EXTENSIONS OF THE THEORY OF POSITIVE OPERATORS AND THEIR RELATIONSHIP TO MINIMAX GAMES

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P R E F A C E

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The thesis is concerned with the extensions of the theory of positive operators to operators that leave invariant a convex set in a normed linear space and their applications to minimax games. Some of the results of the thesis have already appeared in the articles [26] and [27] of the author.

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C O N T E N T S

INTRODUCTION

The theory of positive matrices (matrices with non-negative entries) and the theory of positive operators are used extensively in the study of vibrations of mechanical systems [16], stochastic processes [36] and mathematical economics ([9] and [20].) It is well known from the classical theorems of Perron and Frobenius ([13], [14] and [25]) that any non-singular square matrix with non-negative entries has a positive eigenvalue which is maximal in modulus among all the eigenvalues of the matrix. Further for this positive eigenvalue, it has a non-null eigenvector with all components non-negative. If the matrix is also irreducible[1] it was shown by Frobenius that the positive eigenvalue, maximal in modulus, is algebraically and geometrically simple. Further if there are exactly h eigenvalues with an absolute value equal to the above positive eigenvalue, say r, then they are precisely the roots of the equation $\lambda^h - r^h = 0$.

(1) A non-negative matrix A is reducible whenever for some permutation matrix π, $\pi A \pi' = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$ where B and D are square matrices. A is said to be irreducible when it is not reducible.

The theorem of Perron-Frobenius was established by Jentsch [17] for integral equations

$$\varphi(s) = \lambda \int_a^b k(t, s)\, \varphi(s)\, ds$$

with positive kernel (i.e. $k(t, s) > 0$ for $a \leq t, s \leq b$).

When the positive kernel $k(t, s)$ is continuous, it was shown by Jentsch, that there exists a positive continuous solution φ for the above integral equation for a positive parameter λ. Further the parameter λ is simple and greatest in modulus of the roots of the Fredholm's determinant.

Since the positive orthant is a cone (2) and since matrices with non-negative entries leave this cone invariant, Krein and Rutman [23] extended the theorems of Perron and Frobenius to arbitrary real Banach spaces, by studying the spectral theory of operators that leave a cone invariant.

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(2) K is a cone in a linear space E whenever $x, y \in K$ implies $\lambda x + \mu y \in K$ for any $\lambda, \mu \geq 0$ and $x, -x \in K$ implies $x = \theta$, the origin.

Since they are the infinite analogues of matrices with positive entries, they are also known as positive operators.

When a cone in a real Banach space has a dense linear hull and when a compact linear operator is positive with respect to this cone with positive spectral radius, it was shown by Krein and Rutman [23], that the spectral radius is actually an eigenvalue of the operator and it has a non-null eigenvector in the cone. Further in the case of cones with interior if the compact linear operator is strongly positive(3), they proved that the spectral radius is a simple eigenvalue and the eigenvector in the cone is a simple eigenvector. It was also proved by them that if a positive linear operator, positive with respect to a normal cone(4), has a positive eigenvalue with an eigenvector in the interior of the cone, then this eigenvalue is actually the

(3) An operator A is strongly positive with respect to a cone K, if for any $x \in K - \theta$ $A^{n(x)} x$ belongs to the interior of K for some suitable power $n(x)$. and $AK \subseteq K$.

(4) A cone K in a normed linear space is normal if $x, y \in K$ implies $| x + y | \geq \partial \max \{|x|, |y|\}$ where $\partial > 0$ and $|x|$ is the norm of x.

spectral radius of the operator.

Suppose if we go back to their motivation, then we find that the positive orthant, instead of being viewed as a cone, can as well be viewed as a closed convex set with origin as an extreme point and square matrices with non-negative entries leave this convex set invariant. Thus the following question naturally arises.

What can we say about the spectrum of linear operators which leave a closed convex set with origin as an extreme point invariant in a real Banach space ? Can we extend the theorems of Krein and Rutman without any restrictions ?

In the first chapter we extend the theorems of Krein and Rutman to this general case. We show that if a convex set of the above type has a dense linear hull in a real Banach space, then for any compact linear operator which has spectral radius strictly greater than unity and which leaves the convex set invariant, the spectral radius is an eigenvalue of the operator, with a non-null eigenvector in the convex set. Further this result is extended to the more general case where the linear operator is compact[5]

(5) An operator is compact in the convex set when it sends bounded subsets of the convex set to conditionally compact subsets of the convex set. Such operators neet not even be bounded.

when restricted to the convex set. Here, if the partial spectral radius(6) exists and if it is strictly greater than unity, then it is shown that the partial spectral radius is an eigenvalue of the operator with a non-null eigenvector in the convex set. Here the basic normed linear space need not even be complete. We just demand that the convex set is closed and complete in the normed linear space. In the real finite dimensional case we show that if a linear transformation leaves a closed convex set with origin as an extreme point invariant and if there exists an eigenvector in the interior of the convex set for some eigenvalue which is not less than unity then the eigenvalue is actually the spectral radius of the transformation.

For the case of cones our assumption on the spectral radius is only apparently restrictive, but since positive multiples of positive operators are also positive, we can assume that the spectral radius is strictly greater than unity when it is positive.

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(6) If A is a linear operator with $AK \leq K$ for some closed convex set with $\theta \in K$, then the partial norm norm $= p(A) = \sup_{x \in K\, |x| \leq 1} |Ax|$ and the partial

spectral radius $= \lim_{n \to \infty} \sqrt[n]{p(A^n)}$ whenever it exists.

It is important to note that for the following reasons the theorems stated above are not deducible from the theorems of Krein by taking the generated cone.

Firstly, when we take the cone generated by the convex set and take its closure, origin may cease to be an extreme point of the generated cone and secondly even when the origin is an extreme point of the generated cone, the eigenvector for the spectral radius need not belong to the convex set. Such a situation happens in many cases where the spectral radius is strictly less than unity. The problems corresponding to non-linear compact operators which leave such convex sets invariant are still open.

In the second chapter we show how the abstract minimax theorem of Ky Fan [11] can be used to obtain all the known results on strongly positive operators [23] in reflexive Banach spaces. By this method we also deduce a theorem of H. Schaefer [30] on strongly positive operators in general Banach spaces.

In the third chapter, we continue with our study of the relations between minimax theorem and positive operators. Here we apply the known theorems on positive operators to study the nature of the optimal strategies of players in a zero-sum two person game. We show that for a commuting family of positive continuous symmetric pay-off kernels on the unit square, having a common optimal strategy for player I for every finite subfamily, the ratio of the game values and the ratio of the spectral radii are same provided there are optimal strategies for player II with the whole unit interval as the spectrum for every game kernel. Here continuity can be dispensed with if the kernel is bounded and the game value exists with minimax solutions. Symmetry can be replaced by the weaker condition that the associated operators are normal in L_2 [0 1].

In the fourth chapter we obtain a co-ordinate free characterization of stochastic matrices. We show that any linear transformation that leaves a lattice cone invariant in a real finite dimensional vector space with a vector fixed in the interior of the cone can be represented by a stochastic matrix with respect to some basis. The infinite analogue of stochastic matrices are characterized to within an operator isomorphism by operators positive with respect to a lattice normal cone with a fixed vector in the interior of the cone.

1. LINEAR OPERATORS LEAVING INVARIANT A CONVEX SET IN A NORMED LINEAR SPACES.

1.a. Preliminaries:

Let K be a closed convex set with the origin as an extreme point in a real normed linear space E. Let A be a bounded linear operator which leaves K invariant. Further, let the spectral radius μ be positive. We would like to know whether the spectral radius μ is an eigenvalue of A. Also we are interested in the eigensubspace corresponding to the spectral radius, whenever it is an eigenvalue of A.

When K is a closed cone whose linear hull is dense in E and when E is complete, then for all compact operators A with a positive spectral radius, the spectral radius μ is actually an eigenvalue of A. Further there exists for μ a non-null eigenvector y in K.

Suppose; for the convex set K, if we take the closed cone $\bar{C}$ generated by K, we have $A\bar{C} \subseteq \bar{C}$. Even if A is compact and $\mu > 0$, it does not follow from the theorems of Krein-Rutman [23] that μ is an eigenvalue of A. Even if it were an eigenvalue, there may not be any non-null eigenvector y in K for μ. This happens because of the following two reasons. Firstly the closed generated cone may not be a cone in our sense since θ may cease to be an extreme point. Even if it were so, no eigenvector y of μ need belong to K but might lie in the cone $\bar{C}$. The following two examples demonstrate this fact even when E is finite dimensional.

<u>Example 1</u>: Let $E = R^2$, the real Euclidean 2-space. Let $K = \{(x, y)': x \geq y^2, \; y \geq 0\}$. Let A be given by the matrix

$$A = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with $AX = (a_{11}x + a_{12}y \, , \, a_{21}x + a_{22}y)$ for $X = (x, y)'$

Here we have $AK \subseteq K$ and $K - K = R^2$. The spectral radius is $1/2$ and all the eigenvectors for the eigenvalue $1/2$ lie on the y-axis. Since θ is the only common element for the convex set K and the y-axis, we don't have any non-null eigenvector in K for the eigenvalue $1/2$.

<u>Example 1.2</u>: Let $E : R^4$, the real Euclidean 4-space. Let $K = \{(x_1, x_2, x_3, x_4)': \; x_1 + x_3 \geq (x_2 - x_4)^2, \; x_1, x_2, x_3, x_4 \geq 0\}$.

Let A be the linear transformation given by the matrix

$$A = \begin{bmatrix} 1/8 & 0 & 1/8 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/8 & 0 & 1/8 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

Let $X \to AX$ be defined as in example 1.1. Here also we find that $AK \subseteq K$ with $K - K = R^4$. The spectral radius is 1/2 and its eigenvectors lie on the subspace $S = \left\{ (x_1, x_2, x_3, x_4)' : x_1 = 0, \ x_3 = 0 \right\}$. Since $S \cap K = \theta$, no non-trivial eigenvector for the eigenvalue 1/2 lies in K.

Thus in both of these examples the eigenvectors belong to $\bar{C} - C$. Further we observe that the spectral radius μ is strictly less than unity.

But this does not happen, when the spectral radius μ is strictly greater than unity and we have the following theorems in finite and infinite dimensions.

