m}(x)\mu(\sigma,\lambda)d\lambda + \left(\frac{1}{2\pi}\right) \sum_{k\in L_\sigma^{m,n}} \phi(k)\Phi_{\sigma,k}^{n,m}(x)|k|, \tag{5.3.2}$$

where, $L_\sigma^{m,n} = \{k \in \mathbb{Z}^{\sigma_-} \mid 0 < k < \min\{m,n\}$ or $\max\{m,n\} < k < 0\}$ and the Plancherel weight $\mu(\sigma,\lambda)$ is the meromorphic function

$$\mu(\sigma,\lambda) = \begin{cases} (\lambda\pi i/2)\tan\lambda\pi/2, & \text{if } \sigma = \sigma_+ \\ (-\lambda\pi i/2)\cot\lambda\pi/2, & \text{if } \sigma = \sigma_-. \end{cases} \tag{5.3.3}$$

It is clear from (5.3.3) that there exists a constant c such that for all σ and $\lambda \in i\mathbb{R}$, $|\mu(\sigma,\lambda)| \leq c(1+|\lambda|)$. For a proof of the above Paley-Wiener theorem one can see [11].
When $|m| = |n|$, the function $\lambda \mapsto \Phi_{\sigma,\lambda}^{m,n}$ is either even or odd function, hence this case is simpler to deal with. When $|m| \neq |n|$, to avoid repetition of the arguments, we shall confine ourselves the case $|m| > |n|$. One can mimic the argument, with suitable changes, to tackle the case $|m| < |n|$. From the definition (5.3.1) of the *Kostant function* $\phi_{\sigma,\lambda}^{m,n}$ and from (iii), (iv), (v) of the above Proposition 5.3.1 it follows that: for $|m| > |n|$,

$$\frac{1}{(|m|-1-\lambda)(|m|-3-\lambda)\cdots(|n|+1-\lambda)}\Phi_{\sigma,\lambda}^{m,n}(x) = \frac{1}{(|m|-1+\lambda)(|m|-3+\lambda)\cdots(|n|+1+\lambda)}\Phi_{\sigma,-\lambda}^{m,n}(x), \quad x \in G, \lambda \in \mathbb{C}. \tag{5.3.4}$$

Moreover, both sides of (5.3.4) are holomorphic in λ. We denote

$$P_{m,n}(\lambda) = (|m|-1+\lambda)(|m|-3+\lambda)\cdots(|n|+1+\lambda). \tag{5.3.5}$$

From (5.3.4) and from the definition (5.2.14) of the continuous part of the

Fourier transform it follows that for each $f \in \mathcal{D}_{m,n}(G)$, $\mathcal{F}_H^{m,n} f$ satisfies

$$\frac{1}{P_{m,n}(-\lambda)}\mathcal{F}_H^{m,n} f(\lambda) = \frac{1}{P_{m,n}(\lambda)}\mathcal{F}_H^{m,n} f(-\lambda), \tag{5.3.6}$$

and both sides of the above equation are entire functions.

Proposition 5.3.4. *Let $\varphi \in \mathcal{D}^T_{H;m,n}(\widehat{G})$ where, $m, n \in \mathbb{Z}^\sigma$ and $|m| > |n|$, then the map $\lambda \mapsto \psi(\lambda) = P_{m,n}(-\lambda)^{-1}\varphi(\lambda)$ ($\lambda \in \mathbb{C}$) is an even entire function of exponential type-T.*
Moreover, if m and n are of opposite signs then ψ vanishes on the points $k \in \mathbb{Z}^{\sigma-}$ where $|k| < |n|$.

Proof. As $\varphi \in \mathcal{D}^T_{H;m,n}(\widehat{G})$, so φ is an entire function of exponential type-T. Now from (5.3.6) we already know that ψ is an even entire function. What remains is to show that it is of exponential type which can be done by elementary means.
It can be seen, moreover that ψ is of exponential type-T as φ is so.
When m and n are of opposite signs, then a careful observation of the zeros of $\Phi^{m,n}_{\sigma,\lambda}$ given in (i) of Remark 5.2.1 and the zeros of the polynomial $P_{m,n}$ shows that ψ vanishes on the points $k \in \mathbb{Z}^{\sigma-}$ where $|k| < |n|$. ∎

Lemma 5.3.5. *Let $m, n \in \widehat{M}$ and $|m| > |n|$ then*

(i) when $m > 0$, $P_{m,n}(-\lambda)\Phi^{n,m}_{\sigma,\lambda}(x) = \Phi^{n,|n|}_{\sigma,\lambda}(\mathfrak{E}_{m,n}; x)$;

(ii) when $m < 0$ then $P_{m,n}(-\lambda)\Phi^{n,m}_{\sigma,\lambda}(x) = (-1)^{\frac{|m|-|n|}{2}}\Phi^{n,-|n|}_{\sigma,\lambda}(\mathfrak{F}_{m,n}; x)$

for all $x \in G$ and $\lambda \in \mathbb{C}$. Here $\mathfrak{E}_{m,n}$ and $\mathfrak{F}_{m,n}$ are certain differential operators on the group.

Proof. Let us first consider $m > 0$. From the definition (5.3.5) of the polynomial $P_{m,n}$ and the matrix entries $\Phi^{m,n}_{\sigma,\lambda}(\cdot)$ of the principle series representation we get:

$$\begin{aligned}
&P_{m,n}(-\lambda)\Phi^{n,m}_{\sigma,\lambda}(x)\\
&\qquad = (m-1-\lambda)(m-3-\lambda)\cdots(|n|+1-\lambda)\langle \pi_{\sigma,\lambda}(x)\tau_n\ ,\ \tau_m\rangle\\
&\qquad = (m-3-\lambda)\cdots(|n|+1-\lambda)\langle \pi_{\sigma,\lambda}(x)\tau_n\ ,\ (m-1-\overline{\lambda})\tau_m\rangle\\
&\qquad = (m-3-\lambda)\cdots(|n|+1-\lambda)\langle \pi_{\sigma,\lambda}(x)\tau_n\ ,\ \pi_{\sigma,\overline{(-\lambda)}}(E)\tau_{m-2}\rangle\\
&\qquad = \left\langle \pi_{\sigma,\lambda}(x)\tau_n\ ,\ \pi_{\sigma,\overline{(-\lambda)}}(E^{t_{mn}})\tau_{|n|}\right\rangle,\ \text{where } t_{mn} = \frac{m-|n|}{2}\\
&\qquad = \left\langle \pi_{\sigma,\lambda}(\mathfrak{E}_{m,n})\pi_{\sigma,\lambda}(x)\tau_n\ ,\ \tau_{|n|}\right\rangle = \Phi^{n,|n|}_{\sigma,\lambda}(\mathfrak{E}_{m,n}; x).
\end{aligned} \tag{5.3.7}$$

