Intersection numbers, Embedded spheres and Geosphere laminations for free groups

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CONTENTS

0. PREFACE

Topological and geometric methods have played a major role in the study of infinite groups since the time of Poincaré and Klein, with the work of Nielsen, Dehn, Stallings and Gromov showing particularly deep connections with the topology of surfaces and three-manifolds. This is in part because a surface or a 3-manifold is essentially determined by its fundamental group, and has a geometric structure due to the Poincaré-Köbe-Klein uniformisation theorem for surfaces and Thurston's geometrisation conjecture, which is now a theorem of Perelman, for 3-manifolds.

A particularly fruitful instance of such an interplay is the relation between intersection numbers of simple curves on a surface and the hyperbolic geometry and topology of the surface. This has reached its climax in the classification of finitely generated Kleinian groups by Yair Minsky and his collaborators, who along the way developed a deep understanding of the geometry of the *curve complex.*

Free (nonabelian) groups and the group of their outer automorphisms have been extensively studied in analogy with (fundamental groups of) surfaces and the mapping class groups of surfaces.

In my thesis, we study the analogue of intersection numbers of simple curves, namely the Scott-Swarup *algebraic intersection number* of splittings of a free group and we also study embedded spheres in 3-manifold of the form $M = \sharp_n S^2 \times S^1$. The fundamental group of M is a free group of rank n. This 3-manifold will be our model for free groups. We construct geosphere laminations in free group which are analogues of geodesic laminations on a surface.

<u>Chapter 1</u> In this chapter, we introduce basic concepts related to free product, free groups and splittings of groups.

<u>Chapter 2</u> In this chapter, we study geometric intersection number of simple closed curves on a surface. In particular, we see its applications to study geometric properties of curve complex of the surface. We also study topological properties of curve complex. We shall see how curve complex is used to study mapping class group of surfaces. The geometric intersection number has been used to study Thurston's compactification of Teichmüller space of surface and the boundary of Teichmüller space, namely the space of projectivized measured laminations. At the end of this chapter, we study its analogue sphere complex of a 3-manifold and its topological properties.

<u>Chapter 3</u> In this chapter, we study the model 3-manifold $M = \sharp_k S^2 \times S^1$. We also see how a partition of ends of the space $\widetilde{M}$, the universal cover of M, corresponds to an embedded spheres in $\widetilde{M}$. We also discuss the intersection number of a proper path in $\widetilde{M}$ with a homology class in $H_2(\widetilde{M})$. At the end of this chapter, we see how embedded spheres in M correspond to splittings of the fundamental group of M.

<u>Chapter 4</u> Scott and Swarup [39] introduced an algebraic analogue, called the *algebraic intersection number*, for a pair of *splittings* of groups. This is based on the associated partition of the ends of a group [42]. Splittings of groups are the natural analogue of simple closed curves on a surface F – splittings of $\pi_1(F)$ corresponding to homotopy classes of simple closed curves on F. Scott and Swarup showed that, in the case of surfaces, the algebraic and geometric intersection numbers coincide.

Embedded spheres in M correspond to splittings of the free group. Hence, given a pair of embedded spheres in M, we can consider their *geometric intersection number* as well as the *algebraic intersection number* of Scott and Swarup for the corresponding splittings. Our main result is that, for embedded spheres in M these two intersection numbers coincide. The principal method we use is the normal form for embedded spheres developed by Hatcher. The results in this chapter are the outcome of joint work with my adviser Siddhartha Gadgil.

<u>Chapter 5</u> In this chapter, we study embedded spheres in $M = \sharp_k S^2 \times S^1$ and $\widetilde{M}$, the universal cover of M. In the Section 5.1, we see how a partition A of the set of ends of $\widetilde{M}$ corresponds to an embedded sphere in $\widetilde{M}$ which is in normal form in the sense of Hatcher, by specifying the data determining the partition A and the normal sphere. Given a properly embedded path $c : \mathbb{R} \to \widetilde{M}$ and a homology class $A \in H_2(\widetilde{M})$, we have an intersection number $c \cdot A$. Further, this depends only on the ends $c_\pm$ of the path c. In the Section 5.2, we prove that the class $A \in H_2(\widetilde{M})$ can be represented by an embedded sphere in $\widetilde{M}$ if and only if, for each proper map $c : \mathbb{R} \to \widetilde{M}$, $c \cdot A \in \{0, 1, -1\}$. We also constructively prove that the class $A \in \pi_2(M)$ can be represented by an embedded sphere in M if and only if A can be represented by an embedded sphere in $\widetilde{M}$ and for all deck transformations $g \in \pi_1(M)$, A and gA do not cross. The results in this chapter are the outcome of joint work with my adviser Siddhartha Gadgil.

<u>Chapter 6</u> Geodesic laminations (and measured laminations) on surfaces have proved to be very fruitful in three-manifold topology, Teichmüller theory and related areas. In this chapter, we construct analogously *geosphere laminations* for free groups. They have the same relation to (disjoint unions of) embedded spheres in the connected sum $M = \sharp_n S^2 \times S^1$ of n copies of $S^2 \times S^1$ as geodesic laminations on surfaces have to (disjoint unions of) simple closed curves on surfaces. The manifold M has fundamental group the free group on n generators, and is a natural model for the study of free groups.

Laminations for groups (including free groups) have been constructed and studied in various contexts. However, they are one-dimensional objects, corresponding to geodesics. We study here objects of codimension one, which correspond to splittings. In the case of surfaces, dimension one and codimension one coincide. Our main result is a compactness theorem for the space of (non-trivial) geosphere laminations. We also show that embedded spheres in M are geosphere laminations. Hence sequences of spheres, in particular under iterations of an outer automorphism of the free group, have subsequences converging to geosphere laminations. It is such limiting constructions that make geodesic laminations for surfaces a very useful construction.

Our construction is based on the normal form for disjoint unions of spheres in M due to Hatcher. The normal form is relative to a decomposition of M with respect to a maximal collection of spheres in M. This is in many respects analogous to a normal form with respect to an ideal triangulation of a punctured surface. In particular, isotopy for spheres in normal form implies normal isotopy, i.e., the normal form is unique. As in the case of normal curves on surfaces and normal surfaces in three-manifolds, we can associate the number of pieces of each type to a collection of spheres in Hatcher's normal form. However, these numbers do not determine the (collection of) spheres up to isotopy. We instead proceed by considering lifts of normal spheres to the universal cover $\widetilde{M}$ of M. In the universal cover $\widetilde{M}$, a normal sphere is determined by a *finite* subtree τ of a tree T associated to $\widetilde{M}$ together with some additional data. We construct geospheres in $\widetilde{M}$ by dropping the finiteness condition. We construct an appropriate topology on the space of geospheres and show that the space is locally compact and totally disconnected. The lift of a normal sphere in M to its universal cover satisfies an additional condition, namely it is disjoint from all its translates. This can be reformulated in terms of the notion of *crossing* of spheres in $\widetilde{M}$, following Scott-Swarup, which depends on the corresponding partition of ends of $\widetilde{M}$. We show that there is an appropriate notion of crossing for geospheres, which is defined in terms of the appropriate partition of ends (into three sets in this case). Our main technical result is that crossing is an open condition. We recall that this is the case for crossing of geodesics in hyperbolic space, and that this plays a central role in the study of geodesic laminations. The proof of compactness of the space of geospheres uses the result that crossing is open. The construction based on normal forms is not intrinsic, as it depends on the maximal collection of spheres with respect to which M is decomposed. However, we show that geospheres can be described in terms of their associated partitions. This gives an intrinsic definition. The results in this chapter are the outcome of joint work with my adviser Siddhartha Gadgil.

CHAPTER 7 In this chapter, we discuss the natural questions arising out of this thesis and further directions for research.

1. FREE PRODUCTS, FREE GROUPS AND SPLITTINGS OF GROUPS

In this chapter, we introduce basic concepts related to free products, free groups and splittings of groups.

1.1 Free Products of Groups

We shall see the concept of the free product of groups. For more details, see [38].

Let G be a group. If $\{G_\alpha\}_{\alpha \in J}$ is a family of subgroups of G, we say that these groups generate G if every element x of G can be written as a finite product of elements of the groups G_α. This means that there is a finite sequence $(x_1, \ldots, x_n)$ of elements of G_α such that $x = x_1 \cdots x_n$. Such a sequence is called a word of length n in groups G_α; it is said to represent the element x of G. As we lack commutativity, we can not rearrange the factors in the expression for x so as to group together factors that belong to a single one of the groups G_α. However, if in the expression for x, x_i and x_{i+1} both belong to the same group G_α, we can group them together, thereby obtaining the word $(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n)$ of length $n-1$, which also represents x. Furthermore, if any x_i equals 1, we can delete x_i from the sequence, again obtaining a shorter word that represents x.

Applying these reduction operations repeatedly, one can in general obtain a word representing x of the form $(y_1, \ldots, y_m)$, where no group G_α contains both y_i and y_{i+1}, and $y_i \neq 1$, for all i. Such a word is called reduced word. This discussion does not apply, however, if x is the identity element of G. For, in that case, one might represent x by a word such as (a, a^{-1}), which reduces successively to the word (aa^{-1}) of length 1, and then disappear altogether. Accordingly, we make the convention that the empty set is considered to be reduced word of length zero that represents the identity element of G. With this convention, it is true that if the groups G_α generate G, then every element of G can be represented by a reduced word in the elements of group G_α. If $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ are words representing x and y, respectively, then $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a word representing xy. Even if two words are reduced words, however, the third will not be a reduced word unless none of the groups contains both x_n and y_1.

Definition 1.1.1. Let G be a group, let $\{G_\alpha\}_{\alpha \in J}$ be a family of subgroups of G that generates G. Suppose that $G_\alpha \cap G_\beta$ consists of identity alone whenever $\alpha \neq \beta$. We say that G is the free product of the groups G_α if for each $x \in G$, there is only one reduced word in the groups G_α that represents x. In this case, we write $G = *_{\alpha \in J} G_\alpha$ or in the finite case, $G = G_1 * \cdots * G_n$.

The free product satisfies an extension condition:

Proposition 1.1.2. *Let G be a group, let $\{G_\alpha\}$ be a family of subgroups of G. If G is the free product of the groups G_α, then G satisfies the following condition:*

Given any group H and any family of homomorphisms $h_\alpha : G_\alpha \to H$, there exists a homomorphism $h : G \to H$ whose restriction to G_α equals h_α, for each α.

Furthermore, h is unique.

For proof, see [38, Lemma 68.1].

We now consider the problem of taking an arbitrary family of groups $\{G_\alpha\}$ and finding a group G that contains subgroup G'_α isomorphic to the groups G_α, such that G is free product of the groups G'_α.

Definition 1.1.3. Let $\{G_\alpha\}$ be an indexed family of groups. Suppose that G is a group and that $i_\alpha : G_\alpha \to G$ is a family of monomorphisms, such that G is the free product of the groups $i_\alpha(G_\alpha)$. Then, we say that G is the external free product of the groups G_α, relative to the monomorphisms i_α.

The group G is not unique. We shall see later that it is unique up to isomorphism. Now, we shall see a construction of G.

Theorem 1.1.4. *Given a family $\{G_\alpha\}_{\alpha \in J}$ of groups, there exists a group G and a family of monomorphisms $i_\alpha : G_\alpha \to G$ such that G is the free product of the groups $i_\alpha(G\alpha)$.*

We can assume that the groups G_α are disjoint as sets. Then as before, we define a word (of length n) in the elements of the groups G_α to be an n-tuple $w = (x_1, \ldots, x_n)$ of elements of $\cup G_\alpha$. It is called a reduced word if $\alpha_i \neq \alpha_{i+1}$, for all i, where α_i is the index such that $x_i \in G_\alpha$, and if for each i, x_i is not the identity element of G_{α_i}. We define the empty set to be the unique reduced word of length zero. We denote the element w as $w = x_1 \cdots x_n$.

Let W denote the set of all reduced words in the elements of the groups G_α. We define the group operation in W as juxtaposition,

$$(x_1 \cdots x_n)(y_1 \cdots y_m) = x_1 \cdots x_n y_1 \cdots y_m.$$

This product may not be reduced, however: if x_n and y_1 belong to the the same G_α, then they should be combined into single letter $(x_n y_1)$ according to the multiplication in G_α and if this new letter $x_n y_1$ happens to be the identity of G_α, then it should be canceled from the product. This may allow x_{n-1} and y_2 to be combined, and possibly canceled too. Repetition of this process eventually produces a reduced word. For example, in the product $(x_1 \cdots x_m)(x_m^{-1} \cdots x_1^{-1})$ everything cancels and we get the identity element of W, the empty word. One can easily see that W with this group operation forms a group. For detailed proof of this, see [38, Theorem 68.2]. We denote $W = G = *_\alpha G_\alpha$. Each group G_α is naturally identified with a subgroup of G, the subgroup consisting of the empty word and the nonidentity one-letter word $x \in G_\alpha$. From this point of view, the empty word is the common identity element for all the subgroups G_α, which are otherwise disjoint. Thus, we can easily see that we get a family of monomorphisms $i_\alpha : G_\alpha \to G$ such that G is the free product of the groups $i_\alpha(G\alpha)$.

The extension condition for ordinary free products translates immediately into an extension condition for external free product. For proof, see [38, Lemma 68.3].

Lemma 1.1.5. *Let $\{G_\alpha\}$ be a family of groups; let G be a group; let $i_\alpha : G_\alpha \to G$ be a family of homomorphisms. If each i_α is a monomorphism and G is the free product of the groups $i_\alpha(G_\alpha)$, then G satisfies the following condition:*

Given a group H and a family of homomorphisms $h_\alpha : G_\alpha \to H$, there exists a homomorphism $h : G \to H$ such that $h \circ i_\alpha = h_\alpha$ for each α.

Furthermore, h is unique.

An immediate consequence is a uniqueness theorem for (external) free products:

Theorem 1.1.6. *Let $\{G_\alpha\}$ be a family of groups. Suppose G and G' are groups and $i_\alpha : G_\alpha \to G$ and $i'_\alpha : G_\alpha \to G'$ are families of monomorphisms, such that the families $\{i_\alpha(G_\alpha)\}$ and $\{i'_\alpha(G_\alpha)\}$ generate G and G', respectively. If both G and G' have the extension property stated in the preceding lemma, then there is a unique isomorphism $\phi' : G \to G'$ such that $\phi' \circ i_\alpha = i'_\alpha$, for all α.*

For proof, see [38, Theorem 68.4].

Now, we state the following result which shows that the extension condition characterizes free products:

Theorem 1.1.7. *Let $\{G_\alpha\}$ be a family of groups; let G be a group; let $i_\alpha : G_\alpha \to G$ be a family of homomorphisms. If the extension condition of the Lemma 1.1.5 holds, then each i_α is a monomorphism and G is the free product of the groups $i_\alpha(G_\alpha)$.*

For detailed proof, see [38, Lemma 68.5].

1.2 Free Groups

Let G be a group; let $\{a_\alpha\}$ be a family of elements of G, for $\alpha \in J$, where J is some index set. We say that the elements $\{a_\alpha\}$ generate G if every element of G can be written as a product of powers of the elements a_α. If the family $\{a_\alpha\}$ is finite, we say G is finitely generated.

Definition 1.2.1. Let $\{a_\alpha\}$ be a family of elements of a group G. Suppose each a_α generates an infinite cyclic subgroup G_α of G. If G is the free product of the groups $\{G_\alpha\}$, then G is said to be a free group, and the family $\{a_\alpha\}$ is called a system of free generators for G.

In this case, for each element x of G, there is a unique reduced word in the elements of the groups G_α that represents x. This says that if $x \neq 1$, then x can be written uniquely in the form $x = (a_{\alpha_1}^{n_1}) \cdots (a_{\alpha_k}^{n_k})$, where $\alpha_i \neq \alpha_{i+1}$ and $n_i \neq 0$, for each i. The integers n_i may be negative.

Free groups are characterized by the following extension property:

Lemma 1.2.2. *Let G be a group; let $\{a_\alpha\}$ be a family of elements of G. If G is a free group with system of free generators $\{a_\alpha\}$, then G satisfies the following condition:*

Given any group H and any family $\{y_\alpha\}$ of elements of H, there is a homomorphism $h : G \to H$ such that $h_\alpha(a_\alpha) = y_\alpha$ for each α.

Furthermore, h is unique. Conversely, if the above extension condition holds, then G is a free group with system of free generators $\{a_\alpha\}$.

For the proof see [38, Lemma 68.1].

In other words, a free group is the free product of any number of copies of $\mathbb{Z}$, finite or infinite, where $\mathbb{Z}$ is the group of integers. The elements of a free group are uniquely representable as reduced words in the powers of generators of the various copies $\mathbb{Z}$, with one generator of each $\mathbb{Z}$. These generators are called basis for the free group, and the number of basis elements is the rank of the free group. The abelianization of a free group is the a free abelian group with basis the same set of generators (images in the abelianization), so since the rank of a free abelian group is well defined, independent of the choice of basis, the same is true for the rank of a free group. For details, see [38, section 69].

An example of a free product that is not a free group is $\mathbb{Z}_2 * \mathbb{Z}_2$.

We have the following result for subgroups of a free group.

Proposition 1.2.3. *Every subgroup of a free group is free.*

For proof, see [38, Theorem 85.1].

1.3 Presentation of a group

One method of defining a group is by a presentation. One specifies a set S of generators so that every element of the group can be written as a product of some of these generators, and a set R of relations among those generators. We then say G has presentation $\langle S \mid R\rangle$.

Informally, G has the above presentation if it is the "freest group" generated by S subject only to the relations R. Formally, the group G is said to have the above presentation if it is isomorphic to the quotient of a free group on S by the normal subgroup generated by the relations R.

As a simple example, the cyclic group of order n has the presentation $\langle a \mid a^n = 1\rangle$, where 1 is the group identity. This may be written equivalently as $\langle a \mid a^n\rangle$, since terms that don't include an equals sign are taken to be equal to the group identity.

Every group G has a presentation. To see this consider the free group $\langle G\rangle$ on G. Since G clearly generates itself, one should be able to obtain it by a quotient of $\langle G\rangle$. Indeed, by the universal property of free groups, there exists a unique group homomorphism $\phi : \langle G\rangle \to G$ which covers the identity map. Let K be the kernel of this homomorphism. Then, G clearly has the presentation $\langle G \mid K\rangle$.

Every finite group has a finite presentation, in fact, many different presentations.

A presentation is said to be finitely generated if S is finite and finitely related if R is finite. If both are finite it is said to be a finite presentation. A group is finitely generated (respectively, finitely related, finitely presented) if it has a presentation that is finitely generated (respectively, finitely related, a finite presented).

Some more examples of group presentations include the following.

1. The presentation $\langle x, y \mid x^2 = 1, y^n = 1, (xy)^n = 1\rangle$ defines a group, isomorphic to the dihedral group D_n of finite order $2n$, which is the group of symmetries of a regular n-gon.

2. The fundamental group of a surface of genus g has the presentation:
$\langle x_1, y_1, x_2, ..., x_g, y_g \mid [x_1, y_1][x_2, y_2]...[x_g, y_g] = 1\rangle$.

1.4 Amalgamated Free products and HNN-Extension

Free products of groups are generalized by a notion of amalgamated products of groups joined together along specified subgroups. For the sake of concreteness, we will carry out this construction for an amalgamated product of two groups. Suppose, we have two groups G_1 and G_2 and homomorphisms $f_1 : H \to G_1$ and $f_2 : H \to G_2$. We define:

Definition 1.4.1. The amalgamated product $G_1 *_H G_2$ is defined as follows: let N be the normal subgroup of $G_1 * G_2$ generated by elements of the form $f_1(h)(f_2(h))^{-1}$ for $h \in H$; then

$$G_1 *_H G_2 := (G_1 * G_2)/N.$$

Note that $G_1 * G_2$ can be expressed as the special case of the amalgamated product where H is trivial. The amalgamated product satisfies a natural universal property generalizing the one for the free product:

Proposition 1.4.2. *For a group G', write $Hom(G_1, G') \times_H Hom(G_2, G')$ for $\{(g_1, g_2) \in Hom(G_1, G') \times Hom(G_2, G') : f_1 \circ g_1 = f_2 \circ g_2\}$. Then, the natural map induced by composition with $G_1 \to G_1 *_H G_2$ and $G_2 \to G_1 *_H G_2$ induces a bijection $Hom(G_1 *_H G_2, G') \to Hom(G_1, G') \times_H Hom(G_2, G')$.*

For a proof, see [41].

The amalgamated product also arises naturally in topology: the fundamental group of the gluing of two topological spaces along given subspaces is the amalgamated product of the fundamental groups of the two spaces, over the fundamental group of the subspaces being glued.

Definition 1.4.3. Let G be a group with presentation $G = \langle S \mid R\rangle$, and let α be an isomorphism between two subgroups H and K of G. Let t be a new symbol not in S, and define

$G*_\alpha = \langle S, t \mid R, tht^{-1} = \alpha(h), \forall h \in H\rangle$

The group $G*_\alpha$ is called the HNN- extension of G relative to α. The original group G is called the base group for the construction, while the subgroups H and K are the associated subgroups. The new generator t is called the stable letter. Sometimes, we also write $G*_H$ for $G*_\alpha$.

Since the presentation for $G*_\alpha$ contains all the generators and relations from the presentation for G, there is a natural homomorphism, induced by the identification of generators, which takes G to $G*_\alpha$.

Higman, Neumann and Neumann proved that this homomorphism is injective, that is, an embedding of G into $G*_\alpha$. A consequence is that two isomorphic subgroups of a given group are always conjugate in some over group; the desire to show this was the original motivation for the construction. In terms of the fundamental group in algebraic topology, the HNN- extension is the construction required to understand the fundamental group of a topological space X that has been 'glued back' on itself by a mapping f.

1.5 Graph of groups

We now introduce the terminology, due to Serre, of a graph of groups. A graph Γ is a 1-dimensional CW-complex, so that a it may contain a loop, i.e., an edge with its two endpoints identified. This gives rise to difficulties with orientations of such an edge. In order to avoid these difficulties, we first introduce the idea of an abstract graph. Essentially this has twice many edges as Γ, one for each orientation of an edge of Γ.

Definition 1.5.1. An abstract graph Γ consists of two sets $E(\Gamma)$ and $V(\Gamma)$ called the edges and vertices of Γ, an involution on $E(\Gamma)$ which sends e to $\bar{e}$, where $\bar{e} \neq e$ and a map $\partial_0 : E(\Gamma) \to V(\Gamma)$.

We define $\partial_1 e = \partial_0 \bar{e}$ and say that e joins $\partial_0 e$ to $\partial_1 e$.

An abstract graph Γ has an obvious geometric realization $|\Gamma|$ with vertices $V(\Gamma)$ and edges corresponding to pairs $(e, \bar{e})$. When we say that Γ is connected or has some topological property, we shall mean that the realization of Γ has the appropriate property. An orientation of an abstract graph is a choice of one edge out of each pair $(e, \bar{e})$.

A graph of groups consists of an abstract graph Γ together with a function assigning to each vertex v of Γ a group G_v and to each edge e a group G_e, with $G_{\bar{e}} = G_e$, and an injective homomorphism $f_e : G_e \to G_{\partial_0 e}$.

Similarly, we may define a graph χ of topological spaces, or of spaces with preferred base point: here, it is not necessary for the map $X_e \to X_{\partial_0 e}$ to be injective, as we can use the mapping cylinder construction to replace the maps by inclusions and this does not alter the total space defined below. But, we will suppose for the convenience that the spaces are CW-complexes and maps are cellular.

Given a graph χ of spaces, we can define total space χ_Γ as the quotient of $\cup\{X_v : v \in V(\Gamma)\} \cup \{\cup\{X_e \times I : e \in E(\Gamma)\}\}$ by identifications,

$$X_e \times I \to X_{\bar{e}} \times I \text{ by } (x,t) \to (x, 1-t)$$

$$X_e \to X_{\partial_0 e} \text{ by } (x,0) \to f_e(x)$$

.

If χ is a graph of (connected) based spaces, then by taking fundamental groups we obtain a graph Σ of groups (with the same underlying abstract graph Γ). The fundamental group G_Γ of the graph of groups is defined to be the fundamental group of the total space χ_Γ. One can show that G_Γ is independent of the choice of χ. Observe that in the case when Γ has just one pair $(e, \bar{e})$ of edges and two vertices v_1 and v_2, if groups associated to v_1, v_2 and $(e, \bar{e})$ are A, B and C, respectively, the fundamental group G_Γ is $A *_C B$. In the case when Γ has just one pair $(e, \bar{e})$ of edges and one vertex v, if the associated groups are C and A, respectively, then the fundamental group G_Γ is $A*_C$. For more details, see [40].

1.6 Splittings of a group

A group G is said to split over a subgroup H if G is isomorphic to $A*_H$ or to $A *_H B$, with $A \neq H \neq B$. We will need a precise definition of a splitting of G.

Definition 1.6.1. We shall say that a splitting of G consists either of proper subgroups A and B of G and a subgroup H of $A \cap B$ such that the natural map $A *_H B \to G$ is an isomorphism, or it consists of a subgroup A of G and subgroups H_0 and H_1 of A such that there is an element t of G which conjugates H_0 to H_1 and the natural map $A*_H \to G$ is an isomorphism.

If G splits over some subgroup, we say G is splittable. For example, Z is splittable as $Z = \{1\}*_{\{1\}}$.

A collection of n splittings of a group G is compatible if G can be expressed as the fundamental group of graph of groups with n edges, such that, for each i, collapsing all edges but i-th, yields the i-th splitting of G. For more details, see [39].

1.7 Some Important theorems

Two of most important theorems about free products are the theorems of Grushko (1940) and Neumann (1943) and that of Kurosh (1934) [33].

Theorem 1.7.1. *Let F be a free group, and let $\phi : F \to *A_\alpha$. Then, there is a factorization of F as a free product, $F = *F_\alpha$ such that $\phi(F_\alpha) = A_\alpha$.*

It has a following important corollary:

Corollary 1.7.2. *If $G = A_1 * ... * A_n$ and the rank (minimal number of generators) of A_i is r_i, then the rank of G is $r_1 + \cdots + r_n$.*

Theorem 1.7.3. *Let $G = *A_\alpha$, and let H be a subgroup of G. Then, H is a free product, $H = F * (*H_\beta)$, where F is a free group and each H_β is the intersection of H with a conjugate of some factor A_α of G.*

1.8 Kneser conjecture on free products

Now, we shall prove that each splitting of the fundamental group of a 3-manifold as a free product is induced by splitting of the manifold as a connected sum. We need the following definitions:

Definition 1.8.1. The connected sum $M_1 \sharp M_2$ of n-manifolds M_1 and M_2 is formed by deleting the interiors of n-balls B_i^n in M_i^n and attaching the resulting punctured manifolds $M_i - int(B_i)$ to each other by a homeomorphism $h : \partial B_2 \to \partial B_1$, so $M_1 \sharp M_2 = (M_1 - int(B_1)) \cup_h (M_2 - int(B_2))$.

The n-balls B_i is required to be interior to M_i and ∂B_i bicollared in M_i to ensure that the connected sum is a manifold.

An incompressible surface, heuristically, is a surface, embedded in a 3-manifold, which has been simplified as much as possible while remaining "nontrivial" inside the 3-manifold.

Definition 1.8.2. Suppose that S is a compact surface properly embedded in a 3-manifold M. Suppose that D is a disk, also embedded in M, with $D \cap S = \partial D$.

Suppose that the curve ∂D in S does not bound a disk inside of S. Then, D is called a compressing disk for S and we also call S a compressible surface in M. If no such disk exists and S is not the 2-sphere, then we call S incompressible (or geometrically incompressible).

There is also an algebraic version of incompressibility: Suppose $\iota : S \to M$ is a proper embedding of a compact surface. Then, S is π_1-injective (or algebraically incompressible) if the induced map on fundamental groups $\iota_\star : \pi_1(S) \to \pi_1(M)$ is injective. The loop theorem then implies that a two-sided, properly embedded, compact surface (not a 2-sphere) is incompressible if and only if it is π_1-injective.

An incompressible sphere is a 2-sphere in a 3-manifold that does not bound a 3-ball. Thus, such a sphere either does not separate the 3-manifold or gives a nontrivial connected sum decomposition. Since this notion of incompressibility for a sphere is quite different from the above definition for surfaces, often an incompressible sphere is instead referred to as an essential sphere or reducing sphere.

Definition 1.8.3. For a 3-manifold M and a space X, we say that two maps $f, g : M \to X$ are *C-equivalent* if there are maps $f = f_0, ..., f_n = g$ of M to X with either f_i homotopic to f_{i-1} or f_i agreeing with f_{i-1} on $M - B$ for homotopy 3-cell $B \subset M$ with $B \cap \partial M$ empty or a 2-cell.

If $\pi_3(X) = 0$, C-equivalent maps are homotopic. In any case, C-equivalent maps induce the same homomorphism $\pi_1(M) \to \pi_1(X)$ up to choices of base point and inner automorphisms. Now, we see the following theorem from [25].

Theorem 1.8.4. *Let M be a compact 3-manifold such that each component of ∂M (possibly empty) is incompressible in M. If $\pi_1(M) \cong G_1 * G_2$, then $M = M_1 \sharp M_2$, where $\pi_1(M_i) \cong G_i$, for $i = 1, 2$.*

Proof. Choose complexes X_1 and X_2 with $\pi_1(X_i) \cong G_i$ and $\pi_2(X_i) = 0$. Join a point of X_1 to a point of X_2 by a 1-simplex A to form a complex $X = X_1 \cup A \cup X_2$. Note that $\pi_1(X) \cong G_1 * G_2$ and $\pi_2(X) = 0$. Thus, we can construct a map $f : M \to X$ such that $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism (which can be preassigned). Choose $x_0 \in int(A)$. We may assume that each component of $f^{-1}(x_0)$ is a 2-sided incompressible surface properly embedded in M. If F is a component of $f^{-1}(x_0)$, then since $\ker(\pi_1(F) \to \pi_1(M)) = 1$, f_* is injective, and $f(F) = x_0$, we must have $\pi_1(F) = 1$. If some component F of $f^{-1}(x_0)$

is a (incompressible) 2-cell, then by hypothesis ∂F bounds a 2-cell $D \subset \partial M$. The 2-sphere $F \cup D$ can be pushed slightly into $int(M)$ to obtain an incompressible 2-sphere F'. Since, $\pi_2(X_i) = 0$, f can be modified by a C-equivalence, to a map which replaces F by F' as a component of the inverse of x_0. By this reasoning, we may now assume that each component of $f^{-1}(x_0)$ is an incompressible 2-sphere in $int(M)$. If $f^{-1}(x_0)$ is connected, we are done. If not, there is a path $\beta : I \to M$ such that $\beta(0)$ and $\beta(1)$ lie in different components of $f^{-1}(x_0)$. Now, $f \circ \beta$ is a loop in X and since f_* is surjective, there is a loop γ based at $\beta(1)$ such that $[f \circ \gamma] = [f \circ \beta]^{-1}$. Then, $\alpha = \beta\gamma$ is a path satisfying

1. $\alpha(0)$ and $\alpha(1)$ are in different components of $f^{-1}(x_0)$,
2. $[f \circ \alpha] = 1 \in \pi_1(X)$.

We may assume that α is a simple path which crosses $f^{-1}(x_0)$ transversely at each point of $\alpha(int(I)$. Of all such paths satisfying the above conditions, we assume that $\sharp(\alpha^{-1}(f^{-1}(x_0)))$ is minimal. We must have $\alpha(int(I)) \cap f^{-1}(x_0) = \emptyset$. For if not, we can write $\alpha = \alpha_1\alpha_2 \cdots \alpha_k \quad (k \geq 2)$ where for each i, $\alpha_i(int(I) \cap f^{-1}(x_0) = \emptyset$ and $\alpha_i(\partial I) \subset f^{-1}(x_0)$. Then, $[f \circ \alpha_1][f \circ \alpha_2] \cdots [f \circ \alpha_k]$ is a representation of the identity element as an alternating product in the free product $G_1 * G_2$. Thus, for some i, $[f \circ \alpha_i] = 1$. If $\alpha_i(0)$ and $\alpha_i(1)$ lie in the same component of $f^{-1}(x_0)$, we could reduce $\sharp\alpha^{-1}(f^{-1}(x_0))$. If not, we contradict our minimality assumption. Thus, we have $\alpha(int(I)) \cap f^{-1}(x_0) = \emptyset$. Let F_j $(j = 0, 1)$ be the component of $f^{-1}(x_0)$ containing $\alpha(j)$. Let C be a small regular neighborhood of $\alpha(I)$ such that $C \cap F_j = D_j$ is a spanning 2-cell of C and $C \cap f^{-1}(x_0) = D_0 \cup D_1$. Let B be the annulus in ∂C bounded by $\partial D_0 \cup \partial D_1$. Push $int(B)$ slightly into $int(C)$ to obtain an annulus B' with $\partial B' = \partial B$ and $B \cup B'$ the boundary of a solid torus T. We define a map $f_1 : M \to X$ as follows. Put $f_1|M - int(C) = f|M - int(C)$ and $f_1(B') = x_0$. Since, $[f \circ \alpha] = 1$, we can extend f_1 across a meridional 2-cell E of T. Now, it remains to extend f_1 across the remaining two open 3-cells; this can be done since $\pi_2(X_i) = 0$, for $i = 1, 2$. The extension can be done so that $f_1^{-1}(x_0) \cap C = B'$. Thus, f_1 is C-equivalent to f and $f_1^{-1}(x_0) = (F^{-1}(x_0) - (D_0 \cup D_1)) \cup B'$ has one less component than $f^{-1}(x_0)$. The proof is completed by induction. □

1.9 The mapping class group of a surface and $Out(\mathbb{F}_n)$

Definition 1.9.1. Let $\Sigma = \Sigma_{g,n}$ be a compact oriented surface of genus g and with n boundary components. The mapping class group $M_{g,n} = M(\Sigma)$ is the group of isotopy classes of homeomorphisms of Σ.

