STUDIES IN BOREL STRUCTURES

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A thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements for the
degree of Doctor of Philosophy

Calcutta
1969

ACKNOWLEDGEMENTS

I am grateful to Professor C.R.Rao, F.R.S., Director of the Research and Training School of the Indian Statistical Institute for extending to me all the facilities to carry out the research work and for his constant encouragement. I am grateful to Professor J.K.Ghosh for supervising this work.

My heartfelt and sincere thanks are to Prof. J.K.Ghosh and Prof. Ashok Maitra for the time they have devoted to me - in the many useful discussions; in carefully and patiently going through the entire manuscript at its various stages; in offering valuable advice and in encouraging to pursue the problems treated in this thesis. How much I am indebted to these two persons is known, perhaps, only to the three of us!

I take this opportunity to thank all my colleagues with whom I had useful discussions.

Finally, it is a pleasure to thank Mr. Gour Mohon Das for his neat, careful and quick typing of this thesis.

Dated 23 September 1969
Calcutta.

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Page

CHAPTER 3: ON ANALYTIC AND BOREL STRUCTURES

INTRODUCTION

This thesis consists of three chapters. The first chapter 'On Analytic structures' has as its main object of study the σ-algebra generated by analytic sets on the unit interval. In the same chapter we discuss the problem about the product of discrete spaces being discrete and draw many interesting conclusions. The second chapter 'On Borel structures' has for its main object of study the Borel fields on a set. Much light is thrown in this chapter on the existence of certain special types of Borel structures. The third chapter 'On Analytic and Borel structures' lists - with discussions wherever possible - a few problems raised in or connected with the first two chapters. Since each chapter has its own summary at the beginning, we shall not give a detailed chapter-wise summary here. Instead we shall spend some time on saying a few words about the mathematical objects investigated in this thesis.

The theory of analytic sets was started by two Russian mathematicians N. Lusin and M. Souslin in the year 1917, the idea being suggested to Souslin after he discovered a false argument by H. Lebesgue in proving a correct theorem. It was developed, among others, by the two famous Polish mathematicians C. Kuratowski and W. Sierpiński and brought to the attention of the modern mathematical public by three different persons in three different areas (as far as the author is aware). First is D. Blackwell, who showed the importance of these sets in avoiding certain inherent pathologies in Kolmogorov's foundations of the Probability Theory (of course, it is now known that more general sets can be used for that purpose) as well as in dynamic programming problems. Second is G. W. Mackey who got into these sets in problems regarding group representations and in particular in defining topology on measurable groups. Third is G. Choquet who observed the uses of these sets (better to say - niceness of these sets) in Potential Theory. These three works - all appearing between 1954-56 - activated the study of analytic sets in current times.

A set with a nice and neat mathematical structure is always of interest in many ways. Just as for a nonempty set a family of sub-sets closed under the formation of arbitrary unions and finite intersections, containing empty set and whole-space is called a Topology; a set with a nonvoid family of subsets closed under the formation of countable unions and complementation is called a Borel space and this structure on the set is called a Borel structure - rather than the customary term 'Measurable space' which sounds as if some measures are involved. This fashionable nomenclature is due to G.W.Mackey, who was perhaps the first to realize that a systematic theory of Borel spaces can (and in fact, must) be developed analogous to the theory of Topological spaces. Borel spaces - as one knows - play a fundamental rôle in many areas, either directly or indirectly. For instance; all the probabilities of a probabilist live on such spaces, all the transformations of interest for an ergodic theorist travel in such spaces, and for a functional analyst most of his concrete houses (like L^p-spaces) are built through measures on such spaces and so on. Thus though one can not deny the importance of Borel spaces, it is quite surprising that the theory of these spaces is far less developed when compared to its cousin - the theory of topological spaces. Two reasons may be stated for this state of affairs. Firstly, in most of the areas what one needs is the theories built on Borel spaces rather than the Borel spaces themselves. This, as the author feels, is not completely true.

As a matter of fact in almost all game theoretic and dynamic programming problems one needs measurable strategies - which of course brings in the Borel structures onto the stage. In fact one does not need to go so far. Look at the beautiful and powerful theorem of Sierpiński. 'Any Lebesgue measurable convex function on the real line is continuous'. Secondly, perhaps, many persons feel that these spaces are just special cases of the more sophisticated (interesting and useful too) and much developed theory - namely Boolean Algebras. We do not share these feelings. We hope that chapter 2 of this thesis will convince others that there are many interesting questions in Borel spaces some of which can not be formulated for Boolean Algebras and some others even if formulated, are not much informative.