1.b. <u>Linear transformations leaving invariant a convex set in finite dimensions:</u>

<u>Theorem</u> 1.1: Let K be a closed convex set with the origin θ as an extreme point in a real finite dimensional

vector space E. Let $A^*f - f$ be non-negative on K (i.e. $(A^*f - f)(x) \geq 0$ for $x \in K$) for some f in E^*. Further let $T = \{x : f(x) = c\} \cap K$ be bounded for some $c > 0$. Then

1. A has an eigenvalue $\lambda_0 \geq 1$.

2. A has a non-null eigenvector y_0 in K for λ_0.

Proof: Since T is a closed bounded set, T is compact in E. Moreover T is convex. Since $A^*f - f$ is non-negative on K

$$(A^*f - f)(x) = f(Ax) - f(x) \geq 0 \quad \text{for all } x \in K.$$

Therefore $f(Ax) \geq c$ for all $x \in T$. If $K_1 = \{x : f(x) \leq c\} \cap K$, then θ is evidently in K_1 and $Ax \notin K_1 - T$, for $x \in T$. Thus for any $x \in T$, the line joining θ and Ax cuts T at a unique point, say $\mu_x . Ax$. Consider the map $\phi : x \to \mu_x . Ax$ of T into itself. Since $Ax \notin K_1 - T$, $0 < \mu_x \leq 1$. It is easily seen that ϕ is also continuous. Therefore by Brouwer's fixed point theorem [10] $\phi(y_0) = y_0$ for some $y_0 \in T$. i.e. $\mu_{y_0} Ay_0 = y_0$ which shows that

$$Ay_0 = \lambda_0 y_0, \quad \lambda_0 = \frac{1}{\mu_{y_0}} \geq 1, \quad y_0 \in T.$$

This completes the proof of the theorem.

<u>Theorem</u> 1.2: Let K be a closed convex set with interior in a real finite dimensional vector space E and let θ be an extreme point of K. If A is a linear transformation with $AK \subseteq K$ and $Ay = \mu y$ for some y in the interior of K and $\mu \geq 1$, then μ is the spectral radius of A. If K is normal(7), the condition $\mu \geq 1$ can be relaxed to $\mu > 0$, even when E is an infinite dimensional Banach space.

Proof: We shall omit the proof for the case when K is normal as it follows directly from a lemma of Krein-Rutuman ([23], lemma 4 .2) applied to the closed cone generated by K. Since y is in the interior of K, $S(y, \rho) \subseteq K$ for a closed sphere of radius $\rho > 0$. i.e. $y \pm \rho \frac{x}{|x|} \in K$ for all x in E. Thus we have vectors of the form $y \pm z$ in K. Further for all such z, $|z| \leq N$ for some N. Suppose to the contrary, let $y \pm z_n \in K$ with $|z_n| > n$ for all $n = 1, 2, 3, \ldots$ Since $\frac{1}{2} \leq \left|\frac{y + z_n}{|z_n|}\right| \leq 2$ for all large n

(7) See footnote (4) in page 3

and since E is finite dimensional, there exists a subsequence $\frac{y+z_{n_j}}{|z_{n_j}|}$ of $\frac{y+z_n}{|z_n|}$ with $\frac{y+z_{n_j}}{|z_{n_j}|} \to z_o \neq \theta$.

Therefore, $\frac{y-z_{n_j}}{|z_{n_j}|} \to -z_o \neq \theta$.

Further since K is closed $z_o, -z_o \in K$. This contradicts the assumption that θ is an extreme point of the convex set K.

Thus $|z| \leq N$ for some large N. Since $A^nK \subseteq K$, $A^n(y \pm z) = \mu^n y \pm A^n z \in K$, whenever $y \pm z \in K$. Since $\mu \geq 1$, $(y \pm \frac{A^n z}{\mu^n}) \in K$ and therefore $\left|\frac{A^n y}{\mu^n}\right| \leq N$ for all n. Thus for any α with $|\alpha| > \mu$ and $y \pm z \in K$

$$\left|\sum_0^\infty \frac{A^n z}{\alpha^{n+1}}\right| \leq \sum_0^\infty \frac{|A^n z|}{\mu^n} \cdot \frac{\mu^n}{|\alpha^{n+1}|} \leq N \sum_0^\infty \frac{\mu^n}{|\alpha|^{n+1}} < \infty .$$

This shows that $\sum_0^\infty \frac{A^n z}{\alpha^{n+1}}$ exists as a norm convergent series. But for any $x \in B$, $x \neq \theta$ $y \pm \epsilon \frac{x}{|x|} \in K$ for some $\epsilon > 0$. Therefore $\frac{\epsilon}{|x|} \sum_0^\infty \frac{A^n x}{\alpha^{n+1}}$ exists and hence $\Sigma \frac{A^n x}{\alpha^{n+1}}$ exists.

This shows that $(A - \alpha I)$ is invertible as a linear transformation for $|\alpha| > \mu$. Further μ is also an eigenvalue of A. This completes the proof of the theorem.

Remark 1.1: The condition $\mu \geq 1$ cannot be relaxed for the general convex set even in finite dimensions, and it is seen from the example given below.

Example 1.3: Let $K = \{(x, y)' : x \geq y^2\}$ in $E = R^2$. Let A be the linear transformation as defined in example 1.1 Evidently $AK \subseteq K$ and K has non-null interior. The vector (1,0) is an interior point of K and it is the eigenvector corresponding to the eigenvalue 1/4. But 1/2 is the spectral radius of A.

1.c. Linear operators leaving invariant a convex set in a Banach Space:

Theorem 1.3: Let K be a closed convex set with a dense linear hull in a real Banach space E. Further let θ be an extreme point of K. If $AK \subseteq K$ for a compact linear operator A with spectral radius $\mu > 1$, then $Ay = \mu y$ for some non-null y in K.

We shall prove the theorem under different cases.

Case 1: The spectral radius $\mu > 2$ and it belongs to the spectrum $\sigma(A)$.

Proof: By the well known theory of compact operators in Banach spaces [1], we have for any λ in the neighbourhood of μ

$$(A - \lambda I)^{-1} = R(\lambda, A) = \sum_{r=-n}^{\infty} (\lambda - \mu)^r B_r$$

whenever $R(\lambda, A)$ exists. This Laurent's series as an analytic operator function has only finite poles. Further B_r: $r = -n, -\overline{n-1}, \ldots$ are bounded linear operators. Since the linear hull of K is dense in E, there exists a $u \in K$ with $B_{-n} u \neq \theta$. Now $R(\lambda, A) u = -\sum_{0}^{\infty} \frac{A^n u}{\lambda^{n+1}}$ exists for $\lambda > \mu \geq 2$. Since $AK \subseteq K$, $A^n u \in K$ for all n. For any finite N, $\sum_{0}^{N} \frac{A^n u}{\lambda^{n+1}} + (1 - \sum_{0}^{N} \frac{1}{\lambda^{n+1}})\,\theta$ is a convex combination of elements of K. This follows from the fact that $\sum_{0}^{N} \frac{1}{\lambda^{n+1}} \leq \sum_{0}^{N} \frac{1}{2^{n+1}} < 1$. Thus $\sum_{0}^{N} \frac{A^n u}{\lambda^{n+1}} \in K$ for all N. Since K is closed $\lim_{n \to \infty} \sum_{0}^{N} \frac{A^n u}{\lambda^{n+1}} \in K$ where the limit is taken in the strong sense.

i.e. $-R(\lambda, A) u \in K$.

Thus $\lim_{\lambda \to \mu+0} -(\lambda - \mu)^n R(\lambda, A)u = - B_{-n}u \in K - \theta$.

Let $y = - B_{-n}u$.

Now, we have

$$AR(\lambda, A)u = \lambda R(\lambda, A)u + u \quad \text{and}$$

$$- \lim_{\lambda \to \mu+0} A(\lambda - \mu)^n R(\lambda, A)u = - \lim_{\lambda \to \mu+0} \lambda(\lambda - \mu)^n R(\lambda, A)u$$

$$- \lim_{\lambda \to \mu+0} (\lambda - \mu)^n u.$$

The last term on the right hand side of the above expression is θ. Thus we have $Ay = \mu y$ where $y = - B_{-n} u \in K - \theta$.

Case 2: The spectral radius is strictly greater than unity and among the characteristic numbers of A of maximal modulus there is a root of a positive number.

Let $Au = \lambda u$ with $|\lambda| = \mu > 1$, the spectral radius and $\lambda^{n_o} > 0$ for some n_o. Since λ^n is an eigenvalue of A^n and further since λ^n is the spectral radius of A^n we have $\lambda^n > 2$ for some n which is a multiple of n_o. Now by case 1,

$$A^n y_o = \lambda^n y_o \quad \text{for some non-null } y_o \text{ in } K.$$

Consider

$$y = \frac{A^{2n-1} y_0}{|\lambda^{2n}|} + \frac{A^{2n-2} y_0}{|\lambda^{2n-1}|} + \ldots + \frac{A^{n+1} y_0}{|\lambda|^{n+2}} + \frac{A^n y_0}{|\lambda|^{n+1}} .$$

Then $$Ay = \frac{A^{2n} y_0}{|\lambda|^{2n}} + \frac{A^{2n-1} y_0}{|\lambda|^{2n-1}} + \ldots + \frac{A^{n+1} y_0}{|\lambda|^{n+1}}$$

$$= |\lambda| \left(\frac{A^{2n-1} y_0}{|\lambda|^{2n}} + \frac{A^{2n-2} y_0}{|\lambda|^{2n-1}} + \ldots + \frac{|\lambda^{2n} y_0}{|\lambda|^{2n+1}} \right)$$

(Here $\dfrac{A^{2n} y_0}{|\lambda|^{2n}} = |\lambda| \dfrac{\lambda^{2n} y_0}{|\lambda|^{2n+1}}$.

Therefore $Ay = |\lambda| y$.

Here, since θ is an extreme point of K and since $y_0 \neq \theta$, $y \neq \theta$.

Since $|\lambda|^n = \lambda^n > 2$, $y \in K$ and $|\lambda|$ is the spectral radius. This completes the proof of the theorem for case 2.

Case 3: The spectral radius is strictly greater than unity and no eigenvalue of maximal modulus is a root of a positive number.

Let λ_0 be an eigenvalue with maximal real part among roots of maximal modulus.

$$\text{i.e.}\quad A\, y_0 = \lambda_0\, y_0, \quad \lambda_0 = \mu \exp\,(i\,\varphi_0),$$
$$0 < \varphi_0 < 2\,\lambda$$

and $\exp\,(in\,\varphi_0) \neq 1$ for any integer n.

By the spectral mapping theorem [10] we have
$\sigma((1-\varepsilon)\,A + \varepsilon\,A^2) = \{(1-\varepsilon)\lambda + \varepsilon\,\lambda^2; \lambda \in \sigma(A)\}$ where $\sigma(A)$ is the spectrum of A.