Definition 1.9.2. The outer automorphism group $Out(\mathbb{F}_n)$ is group whose elements are equivalence classes of automorphisms $\Phi : \mathbb{F}_n \to \mathbb{F}_n$, where two automorphism are equivalent if they differ by an inner automorphism.

The outer automorphism group $Out(\mathbb{F}_n)$ of the free group of rank n is naturally maps onto $GL_n(Z)$ and contains as a subgroup of the mapping class group of a compact surface with fundamental group $\mathbb{F}_n$. It is not surprising then to expect $Out(\mathbb{F}_n)$ to exhibit the phenomena present in both linear groups and mapping class groups. Much of the recent of $Out(\mathbb{F}_n)$ has focused on developing tools and proving results known in other two categories.

1.9.1 Dehn-Nielsen-Baer theorem

Theorem 1.9.3. *Let S be a closed surface of positive genus. Then, the mapping class group of S is isomorphic to the group of outer automorphisms of $\pi_1(S)$.*

This is a beautiful example of the interplay between topology and algebra in the mapping class group. For proof, see [29].

2. GEOMETRIC INTERSECTION NUMBER, CURVE COMPLEX AND SPHERE COMPLEX

2.1 Introduction

In this chapter, we study geometric intersection number of simple closed curves on a surface. In particular, we see its applications to study geometric properties of curve complex of the surface. We also study topological properties of curve complex. We shall see how curve complex is used to study mapping class group of surfaces. The geometric intersection number of curves on surfaces has been used to study Thurston compactification of Teichmüller space of a surface and the boundary of Teichmüller space, namely the space of projectivized measured laminations. At the end of this chapter, we study its analogue sphere complex of a 3-manifold and its topological properties.

2.2 Intersection numbers of curves on surfaces

(1) Let Σ be an orientable surface.

Definition 2.2.1. A simple closed curve in Σ is said to be essential if it does not bound a disk in Σ.

Henceforth, we shall deal with essential simple closed curves only.

Definition 2.2.2. Given two isotopy classes α and β of essential simple closed curves in Σ, we define the geometric intersection number $I(\alpha, \beta)$ as the minimal of the cardinality of $|\alpha \cap \beta|$ among all the realizations of α and β in Σ, i.e.,

$$I(\alpha, \beta) = min\{|a \cap b||a \in \alpha, b \in \beta\}.$$

Here, a and b are simple closed curves on Σ representing the isotopy classes α and β respectively.

It is clear that this number is symmetric in the sense that it is independent of the order of α and β. Also, $I(\alpha, \beta) = 0$ if and only if there exists representatives a and b of α and β, respectively, such that a and b are disjoint simple closed curves in Σ.

(2) We can also define intersection number $\acute{I}(\alpha, \beta)$ of α and β as follows:

One can always choose representatives a and b of α and β respectively, to be shortest closed geodesic in some Riemannian metric with negative curvature on Σ so that they automatically intersect minimally.

Let G denote $\pi_1(\Sigma)$. Let H denote the infinite cyclic subgroup of G carried by a, and let Σ_H denote the cover of Σ with fundamental group equal to H. Then a lifts to Σ_H and we denote its lift by a again. Let $\widetilde{a}$ denote the pre-image of this lift in the universal cover $\widetilde{\Sigma}$ of Σ . The full pre-image of a in $\widetilde{\Sigma}$ consists of disjoint lines which we call a-lines, which are all translates of $\widetilde{a}$ by the left action of G. Similarly, we define K, Σ_K , the line $\widetilde{b}$ and b-lines in $\widetilde{\Sigma}$. Now, we consider the images of the a-lines in Σ_K. Each a-line has image in Σ_K which is a line or circle. Then we define $I'(\alpha, \beta)$ to be the number of images of a-lines in Σ_k which meet $\widetilde{b}$. Similarly, we define $I(\beta, \alpha)$ to be the number of images of b-lines in Σ_H which meet a. Using the assumption that a and b are shortest closed geodesics, that each a-line in Σ_k crosses b at most once, and similarly for b-lines in Σ_H . It follows that $I'(\alpha, \beta)$ and $I'(\beta, \alpha)$ are each equal to the number of points of $a \cap b$, and so they are equal to each other.

(3) We can define geometric intersection number for surfaces with nonempty boundary as follows:

Given a compact orientable surface $\Sigma = \Sigma_{g,n}$ of genus g with n boundary components, a curve system on Σ is a proper 1-dimensional sub-manifold so that each component of it is not null homotopic and not relatively homotopic into the boundary. The space of all isotopy classes of curve systems on Σ is denoted by $CS(\Sigma)$. This space was introduced by Max Dehn in 1938 who called it the arithmetic field of the topological surface.

Definition 2.2.3. Given two classes α and β in $CS(\Sigma)$, their geometric intersection number $I(\alpha, \beta)$ is defined to be $\min\{|a \cap b| | a \in \alpha, b \in \beta\}$.

2.3 Curve complex

The complex of curves of a surface Σ is the simplicial complex with vertices isotopy classes of simple closed curves on Σ and simplices disjoint families of simple closed curves on Σ. The complex of curves is used in the study of 3-manifolds and mapping class groups. This complex was considered by Harer from homological point of view (with applications to the homology of the mapping class group). In particular, Harer determined the homotopy type of the curve complex [15], [16]. Ivanov used the curve complex to determine the structure of the mapping class group [27]. Masur and Minsky [36] showed that the curve complex is $\delta - hyperbolic$ in the sense of Gromov. Hempel and others used the curve complex for studying 3-manifolds.

A particularly useful tool in studying the complex of curves is *intersection numbers*. For instance, these have been used to prove geometric property of curve complex like hyperbolicity of the curve complex. Feng Luo has been used intersection number of curves on a surface to study Thurston's compactification of Teichmüller space of a surface [35]. The intersection numbers of curves on a surface has been used to give important constructions like Thurston's space of measured laminations. Now, we shall see precise definitions.

2.3.1 The curve complex

Let Σ be a closed orientable surface and let $\pi \subset \Sigma$ be a (possibly empty) finite set. Harvey associated a curve complex to (Σ, π) as follows:

The vertex set $X = X(\Sigma, \pi)$, consists of the set of isotopy classes of essential simple closed curves in $\Sigma \backslash \pi$ (which we refer to simply as curves). A set of curves is deemed to span a simplex in the curve complex if they can be realized disjointly in $\Sigma \backslash \pi$.

There are a few exceptional cases (sporadic cases) namely,

(1) If Σ is a 2-sphere and $|\pi| \leq 3$, then X= ϕ.

(2) If Σ is either a 2- sphere with $|\pi| = 4$ or a torus with $|\pi| = 1$, then the associated curve complex is just a countable set of points.

For non-exceptional cases (Σ, π), one can see that the curve complex is connected and has dimension $3g(\Sigma) + |\pi| - 4$, where $g(\Sigma)$ = genus of Σ. We define complexity of C(Σ, π)= $3g(\Sigma) + |\pi| - 4$, where C(Σ, π) is the curve complex associated to (Σ, π).

The curve complex is locally infinite. The finiteness of dimension follows by an Euler characteristic argument. The maximal dimensional simplex in the curve complex is called Fenchel- Nielsen system (or pants decomposition).

People have used topology and geometric properties of the curve complex to study various objects like mapping class groups and Teichmüller spaces. Now, we shall see how topology of curve complex has been used.

2.4 Topology of curve complex

The homotopy type of the curve complex was determined by Harer [16].

Theorem 2.4.1. *Let $\Sigma = \Sigma_{g,n}$ be compact orientable surface with genus g and n boundary components, then the curve complex associated to it is homotopically equivalent to a wedge of spheres of dimension r, where*

$(i)r = 2g + n - 3$ if $g > 0$ and $n > 0$.

$(ii)r = 2g - 2$ if $n = 0$.

$(iii)r = n - 4$ if $g = 0$.

This shows that the curve complex is simply connected and not contractible. Topology of curve complex has been used by Harer to compute the virtual cohomological dimension of the mapping class group of surface $\Sigma = \Sigma^r_{g,n}$ of genus g with n boundary components and r punctures.

Theorem 2.4.2. *For $2g + s + r > 2$, the mapping class group $M^r_{g,n} = M(\Sigma = \Sigma^r_{g,n})$ is a virtual duality group of dimension $d(g, r, s)$, where $d(g, 0, 0) = 4g - 5$, $d(g, r, s) = 4g + 2r + s - 4$, $g > 0$ and $r + s > 0$, and $d(O, r, s) = 2r + s - 3$. In particular, the virtual cohomological dimension of $M^r_{g,n}$ is $d(g, r, s)$.*

For proof, see [16].

2.5 Mapping class group and the curve complex

We recall the definition of mapping class group of surfaces.

2.5.1 Mapping class group:

Let $\Sigma = \Sigma_{g,n}$ be a compact oriented surface of genus g and n boundary components. The mapping class groups $M_{g,n} = M(\Sigma)$ is the group of homeomorphisms of Σ which are identity on boundary $\partial\Sigma$ modulo isotopy. Here, isotopies leave points on $\partial\Sigma$ fixed.

The mapping class group has a natural simplicial action on the curve complex $C(\Sigma)$, where vertices are isotopy classes of essential unoriented non boundary parallel simple loops in Σ.

If $[h] \in M(\Sigma)$ and $\alpha = [a] \in C(\Sigma)$, then $[h] \cdot \alpha = [h(a)]$. Here, simplicial action means simplicial structure preserving action.

A natural question one would like to ask is whether every automorphism of the curve complex is induced by a homeomorphism of the surface.

In 1989, Ivanov [28] sketched a proof the result that if the genus of a surface is at least 2, then any automorphism of the curve complex $C(\Sigma)$ is induced by a homeomorphism of the surface.

Feng Luo [32] has settled the automorphism problem for the rest of the surfaces. His proof does not distinguish the case genus $g \geq 2$ from the case genus $g \leq 1$.

Theorem 2.5.1. *(a)If the dimension $3g+n-4$ of the curve complex is at least 1 and $(g,n) \neq (1,2)$, then any automorphism of $C(\Sigma_{g,n})$ is induced by a self homeomorphism of the surface.*

(b)Any automorphism of $C(\Sigma_{1,2})$ preserving the set of vertices represented by separating loops is induced by the self homeomorphism of the surface.

(c)There is an automorphism of $C(\Sigma_{1,2})$ which is not induced by any homeomorphism of the surface $\Sigma_{1,2}$.

This proof uses the work of Harer on homotopy type of the curve complex. An important step is to show that any automorphism of $C(\Sigma)$ preserving the multiplicative structure (See [32]) on $C(\Sigma)$ is induced by the homeomorphism of the surface. For proof, see [32].

2.6 Geometric properties of the curve complex

Among others, Masur, Minsky, Bowditch, Feng Luo have studied geometric properties of curve complex. Geometry of curve complex plays a central role in recent work on the geometry of non-compact hyperbolic 3- manifolds, in particular by Minsky and his collaborators towards proving Thurston's ending lamination conjecture. Now, we see some of the geometric properties of curve complex and how these are used.

2.6.1 Intersection numbers and Hyperbolicity of the Curve Complex

Let Σ be a closed orientable surface and π be a (possibly empty) finite. The 1-skeleton of the curve complex $C(\Sigma)$ is a graph which we denote by $G = G(\Sigma, \pi)$. We write d for the induced combinatorial path metric on X which assigns unit length to each edge of G. Thus, (G, d) is a metric space, which is actually a path connected metric space. Mazur and Minsky [36] showed that the curve complex $C(\Sigma)$ associated with the surface is hyperbolic in the sense of Gromov. This geometric property of curve complex is useful in studying mapping class group of surfaces. To prove hyperbolicity of the curve complex, we require a simple inequality relating intersection number to distances in the curve complex. The inequality is :

Lemma 2.6.1. *If the complexity of $C(\Sigma)$ is positive, then $\forall \alpha, \beta \in X$ we have,*

$$d(\alpha, \beta) \leq I(\alpha, \beta) + 1$$

Now, we recall notions of geodesic metric space and hyperbolicity. The notion of hyperbolic metric space is due to Gromov.

Hyperbolicity :

1.A geodesic metric space X is a path-connected metric space in which any two points x and y are connected by an isometric image of an interval in the real line, called a geodesic and denoted by $[xy]$.

2. We say that X satisfies the " thin triangle condition " if there exists some δ such that for any geodesic triangle $[xy] \cup [yz] \cup [xz]$ in X each side is contained in a δ- neighborhood of the other two. This is one of the several equivalent conditions for X to be δ hyperbolic in the sense of Gromov or negatively curved in the sense of Cannon.

Examples :

1. Classical Hyperbolic Spaces.
2. All simplicial trees.
3. Cayley Graphs of the fundamental groups of a closed negatively curved manifolds.
4. Every finite diameter space is trivially hyperbolic space with δ equal to diameter.

Bowditch [5] has given another proof of the same result. The constructions in his proof are more combinatorial in nature and allow for certain refinements and elaborations. Mazur and Minsky has not given an explicit estimate of the hyperbolicity constant, but Bowditch has shown that the hyperbolicity constant is bounded by a logarithmic function of complexity. Thus, hyperbolic constant depends on (Σ, π).

Any upper bound on $d(\alpha, \beta)$ in terms of $I(\alpha, \beta)$ is enough to prove hyperbolicity.

The logarithmic bound on the hyperbolicity constant is obtained by the bound on $d(\alpha, \beta)$ in the following lemma:

Lemma 2.6.2. *There is a function $F : N \to N$ with $F(n) = O(logn)$ such that if complexity of curve complex is positive and $\alpha, \beta \in X$, then*

$$d(\alpha, \beta) \leq F(I(\alpha, \beta))$$

.

2.6.2 Infinite diameter of the curve complex

All this would be rather trivial if the curve complex had finite diameter because a space of finite diameter is obviously hyperbolic. Feng Luo has given a simple argument which shows that any non-exceptional curve complex has infinite diameter [36]. We will see the sketch of this proof.

The sketch of the proof: Let μ be a maximal geodesic lamination and λ_i be any sequence of closed geodesics converging geometrically to μ. Then, if $d(\gamma_0, \gamma_n)$ remains bounded, then after restricting to a subsequence, we may assume that $d(\gamma_0, \gamma_n) = N, \forall n \geq 0$. For each γ_n, we may then find β_n such that $d(\beta_n, \alpha_n) = 1$ and $d(\gamma_0, \beta_n) = N - 1$. But $\gamma_n \to \mu$ and μ is maximal implies that $\beta_n \to \mu$ as well, since γ_n and β_n are disjoint in Σ. Proceeding inductively, we arrive at the case $N = 1$ and in this case the conclusion is that $\beta_n \to \mu$ and $\beta_n = \gamma_0$, which is a contradiction .

The basic idea to prove hyperbolicity of curve complex is to construct a preferred family of of paths connecting any pair of vertices in G. Thus, if $\alpha, \beta \in X$, we have a path π_{ab} in G from α to β. Then, we show that any triangle formed by three paths $\pi_{\alpha\beta}$, $\pi_{\beta\gamma}$ and $\pi_{\gamma\alpha}$ is "thin" in an appropriate sense. In particular, there is a "center", $\phi(\alpha, \beta, \gamma) \in X$, which is a bounded distance from all three sides. A key point in the argument is to show that if $\gamma, \delta \in X$ are adjacent, then $d(\phi(\alpha, \beta, \gamma), \phi(\alpha, \beta, \delta))$ is bounded. Given this one sees that the paths $\pi_{\alpha,\beta}$ are uniformly quasigeodesic. From this the hyperbolicity of G follows via a subquadratic isoperimetric inequality .

The curve complex encodes the asymptotic geometry of the Teichmüller space of a surface. We shall also see how geometric intersection number of curve curves on a surface is used to give various important constructions like Thurston's space of measured laminations. Now, we shall see what is the Teichmüller space of a surface.

2.7 Teichmüller space of surface and Thurston's compactification of Teichmüller space

Let $\Sigma = \Sigma_{g,n}$ be a compact, connected, orientable surface of genus g and n boundary components (n may be 0) and of negative Euler characteristic. By a hyperbolic metric on the surface Σ, we mean a Riemannian metric of curvature -1 on the surface Σ so that its boundary components are geodesics. The Teichmüller space $T(\Sigma)$ is the space of all isotopy classes of hyperbolic metrics on the surface Σ. Two hyperbolic metrics are *isotopic* if there is an isometry between the two metrics which is isotopic to identity.

Thurston introduced the space of projective measured laminations on Σ, which will be denoted by $PML(\Sigma)$, and a compactification of $T(\Sigma)$ whose boundary is equal to $PML(\Sigma)$. Thurston boundary $PML(\Sigma)$ is a natural boundary of $T(\Sigma)$, in the sense that the action of mapping class group of Σ extends continuously to the Thurston compactification $\overline{T(\Sigma)} = T(\Sigma) \cup PML(\Sigma)$.

Intersection number and Thurston's space of measured laminations

Recall from [35], given a compact, orientable surface Σ with possibly non empty boundary, space of all isotopy classes of curve system on Σ is denoted by $CS(\Sigma)$. Thurston observed that the pairing $I(,) : CS(\Sigma) \times CS(\Sigma) \to \mathbb{Z}$ behaves like a non-degenerate bilinear form in the sense that

(1) Given any α in $CS(\Sigma)$, there is β in $CS(\Sigma)$ so that their intersection number $I(\alpha, \beta)$ is non-zero.

(2) $I(k_1\alpha_1, k_2\alpha_2) = k_1k_2I(\alpha_1, \alpha_2)$, for $k_i \in \mathbb{Z}_{\geq 0}, \alpha_i \in CS(\Sigma)$, where $k_i\alpha_i$ is the collection of k_i copies of α_i.

Thurston's space of measured laminations on the surface Σ, denoted by $ML(\Sigma)$ is defined to be the completion of the pair $(CS(\Sigma), I(,))$ in the following sense : Given α in $CS(\Sigma)$, let $\pi(\alpha)$ be the map sending β to $I(\alpha, \beta)$. This gives an embedding $\pi : CS(\Sigma) \to \mathbb{R}^{CS(\Sigma)}$, where the target has product topology. The space $ML(\Sigma)$ is defined to be the closure of $\mathbb{Q}_{>0} \times \pi(CS(\Sigma)) = \{r\pi(x) : r \in \mathbb{Q}_{>o}, x \in CS(\Sigma)\}$

Using notion of train tracks, Thurston showed that $ML(\Sigma)$ is homeomorphic to a Euclidean space and intersection pairing $I(,)$ extends to a continuous homogeneous map from $ML(\Sigma) \times ML(\Sigma)$ to $\mathbb{R}$. See [35].

2.7.1 Thurston's compactification of Teichmüller spaces

Consider a fixed hyperbolic structure σ on Σ.

Definition 2.7.1. A geodesic lamination μ is a closed subset of Σ, which is a disjoint union of simple geodesics which are called leaves of μ. The leaves of a geodesic lamination are complete, i.e., each leaf is either closed or has infinite length in both of its ends, and a geodesic lamination is determined by its support, i.e., a geodesic lamination is a union of geodesics in just one way.

We write $GL(\Sigma)$ to denote the space of geodesic laminations on Σ, which is equipped with the Hausdorff metric on closed subsets. Note that $GL(\Sigma)$ is compact and therefore, in particular, every infinite sequence of nontrivial simple closed geodesics has a convergent subsequence.

A transverse measure on a geodesic lamination μ is a rule, which assigns to each transverse arc α a measure that is supported on $\mu \cap \alpha$, which is invariant under a map from α to another arc β if it takes each point of intersection of α with a leaf of μ to a point of intersection of β with the same leaf. A measured lamination on Σ is a geodesic lamination μ with a transverse measure of full support, i.e., if $(\alpha \cap \mu) \neq \phi$ then α has nonzero measure for any transverse arc α. For example, a simple closed geodesic equipped with counting measure is a measured lamination. We write $ML(\Sigma)$ to denote the space of measured laminations on Σ. There is a natural action of $\mathbb{R}^+$ on $ML(\Sigma)$. Suppose that $r > 0$. The measured lamination $r\mu$ has the same geodesic lamination as μ with the transverse measure scaled by r. We write $PML(S)$ to denote the set of equivalence classes of projective measured laminations.

Then, Thurston's compactification of $T(\Sigma)$ is $\overline{T(\Sigma)} = T(\Sigma) \cup PML(\Sigma)$, with appropriate topology. See [30].

Now, we study one dimensional higher analogue of curve complex, namely sphere complex.

2.8 Sphere complex

The sphere complex associated to $M = \sharp_k S^2 \times S^1$, i.e., the connected sum of k copies of $S^2 \times S^1$, is a simplicial complex whose vertices are the isotopy classes of embedded spheres in M. A set of isotopy classes of embedded spheres in M is deemed to span a simplex if they can be realized disjointly in M. This is an analogue of the curve complex associated to a surface. The topological properties of the sphere complex have been studied by Hatcher, Hatcher-Vogtmann and Hatcher-Wahl in [17], [20], [21], [22], [23], [24].

Definition 2.8.1. A smooth, embedded 2-sphere in M is said to be essential if it does not bound a 3-ball in M.

Definition 2.8.2. A *system of 2-spheres* in M is defined as a finite collection of disjointly embedded, pair-wise non-isotopic, essential smooth 2-spheres $S_i \subset M$.

Definition 2.8.3. The sphere complex $\mathbb{S}(M)$ associated to M is a simplicial complex whose vertices are the isotopy classes of essential embedded 2-spheres in M. A set of isotopy classes of embedded spheres in M is deemed to span a simplex in the sphere complex if they can be realized disjointly in M.

The maximal simplices of $\mathbb{S}(M)$ all have the same dimension, namely $3n + s - 4$, as one sees by Euler characteristic considerations using the fact that the complementary regions of a maximal system of 2-spheres are all 3-punctured spheres.

2.8.1 Topology of sphere complex

In [17], Hatcher has proved that the sphere complex $\mathbb{S}(M)$ is contractible. This is proved by imitating the simple proof in [19] of contractibility of the analogous complex of arcs on a punctured surface. However, for this scheme to work one needs the fact that sphere systems can be isotoped into a fairly canonical normal form with respect to a decomposition of M into "pairs of pants", i.e., 3-punctured S^3's. This normal form is analogue of a well-known property of curves on a surface. We shall discuss "*normal forms of sphere systems*" in details in the Chapter 4. Culler and Vogtmann [7], introduced a space X_n on which the group $Out(\mathbb{F}_n)$ acts with finite point stabilizers, and proved that X_n is contractible. Peter Shalen later invented the name " *Outer space*" for X_n. Outer space with the action of $Out(\mathbb{F}_n)$ can be thought of as free group analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface. Culler and Morgan have constructed a compactification of Outer space much like Thurston's compactification of Teichmüller space [6].

Now, we see the connection between the sphere complex and the outer space.

2.8.2 Sphere complex and Outer space

The points of the rank n Outer Space X_n of Culler-Vogtmann are equivalence classes of homotopy equivalences $f : X_0 \to X$, where X_0 is a bouquet of n circles and X is a metric graph which doesn't deformation retract onto any subgraph, the metric being normalized so that the total length of all the edges is 1. The equivalence relation on such "marked metric graphs" $f : X_0 \to X$ is given by homotopy of f and composition with isometries $X \to X'$. Fixing the topological type of X and varying only the lengths of its edges traces out an open simplex in X_n. Passing to faces of this simplex corresponds to letting the lengths of some edges go to zero. Depending on which edges are collapsing in this way, the face might or might not belong to X_n. Let $\mathbb{S} = \mathbb{S}(M)$, and let $\mathbb{S}_\infty$ be the subcomplex of $\mathbb{S}$ consisting of sphere systems having at least one non simply-connected complementary component in M. A sphere System S has a dual graph $G(S)$ having vertices the components of $M - S$ and edges the spheres of S. We may view $G(S)$ as embedded in M by choosing a vertex point in each component of $M - S$ and connecting these vertices by edges crossing the spheres of S, each sphere having a single edge crossing it exactly once. Some what more canonically, $G(S)$ is also a quotient of M, obtained by thickening S to a product $S \times [-1, 1] \subset M$, then collapsing the components of $M - (S \times (0, 1))$ to points and also the components of $S \times t$, for each $t \in (0, 1)$. If S is in $\mathbb{S} - \mathbb{S}_\infty$, then both maps $G(S) \to M$ and $M \to G(S)$ are isomorphisms on π_1.

Fixing a System S_0 with $G(S_0) = X_0$, the composition $G(S_0) \to M \to G(S)$ is then a homotopy equivalence. The barycentric coordinates of a point in the open simplex of $\mathbb{S}$ determined by S give weights on the components of S and hence lengths on the corresponding edges of $G(S)$. In this way we obtain a map $\Theta : \mathbb{S} - \mathbb{S}_\infty \to X_n$ sending the weighted system S to $G(S_0) \to G(S)$. On each open simplex of $\mathbb{S} - \mathbb{S}_\infty$,$\Theta$ is a linear homeomorphism onto an open simplex of X_n, and Θ is continuous when we pass to faces of simplices, hence Θ is continuous everywhere. Also, Θ is equivariant with respect to the natural action of $Out(\mathbb{F}_n)$ on $\mathbb{S} - \mathbb{S}_\infty$ and X_n. This maps actually turns out be a homeomorphism. See [17].

The space X_n has dimension $3n - 4$, and Culler-Vogtmann describe a nice "spine" of X_n which is a contractible subcomplex of dimension $2n-3$ on which $Out(\mathbb{F}_n)$ acts with finite stabilizers and finite quotient. Using this they prove that $Out(\mathbb{F}_n)$ has finitely generated homology groups and virtual cohomological dimension $2n - 3$. See [7].

3. THE MODEL 3-MANIFOLD M AND ENDS

3.1 Introduction

In this chapter, we study the model 3-manifold $M = \sharp_k S^2 \times S^1$. We also see how a partition of ends of the space $\widetilde{M}$, the universal cover of M, corresponds to an embedded sphere in $\widetilde{M}$. We also discuss the intersection number of a proper path in $\widetilde{M}$ with a homology class in $H_2(\widetilde{M})$. In the last section of this chapter, we discuss splittings of the fundamental group of M.

3.2 The model 3-manifold M

Consider the 3-manifold $M = \sharp_k S^2 \times S^1$, i.e., the connected sum of k copies of $S^2 \times S^1$. A description of M can be given as follows: Consider the sphere S^3 and let $A_i, B_i, 1 \leq i \leq k$, be a collection of $2k$ disjoint embedded balls in S^3. Let P be the complement of the union of the interiors of these balls and let S_i (respectively, T_i) denote the boundary of A_i (respectively, B_i). Then, M is obtained from P by gluing together S_i and T_i with an orientation reversing diffeomorphism φ_i for each $i, 1 \leq i \leq k$. Let $\Sigma_i' = S_i \bigsqcup_{\varphi_i} T_i$, for $1 \leq i \leq k$. The fundamental group $\pi_1(M) = G$ of M, which is a free group of rank k, acts freely on the universal cover $\widetilde{M}$ of M by deck transformations.

Let $\Sigma = \cup_j \Sigma_j$ be a maximal system of 2-sphere in M. Splitting M along Σ, then produces a finite collection of 3-punctured 3-spheres P_k. Here, a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere.

We recall some constructions from [17]. First, we associate a tree T to $\widetilde{M}$ corresponding to the decomposition of M by Σ. Let $\widetilde{\Sigma}$ be the pre-image of Σ in $\widetilde{M}$. The closure of each component of $\widetilde{M} - \widetilde{\Sigma}$ is a 3-punctured 3-sphere $\widetilde{P_k}$ which is a lift of a P_k. The vertices of the tree are of two types, with one vertex corresponding to the closure of each component of $\widetilde{M} - \widetilde{\Sigma}$ and one vertex for each component of $\widetilde{\Sigma}$. An edge of T joins a pair of vertices if one of the vertices corresponds to the closure of a component X of $\widetilde{M} - \widetilde{\Sigma}$ and the other vertex corresponds to a component of $\widetilde{\Sigma}$ that is in the boundary of X. Thus, we have a Y-shaped subtree corresponding to each complementary component. We pick an embedding of T in $\widetilde{M}$ respecting the correspondences. This tree has bivalent and trivalent vertices. Bivalent vertices correspond to components of $\widetilde{\Sigma}$. We call components of $\widetilde{\Sigma}$ as standard spheres in $\widetilde{M}$.

Let $\tau = \tau_1 \subset \tau_2 \subset \dots$ be an exhaustion of T by finite subtrees of T such that all the terminal vertices of each τ_i are bivalent in T. Let K_τ be the union of closures of $\widetilde{P_k}$'s which corresponds to vertices in τ which are trivalent in T. Then, one can easily see that K_τ is a compact, simply-connected space homeomorphic to a space of the form $S^3 - \cup_{j=1}^{n} int(D_j)$ with D_j disjoint embedded balls in S^3.

We observe that $\pi_2(M) = \pi_2(\widetilde{M}) = H_2(\widetilde{M})$. This follows from Hurewicz theorem which we state below:

Theorem 3.2.1. *If a CW-complex X is $(n-1)$-connected, $n \geq 2$,then $\pi_n(X)$ is isomorphic to $H_n(X)$.*

A topological space X is said to be m-connected if and only if it is path-connected and its first m homotopy groups vanish identically, that is,

$$\pi_i(X) = 0, 1 \leq i \leq m.$$

So a class in $\pi_2(M)$ can be considered as a class in $\pi_2(\widetilde{M})$ as well as a class in $H_2(\widetilde{M})$. We shall implicitly use this identification throughout. For reference, see [18].

3.3 Ends of $\widetilde{M}$

We recall the notion of ends of a topological space: Let X be a topological space. For a compact set $K \subset X$, let $C(K)$ denote the set of components of $X - K$. For L compact with $K \subset L$, we have a natural map $C(L) \to C(K)$. Thus, as compact subsets of X define a directed system under inclusion, we can define the set of ends $E(X)$ as the inverse limit of the sets $C(K)$. Further, we can compute the inverse limit with respect to any exhaustion by compact sets.

It is easy to see that a proper map $f : X \to Y$ induces a map $E(f) : E(X) \to E(Y)$ and that if $f : X \to Y$ and $g : Y \to Z$ are proper maps, then $E(g \circ f) = E(g) \circ E(f)$. In particular, the real line $\mathbb{R}$ has two ends which can be regarded as ∞ and $-\infty$. Hence, a proper map $c : \mathbb{R} \to X$ gives a pair of ends c_- and c_+ of X which may be equal.

Now, consider proper maps $c : \mathbb{R} \to \widetilde{M}$. As $\widetilde{M}$ is a union of the simply-connected compact sets K_τ, the following lemma is straightforward.

Lemma 3.3.1. *There is a one-one correspondence between proper homotopy classes of maps $c : \mathbb{R} \to \widetilde{M}$ and pairs $(c_-, c_+) \in E(\widetilde{M}) \times E(\widetilde{M})$.*

3.3.1 Topology on the set $E(\widetilde{M})$

To define topology on $E(\widetilde{M})$, we use compact subsets of $\widetilde{M}$. If K is any compact subset of $\widetilde{M}$, then $\widetilde{M} - K$ has finitely many components. Then, we have the set of ends of a component of $\widetilde{M} - K$ whose closure is non-compact to be a basis element. We can easily see that the collection of all the sets of ends of components of $\widetilde{M} - K$ whose closures are non-compact, for all compact subsets K of $\widetilde{M}$, forms a basis for a topology on $E(\widetilde{M})$.

The set $E(\widetilde{M})$ is homeomorphic to a Cantor set, hence compact. Note that the set $E(T)$ of ends of T can be identified with the set $E(\widetilde{M})$.

3.4 Embedded spheres in $\widetilde{M}$ and partitions of ends of $\widetilde{M}$

Fix an orientation of M and hence, of $\widetilde{M}$.