Finally chapter 3 suggests some lines along which the Borel space theory needs immediate attention. Let us hope that in the next few years this theory builds a house of its own.

CHAPTER 1

ON ANALYTIC STRUCTURES

§ 0. Summary.

This chapter consists of eight sections. In section 1, we fix some of the notation to be used in this chapter - especially for sections 2 and 3. In section 2, we show that the σ- algebra on the unit interval generated by its analytic sets is not countably generated; from which we deduce that the product of this σ- algebra with itself does not pick up all analytic subsets of the unit square. In section 3, we show - under the continuum hypothesis - that the product of discrete σ- algebra on the unit interval with itself is the discrete σ- algebra of the square. From this we deduce a number of interesting consequences including the fact that there is a separable σ- algebra on the unit interval containing its analytic sets - in fact, one containing all its projective sets. In section 4, after introducing some notation, we recall the axiom of determinateness along with some of its interesting consequences; thus paving the way for partial generalizations of and for partial contradictions to, our

previous results. In section 5, we generalize partially, the results of sections 2 and 3; whereas in section 6 we use some of the known results in conjunction with the axiom of determinateness. In this later section we also observe that the results of section 2 combined with the axiom of determinateness contradict some of the results of section 3. In section 7, we shall put the results of sections 2 and 6 in an abstract setting involving only σ-algebras but not topologies. Finally in section 8, we recall the histories of the problems treated in this chapter.

§ 1. Notation.

I denotes the closed unit interval [0, 1]. $\underline{\underline{B}}$, $\underline{\underline{A}}$, $\underline{\underline{L}}$, $\underline{\underline{C}}$ denote the Borel σ-algebra; the σ-algebra generated by analytic sets; the σ-algebra of Lebesgue measurable sets; and the σ-algebra consisting of the class of all subsets respectively on I. For most of the purposes, $\underline{\underline{L}}$ could consist of subsets of I measurable w.r.t. a fixed non-atomic probability measure (for that matter, even a σ-finite measure) on $\underline{\underline{B}}$. That this will suffice is of importance in some considerations in sections 4, 5 and 6. A symbol like $\underline{\underline{C}} \times \underline{\underline{L}}$ stands for the usual product σ-algebra on the unit square generated by rectangles of the form X × Y; X in $\underline{\underline{C}}$ and Y in $\underline{\underline{L}}$.

U denotes any analytic subset of $I \times I$ universal w.r.t. the analytic subsets of I. For definiteness, we suppose that the vertical sections of U give all analytic subsets of I.

Recall that a subset of I is said to have the Baire property if it is open modulo sets of first category (Kuratowski [2], page 87). Note that the collection of all sets having the Baire property is a σ-algebra (ibid, page 88). We shall denote this σ-algebra by $\underline{\underline{Q}}$. Since $\underline{\underline{Q}}$ is closed under Souslin operation (ibid, page 94). We have $\underline{\underline{A}} \subset \underline{\underline{Q}}$.

Let $\underline{\underline{F}}$ be any class of subsets of I. The projections to I of sets of the σ-algebra on $I \times I$ over the rectangles with sides in $\underline{\underline{F}}$ are called $\underline{\underline{F}}$ - analytic sets or sets of the first projective class $\underline{\underline{P}}_1$ over $\underline{\underline{F}}$. Let Ω denote the first uncountable ordinal. Having defined $\underline{\underline{P}}_\alpha$ for $\alpha < \gamma < \Omega$; we define $\underline{\underline{P}}_\gamma$ as the projections on I of the sets of the σ-algebra on $I \times I$ over the rectangles with sides in the previous projective classes. The collection of all these classes $\underline{\underline{P}}_\alpha$, $\alpha < \Omega$; is called the generalized projective class $G(\underline{\underline{F}})$ over $\underline{\underline{F}}$. Clearly one need not proceed after the first uncountable ordinal. That is, if we take the projections on I of the sets of the σ-algebra on $I \times I$ over the rectangles with sides in $G(\underline{\underline{F}})$ we get again elements of $G(\underline{\underline{F}})$.