The third equality follows from (5.2.6), the forth by iterating the same method and in the last line $\pi_{\sigma,\lambda}(\mathfrak{E}_{m,n})$ is the adjoint of $\pi_{\sigma,\overline{(-\lambda)}}(E^{t_{mn}})$. When $m<0$ we take $m=-\mu$ $(\mu>0)$. Hence in this case we get:

$$\begin{aligned}
&P_{m,n}(-\lambda)\Phi^{n,m}_{\sigma,\lambda}(x)\\
&\qquad=(\mu-\lambda-1)(\mu-\lambda-3)\cdots(|n|+1-\lambda)\langle\pi_{\sigma,\lambda}(x)\tau_n\ ,\ \tau_{-\mu}\rangle\\
&\qquad=(\mu-3-\lambda)\cdots(|n|+1-\lambda)\langle\pi_{\sigma,\lambda}(x)\tau_n\ ,\ -(-\mu+1+\overline{(-\lambda)})\tau_{-\mu}\rangle\\
&\qquad=(\mu-3-\lambda)\cdots(|n|+1-\lambda)\langle\pi_{\sigma,\lambda}(x)\tau_n\ ,\ -\pi_{\sigma,\overline{(-\lambda)}}(F)\tau_{-\mu+2}\rangle\\
&\qquad=(-1)^{\frac{|m|-|n|}{2}}\langle\pi_{\sigma,\lambda}(x)\tau_n\ ,\ \pi_{\sigma,\overline{(-\lambda)}}(F^{t_{mn}})\tau_{-|n|}\rangle\\
&\qquad=(-1)^{\frac{|m|-|n|}{2}}\langle\pi_{\sigma,\lambda}(\mathfrak{F}_{m,n};x)\tau_n\ ,\ \tau_{-|n|}\rangle=(-1)^{\frac{|m|-|n|}{2}}\Phi^{n,-|n|}_{\sigma,\lambda}(\mathfrak{F}_{m,n};x).
\end{aligned}\tag{5.3.8}$$

Again the third and the forth line in the above chain of equality follows from (5.2.6) and in the last line $\pi_{\sigma,\lambda}(\mathfrak{F}_{m,n})$ is the adjoint of $\pi_{\sigma,(-\overline{\lambda})}(F^{t_{m,n}})$. ∎

Remark 5.3.6. *When $|m|<|n|$, an analogue of the above Lemma can be obtained. The proof will be similar to that of the above Lemma.*

The following Proposition is a nice corollary to the above Lemma. This proposition will play a very crucial role for proving the main result of this section.

Proposition 5.3.7. *Let $m,n\in\mathbb{Z}^\sigma$ where $|m|>|n|$ and $T>0$. Then, each $f\in\mathcal{D}^T_{m,n}(G)$ can be represented as follows:*

(i) if $m>0$ then there exists an unique $\psi\in\mathcal{D}^T_{|n|,n}(G)$ such that $f=\mathfrak{E}_{m,n}\psi$;

(ii) when $m<0$ then there exists an unique $\varphi\in\mathcal{D}^T_{-|n|,n}(G)$ such that $f=(-1)^{\frac{|m|-|n|}{2}}\mathfrak{F}_{m,n}\varphi$;

where, $\mathfrak{E}_{m,n}$ and $\mathfrak{F}_{m,n}$ are differential operators as defined in Lemma 5.3.5.

Proof. Let $f\in\mathcal{D}^T_{m,n}(G)$ and $\widetilde{f}$ be the image of f under the transform $\mathcal{F}^{m,n}_H$. Then by the inversion formula we can write:

$$f(x)=\left(\frac{1}{2\pi}\right)^2\int_{\Re\lambda=0}\widetilde{f}(\lambda)\Phi^{n,m}_{\sigma,\lambda}(x)\mu(\sigma,\lambda)d\lambda+\frac{1}{2\pi}\sum_{\ell\in L^{m,n}_\sigma}\widetilde{f}(\ell)\Phi^{n,m}_{\sigma,\ell}(x)|\ell|.\tag{5.3.9}$$

We have already noticed in Lemma 5.3.4 that $\widetilde{f}(\lambda)$ is a multiple of $P_{m,n}(-\lambda)$ for all $\lambda \in \mathbb{C}$ and $\Psi(\lambda) = \frac{\widetilde{f}(\lambda)}{P_{m,n}(-\lambda)}$ is an even entire function of exponential type-T. Also as $|m| > |n|$ so for $m > 0$, $L_\sigma^{m,n} = L_\sigma^{|n|,n}$ and for $m < 0$, $L_\sigma^{m,n} = L_\sigma^{-|n|,n}$. Hence we may rewrite (5.3.9) as follows

$$f(x) = \left(\frac{1}{2\pi}\right)^2 \int_{\Re\lambda=0} \frac{\widetilde{f}(\lambda)}{P_{m,n}(-\lambda)} P_{m,n}(-\lambda)\Phi_{\sigma,\lambda}^{n,m}(x)\mu(\sigma,\lambda)d\lambda + \frac{1}{2\pi}\sum_{\ell\in L_\sigma^{m,n}} \frac{\widetilde{f}(\ell)}{P_{m,n}(-\ell)} P_{m,n}(-\ell)\Phi_{\sigma,\ell}^{n,m}(x)|\ell|. \tag{5.3.10}$$

By the Lemma 5.3.5 we get: for $m > 0$

$$f(x) = \left(\frac{1}{2\pi}\right)^2 \int_{\Re\lambda=0} \Psi(\lambda)\Phi_{\sigma,\lambda}^{n,|n|}(\mathfrak{E}_{m,n};x)\mu(\sigma,\lambda)d\lambda + \frac{1}{2\pi}\sum_{\ell\in L_\sigma^{|n|,n}} \Psi(\ell)\Phi_{\sigma,\ell}^{n,|n|}(\mathfrak{E}_{m,n};x)|\ell|; \tag{5.3.11}$$

and for $m < 0$

$$f(x) = \left(\frac{1}{2\pi}\right)^2 \int_{\Re\lambda=0} (-1)^{\frac{|m|-|n|}{2}}\Psi(\lambda)\Phi_{\sigma,\lambda}^{n,-|n|}(\mathfrak{E}_{m,n};x)\mu(\sigma,\lambda)d\lambda + \frac{1}{2\pi}\sum_{\ell\in L_\sigma^{-|n|,n}} (-1)^{\frac{|m|-|n|}{2}}\Psi(\ell)\Phi_{\sigma,\ell}^{n,-|n|}(\mathfrak{E}_{m,n};x)|\ell|. \tag{5.3.12}$$

Now by the Paley-Wiener theorem (Theorem 5.3.3) there exists unique $\psi \in \mathcal{D}^T_{|n|,n}(G)$ and $\varphi \in \mathcal{D}^T_{-|n|,n}(G)$ in the cases (5.3.10) and (5.3.12) respectively such that for all $x \in G$

$$f(x) = \begin{cases} \psi(\mathfrak{E}_{m,n};x), & \text{when } \ m > 0 \\ (-1)^{\frac{|m|-|n|}{2}}\varphi(\mathfrak{F}_{m,n};x), & \text{when } \ m < 0. \end{cases}$$