Lemma 3.4.1. *Let S is an embedded sphere in $\widetilde{M}$. Then, S separates $\widetilde{M}$.*

Proof. Suppose S is non-separating. Choose a regular neighborhood $V = S^2 \times [-1, 1]$ of S and an embedded path γ in $M - V$ from a point of $S^2 \times -1$ to a point $S^2 \times 1$. The sphere S' which is the connected sum of $S^2 \times -1$ with $S^2 \times 1$ along with the boundary of a regular neighborhood U of γ, clearly bounds $U \cup V$ in $\widetilde{M}$. Thus, $\widetilde{M} = (U \cup V) \cup (M - (U \cup V))$. Then, $U \cup V$ is $(S^2 \times S^1) - B^3$ with boundary S', where B^3 is a 3-ball and $(M - (U \cup V))$ is a 3-manifold with boundary S'. Thus, $\widetilde{M}$ is a connected sum of $S^2 \times S^1$ with some three manifold. This implies, by applying Van-Kampen theorem, that the fundamental group of $\widetilde{M}$ is non-trivial, which is a contradiction. So, S separates $\widetilde{M}$.

□

If S is an embedded sphere in $\widetilde{M}$, then S separates $\widetilde{M}$ into two components, say V^+ and V^-, with V^+ on the positive side of S according to the given orientations on S and $\widetilde{M}$. If the closure of one of these components is compact, then S is homologically trivial. If the closures of both the components are non-compact, then we get a partition of the set $E(\widetilde{M})$ of ends of $\widetilde{M}$ into two non-empty subsets $E^{\pm}(S)$ of $E(\widetilde{M})$. The sets $E^{\pm}(S)$ are the sets $E(V^{\pm})$ of ends components $V^{\pm}$.

Proposition 3.4.2. *The sets $E^{\pm}(S)$ are open in $E(\widetilde{M})$.*

Proof. Suppose τ is finite subtree of T with all of its terminal vertices bivalent such that S is contained in K_τ. Then, $K = K_\tau$ is a compact, 3-dimensional, connected manifold contained in $\widetilde{M}$ such that the closure W_i of each complementary component of K is non-compact. As $\widetilde{M}$ is simply-connected and K is connected, $N_i = \partial W_i$ is connected for each W_i. Note that there are finitely many sets W_i and $E(\widetilde{M})$ is partitioned into the sets $E(W_i)$. The space K is S^3- interior of finitely many disjointly embedded 3-balls with boundary spheres N_i. The sphere S separates K and gives a partition of the collection $\{N_i\}$). Note that interior of each W_i is completely contained either in V^+ or V^- and hence each set $E(W_i)$ lies entirely either inside E^+ or inside E^-. Thus, both E^+ and E^- are unions of basis elements, hence they are open. □

As the sets $E^{\pm}(S)$ give partition of $E(\widetilde{M})$, both E^+ and E^- are closed subsets of $E(\widetilde{M})$. As $E(\widetilde{M})$ is compact, both E^+ and E^- are compact subsets of $E(\widetilde{M})$.

Proposition 3.4.3. *If S' is an embedded sphere in $\widetilde{M}$, homologous to S, then both S and S' give the same partition of the set of ends of $\widetilde{M}$.*

Proof. As S and S' are embedded spheres in $\widetilde{M}$, there exist a finite subtree τ of T with all of its terminal vertices bivalent in T, such that both S and S' are contained in K_τ. The space $K = K_\tau$ is a compact, 3-dimensional, connected manifold contained in $\widetilde{M}$ such that the closure W_i of each complementary component of K is non-compact. As $\widetilde{M}$ is simply-connected and K is connected, $N_i = \partial W_i$ is connected for each W_i. There are finitely many sets W_i and $E(\widetilde{M})$ is partitioned into the sets $E(W_i)$. The space K is S^3- interior of finitely many 3-balls with boundary spheres N_i. As each W_i is non-compact and the boundary sphere N_i is non-trivial in $H_2(W_i)$, an algebraic topology argument implies that spheres S and S' are homologous in $\widetilde{M}$ if and only if they are homologous in K. Now, we claim the following:

The embedded spheres S and S' in K are homologous in K if and only if they give the same partition of the collection of boundary spheres N_i.

Let $K-S = K_1 \cup K_2$ and $K-S' = K_1' \cup K_2'$. Let $\partial \bar{K}_1 - S = N_1 \cup \cdots \cup N_r$ and $\partial \bar{K}_2 - S = N_{r+1} \cup \cdots \cup N_k$. Let $A_1 = N_1 \cup \cdots \cup N_r$ and $A_2 = N_{r+1} \cup \cdots \cup N_k$. We have $\partial K = A_1 \bigsqcup A_2$. Similarly, let $\partial \bar{K}_1' - S' = A_1'$ and $\partial \bar{K}_2 - S = A_2'$, where each A_i' is a disjoint union boundary spheres and $\partial K = A_1' \bigsqcup A_2'$. It follows that S is homologous to A_1 and also to A_2. Similarly, S' is homologous to A_1' and also to A_2'.

If S and S' are homologous in K, then A_i is homologous A_j', for all $1 \leq j,k \leq 2$. Note $H_2(K)$ is generated by the homology classes $[N_i]$ of the boundary spheres N_i with the relation $\sum_i [N_i] = 0$. Now, A_1 is homologous to A_1' and if $A_1 \neq A_1'$, then $A_1' = A_2$, as the class $B = [A_1] - [A_2] = 0$ can be represented by union of boundary spheres N_i. From this, it clear that both S and S' give the same partition of the collection of boundary spheres.

Conversely, if S and S' give the same partition $\{A_1, A_2\}$ of the collection of boundary spheres N_i of K. Then, both S and S' are homologous to A_1. Hence, S and S' are homologous in K.

Now, if S and S' are homologous in $\widetilde{M}$, then they are homologous in K. So, they give the same partition of the collection of boundary spheres N_i of K. Therefore, they give the same partition of the collection $\{E(W_i)\}$ and hence, give the same partition of the set $E(\widetilde{M})$. □

Conversely,

Proposition 3.4.4. *If S and S' are two embedded spheres in $\widetilde{M}$ such that they give the same partition (E^+, E^-) of the set $E(\widetilde{M})$ of ends of $\widetilde{M}$, then S and S' are homologous in $\widetilde{M}$.*

Proof. As S and S' are embedded spheres in $\widetilde{M}$, there exist a finite subtree τ of T with all of its terminal vertices bivalent in T, such that both S and S' are contained in K_τ. The space $K = K_\tau$ is a compact, 3-dimensional, connected manifold contained in $\widetilde{M}$ such that the closure W_i of each complementary component of K is non-compact. Let $N_i = \partial W_i$. There are finitely many sets W_i and $E(\widetilde{M})$ is partitioned into the sets $E(W_i)$. The space K is S^3- interior of finitely many 3-balls with boundary spheres N_i. The sphere S separates K and gives a partition of the boundary spheres of K into two sets. This partition of boundary sphere into two sets gives a partition of the collection $\{E(W_i)\}$ into two sub collections. Each set $E(W_i)$ lies entirely either inside E^+ or E^- and theses two sub collections of $\{E(W_i)\}$ determine the sets E^+ and E^-. Similarly, this is true for S'. This implies S and S' give the same partition of the boundary spheres of K. Hence, S and S' are homologous in K and therefore, homologous in $\widetilde{M}$. □

Proposition 3.4.5. *Given a partition of the set $E(\widetilde{M})$ into two infinite closed (hence open) sets E^+ and E^-, there exist an embedded sphere S in $\widetilde{M}$, which gives the same partition of $E(\widetilde{M})$.*

Proof. Suppose that E^+ and E^- are two disjoint closed subsets of $E(\widetilde{M})$, which give a partition of $E(\widetilde{M})$. If $\tau \subset T$ is a tree such that all the terminal vertices of τ are bivalent in T, then $K = K_\tau$ is a compact, 3-dimensional, connected manifold contained in $\widetilde{M}$ such that the closure W_i of each complementary component of K is non-compact. Let $N_i = \partial W_i$.

Suppose that we can choose a subtree $\tau \subset T$ with all the terminal vertices of τ bivalent in T such that each $E(W_i)$ lies entirely either in E^+ or E^-. We can assign signs to N_i, $+$ or $-$ depending upon whether $E(W_i)$ lies inside E^+ or E^-. Then, we can choose a sphere S in K which separates all positive signed N_i from all negative signed N_i, as K is S^3- interior of finitely many 3-balls with boundary spheres N_i. Then, one can easily see that S gives the partition of $E(\widetilde{M})$ into the sets E^+ and E^-.

Now, we see how to choose such a tree τ. We have both E^+ and E^- are both open and closed. As the set $E(\widetilde{M})$ is compact, both E^+ and E^- are compact. Let $e \in E^+$. As E^+ is open, we can choose a finite tree $\tau \subset T$ such that e is an element of the set U_e of the ends of a component of $\widetilde{M} - K_\tau$ and $U_e \subset E^+$. We can choose such a basic open set U_e for each $e \in E^+$. As E^+ is compact, there exists finitely many basic open sets $U_{e_1}, \ldots, U_{e_n}$ such that $E^+ = \cup_{i=1}^n U_{e_i}$. Let τ_i be finite subtree of T with all their terminal vertices bivalent such that U_{e_i} is the set of ends of a component of $\widetilde{M} - K_{\tau_i}$, for each $i = 1, \ldots, n$. Let W_i be the closure of the component of $\widetilde{M} - K_{\tau_i}$ such that $U_{e_i} = E(W_i)$, the set of ends of W_i. Let $N_i = \partial W_i$. Then, N_i corresponds to a terminal vertex v_i of τ_i. If for some $1 \leq j, k \leq n$, N_k lies in W_j, then we have $W_k \subset W_j$ and $U_k = E(W_k) \subset E(W_j) = U_j$. Then, we discard the vertex v_k corresponding to N_k from the collection of bivalent vertices $\{v_1, \ldots, v_n\}$ of T. So finally, we get bivalent vertices $v_{j_1}, \ldots, v_{j_m}$ such that the sets $W_{j_1}, \ldots, W_{j_m}$ are disjoint. Then, we have the sets $U_{j_1}, \ldots, U_{j_m}$ are disjoint and $E^+ = \cup_{i=1}^m U_{j_i}$. Note that given any two bivalent vertices v_{j_i} and v_{j_k}, $1 \leq i, k \leq m$, the reduced path joining them does not contain any other v_{j_l}, $j \neq l \neq k$. We consider the subtree τ' of T which is the span of the vertices v_i. Then, all v_{j_i}, $1 \leq i \leq m$, are the terminal vertices of τ'. We enlarge the tree τ' to the subtree τ of T by taking unions of those Y's which have non-empty intersection with the interior of the tree τ'. The tree τ is a subtree of T with all the terminal vertices of τ are bivalent in T and all the V_{j_i}'s are terminal vertices of τ. Now, one can easily see that τ has the required property. For, if $K = K_\tau$, K is S^3- interior of finitely many 3-balls with boundary spheres $N_{j_1}, \ldots, N_{j_m}, N'_1, \ldots, N'_l$. The boundary sphere N_{j_i} corresponds to the vertex v_{j_i}. The space $\widetilde{M} - K$ has components with closures $W_{j_1}, \ldots, W_{j_m}, W'_1, \ldots, W'_l$. For each $1 \leq i \leq m$, $N_{j_i} = \partial W_{j_i}$ and for each $1 \leq k \leq l$, $N'_k = \partial W'_k$. As the sets $U_{j_1}, \ldots, U_{j_m}$ are disjoint and $E^+ = \cup_{i=1}^m U_{j_i}$, for each $1 \leq k \leq l$, $E(W'_k) \subset E^-$. Thus, we have $\tau \subset T$ with all the terminal vertices of τ bivalent in T such that each the set of ends of the closure of each complementary component of K lies entirely either in E^+ or E^-. $\square$

3.5 Crossings of spheres in $\widetilde{M}$

Let A and B be two homology classes in $H_2(\widetilde{M})$ represented by embedded spheres in $\widetilde{M}$. We saw a homology classes A of embedded spheres S in $\widetilde{M}$ is completely determined by a partition of $E(\widetilde{M})$ into two open subsets of $E(\widetilde{M})$. If S gives partition of $E(\widetilde{M})$ into two open subsets $E^+(S)$ and $E^-(S)$ of $E(\widetilde{M})$, then we can write $E^+(A) = E^+(S)$ and $E^-(A) = E^-(S)$.

Definition 3.5.1. We say that A and B *cross* each of the four sets $E^\varepsilon(A) \cap E^\eta(B) \neq \phi$, for ε and η obtained by choosing signs ε and η in $\{+, -\}$ is non-empty.

Suppose A and B do not cross, then for some choice of sign $E^\varepsilon(A) \supset E^\eta(B)$. It follows that $E^{\bar{\varepsilon}}(A) \subset E^{\bar{\eta}}(B)$, where $\bar{\varepsilon}$ and $\bar{\eta}$ denote the opposite signs. Further, if $A \neq B$, then the inequalities are strict.

Definition 3.5.2. We say that B is on the positive side of A if $E^+(A) \supset E^\eta(B)$ for some sign η. Otherwise, we say that B is on the negative side of A. In general, we say that B is on the ε-side of A for the appropriate sign ε.

Proposition 3.5.3. *Let A and B be two homology classes in $H_2(\widetilde{M})$ represented by embedded spheres in $\widetilde{M}$. Then A and B can be represented by disjoint embedded spheres in $\widetilde{M}$ if and only if A and B do not cross.*

Proof. Suppose A and B can be represented by embedded spheres S and S' respectively. Denote the closures of the components of the complement of S (respectively, S') by X_1 and X_2 (respectively, Y_1 and Y_2) so that $E(X_1) = E^+(A) = E^+(S)$ and $E(X_2) = E^-(A) = E^-(S)$ (respectively, $E(Y_1) = E^+(B) = E^+(S')$ and $E(Y_2) = E^-(B) = E^-(S')$). Suppose S and S' are disjoint embedded spheres in $\widetilde{M}$. Suppose S' is contained in the interior of X_1. Then, the component Y_i which does not intersect S is completely contained inside X_1. Let this component be Y_1. Then, $E^+(B) = E(Y_1) \subseteq E(X_1) = E^+(A)$. This implies $E^+(B) \cap E^-(A) = \phi$. Similarly, we can see in all other cases at least one of the four sets $E^\varepsilon(A) \cap E^\eta(B)$, for ε and η obtained by choosing signs ϵ and η in $\{+, -\}$, is empty.

Conversely, suppose A and B do not cross. We shall show that A and B can be represented by disjoint embedded spheres in $\widetilde{M}$. Let τ be a finite subtree of T with all of its terminal vertices bivalent in T, such that both A and B are supported in $K = K_\tau$. The space $K = K_\tau$ is a compact, 3-dimensional, connected manifold contained in $\widetilde{M}$ such that the closure W_i of each complementary component of K is non-compact. Let $N_i = \partial W_i$. Note that there are finitely many sets W_i and $E(\widetilde{M})$ is partitioned into the sets $E(W_i)$. Note K is S^3- interior of finitely many 3-balls with boundary spheres N_i. Each set $E(W_i)$ is completely contained either in $E^+(A)$ or $E^-(A)$ (respectively, each set $E(W_i)$ is completely contained either in $E^+(B)$ or $E^-(B)$). We can assign signs to N_i, $+_A$ or $-_A$ depending upon whether $E(W_i)$ lies inside $E^+(A)$ or $E^-(A)$ (respectively, we can assign signs to N_i, $+_B$ or $-_B$ depending upon whether $E(W_i)$ lies inside $E^+(B)$ or $E^-(B)$). Thus, the collection of boundary spheres N_i of K get partitioned into two sets U_A^+ and U_A^- containing $+_A$ signed and $-_A$ signed boundary spheres, respectively. Similarly, the collection of boundary spheres N_i of K get partitioned into two sets U_B^+ and U_B^- containing $+_B$ signed and $-_B$ signed boundary spheres, respectively. As A and B do not cross, we can assume that for some choice of sign, $E^{\varepsilon}(A) \supset E^{\eta}(B)$. Suppose $E^{+}(A) \subset E^{+}(B)$. This implies $U_A^{+} \subset U_B^{+}$. Then, inside K, we can choose disjointly embedded spheres S and S' which give partitions $\{U_A^{+}, U_A^{-}\}$ and $\{U_B^{+}, U_B^{-}\}$ of boundary spheres N_i of K, respectively. Thus, we get two disjointly embedded spheres S and S' representing the homology classes A and B, respectively. Similarly, we can consider all the other cases.

□

3.6 Intersection number of a proper path and homology classes

Let $A \in H_2(\widetilde{M}) = \pi_2(\widetilde{M})$. Represent A by a (not necessarily connected) surface in $\widetilde{M}$ (also denoted A). Given a proper map $c : \mathbb{R} \to \widetilde{M}$ which is transversal to A, we consider the algebraic intersection number $c \cdot A$. This depends only on the homology class of A and the proper homotopy class of c. Now we shall discuss this intersection number in details: The proper map $c : \mathbb{R} \to \widetilde{M}$ gives a pair of ends c_- and c_+ of $\widetilde{M}$. We shall refer c as a proper path from c_- to c_+ or as a proper path joining c_- and c_+. We denote such a path c by (c_-, c_+). This is well defined up to proper homotopy. In particular, for a homology class $A \in H_2(\widetilde{M})$, the intersection number $(c_-, c_+) \cdot A$ (which we define in detail below) is well defined and can be computed using any proper path joining c_- and c_+. We shall use this implicitly throughout.

For a proper path $c : \mathbb{R} \to \widetilde{M}$ and an element $A \in H_2(\widetilde{M})$, we can define the algebraic intersection number $c \cdot A$ by making c transversal to A and computing the intersection number. We formalize this using the exhaustion of $\widetilde{M}$ by the sets K_τ. Namely, if $c : \mathbb{R} \to \widetilde{M}$ is a proper path, then there is an interval $[-L, L]$ such that $c^{-1}(K_\tau) \subset [-L, L]$. It follows that $c|_{[-L,L]}$ gives an element in $H_1(\widetilde{M}, \widetilde{M} - int(K_\tau)) = H_1(K_\tau, \partial K_\tau) = H^2(K_\tau)$, where the first isomorphism is by excision and the second by Poincaré duality. On passing to inverse limits, we see that c gives an element of $H^2(\widetilde{M})$. Evaluating this element on A gives $c \cdot A$.

Note that every class $A \in H_2(\widetilde{M})$ is supported in K_τ for some finite tree τ, and a proper path c gives an element of $H^2(K_\tau)$. Further, as the closures of the complementary components of K_τ in M are all non-compact, any proper path $\alpha : [0,1] \to K_\tau$ can be extended to a proper path $c : \mathbb{R} \to \widetilde{M}$ whose intersection with K_τ is α. In particular, the cohomology class in $H^2(K_\tau) = H_1(K_\tau, \partial K_\tau)$ corresponding to α is the image under the map induced by inclusion of the class corresponding to c. It follows that $\alpha \cdot A = c \cdot A$ for $A \in H_2(K_\tau)$.

We use the above observations and the fact that K_τ is a compact, simply-connected space homeomorphic to S^3 with finitely many balls deleted, with the boundary components corresponding to the edges in $\delta\tau$ to deduce some elementary results concerning the homology of $\widetilde{M}$.

As $H_2(K_\tau)$ is generated by its boundary components of K_τ, it follows that these spheres generate $H_2(\widetilde{M})$.

Next, note that if A and B are two homology classes, then for some finite subtree $\tau \subset T$, they are both supported by K_τ. If A is not homologous to B, then as $H_1(K_\tau) = 0$, by Poincaré duality there exists a proper path α in K_τ such that $\alpha \cdot A \neq \alpha \cdot B$. By extending α to a proper path $c : \mathbb{R} \to \widetilde{M}$, we deduce that there is a proper path $c : \mathbb{R} \to \widetilde{M}$ with $c \cdot A \neq c \cdot B$. Thus, an element $A \in H_2(\widetilde{M})$ is determined by the intersection numbers $c \cdot A$, for proper paths $c : \mathbb{R} \to \widetilde{M}$.

If S is an embedded sphere in M, then S separates $\widetilde{M}$ into two components. If the closure of one of these is compact, then S is homologically trivial. Otherwise, we can find a proper path $c : \mathbb{R} \to \widetilde{M}$ with $c \cdot S = 1$, from which it follows that S is primitive.

3.7 Splitting of the fundamental group and embedded spheres

Recall from section 3.2, the description of M. Fix a base point x_0 away from Σ'_i. For each $1 \leq i \leq k$, consider the element $\alpha_i \in \pi_1(M)$ represented by a closed path γ_i starting from x_0 of M, going to A_i, piercing Σ'_i, and returning to the base point from B_i. We choose this closed path γ_i such that it does not intersect any Σ'_j, $j \neq i$. Then, the collection $\{\alpha_1, \ldots, \alpha_k\}$ forms a free basis of $G = \pi_1(M) = \langle \alpha_1, \ldots, \alpha_k \rangle$ which is a free group of rank k. Any directed closed path in M hitting the Σ'_i transversely represents a word in $\{\alpha_1, \ldots, \alpha_k\}$ by the way it pierces each Σ'_i and the order in which it does so. Without a base point chosen, such a closed path represents a conjugacy class, or equivalently the cyclic word. We call the basis $\{\alpha_1, \ldots, \alpha_k\}$ as a standard basis and spheres $\Sigma'_1, \ldots, \Sigma'_k$ as standard basic spheres.

Group theoretically, embedded spheres in M correspond to splittings of the fundamental group of M. Now, we shall see how an embedded sphere in M corresponds to a splitting of G.

Let S is an embedded sphere in M . If S separates M, then using Van-Kampen's theorem, we can easily get a splitting of the fundamental group G of M. Now, suppose S is non-separating. Choose a regular neighborhood $V = S^2 \times [-1, 1]$ of S and an embedded path γ in $M - V$ from a point of $S^2 \times -1$ to a point $S^2 \times 1$. The sphere S' which is the connected sum of $S^2 \times -1$ with $S^2 \times 1$ along with the boundary of a regular neighborhood U of γ, clearly bounds $U \cup V$ in M. We have $M = U \cup V \cup (M - (U \cup V))$. The set $U \cup V$ is $(S^2 \times S^1) - B^3$ with boundary S' and $(M - (U \cup V))$ is a 3-manifold $M' - B^3$ with boundary S'. Thus, M is a connected sum of $S^2 \times S^1$ with the three manifold M'. Then, we get a spitting of $G = G' * \langle t \rangle$, where $\pi_1(M') = G'$ and $\pi_1(S^2 \times S^1) = \langle t \rangle$. Thus, G can be viewed as an HNN-extension of G' over the trivial subgroup $\{1\}$ of G'.

Now, we shall see that given a splitting of G, there exists an embedded sphere S in M which gives that splitting of G. It follows from 1.8. Here, we give another proof of this in M. There are two cases depending on whether the splitting is an amalgamated free product or a HNN extension over the trivial group.

Suppose $G = F_1 * F_2$. As subgroups of free group are free, both F_1 and F_2 are free. Choose free bases $\{a_1, \ldots, a_m\}$ and $\{b_{m+1}, \ldots, b_{m+n}\}$ of F_1 and F_2, respectively. The set $\{a_1, \ldots, a_m, b_{m+1}, \ldots, b_n\}$ forms a free basis for G. Therefore, $m + n = k$. Any two bases of a free group are equivalent in the sense that there exists an automorphism of that free group sending one basis to another. So, we have an automorphism ϕ of G sending the basis $\{a_1, \ldots, a_m, b_{m+1}, \ldots, b_k\}$ to the standard basis with $\phi(a_i) = \alpha_i$, for $1 \leq i \leq m$

and $\phi(b_{m+j}) = \alpha_{m+j}$, for $1 \leq j \leq n$. Every automorphism of a free group is finite composition of Nielsen automorphisms and every Nielsen automorphism of G is induced by a homeomorphism of M, [33], [42]. Thus, every automorphism of G is induced by a homeomorphism of M fixing the base point. Let h be a homeomorphism of M which fixes the base point and induces the automorphism ϕ on G. The element $\phi(a_i) = h_*(a_i) = \alpha_i$, for $1 \leq i \leq m$, corresponds to the basic standard sphere Σ'_i and $\phi(b_{m+j}) = h_*(b_{m+j}) = \alpha_{m+j}$, for $1 \leq j \leq n$ corresponds to the basic standard sphere Σ'_{m+j}. We can choose an embedded sphere S, disjoint from all Σ'_i such that it partitions the collection of basic standard spheres into two sets, namely, $\{\Sigma'_i, \ldots, \Sigma'_m\}$ and $\{\Sigma'_{m+1}, \ldots, \Sigma'_{m+n}\}$. Then, S gives a free splitting of $G = A * B$, where $A = \langle \alpha_1, \ldots, \alpha_m \rangle$ and $B = \langle \alpha_{m+1}, \ldots, \alpha_{m+n} \rangle$. Now, the sphere $h^{-1}(S) = S'$ gives partition of the collection of spheres $\{h^{-1}(\Sigma'_1), \ldots, h^{-1}(\Sigma'_{m+n})\}$ into two sets $\{h^{-1}(\Sigma'_i), \ldots, h^{-1}(\Sigma'_m)\}$ and $\{h^{-1}(\Sigma'_{m+1}), \ldots, h^{-1}(\Sigma'_{m+n})\}$. The sphere structure $\{h^{-1}(\Sigma'_1), \ldots, h^{-1}(\Sigma'_{m+n})\}$ corresponds to the basis $\{a_1, \ldots, a_m, b_{m+1}, \ldots, b_{m+n}\}$. Then, by applying Van-Kampen theorem, we can see that S' gives the splitting $G = F_1 * F_2$.

Note that if G is an HNN-extension of a subgroup G' of G relative to the subgroups H, K of G' and an isomorphism $\theta : H \to K$, then $H = K = \{1\}$ and G' is a subgroup of rank $n-1$. Thus, $G = G' * \langle t \rangle$, where $t \in G$. We choose a basis $\{c_1, \ldots, c_{k-1}\}$ of G'. The set $\{c_1, \ldots, c_{k-1}, t\}$ forms a basis of G. Then, we have an isomorphism ϕ' of G sending the basis $\{c_1, \ldots, c_{k-1}, t\}$ to the standard basis with $\phi'(c_i) = \alpha_i$, for $1 \leq i \leq k-1$ and $\phi'(t) = \alpha_k$. Let h' be a homeomorphism of M which fixes the base point and induces the automorphism ϕ' on G. The element $\phi'(c_i) = h'_*(c_i) = \alpha_i$, for $1 \leq i \leq k-1$, corresponds to the basic standard sphere Σ'_i and $\phi'(t) = h'_*(t) = \alpha_k$ corresponds to the basic sphere Σ'_k. The sphere structure $\{h'^{-1}(\Sigma'_1), \ldots, h'^{-1}(\Sigma'_{k-1}), h'^{-1}(\Sigma'_k)\}$ corresponds to the basis $\{c_1, \ldots, c_{k-1}, t\}$ of G. Now, one can easily see that the sphere $h'^{-1}(\Sigma'_k)$ gives a splitting of G as an HNN-extension of G' over the trivial subgroup $\{1\}$. Thus, embedded spheres in M corresponds to splittings of the fundamental group of M.

4. ALGEBRAIC AND GEOMETRIC INTERSECTION NUMBERS FOR FREE GROUPS

The *geometric intersection number* of homotopy classes of (simple) closed curves on a surface is the minimum number of intersection points of curves in the homotopy classes. In Chapter 2, we saw that this is a much studied concept and has proved to be extremely useful in low-dimensional topology.

Scott and Swarup [39] introduced an algebraic analogue, called the *algebraic intersection number*, for a pair of *splittings* of groups. This is based on the associated partition of the ends of a group [42]. Splittings of groups are the natural analogue of simple closed curves on a surface F – splittings of $\pi_1(F)$ corresponding to homotopy classes of simple closed curves on F. Scott and Swarup showed that, in the case of surfaces, the algebraic and geometric intersection numbers coincide.

We show here that the analogous result holds for free groups, viewed as the fundamental group of the connected sum $M = \sharp_n S^2 \times S^1$ of n copies of $S^2 \times S^1$. Thus, the manifold M can be regarded as a model for studying the free group and its automorphisms.

Embedded spheres in M correspond to splittings of the free group. Hence, given a pair of embedded spheres in M, we can consider their *geometric intersection number* (defined below) as well as the *algebraic intersection number* of Scott and Swarup for the corresponding splittings. Our main result is that, for embedded spheres in M these two intersection numbers coincide. The principal method we use is the normal form for embedded spheres developed by Hatcher.

Before stating our result, we recall the definition of the intersection numbers.

4.1 Intersection numbers

Definition 4.1.1. Let A and B be two isotopy classes of embedded spheres S and S', respectively, in M. The *geometric intersection number* $I(A, B)$ of A and B is defined as the minimum of the number of components $|S \cap S'|$ of $S \cap S'$ over embedded transversal spheres S and S' representing the isotopy classes A and B, respectively.

This is clearly symmetric. Further, for an embedded sphere S, if $A = [S]$, then $I(A, A) = 0$.

We consider next the algebraic intersection number. Let $\widetilde{M}$ be the universal cover of M. Observe that $\pi_2(M) = \pi_2(\widetilde{M}) = H_2(\widetilde{M})$. The fundamental group $\pi_1(M) = G$ of M, which is a free group of rank n, acts freely on the universal cover $\widetilde{M}$ of M by deck transformations. Homotopy classes of spheres in M

correspond to equivalence classes of elements in $H_2(\widetilde{M})$ up to the action of deck transformations. For embedded spheres, we can consider isotopy classes instead of homotopy classes as the homotopy classes of embedded spheres are the same as isotopy classes of embedded spheres [31].

For an embedded sphere $S \in M$ with lift $\widetilde{S} \in \widetilde{M}$, all the translates of $\widetilde{S}$ are embedded and disjoint from $\widetilde{S}$. In particular, if $\widetilde{A} = [\widetilde{S}]$ is the isotopy class represented by $\widetilde{S}$, then $\widetilde{A}$ and $g\widetilde{A}$ can be represented by disjoint embedded spheres for each deck transformation $g \in G$.

Definition 4.1.2. Let $A = [S]$ and $B = [S']$ be two isotopy classes of embedded spheres S and S', respectively, in M. Let $\widetilde{A} = [\widetilde{S}]$ and $\widetilde{B} = [\widetilde{S'}]$, where $\widetilde{S}$ and $\widetilde{S'}$ are the lifts of S and S', respectively, to $\widetilde{M}$. The *algebraic intersection number* $\widetilde{I}(A,B)$ of A and B is defined as the number of translates $g\widetilde{B}$ of $\widetilde{B}$ such that $\widetilde{A}$ and $g\widetilde{B}$ can not be represented by disjoint embedded spheres in $\widetilde{M}$.

It was shown in [11] that this coincides with the algebraic intersection number of Scott and Swarup.

Definition 4.1.3. We say that two isotopy classes $\widetilde{A} = [\widetilde{S}]$ and $\widetilde{B} = [\widetilde{S'}]$ of embedded spheres in $\widetilde{M}$ *cross* if they cannot be represented by disjoint embedded spheres.

Thus, the algebraic intersection number is the number of elements $g \in \pi_1(M)$ such that $\widetilde{A}$ and $g\widetilde{B}$ cross. We shall also say that $\widetilde{S}$ and $\widetilde{S'}$ cross if the classes they represent cross.

It is immediate that $\widetilde{A}$ and $g\widetilde{B}$ cross if and only if $g^{-1}\widetilde{A}$ and $\widetilde{B}$ cross. It follows that $\tilde{I}(A,B) = \tilde{I}(B,A)$. Thus, the *algebraic intersection number* is symmetric.

Clearly, for all but finitely many translates $g\widetilde{B}$ of $\widetilde{B}$, $\widetilde{A}$ and $g\widetilde{B}$ can be represented by disjoint embedded spheres in $\widetilde{M}$. This is because, for any pair of embedded spheres S and S' in M, all but finitely many translates of $\widetilde{S'}$ are disjoint from $\widetilde{S}$ in $\widetilde{M}$. Hence, $\widetilde{I}(A,B)$ is finite for all isotopy classes A and B of embedded spheres in M.

As was shown in [11], it follows from results of Scott and Swarup that if the algebraic intersection number between classes A and B as above vanishes, then they can be represented by disjoint embedded spheres, i.e., their geometric intersection number vanishes. The converse is an easy observation.

We prove here a much stronger result that the algebraic and geometric intersection numbers are equal.