For the theory of classical analytic sets - especially for the Borel isomorphism theorem and the existence of universal analytic sets, to be used later - we refer the reader to the classical treatises of Kuratowski ([1] and [2]) and Sierpiński [4]. For a short discussion of the generalized analytic sets see Ulam [3] and also sections 4 and 5 of this chapter. In few places in this chapter we make use of the Marczewski function. This is defined at those places. For a full treatment, see Marczewski ([1] and [2]) and for a short account see also chapter 2, Section 1.

UNLESS EXPLICITLY STATED TO THE CONTRARY (WHICH WILL BE DONE IN SECTION 6) WE MAKE USE OF THE AXIOM OF CHOICE IN THIS CHAPTER.

'c' denotes the cardinality of the real number system.

UNLESS EXPLICITLY STATED (WHICH WILL BE DONE IN SECTION 3) CONTINUUM HYPOTHESIS IS NOT ASSUMED.

For typographical convenience we use N for the Hebrew letter Aleph.

§ 2. σ-algebra generated by analytic sets.

Let $\underline{\underline{E}}$ be any σ-algebra on I such that $\underline{\underline{A}} \subset \underline{\underline{E}} \subset \underline{\underline{L}}$. In the first part of this section we shall prove the following theorems:

Theorem 1: $\underline{\underline{E}}$ is not countably generated.

Theorem 2: $U \not\in \underline{\underline{C}} \times \underline{\underline{L}}$.

Before proving these theorems we make some remarks. There is no general way of showing that a σ-algebra is not countably generated. The first method available in the literature is a simple cardinality argument. For instance if cardinality of $\underline{\underline{E}}$ is greater than c then $\underline{\underline{E}}$ is not countably generated, though not conversely. This argument obviously fails here because the cardinality of $\underline{\underline{E}}$ can be c. In fact there are exactly 2^c distinct σ-algebras between $\underline{\underline{A}}$ and $\underline{\underline{L}}$ each with cardinality c. [Reason: Since $(2^c)^c = 2^c$ (Sierpiński [5], page 140), there are at most 2^c such σ-algebras. Fix c - many pairwise disjoint Borel sets B_α from I, each with Lebesgue measure zero, and uncountable. Inside each B_α, fix a non - $\underline{\underline{A}}$ set $Z_\alpha \in \underline{\underline{L}}$. Extend $\underline{\underline{A}}$ by adding any number of Z_α's. Thus we get 2^c distinct σ-algebras]. The second method is to exhibit a probability measure on $\underline{\underline{E}}$, giving zero mass to singletons and taking only two values zero and one. Because if there is a two valued probability measure on a countably generated σ-algebra then it is concentrated on an atom. Such a measure does not exist here because probability measures on $\underline{\underline{E}}$ give rise to corresponding probability measures (restrictions) on $\underline{\underline{B}}$ and $\underline{\underline{B}}$ is countably generated.

Proof of Theorem 1: Suppose $\underline{\underline{E}}$ has a countable generator say $(A_n ; n \geq 1)$. Consider the Marczewski function f on I defined by

$$f(x) = \Sigma \frac{2 \chi_{A_i}(x)}{3^i}$$

with range say $X \subset I$. If $\underline{\underline{B}}_X$ is the relativized Borel σ-algebra on X then f is a Borel - isomorphism of $(I, \underline{\underline{E}})$ onto $(X, \underline{\underline{B}}_X)$. Since the Lebesgue measure λ on $(I, \underline{\underline{E}})$ is compact (Marczewski [3]) and hence perfect (Ryll-Nardzewski [1]), there is a Borel subset B of I with

$$B \subset X ; \quad \lambda (f^{-1}B) = 1.$$

Denoting by Y the set $f^{-1}B$; by $\underline{\underline{E}}_Y$ the σ-algebra $\underline{\underline{E}}$ on I restricted to Y; by f_1 the map f restricted to Y; we observe that f_1 is a Borel isomorphism on $(Y, \underline{\underline{E}}_Y)$ onto

$(B, \underline{\underline{B}}_B)$. If C is a Borel subset of I, contained in X then the map f^{-1} restricted to C being one-one and Borel we have in view of a famous theorem of Lusin (Kuratowski [2], page 489) that $f^{-1}(C)$ is not only in $\underline{\underline{E}}$ but is a Borel subset of I. Thus in effect f_1^{-1} of every set in $\underline{\underline{B}}_B$ is a Borel subset of Y and that in particular Y itself is a Borel subset of I. Since $\lambda(Y) = 1$, Y is uncountable and hence contains non-Borel analytic sets (Kuratowski [2], page 460). Thus there are sets in $\underline{\underline{E}}_Y$ which are not of the form $f_1^{-1}(C)$ for any C in $\underline{\underline{B}}_B$. This contradicts the fact that f_1 is an isomorphism on Y onto B. This completes the proof of Theorem 1.

