Geometric Invariants for a Class of Semi-Fredholm Hilbert Modules

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In the memory of my grandfather

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Contents

Notation

$\mathbb{C}[\underline{z}]$	the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$ of m - complex variables
$\mathbb{Z}_+$	the set of non-negative integers
$\mathcal{GL}(\mathbb{C}^k)$	the group of all invertible linear transformations on $\mathbb{C}^k$
$\mathfrak{m}_w$	the maximal ideal of $\mathbb{C}[\underline{z}]$ at the point $w\in\mathbb{C}^m$
Ω	a bounded domain in $\mathbb{C}^m$
Ω^*	$\{\bar{z}: z\in\Omega\}$
$\mathbb{D}$	the open unit disc in $\mathbb{C}$
$\mathbb{D}^m$	the poly-disc $\{z\in\mathbb{C}^m: \lvert z_i\rvert<1,\, 1\leq i\leq t\}, m\geq 1$
M_i	the module multiplication by the co-ordinate function z_i, $1\leq i\leq m$
M_i^*	the adjoint of M_i
$D_{(\mathbf{M}-w)^*}$	the operator $\mathcal{M}\to\mathcal{M}\oplus\ldots\oplus\mathcal{M}$ defined by $f\mapsto((M_j-w_j)^*f)_{j=1}^m$
$\hat{\mathcal{H}}$	the analytic localization $\mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H}$ of the Hilbert module $\mathcal{H}$
$\mathrm{B}_n(\Omega)$	Cowen-Douglas class of operators of rank n, also
	Hilbert modules such that $\mathbf{M}^*=(M_1^*,\ldots,M_m^*)\in\mathrm{B}_n(\Omega^*)$
$\alpha, \lvert\alpha\rvert, \alpha!$	the multi index $(\alpha_1,\ldots,\alpha_m)$, $\lvert\alpha\rvert=\sum_{i=1}^m\alpha_i$ and $\alpha!=\alpha_1!\ldots\alpha_m!$
$\binom{\alpha}{k}$	$=\prod_{i=1}^m\binom{\alpha_i}{k_i}$ for $\alpha=(\alpha_1,\ldots,\alpha_m)$ and $k=(k_1,\ldots,k_m)$
$k\leq\alpha$	if $k_i\leq\alpha_i$, $1\leq i\leq m$.
z^α	$z_1^{\alpha_1}\ldots z_m^{\alpha_m}$
q^*	$q^*(z)=\overline{q(\bar{z})}(=\sum_\alpha\bar{a}_\alpha z^\alpha$, for q of the form $\sum_\alpha a_\alpha z^\alpha)$
$\partial^\alpha,\bar{\partial}^\alpha$	$\partial^\alpha=\frac{\partial^{\lvert\alpha\rvert}}{\partial z_1^{\alpha_1}\cdots z_m^{\alpha_m}},\bar{\partial}^\alpha=\frac{\partial^{\lvert\alpha\rvert}}{\partial\bar{z}_1^{\alpha_1}\cdots\bar{z}_m^{\alpha_m}}$ for $\alpha\in\mathbb{Z}^+\times\cdots\times\mathbb{Z}^+$
$q(D)$	the differential operator $q(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_m})$ ($=\sum_\alpha a_\alpha\partial^\alpha$, where $q=\sum_\alpha a_\alpha z^\alpha$)
$K(z,w)$	a reproducing kernel
$\mathcal{O}_\Omega$, $\mathcal{O}(\Omega)$	the sheaf of holomorphic functions on Ω
$\mathcal{O}_w$	the germs of holomorphic function at the point $w\in\mathbb{C}^m$
g_0	germ of the holomorphic function g at 0
$\mathcal{S}^{\mathcal{M}}$	the analytic subsheaf of $\mathcal{O}_\Omega$, corresponding to the Hilbert module $\mathcal{M}\in\mathfrak{B}_1(\Omega)$
$E(w)$	the evaluation functional (the linear functional induced by $K(\cdot,w)$)
$\lVert\cdot\rVert_{\bar{\Delta}(0;r)}$	supremum norm

$\\| \cdot \\|_2$	the L^2 norm with respect to the volume measure
$V(\mathcal{F})$	$\{z \in \Omega : f(z) = 0 \text{ for all } f \in \mathcal{F}\}$, where $\mathcal{F} \subset \mathcal{O}(\Omega)$
$\mathbb{V}_w(\mathcal{F})$	$\{q \in \mathbb{C}[\underline{z}] : q(D)f\big\|_w = 0,\ f \in \mathcal{F}\}$ is the characteristic space of $\mathcal{F} \subseteq \mathcal{O}_w$ at w
$\widetilde{\mathbb{V}}_w(\mathcal{F})$	$\{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_w(\mathcal{F}),\ 1 \leq i \leq m\}$, where $\mathcal{F} \subseteq \mathcal{O}_w$
$[\mathcal{I}]$	the completion of a polynomial ideal $\mathcal{I}$ in some Hilbert module
$K_{[\mathcal{I}]}$	the reproducing kernel of $[\mathcal{I}]$
$\langle\ ,\ \rangle_{w_0}$	the Fock inner product at w_0, defined by $\langle p, q\rangle_{w_0} := q^*(D)p\|_{w_0} = (q^*(D)p)(w_0)$
$\mathbb{P}_0$	the orthogonal projection onto $ran\ D_{(\mathbf{M}-w_0)^*}$
$\mathcal{P}_w$	$\ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$ for $w \in \Omega$

0. Overview

One of the basic problem in the study of a Hilbert module $\mathcal{H}$ over the ring of polynomials $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \ldots, z_m]$ is to find unitary invariants (cf. [15, 7]) for $\mathcal{H}$. It is not always possible to find invariants that are complete and yet easy to compute. There are very few instances where a set of complete invariants have been identified. Examples are Hilbert modules over continuous functions (spectral theory of normal operator), contractive modules over the disc algebra (model theory for contractive operator) and Hilbert modules in the class $\mathrm{B}_n(\Omega)$ for a bounded domain $\Omega \subseteq \mathbb{C}^m$ (adjoint of multiplication operators on reproducing kernel Hilbert spaces). In this thesis, we study Hilbert modules consisting of holomorphic functions on some bounded domain possessing a reproducing kernel. Our methods apply, in particular, to submodules of Hilbert modules in $\mathrm{B}_1(\Omega)$.

Another important aspect of operator theory starts from the work of Beurling [4]. Beurling's theorem describing the invariant subspaces of the multiplication (by the coordinate function) operator on the Hardy space of the unit disc is essential to the Sz.-Nagy – Foias model theory and several other developments in modern operator theory. In the language of Hilbert modules, Beurling's theorem says that all submodules of the Hardy module of the unit disc are equivalent (in particular, equivalent to the Hardy module). This observation, due to Cowen and Douglas [9], is peculiar to the case of one-variable operator theory. The submodule of functions vanishing at the origin of the Hardy module $H_0^2(\mathbb{D}^2)$ of the bi-disc is not equivalent to the Hardy module $H^2(\mathbb{D}^2)$. To see this, it is enough to note that the joint kernel of the adjoint of the multiplication by the two co-ordinate functions on the Hardy module of the bi-disc is 1 - dimensional (it is spanned by the constant function 1) while the joint kernel of these operators restricted to the submodule is 2 - dimensional (it is spanned by the two functions z_1 and z_2).

There has been a systematic study of this phenomenon in the recent past [1, 16] resulting in a number of "Rigidity theorems" for submodules of a Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ of the form $[\mathcal{I}]$ obtained by taking the norm closure of a polynomial ideal $\mathcal{I}$ in the Hilbert module. For a large class of polynomial ideals, these theorems often take the form: two submodules $[\mathcal{I}]$ and $[\mathcal{J}]$ in some Hilbert module $\mathcal{M}$ are equivalent if and only if the two ideals $\mathcal{I}$ and $\mathcal{J}$ are equal. More generally

Theorem 0.1. *Let $\mathcal{I}$, $\widetilde{\mathcal{I}}$ be any two polynomial ideals and $\mathcal{M}$, $\widetilde{\mathcal{M}}$ be two Hilbert modules of the form $[\mathcal{I}]$ and $[\widetilde{\mathcal{I}}]$ respectively. Assume that $\mathcal{M}$, $\widetilde{\mathcal{M}}$ are in $\mathfrak{B}_1(\Omega)$ and that the dimension of the zero set of these modules is at most $m-2$. Also, assume that every algebraic component of zero* *sets intersects* Ω. *If* $\mathcal{M}$ *and* $\widetilde{\mathcal{M}}$ *are equivalent, then* $\mathcal{I} = \widetilde{\mathcal{I}}$.

We give a short proof of this theorem using the sheaf theoretic model developed in this thesis and construct tractable invariants for Hilbert modules over $\mathbb{C}[\underline{z}]$.

Let $\mathcal{M}$ be a Hilbert module of holomorphic functions on a bounded open connected subset Ω of $\mathbb{C}^m$ possessing a reproducing kernel K. Assume that $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ is the singly generated ideal $\langle p \rangle$. Then the reproducing kernel $K_{[\mathcal{I}]}$ of $[\mathcal{I}]$ vanishes on the zero set $V(\mathcal{I})$ and the map $w \mapsto K_{[\mathcal{I}]}(\cdot, w)$ defines a holomorphic Hermitian line bundle on the open set $\Omega^*_{\mathcal{I}} = \{w \in \mathbb{C}^m : \bar{w} \in \Omega \setminus V(\mathcal{I})\}$ which naturally extends to all of Ω^*. As is well known, the curvature of this line bundle completely determines the equivalence class of the Hilbert module $[\mathcal{I}]$. However, if $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ is not a principal ideal, then the corresponding line bundle defined on $\Omega^*_{\mathcal{I}}$ no longer extends to all of Ω^*. For example, $H^2_0(\mathbb{D}^2)$ is in the Cowen-Douglas class $\mathrm{B}_1(\mathbb{D}^2 \setminus \{(0,0)\})$ but it does not belong to $\mathrm{B}_1(\mathbb{D}^2)$. Indeed, it was conjectured in [14] that the dimension of the joint kernel of the Hilbert module $[\mathcal{I}]$ at w is 1 for points w not in $V(\mathcal{I})$, otherwise it is the codimension of $V(\mathcal{I})$. Assuming that

(a) $\mathcal{I}$ is a principal ideal or

(b) w is a smooth point of $V(\mathcal{I})$,

Duan and Guo verify the validity of this conjecture in [17]. Furthermore, they show that if $m = 2$ and $\mathcal{I}$ is prime then the conjecture is valid.

To systematically study examples of submodules like $H^2_0(\mathbb{D}^2)$, or more generally a submodule $[\mathcal{I}]$ of a Hilbert module $\mathcal{M}$ in the Cowen-Douglas class $\mathrm{B}_1(\Omega)$, we make the following definition (cf. [6]).

Definition 0.2. Fix a bounded domain $\Omega \subseteq \mathbb{C}^m$. A Hilbert module $\mathcal{M} \subseteq \mathcal{O}(\Omega)$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the class $\mathfrak{B}_1(\Omega)$ if

(rk) it possess a reproducing kernel K (we don't rule out the possibility: $K(w,w) = 0$ for w in some closed subset X of Ω) and

(fin) the dimension of $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ is finite for all $w \in \Omega$.

For a Hilbert modules $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$ we have proved the following Lemma.

Lemma 0.3. *Suppose* $\mathcal{M}$ *is a Hilbert modules in* $\mathfrak{B}_1(\Omega)$ *which is of the form* $[\mathcal{I}]$ *for some polynomial ideal* $\mathcal{I}$. *Then* $\mathcal{M}$ *is in* $\mathrm{B}_1(\Omega)$ *if the ideal* $\mathcal{I}$ *is singly generated while if the cardinality of the minimal set of generators is not* 1, *then* $\mathcal{M}$ *is in* $\mathrm{B}_1(\Omega_{\mathcal{I}})$.

This ensures that to a Hilbert module in $\mathfrak{B}_1(\Omega)$ of the form $[\mathcal{I}]$, there corresponds a holomorphic Hermitian line bundle over $\Omega^*_{\mathcal{I}}$ defined by the joint kernel. However, since the map $w \mapsto \dim(\mathcal{M}/\mathfrak{m}_w\mathcal{M})$ is only upper semi-continuous (the jump locus, which is $V(\mathcal{I})$, is an analytic set), it is not always possible to extend the holomorphic Hermitian line bundle defined on $\Omega^*_{\mathcal{I}}$ to all of Ω^*.

Refining the correspondence of locally free sheaf of modules over the analytic sheaf $\mathcal{O}(\Omega)$ on Ω with holomorphic vector bundles on Ω (cf. [30]), we construct a coherent analytic sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ which reflects a number of properties of the Hilbert module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega)$. Let $\mathcal{O}_w$ denotes the germs of holomorphic function at the point $w \in \mathbb{C}^m$. The sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ whose stalk at $w \in \Omega$ is

$$\big\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\big\},$$

or equivalently,

$$\mathcal{S}^{\mathcal{M}}(U) = \Big\{\sum_{i=1}^{n}\big(f_{i|U}\big)g_i : f_i \in \mathcal{M}, g_i \in \mathcal{O}(U)\Big\}$$

for U open in Ω.

Lemma 0.4. *For a Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, the sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is coherent.*

In the paper [6], we isolate circumstances when the sheaf $\mathcal{S}^{\mathcal{M}}$ agrees with a very useful but somewhat different sheaf model described in [18, Chapter 4].

It is well known that if the ideal $\mathcal{I}$ is principal, say $< p >$, then the reproducing kernel $K_{[\mathcal{I}]}$ factors as $K_{[\mathcal{I}]}(z,w) = p(z)\chi(z,w)\overline{p(w)}$ where $\chi(w,w) \neq 0$ for $w \in \Omega$. However if the ideal $\mathcal{I}$ is not principal, then no such factorization is possible. Nevertheless, using the Lemma 0.4, it is possible to give a description of the reproducing kernel K in terms of the generators of the stalk $\mathcal{S}^{\mathcal{M}}_w$. For any fixed point w_0 in Ω, we find a neighborhood Ω_0 of w_0 such that the reproducing kernel K for $\mathcal{M} \in \mathfrak{B}_1(\Omega)$, admits a useful decomposition described precisely in the following theorem.

Theorem 0.5. *Suppose g^0_i, $1 \leq i \leq d$, be a minimal set of generators for the stalk $\mathcal{S}^{\mathcal{M}}_{w_0}$. Then*

(i) there exists an open neighborhood Ω_0 of w_0 such that

$$K(\cdot, w) := K_w = \overline{g^0_1(w)}K^{(1)}_w + \cdots + \overline{g^0_d(w)}K^{(d)}_w,\ w \in \Omega_0$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathcal{M}$,

(ii) the vectors $K^{(i)}_w$, $1 \leq i \leq d$, are linearly independent in $\mathcal{M}$ for w in some neighborhood of w_0,

(iii) the vectors $\{K^{(i)}_{w_0} \mid 1 \leq i \leq d\}$ are uniquely determined by these generators $g^0_1, \ldots, g^0_d$,

(iv) the linear span of the set of vectors $\{K^{(i)}_{w_0} \mid 1 \leq i \leq d\}$ in $\mathcal{M}$ is independent of the generators $g^0_1, \ldots, g^0_d$, and

(v) $M^*_p K^{(i)}_{w_0} = \overline{p(w_0)}K^{(i)}_{w_0}$ for all i, $1 \leq i \leq d$, where M_p denotes the module multiplication by the polynomial p.

It is evident from the part (v) of Theorem 0.5 that the dimension of the joint kernel of the adjoint of the multiplication operator $D_{\mathbf{M}^*}$ at a point w_0 is greater or equal to the number of minimal generators of the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$ at $w_0 \in \Omega$, that is,

$$\dim \mathcal{M}/(\mathfrak{m}_{w_0}\mathcal{M}) \geq \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}. \tag{0.0.1}$$

It would be interesting to produce a Hilbert module $\mathcal{M}$ for which the inequality of (0.0.1) is strict. We identify several classes of Hilbert modules for which equality is forced in (0.0.1).

Definition 0.6. A Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be an *analytic Hilbert module* (cf. [7]) if we assume that

(rk) it consists of holomorphic functions on a bounded domain $\Omega \subseteq \mathbb{C}^m$ and possesses a reproducing kernel K,

(dense) the polynomial ring $\mathbb{C}[\underline{z}]$ is dense in it,

(vp) the set of virtual points which is $\{w \in \mathbb{C}^m : p \mapsto p(w),\ p \in \mathbb{C}[\underline{z}],$ extends continuously to $\mathcal{M}\}$ equals Ω.

We apply Lemma 0.3 to analytic Hilbert modules, which are singly generated by the constant function 1, to conclude that they must be in the class $\mathrm{B}_1(\Omega^*)$, where Ω is the set of virtual points of $\mathcal{H}$. Evidently, in this case, we have equality in (0.0.1). However, we have equality in many more cases.

Proposition 0.7. *Let $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$, where $\mathcal{I}$ is an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$. Then*

$$\dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} = \sharp\{minimal\ set\ of\ generators\ for\ \mathcal{S}_{w_0}^{\mathcal{M}}\} = \dim \mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M}.$$

More generally, consider the map $i_w : \mathcal{M} \longrightarrow \mathcal{M}_w$ defined by $f \mapsto f_w$, where f_w is the germ of the function f at w. Clearly, this map is a vector space isomorphism onto its image. The linear space $\mathcal{M}^{(w)} := \sum_{j=1}^m (z_j - w_j)\mathcal{M} = \mathfrak{m}_w\mathcal{M}$ is closed since $\mathcal{M}$ is assumed to be in $\mathfrak{B}_1(\Omega)$. Then the map $f \mapsto f_w$ restricted to $\mathcal{M}^{(w)}$ is a linear isomorphism from $\mathcal{M}^{(w)}$ to $(\mathcal{M}^{(w)})_w$. Consider

$$\mathcal{M} \xrightarrow{i_w} \mathcal{S}_w^{\mathcal{M}} \xrightarrow{\pi} \mathcal{S}_w^{\mathcal{M}}/\mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}},$$

where π is the quotient map. Now we have a map $\psi : \mathcal{M}_w/(\mathcal{M}^{(w)})_w \longrightarrow \mathcal{S}_w^{\mathcal{M}}/\{\mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}}\}$ which is well defined because $(\mathcal{M}^{(w)})_w \subseteq \mathcal{M}_w \cap \mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}}$. The question of equality in (0.0.1) is same as the question of whether the map ψ is an isomorphism and can be interpreted as a global factorization problem. To be more specific, we say that the module $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ possesses *Gleason's property at a point* $w_0 \in \Omega$ if for every element $f \in \mathcal{M}$ vanishing at w_0 there are $f_1, ..., f_m \in \mathcal{M}$ such that $f = \sum_{i=1}^m (z_i - w_{0i})f_i$. We further assume here $\mathcal{M}$ is a AF-cosubmodule (cf. [7, page - 38]).

Proposition 0.8. *Any AF-cosubmodule $\mathcal{M}$ has Gleason's property at w_0 if and only if*

$$\dim \mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M} = \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

Proposition 0.9. *Let $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$ on a bounded domain Ω, where $\mathcal{I}$ is a polynomial ideal, each of whose algebraic component intersects Ω. Then*

$$\dim \mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M} = \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0},\ w_0 \in \Omega.$$

Corollary 0.10. *If $\mathcal{M}$ is a submodule of an analytic Hilbert module of finite co-dimension with the zero set $V(\mathcal{M}) \subset \Omega$, then the Gleason problem is solvable for $\mathcal{M}$.*

Corollary 0.11. *Suppose $\mathcal{M}$ is a submodule of an analytic Hilbert module given by closure of a polynomial ideal and $w_0 \in V(\mathcal{I})$ is a smooth point then,*

$$\dim \ker D_{(\mathbf{M}-w_0)^*} = \text{codimension of } \mathrm{V}(\mathcal{I}).$$

Next, we obtain invariants for those modules in $\mathfrak{B}_1(\Omega)$ for which equality holds in (0.0.1). Since $H^2_0(\mathbb{D}^2)$ is in $\mathrm{B}_1(\mathbb{D}^2 \setminus \{(0,0)\})$, the curvature of the associated Hermitian holomorphic line bundle is a complete invariant (cf. [8]). However explicit computation of the curvature, even in this simple case is difficult. An example is provided in the appendix (section 6.2). As was pointed out in [12], the dimension of $\ker D_{(\mathbf{M}-w_0)^*}$, $w_0 \in \mathbb{D}^2$ is an invariant of the module $H^2_0(\mathbb{D}^2)$. Therefore, it may not be desirable to exclude the point $(0,0)$ altogether in any attempt to study the module $H^2_0(\mathbb{D}^2)$. Fortunately, implicit in the proof of Theorem 2.2 in [11], there is a construction which makes it possible to write down invariants on all of $\mathbb{D}^2$. This theorem assumes only that the module multiplication has closed range as in Definition 0.2. Therefore, it plays a significant role in the study of the class of Hilbert modules $\mathfrak{B}_1(\Omega)$.

We also note, from the Theorem 0.5, that the map $\Gamma_K : \Omega^*_0 \to \mathrm{Gr}(\mathcal{M}, d)$ defined by $\Gamma_K(\bar{w}) = (K^{(1)}_w, \ldots, K^{(d)}_w)$ is holomorphic. The pull-back of the canonical bundle on $\mathrm{G}r(\mathcal{M}, d)$ under Γ_K defines a holomorphic Hermitian vector bundle on the open set Ω^*_0. Unfortunately, the decomposition of the reproducing kernel given in Theorem above, is not canonical except when the stalk is singly generated. In this special case, the holomorphic Hermitian bundle obtained in this manner is indeed canonical. However, in general, it is not clear if this vector bundle contains any useful information. Suppose we have equality in (0.0.1) for a Hilbert module $\mathcal{M}$. Then it is possible to obtain a canonical decomposition following [11], which leads in the same manner as above, to the construction of a Hermitian holomorphic vector bundle in a neighborhood of each point $w \in \Omega$.

For any fixed but arbitrary $w_0 \in \Omega$ and a small enough neighborhood Ω_0 of w_0, the proof of Theorem 2.2 from [11] shows the existence of a holomorphic function $P_{\bar{w}_0} : \Omega^*_0 \to \mathcal{B}(\mathcal{M})$ with the property that the operator $P_{\bar{w}_0}$ restricted to the subspace $\ker D_{(\mathbf{M}-w_0)^*}$ is invertible. The range of $P_{\bar{w}_0}$ can then be seen to be equal to the kernel of the operator $\mathbb{P}_0 D_{(\mathbf{M}-w)^*}$, where $\mathbb{P}_0$ is the orthogonal projection onto $\mathrm{ran} D_{(\mathbf{M}-w_0)^*}$.

Lemma 0.12. *The dimension of* $\ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$ *is constant in a suitably small neighborhood* Ω_0 *of* $w_0 \in \Omega$.

Let $\{e_0, \ldots, e_k\}$ be a basis for $\ker D_{(\mathbf{M}-w_0)^*}$. Since $P_{\bar{w}_0}$ is holomorphic on Ω_0^*, it follows that $\gamma_1(\bar{w}) := P_{\bar{w}_0}(\bar{w})e_1, \ldots, \gamma_k(\bar{w}) := P_{\bar{w}_0}(\bar{w})e_k$ are holomorphic on Ω_0^*. Thus from Lemma 0.8, $\Gamma : \Omega_0^* \to \mathrm{Gr}(\mathcal{M}, k)$, given by $\Gamma(\bar{w}) = \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$, $w \in \Omega_0$, defines a holomorphic Hermitian vector bundle $\mathcal{P}_0$ on Ω_0^* of rank k corresponding to the Hilbert module $\mathcal{M}$.

Theorem 0.13. *If any two Hilbert modules* $\mathcal{M}$ *and* $\widetilde{\mathcal{M}}$ *belonging to the class* $\mathfrak{B}_1(\Omega)$ *are isomorphic via an unitary module map, then the corresponding vector bundles* $\mathcal{P}_0$ *and* $\widetilde{\mathcal{P}}_0$ *on* Ω_0^* *are equivalent as holomorphic Hermitian vector bundles.*

So the theorem above says that the equivalence class of the corresponding vector bundle P_0 obtained from this canonical decomposition is an invariant for the isomorphism class of the Hilbert module $\mathcal{M}$. These invariants, are by no means easy to compute either. We give computation of these invariants for the submodule $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ consisting of function vanishing at the origin of the weighted Bergman module $H^{(\lambda,\mu)}(\mathbb{D}^2)$ determined by the reproducing kernel

$$K^{(\lambda,\mu)}(z,w) = \frac{1}{(1 - z_1\bar{w}_1)^{\lambda}(1 - z_2\bar{w}_2)^{\mu}}, \; z, w \in \mathbb{D}^2.$$

It is therefore desirable to construct invariants which are more easily computable. In this context, we show that the holomorphic Hermitian line bundle on $\Omega_{\mathcal{I}}^*$ extends to a holomorphic Hermitian line bundle $\mathcal{L}(\mathcal{M})$ on the "blow-up" space $\hat{\Omega}^*$ via the monoidal transform under mild hypothesis on the zero set $V(\mathcal{I})$. We also show that this line bundle determines the equivalence class of the module $[\mathcal{I}]$ and therefore its curvature is a complete invariant.

Theorem 0.14. *Let* $\mathcal{M} \subseteq \mathfrak{B}_1(\Omega)$ *and* $\widetilde{\mathcal{M}} \subseteq \mathfrak{B}_1(\Omega)$ *be two Hilbert modules of the form* $[\mathcal{I}]$ *and* $[\widetilde{\mathcal{I}}]$*, respectively, where* $\mathcal{I}, \widetilde{\mathcal{I}}$ *are polynomial ideals. Assume that the dimension of the zero set of these modules is at most* $m-2$*. Then* $\mathcal{M}$ *and* $\widetilde{\mathcal{M}}$ *are equivalent if and only if the line bundles* $\mathcal{L}(\mathcal{M})$ *and* $\mathcal{L}(\widetilde{\mathcal{M}})$ *are equivalent as Hermitian holomorphic line bundle on* $\widehat{\Delta}(w_0; r)^*$.

However, computing it explicitly on all of $\hat{\Omega}^*$ is difficult again. However if we restrict the line bundle on $\hat{\Omega}^*$ to the exceptional subset of $\hat{\Omega}^*$, then the curvature invariant is easy to compute. We have calculated these invariant for a class of submodules of weighted Begman module $\mathcal{A}_{\alpha,\beta,\gamma}(\mathbb{B}^2)$ on the unit ball of $\mathbb{C}^2$, appeared in [26]. Also one can use the quadratic transform to calculate the curvature invariant in the same way as above. Finally, we calculate these invariants for a class of subspace of the weighted Bergman module $H^{(\lambda,\mu)}(\mathbb{D}^2)$. We show, using quadratic transform, that for fixed $n \in \mathbb{N}$, the submodules

$$\{[\mathcal{I}_k] \subset H^2(\mathbb{D}^2) : \mathcal{I}_k =< z_1^n, z_1^k z_2^{n-k} >, 1 \le k \le n\}$$

of the Hardy module $H^2(\mathbb{D}^2)$ are equivalent if and only if $k = k'$.

A line bundle is completely determined by its sections on open subsets. To write down the sections, we use the decomposition theorem for the reproducing kernel [6, Theorem 1.5]. The actual computation of the curvature invariant require the explicit calculation of norm of these sections. Thus it is essential to obtain a concrete description of the eigenvectors $K^{(i)}$, $1 \leq i \leq d$, in terms of the reproducing kernel. We give two examples which, we hope, will motivate the results that follow. Let $H^2(\mathbb{D}^2)$ be the Hardy module over the bi-disc algebra. The reproducing kernel for $H^2(\mathbb{D}^2)$ is the Sz̈ego kernel $\mathbb{S}(z,w) = \frac{1}{1-z_1\bar{w}_2}\frac{1}{1-z_2\bar{w}_2}$. Let $\mathcal{I}_0$ be the polynomial ideal $\langle z_1, z_2\rangle$ and let $[\mathcal{I}_0]$ denote the minimal closed submodule of the Hardy module $H^2(\mathbb{D}^2)$ containing $\mathcal{I}_0$. Then the joint kernel of the adjoint of the multiplication operators M_1 and M_2 is spanned by the two linearly independent vectors: $z_1 = p_1(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$ and $z_2 = p_2(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$, where p_1, p_2 are the generators of the ideal $\mathcal{I}_0$. For a second example, take the ideal $\mathcal{I}_1 = \langle z_1 - z_2, z_2^2\rangle$ and let $[\mathcal{I}_1]$ be the minimal closed submodule of the Hardy module $H_0^2(\mathbb{D}^2)$ containing $\mathcal{I}_1$. The joint kernel is not hard to compute. A set of two linearly independent vectors which span it are $p_1(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$ and $p_2(\bar{\partial}_1, \bar{\partial}_2)\mathbb{S}(z,w)_{|w_1=0=w_2}$, where $p_1 = z_1 - z_2$ and $p_2 = (z_1+z_2)^2$. Unlike the first example, the two polynomials p_1, p_2 are not the generators for the ideal $\mathcal{I}_1$ that were given at the start, never the less, they are easily seen to be a set of generators for the ideal $\mathcal{I}_1$ as well. This prompts the question:

Question: Let $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ be a Hilbert module and $\mathcal{I} \subseteq \mathcal{M}$ be a polynomial ideal. Assume without loss of generality that $0 \in V(\mathcal{I})$. We ask

1. if there exists a set of polynomials $p_1, \ldots, p_t$ such that

$$p_i(\tfrac{\partial}{\partial \bar{w}_1}, \ldots, \tfrac{\partial}{\partial \bar{w}_m})K_{[\mathcal{I}]}(z,w)|_{w=0},\ i = 1, \ldots, t,$$

spans the joint kernel $\{\gamma_w : (M_p - p(w))^*\gamma_w = 0, p \in \mathbb{C}[z]\}$ of $[\mathcal{I}]$;

2. what conditions, if any, will ensure that the polynomials $p_1, \ldots, p_t$, as above, is a generating set for $\mathcal{I}$?

We show that the answer to the Question (1) is affirmative, that is, there is a natural basis for the joint eigenspace of the Hilbert module $[\mathcal{I}]$, which is obtained by applying a differential operator to the reproducing kernel $K_{[\mathcal{I}]}$ of the Hilbert module $[\mathcal{I}]$. To facilitate this description, we make the following definition. For $w_0 \in \Omega$, let

$$\mathbb{V}_{w_0}(\mathcal{I}) := \{q \in \mathbb{C}[\underline{z}] : q(D)p|_{w_0} = 0 \text{ for all } p \in \mathcal{I}\}$$

and let

$$\widetilde{\mathbb{V}}_{w_0}(\mathcal{I}) := \{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_{w_0}(\mathcal{I}),\ 1 \leq i \leq m\}.$$

Lemma 0.15. *Fix $w_0 \in \Omega$ and polynomials $q_1, \ldots, q_t$. Let $\mathcal{I}$ be a polynomial ideal and K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then the vectors*

$$q_1^*(\bar{D})K(\cdot, w)|_{w=w_0}, \ldots, q_t^*(\bar{D})K(\cdot, w)|_{w=w_0}$$

form a basis of the joint kernel at w_0 of the adjoint of the multiplication operator if and only if the classes $[q_1], \ldots, [q_t]$ form a basis of $\widetilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$.

Often, these differential operators encode an algorithm for producing a set of generators for the ideal $\mathcal{I}$ with additional properties. It is shown that there is an affirmative answer to the Question (2) as well, if the ideal is assumed to be homogeneous.

Theorem 0.16. *Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be a homogeneous ideal and $\{p_1, \ldots, p_v\}$ be a minimal set of generators for $\mathcal{I}$ consisting of homogeneous polynomials. Let K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then there exists a set of generators $q_1, ..., q_v$ for the ideal $\mathcal{I}$ such that the set $\{q_i(\bar{D})K(\cdot, w)|_{w=0} : 1 \leq i \leq v\}$ is a basis for* $\ker D_{\mathbf{M}^*}$.

It then follows that if there were two sets of generators which serve to describe the joint kernel, as above, then these generators must be linear combinations of each other, that is, the sets of generators are determined modulo a linear transformation. We call such a generating set, *a canonical set of generators.* The canonical generators provide an effective tool to determine if two ideal are equal. A number of examples illustrating this phenomenon is given. For instance, consider the ideals $\mathcal{I}_1 :=< z_1, z_2^2 >$ and $\mathcal{I}_2 :=< z_1 - z_2, z_2^2 >$. They are easily seen to be distinct: A canonical set of generators for $\mathcal{I}_1$ is $\{z_1, z_2^2\}$ while for $\mathcal{I}_1$ it is $\{z_1 - z_2, (z_1 + z_2)^2\}$. A brief description of the chapters in this thesis follows.

In the Chapter Preliminaries, we recall the notion of a reproducing kernel and a functional Hilbert space. Following [8] and [11], we show that operators in Cowen- Douglas class can be realized as the adjoint of the multiplication operator defined by the co-ordinate functions. These operators then define a natural action of the polynomial ring $\mathbb{C}[\underline{z}]$ on the Hilbert space, making it a "Hilbert module". These Hilbert modules are semi-Fredholm but they also possess an additional property, namely the dimension of $\mathcal{H}/\mathfrak{m}_w\mathcal{H}$ is constant for w in some open set. We point out that in many natural example this additional property is absent making a case for study of semi-Fredholm Hilbert modules.

In Chapter 2, we develop the sheaf model for a Hilbert module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega)$. We prove the decomposition theorem (Theorem 0.5). A relationship between the joint kernel $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ and the stalk $\mathcal{S}_w^{\mathcal{M}}$ is established. We solve the Gleason problem for an analytic Hilbert module (Proposition 0.8 and Corollary 0.10). An alternative proof of the rigidity theorem is given, again, using the sheaf model (Theorem 0.1).

Chapter 3 provides a canonical decomposition for the reproducing kernel using [11, Theorem 2.2]. We show that the canonical decomposition guarantees the existence of a vector bundle of rank r (r possibly > 1). We extract invariants for the Hilbert module from this vector bundle (Theorem 0.13). An explicit calculation of these invariants for a submodule of weighted Bergman modules is given at the end of this chapter.

We address the questions (1) and (2) in Chapter 4 and prove Theorem 0.16. In this chapter, the notion of canonical generators is introduced and several explicit examples are given.

In Chapter 5, we use the familiar technique of 'resolution of singularities' to construct the blow-up space of Ω along an ideal $\mathcal{I}$. Applying the monoidal transform, we construct a Hermitian holomorphic line bundle on the blow-up space and prove Theorem 0.14. We also describe the construction of a Hermitian holomorphic line bundle using the quadratic transform. We have given various examples which illustrate the utility of some of these results.

Most of the results in Chapters 2 and 3 are from [6] and those in Chapters 4 and 5 are from [5].

1. Preliminaries

In this chapter, first recall the definition of the Cowen- Douglas class of operators and then recast this definition in the language of Hilbert modules over the polynomial ring $\mathbb{C}[\underline{z}]$. We discuss the notion of a reproducing kernel and the important role it plays in the study of Hilbert modules over polynomial rings. Beyond the Hilbert modules defined by the action of adjoint of a commuting tuple of operators in the Cowen-Douglas class, which have been studied vigorously over the last two three decades, lies the semi-Fredholm modules. Submodules of Analytic Hilbert modules provide large class of examples of semi-Fredholm Hilbert module. Following Chen and Guo [7], we discuss the characteristic space of a polynomial ideal. We record a number of of well known results on polynomial ideals which are used frequently in this thesis.

1.1 The reproducing kernel

Let Ω be an open connected subset of $\mathbb{C}^m$. Also let $\mathcal{M}_n(\mathbb{C})$ denotes the vector space of all $n \times n$ complex matrices and $\langle\ ,\ \rangle_{\mathbb{C}^n}$ be the standard inner product in $\mathbb{C}^n$ (though we will mention it only when it is not clear from the context or to distinguish from other inner products).

Definition 1.1. A function $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ holomorphic in the first and anti-holomorphic in the second variable, satisfying

$$\sum_{i,j=1}^{p} \langle K(w^{(i)}, w^{(j)})\zeta_j, \zeta_i\rangle \geq 0,\ w^{(1)}, \ldots, w^{(p)} \in \Omega,\ \zeta_1, \ldots, \zeta_p \in \mathbb{C}^n,\ p \geq 1 \tag{1.1.1}$$

is said to be a *non negative definite kernel* on Ω.

Given a non negative definite kernel K, let $\mathcal{H}^0$ be the linear span of all vectors from the set

$$S := \{K(\cdot, w)\zeta,\ w \in \Omega,\ \zeta \in \mathbb{C}^n\}.$$

Define an inner product between two of the vectors from the set S by setting

$$\langle K(\cdot, w)\zeta, K(\cdot, w')\eta\rangle = \langle K(w', w)\zeta, \eta\rangle_{\mathbb{C}^n},\ \text{for } \zeta, \eta \in \mathbb{C}^n,\ \text{and } w, w' \in \Omega, \tag{1.1.2}$$

and extend it to the linear space $\mathcal{H}^0$. The completion $\mathcal{H}$ of the inner product space $\mathcal{H}^0$ is a Hilbert space. It is evident that it has the reproducing property, namely,

$$\langle f(w), \zeta\rangle_{\mathbb{C}^n} = \langle f, K(\cdot, w)\zeta\rangle_{\mathcal{H}},\ w \in \Omega,\ \zeta \in \mathbb{C}^n,\ f \in \mathcal{H}. \tag{1.1.3}$$

Remark 1.2. Although, in the definition of the kernel K, it is merely required to be non negative definite, the equation (1.1.2) defines a *positive definite* sesqui-linear form as is easy to see: $|\langle f(w), \zeta\rangle| = |\langle f, K(\cdot, w)\zeta\rangle|$ which is at most $\|f\|\langle K(w,w)\zeta,\zeta\rangle^{1/2}$ by the Cauchy - Schwarz inequality. It follows that if $\|f\|^2 = 0$ then $f = 0$. Another application of the Cauchy-Schwarz inequality shows that the linear transformation $e_w : \mathcal{H} \to \mathbb{C}^n$, defined by $e_w(f) = f(w)$, is bounded for all $w \in \Omega$, $f \in \mathcal{H}$, that is,

$$|e_w(f)| = |\sum_{i=1}^{n} \langle f(w), e_i\rangle e_i| \leq \sum_{i=1}^{n} |\langle f(w), e_i\rangle| \|e_i\| \leq \|f\|(\sum_{i=1}^{n} \langle K(w,w)e_i, e_i\rangle^{1/2}),$$

$e_i = (0, .., 1, .., 0) \in \mathbb{C}^n$ with 1 in the i-th co-ordinate.

Conversely, let $\mathcal{H}$ be a Hilbert space of holomorphic functions on Ω taking values in $\mathbb{C}^n$. If the linear transformation $e_w : \mathcal{H} \to \mathbb{C}^n$ of evaluation at w is bounded for all $w \in \Omega$. Then e_w admits a bounded adjoint $e_w^* : \mathbb{C}^n \to \mathcal{H}$ such that $\langle e_w(f), \zeta\rangle_{\mathbb{C}^n} = \langle f, e_w^*\zeta\rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$ and $\zeta \in \mathbb{C}^n$. A function f in $\mathcal{H}$ is then orthogonal to $e_w^*(\mathbb{C}^n)$ if and only if $f = 0$. Thus $f = \sum_{i=1}^{p} e_{w^{(i)}}^* \zeta_i$ with $w^{(1)}, \ldots, w^{(p)} \in \Omega,\ \zeta_1, \ldots, \zeta_p \in \mathbb{C}^n$, $p > 0$, form a dense set in $\mathcal{H}$. Therefore we have

$$\|f\|^2 = \sum_{i,j=1}^{p} \langle e_{w^{(i)}} e_{w^{(j)}}^* \zeta_j, \zeta_i\rangle,$$

where $f = \sum_{i=1}^{p} e_{w^{(i)}}^* \zeta_i,\ w^{(i)} \in \Omega$ and $\zeta_i \in \mathbb{C}^n$ for $1 \leq i \leq p$. Since $\|f\|^2 \geq 0$, it follows that the kernel $K(z,w) = e_z e_w^*$ is non-negative definite as in (1.1.1). Clearly, $K(\cdot, w)\zeta$ is in $\mathcal{H}$ for each $w \in \Omega$ and $\zeta \in \mathbb{C}^n$ and that it has the reproducing property (1.1.3). It is not hard to see that such a kernel is uniquely determined.

A Hilbert space of holomorphic functions on some bounded domain $\Omega \subseteq \mathbb{C}^m$ will be called a *reproducing kernel Hilbert space* if the evaluation e_w at w is bounded for w in some open subset of Ω. Thus if K is the reproducing kernel for some Hilbert space $\mathcal{H}$, then $\mathcal{H} = \overline{\text{span}}\{K(\cdot, w)\zeta : w \in \Omega, \zeta \in \mathbb{C}^n\}$.

There is a useful alternative description of the reproducing kernel K in terms of the orthonormal basis $\{e_k : k \geq 0\}$ of the Hilbert space $\mathcal{H}$. We think of the vector $e_k(w) \in \mathbb{C}^n$ as a column vector for a fixed $w \in \Omega$ and let $e_k(w)^*$ be the row vector $(\overline{e_k^1(w)}, \ldots, \overline{e_k^n(w)})$. We see that

$$\begin{aligned}\langle K(z,w)\zeta, \eta\rangle &= \langle K(\cdot, w)\zeta, K(\cdot, z)\eta\rangle \;=\; \langle \sum_{j=0}^{\infty} \langle K(\cdot, w)\zeta, e_j\rangle e_j, \sum_{k=0}^{\infty} \langle K(\cdot, z)\eta, e_k\rangle e_k\rangle \\ &= \sum_{k=0}^{\infty} \langle K(\cdot, w)\zeta, e_k\rangle \langle K(\cdot, z)\eta, e_k\rangle \;=\; \sum_{k=0}^{\infty} \overline{\langle e_k(w), \zeta\rangle} \langle e_k(z), \eta\rangle \\ &= \sum_{k=0}^{\infty} \langle e_k(z) e_k(w)^* \zeta, \eta\rangle\end{aligned}$$

for any pair of vectors $\zeta, \eta \in \mathbb{C}^n$. Therefore, we have the following very useful representation for the reproducing kernel K:

$$K(z,w)=\sum_{k=0}^{\infty}e_k(z)e_k(w)^*, \tag{1.1.4}$$

where $\{e_k : k \geq 0\}$ is any orthonormal basis in $\mathcal{H}$.