Theorem 4.1.4. *For isotopy classes A and B of embedded spheres in M, $\widetilde{I}(A,B) = I(A,B)$.*

Our proof is based on the normal form for spheres in M due to Hatcher [17], which we recall in Section 4.2. We extend a sphere Σ in the isotopy class B to a maximal system of spheres and consider a sphere S in the isotopy class of A in normal form with respect to this system. We then show in Section 4.3 that, when S is in normal form, the number of components of intersection between S and Σ is the algebraic intersection number between the isotopy classes $A = [S]$ and B.

Our methods also show that, if $A_1, \dots A_n$ is a collection of isotopy classes of embedded spheres, each pair of which can be represented by disjoint spheres, then all the classes A_i can be simultaneously represented by disjoint spheres. We prove this in Theorem 4.3.3.

An important ingredient of our proofs is the observation that if S and S' are embedded spheres in M and S is in normal form with respect to a maximal system of spheres containing S', then S and S' intersect minimally. This is somewhat analogous to results for geodesics and least-area surfaces [9],[10]. Further the components of intersection correspond to *crossing*. This is very similar to the case of geodesics, where intersections correspond to linking of end points.

4.2 Normal spheres

We recall the notion of *normal sphere systems* from [17].

Let $\Sigma = \cup_j \Sigma_j$ be a maximal system of 2-sphere in M. Splitting M along Σ, then produces a finite collection of 3-punctured 3-spheres P_k. Here a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere.

Definition 4.2.1. A system of 2-spheres $S = \cup_i S_i$ in M is said to be in *normal form with respect to* Σ if each S_i either coincides with a sphere Σ_j or meets Σ transversely in a non empty finite collection of circles splitting S_i into components called pieces, such that the following two conditions hold in each P_k:

1. Each piece in P_k meets each component of ∂P_k in at most one circle.
2. No piece in P_k is a disk which is isotopic, fixing its boundary, to a disk in ∂P_k.

From (1), it follows that the pieces are either disks, cylinders, or pairs of pants. A cylinder piece connects two components of ∂P_k and a pants piece connects all three components of ∂P_k. A disk piece has boundary on one component of ∂P_k and separates the other two components of ∂P_k, by (2). Hence a P_k can not contain both disk and pants pieces, and all the disk pieces in a P_k must have their boundaries on the same component of ∂P_k. Each individual cylinder or pants piece in a P_k must be unknotted in P_k since its boundary circles lie on different components of ∂P_k, but a collection of cylinder and pants pieces can be knotted and linked in a complicated fashion. However, since homotopic systems are isotopic, such knotting and linking can always be eliminated by an isotopy of the system in M, though the isotopy will generally have to move outside P_k.

Recall the following result from [17].

Proposition 4.2.2 (Hatcher)**.** *Every system $S \subset M$ can be isotoped to be in normal form with respect to Σ. In particular, every essential embedded sphere S in M can be isotoped to be in normal form with respect to Σ.*

We recall some constructions from [17]. First, we associate a tree T to $\widetilde{M}$ corresponding to the decomposition of M by Σ. Let $\widetilde{\Sigma}$ be the pre-image of Σ in $\widetilde{M}$. The closure of each component of $\widetilde{M} - \widetilde{\Sigma}$ is a 3-punctured 3-sphere. The vertices of the tree are of two types, with one vertex corresponding to the closure of each component of $\widetilde{M} - \widetilde{\Sigma}$ and one vertex for each component of $\widetilde{\Sigma}$. An edge of T joins a pair of vertices if one of the vertices corresponds to the closure of a component X of $\widetilde{M} - \widetilde{\Sigma}$ and the other vertex corresponds to a component of $\widetilde{\Sigma}$ that is in the boundary of X. Thus, we have a Y-shaped subtree corresponding to each complementary component. We pick an embedding of T in $\widetilde{M}$ respecting the correspondences.

Given a sphere S in normal form with respect to Σ and a lift $\widetilde{S}$ of S to $\widetilde{M}$, we associate a tree $T(\widetilde{S})$ corresponding to the decomposition of $\widetilde{S}$ into pieces. The tree has two types of vertices, vertices corresponding to closures of components of $\widetilde{S}-\widetilde{\Sigma}$ (i.e., pieces) and vertices corresponding to each component of $\widetilde{S}\cap\widetilde{\Sigma}$. Edges join a pair of vertices if one of the vertices corresponds to a piece and the other to a boundary component of the piece.

In [17], it is shown that $T(\widetilde{S})$ is a tree. Moreover, the inclusion $\widetilde{S} \hookrightarrow \widetilde{M}$ induces a natural inclusion map $T(\widetilde{S}) \hookrightarrow T$. So, we can view $T(\widetilde{S})$ as a subtree of T. The bivalent vertices of T correspond to spheres components in $\widetilde{\Sigma}$, i.e., lifts of the spheres Σ_j and their translates.

4.3 Algebraic and Geometric Intersection numbers

Consider now two isotopy classes A and B of embedded spheres in M. Choose an embedded sphere Σ_1 in the isotopy class B and extend this to a maximal collection Σ of spheres. Let S be a representative for A in normal form with respect to Σ. Theorem 4.1.4 is equivalent to showing that $\widetilde{I}(A,[\Sigma_j]) = I(A,[\Sigma_j])$ for $j = 1$. We begin by showing the non-trivial inequality here.

Lemma 4.3.1. *If $A = [S]$ is the isotopy class of the embedded sphere S in M, then for the isotopy class $[\Sigma_j]$ of Σ_j in M, $\widetilde{I}(A,[\Sigma_j]) \geq I(A,[\Sigma_j])$.*

Proof. The sphere S, which is in normal form with respect to Σ, represents the class A. We shall show that the number of components of intersection of S with Σ_j is $\widetilde{I}(A,[\Sigma_j])$. As the geometric intersection number is the minimum of the number of components of intersection of spheres in the isotopy classes, the lemma is an immediate consequence of this claim.

Fix a lift $\widetilde{S}$ of S. The components of $S \cap \Sigma_j$ are homotopically trivial circles in M. These lift to circles of intersection between $\widetilde{S}$ and components of the pre-image of Σ_j. These correspond to vertices of $T(\widetilde{S})$. As $T(\widetilde{S})$ is a tree which embeds in T, different circles of intersection of S and Σ_j correspond to intersections of $\tilde{S}$ with different components of the pre-image of Σ_j. It follows that the number of components of intersection of S with Σ_j is the number of components of the pre-image of Σ_j that intersect $\widetilde{S}$.

The main observation needed is the following lemma.

Lemma 4.3.2. *If $\widetilde{S}$ intersects a component $\widetilde{\Sigma_j}$ of the pre-image of Σ_j, then the spheres $\widetilde{S}$ and $\widetilde{\Sigma_j}$ cross.*

Proof. Assume that $\widetilde{S}$ intersects the component $\widetilde{\Sigma_j}$ of the pre-image of Σ_j. The sphere $\widetilde{\Sigma_j}$ corresponds to a vertex v_0 of T. As $\widetilde{S}$ intersects $\widetilde{\Sigma_j}$ and S is in normal form, the vertex v_0 is an interior vertex of $T(\widetilde{S})$.

We recall the notion of crossing due to Scott and Swarup, which by [11] is equivalent to the notion we use. The spheres $\widetilde{S}$ and $\widetilde{\Sigma_j}$ partition the ends of $\widetilde{M}$ into pairs of complementary subsets $E^{\pm}_S$ and $E^{\pm}_{\Sigma}$, corresponding to the components of the complement of the respective spheres in $\widetilde{M}$. The spheres $\widetilde{S}$ and $\widetilde{\Sigma_j}$ cross if all the four intersections $E^{\pm}_{\widetilde{S}} \cap E^{\pm}_{\widetilde{\Sigma}}$ are non-empty.

A properly embedded path $c : \mathbb{R} \to \widetilde{M}$ induces a map from the ends $\pm\infty$ of $\mathbb{R}$ to the ends of $\widetilde{M}$. Thus, we can associate to c a pair of ends $c_{\pm}$. We say that the path c is a path from c_- to c_+. Poincaré duality gives a useful criterion for when two ends E and E' of $\widetilde{M}$ are in different equivalence classes with respect to the partition corresponding to $\widetilde{S}$. Namely, E and E' are in different equivalence classes if and only if there is a proper path c from E to E' so that $c \cdot \widetilde{S} = \pm 1$, with $c \cdot \widetilde{S}$ the intersection pairing obtained from the cup product using the duality between homology and cohomology with compact support.

The ends of $\widetilde{M}$ can be naturally identified with the ends of the tree T. The sets $E^{\pm}_{\widetilde{\Sigma}}$ correspond to the ends of the two components of $T - \{v_0\}$. It is easy to see that $\widetilde{\Sigma_j}$ and $\widetilde{S}$ cross if and only if each of the sets $E^{\pm}_{\widetilde{\Sigma}}$ contain pairs of ends E_1 and E_2 which are in different equivalence classes with respect to the partition corresponding to $\widetilde{S}$. By symmetry, it suffices to consider the case of $E^{+}_{\widetilde{\Sigma}}$. Let X denote the closure of the component of $\widetilde{M} - \widetilde{\Sigma}_j$ with $ends(X) = E^{+}_{\widetilde{\Sigma}}$.

As v_0 is an internal vertex of the tree $T(\widetilde{S})$, there is a terminal vertex w of $T(\widetilde{S})$ contained in X. A terminal vertex of $T(\widetilde{S})$ corresponds to a piece which is a disc D in a 3-punctured sphere P, with P the closure of a component of $\widetilde{M} - \widetilde{\Sigma}$. Let Q_1 and Q_2 denote the boundary components of P disjoint from D (hence from S). Then D separates Q_1 and Q_2.

For $i = 1, 2$, let W_i denote the closure of the component of $\widetilde{M} - Q_i$ which does not contain S. As Q_i is the lift of an essential sphere, and $\widetilde{M}$ is simply-connected, Q_i is non-trivial as an element of $H_2(\widetilde{M})$. Hence W_i is non-compact. By construction $W_i \subset X$, hence the ends of W_i are contained in $E^{+}_{\widetilde{\Sigma}}$.

As D separates Q_1 and Q_2, (after possibly interchanging Q_1 and Q_2) there is a path $c : [0, 1] \to P$ intersecting S transversely in one point (with the sign of the intersection $+1$) so that $c(0) \in Q_1$ and $c(1) \in Q_2$. As W_1 and W_2 are non-compact, we can extend c to a proper function $c : \mathbb{R} \to \widetilde{M}$ with $c((-\infty, 0)) \subset W_1$ and $c((1, \infty)) \subset W_2$.

The ends E_1 and E_2 of c are ends of X (as $W_i \subset X$ for $i = 1, 2$). Further, by construction $c \cdot \widetilde{S} = 1$. It follows that the ends $E_1, E_2 \subset E^{+}_{\widetilde{\Sigma}}$ are in different components with respect to the partition corresponding to $\widetilde{S}$. By symmetry, we can find a similar pair of ends in $E^{-}_{\widetilde{\Sigma}}$. It follows that $\widetilde{S}$ and $\widetilde{\Sigma}$ cross.

□

We now complete the proof of Lemma 4.3.1. We have seen that the number of components of $S \cap \Sigma_j$ is the number of components of the pre-image of Σ_j which intersect $\widetilde{S}$. For a fixed lift $\widetilde{\Sigma_j}$ of Σ_j, the components of the pre-images of Σ_j are the translates $g\widetilde{\Sigma_j}$ of $\widetilde{\Sigma_j}$.

By Lemma 4.3.2, it follows that if $\widetilde{S}$ intersects $g\widetilde{\Sigma_j}$, then $\widetilde{S}$ crosses $g\widetilde{\Sigma_j}$. The converse of this is obvious. By the definition of algebraic intersection number, Lemma 4.3.1 follows.

□

Proof of Theorem 4.1.4. We have seen that it suffices to consider the case when $A = [S]$, $B = [\Sigma_1]$ and S is in normal form with respect to Σ. By Lemma 4.3.1, $\widetilde{I}(A, B) \geq I(A, B)$.

Conversely, let S and Σ_1 be embedded spheres with $A = [S]$, $B = [\Sigma_1]$ and $I(A, B) = |S \cap \Sigma_1|$. Let $\widetilde{S}$ and $\widetilde{\Sigma_1}$ be lifts of S and Σ_1, respectively, to $\widetilde{M}$. Observe that (distinct) components of intersection of S with Σ_1 lift to (distinct) components of intersection of $\widetilde{S}$ with translates of $\widetilde{\Sigma_1}$. Hence the number of translates of $\widetilde{\Sigma_1}$ that intersect $\widetilde{S}$ is at most $I(A, B)$. As $\tilde{I}(A, B)$ is the number of translates of $\widetilde{\Sigma_1}$ that *cross* $\widetilde{S}$, and components that cross must intersect, it follows that $\tilde{I}(A, B) \leq I(A, B)$.

This completes the proof of the theorem.

□

Our methods also yield the following result. This also follows from the work of Scott and Swarup, see [39].

Theorem 4.3.3. *If $A_1, \ldots, A_n$ are isotopy classes of embedded spheres in M such that, for $1 \leq i, j \leq n$, A_i and A_j can be represented by disjoint spheres, then there exist disjointly embedded spheres S_i, $1 \leq i \leq n$, such that $A_i = [S_i]$.*

Proof. We prove this by induction on n. For $n = 1, 2$, the conclusion is immediate from the hypothesis. Assume that the result holds for $n = k$ and consider a collection A_i as in the hypothesis with $n = k + 1$.

Suppose one of the spheres, which we can assume without loss of generality is A_n, is not essential. By the induction hypothesis, there are disjoint embedded spheres S_i, $1 \leq i < n$, with $[S_i] = A_i$. Choose a 3-ball disjoint from the spheres S_i, $1 \leq i < n$ and let S_n be its boundary. Then, the spheres S_i, $1 \leq i \leq n$, give the required collection.

Thus, we may assume that all the isotopy classes A_i of spheres are essential. By induction hypothesis, there are disjoint embedded spheres S_i, $1 \leq i < n$, with $[S_i] = A_i$. As these are essential by our assumption, we can extend the collection S_i to a maximal system of spheres. We let S_n be a sphere in normal form with respect to this collection. By hypothesis, $I(S_n, S_i) = 0$ for $1 \leq i \leq n$. By the proof of Lemma 4.3.1, it follows that S_n is disjoint from S_i. Thus, S_i, $1 \leq i \leq n$, is a collection of disjoint embedded spheres with $A_i = [S_i]$.

□

Remark 4.3.4. The above theorem shows that the sphere complex associated to M is a full complex in the sense that if $V_1, V_2, \ldots., V_k$ are the vertices of the sphere complex and if there is an edge between every pair V_i, V_j of vertices, where $1 \leq i, j \leq k$, then these vertices bound a simplex in the sphere complex.

The *geometric intersection number* of curves on a surface has been used to give constructions like the space of measured laminations whose projectivization is the boundary of Teichmüller space, [35], as well as to study geometric properties, including hyperbolicity of the curve complex in [5], [36]. One may hope that the *geometric intersection number* of embedded spheres in M might be useful to give such constructions in case sphere complex and Outer space.

5. EMBEDDED SPHERES, NORMAL FORM AND PARTITIONS OF ENDS

In this chapter, we study embedded spheres in $M = \sharp_k S^2 \times S^1$ and $\widetilde{M}$, the universal cover of M. In the Section 5.1, we see how a partition A of the set of ends of $\widetilde{M}$ corresponds to an embedded sphere in $\widetilde{M}$ which is in normal form in the sense of Hatcher, by specifying the data determining the partition A and the normal sphere. Given a properly embedded path $c : \mathbb{R} \to \widetilde{M}$ and a homology class $A \in H_2(\widetilde{M})$, we have an intersection number $c \cdot A$. Further, this depends only on the ends $c_\pm$ of the path c. In the Section 5.2, we prove that the class $A \in H_2(\widetilde{M})$ can be represented by an embedded sphere in $\widetilde{M}$ if and only if, for each proper map $c : \mathbb{R} \to \widetilde{M}$, $c \cdot A \in \{0, 1, -1\}$. We also constructively prove that the class $A \in \pi_2(M)$ can be represented by an embedded sphere in M if and only if A can be represented by an embedded sphere in $\widetilde{M}$ and for all deck transformations $g \in \pi_1(M)$, A and gA do not cross.

5.1 *Partition of ends and normal forms*

This section is devoted to associating to a partition $A = (E^+(A), E^-(A))$ of the space $E(\widetilde{M})$ of ends of $\widetilde{M}$ into open sets, an embedded sphere S in $\widetilde{M}$ which is in normal form in the sense of Hatcher, so that $E^\pm(S) = E^\pm(A)$. Along the way, we see what data determines a sphere in normal form in $\widetilde{M}$ and the relation between this data, partitions of ends and crossings. Specifically, we prove the following:

Theorem 5.1.1. *Given a partition $A = (E^+(A), E^-(A))$ of the set $E(\widetilde{M})$ of ends of $\widetilde{M}$ into two open sets, there is a normal sphere S in $\widetilde{M}$ so that A is the partition given by the ends of the components of $\widetilde{M} - S$.*

In this case, we say S represents A. We recall the notion of crossing of two such partitions A and B.

Definition 5.1.2. We say that A and B *cross* if each of the four sets $E^\epsilon(A) \cap E^\eta(B)$ is non-empty, for ϵ and η obtained by choosing signs ϵ and η in $\{+, -\}$.

Suppose A and B do not cross, then for some choice of sign, $E^\epsilon(A) \supset E^\eta(B)$. It follows that $E^{\bar{\epsilon}}(A) \subset E^{\bar{\eta}}(B)$, where $\bar{\epsilon}$ and $\bar{\eta}$ denote the opposite signs. Further, if $A \neq B$, then the inequalities are strict.

Definition 5.1.3. We say that B is on the positive side of A if $E^+(A) \supset E^\eta(B)$ for some sign η. Otherwise, we say that B is on the negative side of A. In general, we say that B is on the ϵ-side of A, for the appropriate sign ϵ.

Note also that if A and B do not cross, then either $E^+(A) \subset E^\eta(B)$ for some sign η or $E^-(A) \subset E^\eta(B)$ for some sign η. Further, these are exclusive unless $A = B$.

We shall need the following elementary observation.

Lemma 5.1.4. *If B is on the ϵ-side of A and C is on the $\bar{\epsilon}$-side, then B and C do not cross.*

Proof. For appropriate signs η and ξ, $E^\eta(B) \subset E^\epsilon(A)$ and $E^\xi(C) \subset E^{\bar{\epsilon}}(A)$. Hence, $E^\eta(B) \cap E^\xi(C) = \phi$, which shows that B and C do not cross. □

If B is on the ϵ-side of A and C is on the $\bar{\epsilon}$-side, then we say that A is *between* B and C.

Given a sphere S in $\widetilde{M}$, we have a natural partition of ends of $\widetilde{M}$ associated to it. So, we can talk about crossing of a partition A of $E(\widetilde{M})$ and sphere S. We now turn to the construction of the normal sphere in $\widetilde{M}$ representing A.

We recall the notion of *normal sphere systems* from [17].

Let $\Sigma = \cup_j \Sigma_j$ be a maximal system of 2-sphere in M. We recall that splitting M along Σ, then produces a finite collection of 3-punctured 3-spheres P_k. Here, a 3-punctured 3-sphere is the complement of the interiors of three disjointly embedded 3-balls in a 3-sphere.

Definition 5.1.5. A system of 2-spheres $S = \cup_i S_i$ in M is said to be in *normal form with respect to* Σ if each S_i either coincides with a sphere Σ_j or meets Σ transversely in a non-empty finite collection of circles splitting S_i into components called pieces, such that the following two conditions hold in each P_k:

1. Each piece in P_k meets each component of ∂P_k in at most one circle.
2. No piece in P_k is a disk which is isotopic, fixing its boundary, to a disk in ∂P_k.

Similarly, we can define sphere systems in *normal form* with respect to the pre-image $\widetilde{\Sigma}$ of Σ in $\widetilde{M}$.

We call each sphere Σ_i as a standard sphere in M and $\widetilde{\Sigma_i}$ as standard sphere in $\widetilde{M}$.

Given a sphere S in normal form with respect to Σ in M and a lift $\widetilde{S}$ of S to $\widetilde{M}$, we associate a tree $T(\widetilde{S})$ corresponding to the decomposition of $\widetilde{S}$ into pieces. We consider the dual tree $T(\tilde{S})$ to $\tilde{S} \cap \tilde{\Sigma}$ in $\tilde{S}$, having a vertex for each component of $\tilde{S} \setminus \tilde{\Sigma}$ and an edge for each circle of $\tilde{S} \cap \tilde{\Sigma}$.

In [17], it is shown that $T(\widetilde{S})$ is a tree. Moreover, the inclusion $\widetilde{S} \hookrightarrow \widetilde{M}$ induces a natural inclusion map $T(\widetilde{S}) \hookrightarrow T$. So, we can view $T(\widetilde{S})$ as a subtree of T. The bivalent vertices of T correspond to spheres components in $\widetilde{\Sigma}$, i.e., lifts of the spheres Σ_j and their translates. This shows that each $\widetilde{P_k}$ contains at the most one piece of $\widetilde{S}$. Similarly, one can easily see that if S' is a normal sphere in $\widetilde{M}$, then each $\widetilde{P_k}$ contains at the most on piece of S'. If S is a standard sphere (or can be isotoped to standard sphere), then the associated tree $T(\widetilde{S})$ is single vertex in T corresponding to that standard vertex.

Our construction is motivated by the following lemma from [12].

Lemma 5.1.6. *Let S be a normal sphere in $\widetilde{M}$ and let $\widetilde{\Sigma}_i$ be a standard sphere.*

- *The spheres S and $\widetilde{\Sigma}_i$ intersect if and only if they cross.*

- *If S and $\widetilde{\Sigma}_i$ intersect, they intersect transversally in a circle S^1.*

Thus, if A is represented by a normal sphere S in $\widetilde{M}$, we can determine the intersection of S with each standard sphere in $\widetilde{M}$, in terms of crossings. The standard spheres in $\widetilde{M}$ correspond to the bivalent vertices of the tree T. We call them as "*standard vertices*". We can talk about the crossing of A with the classes of standard spheres in $\widetilde{M}$. Now, we associate a subgraph τ of T to A as follows: If A crosses standard sphere $\widetilde{\Sigma_i}$, then τ contains the bivalent vertex v_i corresponding to $\widetilde{\Sigma_i}$ and the edges e_1^i and e_2^i containing that vertex v_i. The other end vertex v_j^i of each edge $e_j^i, j = 1, 2$ is a trivalent vertex in T which corresponds to a component of $\widetilde{M} - \widetilde{\Sigma}$. Each v_j^i may be a bivalent or univalent or a trivalent vertex in τ. If A does not cross some standard sphere in $\widetilde{M}$, then τ does not contain the standard vertex corresponding this standard sphere and hence, it does not contain the edges containing this standard vertex. Note that any edge e in T has a unique end vertex which is a standard bivalent vertex in T.

Lemma 5.1.7. *Suppose $A = (E^+, E^-)$ is a partition of $E(\widetilde{M})$ into two open sets such that $E^+ \neq \phi \neq E^-$. Suppose no standard sphere crosses A. Then, there exists a standard sphere Σ_0 such that $E^\pm = E^\pm(\Sigma_0)$.*

Proof. By hypothesis, if v is a standard bivalent vertex of T, the standard sphere $\Sigma(v)$ corresponding to v does not cross A. Hence, after choosing orientations appropriately, either $E^+(\Sigma(v)) \subset E^+$ or $E^-(\Sigma(v)) \subset E^-$. If $\Sigma(v) = \Sigma_0$ satisfies both the conditions, then $E^\pm = E^\pm(\Sigma_0)$.

Suppose no $\Sigma(v)$ satisfies both the above conditions, we get a partition of bivalent vertices of T as

$$V^+ = \{v : E^+(\Sigma(v)) \subset E^+\}$$

and

$$V^- = \{v : E^-(\Sigma(v)) \subset E^-\}.$$

Let $X^\pm$ is the union of all the edges e in T such that the bivalent vertex of e lies in $V^\pm$. Then $X^\pm$ are closed and $T = X^+ \cup X^-$. Hence, $X^+ \cap X^- \neq \phi$. By construction, $X^+ \cap X^-$ consists of trivalent vertices of T. Let $w \in X^+ \cap X^-$ and let v_1, v_2 and v_3 be bivalent vertices adjacent to w. Note that at least one $v_i \in X^+$ and at least one $v_j \in X^-$. Without loss of generality, suppose $v_1, v_2 \in X^+$ and $v_3 \in X^-$. Let $N(w)$ denote the set of all the points in T distance at most 1 from w. Then, $T - N(w)$ has three components V_1, V_2 and V_3 whose closures contain the vertices v_1, v_2 and v_3, respectively. It is easy to see that $E(V_1) \subset E^+, E(V_2) \subset E^+$ and $E(V_3) \subset E^-$. It follows that $E^+(\Sigma(v_3)) = E^+(\Sigma(v_1)) \cup E^+(\Sigma(v_2))$. This implies $E^+(\Sigma(v_3)) \subset E^+$. As $v_3 \in X^-$, $E^-(\Sigma(v_3)) \subset E^-$. But then, $v_3 \in V^+ \cap V^-$. This is a contradiction as V^+ and V^- are disjoint. Hence, there must exist a standard sphere Σ_0 such that $E^\pm = E^\pm(\Sigma_0)$. □

Thus, if A does not cross any standard sphere, the tree τ associated to A is a standard vertex corresponding to the standard sphere representing A.

A normal sphere S in $\widetilde{M}$ has connected intersection with each set $\widetilde{P_k}$. The set $S \cap \widetilde{P_k}$ (which we call a piece) is a disc D, an annulus (which we call a tube) A or a thrice punctured 2-sphere Y (which we call a

Y-piece or pant piece) according as the number of edges in $\tau(S)$ adjoining v corresponding to $\widetilde{P_k}$ is 1, 2 or 3. We first make some observations about these cases.

Firstly, suppose v is a vertex of τ adjacent to a single edge $e_0 \in \tau$, i.e., a terminal vertex of τ. Let $v_0 \in \tau$ be the other end vertex of e_0 and Σ_0 be the standard sphere in $\widetilde{M}$ corresponding to v_0. Then, A crosses Σ_0. Let the other edges adjacent to v in T be e_1 and e_2 with other end vertices v_1 and v_2, respectively. Consider the standard spheres $\tilde{\Sigma}_i = \tilde{\Sigma}(v_i)$ corresponding to vertices v_i, with orientations chosen so that for $i = 1, 2$, the set $E^+(\tilde{\Sigma}_i)$ is the set of ends of the component of $\widetilde{M} - \tilde{\Sigma}_i$ that does not contain $\tilde{\Sigma}_0$. We can orient $\tilde{\Sigma}_0$ so that $E^+(\tilde{\Sigma}_0) = E^+(\tilde{\Sigma}_1) \cup E^+(\tilde{\Sigma}_2)$.

Lemma 5.1.8. *For some sign ϵ, $E^\epsilon(A) \supset E^+(\tilde{\Sigma}_1)$ and $E^{\bar{\epsilon}}(A) \supset E^+(\tilde{\Sigma}_2)$.*

Proof. First note that we cannot have $E^+(\tilde{\Sigma}_i) \supset E^\eta(A)$, for $i = 1, 2$, as this would imply that $E^+(\tilde{\Sigma}_0) \supset E^\eta(A)$, contradicting the hypothesis that A crosses $[\tilde{\Sigma}_0]$. Hence, as A does not cross the spheres $\tilde{\Sigma}_i$, for appropriate signs ϵ_i, $E^{\epsilon_i}(A) \supset E^+(\tilde{\Sigma}_i)$ for $i = 1, 2$. Finally, if $\epsilon_1 = \epsilon_2 = \epsilon$, then $E^\epsilon(A) \supset E^+(\tilde{\Sigma}_0)$ as $E^+(\tilde{\Sigma}_0) = E^+(\tilde{\Sigma}_1) \cup E^+(\tilde{\Sigma}_2)$, contradicting the hypothesis that A crosses $\tilde{\Sigma}_0$. □

Thus, one of the spheres $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ is on the positive side of A and the other on the negative side. In the case of a vertex v of valence 2 of τ, either it is a bivalent vertex (standard vertex) of T or there is an edge e_v of T adjacent to v which is not in τ. The standard sphere $\tilde{\Sigma}(e_v)$ corresponding to the other end vertex of the edge e_v is either on the positive side of A or on the negative side.

We shall see that τ is a tree and the partition A is determined by τ together with data of the above form at terminal and non-standard bivalent vertices of τ. The standard bivalent vertices of τ are standard (bivalent) vertices of T. We begin by showing that τ is a tree. As this is a subgraph of a tree T, it suffices to show that τ is connected.

Lemma 5.1.9. *The subgraph $\tau \subset T$ is connected, hence a tree.*

Proof. Suppose τ is not connected. As τ is a subgraph of T, there is a standard vertex $v \notin \tau$ such that both components X_1 and X_2 of $T - v$ intersect τ. Let $\tau_i = \tau \cap X_i$, for $i = 1, 2$. Let $\tilde{\Sigma}_0 = \tilde{\Sigma}(v)$.

As $v \notin \tau$, A does not cross $\tilde{\Sigma}$. Hence, we can orient $\tilde{\Sigma}_0$ so that for some sign ϵ, $E^+(\tilde{\Sigma}_0) \subset E^\epsilon(A)$. Without loss of generality, $E^+(\tilde{\Sigma}_0)$ is the set of ends of X_1.

Let τ' be the convex hull of the vertex v and τ_1. As τ is a finite graph, τ' is a finite tree. Let w be a terminal vertex of τ' distinct from v. Then, w is a vertex of τ by Lemma 5.1.8, there is an adjacent edge $e' \notin \tau$ with its other end vertex v' such that $E^+(\tilde{\Sigma}(v')) \subset E^{\bar{\epsilon}}(A)$, with the orientation chosen so that $E^+(\tilde{\Sigma}(v'))$ is the set of ends of the component of $\widetilde{M} - \tilde{\Sigma}(v')$ that does not intersect τ. It follows that $E^+(\tilde{\Sigma}(v')) \subset E^+(\tilde{\Sigma}_0)$, and hence, $E^+(\tilde{\Sigma}_0) \cap E^{\bar{\epsilon}}(A) \neq \phi$, a contradiction. □

We next see that τ is a finite tree.

Lemma 5.1.10. *The tree τ is compact, hence finite.*

Proof. If τ is not compact, then some end $P \in E(\widetilde{M}) = E(T)$ is an end of τ. Without loss of generality $P \in E^+(A)$. As $E^+(A)$ is open in the space of ends of T, there is a finite connected tree $\kappa \subset T$ and a component V of $T - \kappa$ so that $P \in E(V) \subset E^+(A)$. We shall show that no edge of V is contained in τ, contradicting the assumption that P is an end of τ.

Let e be an edge of T contained in $V = T - \kappa$. Then, as κ is connected, some component W of $T - e$ is disjoint from κ, and hence contained in V. Suppose v is the end vertex of e such that v is a standard bivalent vertex in T. Let $\Sigma(v)$ be the standard sphere corresponding to v, then it follows that for some sign ε, $E^\varepsilon(\Sigma(v)) \subset E(V) \subset E^+(A)$, and hence, $\Sigma(v)$ does not cross A. This implies v is not in τ. It follows that e is not in τ. Thus, no edge of V is in τ, as required. □

Thus, we have a finite tree τ associated to the partition A. Note that the terminal vertices of τ are trivalent vertices in T. We shall next show that the partition A is determined by the tree together with additional data for vertices adjoining the tree.

Let $N(\tau)$ be the subgraph of T consisting of points with distance at most 1 from τ. Then, $N(\tau)$ is a tree, which is the union of τ with the following two kinds of edges:

1. For each terminal vertex v of τ, we have a pair of edges $e_1(v) \notin \tau$ and $e_2(v) \notin \tau$ with v as an end-vertex. Let v_1 and v_2 be the other end vertices of e_1 and e_2, respectively.

2. For each non-standard bivalent vertex w of τ, we have an edge $e(w) \notin \tau$ with w as an end-vertex. Let w_1 be its other end vertex.

By Lemma 5.1.8, for a terminal vertex v, the sphere corresponding to one of v_1 and v_2 is on the positive side of τ (positive side of A). The vertices v_1 and v_2 are end vertices of e_1 and e_2 respectively. So, we can assign positive or negative signs to these edges accordingly. We denote this by $e_+(v)$ and denote the other edge (which is on the negative side) by $e_-(v)$. We denote the standard spheres corresponding to v_1 and v_2 by $\tilde{\Sigma}(v_1) = \tilde{\Sigma}(e_1)$ and $\tilde{\Sigma}(v_2) = \tilde{\Sigma}(e_2)$, respectively. For a non-standard bivalent vertex w of τ, we can associate a sign $\epsilon(w)$ so that $\tilde{\Sigma}(w_1) = \tilde{\Sigma}(e(w))$ is on the $\epsilon(w)$-side of A.