<u>Proof of Theorem 2:</u> If $U \in \underline{\underline{C}} \times \underline{\underline{L}}$; then obviously there exist a countable number of rectangles $(E_n \times F_n;\ n \geq 1)$ such that U is in the σ-algebra on $I \times I$ generated by these rectangles. Define $\underline{\underline{E}}$ to be the σ-algebra on I generated by $(F_n;\ n \geq 1)$. Since each of the rectangles $E_n \times F_n$ belongs to $\underline{\underline{C}} \times \underline{\underline{E}}$; we have $U \in \underline{\underline{C}} \times \underline{\underline{E}}$. Since each $F_n \in \underline{\underline{L}}$, we have $\underline{\underline{E}} \subset L$. Since any analytic subset of I is an x - section for some point x, and since $U \in \underline{\underline{C}} \times \underline{\underline{E}}$; we have every x- section of U, that is, every analytic subset of I in $\underline{\underline{E}}$. Consequently $\underline{\underline{A}} \subset \underline{\underline{E}}$. By definition $\underline{\underline{E}}$ is countably generated. Since we have already observed that $\underline{\underline{A}} \subset \underline{\underline{E}} \subset \underline{\underline{L}}$ we have here a contradiction to Theorem 1, to complete the proof of Theorem 2.

Of course a dual to Theorem 2 can also be stated: $V \notin \underline{\underline{L}} \times \underline{\underline{C}}$, where V is a subset of the square whose horizontal sections give all analytic subsets of I. It follows in particular that U (and V) $\notin \underline{\underline{A}} \times \underline{\underline{A}}$. We show in the next section that $\underline{\underline{L}}$ can not be replaced by $\underline{\underline{C}}$ in the above Theorems without additional axioms.

We conclude this section by proving two theorems which are similar to the above two theorems.

<u>Theorem 3:</u> If $\underline{\underline{E}}$ is any σ-algebra on I and $\underline{\underline{A}} \subset \underline{\underline{E}} \subset \underline{\underline{O}}$; then $\underline{\underline{E}}$ is not countably generated.

<u>Theorem 4:</u> $U \notin \underline{\underline{O}} \times \underline{\underline{O}}$.

Before proving Theorem 3 we shall make a few remarks. Observe that there are Lebesgue measurable sets of measure one which are of the first category. Consequently from the well-known results (Kuratowski [2], page 91 and also section 40) it follows that there are non-Lebesgue measurable sets with the Baire property and Lebesgue measurable sets without the Baire property. In other words neither $\underline{\underline{O}}$ is contained in $\underline{\underline{L}}$, nor $\underline{\underline{L}}$ is contained in $\underline{\underline{O}}$. Thus Theorem 3 neither implies nor is implied by Theorem 1. Well, if one assumes the axiom of determinateness, the situation is entirely different to which we return in a later section. Theorems 1 and 3 - whose statements and proofs are similar - will convince the reader once again as to how deep (unexpectedly!) the remark of Kuratowski ([2], page 87) goes: 'The role played by the Baire property in topology is analogous to that of measurability (of sets or functions) in analysis'. It is worth noting that the proof of Theorem 3, unlike that of Theorem 1, does not depend on measure theoretic considerations.

<u>Proof of Theorem 3:</u> If there is such a σ-algebra $\underline{\underline{E}}$ with a countable generator say $(A_n ; n \geq 1)$ then consider the Marczewski function f defined by