This completes the proof of the Proposition. ■

5.4 Schwartz Spaces and Schwartz space isomorphism

Let us now come to the L^p-Schwartz space $(1 < p \le 2)$ $\mathcal{S}^p_{m,n}(G)$. The space $\mathcal{S}^p_{m,n}(G)$ is the space of all smooth functions f on G such that $f = f^{m,n}$ is of

(m,n)-type and for all
$\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ and $r \in \mathbb{R}^+$

$$\rho^p_{\mathbf{g}_1,\mathbf{g}_2,r}(f) = sup_{x\in G}|f(\mathbf{g}_1;x;\mathbf{g}_2)|(1+|x|)^r \varphi_0^{-\frac{2}{p}}(x) < +\infty. \tag{5.4.1}$$

$\mathcal{S}^p_{m,n}(G)$ is a Fréchet space with the topology induced by the family of seminorms $\{\rho_{\mathbf{g}_1,\mathbf{g}_2,r}\}$. Let $\mathcal{S}^p_{m,n}(G) = \mathcal{P}_{m,n}(\mathcal{S}^p(G))$. For each p in $1 < p \leq 2$ we assign as before a positive real number $\varepsilon = \left(\frac{2}{p} - 1\right)$. We denote $S^\varepsilon = \{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq \varepsilon\}$, $L^{m,n}_\sigma(\varepsilon) = \{k \in L^{m,n}_\sigma \mid |k| \leq \varepsilon\}$ and $L^{m,n}_\sigma(\varepsilon)^c = \{k \in Ł^{m,n}_\sigma \mid |k| > \varepsilon\}$. We note that for each $m,n \in \mathbb{Z}^\sigma$ the set $L^{m,n}_\sigma(\varepsilon)^c$ is a finite set.

Definition 5.4.1. *For each $m,n \in \mathbb{Z}^\sigma$ we denote by $\mathcal{S}^p_{B,m,n}(\widehat{G})$ the set of functions $\mathbb{C}^{L^{m,n}_\sigma(\varepsilon)^c}$.*

The finite dimensional space $\mathcal{S}^p_{B,m,n}(\widehat{G})$ is given a linear topology.

Definition 5.4.2. *For $1 < p \leq 2$ and $m,n \in \mathbb{Z}^\sigma$ we denote $\mathcal{S}^p_{H;m,n}(\widehat{G})$ for the space of functions $f : S^\varepsilon \mapsto \mathbb{C}$, such that,*

(i) the map $\lambda \mapsto f(\sigma,\lambda)$ is holomorphic on $IntS^\varepsilon$ and it extends as a continuous function on the closed strip S^ε;

(ii) for all $\lambda \in S^\varepsilon$, $f(-\lambda) = \phi^{n,m}_\lambda f(\sigma,\lambda)$;

(iii) for all $r_1, r_2 \in \mathbb{Z}^+ \cup \{0\}$

$$\overline{\tau}^p_{(m,n)_{r_1,r_2}}(f) = \sup_{\lambda\in IntS^\varepsilon} \left| \left(\frac{d}{d\lambda}\right)^{r_1} f(\lambda) \right| (1+|\lambda|)^{r_2} < +\infty; \tag{5.4.2}$$

(iv) if $mn < 0$ then $f(k) = 0$ for all $k \in \mathbb{Z}^{-\sigma}$ with $|k| \leq \min(|m|,|n|,\varepsilon)$.

$\mathcal{S}^p_{H;m,n}(\widehat{G})$ becomes a Fréchet space with the topology induced by the family of seminorms $\left\{\overline{\tau}^p_{(m,n)_{r_1,r_2}}\right\}_{r_1,r_2\in\mathbb{Z}^+\cup\{0\}}$. For $f \in \mathcal{S}^p_{m,n}(G)$, we define the Fourier transform $(\mathcal{F}^{m,n}_H, \mathcal{F}^{m,n}_B)$ in consistency with (5.2.14) and (5.2.15):

$$\mathcal{F}^{m,n}_H(f)(\sigma,\lambda) = \int_G f(x)\Phi^{m,n}_{\sigma,\lambda}(x^{-1})dx \ \text{ and } \ \mathcal{F}^{m,n}_B(f)(k) = \int_G f(x)\Psi^{m,n}_k(x^{-1}).$$

It can be shown that the integrals converge absolutely and that $\mathcal{F}^{m,n}_H(f)$ is a holomorphic function on $IntS^\varepsilon$. It may be also be noted that the function

$\Psi_k^{m,n}$, where $|k| > \varepsilon$, belong to the space $\mathcal{S}^p_{m,n}(G)$, using the Barker's estimate (5.2.12).

Proposition 5.4.3. *For each $f \in \mathcal{S}^p_{m,n}(G)$, $\mathcal{F}_H^{m,n}(f) \in \mathcal{S}^p_{H;m,n}(\widehat{G})$ and $\mathcal{F}_B^{m,n}(f) \in \mathcal{S}^p_{B;m,n}(\widehat{G})$. The transforms $\mathcal{F}_H^{m,n}$ and $\mathcal{F}_B^{m,n}$ are continuous maps and $\mathcal{F}_B^{m,n}$ is onto.*

Proof. For a proof of this Proposition we refer to Theorem 9.1 and Theorem 9.6 of [7]. The remark about the functions $\Psi_k^{m,n}$ and the fact that $\Psi_k^{m,n}$ and $\Psi_{k'}^{m,n}$ are orthogonal if $k \neq k'$ ensure that $\mathcal{F}_B^{m,n}$ is onto. ∎

Let us denote $\mathcal{S}^p_{m,n}(\widehat{G}) \simeq \mathcal{S}^p_{H,m,n}(\widehat{G}) \times \mathcal{S}^p_{B,m,n}(\widehat{G})$. Now $\mathcal{F}^{m,n} = (\mathcal{F}_H^{m,n}, \mathcal{F}_B^{m,n})$ is a continuous function of $\mathcal{S}^p_{m,n}(G)$ into $\mathcal{S}^p_{m,n}(\widehat{G})$.