Differentiating (1.1.3), we also obtain the following extension of the reproducing property:

$$\langle(\partial_i^j f)(w),\eta\rangle=\langle f,\bar{\partial}_i^j K(\cdot,w)\eta\rangle \text{ for } 1\leq i\leq m,\ \ j\geq 0,\ w\in\Omega,\ \eta\in\mathbb{C}^k,\ f\in\mathcal{H}. \tag{1.1.5}$$

Familiar examples of reproducing kernel Hilbert spaces are the Hardy and the Bergman spaces over the Euclidean ball and the polydisc. A detailed discussion of reproducing kernel can be found in [3].

1.2 The Cowen-Douglas class

Let $\mathbf{T}=(T_1,\ldots,T_m)$ be an m-tuple of commuting bounded linear operator on a separable complex Hilbert space $\mathcal{H}$. The operator $D_{\mathbf{T}}:\mathcal{H}\to\mathcal{H}\oplus\ldots\oplus\mathcal{H}$ is defined by $D_T(x)=(T_1x,\ldots,T_mx)$, $x\in\mathcal{H}$. Let Ω be a bounded domain in $\mathbb{C}^n$. For $w=(w_1,\ldots,w_m)\in\Omega$, let $\mathbf{T}-w$ denote the operator tuple $(T_1-w_1,\ldots,T_m-w_m)$. Note that $\ker D_{\mathbf{T}-w}=\cap_{j=1}^m\ker(T_j-w_j)$. Let k be positive integer

Definition 1.3. The m-tuple $\mathbf{T}$ is said to be in the Cowen-Douglas class $\mathrm{B}_k(\Omega)$ if

(1) ran $D_{\mathbf{T}-w}$ is closed for all $w\in\Omega$;

(2) span$\{\ker\ D_{\mathbf{T}-w}: w\in\Omega\}$ is dense in $\mathcal{H}$; and

(3) dim ker $D_{\mathbf{T}-w}=k$ for all $w\in\Omega$.

For a commuting tuple of operators $\mathbf{T}$ in $\mathrm{B}_k(\Omega)$, let

$$E_{\mathbf{T}}=\{(w,x)\in\Omega\times\mathcal{H}: x\in\ker\ D_{\mathbf{T}-w}\}$$

with $\pi(w,x)=w$ be the sub-bundle of the trivial bundle $\Omega\times\mathcal{H}$. For $\mathbf{T}\in\mathrm{B}_k(\Omega)$, we recall from [10] that the map $w\mapsto\ker\ D_{\mathbf{T}-w}$ defines a holomorphic Hermitian vector bundle $E_{\mathbf{T}}$ of rank k over Ω.

Theorem 1.4. *[8, Theorem 1.14] Two commuting tuples of operators $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ in $\mathrm{B}_k(\Omega)$ are unitarily equivalent if and only if the vector bundle $E_{\mathbf{T}}$ and $E_{\widetilde{\mathbf{T}}}$ are equivalent as holomorphic Hermitian vector bundle.*

Deciding when two holomorphic Hermitian vector bundles are equivalent is not an easy task except when the rank of these bundles are 1. In this case, the curvature

$$\mathcal{K}(\omega)=-\sum_{i,j=1}^{m}\frac{\partial^2\log\parallel\gamma(w)\parallel^2}{\partial w_i\partial\bar{w}_j}dw_i\wedge d\bar{w}_j,\ w=(w_1,\ldots,w_m)\in\Omega$$

of the line bundle E defined with respect to a non-zero holomorphic section γ is a complete invariant. (It is not hard to see that the definition of the curvature does not depend on the choice of the particular section γ: If γ_0 is another holomorphic section of E, then $\gamma_0 = \phi\gamma$ for some holomorphic function ϕ on Ω and the harmonicity of $\log|\phi|$ completes the verification.)

Thus Theorem 1.4 says that two commuting tuples of operators $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ in $\mathrm{B}_1(\Omega)$ are unitarily equivalent if and only if the curvature of the corresponding line bundles $E_{\mathbf{T}}$ and $E_{\tilde{\mathbf{T}}}$ are equal on some open subset of Ω. In general (Cf. [8] and [10]), the curvature of the bundle E_T along with a certain number of derivatives forms a complete set of unitary invariants for the operator T.

Every commuting m-tuple of operators in $\mathrm{B}_k(\Omega)$ can be realized as the m-tuple of the adjoint of multiplication by coordinate functions on a Hilbert space of holomorphic functions defined on an open subset of $\Omega^* = \{w \in \mathbb{C}^m : w \in \Omega\}$: Pick a holomorphic frame $\gamma_1(w), \ldots, \gamma_k(w)$ of the vector bundle $E_{\mathbf{T}}$ on some open subset Ω_0 of Ω. The map $\Gamma : \Omega_0 \to \mathcal{L}(\mathbb{C}^k, \mathcal{H})$ defined by the rule

$$\Gamma(w)\zeta = \sum_{i=0}^{k} \zeta_i \gamma_i(w),\ \zeta = (\zeta_1, \ldots, \zeta_k)$$

is holomorphic. Let $\mathcal{O}(\Omega_0^*, \mathbb{C}^k)$ be the algebra of holomorphic functions on Ω_0^* taking values in $\mathbb{C}^k$ and $U_\Gamma : \mathcal{H} \to \mathcal{O}(\Omega_0^*, \mathbb{C}^k)$ be the map defined by

$$(U_\Gamma f)(w) = \Gamma(\bar{w})^* f, \quad f \in \mathcal{H},\ w \in \Omega_0. \tag{1.2.1}$$

The map U_Γ is linear and injective. Therefore, it defines an inner product on $\mathcal{H}_\Gamma := \operatorname{ran} U_\Gamma$:

$$\langle U_\Gamma f, U_\Gamma g\rangle_\Gamma = \langle f, g\rangle,\ f, g \in \mathcal{H}.$$

Equipped with this inner product $\mathcal{H}_\Gamma$ consisting of $\mathbb{C}^k$-valued holomorphic functions on Ω_0^* becomes a Hilbert space. It is then shown in [11, Remarks 2.6] that

(a) $K(z, w) = \Gamma(\bar{z})^*\Gamma(\bar{w})$, $z, w \in \Omega_0^*$ is the reproducing kernel for the Hilbert space $\mathcal{H}_\Gamma$ and

(b) $M_i^* U_\Gamma = U_\Gamma T_i$, where $(M_i f)(z) = z_i f(z)$, $z = (z_1, \ldots, z_m) \in \Omega_0^*$.

The map $\mathbb{C}[\underline{z}] \times \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ defined by $(p, f) \mapsto p \cdot f$, $p \in \mathbb{C}[\underline{z}]$, $f \in \mathcal{H}_\Gamma$ is a module map. Here $p \cdot f$ is the function obtained by pointwise multiplication of the two functions p and f. Thus we think of $\mathcal{H}_\Gamma$ as a module over the polynomial ring.

Clearly, the representation of the commuting m-tuple $\mathbf{T}$ as the adjoint of the multiplication tuple $\mathbf{M} = (M_1, \ldots, M_m)$ on the space $\mathcal{H}_\Gamma$ depends on the initial choice of the frame γ. It is shown in [11] that there is a canonical choice for the Hilbert module $\mathcal{H}_\Gamma$, namely, one where one may assume that the kernel K is normalized.

Definition 1.5. A non negative definite kernel K is said to be normalized at w_0 if $K(z, w_0) = I$ for z in some open subset Ω_0^* of Ω^*.

Fix $w_0 \in \Omega^*$ and note that $K(z, w_0)$ is invertible for z in some neighborhood $\Delta_0^* \subseteq \Omega^*$ of w_0. Let K_{res} be the restriction of K to $\Delta_0^* \times \Delta_0^*$. Define a kernel function K_0 on Δ_0^* by

$$K_0(z, w) = \phi(z)K(z, w)\phi(w)^*, \ z, w \in \Delta_0^*, \tag{1.2.2}$$

where $\phi(z) = K_{\text{res}}(w_0, w_0)^{1/2}K_{\text{res}}(z, w_0)^{-1}$. Clearly the kernel K_0 is normalized at w_0. Let $\mathbf{M_0}$ denote the m-tuple of multiplication operators on the Hilbert space $\mathcal{H}$. It is not hard to establish the unitary equivalence of the two m - tuples $\mathbf{M}$ and $\mathbf{M_0}$ as in (cf. [11, Lemma 3.9 and Remark 3.8]). First, the restriction map res : $f \to f_{\text{res}}$, which restricts a function in $\mathcal{H}$ to Δ_0^* is a unitary map intertwining the m-tuple $\mathbf{M}$ on $\mathcal{H}$ with the m-tuple $\mathbf{M}$ on $\mathcal{H}_{\text{res}} = \text{ran res}$. The Hilbert space $\mathcal{H}_{\text{res}}$ is a reproducing kernel Hilbert space with reproducing kernel K_{res}. Second, suppose that the m-tuples $\mathbf{M}$ defined on two different reproducing kernel Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are in $\mathrm{B}_k(\Omega^*)$ and $X : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator intertwining these two operator tuples. Then X must map the joint kernel of one tuple in to the other, that is, $XK_1(\cdot, w)\xi = K_2(\cdot, w)\varphi(w)\xi$, $\xi \in \mathbb{C}^k$, for some function $\varphi : \Omega^* \to \mathbb{C}^{k\times k}$. Assuming that the kernel functions K_1 and K_2 are holomorphic in the first and anti-holomorphic in the second variable, it follows, again as in [11, pp. 472], that φ is anti-holomorphic. An easy calculation then shows that X^* is the multiplication operator M_{φ^*}, where $\varphi(w)^* = \overline{\varphi(w)}^{\text{tr}}$. If the two operator tuples are unitarily equivalent then there exists an unitary operator U intertwining them. Hence U^* must be of the form M_ψ for some holomorphic function ψ. Also, the operator U must map the kernel of $D_{(\mathbf{M}-w)^*}$ acting on $\mathcal{H}_1$ isometrically onto the kernel of $D_{(\mathbf{M}-w)^*}$ acting on $\mathcal{H}_2$ for all $w \in \Omega^*$. The unitarity of U is equivalent to the relation $K_1(\cdot, w)\xi = U^*K_2(\cdot, w)\psi(w)^*\xi$ for all $w \in \Omega$ and $\xi \in \mathbb{C}^k$. It then follows that

$$K_1(z, w) = \psi(z)K_2(z, w)\psi(w)^*, \tag{1.2.3}$$

where $\psi : \Omega^* \to \mathcal{GL}(\mathbb{C}^k)$ is some holomorphic function. Here, $\mathcal{GL}(\mathbb{C}^k)$ denotes the group of all invertible linear transformations on $\mathbb{C}^k$.

Conversely, if two kernels are related as in equation (1.2.3), then the corresponding tuples of multiplication operators are unitarily equivalent since

$$M_i^*K(\cdot, w)\zeta = \bar{w}_iK(\cdot, w)\zeta, \quad w \in \Omega, \ \zeta \in \mathbb{C}^k,$$

where $(M_if)(z) = z_if(z)$, $f \in \mathcal{H}$ for $1 \leq i \leq m$.

In general, the adjoint of the multiplication tuple $\mathbf{M}$ on a reproducing kernel Hilbert space need not be in the Cowen-Douglas class $\mathrm{B}_k(\Omega)$. However, one may impose additional conditions (cf. [11]) on K to ensure this. The normalized kernel K (modulo conjugation by a constant unitary on $\mathbb{C}^m$) then determines the unitary equivalence class of the multiplication tuple $\mathbf{M}$.

In conclusion, it is possible to answer a number of questions regarding the m-tuple of operators $\mathbf{T}$ using either the corresponding vector bundle or the normalized kernel. An elementary discussion on curvature invariant is given in appendix (section 6.1).

1.3 Hilbert modules over polynomial ring and semi-Fredholmness

The notion of a Hilbert module was formulated and studied in [15]. This was introduced to emphasize algebraic methods in the study of Hilbert space operators and more generally algebras of operators on Hilbert space.

Definition 1.6. A Hilbert module $\mathcal{H}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is a Hilbert space $\mathcal{H}$ together with a unital module multiplication $\mathbb{C}[\underline{z}] \times \mathcal{H} \to \mathcal{H}$ which is assumed to define a bounded operator for each p, that is, the map $M_p : \mathcal{H} \to \mathcal{H}$ defined by $h \mapsto p \cdot h$ is bounded for $p \in \mathbb{C}[\underline{z}]$.

We note that given a commuting m-tuple $(T_1, \ldots, T_m)$ on a Hilbert space $\mathcal{H}$, it can be naturally endowed with a module structure over the polynomial ring $\mathbb{C}[\underline{z}]$ by setting $p \cdot h = p(T_1, \ldots T_m)h$. We say two Hilbert modules $\mathcal{H}_1$ and $\mathcal{H}_2$ are *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines the module action, that is, $UM_p = M_pU$ for all $p \in \mathbb{C}[\underline{z}]$. Note for equivalence of two Hilbert modules, it is enough to check that $UM_{z_i} = M_{z_i}U$, $1 \leq i \leq m$.

If $\mathcal{H}$ is a Hilbert module over $\mathbb{C}[\underline{z}]$, then a set $\{h_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{H}$ is called a *generating set* for $\mathcal{H}$ if finite linear sum of the form

$$\sum_i p_i h_{\lambda_i},\ p_i \in \mathbb{C}[\underline{z}],\ \lambda_i \in \Lambda$$

are dense in $\mathcal{H}$.

Definition 1.7. If $\mathcal{H}$ is a Hilbert module over $\mathbb{C}[\underline{z}]$, then $\text{rank}_{\mathbb{C}[\underline{z}]}\mathcal{H}$, the rank of $\mathcal{H}$ over $\mathbb{C}[\underline{z}]$, is the minimum cardinality of a generating set for $\mathcal{H}$.

A Hilbert module $\mathcal{H}$ over $\mathbb{C}[\underline{z}]$ is said to be *finitely generated* if $\text{rank}_{\mathbb{C}[\underline{z}]}\mathcal{H} < \infty$.

Definition 1.8. A Hilbert module $\mathcal{H}$ is said to be semi-Fredholm at the point w if

$$\dim \mathcal{H}/\mathfrak{m}_w\mathcal{H} < \infty,$$

where $\mathfrak{m}_w$ is the maximal ideal of $\mathbb{C}[\underline{z}]$ at w.

We study the class of semi-Fredholm Hilbert modules which includes the finitely generated ones (see [15, page - 89]). In particular, any submodule of an analytic Hilbert module $\mathcal{M}$ of the form $[\mathcal{I}]$ for some ideal $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ is semi-Fredholm.

Recall that if $\mathfrak{m}_w\mathcal{H}$ has finite codimension then $\mathfrak{m}_w\mathcal{H}$ is a closed subspace of $\mathcal{H}$. A Hilbert module $\mathcal{H}$ semi-Fredholm on Ω if it is semi-Fredholm for every $w \in \Omega$.

Definition 1.9. Consider the semi-Fredholm modules for which the two conditions

(const) $\dim \mathcal{H}/\mathfrak{m}_w\mathcal{H} = n < \infty$ for all $w \in \Omega$;

(span) $\cap_{w \in \Omega}\mathfrak{m}_w\mathcal{H} = 0$, hold. We will say these Hilbert modules are in the Cowen-Douglas class $\mathrm{B}_n(\Omega)$. (The adjoint of the multiplication tuple defined on $\mathcal{H}$ is in $\mathrm{B}_n(\Omega^*)$.)

For any Hilbert module $\mathcal{H}$ in $\mathrm{B}_n(\Omega)$, the analytic localization $\mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H}$ is a locally free module when restricted to Ω, see [18] for details. Let us denote, in short,

$$\hat{\mathcal{H}} := \mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H}\big|_{\Omega},$$

and let $E_{\mathcal{H}} = \hat{\mathcal{H}}|_{\Omega}$ be the associated holomorphic vector bundle. Fix $w \in \Omega$. The minimal projective resolution of the maximal ideal at the point w is given by the Koszul complex $K_{\bullet}(z - w, \mathcal{H})$, where $K_p(z - w, \mathcal{H}) = \mathcal{H} \otimes \wedge^p(\mathbb{C}^m)$ and the connecting maps $\delta_p(w) : K_p \to K_{p-1}$ are defined, using the standard basis vectors e_i, $1 \le i \le m$ for $\mathbb{C}^m$, by

$$\delta_p(w)(fe_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p}(-1)^{j-1}(z_j - w_j) \cdot fe_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_p}.$$

Here, $z_i \cdot f$ is the module multiplication. In particular $\delta_1(w) : \mathcal{H} \oplus \ldots \oplus \mathcal{H} \to \mathcal{H}$ is defined by $(f_1, \ldots, f_m) \mapsto \sum_{j=1}^m (M_j - w_j)f_j$, where M_i is the operator $M_j : f \mapsto z_j \cdot f$, for $1 \le j \le m$ and $f \in \mathcal{H}$. The 0-th homology group of the complex, $H_0(K_{\bullet}(z - w, \mathcal{H}))$ is same as $\mathcal{H}/\mathfrak{m}_w\mathcal{H}$. For $w \in \Omega$, the map $\delta_1(w)$ induces a map localized at w,

$$K_1(z - w, \hat{\mathcal{H}}_w) \xrightarrow{\delta_{1w}(w)} K_0(z - w, \hat{\mathcal{H}}_w).$$

Then $\hat{\mathcal{H}}_w = \text{coker } \delta_{1w}(w)$ is a locally free $\mathcal{O}_w$ module and the fiber of the associated holomorphic vector bundle $E_{\mathcal{H}}$ is given by

$$E_{\mathcal{H},w} = \hat{\mathcal{H}}_w \otimes_{\mathcal{O}_w} \mathcal{O}_w/\mathfrak{m}_w\mathcal{O}_w.$$

We identify $E^*_{\mathcal{H},w}$ with $\ker\ \delta_1(w)^*$. Thus $E^*_{\mathcal{H}}$ is a Hermitian holomorphic vector bundle on $\Omega^* := \{\bar{z} : z \in \Omega\}$. Let $D_{\mathbf{M}^*}$ be the commuting m-tuple $({M_1}^*, \ldots, {M_m}^*)$ from $\mathcal{H}$ to $\mathcal{H} \oplus \ldots \oplus \mathcal{H}$. Clearly $\delta_1(w)^* = D_{(\mathbf{M}-w)^*}$ and $\ker\ \delta_1(w)^* = \ker D_{(\mathbf{M}-w)^*} = \cap_{j=1}^m \ker(M_j - w_j)^*$ for $w \in \Omega$.

Let $\mathrm{Gr}(\mathcal{H}, n)$ be the rank n Grassmanian on the Hilbert module $\mathcal{H}$. The map $\Gamma : \Omega^* \to \mathrm{Gr}(\mathcal{H}, n)$ defined by $\bar{w} \mapsto \ker D_{(\mathbf{M}-w)^*}$ is shown to be holomorphic in [8]. The pull-back of the canonical vector bundle on $\mathrm{Gr}(\mathcal{H}, n)$ under Γ is then the holomorphic Hermitian vector bundle $E^*_{\mathcal{H}}$ on the open set Ω^*. A restatement of Theorem 1.4 is that equivalent Hilbert modules correspond to equivalent vector bundles and vice-versa. Examples are the Hardy and the Bergman modules over the Euclidean ball and the poly-disc.

We recall, from section 1.2, that a Hilbert module in the Cowen-Douglass class $\mathrm{B}_1(\Omega)$ consists of

- a Hilbert space $\mathcal{H}$ of holomorphic functions on some bounded domain Ω_0 in $\mathbb{C}^m$,
- a reproducing kernel K for $\mathcal{H}$ on the Ω_0 for $\mathcal{H}$ which is non-degenerate, that is, $K(w, w) \neq 0$, $w \in \Omega_0$,

- the module multiplication is the pointwise multiplication.

For Hilbert modules as above, $E^*_{\mathcal{H}} \cong \mathcal{O}_{\Omega^*}$, that is, the associate holomorphic vector bundle is trivial, with $K_w := K(\cdot, w)$ as a non-vanishing global section. For modules in $\mathrm{B}_1(\Omega)$, the curvature of the vector bundle $E^*_{\mathcal{H}}$ is a complete invariant. However, in many natural examples of submodules of Hilbert modules from the class $\mathrm{B}_1(\Omega)$, the dimension of the joint kernel does not remain constant. Let us look at an example. Let $H^2(\mathbb{D}^2)$ be the Hardy space on the bi-disc. This may be thought of as a Hilbert space of holomorphic functions defined on $\mathbb{D}^2$ determined by the reproducing kernel

$$K(z,w) = (1 - z_1\overline{w}_1)^{-1}(1 - z_2\bar{w}_2)^{-1},\ z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{D}^2.$$

This follows from (1.1.4) as $\{{z_1}^i {z_2}^j\}_{i,j\geq 0}$ forms an orthonormal basis for $\mathcal{H}$. Let

$$H^2_0(\mathbb{D}^2) = \{f \in H^2(\mathbb{D}^2) : f(0,0) = 0\}$$

be the submodule of functions vanishing at the origin. Using (1.1.4), we see that the reproducing kernel K_0 for $H^2_0(\mathbb{D}^2)$ is

$$\begin{aligned} K_0(z,w) &= K(z,w) - 1 \\ &= (z_1\bar{w}_1 + z_2\bar{w}_2 - z_1z_2\bar{w}_1\bar{w}_2)K(z,w) \end{aligned}$$

where $z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{D}^2$. We have

$$\dim\ker D_{(\mathbf{M}-w)^*} = \begin{cases} 1 & \text{if } w \neq (0,0) \\ 2 & \text{if } w = (0,0). \end{cases} \tag{1.3.1}$$

Clearly, the map $\bar{w} \mapsto \ker D_{(\mathbf{M}-w)^*}$ is not holomorphic on all of $\mathbb{D}^2$ but only on $\mathbb{D}^2 \setminus \{(0,0)\}$. To extract invariants for Hilbert modules as above, we begin a systematic study of a class of submodules of kernel Hilbert modules (over the polynomial ring $\mathbb{C}[\underline{z}]$) which are semi-Fredholm on Ω.

Definition 1.10. A Hilbert module $\mathcal{M} \subseteq \mathcal{O}(\Omega)$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the class $\mathfrak{B}_1(\Omega)$ if

(rk) it possess a reproducing kernel K (we don't rule out the possibility: $K(w,w) = 0$ for w in some closed subset X of Ω) and

(fin) the dimension of $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ is finite for all $w \in \Omega$.

The following Lemma isolates a large class of elements from $\mathfrak{B}_1(\Omega)$ which belong to $\mathrm{B}_1(\Omega_0)$ for some open subset $\Omega_0 \subseteq \Omega$.

Lemma 1.11. *Suppose $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ is the closure of a polynomial ideal $\mathcal{I}$. Then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega)$ if the ideal $\mathcal{I}$ is principal while if $p_1, p_2, \ldots, p_t \,(t > 1)$ is a minimal set of generators for $\mathcal{I}$, then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega \setminus X)$ for $X = \cap_{i=1}^t \{z : p_i(z) = 0\} \cap \Omega$.*

Proof. The proof is a refinement of the argument given in [13, pp. 285]. Let γ_w be any eigenvector at w for the adjoint of the module multiplication, that is, $M_p^* \gamma_w = \overline{p(w)} \gamma_w$ for $p \in \mathbb{C}[\underline{z}]$.

First, assume that the module $\mathcal{M}$ is generated by the single polynomial, say p. In this case, $K(z, w) = p(z)\chi(z, w)\overline{p(w)}$ for some positive definite kernel χ on all of Ω. Set $K_1(z, w) = p(z)\chi(z, w)$ and note that $K_1(\cdot, w)$ is a non-zero eigenvector at $w \in \Omega$. We have

$$\langle pq, \gamma_w \rangle = \langle p, M_q^* \gamma_w \rangle = \langle p, \overline{q(w)} \gamma_w \rangle = q(w) \langle p, \gamma_w \rangle.$$

Also, we have

$$p(w)q(w)\langle p, \gamma_w \rangle = \langle pq, K(\cdot, w)\rangle \langle p, \gamma_w \rangle = p(w)\langle pq, \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w)\rangle.$$

The analytic function $q(w)\langle p, \gamma_w \rangle - \langle pq, \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w)\rangle$ on Ω is equal to 0 on $\Omega \setminus \{z : p(z) = 0\}$ and hence is 0 on Ω (as Ω is connected). Thus

$$\langle pq, \gamma_w \rangle = \langle pq, \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w)\rangle.$$

Since vectors of the form $\{pq : q \in \mathbb{C}[\underline{z}]\}$ are dense in $\mathcal{M}$, it follows that $\gamma_w = \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w)$ and the proof is complete in this case.

Now, assume that $p_1, \ldots, p_t$ is a set of generators for the ideal $\mathcal{I}$. Then for $w \notin X$, there exist a $k \in \{1, \ldots, t\}$ such that $p_k(w) \neq 0$. We note that for any i, $1 \leq i \leq m$,

$$p_k(w)\langle p_i, \gamma_w \rangle = \langle p_i, M_{p_k}^* \gamma_w \rangle = \langle p_i p_k, \gamma_w \rangle = \langle p_k, M_{p_i}^* \gamma_w \rangle = p_i(w)\langle p_k, \gamma_w \rangle.$$

Therefore we have

$$\begin{aligned} \langle \sum_{i=1}^t p_i q_i, \gamma_w \rangle &= \sum_{i=1}^t \langle p_i, M_{q_i}^* \gamma_w \rangle \\ &= \sum_{i=1}^t q_i(w) \langle p_i, \gamma_w \rangle \\ &= \sum_{i=1}^t \langle p_i q_i, \frac{\overline{\langle p_k, \gamma_w \rangle} K(\cdot, w)}{\overline{p_k(w)}} \rangle. \end{aligned}$$

Setting $c(w) = \frac{\langle p_k, \gamma_w \rangle}{p_k(w)}$, we have

$$\langle \sum_{i=1}^t p_i q_i, \gamma_w \rangle = \langle \sum_{i=1}^t p_i q_i, \overline{c(w)} K(\cdot, w)\rangle.$$

Since vectors of the form $\{\sum_{i=1}^t p_i q_i : q_i \in \mathbb{C}[\underline{z}], 1 \leq i \leq t\}$ are dense in $\mathcal{M}$, it follows that $\gamma_w = \overline{c(w)} K(\cdot, w)$ completing the proof of the second half. □

Note that the lemma given above only says what happens to the dimension of the joint kernel for points outside the zero set X. A complete formula for the dimension of the joint kernel (in some cases) is given in [17] which we reproduced below.

Theorem 1.12 (Duan-Guo). *Let $[\mathcal{I}]$ be a polynomialideal and $V(\mathcal{I})$ be the common zero of the ideal $\mathcal{I}$. Let $\mathcal{H}$ be an analytic Hilbert module over Ω. Suppose $\mathcal{H}_0$ is a submodule of $\mathcal{H}$ which is the completion of the ideal $\mathcal{I}$ in $\mathcal{H}$. Then assuming that the ideal $\mathcal{I}$ satisfies one of the following conditions*

(1) is singly generated

(2) is prime ideal of $\mathbb{C}[z_1, z_2]$

(3) is prime ideal of $\mathbb{C}[z_1, \dots, z_m]$, $m > 2$ and w is a smooth point of $V(\mathcal{I})$,

we have

$$\dim \cap_{i=1}^m \ker(M_j|_{\mathcal{H}_0} - w_j)^* = \begin{cases} 1 & \text{for } w \notin V(\mathcal{I}) \cap \Omega; \\ \text{codimension of } V(\mathcal{I}) & \text{for } w \in V(\mathcal{I}) \cap \Omega. \end{cases}$$

We note that $H_0^2(\mathbb{D}^2) = [\mathfrak{m}_0]$, where $\mathfrak{m}_0 =< z_1, z_2 >$, that is, the ideal generated by z_1 and z_2 in $\mathbb{C}[z_1, z_2]$. Consequently, the equation (1.3.1) follows from the theorem of Duan and Guo.

1.4 Some results on polynomial ideals and analytic Hilbert modules

Let $\alpha = (\alpha_1, \dots \alpha_m) \in (\mathbb{Z}_+)^m$ be a multi index and $z^\alpha = z_1^{\alpha_1} \dots z_m^{\alpha_m}$. For $q \in \mathbb{C}[\underline{z}]$ of the form $q(z) = \sum_\alpha a_\alpha z^\alpha$, let $q(D)$ denote the linear partial differential operator

$$q(D) \;=\; \sum_\alpha a_\alpha \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_m^{\alpha_m}}$$

where $|\alpha| = \sum_i \alpha_i$. For an ideal $\mathcal{I}$, the characteristic space at w is the linear space

$$\mathbb{V}_w(\mathcal{I}) = \{q \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0,\ p \in \mathcal{I}\}.$$

Here $q(D)p|_w = (q(D)p)(w)$. The following identity is easily verified:

$$q(D)(z_j f)|_w \;=\; w_j q(D)p|_w + \frac{\partial q}{\partial z_j}(D) f|_w,\ j = 1, \dots, m$$

for any analytic function f defined in a small neighborhood of w. The characteristic space $\mathbb{V}_w(\mathcal{I})$ is invariant under the action of the partial differential operators $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_m}\}$ and $\mathbb{V}_w(\mathcal{I}) \neq \{0\}$ if and only if $w \in V(\mathcal{I})$. The envelope, $\mathcal{I}_w^e$ of $\mathcal{I}$ at w is the ideal

$$\mathcal{I}_w^e := \{q \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0 \text{ for all } q \in \mathbb{V}_w(\mathcal{I})\}, \tag{1.4.1}$$

containing $\mathcal{I}$. Let $\mathcal{I} = \cap_{j=1}^{n}\mathcal{I}_j$ be an irredundant primary decomposition of the ideal $\mathcal{I}$. Thus each ideal is P_j-primary for some prime ideal P_j. The set $\{P_j : 1 \leq j \leq n\}$ is uniquely determined by $\mathcal{I}$ while the set $\{\mathcal{I}_j : 1 \leq j \leq n\}$ is not. Note that

$$V(\mathcal{I}) = \cap_{j=1}^{n} V(P_j).$$

For $1 \leq j \leq n$, the set $V(P_j)$ is called an algebraic component of $\mathcal{I}$.

Theorem 1.13. *[25, Corollary 2.2] Let Ω be a subset of $\mathbb{C}^m$. If each algebraic component of the ideal $\mathcal{I}$ intersects Ω, then*

$$\mathcal{I} = \cap_{w\in\Omega}\mathcal{I}_w^e.$$

For polynomial ideals $\mathcal{I}_1, \mathcal{I}_2$ satisfying $\mathcal{I}_1 \supseteq \mathcal{I}_2$, we note that $\mathcal{I}_{1w} \subseteq \mathcal{I}_{2w}$ for all $w \in \mathbb{C}^m$. Let

$$V(\mathcal{I}_2) \setminus V(\mathcal{I}_1) := \{w \in V(\mathcal{I}_2) : \mathcal{I}_{2w} \neq \mathcal{I}_{1w}\}.$$

Lemma 1.14. *[24, Corollary 2.5] If $\mathcal{I}_1, \mathcal{I}_2$ are two ideals in $\mathbb{C}[\underline{z}]$, $\mathcal{I}_1 \supseteq \mathcal{I}_2$, and $\dim \mathcal{I}_1/\mathcal{I}_2 < \infty$, then*

$$\dim \mathcal{I}_1/\mathcal{I}_2 = \sum_{w\in V(\mathcal{I}_2)\setminus V(\mathcal{I}_1)} \dim \mathcal{I}_{2w}/\mathcal{I}_{1w}.$$

We now state two important theorems about analytic Hilbert module. The first of these theorems is a generalization of the result of Ahern and Clark [2].

Theorem 1.15. *[16, Corollary 2.8] Let $\mathcal{H}$ be an analytic Hilbert module on a bounded domain Ω in $\mathbb{C}^m$. Then the maps $\mathcal{I} \mapsto [\mathcal{I}]$ and $\mathcal{M} \mapsto \mathcal{M} \cap \mathbb{C}[\underline{z}]$ define bijective correspondence between the ideal $\mathcal{I}$ of $\mathbb{C}[\underline{z}]$ with $V(\mathcal{I}) \subset \Omega$ and the submodule $\mathcal{M}$ of $\mathcal{H}$ of finite codimension.*

Theorem 1.16. *[24, Theorem 3.1] Le $\mathcal{H}$ be an analytic Hilbert module on the domain $\Omega \subseteq \mathbb{C}^m$, and $\mathcal{I}_1$, $\mathcal{I}_2$ be two polynomial ideal satisfying $\mathcal{I}_1 \supseteq \mathcal{I}_2$, and $V(\mathcal{I}_2) \setminus V(\mathcal{I}_1) \subset \Omega$. Let $[\mathcal{I}_1], [\mathcal{I}_2]$ be the closures of $\mathcal{I}_1, \mathcal{I}_2$ respectively in $\mathcal{H}$. Then*

$$\dim[\mathcal{I}_1]/[\mathcal{I}_2] \;=\; \dim \mathcal{I}_1/\mathcal{I}_2.$$

2. The sheaf model

In this chapter, we develop the sheaf model for a Hilbert module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega)$. We prove the decomposition theorem. A relationship between the joint kernel $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ and the stalk $\mathcal{S}_w^{\mathcal{M}}$ is established. We solve the Gleason problem for an analytic Hilbert module and its finite codimensional submodules. An alternative proof of the rigidity theorem is given.

2.1 The sheaf construction and decomposition theorem

Let us consider a Hilbert module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega)$ which is a submodule of some Hilbert module $\mathcal{H}$ in $\mathrm{B}_1(\Omega)$, possessing a nondegenerate reproducing kernel K. Clearly then we have the following module map

$$\mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{M} \longrightarrow \mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H} \cong \mathcal{O}_\Omega. \tag{2.1.1}$$

Let $\mathcal{S}^{\mathcal{M}}$ denotes the range of the composition map in the above equation. Then the stalk of $\mathcal{S}^{\mathcal{M}}$ at $w \in \Omega$ is given by $\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\}$

Motivated by the construction above and the analogy with the correspondence of a vector bundle with a locally free sheaf [30, page-40], we construct a sheaf $\mathcal{S}^{\mathcal{M}}$ for the Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[\underline{z}]$, in the class $\mathfrak{B}_1(\Omega)$. The sheaf $\mathcal{S}^{\mathcal{M}}$ is the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ whose stalk $\mathcal{S}_w^{\mathcal{M}}$ at $w \in \Omega$ is

$$\big\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\big\},$$

or equivalently,

$$\mathcal{S}^{\mathcal{M}}(U) = \Big\{\sum_{i=1}^{n}\big(f_{i|U}\big)g_i : f_i \in \mathcal{M}, g_i \in \mathcal{O}(U)\Big\}$$

for U open in Ω.

For any two Hilbert module $\mathcal{M}_1$ and $\mathcal{M}_2$ in the class $\mathfrak{B}_1(\Omega)$ and $L : \mathcal{M}_1 \to \mathcal{M}_2$ a module map between them, let $\mathcal{S}^L : \mathcal{S}^{\mathcal{M}_1}(V) \to \mathcal{S}^{\mathcal{M}_2}(V)$ be the map defined by

$$\mathcal{S}^L\sum_{i=1}^{n} f_i|_V g_i := \sum_{i=1}^{n} Lf_i|_V g_i, \text{ for } f_i \in \mathcal{M}_1, g_i \in \mathcal{O}(V),\ n \in \mathbb{N}.$$

The map $\mathcal{S}^L$ is well defined: if $\sum_{i=1}^n f_i|_V g_i = \sum_{i=1}^n \widetilde{f}_i|_V \widetilde{g}_i$, then $\sum_{i=1}^n Lf_i|_V g_i = \sum_{i=1}^n L\widetilde{f}_i|_V \widetilde{g}_i$. Suppose $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$ via the unitary module map L. Now, it is easy to verify that

$(\mathcal{S}^L)^{-1} = \mathcal{S}^{L^*}$. It then follows that $\mathcal{S}^{M_1}$ is isomorphic, as sheaves of modules over $\mathcal{O}(\Omega)$, to $\mathcal{S}^{M_2}$ via the map $\mathcal{S}^L$.

It is clear that if the Hilbert module $\mathcal{M}$ is in the class $\mathrm{B}_1(\Omega)$, then the sheaf $\mathcal{S}^{\mathcal{M}}$ is locally free. Also, if the Hilbert module is taken to be the maximal set of functions vanishing on an analytic hyper-surface $\mathcal{Z}$, then the sheaf $S^{\mathcal{M}}$ coincides with the ideal sheaf $\mathcal{I}_{\mathcal{Z}}(\Omega)$ and therefore it is coherent (cf.[22]). However, much more is true

Proposition 2.1. *For any Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, the sheaf $\mathcal{S}^{\mathcal{M}}$ is coherent.*

Proof. The sheaf $\mathcal{S}^{\mathcal{M}}$ is generated by the family $\{f : f \in \mathcal{M}\}$ of global sections of the sheaf $\mathcal{O}(\Omega)$. Let J be a finite subset of $\mathcal{M}$ and $\mathcal{S}_J^{\mathcal{M}} \subseteq \mathcal{O}(\Omega)$ be the subsheaf generated by the sections $f,\ f \in J$. It follows (see [23, Corollary 9, page. 130]) that $\mathcal{S}_J^{\mathcal{M}}$ is coherent. The family $\{\mathcal{S}_J^{\mathcal{M}} :$ $J$is a finite subset of$\mathcal{M}\}$ is increasingly filtered, that is, for any two finite subset I and J of $\mathcal{M}$, the union $I \cup J$ is again a finite subset of $\mathcal{M}$ and $\mathcal{S}_I^{\mathcal{M}} \cup \mathcal{S}_J^{\mathcal{M}} \subset \mathcal{S}_{I\cup J}^{\mathcal{M}}$. Also, clearly $\mathcal{S}^{\mathcal{M}} = \bigcup_J \mathcal{S}_J^{\mathcal{M}}$. Using Noether's lemma [22, page. 111] which says that every increasingly filtered family of coherent sheaves must be stationary, we conclude that the analytic sheaf $\mathcal{S}^{\mathcal{M}}$ is coherent. □

For $w \in \Omega$, the coherence of $\mathcal{S}^{\mathcal{M}}$ ensures the existence of $m, n \in \mathbb{N}$ and an open neighborhood U of w such that

$$(\mathcal{O}^m)_{|U} \to (\mathcal{O}^n)_{|U} \to (\mathcal{S}^{\mathcal{M}})_{|U} \to 0$$

is an exact sequence. Thus

$$\left\{\left(\mathcal{S}_w^{\mathcal{M}}/\mathfrak{m}_w \mathcal{S}_w^{\mathcal{M}}\right)^* : w \in \Omega\right\}$$

defines a holomorphic linear space on Ω (cf. [20, 1.8 (page. 54)]). Although, we have not used this correspondence in any essential manner in this thesis, we expect it to be a useful tool in the investigation of some of the questions we raise here.

Remark 2.2. Let $\mathcal{M}$ is any module in $\mathfrak{B}_1(\Omega)$ with Ω pseudoconvex and a finite set of generators $\{f_1, \ldots, f_t\}$. From [7, Lemma 2.3.2], it follows that the associated sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is not only coherent, it has global generators $\{f_1, \ldots, f_t\}$, that is, $\{f_{1w}, \ldots, f_{tw}\}$ generates the stalk $\mathcal{S}_w^{\mathcal{M}}$ for every $w \in \Omega$. Theorem 2.3.3 of [7] (or equivalently [27, Theorem 7.2.5]) is a consequence of the Cartan's Theorem B(cf. [27, Theorem 7.1.7]) together with the coherence of every locally finitely generated subsheaf of $\mathcal{O}^k$ (cf. [27, Theorem 7.1.8]). It is then easy to verify that if $\mathcal{M}$ is any module in $\mathfrak{B}_1(\Omega)$ and if $\{f_1, \ldots, f_t\}$ is finite set of generators for $\mathcal{M}$, then for $f \in \mathcal{M}$, there exist $g_1, \ldots, g_t \in \mathcal{O}(\Omega)$ such that

$$f = f_1 g_1 + \cdots + f_t g_t. \tag{2.1.2}$$

More generally, if $f \in \mathcal{S}^{\mathcal{M}}(U)$, then $f = \sum_{i=1}^t f_i g_i$, with $g_i \in \mathcal{O}(U)$.

The coherence of the sheaf $\mathcal{S}^{\mathcal{M}}$ implies, in particular, that the stalk $(\mathcal{S}^{\mathcal{M}})_w$ at $w \in \Omega$ is generated by a finite number of elements $g_1, \ldots, g_d$ from $\mathcal{O}_w$. Sometimes we also write g_i to denote a holomorphic function as a representative of the germ g_i at $w \in \mathbb{C}^m$. If K is the reproducing kernel for $\mathcal{M}$ and $w_0 \in \Omega$ is a fixed but arbitrary point, then for w in a small neighborhood Ω_0 of w_0, we obtain the following decomposition theorem.

Theorem 2.3. *Suppose g_i^0, $1 \le i \le d$, be a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$. Then*

(i) there exists an open neighborhood Ω_0 of w_0 such that

$$K(\cdot, w) := K_w = \overline{g_1^0(w)} K_w^{(1)} + \cdots + \overline{g_d^0(w)} K_w^{(d)},\ w \in \Omega_0$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathcal{M}$,

(ii) the vectors $K_w^{(i)}$, $1 \le i \le d$, are linearly independent in $\mathcal{M}$ for w in some neighborhood of w_0,

(iii) the vectors $\{K_{w_0}^{(i)} \mid 1 \le i \le d\}$ are uniquely determined by these generators $g_1^0, \ldots, g_d^0$,

(iv) the linear span of the set of vectors $\{K_{w_0}^{(i)} \mid 1 \le i \le d\}$ in $\mathcal{M}$ is independent of the generators $g_1^0, \ldots, g_d^0$, and

(v) $M_p^* K_{w_0}^{(i)} = \overline{p(w_0)} K_{w_0}^{(i)}$ for all i, $1 \le i \le d$, where M_p denotes the module multiplication by the polynomial p.