$$f(x) = \Sigma \frac{2 \chi_{A_n}(x)}{3^n}$$

with range say X a subset of I. Recall that f is a Borel isomorphism of $(I, \underline{\underline{E}})$ onto $(X, \underline{\underline{B}}_X)$. Since $\underline{\underline{E}} \subset \underline{\underline{O}}$, the function f has the Baire property' (Kuratowski [2], page 399). Recall that f has Baire property if for any Borel set Z, $f^{-1}(Z)$ has Baire property, that is, it differs from an open set by a set of the first category. In view of a famous theorem of Baire (Kuratowski [2], page 400) there is a set of the first category P in I such that f restricted to $I - P$ is continuous. By enlarging P if necessary, we can suppose that P is an F_σ and hence that $I - P$ is a G_δ (Borel is enough for our purpose). Denoting $I - P$ by Y; $\underline{\underline{E}}$ restricted to Y by $\underline{\underline{E}}_Y$; f restricted to Y by f_1; the set $f(Y) \subset X$ by B and its relativized Borel algebra by $\underline{\underline{B}}_B$; observe that f_1 is an isomorphism of $(Y, \underline{\underline{E}}_Y)$ onto $(B, \underline{\underline{B}}_B)$. Since f_1 is a one to one continuous map on the G_δ set Y, we have B to be a Borel subset of I (Kuratowski [2], page 487). Observe that if C is a Borel subset of I and is contained in X the map f^{-1} restricted to C being one one and Borel we have in view of a famous theorem of Souslin (Kuratowski [2], page 489) that $f^{-1}(C)$ to be Borel. Now it is sufficient to observe that there are analytic non-Borel sets in $\underline{\underline{E}}_Y$ where as every set in $\underline{\underline{B}}_B$ is Borel. This contradicts the fact that f_1 is an isomorphism and proves the Theorem.

Since the proof of Theorem 4 is similar to that of Theorem 2 we omit the proof.

One interesting feature of Theorem 3 is, as remarked in the beginning, that it shows that '$\underline{\underline{A}}$ is not countably generated' and '$U \notin \underline{\underline{A}} \times \underline{\underline{A}}$' without using the concept of measure.

In later sections we recall Theorem 1 and prove partial generalizations of it. In those situations it is not difficult to obtain analogues by using Theorem 3 also.

Though the proofs of Theorems 1 and 3 are apparently different; the reader will see in section 7, that the main idea is essentially the same and that these are trivial consequences of a much more general and abstract version.

§ 3. Discrete Borel spaces.

The question which we ask ourselves now is the following: Is the product with itself of discrete σ-algebra on a set X, the discrete σ-algebra on $X \times X$? When we say discrete, we mean the class of all subsets. The answer is trivially 'yes' if card $(X) \leq \aleph_0$. If card $(X) > c$, the answer is in the negative. This follows from a lemma. Since this lemma is itself interesting we elevate it to a theorem:

Theorem 5: Let $\underline{\underline{E}}$ be a σ-algebra on a set X. The diagonal D of $X \times X$ belongs to $\underline{\underline{E}} \times \underline{\underline{E}}$ if and only if there is a countably generated σ-algebra $\underline{\underline{E}}_0 \subset \underline{\underline{E}}$ with singletons as atoms.

Proof: Suppose $D \in \underline{\underline{E}} \times \underline{\underline{E}}$. Then there exist countable number of rectangles $(B_i \times C_i ; i \geq 1)$ such that D is in the σ-algebra generated by these rectangles. Let $\underline{\underline{E}}_0$ be the σ-algebra on X generated by $(B_i, C_i ; i \geq 1)$. Clearly $\underline{\underline{E}}_0$

is countably generated. Since each of the rectangles $B_i \times C_i$ is in $\underline{\underline{E}}_0 \times \underline{\underline{E}}_0$, $D \in \underline{\underline{E}}_0 \times \underline{\underline{E}}_0$ and hence every section of D - namely singletons - belong to $\underline{\underline{E}}_0$. Thus $\underline{\underline{E}}_0$ has the desired properties. The reader should note that this part of the proof resembles the proof of Theorem 2. (There we had taken only vertical sides of the rectangles where as here both sides are used). Arguments similar to these will be omitted in the next sections.

To prove the converse part of the theorem, suppose that there is a $\underline{\underline{E}}_0$ with the stated properties. We shall express D as a countable intersection of sets each being a finite union of rectangles with sides in $\underline{\underline{E}}_0$. In that case D belongs to $\underline{\underline{E}}_0 \times \underline{\underline{E}}_0$, and hence a fortiori to $\underline{\underline{E}} \times \underline{\underline{E}}$. To do this let $(B_i \,;\, i \geq 1)$ be a generator for $\underline{\underline{E}}_0$. For each $n \geq 1$ let $\underline{\underline{P}}_n$ be the partition of the space induced by $B_1, \ldots, B_n$. That is,

$$\underline{\underline{P}}_n = \left\{ C_1 \cap \cdots \cap C_n \mid C_i = B_i \text{ or } X - B_i \right\}.$$

It is not difficult to see that,

$$D = \bigcap_n \bigcup_{P \in \underline{\underline{P}}_n} P \times P$$

This completes the proof of the theorem.