For each pair $F = (\vartheta, \xi) \in \mathcal{S}^p_{m,n}(\widehat{G})$, we define the following functions on the group G

$$\mathcal{J}^p_{H;m,n}(\vartheta)(x) = \left(\frac{1}{2\pi}\right)^2 \int_{\Re\lambda=0} \vartheta(\lambda)\Phi^{n,m}_{\sigma,\lambda}(x)\mu(\sigma,\lambda)d\lambda; \tag{5.4.3}$$

$$\mathcal{J}^p_{B;m,n}(\xi)(x) = \sum_{k\in L^{m,n}_\sigma(\varepsilon)^c} \xi(k)\Psi_k^{m,n}(x). \tag{5.4.4}$$

We write $\mathcal{J}^p_{m,n}(\vartheta,\xi)(x) = \mathcal{J}^p_{H;m,n}(\vartheta)(x) + \mathcal{J}^p_{B;m,n}(\xi)(x)$

Lemma 5.4.4. *(i) For each $\vartheta \in \mathcal{S}^p_{H;m,n}(\widehat{G})$, the inversion $\mathcal{J}_{H;m,n}\vartheta$ is a $\mathcal{C}^\infty$ (m,n)-type function on G.*

(ii) For each $\xi \in \mathcal{S}^p_{B;m,n}(\widehat{G})$, the function $\mathcal{J}_{B;m,n}\xi \in \mathcal{S}^p_{m,n}(G)$.

Proof. (i) For each $\vartheta \in \mathcal{S}^p_{H;m,n}(\widehat{G})$, the convergence of the integral in the definition (5.4.3) of the inversion map can be shown by using the decay (5.4.2) of ϑ and the estimate (5.2.10) of the spherical function.

The statement (ii) is a consequence of the remark about $\Psi_k^{m,n}$ made above. ∎

Our main aim now is to show that $\mathcal{J}^p_{m,n}$ is a continuous map from $\mathcal{S}^p_{m,n}(\widehat{G})$ to $\mathcal{S}^p_{m,n}(G)$. Since $\mathcal{S}^p_{B;m,n}(\widehat{G})$ is finite dimensional, we need only to show that the continuous part $\mathcal{J}^p_{H;m,n}$ is a continuous map into $\mathcal{S}^p_{m,n}(G)$. To prove this, the main tool that we shall be using is the *Abel transform.* For $f \in \mathcal{S}^p_{m,n}(G)$ we define the Abel transform by the following

$$\mathcal{A}f(t) = e^t \int_N f(a_t n)dn. \tag{5.4.5}$$

The following Lemma, due to Sarkar and Sengupta [40, Lemma 3.4], shows that the Abel transform (5.4.5) gives a commutative-as in the bi-K-invariant case- diagram under the continuous Fourier transform $\mathcal{F}_H^{m,n}$. For the sake of completeness we also reproduce their proof

Lemma 5.4.5. *Let $\sigma \in \widehat{M}$ and let $f \in \mathcal{S}^p_{m,n}(G)$ for some $m, n \in \mathbb{Z}^\sigma$. Then*

$$\mathcal{F}_H^{m,n}(f)(\lambda) = \widetilde{\mathcal{A}f}(-i\lambda), \tag{5.4.6}$$

for $\lambda \in i\mathbb{R}$ where, $\widetilde{\mathcal{A}f}(\nu) = \int_{\mathbb{R}} \mathcal{A}f(t)e^{-i\nu t}dt$.

Proof. Using the definition (5.2.14) of the transform $\mathcal{F}_H^{m,n}$ and the integral representation (5.2.8) of the matrix entry $\Phi_{\sigma,\lambda}^{m,n}$ of the principal series representation we write: for $\lambda \in i\mathbb{R}$

$$\mathcal{F}_H^{m,n}(f)(\lambda) = \int_G f(x) \int_K e^{-(\lambda+1)H(xk^{-1})}\tau_{-m}(K(xk^{-1}))\tau_n(k^{-1})dk\ dx,$$

as the repeated integral converges absolutely, we can interchange the integrals. Then we by substituting $k^{-1}xk$ for x and also by using the fact that the Haar measure of the group G is invariant under the action of K, we get:

$$\int_K \int_G f(k^{-1}yk)e^{-(\lambda+1)H(k^{-1}x)}\tau_{-m}(K(k^{-1}y))\tau_n(k^{-1})dy\ dk. \tag{5.4.7}$$

As, the function f is of spherical (m,n)-type, so $f(k^{-1}yk) = f(y)\tau_m(k^{-1})\tau_n(k)$. Also we know that $\tau_{-m}(K(k^{-1}y)) = \tau_m(k)\tau_{-m}(K(y))$. Hence (5.4.7) reduces to:

$$\mathcal{F}_H^{m,n}(f)(\lambda) = \int_G f(y)e^{-(\lambda+1)H(y)}\tau_{-m}(K(y))dy. \tag{5.4.8}$$

Now we take the Iwasawa $G = KAN$ decomposition of the group and also the corresponding decomposition (as given in section 5.2) of the Haar measure to get:

$$\begin{aligned} &= \int_K \int_A \int_N f(ka_t n)e^{-(\lambda+1)t}\tau_{-m}(k)dke^{2t}dtdn \\ &= \int_K \tau_m(k)\tau_{-m}(k)dk \int_A \int_N f(a_t n)e^t e^{-\lambda t}dndt \end{aligned}$$

$$= \int_A \left\{ e^t \int_N f(a_t n) dn \right\} e^{-\lambda t} dt$$
$$= \int_A \mathcal{A}f(t) e^{-i(-i\lambda)t} dt, \qquad \text{by the definition (5.4.5)}$$
$$= \widetilde{\mathcal{A}f}(-i\lambda). \tag{5.4.9}$$

■

Remark 5.4.6. *We should note that, the above Lemma also holds for all $\lambda \in \mathcal{S}^\varepsilon$ for which both sides of the equality (5.4.6) are well-defined and analytic.*

5.4.1 $|m| = |n|$ case.

In the first stage we shall be interested in the class of (m, n)-type functions, where $|m| = |n|$. This class of function is very close to the class of bi-K-invariant functions. Let us first fix one $\sigma \in \widehat{M}$ and take some $m, n \in \mathbb{Z}^\sigma$ such that $|m| = |n|$. In this case each $\vartheta \in \mathcal{S}^p_{H;m,n}(\widehat{G})$ is either an even or an odd function on $\mathcal{S}^\varepsilon$, holomorphic in the interior of $\mathcal{S}^\varepsilon$ and continuous on the closed strip and also satisfies the decay condition

$$\overline{\tau}^p_{r_1,r_2}(\vartheta) = \sup_{\lambda \in Int\mathcal{S}^\varepsilon} \left| \left(\frac{d}{d\lambda} \right)^{r_1} \vartheta(\lambda) \right| (1 + |\lambda|)^{r_2} < +\infty. \tag{5.4.10}$$

Lemma 5.4.7. *The Abel transform (5.4.5) is a topological isomorphism between the spaces $\mathcal{D}^T_{m,n}(G)$, where $m, n \in \mathbb{Z}^\sigma$ with $|m| = |n|$ $T \in \mathbb{R}^+$, and the space $\mathcal{D}^T(\mathbb{R})_{even}$ $(\mathcal{D}^T(\mathbb{R})_{odd})$ of all even (odd) $\mathcal{C}^\infty$ functions on $\mathbb{R}$ supported in $[-T, T]$.*