Proof. For simplicity of notation, without loss of generality, we assume that $0 = w_0 \in \Omega$. Let $\{e_n\}_{n=0}^\infty$ be a orthonormal basis for $\mathcal{M}$. From the equation (1.1.4), we write

$$K(z, w) = \sum_{n=0}^\infty e_n(z) \overline{e_n(w)},\ z, w \in \Omega.$$

It follows from [23, Theorem 2, page. 82] that for every element f in $\mathcal{S}_0^{\mathcal{M}}$, and therefore in particular for every e_n, we have

$$e_n(z) = \sum_{i=1}^d g_i^0(z) h_i^{(n)}(z),\ z \in \Delta(0; r)$$

for some holomorphic functions $h_i^{(n)}$ defined on the closed polydisc $\bar{\Delta}(0; r) \subseteq \Omega$. Furthermore, these functions can be chosen with the bound $\| h_i^{(n)} \|_{\bar{\Delta}(0;r)} \le C \| e_n \|_{\bar{\Delta}(0;r)}$ for some positive constant C independent of n. Although, the decomposition is not necessarily with respect to the standard coordinate system at 0, we will be using only a point wise estimate. Consequently, in the equation given above, we have chosen not to emphasize the change of variable involved and we have,

$$K(z, w) = \sum_{n=0}^\infty \{ \sum_{i=1}^d \overline{g_i^0(w)} \overline{h_i^{(n)}(w)} \} e_n(z) = \sum_{i=1}^d \overline{g_i^0(w)} \{ \sum_{n=0}^\infty \overline{h_i^{(n)}(w)} e_n(z) \}.$$

Setting $K_w^{(i)}(z)(= K_i(z,w))$ to be the sum $\sum_{n=0}^{\infty} \overline{h_i^{(n)}(w)} e_n(z)$, we can write

$$K(z,w) = \sum_{i=1}^{d} \overline{g_i^0(w)} K_w^{(i)}(z),\ w \in \Delta(0;r).$$

The function K_i is holomorphic in the first variable and antiholomorphic in the second by construction. For the proof of part (i), we need to show that $K_w^{(i)} \in \mathcal{M}$ where $w \in \Delta(0;r)$. Or, equivalently, we have to show that $\sum_{n=0}^{\infty} |h_i^{(n)}(w)|^2 < \infty$ for each $w \in \Delta(0;r)$. First, using the estimate on $h_i^{(n)}$, we have

$$|h_i^{(n)}(w)| \leq \| h_i^{(n)} \|_{\bar{\Delta}(0;r)} \leq C \| e_n \|_{\bar{\Delta}(0;r)}.$$

We prove below, the inequality $\sum_{n=0}^{\infty} \| e_n \|_{\bar{\Delta}(0;r)}^2 < \infty$ completing the proof of part (i). We prove, more generally, that for $f \in \mathcal{M}$,

$$\| f \|_{\bar{\Delta}(0;r)} \leq C' \| f \|_{2,\bar{\Delta}(0;r)}, \tag{2.1.3}$$

where $\| \,.\, \|_2$ denotes the L^2 norm with respect to the volume measure on $\bar{\Delta}(0;r)$. It is evident from the proof that the constant C' may be chosen to be independent of the functions f. We will give two proofs, of which the second one, although long, has the advantage of being elementary.

First Proof. Any function f holomorphic on Ω belongs to the Bergman space $L_a^2(\Delta(0;r+\varepsilon))$ as long as $\Delta(0;r+\varepsilon) \subseteq \Omega$. We can surely pick $\varepsilon > 0$ small enough to ensure $\Delta(0;r+\varepsilon) \subseteq \Omega$. Let B be the Bergman kernel of the Bergman space $L_a^2(\Delta(0;r+\varepsilon))$. Thus we have

$$|\, f(w) \,| \;=\; |\, \langle f, B(\cdot,w) \rangle \,| \;\leq\; \| f \|_{2,\Delta(0;r+\varepsilon)} B(w,w)^{\frac{1}{2}},\ w \in \Delta(0;r+\varepsilon).$$

Since the function $B(w,w)$ is bounded on compact subsets of $\Delta(0;r+\varepsilon)$, it follows that $C'^2 := \sup\{B(w,w) : w \in \bar{\Delta}(0;r)\}$ is finite. We therefore see that

$$\| f \|_{\bar{\Delta}(0;r)} = \sup\{|\, f(w) \,| : w \in \bar{\Delta}(0;r)\} \leq C' \| f \|_{2,\Delta(0;r+\varepsilon)}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, we infer the inequality (2.1.3).

Second Proof. Let us take $w \in \Delta(0;r)$. Let $\delta_j = r_j - |\, w_j \,|$. Consider the neighborhood $\Delta(w;\varepsilon)$ of polyradius $\varepsilon = (\varepsilon_1, ..., \varepsilon_m), \varepsilon_j > 0, 1 \leq j \leq m$, around w such that $\Delta(w;\varepsilon) \subset \Delta(0;r)$. Now by repeated application of Cauchy's integral formula for holomorphic functions of one variable, we have

$$\begin{aligned} f(w) &= (2\pi i)^{-m} \int_{\partial\Delta(w_1;\varepsilon_1)} \frac{dz_1}{(z_1 - w_1)} \int_{\partial\Delta(w_2;\varepsilon_2)} \frac{dz_2}{(z_2 - w_2)} \cdots \int_{\partial\Delta(w_n;\varepsilon_n)} \frac{dz_n}{(z_m - w_m)} f(z_m) \\ &= (2\pi)^{-m} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f(w_1 + \varepsilon_1 e^{i\theta_1}, ..., w_m + \varepsilon_m e^{i\theta_m}) d\theta_1 d\theta_2 ... d\theta_m \end{aligned}$$

where $z_j = w_j + \varepsilon_j e^{i\theta_j}$ which implies $dz_j = i\varepsilon_j e^{i\theta_j} d\theta_j$ for $1 \leq j \leq m$. Let us denote $(w_1 + \varepsilon_1 e^{i\theta_1}, ..., w_m + \varepsilon_m e^{i\theta_m})$ by $w + \varepsilon e^{i\theta}$. For a fixed point w, the integrand in the integral below is continuous on the compact domain of integration. Hence the iterated integral can be replaced by the single multiple integral

$$\begin{aligned}
&\int_0^{\delta_m}\cdots\int_0^{\delta_1}\varepsilon_m...\varepsilon_1 f(w)d\varepsilon_1...d\varepsilon_m\\
=\;&(2\pi)^{-m}\int_0^{\delta_n}\cdots\int_0^{\delta_1}\varepsilon_m...\varepsilon_1\{\int_0^{2\pi}\cdots\int_0^{2\pi}f(w+\varepsilon e^{i\theta})d\theta_1 d\theta_2...d\theta_m\}d\varepsilon_1...d\varepsilon_m\\
=\;&(2\pi)^{-m}\int_0^{\delta_m}\cdots\int_0^{\delta_1}\int_0^{2\pi}\cdots\int_0^{2\pi}\varepsilon_m...\varepsilon_1 f(w+\varepsilon e^{i\theta})d\theta_1 d\theta_2...d\theta_m d\varepsilon_1...d\varepsilon_m.
\end{aligned}$$

Now $\int_0^{\delta_m}\cdots\int_0^{\delta_1}\varepsilon_m...\varepsilon_1 f(w)d\varepsilon_1...d\varepsilon_m = \frac{\prod_{j=1}^m {\delta_j}^2}{2^m}f(w)$ and by Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\frac{\prod_{j=1}^m {\delta_j}^2}{2^m}\mid f(w)\mid\\
\leq\;&(2\pi)^{-m}\int_0^{\delta_m}\cdots\int_0^{\delta_1}\int_0^{2\pi}\cdots\int_0^{2\pi}\sqrt{\varepsilon_m...\varepsilon_1}f(w+\varepsilon e^{i\theta})\sqrt{\varepsilon_m...\varepsilon_1}d\theta_1 d\theta_2...d\theta_m d\varepsilon_1...d\varepsilon_m\\
\leq\;&(2\pi)^{-m}\{\int_0^{\delta_m}\cdots\int_0^{\delta_1}\int_0^{2\pi}\cdots\int_0^{2\pi}\varepsilon_m...\varepsilon_1\mid f(w+\varepsilon e^{i\theta})\mid^2 d\theta_1 d\theta_2...d\theta_m d\varepsilon_1...d\varepsilon_m\}^{\frac{1}{2}}\times\\
&\{\int_0^{\delta_m}\cdots\int_0^{\delta_1}\int_0^{2\pi}\cdots\int_0^{2\pi}\varepsilon_m...\varepsilon_1 d\theta_1 d\theta_2...d\theta_m d\varepsilon_1...d\varepsilon_m\}^{\frac{1}{2}}\\
\leq\;&(2\pi)^{-m}\{\int_{\bar{\Delta}(w;\varepsilon)}\mid f(z)\mid^2 dz_1\wedge d\bar{z}_1\wedge...\wedge dz_m\wedge d\bar{z}_m\}^{\frac{1}{2}}\{\frac{\prod_{j=1}^m {\delta_j}^2}{2^m}.(2\pi)^m\}^{\frac{1}{2}}\\
\leq\;&\frac{\prod_{j=1}^m \delta_j}{(4\pi)^{\frac{m}{2}}}\{\int_{\bar{\Delta}(w;\varepsilon)}\mid f(z)\mid^2 dz_1\wedge d\bar{z}_1\wedge...\wedge dz_m\wedge d\bar{z}_m\}^{\frac{1}{2}}.
\end{aligned}$$

Now as $\Delta(w;\varepsilon)\subset\Delta(0;r)$, we have

$$\begin{aligned}
\mid f(w)\mid\;&\leq\;\frac{1}{\{(\prod_{j=1}^m {\delta_j}^2)\pi\}^{\frac{m}{2}}}\{\int_{\bar{\Delta}(w;\varepsilon)}\mid f(z)\mid^2 dz_1\wedge d\bar{z}_1\wedge...\wedge dz_m\wedge d\bar{z}_m\}^{\frac{1}{2}}\\
&\leq\;\frac{1}{\{(\prod_{j=1}^m {\delta_j}^2)\pi\}^{\frac{m}{2}}}\{\int_{\bar{\Delta}(0;r)}\mid f(z)\mid^2 dz_1\wedge d\bar{z}_1\wedge...\wedge dz_m\wedge d\bar{z}_m\}^{\frac{1}{2}}.
\end{aligned}$$

The last inequality then implies that $\parallel f\parallel_{\bar{\Delta}(0;r)}\leq C\parallel f\parallel_{2,\bar{\Delta}(0;r)}$, where $C=\frac{1}{\{(\prod_{j=1}^m {\delta_j}^2)\pi\}^{\frac{m}{2}}}$.

The inequality (2.1.3) implies, in particular, that

$$\sum_{n=0}^{\infty}\parallel e_n\parallel^2_{\bar{\Delta}(0;r)}\leq C'^2\sum_{n=0}^{\infty}\int_{\bar{\Delta}(0;r)}\mid e_n(z)\mid^2 dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_m\wedge d\bar{z}_m.$$

Since $K_z=\sum_{n=0}^{\infty}\overline{e_n(z)}e_n$ $(K_z(\tilde{z})=K(\tilde{z},z))$, the function $G(z):=\sum_{n=0}^{\infty}|e_n(z)|^2$ is finite for each $z\in\Omega$. The sequence of positive continuous functions $G_k(z):=\sum_{n=0}^{k}|e_n(z)|^2$ converges uniformly to G on $\bar{\Delta}(0;r)$. To see this, we note that

$$\begin{aligned}
\parallel G_k-G\parallel^2{}_{\bar{\Delta}(0;r)}\;&\leq\;C'^2\int_{\bar{\Delta}(0;r)}|G_k(z)-G(z)|^2 dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_m\wedge d\bar{z}_m\\
&\leq\;C'^2\int_{\bar{\Delta}(0;r)}\{\sum_{n=k+1}^{\infty}\mid e_n(z)\mid^2\}^2 dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_m\wedge d\bar{z}_m,
\end{aligned}$$

which tends to 0 as $k \to \infty$. So, by monotone convergence theorem we can interchange the integral and the infinite sum to conclude

$$\sum_{n=0}^{\infty} \| e_n \|^2_{\bar{\Delta}(0;r)} \leq C \int_{\bar{\Delta}(0;r)} \sum_{n=0}^{\infty} | e_n(z) |^2 dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m < \infty$$

as G is a continuous function on $\bar{\Delta}(0;r)$. This shows that

$$\sum_{n=0}^{\infty} \Big| h_i^{(n)}(w) \Big|^2 \leq K \sum_{n=0}^{\infty} \| e_n \|^2_{\bar{\Delta}(0;r)} < \infty.$$

Hence $K_w^{(i)} \in \mathcal{M}$, $1 \leq i \leq d$.

To prove statement (ii), at 0, we have to show that whenever there exist complex numbers $\alpha_1, \ldots, \alpha_d$ such that $\sum_{i=1}^{d} \alpha_i K_i(z,0) = 0$, then $\alpha_i = 0$ for all i. We assume, on the contrary, that there exists some $i \in \{1, \ldots, d\}$ such that $\alpha_i \neq 0$. Without loss of generality, we assume $\alpha_1 \neq 0$, then $K_1(z,0) = \sum_{i=2}^{d} \beta_i K_i(z,0)$ where $\beta_i = \frac{\alpha_i}{\alpha_1}, 2 \leq i \leq d$. This shows that $K_1(z,w) - \sum_{i=2}^{d} \beta_i K_i(z,w)$ has a zero at $w = 0$. From [27, Theorem 7.2.9], it follows that

$$K_1(z,w) - \sum_{i=2}^{d} \beta_i K_i(z,w) = \sum_{j=1}^{m} \bar{w}_j G_j(z,w)$$

for some function $G_j : \Omega \times \Delta(0;r) \to \mathbb{C}, 1 \leq j \leq m$, which is holomorphic in the first and antiholomorphic in the second variable. So, we can write

$$\begin{aligned} K(z,w) &= \sum_{i=1}^{d} \bar{g}_i^0(w) K_i(z,w) = \bar{g}_1^0(w) K_1(z,w) + \sum_{i=2}^{d} \bar{g}_i^0(w) K_i(z,w) \\ &= \bar{g}_1^0(w) \{ \sum_{i=2}^{d} \beta_i K_i(z,w) + \sum_{j=1}^{m} \bar{w}_j G_j(z,w) \} + \sum_{i=2}^{d} \bar{g}_i^0(w) K_i(z,w) \\ &= \sum_{i=2}^{d} (\bar{g}_i^0(w) + \beta_i \bar{g}_1^0(w)) K_i(z,w) + \sum_{j=1}^{m} \bar{w}_j \bar{g}_1^0(w) G_j(z,w). \end{aligned}$$

For $f \in \mathcal{M}$ and $w \in \Delta(0;r)$, we have

$$f(w) = \langle f, K(\cdot,w) \rangle = \sum_{i=2}^{d} (g_i^0(w) + \bar{\beta}_i g_1^0(w)) \langle f, K_i(z,w) \rangle + g_1^0(w) \langle f, \sum_{j=1}^{m} \bar{w}_j G_j(z,w) \rangle.$$

We note that $\langle f, \sum_{j=1}^{m} \bar{w}_j G_j(z,w) \rangle$ is a holomorphic function in w which vanishes at $w = 0$. It then follows that $\langle f, \sum_{j=1}^{m} \bar{w}_j G_j(z,w) \rangle = \sum_{j=1}^{m} w_j \widetilde{G}_j(w)$ for some holomorphic functions $\widetilde{G}_j$, $1 \leq j \leq m$ on $\Delta(0;r)$. Therefore, we have

$$f(w) = \sum_{i=2}^{d} (g_i^0(w) + \bar{\beta}_i g_1^0(w)) \langle f, K_i(z,w) \rangle + \sum_{j=1}^{m} w_j g_1^0(w) \widetilde{G}_j(w).$$

Since the sheaf $\mathcal{S}^{\mathcal{M}}|_{\Delta(0;r)}$ is generated by the Hilbert module $\mathcal{M}$, it follows that the set $\{g_2^0 + \bar\beta_2 g_1^0, \ldots, g_d^0 + \bar\beta_d g_1^0, z_1 g_1^0, \ldots, z_m g_1^0\}$ also generates $\mathcal{S}^{\mathcal{M}}|_{\Delta(0;r)}$. In particular, they generate the stalk at 0. To arrive at a contradiction, it is enough to show that g_1^0 can not be written in combination of the new set of generators. If possible, suppose

$$g_1^0(z) = \sum_{i=2}^{d} a_i(z)\{g_i^0(z) + \bar\beta_i g_1^0(z)\} + \sum_{j=1}^{m} b_j(z) z_j g_1^0(z), \tag{2.1.4}$$

where a_i, b_j are holomorphic functions on some small enough neighborhood of 0, say U, for $2 \le i \le d, 1 \le j \le m$. First we suppose that $a_i(0) = 0$ for all $i, 2 \le i \le d$. Now, rewrite the equation (2.1.4) as follows

$$\{1 - \sum_{i=2}^{d} \bar\beta_i a_i(z) - \sum_{j=1}^{m} b_j(z) z_j\} g_1^0(z) = \sum_{i=2}^{d} a_i(z) g_i^0(z). \tag{2.1.5}$$

Let $c(z) = 1 - \sum_{i=2}^{d} \bar\beta_i a_i(z) - \sum_{j=1}^{m} b_j(z) z_j$. Since $c(0) = 1$, the germ of c at 0 is a unit in $\mathcal{O}_0$. Then considering the the equation (2.1.5) at the level of germs, we have $g_1^0 = \sum_{i=2}^{d} (c_0^{-1} a_{i0}) g_i^0$, which contradicts the minimality of the generators of the stalk at 0. Hence there exist some $k, 2 \le k \le d$, such that $a_k(0) \ne 0$. So a_{k0} is a unit in $\mathcal{O}_0$. Thus at the level of germs, equation (2.1.5) is of the form

$$g_{k0} = a_{k0}^{-1}\{c_0 g_{10} - \sum_{i=2, i\ne k}^{d} a_{i0} g_{i0}\},$$

which is again a contradiction to the minimality of the generators of the stalk at 0. This contradiction is consequence of the assumption that $\alpha_i \ne 0$ for some $i, 1 \le i \le m$. Therefore $\alpha_i = 0$ for all i and so $\{K_i(z,0)\}_{i=1}^{d}$ are linearly independent.

We point out that this constitute a proof of Nakayama's Lemma (cf.[29, Page - 57]). Clearly we obtain the same result as a consequence of Nakayama's Lemma: Suppose $A \subset \mathcal{S}_0^{\mathcal{M}}$ is generated by germs of the functions $g_2^0 + \bar\beta_2 g_1^0, \ldots, g_d^0 + \bar\beta_d g_1^0$. Let $\mathfrak{m}(\mathcal{O}_0)$ denotes the the only maximal ideal of the local ring $\mathcal{O}_0$, consisting of the germs of functions vanishing at 0. Then it follows that

$$\mathfrak{m}(\mathcal{O}_0)\{\mathcal{S}_0^{\mathcal{M}}/A\} = \mathcal{S}_0^{\mathcal{M}}/A.$$

Using Nakayama's lemma (cf. [29, p.57]), we see that $\mathcal{S}_0^{\mathcal{M}}/A = 0$, that is, $\mathcal{S}_0^{\mathcal{M}} = A$. This contradicts the minimality of the generators of the stalk at 0 completing the proof of first half of (ii).

To prove the slightly stronger statement, namely, the independence of the vectors $K_w^{(i)}$, $1 \le i \le d$ in a small neighborhood of 0, consider the Grammian $\big(\!\big(\langle K_w^{(i)}, K_w^{(j)}\rangle\big)\!\big)_{i,j=1}^{d}$. The determinant of this Grammian is nonzero at 0. Therefore it remains non-zero in a suitably small neighborhood of 0 since it is a real analytic function on Ω_0. Consequently, the vectors $K_w^{(i)}$, $i = 1, \ldots, d$ are linearly independent for all w in this neighborhood.

To prove statement (iii) we have to prove that $K_0^{(i)}$ are uniquely determined by the generators g_i^0 , $1 \leq i \leq d$. We will let g_i^0 denote the germ of g_i^0 at 0 as well. Let $K(z,w) = \sum_{i=1}^d \overline{g_i^0(w)} \widetilde{K}_w^{(i)}$ be another decomposition. Let $\widetilde{K}_w^{(i)} = \sum_{n=0}^\infty \overline{\widetilde{h}_i^n(w)} e_n$ for some holomorphic functions on some small enough neighborhood of 0. Thus we have

$$\sum_{n=0}^\infty \sum_{i=1}^d \overline{g_i^0(w)} \{ \overline{h_i^n(w)} - \overline{\widetilde{h}_i^n(w)} \} e_n = 0.$$

Hence, for each n

$$\sum_{i=1}^d g_i^0(z) \{ h_i^n(z) - \widetilde{h}_i^n(z) \} = 0.$$

Fix n and let $\alpha_i(z) = h_i^n(z) - \widetilde{h}_i^n(z)$. In this notation, $\sum_{i=1}^d g_i^0(z) \alpha_i(z) = 0$. Now we claim that $\alpha_i(0) = 0$ for all $i \in \{1, \ldots, d\}$. If not, we may assume $\alpha_1(0) \neq 0$. Then the germ of α_1 at 0 is a unit in $\mathcal{O}_0$. Hence we can write, in $\mathcal{O}_0$,

$$g_1^0 = -(\sum_{i=2}^d g_i^0 \alpha_{i0}) {\alpha_{10}}^{-1},$$

where α_{i0} denotes the germs of the analytic functions α_i at 0, $1 \leq i \leq d$. This is a contradiction, as $g_1^0, \ldots, g_d^0$ is a minimal set of generators of the stalk $\mathcal{S}_0^{\mathcal{M}}$ by hypothesis. As a result, $h_i^n(0) = \widetilde{h}_i^n(0)$ for all $i \in \{1, \ldots, d\}$ and $n \in \mathbb{N} \cup \{0\}$. This completes the proof of (iii).

To prove statement (iv), let $\{g_1^0, \ldots, g_d^0\}$ and $\{\widetilde{g}_1^0, \ldots, \widetilde{g}_d^0\}$ be two sets of generators for $\mathcal{S}_0^{\mathcal{M}}$ both of which are minimal. Let $K^{(i)}$ and $\widetilde{K}^{(i)}$, $1 \leq i \leq d$, be the corresponding vectors that appear in the decomposition of the reproducing kernel K as in (i). It is enough to show that

$$\mathrm{span}_{\mathbb{C}} \{ K_i(z,0) : 1 \leq i \leq d \} = \mathrm{span}_{\mathbb{C}} \{ \widetilde{K}_i(z,0) : 1 \leq i \leq d \}.$$

There exists holomorphic functions ϕ_{ij}, $1 \leq i,j \leq d$, in a small enough neighborhood of 0 such that $\widetilde{g}_i^0 = \sum_{j=1}^d \phi_{ij} g_j^0$. It now follows that

$$\begin{aligned} K(z,w) &= \sum_{i=1}^d \bar{\widetilde{g}}_i^0(w) \widetilde{K}_i(z,w) = \sum_{i=1}^d (\sum_{j=1}^d \bar{\phi}_{ij}(w) \bar{g}_j^0(w)) \widetilde{K}_i(z,w) \\ &= \sum_{j=1}^d \bar{g}_j^0(w) (\sum_{i=1}^d \bar{\phi}_{ij}(w) \widetilde{K}_i(z,w)) \end{aligned}$$

for w possibly from an even smaller neighborhood of 0. But $K(z,w) = \sum_{j=1}^d \bar{g}_j^0(w) K_j(z,w)$ and uniqueness at the point 0 implies that

$$K_j(z,0) = \sum_{i=1}^d \bar{\phi}_{ij}(0) \widetilde{K}_i(z,0)$$

for $1 \leq j \leq d$. So, we have $\mathrm{span}_{\mathbb{C}} \{ K_i(z,0) : 1 \leq i \leq d \} \subseteq \mathrm{span}_{\mathbb{C}} \{ \widetilde{K}_i(z,0) : 1 \leq i \leq d \}$. Writing g_j^0 in terms of $\widetilde{g}_i^0$, we get the other inclusion.

Finally, to prove statement (v), let us apply ${M_j}^*$ to both sides of the decomposition of the reproducing kernel K given in part (i) to obtain $\bar{w}_j K(z,w) = \sum_{i=1}^d \bar{g}_i^0(w){M_j}^* K_i(z,w)$. Substituting K from the first equation, we get $\sum_{i=1}^d \bar{g}_i^0(w)(M_j - w_j)^* K_i(z,w) = 0$. Let $F_{ij}(z,w) = (M_j - w_j)^* K_i(z,w)$. For a fixed but arbitrary $z_0 \in \Omega$, consider the equation $\sum_{i=1}^d \bar{g}_i^0(w) F_{ij}(z_0,w) = 0$. Suppose there exists $k, 1 \le k \le d$ such that $F_{kj}(z_0,0) \ne 0$. Then

$$g_k^0 = \{\overline{F_{kj}(z_0,\cdot)}_0\}^{-1} \sum_{i=1, i\ne k}^d g_i^0 \overline{F_{ij}(z_0,\cdot)}_0.$$

This is a contradiction. Therefore $F_{ij}(z_0,0) = 0$, $1 \le i \le d$, and for all $z_0 \in \Omega$. So ${M_j}^* K_i(z,0) = 0$, $1 \le i \le d$, $1 \le j \le m$. This completes the proof of the theorem. □

Remark 2.4. Let $\mathcal{I}$ be an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$. Suppose $\mathcal{M} \supset \mathcal{I}$ and that $\mathcal{I}$ is dense in $\mathcal{M}$. Let $\{p_i \in \mathbb{C}[\underline{z}] : 1 \le i \le t\}$ be a minimal set of generators for the ideal $\mathcal{I}$. Let $V(\mathcal{I})$ be the zero variety of the ideal $\mathcal{I}$. If $w \notin V(\mathcal{I})$, then $\mathcal{S}_w^{\mathcal{M}} = \mathcal{O}_w$. Although $p_1, \ldots, p_t$ generate the stalk at every point, they are not necessarily a minimal set of generators. For example, let $\mathcal{I} =< z_1(1+z_1), z_1(1-z_2), z_2^2 >\subset \mathbb{C}[z_1,z_2]$. The polynomials $z_1(1+z_1), z_1(1-z_2), z_2^2$ form a minimal set of generators for the ideal $\mathcal{I}$. Since $1+z_1$ and $1-z_2$ are units in $\mathcal{O}_0$, it follows that the functions z_1 and z_2^2 form a minimal set of generators for the stalk $\mathcal{S}_0^{\mathcal{M}}$.

For simplicity, we have stated the decomposition theorem for Hilbert modules which consists of holomorphic functions taking values in $\mathbb{C}$. However, all the tools that we use for the proof work equally well in the case of vector valued holomorphic functions. Consequently, it is not hard to see that the theorem remains valid in this more general set-up.

2.2 The joint kernel at w_0 and the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$

Let $g_1^0, \ldots, g_d^0$ be a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$ as before. For $f \in \mathcal{S}_{w_0}^{\mathcal{M}}$, we can find holomorphic functions f_i, $1 \le i \le d$ on some small open neighborhood U of w_0 such that $f = \sum_{i=1}^d g_i^0 f_i$ on U. We write

$$f = \sum_{i=1}^d g_i^0 f_i = \sum_{i=1}^d g_i^0 \{f_i - f_i(w_0)\} + \sum_{i=1}^d g_i^0 f_i(w_0).$$

on U. Let $\mathfrak{m}(\mathcal{O}_{w_0})$ be the maximal ideal (consisting of the germs of holomorphic functions vanishing at the point w_0) in the local ring $\mathcal{O}_{w_0}$ and $\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}_{w_0}^{\mathcal{M}} = \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Thus the linear span of the equivalence classes $[g_1^0], \ldots, [g_d^0]$ is the quotient module $\mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Therefore we have

$$\dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} \le d.$$

It turns out that the elements $[g_1^0], \ldots, [g_d^0]$ in the quotient module are linearly independent. Then $\dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} = d$. To prove the linear independence, let us consider the equation

$\sum_{i=1}^d \alpha_i[g_i^0] = 0$ for some complex numbers α_i, $1 \leq i \leq d$, or equivalently, $\sum_{i=1}^d \alpha_i g_i^0 \in \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Thus there exists holomorphic functions f_i, $1 \leq i \leq d$, defined on a small neighborhood of w_0 and vanishing at w_0 such that $\sum_{i=1}^d (\alpha_i - f_i)g_i = 0$. Now suppose $\alpha_k \neq 0$ for some $k, 1 \leq k \leq d$. Then we can write $g_k^0 = -\sum_{i\neq k}(\alpha_k - f_k)_0^{-1}(\alpha_i - f_i)_0 g_i^0$ which is a contradiction. From the decomposition Theorem 2.3, it follows that

$$\dim\ker D_{(\mathbf{M}-w)^*} \geq \sharp\{minimal\ generators\ for\ \mathcal{S}_{w_0}^{\mathcal{M}}\} = \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}. \tag{2.2.1}$$

We will impose natural conditions on the Hilbert module $\mathcal{M}$, which is always assumed to be in the class $\mathfrak{B}_1(\Omega)$, so as to ensure equality in (2.2.1). One such condition is that the module $\mathcal{M}$ is finitely generated. Let $V(\mathcal{M}) := \{w \in \Omega : f(w) = 0, \text{ for all } f \in \mathcal{M}\}$. Then for $w_0 \notin V(\mathcal{M})$, the number of minimal generators for the stalk at w_0 is one, in fact, $\mathcal{S}_{w_0}^{\mathcal{M}} = \mathcal{O}_{w_0}$. Also for $w_0 \notin V(\mathcal{M})$, dim $\ker D_{(\mathbf{M}-w_0)^*} = 1$, following the proof of Lemma 1.11. Therefore, outside the zero set, we have equality in (2.2.1). We will show, for a large class of Hilbert modules, even on the zero set the reverse inequality is valid. For instance, for Hilbert modules of rank 1 over $\mathbb{C}[\underline{z}]$, we have equality everywhere. This is easy to see from [15, page - 89]:

$$1 \geq \dim \mathcal{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{w_0} = \dim\ker D_{(\mathbf{M}-w)^*} \geq \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} \geq 1.$$

Let $\mathcal{H}$ be a Hilbert module in $\mathrm{B}_1(\Omega)$ that possesses a reproducing kernel which is non-degenerate on Ω. Let $\mathcal{M}$ is a submodule of $\mathcal{H}$. Then the module map

$$\mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{M} \longrightarrow \mathcal{S}^{\mathcal{M}}$$

induced from (2.1.1) is surjective. This naturally defines a map

$$\mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M} \cong \mathcal{O}_{w_0}/\mathfrak{m}_{w_0}\mathcal{O}_{w_0} \otimes \mathcal{M} \longrightarrow \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$$

for $w_0 \in \Omega$. To understand the more general case, consider the map $i_{w_0} : \mathcal{M} \longrightarrow \mathcal{M}_{w_0}$ defined by $f \mapsto f_{w_0}$, where f_{w_0} is the germ of the function f at w_0. Clearly, this map is a vector space isomorphism onto its image. The linear space $\mathcal{M}^{(w_0)} := \sum_{j=1}^m (z_j - w_{0j})\mathcal{M} = \mathfrak{m}_{w_0}\mathcal{M}$ is closed since $\mathcal{M}$ is assumed to be in $\mathfrak{B}_1(\Omega)$. The map $f \mapsto f_{w_0}$ restricted to $\mathcal{M}^{(w_0)}$ is a linear isomorphism from $\mathcal{M}^{(w_0)}$ to $(\mathcal{M}^{(w_0)})_{w_0}$. Consider

$$\mathcal{M} \xrightarrow{i_{w_0}} \mathcal{S}_{w_0}^{\mathcal{M}} \xrightarrow{\pi} \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}},$$

where π is the quotient map. Now we have a map $\psi : \mathcal{M}_{w_0}/(\mathcal{M}^{(w_0)})_{w_0} \longrightarrow \mathcal{S}_{w_0}^{\mathcal{M}}/\{\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}\}$ which is well defined because $(\mathcal{M}^{(w_0)})_{w_0} \subseteq \mathcal{M}_{w_0} \cap \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Whenever ψ can be shown to be one-one, equality in (0.0.1) is forced. To see this, note that $\mathcal{M} \ominus \mathcal{M}^{(w_0)} \cong \mathcal{M}/\mathcal{M}^{(w_0)}$ and

$$\ker D_{(\mathbf{M}-w_0)^*} = \cap_{j=1}^m \{\mathrm{ran}(M_j - w_{0j})\}^\perp = \mathcal{M} \ominus \sum_{j=1}^m (z_j - w_{0j})\mathcal{M} = \mathcal{M} \ominus \mathcal{M}^{(w_0)}.$$

Hence

$$d \leq \dim\ker D_{(\mathbf{M}-w_0)^*} = \dim \mathcal{M}/\mathcal{M}^{(w_0)} \leq \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0} = d. \qquad (2.2.2)$$

Suppose $\psi(f) = 0$ for some $f \in \mathcal{M}$. Then $f_{w_0} \in \mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$ and consequently, $f = \sum_{i=1}^m (z_i - w_{0i})f_i$ for holomorphic functions f_i, $1 \leq i \leq m$, on some small open set U. The main question is if the functions f_i, $1 \leq i \leq m$, can be chosen from the Hilbert module $\mathcal{M}$. We isolate below a class of Hilbert modules for which this question has an affirmative answer.

Let $\mathcal{H}$ be a Hilbert module over the polynomial ring $\mathbb{C}[\underline{z}]$ in the class $\mathrm{B}_1(\Omega)$. Pick, for each $w \in \Omega$, a $\mathbb{C}$ - linear subspace $\mathbb{V}_w$ of the polynomial ring $\mathbb{C}[\underline{z}]$ with the property that it is invariant under the action of the partial differential operators $\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_m}\}$ (see [7]). Set

$$\mathcal{M}(w) = \{f \in \mathcal{H} : q(D)f|_w = 0 \text{ for all } q \in \mathbb{V}_w\}.$$

For $f \in \mathcal{M}(w)$ and $q \in \mathbb{V}_w$,

$$q(D)(z_j f)|_w = w_j q(D)f|_w + \frac{\partial q}{\partial z_j}(D)f|_w = 0.$$

Now, the assumption on $\mathbb{V}_w$ ensure that $\mathcal{M}(w)$ is a module. We consider below, the class of (non-trivial) Hilbert modules which are of the form $\mathcal{M} := \bigcap_{w\in\Omega}\mathcal{M}(w)$. It is easy to see that

$$w \notin V(\mathcal{M}) \text{ if and only if } \mathbb{V}_w = \{0\} \text{ if and only if } \mathcal{M}(w) = \mathcal{H}.$$

Therefore, $\mathcal{M} = \bigcap_{w\in V(\mathcal{M})}\mathcal{M}(w)$. These modules are called *AF- cosubmodule*(see [7, page - 38]). Let

$$\mathbb{V}_w(\mathcal{M}) := \{q \in \mathbb{C}[\underline{z}] : q(D)f\big|_w = 0 \text{ for all } f \in \mathcal{M}\}.$$

Note that $\mathbb{V}_w(\mathcal{M}) = \mathbb{V}_w$. Fix a point in $V(\mathcal{M})$, say w_0. Consider

$$\widetilde{\mathbb{V}}_{w_0}(\mathcal{M}) = \{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_{w_0}(\mathcal{M}),\ 1 \leq i \leq m\}.$$

For $w \in V(\mathcal{M})$, let

$$\mathbb{V}^{w_0}_w(\mathcal{M}) = \begin{cases} \mathbb{V}_w(\mathcal{M}) & \text{if } w \neq w_0 \\ \widetilde{\mathbb{V}}_{w_0}(\mathcal{M}) & \text{if } w = w_0. \end{cases}$$

Now, define $\mathcal{M}^{w_0}(w)$ to be the submodule (of $\mathcal{H}$) corresponding to the family of the $\mathbb{C}$-linear subspaces $\mathbb{V}^{w_0}_w(\mathcal{M})$ and let $\mathcal{M}^{w_0} = \bigcap_{w\in V(\mathcal{M})}\mathcal{M}^{w_0}(w)$. So we have $\mathbb{V}_w(\mathcal{M}^{w_0}) = \mathbb{V}^{w_0}_w(\mathcal{M})$. For $f \in \mathcal{M}^{(w_0)}$, we have $f = \sum_{j=1}^m (z_j - w_{0j})f_j$, for some choice of $f_1, \ldots, f_m \in \mathcal{M}$. Now for any $q \in \mathbb{C}[\underline{z}]$, following [7], we have

$$q(D)f = \sum_{j=1}^m q(D)\{(z_j - w_{0j})f_j\} = \sum_{j=1}^m \{(z_j - w_{0j})q(D)f_j + \frac{\partial q}{\partial z_j}(D)f_j\}. \qquad (2.2.3)$$

For $w \in V(\mathcal{M})$ and $f \in \mathcal{M}^{(w_0)}$, it follows from the definitions that

$$q(D)f\big|_w = \begin{cases} \sum_{j=1}^m \{(w_j - w_{0j})q(D)f_j|_w + \frac{\partial q}{\partial z_j}(D)f_j|_w\} = 0 & q \in \mathbb{V}_w^{w_0},\ w \neq w_0 \\ \sum_{j=1}^m \{\frac{\partial q}{\partial z_j}(D)f_j|_{w_0}\} = 0 & q \in \mathbb{V}_{w_0}^{w_0},\ w = w_0. \end{cases}$$

Thus $f \in \mathcal{M}^{(w_0)}$ implies that $f \in \mathcal{M}^{w_0}(w)$ for each $w \in V(\mathcal{M})$. Hence $\mathcal{M}^{(w_0)} \subseteq \mathcal{M}^{w_0}$. Now we describe the Gleason property for $\mathcal{M}$ at a point w_0.

Definition 2.5. We say that an AF- cosubmodule $\mathcal{M}$ has the Gleason property at a point $w_0 \in V(\mathcal{M})$ if $\mathcal{M}^{w_0} = \mathcal{M}^{(w_0)}$.

In analogy with the definition of $\mathbb{V}_{w_0}(\mathcal{M})$ for a Hilbert module $\mathcal{M}$, we define the space

$$\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) = \{q \in \mathbb{C}[\underline{z}] : q(D)f\big|_{w_0} = 0,\ f_{w_0} \in \mathcal{S}_{w_0}^{\mathcal{M}}\}.$$

It will be useful to record the relation between $\mathbb{V}_{w_0}(\mathcal{M})$ and $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$ in a separate lemma.

Lemma 2.6. *For any Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$ and $w_0 \in \Omega$, we have $\mathbb{V}_{w_0}(\mathcal{M}) = \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$.*

Proof. We note that the inclusion $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) \subseteq \mathbb{V}_{w_0}(\mathcal{M})$ follows from $\mathcal{M}_{w_0} \subseteq \mathcal{S}_{w_0}^{\mathcal{M}}$. To prove the reverse inclusion, we need to show that $q(D)h|_{w_0} = 0$ for $h \in \mathcal{S}_{w_0}^{\mathcal{M}}$, for all $q \in \mathbb{V}_{w_0}(\mathcal{M})$. Since $h \in \mathcal{S}_{w_0}^{\mathcal{M}}$, we can find functions $f_1, \ldots, f_n \in \mathcal{M}$ and $g_1, \ldots, g_n \in \mathcal{O}_{w_0}$ such that $h = \sum_{i=1}^n f_i g_i$ in some small open neighborhood of w_0. Therefore, it is enough to show that $q(D)(fg)|_{w_0} = 0$ for $f \in \mathcal{M}$, g holomorphic in a neighborhood, say U_{w_0} of w_0, and $q \in \mathbb{V}_{w_0}(\mathcal{M})$. We can choose U_{w_0} to be a small enough polydisc such that $g = \sum_\alpha a_\alpha (z - w_0)^\alpha$, $z \in U_{w_0}$. Then $q(D)(fg) = \sum_\alpha a_\alpha q(D)\{(z - w_0)^\alpha f\}$ for $z \in U_{w_0}$. Clearly, $(z - w_0)^\alpha f$ belongs to $\mathcal{M}$ whenever $f \in \mathcal{M}$. Hence $q(D)\{(z - w_0)^\alpha f\}|_{w_0} = 0$ and we have $q(D)(fg)|_{w_0} = 0$ completing the proof of $\mathbb{V}_{w_0}(\mathcal{M}) \subseteq \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$.. □

We will show that we have equality in (2.2.1) for all AF - cosubmodules satisfying Gleason's property.

Proposition 2.7. *Any AF-cosubmodule $\mathcal{M}$ has Gleason's property at w_0 if and only if*

$$\dim \mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M} = \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}.$$

Proof. We first show that $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{w_0}$. Showing $\ker(\pi \circ i_{w_0}) \subseteq \mathcal{M}^{w_0}$ is same as showing $\mathcal{M}_{w_0} \cap \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} \subseteq (\mathcal{M}^{w_0})_{w_0}$. We claim that

$$\mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}) = \mathbb{V}_{w_0}^{w_0}(\mathcal{M})\,(= \widetilde{\mathbb{V}}_{w_0}(\mathcal{M})). \tag{2.2.4}$$

If $f \in \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$, then there exists $f_j \in \mathcal{S}_{w_0}^{\mathcal{M}}$, $1 \leq j \leq m$, such that $f = \sum_{j=1}^m (z_j - w_{0j}) f_j$. From equation (2.2.3), we have

$$q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}) \text{ if and only if } \frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) = \mathbb{V}_{w_0}(\mathcal{M})$$

for all $j, 1 \leq j \leq m$. Now, from Lemma 2.6, we see that $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{M})$ $1 \leq j \leq m$, if and only if $q \in \widetilde{\mathbb{V}}_{w_0}(\mathcal{M})$, which proves our claim. So for $f \in \mathcal{M}$, if $f_{w_0} \in \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$, then $f \in \mathcal{M}^{w_0}(w)$ for all $w \in V(\mathcal{M})$. Hence $f \in \mathcal{M}^{w_0}$ and as a result, we find $\mathcal{M}_{w_0} \cap \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} \subseteq (\mathcal{M}^{w_0})_{w_0}$.

Now let $f \in \mathcal{M}^{w_0}$. From (2.2.4) it follows that

$$f \in \{g \in \mathcal{O}_{w_0} : q(D)g\big|_{w_0} = 0 \text{ for all } q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}})\}.$$

According to [7, Proposition 2.3.1] we have $f \in \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Therefore $f \in \ker(\pi \circ i_{w_0})$ and $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{w_0}$.

Next we show that the map $\pi \circ i_{w_0}$ is onto. Let $\sum_{i=1}^n f_i g_i \in \mathcal{S}_{w_0}^{\mathcal{M}}$, where $f_i \in \mathcal{M}$ and g_i's are holomorphic function in some neighborhood of w_0, $1 \leq i \leq n$. We need to show that there exist $f \in \mathcal{M}$ such that the class $[f]$ is equal to $[\sum_{i=1}^n f_i g_i]$ in $\mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Let us take $f = \sum_{i=1}^n f_i g_i(w_0)$. Then

$$\sum_{i=1}^n f_i g_i - f = \sum_{i=1}^n f_i\{g_i - g_i(w_0)\} \in \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}.$$

This completes the proof of surjectivity.

