A STUDY ON ALGEBRAS WITH RETRACTIONS AND ON PLANES OVER A DVR

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Dedicated to my parents

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Notation

Throughout the thesis all rings will be assumed to be commutative rings with unity. For a commutative ring R, a prime ideal P of R, and an R-algebra A, the following notation will be used:

$Spec(R)$	:	The set of all prime ideals of R.
$ht(P)$	:	The height of the ideal P.
$Qt(R)$	:	The field of fractions of R, when R is an integral domain.
$R^{[n]}$	:	Polynomial ring in n variables over R.
R^*	:	Group of units of R.
$k(P)$	:	Residue field R_P/PR_P.
A_P	:	$= S^{-1}A$ where $S = R\backslash P$.
$Sym_R(M)$	:	Symmetric algebra of an R-module M over R.
$Aut_R(A)$	:	The group of R-algebra automorphisms of A.
$tr.deg_R(A)$	:	Transcendence degree of A over R.
$ch(R)$	:	Characteristic of R.
$\Omega_R(A)$	:	The universal module of differential of A over R.
DVR	:	Discrete valuation ring.

Contents

Chapter 1

Introduction

Aim:

The main aim of this thesis is to study the following problems:

1. For a Noetherian ring R, to find a set of minimal sufficient fibre conditions for an R-algebra with a retraction to R to be an $\mathbb{A}^1$-fibration over R.

2. To investigate sufficient conditions for a factorial $\mathbb{A}^1$-form, with a retraction to the base ring, to be $\mathbb{A}^1$.

3. To investigate whether planes of the form $b(X,Y)Z^n - a(X,Y)$ are coordinate planes in the polynomial ring in three variables X, Y and Z over a discrete valuation ring.

The 1^{st} problem will be discussed in Chapter 3 entitled "Codimension-one $\mathbb{A}^1$-fibration with retraction", the 2^{nd} problem will be studied in Chapter 4 under the heading "$\mathbb{A}^1$-form with retraction" and the 3^{rd} problem will be investigated in Chapter 5 which has the title "Planes of the form $b(X,Y)Z^n - a(X,Y)$ over a DVR".

Brief introductions to the topics of the problems and precise statements of the main results obtained are given below:

• Codimension-one $\mathbb{A}^1$-fibration with retraction

Let R be a ring. A finitely generated flat R-algebra A is said to be an $\mathbb{A}^1$-*fibration* over R if $A \otimes_R k(P) = k(P)^{[1]}$ for all prime ideals P of R. A very interesting and important phenomenon is that the generic and codimension-one fibres determine an $\mathbb{A}^1$-fibration. To get a feel for this striking feature of $\mathbb{A}^1$-fibration, here is a nice result by Bhatwadekar-Dutta ([BD95]):

Theorem 1.0.1. *Let R be a Noetherian domain with field of fractions K and A an R-subalgebra of $R[T_1, T_2, \cdots, T_n]$ such that A is flat over R, $A \otimes_R K = K^{[1]}$ and $A \otimes_R k(P)$ is an integral domain for every prime ideal P in R of height one. Then*

(i) *If R is normal, then $A \cong Sym_R(I)$ for an invertible ideal I of R.*

(ii) *If R contains $\mathbb{Q}$, then A is an $\mathbb{A}^1$-fibration over R.*

(iii) *If R is seminormal and contains $\mathbb{Q}$, then $A \cong Sym_R(I)$ for an invertible ideal I of R.*

An analogous result has also been obtained by Dutta ([Dut95]) for finitely generated faithfully flat R-subalgebras:

Theorem 1.0.2. *Let R be a Noetherian domain with field of fractions K and A a faithfully flat finitely generated R-algebra such that $A \otimes_R K = K^{[1]}$ and $A \otimes_R k(P)$ is geometrically integral for every prime ideal P in R of height one. Then*

(i) *If R is normal, then $A \cong Sym_R(I)$ for an invertible ideal I of R.*

(ii) *If R contains $\mathbb{Q}$, then A is an $\mathbb{A}^1$-fibration over R.*

(iii) *If R is seminormal and contains $\mathbb{Q}$, then $A \cong Sym_R(I)$ for an invertible ideal I of R.*

We will call an R-algebra A a *Codimension-one $\mathbb{A}^1$-fibration* if $A \otimes_R k(P) = k(P)^{[1]}$ for each prime ideal P of R with $ht(P) \leq 1$. In view of the above theorems it is easy to see that

1. For a Noetherian normal domain R or a Noetherian domain R containing $\mathbb{Q}$, a flat R-subalgebra A of a polynomial algebra over R is an $\mathbb{A}^1$-fibration over R if and only if A is a codimension-one $\mathbb{A}^1$-fibration over R.

2. For a Noetherian normal domain R or a Noetherian domain R containing $\mathbb{Q}$, a faithfully flat finitely generated R-algebra A is an $\mathbb{A}^1$-fibration over R if and only if A is a codimension-one $\mathbb{A}^1$-fibration over R.

In ([Asa87], Theorem 3.4), Asanuma has given a structure theorem for $\mathbb{A}^r$-fibrations over a Noetherian ring. The statement of Asanuma's theorem shows that

A *necessary* condition for an algebra A over a Noetherian ring R to be $\mathbb{A}^r$-fibration is that A is isomorphic, as an R-algebra, to an R-subalgebra of some polynomial ring over R.

As a consequence of this result we get that any $\mathbb{A}^r$-fibration over a Noetherian ring has a retraction to R. Therefore, when R is Noetherian, it is natural to ask for minimal sufficient fibre conditions which ensure that an R-algebra with a retraction to R will be a codimension-one $\mathbb{A}^1$-fibration over R.

Recently, in [BDO], Bhatwadekar-Dutta-Onoda have shown, as a consequence of a general structure theorem for any faithfully flat R-algebra over a Noetherian normal domain which is locally $\mathbb{A}^1$ in codimension-one, that for a Noetherian normal domain R, a flat R-algebra A with a retraction to R is an $\mathbb{A}^1$-fibration over R (in fact, $Spec(A)$ is an algebraic line bundle over $Spec(R)$) if A is locally $\mathbb{A}^1$ in codimension-one; more precisely,

Theorem 1.0.3. *Let R be a Noetherian normal domain with field of fractions K and A a Noetherian flat R-algebra such that $A_P = {R_P}^{[1]}$ for each prime ideal P of R of height one. Suppose that there exists a retraction $\Phi : A \twoheadrightarrow R$. Then $A \cong Sym_R(I)$ for an invertible ideal I in R.*

In view of the above results, naturally one asks the following questions:

(1) Is Theorem 1.0.1 true when the condition "A is an R-subalgebra of $R[T_1, T_2, \cdots, T_n]$" is replaced by the condition "A has a retraction to R"?

(2) Is Theorem 1.0.2 true when the condition "A is a faithfully flat finitely generated R-algebra" is replaced by the condition "A is a flat R-algebra with a retraction to R"?

(3) How far can the hypothesis "R is normal" in Theorem 1.0.3 be relaxed?

In Chapter 3 of the thesis, we investigate the above questions. We will show that questions (1) and (2) have answers in the affirmative when A is Noetherian; the results also show that Theorem 1.0.3 holds in more generality. The main results of this study are listed below (Proposition 3.3.4, Theorem 3.3.5, Theorem 3.3.7, Theorem 3.3.9):

Proposition A. *Let R be either a Noetherian domain or a Krull domain with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A_P = {R_P}^{[1]}$ for every prime ideal P of R satisfying depth $(R_P) = 1$.*

Then there exists an invertible ideal I of R such that $A \cong Sym_R(I)$.

Theorem A. *Let R be a Krull domain with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R K = K^{[1]}$.*

(3) *$A \otimes_R k(P)$ is an integral domain for each height one prime ideal P of R.*

Then there exists an invertible ideal I of R such that $A \cong Sym_R(I)$.

Theorem B. *Let R be a Noetherian domain with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R K = K^{[1]}$.*

(3) *$A \otimes_R k(P)$ is geometrically integral over $k(P)$ for each height one prime ideal P of R.*

Then A is finitely generated over R and there exists a finite birational extension R' of R and an invertible ideal I of R' such that $A \otimes_R R' \cong Sym_{R'}(I)$.

Theorem C. *Let R be a Noetherian domain containing $\mathbb{Q}$ with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R K = K^{[1]}$.*

(3) *$A \otimes_R k(P)$ is an integral domain for each height one prime ideal P of R.*

Then A is an $\mathbb{A}^1$-fibration over R. Thus, if R is seminormal, then $A \cong Sym_R(I)$ for some invertible ideal I of R.

As a consequence of Theorem A, we get the following Lüroth-type result (see Corollary 3.3.6):

Corollary A. *Let R be a UFD with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R K = K^{[1]}$.*

(3) *$A \otimes_R k(P)$ is an integral domain for each height one prime ideal P of R.*

Then there exists $x \in Ker\ \Phi$ such that $A = R[x] = R^{[1]}$.

• Factorial $\mathbb{A}^1$-form with retraction

Let k be a field with algebraic closure $\bar{k}$ and let $R \hookrightarrow A$ be k-algebras. We shall call A an $\mathbb{A}^1$-form over R if $A \otimes_k \bar{k} = (R \otimes_k \bar{k})^{[1]}$. It is well known that any separable $\mathbb{A}^1$-form over any field is trivial. More generally, the following result ([Dut00], Theorem 7) shows that a separable $\mathbb{A}^1$-form over any arbitrary commutative algebra is trivial.

Theorem 1.0.4. *Let k be a field, L a separable field extension of k, R a k-algebra and A an R-algebra such that $A \otimes_k L \cong Sym_{(R \otimes_k L)}(P')$ for a finitely generated rank one projective module P' over $R \otimes_k L$. Then $A \cong Sym_R(P)$ for a finitely generated rank one projective module P over R.*

If k is not perfect, there exist non-trivial purely inseparable $\mathbb{A}^1$-forms. Asanuma gave a complete structure theorem for purely inseparable $\mathbb{A}^1$-forms over a field k of characteristic $p > 2$ ([Asa05], Theorem 8.1). However, from Asanuma's results, it can be deduced that any factorial $\mathbb{A}^1$-form over a field k with a k-rational point is trivial, i.e, we have the following result:

Theorem 1.0.5. *Let k be a field and A an $\mathbb{A}^1$-form over k such that*

(1) *A is a UFD.*

(2) *A has a k-rational point.*

Then $A = k^{[1]}$.

In Chapter 4 of this thesis we prove the following generalization (see Theorem 4.2.2) of the above result. Our result also gives a simple proof of Theorem 1.0.5 without using Asanuma's intricate structure theorem.

Theorem D. *Let k be a field and let $R \hookrightarrow A$ be k-algebras such that*

(1) *A is a UFD.*

(2) *There is a retraction $\Phi : A \longrightarrow R$.*

(3) *A is an $\mathbb{A}^1$-form over R.*

Then $A = R^{[1]}$.

• Planes of the form $b(X,Y)Z^n - a(X,Y)$ over a DVR

Let k be a field and $g \in k[X_1, X_2, \ldots, X_m](= k^{[m]})$. We say g is a *variable* in $k[X_1, X_2, \ldots, X_m]$ if there exist elements $f_1, f_2, \ldots, f_{m-1}$ such that $k[X_1, X_2, \ldots, X_m] = k[g][f_1, f_2, \ldots, f_{m-1}] = k[g]^{[m-1]}$. It is obvious that if $g \in k[X_1, X_2, \ldots, X_m](= k^{[m]})$ is a variable, then $k[X_1, X_2, \ldots, X_m]/(g) = k^{[m-1]}$. Naturally one asks whether the converse holds:

Problem 1. *Let k be a field, $m \geq 2$ an integer and $g \in k[X_1, X_2, \ldots, X_m](= k^{[m]})$ be such that $k[X_1, X_2, \ldots, X_m]/(g) = k^{[m-1]}$. Is then $k[X_1, X_2, \ldots, X_m] = k[g]^{[m-1]}$?*

In affine algebraic geometry, this problem is generally known as the *Epimorphism problem*. While the problem is open in general, a few special cases have been investigated by some mathematicians. For such cases, one also considers the corresponding generalized epimorphism problem.

Problem 1′. *Let R be an integral domain, $m \geq 2$ an integer and $g \in R[X_1, X_2, \ldots, X_m](= R^{[m]})$ be an element such that $R[X_1, X_2, \ldots, X_m]/(g) = R^{[m-1]}$. Is then $R[X_1, X_2, \ldots, X_m] = R[g]^{[m-1]}$?*

The first major breakthrough in this area was got, independently, by Abhyankar-Moh ([AM75]) and Suzuki ([Suz74]). They showed that Problem 1 has an affirmative answer for the case $m = 2$ when the characteristic of the field k is 0:

Theorem 1.0.6. *Let k be a field of characteristic 0. Suppose that $g \in k[X, Y](= k^{[2]})$ is such that $k[X, Y]/(g) = k^{[1]}$. Then $k[X, Y] = k[g]^{[1]}$.*

This theorem is known as the famous Abhyankar-Moh and Suzuki Epimorphism Theorem. The following well known counter example shows that Theorem 1.0.6 does not hold over fields of positive characteristic.

Example 1.0.7. *Let k be a field of characteristic $p > 0$ and $g = Y^{p^e} - X - X^{sp} \in k[X,Y](= k^{[2]})$ where $p \nmid s$ and $e \geq 2$. Then $k[X,Y]/(g) = k^{[1]}$ but $k[X,Y] \neq k[g]^{[1]}$ (see [Abh77], Example 9.12, pg. 72).*

In ([RS79], Theorem 2.6.2), Russell-Sathaye showed that Theorem 1.0.6 holds over locally factorial Krull domains of characteristic 0. The most generalized version of Theorem 1.0.6 has been obtained by Bhatwadekar. He has shown that the theorem can be extended to any seminormal domain of characteristic 0 and to any integral domain containing a field of characteristic 0 ([Bha88], Theorem 3.7 and Theorem 3.9):

Theorem 1.0.8. *Let R be a seminormal domain of characteristic 0 or an integral domain containing $\mathbb{Q}$. Let $g \in R[X,Y](= R^{[2]})$ be such that $R[X,Y]/(g) = R^{[1]}$. Then $R[X,Y] = R[g]^{[1]}$.*

The case $m = 3$ of Problem 1 is still open in general. Among the partial results in this direction, the following theorem of Kaliman ([Kal02]) deserves a special mention.

Theorem 1.0.9. *Let $g \in \mathbb{C}[X,Y,Z]$ be such that $\mathbb{C}[X,Y,Z]/(g - \lambda)$ for all but finitely many $\lambda \in \mathbb{C}$. Then $\mathbb{C}[X,Y,Z] = \mathbb{C}[g]^{[2]}$.*

For certain specific forms of g, affirmative answers (to the case $m = 3$ of Problem 1) had been obtained by Sathaye, Russell and Wright. In particular, when g is of the form $b(X,Y)Z^n - a(X,Y)$, affirmative answers were obtained in the following cases:

(1) $n = 1$, k a field of characteristic 0 (A. Sathaye, [Sat76]).

(2) $n = 1$, k a field of any characteristic (P. Russell, [Rus76]).