Proof. This Lemma is a simple consequence of the Paley-Wiener theorem (Theorem 5.3.3) and the slicing property of the Abel transform proved in Lemma 5.4.5.
Let $f \in \mathcal{D}^T_{m,n}(G)$ with $|m| = |n|$. Then, by the Definition 5.3.2 of the Paley-Wiener space, $\mathcal{F}^{m,n}_H(f)$ is an entire function of exponential type-T and it is purely even or odd in nature depending on the choice of m and n. Therefore for $\lambda \in \mathbb{C}$,

$$\mathcal{F}^{m,n}_H(f)(-i\lambda) = \mathcal{F}^{m,n}_H(f)(i\lambda) \ \text{ or } \ -\mathcal{F}^{m,n}_H(f)(i\lambda).$$

Hence by the Lemma 5.4.5 and the Remark 5.4.6 we get: for $\lambda \in \mathbb{C}$

$$\widetilde{\mathcal{A}f}(-\lambda) = \widetilde{\mathcal{A}f}(\lambda) \text{ or } -\widetilde{\mathcal{A}f}(\lambda).$$

Also $\widetilde{\mathcal{A}f}$ is entire and of exponential type-T. Hence by the Euclidean Paley-Wiener theorem $\mathcal{A}f \in \mathcal{D}^T(\mathbb{R})_{even}$ or $\mathcal{D}^T(\mathbb{R})_{odd}$. As both the maps $f \mapsto \mathcal{F}_H^{m,n}(f)$ and $\widetilde{\mathcal{A}f} \mapsto \mathcal{A}f$ are topological isomorphisms so by the commutative diagram given in Lemma 5.4.5, $f \mapsto \mathcal{A}f$ is also a topological isomorphism from $\mathcal{D}^T_{m,n}(G)$ onto $\mathcal{D}^T(\mathbb{R})_{even}$ $(\mathcal{D}^T(\mathbb{R})_{odd})$. ■

We have already noted that $\mathcal{D}_{m,n}(G)$ is dense in the Schwartz space $\mathcal{S}^p_{m,n}(G)$.

Lemma 5.4.8. *For $m, n \in \mathbb{Z}^\sigma$ and $|m| = |n|$, the map $\mathcal{I}^p_{H;m,n}$ is a continuous map from $\mathcal{S}^p_{H;m,n}(\widehat{G})$ into $\mathcal{S}^p_{m,n}(G)$.*

Proof. To prove this Lemma we shall almost mimic the proof, due to Anker [2, Lema 15], of the bi-K-invariant analogue of the above Lemma. We shall first consider the spaces $\mathcal{D}_{m,n}(G)$ and $\mathcal{D}_{H;m,n}(\widehat{G}) = \mathcal{F}_H^{m,n}(\mathcal{D}_{m,n}(G))$ with the topologies of the respective Schwartz spaces containing them. We shall first establish the Lemma for these subspaces and then extend the map to the whole space by a density argument.
Let us take $g \in \mathcal{D}_{m,n}(\widehat{G})$, then clearly by the definition there exists some $T > 0$ such that $g \in \mathcal{D}^T_{m,n}(\widehat{G})$. The Paley-Wiener theorem 5.3.3 gives an unique $f \in \mathcal{D}_{m,n}(G)$ such that $g(\lambda) = \mathcal{F}_H^{m,n}(f)(\lambda)$ for all $\lambda \in S^\varepsilon$ or in other words $f(x) = \mathcal{I}^p_{H;m,n}(g)(x)$ for all $x \in G$. Let us choose $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. The following estimate follows from the inversion formula (5.4.3) for the (m,n)-type functions by using the estimate (5.2.11) of the spherical function $\Phi^{n,m}_{\sigma,\lambda}$ with $\lambda \in i\mathbb{R}$ and also the Schwartz space decay of the function g:

$$|f(\mathbf{g}_1; x; \mathbf{g}_2)| \leq cK(m,n)(1+|x|)\varphi_0(x)\ \overline{\tau}^p_{(0)r_1}(g), \tag{5.4.11}$$

where, $K(m,n)$ is a positive constant given by $K(m,n) = (1+|m|)^r(1+|n|)^{r'}$, here r, r' are nonnegative integers depending on the degree of the derivatives $\mathbf{g}_1, \mathbf{g}_2$ and $\overline{\tau}^p_{(0)s}(g) = \sup_{\lambda \in i\mathbb{R}} |g(\lambda)|(1+|\lambda|)^s$. Now we fix a positive integer q and denote $F(x) = |f(\mathbf{g}_1; x; \mathbf{g}_2)|(1+|x|)^q \varphi_0^{-\frac{2}{p}}(x)$. Hence by (5.4.11):

$$F(x) \leq cK(m,n)(1+|x|)^{1+q}\varphi_0^{-\varepsilon}(x)\ \overline{\tau}^p_{(0)r_1}(g). \tag{5.4.12}$$

We shall now break the group G into an increasing sequence of compact sets. Let $G_j = K(\exp[-j,j])K$ for $j = 1, 2, \cdots$. It is easy to see that

$$\sup_{x\in G_2} F(x) \le c(1+|n|)^r \overline{\tau}^p_{(0)r_1}(g) \sup_{x\in G_2}(1+|x|)^{1+q}\varphi_0^{-\varepsilon}(x) < +\infty. \tag{5.4.13}$$

Next we shall be considering the set $G_{j+1}\setminus G_j$ for $j = 2, 3, \cdots$. This is the most crucial step. Let $\omega \in C^\infty$ be a function such that $\omega \equiv 0$ on $(-\infty, 0]$ and $\omega \equiv 1$ on $[1, +\infty)$. We define the 'auxiliary function'

$$\omega_j(t) = \omega(j-t)\omega(j+t) \quad \forall t \in \mathbb{R}. \tag{5.4.14}$$

Clearly, ω_j for $j = 2, 3, \cdots$, is an even C^∞ function on $\mathbb{R}$ furthermore $\omega_j \equiv 1$ on $[-j+1, j-1]$ and $\omega_j \equiv 0$ outside $[-j, j]$. Let us denote $h_j = (1-\omega_j)\mathcal{A}f$. As $f \in \mathcal{D}^T_{m,n}(G)$ so by Lemma 5.4.7, $\mathcal{A}f$ is either purely even or purely odd C^∞ function on $\mathbb{R}$ and of exponential type-T. We note that the function $(1-\omega_j)$ and all its derivatives are uniformly bounded with respect to j and also it is an even function. Thus if $\mathcal{A}f \in \mathcal{D}^T(\mathbb{R})_{even}$ ($\mathcal{D}^T(\mathbb{R})_{odd}$) then, for each j, $h_j \in \mathcal{D}^T(\mathbb{R})_{even}$ ($\mathcal{D}^T(\mathbb{R})_{odd}$). Hence by the characterization obtained in Lemma 5.4.7, there exists unique element $f_j \in \mathcal{D}^j_{m,n}(G)$ such that $h_j = \mathcal{A}f_j$. We denote $g_j = \mathcal{F}^{m,n}_H f_j$. By the Lemma 5.4.7, the functions f_j and f can differ only inside G_j. Hence, if we consider $x \in G_{j+1}\setminus G_j$, the expression $F(x)$ will not change even if we replace f by f_j. The following inequality is obtained from (5.4.12) by using the estimate (2.0.22) of the elementary spherical functions

$$\sup_{x\in G_{j+1}\setminus G_j} F(x) \le c_1(1+|n|)^r j^{r_2}\, e^{\varepsilon j}\, \overline{\tau}^p_{(0)r_1}(g_j). \tag{5.4.15}$$