Suppose Gleason's property holds for $\mathcal{M}$ at w_0. Since $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{w_0}$, it follows from the Gleason's property at w_0 that we have the equality $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{(w_0)}$. We recall then that the map $\psi : \mathcal{M}/\mathcal{M}^{(w_0)} \longrightarrow \mathcal{S}_{w_0}^{\mathcal{M}}/\{\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}\}$ is one to one. The equality in (2.2.1) is established as in the equation (2.2.2).

Now suppose equality holds in (0.0.1). From the above, it is clear that $\mathcal{M}/\mathcal{M}^{w_0}$ is isomorphic to $\mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Thus

$$\dim \mathcal{M}/\mathcal{M}^{w_0} = \dim \mathcal{M}/\mathcal{M}^{(w_0)}.$$

But as $\mathcal{M}^{(w_0)} \subseteq \mathcal{M}^{w_0}$, we have $\mathcal{M}^{(w_0)} = \mathcal{M}^{w_0}$ and hence Gleason's property holds for $\mathcal{M}$ at w_0. □

A class of examples of Hilbert spaces satisfying Gleason's property can be found in [19]. It was shown in [19] that Gleason's property holds for all analytic Hilbert modules. However it is not entirely clear if it continues to hold for submodules of analytic Hilbert modules. Nevertheless we will identify here, a class of submodules for which we have equality in (2.2.1). Let $\mathcal{M}$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$. Assume that $\mathcal{M}$ is a closure of an ideal $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$. From [7, 17], we note that

$$\dim \ker D_{(\mathbf{M}-w_0)^*} = \dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I}.$$

Therefore from (2.2.1) we have

$$\dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I} \geq \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}.$$

So it remains to prove the reverse inequality. Fix a point $w_0 \in \Omega$. Consider the map

$$\mathcal{I} \xrightarrow{i_{w_0}} \mathcal{S}^{\mathcal{M}}_{w_0} \xrightarrow{\pi} \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

We will show that $\ker(\pi \circ i_{w_0}) = \mathfrak{m}_{w_0}\mathcal{I}$. Let $V(\mathcal{I})$ denote the zero set of the ideal $\mathcal{I}$ and $\mathbb{V}_w(\mathcal{I})$ be its characteristic space at w. We begin by proving that the characteristic space of the ideal coincides with that of corresponding Hilbert module.

Lemma 2.8. *Assume that $\mathcal{M} = [\mathcal{I}] \in \mathfrak{B}_1(\Omega)$. Then $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{M})$ for $w_0 \in \Omega$.*

Proof. Clearly $\mathbb{V}_{w_0}(\mathcal{I}) \supseteq \mathbb{V}_{w_0}(\mathcal{M})$, so we have to prove $\mathbb{V}_{w_0}(\mathcal{I}) \subseteq \mathbb{V}_{w_0}(\mathcal{M})$. For $q \in \mathbb{V}_{w_0}(\mathcal{I})$ and $f \in \mathcal{M}$, we show that $q(D)f|_{w_0} = 0$. Now, for each $f \in \mathcal{M}$, there exists a sequence of polynomial $p_n \in \mathcal{I}$ such that $p_n \to f$ in the Hilbert space norm. For $w \in \Omega$ and a compact neighborhood C of w, from equation (1.1.5) we have

$$\begin{aligned} |q(D)p_n(w) - q(D)f(w)| &= |\langle p_n - f, q(\bar{D})K(\cdot, w)\rangle| \leq \| p_n - f \|_{\mathcal{M}} \| q(\bar{D})K(\cdot, w) \|_{\mathcal{M}} \\ &\leq \| p_n - f \|_{\mathcal{M}} \sup_{w \in C} \| q(\bar{D})K(\cdot, w) \|_{\mathcal{M}} . \end{aligned}$$

Therefore $q(D)p_n\big|_{w_0} \longrightarrow q(D)f\big|_{w_0}$ as $n \longrightarrow \infty$. Since $q(D)p_n\big|_{w_0} = 0$ for all n, it follows that $q(D)f\big|_{w_0} = 0$. Hence $q \in \mathbb{V}_{w_0}(\mathcal{M})$ and we are done. □

Now let $\mathcal{J} = \mathfrak{m}_{w_0}\mathcal{I}$. Recall (cf. [17, Proposition 2.3]) that $V(\mathcal{J}) \setminus V(\mathcal{I}) := \{w \in \mathbb{C}^m : \mathbb{V}_w(\mathcal{I}) \subsetneq \mathbb{V}_w(\mathcal{J})\} = \{w_0\}$. Here we will explicitly write down the characteristic space. Let

$$\widetilde{\mathbb{V}}_{w_0}(\mathcal{I}) = \{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_{w_0}(\mathcal{I}),\ 1 \leq i \leq m\}$$

and

$$\mathbb{V}^{w_0}_w(\mathcal{I}) = \begin{cases} \mathbb{V}_w(\mathcal{I}), & w \neq w_0; \\ \widetilde{\mathbb{V}}_{w_0}(\mathcal{I}), & w = w_0. \end{cases}$$

Lemma 2.9. *For $w \in \mathbb{C}^m$, $\mathbb{V}_w(\mathcal{J}) = \mathbb{V}^{w_0}_w(\mathcal{I})$.*

Proof. Since $\mathcal{J} \subset \mathcal{I}$, we have $\mathbb{V}_w(\mathcal{I}) \subseteq \mathbb{V}_w(\mathcal{J})$ for all $w \in \mathbb{C}^m$. Now let $w \neq w_0$. For $f \in \mathcal{I}$ and $q \in \mathbb{V}_w(\mathcal{J})$, we show that $q(D)f\big|_w = 0$ which implies q must be in $\mathbb{V}_w(\mathcal{I})$.

Note that for any $k \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$, $q(D)\{(z_j - w_{0j})^k f\}\big|_w = 0$ as $(z_j - w_{0j})^k f \in \mathcal{J}$. This implies $\sum_{l=0}^k (w_j - w_{0j})^l \binom{k}{l} \frac{\partial^{k-l} q}{\partial z_j^{k-l}}(D)f\big|_w = 0$. Hence (inductively) we have

$$(w_j - w_{0j})^k q(D)f\big|_w = (-1)^k \frac{\partial^k q}{\partial z_j^k}(D)f\big|_w \text{ for all } k \in \mathbb{N} \text{ and } j \in \{1, \ldots, m\}.$$

So, if $w \neq w_0$, then there exists $i \in \{1, \ldots, m\}$ such that $w_i \neq w_{0i}$. Therefore, by choosing k large enough with respect to the degree of q, we can ensure $(w_i - w_{0i})^k q(D)f\big|_w = 0$. Thus $q(D)f\big|_w = 0$. For $w = w_0$, we have $q \in \mathbb{V}_{w_0}(\mathcal{J})$ if and only if $q(D)\{(z_j - w_{0j})f\}\big|_{w_0} = 0$ for all

$f \in \mathcal{I}$ and $j \in \{1, \dots, m\}$ if and only if $\frac{\partial q}{\partial z_j}(D)f\big|_{w_0} = 0$ for all $f \in \mathcal{I}$ and $j \in \{1, \dots, m\}$ if and only if $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{I})$ for all $j \in \{1, \dots, m\}$ if and only if $q \in \widetilde{\mathbb{V}}_{w_0}(\mathcal{I})$. This completes the proof of the lemma. □

We have shown that $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{M}) = \mathbb{V}_{w_0}(\mathcal{S}^{\mathcal{M}}_{w_0})$. The next Lemma provides a relationship between the characteristic space of $\mathcal{J}$ at the point w_0 and the sheaf $\mathcal{S}^{\mathcal{M}}_{w_0}$.

Lemma 2.10. $\mathbb{V}_{w_0}(\mathcal{J}) = \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0})$.

Proof. We have $\mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}) \subseteq \mathbb{V}_{w_0}(\mathcal{J})$. From the previous Lemma, it follows that if $q \in \mathbb{V}_{w_0}(\mathcal{J})$, then $q \in \widetilde{\mathbb{V}}_{w_0}(\mathcal{I})$, that is, $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{S}^{\mathcal{M}}_{w_0})$ for all $j \in \{1, \dots, m\}$. From (2.2.4), it follows that $q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0})$. □

Now, we have all the ingredients to prove that we must have equality in (2.2.1) for submodules of analytic Hilbert modules which are obtained as closure of some polynomial ideal.

Proposition 2.11. *Let $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$ on a bounded domain Ω, where $\mathcal{I}$ is a polynomial ideal, each of whose algebraic component intersects Ω. Then*

$$\dim \mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M} = \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0},\ w_0 \in \Omega.$$

Proof. Let $p \in \mathcal{I}$ such that $\pi \circ i_{w_0}(p) = 0$, that is, $p_{w_0} \in \mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$. The preceding Lemma implies $q(D)p\big|_{w_0} = 0$ for all $q \in \mathbb{V}_{w_0}(\mathcal{J})$. So $p \in \mathcal{J}^e_{w_0}$ (see the definition of envelope of an ideal in the equation 1.4.1). Since each of the algebraic component of $\mathcal{J}$ (see section 1.4) intersects Ω , therefore, from Theorem 1.13, we have $p \in \bigcap_{w \in \Omega} \mathcal{J}^e_w = \mathcal{J}$. Thus $\ker(\pi \circ i_{w_0}) = \mathcal{J} = \mathfrak{m}_{w_0}\mathcal{I}$. Then the map $\pi \circ i_{w_0} : \dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I} \to \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$ is one-one and we have

$$\dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I} \le \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

Therefore, we have equality in (2.2.2). □

The following Corollary, which is an immediate consequence of Theorem 2.7 and the Proposition 2.11.

Corollary 2.12. *If $\mathcal{M}$ is a submodule of an analytic Hilbert module of finite co-dimension with the zero set $V(\mathcal{M}) \subset \Omega$, then the Gleason problem is solvable for $\mathcal{M}$.*

Proof. From Theorem 1.15, it follows that the submodule $\mathcal{M}$ corresponds to an ideal such that $\mathcal{M} = [\mathcal{I}]$. The proof is complete using Propositions 2.7 and 2.11. □

Remark 2.13. In fact, this Corollary is valid for all submodules of the form $[\mathcal{I}]$ whenever it is an AF- cosubmodule for some polynomial ideal $\mathcal{I}$.

The following corollary to Proposition 2.11 proves the conjecture of [14, page - 262]. It was first proved by Duan-Guo [17].

Corollary 2.14. *Suppose $\mathcal{M}$ is a submodule of an analytic Hilbert module given by closure of a polynomial ideal $\mathcal{I}$ and $w_0 \in V(\mathcal{I})$ is a smooth point then,*

$$\dim \ker D_{(\mathbf{M}-w_0)^*} = \text{codimension of V}(\mathcal{I}).$$

Proof. From Remark 2.2, it follows that if $\mathcal{I}$ is generated by $p_1, \ldots, p_t$, then $\mathcal{S}_{w_0}^{\mathcal{M}}$ is generated by $p_{1w_0}, \ldots, p_{tw_0}$. In the course of the proof of the Theorem 2.3 in [17], a change of variable argument is used to show that the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$ at w_0 is isomorphic to the ideal generated by the co-ordinate functions $z_1 - w_{01}, \ldots, z_r - w_{0r}$, where r is the co-dimension of $V(\mathcal{I})$. Therefore, the number of minimal generators for the stalk at a smooth point is equal to r which is the codimension of $V(\mathcal{I})$. The proof is complete by Propositions 2.11. □

2.3 The rigidity theorem

Let K_i be the reproducing kernel corresponding to $\mathcal{M}_i$, $i = 1, 2$. We assume that the dimension of the zero sets $X_i = V(\mathcal{M}_i)$ of the modules $\mathcal{M}_i$, $i = 1, 2$, is less or equal to $m-2$. Recall that the stalk $\mathcal{S}_w^{\mathcal{M}_i}$ is $\mathcal{O}_w$ for $w \in \Omega \setminus X_i$, $i = 1, 2$. Let $X = X_1 \cup X_2$ and assume that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are in $\mathrm{B}_1(\Omega \setminus X)$. From [6, Lemma 1.3] and [11, Theorem 3.7], it follows that there exists a non-vanishing holomorphic function $\phi : \Omega \setminus X \to \mathbb{C}$ such that $LK_1(\cdot, w) = \bar{\phi}(w) K_2(\cdot, w),\ L^* f = \phi f$ and $K_1(z, w) = \phi(z) K_2(z, w) \bar{\phi}(w)$. The function $\psi = 1/\phi$ on $\Omega \setminus X$ (induced by the inverse of L, that is, L^*) is holomorphic. Since $\dim X \leq m - 2$, by Hartog's theorem (cf. [28, Page 198]) there is a unique extension of ϕ to Ω such that ϕ is non-vanishing on Ω (ψ have an extension to Ω and $\phi\psi = 1$ on the open set $\Omega \setminus X$). Thus $X_1 = X_2$. For $w_0 \in X$, the stalks are not just isomorphic but equal:

$$\begin{aligned} \mathcal{S}_{w_0}^{\mathcal{M}_1} &= \{\sum_{i=1}^{n} h_i g_i : g_i \in \mathcal{M}_1, h_i \in \mathcal{O}_{w_0}, 1 \leq i \leq n, n \in \mathbb{N}\} \\ &= \{\sum_{i=1}^{n} h_i \phi f_i : f_i \in \mathcal{M}_2, h_i \in \mathcal{O}_{w_0}, 1 \leq i \leq n, n \in \mathbb{N}\} \\ &= \{\sum_{i=1}^{n} \widetilde{h}_i f_i : f_i \in \mathcal{M}_2, \widetilde{h}_i \in \mathcal{O}_{w_0}, 1 \leq i \leq n, n \in \mathbb{N}\} = \mathcal{S}_{w_0}^{\mathcal{M}_2}. \end{aligned}$$

The following theorem is modeled after the well known rigidity theorem which is obtained by taking $\mathcal{M} = \widetilde{\mathcal{M}}$. While the spirit of the proof that follows is not very different from that of [7] or [16], we believe, passing to the sheaf model obtained in [6], makes the proof of the rigidity theorem somewhat more transparent. Since the dimension of the algebraic variety $V(\mathcal{I})$ for an ideal $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ is the same as the holomorphic dimension (cf. [29, Theorem 5.7.1]), it follows that the hypothesis of [16, Theorem 3.6] coincides with the ones we make here.

Theorem 2.15. *Let $\mathcal{I}, \widetilde{\mathcal{I}}$ be any two polynomial ideals and $\mathcal{M}\,\widetilde{\mathcal{M}}$ be two Hilbert modules of the form $[\mathcal{I}]$ and $[\widetilde{\mathcal{I}}]$ respectively. Assume that $\mathcal{M}$, $\widetilde{\mathcal{M}}$ are in $\mathfrak{B}_1(\Omega)$ and that the dimension of the zero set of these modules is at most $m-2$. Also, assume that every algebraic component of $V(\mathcal{I})$ and $V(\widetilde{\mathcal{I}})$ intersects Ω. If $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent, then $\mathcal{I} = \widetilde{\mathcal{I}}$.*

Proof. For $w_0 \in \Omega$, we have $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{S}^{\mathcal{M}}_{w_0})$ from Lemma 2.6 and 2.8, and $\mathcal{S}^{\mathcal{M}}_{w_0} = \mathcal{S}^{\widetilde{\mathcal{M}}}_{w_0}$. Therefore $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\widetilde{\mathcal{I}})$. From definition of envelope 1.4, we see that $\mathcal{I}^e_{w_0} = \widetilde{\mathcal{I}}^e_{w_0}$ for all $w_0 \in \Omega$. The proof is now complete since $\mathcal{I} = \cap_{w_0\in\Omega}\mathcal{I}^e_{w_0}$ (see Theorem 1.13). □

Example 2.16. For $j = 1, 2$, let $\mathcal{I}_j \subset \mathbb{C}[z_1, \ldots, z_m]$, $m > 2$, be the ideals generated by z_1^n and $z_1^{k_j} z_2^{n-k_j}$. Let $[\mathcal{I}_j]$ be the submodule in the Hardy module $H^2(\mathbb{D}^m)$. Now, from the Theorem proved above, it follows that $[\mathcal{I}_1]$ is equivalent to $[\mathcal{I}_2]$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$. We will see, by using the notion of canonical generators (Proposition 4.11), that these two ideals are same only if $k_1 = k_2$.

3. The Curto - Salinas vector bundle

In this chapter, we give a canonical decomposition for the reproducing kernel for a Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, using [11, Theorem 2.2]. This naturally leads to the existence of a vector bundle of rank possibly > 1. It is shown that if two Hilbert modules $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ in $\mathfrak{B}_1(\Omega)$ are equivalent, then the corresponding holomorphic Hermitian vector bundles obtained from the decomposition of the reproducing kernel are equivalent. Thus the curvature of these bundles, among others, is an invariant for a Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$. We explicitly calculate the curvature invariant for some submodule of the weighted Bergman module.

3.1 *Existence of a canonical decomposition*

Let $\mathcal{M}$ be a Hilbert module in $\mathfrak{B}_1(\Omega)$ and $w_0 \in \Omega$ be fixed. The vectors $K_w^{(i)} \in \mathcal{M}$, $1 \leq i \leq d$, produced in part (ii) of the decomposition Theorem 2.3 are independent in some small neighborhood, say Ω_0 of w_0. However, while the choice of these vectors is not canonical, in general, we provide below a recipe for finding the vectors $K_w^{(i)}$, $1 \leq i \leq d$, satisfying

$$K(\cdot, w) = \overline{g_1^0(w)} K_w^{(1)} + \cdots + \overline{g_d^0(w)} K_w^{(d)},\ w \in \Omega_0$$

following [11]. We note that $\mathfrak{m}_w \mathcal{M}$ is a closed submodule of $\mathcal{M}$. We assume that we have equality in (0.0.1) for the module $\mathcal{M}$ at the point $w_0 \in \Omega$, that is, $\text{span}_{\mathbb{C}}\{K_{w_0}^{(i)} : 1 \leq i \leq d\} = \ker D_{(\mathbf{M}-w_0)^*}$.

Let $D_{(\mathbf{M}-w)^*} = V_{\mathbf{M}}(w)|D_{(\mathbf{M}-w)^*}|$ be the polar decomposition of $D_{(\mathbf{M}-w)^*}$, where $|D_{(\mathbf{M}-w)^*}|$ is the positive square root of the operator $\big(D_{(\mathbf{M}-w)^*}\big)^* D_{(\mathbf{M}-w)^*}$ and $V_{\mathbf{M}}(w)$ is the partial isometry mapping $\big(\ker D_{(\mathbf{M}-w)^*}\big)^{\perp}$ isometrically onto $\text{ran} D_{(\mathbf{M}-w)^*}$. Let $Q_{\mathbf{M}}(w)$ be the positive operator:

$$Q_{\mathbf{M}}(w)\big|_{\ker D_{(\mathbf{M}-w)^*}} = 0 \text{ and } Q_{\mathbf{M}}(w)\big|_{(\ker D_{(\mathbf{M}-w)^*})^{\perp}} = \big(|D_{(\mathbf{M}-w)^*}|\big|_{(\ker D_{(\mathbf{M}-w)^*})^{\perp}}\big)^{-1}.$$

Let $R_{\mathbf{M}}(w) : \mathcal{M} \oplus \cdots \oplus \mathcal{M} \to \mathcal{M}$ be the operator $R_{\mathbf{M}}(w) = Q_{\mathbf{M}}(w) V_{\mathbf{M}}(w)^*$. The two equations, involving the operator $D_{(\mathbf{M}-w)^*}$, stated below are analogous to the semi-Fredholmness property of a single operator (cf. [8, Proposition 1.11]):

$$R_{\mathbf{M}}(w) D_{(\mathbf{M}-w)^*} = I - P_{\ker D_{(\mathbf{M}-w)^*}} \tag{3.1.1}$$

$$D_{(\mathbf{M}-w)^*} R_{\mathbf{M}}(w) = P_{\text{ran}\, D_{(\mathbf{M}-w)^*}}, \tag{3.1.2}$$

where $P_{\ker D_{(\mathbf{M}-w)^*}}, P_{\text{ran} D_{(\mathbf{M}-w)^*}}$ are orthogonal projection onto $\ker D_{(\mathbf{M}-w)^*}$ and $\text{ran} D_{(\mathbf{M}-w)^*}$ respectively. Consider the operator

$$P(\bar{w}, \bar{w}_0) = I - \{I - R_{\mathbf{M}}(w_0) D_{\bar{w}-\bar{w}_0}\}^{-1} R_{\mathbf{M}}(w_0) D_{(\mathbf{M}-w)^*},\ w \in B(w_0; \| R(w_0) \|^{-1}),$$

where $B(w_0; \| R(w_0) \|^{-1})$ is the ball of radius $\| R(w_0) \|^{-1}$ around w_0. Using the equations (3.1.1) and (3.1.2) given above, we write

$$P(\bar{w}, \bar{w}_0) = \{I - R_{\mathbf{M}}(w_0) D_{\bar{w}-\bar{w}_0}\}^{-1} P_{\ker D_{(\mathbf{M}-w_0)^*}}, \tag{3.1.3}$$

where $D_{\bar{w}-\bar{w}_0} f = ((\bar{w}_1 - \bar{w}_{01}) f_1, \ldots, (\bar{w}_m - \bar{w}_{0m}) f_m)$. The details can be found in [11, page - 452]. From the definition of $P(\bar{w}, \bar{w}_0)$, it follows that $P(\bar{w}, \bar{w}_0) P_{\ker D_{(\mathbf{M}-w)^*}} = P_{\ker D_{(\mathbf{M}-w)^*}}$. This implies $\ker D_{(\mathbf{M}-w)^*} \subset \operatorname{ran} P(\bar{w}, \bar{w}_0)$ for $w \in \Delta(w_0; \varepsilon)$. Consequently $K(\cdot, w) \in \operatorname{ran} P(\bar{w}, \bar{w}_0)$ and therefore

$$K(\cdot, w) = \sum_{i=1}^{d} \overline{a_i(w)} P(\bar{w}, \bar{w}_0) K_{w_0}^{(i)},$$

for some complex valued functions $a_1, \ldots, a_d$ on $\Delta(w_0; \varepsilon)$. We will show that the functions a_i, $1 \le i \le d$, are holomorphic and their germs form a minimal set of generators for $S_{w_0}^{\mathcal{M}}$. Now

$$R_{\mathbf{M}}(w_0) D_{\bar{w}-\bar{w}_0} K(\cdot, w) = R_{\mathbf{M}}(w_0) D_{(\mathbf{M}-w_0)^*} K(\cdot, w) = (I - P_{\ker D_{(\mathbf{M}-w_0)^*}}) K(\cdot, w).$$

Hence we have,

$$\{I - R_{\mathbf{M}}(w_0) D_{\bar{w}-\bar{w}_0}\} K(\cdot, w) = P_{\ker D_{(\mathbf{M}-w_0)^*}} K(\cdot, w).$$

Since $K(\cdot, w) \in \operatorname{ran} P(\bar{w}, \bar{w}_0)$, we also have

$$P(\bar{w}, \bar{w}_0)^{-1} K(\cdot, w) = P_{\ker D_{(\mathbf{M}-w_0)^*}} K(\cdot, w).$$

Let $v_1, \ldots, v_d$ be the orthonormal basis for $\ker D_{(\mathbf{M}-w_0)^*}$. Let $g_1, \ldots, g_d$ denotes the minimal set of generators for the stalk at $\mathcal{S}_{w_0}^{\mathcal{M}}$. Then there exist a neighborhood U, small enough such that $v_j = \sum_{i=1}^{d} g_i f_i^j$, $1 \le j \le d$, and for some holomorphic functions $f_i^j, 1 \le i, j \le d$, on U. We then have

$$\begin{aligned} P(\bar{w}, \bar{w}_0)^{-1} K(\cdot, w) &= P_{\ker D_{(\mathbf{M}-w_0)^*}} K(\cdot, w) = \sum_{j=1}^{d} \langle K(\cdot, w), v_j \rangle v_j \\ &= \sum_{j=1}^{d} \langle K(\cdot, w), \sum_{i=1}^{d} g_i f_i^j \rangle v_j = \sum_{i=1}^{d} \sum_{j=1}^{d} \overline{g_i(w) f_i^j(w)} v_j \\ &= \sum_{i=1}^{d} \overline{g_i(w)} \{ \sum_{j=1}^{d} \overline{f_i^j(w)} v_j \}. \end{aligned}$$

So $K(z, w) = \sum_{i=1}^{d} \overline{g_i(w)} \{ \sum_{j=1}^{d} \overline{f_i^j(w)} P(\bar{w}, \bar{w}_0) v_j(z) \}$. Let

$$\widetilde{K}_w^{(i)} = \sum_{j=1}^{d} \overline{f_i^j(w)} P(\bar{w}, \bar{w}_0) v_j.$$

Since the vectors $K_{w_0}^{(i)}$, $1 \le i \le d$ are uniquely determined as long as $g_1, \ldots, g_d$ are fixed and $P(\bar{w}_0, \bar{w}_0) = P_{\ker D_{(\mathbf{M}-w_0)^*}}$, it follows that $K_{w_0}^{(i)} = \widetilde{K}_{w_0}^{(i)} = \sum_{j=1}^{d} \overline{f_i^j(w_0)} v_j$, $1 \le i \le d$. Therefore, the

$d\times d$ matrix $(\overline{f_i^j(w_0)})_{i,j=1}^d$ has a non-zero determinant. As $Det\ (\overline{f_i^j(w)})_{i,j=1}^d$ is an anti-holomorphic function, there exist a neighborhood of w_0, say $\Delta(w_0;\varepsilon), \varepsilon > 0$, such that $Det\ (\overline{f_i^j(w)})_{i,j=1}^d \neq 0$ for all $w \in \Delta(w_0;\varepsilon)$. The set of vectors $\{P(\bar{w},\bar{w}_0)v_j\}_{j=1}^n$ is linearly independent since $P(\bar{w},\bar{w}_0)$ is injective on $\ker D_{(\mathbf{M}-w_0)^*}$. Let $(\alpha_{ij})_{i,j=1}^d = \{(\overline{f_i^j(w_0)})_{i,j=1}^d\}^{-1}$, in consequence, $v_j = \sum_{l=1}^d \alpha_{jl} K_{w_0}^{(l)}$. We then have

$$\begin{aligned} K(\cdot,w) &= \sum_{i=1}^d \overline{g_i(w)}\{\sum_{j=1}^d \overline{f_i^j(w)} P(\bar{w},\bar{w}_0)(\sum_{l=1}^d \alpha_{jl} K_{w_0}^{(l)})\} \\ &= \sum_{l=1}^d \{\sum_{i,j=1}^d \overline{g_i(w)}\,\overline{f_i^j(w)}\alpha_{jl}\} P(\bar{w},\bar{w}_0) K_{w_0}^{(l)}). \end{aligned}$$

Since the matrices $(\overline{f_i^j(w)})_{i,j=1}^d$ and $(\alpha_{ij})_{i,j=1}^d$ are invertible, the functions

$$a_l(z) = \sum_{i,j=1}^d g_i(z) f_i^j(z)\alpha_{jl},\ 1 \le l \le d,$$

form a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$ and hence we have the canonical decomposition,

$$K(\cdot,w) = \sum_{i=1}^d \overline{a_i(w)} P(\bar{w},\bar{w}_0) K_{w_0}^{(i)}.$$

3.2 *Construction of higher rank bundle and equivalence*

Let $\mathcal{P}_w = \text{ran} P(\bar{w},\bar{w}_0) P_{\ker D_{(\mathbf{M}-w_0)^*}}$ for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. Since $P(\bar{w},\bar{w}_0)$ restricted to the $\ker D_{(\mathbf{M}-w_0)^*}$ is one-one, $\dim \mathcal{P}_w$ is constant for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. Thus to prove the following lemma, we will show that $\mathcal{P}_w = \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$, where $\mathbb{P}_0$ is the orthogonal projection onto $\text{ran} D_{(\mathbf{M}-w_0)^*}$.

Lemma 3.1. *The dimension of* $\ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$ *is constant in a suitably small neighborhood of* $w_0 \in \Omega$, *say* Ω_0.

Proof. From [11, pp. 453], it follows that $\mathbb{P}_0 D_{(\mathbf{M}-w)^*} P(\bar{w},\bar{w}_0) = 0$. So, $\mathcal{P}_w \subseteq \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$. Using (3.1.1) and (3.1.2), we can write

$$\begin{aligned} \mathbb{P}_0 D_{(\mathbf{M}-w)^*} &= D_{(\mathbf{M}-w_0)^*} R_{\mathbf{M}}(w_0)\{D_{(\mathbf{M}-w_0)^*} - D_{(\bar{w}-\bar{w}_0)}\} \\ &= D_{(\mathbf{M}-w_0)^*}\{I - P_{\ker D_{(\mathbf{M}-w_0)^*}} - R_{\mathbf{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\} \\ &= D_{(\mathbf{M}-w_0)^*}\{I - R_{\mathbf{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\}. \end{aligned}$$

Since $\{I - R_{\mathbf{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\}$ is invertible for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$, we have

$$\dim \mathcal{P}_w = \dim \ker D_{(\mathbf{M}-w_0)^*} \geq \dim \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}.$$

This completes the proof. $\square$

From the construction of the operator $P(\bar{w}, \bar{w}_0)$, it follows that, the association $w \to \mathcal{P}_w$ forms a Hermitian holomorphic vector bundle of rank m over $\Omega_0^* = \{\bar{z} : z \in \Omega_0\}$ where $\Omega_0 = B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. Let $\mathcal{P}$ denote this Hermitian holomorphic vector bundle.

Theorem 3.2. *If any two Hilbert modules $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ belonging to the class $\mathfrak{B}_1(\Omega)$ are isomorphic via an unitary module map, then the corresponding vector bundles $\mathcal{P}_0$ and $\widetilde{\mathcal{P}}_0$ on Ω_0^* are equivalent as holomorphic Hermitian vector bundles.*

Proof. Since $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent Hilbert modules, there exist a unitary $U : \mathcal{M} \to \widetilde{\mathcal{M}}$ intertwining the adjoint of the module multiplication, that is, $UM_j{}^* = \widetilde{M}_j^* U$, $1 \le j \le m$. Here $\widetilde{M}_j$ denotes the multiplication by co-ordinate function $z_j, 1 \le j \le m$ on $\widetilde{\mathcal{M}}$. It is enough to show that $UP(\bar{w}, \bar{w}_0) = \widetilde{P}(\bar{w}, \bar{w}_0)U$ for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$.

Let $\mid D_{\mathbf{M}^*} \mid = \{\sum_{j=1}^m M_j M_j{}^*\}^{\frac{1}{2}}$, that is, the positive square root of $(D_{\mathbf{M}^*})^* D_{\mathbf{M}^*}$. We have

$$\sum_{j=1}^m M_j M_j{}^* = U^* (\sum_{j=1}^m \widetilde{M}_j \widetilde{M}_j^*) U = (U^* \mid D_{\widetilde{\mathbf{M}}^*} \mid U)^2.$$

Clearly, $\mid D_{\mathbf{M}^*} \mid = U^* \mid D_{\widetilde{\mathbf{M}}^*} \mid U$. Similar calculation gives $\mid D_{(\mathbf{M}-w_0)^*} \mid = U^* \mid D_{(\widetilde{\mathbf{M}}-w_0)^*} \mid U$. Let $P_i : \mathcal{M} \oplus \mathcal{M} \cdots \oplus \mathcal{M}(\text{ m times}) \longrightarrow \mathcal{M}$ be the orthogonal projection on the i-th component. In this notation, we have $P_j D_{\mathbf{M}^*} = M_j{}^*, 1 \le j \le m$. Then,

$$\begin{aligned} \widetilde{P}_j D_{(\widetilde{\mathbf{M}}-w_0)^*} &= UP_j D_{(\mathbf{M}-w_0)^*} U^* = UP_j V_{\mathbf{M}}(w_0) U^* U \mid D_{(\mathbf{M}-w_0)^*} \mid U^* \\ &= UP_j V_{\mathbf{M}}(w_0) U^* \mid D_{(\widetilde{\mathbf{M}}-w_0)^*} \mid . \end{aligned}$$

But $\widetilde{P}_j D_{(\widetilde{\mathbf{M}}-w_0)^*} = \widetilde{P}_j V_{\widetilde{\mathbf{M}}}(w_0) \mid D_{(\widetilde{\mathbf{M}}-w_0)^*} \mid$. The uniqueness of the polar decomposition implies that $\widetilde{P}_j V_{\widetilde{\mathbf{M}}}(w_0) = UP_j V_{\mathbf{M}}(w_0) U^*$, $1 \le j \le m$. It follows that $Q_{\widetilde{\mathbf{M}}}(w_0) = UQ_{\mathbf{M}}(w_0) U^*$.

Note that $P_j{}^* : \mathcal{M} \longrightarrow \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ is given by $P_j{}^* h = (0, \ldots, h, \ldots, 0)$, $h \in \mathcal{M}, 1 \le j \le m$. So we have $V_{\widetilde{\mathbf{M}}}(w_0)^* \widetilde{P}_j^* = UV_{\mathbf{M}}(w_0)^* P_j{}^* U^*$, $1 \le j \le m$. Let $\widetilde{D}_{\bar{w}} : \mathcal{M} \longrightarrow \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ be the operator: $\widetilde{D}_{\bar{w}} f = (\bar{w}_1 f, \ldots, \bar{w}_m f)$, $f \in \widetilde{\mathcal{M}}$. Clearly, $\widetilde{D}_{\bar{w}} = UD_{\bar{w}} U^*$, that is, $U^* \widetilde{P}_j \widetilde{D}_{\bar{w}} = P_j D_{\bar{w}} U^*$, $1 \le j \le m$. Finally,

$$\begin{aligned} & R_{\widetilde{\mathbf{M}}}(w_0) \widetilde{D}_{\bar{w}-\bar{w}_0} \\ &= Q_{\widetilde{\mathbf{M}}}(w_0) V_{\widetilde{\mathbf{M}}}(w_0)^* \widetilde{D}_{\bar{w}-\bar{w}_0} \;=\; Q_{\widetilde{\mathbf{M}}}(w_0) V_{\widetilde{\mathbf{M}}}(w_0)^* (\widetilde{P}_1 \widetilde{D}_{\bar{w}-\bar{w}_0}, \ldots, \widetilde{P}_m \widetilde{D}_{\bar{w}-\bar{w}_0}) \\ &= Q_{\widetilde{\mathbf{M}}}(w_0) V_{\widetilde{\mathbf{M}}}(w_0)^* (\sum_{j=1}^m \widetilde{P}_j^* \widetilde{P}_j \widetilde{D}_{\bar{w}-\bar{w}_0}) \\ &= Q_{\widetilde{\mathbf{M}}}(w_0) U V_{\mathbf{M}}(w_0)^* (\sum_{j=1}^m P_j{}^* U^* \widetilde{P}_j \widetilde{D}_{\bar{w}-\bar{w}_0}) \\ &= UQ_{\mathbf{M}}(w_0) V_{\mathbf{M}}(w_0)^* (\sum_{j=1}^m P_j{}^* P_j D_{\bar{w}-\bar{w}_0} U^*) \;=\; UQ_{\mathbf{M}}(w_0) V_{\mathbf{M}}(w_0)^* D_{\bar{w}-\bar{w}_0} U^* \\ &= UR_{\mathbf{M}}(w_0) D_{\bar{w}-\bar{w}_0} U^*. \end{aligned}$$

Hence $\{R_{\widetilde{\mathbf{M}}}(w_0)\widetilde{D}_{\bar{w}-\bar{w}_0}\}^k = U\{R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^k U^*$ for all $k \in \mathbb{N}$. From (3.1.3), $P(\bar{w},\bar{w}_0) = \sum_{k=0}^{\infty}\{R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^k P_{\ker D_{(\mathbf{M}-w_0)^*}}$. Also as U maps $\ker D_{(\mathbf{M}-w)^*}$ onto $\ker D_{(\widetilde{\mathbf{M}}-w)^*}$ for each w, we have in particular, $UP_{\ker D_{(\mathbf{M}-w_0)^*}} = P_{\ker D_{(\widetilde{M}-w_0)^*}}U$. Therefore,

$$\begin{aligned} &UP(\bar{w},\bar{w}_0) \\ &= \sum_{k=0}^{\infty} U\{R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^k P_{\ker D_{(\mathbf{M}-w_0)^*}} = \sum_{k=0}^{\infty}\{R_{\widetilde{\mathbf{M}}}(w_0)\widetilde{D}_{\bar{w}-\bar{w}_0}\}^k UP_{\ker D_{(\mathbf{M}-w_0)^*}} \\ &= \sum_{k=0}^{\infty}\{R_{\widetilde{\mathbf{M}}}(w_0)\widetilde{D}_{\bar{w}-\bar{w}_0}\}^k P_{\ker D_{(\widetilde{\mathbf{M}}-w_0)^*}}U = \widetilde{P}(\bar{w},\bar{w}_0)U, \end{aligned}$$

for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. □

Remark 3.3. For any commuting m-tuple $D_{\mathbf{T}} = (T_1, \ldots, T_m)$ of operator on $\mathcal{H}$, the construction given above, of the Hermitian holomorphic vector bundle, provides a unitary invariant, assuming only that $\operatorname{ran} D_{\mathbf{T}-w}$ is closed for w in $\Omega \subseteq \mathbb{C}^m$. Consequently, the class of this Hermitian holomorphic vector bundle is an invariant for any semi-Fredholm Hilbert module over $\mathbb{C}[\underline{z}]$.

3.3 Examples

Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules in $\mathrm{B}_1(\Omega)$ and $\mathcal{I}$, $\mathcal{J}$ be two ideals in $\mathbb{C}[\underline{z}]$. Let $\mathcal{M}_{\mathcal{I}} := [\mathcal{I}] \subseteq \mathcal{M}$ (resp. $\widetilde{\mathcal{M}}_{\mathcal{J}} := [\mathcal{J}] \subset \widetilde{\mathcal{M}}$) denote the closure of $\mathcal{I}$ in $\mathcal{M}$ (resp. closure of $\mathcal{J}$ in $\widetilde{\mathcal{M}}$). Also we let $\dim V(\mathcal{I}), \dim V(\mathcal{J}) \leq m-2$. The rigidity Theorem of section 2.3, says that if $\mathcal{M}_{\mathcal{I}}$ and $\widetilde{\mathcal{M}}_{\mathcal{J}}$ are equivalent, then $\mathcal{I} = \mathcal{J}$. We ask if $\mathcal{I} = \mathcal{J}$, whether $\mathcal{M}_{\mathcal{I}}$ is equivalent to $\widetilde{\mathcal{M}}_{\mathcal{I}}$. Also if we assume that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are minimal extensions of the two modules $\mathcal{M}_{\mathcal{I}}$ and $\widetilde{\mathcal{M}}_{\mathcal{I}}$ respectively and that $\mathcal{M}_{\mathcal{I}}$ is equivalent to $\widetilde{\mathcal{M}}_{\mathcal{I}}$, then does it follow that the extensions $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent? The answers for a class of examples is given below.

For $\lambda, \mu > 0$, let $H^{(\lambda,\mu)}(\mathbb{D}^2)$ be the reproducing kernel Hilbert space on the bi-disc determined by the positive definite kernel

$$K^{(\lambda,\mu)}(z,w) = \frac{1}{(1-z_1\bar{w}_1)^{\lambda}(1-z_2\bar{w}_2)^{\mu}},\ z,w \in \mathbb{D}^2.$$

As is well-known, $H^{(\lambda,\mu)}(\mathbb{D}^2)$ is in $\mathrm{B}_1(\mathbb{D}^2)$. Let I be the maximal ideal in $\mathbb{C}[z_1,z_2]$ of polynomials vanishing at $(0,0)$. Let $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := [I]$. For any other pair of positive numbers λ', μ', we let $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ denote the closure of I in the reproducing kernel Hilbert space $H^{(\lambda',\mu')}(\mathbb{D}^2)$. Let $K^{(\lambda',\mu')}$ denote the corresponding reproducing kernel. The modules $H^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H^{(\lambda',\mu')}(\mathbb{D}^2)$ are in $\mathrm{B}_1(\mathbb{D}^2 \setminus \{(0,0)\})$ but not in $\mathrm{B}_1(\mathbb{D}^2)$. So, there is no easy computation to determine when they are equivalent. We compute the curvature, at $(0,0)$, of the holomorphic Hermitian bundle $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ of rank 2 corresponding to the modules $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ respectively. The calculation of the curvature show that if these modules are equivalent then $\lambda = \lambda'$ and $\mu = \mu'$, that is, the extensions $H^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H^{(\lambda',\mu')}(\mathbb{D}^2)$ are then equal.