(3) $n \geq 2$ and k an algebraically closed field of characteristic $p \geq 0$ with $p \nmid n$ (D. Wright, [Wri78]).

In Chapter 5 of the thesis, we first show that the result (3) of D. Wright can be generalized to any field, not necessarily algebraically closed, in the following form (see Theorem 5.2.5):

Theorem E. *Let k be a field of characteristic $p \geq 0$ and let $g \in k[X, Y, Z]$ be of the form $bZ^n - a$ where $a, b \in k[X, Y]$ with $b \neq 0$ and n is an integer ≥ 2 not divisible by p. Suppose that $B := k[X, Y, Z]/(g) = k^{[2]}$ and identify $k[X, Y]$ with its image in B. Then there exist variables U, V in B such that V is the image of Z in B, $U \in k[X, Y]$, $b \in k[U]$, $k[X, Y] = k[U, a]$ and $k[X, Y, Z] = k[U, g, Z]$.*

We will then discuss how far the result of David Wright can be generalized to the case of DVR and more general rings so that we can get some answers to Problem $1'$ for $m = 3$ when $g = b(X, Y)Z^n - a(X, Y)$, $n \geq 2$.

The study of Epimorphism problem (Problem $1'$) for $m = 3$ over a DVR containing $\mathbb{Q}$ has an additional importance in that it is closely related to the study of $\mathbb{A}^2$-fibration over a regular local ring of dimension 2. We recall below the connection.

Let R be a ring and A an R-algebra. If $A = R^{[2]}$, it is obvious that A is an $\mathbb{A}^2$-fibration over R. Now, what about the converse? If A is an $\mathbb{A}^2$-fibration over R, is then $A = R^{[2]}$? Till now this is an open problem when R is a regular local ring containing $\mathbb{Q}$. However, some partial results have been obtained in this direction. In ([Sat83]), Sathaye showed that an $\mathbb{A}^2$-fibration over a DVR containing $\mathbb{Q}$ is $\mathbb{A}^2$. It can be seen by a result of Bass-Connell-Wright ([BCW77]) that over a PID containing $\mathbb{Q}$, an $\mathbb{A}^2$-fibration is $\mathbb{A}^2$. An immediate question occurring after this result is the following:

Problem 2. *Let R be a regular local ring of dimension two containing $\mathbb{Q}$. Suppose A is an $\mathbb{A}^2$-fibration over R. Is then $A = R^{[2]}$?*

Though Problem 2 is open till now, Bhatwadekar-Dutta showed in ([BD94b], section 4) that this problem is closely related to the following Epimorphism problem (a special case of Problem $1'$) in the sense that a counter example to this Epimorphism problem (Problem 3) will give rise to a counter example to Problem 2:

Problem 3. *Let (R, t) be a DVR containing $\mathbb{Q}$ and let $g \in R[X, Y, Z] (= R^{[3]})$ be such that $R[X, Y, Z]/(g) = R^{[2]}$. Is then $R[X, Y, Z] = R[g]^{[2]}$?*

Hence, to explore Problem 2, it is relevant to explore Problem 3 at least for polynomials like $g = b(X, Y)Z^n - a(X, Y)$ for which the corresponding Problem 1 (with $m = 3$) has already been settled.

The first investigation in this direction was made by Bhatwadekar-Dutta in [BD94a]. They showed ([BD94a], Theorem 3.5) that Problem 3 has an affirmative answer (in any characteristic) when $g = b(X,Y)Z - a(X,Y)$ with $t \nmid b(X,Y)$, thereby partially generalizing A. Sathaye's theorem on linear planes over a field ([Sat76]).

In Chapter 5 we will show that Problem 3 has an affirmative answer for polynomials of the form $g = b(X,Y)Z^n - a(X,Y)$, where $n \geq 2$ is an integer not divisible by the characteristic of R/tR, thereby obtaining a generalization of D. Wright's theorem ([Wri78], Theorem). More precisely, we will prove the following (see Theorem 5.3.3):

Theorem F. Let (R, t) be a DVR with residue field k. Let $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 such that n is not divisible by the characteristic of R/tR. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g,Z]^{[1]}$, $R[X,Y] = R[a]^{[1]}$ and $b \in R[X_0]$ where $K[X,Y] = K[X_0, a]$.

The proof of Bhatwadekar-Dutta's theorem on linear planes over a DVR is highly technical. However, in the case of planes of the form $bZ^n - a$ with $n \geq 2$, the proof turns out to be much simpler due to the fact that g is a variable *along with* Z.

Using theorems on residual variables of Bhatwadekar-Dutta ([BD93]), we shall show that Theorem F can be further generalized over (i) any integral domain containing $\mathbb{Q}$ and (ii) any Noetherian UFD containing a field of characteristic $p \geq 0$ where $p \nmid n$. We shall prove (see Theorem 5.4.1 and Theprem 5.4.2):

Theorem G. *Let R be an integral domain containing $\mathbb{Q}$. Let $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ and n is an integer ≥ 2. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.*

Theorem H. *Let R be a Noetherian UFD containing a field of characteristic $p \geq 0$ and $g \in R[X,Y,Z](= R^{[3]})$ be of the form $bZ^n - a$ where $a, b \in R[X,Y]$, $b \neq 0$ and n is an integer ≥ 2 such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.*

The results obtained in Chapter 3 and Chapter 5 were obtained in two joint works with my supervisor Dr. Amartya K. Dutta ([DDa], [DDb]); and the results of Chapter 4 was obtained in my independent work [Das].

Chapter 2

Preliminaries

Throughout the thesis R will denote a commutative ring with unity. The notation $A = R^{[n]}$ will mean that A is isomorphic, as an R-algebra, to a polynomial ring in n variables over R.

Definitions

1. An R-algebra A is said to be an A^r-*fibration* over R if

   (i) A is finitely generated over R.

   (ii) A is flat over R.

   (iii) $A \otimes_R k(P) = k(P)^{[r]}$ for all prime ideals P of R.

2. Let k be a field, $\bar{k}$ denote the algebraic closure of k and R be a k-algebra. An R-algebra A is said to be an $\mathbb{A}^r$-*form* over R (with respect to k) if $A \otimes_k \bar{k} = (R \otimes_k \bar{k})^{[r]}$.

3. Let k be a field and $\bar{k}$ denote the algebraic closure of k. A k-algebra R is said to be *geometrically integral over* k if $R \otimes_k \bar{k}$ is an integral domain.

4. Let k be a field. A k-algebra A is said to be *geometrically normal* if $A \otimes_k \bar{k}$ is a normal domain.

5. A reduced ring R is said to be *seminormal* if it satisfies the condition : for $a, b \in R$ with $a^2 = b^3$, there exists $t \in R$ such that $t^3 = a$ and $t^2 = b$.

6. Let A be a ring and R be a subring of A. An R-algebra homomorphism $\alpha : A \twoheadrightarrow R$ is called a *retraction* from A to R and R is called a *retract* of A.

7. Let k be a field. A k-algebra A is said to have a k-rational point if there is a retraction from A to k.

Results

We state some results which have been used subsequently. The first result occurs in ([BD95], Lemma 3.4).

Lemma 2.0.10. *Let R be a Noetherian ring and R_1 a ring containing R which is finitely generated as an R-module. If A is a flat R-algebra such that $A \otimes_R R_1$ is a finitely generated R_1-algebra, then A is a finitely generated R-algebra.*

The following result follows from ([BD95], Lemma 3.3 and Corollary 3.5).

Lemma 2.0.11. *Let R be a Noetherian ring and A a flat R-algebra such that, for every minimal prime ideal P of R, PA is a prime ideal of A, $PA \cap R = P$ and A/PA is finitely generated over R/P. Then A is finitely generated over R.*

We now quote a theorem on finite generation due to N. Onoda ([Ono84], Theorem 2.20).

Theorem 2.0.12. *Let R be a Noetherian domain and let A be an integral domain containing R such that*

(1) *There exists a non zero element $t \in A$ for which $A[1/t]$ is a finitely generated R-algebra.*

(2) *$A_{\mathfrak{m}}$ is a finitely generated $R_{\mathfrak{m}}$-algebra for each maximal ideal $\mathfrak{m}$ of R.*

Then A is a finitely generated R-algebra.

The results on $\mathbb{A}^1$-fibrations in ([BD95], [Dut95], [DO07]) crucially involve certain patching techniques. We state below one such "patching lemma" ([DO07], Corollary 3.2).

Lemma 2.0.13. *Let $R \subset A$ be integral domains with A being faithfully flat over R. Suppose that there exists a non-zero element $t \in R$ such that*

(1) $A[1/t] = R[1/t]^{[1]}$.

(2) $S^{-1}A = (S^{-1}R)^{[1]}$, *where* $S = \{r \in R |\ r$ *is not a zero-divisor in* $R/tR\}$.

Then there exists an invertible ideal I in R such that $A \cong Sym_R(I)$.

Now, we state the result of D. Wright ([Wri78], Pg. 95) which we will generalize in Chapter 5.

Theorem 2.0.14. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let $g \in k[X,Y,Z](= k^{[3]})$ be of the form $bZ^n - a$ where $a, b \in k[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 not divisible by p. Suppose that $k[X,Y,Z]/(g) = k^{[2]}$. Then there exist variables $\widetilde{X}, \widetilde{Y}$ in $k[X,Y]$ such that $a = \widetilde{Y}$ and $b \in k[\widetilde{X}]$ and $k[X,Y,Z] = k[\widetilde{X}, g, Z]$.*

We also mention some relevant result on $Aut_k(k^{[2]})$ over a field k (see [Wri78], Appendix, Theorems 2 and 3).

Theorem 2.0.15. *Let k be a field and $A = k[U,V](= k^{[2]})$. Let $GA_2(k)$ denote the group of k-automorphisms of A, $Af_2(k)$ the subgroup of $GA_2(k)$ defined by $Af_2(k) = \{(U,V) \mapsto (\alpha_1 U + \beta_1 V + \gamma_1, \alpha_2 U + \beta_2 V + \gamma_2) |\ \alpha_i, \beta_i, \gamma_i \in k$ and $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0\}$, $\mathcal{E}_2(k)$ the subgroup of $GA_2(k)$ defined by $\mathcal{E}_2(k) = \{(U,V) \mapsto (\alpha U + h(V), \beta V + \gamma) |\ \alpha, \beta \in k^*, \gamma \in k$ and $h(V) \in k[V]\}$ and $Bf_2(k) = Af_2(k) \cap \mathcal{E}_2(k)$. Then $GA_2(k) = Af_2(k) *_{Bf_2(k)} \mathcal{E}_2(k)$. Moreover, if $\sigma \in GA_2(k)$ is of finite order, then there exists $\tau \in GA_2(k)$ such that either $\tau\sigma\tau^{-1} \in Af_2(k)$ or $\tau\sigma\tau^{-1} \in \mathcal{E}_2(k)$.*

The next result is due to A. Sathaye ([Sat76], Corollary 1). We will use it to prove Lemma 5.2.2.

Theorem 2.0.16. *Let $L|_k$ be a separable field extension. Assume that there exist $h \in k[X,Y]$ and elements $u_i \in L[X,Y]$ for $1 \leq i \leq s$ such that*

1 $L[X,Y]/(u_i) = L^{[1]}$ for each i.

2 $(u_i, u_j)L[X,Y] = L[X,Y]$ for $i \neq j$.

3 $h = \prod_{i=1}^{s} {u_i}^{r_i}$, $r_i > 0$.

Then there exist $u \in k[X,Y]$, $\lambda_i \in L^*$ and $\mu_i \in L$ such that $u_i = \lambda_i u + \mu_i$ for $1 \leq i \leq s$.

We will also use the following special case of the result ([Dut00], Theorem 7).

Theorem 2.0.17. *Let k be a field, L a separable field extension of k, A a factorial k-domain and B an A-algebra such that $B \otimes_k L = (A \otimes_k L)^{[1]}$. Then $B = A^{[1]}$.*

The following version of Abhyankar-Eakin-Heinzer's cancellation theorem ([AEH72], Theorem 3.3) will be used in the proofs.

Theorem 2.0.18. *Let A be an affine domain over a field k such that k is algebraically closed in A and $tr.deg_k(A) = 1$. Suppose that B is a k-algebra such that $A^{[n]} = B^{[n]}$ for some $n \geq 1$. Then either $B = A$ or $B \cong A = k^{[1]}$.*

We now state a version of the Russell-Sathaye criterion ([RS79], Theorem 2.3.1) for a ring to be a polynomial algebra over a subring (see [BD94a], Theorem 2.6).

Theorem 2.0.19. *Let $R \subset A$ be integral domains with A being finitely generated over R. Suppose that there exist primes $p_1, p_2, \ldots, p_n$ in R such that for each i, $1 \leq i \leq n$,*

(1) p_i remains prime in A,

(2) $p_i A \cap R = p_i R$,

(3) $A[\frac{1}{p_1 p_2 \cdots p_n}] = R[\frac{1}{p_1 p_2 \cdots p_n}]^{[1]}$ and

(4) $R/p_i R$ is algebraically closed in $A/p_i A$.

Then $A = R^{[1]}$.

The following result from ([BD94a], Lemma 2.5) will enable us to apply Theorem 2.0.19.

Lemma 2.0.20. *Let R be an integral domain and $F \in R[X,Y](= R^{[2]})$ be such that $R[X,Y]/(F) = R^{[1]}$. Then $R[F]$ is algebraically closed in $R[X,Y]$.*

We now quote a result of E. Hamann ([Ham75], Theorem 2.6).

Theorem 2.0.21. *Let R be a Noetherian ring such that R_{red} is seminormal. Then $R^{[1]}$ is R-invariant, i.e., if A is an R-algebra such that $A^{[m]} = R^{[m+1]}$ as R-algebras, then $A = R^{[1]}$.*

Finally, we state a result on residual variables which will be our main tool to prove Theorem G and Theorem H. It comes as a direct consequence of Theorem 3.1, Theorem 3.2 and Remark 3.4 in [BD93].

Theorem 2.0.22. *Let R be a Noetherian domain such that either R contains $\mathbb{Q}$ or R is seminormal, A be a polynomial algebra in n variables over R and $W_1, W_2, \ldots, W_{n-1} \in A$. Then the following are equivalent:*

1. $A = R[W_1, W_2, \ldots, W_{n-1}]^{[1]}$.