Now by Lemma 5.4.7, for $\lambda \in \mathbb{C}$, $g_j(\lambda) = \widetilde{h_j}(i\lambda) = \int_{\mathbb{R}} h_j(t)e^{\lambda t}dt$. Hence we get the following inequality.

$$\begin{aligned}\overline{\tau}^p_{(0)r_1}(g_j) &\le c_2 \sum_{\ell=0}^{r_1}\int_{\mathbb{R}}\left|\left(\frac{d}{dt}\right)^\ell h_j(t)\right| dt \\ &\le c_3 \sum_{\ell=0}^{r_1}\sup_{t\in\mathbb{R}^+}(1+t)^2\left|\left(\frac{d}{dt}\right)^\ell h_j(t)\right|. \end{aligned} \tag{5.4.16}$$

Now we know that $h_j = (1-\omega_j)\mathcal{A}f$. It follows easily that the function $(1-\omega_j)$ vanishes on $[-j+1, j-1]$ and it is bounded uniformly along with all its derivatives with respect to j. Thus we have

$$\overline{\tau}^p_{(0)r_1}(g_j) \leq c_4 \sum_{\ell=0}^{r_1} \sup_{t\in\mathbb{R}^+\setminus[-j+1,j-1]} \left|(1+t)^2 \left(\frac{d}{dt}\right)^\ell \mathcal{A}f(t)\right|. \tag{5.4.17}$$

Consequently, we get

$$\begin{aligned} j^{r_2}\ e^{\varepsilon j}\ \overline{\tau}^p_{(0)r_1}(g_j) &\leq c_5 \sum_{\ell=0}^{r_1} \sup_{t\in\mathbb{R}^+\setminus[-j+1,j-1]} \left|(1+t)^{2+r_2}\ e^{\varepsilon t} \left(\frac{d}{dt}\right)^\ell \mathcal{A}f(t)\right| \\ &\leq c_6 \sum_{\ell=0}^{r_1} \sup_{t\in\mathbb{R}^+} \left|(1+t)^{2+r_2}\ e^{\varepsilon t} \left(\frac{d}{dt}\right)^\ell \mathcal{A}f(t)\right|. \end{aligned} \tag{5.4.18}$$

We now use the commutative diagram given in Lemma 5.4.5 to write $\mathcal{A}f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(-i\lambda)e^{i\lambda t}d\lambda$ and hence from (5.4.18) we get

$$\begin{aligned} j^{r_2}\ e^{\varepsilon j}\ \overline{\tau}^p_{(0)r_1}(g_j) &\leq c_7 \sum_{\ell=0}^{r_1} \sup_{t\in\mathbb{R}^+} \left|(1+t)^{2+r_2}\ e^{\varepsilon t} \int_{\mathbb{R}} (i\lambda)^\ell g(-i\lambda)e^{i\lambda t}d\lambda\right| \\ &\leq c_7 \sum_{\ell=0}^{r_1}\sum_{m=0}^{2+r_2} \binom{2-r_2}{m} \sup_{t\in\mathbb{R}^+} \left|\int_{\mathbb{R}} (i\lambda)^\ell \left(\frac{d}{d\lambda}\right)^m g(-i\lambda)e^{(\varepsilon+i\lambda)t}d\lambda\right| \\ &\leq c_7 \sum_{\ell=0}^{r_1}\sum_{m=0}^{2+r_2} \binom{2-r_2}{m} \sup_{t\in\mathbb{R}^+} \left|\int_{\mathbb{R}} (i\lambda-\varepsilon)^\ell \left(\frac{d}{d\lambda}\right)^m g(\varepsilon-i\lambda)e^{(\varepsilon+i\lambda)t}d\lambda\right| \\ &\leq c_8 \sum_{\ell=0}^{2+r_2} \int_{\mathbb{R}} (1+|\lambda|)^{r_3} \left|\left(\frac{d}{d\lambda}\right)^\ell g(\varepsilon-i\lambda)\right| d\lambda, \end{aligned} \tag{5.4.19}$$

where the third inequality in this chain follows by Cauchy's theorem and the last one by choosing a suitable $r_3 \in \mathbb{Z}^+$ depending on r_1. From (5.4.15) and (5.4.19) it follows that:

$$\sup_{x\in G_{j+1}\setminus G_j} F(x) \leq c_9(1+|n|)^r \sum_{\ell=0}^{2+r_2} \sup_{\lambda\in S^\varepsilon} (1+|\lambda|)^{r_3+2} \left|\left(\frac{d}{d\lambda}\right)^\ell g(\varepsilon-i\lambda)\right|. \tag{5.4.20}$$

The right hand side of (5.4.20) is obviously finite and independent of j. Thus we get the Lemma. ■

Using Proposition 5.4.3, Lemma 5.4.8 and the fact that $\mathcal{J}^p_{B;m,n}$ is a continuous map from the space $\mathcal{S}^p_{B;m,n}(\widehat{G})$ into $\mathcal{S}^p_{m,n}(G)$ we conclude: for $|m| = |n|$, the Fourier transform $(\mathcal{F}^{m,n}_H, \mathcal{F}^{m,n}_B)$ is a topological isomorphism between the spaces $\mathcal{S}^p_{m,n}(G)$ and $\mathcal{S}^p_{m,n}(\widehat{G})$.

Remark 5.4.9. *We should note that we have not included $p = 1$ case in the above section. This restriction can also be removed in some choice of m and n. To make it precise we have to recall the general inversion formula on $\mathcal{S}^p_{H;m,n}(\widehat{G})$, which is given by:*

$$\mathcal{J}^p_{m,n}(h)(x) = \left(\frac{1}{2\pi}\right)^2 \int_{i\mathbb{R}} h(\lambda)\Phi^{n,m}_{\sigma,\lambda}(x)\mu(\sigma,\lambda)d\lambda + \sum_{k\in L^{m,n}_\sigma(\varepsilon)^c} h(k)\Phi^{n,m}_{\sigma,k}(x)|k|, \quad (5.4.21)$$

where, $\mu(\sigma,\lambda)$ and $L^{m,n}_\sigma(\varepsilon)$ are as defined earlier and $L^{m,n}_\sigma(\varepsilon)^c = \{k \in L^{m,n}_\sigma \mid |k| > \varepsilon\}$.

