Sankhyā : The Indian Journal of Statistics
1993, Volume 55, Series A, Pt. 1, pp. 150–152.

NOTES

MOMENT BOUNDS FOR SOME STOCHASTIC PROCESSES

By RATAN DASGUPTA

Indian Statistical Institute

SUMMARY. For a general class of stochastic processes including martingales a sharp moment bound is obtained.

1. Introduction and the result

Moment bounds for sum of stochastic processes including martingale differences are of interest in application to the limit theorems and for computing rates of such convergences. For martingale summands, Dharmadhikari, Fabian and Jogdeo (1968) computed such bounds and later Bickel (1974) sharpened it under some restriction on the order of the moment. In this note these bounds are further sharpened for a general class of stochastic processes which includes martingales. The result states the effect of the sample size and the order of the moment on the bound. The following theorem is proved.

Theorem. *Let* $\{X_i, i \geqslant 1\}$ *be a stochastic process with* $E[sgn\,(S_{i-1})X_i \mid |S_{i-1}|] \leqslant 0$, $E\left(\sum_{i=1}^{n} \pm X_i\right)^2 \leqslant n\beta^*_{2,n}$ *where* $S_i = \sum_{j=1}^{i} X_j$, $\gamma_{\nu,n} = E|X_n|^\nu$, $\beta^*_{\nu,n} = \max_{1 \leqslant j \leqslant n} \gamma_{\nu,j}$. *If the l.h.s. of* (1.1) *is finite, then for* $\nu \geqslant 2$

$$E|S_n|^\nu \leqslant c_\nu\, n^{\nu/2}\beta^*_{\nu,n} \quad \ldots \quad (1.1)$$

where $c_\nu = [(\nu-1)\delta]^{\nu/2}$ *and for large n,* $\delta \simeq \left(1+\frac{\nu}{2n}\right)$.

One may note that the above set up includes martingales as a special case since then $E(X_i \mid S_{i-1}) = 0$. In fact our technique of proof will be a modification of the martingale case proved in Dharmadhikari, Fabian and Jogdeo (1968) with a large c_ν, viz. $c_\nu = [8(\nu-1)\ \max\ (1,\ 2^{\nu-3})]^\nu$. Bickel (1974) obtained $c_\nu = (2e\nu)^{\nu/2}$ for martingales where ν is an *even integer*, $\nu \leqslant 2n$. As noted therein, these type of bounds are quite sharp. The condition $E[\text{sign}\,(S_{i-1})X_i \mid |S_{i-1}|] \leqslant 0$ used in the theorem suggests some type of negative (conditional) association between sgn (S_{i-1}) and X_i. This is implied

Paper received. January 1988; revised November 1990.

AMS (1980) *subject classification.* 60G99

Key words and phrases. Limit therem, Ornstein-Ulenback process.

by a more stringent assumption sgn $(S_{i-1})\, E[X_i \mid S_{i-1}] \leqslant 0$. As for example if sgn $(S_{i-1}) \geqslant 0$ i.e., $S_{i-1} \geqslant 0$ then X_i given S_{i-1} is expected to be less than or equal to zero so that the sum S_i is drifted towards zero. Similarly if $S_{i-1} \leqslant 0$ then $E[X_i \mid S_{i-1}] \geqslant 0$ causing S_i a drift towards zero.

A similar assumption holds for Ornstein-Uhlenback process in continuous time parameter where there is a drift towards origin when considered relative to wiener process.

2. Proof of theorem and some remarks

As in (3.1) of Dharmadhikari *et al.*, (1968), write

$$|S_n|^\nu = |S_{n-1}|^\nu + \nu \operatorname{sgn}(S_{n-1})\, |S_{n-1}|^{\nu-1} X_n + \tfrac{1}{2}\nu(\nu-1)\, |S_{n-1}+\theta X_n|^{\nu-2} X_n^2 \quad \ldots \quad (2.1)$$

where $0 < \theta < 1$.

Now

$$|S_{n-1}+\theta X_n| \leqslant \max\left(|S_{n-1}+X_n|,\ |S_{n-1}-X_n|\right) \quad \ldots \quad (2.2)$$

The first term in the max is S_n, the second term also behave similarly, since $\gamma_{\cdot, n}$ remains unchanged when X_n is replaced by $-X_n$ and since in the induction hypothesis the theorem is assumed for ν_0 ($\geqslant 2$) and for all n.