Suppose $\lambda \in \sigma(A)$ and $|\lambda| = |\lambda_0| = \mu$. Then $\lambda = \mu \exp\,(i\,\varphi)$ and $\cos\varphi \leq \cos\varphi_0$ by assumption. Further

$$(1-\varepsilon)\lambda + \varepsilon\,\lambda^2 = (1-\varepsilon)\,\mu\, e^{i\varphi} + \varepsilon\,\mu^2\, e^{2\,i\,\varphi} \quad \text{and}$$

$$|(1-\varepsilon)\lambda + \varepsilon\,\lambda^2| = \mu\sqrt{(1-\varepsilon)^2 + \varepsilon^2\mu^2 + 2\varepsilon\,(1-\varepsilon)\cos\varphi}$$
$$\leq |(1-\varepsilon)\lambda_0 + \varepsilon\,\lambda_0^2| \quad \text{by the choice of } \lambda_0.$$

Since A is defined on a real Banach space both $(1-\varepsilon)\lambda_0 + \varepsilon\,\lambda_0^2$, $(1-\varepsilon)\bar{\lambda}_0 + \varepsilon\,\bar{\lambda}_0^2$ are eigenvalues of $A_\varepsilon = (1-\varepsilon)A + \varepsilon\,A^2$. By a proper choice of $\varepsilon > 0$ we can make the argument of $(1-\varepsilon)\lambda_0 + \varepsilon\,\lambda_0^2$ commensurable with 2π. Further, under the assumption in this case, for arbitrarily small $\varepsilon > 0$ we can assume $|(1-\varepsilon)\lambda_0 + \varepsilon\,\lambda_0^2| > 1$ and therefore by case 2 which is applicable here, $(1-\epsilon)\lambda_o + \epsilon\lambda_o^2 > 0$. Actually for a suitable positive ϵ satisfying the commensurability condition $(1-\epsilon)A + \epsilon A^2$ leaves K invariant. As ϵ is arbitrary, it follows that $\lambda_o > 1$. This completes the proof of the theorem under case 3 and hence the proof of the theorem is complete.

Theorem 1.4: Let E be a real Banach space and let K be a closed convex set with Θ as an extreme point of K. Let the linear hull of K be dense in E. If A is a compact operator in E and

a) $A K \subset K$,

b) $Ay_o - cy_o \in K$, $y_o \in K$, $|y_o| = 1$, $c \geq 1$,

c) There exists an $f \in E^*$ with $f(y) > 0$ for all $y \in K - \Theta$,

then

$$Ay = \mu y \text{ , } y \in K - \Theta, \ \mu \geq c \text{ where } \mu \text{ is the}$$

spectral radius of A.

Proof: Since $Ay_o - cy_o \in K$, $A^n y_o - cA^{n-1}y_o \in K$ for all n.

Let $a_1, a_2, \ldots, a_N$ be N positive numbers with

i) $0 < a_i < 1 \qquad i = 1, 2, \ldots, N.$

ii) $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = \frac{a_{N-1}}{a_N} = c$

iii) $\Sigma\, a_i < 1.$

Since $A^n y_o - cA^{n-1} y_o \in K$ for all n, by the convexity of K,

$$\sum_1^N a_i (A^i y_o - cA^{i-1} y_o) + (1 - \Sigma\, a_i)\, \theta \in K.$$

i.e. $a_N A^N y_o - a_1 c y_o \in K$ which gives $a_N (A^N y_o - c^N y_o) \in K.$ Since $f(y) > 0$ for all $y \in K - \theta$

$$f(a_N (A^N y_o - c^N y_o)) = a_N\, f(A^n y_o - c^N y_o) \geq 0 .$$

By the positivity of a_N we have $f(A^N y_o) \geq c^N f(y_o) > 0$ Thus $|f|\ |A^N| \geq c^N f\ (y_o)$ which shows that

$$|A^N| \geq \frac{c^N f(y_o)}{|f|}$$

Since $0 < \frac{f(y_o)}{|f|} \leq 1 \quad \overline{\lim}_{n \to \infty} \sqrt[N]{|A^N|} \geq c > 1.$

i.e. the spectral radius of A is strictly greater than unity. Hence by theorem 1.3 we have the required assertion of the theorem.

1.d. Some applications of the extensions:

Let us give some applications of the theorems proved now.

Example 1.3: Let $E = R^4$ (Real Euclidean 4 - space). Let $K = \{(x_1, x_2, x_3, x_4)' : x_1 \geq (x_2 + x_3 - x_4)^2$

$x_1, x_2, x_3, x_4 \geq 0\}$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

with $a_{ij} \geq 0 \quad a_{11} > 1, \quad a_{21} + a_{31} - a_{41} = 0$

$(a_{22} + a_{32} - a_{42}) = (a_{23} + a_{33} - a_{43}) = -(a_{24} + a_{34} - a_{44})$

and $a_{11} \geq (a_{22} + a_{32} - a_{42})^2$.

Then it is easily seen that the points of K are mapped into K by A. Further $(1, 0, 0, 0) = y \in K$ with $|y| = 1$.

Thus for a suitable $\epsilon > 0$ $Ay - c\, y \in K$ with $c = (1+\epsilon) > 1$. Further it is easily seen that $K - K = R^4$ and since the convex set is contained in the positive orthant there also exists a strictly positive functional.

Therefore by theorem 1.4 there exists an eigenvector $x^o = (x_1^o, x_2^o, x_3^o, x_4^o)$ for the spectral radius λ^o which is an eigenvalue of A.

Say, for example, if

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 3 & 2 & 3 & 1 \end{bmatrix}$$

then $Ax^o = \lambda^o x^o$, $\lambda^o \simeq 4.294$ and

$$x^o \simeq (1,\ 1.294,\ 4.5475,\ 5.8399)'$$

is an eigenvector for λ^o. Evidently $x^o \in K$.

As an example of a reducible matrix A which leaves K invariant we consider

$$A = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 3 \end{bmatrix}$$

Since the matrix is reducible no eigenvector for the spectral radius need lie in the interior of the positive orthant. Still we have

$$Ax^o = \lambda^o x^o$$

where the spectral radius $\lambda^o \simeq 5.302$ with x $x^o \simeq (1,\ 0.302,\ .6971,\ 1)'$ clearly $x^o \in K$.

Example 1.4: Let $E = c[0,\ 1]$, real continuous functions on the closed interval $[0\ 1]$.

Let $K = \{ x(t)\colon\ x(t) \geq 0 \quad x(t) \in E,$

$$\int_0^{1/4} x(t)dt \geq (x(1/2) - x(3/4))^2 + (x(7/8) - x(1))^2 \}$$

K is evidently a closed convex set with the constant function 1 as an interior point. θ is an extreme point of K as the convex set is contained in the cone of non-negative continuous functions. Further since E is separable [23]

There exists a strictly positive functional for the cone and hence for the convex set. Let

$K(t, s)$ be a non-negative continuous kernel in $0 \leq t, \ s \leq 1$ with $K(\frac{1}{2}, s) = K(\frac{3}{4}, s)$ and $k(\frac{7}{8}, s) = K(1, s)$.

We are interested in the solutions $f^0(t)$ of the integral equation

$$\lambda^0 f^0(t) = \int_0^1 K(t,s)\, f^0(s)ds.$$

If $K(t, s) > 4$ for $0 \leq t, s \leq \frac{1}{2}$, then

$\int_0^1 K(t, s)\, u(s)ds - (1+\epsilon)u(t)$ belongs to K

for a suitable $\epsilon > 0$, with $|u| = 1$. Choose $u \in K$ with

$$\begin{aligned} u(t) &= 4t \quad 0 \leq t \leq \tfrac{1}{4} \\ &= (2 - 4t) \quad \tfrac{1}{4} \leq t \leq \tfrac{1}{2} \\ &= 0 \quad \text{outside.} \end{aligned}$$

Thus $u(t) \in K$ with $Au - cu \subset K$ where $c = (1+\epsilon)$ where A is the compact operator given by

$$(Af)(t) = \int_0^1 K(t, s)\, f(s)ds$$

Thus, since the conditions of Theorem 1.4 are satisfied, we have

$$\lambda^o f^o(t) = \int_0^1 K(t, s) f^o(s) ds$$

where $\lambda^o (\geq 1)$ is the maximal parameter for such integral equations and f^o is a non-negative continuous function with

$$\int_0^{1/4} f^o(t)dt \geq (f^o(\tfrac{1}{2}) - f^o(\tfrac{3}{4}))^2 + (f^o(\tfrac{7}{8}) - f^o(1))^2.$$

i.e. $f^o \in K$.

1.e. Linear operators in complete convex sets.

Definition 1: A linear operator A is compact in a convex set K if A sends bounded subsets of K to conditionally compact subsets of K.
(Hence $AK \subseteq K$ is also a consequence of the definition).

Theorem 2.5: Let E be a real normed linear space and let K be a closed complete convex set with θ as an extreme point of K. If A is a linear operator (not necessarily bounded in E) which is compact in K and if the partial spectral radius $\mu^{(7)}$ exists with $\mu > 1$, then $Ay = \mu y$ for some non-null y in K.

We first state the following version of the uniform boundedness theorem which will be used in the proof of this theorem.

<u>Lemma 1.1</u>: Let $A_t, t \in T$ be a family of linear operators with $A_t K \subseteq K$ and continuous in a closed convex set K which is complete in a real normed linear space E.

If $\{A_t x : t \in T\}$ is bounded for each x in K, then $p(A_t): t \in T$, the collection of partial norms, is it self a bounded set.

Since this follows directly from uniform boundedness theorem, the proof is omitted.

Proof of the theorem:

Case 1: The partial spectral radius $\mu > 2$.

Since K is complete, for any $x \in K$ and $\alpha > \mu (> 2)$ the infinite sum

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(7) $\mu = \lim_{n \to \infty} \sqrt[n]{p(A^n)}$ where $p(A) = \sup_{x \in K, |x| \leq 1} |Ax|$

whenever μ exists. (This is defined earlier in the introduction).

$$R(\alpha, A)x = \frac{x}{\alpha} + \frac{A x}{\alpha^2} + \dots + \frac{A^{n-1} x}{\alpha^n} + \dots \quad \text{exists.}$$

Further $R(\alpha, A)$ as a linear transformation maps K into itself. Now, let us establish

$$\limsup_{\alpha \downarrow \mu} p(R(\alpha, A)) = \infty .$$

Suppose $p(R(\alpha, A)) \leq M$ for all α with $2 < \mu < \alpha < \mu + \epsilon$ for some ϵ in $0 < \epsilon < \frac{1}{2M}$ choose $2 < \lambda < \mu$ with $\Sigma (\alpha - \lambda)^n M^{n+1} < \infty$ and $0 < \sum_1^\infty (\alpha - \lambda)^n \leq \partial < 1$. Such a choice of λ is always possible.

Fix an α and define for each x in K

$$B(\lambda, \partial)x = (1-\partial)R(\alpha, A)x + (\alpha-\lambda) R^2(\alpha,A)x + (\alpha-\lambda)^2 R^3(\alpha,A)x + \dots$$

Since $0 < \lambda < 1$, $B(\lambda, \partial)x$ belongs to K by the choice of λ. Further

$$\begin{aligned}(\lambda-A)B(\lambda,\partial)x &= ((\alpha-A)-(\alpha-\lambda))\, B(\lambda,\partial)x \\ &= (1-\partial)x + \partial(\alpha-\lambda)R(\alpha, A)x .\end{aligned}$$

Therefore

$$B(\lambda, \partial)x = \frac{(1-\partial)}{\lambda}\left[\frac{x}{\lambda} + \frac{Ax}{\lambda^2} + \dots + \frac{A^n x}{\lambda^{n+1}}\right] +$$

$$+ \partial(\alpha-\lambda)\left[\frac{1}{\lambda}\cdot R(\alpha,A)x + \frac{1}{\lambda^2}\cdot A\cdot R(\alpha,A)x + \right.$$

$$\left. + \dots + \frac{A^n}{\lambda^{n+1}}\cdot R(\alpha,A)x\right] + \frac{A^{n+1}}{\lambda^{n+1}} B(\lambda, \partial)x$$

Since $\lambda > 2$, $0 < \partial < 1$ and $(1-\partial) + \partial(\alpha-\lambda) + \frac{1}{\lambda^{n+1}} \leq 1$ for large n, $B(\lambda, \partial)x - \frac{(1-\partial)}{\lambda^{n+1}} \cdot A^n x \in K$ for sufficiently large n.