(i) If m and n are of opposite signs then by the definition $L^{m,n}_\sigma$ is an empty set. Hence in this case (5.4.21) will reduce to (5.4.3) for all $0 < p \leq 2$. Furthermore if we impose the condition $|m| = |n|$, then Lemma 5.4.8 holds for all p in $0 < p \leq 2$.

(ii) If we take $m, n \in \mathbb{Z}^{\sigma_-}$, for which both m and n are odd integers. Then members of $L^{m,n}_\sigma$ are all even integers. Hence in this situation for all $1 \leq p \leq 2$, $L^{m,n}_\sigma(\varepsilon)^c$ is empty. Therefore, (5.4.21) reduces to (5.4.3). Similarly as above, for $m, n \in \mathbb{Z}^{\sigma_-}$ and $m = n$ the Lemma 5.4.8 holds for all $1 \leq p \leq 2$.

Note that the assumption $|m| = |n|$ in this section has been used only to conclude that the functions in $\mathcal{S}^p_{H;m,n}(\widehat{G})$ are either purely even or purely odd. This restriction will be removed in the next section. Our next aim is to extend this result for general (m, n) case.

5.4.2 $|m| \neq |n|$ case.

As proposed in Sectio 5.2, we confine ourselves to the case $|m| > |n|$. Analogous results for $|m| < |n|$ will have a similar proof.

Proposition 5.4.10. *For $1 < p \leq 2$, $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^\sigma$ with $|m| > |n|$, the space $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$ (respectively $\mathcal{S}^p_{H;-|n|,|n|}(\widehat{G})$) is topologically isomorphic to the Schwartz space $\mathcal{S}^p_{H;m,n}(\widehat{G})$ if $m \geq 0$ (respectively, if $m < 0$).*

Proof. When n is even or a non-negative integer:
We suppose $m \geq 0$. Let $\psi \in \mathcal{S}^p_{H;|n|,n}(\widehat{G})$. We define the map

$$\psi(\lambda) \mapsto P_{m,n}(-\lambda)\psi(\lambda) = \xi(\lambda) \quad \lambda \in S^\varepsilon. \tag{5.4.22}$$

We show that $\xi \in \mathcal{S}^p_{H;m,n}(\widehat{G})$ by verifying conditions $(i) - (iv)$ of the Definition 5.4.2. The conditions (i), (iii) and (iv) present no difficulty. For (ii) what we need is $\xi(-\lambda) = \phi^{n,m}_{\sigma,\lambda}\xi(\lambda)$, $\lambda \in S^\varepsilon$.
We have $\psi(-\lambda) = (-1)^{\frac{|n|-n}{2}}\psi(\lambda)$ as $\psi \in \mathcal{S}^p_{H;|n|,n}(\widehat{G})$. Thus

$$\begin{aligned}
\xi(-\lambda) &= P_{m,n}(\lambda)\psi(-\lambda) \\
&= P_{m,n}(\lambda)(-1)^{\frac{|n|-n}{2}}\psi(\lambda) \\
&= \frac{(m-1+\lambda)\cdots(|n|+1+\lambda)}{(m-1-\lambda)\cdots(|n|+1-\lambda)}P_{m,n}(-\lambda)\psi(\lambda), \quad \text{as } \frac{|n|-n}{2} \text{ is even} \\
&= \phi^{n,m}_{\sigma,\lambda}\xi(\lambda).
\end{aligned} \tag{5.4.23}$$

As to condition (iv), since $\min(|m|,|n|) = |n|$, the condition is the same for $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$ and $\mathcal{S}^p_{H;m,n}(\widehat{G})$. We denote by $\overline{\tau}^p_{(m,n)}$ and $\overline{\tau}^p_{(|n|,n)}$ respectively for the seminorms on the Schwartz spaces $\mathcal{S}^p_{H;m,n}(\widehat{G})$ and $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$.

$$\begin{aligned}
\overline{\tau}^p_{(m,n)_{r_1,r_2}}(\xi) &= \sup_{\lambda \in IntS^\varepsilon}\left|\left(\frac{d}{d\lambda}\right)^{r_1}\xi(\lambda)\right|(1+|\lambda|)^{r_2} \\
&= \sup_{\lambda \in IntS^\varepsilon}\left|\left(\frac{d}{d\lambda}\right)^{r_1}\psi(\lambda)P_{m,n}(-\lambda)\right|(1+|\lambda|)^{r_2} \\
&\leq c \sup_{\lambda \in IntS^\varepsilon}\left|\left(\frac{d}{d\lambda}\right)^{r_1}\psi(\lambda)\right|(1+|\lambda|)^{r_{m,n}} \\
&= c\, \overline{\tau}^p_{(|n|,n)_{r_1,r_{m,n}}}(\psi).
\end{aligned} \tag{5.4.24}$$

Thus $\xi \in \mathcal{S}^p_{H;m,n}(\widehat{G})$ and so our map (5.4.22) is continuous from $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$ into $\mathcal{S}^p_{H;m,n}(\widehat{G})$. What remains is to show that the inversion map $\xi(\lambda) \mapsto (P_{m,n}(-\lambda))^{-1}\xi(\lambda) = \psi(\lambda)$, $\lambda \in S^\varepsilon$ is a continuous map from $\mathcal{S}^p_{H;m,n}(\widehat{G})$ into $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$.
At this point we note that the zeros of the polynomial $P_{m,n}(\lambda)$ are lying outside of the closed strip S^ε, hence there exists a positive constant $d(m,n)$ such that $|P_{m,n}(\lambda)| \geq d(m,n)$ for all $\lambda \in S^\varepsilon$. By an argument, similar to (5.4.23), one can show that $\psi \in \mathcal{S}^p_{H;|n|,n}(\widehat{G})$.

To show that $\xi \mapsto \psi$ is continuous, one has to show that for each $r_1, r_2 \in \mathbb{Z}^+$ there exists $s(m,n), r(m,n) \in \mathbb{Z}^+$ such that

$$\overline{\tau}^p_{(|n|,n)_{r_1,r_2}}(\psi) \leq c\overline{\tau}^p_{(m,n)_{s(m,n),r(m,n)}}(\xi).$$

This follows easily from the fact that the polynomial $|P_{m,n}(-\lambda)|$ admits a lower bound in the strip $\lambda \in S^\varepsilon$.