2. $A \otimes_R k(P) = (R[W_1, W_2, \ldots, W_{n-1}] \otimes_R k(P))^{[1]}$ *for every prime ideal P of R.*

Chapter 3

Codimension-one $\mathbb{A}^1$-fibration with retraction

3.1 Preview

The following result on $\mathbb{A}^1$-fibrations was proved in ([Dut95], Theorem 3.4, Theorem 3.5):

Theorem 3.1.1. *Let R be a Noetherian domain with field of fractions K and A a faithfully flat finitely generated R-algebra such that $A \otimes_R K = K^{[1]}$ and $A \otimes_R k(P)$ is geometrically integral over $k(P)$ for each height one prime ideal P of R. Under these hypotheses, we have the following results:*

(i) *If R is normal, then $A \cong Sym_R(I)$ for an invertible ideal I of R.*

(ii) *If R contains $\mathbb{Q}$, then A is an $\mathbb{A}^1$-fibration over R.*

A striking feature of this result is that conditions on merely the generic and codimension-one fibres imply that all fibres are $\mathbb{A}^1$. Analogous results were proved for subalgebras of polynomial algebras ([BD95], 3.10, 3.12) without the hypothesis "A is finitely generated over R". In this chapter we investigate whether the condition "A is finitely generated" in Theorem 3.1.1 can be replaced by a weaker hypothesis like "A is Noetherian" when the R algebra A is known to have a retraction to R. Recently, in [BDO], Bhatwadekar-Dutta-Onoda have shown the following:

Theorem 3.1.2. *Let R be a Noetherian normal domain with field of fractions K and A a Noetherian flat R-algebra such that $A_P = {R_P}^{[1]}$ for each prime* *ideal P of R of height one. Suppose that there exists a retraction $\Phi : A \twoheadrightarrow R$. Then $A \cong Sym_R(I)$ for an invertible ideal I in R.*

The above theorem occurs in [BDO] as a consequence of a general structure theorem for any faithfully flat algebra over a Noetherian normal domain R which is locally $\mathbb{A}^1$ in codimension-one. The statements and proofs in [BDO] are quite technical. In this chapter, we will first prove (see Theorem 3.3.5) an analogue of Theorem 3.1.1 (i). Our approach, which is more in the spirit of the proof in ([Dut95], 3.4), will provide a short and direct proof of Theorem 3.1.2. Next we will prove an analogous version of Theorem 3.1.1 (ii) (see Theorem 3.3.9).

3.2 A version of Russell-Sathaye criterion for an algebra to be a polynomial algebra

In this section we present a version of Russell-Sathaye criterion ([RS79], Theorem 2.3.1) for an algebra to be a polynomial algebra. Our version is an extension of the version given by Dutta-Onoda ([DO07], Theorem 2.4) and suitable for algebras which are known to have retractions to the base ring. For convenience, we first record a few preliminary results. The first result is easy.

Lemma 3.2.1. *Let $B \subset A$ be integral domains. Suppose that there exists a non-zero element p in B such that $B[1/p] = A[1/p]$ and $pA \cap B = pB$. Then $B = A$.*

Lemma 3.2.2. *Let C be a D-algebra such that D is a retract of C. Then the following hold:*

(I) *$pC \cap D = pD$ for all $p \in D$.*

(II) *If $D \subset C$ are domains, then D is algebraically closed in C.*

Proof. Proof of (I): Let $p \in D$. Note that $pC \cap D = pD$ is equivalent to say that the map $D/pD \longrightarrow C/pC$ is injective. Now since D is a retract of C, the composite map $D/pD \longrightarrow C/pC \twoheadrightarrow D/pD$ is identity and hence the map $D/pD \longrightarrow C/pC$ is injective.

Proof of (II): Let $\phi : C \twoheadrightarrow D$ be the retraction and let $t \in C\backslash\{0\}$ be algebraic over D. Then there exits a polynomial $f(X) \in D[X]$ (unique upto a constant multiple) of least degree such that $f(t) = 0$. Note that $\phi(t) \neq 0$. Since $\phi(f(t)) = f(\phi(t)) = 0$ and since $\phi(t) \in D$, we must have $f(X) = (X - \phi(t))g(X)$ where $deg(g(X)) < deg(f(X))$. Now, since $f(X)$ is a polynomial of least degree such that $f(t) = 0$, we get $g(t) \neq 0$ and hence from the relation $f(t) = (t - \phi(t))g(t) = 0$ we have $t = \phi(t)$, i.e., $t \in D$. Thus D is algebraically closed in C. $\square$

Lemma 3.2.3. *Let R be a ring and A be an R-algebra with a generating set $S = \{x_i : i \in \Lambda\}$ where Λ is some indexing set. Suppose that there is a retraction $\Phi : A \twoheadrightarrow R$. Then $Ker\ \Phi = (\{x_i - r_i : i \in \Lambda\})A$ where $r_i = \Phi(x_i)$ for each $i \in \Lambda$.*

Proof. Let $\widetilde{S} = \{x_i - r_i : i \in \Lambda\}$ and I be the ideal of A generated by $\widetilde{S}$. Note that $R[S] = R[\widetilde{S}]$. It is easy to see that $A = R \oplus Ker\ \Phi = R \oplus I$. Since $I \subseteq Ker\ \Phi$, it follows that $Ker\ \Phi = I$. $\square$

Lemma 3.2.4. *Let $R \subset A$ be integral domains and $\Phi : A \twoheadrightarrow R$ be a retraction with finitely generated kernel. Suppose that there exists an element p which is a non-zero non-unit in R such that $A[1/p] = R[1/p]^{[1]}$. Then there exists $x \in Ker\ \Phi$ such that $x \notin pA$ and $A[1/p] = R[1/p][x]$.*

Proof. Suppose, if possible, that $x \in pA$ for every $x \in Ker\ \Phi$ for which $A[1/p] = R[1/p][x]$.

Let $Ker\ \Phi = (a_1, a_2, \ldots, a_m)A$. Choose $x_0 \in Ker\ \Phi$ such that $A[1/p] = R[1/p][x_0]$. Note that Φ extends to a retraction $\Phi_p : A[1/p] \twoheadrightarrow R[1/p]$ with kernel $x_0(A[1/p])$. By our assumption, $x_0 = px_1$ for some $x_1 \in A$. Obviously, $x_1 \in Ker\ \Phi$ and $A[1/p] = R[1/p][x_1]$ and hence $x_1 \in pA$. Arguing in a similar manner, we get $x_2 \in Ker\ \Phi$ such that $x_1 = px_2$, $A[1/p] = R[1/p][x_2]$ and $x_2 \in pA$. Continuing this process we get a sequence $\{x_n\}_{n \geq 0}$ such that $x_n \in Ker\ \Phi$, $A[1/p] = R[1/p][x_n]$ and $x_n = px_{n+1}$. Thus $x_0 = p^n x_n$ for all $n \geq 1$.

Note that $(x_0, x_1, \ldots, x_n, \ldots)A \subseteq (a_1, a_2, \ldots, a_m)A$. But since $a_i \in A \subset A[1/p] = R[1/p][x_0]$, there exist $n_i \in \mathbb{N}$ and $\alpha_{ij} \in R[1/p]$ such that $a_i = \sum\limits_{j=0}^{n_i} \alpha_{ij} {x_0}^j$. Choose $N \in \mathbb{N}$ such that $\alpha_{ij} p^{jN} \in R$ for all i, j and set $\lambda_{ij} := \alpha_{ij} p^{jN}$. Now since $x_0, a_i \in Ker\ \Phi_p$, we have $\alpha_{i0} = 0$ for all i and hence $a_i = \sum\limits_{j=1}^{n_i} \alpha_{ij} {x_0}^j$. Thus $a_i = \sum\limits_{j=1}^{n_i} \lambda_{ij} {x_N}^j \in x_N R[x_N] \subseteq x_N A$ for all i, $1 \leq i \leq m$. So, we have $Ker\ \Phi = (a_1, a_2, \ldots, a_m)A = x_N A$. Now $x_{N+1} \in Ker\ \Phi = x_N A$, which implies that $x_{N+1} = \alpha x_N$ for some $\alpha \in A$. Since $x_N = px_{N+1}$, it follows that $\alpha p = 1$, which is a contradiction to the fact that p is not a unit in A. Thus there exists $x \in Ker\ \Phi$ such that $x \notin pA$ and $A[1/p] = R[1/p][x]$. $\square$

Now we present a version of Russell-Sathaye criterion when there exists a retraction.

Proposition 3.2.5. *Let $R \subset A$ be integral domains such that there exists a retraction $\Phi : A \twoheadrightarrow R$. Suppose that there exists a prime p in R such that*

(1) *p is a prime in A.*

(2) *$A[1/p] = R[1/p]^{[1]}$.*

Then $pA \cap R = pR$, R/pR is algebraically closed in A/pA and there exists an increasing chain $A_0 \subseteq A_1 \subseteq A_2 \ldots \subseteq A_n \subseteq \ldots$ of subrings of A and a sequence of elements $\{x_n\}_{n \geq 0}$ in $Ker\ \Phi$ with $x_0A \subseteq x_1A \subseteq \cdots \subseteq x_nA \subseteq \ldots$ such that

(a) *$A_n = R[x_n] = R^{[1]}$ for all $n \geq 0$.*

(b) *$A[1/p] = A_n[1/p]$ for all $n \geq 0$.*

(c) *$pA \cap A_n \subseteq pA_{n+1}$ for all $n \geq 0$.*

(d) *$A = \bigcup_{n \geq 0} A_n = R[x_1, x_2, \ldots, x_n, \ldots]$.*

(e) *$Ker\ \Phi = (x_0, x_1, x_2, \ldots, x_n, \ldots)A$.*

Moreover the following are equivalent:

(i) *$Ker\ \Phi$ is finitely generated.*

(ii) *$Ker\ \Phi = x_NA$ for some $N \geq 0$.*

(iii) *A is finitely generated over R.*

(iv) *$A = R[x_N]$ for some $N \geq 0$.*

(v) *There exists $x \in Ker\ \Phi \backslash pA$ such that $A = R[x] = R^{[1]}$.*

The conditions (i)–(v) will be satisfied if $\bigcap_{n \geq 0} p^nA = (0)$.

Proof. $pA \cap R = pR$ by Lemma 3.2.2. Since Φ induces a retraction $\Phi_p : A/pA \twoheadrightarrow R/pR$, R/pR is algebraically closed in A/pA by Lemma 3.2.2.

By condition (2), there exists ${x'}_0 \in A$ such that $A[1/p] = R[1/p][{x'}_0]$. Let $x_0 = {x'}_0 - \Phi({x'}_0)$. Then $x_0 \in Ker\ \Phi$ and $A[1/p] = R[1/p][x_0] = R[1/p]^{[1]}$. Set $A_0 := R[x_0](= R^{[1]})$. Then $A_0 \subseteq A$ and $A[1/p] = A_0[1/p] = R[1/p][x_0]$.

Now suppose that we have obtained elements $x_0, x_1, \dots, x_n \in Ker\ \Phi$ such that setting $A_m := R[x_m](= R^{[1]})$ for all m, $0 \le m \le n$, we have $A_0 \subseteq A_1 \subseteq A_2 \cdots \subseteq A_n \subseteq A$ and $A_m[1/p] = A[1/p]$; $0 \le m \le n$.

We now describe our choice of x_{n+1}:

Let $\overline{x_n}$ denote the image of x_n in A/pA. We consider separately the two possibilities:

(I) $\overline{x_n}$ is transcendental over R/pR.

(II) $\overline{x_n}$ is algebraic over R/pR.

Case I : $\overline{x_n}$ is transcendental over R/pR. In this case the map $A_n/pA_n(= R[x_n]/pR[x_n]) \longrightarrow A/pA$ is injective, i.e., $pA_n = pA \cap A_n$. Since $A_n[1/p] = A[1/p]$, we get $A_n = A$ by Lemma 3.2.1. Now we set $x_{n+1} := x_n$ and $A_{n+1} := R[x_{n+1}](= A_n = A)$.

Case II: $\overline{x_n}$ is algebraic over R/pR. Since R/pR is algebraically closed in A/pA, we see that $\overline{x_n} \in R/pR$. Thus $x_n = pu_n + c_n$ for some $u_n \in A$ and $c_n \in R$. Applying Φ, we get $0 = \Phi(x_n) = p\Phi(u_n) + c_n$ showing that $c_n \in pR$ and hence $x_n \in pA$. Set $x_{n+1} := x_n/p(\in A)$. Clearly $x_{n+1} \in Ker\ \Phi$. Now setting $A_{n+1} := R[x_{n+1}](= R^{[1]})$, we see that $A_0 \subseteq A_1 \subseteq A_2 \cdots \subseteq A_n \subseteq A_{n+1} \subseteq A$ and $A_{n+1}[1/p] = A_n[1/p] = A[1/p]$.

Thus we set $x_{n+1} := x_n$ or $x_{n+1} := x_n/p$ depending on whether the image of x_n in A/pA is transcendental or algebraic over R/pR. By construction, conditions (a) and (b) hold. We now verify (c).

If $x_n = x_{n+1}$, i.e., $A_{n+1} = A_n = A$, then $pA \cap A_n = pA = pA_{n+1}$. Now consider the case $x_n = px_{n+1} \in pA_{n+1}$. Let $a \in pA \cap A_n$. Then $a = r_0 + r_1(px_{n+1}) + \cdots + r_l(px_{n+1})^l$ for some $l \ge 0$ and $r_0, r_1, \dots, r_l \in R$. Then $r_0 \in pA \cap R = pR \subset pA_{n+1}$. Therefore, $a \in pA_{n+1}$. Thus $pA \cap A_n \subseteq A_{n+1}$.

We now prove (d). Let $B = \underset{n \ge 0}{\cup} A_n$. Obviously, $B \subseteq A$ and $B[1/p] = A[1/p]$. Hence, by Lemma 3.2.1, it is enough to show that $pA \cap B = pB$.

Clearly, $pB \subseteq pA \cap B$. Now let $y \in pA \cap B$. Then there exists $i \geq 0$ such that $y \in pA \cap A_i \subseteq pA_{i+1} \subseteq pB$. Thus $pA \cap B = pB$.

(e) follows from Lemma 3.2.3.

(i) $\implies$ (v) follows from Lemma 3.2.4. Our construction shows that (iii) $\implies$ (iv). The implications (v) $\implies$ (iii) and (iv) $\implies$ (ii) $\implies$ (i) are easy.

Note that our construction shows that the sequence $\{x_n\}_{n\geq 0}$ is eventually a constant sequence (i.e., there exists $N \geq 0$ such that $x_{N+r} = x_N$ for all $r \geq 0$) if and only if there exists $N \geq 0$ such that the image of x_N in A/pA is transcendental over R/pR. It is easy to see that each of the conditions (i)-(v) is equivalent to the above condition.

If the image of x_m in A/pA is algebraic over R/pR for $1 \leq m \leq n$, then $x_n = p^n x_0 \in p^n A$. Therefore, if $\bigcap\limits_{n\geq 0} p^n A = (0)$, then the sequence $\{x_n\}_{n\geq 0}$ must be eventually constant and hence the conditions (i)–(v) hold. □

Proposition 3.2.5 shows that we can extend the Dutta-Onoda version ([DO07], 2.4) of Russell-Sathaye criterion for A to be $R^{[1]}$ as follows:

Corollary 3.2.6. *Let $R \subset A$ be integral domains. Suppose that there exists a prime p in R such that*

(1) *p is a prime in A.*

(2) *$pA \cap R = pR$.*

(3) *$A[1/p] = R[1/p]^{[1]}$.*

(4) *R/pR is algebraically closed in A/pA.*

Then the following are equivalent:

(i) *A is finitely generated over R.*

(ii) *A has a retraction to R with finitely generated kernel.*

(iii) *$tr.deg_{R/pR}(A/pA) > 0$.*

(iv) *$A = R^{[1]}$.*

Proof. Follows from ([DO07], 2.4) and Proposition 3.2.5. □

By repeated application of Proposition 3.2.5 we get the following:

Corollary 3.2.7. *Let $R \subset A$ be integral domains with a retraction $\Phi : A \twoheadrightarrow R$. Suppose that there exist primes $p_1, p_2, \ldots, p_n$ in R such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$p_1, p_2, \ldots, p_n$ are primes in A.*

(3) *$A[\frac{1}{p_1 p_2 \cdots p_n}] = R[\frac{1}{p_1 p_2 \cdots p_n}]^{[1]}$.*

Then there exists $x \in Ker\ \Phi$ such that $A = R[x] = R^{[1]}$.