Further since $\lambda < \mu$, there exists a y in K with $\left| \frac{A^n y}{\lambda^{n+1}} \right|$ unbounded. This follows from the fact that for any k with $2 < \lambda < k < \mu$

$$p\left(\frac{A^n}{\lambda^{n+1}}\right) > \frac{k^n}{\lambda^{n+1}} \quad \text{for sufficiently large}$$

n, and therefore by lemma 1 we have a y in K with $\left|\frac{A^n y}{\lambda^{n+1}}\right|$ unbounded. From this unbounded sequence, we can choose a subsequence n_j of integers from the sequence n of positive integers, with

i) $\lim_{j \to \infty} \left| \frac{A^{n_j} y}{\lambda^{n_j+1}} \right| = \infty$

ii) $\left| \frac{A^{n_j} y}{\lambda^{n_j+1}} \right| > \left| \frac{A^{n_j-1} y}{\lambda^{n_j}} \right|$

iii) $\lim_{n \to \infty} \left| \frac{A^{n_j-1} y}{\lambda^{n_j}} \right| = \infty$.

Also we have $B(\lambda, \partial)y - \frac{(1-\lambda)}{\partial} A \cdot \frac{A^{n_j-1} y}{\lambda^{n_j}} \in K$ for sufficiently large j.

If we define $y_j = |A^{n_j-1}|^{-1} \cdot A^{n_j-1} y$ then,

$\left| \frac{1}{\lambda^{n_j}} \cdot A^{n_j-1} y \right|^{-1} B(\lambda, \partial)y = \frac{(1-\partial)}{\lambda} A\, y_j + z_j$ where $z_j \in K$.

Since $|y_j| \leq 1$, by the compactness of A in K we have a subsequence $A\, y_{j_k}$ converging to a vector in K. Without loss of generality we can assume it to be the original sequence so that $A\, y_j \to y_0$ in K.

But however by (i)

$$\lim_{j \to \infty} \left| \frac{(1-\partial)}{\lambda} \cdot A\, y_j + z_j \right| = 0$$

Further since $\lambda > 2$ and $0 < \partial < 1$, $\frac{(1-\partial)}{\lambda} \cdot y_j \to \theta$ and $z_j \to \theta$ for otherwise θ will *not* be an extreme point of K.

But $\left|\frac{(1-\partial)}{\lambda} \cdot y_0\right| > (1-\partial)$ by (ii)

This contradiction shows that

$$\lim_{\alpha \downarrow \mu} \sup \; p(R(\alpha, A)) = \infty .$$

Thus there exists a sequence $\alpha_n \downarrow \mu$ and a sequence x_n in K with $|x_n| \leq 1$ and

$$|R(\alpha_n, A)x_n| > \frac{1}{2}\, p(R(\alpha_n, A)), \quad p(R(\alpha_n, A) \to \infty$$

If we define $u_n = (p(R(\alpha_n, A)))^{-1} R(\alpha_n, A)\, x_n$ then $u_n \in K$ for sufficiently large n.

$$\text{Now } (\mu u_n - Au_n) = (\alpha_n - A)u_n - (\alpha_n - \mu)\, u_n$$
$$= -(\alpha_n - \mu)u_n + p(R(\alpha_n, A))^{-1}\, x_n .$$

By the choice of α_n and by the boundedness of x_n, $p(R(\alpha_n, A)^{-1}\, x_n \to \theta$. Thus we have

$$\lim_{n \to \infty} (\mu\, u_n - A\, u_n) = \theta .$$

By the compactness of A in K, $Au_{n_k} \to u_0$ in K for some subsequence u_{n_k} of u_n and thus $\mu u_{n_k} \to u_0$. But $A\mu u_{n_k} \to Au_0 = \mu u_0$. Since $|u_n| \geq \frac{1}{2}$ for all n, $u_0 \neq \theta$. This completes the proof for case 1.

Case 2: The partial spectral radius lies strictly between 1 and 2.

$$\text{Since } \mu = \lim_{n \to \infty} \sqrt[n]{p(A^n)} > 1$$

$$= \lim_{n \to \infty} \sqrt[nk]{p(A^{nk})}$$

for any positive integer k

$$\lim_{n \to \infty} \sqrt[n]{p(A^{nk})} = \mu^k > 2 \text{ for a suitable } k.$$

Since $A^k K \subseteq K$ with partial spectral radius $\mu^k > 2$, case 1 can be applied and in fact for the operator A^k which is also compact in K we have $A^k u = \mu^k u$ for some non-null u in K.

Let $y = \beta\left(\frac{1}{\mu} u + \frac{1}{\mu^2} Au + \dots + \frac{1}{\mu^{k-1}} \cdot A^{k-1} u\right)$ with $\beta > 0$ and $\beta\left(\frac{1}{\mu} + \dots + \frac{1}{\mu^{k-1}}\right) = 1$. Then

$Ay = \beta(\frac{1}{\mu} Au + \dots + \frac{1}{\mu^k} A^k u) = \mu y$. $y \in K$ is evidently non-null as θ is an extreme point of K. This completes the proof for case 2 and the proof of the theorem is complete.

ABSTRACT MINIMAX THEOREM AND STRONGLY POSITIVE OPERATORS IN BANACH SPACES

2.a. Preliminaries:

It was observed by Alexandroff and Hopf [23] that if a square matrix of order n with non-negative components is n-singular, then by Brower's fixed point theorem [10] applied to a bounded cross-section of the positive orthant in R^n by a hyperplane, one can prove the existence of a positive eigenvalue with a non-negative eigenvector for it. (Here a vector is non-negative if each component is non-negative). But the complete propositions of Perron and Forbenius on such matrices were not obtained by this method. Recently Blackwell [3] studied this problem in the light of the minimax theorem of von-Neumann and obtained a more complete result.

Here we will obtain all the known results on strongly positive operators from the abstract minimax theorem of Ky Fan. Incidentally, our method yields some further results on such problems for very general Banach spaces.

We state the following theorems of Ky Fan (a modified version) and M.G.Krein, which will be used in the sequel.

Theorem 2.1 (Ky Fan [12]):

If K_1 and K_2 are two compact convex sets in locally convex real linear topological spaces E_1 and E_2

respectively and if $f(x, y)$ is a bilinear functional on $E_1 \times E_2$ continuous in each variable, then

$$\max_{K_1} f(x, y) = \max_{K_1} \min_{K_2} f(x, y)$$
$$= f(x_o, y_o); \quad x_o \in K_1, \; y_o \in K_2.$$

Theorem 2.2 (Krein [23]: Let K be a normal[8] cone in a real Banach space E. If $u \in K$, then for all $x \in E$ with $u \pm x \in K$, $|x| \leq c < \infty$ for some $c > 0$.

Theorem 2.3 [23]: Let K be a closed cone in a real Banach space E. If $x_o \notin K$ then there exists an $f \in K^*$ with $f(x_o) < 0$.

2.b. Minimax solutions and strongly positive operators:

Theorem 2.4: Let be a closed cone in real reflexive Banach space Let K and the conjugate cone K^*[9] have non-null interior. Further let A be a strongly positive operator[10]. Then

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(8) see footnote (3), page 3.

(9) $K^* = \{ f: f \in E^*, f(x) \geq 0 \text{ for all } x \in K \}$.

(10) A is strongly positive if $AK \subseteq K$ and for any $x \in K$, $x \neq \theta$, there exists a positive integer $n(x)$ such that $A^{n(x)} x$ is an interior point of K.

1. $Az = \lambda_0 z$, $\lambda_0 > 0$, $z \in$ interior of K.

2. λ_0 is the spectral radius of A.

3. The subspace $S_{\lambda_0} = \{y : Az = \lambda_0 z\}$ is one dimensional

4. A^* has an eigenvector (φ) for λ_0 which is strictly positive on $K^* - \theta$

5. The subspace $S^*_{\lambda_0} = \{f: A^* f = \lambda_0 f\}$ is dimensional.

6. No other vectors linearly independent of z or (φ) lie in K or K^* respectively.

Proof:

Since $E = E^{**}$ and $K = \bar{K}$, it is easily seen from Theorem 2.3 that $K = K^{**}$. For any u in the interior of K and $f_1, f_2 \in K^*$ we have

$$u \pm \rho \frac{x}{|x|} \in K \quad \text{for all } x \text{ in } E$$

and for some $\rho > 0$. Therefore

$$f_1(u) \geq \rho\, f_1\left(\frac{x}{|x|}\right) \text{ and } f_2(u) \geq \rho\, f_2\left(\frac{x}{|x|}\right)$$

Thus $|f_1 + f_2| \geq f_1(u) + f_2(u) \geq \rho \max(|f_1|, |f_2|)$. Therefore K^* is normal. Similarly K is normal as K^* has interior.

Let us consider the following sets π_1 and π_2.

$$\pi_1: \{f: f(y_0) = 1\} \cap K^*, \quad y_0 \in \text{interior of } K, |y_0| = 1$$

$$\pi_2: \{x: f_0(x) = 1\} \cap K, \quad f_0 \in \text{interior of } K^*, |f_0| = 1$$

Since y_0, f_0 are interior points of K and K^* respectively, as it is shown above $f(y_0) \geq \rho_1 |f|$ and $f_0(x) \geq \rho_2 |x|$ for some $\rho_1, \rho_2 > 0$ and for all x, f in K and K^* respectively. Thus π_1 and π_2 are bounded by $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ respectively. Since they are bounded and weakly closed by the reflexivity of E they are also weakly compact. Moreover they are cross-sections of K and K^* respectively. (i.e. $f \in K^* - \theta \Rightarrow \lambda f \in \pi_1$ for some $\lambda > 0$ and $x \in K - \theta \Rightarrow \mu x \in \pi_2$ for some $\mu > 0$).

Consider the bilinear functional

$$K_\lambda(f, y) = (f, (A - \lambda)y) = f((A - \lambda)y).$$

$K_\lambda(f, y)$ is weakly continuous in each variable f and y for all real λ.

Now by applying Theorem 2.1 of Ky Fan we have

$$v(\lambda) = \min_{\pi_2} \max_{\pi_1} (f, (A - \lambda)y) = \max_{\pi_1} \min_{\pi_2} (f, (A - \lambda)y)$$

$$= (f_0, (A - \lambda)y_0); \quad f_0 \in \pi_1 \quad y_0 \in \pi_2 .$$

We observe the following.

i) $v(0) > 0$

ii) $v(\lambda)$ is continuous and non-decreasing

iii) $v(\lambda) \to -\infty$ as $\lambda \to \infty$.