Hence writing $S_n^* = |S_{n-1}-X_n|$

$$\begin{aligned}\Delta_n(\nu) &= E(|S_n|^\nu - |S_{n-1}|^\nu)\\ &\leqslant \tfrac{1}{2}\nu(\nu-1) \max E(|S_n|^{\nu-2}X_n^2,\ |S_n^*|^{\nu-2} X_n^2) \quad \ldots \quad (2.3)\end{aligned}$$

where one uses

$$E[\operatorname{sgn}(S_{i-1})\, X_i \mid |S_{i-1}|] \leqslant 0.$$

Then following Dharmadhikari, et al. (1968) along the same lines of (3.10), (3.11), (3.6), (3.7) and (3.8) with η (> 1) replacing 2 in the r.h.s. therein, one gets

$$\begin{aligned}E|S_n|^{\nu_1} &= \sum_{j=1}^{n} \Delta_j(\nu_1)\\ &\leqslant \frac{\eta}{2}\, \nu_1(\nu_1-1)\, c_{\nu_0}^{(\nu_2-2)/\nu_2}\, \beta^*_{\nu_1, n} \sum_{j=1}^{n} j^{(\nu_1-2)/2} \quad \ldots \quad (2.4)\end{aligned}$$

Use the approximation

$$\sum_{j=1}^{n} j^{(\nu_1-2)/2} \leqslant \int_1^n X^{\nu_1/2-1}\, dx + n^{\nu_1/2-1} = \left(\frac{2}{\nu_1}+n^{-1}\right) n^{\nu_1/2} \quad \ldots \quad (2.5)$$

Hence

$$E|S_n|^{\nu_1} \leqslant \eta\, \nu_1(\nu_1-1)/2\, c_{\nu_0}^{1-2/\nu_2}\, \beta^*_{\nu_1,n}\, n^{\nu_1/2}\, (n^{-1}+2/\nu_1)$$

$$\leqslant (\nu_1-1)c_{\nu_1}^{1-2/\nu_1}\, \beta^*_{\nu_1,n}\, n^{\nu_1/2}\, (1+\nu_1/2n) \quad \ldots \quad (2.6)$$

observing $c_\nu \uparrow \nu$ and letting $\eta \to 1$, $\nu_2 \to \nu_1$, $\leqslant E|S_n|^{\nu_1} \leqslant c^{\nu_1}\, n^{\nu_1/2}\, \beta^*_{\nu_1,n}$ completing the proof.

Although $ES_n^2 = O(n)$ in many cases where Central Limit Theorem holds, a condition like $ES_n^2 = O(n)$ is not required in the proof. We only assumed $E\left(\sum_{i=1}^{n} \pm X_i\right)^2 \leqslant n\,\beta^*_{2,n}$ where $\beta^*_{2,n} = \max_{1\leqslant j\leqslant n} \gamma_{2,j} = \max_{1\leqslant j\leqslant n} E|X_j|^2$, might very well depend on n.

The improvement here over earlier results is due to (2.2). It will be interesting to prove (1.1) with $\delta = 1$. Further sharpening of (2.2) and (2.5) will then be required.

Acknowledgement. The author thanks the referee for pointing out an error in an earlier version of the paper.

Note added in the proof. It has been possible to sharpen (1.1) further to get

$$E|S_n|^\nu \leqslant c_\nu n^{\nu/2}\, \beta^*_{\nu,n}, \quad \nu \geqslant 2$$

where $$e_\nu = (2\nu)^{\nu/2}$$

Ref : Dasgupta, R. (1993). Sharper moment bound for martingales with applications. preprint, ISI.

References

Bickel, P. J. (1974). Asymptotic expansions in nonparametric statistics. *Ann. Statist*, **2**, 1-20.

Dharmadhikari, S. W., Fabian, V. and Jogdeo, K. (1968). Bounds on the moments of the martingales. *Ann. Math. Statist.*, **5**, 1719-1723.

Statistics and Mathematics Division
Indian Statistical Institute
203 B. T. Road
Calcutta 700 035
India