We show that the tree τ together with the additional data determines a partition of the ends, which coincides with the given partition.

Lemma 5.1.11. *The partition A is determined by τ together with the functions $e_+(v)$ and $\epsilon(w)$, where v's is univalent vertices of τ and w's is non-standard bivalent vertices of τ.*

Proof. We show that the partition of the ends of $\widetilde{M}$ into $E^\pm(A)$ is determined by the given data. The spheres $\tilde{\Sigma}(e_\pm(v))$ for terminal vertices of τ together with $\tilde{\Sigma}(e(w))$ for non-standard bivalent vertices of τ separate $\widetilde{M}$ into a compact component corresponding to τ and one non-compact component for each such sphere $\tilde{\Sigma}$. We can orient the spheres $\tilde{\Sigma}$ so that A is on the negative side of $\tilde{\Sigma}$. Then, the set of ends of the non-compact component is $E^+(\tilde{\Sigma})$. Hence, we have a partition

$$E(\widetilde{M}) = \cup_v (E^+(\tilde{\Sigma}(e_+(v))) \cup E^+(\tilde{\Sigma}(e_-(v)))) \cup_w E^+(\tilde{\Sigma}(e(w)))$$

By construction, $E^+(\tilde{\Sigma}(e_+(v))) \subset E^+(A)$ and $E^+(\tilde{\Sigma}(e_-(v))) \cap E^+(A) = \phi$, for each terminal vertex v of τ. For each non-standard bivalent vertex w of τ, $E^+(\tilde{\Sigma}(e(w))) \subset E^+(A)$ if $\epsilon(w) = +$ and $E^-(\tilde{\Sigma}(e(w))) \cap E^+(A) = \phi$ otherwise. Hence,

$$E(A) = \cup_v E^+(\tilde{\Sigma}(e_+(v))) \cup_{\{w:\epsilon(w)=+\}} E^+(\tilde{\Sigma}(e(w)))$$

This is determined by the given data. Hence, the partition A is determined by the given data.

□

It is now easy to construct a normal sphere S in $\widetilde{M}$ representing the same the partition A. Note that a normal sphere S represents the partition A if and only if $E^\pm(S) = E^\pm(A)$.

We can associate to an oriented normal sphere S data very similar to that associated to the partition A. Firstly, the sphere S has a support which is a subtree τ. A terminal vertex v of τ corresponds to a disc pieces in a thrice-punctured 3-sphere $P(v)$, which separates the two other boundary components of $P(v)$. Exactly one of these lies on the positive side. Thus, as the boundary components correspond to vertices of T adjacent to v, we get a pair of edges $e_\pm(v)$. A non-standard bivalent vertex w of τ corresponds to an annulus piece (cylinder piece) in $P(w)$. The boundary component of $P(w)$ not intersecting the annulus is on either the positive or the negative side of S, giving a sign $\epsilon(w)$. A trivalent vertex w' corresponds to a pant piece in the 3-punctured 3-sphere $P(w')$.

Lemma 5.1.12. *Given a finite tree τ, associated data $e_+(v)$ and $\epsilon(w)$, and a oriented circle on the sphere $\tilde{\Sigma}(V')$ for each non-terminal vertex v' of τ, there is an oriented normal sphere S with corresponding data τ, e_+ and ϵ and whose restriction to each thrice-punctured 3-sphere has boundary the corresponding oriented circles. Further, the partition corresponding to S is the one corresponding to the data τ, e_+ and ϵ.*

Proof. To each standard bivalent vertex of the tree τ, we associate a circle in the corresponding standard sphere which we co-orient according to the given partition. For each trivalent vertex v of τ, we associate a pants piece with boundary the circles in the standard sphere that have been constructed.

Next, if v is a non-standard bivalent vertex of τ, the two adjacent standard vertices correspond to circles on two standard spheres. We join them by an annulus so that the other standard sphere bounding the 3-holed sphere corresponding to v is on the side of the annulus given by $\epsilon(v)$. Similarly, for a terminal vertex of τ we consider a disc so that the standard spheres corresponding to adjacent vertices $e_+(v)$ of T that is not in τ is on the positive side of the disc and the one corresponding to the other adjacent vertex is on the negative side. □

5.2 Embedding classes, normal spheres and graphs of trees

In this section, we give proofs of theorems of [11] using normal forms.

We give a constructive proof of the following result from [11] giving a criterion for a class $A \in \pi_2(M) = H_2(\widetilde{M})$ to be representable by an embedded sphere in M.

As $H_1(\widetilde{M}) = 0$, a homology class A is determined by the intersection numbers $c \cdot A$, where $c : \mathbb{R} \to \widetilde{M}$ is a proper path. The first result of [11] characterizes which classes in $H_2(\widetilde{M})$ can be represented by embedded spheres in $\widetilde{M}$.

Theorem 5.2.1. *The class $A \in H_2(\widetilde{M})$ can be represented by an embedded sphere in $\widetilde{M}$ if and only if, for each proper map $c : \mathbb{R} \to \widetilde{M}$, $c \cdot A \in \{0, 1, -1\}$.*

It is easy to see (for proofs see [11]) that if S is an embedded sphere in $\widetilde{M}$, then S partitions $\widetilde{M}$ into two components with closure of each component non- compact. Hence, the ends of $\widetilde{M}$ are partitioned into components $E^{\pm}(S)$, so that if $c : \mathbb{R} \to \widetilde{M}$ is a proper path, then

- If $c_- \in E^-(S)$ and $c_+ \in E^+(S)$, then $c \cdot S = 1$
- If $c_- \in E^+(S)$ and $c_+ \in E^-(S)$, then $c \cdot S = -1$
- If $c_- \in E^-(S)$ and $c_+ \in E^-(S)$, then $c \cdot S = 0$
- If $c_- \in E^+(S)$ and $c_+ \in E^+(S)$, then $c \cdot S = 0$

In particular, $c \cdot S \in \{0, 1, -1\}$.

Conversely, let A be a homology class satisfying the hypothesis of the theorem. We shall construct a normal sphere in $\widetilde{M}$ that represents A.

The first step is the following Lemma whose proof is in [11].

Lemma 5.2.2. *There is a partition $E^{\pm}(A)$ of the set $E(\widetilde{M})$ of ends $\widetilde{M}$ so that*

- *If $c_- \in E^-(A)$ and $c_+ \in E^+(A)$, then $c \cdot A = 1$*
- *If $c_- \in E^+(A)$ and $c_+ \in E^-(A)$, then $c \cdot A = -1$*
- *If $c_- \in E^-(A)$ and $c_+ \in E^-(A)$, then $c \cdot A = 0$*
- *If $c_- \in E^+(A)$ and $c_+ \in E^+(A)$, then $c \cdot A = 0$*

Thus, we have a partition of the ends just as in the case of embedded spheres. By Theorem 5.1.1, this corresponds to the partition given by a normal sphere S in $\widetilde{M}$. As homology classes are determined by their associated partitions, the sphere S represents the homology class $A \in H_2(\widetilde{M})$ and the result follows.

We now turn to the question of when a class in $\pi_2(M)$ can be represented by an embedded sphere in M.

Theorem 5.2.3. *The class $A \in \pi_2(M)$ can be represented by an embedded sphere in M if and only if A can be represented by an embedded sphere in $\widetilde{M}$ and for all deck transformations $g \in \pi_1(M)$, A and gA do not cross.*

If A can be represented by an embedded sphere S in M, then one can easily see that its lift $\widetilde{S}$ and all of its translates in $\widetilde{M}$ are disjoint. Therefore, A and gA can be represented by disjoint embedded spheres in $\widetilde{M}$, for all $g \in \pi_1(M)$, which implies A and gA do not cross, for all deck transformations $g \in \pi_1(M)$. Now, we give the constructive proof of the converse. This is based on *graph of trees* associated to normal spheres.

We recall from [8] that, a graph $\mathbb{T}$ consists of two sets $E(\mathbb{T})$ and $V(\mathbb{T})$, called the edges and vertices of $\mathbb{T}$, a mapping from $E(\mathbb{T})$ to $E(\mathbb{T})$, with $e \mapsto \bar{e}$, for which $e \neq \bar{e}$ and $\bar{\bar{e}} = e$, and a mapping from $E(\mathbb{T})$ to $V(\mathbb{T}) \times V(\mathbb{T})$, $e \mapsto (o(e), t(e))$ such that $\bar{e} \mapsto (t(e), o(e))$ for every $e \in E(\mathbb{T})$. An edge path in $\mathbb{T}$ is a sequence $e_1, e_2, \ldots, e_n$ of edges, such that $t(e_j) = o(e_{i+1})$, $e_i \neq e_{i+1}$, for $i = 1, 2, ..., n-1$. If $e, f \in E(\mathbb{T})$, we write $e \leq f$ if there is an edge path $e_1, e_2, \ldots, e_n$ for which $e_1 = e$ and $e_n = f$. If $\mathbb{T}$ is a tree, then $\leq$ determines a partial ordering on on $E(\mathbb{T})$. In addition the following conditions are satisfied:

1. if $e \leq f$, then $\bar{f} \leq \bar{e}$;
2. if $e \leq f$, there are only finitely many g for which $e \leq g \leq f$;
3. for any pair e, f, at least one of $e \leq f$, $e \leq \bar{f}$, $\bar{e} \leq f$, $\bar{e} \leq \bar{f}$ holds;
4. for no pair e, f is $e \leq f$ and $e \leq \bar{f}$;
5. for no pair e, f is $e \leq f$ and $\bar{e} \leq f$.

Theorem 5.2.4 (Dunwoody). *Let $(E, \leq)$ be a partially ordered set with a mapping $E \to E$, $e \to \bar{e}$, for which $e = \bar{\bar{e}}$, and suppose above conditions $(1)-(5)$ are satisfied. Then, there exists a tree $\mathbb{T}$ with $E = E(\mathbb{T})$, where $E(\mathbb{T})$ is the set of edges of $\mathbb{T}$ and the order relation on E is precisely that determined by edge paths in $\mathbb{T}$ as above.*

Our first step is to understand data specifying a normal sphere in M (up to homotopy).

Suppose we are given an oriented (hence, *co-oriented*) normal sphere S in M. Let $\widetilde{S}$ be a lift of S in $\widetilde{M}$. Then, S induces an orientation (hence, *co-orientation*) on $\widetilde{S}$ and on each of its translates $g\widetilde{S}$, $g \in \pi_1(M)$. The orientation on $\widetilde{S}$ (respectively, on $g\widetilde{S}$) determines the positive and negative complementary components of $\widetilde{S}$ (respectively, of $g\widetilde{S}$) in $\widetilde{M}$. The sets $E^+(\widetilde{S})$ and $E^-(\widetilde{S})$ (respectively, $E^+(g\widetilde{S})$ and $E^-(g\widetilde{S})$) correspond to the sets of ends of positive and negative complementary components of $\widetilde{S}$ (respectively, of $g\widetilde{S}$) in $\widetilde{M}$ respectively. As S is in normal form with respect to the maximal system of 2-spheres $\Sigma = \cup_i \Sigma_i$, $\widetilde{S}$ and its translates $g\widetilde{S}$ are in normal form with respect to the inverse image $\widetilde{\Sigma}$ of Σ. The sphere $\widetilde{S}$ and all of its translates in $\widetilde{M}$ are disjoint from each other. If we consider the homology classes of any two translates, say $g_1\widetilde{S}$ and $g_2\widetilde{S}$ of $\widetilde{S}$, then they do not cross. So, the class $[g_1\widetilde{S}]$ is either on the positive side or on the negative side of $[g_2\widetilde{S}]$. Then accordingly, the sphere $g_1\widetilde{S}$ is either in the positive or negative complementary component of $g_2\widetilde{S}$ in $\widetilde{M}$. The positive and negative complementary components of a translate $g\widetilde{S}$ of $\widetilde{S}$ in $\widetilde{M}$ determines positive and negative complementary components of a piece inside a $\widetilde{P_k}$. If both the translates $g_1\widetilde{S}$ and $g_2\widetilde{S}$ intersect some $\widetilde{P_k}$, the piece of $g_1\widetilde{S}$ lies either on the positive side or negative side of the piece of $g_2\widetilde{S}$ in $\widetilde{P_k}$, according to the sphere $g_1\widetilde{S}$ lies on the positive or negative side of $g_2\widetilde{S}$. Similarly, orientation on the translate $g\widetilde{S}$ determine the positive and negative side of the circle of intersection of $g\widetilde{S} \cap \tilde{\Sigma}_i$ in $\tilde{\Sigma}_i$, where $\tilde{\Sigma}_i$ is the boundary sphere of $\widetilde{P_k}$. Thus, orientation on S, (hence, *co-orientation*) determines co-orientation of each circle of intersection of $S \cap \Sigma_i$, for each i. The same is true for each piece of S inside P_k.

We associate to S a *graph of trees* structure as follows: The standard spheres Σ_i decompose M into components P_k. We have an associated graph Γ with vertices P_k and edges Σ_i. The graph of trees we consider is analogous to a graph of groups, with trees associated to edges and vertices and appropriate inclusion maps of the trees.

First, let Σ_i be a standard sphere. Then, $S \cap \Sigma_i$ is a collection of disjoint circles. Consider the graph $t(\Sigma_i)$ whose edges e correspond to the circles of intersection and vertices v complementary components, with v a vertex of e if the boundary of the component corresponding to v contains the circle corresponding to e. The co-orientation of S induces co-orientations for each of the circles of intersection, hence for the edges of the graph $t(\Sigma_i)$.

Lemma 5.2.5. *The graph $t(\Sigma_i)$ is a finite tree. Further, given a finite tree t, there is a collection of circles in Σ_i with corresponding tree t.*

Proof. Firstly, we shall prove that the graph $t(\Sigma_i)$ is a tree. We prove this by induction on number of circles of intersection of $S \cap \Sigma_i$ in Σ_i. If there is exactly one circle c_1 of intersection, then as it separates the 2-sphere Σ_i into two components with closure of each component is a disc with boundary c_1. Then, the corresponding graph contains exactly two vertices and an edge joining them which is clearly a tree.

Now, suppose the result is true for any n circles of intersections of $S \cap \Sigma_i$ in Σ_i. A tree with n edges contains $n+1$ vertices. So, these n circles of intersections separates Σ_i into $n+1$ components. Note that the terminal vertices of the tree $t(\Sigma_i)$ corresponds to disc components of the complements of these circles in Σ_i.

Now, suppose that there are $n+1$ circles of intersections in Σ_i, say $c_1, c_2, \ldots, c_n, c_{n+1}$. For circles $c_2, c_3, \ldots, c_n, c_{n+1}$, we have the associated graph $t'(\Sigma_i)$ is a tree with edges $e_2, e_3, \ldots, e_n, e_{n+1}$ and with vertices $v_2, v_3, \ldots, v_{n+1}, v_{n+2}$. These vertices corresponds to the components of $\Sigma \setminus (c_2 \cup c_3 \cup \cdots \cup c_n \cup c_{n+1})$. We denote these components also by $v_2, v_3, \ldots, v_{n+1}, v_{n+2}$. Now, c_1 is disjoint from all $c_2, c_3, \ldots, c_n, c_{n+1}$. So, it lies completely in exactly one component, say v_k. It separates Σ_i in two components such that the closure of each component is a disc with c_1 as boundary. Suppose these components are D_1 and D_2. The circle c_1 separates v_k into components $v'_k = v_k \cap D_1$ and $v''_k = v_k \cap D_2$. So now, $\Sigma_i \setminus ((c_1 \cup c_2 \cup \cdots \cup c_n \cup c_{n+1})$ has components $v_2, v_3, \ldots, v_{k-1}, v'_k, v''_k, \ldots, v_{n+1}, v_{n+2}$. So, the associated graph $t(\Sigma_i)$ can be obtained from $t'(\Sigma_i)$ by replacing the vertex v_k by an edge e joining v'_k and v''_k. Now, each edge in incidenting on v_k in $t'(\Sigma_i)$ either incident at v'_k or at v''_k. The tree $t'(\Sigma_i)$ can be obtained from $t(\Sigma_i)$ by collapsing the edge e to vertex v_k. Then, one can easily see that $t(\Sigma_i)$ is tree as, if it contains any loop or circuit, then collapsing the edge e to a vertex v_k gives a circuit in $t'(\Sigma_i)$ which is impossible. Hence, the proof.

Now, we shall show that given a finite tree t, there is a collection of circles in Σ_i with the corresponding tree t. We again prove this by induction on number of edges in the tree t. Suppose t contains exactly one edge e with end vertices v_1 and v_2. If we consider any circle c on Σ_i, then it separates Σ_i in two boundary components such that closure of each component is a disc with boundary c. Then, one can easily see the corresponding graph is t. Suppose the result is true for any tree t containing n edges. Now, consider a tree t with $n+1$ edges. Let v be a terminal vertex of t. Let e be the edge in t containing v as its end vertex. Let the other end vertex be v'. By deleting the edge e from t, we get a tree t' with n edges. By induction hypothesis, we get disjointly embedded circles $c_1, c_2, \ldots, c_n$ in Σ_i with corresponding graph t'. Now, if we consider a circle c_1 which completely lies in the interior of the component v' corresponding to the vertex v' of t' such that c_1 bounds a disc in v'. Then, one can easily see that the associated graph is t in this case. Thus, for any tree t, we get a collection of disjoint circles in Σ_i with corresponding tree t. $\square$

If the edges of t are oriented, the circles can be co-oriented accordingly.

Let $\tilde{\Sigma}_i$ be a lift of Σ_i in $\widetilde{M}$. We can associate the same oriented tree $t(\Sigma_i)$ to the sphere $\widetilde{\Sigma_i}$, where each edge corresponds to the circle of intersection of $\tilde{\Sigma}_i$ with a translate $g\widetilde{S}$ of $\widetilde{S}$. Let $g_1, \ldots, g_n$, where $g_i \in \pi_1(M)$, for $1 \le i \le n$, such that $g_i\widetilde{S}$ intersects $\tilde{\Sigma}_i$. Let $B_i = [g_i\widetilde{S}]$, for $1 \le i \le n$, be the homology class in $H_2(\widetilde{M})$. Note that each B_i crosses the class $[\tilde{\Sigma}_i]$. As any pair of translates B_i and B_j of $B = [\widetilde{S}]$ do not cross, B_j is either on the positive or the negative side of B_i. Similarly, if B_i, B_j and B_k are translates of B, we can determine whether B_i is between B_j and B_k.

Given a tree τ, we have a notion of when an edge e_1 of τ is between two other edges e_2 and e_3 of τ. If τ is oriented, then we can speak of edges being on the positive side of τ.

Using Theorem 5.2.4, the following lemma is an easy consequence.

Lemma 5.2.6. *Let $B_1, \ldots, B_n$ be translates of B. Then, there is an associated oriented tree with edges e_i corresponding to classes B_i so that e_j is on the positive side of e_i if and only if $E^+(B_j) \subset E^+(B_i)$*

Proof. We choose set E as $\{B_1, \ldots, B_n, \bar{B}_1, \ldots \bar{B}_n\}$, where if B_i is represented by oriented sphere $g_i\widetilde{S}$, then $\bar{B}_i$ is a homology class of $\overline{g\widetilde{S}}$, sphere $g\widetilde{S}$ with opposite orientation. We define $B_i \le B_j$ if $E^+(B_j) \subset E^+(B_i)$. We can easily check "$\le$" turns out be a partial order relation satisfying the hypothesis of theorem 5.2.4. Hence, the result. $\square$

Note that the oriented tree in the above lemma 5.2.6 associated to $B_1, \ldots, B_n$ is the tree $t(\Sigma_i)$.

We can similarly associate an oriented tree $t(P_k)$ to a component P_k (and to a lift $\widetilde{P_k}$), with edges as the pieces of S in P_k and vertices as their complementary components. If Σ_i is a boundary sphere of P_k, there is a natural simplicial inclusion map from $t(\Sigma_i)$ to $t(P_k)$ which respects orientations. Each edge in the tree $t(P_k)$ associated to $\widetilde{P_k}$ corresponds to a piece of a translate of $\widetilde{S}$. There is at the most one piece of any translate of $\widetilde{S}$ in $\widetilde{P_k}$. If we consider the homology classes of the translates of $\widetilde{S}$ intersecting at least one boundary sphere of $\widetilde{P_k}$, then the oriented tree in the lemma 5.2.6 associated to these homology classes is the oriented tree $t(P_k)$. Given an oriented edge e, it has an initial vertex and a ending vertex. We can make the convention that initial and ending vertices of the edge e correspond to negative and positive complementary components of the pieces of S corresponding to e inside P_k, respectively.

Each vertex V of the graph Γ is trivalent. We associate oriented trees to the vertices and edges of Γ by taking the oriented trees associated to the corresponding component P_k's and standard spheres Σ_i respectively. If E_1, E_2 and E_3 are three adjoining edges to V, we have simplicial inclusion maps $i_j : t(E_j) \to t(V)$ respecting the orientation. It is easy to see that the union of the images $i_j(t(E_j))$ is all of $t(V)$. If e is an edge in $t(V)$, then e corresponds to a pants piece in P_k corresponding to V if and only if it has non-empty inverse image under all three inclusion maps i_j's. If e has non-empty inverse image under exactly two inclusion maps, say i_{j_1} and i_{j_2}, then e corresponds to a tube piece in P_k joining the standard spheres corresponding to the $t(E_{j_1})$ and $t(E_{j_2})$. Conversely, if e corresponds to a tube piece in P_k, then e has non-empty inverse image under exactly two inclusion maps i_j's. If e has non-empty inverse image under exactly one inclusion map i_j, then e corresponds to a disc piece with boundary on the standard sphere corresponding to $t(E_j)$ and, conversely.

To the edges in $t(V) = t(P_k)$, we associate the following data: If an edge e in $t(V)$ corresponds to a tube piece in P_k joining the boundary spheres, say Σ_{j_1} and Σ_{j_2}. Then, the third boundary sphere Σ_{j_3} lies either in positive or in negative complementary component of the tube in P_k. If the sphere Σ_{j_3} lies in the positive component, then we assign this sphere to the ending vertex of the edge e, otherwise to the initial vertex. If an edge e in $t(V)$ corresponds to a disc piece in P_k with boundary circle on Σ_{j_1}, then it separates the other two boundary spheres Σ_{j_2} and Σ_{j_3}. The sphere which lies in the positive component, we assign it to the ending vertex of e and the other boundary sphere to the initial vertex of e. If an edge corresponds to a pants piece in P_k, then we do not associate any data to this edge. This is the graph of oriented trees structure associated to S.

Our goal is to associate a graph of oriented trees to a class A satisfying the hypothesis of Theorem 5.2.3 and construct a corresponding normal sphere S. The class A can be represented by an embedded sphere in $\widetilde{M}$, say S'. Fix an orientation of S'. Then, S' determines the the set $E^+(A)$ and $E^-(A)$. The orientation on S' induces an orientation on each translate gS' of S'. It will then determine the sets $E^+(gA)$ and $E^-(gA)$ for each $g \in \pi_1(M)$.

Consider $(\widetilde{A}, \widetilde{\Sigma_i})$, where $\widetilde{A}$ is a lift of A and $\widetilde{\Sigma_i}$ is a lift of Σ_i such that $\widetilde{A}$ crosses $\tilde{\Sigma}_i$. We define $(\widetilde{A}, \widetilde{\Sigma_i})$ is equivalent to $(g\widetilde{A}, g\widetilde{\Sigma_i})$ for all $g \in \pi_1(M)$. Note that this is an equivalence relation. Given any lift $\tilde{\Sigma}_i^0$ of Σ_i and an equivalence class $[(\widetilde{A}, \widetilde{\Sigma_i})]$, there is a unique representative $(\widetilde{A}^0, \widetilde{\Sigma_i}^0)$ equivalent to $(\widetilde{A}, \widetilde{\Sigma_i})$, where $\tilde{\Sigma}_i^0 = g\tilde{\Sigma}_i$ and $\widetilde{A}^0 = g\widetilde{A}$. We define the partial order " $\leq$ " as $[(\widetilde{A}, \widetilde{\Sigma_i})] \leq [(\widetilde{A}', \widetilde{\Sigma_i})]$ if and only if $E^+(\widetilde{A}') \subset E^+(\widetilde{A})$. One can easily see that this is well defined.

Similarly, we can consider pairs $(\widetilde{A}, \widetilde{P_k})$, where $\widetilde{A}$ is a lift of A and $\widetilde{P_k}$ is a lift of P_k such that $\widetilde{A}$ crosses at least one boundary sphere of $\widetilde{P_k}$. We define $(\widetilde{A}, \widetilde{P_k})$ is equivalent to $(g\widetilde{A}, g\widetilde{P_k})$ for all $g \in \pi_1(M)$. Note that this is an equivalence relation. Given any lift $\widetilde{P_k}^0$ of P_k and an equivalence class $[(\widetilde{A}, \widetilde{P_k})]$, there is a unique representative $(\widetilde{A}^0, \widetilde{P_k}^0)$ equivalent to $(\widetilde{A}, \widetilde{P_k})$, where $\widetilde{P_k}^0 = g\widetilde{P_k}$ and $\widetilde{A}^0 = g\widetilde{A}$. We define the partial order " $\leq$ " as $[(\widetilde{A}, \widetilde{P_k})] \leq [(\widetilde{A}', \widetilde{P_k})]$ if and only if $E^+(\widetilde{A}') \subset E^+(\widetilde{A})$. One can easily see that this is well defined.

Let Σ_i be a standard sphere and let $\tilde{\Sigma}_i$ be a lift to $\widetilde{M}$. Then, as A has compact support, at most finitely many translates of A cross $\tilde{\Sigma}_i$. Denote these translates $A_1, \ldots, A_n, \bar{A}_1, \ldots, \bar{A}_n$. By Lemma 5.2.6, we can associate an oriented tree $t(\Sigma_i)$ to the collection $\{[(A_1, \widetilde{\Sigma_i})], \ldots, [(A_n, \widetilde{\Sigma_i})], [(\bar{A}_1, \widetilde{\Sigma_i})], \ldots, [(\bar{A}_n, \widetilde{\Sigma_i})]\}$ respecting the relation "$\leq$". We associate this oriented tree to the edge in Γ corresponding to Σ_i. Next, consider a component P_k and let $\widetilde{P_k}$ be a lift to $\widetilde{M}$. We consider the translates of A that cross at least one of the boundary spheres of $\widetilde{P_k}$. Suppose these translates are $B_1, \ldots, B_r, \bar{B}_1, \ldots, \bar{B}_r$. Once more, we can associate an oriented tree $t(P_k)$ to collection $\{[(B_1, \widetilde{P_k})], \ldots, [(B_r, \widetilde{P_k})], [(\bar{B}_1, \widetilde{P_k})], \ldots, [(\bar{B}_r, \widetilde{P_k})]\}$ respecting the the relation "$\leq$". To each edge e of $t(P_k)$ we associate the following data: Suppose that e corresponds to a translate B_i of A and B_i does not cross some boundary sphere $\widetilde{\Sigma_i}$ of $\widetilde{P_k}$. Then, the sphere $\widetilde{\Sigma_i}$ is either on the positive or negative side of B_i. If $\tilde{\Sigma}_i$ is on the positive side of B_i, we associate the sphere Σ_i (image of $\tilde{\Sigma}_i$) to the ending vertex of e, otherwise to initial vertex of e. We associate this oriented tree to the vertex in Γ corresponding to P_k. If B_i crosses all the boundary spheres of P_k, then we do not associate any data to the edge e.

Lemma 5.2.7. *Let A_i, A_j and A_k be translates of A such that $A_i \leq A_j \leq A_k$. If A_i and A_k cross the homology class of the boundary sphere $\widetilde{\Sigma_i}$ of $\widetilde{P_k}$, so does A_j.*

Proof. As $A_j \leq A_k$, $E^+(A_k) \subset E^+(A_j)$. As A_k crosses the homology class $[\widetilde{\Sigma_i}]$ of the boundary sphere $\tilde{\Sigma}_i$, we have $E^{\pm}([\Sigma_i]) \cap E^+(A_k) \neq \phi$. This implies $E^{\pm}([\Sigma_i]) \cap E^+(A_j) \neq \phi$.

As $A_i \leq A_j$, $E^+(A_j) \subset E^+(A_i)$ and hence, $E^-(A_i) \subset E^-(A_j)$. As A_i crosses the homology class $[\widetilde{\Sigma_i}]$ of the boundary sphere $\tilde{\Sigma}_i$, we have $E^{\pm}([\Sigma_i]) \cap E^-(A_i) \neq \phi$. This implies $E^{\pm}([\Sigma_i]) \cap E^-(A_j) \neq \phi$.

Thus, all the four intersections $E^{\pm}([\Sigma_i]) \cap E^{\pm}(A_j)$ are non-empty. Hence, A_j crosses the homology class of the boundary sphere $\widetilde{\Sigma_i}$ of $\widetilde{P_k}$. □

We shall define a map on vertices of $t(\Sigma_i)$ to vertices of $t(P_k)$, where Σ_i is a boundary sphere of P_k, as follows: Let v be vertex in $t(\Sigma_i)$, Consider an edge e with v as its ending vertex. This edge e corresponds to an equivalence class $[(A_i, \tilde{\Sigma}_i)]$. As A_i crosses $\tilde{\Sigma}_i$, A_i crosses $\widetilde{P_k}$. So, the class $[(A_i, \widetilde{P_k})]$ corresponds to an edge e'in $t(P_k)$. We map v to the terminal vertex of e'. Suppose e'' is another edge with v as a terminal vertex. The edge $\bar{e''}$ corresponds to a class $[(A_j, \widetilde{\Sigma_i})]$. As the classes A_i and A_j cross Σ_i and the edges e and e'' are adjacent edges in $t(\Sigma_i)$, by Lemma 5.2.7, the classes $[(A_i, \widetilde{P_k})]$ and $[(A_j, \widetilde{P_k})]$ correspond to the adjacent edges e' and e''' in $t(P_k)$. Thus, the edges e' and $\bar{e'''}$ have the same terminal vertex. This shows this map on vertices is well-defined. This map can be naturally extended on edges.

Thus, we have a natural simplicial inclusion maps from the trees associated to each boundary component of P_k to the tree associated to P_k respecting the orientation and the image under inclusion of the tree associated to a boundary sphere is a subtree of the tree associated to P_k. Thus, we have a graph of oriented trees associated to A.

We are now in a position to construct the normal sphere in M representing A. By Lemma 5.2.5, we have a collection of disjoint co-oriented circles in each standard sphere Σ_i corresponding to the edges in $t(\Sigma_j)$. We shall extend these to each component P_k using the following lemma. We use the inclusion maps to regard the trees corresponding to the boundary spheres as subtrees of the trees corresponding to P_k.

Lemma 5.2.8. *Given an oriented tree $t = t(P_k)$ associated to P_k and orientation preserving simplicial inclusion maps i_j of oriented trees $t(\Sigma_j)$ associated to the boundary spheres Σ_j of P_k so that the union of the images $i_j(t(\Sigma_j))$ is all of $t(P_k)$, there is a collection of co-oriented disjoint pieces in P_k whose restriction to each boundary sphere Σ_j is the given collection of disjoint circles corresponding to edges in $t(\Sigma_j)$. Furthermore, using the data associated to each edge, we can choose each piece such that the boundary sphere not intersecting that piece lies in any specified component of the complement of that piece.*

Proof. We proceed by induction on the size of the tree $t = t(P_k)$. Thus, if t' is obtained from $t(P_k)$ by deleting a terminal vertex v and its adjoining edge e, there is a collection of co-oriented disjoint pieces in P_k whose restriction to each boundary sphere is the given collection of disjoint circles except those associated to e. Also, using the data associated to each edge, each piece is chosen such that the boundary sphere not intersecting that piece lies in any specified component of the complement of that piece. We shall extend this using one more piece corresponding to the edge e.