Since $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := \{f \in H^{(\lambda,\mu)}(\mathbb{D}^2) : f(0,0) = 0\}$, the corresponding reproducing kernel $K_0^{(\lambda,\mu)}$ is given by the formula

$$K_0^{(\lambda,\mu)}(z,w) = \frac{1}{(1-z_1\bar{w}_1)^{\lambda}(1-z_2\bar{w}_2)^{\mu}} - 1,\ z,w \in \mathbb{D}^2.$$

The set $\{z_1^m z_2^n : m,n \geq 0, (m,n) \neq (0,0)\}$ forms an orthogonal basis for $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$. Also $\langle z_1^l z_2^k, M_1^* z_1^{m+1}\rangle = \langle z_1^{l+1} z_2^k, z_1^{m+1}\rangle = 0$, unless $l = m, k = 0$ and $m > 0$. In consequence,

$$\langle z_1^m, M_1^* z_1^{m+1}\rangle = \langle z_1^{m+1}, z_1^{m+1}\rangle = \frac{1}{(-1)^{m+1}\binom{-\lambda}{m+1}} = \frac{(-1)^m\binom{-\lambda}{m}}{(-1)^{m+1}\binom{-\lambda}{m+1}}\langle z_1^m, z_1^m\rangle.$$

Then

$$\langle z_1^l z_2^k, M_1^* z_1^{m+1} - \frac{m+1}{\lambda+m} z_1^m\rangle = 0 \text{ for all } l,k \geq 0, (l,k) \neq (0,0),$$

where $\binom{-\lambda}{m} = (-1)^m \frac{\lambda(\lambda+1)\ldots(\lambda+m-1)}{m!}$. Now, $\langle z_1^l z_2^k, M_1^* z_1\rangle = \langle z_1^{l+1} z_2^k, z_1\rangle = 0$, $l,k \geq 0$ and $(l,k) \neq (0,0)$. Therefore, we have

$$M_1^* z_1^{m+1} = \begin{cases} \frac{m+1}{\lambda+m} z_1^m & m > 0 \\ 0 & m = 0. \end{cases}$$

Similarly,

$$M_2^* z_2^{n+1} = \begin{cases} \frac{n+1}{\lambda+n} z_1^n & n > 0 \\ 0 & n = 0. \end{cases}$$

We easily verify that $\langle z_1^l z_2^k, M_2^* z_1^{m+1}\rangle = \langle z_1^l z_2^{k+1}, z_1^{m+1}\rangle = 0$. Hence $M_2^* z_1^{m+1} = 0 = M_1^* z_2^{n+1}$ for $m,n \geq 0$. Finally, calculations similar to the one given above, show that

$$M_1^* z_1^{m+1} z_2^{n+1} = \frac{m+1}{\lambda+m} z_1^m z_2^{n+1} \text{ and } M_2^* z_1^{m+1} z_2^{n+1} = \frac{n+1}{\mu+n} z_1^{m+1} z_2^n, m.n \geq 0$$

Therefore we have

$$(M_1M_1^* + M_2M_2^*) : \begin{cases} z_1^{m+1} \longmapsto \frac{m+1}{\lambda+m} z_1^{m+1}, & \text{for } m > 0; \\ z_2^{n+1} \longmapsto \frac{n+1}{\mu+n} z_2^{n+1}, & \text{for } n > 0; \\ z_1^{m+1} z_2^{n+1} \longmapsto (\frac{m+1}{\lambda+m} + \frac{n+1}{\mu+n}) z_1^{m+1} z_2^{n+1}, & \text{for } m,n \geq 0; \\ z_1, z_2 \longmapsto 0. \end{cases}$$

Also, since $D_{\mathbf{M}^*} f = (M_1^* f, M_2^* f)$, we have

$$D_{\mathbf{M}^*} : \begin{cases} z_1^{m+1} \longmapsto (\frac{m+1}{\lambda+m} z_1^m, 0), & \text{for } m > 0; \\ z_2^{n+1} \longmapsto (0, \frac{n+1}{\mu+n} z_2^n), & \text{for } n > 0; \\ z_1^{m+1} z_2^{n+1} \longmapsto (\frac{m+1}{\lambda+m} z_1^m z_2^{n+1}, \frac{n+1}{\mu+n} z_1^{m+1} z_2^n), & \text{for } m,n \geq 0; \\ z_1, z_2 \longmapsto (0,0). \end{cases}$$

It is easy to calculate $V_{\mathbf{M}}(0)$ and $Q_{\mathbf{M}}(0)$ and show that

$$V_{\mathbf{M}}(0):\begin{cases} z_1^{m+1}\longmapsto \sqrt{\frac{m+1}{\lambda+m}}(z_1^m,0), & \text{for } m>0;\\ z_2^{n+1}\longmapsto \sqrt{\frac{n+1}{\mu+n}}(0,z_2^n), & \text{for } n>0;\\ z_1^{m+1}z_2^{n+1}\longmapsto \frac{1}{\sqrt{\frac{m+1}{\lambda+m}+\frac{n+1}{\mu+n}}}(\frac{m+1}{\lambda+m}z_1^m z_2^{n+1},\frac{n+1}{\mu+n}z_1^{m+1}z_2^n), & \text{for } m,n\geq 0;\\ z_1,z_2\longmapsto (0,0), \end{cases}$$

while

$$Q_{\mathbf{M}}(0):\begin{cases} z_1^{m+1}\longmapsto \frac{1}{\sqrt{\frac{m+1}{\lambda+m}}}z_1^{m+1}, & \text{for } m>0;\\ z_2^{n+1}\longmapsto \frac{1}{\sqrt{\frac{n+1}{\mu+n}}}z_2^{n+1}, & \text{for } n>0;\\ z_1^{m+1}z_2^{n+1}\longmapsto \frac{1}{\sqrt{\frac{m+1}{\lambda+m}+\frac{n+1}{\mu+n}}}z_1^{m+1}z_2^{n+1}, & \text{for } m,n\geq 0;\\ z_1,z_2\longmapsto 0. \end{cases}$$

Now for $w\in\Delta(0,\varepsilon)$,

$$P(\bar{w},0)=(I-R_{\mathbf{M}}(0)D_{\bar{w}})^{-1}P_{\ker D_{M^*}}=\sum_{n=0}^{\infty}(R_{\mathbf{M}}(0)D_{\bar{w}})^nP_{\ker D_{M^*}},$$

where $R_{\mathbf{M}}(0)=Q_{\mathbf{M}}(0)V_{\mathbf{M}}(0)^*$. The vectors z_1 and z_2 forms a basis for $\ker D_{\mathbf{M}^*}$ and therefore define a holomorphic frame: $\big(P(\bar{w},0)z_1,P(\bar{w},0)z_2\big)$. Recall that $P(\bar{w},0)z_1=\sum_{n=0}^{\infty}(R_{\mathbf{M}}(0)D_{\bar{w}})^nz_1$ and $P(\bar{w},0)z_2=\sum_{n=0}^{\infty}(R_{\mathbf{M}}(0)D_{\bar{w}})^nz_2$. To describe these explicitly, we calculate $(R_{\mathbf{M}}(0)D_{\bar{w}})z_1$ and $(R_{\mathbf{M}}(0)D_{\bar{w}})z_2$:

$$\begin{aligned}(R_{\mathbf{M}}(0)D_{\bar{w}})z_1 &= R_{\mathbf{M}}(0)(\bar{w}_1,z_1,\bar{w}_2z_2)\\ &= \bar{w}_1R_{\mathbf{M}}(0)(z_1,0)+\bar{w}_2R_{\mathbf{M}}(0)(0,z_2)\\ &= \bar{w}_1Q_{\mathbf{M}}(0)V_{\mathbf{M}}(0)^*(z_1,0)+\bar{w}_2Q_{\mathbf{M}}(0)V_{\mathbf{M}}(0)^*(0,z_2).\end{aligned}$$

We see that

$$V_{\mathbf{M}}(0)^*(z_1,0)=\sum_{l,k\geq 0,\,(l,k)\neq(0,0)}\langle V_{\mathbf{M}}(0)^*(z_1,0),\frac{z_1^lz_2^k}{\parallel z_1^lz_2^k\parallel}\rangle\frac{z_1^lz_2^k}{\parallel z_1^lz_2^k\parallel}.$$

Therefore,

$$\langle V_{\mathbf{M}}(0)^*(z_1,0),z_1^lz_2^k\rangle=\langle(z_1,0),V_{\mathbf{M}}(0)(z_1^lz_2^k)\rangle,\ l,k\geq 0,(l,k)\neq(0,0).$$

From the explicit form of $V_{\mathbf{M}}(0)$, it is clear that the inner product given above is 0 unless $l=2$, $k=0$. For $l=2$, $k=0$, we have

$$\langle(z_1,0),V_{\mathbf{M}}(0)z_1^2\rangle=\sqrt{\frac{2}{\lambda+1}}\parallel z_1\parallel^2=\sqrt{\frac{2}{\lambda+1}}\frac{1}{\lambda}.$$

Hence

$$V_{\mathbf{M}}(0)^*(z_1,0)=\sqrt{\frac{2}{\lambda+1}}\frac{1}{\lambda}\frac{z_1^2}{\parallel z_1^2\parallel^2}=\sqrt{\frac{2}{\lambda+1}}\frac{1}{\lambda}\frac{\lambda(\lambda+1)}{2}z_1^2=\sqrt{\frac{\lambda+1}{2}}z_1^2.$$

Again, to calculate $V_{\mathbf{M}}(0)^*(0, z_1)$, we note that $\langle V_{\mathbf{M}}(0)^*(0, z_1), z_1^l z_2^k\rangle$ is 0 unless $l = 1$, $m = 1$. For $l = 1$, $m = 1$, we have

$$\begin{aligned}\langle V_{\mathbf{M}}(0)^*(0, z_1), z_1 z_2\rangle &= \langle (0, z_1), V_{\mathbf{M}}(0) z_1 z_2\rangle \\ &= \langle \frac{1}{\sqrt{\frac{1}{\lambda} + \frac{1}{\mu}}}(\frac{1}{\lambda} z_2, \frac{1}{\mu} z_1), (0, z_1)\rangle \\ &= \frac{1}{\sqrt{\frac{1}{\lambda} + \frac{1}{\mu}}} \frac{1}{\mu} \parallel z_1 \parallel^2 = \frac{1}{\sqrt{\frac{1}{\lambda} + \frac{1}{\mu}}} \frac{1}{\lambda\mu}.\end{aligned}$$

Thus

$$V_{\mathbf{M}}(0)^*(0, z_1) = \langle V_{\mathbf{M}}(0)^*(0, z_1), z_1 z_2\rangle \frac{z_1 z_2}{\parallel z_1 z_2 \parallel^2} = \frac{1}{\sqrt{\frac{1}{\lambda} + \frac{1}{\mu}}} z_1 z_2.$$

Since

$$\begin{aligned}Q_{\mathbf{M}}(0) z_1^2 &= \sqrt{\frac{\lambda + 1}{2}} z_1^2, \\ Q_{\mathbf{M}}(0) z_1 z_2 &= \frac{1}{\sqrt{\frac{1}{\lambda} + \frac{1}{\mu}}} z_1 z_2, \\ Q_{\mathbf{M}}(0) z_2^2 &= \sqrt{\frac{\mu + 1}{2}} z_2^2,\end{aligned}$$

it follows that

$$R_{\mathbf{M}}(0) D_{\bar{w}} z_1 = \bar{w}_1 \frac{\lambda + 1}{2} z_1^2 + \bar{w}_2 \frac{\lambda\mu}{\lambda + \mu} z_1 z_2.$$

Similarly, we obtain the formula

$$R_{\mathbf{M}}(0) D_{\bar{w}} z_2 = \bar{w}_1 \frac{\lambda\mu}{\lambda + \mu} z_1 z_2 + \bar{w}_2 \frac{\mu + 1}{2} z_2^2.$$

We claim that

$$\langle (R_{\mathbf{M}}(0) D_{\bar{w}})^m z_i, (R_{\mathbf{M}}(0) D_{\bar{w}})^n z_j\rangle = 0 \text{ for all } m \neq n \text{ and } i, j = 1, 2. \tag{3.3.1}$$

This makes the calculation of

$$h(w, w) = \big(\big(\langle P(\bar{w}, 0) z_i, P(\bar{w}, 0) z_j\rangle\big)\big)_{1 \le i,j \le 2}, \; w \in U \subset \mathbb{D}^2,$$

which is the Hermitian metric for the vector bundle $\mathcal{P}$, on some small open set $U \subseteq \mathbb{D}^2$ around $(0, 0)$, corresponding to the module $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$, somewhat easier.

We will prove the claim by showing that $(R_{\mathbf{M}}(0) D_{\bar{w}})^n z_i$ consists of terms of degree $n + 1$. For this, it is enough to calculate $V_{\mathbf{M}}(0)^*(z_1^l z_2^k, 0)$ and $V_{\mathbf{M}}(0)^*(0, z_1^l z_2^k)$ for different l, $k \ge 0$ such that $(l, k) \neq (0, 0)$. Calculations similar to that of $V_{\mathbf{M}}(0)^*$ show that

$$\begin{aligned}&V_{\mathbf{M}}(0)^*(z_1^m, 0) = \sqrt{\frac{\lambda + m}{m + 1}} z_1^{m+1}, V_{\mathbf{M}}(0)^*(0, z_2^n) = \sqrt{\frac{\mu + n}{n + 1}} z_2^{n+1} \text{ and,} \\ &V_{\mathbf{M}}(0)^*(z_1^m z_2^{n+1}, 0) = V_{\mathbf{M}}(0)^*(0, z_1^{m+1} z_2^n) = \frac{1}{\sqrt{\frac{m+1}{\mu+n} + \frac{n+1}{\mu+n}}} z_1^{m+1} z_2^{n+1}.\end{aligned}$$

Recall that $(R_{\mathbf{M}}(0)D_{\bar{w}})z_i$ is of degree 2. From the equations given above, inductively, we see that $(R_{\mathbf{M}}(0)D_{\bar{w}})^n z_i$ is of degree $n+1$. Since monomials are orthogonal in $H^{(\lambda,\mu)}(\mathbb{D}^2)$, the proof of claim (3.3.1) is complete. We then have

$$P(\bar{w},0)z_1 = z_1 + \bar{w}_1\frac{\lambda+1}{2}z_1^2 + \bar{w}_2\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \sum_{n=2}^{\infty}(R_{\mathbf{M}}(0)D_{\bar{w}})^n z_1 \text{ and}$$

$$P(\bar{w},0)z_2 = z_2 + \bar{w}_1\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \bar{w}_2\frac{\mu+1}{2}z_2^2 + \sum_{n=2}^{\infty}(R_{\mathbf{M}}(0)D_{\bar{w}})^n z_2.$$

Putting all of this together, we see that

$$h(w,w) = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} + \sum a_{IJ}w^I\bar{w}^J,$$

where the sum is over all multi-indices I,J satisfying $|I|,|J|>0$ and $w^I = w_1^{i_1}w_2^{i_2}$, $\bar{w}^J = \bar{w}_1^{j_1}\bar{w}_2^{j_2}$. The metric h is (almost) normalized at $(0,0)$, that is, $h(w,0) = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)$. The metric h_0 obtained by conjugating the metric h by the invertible (constant) linear transformation $\left(\begin{smallmatrix}\sqrt{\lambda} & 0\\ 0 & \sqrt{\mu}\end{smallmatrix}\right)$ induces an equivalence of holomorphic Hermitian bundles. The vector bundle $\mathcal{P}$ equipped with the Hermitian metric h_0 has the additional property that the metric is normalized: $h_0(w,0)=I$. The coefficient of $dw_i\wedge d\bar{w}_j$, $i,j=1,2$, in the curvature of the holomorphic Hermitian bundle $\mathcal{P}$ at $(0,0)$ is then the Taylor coefficient of $w_i\,\bar{w}_j$ in the expansion of h_0 around $(0,0)$ (cf. [30, Lemma 2.3]).

Thus the normalized metric $h_0(w,w)$, which is real analytic, is of the form

$$\begin{aligned} h_0(w,w) &= \begin{pmatrix} \lambda\langle P(\bar{w},0)z_1, P(\bar{w},0)z_1\rangle & \sqrt{\lambda\mu}\langle P(\bar{w},0)z_1, P(\bar{w},0)z_2\rangle \\ \sqrt{\lambda\mu}\langle P(\bar{w},0)z_2, P(\bar{w},0)z_1\rangle & \mu\langle P(\bar{w},0)z_2, P(\bar{w},0)z_2\rangle \end{pmatrix} \\ &= I + \begin{pmatrix} \frac{\lambda+1}{2}|w_1|^2 + \frac{\lambda^2\mu}{(\lambda+\mu)^2}|w_2|^2 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_1\bar{w}_2 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_2\bar{w}_1 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}|w_1|^2 + \frac{\mu+1}{2}|w_2|^2 \end{pmatrix} + O(|w|^3), \end{aligned}$$

where $O(|w|^3)_{i,j}$ is of degree ≥ 3. Explicitly, it is of the form

$$\sum_{n=2}^{\infty}\langle (R_{\mathbf{M}}(0)D_{\bar{w}})^n z_i, (R_{\mathbf{M}}(0)D_{\bar{w}})^n z_j\rangle.$$

The curvature at $(0,0)$, as pointed out earlier, is given by $\bar{\partial}\partial h_0(0,0)$. Consequently, if $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent, then the corresponding holomorphic Hermitian vector bundles $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ of rank 2 must be equivalent. Hence their curvatures, in particular, at $(0,0)$, must be unitarily equivalent. The curvature for $\mathcal{P}$ at $(0,0)$ is given by the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\lambda^2\mu}{(\lambda+\mu)^2} & 0 \\ 0 & \frac{\mu+1}{2} \end{pmatrix}.$$

The curvature for $\widetilde{\mathcal{P}}$ has a similar form with λ' and μ' in place of λ and μ respectively. All of them are to be simultaneously equivalent by some unitary map. The only unitary that intertwines the 2×2 matrices

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda'\mu'}}\left(\frac{\lambda'\mu'}{\lambda'+\mu'}\right)^2 \\ 0 & 0 \end{pmatrix}$$

is aI with $|a|=1$. Since this fixes the unitary intertwiner, we see that the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{\lambda'+1}{2} & 0 \\ 0 & \frac{\lambda'\mu'^2}{(\lambda'+\mu')^2} \end{pmatrix}$$

must be equal. Hence we have $\frac{\lambda+1}{2} = \frac{\lambda+1}{2}$, that is $\lambda = \lambda'$. Consequently, $\frac{\lambda\mu^2}{(\lambda+\mu)^2} = \frac{\lambda'\mu'^2}{(\lambda'+\mu')^2}$ gives $\frac{\mu^2}{(\lambda+\mu)^2} = \frac{\mu'^2}{(\lambda+\mu')^2}$ and then $(\mu-\mu')\{\lambda^2(\mu+\mu')+2\lambda\mu\mu'\}=0$. We then have $\mu=\mu'$. Therefore, $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent if and only if $\lambda=\lambda'$ and $\mu=\mu'$.

4. Description of the joint kernel

To compute the curvature invariant for Hilbert modules in $\mathfrak{B}_1(\Omega)$, the explicit description of a basis for the joint kernel is essential. In fact, it will be desirable to obtain such description in terms of derivatives of the reproducing kernel. Let us go back to the example of $H_0^2(\mathbb{D}^2)$. Let K_0 be the reproducing kernel for $H_0^2(\mathbb{D}^2)$. For $\mathcal{H}$ in $\mathfrak{B}_1(\Omega)$, pick any $g \in \mathcal{H}$ and $p \in \mathbb{C}[\underline{z}]$. Then

$$\begin{aligned}\langle g, M_p^*\bar{\partial}_i K(\cdot,w)\rangle &= \langle pg, \bar{\partial}_i K(\cdot,w)\rangle = \partial_i(pg)(w) = \partial_i p(w)g(w) + p(w)\partial_i g(w)\\ &= \partial_i p(w)\langle g, K(\cdot,w)\rangle + p(w)\langle g, \bar{\partial}_i K(\cdot,w)\rangle\\ &= \langle g, \partial_i \bar{p}(w)K(\cdot,w) + \overline{p(w)}\bar{\partial}_i K(\cdot,w)\rangle\end{aligned}$$

which implies that

$$M_p^*\bar{\partial}_i K(\cdot,w) = \partial_i \bar{p}(w)K(\cdot,w) + \overline{p(w)}\bar{\partial}_i K(\cdot,w),\ 1 \le i \le m,$$

So we have $M_p^*\bar{\partial}_i K_0(\cdot,w)|_0 = \overline{p(0)}\bar{\partial}_i K_0(\cdot,w)|_0$. In particular $M_j^*\bar{\partial}_i K_0(\cdot,w)|_0 = 0,\ 1 \le i,j \le 2$. In other words, $\bar{\partial}_i K_0(\cdot,w)|_0$ is in $\ker D_{\mathbf{M}^*}$, $i = 1,2$. Next we check that these vectors are independent. Since

$$\langle f, a_1\bar{\partial}_1 K_0(\cdot,w) + a_2\bar{\partial}_1 K_0(\cdot,w)\rangle = \bar{a}_1\partial_1 f(w) + \bar{a}_2\partial_2 f(w),\ f \in H_0^2(\mathbb{D}^2),$$

assuming $a_1\bar{\partial}_1 K_0(\cdot,w)|_0 + a_2\bar{\partial}_1 K_0(\cdot,w)|_0 = 0$ will force $\bar{a}_1\partial_1 f(0) + \bar{a}_2\partial_2 f(0) = 0$. Choosing $f(z) = z_1$, we conclude that $a_1 = 0$. Similarly by choosing $f(z) = z_2$, $a_2 = 0$. Hence we have proved that $\bar{\partial}_1 K_0(\cdot,w)|_0, \bar{\partial}_2 K_0(\cdot,w)|_0$ are independent. Let $\gamma_w \in \cap_{j=1}^2 \ker M_j^* \subseteq H_0^2(\mathbb{D}^2)$, and let

$$\gamma_w(z) = \sum_{(k,l)\neq(0,0)} a_{kl} z_1^k z_2^l,\ z = (z_1,z_2) \in \mathbb{D}^2.$$

Now $M_1^* z_1^k z_2^l = z_1^{l-1} z_2^l$, and $M_2^* z_1^k z_2^l = z_1^k z_2^{l-1}$ for $k,l \ge 1$, which shows that $z_1^k z_2^l$ can not be in $\ker D_{\mathbf{M}^*}$ for $k,l \ge 1$. So $\gamma_w(z) = a_{10}z_1 + a_{01}z_2$. We note that $\bar{\partial}_1 K_0(z,w)|_0 = z_1$, and $\bar{\partial}_2 K_0(z,w)|_0 = z_2$ (In fact, $\{\bar{\partial}_1^k\bar{\partial}_2^l K_0(\cdot,w)|_0\}_{k,l\ge 0,(k,l)\neq(0,0)}$ generates $H_0^2(\mathbb{D}^2)$.) Thus we have $\gamma_w(z) = a_{10}\bar{\partial}_1 K_0(z,w)|_{w=0} + a_{01}\bar{\partial}_2 K_0(z,w)|_{w=0}$ and hence $\{\bar{\partial}_1 K_0(\cdot,w), \bar{\partial}_1 K_0(\cdot,w)\}$ is a basis of $\ker D_{\mathbf{M}^*}$. Only the last argument is specific to the module $H_0^2(\mathbb{D}^2)$. In general, using Lemma 5.11 in [15], or using Theorem 2.3 along with Remark 2.2, we arrive at the same conclusion. Thus we have the following lemma.

Lemma 4.1. *Let $\mathcal{M}$ be an analytic Hilbert module over $\Omega \subseteq \mathbb{C}^m$, and $\mathcal{M}_n$ be a submodule of $\mathcal{M}$ of the form $[\mathcal{I}]$, where*

$$\mathcal{I} = \langle z^\alpha = z_1^{\alpha_1}\ldots z_m^{\alpha_m} : \alpha_i \in \mathbb{N}\cup\{0\}, |\alpha| = \sum_{i=1}^m \alpha_i = n\rangle.$$

Let K_n be the reproducing kernel corresponding to $\mathcal{M}_n$. We have

(1) $\mathcal{M}_n = \{f \in \mathcal{M} : \partial^\alpha f(0) = 0, \text{ for } \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| \leq n-1\}$

(2) $\ker D_{(\mathbf{M}|_{\mathcal{M}_n} - w)^*} = \begin{cases} \operatorname{span}\{\mathrm{K_n}(\cdot, \mathrm{w})\}, & \text{for } w \neq 0; \\ \operatorname{span}\{\bar\partial^\alpha \mathrm{K_n}(\cdot, w)|_{w=0} : \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| = n\}, & \text{for } w = 0. \end{cases}$

The Lemma given above, describes the joint kernel for a particular class of submodules of analytic Hilbert module. However it is not clear that such explicit calculation are possible for modules which are closures of arbitrary polynomial ideal. In this chapter, we have addressed this issue at length.

Construction of the Fock inner product. The Fock inner product of a pair of polynomials p and q is defined by the rule:

$$\langle p, q\rangle_0 = q^*(\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_m})\, p|_0,\ q^*(z) = \overline{q(\bar z)}.$$

The map $\langle\ ,\ \rangle_0 : \mathbb{C}[\underline{z}] \times \mathbb{C}[\underline{z}] \longrightarrow \mathbb{C}$ is linear in first variable and conjugate linear in the second and for $p = \sum_\alpha a_\alpha z^\alpha,\ q = \sum_\alpha b_\alpha z^\alpha$ in $\mathbb{C}[\underline{z}]$, we have

$$\langle p, q\rangle_0 = \sum_\alpha \alpha! a_\alpha \bar b_\alpha$$

since $z^\alpha(D) z^\beta|_{z=0} = \alpha!$ if $\alpha = \beta$ and 0 otherwise. Also, $\langle p, p\rangle_0 = \sum_\alpha \alpha! |a_\alpha|^2 \geq 0$ and equals 0 only when $a_\alpha = 0$ for all α. The completion of the polynomial ring with this inner product is the well known Fock space $L^2_a(\mathbb{C}^m, d\mu)$, that is, the space of all μ-square integrable entire functions on $\mathbb{C}^m$, where

$$d\mu(z) = \pi^{-m} e^{-|z|^2} d\nu(z)$$

is the Gaussian measure on $\mathbb{C}^m$ ($d\nu$ is the usual Lebesgue measure).

The characteristic space (see section 1.4) of an ideal $\mathcal{I}$ in $\mathbb{C}[\underline{z}]$ at the point w is the vector space

$$\mathbb{V}_w(\mathcal{I}) = \{q \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0,\, p \in \mathcal{I}\} = \{q \in \mathbb{C}[\underline{z}] : \langle p, q^*\rangle_w = 0,\, p \in \mathcal{I}\}.$$

The envelope of the ideal $\mathcal{I}$ at the point w is defined to be the ideal

$$\begin{aligned} \mathcal{I}^e_w &= \{p \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0,\, q \in \mathbb{V}_w(\mathcal{I})\} \\ &= \{p \in \mathbb{C}[\underline{z}] : \langle p, q^*\rangle_w = 0,\, q \in \mathbb{V}_w(\mathcal{I})\}. \end{aligned}$$

It is known [7, Theorem 2.1.1, page 13] that $\mathcal{I} = \cap_{w \in V(\mathcal{I})} \mathcal{I}^e_w$. The proof makes essential use of the well known Krull's intersection theorem. In particular, if $V(\mathcal{I}) = \{w\}$, then $\mathcal{I}^e_w = \mathcal{I}$. It is easy to verify this special case using the Fock inner product. We provide the details below after setting $w = 0$, without loss of generality.

Let $\mathfrak{m}_0$ be the maximal ideal in $\mathbb{C}[\underline{z}]$ at 0. By Hilbert's Nullstellensatz, there exists a positive integer N such that $\mathfrak{m}_0^N \subseteq \mathcal{I}$. We identify $\mathbb{C}[\underline{z}]/\mathfrak{m}_0^N$ with $\operatorname{span}_{\mathbb{C}}\{z^\alpha : |\alpha| < N\}$ which is the same as $(\mathfrak{m}_0^N)^\perp$ in the Fock inner product. Let $\mathcal{I}_N$ be the vector space $\mathcal{I}\cap \text{span}_{\mathbb{C}}\{z^\alpha : |\alpha| < N\}$. Clearly $\mathcal{I}$ is the vector space (orthogonal) direct sum $\mathcal{I}_N \oplus \mathfrak{m}_0^N$. Let

$$\widetilde{V} = \{q \in \mathbb{C}[\underline{z}] : \deg q < N \text{ and } \langle p, q\rangle_0 = 0,\, p \in \mathcal{I}_N\} = \big(\mathfrak{m}_0^N\big)^\perp \ominus \mathcal{I}_N.$$

Evidently, $\mathbb{V}_0(\mathcal{I}) = \widetilde{V}^*$, where $\widetilde{V}^* = \{q \in V : q^* \in \widetilde{V}\}$. It is therefore clear that the definition of $\widetilde{V}$ is independent of N, that is, if $\mathfrak{m}^{N_1} \subset \mathcal{I}$ for some N_1, then $(\mathfrak{m}_0^{N_1})^\perp \ominus \mathcal{I}_{N_1} = (\mathfrak{m}_0^N)^\perp \ominus \mathcal{I}_N$. Thus

$$\begin{aligned}\mathcal{I}_0^e &= \{p \in \mathbb{C}[\underline{z}] : \deg p < N \text{ and } \langle p, q^*\rangle_0 = 0,\, q \in \mathbb{V}_0(\mathcal{I})\} \oplus \mathfrak{m}_0^N \\ &= \big((\mathfrak{m}_0^N)^\perp \ominus \widetilde{V}\big) \oplus \mathfrak{m}_0^N \\ &= \mathcal{I}_N \oplus \mathfrak{m}_0^N\end{aligned}$$

showing that $\mathcal{I}_0^e = \mathcal{I}$.

Let $\mathcal{M}$ be a submodule of an analytic Hilbert module $\mathcal{H}$ on Ω such that $\mathcal{M} = [\mathcal{I}]$, closure of the ideal $\mathcal{I}$ in $\mathcal{H}$. It is known that $\mathbb{V}_0(\mathcal{I}) = \mathbb{V}_0(\mathcal{M})$ (cf. [6, 16]). Since

$$\mathcal{M} \subseteq \mathcal{M}_0^e := \{f \in \mathcal{H} : q(D)f|_0 = 0 \text{ for all } q \in \mathbb{V}_0(\mathcal{M})\},$$

it follows that

$$\begin{aligned}\dim\mathcal{H}/\mathcal{M}_0^e \le \dim\mathcal{H}/\mathcal{M} = \dim \mathbb{C}[\underline{z}]/\mathcal{I} &\le \dim \mathbb{C}[\underline{z}]/\mathfrak{m}_0^N \\ &\le \sum_{k=0}^{N-1} \binom{k+m-1}{m-1} < +\infty.\end{aligned}$$

Therefore, from [16, Corollary 2.8], we have $\mathcal{M}_0^e \cap \mathbb{C}[\underline{z}] = \mathcal{I}_0^e$ and $\mathcal{M} \cap \mathbb{C}[\underline{z}] = \mathcal{I}$, and hence

$$\mathcal{M}_0^e = [\mathcal{I}_0^e] = [\mathcal{I}] = \mathcal{M}. \tag{4.0.1}$$

4.1 Modules of the form $[\mathcal{I}]$

Assumption: We assume that the Hilbert module $\mathcal{M}$ is (i) the completion with respect to some inner product of the ideal $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$ and that (ii) it is in the class $\mathfrak{B}_1(\Omega)$.

For notational convenience, in the following discussion, we let K be the reproducing kernel of $\mathcal{M} = [\mathcal{I}]$, instead of $K_{[\mathcal{I}]}$. To describe the joint kernel $\ker D_{(\mathbf{M}-\mathbf{w})^*}$ using the characteristic space $\mathbb{V}_w(\mathcal{I})$, it will be useful to recall the auxiliary space

$$\widetilde{\mathbb{V}}_w(\mathcal{I}) \;=\; \{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_w(\mathcal{I}),\, 1 \le i \le m\}.$$

From [6, Lemma 3.4], it follows that $V(\mathfrak{m}_w\mathcal{I}) \setminus V(\mathcal{I}) = \{w\}$ and $\mathbb{V}_w(\mathfrak{m}_w\mathcal{I}) = \widetilde{\mathbb{V}}_w(\mathcal{I})$. Therefore,

$$\begin{aligned}\dim\ker D_{(\mathbf{M}-\mathbf{w})^*} &= \dim \mathcal{M}/\mathfrak{m}_w\mathcal{M} \;=\; \dim \mathcal{I}/\mathfrak{m}_w\mathcal{I} \\ &= \sum_{\lambda \in V(\mathfrak{m}_w\mathcal{I})\setminus V(\mathcal{I})} \dim \mathbb{V}_\lambda(\mathfrak{m}_w\mathcal{I})/\mathbb{V}_\lambda(\mathcal{I}) \\ &= \dim \widetilde{\mathbb{V}}_w(\mathcal{I})/\mathbb{V}_w(\mathcal{I}).\end{aligned} \tag{4.1.1}$$

For the second and the third equalities, see [7, Theorem 2.2.5 and 2.1.7]. Since $\widetilde{\mathbb{V}}_w(\mathcal{I})$ is a subspace of the inner product space $\mathbb{C}[\underline{z}]$, we will often identify the quotient space $\widetilde{\mathbb{V}}_w(\mathcal{I})/\mathbb{V}_w(\mathcal{I})$ with the subspace of $\widetilde{\mathbb{V}}_w(\mathcal{I})$ which is the orthogonal complement of $\mathbb{V}_w(\mathcal{I})$ in $\widetilde{\mathbb{V}}_w(\mathcal{I})$. Equation (4.1.1) motivates following lemma describing the basis of the joint kernel of the adjoint of the multiplication operator at a point in Ω. This answers the question (1) of the introduction.

Lemma 4.2. *Fix $w_0 \in \Omega$ and polynomials $q_1, \ldots, q_t$. Let $\mathcal{I}$ be a polynomial ideal and K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then the vectors*

$$q_1(\bar{D})K(\cdot, w)|_{w=w_0}, \ldots, q_t(\bar{D})K(\cdot, w)|_{w=w_0}$$

form a basis of the joint kernel at w_0 of the adjoint of the multiplication operator if and only if the classes $[q_1^*], \ldots, [q_t^*]$ form a basis of $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$.

Proof. Without loss of generality we assume $0 \in \Omega$ and $w_0 = 0$.

Claim 1: For any $q \in \mathbb{C}[\underline{z}]$, the vector $q(\bar{D})K(\cdot, w)|_{w=0} \neq 0$ if and only if $q^* \notin \mathbb{V}_0(\mathcal{I})$.

Using the reproducing property $f(w) = \langle f, K(\cdot, w)\rangle$ of the kernel function K, it is easy to see (cf. [11]) that

$$\partial^\alpha f(w) = \langle f, \bar{\partial}^\alpha K(\cdot, w)\rangle,\ \alpha \in \mathbb{Z}_m^+,\ w \in \Omega,\ f \in \mathcal{M}.$$

Thus

$$\begin{aligned}
\partial^\alpha f(w)|_{w=0} &= \langle f, \bar{\partial}^\alpha K(\cdot, w)\rangle|_{w=0} = \langle f, \bar{\partial}^\alpha\{\sum_\beta \frac{\partial^\beta K(z,0)}{\beta!}\bar{w}^\beta\}\rangle|_{w=0}\\
&= \langle f, \{\sum_{\beta\geq\alpha} \frac{\partial^\beta K(z,0)\alpha!}{\beta!}\bar{w}^{\beta-\alpha}\}\rangle|_{w=0} = \{\sum_{\beta\geq\alpha}\langle f, \frac{\partial^\beta K(z,0)\alpha!}{\beta!}\rangle\bar{w}^{\beta-\alpha}\}|_{w=0}\\
&= \langle f, \bar{\partial}^\alpha K(\cdot, w)|_{w=0}\rangle.
\end{aligned}$$

So for $f \in \mathcal{M}$ and a polynomial $q = \sum a_\alpha z^\alpha$, we have

$$\begin{aligned}
\langle f, q(\bar{D})K(\cdot, w)|_{w=0}\rangle &= \langle q, \sum_\alpha a_\alpha \bar{\partial}^\alpha K(\cdot, w)\rangle|_{w=0} = \sum_\alpha \bar{a}_\alpha\langle f, \bar{\partial}^\alpha K(\cdot, w)\rangle|_{w=0} \quad (4.1.2)\\
&= \{\sum_\alpha \bar{a}_\alpha \partial^\alpha\langle f, K(\cdot, w)\rangle\}|_{w=0} = q^*(D)f|_{w=0}.
\end{aligned}$$

This proves the claim.

Claim 2: For any $q \in \mathbb{C}[\underline{z}]$, the vector $q(\bar{D})K(\cdot, w)|_{w=0}$ is in $\cap_{j=1}^m \ker M_j^*$ if and only if $q^* \in \tilde{\mathbb{V}}_0(\mathcal{I})$.

For any $f \in \mathcal{M}$, we have

$$\begin{aligned}
\langle f, M_j^* q(\bar{D})K(\cdot, w)|_{w=0}\rangle &= \langle M_j f, q(\bar{D})K(\cdot, w)|_{w=0}\rangle = q^*(D)(z_j f)|_{w=0}\\
&= \{z_j q^*(D)f + \frac{\partial q^*}{\partial z_j}(D)f\}|_{w=0} = \frac{\partial q^*}{\partial z_j}(D)f|_{w=0}
\end{aligned}$$

verifying the claim.

As a consequence of claims 1 and 2, we see that $q(\bar{D})K(\cdot,w)|_{w=0}$ is a non-zero vector in the joint kernel if and only if the class $[q^*]$ in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$ is non-zero.

Pick polynomials $q_1,\ldots,q_t$. From the equation (4.1.1) and claim 2, it is enough to show that $q_1(\bar{D})K(\cdot,w)|_{w=0},\ldots,q_t(\bar{D})K(\cdot,w)|_{w=0}$ are linearly independent if and only if $[q_1^*],\ldots,[q_t^*]$ are linearly independent in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$. But from claim 1 and equation (4.1.2), it follows that

$$\sum_{i=1}^{t}\bar{\alpha}_i q_i(\bar{D})K(\cdot,w)|_{w=0}=0$$

if and only if $\sum_{i=1}^{t}\alpha_i[q_i^*]=0$ in $\tilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$ for $\alpha_i\in\mathbb{C}$, $1\le i\le t$. This completes the proof. □

Remark 4.3. The 'if' part of the theorem can also be obtained from the decomposition theorem 2.3. For module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega)$, let $\mathcal{S}^{\mathcal{M}}$ be the subsheaf of the sheaf of holomorphic functions $\mathcal{O}_\Omega$ whose stalk $\mathcal{S}^{\mathcal{M}}_w$ at $w\in\Omega$ is

$$\big\{(f_1)_w\mathcal{O}_w+\cdots+(f_n)_w\mathcal{O}_w : f_1,\ldots,f_n\in\mathcal{M}\big\},$$

and the characteristic space at $w\in\Omega$ is the vector space

$$\mathbb{V}_w(\mathcal{S}^{\mathcal{M}}_w) \;=\; \{q\in\mathbb{C}[\underline{z}] : q(D)f\big|_w=0,\ f_w\in\mathcal{S}^{\mathcal{M}}_w\}.$$

Since

$$\dim\mathcal{S}^{\mathcal{M}}_0/\mathfrak{m}_0\mathcal{S}^{\mathcal{M}}_0=\dim\ker D_{\mathbf{M}^*}=\dim\widetilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})=t,$$

there exists a minimal set of generators $g_1,\cdots,g_t$ of $\mathcal{S}^{\mathcal{M}}_0$ and a $r>0$ such that

$$K(\cdot,w)=\sum_{i=1}^{t}\overline{g_j(w)}K^{(j)}(\cdot,w)\ \text{for all}\ w\in\Delta(0;r)$$

for some choice of anti-holomorphic functions $K^{(1)},\ldots,K^{(t)}:\Delta(0;r)\to\mathcal{M}$. Now for each $w\in\Delta(0;r)$ and j, $1\le j\le t$, we can write

$$K^{(j)}(\cdot,w)=\sum_{\alpha}\frac{1}{\alpha!}\frac{\partial^\alpha K^{(j)}(\cdot,w)}{\partial\bar{w}^\alpha}\Big|_{w=0}\bar{w}^\alpha.$$

Therefore for $w\in\Delta(0;r)$,

$$\begin{aligned}K(\cdot,w) &= \sum_{i=1}^{t}\overline{g_j(w)}\Big(\sum_{\alpha}\frac{1}{\alpha!}\frac{\partial^\alpha K^{(j)}(\cdot,w)}{\partial\bar{w}^\alpha}\Big|_{w=0}\bar{w}^\alpha\Big)\\ &= \sum_{j=1}^{t}\sum_{\alpha}\frac{1}{\alpha!}\frac{\partial^\alpha K^{(j)}(\cdot,w)}{\partial\bar{w}^\alpha}\Big|_{w=0}(\bar{w}^\alpha\overline{g_j(w)}),\end{aligned}$$

and thus

$$q(\bar{D})K(\cdot,w)=\sum_{j=1}^{t}\sum_{\alpha}\frac{1}{\alpha!}\frac{\partial^\alpha K^{(j)}(\cdot,w)}{\partial\bar{w}^\alpha}\Big|_{w=0}q(\bar{D})(\bar{w}^\alpha\overline{g_j(w)}),$$

for any $q \in \mathbb{C}[\underline{z}]$. We can interchange the sum as the convergence is uniform and absolute on compact subsets of $\Delta(0;r)$. Now for $z^\alpha = z_1^{\alpha_1}\cdots z_m^{\alpha_m}$

$$q(D)(z^\alpha g) = \sum_{k\leq\alpha}\binom{\alpha}{k} z^{\alpha-k}\frac{\partial^{|k|}q}{\partial z^k}(D)(g) \tag{4.1.3}$$

where $\binom{\alpha}{k} = \prod_{i=1}^m \binom{\alpha_i}{k_i}$ for the multi indices $\alpha = (\alpha_1,\ldots,\alpha_m), k = (k_1,\ldots,k_m)$. The order $k \leq \alpha$ if and only if $k_i \leq \alpha_i$ for all i, $1 \leq i \leq m$. This shows that at $z = 0$, the only term that survives is when $k_i = \alpha_i$ for all i, $1 \leq i \leq m$, that is, $q(D)(z^\alpha g)|_0 = \frac{\partial^\alpha q}{\partial z^\alpha}(D)(g)$. We note that $\mathbb{V}_0(\mathfrak{m}_0\mathcal{S}_0^{\mathcal{M}}) = \widetilde{\mathbb{V}}_0(\mathcal{I})$ (Lemma 2.10). Therefore for $q_i^* \in \widetilde{\mathbb{V}}_0(\mathcal{I})$, $g_j \in \mathcal{S}_0^{\mathcal{M}}, 1 \leq i,j \leq t$ and $|\alpha| > 0$, we have $q_i(D)(z^\alpha g)|_0 = 0$, since for $|\alpha| > 0$, $\frac{\partial^\alpha q_i}{\partial z^\alpha} \in \mathbb{V}_0(\mathcal{I}) = \mathbb{V}_0(\mathcal{S}_0^{\mathcal{M}})$. Thus for $1 \leq i,j \leq t$,

$$q_i(\bar{D})K(\cdot,w)|_{w=0} = \sum_{j=1}^t \{K^{(j)}(\cdot,w)|_{w=0}\}\{q_i(\bar{D})\overline{g_j(w)}|_{w=0}\}.$$

From part (*ii*) of Theorem 2.3, we note that $\{K^{(j)}(\cdot,w)|_{w=0}\}_{j=1}^t$ is a linearly independent set of vectors. Also $q_i(\bar{D})\,\overline{g_j(w)}_{|\,w=0} = \overline{q_i^*(D)g_{j|\,0}}$. Therefore to prove the set of vectors

$$\{q_i(\bar{D})K(\cdot,w)|_{w=w_0} : 1 \leq i \leq t\}$$

is linearly independent, it is enough to prove that the matrix $A = (a_{ij})_{i,j=1}^t$ is non-singular, where $a_{ij} = q_i^*(D)g_j|_0$, $1 \leq i,j \leq t$. Now the matrix above is singular if and only if there exists scalars α_i, $1 \leq i \leq t$, not all zero, such that $\sum_{i=1}^t \alpha_i a_{ij} = 0$ for all $j, 1 \leq j \leq t$. This shows that

$$(\sum_{i=1}^t \alpha_i q_i)(D)g_j|_0 = 0 \text{ for all } j,\ 1 \leq j \leq t.$$

Since $q_i^* \in \widetilde{\mathbb{V}}_0(\mathcal{I})$ and $g_1,\ldots,g_t$ are generators for $\mathcal{S}_0^{\mathcal{M}}$, it follows that $\sum_{i=1}^t \alpha_i q_i^* \in \mathbb{V}_0(\mathcal{S}_0^{\mathcal{M}}) = \mathbb{V}_0(\mathcal{I})$. Thus $[\sum_{i=1}^t \alpha_i q_i^*] = \sum_{i=1}^t \alpha_i [q_i^*] = 0$. Since the classes $[q_1^*],\ldots,[q_t^*]$ form a basis of the quotient space $\widetilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$, it follows that $\alpha_i = 0$ for all i, $1 \leq i \leq t$. This shows that the matrix A is invertible. Therefore the vectors $q_1(\bar{D})K(\cdot,w)|_{w=0},\ldots,q_t(\bar{D})K(\cdot,w)|_{w=0}$ are linearly independent. The proof is then complete by equation (4.1.1).