It is interesting to observe that the converse part of the theorem can also be proved elegantly by using the Marczewski function. Since we need an argument similar to the above in proving Theorem 6, we have given that proof here.

Thus the problem posed in the beginning of this section remains to be answered in case X has cardinality c. Since the problem is one depending on cardinality and not on the particular set X in question, we can and shall take X to be I. Now we shall solve this problem, assuming the continuum hypothesis to be true.

<u>Theorem 6</u>: If <u>CH</u> is valid then $\underline{\underline{C}} \times \underline{\underline{C}} = \underline{\underline{C}}(I \times I)$ where $\underline{\underline{C}}\,(I \times I)$ is the discrete σ-algebra on $I \times I$.

<u>Proof</u>: First observe that if f is any function defined on a subset of I into I then its graph

$$G = \left\{ (x, y) : \begin{array}{l} x \quad \text{in domain of } f \\ f(x) = y \end{array} \right\}$$

is in $\underline{\underline{C}} \times \underline{\underline{C}}$. For this it suffices to verify that

$$G = \bigcap_{n=1}^{\infty} S_n$$

where, $S_n = \bigcup_{k=1}^{n} (A_{nk} \times B_{nk})$

$$A_{nk} = [x \text{ in domain } f : \frac{k-1}{n} \leq f(x) < \frac{k}{n}]$$

$$B_{nk} = [y \text{ in range } f : \frac{k-1}{n} \leq y < \frac{k}{n}]$$

(For $k = n$ include the right end point as well).

Since this verification is straight forward we omit the details here.

Secondly; if $B \subset I \times I$ be such that every vertical section is at most countable then $B \in \underline{\underline{C}} \times \underline{\underline{C}}$. This follows by realizing B as a countable union of graphs (axiom of choice!).

Thirdly; if $B \subset I \times I$ be such that every horizontal section is at most countable then also $B \in \underline{\underline{C}} \times \underline{\underline{C}}$.

Fourthly; $I \times I = X \cup Y$, where, every horizontal section of Y is at most countable and every vertical section of X is at most countable. This can be done by realizing I as the set of ordinals less than the first uncountable ordinal (CH!) and then taking the portions below and not below the diagonal (see also Sierpiński [6], page 376).

Finally, if $B \subset I \times I$ then by the previous observations, $B \cap X$ and $B \cap Y$ are in $\underline{\underline{C}} \times \underline{\underline{C}}$. This completes the proof of the theorem.

Let Z be any set of cardinality $\aleph_1$ the first uncountable cardinal. An obvious modification of the above proof yields us

Theorem 7: The product with itself of the discrete σ-algebra on Z is the discrete σ-algebra on $Z \times Z$. Consequently if $A \subset I \times I$ be such that card $(A) \leq \aleph_1$ then $A \in \underline{\underline{C}} \times \underline{\underline{C}}$.

Proof: The first part is a straight forward imitation of the previous proof after establishing a correspondence on Z in a one to one way onto a subset of I. For the second part observe that there is a set $A_0 \subset I$ of cardinality $\aleph_1$ such that $A \subset A_0 \times A_0$. (Take A_0 to be simply the union of the horizontal and vertical sections of A). Now apply the first part of the theorem to A_0, in place of Z.

Clearly, Theorem 6 is a consequence of Theorem 7 together with CH.

Theorem 8: Let $(Z_\alpha ; \alpha \in T)$ be any collection of subsets (possibly empty also) of Z, where card $(T) \leq \aleph_1$. Then there is a separable σ-algebra (countably generated and containing singletons) on Z containing the given collection.

Proof: There is no loss in taking $T = Z$ as we do.

Put

$$A = \bigcup_{\alpha \in Z} (\{\alpha\} \times Z_\alpha)$$

By Theorem 7, A is in the product of discrete σ-algebras on Z. Familiar arguments will now complete the proof.

As an immediate consequence of the above theorem we have

Theorem 9: Let CH be valid. Then there is a separable σ-algebra on I containing all its analytic subsets. In fact there is one such containing all its generalized projective sets $G(\underline{\underline{F}})$ over any fixed class $\underline{\underline{F}}$, provided card $(\underline{\underline{F}}) \leq c$.