3.3 Codimension-one $\mathbb{A}^1$-fibration with retraction

In this section we will prove our main theorems (Theorems 3.3.5 and 3.3.9) and auxiliary results (Propositions 3.3.4 and 3.3.7).

We first state a few preliminary results. The next result is easy to prove.

Lemma 3.3.1. *Let R be a ring and A an R-algebra. If R' is a faithfully flat algebra over R such that $A \otimes_R R'$ is finitely generated over R', then A is finitely generated over R.*

We now observe a property of algebras with retractions.

Lemma 3.3.2. *Let R be an integral domain with field of fractions K and A be an integral domain containing R with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R K = K^{[1]}$.*

Then there exists $t \in R$ and $F \in Ker\ \Phi$ such that $A[1/t] = R[1/t][F]$.

Proof. Let $S = R \backslash \{0\}$. By (2), $S^{-1}A = K^{[1]}$. Since A has a retraction Φ, it is easy to see that there exists $F \in Ker\ \Phi$ such that $S^{-1}A = K[F] (= K^{[1]})$ and hence $F(S^{-1}A) = (Ker\ \Phi)S^{-1}A$. Therefore, by (1), there exists $t \in S$ such that $FA[1/t] = (Ker\ \Phi)A[1/t]$. Thus $FA[1/t]$ is the kernel of the induced retraction $\Phi_t : A[1/t] \twoheadrightarrow R[1/t]$. Hence we have

$$\begin{aligned} A[1/t] &= R[1/t] \oplus FA[1/t] \\ &= R[1/t] \oplus FR[1/t] \oplus F^2A[1/t] \\ &\cdots \\ &= R[1/t] \oplus FR[1/t] \oplus F^2R[1/t] \oplus \cdots \oplus F^nR[1/t] \oplus F^{n+1}A[1/t] \quad \forall n \in \mathbb{N}. \end{aligned}$$

As $S^{-1}A = \underset{n\geq 0}{\oplus} KF^n$, it follows that $A[1/t] = R[1/t][F]$. $\square$

Remark 3.3.3. *In Lemma 3.3.2 if we assume that $Ker\ \Phi$ is principal, say, $Ker\ \Phi = (G)$, then $A = R[G]$.*

We now deduce a local-global result. Our approach gives a simpler proof of Theorem 3.1.2 which is obtained in [BDO] as a consequence of a highly technical structure theorem.

Proposition 3.3.4. *Let R be either a Noetherian domain or a Krull domain with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A_P = {R_P}^{[1]}$ for every prime ideal P of R satisfying depth $(R_P) = 1$.*

Then there exists an invertible ideal I of R such that $A \cong Sym_R(I)$.

Proof. The case $dim\ R = 0$ is trivial. So we assume that $dim\ R \geq 1$. Note that A is a faithfully flat R-algebra and an integral domain. By Lemma 3.3.2, $A[1/t] = R[1/t][F]$. If t is a unit in R, then $A = R^{[1]}$ and we would be through. So we assume that t is a non-unit in R.

Let $P_1, P_2, ..., P_s$ be the associated prime ideals of tR. Let $S = R\backslash(\bigcup_{i=1}^{s} P_i) = \{r \in R |\ r$ is not a zero-divisor in $R/tR\}$. By (2), for each maximal ideal $\mathfrak{m}$ of $S^{-1}R$, $(S^{-1}A)_{\mathfrak{m}} = {(S^{-1}R)_{\mathfrak{m}}}^{[1]}$ and hence $S^{-1}A = (S^{-1}R)^{[1]}$, $S^{-1}R$ being a semilocal domain. Hence, by Lemma 2.0.13, $A \cong Sym_R(I)$ for some invertible ideal I of R. $\square$

We now prove Theorem A for the case R is a Krull domain.

Theorem 3.3.5. *Let R be a Krull domain with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R K = K^{[1]}$.*

(3) *$A \otimes_R k(P)$ is an integral domain for each height one prime ideal P of R.*

Then there exists an invertible ideal I of R such that $A \cong Sym_R(I)$.

Proof. Let P be a prime ideal in R for which *depth* $(R_P)(= ht(P)) = 1$. Then R_P is a DVR. Let π_P be the uniformising parameter of R_P. Note that the retraction $\Phi : A \twoheadrightarrow R$ induces a retraction $\Phi_P : A_P \twoheadrightarrow R_P$ with finitely generated kernel, condition (2) ensures that $A_P[1/\pi_P] = R_P[1/\pi_P]^{[1]} = K^{[1]}$, and condition (3) ensures that π_P is a prime in A_P. Hence, by Corollary 3.2.7, $A_P = {R_P}^{[1]}$. Therefore, by Proposition 3.3.4, $A \cong Sym_R(I)$ for some invertible ideal I of R. □

As an immediate consequence we get the following variant of a Lüroth-type result over UFD (see [RS79], 3.4):

Corollary 3.3.6. *Let R be a UFD with field of fractions K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R K = K^{[1]}$.*

(3) *$A \otimes_R k(P)$ is an integral domain for each height one prime ideal P of R.*

Then there exists $x \in Ker\ \Phi$ such that $A = R[x] = R^{[1]}$.

Now we prove Theorem B:

Theorem 3.3.7. *Let R be a Noetherian ring and A be a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R k(P) = k(P)^{[1]}$ for each minimal prime ideal P of R.*

(3) *$A \otimes_R k(P)$ is geometrically integral over $k(P)$ for each height one prime ideal P of R.*

Then:

(I) *$A \otimes_R k(P)$ is an $\mathbb{A}^1$-form over $k(P)$ for each prime ideal P of R.*

(II) *A is finitely generated over R.*

(III) *If R is an integral domain, then there exists a finite birational extension R' of R and an invertible ideal I of R' such that $A \otimes_R R' \cong Sym_{R'}(I)$.*

Proof. (I): Note that for any prime ideal P of R, $A \otimes_R k(P) = A_P \otimes_{R_P} k(P)$. So, to prove the fibre condition (I), we replace R by R_P (and A by A_P) and assume that R is a local ring with maximal ideal $\mathfrak{m}$. We prove that $A \otimes_R k(\mathfrak{m})$ is an $\mathbb{A}^1$-form over $k(\mathfrak{m})$ by induction on height $\mathfrak{m}$, i.e., dim R.

Case : $\underline{\dim R = 0}$.
Trivial.

Case : $\underline{\dim R = 1}$.
Replacing R by R/P_0 for some minimal prime ideal P_0, we may assume that R is a Noetherian one-dimensional local integral domain with field of fractions K. Note that condition (3) implies that $A \otimes_R k(\mathfrak{m})$ is geometrically integral over $k(\mathfrak{m})$.

Let $\widetilde{R}$ be the normalisation of R and let $\widetilde{A} = A \otimes_R \widetilde{R}$. Then, by Krull-Akizuki theorem, $\widetilde{R}$ is a Dedekind domain ([Mat89], p 85); and since R is local, $\widetilde{R}$ is semilocal and hence a PID. Let $\widetilde{\mathfrak{m}}_1, \widetilde{\mathfrak{m}}_2, \dots, \widetilde{\mathfrak{m}}_r$ be the maximal ideals of $\widetilde{R}$. Again, by Krull-Akizuki theorem ([Mat89], p 85), $k(\widetilde{\mathfrak{m}}_i)$ is a finite algebraic extension of $k(\mathfrak{m})$. Clearly, the retraction $\Phi : A \twoheadrightarrow R$ gives rise to a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$. From the split exact sequence $0 \longrightarrow Ker\ \Phi \longrightarrow A \longrightarrow R \longrightarrow 0$, it follows that $Ker\ \widetilde{\Phi} = Ker\ \Phi \otimes_R \widetilde{R} = Ker\ \Phi \otimes_A \widetilde{A} = (Ker\ \Phi)\widetilde{A}$ and hence $Ker\ \widetilde{\Phi}$ is finitely generated.

Thus, from (1), (2) and (3), we have:

(i) $Ker\ \widetilde{\Phi}$ is finitely generated.

(ii) $\widetilde{A} \otimes_{\widetilde{R}} K = K^{[1]}$.

(iii) $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{\mathfrak{m}}_i)$ is geometrically integral over $k(\widetilde{\mathfrak{m}}_i)$ for every maximal ideal $\widetilde{\mathfrak{m}}_i$ of $\widetilde{R}$.

Hence, by Corollary 3.3.6, $\widetilde{A} = \widetilde{R}^{[1]}$. In particular, $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{\mathfrak{m}}_i) = k(\widetilde{\mathfrak{m}}_i)^{[1]}$ for each maximal ideal $\widetilde{\mathfrak{m}}_i$ of $\widetilde{R}$. This shows that $A \otimes_R k(\mathfrak{m})$ is an $\mathbb{A}^1$-form over $k(\mathfrak{m})$.

Case : $\underline{\dim R \geq 2}$.
By induction hypothesis we have that $A \otimes_R k(P)$ is an $\mathbb{A}^1$-form for every non-maximal prime ideal P of R. Let $\widehat{R}$ denote the completion of R and let $\widehat{A} = A \otimes_R \widehat{R}$. Then $\widehat{R}$ is a complete local ring with maximal ideal $\widehat{\mathfrak{m}}$ and $\widehat{R}/\widehat{\mathfrak{m}} \cong R/\mathfrak{m}$. Since R is Noetherian, $\widehat{R}$ is Noetherian and faithfully flat over R and hence $\widehat{A}$ is faithfully flat over both A and $\widehat{R}$. The retraction $\Phi : A \twoheadrightarrow R$ gives rise to a retraction $\widehat{\Phi} : \widehat{A} \twoheadrightarrow \widehat{R}$. Note that $Ker\ \widehat{\Phi} = (Ker\ \Phi)\widehat{A}$ is finitely

generated. Now, for any non-maximal prime ideal $\widehat{P}$ of $\widehat{R}$, $\widehat{P} \cap R \neq \mathfrak{m}$ and hence $\widehat{A} \otimes_{\widehat{R}} k(\widehat{P})$ is an $\mathbb{A}^1$-form over $k(\widehat{P})$.

Replacing R by $\widehat{R}$, we may assume R to be a complete local Noetherian ring. Further, replacing R by R/P_0, where P_0 is a minimal prime ideal of R, we may assume R to be a complete, local, Noetherian domain with maximal ideal $\mathfrak{m}$ and field of fractions K such that

(a) $A \otimes_R K = K^{[1]}$.

(b) $A \otimes_R k(P)$ is $\mathbb{A}^1$-form over $k(P)$ for each non-maximal prime ideal P of R.

Let $\widetilde{R}$ be the normalisation of R. Since R is a complete local ring, $\widetilde{R}$ is a finite R-module ([Mat89], p 263) and hence is a Noetherian normal local domain. Let $\widetilde{A} = A \otimes_R \widetilde{R}$. As before, the retraction $\Phi : A \twoheadrightarrow R$ induces a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel $(Ker\ \Phi)\widetilde{A}$. Now we have the following:

$\widetilde{R}$ is a Noetherian normal local domain with field of fractions K and $\widetilde{A}$ is a faithfully flat $\widetilde{R}$-algebra such that

(1′) There is a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel.

(2′) $\widetilde{A} \otimes_{\widetilde{R}} K = A \otimes_R K = K^{[1]}$.

(3′) $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{P})$ is an $\mathbb{A}^1$-form over $k(\widetilde{P})$ for each height one prime ideal $\widetilde{P}$ of $\widetilde{R}$ (since for any height one prime ideal $\widetilde{P}$ of $\widetilde{R}$, $\widetilde{P} \cap R \neq \mathfrak{m}$).

By Theorem 3.3.5, $\widetilde{A} = \widetilde{R}^{[1]}$; in particular, $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{\mathfrak{m}}) = k(\widetilde{\mathfrak{m}})^{[1]}$. This shows that $A \otimes_R k(\mathfrak{m})$ is an $\mathbb{A}^1$-form over $k(\mathfrak{m})$ and hence $A \otimes_R k(P)$ is an $\mathbb{A}^1$-form over $k(P)$ for every prime ideal P of R.

(II): We now show that A is finitely generated over R. By Lemma 2.0.11, it is enough to take R to be an integral domain; by Theorem 2.0.12 and Lemma 3.3.2, it is enough to assume R to be local and, by Lemma 3.3.1, it is enough to take R to be complete. Thus we assume that R is a Noetherian local complete integral domain. Let $\widetilde{R}$ be the normalisation of R. Then the proof of (I) shows that $A \otimes_R \widetilde{R} = \widetilde{R}^{[1]}$; in particular, $A \otimes_R \widetilde{R}$ is finitely generated over $\widetilde{R}$. Since $\widetilde{R}$ is a finite module over R, by Lemma 2.0.10, A is finitely generated over R.

(III): Now R is given to be an integral domain. Recall that, by (I), we have that $A \otimes_R k(P)$ is an $\mathbb{A}^1$-form over $k(P)$ for every prime ideal P of R.

Let $\widetilde{R}$ be the normalisation of R. Then $\widetilde{R}$ is a Krull domain ([Mat89], p 91). Let $\widetilde{A} = A \otimes_R \widetilde{R}$. As before, there is a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel. We now have the following:

$\widetilde{R}$ is a Krull domain with field of fractions K and $\widetilde{A}$ is a faithfully flat $\widetilde{R}$-algebra such that

(1″) There is a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel.

(2″) $\widetilde{A} \otimes_{\widetilde{R}} K = K^{[1]}$.

(3″) $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{P})$ is an $\mathbb{A}^1$-form over $k(\widetilde{P})$ for each prime ideal $\widetilde{P}$ of $\widetilde{R}$ (since $k(\widetilde{P})$ is algebraic over $k(\widetilde{P} \cap R)$).

Using Theorem 3.3.5, we get that $A \otimes_R \widetilde{R} = \widetilde{R}[\widetilde{I}T]$ for some invertible ideal $\widetilde{I}$ of $\widetilde{R}$. Let $\widetilde{I} = (a_1, a_2, \ldots, a_n)\widetilde{R}$ and let $\alpha_1, \ldots, \alpha_n \in \widetilde{I}^{-1}$ be such that $a_1\alpha_1 + \ldots a_n\alpha_n = 1$. Set $b_{ij} := a_i\alpha_j (\in \widetilde{R})$, i, j, $1 \le i, j \le n$. Let $a_pT = \sum_{q=1}^{s_p} u_{pq} \otimes c_{pq}$ where $c_{pq} \in \widetilde{R}$ and $u_{pq} \in A$.