Remark 4.4. We give details of the case where the ideal $\mathcal{I}$ is singly generated, namely $\mathcal{I} = < p >$. From [14], it follows that the reproducing kernel K admits a global factorization, that is, $K(z,w) = p(z)\chi(z,w)\overline{p(w)}$ for $z,w \in \Omega$ where $\chi(w,w) \neq 0$ for all $w \in \Omega$. So we get $K_1(\cdot,w) = p(\cdot)\chi(\cdot,w)$ for all $w \in \Omega$. We use Lemma 4.2 to write down this section in term of reproducing kernel. Let $0 \in V(\mathcal{I})$. Let q_0 be the lowest degree term in p. We claim that $[q_0^*]$ gives a non-trivial class in $\widetilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$. All partial derivatives of q_0^* have degree less than that of q_0^*. Hence from (4.1.3)

$$q_0^*(D)(z^\alpha g)|_0 = \frac{\partial^{|\alpha||}q_0^*}{\partial z^\alpha}(D)(p)\big|_0 = 0,\ \alpha,\ |\alpha| > 0.$$

Consequently, $\frac{\partial q_0^*}{\partial z_i} \in \mathbb{V}_0(\mathcal{I})$ for all i, $1 \leq i \leq m$, that is, $q_0^* \in \widetilde{\mathbb{V}}_0(\mathcal{I})$. Also as the lowest degree of $p - q_0$ is strictly greater than that of q_0,

$$q_0^*(D)p|_0 = q_0^*(D)(p - q_0 + q_0)|_0 = q_0^*(D)q_0|_0 = \| q_0 \|_0^2 > 0$$

This shows that $q_0^* \notin \mathbb{V}_0(\mathcal{I})$ and hence its class in $\widetilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$ is nontrivial. Therefore, from the proof of Lemma 4.2, we have

$$q_0(\bar{D})K(\cdot, w)|_{w=0} = K_1(\cdot, w)|_{w=0}q_0(\bar{D})\overline{p(w)}|_0 = \| q_0^* \|_0^2 \, K_1(\cdot, w)|_{w=0}.$$

Let q_{w_0} denotes the term of lowest degree in the expression of p around w_0. Then we can write

$$K_1(\cdot, w)|_{w=w_0} = \begin{cases} \frac{K(\cdot,w)|_{w=w_0}}{p(w_0)} & \text{if } w_0 \notin V(\mathcal{I}) \cap \Omega \\ \frac{q_{w_0}(\bar{D})K(\cdot,w)|_{w=w_0}}{\|q_{w_0}^*\|_{w_0}^2} & \text{if } w_0 \in V(\mathcal{I}) \cap \Omega. \end{cases} \tag{4.1.4}$$

For a fixed set of polynomials $q_1, \ldots, q_t$, the next lemma provides a sufficient condition for the classes $[q_1^*], \ldots, [q_t^*]$ to be linearly independent in $\widetilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$. The methods of two proof we give below will be used repeatedly in the sequel.

Lemma 4.5. *Let $q_1, \ldots, q_t$ are linearly independent polynomials in the polynomial ideal $\mathcal{I}$ such that $q_1^*, \ldots, q_t^* \in \widetilde{\mathbb{V}}_{w_0}(\mathcal{I})$. Then $[q_1^*], \ldots, [q_t^*]$ are linearly independent in $\widetilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$.*

First Proof. Suppose $\sum_{i=1}^t \alpha_i[q_i^*] = 0$ in $\widetilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$ for some $\alpha_i \in \mathbb{C}$, $1 \leq i \leq t$. Thus $\sum_{i=1}^t \alpha_i q_i^* = q$ for some $q \in \mathbb{V}_{w_0}(\mathcal{I})$. Taking the inner product of $\sum_{i=1}^t \alpha_i q_i^*$ with q_j for a fixed j, we get

$$\sum_{i=1}^t \alpha_i \langle q_j, q_i \rangle_{w_0} = \big(\sum_{i=1}^t \alpha_i q_i^*\big)(D)q_j|_{w_0} = q(D)q_j|_{w_0} = 0.$$

The Grammian $\big(\!\big(\langle q_j, q_i \rangle_{w_0}\big)\!\big)_{i,j=1}^t$ of the linearly independent polynomials $q_1, \ldots, q_t$ is non-singular. Thus $\alpha_i = 0$, $1 \leq i \leq t$, completing the proof.

Second Proof. If $[q_1^*], \ldots, [q_t^*]$ are not linearly independent, then we may assume without loss of generality that $[q_1^*] = \sum_{i=2}^t \alpha_i[q_i^*]$ for $\alpha_1, \ldots, \alpha_t \in \mathbb{C}$. Therefore $[q_1^* - \sum_{i=2}^t \alpha_i p_i^*] = 0$ in the quotient space $\widetilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$, that is, $q_1^* - \sum_{i=2}^t \alpha_i q_i^* \in \mathbb{V}_{w_0}(\mathcal{I})$. So, we have

$$(q_1^* - \sum_{i=2}^t \alpha_i q_i^*)(D)q|_{w_0} = 0 \text{ for all } q \in \mathcal{I}.$$

Taking $q = q_1 - \sum_{i=2}^t \bar{\alpha}_i q_i$ we have $\| q_1 - \sum_{i=2}^t \bar{\alpha}_i q_i \|_{w_0}^2 = 0$. Hence $q_1 = \sum_{i=2}^t \bar{\alpha}_i q_i$ which is a contradiction. ☐

In the rest of this chapter, we continue our discussions assuming $w_0 = 0$, however the results, properly translated, remain valid in general. Suppose $\{p_1, ..., p_t\}$ is a minimal set of generators for $\mathcal{I}$. Let $\mathcal{M}$ be the completion of $\mathcal{I}$ with respect to some inner product induced by a positive definite kernel. We recall from [15] that $\text{rank}_{\mathbb{C}[\underline{z}]}\mathcal{M} = t$. Let w_0 be a fixed but arbitrary point in Ω. We ask if there exist a choice of generators $q_1, ..., q_t$ such that the vectors $q_1(\bar{D})K(\cdot, w)|_0, \ldots, q_t(\bar{D})K(\cdot, w)|_0$ form a basis for $\cap_{j=1}^m \ker M_j^*$. We isolate some instances where the answer is affirmative. However, this is not always possible (see remark 4.16). From [15, Lemma 5.11, Page-89], we have

$$\dim \cap_{j=1}^m \ker M_j* = \dim\mathcal{M}/\mathfrak{m}_0\mathcal{M} = \dim\mathcal{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_0 \leq \text{rank}_{\mathbb{C}[\underline{z}]}\mathcal{M}.\dim\mathbb{C}_0 \leq t,$$

where $\mathfrak{m}_0$ denotes the maximal ideal of $\mathbb{C}[\underline{z}]$ at 0. So, $\dim \cap_{j=1}^m \ker M_j^* \leq t$. If the germs $p_{10}, \ldots, p_{t0}$ is a minimal set of generators of the stalk $\mathcal{S}_0^{\mathcal{M}}$, then we would have $\dim \cap_{j=1}^m \ker M_j^* = t$. However, the set of generators $p_{10}, \ldots, p_{t0}$ need not be minimal in general.

For example, let $\mathcal{I}$ be the ideal generated by the polynomials $z_1(1 + z_1), z_1(1 - z_2), z_2^2$, which form a minimal set of generators for the ideal $\mathcal{I}$. Hence they also form a minimal set of generators of $\mathcal{M}$, but not of $\mathcal{S}_0^{\mathcal{M}}$. Since $\{z_1, z_2\}$ is a minimal set of generators for $\mathcal{S}_0^{\mathcal{M}}$, it follows that $\{z_1(1 + z_1), z_1(1 - z_2), z_2^2\}$ is not minimal for $\mathcal{S}_0^{\mathcal{M}}$. This was pointed out by R. G. Douglas. However minimality can be assured under some additional hypotheses.

Lemma 4.6. *Let $p_1, \ldots, p_t$ be a minimal set of generators for an ideal $\mathcal{I} \subset \mathbb{C}[\underline{z}]$. Assume that $p_1, \ldots, p_t$ are homogeneous polynomials not necessarily of the same degree. Let $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ be of the form $[\mathcal{I}]$. Then the germs $p_{10}, \ldots, p_{t0}$ at 0 form a minimal set of generators for $\mathcal{S}_0^{\mathcal{M}}$.*

Proof. For $1 \leq i \leq t$, let $\deg p_i = \alpha_i$. Without loss of generality we assume that $\alpha_i \leq \alpha_{i+1}$, $1 \leq i \leq t - 1$. Suppose the germs $p_{10}, \ldots, p_{t0}$ are not minimal. Then we have

$$p_k = \sum_{i:\alpha_i \leq \alpha_k} \phi_{i,\alpha_k - \alpha_i} p_i,$$

where $\phi_{i,\alpha_k-\alpha_i}$ is the Taylor polynomial of degree $\alpha_k - \alpha_i$ of the holomorphic function ϕ_i. Therefore $p_1, \ldots, p_t$ can not be a minimal set of generators for the ideal $\mathcal{I}$. This contradiction completes the proof. $\square$

Consider the ideal $\mathcal{I}$ generated by the polynomials $z_1 + z_2 + z_1^2, z_2^3 - z_1^2$. We will see later that the joint kernel at 0, in this case is spanned by the independent vectors $p(\bar{D})K(\cdot, w)|_{w=0}, q(\bar{D})K(\cdot, w)|_{w=0}$, where $p = z_1 + z_2$ and $q = (z_1 - z_2)^2$. Therefore any vectors in the joint kernel is of the form $(\alpha p + \beta q)(\bar{D})K(\cdot, w)|_{w=0}$ for some $\alpha, \beta \in \mathbb{C}$. It then follows that $\alpha p + \beta q$ and $\alpha' p + \beta' q$ can not be a set of generators of $\mathcal{I}$ for any choice of $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$. However in certain cases, this is possible. We describe below the case where $\{p_1(\bar{D})K(\cdot, w)|_{w=0}, ..., p_t(\bar{D})K(\cdot, w)|_{w=0}\}$ forms a basis for $\cap_{j=1}^n \ker M_j^*$ for an obvious choice of generating set in $\mathcal{I}$.

Lemma 4.7. *Suppose that $\{p_1, \ldots, p_t\}$ is a minimal set of generators for the homogeneous ideal $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ and that $p_1, \ldots, p_t$ be homogeneous polynomials of same degree. Let K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then the set*

$$\{p_1(\bar{D})K(\cdot, w)|_{w=0}, ..., p_t(\bar{D})K(\cdot, w)|_{w=0}\}$$

forms a basis for $\cap_{j=1}^m \ker M_j^*$.

Proof. For $1 \leq i \leq t$, let deg $p_i = k$. It is enough to show, using Lemma 4.2, 4.5 and 4.6, that the polynomials $p_1^*, \ldots, p_t^*$ are in $\widetilde{\mathbb{V}}_0(\mathcal{I})$. The degree of $\frac{\partial p_i^*}{\partial z_j}$ is at most $k-1$, $1 \leq i \leq t$, $1 \leq j \leq m$. The term of lowest degree in each polynomial p in the ideal $\mathcal{I}$ is at least k. It follows that $\frac{\partial p_i^*}{\partial z_j}(D)p|_0 = 0$, $p \in \mathcal{I}$, $1 \leq i \leq t, 1 \leq j \leq m$. This completes the proof. □

Remark 4.8. Lemma 4.1 follows from the proposition above.

A similar description of the joint kernel is possible even if the restrictive assumption of "same degree" is removed. We begin with the simple case of two generators.

Proposition 4.9. *Suppose* $\{p_1, p_2\}$ *is a minimal set of generators for the ideal* $\mathcal{I}$. *and are homogeneous with deg* $p_1 \neq$ *deg* p_2. *Let* K *be the reproducing kernel corresponding the Hilbert module* $[\mathcal{I}]$, *which is assumed to be in* $\mathfrak{B}_1(\Omega)$. *Then there exist polynomials* q_1, q_2 *which generate the ideal* $\mathcal{I}$ *and*

$$\{q_1(\bar{D})K(\cdot, w)|_{w=0},\, q_2(\bar{D})K(\cdot, w)|_{w=0}\}$$

is a basis for $\ker D_{\mathbf{M}^*}$.

Proof. Let deg $p_1 = k$ and deg $p_2 = k+n$ for some $n \geq 1$. The set $\{p_1, p_2 + (\sum_{|i|=n} \gamma_i z^i)p_1\}$ is a minimal set of generators for $\mathcal{I}$, $\gamma_i \in \mathbb{C}$ where $i = (i_1, \ldots, i_m)$ and $|i| = i_1 + \ldots + i_m$. We will take $q_1 = p_1$ and find constants γ_i in $\mathbb{C}$ such that

$$q_2 = p_2 + (\sum_{|i|=n} \gamma_i z^i)p_1.$$

We have to show (cf. Lemma 4.2) that $\{[q_1^*], [q_2^*]\}$ is a basis in $\widetilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$. From the equation (4.1.1) and Lemma 4.5, it is enough to show that q_2^* is a in $\widetilde{\mathbb{V}}_0(\mathcal{I})$. To ensure that $\frac{\partial q_2^*}{\partial z_k} \in \mathbb{V}_0(\mathcal{I})$, $1 \leq k \leq m$, we need to check:

$$\frac{\partial^{|\alpha|} q_2^*}{\partial z^\alpha}(D)p_i|_{w=0} = \langle p_i, \frac{\partial^{|\alpha|} q_2}{\partial z^\alpha}\rangle|_0 = 0,$$

for all multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $1 \leq |\alpha| \leq n$ and $i = 1, 2$. For $|\alpha| > n$, these conditions are evident. Since the degree of the polynomial q_2 is $k+n$, we have $\langle p_2, \frac{\partial^{|\alpha|} q_2}{\partial z^\alpha}\rangle_0 = 0$, $1 \leq |\alpha| \leq n$. If $n > 1$, then $\langle p_1, \frac{\partial^{|\alpha|} q_2}{\partial z^\alpha}\rangle_0 = 0$, $1 \leq |\alpha| < n$. To find γ_i, $i = (i_1, \ldots, i_m)$, we solve the equation $\langle p_1, \frac{\partial^{|\alpha|} q_2}{\partial z^\alpha}\rangle|_0 = 0$ for all α such that $|\alpha| = n$. By the Leibnitz rule,

$$\begin{aligned}
\frac{\partial^{|\alpha|} q_2^*}{\partial z^\alpha} &= \frac{\partial^{|\alpha|} p_2^*}{\partial z^\alpha} + \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \partial^{\alpha-\nu}(\sum_{|i|=n} \bar{\gamma}_i z^i) \frac{\partial^{|\nu|} p_1^*}{\partial z^\nu} \\
&= \frac{\partial^{|\alpha|} p_2^*}{\partial z^\alpha} + \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} (\sum_{|i|=n, i \geq \alpha-\nu} \bar{\gamma}_i \frac{i!}{(i-\alpha+\nu)!} z^{i-\alpha+\nu}) \frac{\partial^{|\nu|} p_1^*}{\partial z^\nu}.
\end{aligned}$$

Now (using (4.1.3)) $\frac{\partial^{|\alpha|}p^*}{\partial z^\alpha}(D)p_i|_{w=0} = 0$ gives

$$0 = \Big(\frac{\partial^{|\alpha|}p_2^*}{\partial z^\alpha} + \sum_{\nu\le\alpha}\binom{\alpha}{\nu}\Big(\sum_{|i|=n, i\ge\alpha-\nu}\bar{\gamma}_i\frac{i!}{(i-\alpha+\nu)!}z^{i-\alpha+\nu}\Big)\frac{\partial^{|\nu|}p_1^*}{\partial z^\nu}\Big)(D)p_1|_{w=0} \tag{4.1.5}$$
$$= \langle p_1, \frac{\partial^{|\alpha|}p_2}{\partial z^\alpha}\rangle_0 + \sum_{r=0}^{n}\sum_{|i|=n}\overline{A_{\alpha i}(r)}\bar{\gamma}_i,$$

where given the multi-indices α, i,

$$A_{\alpha i}(r) = \begin{cases} \sum_\nu \binom{\alpha}{\nu}\frac{i!}{(i-\alpha+\nu)!}\langle\frac{\partial^{|\nu|}p_1}{\partial z^\nu}, \frac{\partial^{|i-\alpha+\nu|}p_1}{\partial z^{i-\alpha+\nu}}\rangle_0 & |\nu| = r, \nu\le\alpha, i\ge\alpha-\nu; \\ 0 & \text{otherwise.}\end{cases} \tag{4.1.6}$$

Let $A(r) = \big(\!\big(A_{\alpha i}(r)\big)\!\big)$ be the $\binom{n+m-1}{m-1}\times\binom{n+m-1}{m-1}$ matrix in colexicographic order on α and i. Let $A = \sum_{r=0}^n A(r)$ and γ_n be the $\binom{n+m-1}{m-1}\times 1$ column vector $(\gamma_i)_{|i|=n}$. Thus the equation (4.1.5) is of the form

$$\bar{A}\bar{\gamma}_n = \Gamma, \tag{4.1.7}$$

where Γ is the $\binom{n+m-1}{m-1}\times 1$ column vector $(-\langle p_1, \frac{\partial^{|\alpha|}p_2}{\partial z^\alpha}\rangle_0)_{|\alpha|=n}$. Invertibility of the coefficient matrix A then guarantees the existence of a solution to the equation (4.1.7). We show that the matrix $A(r)$ is non-negative definite and the matrix $A(0)$ is diagonal:

$$A(0)_{\alpha i} = \begin{cases}\alpha! \parallel p_1 \parallel^2 & \text{if } \alpha = i\\ 0 & \text{if } \alpha\ne i.\end{cases} \tag{4.1.8}$$

and therefore positive definite. Fix a r, $1\le r\le n$. To prove that $A(r)$ is non-negative definite, we show that it is the Grammian with respect to Fock inner product at 0. To each $\mu = (\mu_1,\dots,\mu_m)$ such that $|\mu| = n-r$, we associate a $1\times\binom{n+m-1}{m-1}$ tuple of polynomials X_μ^r, defined as follows

$$X_\mu^r(\beta) = \begin{cases}\mu!\binom{\beta}{\beta-\mu}\frac{\partial^{|\beta-\mu|}p_1}{\partial z^{\beta-\mu}} & \text{if } \beta\ge\mu\\ 0 & \text{otherwise,}\end{cases}$$

where $\beta = (\beta_1,\dots,\beta_m)$, $|\beta| = n$ ($\beta\ge\mu$ if and only if $\beta_i\ge\mu_i$ for all i). By $X_\mu^r\cdot(X_\mu^r)^t$, we denote the $\binom{n+m-1}{m-1}\times\binom{n+m-1}{m-1}$ matrix whose αi-th element is $\langle X_\mu^r(\alpha), X_\mu^r(i)\rangle_0$, $|\alpha| = n = |i|$. We note that

$$\sum_{|\mu|=n-r}\frac{1}{\mu!}(X_\mu^r\cdot(X_\mu^r)^t)_{\alpha i} = \sum_{|\mu|=n-r}\frac{1}{\mu!}\langle X_\mu^r(\alpha), X_\mu^r(i)\rangle_0 \tag{4.1.9}$$
$$= \sum_{|\mu|=n-r,\alpha\ge\mu,i\ge\mu}\frac{1}{\mu!}\langle\mu!\binom{\alpha}{\alpha-\mu}\frac{\partial^{|\alpha-\mu|}p_1}{\partial z^{\alpha-\mu}}, \mu!\binom{i}{i-\mu}\frac{\partial^{|i-\mu|}p_1}{\partial z^{i-\mu}}\rangle_0$$
$$= \sum_{|\nu|=r,\nu\le\alpha,i\ge\alpha-\nu}(\alpha-\nu)!\binom{\alpha}{\nu}\binom{i}{i-\alpha+\nu}\langle\frac{\partial^{|\alpha-\mu|}p_1}{\partial z^{\alpha-\mu}}, \frac{\partial^{|i-\mu|}p_1}{\partial z^{i-\mu}}\rangle_0$$
$$= A_{\alpha i}(r).$$

Since $X^r_\mu \cdot (X^r_\mu)^t$ is the Grammian of the vector tuple X^r_μ, it is non-negative definite. Hence $A(r) = \sum_{|\mu|=n-r} \frac{1}{\mu!}(X^r_\mu \cdot (X^r_\mu)^t)$ is non-negative definite. Therefore A is positive definite and hence equation (4.1.7) admits a solution, completing the proof. □

Let $\mathcal{I}$ be a homogeneous polynomial ideal. As one may expect, the proof in the general case is considerably more involved. However the idea of the proof is similar to the simple case of two generators. Let $\{p_1, \ldots, p_v\}$ be a minimal set of generators, consisting of homogeneous polynomials, for the ideal $\mathcal{I}$. We arrange the set $\{p_1, \ldots, p_v\}$ in blocks of polynomials $P^1, \ldots, P^k$ according to ascending order of their degree, that is,

$$\{P^1, \ldots, P^k\} \;=\; \{p^1_1, \ldots, p^1_{u_1}, p^2_1, \ldots, p^2_{u_2}, \ldots, p^l_1, \ldots, p^l_{u_l}, \ldots, p^k_1, \ldots, p^k_{u_k}\},$$

where each $P^l = \{p^l_1, \ldots, p^l_{u_l}\}$, $1 \le l \le k$ consists of homogeneous polynomials of the same degree, say n_l and $n_{l+1} > n_l$, $1 \le l \le k-1$. As before, for $l = 1$, we take $q^1_j = p^1_j$, $1 \le j \le u_1$ and for $l \ge 2$ take

$$q^l_j = p^l_j + \sum_{f=1}^{l-1}\sum_{s=1}^{u_f} \gamma^{fs}_{lj} p^f_s, \text{ where } \gamma^{fs}_{lj}(z) \;=\; \sum_{|i|=n_l-n_f} \gamma^{fs}_{lj}(i) z^i.$$

Each γ^{fs}_{lj} is a polynomial of degree $n_l - n_f$ for some choice of $\gamma^{fs}_{lj}(i)$ in $\mathbb{C}$. So we obtain another set of polynomials $\{Q^1, \ldots, Q^k\}$ with $Q^l = \{q^l_1, \ldots, q^l_{u_l}\}$, $1 \le l \le k$ satisfying the the same property as the set of polynomials $\{P^1, \ldots, P^k\}$. From Lemma 4.2 and 4.5, it is enough to check q^{l*}_j is in $\widetilde{\mathbb{V}}_0(\mathcal{I})$. This condition yields a linear system of equation as in the proof of Proposition 4.9, except that the co-efficient matrix is a block matrix with each block similar to A defined by the equation (4.1.6). For q^{l*}_j in $\widetilde{\mathbb{V}}_0(\mathcal{I})$, the constants $\gamma^{fs}_{lj}(i)$ must satisfy:

$$\begin{aligned} 0 &= \frac{\partial^{|\alpha|} q^{l*}_j}{\partial z^\alpha}(D) p^e_t|_0 \\ &= \langle p^e_t, \frac{\partial^{|\alpha|} p^l_j}{\partial z^\alpha}\rangle_0 + \sum_{f=1}^{l-1}\sum_{s=1}^{u_f}\sum_{\nu\le\alpha}\binom{\alpha}{\nu}\sum_{|i|=n_l-n_f, i\ge\alpha-\nu} \overline{\gamma^{fs}_{lj}(i)} \frac{i!}{(i-\alpha+\nu)!}\langle \frac{\partial^{|i-\alpha+\nu|} p^e_t}{\partial z^{i-\alpha+\nu}}, \frac{\partial^{|\nu|} p^f_s}{\partial z^\nu}\rangle_0 \end{aligned}$$

All the terms in the equation are zero except when $|\alpha| = n_l - n_d$, $1 \le d \le l-1$. For $e = d = f$, we have the equations

$$-\langle p^d_t, \frac{\partial^{|\alpha|} p^l_j}{\partial z^\alpha}\rangle_0 \;=\; \sum_{s=1}^{u_d}\sum_{r=0}^{n_l-n_d}\sum_{|i|=n_l-n_d} \overline{\left(A^d_{st}(r)\right)_{\alpha i}\gamma^{ds}_{lj}(i)}, \tag{4.1.10}$$

where

$$\left(A^d_{st}(r)\right)_{\alpha i} \;=\; \begin{cases} \sum_\nu \binom{\alpha}{\nu}\frac{i!}{(i-\alpha+\nu)!}\langle \frac{\partial^{|\nu|} p^d_s}{\partial z^\nu}, \frac{\partial^{|i-\alpha+\nu|} p^d_t}{\partial z^{i-\alpha+\nu}}\rangle_0 & |\nu| = r, \nu \le \alpha, i \ge \alpha - \nu; \\ 0 & \text{otherwise.} \end{cases}$$

Let $A^d_{st}(r)$ be the $\binom{n_l-n_{d-1}+m-1}{m-1} \times \binom{n_l-n_{d-1}+m-1}{m-1}$ matrix whose αi-th element is $\left(A^d_{st}(r)\right)_{\alpha i}$. We consider the block-matrix $A^d(r) = (A^d_{st}(r))$, $1 \le s, t \le u_d$.

Fix a r, $1 \leq r \leq n_l - n_d$. To each $\mu = (\mu_1, \ldots, \mu_m)$ such that $|\mu| = n_l - n_d - r$, associate a $1 \times \binom{n_l - n_d + m - 1}{m-1}$ tuple of polynomials $X^{ds}_{\mu r}$, where

$$X^{ds}_{\mu r}(\beta) = \begin{cases} \mu! \binom{\beta}{\beta - \mu} \frac{\partial^{|\beta - \mu|} p^d_s}{\partial z^{\beta - \mu}} & \text{if } \beta \geq \mu \\ 0 & \text{otherwise,} \end{cases}$$

with $\beta = (\beta_1, \ldots, \beta_m)$, $|\beta| = n_l - n_d$. Let $X^d_{\mu r} = (X^{d1}_{\mu r}, \ldots, X^{d(n_l - n_d)}_{\mu r})$. Using same argument as in (4.1.8) and (4.1.9), we see that the matrix

$$A^d(r) = \sum_{|\mu| = n - r} \frac{1}{\mu!} (X^d_{\mu r} \cdot (X^d_{\mu r})^t)$$

is non-negative definite when $r \geq 0$ and $A^d(0)$ is positive definite. Thus $A^d = \sum_{r=0}^{n_l - n_d} A^d(r)$ is positive definite. Let

$$\gamma^d_{lj} = ((\gamma^{d1}_{lj}(i))_{|i| = n_l - n_d}, \ldots, (\gamma^{d(n_l - n_d)}_{lj}(i))_{|i| = n_l - n_d})^{tr},$$

where each $(\gamma^{ds}_{lj}(i))_{|i| = n_l - n_d}$ is a $\binom{n_l - n_d + m - 1}{m - 1} \times 1$ column vector. Define

$$\Gamma^d_{lj} = ((-\langle p^d_1, \frac{\partial^{|\alpha|} p^l_j}{\partial z^\alpha} \rangle_0)_{|\alpha| = n_l - n_d}, \ldots, (-\langle p^d_{u_d}, \frac{\partial^{|\alpha|} p^l_j}{\partial z^\alpha} \rangle_0)_{|\alpha| = n_l - n_d}).$$

The equation (4.1.10) is then takes the form $\overline{A^d \gamma^d_{lj}} = \Gamma^d_{lj}$, which admits a solution (as A^d is invertible) for each d, l and j. Thus we have proved the following theorem.

Theorem 4.10. *Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be a homogeneous ideal and $\{p_1, \ldots, p_v\}$ be a minimal set of generators for $\mathcal{I}$ consisting of homogeneous polynomials. Let K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then there exists a set of generators $q_1, ..., q_v$ for the ideal $\mathcal{I}$ such that the set $\{q_i(\bar{D}) K(\cdot, w)|_{w=0} : 1 \leq i \leq v\}$ is a basis for* $\ker D_{\mathbf{M}^*}$.

We remark that the new set of generators $q_1, \ldots, q_v$ for $\mathcal{I}$ is more or less "*canonical*"! It is uniquely determined modulo a linear transformation as shown below.

Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be an ideal. Suppose there are two sets of homogeneous polynomials $\{p_1, \ldots, p_v\}$ and $\{\widetilde{p}_1, \ldots, \widetilde{p}_v\}$ both of which are minimal set of generators for $\mathcal{I}$. Theorem 4.10 guarantees the existence of a new set of generators $\{q_1, \ldots, q_v\}$ and $\{\widetilde{q}_1, \ldots, \widetilde{q}_v\}$ corresponding to each of these generating sets with additional properties which ensure that the equality

$$[\widetilde{q}^*_i] = \sum_{j=1}^{v} \alpha_{ij} [q^*_j], \; 1 \leq i \leq v$$

holds in $\widetilde{\mathbb{V}}_0(\mathcal{I}) / \mathbb{V}_0(\mathcal{I})$ for some choice of complex constants α_{ij}, $1 \leq i, j \leq v$. Therefore $\widetilde{q}^*_i - \sum_{i=1}^{v} \bar{\alpha}_{ij} q^*_j \in \mathbb{V}_0(\mathcal{I})$. Since $\widetilde{q}_i - \sum_{i=1}^{v} \alpha_{ij} q_j$ is in $\mathcal{I}$, we have

$$0 = \big((\widetilde{q}^*_i - \sum_{i=1}^{v} \bar{\alpha}_{ij} q^*_j)(D) \big) \big(\widetilde{q}_i - \sum_{i=1}^{v} \alpha_{ij} q_j \big) = \| \widetilde{q}_i - \sum_{i=1}^{v} \alpha_{ij} q_j \|^2_0, \; 1 \leq i \leq v,$$

and hence $\widetilde{q}_i = \sum_{i=1}^{v} \alpha_{ij} q_j$, $1 \leq i \leq v$. We have therefore proved the following.

Proposition 4.11. *Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be a homogeneous ideal. If $\{q_1, \ldots, q_v\}$ is a minimal set of generators for $\mathcal{I}$ with the property that $\{[q_i^*] : 1 \le i \le v\}$ is a basis for $\widetilde{\mathbb{V}}_0(\mathcal{I})/\mathbb{V}_0(\mathcal{I})$, then $q_1, \ldots, q_v$ is unique up to a linear transformation.*

We end this section with the explicit calculation of the joint kernel for a class of submodules of the Hardy module which illustrate the methods of Proposition 4.9.

Example 4.12. Let p_1, p_2 be the minimal set of generators for an ideal $\mathcal{I} \subseteq \mathbb{C}[z_1, z_2]$. Assume that p_1, p_2 are homogeneous, $\deg p_2 = \deg p_1 + 1$ and $V(\mathcal{I}) = \{0\}$. As in Proposition 4.9, set $q_1 = p_1$ and $q_2 = p_2 + (\gamma_{10} z_1 + \gamma_{01} z_2) p_1$ subject to the equations

$$\begin{pmatrix} \| \partial_1 p_1 \|_0^2 + \| p_1 \|_0^2 & \langle \partial_2 p_1, \partial_1 p_1 \rangle_0 \\ \langle \partial_1 p_1, \partial_2 p_1 \rangle_0 & \| \partial_2 p_1 \|_0^2 + \| p_1 \|_0^2 \end{pmatrix} \begin{pmatrix} \gamma_{10} \\ \gamma_{01} \end{pmatrix} = - \begin{pmatrix} \langle p_1, \partial_1 p_2 \rangle_0 \\ \langle p_1, \partial_2 p_2 \rangle_0 \end{pmatrix} \tag{4.1.11}$$

In this special case, the invertibility of the coefficient matrix follows from the positivity (Cauchy - Schwarz inequality) of its determinant

$$\begin{aligned} &\| p_1 \|_0^4 + \| \partial_1 p_1 \|_0^2 \| p_1 \|_0^2 + \| \partial_2 p_1 \|_0^2 \| p_1 \|_0^2 \\ &\qquad + (\| \partial_1 p_1 \|_0^2 \| \partial_2 p_1 \|_0^2 - |\langle \partial_1 p_1, \partial_2 p_1 \rangle_0|^2). \end{aligned}$$

Specifically, if the ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ is generated by $z_1 + z_2$ and z_2^2. We have $V(\mathcal{I}) = \{0\}$. The reproducing kernel K for $[\mathcal{I}] \subseteq H^2(\mathbb{D}^2)$ is

$$\begin{aligned} K_{[\mathcal{I}]}(z, w) &= \frac{1}{(1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)} - \frac{(z_1 - z_2)(\bar{w}_1 - \bar{w}_2)}{2} - 1 \\ &= \frac{(z_1 + z_2)(\bar{w}_1 + \bar{w}_2)}{2} + \sum_{i+j \ge 2}^{\infty} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j. \end{aligned}$$

The vector $\bar{\partial}_2^2 K_{[\mathcal{I}]}(z, w)|_0 = 2 z_2^2$ is not in the joint kernel of $P_{[\mathcal{I}]}(M_1^*, M_2^*)|_{[\mathcal{I}]}$ since $M_2^*(z_2^2) = z_2$ and $P_{[\mathcal{I}]} z_2 = (z_1 + z_2)/2 \neq 0$. However, from the equation (4.1.11), we have $q_1 = z_1 + z_2$ and $q_2 = (z_1 - z_2)^2$, we see that q_1, q_2 generate the ideal $\mathcal{I}$ and $\{(\bar{\partial}_1 + \bar{\partial}_2) K(\cdot, w)|_0, (\bar{\partial}_1 - \bar{\partial}_2)^2 K(\cdot, w)|_0\}$ forms a basis of the joint kernel.

Remark on Example 4.12. Let $\widetilde{\mathcal{I}}$ be the ideal generated by z_1 and z_2^2. Since z_1 is not a linear combination of q_1 and q_2, it follows (Proposition 4.11) that $\mathcal{I} \neq \widetilde{\mathcal{I}}$.

Example 4.13. This example is similar to the previous one except that it is of higher order. Take $\mathcal{I} = < z_1^2 + z_2^2, z_1 z_2^2, z_2^3 >$. The set $\{z_1^2 + z_2^2, z_1 z_2^2, z_2^3\}$ forms a minimal set of generators and $V(\mathcal{I}) = \{0\}$. Now the reproducing kernel is ,

$$\begin{aligned} K(z, w) &= \frac{1}{(1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)} - 1 - z_1 \bar{w}_1 - z_2 \bar{w}_2 - \frac{(z_1^2 - z_2^2)(\bar{w}_1^2 - \bar{w}_2^2)}{2} - z_1 z_2 \bar{w}_1 \bar{w}_2 \\ &= \frac{(z_1^2 + z_2^2)(\bar{w}_1^2 + \bar{w}_2^2)}{2} + \sum_{i,j=3}^{\infty} z_1^i z_2^i \bar{w}_1^i \bar{w}_2^j. \end{aligned}$$

Note then $\bar\partial_2^3 K(z,w)|_0 = 6z_2^3$ and $M_2^*(6z_2^3) = 6\langle z_2^2, \frac{z_1^2+z_2^2}{\sqrt{2}}\rangle \frac{z_1^2+z_2^2}{\sqrt{2}} = 3(z_1^2+z_2^2) \neq 0$. Taking $p_1^1 = z_1^2+z_2^2$, $p_1^2 = z_1 z_2^2$ and $p_2^2 = z_2^3$, as in Theorem 4.10. The new set of generators are

$$q_1^1 = z_1^2 + z_2^2,$$

$$q_1^2 = z_1 z_2^2 + (\gamma_{21}^{11}(10)z_1 + \gamma_{21}^{11}(01)z_2)(z_1^2+z_2^2),$$
$$q_2^2 = z_2^3 + (\gamma_{22}^{11}(10)z_1 + \gamma_{22}^{11}(01)z_2)(z_1^2+z_2^2).$$

Coefficient of these polynomials then satisfy the following equations:

$$\partial_2^2(z_1^2+z_2^2)|_0 + (3\partial_1^2+\partial_2^2)(z_1^2+z_2^2)|_0\gamma_{21}^{11}(10) + 2\partial_1\partial_2(z_1^2+z_2^2)|_0\gamma_{21}^{11}(01) = 0,,$$
$$2\partial_1\partial_2(z_1^2+z_2^2)|_0 + 2\partial_1\partial_2(z_1^2+z_2^2)|_0\gamma_{21}^{11}(10) + (\partial_1^2+3\partial_2^2)(z_1^2+z_2^2)|_0\gamma_{21}^{11}(01) = 0,$$

and

$$(3\partial_1^2+\partial_2^2)(z_1^2+z_2^2)|_0\gamma_{22}^{11}(10) + 2\partial_1\partial_2(z_1^2+z_2^2)|_0\gamma_{22}^{11}(01) = 0,,$$
$$3\partial_2^2(z_1^2+z_2^2)|_0 + 2\partial_1\partial_2(z_1^2+z_2^2)|_0\gamma_{22}^{11}(10) + (\partial_1^2+3\partial_2^2)(z_1^2+z_2^2)|_0\gamma_{22}^{11}(01) = 0.$$

This amounts to solving the following matrix equation

$$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} \gamma_{21}^{11}(10) \\ \gamma_{21}^{11}(01) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} \gamma_{22}^{11}(10) \\ \gamma_{22}^{11}(01) \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$$

Ignoring the constants, we get $q_1^1 = z_1^2+z_2^2$, $q_1^2 = z_1^3 - 3z_1z_2^2$, $q_2^2 = z_2^3 - 3z_1^2z_2$ which will then generate the ideal $\mathcal{I}$ and $\{(\bar\partial_1^2+\bar\partial_2^2)K(\cdot,w)|_0, (\bar\partial_1^3 - 3\bar\partial_1\bar\partial_2^2)K(\cdot,w)|_0, (\bar\partial_2^3 - 3\bar\partial_1^2\bar\partial_2)K(\cdot,w)|_0\}$ forms a basis of $\ker D_{\mathbf{M}^*}$.

Example 4.14. Take $\mathcal{I} =< z_1^3+2z_2^3, 3z_1^2z_2 - z_1z_2^2, z_2^4 >$,. The set $z_1^3+2z_2^3, 3z_1^2z_2 - z_1z_2^2, z_2^4$ forms a minimal set of generators and $V(\mathcal{I}) = \{0\}$. Now the reproducing kernel is given by the formula

$$K(z,w) = \frac{(z_1^3+2z_2^3)(\bar w_1^3 + 2\bar w_2^3)}{5} + \frac{(3z_1z_2^2 - z_1z_2^2)(3\bar w_1\bar w_2^2 - \bar w_1\bar w_2^2)}{10} + \sum_{i,j=4}^{\infty} z_1^i z_2^i \bar w_1^i \bar w_2^j.$$

Again we have,

$$p_1^1 = z_1^3 + 2z_2^3,\ p_2^1 = 3z_1z_2^2 - z_1z_2^2,\ p_1^2 = z_2^4$$

The new set of generators are $q_1^1 = p_1^1$, $q_2^1 = p_2^1$ and

$$q_1^2 = z_2^4 + (\gamma_{21}^{11}(10)z_1 + \gamma_{21}^{11}(01)z_2)(z_1^3+2z_2^3) + (\gamma_{21}^{12}(10)z_1 + \gamma_{21}^{12}(01)z_2)(3z_1^2 - z_1z_2^2).$$

The corresponding matrix equation is

$$\begin{pmatrix} 36 & 0 & 0 & -12 \\ 0 & 90 & 18 & 0 \\ 0 & 18 & 58 & -12 \\ -12 & 0 & -12 & 42 \end{pmatrix}\begin{pmatrix} \gamma_{21}^{11}(10) \\ \gamma_{21}^{11}(01) \\ \gamma_{21}^{12}(10) \\ \gamma_{21}^{12}(01) \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is 6231168 showing that it is invertible. So the solution to the linear system of equation produces the polynomials q_1^1, q_2^1, q_2^2 generates the ideal $\mathcal{I}$ and the set

$$\{q_1^1(\bar{D})K(\cdot,w)|_{w=0}, q_2^1(\bar{D})K(\cdot,w)|_{w=0}, q_1^2(\bar{D})K(\cdot,w)|_{w=0}\}$$

forms a basis for $\ker D_{\mathbf{M}^*}$.

Example 4.15. Take $\mathcal{I} =< z_1 - z_2, z_2^3 >$. The set $\{z_1 - z_2, z_2^3\}$ forms a minimal set of generators and $V(\mathcal{I}) = \{0\}$. Recall that the reproducing kernel K (Theorem 2.3) can be written as

$$K(z,w) = (\bar{w}_1 - \bar{w}_2)K_1(z,w) + \bar{w}_2^3 K_2(z,w).$$

Differentiating this relationship repeatedly and evaluating at 0, we see that

$$K_1(z,w) = (\bar{\partial}_1 - \bar{\partial}_2)K(z,w)|_0 \text{ and } k_2(z,w) = (2\bar{\partial}_1^3 + 6\bar{\partial}_1^2\bar{\partial}_2 + 3\bar{\partial}_1\bar{\partial}_2^2 + \bar{\partial}_2^3)K(z,w)|_0.$$

It then easily follows that $z_1 - z_2$ and $2z_1^3 + 6z_1^2 z_2 + 3z_1 z_2^2 + z_2^3$ also generate the ideal $\mathcal{I}$.