Proof: Observe, by transfinite induction, that there are not more than c elements in each projective class $\underline{\underline{P}}_\alpha$, $\alpha < \Omega$. Consequently there are at most c many elements in $G(\underline{\underline{F}})$. Now the result is a consequence of Theorem 6 with the continuum Hypothesis.

In the terminology of Marczewski (Marczewski [1], page 222) we can state this theorem in an equivalent and interesting way:

<u>Theorem 10</u>: Let <u>CH</u> be valid. There is a one-one transformation $\emptyset$ of I into I, transforming each set analytic in I into a set Borel in $\emptyset(I)$. In fact if $\underline{\underline{F}}$ is any class of subsets of I with card $(\underline{\underline{F}}) \leq \mathfrak{a}$, then there is one such $\emptyset$ transforming each set in $G(\underline{\underline{F}})$ into a set Borel in $\emptyset(I)$.

<u>Proof</u>: Fix any generator $(A_n; n \geq 1)$ for any separable σ-algebra on I containing $G(\underline{\underline{F}})$. Look at the Marczewski function

$$\emptyset(x) = \Sigma \frac{2 \chi_{A_n}(x)}{3^n} .$$

To prove that this $\emptyset$ satisfies the requirements is a straight forward verification.

Observe that theorem 9 also shows that the $\underline{\underline{L}}$ in theorem 1 can not be replaced by $\underline{\underline{C}}$. Theorem 6 shows that $\underline{\underline{L}}$ in theorem 2 can not be replaced by $\underline{\underline{C}}$. It is interesting to note that at the end of Section 6, we arrive at conclusions exactly opposite to these.

<u>Theorem 11</u>: Let <u>CH</u> be valid. Fix any separable σ-algebra $\underline{\underline{A}}_0$ on I containing all the analytic subsets of I. For every non-atomic probability measure on $\underline{\underline{B}}$, there is at least one set in $\underline{\underline{A}}_0$ which is not measurable.

Proof: Obvious from Theorem 1. Suffices to see as remarked in section 2, that Theorem 1 remains true even if $\underline{\underline{L}}$ is the collection of sets measurable w.r.t. a fixed nonatomic probability measure on $\underline{\underline{B}}$ - not necessarily the Lebesgue measure.

Another interesting consequence of Theorem 8 is the following:

Theorem 12: There exists a separable σ-algebra on Z which supports no continuous probability measure.

Proof: Following Ulam [1], we shall first associate with each finite ordinal n and countable ordinal α a subset $K(n, \alpha)$ of Z satisfying the following conditions:

i) For fixed α; $\bigcup_n K(n, \alpha)$ is a co-countable subset of Z, and

ii) For fixed n; $[K(n, \alpha) : \alpha$ countable ordinal$]$ is a pairwise disjoint family.

To do this there is no loss in assuming Z to be the set of all ordinals less than the first uncountable ordinal Ω. For each $\alpha < \Omega$, fix any one to one map f_α on $[1, \alpha]$ into N, the set of natural numbers. Define

$$K(n, \alpha) = \left\{ \beta : f_\beta(\alpha) = n \right\}.$$

These sets satisfy our requirements. By using Theorem 8, we can find a separable σ-algebra $\underline{\underline{A}}_0$ on Z containing all these sets. If there were a continuous measure μ on $\underline{\underline{A}}_0$ then for each α - since $\bigcup_n K(n, \alpha)$ is cocountable - there is an n_α such that $\mu(K(n_\alpha, \alpha)) > 0$. Since there are countably many n's and uncountably many α's, there is an integer p such that $S = (\alpha : n_\alpha = p)$ is uncountable. In other words there are uncountable number $[K(p, \alpha): \alpha \in S]$ of pairwise disjoint sets each with positive measure. Thus μ can not be finite (not even σ-finite) to complete the proof.