By (II), A is finitely generated; let $A = R[y_1, y_2, \ldots, y_t]$ where each $y_\ell \in Ker\ \Phi$. Then

$$y_\ell \otimes 1 = \sum_{m=0}^{r_\ell} \sum_{m_1+m_2+\cdots+m_n=m} d_{\ell\ m_1m_2\ldots m_n}\ {a_1}^{m_1}{a_2}^{m_2}\ldots{a_n}^{m_n}\ T^m$$

for some $d_{\ell\ m_1m_2\ldots m_n} \in \widetilde{R}$.

Now, let R' be the R-subalgebra of $\widetilde{R}$ generated by the elements $a_1, a_2, \ldots, a_n$; b_{ij} where $1 \le i, j \le n$; c_{pq} where $1 \le q \le s_p$, $1 \le p \le n$; and $d_{\ell\ m_1m_2\ldots m_n}$ where $m_1 + m_2 + \cdots + m_n = m$, $0 \le m \le r_\ell$, $1 \le \ell \le t$. Let I be the ideal $(a_1, a_2, \ldots, a_n)R'$. Then R' is a finite birational extension of R and I is an invertible ideal of R'.

Since A is faithfully flat over R, we have $A \otimes_R R' \subseteq A \otimes_R \widetilde{R} \subseteq A \otimes_R K = K[T]$. Now considering $A \otimes_R R'$ and $R'[IT]$ as subrings of $A \otimes_R K$, it is easy to see that $A \otimes_R R' = R'[IT]$.

This completes the proof. □

Remark 3.3.8. *The above proof shows that in the statement of Theorem 3.3.7 it is enough to assume in (2) that the generic fibres are $\mathbb{A}^1$-forms. (In the proof take $\widetilde{R}$ to be the integral closure of R in L where L is finite extension of K such that $A \otimes_R L = L^{[1]}$.)*

We now prove Theorem C.

Theorem 3.3.9. *Let $\mathbb{Q} \hookrightarrow R$ be a Noetherian ring and A be a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$Ker\ \Phi$ is finitely generated.*

(2) *$A \otimes_R k(P) = k(P)^{[1]}$ at each minimal prime ideal P of R.*

(3) *$A \otimes_R k(P)$ is an integral domain at each height one prime ideal P of R.*

Then:

(I) *A is an $\mathbb{A}^1$-fibration over R.*

(II) *If R is an integral domain, then there exists a finite birational extension R' of R and an invertible ideal I of R' such that $A \otimes_R R' \cong Sym_{R'}(I)$.*

(III) *If R_{red} is seminormal, then $A \cong Sym_R(I)$ for some finitely generated rank one projective R-module I.*

Proof. (I): By Theorem 3.3.7, it is enough to show that $A \otimes_R k(P) = k(P)^{[1]}$ for each prime ideal P in R of height one.

Fix a prime ideal P in R of height one. Replacing R by R_P, we assume that R is a one-dimensional Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Moreover, replacing R by R/P_0 for some minimal prime ideal P_0, we may further assume that R is an integral domain with field of fractions K. We show that $A \otimes_R k = k^{[1]}$.

Note that k is a field of characteristic 0. By Krull-Akizuki theorem, there exists a discrete valuation ring $(\widetilde{R}, \pi, \widetilde{k})$ such that $R \subset \widetilde{R} \subset K$ and $\widetilde{k}$ is a finite separable extension of k. Let $\widetilde{A} = A \otimes_R \widetilde{R}$. Since separable $\mathbb{A}^1$-forms are $\mathbb{A}^1$, to show that $A \otimes_R k = k^{[1]}$, it is enough to show that $\widetilde{A}/\pi\widetilde{A} (= A \otimes_R \widetilde{k}) = \widetilde{k}^{[1]}$ and hence enough to show that $\widetilde{A} = \widetilde{R}^{[1]}$.

Now, the retraction $\Phi : A \twoheadrightarrow R$ with finitely generated kernel induces a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel. Also $\widetilde{A}[1/\pi] = K^{[1]}$. Using Lemma 3.2.4, we get $x \in Ker\ \widetilde{\Phi} \backslash \pi\widetilde{A}$ such that $\widetilde{A}[1/\pi] = K[x]$.

Let $B = \widetilde{R}[x] \subset \widetilde{A}$. We will show that $\widetilde{A} = B$. Since π is a non-zero divisor and since $\widetilde{A}_\pi = B_\pi$, by Lemma 3.2.1, it suffices to show that $\pi\widetilde{A} \cap B = \pi B$.

Let $D = A \otimes_R k$. Then $\widetilde{A}/\pi\widetilde{A} = \widetilde{A} \otimes_{\widetilde{R}} \widetilde{k} = (A \otimes_R k) \otimes_k \widetilde{k} = D \otimes_k \widetilde{k}$. By hypothesis, D is an integral domain and hence, as $\widetilde{k}|_k$ is separable, $\widetilde{A}/\pi\widetilde{A} =$

$D \otimes_k \widetilde{k}$ is a reduced ring. Note that $\widetilde{A}/\pi\widetilde{A}$ is a finite flat module over D and hence $\widetilde{A}$ has only finitely many minimal prime ideals $P_1, P_2, \ldots, P_n$ containing $\pi\widetilde{A}$. To show that $\pi\widetilde{A} \cap B = \pi B$, it is enough to show that $P_i \cap B = \pi B$ for some i.

Suppose, if possible, that $P_i \cap B \neq \pi B$ for all i. Let $P_i \cap B = Q_i$. Then $ht(Q_i) > 1$, i.e., Q_is are maximal ideals of B (since $dim\ B = 2$). Let t be the number of distinct ideals in the family $\{Q_1, Q_2, \ldots, Q_n\}$. By reindexing, if necessary, we assume that $Q_1, Q_2, \ldots, Q_t$ are all distinct. Let $I_i = \bigcap_{P_j \cap B = Q_i} P_j$. Since Q_is are pairwise comaximal, I_is are pairwise comaximal. Thus $\widetilde{A}/\pi\widetilde{A} = \widetilde{A}/I_1 \times \widetilde{A}/I_2 \times \cdots \times \widetilde{A}/I_t$.

Since $D = A \otimes_R k$, the retraction $\Phi : A \twoheadrightarrow R$ induces a retraction $\Phi_k : D \twoheadrightarrow k$. Let $\mathfrak{m}_0$ be a maximal ideal of D such that $D/\mathfrak{m}_0 = k$. Note that $D \hookrightarrow D_{\mathfrak{m}_0}$ and hence due to flatness, $D \otimes_k \widetilde{k} \hookrightarrow D_{\mathfrak{m}_0} \otimes_k \widetilde{k}$. Since $D_{\mathfrak{m}_0}$ is local and since $\widetilde{k}|_k$ is a finite extension, $D_{\mathfrak{m}_0} \otimes_k \widetilde{k}$ is also local with maximal ideal $\mathfrak{m}_0(D_{\mathfrak{m}_0} \otimes_k \widetilde{k})$ and residue field $\widetilde{k}$. As the local ring $D_{\mathfrak{m}_0} \otimes_k \widetilde{k}$ is a localisation of $D \otimes_k \widetilde{k} = \widetilde{A}/\pi\widetilde{A}$, it follows that there exists a prime ideal $\mathfrak{p}$ of $\widetilde{A}/\pi\widetilde{A}$ such that $D_{\mathfrak{m}_0} \otimes_k \widetilde{k} = (\widetilde{A}/\pi\widetilde{A})_{\mathfrak{p}}$.

Note that $\widetilde{A}/\pi\widetilde{A} = D \otimes_k \widetilde{k} \hookrightarrow D_{\mathfrak{m}_0} \otimes_k \widetilde{k} = (\widetilde{A}/\pi\widetilde{A})_{\mathfrak{p}}$. As the map $\widetilde{A}/\pi\widetilde{A} \longrightarrow (\widetilde{A}/\pi\widetilde{A})_{\mathfrak{p}}$ is one-one, it follows that the zero divisors of $\widetilde{A}/\pi\widetilde{A}$ are contained in $\mathfrak{p}$. Consequently, $\overline{P_i} \subset \mathfrak{p}$ where $\overline{P_i}$ is the image of P_i in $\widetilde{A}/\pi\widetilde{A}$. But this would imply that the local ring $(\widetilde{A}/\pi\widetilde{A})_{\mathfrak{p}}$ is a product of t rings which is possible only if $t = 1$. So $P_i \cap B = Q$ for all i, which implies that $\pi\widetilde{A} \cap B = Q$. Note that the retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ induces a retraction $\widetilde{\Phi}_\pi : \widetilde{A}/\pi\widetilde{A} \twoheadrightarrow \widetilde{R}/\pi\widetilde{R}$. Now since $\pi\widetilde{A} \cap B = Q$, the retraction $\widetilde{\Phi}_\pi$ induces a retraction $\widetilde{\Phi}'_\pi : B/Q \twoheadrightarrow \widetilde{k}$. But Q is a maximal ideal of B, i.e., B/Q is a field. Hence $\widetilde{\Phi}'_\pi$ is an isomorphism. As $x \in Ker\ \widetilde{\Phi}$, it then follows that that $x \in Q \subset \pi\widetilde{A}$ and hence $x \in \pi\widetilde{A}$, a contradiction.

Thus $\pi\widetilde{A} \cap \widetilde{R}[x] = \pi\widetilde{R}[x]$ and hence $\widetilde{A} = \widetilde{R}^{[1]}$ showing that $A \otimes_R k = k^{[1]}$.

(II): Follows from (III) of Theorem 3.3.7.

(III): Follows from (I) and the result ([Asa87], 3.4) of Asanuma, using results of Hamann ([Ham75], 2.6 or 2.8) and Swan ([Swa80], 6.1); also see ([Gre86]). □

Remark 3.3.10. *Examples from existing literature would show that the hypotheses in our results cannot be relaxed. For instance, the hypothesis that*

"Ker Φ is finitely generated" is necessary in all the results as can be seen from the example: Let (R, π) be a DVR and $A = R[X, X/\pi, X/\pi^2, \ldots, X/\pi^n, \ldots]$.

An example of Eakin-Silver ([ES72], 3.15) shows that the hypothesis "A has a retraction to R" is necessary in Proposition 3.3.4. Even if R is local and factorial and A Noetherian, the hypothesis "A has a retraction to R" would still be necessary in Theorem 3.3.5 even to conclude that A is finitely generated as has been shown recently in [BDO]. Even if A is finitely generated, the hypothesis "A has retraction to R" would still be necessary in Theorem 3.3.5 to conclude that A is a symmetric algebra (consider $R = k[[t_1, t_2]]$ where k is any field and $A = R[X, Y]/(t_1X + t_2Y - 1)$).

The following example of Yanik ([Yan81], 4.1) shows the necessity of seminormality hypothesis in the passage from (I) to (III) in Theorem 3.3.9: Let k be a field of characteristic zero, $R = k[[t^2, t^3]]$ and $A = R[X, tX^2] + (t^2, t^3)R[X]$; also see [Gre86].

For other examples (e.g. necessity of "flatness" or the necessity of "geometrically integral" in Theorem 3.3.7), and the necessity of $\mathbb{Q} \hookrightarrow R$ in Theorem 3.3.9, see ([BD95], section 4).

Chapter 4

Factorial $\mathbb{A}^1$-form with retraction

4.1 Preview

Let k be a field. We call a k-algebra D a k-form of a k-algebra C if there exists an algebraic field extension $k'|_k$ such that $D \otimes_k k' \cong C \otimes_k k'$. It is to be noted that a k-algebra A is a k-form of $k[X]$ if and only if A is an $\mathbb{A}^1$-form over k. It is easy to see that if A is an $\mathbb{A}^1$-form over k, there exists a finite algebraic extension $k'|_k$ such that $A \otimes_k k' = k'^{[1]}$. A k-form, or an $\mathbb{A}^1$-form, is called purely inseparable (resp. separable), if we can take the field extension $k'|_k$ to be a purely inseparable (resp. separable) extension. It is well known that there exist k-domains A such that all separable k-forms of A are trivial i.e., isomorphic to A. For example,

(i) For any field k, separable $\mathbb{A}^1$-forms over k are trivial([Dut00]). In fact, in ([Dut00], Theorem 2.0.17) it has been shown that separable $\mathbb{A}^1$-forms over arbitrary domains are trivial.

(ii) For any perfect field k, $\mathbb{A}^r$-forms over k ($r = 1, 2$) are trivial ([Kam75]).

However if k is not perfect, Rosenlicht showed that non-trivial $\mathbb{A}^r$-forms over k always exists even if $r = 1$ ([Ros63]). Asanuma gave a structure theorem for the purely inseparable k-forms of geometrically normal affine plane curves and hence a structure of purely inseparable $\mathbb{A}^1$-forms over field k of characteristic $p > 2$ ([Asa05]). But the case characteristic $p = 2$ is still unsolved! Asanuma conjectured that if k is not perfect, any one dimensional affine k-domain has a non-trivial k-form. It is not known if there exists an one dimensional affine k-domain all of whose k-forms are trivial!

One of the discussed questions in this area is: Under what conditions will a k-form be trivial? Theorem 4.1.5 addresses this question. A more general result will be presented in section 4.2.

We now quote some definitions and results of Asanuma ([Asa05]) which are related to our work.

Definition 4.1.1. *Let k be a field of characteristic $p > 0$. A polynomial $f(X) \in k[X](= k^{[1]})$ is called a p-polynomial if $f(X)$ is of the form $f(X) = a_0 + X + a_1X^p + a_2X^{2p} + \cdots + a_nX^{np}$, $(a_i \in k)$.*

Note that $f(X) \in k[X]$ is a p-polynomial if and only if $\partial(f(X)) = 1$ where $\partial(f(X))$ denotes the derivative of $f(X)$ with respect to X.

Asanuma showed that corresponding to each p-polynomial we can construct a non-trivial purely inseparable $\mathbb{A}^1$-form ([Asa05],Proposition 4.4).

Proposition 4.1.2. *Let k be a field of characteristic $p \geq 0$. Let $k[X^{p^e}, f(X)]$, $(e \geq 0)$ for a p-polynomial $f(X) \in k^{1/p^e}[X]$. Then A is a purely inseparable $\mathbb{A}^1$-form over k.*

Definition 4.1.3. *A k-form of $k^{[1]}$ which is k-isomorphic to $A = k[X^{p^e}, f(X)]$ as in Proposition 4.1.2 is said to an $\mathbb{A}^1$-form of p-polynomial type.*

It is easy to observe that if A is an $\mathbb{A}^1$-form over a field k, then $\Omega_k(A)$ is a projective module over A. The next result ([Asa05], Theorem 4.8) of Asanuma shows that that for an $\mathbb{A}^1$-form A over k, $\Omega_k(A)$ is a free A-module if and only if A is of p-polynomial type.

Theorem 4.1.4. *Let k be a field of characteristic $p > 0$ and let A be a k-form of $k^{[1]}$. Then the following are equivalent:*

(I) *A is generated by two elements over k.*

(II) *$\Omega_k(A)$ is free A-module of rank one.*

(III) *A is of p-polynomial type.*

Now it is to be noted that if A is a factorial $\mathbb{A}^1$-form over k, then $\Omega_k(A)$ is a free A-module. So it follows directly from Theorem 4.1.4 that an $\mathbb{A}^1$-form over k is of p-polynomial type if A is an UFD. It it obvious that if an $\mathbb{A}^1$-form is trivial, it must be a UFD. So, the immediate question is: "Under what other conditions will a factorial $\mathbb{A}^1$-form over k be trivial?" It can be seen, as a consequence of the Theorem 4.1.4, that a factorial $\mathbb{A}^1$-form over k is trivial if it has a k-rational point.