Remark 4.16. If the generators of the ideal are not homogeneous then the conclusion of Theorem 4.10 is not valid. For instance, take the ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ generated by $z_1(1 + z_1), z_1(1 - z_2), z_2^2$ which is also minimal for $\mathcal{I}$. We have $V(\mathcal{I}) = \{0\}$. We note that the stalk $\mathcal{S}_0^{\mathcal{M}}$ at 0 is generated by z_1 and z_2^2. Similar calculations, as above, shows that $\{\bar{\partial}_1 K(\cdot,w)|_0, \bar{\partial}_2^2 K(\cdot,w)|_0\}$ is a basis of $\ker D_{\mathbf{M}^*}$. But z_1 and z_2^2 can not be a set of generators for $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ which has rank 3. On the other hand, let $\mathcal{I}$ be the ideal generated by $z_1 + z_2 + z_1^2, z_2^3 - z_1^2$ which is minimal and $V(\mathcal{I}) = \{0\}$. In this case $\{(\bar{\partial}_1 + \bar{\partial}_2)K(\cdot,w)|_0, (\bar{\partial}_1 - \bar{\partial}_2)^2 K(\cdot,w)|_0\}$ is a basis of $\ker D_{\mathbf{M}^*}$. But $z_1 + z_2$ and $(z_1 - z_2)^2$ is not a generating set for the stalk at 0.

5. Invariants using resolution of singularities

We will use the familiar technique of 'resolution of singularities' and construct the blow-up space of Ω along an ideal $\mathcal{I}$, which we will denote by $\hat{\Omega}$. There is a map $\pi : \hat{\Omega} \to \Omega$ which is biholomorphic on $\hat{\Omega} \setminus \pi^{-1}(V(\mathcal{I}))$. However, in general, $\hat{\Omega}$ need not even be a complex manifold. Abstractly, the inverse image sheaf of $\mathcal{S}^{\mathcal{M}}$ under π is locally principal and therefore corresponds to a line bundle on $\hat{\Omega}$. Here, we explicitly construct a holomorphic line bundle, via the monoidal transformation, on $\pi^{-1}(w_0)$, $w_0 \in V(\mathcal{I})$, and show that the equivalence class of these Hermitian holomorphic vector bundles are invariants for the Hilbert module $\mathcal{M}$.

In the paper [14], submodules of functions vanishing at the origin of $H^{(\lambda,\mu)}(\mathbb{D}^2)$ were studied using the blow-up $\mathbb{D}^2 \setminus (0,0) \cup \mathbb{P}^1$ of the bi-disc. This is also known as the quadratic transform. However, this technique yields useful information only if the generators of the submodule are homogeneous polynomials of same degree. We will compute invariants via quadratic transform for submodules of Hardy module. The monoidal transform, as we will see below, has wider applicability.

5.1 The monoidal transformation

Let $\mathcal{M} = [\mathcal{I}]$ be a Hilbert module in $\mathfrak{B}_1(\Omega)$ for some polynomial ideal $\mathcal{I}$. Assume that the dimension of the zero set $V(\mathcal{I})$ is at most $m-2$. Let K denote the corresponding reproducing kernel. Let $w_0 \in V(\mathcal{M})$. Set

$$t = \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0} = \dim \cap_{j=1}^{m} \ker(M_j - w_{0j})^* = \dim \widetilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I}).$$

By the decomposition Theorem [6, Theorem 1.4], there exists a minimal set of generators $g_1, \cdots, g_t$ of $\mathcal{S}^{\mathcal{M}_1}_{w_0}$ and a $r > 0$ such that

$$K(\cdot, w) = \sum_{i=1}^{t} \overline{g_j(w)} K^{(j)}(\cdot, w), \; w \in \Delta(w_0; r) \tag{5.1.1}$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(t)} : \Delta(w_0; r) \to \mathcal{M}$.

Assume that $Z := V(g_1, \ldots, g_t) \cap \Omega$ be a singularity free analytic subset of $\mathbb{C}^m$ of codimension t. We point out that Z depends on $\mathcal{M}$ as well as w_0. Define

$$\widehat{\Delta}(w_0; r) := \{(w, \pi(u)) \in \Delta(w_0; r) \times \mathbb{P}^{t-1} : u_i g_j(w) - u_j g_i(w) = 0, \, 1 \le i, j \le t\}.$$

Here the map $\pi : \mathbb{C}^t \setminus \{\underline{0}\} \to \mathbb{P}^{t-1}$ is given by $\pi(u) = (u_1 : \ldots : u_t)$, the corresponding projective coordinate. The space $\widehat{\Delta}(w_0; r)$ is the monoidal transformation with center Z ([21, page 241]). Consider the map $p := \mathrm{pr}_1 : \widehat{\Delta}(w_0; r) \to \Delta(w_0; r)$ given by $(w, \pi(z)) \mapsto w$. For $w \in Z$, we have $p^{-1}(w) = \{w\} \times \mathbb{P}^{t-1}$. This map is holomorphic and proper. Actually $p : \widehat{\Delta}(w_0; r) \setminus p^{-1}(Z) \to \Delta(w_0; r) \setminus Z$ is biholomorphic with $p^{-1} : w \mapsto (w, (g_1(w) : \ldots : g_t(w)))$. The set $E(\mathcal{M}) := p^{-1}(Z)$ which is $Z \times \mathbb{P}^{t-1}$, is called the exceptional set.

We describe a natural line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$. Consider the open set $U_1 = (\Delta(w_0; r) \times \{u_1 \neq 0\}) \cap \widehat{\Delta}(w_0; r)$. Let $\frac{u_j}{u_1} = \theta_j^1$, $2 \le j \le t$. On this chart $g_j(w) = \theta_j^1 g_j(w)$. From the decomposition given in the equation (5.1.1), we have

$$K(\cdot, w) = \overline{g_1(w)}\{K^{(1)}(\cdot, w) + \sum_{j=2}^{t} \bar{\theta}_j^1 K^{(j)}(\cdot, w)\}.$$

This decomposition then yields a section on the chart U_1 of the line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$:

$$s_1(w, \theta) = K^{(1)}(\cdot, w) + \sum_{j=2}^{t} \bar{\theta}_j^1 K^{(j)}(\cdot, w).$$

The vectors $K^{(j)}(\cdot, w)$ are not uniquely determined. However, there exists a canonical choice of these vectors starting from a basis $\{v_1, \ldots, v_t\}$ of the joint kernel $\cap_{i=1}^n \ker(M_j - w_j)^*$:

$$K(\cdot, w) = \sum_{j=1}^{t} \overline{g_j(w)} P(\bar{w}, \bar{w}_0) v_j,\ w \in \Delta(w_0; r)$$

for some $r > 0$ and generators $g_1, \ldots, g_t$ of the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$. Thus we obtain the canonical choice $K^{(j)}(\cdot, w) = P(\bar{w}, \bar{w}_0) v_j$, $1 \le j \le t$ (cf. [6, Section 6]). Let $\mathcal{L}(\mathcal{M})$ be the line bundle on the blow-up space $\widehat{\Delta}(w_0; r)$ determined by the section $(w, \theta) \mapsto s_1(w, \theta)$, where

$$s_1(w, \theta) = P(\bar{w}, \bar{w}_0) v_1 + \sum_{j=2}^{t} \bar{\theta}_j^1 P(\bar{w}, \bar{w}_0) v_j,\ (w, \theta) \in U_1.$$

Let $\widetilde{\mathcal{M}}$ be a second Hilbert module in $\mathfrak{B}_1(\Omega)$, which is again the closure of some polynomial ideal $\mathcal{I}$ but with respect to a second inner product. Assume that $\widetilde{\mathcal{M}}$ is equivalent to $\mathcal{M}$ via a unitary module map L, that is, $LK(\cdot, w) = \overline{\varphi(w)} \widetilde{K}(\cdot, w), w \in \Omega$ for some nonzero holomorphic function ϕ on Ω. In the proof of Theorem 1.10 in [6], we have shown that $LP(\bar{w}, \bar{w}_0) = \widetilde{P}(\bar{w}, \bar{w}_0)L$. Thus

$$\overline{\phi(w)} \widetilde{K}(\cdot, w) = LK(\cdot, w) = \sum_{j=1}^{t} \overline{g_j(w)} LP(\bar{w}, \bar{w}_0) v_j = \sum_{j=1}^{t} \overline{g_j(w)} \widetilde{P}(\bar{w}, \bar{w}_0) L v_j.$$

Therefore,

(i) $\widetilde{s}_1(w, \theta) = \frac{1}{\overline{\phi(w)}} (\widetilde{P}(\bar{w}, \bar{w}_0) L v_1 + \sum_{j=2}^{t} \bar{\theta}_j^1 \widetilde{P}(\bar{w}, \bar{w}_0) L v_j)$ and

(ii) $Ls_1(w, \theta) = \overline{\phi(w)} \widetilde{s}_1(w, \theta)$.

Hence the line bundles $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\widetilde{\mathcal{M}})$ are equivalent as Hermitian holomorphic line bundle on $\widehat{\Delta}(w_0;r)^* = \{(\bar{w},\pi(\bar{u})) : (w,\pi(u)) \in \widehat{\Delta}(w_0;r)\}$. Since the vectors $K^{(j)}(\cdot,w), 1 \le j \le t$, are linearly independent [6, Theorem 1.5], it follows that $V(\mathcal{M}) \cap \Delta(w_0;r) = Z$. Thus if $w \in \Delta(w_0;r) \setminus Z$, then $g_i(w) \neq 0$ for some i, $1 \le i \le t$. Hence $s_i(w,\theta) = \frac{K(\cdot,w)}{g_i(w)}$ on $(\Delta(w_0;r) \times \{u_i \neq 0\}) \cap \widehat{\Delta}(w_0;r)$. Therefore the restriction of the bundle $\mathcal{L}(\mathcal{M})$ to $\widehat{\Delta}(w_0;r) \setminus p^{-1}(Z)$ is the pull back of the Cowen-Douglas bundle for $\mathcal{M}$ on $\Delta(w_0;r) \setminus Z$ via the biholomorphic map π on $\widehat{\Delta}(w_0;r) \setminus p^{-1}(Z)$. We have therefore proved the following Theorem.

Theorem 5.1. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules of the form $[\mathcal{I}]$ and $[\widetilde{\mathcal{I}}]$ ($\mathcal{I}, \widetilde{\mathcal{I}}$ are polynomial ideals), respectively. Assume that they are in $\mathfrak{B}_1(\Omega)$ and that the dimension of the zero set of these modules is at most $m-2$. Then $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent if and only if the line bundles $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\widetilde{\mathcal{M}})$ are equivalent as Hermitian holomorphic line bundles on $\widehat{\Delta}(w_0;r)^*$.*

Although, in general, Z need not be a complex manifold, the restriction of s_1 to $p^{-1}(w_0)$ for $w_0 \in Z$ determines a holomorphic line bundle on $p^{-1}(w_0)^* := \{(w_0,\pi(\bar{u})) : (\bar{w}_0,\pi(u)) \in p^{-1}(w_0)\}$, which we denote by $\mathcal{L}_0(\mathcal{M})$. Thus $s_1 = s_1(w,\theta)|_{\{w_0\}\times\{u_i\neq 0\}}$ is given by the formula

$$s_1(\theta) = K^{(1)}(\cdot,w_0) + \sum_{j=2}^{t} \bar{\theta}_j^1 K^{(j)}(\cdot,w_0).$$

Since the vectors $K^{(j)}(\cdot,w_0)$, $1 \le j \le t$, are uniquely determined by the generators $g_1,\dots,g_t$, we infer that s_1 is well defined.

The following theorem follows from the one we have just proved. All we have to do is to restrict the line bundles to suitable subsets of the exceptional set. However, the details given in the proof below will be useful in studying exlicit examples in the next section.

Theorem 5.2. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules of the form $[\mathcal{I}]$ and $[\widetilde{\mathcal{I}}]$ ($\mathcal{I}, \widetilde{\mathcal{I}}$ are polynomial ideals), respectively. Assume that they are in $\mathfrak{B}_1(\Omega)$ and that the dimension of the zero set of these modules is at most $m-2$. If the modules $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent, then the corresponding bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\widetilde{\mathcal{M}})$ they determine on the projective space $p^{-1}(w_0)^*$ for $w_0 \in Z$, are equivalent as Hermitian holomorphic line bundles.*

Proof. Let $L : \mathcal{M} \to \widetilde{\mathcal{M}}$ be the unitary module map and K and $\widetilde{K}$ be the reproducing kernels corresponding to $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ respectively. From [6, Lemma 1.3] and [11, Theorem 3.7], it follows that (i) $LK(\cdot,w) = \overline{\phi(w)}\widetilde{K}(\cdot,w)$, (ii) $L^*f = \phi f$ and (iii) $K(z,w) = \phi(z)\widetilde{K}(z,w)\overline{\phi(w)}$ for some holomorphic function ϕ on $\Omega \setminus V(\mathcal{M})$. As we have pointed out earlier, ϕ extends to a non-vanishing holomorphic function on Ω.

Since $\mathcal{M}$ is in $\mathfrak{B}_1(\Omega)$, it admits a decomposition as given in equation (5.1.1), with respect the generators $\widetilde{g}_1,\dots,\widetilde{g}_t$ of $\mathcal{S}_{w_0}^{\widetilde{\mathcal{M}}}$. However, we may assume that $\widetilde{g}_i = g_i$ for $1 \le i \le t$, because

$\mathcal{S}^{\mathcal{M}}_{w_0} = \mathcal{S}^{\widetilde{\mathcal{M}}}_{w_0}$ for all $w_0 \in \Omega$. Thus for some $r > 0$, we have

$$\widetilde{K}(\cdot, w) = \sum_{i=1}^{t} \overline{g_j(w)}\widetilde{K}^{(j)}(\cdot, w),\ \ w \in \Delta(w_0; r).$$

By applying the unitary L to equation (5.1.1), we get

$$\overline{\phi(w)}\widetilde{K}(\cdot, w) = LK(\cdot, w) \ = \ \sum_{i=1}^{t} \overline{g_j(w)} LK^{(j)}(\cdot, w).$$

Since ϕ does not vanish on Ω, we may choose

$$\widetilde{K}^{(j)}(\cdot, w) = \frac{LK^{(j)}(\cdot, w)}{\overline{\phi(w)}},\ 1 \le j \le t,\ w \in \Delta(w_0; r).$$

From part (iii) of the decomposition Theorem ([6, Theorem 1.4]), the vectors $\widetilde{K}^{(j)}(\cdot, w_0)$, $1 \le j \le t$, are uniquely determined by the generators $g_1, \ldots, g_t$. Therefore $\widetilde{K}^{(j)}(\cdot, w_0) = \frac{LK^{(j)}(\cdot, w_0)}{\overline{\phi(w_0)}}$. Now the decomposition for $\widetilde{K}$ yields a holomorphic section $\widetilde{s}_1(\theta) = \widetilde{K}^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \theta^1_j \widetilde{K}^{(j)}(\cdot, w_0)$ for the holomorphic line bundle $\mathcal{L}_0(\widetilde{\mathcal{M}})$ on the projective space $p^{-1}(w_0)^*$. Therefore

$$\begin{aligned} Ls_1(\theta) &= LK^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \bar{\theta}^1_j LK^{(j)}(\cdot, w_0) \\ &= \overline{\phi(w_0)}\{\widetilde{K}^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \bar{\theta}^1_j \widetilde{K}^{(j)}(\cdot, w_0)\} \ = \ \overline{\phi(w_0)}\widetilde{s}_1(\theta). \end{aligned}$$

From the unitarity of L, it follows that

$$\| s_1(\theta) \|^2 = \| Ls_1(\theta) \|^2 = |\phi(w_0)|^2 \ \| \widetilde{s}_1(\theta) \|^2 \tag{5.1.2}$$

and consequently the Hermitian holomorphic line bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\widetilde{\mathcal{M}})$ on the projective space $p^{-1}(w_0)^*$ are equivalent. □

Remark 5.3 (The case, where the dimension of the zero set $V(\mathcal{I})$ is $m-1$)**.** Let $\mathcal{M}$ Be a Hilbert module in $\mathfrak{B}_1(\Omega)$. Assume that $\mathcal{M} = [\mathcal{I}]_{\mathcal{M}}$ for some polynomial ideal $\mathcal{I}$ and the dimension of the zero set of $\mathcal{M}$ is $m-1$. Let the polynomials $p_1, \ldots, p_t$ be a minimal set of generators for $\mathcal{M}$. Let $q = \text{g.c.d}\{p_1, \ldots, p_t\}$. Then the Beurling form (cf. [7]) of $\mathcal{I}$ is $q\mathcal{J}$, where $\mathcal{J}$ is generated by $\{p_1/q, \ldots, p_t/q\}$. From [7, Corollary 3.1.12], dim $V(\mathcal{J}) \le m-2$ unless $\mathcal{J} = \mathbb{C}[\underline{z}]$. The reproducing kernels K of $\mathcal{M}$ is of the form $K(z, w) = q(z)\chi(z, w)\overline{q(w)}$. Let $\mathcal{M}_1$ be the Hilbert module determined by the non-negative definite kernel χ. The Hilbert module $\mathcal{M}$ is equivalent to $\mathcal{M}_1$. Now $\mathcal{M}_1 = [\mathcal{J}]$ and $V(\mathcal{M}_1) = V(\mathcal{J})$. If $V(\mathcal{J}) = \phi$, then the modules $\mathcal{M}_1$ belongs to Cowen-Douglas class of rank 1. Otherwise, dim $V(\mathcal{J}) \le m-2$ and Theorem 5.1 determines its equivalence class.

The existence of the polynomials $q_1, ..., q_t$ such that $K^{(j)}(\cdot, w)|_{w=w_0} = q_j^*(\bar{D})K(\cdot, w)|_{w=w_0}$, $1 \leq j \leq t$, is guaranteed by Lemma 4.2. From the decomposition Theorem ([6, Theorem 1.4]) and Lemma 2.1 that $\widetilde{K}^{(j)}(\cdot, w)|_{w=w_0}$ is a linear combination of $q_i(\bar{D})K(\cdot, w)|_{w=w_0}$, $1 \leq i \leq t$. But the following Lemma shows that

$$\widetilde{K}^{(j)}(\cdot, w)|_{w=w_0} = q_j^*(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0},\ 1 \leq j \leq t,$$

which makes it possible to calculate the section for the line bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\widetilde{\mathcal{M}})$ without any explicit reference to the generators of the stalks at w_0. In the following lemma, the decomposition of the reproducing kernels K and $\widetilde{K}$ are with respect to a common set of generators.

Lemma 5.4. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules both of which are completion of some polynomial ideal $\mathcal{I}$ with respect to two different inner products on the polynomial ring. Assume that they belong to the class $\mathfrak{B}_1(\Omega)$ and $\dim V(\mathcal{I}) \leq m-2$. Let K and $\widetilde{K}$ be the corresponding reproducing kernels. Find polynomials $q_1, \ldots, q_t$, for which the vectors $K^{(j)}(\cdot, w) = q_j^*(\bar{D})K(\cdot, w)$ form a basis for the joint kernel at $w = w_0$. Then $\widetilde{K}^{(j)}(\cdot, w) = q_j^*(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}$ is a basis for the joint kernel at w_0 in $\widetilde{\mathcal{M}}$.*

Proof. For $f \in \mathcal{M}$ and $1 \leq i \leq m$, we have

$$\langle f, \bar{\partial}_i LK(\cdot, w)\rangle = \partial_i\langle f, LK(\cdot, w)\rangle = \partial_i\langle L^*f, K(\cdot, w)\rangle = \langle L^*f, \bar{\partial}_i K(\cdot, w)\rangle = \langle f, L\bar{\partial}_i K(\cdot, w)\rangle,$$

implying $\bar{\partial}_i LK(\cdot, w) = L\bar{\partial}_i K(\cdot, w)$. Thus

$$p(\bar{D})LK(\cdot, w) = Lp(\bar{D})K(\cdot, w) \text{ for any } p \in \mathbb{C}[\underline{z}].$$

From equation (4.1.3), it follows that

$$\begin{aligned}
LK^{(j)}(\cdot, w_0) &= L\{q_j(\bar{D})K(\cdot, w)|_{w=w_0}\} = \{Lq_j(\bar{D})K(\cdot, w)\}|_{w=w_0} \\
&= \{q_j(\bar{D})LK(\cdot, w)\}|_{w=w_0} = \{q_j(\bar{D})\overline{\phi(w)}\widetilde{K}(\cdot, w)\}|_{w=w_0} \\
&= [\sum_\alpha \bar{a}_\alpha\{q_j(\bar{D})(\bar{w} - \bar{w}_0)^\alpha \widetilde{K}(\cdot, w)\}]|_{w=w_0} \\
&= \sum_\alpha \bar{a}_\alpha \frac{\partial^\alpha q_j}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0},
\end{aligned}$$

where $\phi(w) = \sum_\alpha a_\alpha (w - w_0)^\alpha$, the power series expansion of ϕ around w_0. Now for any $p \in \mathcal{I}$ we have

$$\begin{aligned}
\langle p, \frac{\partial^\alpha q_j}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}\rangle &= \langle p, \frac{\partial^\alpha q_j}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)\rangle|_{w=w_0} \\
&= \frac{\partial^\alpha q_j^*}{\partial z^\alpha}(D)p(w)|_{w=w_0}.
\end{aligned}$$

Since Lemma 4.2 ensures that $\{[q_1^*], \ldots, [q_t^*]\}$ is a basis for $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$, it follows that

$$\langle p, \frac{\partial^\alpha q_j}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot, w)|_{w=w_0}\rangle = 0 \text{ for all } p \in \mathcal{I} \text{ and } |\alpha| > 0.$$

Therefore, we have $\frac{\partial^\alpha q_j}{\partial z^\alpha}(\bar{D})\widetilde{K}(\cdot,w)|_{w=w_0} = 0$ for $|\alpha| > 0$. Hence $LK^{(j)}(\cdot,w_0) = \bar{a}_0 q_j(\bar{D})\widetilde{K}(\cdot,w)|_{w=w_0} = \overline{\phi(w_0)}q_j(\bar{D})\widetilde{K}(\cdot,w)|_{w=w_0}$ and consequently $\widetilde{K}^{(j)}(\cdot,w)|_{w=w_0} = q_j(\bar{D})\widetilde{K}(\cdot,w)|_{w=w_0}$, $1 \le j \le t$. □

We illustrate, by means of some examples, the nature of the invariants we obtain from the line bundle $\mathcal{L}_0$ that lives on the projective space. From Theorem 5.2, it follows that the curvature of the line bundle $\mathcal{L}_0$ is an invariant for the submodule. An example was given in [14] showing that the curvature is is not a complete invariant. However the following lemma is useful for obtaining complete invariant in a large class of examples.

Lemma 5.5. *Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are Hilbert modules in $\mathfrak{B}_1(\Omega)$, for some bounded domain Ω in $\mathbb{C}^m$. Suppose that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are such that they are in the Cowen-Douglas class $B_1(\Omega \setminus X)$ where $\dim X \le m-2$. Let $\mathcal{M} \subseteq \mathcal{H}$ and $\widetilde{\mathcal{M}} \subseteq \widetilde{\mathcal{H}}$ be submodules satisfying the following conditions:*

(i) $\mathbb{V}_w(\mathcal{M}) = \mathbb{V}_w(\widetilde{\mathcal{M}})$ for all $w \in \Omega$ and

(ii) $\mathcal{M} = \cap_{w\in\Omega}\mathcal{M}^e_w$ and $\widetilde{\mathcal{M}} = \cap_{w\in\Omega}\widetilde{\mathcal{M}}^e_w$, where as before $\mathcal{M}^e_w := \{f \in \mathcal{H} : q(D)f|_w = 0$ for all $q \in \mathbb{V}_w(\mathcal{M})\}$.

If $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are equivalent, then $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent.

Proof. Suppose $U : \mathcal{H} \to \widetilde{\mathcal{H}}$ is a unitary module map. Then U is induced by a non-vanishing holomorphic function, say ψ, on $\Omega \setminus X$ (cf. [11]). This function ψ extends to all of Ω by Hartog's Theorem. As before, this extension does not vanish on Ω. Let $w_0 \in \Omega$ and $q \in \mathbb{V}_{w_0}(\mathcal{M}) = \mathbb{V}_{w_0}(\widetilde{\mathcal{M}})$. Also let $\psi(w) = \sum_\alpha a_\alpha (w-w_0)^\alpha$ be the power series expansion around w_0. For $f \in \mathcal{M}$, we have

$$\begin{aligned} q(D)(Uf)|_{w=w_0} &= q(D)(\psi f)|_{w=w_0} = q(D)\{\sum_\alpha a_\alpha (w-w_0)^\alpha f\}|_{w=w_0} \\ &= \sum_\alpha a_\alpha q(D)\{(w-w_0)^\alpha f\}]|_{w=w_0} = \{\sum_{k\le\alpha}\binom{\alpha}{k}(w-w_0)^{\alpha-k}\frac{\partial^k q}{\partial z^k}(D)(f)\}_{w=w_0} \\ &= 0 \end{aligned}$$

since $\frac{\partial^k q}{\partial z^k} \in \mathbb{V}_{w_0}(\mathcal{M})$ for any multi index k whenever $q \in \mathbb{V}_{w_0}(\mathcal{M})$. Therefore it follows that $Uf \in \widetilde{\mathcal{M}}$. A similar arguments shows that $U^*\widetilde{\mathcal{M}} \subseteq \mathcal{M}$. The result follows from unitarity of U. □

5.1.1 The (α, β, θ) examples: Weighted Bergman modules in the unit ball

Let $\mathbb{B}^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ be the unit ball in $\mathbb{C}^2$. Let $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ be the Hilbert space of all (equivalence classes of) Borel measurable functions on $\mathbb{B}^2$ satisfying

$$\| f \|^2_{\alpha,\beta,\theta} = \int_{\mathbb{B}^2} |f(z)|^2 d\mu(z_1, z_2) < +\infty,$$

where the measure is

$$d\mu(z_1,z_2) = (\alpha+\beta+\theta+2)|z_2|^{2\theta}(1-|z_1|^2-|z_2|^2)^\alpha(1-|z_2|^2)^\beta dA(z_1,z_2)$$

for $(z_1,z_2)\in\mathbb{B}^2$, $-1<\alpha,\beta,\theta<+\infty$ and $dA(z_1,z_2) = dA(z_1)dA(z_2)$. Here dA denote the normalized area measure in the plane, that is $dA(z) = \frac{1}{\pi}dxdy$ for $z = x+iy$. The weighted Bergman space $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is the subspace of $L^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ consisting of the holomorphic functions on $\mathbb{B}^2$. The Hilbert space $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is non-trivial if we assume that the parameters α,β,θ satisfy the additional condition:

$$\alpha+\beta+\theta+2>0.$$

The reproducing kernel $K_{\alpha,\beta,\theta}$ of $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$ is given by

$$\begin{aligned} K_{\alpha,\beta,\theta}(z,w) &= \frac{1}{\alpha+\beta+\theta+2}\frac{1}{(1-z_1\bar{w}_1)^{\alpha+\beta+\theta+3}} \\ &\quad\times\left\{\sum_{k=0}^{+\infty}\frac{(\alpha+\beta+\theta+k+2)(\alpha+\theta+2)_k}{(\theta+1)_k}\left(\frac{z_2\bar{w}_2}{1-z_1\bar{w}_1}\right)^k\right\}, \end{aligned}$$

where $z=(z_1,z_2), w=(w_1,w_2)\in\mathbb{B}^2$ and $(a)_k = a(a+1)\dots(a+k-1)$ is the Pochhammer symbol. This kernel differs from the kernel $P_{\alpha,\beta,\theta}$ given in [26] only by a multiplicative constant. The reader may consult [26] for a detailed discussion of these Hilbert modules.

Let $\mathcal{I}_P$ be an ideal in $\mathbb{C}[z_1,z_2]$ such that $V(\mathcal{I}_P)=\{P\}\subset\mathbb{B}^2$. We have

$$\dim\ker D_{(\mathbf{M}-\mathbf{w})^*} = \begin{cases} 1 & \text{for } w\in\mathbb{B}^2\setminus\{P\};\\ \dim\mathcal{I}_P/\mathfrak{m}_P\mathcal{I}_P\,(>1) & \text{for } w=P.\end{cases}$$

Hence $[\mathcal{I}_P]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ (the completion of $\mathcal{I}_P$ in $\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)$) is not equivalent to $[\mathcal{I}_{P'}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ (the completion of $\mathcal{I}_P'$ in $\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)$) if $P\neq P'$. Now let us determine when two modules in the set

$$\{[\mathcal{I}_P]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)} : -1<\alpha,\beta,\theta<+\infty \text{ and } \alpha+\beta+\theta+2>0\}.$$

are equivalent. In the following proposition, without loss of generality, we have assumed $P=0$.

Proposition 5.6. *Suppose $\mathcal{I}$ is an ideal in $\mathbb{C}[z_1,z_2]$ with $V(\mathcal{I})=\{0\}$. Then the Hilbert modules $[\mathcal{I}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ and $[\mathcal{I}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ are unitarily equivalent if and only if $\alpha=\alpha',\beta=\beta'$ and $\theta=\theta'$.*

Proof. From the Hilbert Nullstellensatz, it follows that there exist an natural number N such that $\mathfrak{m}_0^N\subset\mathcal{I}$. Let $\mathcal{I}_{m,n}$ be the polynomial ideal generated by z_1^m and z_2^n. Combining (4.0.1) with Lemma 5.5 we see, in particular, that the submodules $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ and $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ are unitarily equivalent for $m,n\geq N$. Let $K_{m,n}$ be the reproducing kernel for $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$. We write $K_{\alpha,\beta,\theta}(z,w)=\sum_{i,j\geq 0}b_{ij}z_1^iz_2^j$ where

$$b_{ij} = \frac{\alpha+\beta+\theta+j+2}{\alpha+\beta+\theta+2}\cdot\frac{(\alpha+\theta+2)_j}{(\theta+1)_j}\cdot\frac{(\alpha+\beta+\theta+j+3)_i}{i!}. \tag{5.1.3}$$

Let $I_{m,n} := \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : i,j \geq 0, i \geq m \text{ or } j \geq n\}$. We note that

$$K_{m,n}(z,w) = \sum_{(i,j)\in I_{m,n}} b_{ij} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j.$$

One easily see that the set $\{z_1^m, z_2^n\}$ forms a minimal set of generators for the sheaf corresponding to $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$. The reproducing kernel then can be decomposed as

$$K_{m,n}(z,w) = \bar{w}_1^m K_1^{m,n}(z,w) + \bar{w}_2^n K_2^{m,n}(z,w), \text{ for some } r>0 \text{ and } w \in \Delta(0;r).$$

Successive differentiation, using Leibnitz rule, gives

$$\begin{aligned} K_1^{m,n}(z,w)|_{w=0} &= \frac{1}{m!}\bar{\partial}_1^m K_{m,n}(\cdot,w)\}|_{w=(0,0)} = b_{m0} z_1^m \text{ and} \\ K_2^{m,n}(z,w)|_{w=0} &= \frac{1}{n!}\bar{\partial}_2^n K_{m,n}(\cdot,w)\}|_{w=(0,0)} = b_{0n} z_2^n. \end{aligned}$$

Therefore

$$s_1(\theta_1) = b_{m0} z_1^m + \theta_1 b_{0n} z_2^n,$$

where θ_1 denotes co-ordinate for the corresponding open chart in $\mathbb{P}^1$. Thus

$$\| s_1(\theta_1) \|^2 = b_{m0}^2 \| z_1^m \|^2 + b_{0n}^2 \| z_2^n \|^2 |\theta_1|^2 = b_{m0} + b_{0n}|\theta_1|^2.$$

Let $a_{m,n} = b_{0n}/b_{m0}$. Let $\mathcal{K}_{m,n}$ denote the curvature corresponding to the bundle $\mathcal{L}_{0,m,n}$ which is determined on the projective space $\mathbb{P}^1$ by the module $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$. Thus we have

$$\begin{aligned} \mathcal{K}_{m,n}(\theta_1) &= \partial_{\theta_1}\partial_{\bar{\theta}_1} \ln\| s_1(\theta_1) \|^2 = \partial_{\theta_1}\partial_{\bar{\theta}_1} \ln(1 + a_{m,n}|\theta_1|^2) \\ &= \partial_{\theta_1} \frac{a_{m,n}\theta_1}{1 + a_{m,n}|\theta_1|^2} = \frac{a_{m,n}}{(1 + a_{m,n}|\theta_1|^2)^2}. \end{aligned}$$

Let $\mathcal{K}'_{m,n}$ denote the curvature corresponding to the bundle $\mathcal{L}'_{0,m,n}$ which is determined on the projective space $\mathbb{P}^1$ by the module $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$. As above, from Lemma 5.4, we have

$$\mathcal{K}'_{m,n}(\theta_1) = \frac{a'_{m,n}}{(1 + a'_{m,n}|\theta_1|^2)^2}.$$

This easily follows from Lemma 5.5. Since the submodules $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha,\beta,\theta}(\mathbb{B}^2)}$ and $[\mathcal{I}_{m,n}]_{\mathcal{A}^2_{\alpha',\beta',\theta'}(\mathbb{B}^2)}$ are unitarily equivalent, from Theorem 5.2, it follows that $\mathcal{K}_{m,n}(\theta_1) = \mathcal{K}'_{m,n}(\theta_1)$ for θ_1 in an open chart $\mathbb{P}^1$ and $m,n \geq N$. Thus

$$\frac{a_{m,n}}{(1 + a_{m,n}|\theta_1|^2)^2} = \frac{a'_{m,n}}{(1 + a'_{m,n}|\theta_1|^2)^2}.$$

This shows that $(a_{m,n} - a'_{m,n})(1 + a_{m,n}a'_{m,n}|\theta_1|^2) = 0$. So $a_{m,n} = a'_{m,n}$ and hence

$$\frac{b_{0n}}{b_{m0}} = \frac{b'_{0n}}{b'_{m0}} \tag{5.1.4}$$

for all $m, n \geq N$. This also follows directly from the equation (5.1.2). It is enough to consider the cases $(m,n) = (N,N), (N,N+1), (N,N+2)$ and $(N+1,N)$ to prove the Proposition. From equation (5.1.4), we have

$$\frac{b_{(N+1)0}}{b_{N0}} = \frac{b'_{(N+1)0}}{b'_{N0}}, \quad \frac{b_{0(N+1)}}{b_{0N}} = \frac{b'_{0(N+1)}}{b'_{0N}} \text{ and } \frac{b_{0(N+2)}}{b_{0(N+1)}} = \frac{b'_{0(N+2)}}{b'_{0(N+1)}}. \tag{5.1.5}$$

Let $A = \alpha + \beta + \theta, B = \alpha + \theta$ and $C = \theta$. From equation (5.1.3), we have

$$\frac{b_{(N+1)0}}{b_{N0}} = \frac{A+N+3}{N+1}, \quad \frac{b_{0(N+1)}}{b_{0N}} = \frac{A+N+3}{A+N+2} \cdot \frac{B+N+2}{C+N+1}$$

and

$$\frac{b_{0(N+2)}}{b_{0(N+1)}} = \frac{A+N+4}{A+N+3} \cdot \frac{B+N+3}{C+N+2}.$$

From (5.1.5), it follows that $A = A'$ and

$$BC' + B(N+1) + C'(N+2) = B'C + B'(N+1) + C(N+2), \tag{5.1.6}$$

$$BC' + B(N+2) + C'(N+3) = B'C + B'(N+2) + C(N+3). \tag{5.1.7}$$

Subtracting (5.1.7) from (5.1.6), we get $B - C = B' - C'$ and thus $\theta = \theta'$. Therefore $\frac{b_{0(N+1)}}{b_{0N}} = \frac{b'_{0(N+1)}}{b'_{0N}}$ implying $B = B'$ and hence $\alpha = \alpha'$. Lastly $A = A'$ and in consequence $\beta = \beta'$. □

5.2 The quadratic transformation

For a homogeneous ideal $\mathcal{I}$, let $\mathcal{M}$ be a Hilbert module in $\mathfrak{B}_1(\Omega)$ is of the form $[\mathcal{I}]$. Assume that $\{p_1, \ldots, p_t\}$ be a minimal set of generators for $\mathcal{I}$ consisting of homogeneous polynomials of same degree, say k. From Lemma 4.6, we knew that $\{p_{10}, \ldots, p_{t0}\}$ is a minimal set of generators for $\mathcal{S}_0^{\mathcal{M}}$. Then on a neighborhood $\Delta(0;\varepsilon)$ of 0, the reproducing kernel K of $\mathcal{M}$ admits a decomposition:

$$K(\cdot, w) = \sum_{i=1}^{t} \overline{p_i(w)} K_i(\cdot, w)$$

as in Theorem 2.3. The set

$$\widehat{\Delta}_Q(0;\varepsilon) := \{(w, \pi(u)) \in \Delta(0;\varepsilon) \times \mathbb{P}^{m-1} : u_i w_j - u_j w_i = 0,\ 1 \leq i, j \leq m\}$$

is called the blow up of the poly-disc $\Delta(0;\varepsilon)$ at the point 0 (also called the *quadratic transformation* of $\Delta(0;\varepsilon)$ at the point 0).

There is a natural line bundle on the blow-up space $\widehat{\Delta}_Q(0;\varepsilon)$, which we describe below. Consider the open chart where $\widehat{U}_1 = (\Delta(0;r) \times \{u_1 \neq 0\}) \cap \widehat{\Delta}_Q(0;\varepsilon)$. On $\widehat{U}_1$, let $\frac{u_j}{u_1} = \theta_j^1$, $2 \leq j \leq m$. Thus $w_j = \theta_j^1 w_1$, $2 \leq j \leq m$ on $\widehat{U}_1$ and we have

$$K(\cdot, w) = \overline{w}_1{}^k \{\sum_{i=1}^{t} \overline{\widetilde{p}_i(\theta)} K_i(\cdot, w)\}.$$

Set

$$s_1(\theta) = \sum_{i=1}^{t} \widetilde{p}_i(\theta) K_i(\cdot, 0), \;\; \theta \in \mathbb{P}^{m-1} \cap \{\pi(u) : u_1 \neq 0\}.$$

For $2 \leq i \leq m$, define s_i on $U_i = \mathbb{P}^{m-1} \cap \{\pi(u) : u_i \neq 0\}$ similarly. $s_i(\theta)$ is called *quadratic transformation* of the reproducing kernel K on U_i. The set $\{s_1, \ldots, s_m\}$ defines a holomorphic Hermitian line bundle on $\mathbb{P}^{m-1}$. Let us denote this line bundle by $\mathcal{Q}(\mathcal{M})$. The blow up space along a linear subspace is defined similarly (cf. [21, Example 2.5.2]). Let the linear subspace be $V = \{z_{r+1} = \ldots = z_m = 0\}$ and the blow up of $\Delta(0; r)$ along V is

$$\widehat{\Delta}_Q^V(0; \varepsilon) := \{(w, \pi(u)) \in \Delta(0; \varepsilon) \times \mathbb{P}^{m-r-1} : u_i w_j - u_j w_i = 0, \, r+1 \leq i, j \leq m, \, m - r \geq 2\}.$$

We illustrate, by means of a number of examples, the nature of the invariants we obtain from the line bundle $\mathcal{Q}$ that lives on the projective space.

Example of blowing up along a linear subspace. Let $\mathcal{H}$ be an analytic Hilbert module over $\Omega \subset \mathbb{C}^m$ containing the origin. Let $\mathcal{H}_0^{(n)}$ be the submodule of $\mathcal{H}$ denoting the closure of the polynomial ideal $\mathcal{I}$ generated by

$$\{z_{r+1}^{i_{r+1}} \ldots z_m^{i_m} : i_j \in \mathbb{N} \cup \{0\}, r+1 \leq j \leq m. i_{r+1} + \ldots + i_m = n, \, m - r \geq 2\}.$$

Let $K_0^{(n)}$ be the reproducing kernel corresponding to $\mathcal{H}_0^{(n)}$. Let us fix a point $(p, 0) \in \mathbb{C}^r \times \mathbb{C}^{m-r}$ in Ω. From decomposition theorem and Lemma 4.6, $K_0^{(n)}$ admits a decomposition:

$$K_0^{(n)}(\cdot, w) = \sum_{i_{r+1} + \ldots + i_m = n} \bar{w}_{r+1}^{i_{r+1}} \ldots \bar{w}_m^{i_m} K_{i_{r+1} \ldots i_m}(\cdot, w),$$

in some neighborhood of the point $(p, 0)$. Clearly for $i = (i_{r+1}, \ldots, i_m)$, we have

$$\bar{\partial}^i K_0^{(n)}(\cdot, w)|_{(p,0)} = i! K_i(\cdot, w)|_{(p,0)}.$$

Let $\theta = (\theta_{r+2}, \ldots, \theta_m)$ be the usual homogeneous coordinates on the open sets $U_{r+1} = \{\pi(u) : u_{r+1} \neq 0\}$ in the complex projective space $\mathbb{P}^{m-r-1}$. Thus, following the construction given above,

$$s_1(\theta) = \{(\bar{\partial}_{r+1} + \theta_{r+2} \bar{\partial}_{r+2} + \ldots + \theta_m \bar{\partial}_m)^n K_0^{(n)}(\cdot, w)\}|_{w=(p,0)}, \tag{5.2.1}$$

determines a section of the line bundle $\mathcal{Q}(\mathcal{M})$ over U_{r+1}, with respect to the point $(p, 0)$. The proposition below and its proof is a straightforward generalization of [14, Theorem 5.1].

Proposition 5.7. *Let $\mathcal{H}_0^{(n)} \subset \mathcal{H}$ and $\widetilde{\mathcal{H}}_0^{(n)} \subset \widetilde{\mathcal{H}}$ be two analytic Hilbert submodules consisting of Holomorphic functions on Ω vanishing to order n. If $\mathcal{H}_0^{(n)}$ and $\widetilde{\mathcal{H}}_0^{(n)}$ are equivalent via a unitary module map, then the corresponding bundles $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are equivalent.*

Proof. Let $L : \mathcal{H}_0^{(n)} \to \widetilde{\mathcal{H}}_0^{(n)}$ be the unitary module map and $K_0^{(n)}$ and $\widetilde{K}_0^{(n)}$ be the reproducing kernels corresponding to $\mathcal{H}_0^{(n)}$ and $\widetilde{\mathcal{H}}_0^{(n)}$ respectively. The existence of a holomorphic function ϕ on

$\Omega\backslash V(\mathcal{I})$ such that $LK_0^{(n)}(\cdot,w)=\overline{\phi(w)}\widetilde{K}_0^{(n)}(\cdot,w),\ L^*f=\phi f$ and $K_0^{(n)}(z,w)=\phi(z)\widetilde{K}_0^{(n)}(z,w)\overline{\phi(w)}$ follows from Lemma 1.11 and [11, Theorem 3.7]. As we have seen before, since $m-r\geq 2$, ϕ extends to a non-vanishing holomorphic function on Ω.