The above assertions i, ii and iii are proved as follows.

i) $v(0) = (f', Ay')$ for some $f' \in \pi_1$ and $y' \in \pi_2$ we have $(f', Ay') \geq (f_0, Ay') > 0$ by the strong positivity of A and further by the property that f_0 is an interior point of K^*.

ii) Let f_1, y_1, f_2, y_2 be minimax solutions for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively.

Therefore

$$v(\lambda_2) \leq (f_2, (A - \lambda_2)y_1)$$

$$v(\lambda_1) \geq (f_2, (A - \lambda_1)y_1)$$

Thus $v(\lambda_2) - v(\lambda_1) \leq (f_2, (\lambda_1 - \lambda_2)y_1)$

$$\leq |\lambda_1 - \lambda_2| \, |f_2| \, |y_1|$$

where norms are taken in the respective Banach spaces.
Changing the suffixes 2 and 1

$$v(\lambda_1) - v(\lambda_2) \leq |\lambda_1 - \lambda_2| \, |f_1| \, |y_2|.$$

Therefore, since π_1 and π_2 are bounded

$$|v(\lambda_1) - v(\lambda_2)| \leq |\lambda_1 - \lambda_2|.C .$$

Therefore $v(\lambda)$ is continuous and even satisfies a Lefschetz condition.

iii) Let $\lambda_1 < \lambda_2$. Then,

$$\lambda_2) \le (f_2, (A - \lambda_2)y_1)$$
$$\le (f_2, (A - \lambda_1)y_1) + (\lambda_1 - \lambda_2)(f_2, y_1)$$
$$\le (f_1, (A - \lambda_1)y_1) + (\lambda_1 - \lambda_2)(f_2, y_1)$$
$$\le v(\lambda_1) - C \quad \text{where} \quad C \ge 0 \quad \text{as} \quad \lambda_1 - \lambda_2 \le 0$$

Thus $v(\lambda_2) \le v(\lambda_1)$.

iii) For any λ if f_λ is the maximal solution, then

$$v(\lambda) \le (f_\lambda, (A - \lambda)y_0) = (f_\lambda, Ay_0) - \lambda(f_\lambda, y_0)$$

Further, since y_0 is an interior point of K
$(f_\lambda, y_0) \ge \delta_1 |f|$. Therefore

$$v(\lambda) \le (f_\lambda, Ay_0) - \lambda \delta_1 |f| \to -\infty \quad \text{as} \quad \lambda \to \infty$$

This shows that $v(\lambda)$ as a real continuous function of λ is positive for $\lambda = 0$ and $v(\lambda) \to -\infty$ as $\lambda \to +\infty$. Therefore there exists a λ_0 with $v(\lambda_0) = 0$, for some $\lambda_0 > 0$.

Let z, (φ) be the optimal minimax solutions for $\lambda = \lambda_0$. Since π_1 and π_2 generate K^* and K respectively, we have

$$((\phi), (A - \lambda_0)y) = ((A - \lambda_0)^* (\phi), y) \geq 0 \quad \text{for all } y \in K$$

and $(f, (A - \lambda_0)z) \leq 0$ for all $f \in K^*$.

Therefore, by Theorem 2.3, we get $(A - \lambda_0)^*(\phi) \in K^*$ and $(\lambda_0 - A)\, z \in K$. Therefore $A(\lambda_0 - A)\, z + \lambda_0(\lambda_0 - A)z \in K$ i.e. $\lambda_0^2 z - A^2 z \in K$. Proceeding inductivity $\lambda_0^{n(z)} z - A^{n(z)} z \in K$. Since $A^{n(z)} z$ is an interior point of K by the strong positivity of A, we have $\lambda_0^{n(z)} z$ and hence z as an interior point of K.

If $(A - \lambda_0)^*(\phi) \neq \theta$, then since z is interior to K and since $(A - \lambda_0)^*(\phi) \in K^*$, $((A - \lambda_0)^*(\phi), z) \geq \rho\,|(A - \lambda_0)^*(\phi)| > 0$ where ρ is some positive number. This contradicts the optimality of z. Therefore $(A - \lambda_0)^*(\phi) = 0$. Thus

$$A^*(\phi) = \lambda_0(\phi).$$

Further (ϕ) is a strictly positive functional. For if $y \in K - \theta$, then $\lambda^{n(y)}(\phi)(y) = (A^{*n(y)}(\phi), y) = (\phi)(A^{n(y)} y) > 0$ by the strong positivity of A. Thus $(\phi)(y) > 0$ for all $y \in K - \theta$. If $(\lambda_0 - A)z \neq \theta$, since (ϕ) is strongly positive $((\phi), (\lambda_0 - A)z) > 0$. i.e. $((\phi), (A - \lambda_0)z) < 0$.

This contradicts the optimality of $(\perp)$. Thus $(\lambda_o - A)z = \theta$

i.e. $$A z = \lambda_o z .$$

Since $\lambda_o^n z = A^n z$ is an interior point of K for some n, z is interior to K. This proves assertions 1 and 4 of the theorem.

Now, if $A y = \lambda_o y$ for any $y \in E$ linearly independent of z in some complex extension E of the real Banach space E where A is the extension of A to E, we can assume $y \in E$ without loss of generality by the positivity of λ_o. We can choose α real with $y + \alpha z$ lying on the boundary of K. This shows that an eigenvector belongs to the boundary of K which contradicts the strong positivity of A. Thus S_{λ_o} and $S^*_{\lambda_o}$ are one-dimensional.

Let $Av = \alpha v$ $v \in K - \theta$, $\alpha \neq \lambda_o$. Then

$$\alpha((\perp), v) = ((\perp), \alpha v) = ((\perp), Av) = (A^*(\perp), v)$$
$$= \lambda_o((\perp), v).$$

Since $\alpha \neq \lambda_o$, $((\perp), v) = 0$. This contradicts the strict positivity of $(\perp)$. Thus we have assertion 6 of the theorem.

For any $y \in E$ with $z \pm y \in K$, we have by Theorem 2.2, $|y| \leq \sigma$. Since

$AK \subseteq K$, $\frac{1}{\lambda_o^n} A^n(z \pm y) = z \pm \frac{A^n y}{\lambda_o^n} \in K$ and for any positive integer n, $\left|\frac{A^n y}{\lambda_o^n}\right| \leq c$. Therefore if $|\lambda| > \lambda_o$ then

$$\sum_0^\infty \frac{|A^n y|}{|\lambda^{n+1}|} = \sum \frac{|A^n y|}{\lambda_o^n} \cdot \frac{\lambda_o^n}{|\lambda^{n+1}|} \leq c \sum_0^\infty \frac{\lambda_o^n}{|\lambda|^{n+1}} < \infty .$$

Thus $\sum_0^\infty \frac{A^n y}{\lambda^{n+1}}$ exists for all y with $z \pm y \in K$.

As z is interior to K for any $y \in E$ $z \pm \epsilon \frac{y}{|y|} \in K$ for some $\epsilon > 0$. Therefore the above series exists for all $y \in E$ and for all λ with $|\lambda| > \lambda_o$. Since λ_o is also an eigenvalue,

$$R(\lambda, A)y = (A - \lambda)^{-1} y = - \sum_0^\infty \frac{A^n y}{\lambda^{n+1}}$$ shows that

λ_o is the spectral radius of A.

This proves assertion 2 of the theorem and the proof of the theorem is complete.

Remark 2.1: Even if E is not reflexive and K^* has no interior we can by the same method prove all the assertions of the theorem except the fact that λ_0 is the spectral radius, provided the cone K has a weakly compact cross-section or even a compact cross-section in some locally-convex topology in which the operator A is also continuous when restricted to the cross-section.

Thus the general minimax theorem, yields also theorem 1 of Schaefer [30].

3. ON POSITIVE GAME MATRICES AND THEIR EXTENSIONS.

3.a. Preliminaries:

Previously we have seen that for a certain class of positive operators in reflexive Banach spaces, the eigenvectors exist for the spectral radius as the eigenvalue. Further, the spectral radius is the minimax value of a suitable bilinear functional and the eigenvectors for the spectral radius are the minimax solutions of this payoff function. We are interested in the converse problem. Namely, can we in some sense analyse the nature of minimax solutions with our theory of positive operators?

In this chapter, we shall study some relations between the optimal strategies, the eigenvalues and the eigenvectors of a square matrix with positive entries, whose rows and columns correspond to the pure strategy spaces of players in a zero-sum two person game. Further we study a property of the game values of commuting family of matrices with positive entries. The results are extended to infinite games on the unit square, with positive kernels as pay-off functions.

It is well known that any two zero-sum two-person games have the same sets of optimal strategies for the two players when their pay-off kernals differ just by a constant. The optimal strategy sets could therefore be analysed by considering a new game with a positive pay-off that differs from the original one just by a fixed constant in each of the entry.

In the particular case where the pure strategy spaces are finite we can further reduce the positive pay-off matrix to a positive square matrix. This reduction is just the restatement of theorem 3 of Kaplansky [19]. By such a reduction, we get the optimal strategies of the new game to be a subset of the optimal strategies of the old game.

As Karlin puts it, in his book [20]: 'What is far less understood is the relationship of the form of the pay-off kernel to the form of the solutions.' So, it seems possible to analyse the interrelations between the optimal strategy spaces and the kernel of a zero-sum-two-person game, when we study the positive game kernels.

3.b. <u>Definitions and notations:</u>

In this section we shall consider only pay-off matrices of the type

$$A = (a_{ij})_{n \times n} \quad a_{ij} > 0, \quad i, j = 1,2,\dots, n,$$

with

$$R = \left\{(\xi_1, \xi_2, \dots, \xi_n) : \xi_i \geq 0, \ \sum_i \xi_i = 1\right\}$$

$$S = \left\{(\eta_1, \eta_2, \dots \eta_n) : \eta_j \geq 0 \ \sum_j \eta_j = 1 \right\}$$

as the respective mixed strategy spaces of player I and Player II. As usual the rows and columns of the given pay-off matrix will be the pure strategy spaces of player I and player II respectively.

The vector $(\xi_1, \xi_2, \dots, \xi_n)$ is also denoted as ξ without any lower suffix. But ξ_j for example will denote the j^{th} component of the vector ξ. Upper suffix will be used to distinguish vectors. All the above notations will be used also for the vector $(\eta_1, \eta_2, \dots, \eta_n)$.