Theorem 4.1.5. *Let k be a field and A an $\mathbb{A}^1$-form over k such that*

(1) *A is a UFD.*

(2) *A has a k-rational point.*

Then $A = k^{[1]}$.

We started investigating whether Theorem 4.1.5 holds in more generality. In section 4.2 we will show that the above result can be generalized over arbitrary k-algebras (Theorem 4.2.2). The proof of our result gives an independent proof to Theorem 4.1.5 (see Theorem 4.2.2, when $B = k$) which does not involve Asanuma's intricate structure theorem.

4.2 Main Result

First we note the following property of polynomial algebra:

Lemma 4.2.1. *Let R be a ring and X a transcendental element over R. Then $ht(XR[X]) = 1$*

Proof. Note that if P is a minimal prime ideal of $R[X]$, then $P = P_0[X]$ for some minimal prime ideal P_0 of R. Now if $P_0[X]$ is a minimal prime ideal in a chain of prime ideals determining the height of $XR[X]$, then $ht(XR[X]) = ht(X(R/P_0)[X])$. Since $\bigcap_{n\geq 1} X^n(R/P_0)[X] = 0$ and since R/P_0 is a domain, $ht(X(R/P_0)[X]) = 1$. Thus $ht(XR[X]) = 1$. □

We now prove Theorem D, the main theorem of this chapter.

Theorem 4.2.2. *Let k be a field and let $R \hookrightarrow A$ be k-algebras such that*

(1) *A is a UFD.*

(2) *There is a retraction $\Phi : A \longrightarrow R$.*

(3) *A is an $\mathbb{A}^1$-form over R.*

Then $A = R^{[1]}$.

Proof. Since A is an $\mathbb{A}^1$-form over R, there exists a finite algebraic extension $L|_k$ such that $A \otimes_k L = (R \otimes_k L)[X] = (R \otimes_k L)^{[1]}$. Let $P = ker\ \Phi$. Then we have a short exact sequence $\mathcal{S}$:

$$0 \longrightarrow P \longrightarrow A \xrightarrow{\Phi} R \longrightarrow 0. \tag{4.2.1}$$

Tensoring $\mathcal{S}$ with L with respect to k gives the short exact sequence $\mathcal{S} \otimes L$:

$$0 \longrightarrow P \otimes_k L \longrightarrow A \otimes_k L \xrightarrow{\widetilde{\Phi}} R \otimes_k L \longrightarrow 0 \tag{4.2.2}$$

where $\widetilde{\Phi} := \Phi \otimes 1$.

Let $\widetilde{P} = P \otimes_k L$. Note that Φ being a retraction from A onto R, $\widetilde{\Phi}$ is also a retraction from $(R \otimes_k L)[X](= A \otimes_k L)$ onto $R \otimes_k L$ with kernel $\widetilde{P} = (X - \widetilde{\Phi}(X))(R \otimes_k L)[X]$. Replacing X by $X - \widetilde{\Phi}(X)$, we assume $\widetilde{P} = X(A \otimes_k L)$.

From the short exact sequence (4.2.2), we see that $P(A \otimes_k L) = \widetilde{P}$; and by faithful flatness of $A \otimes_k L$ over A, we have $\widetilde{P} \cap A = P$ ([Mat89], Pg. 49, Theorem 7.5). Since $A \otimes_k L = (R \otimes_k L)[X]$ is a polynomial ring over $R \otimes_k L$, by Lemma 4.2.1, $ht(\widetilde{P}) = 1$. $A \otimes_k L$ being faithfully flat and integral over A, by the going down theorem ([Mat89], Pg. 68, Theorem 9.5), $ht(P) = ht(\widetilde{P}) = 1$ and since A is a UFD, there exists $g \in A$ such that $P = gA$.

Thus we get $g(A \otimes_k L) = \widetilde{P} = X(A \otimes_k L)$. Since X is a non-zero divisor in $A \otimes_k L(= (R \otimes_k L)[X])$, it follows that $g = \lambda X$ for some $\lambda \in (A \otimes_k L)^*$. Let $\lambda = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$ where $a_0 \in (R \otimes_k L)^*$ and $a_i \in nil(R \otimes_k L)$, for all $i = 1, 2, \cdots, n$. Let $I = (a_1, a_2, \cdots, a_n)(R \otimes_k L)$. Then I is a nilpotent ideal of $R \otimes_k L$. Let N be the least positive integer such that $I^N = (0)$. Since $g \equiv a_0 X (\text{mod } I)$, we have $(R \otimes_k L)[X] = (R \otimes_k L)[g] + I(R \otimes_k L)[X] = \cdots = (R \otimes_k L)[g] + I^N(R \otimes_k L)[X] = (R \otimes_k L)[g]$.

So, we get $R[g] \otimes_k L = A \otimes_k L$ and hence $R[g] = A$, since $L|_k$ is faithfully flat. Thus $A = R^{[1]}$. □

The following two well-known examples ([KMT74], Pg. 70–71, Remark 6.6(a), Examples (i) and (ii)) respectively show that in Theorem 2 (and hence in Theorem 3), the hypothesis on the existence of a retraction and the hypothesis "A is a UFD" are necessary.

Example 4.2.3. *Let $\mathbb{F}_p$ be the prime field of characteristic p and let $k = \mathbb{F}_p(t, u)$ be a purely transcendental extension of $\mathbb{F}_p$ with variables t and u.*

Then $A = k[X,Y]/(Y^p - t - X - uX^p)$ *is a factorial non-trivial* $\mathbb{A}^1$*-form over* k *which does not have a retraction to* k*.*

Example 4.2.4. *Let* k *be a field of characteristic* $p \geq 2$ *and* $A = k[X,Y]/(Y^p - X - aX^p)$ *where* $a \in k \backslash k^p$*. Then* A *is a non-trivial* $\mathbb{A}^1$*-form over* k *with a retraction to* k*. Here* A *is not a UFD.*

Chapter 5

Planes of the form $b(X,Y)Z^n - a(X,Y)$ over a DVR

5.1 Preview

An important question in affine algebraic geometry is the following epimorphism problem:

Question 1. *Let K be a field of characteristic 0. Let $g \in K[X,Y,Z](= K^{[3]})$ be such that $K[X,Y,Z]/(g) = K^{[2]}$. Is then $K[X,Y,Z] = K[g]^{[2]}$?*

While the problem is open in general, a few special cases have been investigated by Sathaye, Russell and Wright in [Sat76], [Rus76], [Wri78] and [RS79]; in some of these cases, Question 1 has an affirmative answer even when K is a field of positive characteristic. In particular, they considered polynomials of the form $b(X,Y)Z^n - a(X,Y)$ and obtained affirmative answers when

(1) $n = 1$, K a field of characteristic 0 (A. Sathaye, [Sat76]).

(2) $n = 1$, K a field of any characteristic (P. Russell, [Rus76]).

(3) $n \geq 2$ and K an algebraically closed field of characteristic $p \geq 0$ with $p \nmid n$ (D. Wright, [Wri78]).

In this chapter we shall first show (see Theorem 5.2.5) that the above result (3) of D. Wright (see Theorem 2.0.14) holds even when K is not necessarily algebraically closed.

We now consider the corresponding question over a DVR.

Question 2. *Let (R,t) be a DVR containing $\mathbb{Q}$ and $g \in R[X,Y,Z](= R^{[3]})$ be such that $R[X,Y,Z]/(g) = R^{[2]}$. Is then $R[X,Y,Z] = R[g]^{[2]}$?*

As shown by Bhatwadekar-Dutta in ([BD94b], section 4), this problem is closely related to the problem of $\mathbb{A}^2$-fibration over a regular two-dimensional affine spot over a field of characteristic zero. Hence, one could explore Question 2 at least for polynomials like $g = b(X,Y)Z^n - a(X,Y)$ for which the corresponding Question 1 has been settled. For such polynomials, in view of the corresponding results over fields, one could extend the investigation of Question 2 even to the positive characteristic case.

The first investigation in this direction was made by Bhatwadekar-Dutta in [BD94a]. They showed (Theorem 3.5, [BD94a]) that Question 2 has an affirmative answer (in any characteristic) when $g = b(X,Y)Z - a(X,Y)$ with $t \nmid b(X,Y)$, thereby partially generalizing A. Sathaye's theorem on linear planes over a field ([Sat76]).

We will show that Question 2 has an affirmative answer (see Theorem 5.3.3) for polynomials of the form $g = b(X,Y)Z^n - a(X,Y)$ where $n \geq 2$ is an integer not divisible by the characteristic of R/tR, thereby obtaining a generalization of D. Wright's theorem (Theorem 2.0.14).

Using theorems on residual variables of Bhatwadekar-Dutta ([BD93]), we will further show that the result for $n \geq 2$ holds over (i) any integral domain containing $\mathbb{Q}$ (see Theorem 5.4.1) and (ii) any Noetherian UFD domain containing a field of characteristic $p \geq 0$, if $p \nmid n$ (see Theorem 5.4.2).

In sections 5.2 and 5.3, we prove our main results over a field and DVR respectively and in section 5.4, we prove our result for rings containing a field.

5.2 Planes of the form $bZ^n - a$ over a field

In this section we will show that Wright's arguments in ([Wri78]) can be modified to show that his result (Theorem 2.0.14) can be extended over any field. We first prove a few auxiliary results (Lemmas 5.2.1 and 5.2.2), then consider the case when the field k contains all nth roots of unity (Proposition 5.2.4) and finally show that Theorem 2.0.14 holds over any field (Theorem 5.2.5).

We first record a result on $\mathrm{A}ut_k(k^{[2]})$.

Lemma 5.2.1. *Let k be a field of characteristic $p \geq 0$ and σ a k-automorphism of $B = k^{[2]}$ of order n such that $p \nmid n$. Suppose that k contains all the n^{th} roots of unity. Then there exist elements $U, V \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U, V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$, where $\alpha^n = \beta^n = 1$.*

Proof. By Theorem 2.0.15, one can choose coordinates U', V' of B such that either $\sigma \in \mathcal{E}_2(k)$ or $\sigma \in Af_2(k)$.

Case: $\sigma \in \mathcal{E}_2(k)$.

In this case $\sigma(U') = \alpha U' + \mu$ and $\sigma(V') = \beta V' + f_1(U')$, where $\alpha, \beta \in k^*$, $\mu \in k$ and $f_1(U') \in k[U']$. Since σ is of order n, we have $\alpha^n = \beta^n = 1$.

If $\alpha = 1$, then $U' = \sigma^n(U') = U' + n\mu$ and hence $\mu = 0$, as $p \nmid n$. Set $U := U'$ if $\alpha = 1$ and $U := U' + \frac{\mu}{\alpha - 1}$ if $\alpha \neq 1$. Then $K[U', V'] = K[U, V']$, $\sigma(U) = \alpha U$ and $\sigma(V') = \beta V' + f(U)$ for some $f(U) \in k[U]$. We will now show that we can choose $g(U) \in k[U]$ such that $\sigma(V' + g(U)) = \beta(V' + g(U))$. Let $f(U) = \sum\limits_{i=0}^{r} a_i U^i$. First we show that for any i, $1 \leq i \leq r$, if $a_i \neq 0$, then $\alpha^i \neq \beta$. Suppose $\beta = \alpha^i$. Now, since $V' = \sigma^n(V') = \beta^n V' + \beta^{n-1} f(U) + \beta^{n-2} f(\alpha U) + \cdots + f(\alpha^{n-1} U)$, we get $\beta^{n-1} f(U) + \beta^{n-2} f(\alpha U) + \cdots + f(\alpha^{n-1} U) = 0$, which implies that $\beta^{n-1} a_i + \beta^{n-2} \alpha^i a_i + \beta^{n-3} \alpha^{2i} a_i + \cdots + \alpha^{(n-1)i} a_i = 0$, i.e., $n\beta^{n-1} a_i = 0$, and hence $a_i = 0$ (as $p \nmid n$ and $\beta \neq 0$). Thus $\alpha^i \neq \beta$ if $a_i \neq 0$.

Now we define b_i for each $i = 1, 2, \cdots, r$ as follows:

$$b_i = \begin{cases} 0 & \text{if} \quad a_i = 0. \\ a_i/(\beta - \alpha^i) & \text{if} \quad a_i \neq 0. \end{cases}$$

Now let $g(U) = \sum\limits_{i=0}^{r} b_i U^i$ and set $V := V' + g(U)$. Then

$$\begin{aligned} \sigma(V) &= \beta V' + f_1(U) + g(\alpha U) \\ &= \beta V' + \sum_{i=0}^{r} a_i U^i + \sum_{i=0}^{r} b_i (\alpha U)^i \\ &= \beta V' + \sum_{i=0}^{r} (a_i + \alpha^i b_i) U^i \\ &= \beta (V' + \sum_{i=0}^{r} b_i U^i) \\ &= \beta (V' + g(U)) \\ &= \beta V. \end{aligned}$$

Thus $k[U', V'] = k[U, V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$.

Case: $\sigma \in Af_2(k)$.

In this case $\sigma(U') = \alpha_1 U' + \beta_1 V' + \gamma_1$ and $\sigma(V') = \alpha_2 U' + \beta_2 V' + \gamma_2$ for some $\alpha_i, \beta_i, \gamma_i \in k$ $(i = 1, 2)$ with $\alpha_1\beta_2 \neq \beta_1\alpha_2$. If $\beta_1 = 0$ or $\alpha_2 = 0$, then it reduces to the previous case after applying some change of variables. So we assume that $\beta_1 \neq 0$ and $\alpha_2 \neq 0$.

Choose $\lambda \in \bar{k}$ such that $(\alpha_1 - \lambda)(\beta_2 - \lambda) - \alpha_2\beta_1 = 0$. Then λ is an eigen value of the linear transformation $(X, Y) \mapsto (\alpha_1 X + \alpha_2 Y, \beta_1 X + \beta_2 Y)$ of $\bar{k}^2$. Let $(\nu_1, \nu_2) \in \bar{k}^2$, not both zero, be an eigen vector corresponding to the eigen value λ. Then we have

$$\begin{aligned} \alpha_1\nu_1 + \alpha_2\nu_2 &= \lambda\nu_1 \\ \beta_1\nu_1 + \beta_2\nu_2 &= \lambda\nu_2 \end{aligned}$$

Therefore, $\sigma(\nu_1 U' + \nu_2 V') = \lambda(\nu_1 U' + \nu_2 V') + \mu$ where $\mu = \nu_1\gamma_1 + \nu_2\gamma_2$. Since σ is of order n, we have $\lambda^n = 1$ and hence $\lambda \in k^*$. Thus we may choose $\nu_1, \nu_2 \in k$. Therefore, setting $U := \nu_1 U' + \nu_2 V'$, we have $\sigma(U) = \lambda U + \mu$ and hence $\sigma(V') = \kappa V' + h(U)$ for some $\kappa \in k^*$ and $h(U) \in k[U]$. Now, by taking U and V' to be the coordinates for B, the problem reduces to the earlier case: $\sigma \in \mathcal{E}_2(k)$.