Let v' be the vertex of e in t'. The edge e has non-empty inverse image under the inclusion maps i_j in one or two or all three oriented trees $t(\Sigma_j)$ associated to the boundary spheres Σ_j of P_k. The same is true for vertex the v. Then, e corresponds to a circle in each of those boundary components Σ_j of P_k for which e has non-empty inverse image in $t(\Sigma_j)$. Consider such boundary spheres Σ_j's of P_k. As v' is a vertex of the tree $t(\Sigma_j)$, by the correspondence between circles and trees, the circle on Σ_j corresponding to e is in the component corresponding to v' in Σ_j and it bounds a disc corresponding to v in the component corresponding to v' in Σ_j. Note that $t(\Sigma_j)$ is a subtree of $t(P_k)$ and each component in Σ_j corresponding to a vertex in $t(\Sigma_j)$ is the intersection of the component of P_k corresponding to the same vertex with Σ_j. So, the circles corresponding to e in Σ_j's and the discs corresponding to v in Σ_j's lie inside the component of P_k corresponding to v'. Now, we can construct the piece as the neighborhood of a graph with terminal vertices in the discs corresponding to the vertex v, inside the component corresponding to v' in P_k. Further, using the data associated to the vertex v or v', we can construct the piece so that any boundary sphere Σ_i not intersecting the piece lies inside the appropriate component of $P_k - S$. □

Now, we shall show that the graph of trees associated to the class $A \in \pi_2(M)$ represents a normal sphere S in M such that S represents the class A. Fix a lift $\widetilde{A}$ of A. We associate a subgraph τ of T to $\widetilde{A}$ as follows: If $\widetilde{A}$ crosses standard sphere $\widetilde{\Sigma_i}$, then τ contains the bivalent vertex v_i corresponding to $\widetilde{\Sigma_i}$ and the edges e^i_1 and e^i_2 containing that vertex v_i. The other end vertex v^i_j of each edge $e^i_j, j = 1, 2$ is a trivalent vertex in T which corresponds to a component of $\widetilde{M} - \widetilde{\Sigma}$. Each v^i_j may be a bivalent or univalent or a trivalent vertex in τ. If $\widetilde{A}$ does not cross some standard sphere, then τ does not contain the standard vertex corresponding this standard sphere. Hence, τ contains no edges containing this standard vertex. By Lemmas 5.1.9, 5.1.10, τ is a finite tree. Consider a standard bivalent vertex of τ. It corresponds to a standard sphere $\tilde{\Sigma}_i$ in $\widetilde{M}$. Consider the image Σ_i of $\tilde{\Sigma}_i$. We associated a tree $t(\Sigma_i)$ to Σ_i. As $\widetilde{A}$ crosses $\tilde{\Sigma}_i$, there is an edge e in $t(\Sigma_i)$ which corresponds to $[(\widetilde{A}, \tilde{\Sigma}_i)]$ and a circle in Σ_i. We consider lift of this circle to $\tilde{\Sigma}_i$ (also to all the other translates of $\tilde{\Sigma}_i$). The edge e is also in $t(P_k)$ and it corresponds to the class $[(\widetilde{A}, \widetilde{P_k})]$, where P_k is a 3-punctured 3-sphere with Σ_i as a boundary sphere. If the vertex $v \in \tau$

corresponding to $\widetilde{P_k}$ of which $\tilde{\Sigma}_i$ is a boundary sphere, is a non-standard bivalent vertex of τ, then data associated e in $t(P_k)$ determines the value of the function ϵ on the vertex v. If v is terminal vertex of τ, then data associated to e determines the value of the function e_+ on the vertex v. Thus, we have the triple (τ, ϵ, e_+) representing $\widetilde{A}$ in $\widetilde{M}$. We can construct pieces of a normal sphere in $\widetilde{M}$ representing $\widetilde{A}$ as follows: We have chosen a circle which correspond to the class $[(\widetilde{A}, \tilde{\Sigma}_i)]$ on each $\widetilde{\Sigma_i}$ as described above. We consider a 3-punctured 3-sphere $\widetilde{P_k}$ such that $\widetilde{A}$ crosses at least one boundary sphere of $\widetilde{P_k}$. Consider its image P_k. Consider the images (in P_k) of the chosen circles in the boundary spheres of $\widetilde{P_k}$. For each boundary sphere Σ_i of P_k, we have a circle and this circle corresponds to an edge e_i in $t(\Sigma_i)$. For each i, under the inclusion map from $t(\Sigma_i)$ to $t(P_k)$, the edge e_i gets mapped to the same edge e in $t(P_k)$. Corresponding to this edge e, there is a piece inside P_k with boundary circles of the piece coinciding with the images of the chosen circles in the boundary spheres of $\widetilde{P_k}$, by Lemma 5.2.8. Consider the lift of this piece in $\widetilde{P_k}$ (and also to the translates of $\widetilde{P_k}$). Thus, we get a normal sphere $\widetilde{S}$ in $\widetilde{M}$ representing the class $\widetilde{A}$. The pieces of $\widetilde{S}$ get mapped to the pieces given by the the graph Γ of trees associated to A. Similarly, we get get normal sphere for each translate of $\widetilde{A}$ such that the normal sphere representing $g\widetilde{A}$ is a translate $g\widetilde{S}$. Now, any piece $\mathbb{P}$ given by Γ corresponds to an edge in $t(V) = t(P_k)$, for some k. This edge corresponds to some class $[(\widetilde{A}', \widetilde{P_k})]$. If $g\widetilde{A'} = \widetilde{A}$, then the lift of this piece $\mathbb{P}$ to $\widetilde{P_k}$ is a piece of the normal sphere $g\widetilde{S}$ representing $g\widetilde{A'}$. Therefore, there is a piece $g^{-1}\mathbb{P}$ of $\widetilde{S}$ which is a g^{-1} translate of $\mathbb{P}$, is mapped to the piece $\mathbb{P}$. Thus, each piece in M given by Γ is the image of a piece of the normal sphere $\widetilde{S}$. Hence, we get a normal sphere in M representing the class A.

6. GEOSPHERE LAMINATIONS FOR FREE GROUPS

Geodesic laminations (and measured laminations) on surfaces have proved to be very fruitful in three-manifold topology, Teichmüller theory and related areas. In this chapter, we construct analogously *geosphere laminations* for free groups. They have the same relation to (disjoint unions of) embedded spheres in the connected sum $M = \sharp_n S^2 \times S^1$ of n copies of $S^2 \times S^1$ as geodesic laminations on surfaces have to (disjoint unions of) simple closed curves on surfaces. The manifold M has fundamental group the free group on n generators, and is a natural model for the study of free groups.

Laminations for groups (including free groups) have been constructed and studied in various contexts. However, they are one-dimensional objects, corresponding to geodesics. We study here objects of codimension one, which correspond to splittings. In the case of surfaces, dimension one and codimension one coincide.

Our main result is a compactness theorem for the space of (non-trivial) geosphere laminations. We also show that embedded spheres in M are geosphere laminations. Hence sequences of spheres, in particular under iterations of an outer automorphism of the free group, have subsequences converging to geosphere laminations. It is such limiting constructions that make geodesic laminations for surfaces a very useful construction.

Our construction is based on the normal form for disjoint unions of spheres in M due to Hatcher. The normal form is relative to a decomposition of M with respect to a maximal collection of spheres in M. This is in many respects analogous to a normal form with respect to an ideal triangulation of a punctured surface. In particular, isotopy for spheres in normal form implies normal isotopy, i.e., the normal form is unique.

As in the case of normal curves on surfaces and normal surfaces in three-manifolds, we can associate the number of pieces of each type to a collection of spheres in Hatcher's normal form. However, these numbers do not determine the (collection of) spheres up to isotopy. We instead proceed by considering lifts of normal spheres to the universal cover $\widetilde{M}$ of M.

In the universal cover $\widetilde{M}$, a normal sphere is determined by a *finite* subtree τ of a tree T associated to $\widetilde{M}$ together with some additional data. We construct geospheres in $\widetilde{M}$ by dropping the finiteness condition. We construct an appropriate topology on the space of geospheres and show that the space is locally compact and totally disconnected.

The lift of a normal sphere in M to its universal cover satisfies an additional condition, namely it is disjoint from all its translates. This can be reformulated in terms of the notion of *crossing* of spheres in $\widetilde{M}$, following Scott-Swarup, which depends on the corresponding partitions of ends of $\widetilde{M}$. We show that there is an appropriate notion of crossing for geospheres, which is defined in terms of the appropriate partition of ends (into three sets in this case).

Our main technical result is that crossing is an open condition. We recall that this is the case for crossing of geodesics in hyperbolic space, and that this plays a central role in the study of geodesic laminations. The proof of compactness of the space of geospheres uses the result that crossing is open.

The construction based on normal forms is not intrinsic, as it depends on the maximal collection of spheres with respect to which M is decomposed. However, we show that geospheres can be described in terms of their associated partitions. This gives an intrinsic definition.

6.1 Geospheres

To construct geosphere laminations in M, we first need the analogue of (not necessarily closed) geodesics in M. We first construct the analogue of geodesics in $\widetilde{M}$, which we call *geospheres*. We then consider when two such geospheres cross, and deduce basic properties of crossing. This allows us to study the appropriate notion of geospheres embedded in M. Our main technical lemma says that crossing is an open condition. This allows us to construct limiting laminations and prove a compactness theorem for geosphere laminations in M.

In Chapter 5, we have seen that a normal sphere in $\widetilde{M}$ is determined by a triple (τ, ϵ, e_+), with τ a finite subtree of T with each univalent vertex of τ is a trivalent vertex of T or τ is a standard vertex, ϵ an assignment of sign to each non-standard bivalent vertex of τ and e_+ an assignment to each univalent vertex v of τ an edge containing v and not contained in τ.

Geospheres are generalizations of such spheres where we drop the condition that τ is finite.

Definition 6.1.1. A geosphere σ in $\widetilde{M}$ is a triple $\sigma = (\tau, \epsilon, e_+)$ with

- τ a subtree of T such that univalent (terminal) vertices of τ are trivalent vertices in T or τ is a standard vertex of T.
- If $B(\tau)$ is the set of non-standard bivalent vertices of τ, ϵ is a function $\epsilon : B(\tau) \to \{+, -\}$.
- If $C(\tau)$ the set of univalent vertices of τ, $e_+ : C(\tau) \to EDGE(T)$ is a function to the edges of T so that for $v \in C(\tau)$, $e_+(v) \in EDGE(T)$ is an edge containing v and not contained in τ.

Let $GS(\widetilde{M})$ be the set of such geospheres in $\widetilde{M}$. To construct a topology on $GS(\widetilde{M})$, we consider restrictions to compact subtrees $\kappa \subset T$ such that each of its univalent vertex is a trivalent vertex in T. We call a tree containing no edge as a trivial tree. We define for a non-trivial tree κ, $N(\kappa)$ is the set of points of distance at most 1 from κ. For a trivial tree κ, we define $N(\kappa) = \kappa$.

Henceforth, we consider only subtrees κ of T such that all univalent vertices of κ are trivalent in T or κ is a trivial tree.

Definition 6.1.2. If $\sigma = (\tau, \epsilon, e_+)$ is a geosphere, then the restriction $res_\kappa(\sigma)$ of σ to κ is the triple $\sigma|_\kappa = (\tau \cap N(\kappa), \epsilon|_{B(\tau)\cap\kappa}, e_+|_{C(\tau)\cap\kappa})$.

Note that the valence of a vertex v of τ such that $v \in \kappa$ is determined by $\tau \cap N(\kappa)$. Further, for univalent vertices v of $\tau \cap \kappa$, the edges $e_+(v)$ (and $e_-(v)$) are in $N(\tau)$. Thus, we can view $res|_\kappa$ as a map from $GS(\widetilde{M})$ to the set $GS(\kappa)$ defined as below:

Definition 6.1.3. For a subtree $\kappa \subset T$, we define $GS(\kappa)$ to be the set of triples $\sigma = (\tau, \epsilon, e_+)$ with

- τ a subtree of $N(\kappa)$ or the empty graph.
- If $B(\tau)$ is the set of vertices $\tau \cap \kappa$ which are non-standard bivalent vertices in τ, ϵ is a function $\epsilon : B(\tau) \to \{+, -\}$.
- If $C(\tau)$ is the set of vertices of $\tau \cap \kappa$ which are not standard vertices in T and univalent in τ, $e_+ : C(\tau) \to EDGE(T)$ is a map to the edges of T so that for $v \in C(\tau)$, $e_+(v)$ is an edge containing v and not contained in τ.

Note that if κ is a finite subtree of T, then the set $GS(\kappa)$ is finite. We say that an element $\sigma = (\tau, \epsilon, e_+)$ of $GS(\kappa)$ is *non-trivial* if τ is non-empty.

Suppose κ' is a subtree of T such that $\kappa' \supset \kappa$, then we can similarly define a restriction map $res_{\kappa,\kappa'} : GS(\kappa') \to GS(\kappa)$. Further, $res_\kappa = res_{\kappa,\kappa'} \circ res_{\kappa'}$. In particular, we can denote without ambiguity the map $res_{\kappa,\kappa'}$ as simply res_κ.

We define a topology on $GS(\widetilde{M})$ using the restriction maps. Namely, for each subtree κ of T and each $\sigma_0 \in GS(\kappa)$, consider the set

$$U(\kappa, \sigma_0) = \{\sigma \in GS(\widetilde{M}) : res_\kappa(\sigma) = \sigma_0\}$$

Lemma 6.1.4. *The sets $U(\kappa, \sigma_0)$ for finite subtrees κ of T form a basis for a topology on $GS(\widetilde{M})$.*

Proof. Showing that the sets $U(\kappa, \sigma_0)$ form a basis for a topology on $GS(\widetilde{M})$ is equivalent to showing that if $U(\kappa^i, \sigma_0^i)$, $1 \le i \le n$ is a finite collection of basic open sets and $\sigma \in \cap_i U(\kappa^i, \sigma_0^i)$, then there is a basic open set containing σ and contained in each of the sets $U(\kappa^i, \sigma_0^i)$.

To show this, let κ be the finite subtree of T spanned by the subtrees κ^i, and let $\sigma_0 = \sigma|_\kappa$. Note that as $\sigma \in U(\kappa^i, \sigma_0^i)$, $res_{\kappa^i}(\sigma) = \sigma_0^i$. Hence, if $\sigma' \in U(\kappa, \sigma|_\kappa)$, as $\kappa \supset \kappa^i$, $res_{\kappa^i}(\sigma') = res_{\kappa^i}(\sigma) = \sigma_0^i$. Thus, $U(\kappa, \sigma|_\kappa) \subset U(\kappa^i, \sigma_0^i)$, for each i as required. □

Henceforth, consider $GS(\widetilde{M})$ with the topology whose basis is given by the sets $U(\kappa, \sigma_0)$ as above. By construction, $GS(\widetilde{M})$ is second countable. If $\kappa = \kappa_1 \subset \kappa_2 \subset \dots$ is an exhaustion of T by finite subtrees of T, then for each i, the collection $\{U(\kappa_i, \sigma) : \sigma \in GS(\kappa_i)\}$ is finite. Hence, one can easily see that the collection $\cup_i \{U(\kappa_i, \sigma) : \sigma \in GS(\kappa_i)\}_i$ gives a countable basis for the topology on $GS(\widetilde{M})$.

If $\kappa \subset T$ is a finite tree and σ_1, σ_2 are elements of $GS(\kappa)$ such that $\sigma_1 \neq \sigma_2$, then $U(\kappa, \sigma_1) \cap U(\kappa, \sigma_2) = \phi$ and $GS(\widetilde{M}) = \amalg U(\kappa, \sigma_i)$, where $\sigma_i \in GS(\kappa)$.

We see that the space $GS(\widetilde{M})$ is Hausdorff, in fact totally disconnected.

Lemma 6.1.5. *The space $GS(\widetilde{M})$ is totally disconnected.*

Proof. Let $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$, be two distinct points in $GS(\widetilde{M})$. It is easy to see that for some finite tree κ, $res_\kappa(\sigma^1) \neq res_\kappa(\sigma^2)$. As $GS(\kappa)$ is a finite set, it follows that we can partition $GS(\kappa)$ into finite sets S_1 and S_2 with $res_\kappa(\sigma^i) \in S_i$, for $i = 1, 2$.

Let $U_i = \{\sigma \in GS(\widetilde{M}) : res_\kappa(\sigma) \in S_i\}$, $i = 1, 2$. Then, U_i are disjoint (finite) unions of basic open sets with $\sigma^i \in U_i$. This shows $GS(\widetilde{M})$ is Hausdorff.

If A is any subset of $GS(\widetilde{M})$ containing more than one point, then we can consider two distinct points $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$ in A. We can find open sets U_i, $i = 1, 2$, as above with $\sigma^i \in U_i$. Then, the sets $A \cap U_1$ and $A \cap U_2$ gives separation of A. □

In fact, we can see that if $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$, are two distinct points in $GS(\widetilde{M})$, then we can find disjoint open sets U_i, $i = 1, 2$ with $\sigma^i \in U_i$ and $U_1 \cup U_2 = GS(\widetilde{M})$.

The main result we need about the topology is the following compactness theorem. This is the analogue of the fact that the set of geodesics in hyperbolic space (more generally, in any Riemannian manifold) that intersect a fixed compact set is compact.

Theorem 6.1.6. *For a fixed finite subtree $\kappa \subset T$, the set of all geospheres whose restriction to κ is non-trivial is compact.*

Proof. Let A be the set of all geospheres whose restriction to κ is non-trivial. As $GS(\widetilde{M})$ is second countable and Hausdorff, it is metrizable. Hence, it suffices to show that every sequence in the given subspace A has a convergent subsequence in A.

Let $\kappa = \kappa_1 \subset \kappa_2 \subset \dots$ be an exhaustion of T by finite subtrees of T. Let σ_i be a sequence of geospheres in $\widetilde{M}$ whose restriction to κ is non-trivial. We construct a convergent subsequence of σ_i.

Firstly, for each i, $res_{\kappa_1}(\sigma_i) \in GS(\kappa_1)$ and $GS(\kappa)$ finite set. Hence, on passing to a subsequence (which we denote by σ_i), we can assume that $res_{\kappa_1}(\sigma_i)$ is constant. Similarly, passing to a further subsequence, we can assume that $res_{\kappa_2}(\sigma_i)$ is constant. Iterating this and passing to a diagonal subsequence, we obtain a sequence, which we also denote σ_i, so that the restriction of σ_i to each of the sets κ_i is *eventually* constant. More concretely, we can assume that for $j, k \geq i$, $res_{\kappa_i}(\sigma_j) = res_{\kappa_i}(\sigma_k)$.

We claim that the subsequence σ_i constructed as above has a limit $\sigma = (\tau, \epsilon, e_+)$. Namely, to determine whether an edge e is in τ, consider i large enough that $e \in \kappa_i$. Then, as $res_{\kappa_i}(\sigma_j) = res_{\kappa_i}(\sigma_i)$ for $j \geq i$ (taking $k = i$), either $e \in \tau_j$ for all j large enough or $e \notin \tau_j$ for all $j \geq i$, where τ_j is the tree corresponding to σ_j. In the former case, we declare $e \in \tau$ and in the latter case $e \notin \tau$. We can see $\tau_1 \subset \tau_2 \subset \dots$ is an exhaustion of τ. We similarly can decide what vertices are in τ and also the values of the functions ϵ and e_+.

As the restriction of each σ_i is non-empty, the limiting subgraph τ is non-empty.

One can show that τ is connected. Namely, is v and w are vertices of τ, for j sufficiently large, v and w are contained in the tree τ_j, hence there is a unique reduced path λ between them. It follows that $\lambda \subset \tau$ by the definition of τ.

Thus, $\sigma \in GS(\widetilde{M})$. Finally, as κ_i is an exhaustion of T by compact subtrees, any compact subtree κ' is contained in κ_j for some j. Hence, for $k > j$, $res_{\kappa'}(\sigma_k) = res_{\kappa'}(\sigma)$. By the definition of the topology on $GS(\widetilde{M})$, we see that $\sigma_i \to \sigma$.

□

As a corollary, we see that $GS(\widetilde{M})$ is locally compact. In fact, every geosphere σ is contained in a compact open subset of $GS(\widetilde{M})$.

Proposition 6.1.7. *Any geosphere σ is contained in a compact open subset U of $GS(\widetilde{M})$.*

Proof. It is easy to see that there is a finite tree κ such that $res_\kappa(\sigma)$ is non-trivial. Let

$$U = \{\sigma' \in GS(\widetilde{M}) : res_\kappa(\sigma) = res_\kappa(\sigma')\}$$

By Theorem 6.1.6, U is compact. The set U is open by definition of the topology on $GS(\widetilde{M})$. □

In Chapter 5, we have seen that a normal sphere S in $\widetilde{M}$ is determined by triple $\sigma = (\tau, \epsilon, e_+)$, with τ is a finite subtree of T, ϵ is an assignment of sign to each non-standard bivalent vertex of τ and e_+ an assignment to each univalent vertex of τ an edge containing v and not contained in τ. Hence, normal spheres $\widetilde{M}$ are geospheres.

Let $S(\widetilde{M})$ be the set of all normal spheres in $\widetilde{M}$, i.e., $S(\widetilde{M})$ is the set of all geospheres $\sigma = (\tau, \epsilon, e_+)$, where τ is a finite subtree of T.

Proposition 6.1.8. *The set $S(\widetilde{M})$ is the set of isolated points of $GS(\widetilde{M})$ and is dense in $GS(\widetilde{M})$.*

Proof. Let $\sigma^0 = (\tau^0, \epsilon^o, e^0_+)$ be a normal sphere in $\widetilde{M}$. We see that $res^0_\tau(\sigma^0) = \sigma^0$. Consider $U(\tau^0, \sigma^o)$. If $\sigma' = (\tau', \epsilon', e'_+) \in U(\tau^0, \sigma^0)$, then $res^0_\tau(\sigma') = \sigma^0$. By definition of res,

$$res^0_\tau(\sigma') = (\tau' \cap N(\tau^0), \epsilon' | B(\tau') \cap \tau^0, e'_+|_{C(\tau') \cap \tau^0}) = (\tau^0, \epsilon^0, e^0_+).$$

As $\tau' \cap N(\tau^0) = \tau^0$, we have $\tau' = \tau^0$ and $\epsilon' = \epsilon^0, e'_+ = e^0_+$. Thus, $\sigma' = \sigma^0$. This implies $U(\tau^0, \sigma^o) = \{\sigma^0\}$ and hence, σ_0 is an isolated point in $GS(\widetilde{M})$.

Let $\sigma = (\tau, \epsilon, e_+)$ be a geosphere in $\widetilde{M}$, where τ is a subtree of T with each univalent vertex of τ a trivalent vertex in T. We call such a geosphere as a non-trivial geosphere. Let κ be subtree of T. We define $res^\kappa(\sigma)$ to be the triple $(\tau \cap \kappa, \epsilon|_{B(\tau) \cap \kappa}, e_+|_{C(\tau) \cap \kappa})$. Suppose κ is a finite subtree of T such that $res^\kappa(\sigma)$ is non-trivial in the sense that $\tau \cap \kappa$ contains at least one edge. Let $\kappa = \kappa_1 \subset \kappa_2 \subset \dots$ be an exhaustion of T by finite subtrees of T. Let $\sigma_i = res^{\kappa_i}(\sigma) = (\tau \cap \kappa_i, \epsilon|_{B(T) \cap \kappa_i}, e_+|_{C(T) \cap \kappa_i})$. To each univalent vertex $v \notin C(\tau)$ of $\tau \cap \kappa_i$, we assign an edge $e'_+(v)$ containing v and contained in $\tau \setminus \kappa_i$. Note that a univalent vertex of $\tau \cap \kappa_i$ is a trivalent vertex in T. Thus, we have a function $e'_+ : C(\tau \cap \kappa_i) \to EDGE(T)$ from univalent vertices of $\tau \cap \kappa_i$ to edge set of T whose restriction on $C(\tau) \cap \kappa_i$ ie equal to $e_+|_{C(\tau)\cap\kappa}$. Let $\sigma'_i = (\tau \cap \kappa_i, \epsilon|_{B(T)\cap\kappa_i}, e'_+|_{C(T)\cap\kappa_i})$. Then, for each i, $\sigma'_i \in S(\widetilde{M}) \subset GS(\widetilde{M})$ and $res_{\kappa_i}(\sigma'_i) = res_{\kappa_i}(\sigma)$. Therefore, $\sigma' \in U(\kappa_i, res_{\kappa_i}(\sigma))$, for each i.

Now, we claim that the sequence σ_i converge to σ in $GS(\widetilde{M})$. Let κ' be subtree of T. Consider the basic neighborhood $U(\kappa', res_{\kappa'}(\sigma))$ of σ in $GS(\widetilde{M})$. For large enough i, $\kappa' \subset \kappa_i$. Then, $U(\kappa_j, res_{\kappa_j}(\sigma)) \subset U(\kappa', res_{\kappa'}(\sigma))$ for all $j \geq i$. This implies $\sigma'_j \in U(\kappa', res_{\kappa'}(\sigma))$ for all $j \geq i$. Hence, the sequence σ_i converge to σ in $GS(\widetilde{M})$. This implies every geosphere $\sigma \notin S(\widetilde{M})$, is the limit of a sequence of points of $S(\widetilde{M})$ and hence, it is not an isolated point in $GS(\widetilde{M})$. This shows that the set $S(\widetilde{M})$ is the set of isolated points of $GS(\widetilde{M})$ and is dense in $GS(\widetilde{M})$. □

6.2 Crossing of geospheres

As in the case of spheres, we can associate to a geosphere a partition of the ends of $\widetilde{M}$, which can be identified with the set of ends $E(T)$. However, in the case of a geosphere $\sigma = (\tau, \epsilon, e_+)$, we get a partition into three sets

$$E(T) = E^\infty(\sigma) \amalg E^+(\sigma) \amalg E^-(\sigma)$$

with $E^\infty(\sigma)$ closed and $E^\pm(\sigma)$ open.

The set $E^\infty(\sigma)$ is defined to be the set of ends of τ. It is easy to see that, as τ is a subtree of T, τ is closed. Hence, $E^\infty(\sigma)$ is closed in $E(T)$. Observe that $E^\infty(\sigma)$ can also be interpreted as the set of ends of $N(\tau)$.

The complement $V(\sigma) = T - N(\tau)$ of $N(\tau)$ is an open set. We shall partition the components of $V(\sigma)$ into sets $V^+(\sigma)$ and $V^-(\sigma)$ using the data for σ, in analogy with the case of spheres. We shall define $E^\pm(\sigma)$ as the set of ends of $V^\pm(\sigma)$.

Let V_0 be a component of $T - N(\tau)$. Then, as τ is a tree, the closure of V_0 contains exactly one vertex w of $N(\tau)$, which in turn is a distance 1 from a unique vertex v of τ which is either bivalent or univalent. If v is bivalent, we say that V_0 is positive (and w is on the positive side of v) if $\epsilon(v) = +$ and say that V_0 is negative otherwise. If v is univalent, we say that V_0 is positive (and w is on the positive side of v) if the edge $e_+(v)$ joins v to w and say that V_0 is negative otherwise.

By the above rule, each component of $V(\sigma)$ is assigned a sign. We define $V^+(\sigma)$ to be the union of the positive components and $V^-(\sigma)$ the union of negative components. We define $E^\pm(\sigma)$ as the set of ends of $V^\pm(\sigma)$.

Given two geospheres σ_1 and σ_2, we can define when they cross.

Definition 6.2.1. The geospheres $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$ cross if either each of the four sets

$$E^\pm(\sigma^1) \cap (E^\pm(\sigma^2) \cup E^\infty(\sigma^2))$$

is non-empty or if each of the four sets

$$E^{\pm}(\sigma^2) \cap (E^{\pm}(\sigma^1) \cup E^{\infty}(\sigma^1))$$

is non-empty.

We remark that it is necessary to consider both the above collections of four sets separately.

The above definition is motivated by the observation that if, for instance, σ^2 is on the positive side of σ^1, then all ends (in fact points) on either the negative side of σ^2 or the positive side of σ^2 (the side away from σ^1) are on the positive side of σ^1. Hence, one of the intersections $E^-(\sigma^1) \cap (E^{\pm}(\sigma^2) \cup E^{\infty}(\sigma^2))$ is empty.

Lemma 6.2.2. *Let $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$ be geospheres. If $\tau^1 \cap \tau^2 = \phi$, then σ^1 and σ^2 do not cross.*

Proof. As τ^1 and τ^2 are subtrees of T and $\tau^1 \cap \tau^2 = \phi$, for some component V^1_0 of $T - N(\tau^1)$, τ^2 is contained in V^1_0. Let v^1 be the point in τ^1 that is unit distance from V^1_0. Without loss of generality assume V^1_0 is positive.

As τ^2 is contained in the closure of V^1_0, $E^{\infty}(\sigma^2)$ is contained in the ends of V^1_0, and hence is contained in $E^+(\sigma^1)$. Further, as τ^1 is a tree, τ^1 is contained in a component V^2_0 of $T - N(\tau^2)$ and all other components of $T - N(\tau^2)$ are contained in V^1_0. Hence, if V^2_0 is positive, then $E^-(\sigma^2)$ is contained in the ends of V^1_0, and hence is contained in $E^+(\sigma^1)$.

Thus, as V^1_0 and V^2_0 are positive, the intersection $E^-(\sigma^1) \cap (E^-(\sigma^2) \cup E^{\infty}(\sigma^2))$ is empty. Considering other cases similarly, we see that in each case, at least one of the intersections $E^{\pm}(\sigma^1) \cap (E^{\pm}(\sigma^2) \cup E^{\infty}(\sigma^2))$ is empty.

Reversing the roles of τ^1 and τ^2, we see that one of the four intersections $E^{\pm}(\sigma^2) \cap (E^{\pm}(\sigma^1) \cup E^{\infty}(\sigma^1))$ is also empty. Thus, σ^1 and σ^2 do not cross. □

Our main technical result is that crossing is an open condition.

Lemma 6.2.3. *Suppose $\sigma^i = (\tau^i, \epsilon^i, e^i_+)$, $i = 1, 2$ cross, then there are open sets U^i, $i = 1, 2$, with $\sigma^i \in U^i$ so that if $s^i \in U^i$ for $i = 1, 2$, then s^1 crosses s^2.*

Proof. Without loss of generality, we assume that each of the four intersections

$$E^{\pm}(\sigma^1) \cap (E^{\pm}(\sigma^2) \cup E^{\infty}(\sigma^2))$$

is non-empty. We shall construct open sets U^i containing σ^i so that for $s^i \in U^i$,

$$E^+(s^1) \cap (E^+(s^2) \cup E^{\infty}(s^2)) \neq \phi$$

We can similarly construct open sets for which each of the other three intersections is non-empty. The intersections of the four pairs of open sets thus constructed give the required neighborhoods of σ^i.

We first make some observations. Suppose $\xi \in E^+(\sigma^1)$ is an end. Then, there is a component V_0 of $T - N(\tau^1)$ so that $\xi \in E(V_0)$. The intersection of the closure of V with $N(\tau^1)$ is a vertex w, which is unit distance from a unique vertex v of τ^1. Further, the vertex is bivalent or univalent, with w on the positive side of v.

Let κ be a finite tree containing v. Then, if $(\tau^0, \epsilon^0, e^0_+)$ is another geosphere with $res_\kappa(\sigma^0) = res_\kappa(\sigma^1)$, then as $N(\kappa) \cap \tau^0 = N(\kappa) \cap \tau^1$, w is a vertex of $N(\tau^0) - \tau^0$, v is in τ^0. As $\epsilon^0 = \epsilon^1$ and $e^0_+ = e^+_1$, w is on the positive side of v with respect to σ^0. It follows, as τ^0 is connected, that V_0 is a component of $T - N(\tau^0)$ which is positive.

Suppose now that ξ is an end in $E^+(\sigma^1) \cap (E^+(\sigma^2) \cup E^\infty(\sigma^2))$. We consider two cases. Firstly, if $\xi \in E^+(\sigma^1) \cap E^+(\sigma^2)$, then as above we have positive components V^i_0 of $T - N(\tau^i)$ containing ξ and corresponding vertices v^i and w^i. Let κ be a finite tree containing v^1 and v^2 and let $U^i = U(\kappa, res_\kappa(\sigma^i))$.

Suppose $s^i = (t^i, \epsilon^i, e^i_+) \in U^i$, $i = 1, 2$, then, as above, V^i_0 is a component of $T - N(t^i)$ and is positive. Hence, $\xi \in E^+(s^i)$ for $i = 1, 2$, i.e., $\xi \in E^+(s^1) \cap E^+(s^2) \subset E^+(s^1) \cap (E^+(s^2) \cup E^\infty(s^2))$.

Next, consider the case when $\xi \in E^+(\sigma^1) \cap E^\infty(\sigma^2)$. Let V_0 be the component of $T - N(\sigma^1)$ that has ξ as an end and let v and w be as above. As $\xi \in E^\infty(\sigma^2)$, the intersection $\tau^2 \cap V_0$ is infinite.

Note that as σ^1 and σ^2 cross, we cannot have $\tau^1 \cap \tau^2 = \phi$, as this would imply that one of the intersections $E^-(\sigma^1) \cap (E^\pm(\sigma^2) \cup E^\infty(\sigma^2))$ is empty. As τ^2 is connected and $\tau^1 \cap \tau^2 \neq \phi \neq V_0 \cap \tau^2$, it follows that v and w are vertices of τ^2.

Let κ be a finite tree containing v and w and let $U^i = U(\kappa, res_\kappa(\sigma^i))$ and s^i be as before. As in the first case, if $s^1 \in U^1$, then V_0 is a positive component of $T - N(t^1)$. To complete the proof, we show that if $s^2 \in U^2$, then the set of ends of V_0 contains either a point of $E^\infty(s_2)$ or a point of $E^+(s_2)$.