The value for the pay-off matrix $A = (a_{ij})_{n \times n}$ is defined as

$$v = \min_{\eta \in S} \max_{\xi \in R} \Sigma \Sigma a_{ij} \xi_i \eta_j$$

The optimal strategy sets R^o and S^o of players I and II are defined as

$$R^o = \left\{ \xi^o = (\xi_1^o, \xi_2^o, \ldots, \xi_n^o) : \sum_i a_{ij} \xi_i^o \geq v, \quad j = 1, 2, \ldots, n, \xi \in R \right\}$$

$$S^o = \left\{ \eta^o = (\eta_1^o, \eta_2^o, \ldots, \eta_n^o) : \sum_j a_{ij} \eta_j^o \leq v, \quad i = 1, 2, \ldots, n, \eta \in S \right\}$$

By von-Neumann's minimax theorem [34]

$$v = \max_{\xi} \min_{\eta} \Sigma \Sigma a_{ij} \xi_i \eta_j = \sum_i \sum_j a_{ij} \xi_i^o \eta_j^o$$

$$= \min_{\eta} \max_{\xi} \Sigma \Sigma a_{ij} \xi_i \eta_j.$$

3.c. Optimal strategies of the positive game matrix, the positive eigenvalue and eigenvector.

Definition 3.1 A strategy $\xi \in R$ for player I is completely mixed if $\xi_i > 0$ for $i = 1, 2, \ldots, n$. Similarly it is defined for $\eta \in R$.

We need the following theorem in the sequel.

<u>Theorem 3.1</u> (Kaplansky [19])

If $A = (a_{ij})_{n \times n}$, $a_{ij} > 0$, $i, j = 1,2,\dots, n$ has a completely mixed strategy for player II, then for any optimal $\xi \in R^o$

$$\sum_i a_{ij}\, \xi_i = v \quad \text{for} \quad j = 1, 2, \dots, n$$

Applying theorem 2.4 let λ^* be the spectral radius of A and ξ^*, the unique eigenvector with $\sum_i \xi_i^* = 1$, $\xi_i^* > 0$ for the eigenvalue λ^*.

<u>Theorem</u> 3.2: If A is symmetric and if there exists a $\xi^o \in R^o$ and a column j_o such that $\sum_i a_{ij_o}\, \xi_i^o > v$, then ξ^* separates the convex sets of optimal strategy spaces R^o and S^o.

i.e.
$$\begin{aligned} \xi^* . x &\geq c \quad \text{for} \quad x \in R^o \\ &\leq c \quad \text{for} \quad x \in S^o \end{aligned}$$

for some c.

Proof: For the given $\xi^o \in R^o$ and any $\eta \in S^o$

$$\sum_i a_{ij}\, \xi_i^o \geq v \quad \text{for all } j \text{ with strict inequality for } j = j_o$$

$$\sum_i a_{ij} \xi_i \geq v \quad \text{for all } j \text{ and for all } \xi \in R^o$$

$$\sum_j a_{ij} \eta_j \leq v \quad \text{for all } i \text{ and for all } \eta \in S^o$$

Therefore

$$\Sigma \Sigma a_{ij} \xi_i^o \xi_j^* > v \quad (\xi_j^* > 0, \; \Sigma \xi_j^* = 1)$$

i.e. $\lambda^* (\xi^o, \xi^*) = \xi^{o'} A \xi^* > v$

where $(\xi^o, \xi^*) = \Sigma \xi_i^o \xi_i^*$.

Therefore $(\xi^o, \xi^*) > \frac{v}{\lambda^*}$ (*)

and $(\xi, \xi^*) \geq \frac{v}{\lambda^*}$ for all other $\xi \in R^o$

By the symmetry of the matrix A we have

$$\lambda^* (\eta, \xi^*) = \lambda^* (\xi^*, \eta) = (A\xi^*, \eta) = (\xi^*, A\eta) \leq v$$

Since R^o and S^o cannot lie completely on the hyperplane $\xi^* . x = \frac{v}{\lambda^*}$ by (*) ξ^* separates the sets R^o and S^o which are evidently convex.

It is important to notice that the inequalities are not affected because of the positivity (or non-negativity) of the vector ξ^*.

Remark: A necessary condition for ξ^* to separate R^0 and S^0 is that any optimal strategy for player II omits choosing some j_0^{th} column.

Theorem 3.3 If player II has a completely mixed strategy $\eta \in S^0$ and $\lambda \neq 0$ is an eigenvalue of the positive matrix A, then any eigenvector of λ whose component sum is zero is orthogonal to R^0.

Proof: Let $\xi^0 \in R^0$ and let $A\xi = \lambda\xi$ with $\lambda \neq 0$ and $\sum_j \xi_j = 0$. Since player II has a completely mixed strategy, by theorem 3.1 stated earlier

$$\sum_i a_{ij}\, \xi_i^0 = v \quad \text{for all } j,$$

$$\sum_i \sum_j a_{ij}\, \xi_i^0\, \xi_j = v \sum \xi_j = \lambda \sum_i \xi_i^0\, \xi_i$$

Since $\sum_j \xi_j = 0$ and since $\lambda \neq 0$, $\sum \xi_i^0\, \xi_i = 0$. i.e. ξ^0 is orthogonal to ξ. Since ξ^0 is an arbitrary element of R^0, the theorem is complete.

We also note that if $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \in R^0$ and player II has an optimal completely mixed strategy, then $v = \frac{\lambda^*}{n}$.

We need the following two propositions for the study of a commuting family of positive pay-off matrices.

<u>Theorem</u> 3.4 (Helly [2]) : If $\{R_t^0,\ t \in T\}$ is a family of compact convex sets in $(n-1)$ dimensional Euclidean space R^{n-1} such that any n of them have a common intersection, then they all have a point in common.

<u>Theorem</u> 3.5 (Krein-Rutman [23]): Let K be a cone with interior in a real Banach space E. Let $\{A_t,\ t \in T\}$ be a commuting family of bounded linear operators mapping the interior of K into itself. Then there exists a strictly positive linear functional $(\phi) \in K^*$ (i.e. $(\phi)(x) > 0$ for $x \in K - \theta$) which is an eigenvector of all conjugate

$$A_t^* \ (\phi) = \lambda_t \ (\phi) \ (\lambda_t > 0, \quad t \in T).$$

Now we prove the following theorem.

<u>Theorem</u> 3.6: Let $\{A_t,\ t \in T\}$ be a commuting family of $(n-1) \times (n-1)$ positive square matrices (with entries positive) with the following properties.

i) For every pay-off A_t, there exist a completely mixed strategy $\xi^{(t)} \in R_t^o$ and a completely mixed $\eta^{(t)} \in S_t^o$ where R_t^o, S_t^o are the optimal strategy spaces of player I and player II respectively.

ii) For any n matrices $A_{t_1}, A_{t_2}, \ldots, A_{t_n}$, $t_1, t_2, \ldots, t_n \in T$, let there be a common optimal strategy for player I

Under these two conditions

$$\frac{v_t}{v_{t'}} = \frac{\lambda_t^*}{\lambda_{t'}^*}$$

where v_t, v_t', λ_t^* and $\lambda_{t'}^*$ are the values and eigenvalues of maximal absolute value for A_t and $A_{t'}$.

<u>Proof</u>: By assumption (i) the family R_t^o of optimal strategies for player I constitutes a family of compact convex sets in the (n-1) dimensional space R^{n-1}. By Helly's theorem, there exists a common optimal strategy for player I for all the pay-off matrices $\{A_t, t \in T\}$, i.e.,

$\Sigma a_{ij}^t \xi_i^o = v_t$ for all j and for all $t \in T$ where $A_t = (a_{ij}^t)_{n \times n}$.

R^n is a self-congugate Banach space and **theorem 3.5 of Krein** can be applied to $\{A_t^*, t \in T\}$ where A_t^* are the linear transformations corresponding to the transpose matrices A_t^* with respect to a fixed basis $e_1, e_2, \ldots, e_n$. Since A_t^*s are posive matrices all the conditions of Theorem 3.5 are satisfied and so, we have an eigenvector $z = (z_1, z_2, \ldots, z_n)$, $z_1 > 0$, $z_2 > 0, \ldots, z_n > 0$ for all A_t s.t.

$$A_t z = \lambda_t z \ (\lambda_t > 0,\ z = (z_1 \ldots z_n),\ z_i > 0,\ i=1,2,\ldots,n)$$

Without loss of generality $\sum_i z_i = 1$.

Now $\sum_i \sum_j a_{ij}^t \xi_i^o z_j = \lambda_t (z, \xi^o) = v_t$ by condition (i)

Hence $\dfrac{v_t}{v_{t'}} = \dfrac{\lambda_t (z, \xi^o)}{\lambda_{t'}(z, \xi^o)} = \dfrac{\lambda_t}{\lambda_{t'}}$.

We shall prove $\lambda_t = \lambda_t^*$, the positive eigenvalue of A_t which is maximal in modulus.

Suppose z is not eigenvector for some A_{t_o} corresponding to the eigenvalue $\lambda_{t_o}^*$ maximal in modulus.

Let u be the positive eigenvector of $A_{t_o}^*$ for the eigenvalue $\lambda_{t_o}^*$. We have $A_{t_o}^* u = \lambda_{t_o}^* u$ and

$$\lambda^*_{t_0}(z, u) = (z, \lambda^*_{t_0} u) = (A_{t_0} z, u) = \lambda_{t_0}(z, u)$$

By assumption $\lambda^*_{t_0} \neq \lambda_{t_0}$. Thus $(z, u) = 0$ which contradicts positivity of z and u. This completes the proof of the theorem.

3.d. Games on the unit square with positive pay-off kernels.

All the above mentioned theorems on positive game matrices could at once be extended to the infinite case with some changes.

Let the unit square $0 \leq x \leq y \leq 1$ be the pure strategy spaces, for two players in a zero-sum two-person game with $k(x, y) > 0$ as the pay-off, kernel. Let F and G be probability distributions on the Borel sets of $0 \leq x \leq 1$ and $0 \leq y \leq 1$ respectively. If $K(x, y)$ is jointly continuous in x and y then

$$v = \min_G \max_F \int_0^1 \int_0^1 K(x,y)dF(x)dG(y) =$$

$$= \max \min \int_0^1 \int_0^1 K(x,y)dF(x)dG(y)$$

where F and G range over all probability distributions on $[0\ 1]$. v is called the value and any F^o with $\int_0^1 K(x, y)\, dF^o(x) \geq v$ for all y is called an optimal strategy for player I. Similarly the optimal strategies are defined for player II. By the joint continuity of the kernel we have

$$\int_0^1 \int_0^1 |K(x, y)|^2\, dx\, dy < \infty \qquad (*)$$

The mapping

$$A : f(x) \to \int_0^1 K(x, y)\, f(y) d\, y,$$

by condition (*) maps bounded subsets of $L_2[0\ 1]$ to sets with compact closure in $L_2[0\ 1]$. So this integral operator A is a compact operator in $H = L_2[0\ 1]$. Further if $K(x, y) > 0$ we have the following.

<u>Theorem 3.7</u> (Krein-Rutman [23]): If $K(x, y) > 0$ is a bounded measurable function, then there exists an a.e. positive eigenfunction ϕ with

$$\lambda \int_0^1 K(x, y)\, \phi(y) d\, y = \phi(x)$$

with $\lambda > 0$. Further ϕ is unique upto a multiplicative constant a.e. for the fixed λ. λ is further the least parameter among all such solutions ϕ of the above equation.