Thus in both the cases we get $U, V \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U, V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$. This completes the proof. $\square$

We now record a consequence of Sathaye's result (Theorem 2.0.16).

Lemma 5.2.2. *Let k be a field, $B = k^{[2]}$ and $b \in B \backslash k$. Suppose that there exist a separable algebraic extension $E|_k$ and an element $X' \in B \otimes_k E$ such that $B \otimes_k E = E[X']^{[1]}$ and $b \in E[X']$. Then there exists $X \in B$ such that $b \in k[X]$, $B = k[X]^{[1]}$ and $E[X'] = E[X]$.*

Proof. Without loss of generality, we assume $E|_k$ to be a finite separable extension.

Let $B = k[X_1, Y_1]$. Then $B \otimes_k E = E[X_1, Y_1] = E[X']^{[1]}$. Let $X' = \phi(X_1, Y_1)$. Interchanging X_1 and Y_1 if necessary, we may assume that the X_1-degree of $\phi(X_1, Y_1)$ is positive. Hence the leading coefficient of X_1 in $\phi(X_1, Y_1)$ is a non-zero element $\lambda \in E$ ([Abh77], Proposition 11.12, pg. 85). Let $X'' = X'/\lambda$.

Let $G = \{\sigma_i \mid i = 1, 2, \cdots, m\}$ be the group of k-automorphisms of $E|_k$. We extend each $\sigma \in G$ to a B-automorphism of $B \otimes_k E$. Let $\bar{k}$ be an algebraic closure of k containing E and $b = \prod_{i=1}^{s}(\lambda_i X'' + \mu_i)^{n_i}$ be the prime decomposition of b in $\bar{k}[X'']$, where $\lambda_i \in \bar{k}^*$, $\mu_i \in \bar{k}$ and $n_i \in \mathbb{N}$, $1 \leq i \leq s$. Since $\sigma(b) = b$ for each $\sigma \in G$, $b = \prod_{i=1}^{s}(\sigma(\lambda_i)\sigma(X'') + \sigma(\mu_i))^{n_i}$ is also a prime decomposition of b in $\bar{k}[X'']$. This shows that for each $\sigma \in G$, $\exists\ \alpha \in \bar{k}^*$ and $\beta \in \bar{k}$ such that $\sigma(X'') = \alpha X'' + \beta$. Since X'' and $\sigma(X'')$ are both monic in X_1, it follows that $\alpha = 1$.

Since X'' is a variable of $B \otimes_k E$, we have $(B \otimes_k E)/(\sigma(X'')) = E^{[1]}$ for each $\sigma \in G$. It is also easy to see that if $\sigma_i(X'') \neq \sigma_j(X'')$ for $\sigma_i, \sigma_j \in G$, then $\sigma_i(X'')$ and $\sigma_j(X'')$ are comaximal in $B \otimes_k \bar{k}$ and hence comaximal in $B \otimes_k E$. Let $u_1, \cdots, u_t$ be the distinct elements of the set $\{\sigma(X'') | \sigma \in G\}$. Then, for each $i = 1, 2, \cdots, t$, there exists $m_i \in \mathbb{N}$ such that $\prod_{\sigma \in G} \sigma(X'') = \prod_{i=1}^{t} u_i^{m_i} \in B$, $(B \otimes_k E)/(u_i) = E^{[1]}$ and for $i \neq j$, u_i and u_j are comaximal in $B \otimes_k L$. Since $B = k^{[2]}$, applying Theorem 2.0.16, we get that for each $\sigma \in G$ there exist $\lambda \in E^*$ and $\mu \in E$ such that $\lambda\sigma(X'') + \mu \in B$. Fix a $\sigma \in G$ and let $X = \lambda\sigma(X'') + \mu \in B$. Then $E[X''] = E[\sigma(X'')] = E[X]$ and $b \in E[X] \cap B$. Since $B = k^{[2]}$, $X \in B$, and $E|_k$ is separable, we must have $E[X] \cap B = k[X]$. Hence $b \in k[X] \subset B$. Now since $B = k^{[2]}$ and $B \otimes_k E = E[X'']^{[1]} = E[X]^{[1]}$, by Theorem 2.0.17, we see that $B = k[X]^{[1]}$. By construction, $E[X] = E[X''] = E[X']$. $\square$

For convenience, we state below a result which follows from a lemma of A. Sathaye ([Sat76], Lemma 1).

Lemma 5.2.3. *Let k be a field and X' a variable in $k[X_1, X_2, \cdots, X_n](= k^{[n]})$ which is comaximal with X_1. Then $X' = \alpha X_1 + \beta$ with $\alpha, \beta \in k$, $\alpha \neq 0$.*

We now show that the Theorem 2.0.14 can be extended to the case of fields containing n^{th} roots of unity.

Proposition 5.2.4. *Let k be a field of characteristic $p \geq 0$ containing the n^{th} roots of unity and $g \in k[X, Y, Z] = k^{[3]}$ be of the form $bZ^n - a$ where $a, b \in k[X, Y]$ with $b \neq 0$ and n is an integer ≥ 2 not divisible by p. Suppose that $B := k[X, Y, Z]/(g) = k^{[2]}$ and identify $k[X, Y]$ with its image in B. Then there exist variables U, V in B such that V is the image of Z in B, $U \in k[X, Y]$, $b \in k[U]$ and $k[X, Y] = k[U, a] = k^{[2]}$.*

Proof. Let σ be the k-automorphism of B induced by the k-automorphism $\widetilde{\sigma}$ of $k[X,Y,Z]$ defined by $\widetilde{\sigma}((X,Y,Z)) = (X,Y,\omega Z)$ where ω is a primitive n^{th} root of unity. Obviously, σ has order n.

Since $B = k^{[2]}$, by Lemma 5.2.1, there exist elements $U', V' \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U', V']$, $\sigma(U') = \alpha U'$ and $\sigma(V') = \beta V'$, where $\alpha^n = \beta^n = 1$. Let $\mathfrak{z}$ be the image of Z in B and $A = k[X,Y][a/b]$. Then $\mathfrak{z}^n = a/b$ and $B = A[\mathfrak{z}] = k[X,Y][\mathfrak{z}] = A \oplus \mathfrak{z}A \oplus \mathfrak{z}^2 A \oplus \cdots \oplus \mathfrak{z}^{n-1}A$ so that, for any $x \in B$, $\mathfrak{z} \mid (x - \sigma(x))$. Thus $\mathfrak{z} \mid (1-\alpha)U'$ and $\mathfrak{z} \mid (1-\beta)V'$. But since U' and V' can not have common (non-unit) factor and $\mathfrak{z} \notin k^*$, we have either $\alpha = 1$ or $\beta = 1$. Interchanging U' and V' if necessary, we assume that $\alpha = 1$. Then the ring of invariants of σ is $A = k[X,Y][a/b] = k[U', a/b] (= k^{[2]})$. Note that V' is a unit multiple of $\mathfrak{z}$. Thus $B = k[U', \mathfrak{z}]$. Set $V := \mathfrak{z}$.

Now we show that we can choose U from $k[X,Y]$ such that $B = k[U,V]$, $b \in k[U]$ and $k[X,Y] = k[U,a]$. If $b \in k^*$, then $k[X,Y] = k[X,Y][a/b] = k[U', a/b]$, so that, in this case, we may set $U := U'$. We now consider the case $b \notin k^*$. Let $p_1, p_2, \ldots, p_m$ be the distinct irreducible factors of b in $A (= k^{[2]})$, and set $\mathfrak{p}_i := k[X,Y] \cap p_i A$. Note that for each $i = 1, 2, \ldots, m$, both b and $a (= b.a/b) \in k[X,Y] \cap bA \subseteq \mathfrak{p}_i$. This shows that $(bZ^n - a)k[X,Y,Z] \subsetneqq \mathfrak{p}_i[Z]$ which implies $ht\ \mathfrak{p}_i > 1$. Thus each $\mathfrak{p}_i$ is a maximal ideal of $k[X,Y]$.

Let $\bar{k}$ denote an algebraic closure of k, L_i be the subfield of $\bar{k}$ isomorphic to $k[X,Y]/\mathfrak{p}_i$ and let L be the subfield of $\bar{k}$ generated by the fields $L_1, L_2, \ldots, L_m$. Then L_i is an algebraic extension of k and $A/\mathfrak{p}_i A = (k[X,Y]/\mathfrak{p}_i)[\zeta_i] = L_i[\zeta_i]$ where ζ_i is the image of a/b in $A/\mathfrak{p}_i A$. Since $\mathfrak{p}_i A \subseteq p_i A$, it follows that ζ_i is transcendental over L_i and $\mathfrak{p}_i A$ is a prime ideal of A. As $ht\ p_i A = 1$ and $\mathfrak{p}_i A \neq 0$, we have $p_i A = \mathfrak{p}_i A$. This shows that p_i's are pairwise comaximal in A and hence in B.

Let $g(\zeta_i)$ be the image of U' in $A/p_i A = L_i[\zeta_i]$. Then $U' - g(a/b)$ is divisible by p_i in $A \otimes_k L_i$. But $U' - g(a/b) = U' - g(V^n)$ is a variable in both $A \otimes_k L_i$ and $B \otimes_k L_i$. Hence $U' - g(a/b)$ is a constant multiple of p_i. Thus $A \otimes_k L_i = L_i[p_i, a/b]$, $B \otimes_k L_i = L_i[p_i, V]$, and for $i \neq j$, $(p_i, p_j)B \otimes_k L = B \otimes_k L$. Set $U := p_1$. Using Lemma 5.2.3, we have $p_i = \lambda_i U + \mu_i$ for $\lambda_i \in L^*$ and $\mu_i \in L$. So, we have $b \in L[U]$. This shows that U is integral over $L[X,Y]$ and hence over $k[X,Y]$. As $U \in k[X,Y][a/b]$ and $k[X,Y]$ is a normal domain, we have $U \in k[X,Y]$. Since $L|_k$ is faithfully flat, it follows that $B = k[U,V]$ with $U \in k[X,Y], V = \mathfrak{z}$ and $b \in k[U]$.

Now, to complete the proof, we are only left to show that $k[X,Y] = k[U,a]$.

We repeat the argument in ([Wri78], pg. 98) to prove it. First we claim that whenever $h \in k[U,a]$ and $h \in bk[X,Y]$, then $h \in bk[U,a]$. To see this, write

$$h = h_0(U) + h_1(U)a + \cdots + h_d(U)a^d.$$

Since $a \in bA$, it follows that $h_0(U) \in bA$. But since $A = k[U, a/b](= k^{[2]})$, we get $h_0(U) \in bk[U]$ and hence $h_0(U) \in bk[U,a]$. So, we may replace h by $h - h_0(U) = h_1(U)a + h_2(U)a^2 + \cdots + h_d(U)a^d$. Let $h' = h_1(U) + h_2(U)a + \cdots + h_d(U)a^{d-1}$. Then $h = h'a$. Since there is no height one prime ideal of $k[X,Y]$ which contains both a and b, and since $h'a \in bk[X,Y]$, it follows from the normality of $k[X,Y]$ that (the associative prime ideals of a are of height one) $h'/b \in \bigcap_{\mathfrak{p} \in Spec(R);\ ht(\mathfrak{p})=1} k[X,Y]_{\mathfrak{p}} = k[X,Y]$. Therefore, $h' \in bk[X,Y]$. Now we argue as before that $h_1(U) \in bk[U,a]$. We continue this process to conclude that $h_i(U) \in bk[U,a]$ for $0 \leq i \leq d$, which proves the claim. Now, let $f \in k[X,Y]$. then $f \in k[U, a/b] = A$. We claim that whenever f can be written as

$$f = f_0 + f_1.(a/b) + \cdots + f_s.(a/b)^s$$

with $f_0, f_1, \cdots, f_s \in k[U,a]$, $s > 0$, then we can express f as such a polynomial of lower degree. Multiplying by b^s, one sees that $f_s a^s \in k[X,Y]$. But since no height one prime ideal of $k[X,Y]$ contains both a and b, it follows that $f_s \in bk[U,a]$. Writing $f_s = bf'$ we get

$$f = f_0 + f_1.(a/b) + \cdots + f_{s-2}.(a/b)^{s-2} + (f_{s-1} + af').(a/b)^{s-1}$$

with $f_{s-1} + af' \in k[U,a]$. Continuing this process we get that $f \in k[U,a]$. Thus, we have shown that $k[X,Y] = k[U,a]$. □

We now prove Theorem E, which essentially shows that the result of D. Wright (Theorem 2.0.14) holds over any field.

Theorem 5.2.5. *Let k be a field of characteristic $p \geq 0$ and $g \in k[X,Y,Z]$ be of the form $bZ^n - a$ where $a, b \in k[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 not divisible by p. Suppose that $B := k[X,Y,Z]/(g) = k^{[2]}$ and identify $k[X,Y]$ with its image in B. Then there exist variables U, V in B such that V is the image of Z in B, $U \in k[X,Y]$, $b \in k[U]$, $k[X,Y] = k[U,a]$ and $k[X,Y,Z] = k[U,g,Z]$.*

Proof. Let E be the field obtained by adjoining all the n^{th} roots of unity to k.

Since $p \nmid n$, E is Galois over k. By Proposition 5.2.4, we get variables U' and V' of $B \otimes_k E$ $(= k[X,Y,Z]/(g) = E^{[2]})$ such that V' is the image of Z, $b \in E[U']$ and $E[X,Y] = E[U', a]$. As $E|_k$ is separable, we have $k[X,Y] = k[a]^{[1]}$ by Theorem 2.0.17. If $b \in k[X,Y] \backslash k$, then, by Lemma 5.2.2, we get $U \in k[X,Y]$ such that $k[X,Y] = k[U]^{[1]}$, $b \in k[U]$ and $E[U] = E[U']$. Since $E|_k$ is faithfully flat, $E[U', a] = E[U, a]$ and $k[U, a] \subseteq k[X,Y]$, we have $k[U,a] = k[X,Y]$. If $b \in k$, then we choose U to be any complementary variable of a in $k[X,Y]$.

From the relation $k[U,a] = k[X,Y]$, we have

$$k[X,Y,Z] = k[U,a,Z] = k[U, bZ^n - a, Z] = k[U,g,Z].$$

The relation $k[X,Y,Z] = k[U,g,Z]$ shows that B is generated by the images of U and Z. This completes the proof. $\square$

Remark 5.2.6. *Theorem 5.2.5 does not hold if $p \mid n$. We reconsider Example 1.0.7: Let k be a field of characteristic $p > 0$ and $g = Z^{p^e} - Y - X^{sp} \in k[Y,Z]$ where $p \nmid s$ and $e \geq 2$. It is known that $k[Y,Z]/(g) = k^{[1]}$ but $k[Y,Z] \neq k[g]^{[1]}$ (see [Abh77], Example 9.12, pg. 72). Therefore $k[X,Y,Z]/(g) = k^{[2]}$. But $k[X,Y,Z](= k[Y,Z][X]) \neq k[g]^{[2]}$ by Theorem 2.0.21.*

5.3 Planes of the form $bZ^n - a$ over a DVR

For convenience, we first record an observation.