Fix $p'=(p,0)\in\mathbb{C}^r\times\mathbb{C}^{m-r}$ in $V(\mathcal{I})$. Now we have

$$\langle f,\bar{\partial}_i LK_0^{(n)}(\cdot,w)\rangle=\bar{\partial}_i\langle f,LK_0^{(n)}(\cdot,w)\rangle=\bar{\partial}_i\langle L^*f,K_0^{(n)}(\cdot,w)\rangle=\langle f,L\bar{\partial}_iK_0^{(n)}(\cdot,w)\rangle.$$

Since f is arbitrary in $\mathcal{H}_0^{(n)}$, it follows that $\bar{\partial}_iLK_0^{(n)}(\cdot,w)=L\bar{\partial}_iK_0^{(n)}(\cdot,w)$, $i=1,2$. Let s_1 and $\tilde{s}_1$ be sections of $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ respectively, of the form (5.2.1), on $U_{r+1}\subseteq\mathbb{P}^1$. As L commutes with differentiation with respect to w, we have,

$$\begin{aligned}
Ls_1(\theta) &= L(\bar{\partial}_{r+1}+\theta_{r+2}\bar{\partial}_{r+2}+...+\theta_m\bar{\partial}_m)^nK_0^{(n)}(\cdot,w)|_{w=p'}\\
&= (\bar{\partial}_{r+1}+\theta_{r+2}\bar{\partial}_{r+2}+...+\theta_m\bar{\partial}_m)^nLK_0^{(n)}(\cdot,w)|_{w=p'}\\
&= \{(\bar{\partial}_{r+1}+\theta_{r+2}\bar{\partial}_{r+2}+...+\theta_m\bar{\partial}_m)^n\overline{\phi(w)}\widetilde{K}_0^{(n)}(\cdot,w)\}|_{w=p'}\\
&= \{\textstyle\sum_{i=0}^n\binom{n}{i}(\bar{\partial}_{r+1}+\theta_{r+2}\bar{\partial}_{r+2}+...+\theta_m\bar{\partial}_m)^i\overline{\phi(w)}(\bar{\partial}_{r+1}+\theta_{r+2}\bar{\partial}_{r+2}+...+\theta_m\bar{\partial}_m)^{n-i}\widetilde{K}_0^{(n)}(\cdot,w)\}|_{w=p'}.
\end{aligned}$$

Since $\widetilde{K}_0^{(n)}(\cdot,w)$ belongs to the canonical subspace $\widetilde{\mathcal{H}}_0^{(n)}$, it follows that

$$(\bar{\partial}_{r+1}+\theta_{r+2}\bar{\partial}_{r+2}+...+\theta_m\bar{\partial}_m)^{n-i}\widetilde{K}_0^{(n)}(\cdot,w)|_{w=p'}=0$$

at $w=p'\in V(\mathcal{I})$ for $i>0$ Hence we have

$$L\tilde{s}_1(\theta_1)=\overline{\phi(p')}(\bar{\partial}_{r+1}+\theta_{r+2}\bar{\partial}_{r+2}+...+\theta_m\bar{\partial}_m)^nK_0^{(n)}(\cdot,w)|_{w=p'}=\overline{\phi(p')}s_1(\theta).$$

From the unitarity of L, we conclude that

$$\|L\tilde{s}_1(\theta)\|^2=|\phi(p')|^2\|s_1(\theta)\|^2.$$

Consequently, the line bundles determined by $\mathcal{H}_0^{(n)}$ and $\widetilde{\mathcal{H}}_0^{(n)}$ on $\mathbb{P}^{m-r-1}$ are equivalent. □

5.2.1 The (λ,μ) examples: Weighted Bergman modules on unit bi-disc

Let $H^{(\lambda,\mu)}(\mathbb{D}^2)$ be the weighted Bergman space determined by the reproducing kernel

$$K^{(\lambda,\mu)}(z,w)=\frac{1}{(1-z_1\bar{w}_1)^\lambda(1-z_2\bar{w}_2)^\mu},\ z,w\in\mathbb{D}^2.$$

Let $\mathcal{H}_{(p,q)}^{(\lambda,\mu,n)}$ be the submodule of $H^{(\lambda,\mu)}(\mathbb{D}^2)$ consists of holomorphic functions vanishing up to order n at the point $(p,q)\in\mathbb{D}^2$, $n\geq 2$. From discussions in the section 1.3, it is clear that the dimension of the joint kernel of $\mathcal{H}_{(p,q)}^{(\lambda,\mu,n)}$ jumps at the point (p,q) and hence $\mathcal{H}_{(p,q)}^{(\lambda,\mu,n)}$ is not equivalent to $\mathcal{H}_{(p',q')}^{(\lambda',\mu',n)}$ if $(p,q)\neq(p',q')$. So for a fixed point $(p,q)\in\mathbb{D}^2$, we want to determine equivalence of any two module in the class $\{\mathcal{H}_{(p,q)}^{(\lambda,\mu,n)}:\lambda,\mu>0\}$. In the following proposition we have done the case when $(p,q)=(0,0)$ using the above theorem. For general (p,q), both the theorem and proposition can be proved similarly with a change in coordinates by Möbius transformation (see [14]).

Proposition 5.8. *For $n \geq 2$, $\mathcal{H}^{(\lambda,\mu,n)}_{(0,0)}$ and $\mathcal{H}^{(\lambda',\mu',n)}_{(0,0')}$ are unitarily equivalent if and only if $\lambda = \lambda'$ and $\mu = \mu'$.*

Proof. The reproducing kernel $K_0^{(n)}(z,w)$ of $\mathcal{H}^{(\lambda,\mu,n)}_{(0,0)}$ is given by

$$\begin{aligned} K_0^{(n)}(z,w) &= (1 - z_1\bar{w}_1)^{-\lambda}(1 - z_2 w_2)^{-\mu} - \sum_{k=0}^{n-1} \sum_{i,j\geq 0, i+j=k} b_{ij} z_1^i z_1^j \bar{w}_1^i \bar{w}_2^j \\ &= \sum_{k=n}^{\infty} \sum_{i,j\geq 0, i+j=k} b_{ij} z_1^i z_1^j \bar{w}_1^i \bar{w}_2^j \end{aligned}$$

where

$$b_{ij} = \binom{\lambda}{i}\binom{\mu}{j} = \frac{1}{\| z_1^i z_1^j \|^2}, \text{ and } \binom{\nu}{l} = \begin{cases} \frac{\nu\ldots(\nu+l-1)}{l!}, & l \geq 1; \\ 1, & l = 0. \end{cases}$$

Then

$$\begin{aligned} s_1(\theta_1) &= \{(\bar{\partial}_1 + \theta_1\bar{\partial}_2)^n K_0^n(\cdot,w)\}|_{w=(0,0)} = \sum_{i,j\geq 0, i+j=n} \binom{n}{i} {\theta_1}^j \bar{\partial}_1^i \bar{\partial}_2^j K_0^n(\cdot,w)\}|_{w=(0,0)} \\ &= \sum_{i,j\geq 0, i+j=n} \binom{n}{i} i!j! b_{ij} z_1^i z_2^j {\theta_1}^j = n! \sum_{i,j\geq 0, i+j=n} b_{ij} z_1^i z_2^j {\theta_1}^j \end{aligned}$$

Let us denote $b_i = b_{in-i}$ for $0 \leq i \leq n$. Hence $s_1(\theta_1) = n! \sum_{i=0}^n b_i z_1^i z_2^{n-i} {\theta_1}^{n-i}$. We note that

$$\| s_1(\theta_1) \|^2 = (n!)^2 \sum_{i=0}^n b_i^2 \| z_1^i z_2^{n-i} \|^2 |\theta_1|^{2(n-i)} = (n!)^2 \sum_{i=0}^n b_i |\theta_1|^{2(n-i)}.$$

Let $a_i = b_i/b_0$. Let $\mathcal{K}$ denote the curvature corresponding to the bundle $\mathcal{Q}$ which is determined by the module $\mathcal{H}^{(\lambda,\mu,n)}_{(0,0)}$. We obtain

$$\begin{aligned} \mathcal{K}(\theta_1) &= -\partial_{\theta_1}\partial_{\bar{\theta}_1} \log(1 + a_1|\theta_1|^2 + \ldots + a_n|\theta_1|^{2n}) = -\partial_{\theta_1} \frac{a_1\theta_1 + \ldots + n a_n {\theta_1}^n \bar{\theta}_1^{n-1}}{1 + a_1|\theta_1|^2 + \ldots + a_n|\theta_1|^{2n}} \\ &= -\frac{ab - |\theta_1|^2 c^2}{b^2} \end{aligned}$$

where $a = a_1 + \ldots + n^2 a_n |\theta_1|^{2(n-1)}$, $b = 1 + a_1|\theta_1|^2 + \ldots + a_n|\theta_1|^{2n}$ and $c = a_1 + \ldots + n a_n |\theta_1|^{2(n-1)}$. The curvature corresponding to the bundle $\mathcal{Q}'$, which is determined by $\mathcal{H}^{(\lambda',\mu',n)}_{(0,0')}$, is given by

$$\mathcal{K}'(\theta_1) = -\frac{a'b' - |\theta_1|^2 c'^2}{b'^2}.$$

This easily follows from Lemma 5.5. If the modules $\mathcal{H}^{(\lambda,\mu,n)}_{(0,0)}$ and $\mathcal{H}^{(\lambda',\mu',n)}_{(0,0')}$ are unitarily equivalent, then $\mathcal{K}(\theta_1) = \mathcal{K}'(\theta_1)$ for $\theta_1 \in \mathbb{P}^1 \cap \{\pi(u) : u_1 \neq 0\}$. Thus

$$\frac{ab - |\theta_1|^2 c^2}{b^2} = \frac{a'b' - |\theta_1|^2 c'^2}{b'^2}$$

which implies

$$bb'(ab'-a'b) = |\theta_1|^2(b'c-bc')(b'c+bc'). \tag{5.2.2}$$

Now we have

$$\begin{aligned} bb' &= (1+a_1|\theta_1|^2+...+a_n|\theta_1|^{2n})(1+a_1'|\theta_1|^2+...+a_n'|\theta_1|^{2n}) \\ &= \{1+(a_1+a_1')|\theta_1|^2+(a_2+a_2'+a_1a_1')|\theta_1|^4...\}, \end{aligned}$$

$$\begin{aligned} ab'-a'b &= (a_1+4a_2|\theta_1|^2+9a_3|\theta_1|^4+...+n^2a_n|\theta_1|^{2(n-1)})(1+a_1'|\theta_1|^2+a_2'|\theta_1|^4+...+a_n'|\theta_1|^{2n}) \\ &\quad -(a_1'+4a_2'|\theta_1|^2+9a_3'|\theta_1|^4+...+n^2a_n'|\theta_1|^{2(n-1)})(1+a_1|\theta_1|^2+a_|\theta_1|^4+...+a_n|\theta_1|^{2n}) \\ &= [(a_1-a_1')+\{4(a_2-a_2')-(a_1-a_1')\}|\theta_1|^2+\{3(a_1'a_2-a_1a_2')+9(a_3-a_3')\}|\theta_1|^4+...], \end{aligned}$$

$$\begin{aligned} b'c-bc' &= (1+a_1'|\theta_1|^2+a_2'|\theta_1|^4+...+a_n'|\theta_1|^{2n})(a_1+2a_2|\theta_1|^2+3a_3|\theta_1|^4+...+na_n|\theta_1|^{2(n-1)}) \\ &\quad -(1+a_1|\theta_1|^2+a_2|\theta_1|^4+...+a_n|\theta_1|^{2n})(a_1'+2a_2'|\theta_1|^2+3a_3'|\theta_1|^4+...+na_n'|\theta_1|^{2(n-1)}) \\ &= [(a_1-a_1')+2(a_1-a_2')|\theta_1|^2+\{3(a_3-a_3')+a_2a_1'-a_2'a_1\}|\theta_1|^4+...], \\ b'c+bc' &= \{(a_1+a_1')+2(a_2+a_2'+a_1a_1')|\theta_1|^2+3(a_3+a_3'+a_1a_2'+a_1'a_2)|\theta_1|^4+...\} \end{aligned}$$

From (5.2.2), equating coefficients of $|\theta_1|^{2n}$, for $n \geq 2$, we find that $\lambda = \lambda'$ and $\mu = \mu'$. Equating the constant term we get

$$a_1 - a_1' = 0, \text{ that is, } a_1 = a_1', \text{ hence } b_1/b_0 = b_1'/b_0'. \tag{5.2.3}$$

Now equating the coefficient of $|\theta_1|^2$ we have, $\{4(a_2-a_2')-(a_1-a_1')\}+({a_1}^2-{a_1'}^2) = ({a_1}^2-{a_1'}^2)$. Thus from (5.2.3), we have

$$a_2 = a_2', \text{ that is, } b_2/b_0 = b_2'/b_0', \text{ and } b_2/b_1 = b_2'/b_1'. \tag{5.2.4}$$

Now

$$\frac{b_1}{b_0} = \frac{b_{1n-1}}{b_{0n}} = \frac{\binom{\lambda}{1}\binom{\mu}{n-1}}{\binom{\lambda}{0}\binom{\mu}{n}} = \frac{\lambda\frac{\mu(\mu+1)...(\mu+n-1-1)}{(n-1)!}}{1\frac{\mu(\mu+1)...(\mu+n-1)}{n!}} = \frac{n\lambda}{\mu+n-1}.$$

Also

$$\frac{b_2}{b_1} = \frac{b_{2n-2}}{b_{1n-1}} = \frac{\binom{\lambda}{2}\binom{\mu}{n-2}}{\binom{\lambda}{1}\binom{\mu}{n-1}} = \frac{\frac{\lambda(\lambda+1)}{2}\frac{\mu(\mu+1)...(\mu+n-2-1)}{(n-2)!}}{\lambda\frac{\mu(\mu+1)...(\mu+n-1-1)}{(n-1)!}} = \frac{(n-1)(\lambda+1)}{2(\mu+n-2)}.$$

From (5.2.3), we have

$$(\lambda\mu' - \lambda'\mu) + (n-1)(\lambda - \lambda') = 0. \tag{5.2.5}$$

Also from (5.2.4), we have

$$(\lambda\mu' - \lambda'\mu) + (n-2)(\lambda - \lambda') = \mu - \mu'. \tag{5.2.6}$$

Subtracting (5.2.6) from (5.2.5), we get $\lambda-\lambda' = -(\mu-\mu') = \kappa(\text{say})$, then $\lambda' = \lambda-\kappa$ and $\mu' = \mu+\kappa$. Again we use (5.2.3) to get $\lambda(\mu+\kappa)-(\lambda-\kappa)\mu+(n-1)\kappa = 0$, that is, $(\lambda+\mu+n-1)\kappa = 0$. Since $\lambda+\mu+n-1 > 0$, we have $\kappa = 0$ and consequently $\lambda = \lambda'$ and $\mu = \mu'$. □

Remark 5.9. From Lemma 5.5, it follows that $\mathcal{H}^{(\lambda,\mu,1)}_{(0,0)}$ and $\mathcal{H}^{(\lambda',\mu',1)}_{(0,0)}$ are unitarily equivalent if and only if $\lambda = \lambda'$ and $\mu = \mu'$.

5.2.2 The (n, k) examples

For a fixed natural number j, let I_j be the polynomial ideal generated by the set $\{z_1^n, z_1^{k_j} z_2^{n-k_j}\}$, $k_j \neq 0$. Let $\mathcal{M}_j$ be the closure of I_j in the Hardy space $H^2(\mathbb{D}^2)$. We claim that $\mathcal{M}_1$ and $\mathcal{M}_2$ are inequivalent as Hilbert module unless $k_1 = k_2$. From Lemma 1.11, it follows that both the modules $\mathcal{M}_1$ and $\mathcal{M}_2$ are in $\mathrm{B}_1(\mathbb{D}^2 \setminus X)$, where $X := \{(0, z) : |z| < 1\}$ is the zero set of the ideal I_j, $j = 1, 2$. However, there is a holomorphic Hermitian line bundle corresponding to these modules on the projectivization of $\mathbb{D}^2 \setminus X$ at $(0, 0)$ (cf. [14, pp. 264]). Following the proof of [14, Theorem 5.1], we see that if these modules are assumed to be equivalent, then the corresponding line bundles they determine must also be equivalent. This leads to contradiction unless $k_1 \neq k_2$.

Suppose $L : \mathcal{M}_1 \to \mathcal{M}_2$ is given to be a unitary module map. Let K_j, $j = 1, 2$, be the corresponding reproducing kernel. From Lemma 1.11, it follows that the joint kernel of M_j at the point $w \in \mathbb{D}^2 \setminus X$ are one dimensional and spanned by the corresponding reproducing kernel K_j, $j = 1, 2$. Since L intertwines module actions, it follows that $M_p^* L K_1(\cdot, w) = \overline{p(w)} L K_1(\cdot, w)$, $p \in \mathbb{C}[\underline{z}]$. Hence,

$$L K_1(\cdot, w) = \overline{\varphi(w)} K_2(\cdot, w), \text{ for } w \notin X. \tag{5.2.7}$$

We conclude that φ must be holomorphic on $\mathbb{D}^2 \setminus X$ since both $LK_1(\cdot, w)$ and $K_2(\cdot, w)$ are anti-holomorphic in w. For $j = 1, 2$, let $\mathcal{Q}_j$ be the holomorphic line bundle on $\mathbb{P}^1$ whose section on the affine chart $U_1 = \{\pi(u) : u_1 \neq 0\}$, by blowing up the origin, is given by

$$s_1^j(\theta) \;=\; z_1^n + \theta^{n-k_j} z_1^{k_j} z_2^{n-k_j}.$$

Consider the co-ordinate change $(w_1, w_2) \to (\rho, \theta)$ where $\bar{w}_1 = \rho$ and $\bar{w}_2 = \rho\theta$ on $\mathbb{D}^2 \setminus X$. Note that

$$\lim_{\frac{\bar{w}_2}{\bar{w}_1} = \theta, w \to 0} |\varphi(\rho, \theta)|^2 = \frac{1 + |\theta^{n-k_1}|^2}{1 + |\theta^{n-k_2}|^2}. \tag{5.2.8}$$

Thus $\varphi(\rho.\theta)$ has a finite limit at $(0, \theta)$, say $\varphi(\theta)$. Then from (5.2.7), and the expression of

$$s_1^j(\theta) = \lim_{\frac{\bar{w}_2}{\bar{w}_1} = \theta, w \to 0} \frac{K_j(\cdot, w)}{\bar{w}_1^n},$$

by a limiting argument, we find that $L s_1^1(\theta) = \varphi(\theta) s_1^2(\theta)$. The unitarity of the map L implies that

$$\|s_1^1(\theta)\|^2 = \|L s_1^1(\theta)\|^2 = |\varphi(\theta)|^2 \|s_1^2(\theta)\|^2.$$

Consequently the line bundles $\mathcal{Q}_j$ determined by $\mathcal{M}_j$, $j = 1, 2$, on $\mathbb{P}^1$ are equivalent. We now calculate the curvature to determine when these line bundles are equivalent. Since the monomials are orthonormal, we note that the square norm of the section is given by

$$\| s_1^j(\theta) \|^2 = 1 + |\theta|^{2(n-k_j)}. \tag{5.2.9}$$

In this case, the equation (5.2.8) is also straight forward from (5.2.9). Consequently the curvature (actually coefficient of the $(1,1)$ form $d\theta \wedge d\bar{\theta}$) of the line bundle on the affine chart U is given by

$$\begin{aligned}
\mathcal{K}_j(\theta) &= -\partial_\theta \partial_{\bar{\theta}} \log \| s_1^j(\theta) \|^2 \;=\; -\partial_\theta \partial_{\bar{\theta}} \log(1 + |\theta|^{2(n-k_j)}) \\
&= -\partial_\theta \frac{(n-k_j)\theta^{(n-k_j)}\bar{\theta}^{(n-k_j-1)}}{1+|\theta|^{2(n-k_j)}} \\
&= -\frac{(n-k_j)^2|\theta|^{2(n-k_j-1)}\{1+|\theta|^{2(n-k_j)}\} - (n-k_j)^2|\theta|^{2(n-k_j)}|\theta|^{2(n-k_j-1)}}{\{1+|\theta|^{2(n-k_j)}\}^2} \\
&= -\frac{(n-k_j)^2|\theta|^{2(n-k_j-1)}}{\{1+|\theta|^{2(n-k_j)}\}^2}.
\end{aligned}$$

So if the bundles are equivalent on $\mathbb{P}^1$, then $\mathcal{K}_1(\theta) = \mathcal{K}_2(\theta)$ for $\theta \in U$, and we obtain

$$\begin{aligned}
&(n-k_1)^2\{|\theta|^{2(n-k_1-1)} + 2|\theta|^{2(n-k_2)}|\theta|^{2(n-k_1-1)} + |\theta|^{4(n-k_2)}|\theta|^{2(n-k_1-1)}\} \\
&- \;(n-k_2)^2\{|\theta|^{2(n-k_2-1)} + 2|\theta|^{2(n-k_1)}|\theta|^{2(n-k_2-1)} + |\theta|^{4(n-k_1)}|\theta|^{2(n-k_2-1)}\} = 0.
\end{aligned}$$

Since the equation given above must be satisfied by all θ corresponding to the affine chart U, it must be an identity. In particular, the coefficient of $|\theta|^{2\{(n-k_1)+(n-k_2)-1\}}$ must be 0 implying $(n-k_1)^2 = (n-k_2)^2$, that is, $k_1 = k_2$. Hence $\mathcal{M}_1$ and $\mathcal{M}_2$ are always inequivalent unless they are equal.

6. Appendix

6.1 The curvature invariant

The usual proof that curvature is a complete invariant for a holomorphic Hermitian line bundle makes crucial use of the existence of harmonic conjugate on a simply connected domain. Here we give a simple proof using the existence of power series expansion for a real analytic function over a domain in $\mathbb{C}$.

Let (E, h) be a holomorphic Hermitian line bundle over $\Omega \subset \mathbb{C}$, where $h(\omega) = (\gamma(\omega), \gamma(\omega))$, $\omega \in \Omega$, is the metric with respect to some nonzero holomorphic cross section γ for E. and $\mathcal{K}$ denote the curvature of E. Let $(\widetilde{E}, \widetilde{h})$ be another holomorphic Hermitian line bundle over Ω. Two vector bundles (E, h) and $(\widetilde{E}, \widetilde{h})$ are said to be locally equivalent if there exists open subset Ω_0 of Ω and a nowhere vanishing holomorphic function ϕ on Ω_0 such that $\widetilde{h}(\omega) = \phi(\omega)h(\omega)\overline{\phi(\omega)}$ for $\omega \in \Omega_0$.

Remark 6.1. Though in general one should get ϕ on all of Ω, since we are dealing with equivalences in Cowen-Douglas class, it is enough to consider this local equivalence.

Let $\widetilde{\mathcal{K}}$ be the curvature of the line bundle $(\widetilde{E}, \widetilde{h})$. Then assuming (E, h) and $(\widetilde{E}, \widetilde{h})$ are locally equivalent, we have

$$\begin{aligned}
\widetilde{\mathcal{K}}(\omega) &= -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \widetilde{h}(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log\{\phi(\omega)h(\omega)\overline{\phi(\omega)}\} = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log\{|\phi(\omega)|^2 h(\omega)\} \\
&= -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log |\phi(\omega)|^2 - \frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log h(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log h(\omega) = \mathcal{K}(\omega),
\end{aligned}$$

in some open subset of Ω.Here, we have $\frac{\partial^2}{\partial w\partial\overline{w}} log|\phi(w)|^2 = 0$, since ϕ is holomorphic. Now we prove the converse.

Proposition 6.2. *If $\mathcal{K} = \widetilde{\mathcal{K}}$ in some open subset of Ω, then (E, h) and $(\widetilde{E}, \widetilde{h})$ are locally equivalent.*

Proof. Since h is real (positive) analytic on Ω, $\log h$ is also real analytic for $\omega \in \Omega$ and admits power series expansion around $w \in \Omega$. Assume $w = 0$ for simplicity. Let $\log h(\omega) = \sum_{m,n=0}^{\infty} a_{mn}\omega^m\overline{\omega}^n$ on some open subset Ω_0 of Ω containing 0. Then

$$\begin{aligned}
\mathcal{K}(\omega) &= -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log h(\omega) = -\sum_{m,n=1}^{\infty} mn\ a_{mn}\omega^{m-1}\overline{\omega}^{n-1} \\
&= \sum_{m,n=0}^{\infty} (m+1)(n+1)a_{m+1,n+1}\omega^m\overline{\omega}^n
\end{aligned}$$

So if, $\mathcal{K}(\omega) = \sum_{m,n=0}^{\infty} k_{mn}\omega^m\overline{\omega}^n$, we get $k_{mn} = -(m+1)(n+1)a_{m+1,n+1}$ for $m, n \geq 0$, which implies,

$$a_{m+1,n+1} = \frac{k_{mn}}{(m+1)(n+1)}, \; m, n \geq 0. \tag{6.1.1}$$

Thus to determine the metric from curvature, we see that all coefficients except those of the form a_{m0} and a_{0n} are known. Since $\log h$ is real analytic, it follows that

$$\sum_{m,n=0}^{\infty} a_{mn}\omega^m\overline{\omega}^n = \sum_{m,n=0}^{\infty} \overline{a_{mn}}\omega^n\overline{\omega}^m$$

Equating coefficients $\omega^m\overline{\omega}^n$, we get $a_{mn} = \overline{a_{nm}}$ for $m, n \geq 0$. In particular, we have $a_{m0} = \overline{a_{0m}}$ for $m \geq 0$. The power series

$$(a_{00}/2) + \sum_{m=1}^{\infty} a_{m0}\omega^m$$

defines a holomorphic function in a neighborhood of 0, say ϕ. Also let

$$h_0(\omega) = \sum_{m,n=0}^{\infty} \frac{k_{mn}}{(m+1)(n+1)}\omega^{m+1}\overline{\omega}^{n+1}$$

for $\omega \in \Omega_0$. From (1) it follows that $\log h(\omega) = \phi(\omega) + \overline{\phi(\omega)} + h_0(\omega)$ implying that

$$h(\omega) = \exp(\phi(\omega)) \; \exp(h_0(\omega)) \; \exp(\overline{\phi(\omega)})$$

for $\omega \in \Omega_0$.

Now $\mathcal{K}(\omega) = \widetilde{\mathcal{K}}(\omega)$ implies that $h_0(\omega) = \widetilde{h}_0(\omega)$ in some small enough neighborhood of $0 \in \Omega_0$. Thus $\widetilde{h}(\omega) = \exp(\widetilde{\phi}(\omega) - \phi(\omega)) \; h(\omega) \; \exp(\overline{\widetilde{\phi}(\omega) - \phi(\omega)})$, that is, $\widetilde{h}(\omega) = \varphi(\omega) \; h(\omega) \; \overline{\varphi(\omega)}$ for the holomorphic function $\varphi(\omega) = \exp(\widetilde{\phi}(\omega) - \phi(\omega))$ on Ω in some small enough neighborhood of 0. This completes the proof. $\square$

Now suppose (E, h) and $(\widetilde{E}, \widetilde{h})$ are holomorphic Hermitian vector bundle of rank n over $\Omega \subset \mathbb{C}$. In this case, (E, h) and $(\widetilde{E}, \widetilde{h})$ are said to be locally equivalent if there exist a holomorphic function $X : \Omega_0 \to \mathcal{GL}(\mathbb{C}^n)$, Ω_0 open subset of Ω, such that $\widetilde{h}(\omega) = X(\omega)^* h(\omega) X(\omega)$. Again, assuming that (E, h) and $(\widetilde{E}, \widetilde{h})$ are locally equivalent, we have

$$\begin{aligned}
\widetilde{\mathcal{K}}(\omega) &= \overline{\partial}\{\widetilde{h}(\omega)^{-1}\partial\widetilde{h}(\omega)\} \\
&= \overline{\partial} \; [X(\omega)^{-1}h(\omega)^{-1}X(\omega)^{*-1}\{X(\omega)^*\partial h(\omega)X(\omega) + X(\omega)^*h(\omega)\partial X(\omega)\}] \\
&= \overline{\partial}\{X(\omega)^{-1}h(\omega)^{-1}\partial h(\omega)X(\omega) + X(\omega)^{-1}\partial X(\omega)^{-1}\} \\
&= X(\omega)^{-1}\mathcal{K}(\omega)X(\omega).
\end{aligned}$$

In this case the curvatures are conjugate to each other rather than being equal. We want to see to what extent it is possible to recover the metric from curvature. We show that if the metric is normalized in the sense of Curto and Salinas[11, page - 473], then it is determined from the curvature.

Since h is a real analytic function on Ω, we can find a positive definite kernel $\widehat{h} : \Omega \times \Omega \longrightarrow \mathcal{M}_n(\mathbb{C})$, holomorphic in the first and anti-holomorphic in the second variable, such that $\overline{\widehat{h}(\omega,\omega)}^{tr} = h(\omega)$, by polarising h. A kernel K is said to be normalized at $w_0 \in \Omega$ if $K(z, w_0) = I$.

Definition 6.3. The metric h is said to be normalized at $w_0 \in \Omega$ if $\widehat{h}$ is normalized at $w_0 \in \Omega$.

Remark 6.4. Assume $w_0 = 0$. If $h(\omega) = \sum_{m,n=0}^{\infty} h_{mn}\omega^m\overline{\omega}^n$, then

$$\widehat{h}(z,w) = \sum_{m,n=0}^{\infty} \overline{h_{nm}}^{tr} z^m \overline{w}^n,$$

where $h_{mn} \in \mathcal{M}_n(\mathbb{C})$. If h is normalized at 0, then $\widehat{h}(z,0) = I$, $z \in \Omega$. Hence $\sum_{m=0}^{\infty} \overline{h_{0m}}^{tr} z^m = I$. Comparing the coefficients both sides we have $\overline{h_{00}}^{tr} = I$ and $\overline{h_{0m}}^{tr} = 0$, that is, $h_{00} = I$ and $h_{0m} = 0$ for all $m \geq 0$. As $\widehat{h}$ is positive definite, we also have $h_{m0} = 0$ for all $m \geq 0$.

Theorem 6.5. *If (E,h) and $(\widetilde{E},\widetilde{h})$ are holomorphic vector bundles equipped with the normalized metric over Ω and $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ be respectively the corresponding curvatures, then (E,H) and $(\widetilde{E},\widetilde{h})$ are locally equivalent if and only if there exist a constant unitary U such that $\widetilde{\mathcal{K}}(\omega) = U^*\mathcal{K}(\omega)U$ for ω in some open subset of Ω.*

Proof. If h and $\mathcal{K}$ be respectively the metric and curvature for the rank n complex bundle E, then we know that $\mathcal{K} = \overline{\partial}(h^{-1}\partial h)$. There exist a real analytic function g on Ω such that

$$hg = I. \tag{6.1.2}$$

Let $h(\omega) = \sum_{i,j=0}^{\infty} h_{ij}\omega^i\overline{\omega}^j$ and $g(\omega) = \sum_{i,j=0}^{\infty} g_{ij}\omega^i\overline{\omega}^j$ for ω in some open subset Ω_0 of Ω, where $h_{ij},\ g_{ij} \in \mathcal{M}_n(\mathbb{C})$ for $i,j \geq 0$. Putting $\omega = 0$, from (6.1.2), we get $h_{00}\ g_{00} = I$. For $l,k \geq 0$, we also have

$$0 = \overline{\partial}^k\partial^l(hg) = \sum_{i=0}^{l}\binom{l}{i}\overline{\partial}^k\ (\partial^{l-i}h\ \partial^i g) = \sum_{i=0}^{l}\sum_{j=0}^{k}\binom{l}{i}\binom{k}{j}\ (\overline{\partial}^{k-j}\partial^{l-i}h)\ (\overline{\partial}^j\partial^i g)$$

Putting $\omega = 0$ we get

$$\sum_{i=0}^{l}\sum_{j=0}^{k}\binom{l}{i}\binom{k}{j} h_{l-i,k-j}\ g_{ij} = 0. \tag{6.1.3}$$

From (6.1.3), for $l = 1$ and $k = 0$ we have

$$g_{10} = -\,{h_{00}}^{-1}\ h_{10}\ {h_{00}}^{-1}$$

and for $l = 0$ and $k = 1$,

$$g_{10} = -\,{h_{00}}^{-1}\ h_{10}\ {h_{00}}^{-1}.$$

Then by inductively we first get g_{m0} (putting $l = m$ and $k = 0$) and g_{0n} (putting $l = 0$ and $k = n$). Recursively then we get g_{mk}'s for $k < m$ and g_{kn}'s for $k < n$ and hence we can calculate g_{mn} for general m and n. Now we have

$$\begin{aligned}
\overline{\partial}^n \partial^m \mathcal{K} &= \overline{\partial}^n \partial^m \{\overline{\partial}(g\ \partial h)\} \;=\; \overline{\partial}^n \partial^m (\overline{\partial} g\ \partial h + g\ \overline{\partial}\partial h) \\
&= \sum_{i=0}^{m} \binom{m}{i} \overline{\partial}^n \{(\overline{\partial}\partial^{m-i} g)(\partial^i h) + (\partial^{m-i} g)(\partial^i \overline{\partial}\partial h)\} \\
&= \sum_{i=0}^{m} \binom{m}{i} [\sum_{j=0}^{n} \binom{n}{j} \{(\overline{\partial}^{n-j}\partial^{i+1} h)(\overline{\partial}^{j+1}\partial^{m-i}) + (\overline{\partial}^{n-j}\partial^{m-i} g)(\overline{\partial}^{j+1}\partial^{m-i} h)\}] \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} \{(\overline{\partial}^{n-j}\partial^{i+1} h)(\overline{\partial}^{j+1}\partial^{m-i}) + (\overline{\partial}^{n-j}\partial^{m-i} g)(\overline{\partial}^{j+1}\partial^{m-i} h)\}
\end{aligned}$$

Let $\mathcal{K}(\omega) = \sum_{i,j=0}^{\infty} k_{ij}\omega^i \overline{\omega}^j$ for w in some small enough neighborhood of 0, where $k_{ij} \in \mathcal{M}_n(\mathbb{C})$. Putting $\omega = 0$, from the above equations we have,

$$\begin{aligned}
m!n!\ k_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} \{(i+1)!(n-j)! h_{i+1,n-j}\ (m-i)!(j+1)! g_{m-i,j+1} + \\
&\qquad (m-i)!(n-j)! g_{m-i,n-j}\ (j+1)!(i+1)! h_{i+1,j+1}\} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} m!n!(i+1)(j+1)(h_{i+1,n-j}\ g_{m-i,j+1} + h_{i+1,j+1}\ g_{m-i,n-j})
\end{aligned}$$

which implies that

$$k_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} (i+1)(j+1)(h_{i+1,n-j}\ g_{m-i,j+1} + h_{i+1,j+1}\ g_{m-i,n-j}) \tag{6.1.4}$$

Now as h is a normalized metric, via Remark 6.4, we have, $h_{00} = Id_n$ and $h_{m0} = 0 = h_{0n}$. Thus from equation (6.1.3), we get $g_{00} = Id_n$ and $g_{m0} = 0 = g_{0n}$. Putting $m = 0 = n$ in (6.1.4), we get $h_{11} = k_{00}$. Then by inductively we first get h_{m1} and h_{1n}. Recursively then we get h_{mk}'s for $k < m$ and h_{kn}'s for $k < n$ and hence we can calculate h_{mn} for general m and n which shows that the metric in this case is determined uniquely.

Following [11] or by comparing coefficients, we note that if both h and $\widetilde{h}$ are normalized then (E, h) and $(\widetilde{E}, \widetilde{h})$ are locally equivalent if there exist a constant unitary U such that $\widetilde{h}(\omega) = U^* h(\omega) U$, for ω in some open subset Ω_0 of Ω. Hence

$$\begin{aligned}
\widetilde{\mathcal{K}}(\omega) &= \frac{\partial}{\partial\overline{\omega}}\{\widetilde{h}(\omega)^{-1}\frac{\partial}{\partial\omega}\widetilde{h}(\omega)\} \;=\; \frac{\partial}{\partial\overline{\omega}}\{(U^* h(\omega) U)^{-1}\frac{\partial}{\partial\omega} U^* h(\omega) U\} \\
&= \frac{\partial}{\partial\overline{\omega}}[U^* h(\omega)^{-1} U U^* \{\frac{\partial}{\partial\omega} h(\omega)\} U] \;=\; U^* \frac{\partial}{\partial\overline{\omega}}\{h(\omega)^{-1}\frac{\partial}{\partial\omega} h(\omega)\} U \\
&= U^* \mathcal{K}(\omega) U.
\end{aligned}$$

Conversely if the corresponding curvatures are equivalent, that is, if $\widetilde{\mathcal{K}}(\omega) = U^*\mathcal{K}(\omega)U$, for ω in some open subset Ω_0 of Ω, then from the preceding computations, it follows that $\widetilde{h}(\omega) = U^* h(\omega) U$, $\omega \in \Omega_0$. ☐

For simplicity, we have given the proof of the theorem above over domains in $\mathbb{C}$. However, similar but somewhat more involved computation show that the proof is valid for domains in $\mathbb{C}^m$, $m > 1$.

6.2 Some curvature calculations

Let $\mathcal{H}^{(\lambda,\mu)}$ be a reproducing kernel Hilbert space of holomorphic functions on $\mathbb{D}^2$ with reproducing kernel

$$K(z,w) = \frac{1}{(1-z_1\bar{w_1})^\lambda(1-z_2\bar{w_2})^\mu}, \text{ for } z=(z_1,z_2), w=(w_1,w_2)\in\mathbb{D}^2.$$

Define $\mathcal{H}^{(\lambda,\mu)}_{(0,0)}$ to be the subspace of functions in $\mathcal{H}^{(\lambda,\mu)}$ which vanish at the point $(0,0)$ in the bidisc, that is, $\mathcal{H}^{(\lambda,\mu)}_{(0,0)} = \{f \in \mathcal{H}^{(\lambda,\mu)} : f(0,0)=0\}$. From Lemma 1.11 and Corollary 2.14, we know that $\mathcal{H}^{(\lambda,\mu)}_{(0,0)}$ does not belong to the class $\mathrm{B}_1(\mathbb{D}^2)$, but it is in $\mathrm{B}_1(\mathbb{D}^2\setminus\{(0,0)\})$ To decide when two modules in the set

$$\{\mathcal{H}^{(\lambda,\mu)}_{(0,0)} : \lambda,\mu>0\} \tag{6.2.1}$$

are unitary equivalent, we calculate curvature of the line bundle corresponding to $\mathcal{H}^{(\lambda,\mu)}_{(0,0)}$, $\lambda,\mu>0$, on $\mathbb{D}^2\setminus\{(0,0)\}$. Let $K_0^{(\lambda,\mu)}$ be the reproducing kernel for $\mathcal{H}^{(\lambda,\mu)}_{(0,0)}$. Then we have

$$K_0^{(\lambda,\mu)}(z,w) = \frac{1}{(1-z_1\bar{w_1})^\lambda(1-z_2\bar{w_2})^\mu} - 1, \text{ for } z=(z_1,z_2), w=(w_1,w_2)\in\mathbb{D}^2.$$

We have $K_0^{(\lambda,\mu)}(P,P) = \frac{1}{(1-|p|^2)^\lambda} - 1 > 0$ for $P=(p,0)\in\mathbb{D}^2\setminus\{(0,0)\}$. We normalize the kernel $K_0^{(\lambda,\mu)}$ at P, as in the equation 1.2.2. Then

$$\begin{aligned}&\widehat{K}_0^{(\lambda,\mu)}(z,w)\\ &= \{\frac{1}{(1-|p|^2)^\lambda}-1\}\{\frac{1}{(1-z_1\bar{p})^\lambda}-1\}^{-1}\{\frac{1}{(1-p\bar{w_1})^\lambda}-1\}^{-1}\{\frac{1}{(1-z_1\bar{w_1})^\lambda(1-z_2\bar{w_2})^\mu}-1\}\end{aligned}$$

for $z=(z_1,z_2), w=(w_1,w_2)\in\Omega_0$, for some neighborhood Ω_0 of P. From [30, Lemma 2.3], to calculate the curvature, it is enough to calculate the coefficients of $|w_1-p|^2, |w_2|^2, (\bar{w_1}-\bar{p})w_2$ and $(w_1-p)\bar{w_2}$ in the expansion of $\widehat{K}_0^{(\lambda,\mu)}(w,w)$ around P. To calculate these coefficients, we note that evaluation of certain number of derivative of $\widehat{K}_0^{(\lambda,\mu)}$ at P will be enough. Let us first calculate the coefficient of $|w_2|^2$, which is

$$= \mu[\{\frac{1}{(1-|p|^2)^\lambda}-1\}(1-|p|^2)^\lambda]^{-1} = \mu\{1-(1-|p|^2)^\lambda\}^{-1}.$$

Hence if the modules $\mathcal{H}^{(\lambda,\mu)}_{(0,0)}$ and $\mathcal{H}^{(\lambda',\mu')}_{(0,0)}$ are equivalent, then

$$\frac{\mu}{\{1-(1-|p|^2)^\lambda\}} = \frac{\mu'}{\{1-(1-|p|^2)^{\lambda'}\}}$$

for arbitrary $p \in \mathbb{D} \setminus \{0\}$. Let us take $p = 1/\sqrt{2}$ and $p = \sqrt{3}/2$. We have the following equations,

$$\mu\{1 - (\frac{1}{2})^{\lambda'}\} = \mu'\{1 - (\frac{1}{2})^{\lambda}\} \text{ and } \mu\{1 - (\frac{1}{4})^{\lambda'}\} = \mu'\{1 - (\frac{1}{4})^{\lambda}\}.$$

Then

$$\frac{\{1 - (\frac{1}{2})^{\lambda'}\}}{\{1 - (\frac{1}{4})^{\lambda'}\}} = \frac{\{1 - (\frac{1}{2})^{\lambda}\}}{\{1 - (\frac{1}{4})^{\lambda}\}}, \text{ which implies } \frac{1}{\{1 + (\frac{1}{2})^{\lambda'}\}} = \frac{1}{\{1 + (\frac{1}{2})^{\lambda}\}}, \text{ and therefore } 2^{\lambda} = 2^{\lambda'}.$$

Thus $\lambda = \lambda'$ and then it follows that $\mu = \mu'$. Clearly, these computions would be impractical if we have to compare two modules vanishing to order k, $k > 1$ or on a variety of positive dimension.

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