To see this, observe that as $\tau^2 \cap V_0$ is infinite and $t^2 \cap N(\kappa) = \tau^2 \cap N(\kappa)$, with κ a tree containing w, $t^2 \cap V_0$ is non-empty. Suppose $t^2 \cap V_0$ is infinite, then an end of $t^2 \cap V_0$ lies in $V_0 \cap E^\infty(s^2)$, as claimed. On the other hand, if $t^2 \cap V_0$ is finite, it has a terminal vertex. By Lemma 5.1.8, a component of $T - N(t^2)$ is positive and contained in V^0. An end of this component gives an element $E^+(s_2)$ which is an end of V_0, hence in $E^+(s^1)$.

Thus, we have shown that in all cases $E^+(s^1) \cap (E^+(s_2) \cup E^\infty(s_2))$ is non-empty for $s^i \in U^i$.

□

6.2.1 Geosphere laminations in M

We are now in a position to define geosphere laminations in M, which are the analogues of embedded geodesic laminations in a surface. Recall that the group $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformations. Geosphere laminations are the natural completion of the inverse image in $\widetilde{M}$ of a sphere (or a collection of spheres) in M.

Definition 6.2.4. A subset $X \subset GS(\widetilde{M})$ is said to be *embedded in M* if for $\sigma_1, \sigma_2 \in X$, σ_1 does not cross σ_2.

Definition 6.2.5. A geosphere lamination in M is a subset $\Gamma \subset GS(\widetilde{M})$ such that

1. Γ is closed in $GS(\widetilde{M})$.

2. Γ is invariant under the action of $\pi_1(M)$.

3. Γ is embedded in M.

We denote the set of geosphere laminations in M by $L(M)$. We shall see that this contains all collections of disjoint, non-parallel spheres in M, and that the space of non-trivial geosphere laminations is compact. This allows us to consider limits of spheres in M.

We first observe that the condition that Γ is closed is easy to achieve.

Lemma 6.2.6. *Suppose $X \subset GS(\widetilde{M})$ is embedded in M, then so is its closure $\bar{X}$.*

Proof. Suppose σ_1 and σ_2 are geospheres in $\bar{X}$ that cross. By Lemma 6.2.3, there are open sets U_i with $\sigma_i \in U_i$ so that if $s_i \in U_i, i = 1, 2$, then s_1 and s_2 cross. As $\sigma_i \in \bar{X}$, there are elements $s_i \in X \cap U_i$, which thus cross. But, this contradicts the hypothesis that X is embedded in M. Thus, $\bar{X}$ is embedded in M. □

It is clear that the closure of a $\pi_1(M)$-invariant set in $GS(\widetilde{M})$ is $\pi_1(M)$-invariant. Thus, if X is not closed but satisfies the other two conditions for being a geosphere lamination, then its closure is a geosphere lamination.

6.2.2 Topology on $L(M)$

We shall make the set $L(M)$ of sphere laminations in M into a topological space by defining a topology on $L(M)$. To do this, we first define a topology on the set of *closed* subsets of $GS(\widetilde{M})$, which we denote by $C(\widetilde{M})$.

The topology we construct is analogous to the Hausdorff topology. Namely, if $\Gamma \subset GS(\widetilde{M})$ is closed and κ is a finite subtree of T, consider the image $res_\kappa(\Gamma)$ of Γ under the restriction map. For $S \subset GS(\kappa)$, consider the set

$$\mathcal{U}(\kappa, S) = \{\Gamma \in C(\widetilde{M}) : res_\kappa(\Gamma) = S\}.$$

Lemma 6.2.7. *The sets $\mathcal{U}(\kappa, S)$ for finite subtrees κ of T form a basis for a topology on $C(\widetilde{M})$.*

Proof. Showing that the sets $\mathcal{U}(\kappa, S)$ form a basis for a topology on $C(\widetilde{M})$ is equivalent to showing that if $\mathcal{U}(\kappa^i, S^i)$, $1 \leq i \leq n$ is a finite collection of basic open sets and $\Gamma \in \cap_i \mathcal{U}(\kappa^i, S^i)$, then there is a basic open set containing Γ and contained in each of the sets $\mathcal{U}(\kappa^i, S^i)$.

To show this, let κ be the finite subtree of T spanned by the subtrees κ^i, and let $S_0 = res_\kappa(\Gamma)$. Note that as $\Gamma \in \mathcal{U}(\kappa^i, S^i)$, $res_{\kappa^i}(\Gamma) = S^i$. Hence, if $\Gamma' \in \mathcal{U}(\kappa, S_0)$, as $\kappa \supset \kappa^i$, $res_{\kappa^i}(\Gamma') = res_{\kappa^i}(\Gamma) = S^i$, for each i. Thus, $\mathcal{U}(\kappa, S_0) \subset \mathcal{U}(\kappa^i, \sigma_0^i)$, for each i as required. □

Thus, the sets $\mathcal{U}(\kappa, S)$ form the basis for a topology, which we take to be the topology on $C(\widetilde{M})$. Note that as $GS(\kappa)$ is finite, so is the collection of subsets of $GS(\kappa)$.

If $\kappa \in T$ is a finite tree and S_1 and S_2 are subsets of $GS(\kappa)$ such that $S_1 \neq S_2$, then $\mathcal{U}(\kappa, S_1) \cap \mathcal{U}(\kappa, S_2) = \phi$ and $C(\widetilde{M}) = \amalg\, \mathcal{U}(\kappa, S_i)$, where S_i is a subset of $GS(\kappa)$.

We can easily see that $C(\widetilde{M})$ is second countable. We see that the topology is Hausdorff, in fact totally disconnected. This is based on the following lemma.

Lemma 6.2.8. *If $\Gamma_1, \Gamma_2 \subset GS(\widetilde{M})$ are closed sets with $\Gamma_1 \neq \Gamma_2$, then for some finite subtree κ of T, $res_\kappa(\Gamma_1) \neq res_\kappa(\Gamma_2)$.*

Proof. As $\Gamma_1 \neq \Gamma_2$, without loss of generality, there is a point $\sigma \in \Gamma_1 \setminus \Gamma_2$. As Γ_2 is closed subset of $GS(\widetilde{M})$, there is a basic open set $\mathcal{U} = \mathcal{U}(\kappa, \sigma_0)$ with $\sigma \in U$ but $\mathcal{U} \cap \Gamma_2 = \phi$. But this means that $res_\kappa(\sigma) \in res_\kappa(\Gamma_1) \setminus res_\kappa(\Gamma_2)$. Hence, $res_\kappa(\Gamma_1) \neq res_\kappa(\Gamma_2)$. □

It is easy to deduce that the topology on $C(\widetilde{M})$ is totally disconnected. The proof is analogous to Lemma 6.1.5.

Lemma 6.2.9. *Given $\Gamma_1, \Gamma_2 \in C(\widetilde{M})$, there are disjoint open sets $\mathcal{U}_1, \mathcal{U}_2 \subset C(\widetilde{M})$ with $\Gamma_i \subset \mathcal{U}_i$ so that $\mathcal{U}_1 \cup \mathcal{U}_2 = C(\widetilde{M})$.*

We can consider $S(\widetilde{M})$ as a subset of $C(\widetilde{M})$. If $\sigma = (\tau, \epsilon, e_+) \in S(\widetilde{M})$, then $\{\sigma\} \in C(\widetilde{M})$ and $res_\tau(\sigma) = \sigma \in GS(\tau)$. One can easily see that $\mathcal{U}(\tau, \{\sigma\}) = \{\{\sigma\}\}$. For, any geosphere whose restriction to τ is σ is equal to σ only. Thus, every point of $S(\widetilde{M})$ is an isolated point of $C(\widetilde{M})$.

The topology on $C(\widetilde{M})$ restricts to one on $L(M)$. To study the restriction, the following lemma is useful.

Lemma 6.2.10. *The subspace $L(M) \subset C(\widetilde{M})$ is closed.*

Proof. As the topology on $C(\widetilde{M})$ is second countable and Hausdorff, it suffices to show that if Γ_0 is the limit of a sequence $\Gamma_i \in L(M)$, then $\Gamma_0 \in L(M)$. Firstly, as $C(\widetilde{M})$ is Hausdorff, limits are well-defined. Hence, if $g \in \pi_1(M)$, as $g\Gamma_i = \Gamma_i$ and $g\Gamma_i \to g\Gamma_0$ (as the deck transformation g is a homeomorphism), $g\Gamma_0 = \Gamma_0$. Thus, Γ_0 is $\pi_1(M)$-invariant. Further, Γ_0 is closed as it is an element of $C(\widetilde{M})$. Thus, to complete the proof it suffices to show that Γ_0 is embedded in M.

Suppose Γ_0 is not embedded in M, then there are elements σ_1, σ_2 in Γ_0 that cross. By Lemma 6.2.3, there are open sets $\mathcal{U}_i$ with $\sigma_i \in \mathcal{U}_i$ so that if $s_i \in \mathcal{U}_i$, then s_1 and s_2 cross. By the definition of the topology on $GS(\widetilde{M})$, for some finite tree κ, $\mathcal{U}_i$ contains the open set $\mathcal{U}(\kappa, res_\kappa(\sigma_i))$. As $\Gamma_i \to \Gamma_0$, for i sufficiently large, $res_\kappa(\Gamma_i) = res_\kappa(\Gamma_0)$, in particular, there are elements $s_i \in \Gamma_i$ with $s_i \in \mathcal{U}_i$. It follows that s_1 and s_2 cross, contradicting the hypothesis that $\Gamma_i \in L(M)$. □

6.3 Constructing Geosphere laminations

In this section, we first see that (collections of) spheres in M have associated geosphere laminations. We then see how limits of spheres give rise to geosphere laminations.

Suppose first that Σ' is a collection of disjoint, non-parallel spheres in M which are in normal form with respect to Σ. Let $\tilde{\Sigma}'$ be the collection of lifts of the spheres in Σ', i.e., the inverse image of Σ' under the covering map $\widetilde{M} \to M$. Each element of $\tilde{\Sigma}'$ is a sphere, and hence, gives a geosphere. Thus, $\tilde{\Sigma}'$ can be viewed as a subset of $GS(\widetilde{M})$.

It is immediate that the set $\tilde{\Sigma}'$ is $\pi_1(M)$-invariant. The set $\tilde{\Sigma}'$ is embedded in M as it is a union of disjoint spheres. To see that $\tilde{\Sigma}'$ gives an element in $L(M))$, it only remains to show that the set $\tilde{\Sigma}'$ is a closed subset of $GS(\widetilde{M})$.

Lemma 6.3.1. *The set $\tilde{\Sigma}'$ is closed in $GS(\widetilde{M})$.*

Proof. The tree τ corresponding to each element $\sigma \in \tilde{\Sigma}'$ is finite, with diameter determined by the corresponding sphere in M. Hence, there is an integer $D > 0$ such that the trees τ corresponding to elements $\sigma \in \tilde{\Sigma}'$ have diameter at most D.

Suppose now σ_0 is in the closure of $\tilde{\Sigma}'$, with τ_0 the tree corresponding to σ_0. Let v be a vertex of τ_0 and let κ be the tree consisting of all points of distance at most D from v.

As σ_0 is in the closure of $\tilde{\Sigma}'$, $res_\kappa(\sigma_0) = res_\kappa(\sigma)$ for some $\sigma \in \tilde{\Sigma}'$. If τ is the tree corresponding to σ, then $v \in \tau$ and τ has diameter at most D. It follows that $\tau \subset \kappa$, and hence, $\tau = \tau \cap N(\kappa)$ and is contained in the interior of $N(\kappa)$. As $\tau_0 \cap N(\kappa) = \tau \cap N(\kappa)$, $\tau_0 \cap N(\kappa)$ is contained in the interior of $N(\kappa)$. Hence, as τ_0 is connected, $\tau_0 = \tau_0 \cap N(\kappa) = \tau \cap N(\kappa) = \tau$. As $res_\kappa(\sigma_0) = res_\kappa(\sigma)$, it follows that $\sigma_0 = \sigma$, hence $\sigma_0 \in \tilde{\Sigma}$. Thus, any element of the closure of $\tilde{\Sigma}'$ is in $\tilde{\Sigma}'$, showing that $\tilde{\Sigma}'$ is closed. □

Thus, given any embedded sphere S in normal form with respect to Σ in M, we have a geosphere lamination associated to it, namely, the inverse image of S in $\widetilde{M}$ under the covering map. So, we can regard S as a geosphere lamination in M. Let $S_0(M)$ be the set of isotopy classes spheres in M. Then, $S_0(M)$ can be considered as subset of $L(M)$.

Definition 6.3.2. Let Γ be geosphere lamination in M. A geosphere $\sigma \in \Gamma$ is called a leaf of Γ.

Definition 6.3.3. A subset Γ' of a geosphere lamination Γ is said to be sublamination of Γ if Γ' itself is a geosphere lamination.

Definition 6.3.4. A geosphere lamination Γ is said to be maximal if Γ is not a proper sublamination of any geosphere lamination in M.

Definition 6.3.5. A geosphere lamination Γ is said to be minimal if no proper subset of Γ sublamination of Γ.

6.3.1 Example of a geosphere lamination as the limit of a sequence of spheres

Consider $M = \sharp_2 S^2 \times S^1$. Then, $\pi_1(M) = G$, which is a free group of rank 2. Fix a basis $\{\alpha_1, \alpha_2\}$ of G. Then, there exists a collection $\Sigma' = \{A, B\}$ of disjoint, embedded 2-spheres in M, related to this basis: Each sphere A and B has two sides, denoted by A^+ and A^- for A and B^+ and B^- for B. The element α_1 is represented by a closed path γ_1 starting from the base point x_0 of M which does not belong to A and B, going to A^- piercing A, and returning to the base point from A^+. We can choose γ_1 such that it does not intersect B. Similarly, the element α_2 is represented by a closed path γ_2 starting from the base point x_0 of M, going to B^- piercing B, and returning to the base point from B^+. Again, we can choose γ_2 such that γ_2 does not intersect A and γ_1.

Extend the collection Σ' to a maximal collection $\Sigma = \{A, B, C\}$ of disjointly embedded 2- spheres in M. The sphere C has two sides, denoted by C^+ and C^-. Cutting M along Σ, then produces two 3-punctured 3-spheres, say P_1 and P_2. Suppose we have chosen sphere C such that P_1 has boundary spheres A^+, B^+, C^+ and P_2 has boundary spheres A^-, B^-, C^-.

6.3.2 The universal cover $\widetilde{M}$ and the related tree T

Let $\widetilde{\Sigma}$ be the inverse image of Σ in $\widetilde{M}$. To the pair $(\widetilde{M}, \tilde{\Sigma})$, we have the tree T associated. For a lift $\widetilde{P_1}$ of P_1, we have Y-shaped subtree of T such that the end vertices of this subtree correspond to lifts of A^+, B^+ and C^+. We denote these end vertices again by A^+, B^+ and C^+. We call such subtrees as Y_{P_1} type of subtree of T. Similarly, For a lift $\widetilde{P_2}$ of P_2, we have Y-shaped subtree of T such that the end vertices of this subtree correspond to lifts of A^-, B^- and C^-. We denote these end vertices again by A^-, B^- and C^-. We call such trees as Y_{P_2} type of subtrees of T.

Consider spheres S_n in M as follows (see figure 6.1): We construct S_n by taking a copy of A^+ in P_1 and a copy of A^- in P_2. We join them by a tube which represents $\alpha_2^n \alpha_1$. We get two disc pieces of S_n, one in P_1 with boundary on C^+ and one disc piece in P_2 with boundary on C^-. S_n has n tube pieces in P_1 joining B^+ to C^+ and n tube pieces joining C^- to B^-. We give an orientation to each S_n such that if we consider the triple $(\tau^n, \epsilon^n, e_+^n)$ associated to a lift $\widetilde{S_n}$ has the following form : The tree τ^n is a finite subtree of T and the terminal vertices of the tree τ^n are trivalent vertices in T. The tree τ^n has two terminal vertices: one terminal vertex v_1 in a Y_{P_1} type of subtree of T such that the edge in τ^n containing v_1 joins the vertex v_1 to the vertex C^+ of Y_{P_1} and the other terminal vertex v_2 in a Y_{P_2} type of subtree of T such that the edge in τ^n containing v_2 joins the vertex v_2 to the vertex C^- of Y_{P_2}. For all the other Y_{P_1} type of subtrees with which τ^n has non-empty intersection with Y_{P_1}, $\tau^n \cap Y_{P_1}$ contains an edge joining the vertex C^+ to the trivalent vertex of Y_{P_1} and an edge joining the trivalent vertex of Y_{P_1} to the vertex B^+ of Y_{P_1}. Similarly, for all the other Y_{P_2} type of subtrees with which τ^n has non-empty intersection, $\tau^n \cap Y_{P_2}$ contains an edge joining the vertex C^- to the trivalent vertex of Y_{P_1} and an edge joining the trivalent vertex of Y_{P_2} to the vertex B^- of Y_{P_2}.

For every non-standard bivalent vertex v of τ^n, $\epsilon^n(v)$ is positive. For terminal vertices $v_1 \in aY_{P_1}$ and $v_2 \in aY_{P_2}$ of τ^n, $e_+^n(v_1)$ is the edge joining the vertex B^+ and $e_+^n(v_2)$ is the joining the vertex B^-.

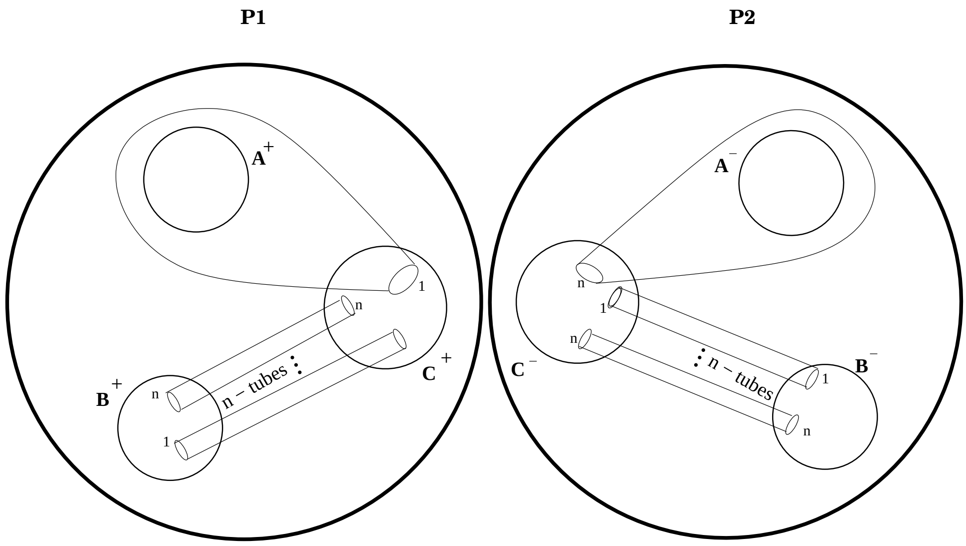

Fig. 6.1: The spheres S_n

6.3.3 The geosphere lamination Γ

Consider the set Γ which consists of the followings geospheres:

1. A geosphere $\sigma = (\tau, \epsilon, e_+)$, where τ has exactly one terminal vertex v in a Y_{P_1} type of subtree of T with $e_+(v)$ is the edge joining v and the vertex B^+ of that Y_{P_1}. The edge in τ containing the terminal vertex v joins the vertex v to the vertex C^+ of Y_{P_1}. For all the other Y_{P_1} type of subtrees of T with which τ has non-empty intersection, $\tau \cap Y_{P_1}$ consists of two edges, one edge joining the vertex B^+ and the trivalent vertex of Y_{P_1} and the other edge joining the the trivalent vertex to C^+. For all the Y_{P_2} type of subtrees of T with which τ has non-empty intersection, $\tau \cap Y_{P_2}$ consists of two edges, one edge joining the vertex C^- and the trivalent vertex of Y_{P_2} and the other edge joining the the trivalent vertex to B^-. For each non-standard bivalent vertex v' of τ, $\epsilon(v')$ is positive. The set Γ contains all the translates of σ.

2. A geosphere $\sigma' = (\tau', \epsilon', e'_+)$, where τ' has exactly one terminal vertex v' in a Y_{P_2} type of subtree of T with $e'_+(v')$ is the edge joining v_0 and the vertex B^- of the Y_{P_2}. The edge in τ' containing the terminal vertex v' joins the vertex v' to the vertex C^- of Y_{P_2}. For all the other Y_{P_2} type of subtrees of T with which τ' has non-empty intersection, $\tau \cap Y_{P_2}$ consists of two edges, one edge joining the vertex C^- and the trivalent vertex of Y_{P_2} and the other edge joining the the trivalent vertex to B^- of Y_{P_2}. For all the Y_{P_1} type of subtrees of T with which τ' has non-empty intersection, $\tau' \cap Y_{P_1}$ consists of two edges, one edge joining the vertex B^+ and the trivalent vertex of Y_{P_1} and the other edge joining the the trivalent vertex to C^+. For each non-standard bivalent vertex v' of τ', $\epsilon'(v')$ is positive. The set Γ contains all the translates of σ'.

3. A geosphere $\sigma'' = (\tau'', \epsilon'', e''_+)$, where τ'' has no terminal vertex. For all the Y_{P_1} type of subtrees of T with which τ'' has non-empty intersection, $\tau'' \cap Y_{P_1}$ consists of two edges, one edge joining the vertex B^+ and the trivalent vertex of Y_{P_1} and the other edge joining the the trivalent vertex to C^+. For all the Y_{P_2} type of subtrees of T with which τ'' has non-empty intersection, $\tau'' \cap Y_{P_2}$ consists of two edges, one edge joining the vertex C^- and the trivalent vertex of Y_{P_2} and the other edge joining the the trivalent vertex to B^-. For each the non-standard bivalent vertex v' of τ'', $\epsilon(v')$ is positive. The set Γ contains all the translates of σ''.

Note that for any geosphere $\beta = (\tau^\beta, \epsilon^\beta, e^\beta_+) \in \Gamma$, the tree τ^β does not contain any vertex of type A^+ and A^-. Therefore, β does not cross sphere $\widetilde{A}$ and its translates, where $\widetilde{A}$ is a lift of the sphere A. Now, clearly Γ is π_1-invariant.

Lemma 6.3.6. *The set Γ is embedded in M.*

Proof. We can easily see that given a type (1) geosphere $\sigma = (\tau, \epsilon, e_+)$, it is the limit of the sequence of geospheres $\widetilde{S_n} = (\tau^n, \epsilon^n, e^n_+)$, where $\widetilde{S_n}$ is a lift of S_n such that each τ^n has a terminal vertex in the same subtree Y_{P_1} of T where τ has its terminal vertex, see Proposition 6.1.8. As crossing of geospheres is an open condition (Lemma 6.2.3), we can see that σ and its translate $g\sigma$ do not cross, for any $g \in \pi_1(M)$. Similarly, we can show that for a type (2) geosphere σ', σ' and $g\sigma'$ do not cross, for any $g \in \pi_1(M)$. Consider a type (1) geosphere $\sigma = (\tau, \epsilon, e_+)$ and a type (2) geosphere $\sigma' = (\tau', \epsilon', e'_+)$. Then, there exists a sequence $\widetilde{S_n} = (\tau^n, \epsilon^n, e^n_+)$, where $\widetilde{S_n}$ is a lift of S_n such that each τ^n has a terminal vertex in the same subtree Y_{P_1} of T where τ has its terminal vertex and the spheres $\widetilde{S_n}$ converges to σ in $GS(\widetilde{M})$. Similarly, there exists a sequence $\widetilde{S'_n} = (\tau'^n, \epsilon'^n, e'^n_+)$, where $\widetilde{S'_n}$ is a lift of S_n such that each τ'^n has a terminal vertex in the same subtree Y_{P_2} of T where τ' has its terminal vertex and the spheres $\widetilde{S'_n}$ converges to σ' in $GS(\widetilde{M})$. Again, using the fact that crossing is an open condition, we see that σ and σ' do not cross. For type (3) geosphere σ'', the set $E^-(\sigma'') = \phi$. Given any translate $g\sigma''$, $(E^\pm(\sigma'') \cup E^\infty(\sigma'')) \cap E^-(g\sigma'') = \phi$ and $(E^\pm(g\sigma'') \cup E^\infty(g\sigma'')) \cap E^-(\sigma'') = \phi$. Hence, σ'' and $g\sigma''$ do not cross, for any $g \in \pi_1(M)$. By similar argument, any geosphere of type (1) and type (3) do not cross, for $i = 1, 2$. Thus, the set Γ is embedded in M. $\square$

Lemma 6.3.7. *The set Γ is a closed subset of $GS(\widetilde{M})$ and is the set of accumulation points of the set $\widetilde{\Sigma''} = \cup_n$ { inverse image of S_n in $\widetilde{M}$}.*

Proof. Suppose $\sigma_0 = (\tau^0, \epsilon^0, e^0_+)$ is a geosphere in $\widetilde{M}$ such that $\sigma_0 \notin \Gamma$. As σ_0 is not in Γ, we have the following possibilities:

1. The geosphere σ_0 crosses some lift $\widetilde{A}$ of A in $\widetilde{M}$: If σ_0 crosses A (i.e., τ^0 contains a vertex of the type A^+ or A^-), then by Lemma 6.2.3, there are open sets U^i, $i = 1, 2$, with $\sigma_0 \in U^1$ and $A \in U^2$ so that if $s^i \in U^i$ for $i = 1, 2$, then s^1 crosses s^2. If σ_0 is a limit point of Γ, there exists a sequence of geosphere $\beta_n \in \Gamma$ converging to σ_0. Therefore, there exists $\beta_n \in U^1$ for large n. But, then this

will imply that β_n crosses A which is absurd. Hence, U^1 is a neighborhood σ_0 in $GS(\widetilde{M})$ which is disjoint from Γ. So, in this case σ_0 can not be limit point of Γ.

2. The geosphere does not cross A (i.e., τ^0 does not contain any A^+ or A^- vertex) and τ^0 has a terminal vertex v in some subtree Y_{P_1} of T such that the edge $e \in \tau^0$ containing v, joins v and the vertex B^+ of Y_{P_1}: Consider a subtree κ of T containing this Y_{P_1}, then $res_\kappa(\sigma_o)$ is the triple $(\tau^0 \cap N(\kappa), \epsilon^0|_{B(\tau^0)\cap\kappa}, e^0_+|_{C(\tau^0)\cap\kappa})$. Then, $\tau^0 \cap N(\kappa)$ contains the edge e and v as a terminal vertex of $\tau^0 \cap N(\kappa)$. For any geosphere $\beta = (\tau^\beta, \epsilon^\beta, e^\beta_+) \in \Gamma$, if we consider $res_\kappa(\beta) = (\tau^\beta \cap N(\kappa), \epsilon^\beta|_{B(\tau^\beta)\cap\kappa}, e^\beta_+|_{C(\tau^\beta)\cap\kappa})$, then $\tau^\beta \cap N(\kappa)$ does not contain edge e with v as terminal vertex of $\tau^\beta \cap N(\kappa)$. So, we have $\beta \notin U(\kappa, res_\kappa(\sigma_0))$, for any $\beta \in \Gamma$. So, we get a neighborhood of σ_0 in $GS(\widetilde{M})$ disjoint from Γ.

3. The geosphere does not cross A (i.e., τ^0 does not contain any A^+ or A^- vertex) and τ^0 has a terminal vertex v in some subtree Y_{P_2} of T such that the edge $e \in \tau^0$ containing v, joins v and the vertex B^- of Y_{P_2}: Consider a subtree κ of T containing this Y_{P_2}, then $res_\kappa(\sigma_o)$ is the triple $(\tau^0 \cap N(\kappa), \epsilon^0|_{B(\tau^0)\cap\kappa}, e^0_+|_{C(\tau^0)\cap\kappa})$. Then, $\tau^0 \cap N(\kappa)$ contains the edge e and v as a terminal vertex of $\tau^0 \cap N(\kappa)$. For any geosphere $\beta = (\tau^\beta, \epsilon^\beta, e^\beta_+) \in \Gamma$, if we consider $res_\kappa(\beta) = (\tau^\beta \cap N(\kappa), \epsilon^\beta|_{B(\tau^\beta)\cap\kappa}, e^\beta_+|_{C(\tau^\beta)\cap\kappa})$, then $\tau^\beta \cap N(\kappa)$ does not contain edge e with v as terminal vertex of $\tau^\beta \cap N(\kappa)$. So, we have $\beta \notin U(\kappa, res_\kappa(\sigma_0))$, for any $\beta \in \Gamma$. So, we get a neighborhood of σ_0 in $GS(\widetilde{M})$ disjoint from Γ.

4. The geosphere does not cross A (i.e., τ^0 does not contain any A^+ or A^- vertex) and τ^0 has a terminal vertex v in some subtree Y_{P_1} of T such that the edge $e \in \tau^0$ containing v, joins v and the vertex B^+ of Y_{P_1} and $e^0_+(v)$ is the edge joining v and the vertex A^+ of Y_{P_1}: Consider a subtree κ of T containing this Y_{P_1}, then $res_\kappa(\sigma_o)$ is the triple $(\tau^0 \cap N(\kappa), \epsilon^0|_{B(\tau^0)\cap\kappa}, e^0_+|_{C(\tau^0)\cap\kappa})$. Then, $\tau^0 \cap N(\kappa)$ contains the edge e and v as a terminal vertex of $\tau^0 \cap N(\kappa)$ and $e^0_+(v)$ is the edge joining v and the vertex A^+ of Y_{P_1}. For any geosphere $\beta = (\tau^\beta, \epsilon^\beta, e^\beta_+) \in \Gamma$, if we consider $res_\kappa(\beta) = (\tau^\beta \cap N(\kappa), \epsilon^\beta|_{B(\tau^\beta)\cap\kappa}, e^\beta_+|_{C(\tau^\beta)\cap\kappa})$, then $\tau^\beta \cap N(\kappa)$ does not contain edge e with v as terminal vertex of $\tau^\beta \cap N(\kappa)$ and $e^\beta_+(v)$ is the edge joining v and the vertex A^+ of Y_{P_1}. So, we have $\beta \notin U(\kappa, res_\kappa(\sigma_0))$, for any $\beta \in \Gamma$. So, we get a neighborhood of σ_0 in $GS(\widetilde{M})$ disjoint from Γ.

5. The geosphere does not cross A (i.e., τ^0 does not contain any A^+ or A^- vertex) and τ^0 has a terminal vertex v in some subtree Y_{P_2} of T such that the edge $e \in \tau^0$ containing v, joins v and the vertex B^+ of Y_{P_2} and $e^0_+(v)$ is the edge joining v and the vertex A^- of Y_{P_2}: Consider a subtree κ of T containing this Y_{P_2}, then $res_\kappa(\sigma_o)$ is the triple $(\tau^0 \cap N(\kappa), \epsilon^0|_{B(\tau^0)\cap\kappa}, e^0_+|_{C(\tau^0)\cap\kappa})$. Then, $\tau^0 \cap N(\kappa)$ contains the edge e and v as a terminal vertex of $\tau^0 \cap N(\kappa)$ and $e^0_+(v)$ is the edge joining v and the vertex A^- of Y_{P_2}. For any geosphere $\beta = (\tau^\beta, \epsilon^\beta, e^\beta_+) \in \Gamma$, if we consider $res_\kappa(\beta) = (\tau^\beta \cap N(\kappa), \epsilon^\beta|_{B(\tau^\beta)\cap\kappa}, e^\beta_+|_{C(\tau^\beta)\cap\kappa})$, then $\tau^\beta \cap N(\kappa)$ does not contain edge e with v as terminal vertex of $\tau^\beta \cap N(\kappa)$ and $e^\beta_+(v)$ is the edge joining v and the vertex A^- of Y_{P_2}. So, we have $\beta \notin U(\kappa, res_\kappa(\sigma_0))$, for any $\beta \in \Gamma$. So, we get a neighborhood of σ_0 in $GS(\widetilde{M})$ disjoint from Γ.

6. The tree τ^0 is finite: In this case Σ_0 is a an isolated point. Hence, σ_o can not be a limit point of Γ.

Thus, for any geosphere $\sigma_0 \notin \Gamma$, we get a neighborhood of σ_0 in $GS(\widetilde{M})$ disjoint from Γ. This shows that Γ is a closed subset of $GS(\widetilde{M})$. Similar, arguments will show that any geosphere $\sigma_o \notin \Gamma$ is not an accumulation point of the set $\widetilde{\Sigma''}$. Hence, Γ is the set of accumulation points of the set $\widetilde{\Sigma''}$.