Lemma 5.3.1. *Let R be a UFD with field of fractions K. Let $U \in R[X,Y]$ be such that $K[X,Y] = K[U]^{[1]}$. Then $K[U] \cap R[X,Y]$ is an inert subring of $R[X,Y]$ and $K[U] \cap R[X,Y] = R[W](= R^{[1]})$, where W is an element of $R[X,Y]$ such that $K[W] = K[U]$.*

Proof. Let $D = K[U] \cap R[X,Y]$. Clearly, D is an inert subring of $R[X,Y]$ and hence a UFD of transcendence degree one over R. Therefore, by ([AEH72], Theorem 4.1), $D = R[W](= R^{[1]})$ for some $W \in R[X,Y]$. Clearly, $K[W] = K[U]$. $\square$

Lemma 5.3.2. *Let R be a UFD of characteristic $p \geq 0$ with field of fractions K and $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then*

(i) $R[a] = K[a] \cap R[X,Y]$.

(ii) $R[a]$ *is an inert subring of* $R[X,Y]$.

(iii) $tR[X,Y] \cap R[a] = tR[a]$.

Proof. (i) By Theorem 5.2.5, $K[X,Y] = K[a]^{[1]}$ and, by Lemma 5.3.1, $K[a] \cap R[X,Y] = R[W]$ for some $W \in R[X,Y]$ satisfying $K[a] = K[W]$. It then follows that $a = \lambda W + \mu$ where $\lambda, \mu \in R$. We claim that $\lambda \in R^*$. Suppose $\lambda \notin R^*$. Let p be a prime factor of λ and L denote the algebraic closure of the field of fractions of R/pR. Let $\bar{a}$ and $\bar{b}$ denote the images of a and b respectively in $L[X,Y,Z]/(g)$. Then we would have $\bar{a}(= \mu) \in L$; in fact, as $L[X,Y,Z]/(g) = L[X,Y,Z]/(\bar{b}Z^n - \bar{a}) = L^{[2]}$, we would have that $\bar{a}$ is a unit in L. Since $L[X,Y] \hookrightarrow L[X,Y,Z]/(\bar{b}Z^n - \bar{a})(= L^{[2]})$, it would follow that $\bar{b} \in L^*$. But then, as $n \geq 2$, $L[X,Y,Z]/(\bar{b}Z^n - \bar{a})$ would not be an integral domain, contradicting that $L[X,Y,Z]/(\bar{b}Z^n - \bar{a}) = L^{[2]}$. Thus $\lambda \in R^*$ and hence $R[a] = R[W] = K[a] \cap R[X,Y]$.

(ii) and (ii) follow from (i) easily. □

We now prove Theorem F.

Theorem 5.3.3. *Let (R,t) be a DVR with residue field k and let $p(\geq 0)$ be the characteristic of k. Let $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y] = R[a]^{[1]}$, $R[X,Y,Z] = R[g,Z]^{[1]}$ and $b \in R[X_0]$ where $K[X,Y] = K[X_0, a]$.*

Proof. Let K and k denote the field of fractions and residue field of (R,t). For any $f \in R[X,Y,Z]$, let $\bar{f}$ denote the image of f in $k[X,Y,Z]$. Note that $k[X,Y,Z]/(bZ^n - a) = k^{[2]}$ and $K[X,Y,Z]/(bZ^n - a) = K^{[2]}$. Hence, by Theorem 5.2.5, we have $K[X,Y] = K[a]^{[1]}$ and $K[X,Y,Z] = K[Z, bZ^n - a]^{[1]}$.

We first suppose $t \nmid b$. In this case, applying Theorem 5.2.5, we get $k[X,Y] = k[\bar{a}]^{[1]}$ and $k[X,Y,Z] = k[Z, \bar{b}Z^n - \bar{a}]^{[1]}$. By Theorem 2.0.19, we get $R[X,Y] = R[a]^{[1]}$ and $R[X,Y,Z] = R[Z, bZ^n - a]^{[1]}$.

We now assume $t \mid b$. In this case, we have:

$$k[X,Y,Z]/(\bar{a}) \ (= k[X,Y]/(\bar{a}))^{[1]} = R[X,Y,Z]/(t, bZ^n - a) = k^{[2]}.$$

Hence, by Theorem 2.0.18, $k[X,Y]/(\bar{a}) = k^{[1]}$. Therefore, by Lemma 2.0.20, we see that $k[\bar{a}]$ is algebraically closed in $k[X,Y]$. Since t is prime in both

$R[a](= R^{[1]})$ and $R[X,Y]$, and since a is a generic variable of $R[X,Y]$, using Theorem 2.0.19, we see that $R[X,Y] = R[a]^{[1]}$. By similar argument, we have $R[X,Y,Z] = R[Z, bZ^n - a]^{[1]}$.

Now, by Theorem 5.2.5, one can choose $U \in R[X,Y]$ such that $K[X,Y] = K[U,a]$ and $b \in K[U]$. By Lemma 5.3.1, $K[U] \cap R[X,Y] = R[X_0]$ for some $X_0 \in R[X,Y]$ satisfying $K[U] = K[X_0]$. Thus $b \in R[X_0]$ where $K[X_0,a] = K[U,a] = K[X,Y]$. Hence the result. □

Note that, in the case R is a $\mathbb{Q}$-algebra, the hypothesis in Theorem 5.3.3 regarding n ($p \nmid n$) is automatically satisfied. Thus, in particular, Theorem 5.3.3 holds when R is a DVR containing $\mathbb{Q}$. In the next section we shall show (Theorem 5.4.2) that the result is, in fact, true for any UFD containing a field of any characteristic $p \nmid n$.

Remark 5.3.4. *Note that, in the notation of Theorem 5.3.3, X_0 need not be a variable in $R[X,Y]$. Consider a DVR (R,t). Let $g = bZ^n - a$ where $a = -Y$ and $b = t^2X + tY^2$, and let $X_0 = tX + Y^2$. Then $R[X,Y,Z]/(g) = R^{[2]}$, $b \in R[X_0]$, $K[X,Y] = K[X_0,Y]$ but $R[X,Y] \neq R[X_0]^{[1]}$.*

The following example shows that the conclusion of Theorem 5.3.3 can fail, even when R is a DVR of characteristic zero, if p divides n.

Example 5.3.5. *Let $R = \mathbb{Z}_{(p)}$ where p is a prime in $\mathbb{Z}$, $K = Qt(R) = \mathbb{Q}$ and $k = R/pR = \mathbb{Z}/p\mathbb{Z}$. Let $a = Y^p + Y + pX$ and $g = Z^p - a \in R[X,Y,Z]$. Then $R[X,Y,Z]/(g) = R^{[2]}$ but $R[X,Y] \neq R[a]^{[1]}$.*

Proof. We shall, in fact, show that $R[X,Y,Z] = R[g]^{[2]}$. Let $Z' = Z - Y$. Then $R[X,Y,Z] = R[X,Y,Z']$ and $g = Z'^p - pf(Z',Y) - Y - pX$ for some $f \in R[Z',Y]$. Let $D = R[g,Z']$. We have $K[X,Y,Z] = K[g,Y,Z] = K[g,Z']^{[1]}$ and $k[X,Y,Z] = k[X,\bar{g},Z'] = k[\bar{g},Z']^{[1]}$ where $\bar{g}$ denotes the image of g in $k[X,Y,Z]$. Since p is prime in R, p is prime in both $R[X,Y,Z]$ and D. Hence, by Theorem 2.0.19, $R[X,Y,Z] = D^{[1]} = R[g]^{[2]}$. Let $\bar{a}$ denote the image of a in $k[X,Y]$. Since $k[\bar{a}] = k[Y + Y^p]$ is not algebraically closed in $k[X,Y]$, $\bar{a}$ is not a variable in $k[X,Y]$ and hence a is not a variable in $R[X,Y]$. □

However the next result shows that Theorem 5.3.3 holds over any DVR (R,t) of characteristic 0 for every $g = bZ^n - a$, for which $(R/tR)[\bar{a}]$ is algebraically closed in $(R/tR)[X,Y]$.

Proposition 5.3.6. *Let (R,t) be a DVR of characteristic 0 and $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ^n - a$ where $a,b \in R[X,Y]$, $b \neq 0$ and n is an integer ≥ 2. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$ and $(R/tR)[\bar{a}]$ is algebraically closed in $(R/tR)[X,Y]$. Then $R[X,Y] = R[a]^{[1]}$ and $R[X,Y,Z] = R[Z,g]^{[1]}$.*

Proof. We see that $R[1/t][X,Y] = R[1/t][a]^{[1]}$ by Theorem 5.2.5, t is prime in both $R[a]$ and $R[X,Y]$, $tR[X,Y] \cap R[a] = tR[a]$ by Lemma 5.3.2 and $(R/tR)[\bar{a}]$ is algebraically closed in $(R/tR)[X,Y]$ by hypothesis. Hence, by Theorem 2.0.19, $R[X,Y] = R[a]^{[1]}$.

Let $B := R[X,Y,Z]/(g)(= R^{[2]})$ and denote the image of Z in B by $\mathfrak{z}$. Then $B/(\mathfrak{z}) = R[X,Y,Z]/(Z, bZ^n - a) = R[X,Y]/(a) = R^{[1]}$ and hence, by the generalized epimorphism theorem of Bhatwadekar (Theorem 1.0.8), we have $B = R[\mathfrak{z}]^{[1]}$. Let $C = R[Z]$. Identifying the image of Z in B with Z itself, we have $C[X,Y]/(g) = C^{[1]}$. Since C is a normal domain of characteristic 0, again by Bhatwadekar's result (Theorem 1.0.8), we have $C[X,Y] = C[g]^{[1]}$, i.e., $R[X,Y,Z] = R[g,Z]^{[1]}$. $\square$

5.4 Planes of the form $bZ^n - a$ over rings containing a field

We now prove Theorem G.

Theorem 5.4.1. *Let R be an integral domain containing $\mathbb{Q}$. Let $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ^n - a$ where $a,b \in R[X,Y]$ and n is an integer ≥ 2. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.*

Proof. Without loss of generality we may assume that R is Noetherian. Fix $P \in Spec(R)$. Let the images of b, a and g in $R[X,Y,Z] \otimes_R k(P)$ be $\bar{b}$, $\bar{a}$ and $\bar{g}$ respectively. We show that $k(P)[X,Y] = k(P)[\bar{a}]^{[1]}$ and $k(P)[X,Y,Z] = k(P)[\bar{g},Z]^{[1]}$ from which by Theorem 2.0.22 the result will follow.

If $\bar{b} \neq 0$, then by Theorem 5.2.5, $k(P)[X,Y,Z] = k(P)[\bar{g},Z]^{[1]}$ and $k(P)[X,Y] = k(P)[\bar{a}]^{[1]}$. If $\bar{b} = 0$ (and hence $\bar{a} = \bar{g}$), then $\frac{k(P)[X,Y]}{(\bar{a})}[z] = \frac{k(P)[X,Y,Z]}{(\bar{g})} = k(P)^{[2]}$ and hence $\frac{k(P)[X,Y]}{(\bar{a})} = k(P)^{[1]}$, by Theorem 2.0.18. Therefore, by the Epimorphism theorem of Abhyankar-Moh and Suzuki (Theorem 1.0.6), $k(P)[X,Y] = k(P)[\bar{a}]^{[1]}$ and hence $k(P)[X,Y,Z] = k(P)[g,Z]^{[1]}$.

This shows that $k(P)[X,Y] = k(P)[\bar{a}]^{[1]}$ and $k(P)[X,Y,Z] = k(P)[g,Z]^{[1]}$ for each prime ideal P in R and hence, by Theorem 2.0.22, $R[X,Y] = R[a]^{[1]}$ and $R[X,Y,Z] = R[g,Z]^{[1]}$. $\square$

Next we will prove Theorem H which will show that when R is a Noetherian UFD, then Theorem G holds even if R contains any field of characteristic $p \geq 0$ where $p \nmid n$.

Theorem 5.4.2. *Let R be a Noetherian UFD containing a field of characteristic $p \geq 0$ and $g \in R[X,Y,Z](= R^{[3]})$ be of the form $bZ^n - a$ where $a, b \in R[X,Y]$, $b \neq 0$ and n is an integer ≥ 2 such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.*

Proof. Fix $P \in Spec(R)$. Let the images of b, a and g in $R[X,Y,Z] \otimes_R k(P)$ be $\bar{b}$, $\bar{a}$ and $\bar{g}$ respectively. Let z denote the image of Z in $k(P)[X,Y,Z]/(\bar{g})$. First we show that $k(P)[X,Y,Z]/(\bar{g}) = k(P)[z]^{[1]}$ and then we show that $k(P)[X,Y] = k(P)[\bar{a}]^{[1]}$ and $k(P)[X,Y,Z] = k(P)[\bar{g},Z]^{[1]}$.

If $\bar{b} \neq 0$, then by Theorem 5.2.5, $k(P)[X,Y,Z] = k(P)[\bar{g},Z]^{[1]}$ and $k(P)[X,Y] = k(P)[\bar{a}]^{[1]}$. If $\bar{b} = 0$, then $\bar{a} = \bar{g}$. Therefore, $\frac{k(P)[X,Y,Z]}{(\bar{g})} = \frac{k(P)[X,Y]}{(\bar{a})}[z] = k(P)^{[2]}$ and hence $\frac{k(P)[X,Y]}{(\bar{a})} = k(P)^{[1]}$, by Theorem 2.0.18. This shows that $\frac{k(P)[X,Y,Z]}{(\bar{g})} = k(P)[z]^{[1]}$ for each prime ideal P in R and hence, by Theorem 2.0.22, $R[X,Y,Z]/(g) = R[Z]^{[1]}$.

We now show that $R[X,Y] = R[a]^{[1]}$ and $R[X,Y,Z] = R[g,Z]^{[1]}$. By Theorem 5.2.5, we have $K[X,Y] = K[a]^{[1]}$ and hence there exists $r \in R$ such that $R[1/r][X,Y] = R[1/r][a]^{[1]}$. Let $p_1, p_2, \cdots, p_t$ be the prime factors of r in R. Then $R[\frac{1}{p_1 p_2 \cdots p_t}][X,Y] = R[\frac{1}{p_1 p_2 \cdots p_t}][a]^{[1]}$. Also, by Lemma 5.3.2, $p_i R[X,Y] \cap R[a] = p_i R[a]$ for all i. Let a_{p_i} denote the image of a in $(R/(p_i))[X,Y]$. Then $(R/(p_i))[X,Y]/(a_{p_i}) = R[X,Y,Z]/(p_i, Z, g) = (R/(p_i))^{[1]}$ and hence by Lemma 2.0.20, $(R/(p_i))[a_{p_i}]$ is algebraically closed in $(R/(p_i))[X,Y]$. Thus by Theorem 2.0.19, $R[X,Y] = R[a]^{[1]}$. By arguing similarly we can show that $R[X,Y,Z] = R[g,Z]^{[1]}$. $\square$

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