Alexandroff's theorem asserts that any $(\dagger) \in C[0\ 1]^*$ is

$$(\dagger)(f) = \int_0^1 f(t)d\ \mu(t)$$

where μ is a unique signed measure of bounded variation on the Borel sets of $[0\ 1]$.

Definition: An element $\phi^* \in C[0\ 1]$ separates two convex sets A and B of $C[0\ 1]^*$ if for some α

$$\int_0^1 \phi^*(x)d\ \mu(x) \leq \alpha \quad \text{for all} \quad \mu \in A$$
$$\geq \alpha \quad \text{for all} \quad \mu \in B$$

where $A, B \subseteq C[0\ 1]^*$.

We prove the following separation theorem.

Theorem 3.8: Let $K(x, y) > 0$ be a continuous symmetric kernel on $0 \leq x, y \leq 1$. If the optimal strategy spaces R^0 of player I and S^0 of player II are disjoint for the game, then an eigenfunction of the minimum modulus parameter λ of the associated integral operator separates R^o and S^o.

Proof: We know

$$\int_0^1 K(x, y)d\,\mu(x) \geq v \quad \text{for all} \quad \mu \subset R^o$$

and for all y in $0 \leq y \leq 1$. Taking any $\mu \in R^o$ and the unique a.e positive eigenfunction ϕ^* of λ with $\int_0^1 \phi^*(y)d\,y = 1$, we have

$$\lambda \int_0^1 d\mu(x) \int_0^1 K(x,y)\phi^*(y) = \int_0^1 \phi^*(x)d\mu\,(x) \geq \lambda\, v$$

The inequalities are not changed because of the positivity of the function ϕ^* and the parameter λ. Further we use Fubini's theorem for the change of order of integration.

By the symmetry of the kernel

$$\int_0^1 \phi^*(x)d\sigma\,(x) \leq \lambda\, v. \quad \text{for all } \sigma \subset S^o$$

Hence ϕ^* separates R^o and S^o.

We need the following generalization of Kaplansky's result [19] in the infinite case.

Theorem 3.9 (Karlin [20]): If pay-off $K(x, y)$ is jointly continuous and if further there exists a $G^0 \in S^0$ with spectrum $\sigma(G^0) = [0\ 1]$, then for any $F^0 \in R^0$

$$\int_0^1 K(x, y)d\,G^0(y) = v \quad \text{for all } y$$

where v is the value of the game.

Theorem 3.10: Let $\lambda^0 \neq 0$ be an eigenvalue of the integral operator

$$A : f(x) \to \int_0^1 K(x, y)f(y)dy$$

Let player II have an optimal strategy with [0 1] as its spectrum.

Then any eigenfunction f^0 of λ^0 with $\int_0^1 f^0(t)dt = 0$ is orthogonal to optimal continuous density strategies of player II.

Proof: Let f be an optimal density for player II and let f be continuous. Then

$$\int_0^1 K(x, y)\, f^o(y)\, d\, y = \lambda^o\, f^o(x),$$

$$\int_0^1\int_0^1 K(x,y)\, f^o(y) f(x) dx\, dy = \lambda^o \int_0^1 f(x) f^o(x) dx$$

$$= v \int_0^1 f^o(x) d\, x$$

$$= 0$$

The above step is a consequence of Fubini's theorem on the change of the order of integration. Since $\lambda^o \neq 0$ the theorem follows at once.

Theorem 3.11: If Lebesgue measure is optimal for player I, and player II has an optimal strategy with the spectrum as [0 1], then the integral of any eigenfunction of any non-zero eigenvalue other than for the one correspon-ding to the positive eigenvalue maximal in modulus, vanishes.

Proof: Since Lebesgue measure is optimal for player I, for any eigenvalue λ and for any eigenfunction f of λ, we have

$$\int_0^1 \int_0^1 K(x,y)f(y)dx\,dy = \int_0^1 f(y)dy = \lambda \int_0^1 f(x)d\,x$$

The above step follows from Fubini's theorem.

If $\lambda = \lambda^*$ the eigenvalue maximal in modulus and $f = f^*$ the a.e positive eigenfunction then $v = \lambda^*$. Since $\lambda \neq \lambda^*$, $v \neq \lambda$ and therefore $\int_0^1 f(x)d\,x = 0$.

While the cone of vectors with non-negative components has non-null interior, the cone of non-negative functions in $L_2[0\ 1]$ has no interior at all. So the infinite analogue of theorem 3.6 is stated as follows.

<u>Theorem</u> 3.12: Let $\{K_t(x, y),\ t \in T\}$ be positive continuous symmetric kernels in $0 \leq x,\ y \leq 1$. Let

$$A_t : f(x) \to \int_0^1 K_t(x, y)f(y)d\,y$$

be a commuting family of operators.

If player II has an optimal strategy with $[0\ 1]$ as the spectrum for each pay-off $K_t(x, y)$ and if for any finite family of these games player I has a common optimal strategy, then

$$\frac{v_t}{v_{t'}} = \frac{\lambda_t^*}{\lambda_{t'}^*}$$

where v_t, $v_{t'}$ are the values corresponding to $K_t(x,y)$ and $K_{t'}(x, y)$ and λ_t^*, $\lambda_{t'}^*$ the spectral radii of operators A_t, $A_{t'}$. (Continuity of the kernels can be dispensed with if the kernel is bounded and if the games have value with minimax solutions).

<u>Proof</u>: Since [0 1] is a compact metric space, the space of all probability measures on [0 1] is itself a compact metric space in the w*-topology [33] of C[0 1]*. The optimal strategy sets are w*-closed and hence, they are compact convex sets. Since there are common optimal strategies for player I for any finite family of them, by the w* compactness of the space of probability measures on [0 1] we have a common optimal for player I for all of them. Let $S = \{f: \|f\| \leq 1, \; f \in L_2 [0 \; 1]\}$. S is compact in the weak topology. We know by theorem 3.5 that there exists an a.e positive eigenfunction $(\psi)_t^*$ with $\|(\psi)_t^*\| \leq 1$ for the associated positive eigenvalue λ_t^* for each A_t. Further λ_t^* is the spectral radius.

Consider the operator $B_t = \frac{A_t}{\lambda_t^*}$. Since $K_t(x, y)$ is symmetric in x and y, $\frac{A_t}{\lambda_t^*}$ is a symmetric operator in L_2 [0 1] and hence

$$|B_t| = \left| \frac{A_t}{\lambda_t^*} \right| \leq 1 .$$

Let $H_t = \left\{ f: B_t f = f \right\} \cap S$. By the positivity and symmetry of the kernels, the eigen subspace corresponding to the eigenvalue 1 is one-dimensional for each B_t.

Let $f \in H_t$. Then

$$B_t B_{t'} f = B_{t'} B_t f = B_{t'} f$$

Therefore $B_{t'} f$ belongs to the eigen subspace corresponding to the eigenvalue 1 of B_t. Since it is true for any t'. We have a common eigenfunction $(\phi)^*$ for all B_t. Further it is easily seen that $(\phi)^*$ is a.e positive and common for all A_t corresponding to the maximal eigenvalue λ_t^*.

Now using a common optimal μ_0 for player I and applying Fubini's theorem we get

$$\int_0^1\int_0^1 K_t(x,\ y)d\ \mu_o(x)\ (\tfrac{1}{\ })^*(y)dy = \int_0^1 (\tfrac{1}{\ })^*(y)dy$$

$$= \lambda_t^* \int_0^1 (\tfrac{1}{\ })^*(x)d\ \mu_o(x).$$

Hence $\frac{v_t}{v_{t'}} = \frac{\lambda_t^*}{\lambda_{t'}^*}$, which completes the proof of the theorem.

3.e. <u>Examples:</u>

Let us give some applications of this theorem in finite and infinite dimensions.

<u>Example</u> 3.1: Let us consider the two games with payoffs

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & 5 \\ 5 & & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 19 & 14 & 24 \\ 29 & 24 & 14 \\ 16 & 26 & 26 \end{bmatrix}$$

Here $AB = BA$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is optimal for player I in both of these games. Further $(\frac{6}{30}, \frac{13}{30}, \frac{11}{30})$ is optimal for player II for the game matrix A and $(\frac{14}{30}, \frac{1}{30}, \frac{15}{30})$ is optimal for player II for the game matrix B. i.e. they are completely mixed. Here one can easily see that

$$v_A = \frac{8}{3} \qquad \text{and} \quad v_B = \frac{64}{3}$$

$$\lambda_A^* = 8 \qquad \text{and} \quad \lambda_B^* = 64$$

Hence $\quad v_A / v_B = \lambda_A^* / \lambda_B^*$.

<u>Example</u> 3.2: Let $0 \leq t, s \leq 1$ be the pure strategy space of players I and II in a zero-sum two-person game with pay-off kernel

$$A(t,s) = \max\,(2t - 3t^2s^2 + 4, \quad 2s - 3t^2s^2 + 4)$$

Let $$B(t, s) = \int_0^1 \max\,(2t - 3t^2u^2 + 4, \quad 2u - 3t^2u^2 + 4) \max\,(2u - 3u^2s^2 + 4, \quad 2s - 3u^2s^2 + 4)du$$

be another pay-off kernel on the unit square for the two players.

Since $A(t, s)$ is symmetric and continuous in t and s and since

$$B(t, s) = \int A(t, u)\, A(u, s)\, d\,u$$

$A(t, s)$ defines the compact operator

$$A : f(t) \to \int A(t, s) f(s) d s$$

and $B(t, s)$ defines the compact operator

$$B : f(t) \to \int B(t, s) f(s) d s.$$

in $L_2 [0\ 1]$.

Further $B = A^2$. Therefore $A \longleftrightarrow B$ (i.e. A commutes with B).

Here $$\int_0^1 A(t, s) d s$$

$$= \int_0^t (2t - 3t^2 s^2 + 4) d s + \int_t^1 (2s - 3t^2 s^2 + 4) d s$$

$$= 2t^2 - 3t^2 . \frac{t^3}{3} + 4t + (1 - t^2) - t^2(1 - t^3) + 4(1 - t)$$

$$= 5.$$

Thus 5 is the eigenvalue with the eigenfunction $f^0 \equiv 1$. Further the Lebesgue measure is also optimal with value 5.

Evidently $A(t, s) > 0$.

Now since $B(t, s) = \int A(t, u)\, A(u, s)\, d\, u$

$\int B(t, s)\, d\, s = 25$. Since $B(t, s) = B(s, t)$ the value is also 25. Thus $v_A \,/\, v_B = \lambda^*_A \,/\, \lambda^*_B = 5$.

4. ON STOCHASTIC MATRICES AND KERNELS

4.a. Preliminaries:

It is well known that every stochastic matrix has 1 as an eigenvalue with (1, 1,..., 1) as a corresponding eigenvector. Further a stochastic matrix as a linear transformation leaves the positive orthant invariant, of which (1, 1,..., 1) is an interior point. Further, the positive orthant K has the additional property that for any pair $x, y \in K$, $\min(x, y) = z$ with $(z)_i = \min(x_i, y_i)$ for the i^{th} co-ordinate and $z \in K$. Similarly $\max(x,y) \in K$. Motivated by this we give a co-ordinate free characterisation of stochastic matrices, and this helps us to study them as just linear transformations on a real finite dimensional vector space. We also obtain the following extension of this result to the infinite dimensional case. If T is a positive operator positive with respect to a lattice normal cone K with interior in a real Banach space having a fixed vector in the interior of K, then T is essentially a stochastic operator (For a precise statement see page 72).