When n is a negative odd integer:

In this case we take the map

$$\psi(\lambda) \to P_{m,n}(-\lambda)\lambda\psi(\lambda), \text{ where } \psi \in \mathcal{S}^p_{H;|n|,n}(\widehat{G}), \lambda \in S^\varepsilon. \tag{5.4.25}$$

For showing $\xi \in \mathcal{S}^P_{H;m,n}(\widehat{G})$ we verify conditions of the Definition 5.4.2. Again, it is easy to verify condition (i), (ii) and (v). for (iii), we first note that $\psi \in \mathcal{S}^p_{H;|n|,n}(\widehat{G})$ and n being a negative odd integer $\psi(-\lambda) = -\psi(\lambda)$. Thus,

$$\begin{aligned}
\xi(-\lambda) &= P_{m,n}(\lambda)(-\lambda)\psi(-\lambda) \\
&= P_{m,n}(\lambda)\lambda\psi(\lambda) \\
&= \frac{P_{m,n}(\lambda)}{P_{m,n}(-\lambda)} P_{m,n}(-\lambda)\lambda\psi(\lambda) = \phi^{n,m}_{\sigma,\lambda}\xi(\lambda).
\end{aligned} \tag{5.4.26}$$

As the map (5.4.25) is simply multiplication by a polynomial so it must be continuous from $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$ into $\mathcal{S}^p_{H;m,n}(\widehat{G})$.

Let us now take up the inverse map

$$\xi(\lambda) \to (P_{m,n}(-\lambda)\lambda)^{-1}\xi(\lambda) = \psi(\lambda)(\text{say}), \ \lambda \in S^\varepsilon, \tag{5.4.27}$$

where, $\xi \in \mathcal{S}^p_{H;m,n}(\widehat{G})$. As m and n are of opposite signs so the function ξ vanishes at the point $\lambda = 0$ and hence the map (5.4.27) is well-defined. To show $\psi \in \mathcal{S}^p_{H;|n|,n}(\widehat{G})$ we shall only check condition (iii) of definition 5.4.2, other conditions are easy to verify.

$$\psi(-\lambda) = (P_{m,n}(\lambda)(-\lambda))^{-1}\xi(-\lambda) = \frac{-P_{m,n}(\lambda)}{P_{m,n}(-\lambda)P_{m,n}(\lambda)\lambda}\xi(\lambda) = -\psi(\lambda). \tag{5.4.28}$$

The above relation shows that $\psi(0) = 0$ and also proves condition (v) of the definition 5.4.2. The continuity of the map (5.4.27), from the Schwartz space

$\mathcal{S}^p_{H;m,n}(\widehat{G})$ into $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$, can be established by using the facts (i) $\xi(0) = 0$ and (ii) the polynomial $P_{m,n}(\lambda)$ does not vanish on the interior of the strip S^ε.

The case $m < 0$ is exactly similar to the above. The only difference in this case is the space $\mathcal{S}^p_{H;m,n}(\widehat{G})$ is being identified with the space $\mathcal{S}^p_{H,-|n|,n}(\widehat{G})$. ■

Now we shall prove an analogue of Lemma 5.4.8 for m, n-types.

Lemma 5.4.11. *The inverse Fourier transform $\mathcal{I}^p_{H;m,n}$ is a continuous map from $\mathcal{S}^p_{H;m,n}(\widehat{G})$ into $\mathcal{S}^p_{m,n}(G)$, where, $m, n \in \mathbb{Z}^\sigma$ such that $|m| > |n|$.*

Proof. To prove this Lemma we shall use the results that we have proved in Lemma 5.4.8. Like the $|m| = |n|$ case we shall first prove that the inversion map $\mathcal{I}^p_{H;m,n}$ from $\mathcal{D}_{m,n}(\widehat{G})$ into $\mathcal{D}_{m,n}(G)$ is continuous with respect to the topologies of the corresponding Schwartz spaces containing them. Then a density argument will extend the continuity to the whole Schwartz spaces.

Let us take one $\varphi \in \mathcal{D}_{m,n}(\widehat{G})$. By the Paley-Wiener theorem there is an unique $\Phi \in \mathcal{D}_{m,n}(G)$ such that $\mathcal{F}^{m,n}_H \Phi = \varphi$. Now by Proposition 5.3.7 there exists an unique $\Psi \in \mathcal{D}_{|n|,n}(G)$, such that $\Phi = \mathfrak{D}_{m,n}\Psi$ for some differential operator $\mathfrak{D}_{m,n}$. Let $\psi(\lambda) = \mathcal{F}^{|n|,n}_H \Psi(\lambda)$ then it can be shown that $\varphi(\lambda) = P_{m,n}(-\lambda)\psi(\lambda)$. Hence $\psi \in \mathcal{D}_{|n|,n}(\widehat{G})$ which is dense in $\mathcal{S}^p_{H;|n|,n}(\widehat{G})$. Let $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $t \in \mathbb{Z}^+$.

$$\begin{aligned}
\rho^p_{\mathbf{g}_1,\mathbf{g}_2,t}(\Phi) &= \sup_{x\in G} |\Phi(\mathbf{g}_1; x; \mathbf{g}_2)| \, (1 + |x|)^t \varphi_0^{-\frac{2}{p}}(x) \\
&= \sup_{x\in G} |(\mathfrak{D}_{m,n}\Psi)(\mathbf{g}_1; x; \mathbf{g}_2)| \, (1 + |x|)^t \varphi_0^{-\frac{2}{p}}(x), \qquad (5.4.29)
\end{aligned}$$

by Lemma 5.4.8, we can find $t_1, s \in \mathbb{Z}^+$ and a positive constant c such that

$$\leq c \sup_{\lambda\in S^\varepsilon} \left| \left(\frac{d}{d\lambda}\right)^{t_1} \psi(\lambda) \right| \, (1 + |\lambda|)^s,$$

by the Proposition 5.4.10, we get some $t(m, n), s(m, n) \in \mathbb{Z}^+$ and constant $c(m, n)$, all dependent on m, n such that

$$\leq c(m, n) \; \overline{\tau}^p_{(m,n)_{t(m,n),s(m,n)}}(\varphi). \qquad (5.4.30)$$

This proves the Lemma. ■

Finally Lemma 5.4.8 and Lemma 5.4.11 together proves the main theorem (stated below) of this chapter.

Theorem 5.4.12. *Let $1 < p \leq 2$, $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^\sigma$, then the Fourier transform $(\mathcal{F}_H^{m,n}, \mathcal{F}_B^{m,n})$ is a topological isomorphism between the Schwartz spaces $\mathcal{S}_{m,n}^p(G)$ and $\mathcal{S}_{m,n}^p(\widehat{G})$.*

As we can see the main key to deal with $|m| \neq |n|$ case is to reduce it to the $|m| = |n|$ case. Hence in the lights of Remark 5.4.9 we can conclude the following:

Remark 5.4.13. *(i) For all $m, n \in \mathbb{Z}^\sigma$ or $\mathbb{Z}^{\sigma-}$ with $mn < 0$, this proof of Schwartz space isomorphism between the spaces $\mathcal{S}_{m,n}^p(G)$ and $\mathcal{S}_{m,n}^p(\widehat{G})$ for $0 < p \leq 2$.*

(ii) For all $m, n \in \mathbb{Z}^{\sigma-}$, the above proof can be extended for the L^p-Schwartz spaces with $1 \leq p \leq 2$.

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