□

6.3.4 The set $res_\kappa(\Gamma)$

Suppose κ is a finite nontrivial tree. The set $res_\kappa(\Gamma)$ consists of empty graph together with the following types of elements $(\tau, \epsilon, e_+) \in GS(\kappa)$:

1. A subtree τ of $N(\kappa)$ having a terminal vertex v which is a trivalent vertex in T with $e_+(v)$ is the edge joining v to a vertex B^+. There is an edge in τ joining the vertex v and a vertex C^+. All the other edges in τ are edges joining a vertex C^- to B^- or a vertex B^+ to C^+. For each non-standard bivalent vertex $v' \in B(\tau)$, $\epsilon(v')$ is positive. Note that τ has exactly one terminal vertex.
2. A subtree τ of $N(\kappa)$ having a terminal vertex v which is a trivalent vertex in T with $e_+(v)$ is the edge joining v to a vertex B^-. There is an edge in τ joining the vertex v and a vertex C^-. All the other edges in τ are edges joining a vertex C^- to B^- or a vertex B^+ to C^+. For each non-standard bivalent vertex $v' \in B(\tau)$, $\epsilon(v')$ is positive. note that τ has exactly one terminal vertex.
3. A subtree τ of $N(\kappa)$ with all the edges are edges joining a vertex C^- to B^- and a vertex B^+ to C^+. For each non-standard bivalent vertex $v' \in B(\tau)$, $\epsilon(v')$ is positive. Note that τ has two terminal vertices.

Proposition 6.3.8. *The sequence S_n of geosphere laminations in $L(M)$ converges to the the set Γ in $C(\widetilde{M})$.*

Proof. Let κ be any subtree of T such that each terminal vertex κ is a trivalent vertex of T. If κ is trivial, then κ is a vertex v of T which is trivalent T. For such κ, we have $N(\kappa) = \kappa$. Then, the set $res_\kappa(\Gamma) = \{\kappa, \phi\} = \{\{v\}, \phi\}$. Then, for any geosphere lamination S_n, we have a lift $\widetilde{S_n} = (\tau^n, \epsilon^n, e_+^n)$ such that v is a terminal vertex of τ^n. So, the set $res_\kappa(S_n) = \{\{v\}, \phi\}$ and hence, $S_n \in \mathcal{U}(\kappa, res_\kappa(\Gamma))$, for all n.

Now, for a non-trivial finite subtree κ of T, $N(\kappa)$ contains both Y_{P_1} and Y_{P_2} type of subtrees of T. If diameter of $N(\kappa)$ is D, then we consider all $n \geq 2D + 4$.

Let $\beta = (\tau^\beta, \epsilon^\beta, e_+^\beta)$ be an element of type (1) in $res_\kappa(\Gamma)$. We choose a lift $\widetilde{S_n} = (\tau^n, \epsilon^n, e_+^n)$ of S_n such that τ^n has a terminal vertex (which is a trivalent vertex) in a Y_{P_1} type of subtree of $N(\kappa)$ where τ^β has its terminal vertex. Then, as diameter of $\tau^n \geq 2D + 4$, the other terminal vertex of τ^n does not lie inside $N(\kappa)$ and $res_\kappa(\widetilde{S_n}) = \beta$. Similarly, given an element $\beta = (\tau^\beta, \epsilon^\beta, e_+^\beta)$ of type (2) in $res_\kappa(\Gamma)$, if we choose a lift $\widetilde{S_n} = (\tau^n, \epsilon^n, e_+^n)$ of S_n such that τ^n has a terminal vertex (which is a trivalent vertex) in a Y_{P_2} type of subtree of $N(\kappa)$ where τ^β has its terminal vertex, then $res_\kappa(\widetilde{S_n}) = \beta$.

Let $\beta = (\tau^\beta, \epsilon^\beta, e_+^\beta)$ be an element of type (3) in $res_\kappa(\Gamma)$. Let v be terminal vertex of τ^β. Note that v is a standard bivalent vertex of T and also a terminal vertex of $N(\kappa)$. If v is a terminal vertex of a Y_{P_1} type of subtree contained inside $N(\kappa)$, then v is either a B^+ or a C^+ vertex in $N(\kappa) \cap Y_{P_1}$. If v is a C^+ vertex, then consider the Y_{P_2} type subtree P' of T containing v. Note that P' is such a unique Y_{P_2} type of subtree. If we consider a lift $\widetilde{S_n}$ such that τ^n has a terminal vertex (which is a trivalent vertex) in P', then as diameter of $\tau^n \geq 2D+4$, the other terminal vertex of τ^n also does not lie inside $N(\kappa)$ and $res_\kappa(\widetilde{S_n}) = \beta$. Now, Suppose v is a B^+ vertex. Let P'' be a Y_{P_2} type of subtree of T containing v and let P''' be a Y_{P_1} type of subtree of T such that P'' and P''' share a vertex v' which corresponds to a C^- vertex in P'' and C^+ vertex in P'''. If we choose a lift $\widetilde{S_n}$ such that τ^n has a terminal vertex (which is a trivalent vertex) in P''', then as diameter of $\tau^n \geq 2D+4$, the other terminal vertex of τ^n also does not lie inside $N(\kappa)$ and $res_\kappa(\widetilde{S_n}) = \beta$.

Similarly, we consider the cases where v corresponds to B^- and C^- type of vertices of some Y_{P_2} contained in $N(K)$. As for a lift $\widetilde{S_n} = (\tau^n, \epsilon^n, e_+^n)$ of S_n only finitely many translates of τ^n intersects $N(\kappa,$ empty graph is also an element of $res_\kappa(\widetilde{S_n})$. Thus, for any $n \geq 2D+4$, $res_\kappa(\widetilde{S_n}) = res_\kappa(\Gamma)$. This implies $S_n \in \mathcal{U}(\kappa, res_\kappa(\Gamma))$, for all $n \geq 4D+4$. Hence, the sequence S_n of geosphere converges to Γ in $C(\widetilde{M})$.

□

In the above example, the geosphere lamination Γ is not minimal as it contains a sublamination Γ' which consists of all the geospheres of type (3). It is not maximal as it is a sublamination of the geosphere lamination $\Gamma \cup A$.

6.3.5 Example of a geosphere lamination not in the closure of $S_0(M)$

Now, consider a subset $\Gamma_0 = \{\sigma_o\}$, where $\sigma_o = (\tau_0, \epsilon_0, e_{o+})$ is geosphere such that $\tau_0 = T$. Then, τ_0 has no terminal as well as non-standard bivalent vertices. The set Γ_0 is clearly a geosphere lamination and it is minimal. For σ_o, $E^\infty(\sigma_0) = E(T)$ and $E^+(\sigma_0) = \phi = E^-(\sigma_0)$. For any type (3) geosphere σ'', $E^\infty(\sigma'')$ contains only two elements. The set $E^+(\sigma'')$ is non-empty and $E^-(\sigma'') = \phi$. Then, we have

$$E^\pm(\sigma_0) \cap (E^\pm(\sigma) \cup E^\infty(\sigma^2)) = \phi$$

and

$$E^-(\sigma'') \cap (E^\pm(\sigma_0) \cup E^\infty(\sigma_0)) = \phi.$$

This implies σ_0 and any geosphere of type (3) do not cross. Thus, the geosphere lamination Γ_0 not maximal as the it is a sublamination of the geosphere lamination $\Gamma_o \cup \Gamma'$.

For any subtree κ of T, $\tau_0 \cap N(\kappa) = N(\kappa)$. The set $res_\kappa(\Gamma_o)$ contains exactly one element which is not an empty graph of $GS(\kappa)$. But for any normal sphere S in M, the set $res_\kappa(S)$, restriction of the geosphere lamination S to κ, contains the element empty graph of $GS(\kappa)$. Thus, for any subtree κ of T, $\mathcal{U}(\kappa, res_\kappa(\Gamma_0))$ does not contain any geosphere lamination given by a sphere in M. Hence, Γ_0 can not be limit of a sequence of geosphere laminations in $S_0(M) \subset L(M)$.

6.4 Compactness for geosphere laminations

Our main result concerning geosphere laminations is the following compactness theorem.

Theorem 6.4.1. *The spaces $L(M)$ and $C(\widetilde{M})$ are compact.*

Proof. First observe that as $L(M)$ is a closed subset of $C(\widetilde{M})$, it suffices to show that $C(\widetilde{M})$ is compact. Further, as $C(\widetilde{M})$ is second countable and Hausdorff, it suffices to show that any sequence $\Gamma_i \in C(\widetilde{M})$ has a convergent subsequence.

As in the proof of Theorem 6.1.6, let κ_i be an exhaustion of T by finite subtrees. Observe that $res_{\kappa_1}(\Gamma_i) \in GS(\kappa_i)$ is contained in a finite set, namely the set of subsets of $GS(\kappa_i)$. Hence, passing to a subsequence, we can assume that this is constant. Similarly, passing to a further subsequence, we can assume that $res_{\kappa_j}(\Gamma_i)$ is constant for each successive integer j. Iterating this and passing to a diagonal subsequence, we obtain a sequence, which we also denote Γ_i, so that the restriction of Γ_i to each of the sets κ_i is *eventually* constant. More concretely, we can assume that for $j, k \geq i$, $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_k) = res_{\kappa_i}(\Gamma_i)$.

We claim that the subsequence Γ_i constructed as above has a limit Γ_0. Let $X_i = \{\sigma \in GS(\widetilde{M}) : res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_i)\}$. It is immediate that $\Gamma_i \subset X_i$. We let $\Gamma_0 = \cap_i X_i$.

We claim that $\Gamma_i \to \Gamma_0$. As the finite trees κ_i form an exhaustion, it suffices to show that for j sufficiently large, $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_0)$. We show this for $j \geq i$.

Observe that for $j \geq i$, $X_j \subset X_i$. This is because if $\sigma \in X_j$, by definition there is a geosphere $\sigma' \in \Gamma_j$ with $res_{\kappa_j}(\sigma) = res_{\kappa_j}(\sigma')$. As $\kappa_i \subset \kappa_j$, it follows that $res_{\kappa_i}(\sigma) = res_{\kappa_i}(\sigma')$ and hence $res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_i)$, hence $\sigma \in X_i$. As $\sigma \in X_j$ was arbitrary, $X_j \subset X_i$.

Next, note that $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_i)$ for $j \geq i$. Hence, we are reduced to showing that $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_0)$. Firstly, as $\Gamma_0 \subset X_i$ and for $\sigma \in X_i$, $res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_i)$, we have $res_{\kappa_i}(\Gamma_0) \subset res_{\kappa_i}(\Gamma_i)$.

Conversely, suppose $\sigma_0 \in res_{\kappa_i}(\Gamma_i)$, and without loss of generality, σ_0 is not trivial. Then, as $res_{\kappa_i}(\Gamma_j) = res_{\kappa_i}(\Gamma_i)$ for $j \geq i$ and $\Gamma_j \subset X_j$, $\sigma_0 \in res_{\kappa_i}(X_j)$. Hence, for $j \geq i$, there is an element $\sigma_j \in X_j$ with $res_{\kappa_i}(\sigma_j) = \sigma_0$.

By the compactness theorem, Theorem 6.1.6, there is a subsequence σ_{n_j} that converges to a geosphere σ. By construction $res_{\kappa_i}(\sigma) = \sigma_0$. We finish the proof by showing that $\sigma \in \Gamma_0$, hence $\sigma_0 \in res_{\kappa_i}(\Gamma_0)$.

Assume without loss of generality that $n_j \geq j$ for all j. Hence, if $j \geq i$ is fixed, for $k \geq j$, $\sigma_{n_k} \in X_{n_k} \subset X_j$. As X_j is closed and $\sigma_{n_k} \to \sigma$, $\sigma \in X_j$. As $j \geq i$ was arbitrary, $\sigma \in \cap_j X_j = \Gamma_0$. Thus, $\sigma_0 = res_{\kappa_i}(\sigma) \in res_{\kappa_i}(\Gamma_0)$.

□

Thus, we can extract limits of geosphere laminations, in particular those of collections of spheres. For this construction to be useful, one would like the limit to be non-trivial. This turns out to be automatic for geosphere laminations in M.

Proposition 6.4.2. *The empty subset $\phi \in L(M)$ is an isolated point.*

Proof. As $\pi_1(M)$ acts cocompactly on T, there is a finite tree κ such that the translates of κ cover T. Let $\mathcal{U}$ be the open set in $C(\widetilde{M})$ given by $\mathcal{U} = \{\Gamma \in C(\widetilde{M}) : res_\kappa(\sigma) = \phi\}$. Clearly, $\phi \in \mathcal{U}$ for the empty lamination ϕ. We shall show that if $\Gamma \in L(M)$ and $\Gamma \neq \phi$, then $\Gamma \notin \mathcal{U}$.

Suppose $\Gamma \in L(M)$ is non-trivial, and let $\sigma \in \Gamma$ be a geosphere. Let v be a vertex in the tree τ corresponding to σ. Then, as the translates of κ cover T, $v \in g\kappa$ for some $g \in \pi_1(M)$. Hence, $g^{-1}v \in \kappa$, which implies that $g^{-1}\tau \cap \kappa \neq \phi$.

It follows that $res_\kappa(g^{-1}\Gamma) \neq \phi$. But, as $\Gamma \in L(M)$, $g^{-1}\Gamma = \Gamma$ and hence, $res_\kappa(\Gamma) \neq \phi$, i.e., $\Gamma \notin \mathcal{U}$ as claimed. □

6.5 Geospheres and partitions

The definition of geospheres *a priori* depends on the choice of standard spheres for M. However, we show that geospheres can be defined intrinsically by showing that they are determined by the partition of the space of ends.

As we have seen that every geosphere $\sigma = (\tau, \epsilon, e_+)$ corresponds to a partition of the set of $E(\widetilde{M})$ of ends of $\widetilde{M}$ in to three sets $E^+(\sigma), E^-(\sigma)$ and $E^\infty(\sigma)$. If τ is a finite tree, then $E^\infty(\sigma) = \phi$. If $\tau = T$, then $E^\infty = E(\widetilde{M})$ and $E^+(\sigma) = E^-(\sigma) = \phi$. In general, we get a partition with $E^\pm(\Sigma)$ open sets and $E^\infty(\Sigma)$ a closed set.

We show that any such partition corresponds to a geosphere.

Theorem 6.5.1. *Given a partition $E(\widetilde{M}) = E^+ \cup E^- \cup E^\infty$ of the ends of M (hence of T) into disjoint sets so that $E^\pm$ are open (and hence E^∞ is closed) so that either E^∞ has at least two points or both E^+ and E^- are non-empty, there is a geosphere $\sigma = (\tau, \epsilon, e_+)$ so that $E^\pm(\Sigma) = E^\pm$ and $E^\infty(\Sigma) = E^\infty$*

The proof is a slight extension of the proof of Theorem 5.1.1. We denote this partition of $E(\widetilde{M})$ by $A = (E^+, E^-, E^\infty) = (E^+(A), E^-(A), E^\infty(A))$. We note that it makes sense to talk of partitions crossing (as in the Definition 6.2.1).

Firstly, we associate a subgraph τ of T to A as in the Section 5.1 as follows: If A crosses standard sphere $\widetilde{\Sigma_i}$, then τ contains the bivalent vertex v_i corresponding to $\widetilde{\Sigma_i}$ and the edges e_1^i and e_2^i containing that vertex v_i. The other end vertex v_j^i of each edge $e_j^i, j = 1, 2$, is a trivalent vertex in T which corresponds to a component of $\widetilde{M} - \widetilde{\Sigma}$. Each v_j^i may be a bivalent or univalent or a trivalent vertex in τ. If A does not cross some standard sphere in $\widetilde{M}$, then τ does not contain the standard vertex corresponding this standard sphere and hence, it does not contain the edges containing this standard vertex.

Lemma 6.5.2. *If the partition A does not cross any standard sphere in $\widetilde{M}$, then $E^\infty(A) = \phi$ and there exists a standard sphere Σ_0 such that $E^\pm = E^\pm(\Sigma_0)$.*

Proof. Firstly we shall show that $E^\infty(A) = \phi$. Suppose $E^\infty(A) \neq \phi$. Let $P \in E^\infty$. Suppose E^∞ has another point Q, we consider the geodesic $\gamma \subset T$ from P to Q. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^-(\Sigma(e))$ and $Q \in E^+(\Sigma(e))$. Hence, $\Sigma(e)$ crosses the given partition A. This contradiction to the hypothesis as A does not cross any standard sphere.

On the other hand, if P is the only point in $E^\infty(A)$, then there are points $Q^\pm \in E^\pm(A)$. Let α be the geodesic from Q^- to Q^+ and let γ be the unique geodesic ray from a point of α to P with the property that its interior is disjoint from α. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^-(\Sigma(e))$ and $Q^\pm \in E^+(\Sigma(e))$. Hence, $\Sigma(e)$ crosses the given partition A. This is contradiction to hypothesis. Therefore, $E^\infty(A) = \phi$.

Now, by hypothesis, if v is a standard bivalent vertex of T, the standard sphere $\Sigma(v)$ corresponding to v does not cross A. Hence, after choosing orientations appropriately, either $E^+(\Sigma(v)) \subset E^+(A)$ or $E^-(\Sigma(V)) \subset E^-(A)$. If $\Sigma(v) = \Sigma_0$ satisfies both the conditions, then $E^\pm(A) = E^\pm(\Sigma_0)$.

Suppose no $\Sigma(v)$ satisfies both the above conditions, we get a partition of bivalent vertices of T as

$$V^+ = \{v : E^+(\Sigma(v)) \subset E^+(A)\}$$

and

$$V^- = \{v : E^-(\Sigma(v)) \subset E^-(A)\}.$$

Let $X^\pm$ is the union of all the edges e in T such that the bivalent vertex of e lies in $V^\pm$. Then, $X^\pm$ are closed and $T = X^+ \cup X^-$. Hence, $X^+ \cap X^- \neq \phi$. By construction, $X^+ \cap X^-$ consists of trivalent vertices of T. Let $w \in X^+ \cap X^-$ and let v_1, v_2 and v_3 be bivalent vertices adjacent to w. Note that at least one $v_i \in X^+$ and at least one $v_j \in X^-$. Without loss of generality, suppose $v_1, v_2 \in X^+$ and $v_3 \in X^-$. Let $N(w)$ denote the set of all the points in T distance at most 1 from w. Then, $T - N(w)$ has three components V_1, V_2 and V_3 whose closures contain the vertices v_1, v_2 and v_3, respectively. It is easy to see that $E(V_1) \subset E^+, E(V_2) \subset E^+$ and $E(V_3) \subset E^-$. It follows that $E^+(\Sigma(v_3)) = E^+(\Sigma(v_1)) \cup E^+(\Sigma(v_2))$. This implies $E^+(\Sigma(v_3)) \subset E^+$. As $v_3 \in X^-$, $E^-(\Sigma(v_3)) \subset E^-$. But then, $v_3 \in V^+ \cap V^-$. This is a contradiction as V^+ and V^- are disjoint. Hence, there must exist a standard sphere Σ_0 such that $E^\pm(A) = E^\pm(\Sigma_0)$. □

If A does not cross any standard sphere, the tree τ associated to A is a standard vertex corresponding to the standard sphere representing A. Note that any edge e in T has a unique end vertex which is a standard bivalent vertex in T.

We make the following observations :

If the partition $A = (E^+, E^-, E^\infty)$ of $E(\widetilde{M})$ crosses a sphere $S = (E^+(S), E^-(S))$ in $\widetilde{M}$, where $(E^+(S), E^-(S))$ is a partition of $E(\widetilde{M})$ given by S, then all the four intersections $E^\pm(S) \cap (E^\pm \cup E^\infty)$ are non-empty. For, if $E^\varepsilon(S) \cap (E^\eta(A) \cup E^\infty(A)) = \phi$, for some sign ε and η, then $E^\eta(A) \subset E^{\bar{\varepsilon}}(S)$ and hence, $E^\eta(A) \cap E^\varepsilon(S) = \phi$. This is a contradiction to the fact the partition A crosses S.

Lemma 6.5.3. *The graph τ is connected and hence, a subtree of T.*

Proof. Suppose S, S' and S'' are standard spheres in $\widetilde{M}$ such that the standard bivalent vertex v' in T corresponding to S' lies on the reduced path in T joining the standard bivalent vertices v and v'' in T corresponding to S and S'', respectively. By giving appropriate orientations to S, S' and S'', we can assume that $E^+(S'') \subset E^+(S') \subset E^+(S)$ and $E^-(S) \subset E^-(S') \subset E^-(S'')$. Now, if A crosses S and S'', then we can easily see that A crosses S'. This shows that the reduced path in T joining v and v'' in T is completely contained in τ. From this, one easily see that τ is connected and hence a subtree of T. □

Note that the terminal vertices of τ are trivalent vertices in T.

If A does not cross a sphere $S = (E^+(S), E^-(S))$ in $\widetilde{M}$, where $(E^+(S), E^-(S))$ is a partition of $E(\widetilde{M})$ given by S, then $E^{\varepsilon}(S) \cap (E^{\eta}(A) \cup E^{\infty}(A)) = \phi$, for some sign ε and η obtained by choosing signs ε and η in $\{+, -\}$. Then, $E^{\varepsilon}(S) \subset E^{\bar{\eta}}(A)$ and $(E^{\eta}(A) \cup E^{\infty}(A)) \subset E^{\bar{\varepsilon}}(S)$. In this case, we say S is on the $\bar{\eta}$-side of A and A is on $\bar{\varepsilon}$-side of S.

Note that the tree τ may or may not have terminal vertices. Suppose v is a vertex of τ adjacent to a single edge $e_0 \in \tau$, i.e., a terminal vertex of τ. Let $v_0 \in \tau$ be the other end vertex of e_0 and Σ_0 be the standard sphere in $\widetilde{M}$ corresponding to v_0. Then, A crosses Σ_0. Let the other edges adjacent to v in T be e_1 and e_2 with other end vertices v_1 and v_2, respectively. Consider the standard spheres $\tilde{\Sigma}_i = \tilde{\Sigma}(v_i)$ corresponding to vertices v_i, with orientations chosen so that for $i = 1, 2$, the set $E^+(\tilde{\Sigma}_i)$ is the set of ends of the component of $\widetilde{M} - \tilde{\Sigma}_i$ that does not contain $\tilde{\Sigma}_0$. We can orient $\tilde{\Sigma}_0$ so that $E^+(\tilde{\Sigma}_0) = E^+(\tilde{\Sigma}_1) \cup E^+(\tilde{\Sigma}_2)$.

Lemma 6.5.4. *For some sign ε, $E^{\varepsilon}(A) \supset E^+(\tilde{\Sigma}_1)$ and $E^{\bar{\varepsilon}}(A) \supset E^+(\tilde{\Sigma}_2)$.*

Proof. First note that for each i, $i = 1, 2$, $E^+(\tilde{\Sigma}_i) \cap E^{\infty}(A) = \phi$. For, if $E^+(\tilde{\Sigma}_i) \cap E^{\infty}(A) \neq \phi$, then $E^+(\tilde{\Sigma}_i) \cap (E^{\pm}(A) \cup E^{\infty}(A)) \neq \phi$. As $E^-(\tilde{\Sigma}_0) \subset E^-(\tilde{\Sigma}_i)$ and A crosses $\tilde{\Sigma}_0$, we have $E^-(\tilde{\Sigma}_i) \cap (E^{\pm}(A) \cup E^{\infty}(A)) \neq \phi$. This implies that A crosses $\tilde{\Sigma}_i$, which is a contradiction. Thus, $E^+(\tilde{\Sigma}_0) \cap E^{\infty}(A) = \phi$.

As A does not cross the spheres $\tilde{\Sigma}_i$, for appropriate signs ε_i, $(E^{\varepsilon_i}(A) \cup E^{\infty}(A)) \cap E^+(\tilde{\Sigma}_i) = \phi$. Then, we have $E^+(\tilde{\Sigma}_i) \subset E^{\bar{\varepsilon}_i}(A)$, for $i = 1, 2$. Finally, if $\varepsilon_1 = \varepsilon_2 = \varepsilon$, then $E^{\bar{\varepsilon}}(A) \supset E^+(\tilde{\Sigma}_0)$ as $E^+(\tilde{\Sigma}_0) = E^+(\tilde{\Sigma}_1) \cup E^+(\tilde{\Sigma}_2)$. As $E^{\infty}(A) \cap E^{(}\tilde{\Sigma}_0) = \phi$, we get $E^+(\tilde{\Sigma}_0) \cap (E^{\varepsilon}(A) \cup E^{\infty}(A)) = \phi$, contradicting the hypothesis that A crosses $\tilde{\Sigma}_0$. Therefore, $\varepsilon_1 \neq \varepsilon_2$. Hence the result. □

Thus, one of the spheres $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ is on the positive side of A and the other on the negative side. In the case of a vertex v of valence 2 of τ, either it is a bivalent vertex (standard vertex) of T or there is an edge e_v of T adjacent to v which is not in τ. The standard sphere $\tilde{\Sigma}(e_v)$ corresponding to the other end vertex of the edge e_v is either on the positive side of A or on the negative side.

Let $N(\tau)$ be the subgraph of T consisting of points with distance at most 1 from τ. Then, $N(\tau)$ is a tree, which is the union of τ with the following two kinds of edges:

1. For each terminal vertex v of τ, we have a pair of edges $e_1(v) \notin \tau$ and $e_2(v) \notin \tau$ with v as an end-vertex. Let v_1 and v_2 be the other end vertices of e_1 and e_2, respectively.

2. For each non-standard bivalent vertex w of τ, we have an edge $e(w) \notin \tau$ with w as an end-vertex. Let w_1 be its other end vertex.

By Lemma 6.5.4, for a terminal vertex v, the sphere corresponding to one of v_1 and v_2 is on the positive side of τ (positive side of A). The vertices v_1 and v_2 are end vertices of e_1 and e_2 respectively. So, we can assign positive or negative signs to these edges accordingly. We denote this by $e_+(v)$ and denote the other edge (which is on the negative side) by $e_-(v)$. We denote the standard spheres corresponding to v_1 and v_2 by $\tilde{\Sigma}(v_1) = \tilde{\Sigma}(e_1)$ and $\tilde{\Sigma}(v_2) = \tilde{\Sigma}(e_2)$, respectively. For a non-standard bivalent vertex w of τ, we can associate a sign $\epsilon(w)$ so that $\tilde{\Sigma}(w_1) = \tilde{\Sigma}(e(w))$ is on the $\epsilon(w)$-side of A. Thus, we have a triple $\sigma = (\tau, \epsilon, e_+)$ which is geosphere in $\widetilde{M}$.

Now we shall show that σ gives the partition A of $E(\widetilde{M})$.

Lemma 6.5.5. *The partition $(E^+(\sigma), E^-(\sigma), E^\infty(\sigma))$ of $E(\widetilde{M})$ given by the geosphere σ is the same as the partition A of $E(\widetilde{M})$.*

Proof. Let $P \in E^+(A)$. As $E^+(A)$ is open in the space of ends of T, there is a finite connected tree $\kappa \subset T$ and a component V of $T - \kappa$ so that $P \in E(V) \subset E^+(A)$. We shall show that no edge of V is contained in τ. Let e be an edge of T contained in $V = T - \kappa$. Then, as κ is connected, some component W of $T - e$ is disjoint from κ, and hence contained in V. Suppose v is the end vertex of e such that v is a standard bivalent vertex in T. Let $\Sigma(v)$ be the standard sphere corresponding to v, then it follows that for some sign ε, $E^\varepsilon(\Sigma(v)) \subset E(V) \subset E^+(A)$, and hence, $\Sigma(v)$ does not cross A. This implies v is not in τ. It follows that e is not in τ. Thus, no edge of V is in τ, as required.

Let W_0 be the component of $T - \tau$ that contains V. Then, the closure of W_0 intersects τ in a single vertex, which is either a terminal vertex or a non-standard bivalent vertex. In either case, $E(W_0) \subset E^+(\sigma)$ by construction of the partition associated to a geosphere. Then, as $P \in E(V) \subset E(W_0)$, $P \in E^+(\sigma)$. Thus, $E^\pm \subset E^\pm(\sigma)$.

We next show that $E^\infty(A) \subset E^\infty(\sigma)$. Let $P \in E^\infty(A)$. Suppose $E^\infty(A)$ has another point Q, we consider the geodesic $\gamma \subset T$ from P to Q. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^-(\Sigma(e))$ and $Q \in E^+(\Sigma(e))$. Hence, $\Sigma(e)$ crosses the given partition A, so $v \in \tau$ and hence, $e \in \tau$. Thus, $\gamma \subset \tau$ and hence $P \in E^\infty(\tau)$.

On the other hand, if P is the only point in $E^\infty(A)$, then there are points $Q^\pm \in E^\pm(A)$. Let α be the geodesic from Q^- to Q^+ and let γ be the unique geodesic ray from a point of α to P with the property that its interior is disjoint from α. Given any edge e of γ, if $\Sigma(e)$ is the standard sphere corresponding to the standard vertex of e oriented appropriately, then $P \in E^-(\Sigma(e))$ and $Q^\pm \in E^+(\Sigma(e))$. Hence, $\Sigma(e)$ crosses the given partition A, so $e \in \tau$. Thus, $\gamma \subset \tau$ and hence, $P \in E^\infty(\tau)$.

This shows that $E^\infty(A) \subset E^\infty(\sigma)$. Thus, as $(E^+(\sigma), E^-(\sigma), E^\infty(\sigma))$ and A form partitions of $E(\widetilde{M})$, both are the same. □

From this, Theorem 6.5.1 follows.

7. FURTHER DIRECTIONS...

In this chapter, we see our further plans of work.

The geometric intersection number of curves on surfaces has been used to study Thurston compactification of Teichmüller space of a surface and the boundary of Teichmüller space, namely the space of projectivised measured laminations. Geodesic laminations (and measured laminations) on surfaces have proved to be very fruitful in three-manifold topology, Teichmüller theory and related areas and mapping class group of a surface. By *Dehn-Nielsen-Baer theorem*, the mapping class group of a surface S of positive genus is isomorphic to the group of outer automorphisms of $\pi_1(S)$.

Culler and Vogtmann [7], introduced a space X_n on which the group $Out(\mathbb{F}_n)$ acts with finite point stabilizers, and proved that X_n is contractible. Peter Shalen later invented the name " *Outer space*" for X_n. Outer space with the action of $Out(\mathbb{F}_n)$ can be thought of as free group analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface. Culler and Morgan have constructed a compactification of Outer space much like Thurston compactification of Teichmüller space [6].

We are trying to develop techniques to study sphere complex, $\mathrm{Out}(\mathbb{F}_n)$, Outer space of a free group analogous to simple closed curves on a surface, intersection numbers, geodesic laminations, measured laminations, curve complex used to study mapping class group of a surface.

We can ask the following questions:

(1) What are the isolated points of the space $L(M)$ of geosphere laminations of M? Given a space X, we can define X_w to be the set of accumulation points of X. This inductively gives sequences $X \supset X_w \supset (X_w)_w \supset \cdots$. What is this for $L(M)$ and for the space $GS(\widetilde{M})$ of geospheres in $\widetilde{M}$?

(2) Given any embedded sphere S in normal form with respect to Σ in M, we have a geosphere lamination associated to it, namely, the inverse image of S in $\widetilde{M}$ under the covering map. So, we can regard S as a geosphere lamination in M. Let $S_0(M)$ be the set of isotopy classes spheres in M. Then, $S_0(M)$ can be considered as subset of $L(M)$. What is the closure of $S_0(M)$ in $L(M)$?

(3) Geosphere can be defined as a partition of the set of ends of $\widetilde{M}$. Put appropriate topology on the set of such partitions and show that the topology on the geosphere defined earlier and this topology are the same. Define notion of geosphere laminations independent of the maximal sphere system in M. Study geosphere laminations in this set up.

(4) Define intersection number for geospheres and geosphere laminations and study this intersection

number.

(5) Introduce concept of measured geosphere laminations and study its connection with the boundary of Outer space of a free group.

(6) Use geosphere laminations and measured geosphere laminations to study $Out(\mathbb{F}_n)$ and try to connect this with the work of Bestivina and Handle, [1], [3], [2], [4].

(7) Embedded sphere in M corresponds to splittings of free group. What are geosphere in algebraic setting?

(8) The *geometric intersection number* of curves on a surface has been used to give constructions like the space of measured laminations whose projectivization is the boundary of Teichmüller space, [35], as well as to study geometric properties, including hyperbolicity of the curve complex in [5], [36]. One may hope that the *geometric intersection number* of embedded spheres in M might be useful to give such constructions in case of the sphere complex and Outer space.

Study Scott-Swarup intersection number of spheres in more details.

(9) Define multiplicative structure of spheres and study it.

(10) Study hyperbolicity of sphere complex. See whether sphere complex is δ-hyperbolic in the sense of Gromov or not.

(11) Define the analogue of geodesic currents so that the geosphere laminations are geodesic currents with self intersection number zero. Is there an analogue of Teichmüller space? (12) What is the structure of a geosphere lamination, in particular in terms of its sublaminations?

(13) Given $\phi \in Out(\pi_1(M))$, relate limits of $\phi^n(\Sigma)$, where Σ is sphere in M, with the structure of outer automorphism.

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