Definition: A cone K in a real Banach space E is a lattice cone, when $x, y \in K$ implies $\sup(x, y)$, $\inf(x, y) \in K$ where sup, inf are with respect to the partial ordering $\leq$ induced by K on E.

Remark: If K is a lattice with interior $\sup(x, y)$, $\inf(x, y)$ exists for all $x, y \in E$. Now we shall state the following propositions of Yudin and Krein which will be used in the sequel.

Theorem 4.1. (Yudin [37]): If K is a lattice cone with interior in a real finite (n) dimensional vector space E, then there exists a basis $e_1, e_2, \ldots, e_n$ in K with

$$K = \left\{ x : \qquad x = \sum_{1}^{n} \xi_i e_i \qquad \xi_i \geq 0 \quad i=1,2,\ldots,n \right\}$$

Theorem 4.2 (Krein-Kakutani [18]: If K is a lattice normal cone with interior in a real Banach space E, then there exists a one-one bicontinuous isometric lattice isomorphism of E onto $C(S)$ where $C(S)$ is the space of continuous functions on some compact Hausdorff space.

4.b. <u>On co-ordinate free characterisation of stochastic matrices:</u>

<u>Theorem</u> 4.3: Let K be a lattice cone with interior in a real finite (n) dimensional vector space. Let $AK \subseteq K$ for a linear transformation A with $Ax = x$ for some x in the interior of K. Then there exists a basis $e_1, e_2, \ldots, e_n$ with respect to which A has a matrix representation which is stochastic.

<u>Proof:</u> Since K satisfies conditions of theorem 1.2 1 is the eigenvalue maximal in modulus. Further by theorem 4.1, there exists a basis $e_1', e_2', \ldots, e_n'$ in K with

$$K = \left\{ y: \quad y = \Sigma\, y_i\, e_i' \;;\quad y_i \geq 0 \quad i = 1, 2, \ldots, n \right\}.$$

Since $x = \Sigma\, x_i\, e_i'$ is an interior point of K, $x_i > 0$ for each i.

If, $A\, e_j' = \Sigma\, a_{ij}'\, e_i' \qquad j = 1,2,\ldots, n$

then since $AK \subseteq K$, $a_{ij}' \geq 0$ for $i, j = 1,2,\ldots, n$. Consider the new basis $e_1, e_2, \ldots, e_n$ defined by

$$e_j = x_j \, e'_j \qquad j = 1,2,\ldots, n.$$

Let $A\, e_j = \sum_i a_{ij}\, e_i \qquad j = 1,2,\ldots, n$

i.e. $A\, x_j\, e'_j = \Sigma\, a_{ij}\, x_i\, e'_i$.

Therefore $A\, e'_j = \sum_j a_{ij} \dfrac{x_i}{x_j} \cdot e'_i$ for $j = 1,2,\ldots, n$.

But we have $A\, e'_j = \Sigma\, a_{ij}\, e'_i$.

Therefore, $a_{ij} = a'_{ij} \dfrac{x_i}{x_j}$ for $i, j = 1,2,\ldots, n$.

Thus $a_{ij} \geq 0$. Since $Ax = x$, $\Sigma_j\, a'_{ij}\, x_j = x_i$ so that

$\sum_j a_{ij} = \dfrac{1}{x_i} \sum_j a'_{ij}\, x_j = 1$ for $i = 1,2,\ldots, n$. Thus the transformation A corresponds to the stochastic matrix (a_{ij}) in the basis $e_1, e_2,\ldots, e_n$.

Similarly for the doubly stochastic case we have the following.

4.c. A characterisation of doubly stochastic matrices:

Theorem 4.4 : *A necessary and sufficient condition that the matrix A of theorem 4.3 be doubly stochastic* (i.e. $\sum_i a_{ij} = \sum_j a_{ij} = 1$) is that $A^*\sigma(x) = \sigma(x)$ where σ is the canonical isomorphism of V onto V^* corresponding to the basis $e_1, e_2, \ldots, e_n$.

<u>Proof</u>: (Sufficiency). Since $x = \sum_i e_i$, $\sigma(x) = \sum f_i$ where $f_1, f_2 \ldots f_n$ is the basis in V^* dual to $e_1\ e_2 \ldots e_n$.

Also $$A^* f_j = \sum_i a_{ji} f_i .$$

Thus $A^* \sigma(x) = \sigma(x)$ gives $\sum_j a_{ji} = 1$ for $i = 1,2,\ldots, n$ i.e. the matrix (a_{ij}) is doubly stochastic. This completes the proof of the theorem.

4.d. <u>Stochastic operators and permutation operators</u>:

A stochastic matrix may be thought of as associating a probability measure on every point of a finite set. This leads us to the following definition of stochastic kernels. We restrict ourselves to compact Hausdorff spaces.

Let S be a compact Hausdorff space, B the class of Borel subsets of S, $C(S)$ the space of real valued continuous functions on S and $C^+(S)$ the cone of non-negative functions in $C(S)$. Recall that $C^+(S)$ is a lattice normal cone with interior.

<u>Definition</u> 4: A function $K(s, E)$ defined on $S \times \mathcal{B}$ is a stochastic kernel if $K(s, \cdot)$ is a regular probability measure on $(S, \mathcal{B})$ for each $s \in S$, such that for every f in $C(S)$, the function g defined by

$$g(S) = \int_S f(t) \ K(s, d\,t)$$

is in $C(S)$.

Observe that our definition reduces to the usual definition (see e.g. [23]) if there exists a measure μ on $(S, \mathcal{B})$ such that $K(s, \cdot) << \mu$ for every $s \in S$.

It is easily seen that the linear transformation T on $C(S)$ (to be called a stochastic operator) defined by

$$(Tf)(s) = \int_S f(t) \, k(s, dt)$$

is bounded and has norm 1. In fact, the spectral radius of T is 1. In case $K(s, E)$ is measurable in s for every fixed $E \in \mathcal{B}$ it can be considered to be the transition function of a Markov process. The corresponding operator on $C(S)$ has been studied in detail by Rosenblatt.

It is interesting to note that with our definition, the identity operator is a stochastic operator and corresponds to the stochastic kernel $K(s, \cdot) = \delta_s(\cdot)$ where $\delta_s(\cdot)$ is the Dirac measure at s defined by

$$\delta_s(E) = 1 \qquad \text{if} \quad s \in E$$
$$= 0 \qquad \text{if} \quad s \notin E$$

Further we may define a large class of stochastic operators which are analogous to the permutation matrices in finite dimensions. The motivation for this definition is the observation that the effect of a permutation matrix on a vector is only to permute the components of a vector.

<u>Definition:</u> An operator P on $C(S)$ is called a <u>permutation</u> if there exists a homeomorphism π of S onto S such that

$$(Pf)(s) = f(\pi^{-1}(s)).$$

Clearly for all f in $C(S)$

$$(Pf)(s) = f(\pi^{-1}(s)) = \int_S f(t)K(s,dt)$$

where $K(s.,) = \partial_{\pi^{-1}(s)}(.)$. i.e., P is a stochastic operator with kernel $\partial_{\pi^{-1}(s)}$.

Every stochastic operator leaves the lattice normal cone $C^+(S)$ invariant and has a fixed vector in its interior, namely the function $f(s) \equiv 1$. The following theorem goes in the opposite direction.

<u>Theorem</u> 4.5 : Let K be a normal lattice cone with interior in a real Banach space E. If, T is a bounded positive linear operator on E (i.e., $TK \subseteq K$) and $Tx = x$ for some x in the interior of K, then there exists a compact Hausdorff space S and a bicontinuous isomorphism σ of E onto $C(S)$ such that T is carried to a stochastic operator by σ.

<u>Proof:</u> By the well-known theorem of Kakutani [18], (Theorem there exists a compact Hausdorff space and an isometric lattice isomorphism $\mathcal{T}$ of E onto $C(S)$ such that

$\mathcal{T}(K) = C^+(S)$.

$$\text{If } T_0 = \mathcal{T} T \mathcal{T}^{-1},$$

it is clear that T_0 is a bounded linear operator on $C(S)$ with $T_0\, C^+(S) \subseteq C^+(S)$ and $T_0\, f_0 = f_0$ where $f_0 (= \mathcal{T}(x))$ lies in the interior of $C^+(S)$. Hence $f_0 > 0$. If $\phi_s(f) = (T_0 f)(s)$, then for fixed s, $\phi_s(f)$ is a bounded linear functional on $C(S)$ which is non-negative by the positivity of T_0.

Hence we have, by Riesz's theorem, [10]

$$(T_0 f)(s) = \int f(t)\, k(s, dt) \ , \quad f \in C(S)$$

where, for every fixed s, $K(s.,)$ is a finite positive regular measure on $(S, \mathcal{B})$.

Put $K_1(s, E) = \frac{1}{f_0(s)} \int_E f_0(t)\, K(s, dt)$

Clearly $K_1(s, E) \geq 0$ for all $s \in S$, $E \in \mathcal{B}$.

Further, $K_1(s, S) = \frac{1}{f_0(s)} \int_S f_0(t)\, K(s, dt)$

$$= \frac{1}{f_0(s)} (T_0\, f_0)(s)$$

$$= 1.$$

i.e., $K_1(s.,)$ is a probability measure for fixed s. $K_1(s. ,)$ is regular because $K_1(s,.) << K(s.)$ and $K(s.)$ is regular by Riesz's theorem.

Also for f in $C(S)$,

$$\int_S f(t)\, K_1(s,dt) = \int_S f(t)\, \frac{f_0(t)}{f_0(s)} \cdot K(s, dt)$$

$$= \frac{1}{f_0(s)} (T_0(f, f_0))(s) \quad \text{which is in } C(S).$$

Thus $K_1(s,\cdot)$ is a stochastic kernel.

Put $(T_1 f)(s) = \int_S f(t)\, K_1(s, dt)$

The map $\tau' : C(S) \to C(S)$ defined by

$$\tau'(f) = \frac{1}{f_0} \cdot f$$

is a bicontinuous isomorphism such that

$$(T_1 f)(s) = \int_S f(t)\, K_1(s,dt)$$

$$= \frac{1}{f_0(s)} \int_S f(t)\, f_0(t)\, K(s,dt)$$

$$= \frac{1}{f_0(s)} \int_S (\tau^{-1} f)(t)\, K(s, dt)$$

$$= (\tau' T_0 \tau'^{-1} f)(s)$$

$$= (\tau' \tau\, T\, \tau^{-1} \tau'^{-1} f)(s)$$

Putting now $\sigma = \tau' \tau$ we see that σ has the required properties and the theorem is proved.

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