If CH is valid, the above theorem says that on I, there is a separable σ-algebra which does not support a continuous probability measure. As is easy to see; if one wishes such a σ-algebra can be taken to contain all analytic subsets of I. It is interesting to observe that there is an alternative way of observing the existence of a separable σ-algebra on I, which does not support a continuous probability measure (of course again under CH). To demonstrate this take a universal null set N of I, that is, a set of cardinality c such that for any nonatomic probability measure μ on $\underline{\underline{B}}$ the μ-outer measure of N is zero. Such sets do exist under the continuum hypothesis. (Kuratowski [2], § 40). Let $\underline{\underline{B}}_N$

be the relativized Borel σ-algebra on N. Fix any one to one map f of I onto N. Then the σ-algebra $f^{-1}(\underline{\underline{B}}_N)$ on I does not support any continuous probability measure. For, if it did then $(N, \underline{\underline{B}}_N)$ also does, say μ. Then $\mu_1(B) = \mu(B \cap N)$ is a probability measure on $\underline{\underline{B}}$ giving outer measure 1 to N, contrary to the hypothesis that N is a universal null set.

We conclude this section with generalizations of Theorem 6. For any set Z, $\underline{\underline{C}}_Z$ stands for the class of all subsets of Z.

<u>Theorem 13</u>: Assume <u>CH</u>. $\underline{\underline{C}}_X \times \underline{\underline{C}}_Y = \underline{\underline{C}}_{X \times Y}$ iff one of the sets is countable or both have cardinalities $\leq c$. More generally

$$\underline{\underline{C}}_{X_1} \times \cdots \times \underline{\underline{C}}_{X_n} = \underline{\underline{C}}_{X_1 \times \cdots \times X_n}$$

iff all X_i have cardinality $\leq c$ or all but one X_i are countable.

<u>Proof</u>: We shall prove only the first sentence of the theorem. The second sentence can be observed along similar lines. Observe that if one of X and Y is countable then trivially $\underline{\underline{C}}_X \times \underline{\underline{C}}_Y = \underline{\underline{C}}_{X \times Y}$. If both X and Y are having cardinalities $\leq c$, then also the same equality holds - but now in view of

Theorem 6.

To prove the converse suppose both X and Y are uncountable. By CH

$$\text{Card } (X) \geq c \; ; \quad \text{Card } (Y) \geq c.$$

Also assume that cardinality of one of them is $> c$. More specifically let card $(X) > c$. **Fix** any cardinal α satisfying,

$$c < \alpha \leq \text{Card } (X) ; \quad \alpha \leq 2^c .$$

Choose α many distinct subsets of X say $(S_x ; x \in X_o)$ indexed by a subset $X_o \subset X$ (possible). Look at

$$S = \bigcup_{x \in X_o} \left(\{ x \} \times S_x \right)$$

claim that $S \notin \underline{\underline{C}}_X \times \underline{\underline{C}}_Y$. For otherwise we can find two countably generated σ-algebras $\underline{\underline{A}}_X$ and $\underline{\underline{A}}_Y$ on X and Y respectively such that S belongs to $\underline{\underline{A}}_X \times \underline{\underline{A}}_Y$. Consequently every x-section of S belongs to $\underline{\underline{A}}_Y$. Thus there is a countably generated σ-algebra on Y containing all the α many sets $(S_x ; x \in X_o)$ which is impossible. This proves the theorem.

Let $(X_\alpha ; \alpha \in T)$ be a collection of nonempty sets and X be their cartesian product, $\underset{\alpha \in T}{X} X_\alpha$. When we have σ-algebras on X_αs, and if we say product σ-algebra on X, we mean the usual σ-algebra on X generated by the coordinate maps or equivalently the σ-algebra generated by finite dimensional measurable cylinders (also called in probability theory the Daniel-Kolmogorov σ-field). Regarding these spaces we have the following consequence of the above theorem.

Theorem 14: Assume CH. $\underline{\underline{C}}_X = \underset{\alpha \in T}{X} \underline{\underline{C}}_{X_\alpha}$ iff there is a finite subset F of T such that

i) $\alpha \notin F$ implies X_α is a singleton, and

ii) either card $(X_\alpha) \leq c$ for all α in F or card $(X_\alpha) \leq \aleph_0$ for all but one α in F.

Proof: To prove the if part, observe that the uncountable product, from (i), is essentially a finite product and hence, from (ii) and the previous theorem, our result follows.

To prove the only if part it suffices to exhibit a finite set $F \subset T$ satisfying (i). For, then the uncountable product is essentially a finite product and (ii) follows from the previous theorem. To this end suppose there are infinitely many X_α s which are not singletons. By the axiom of choice, choose a countably infinite set $S \subset T$ such that for all α in S, X_α has at least two points. Choose a doubleton Y_α contained in X_α for α in S and a singleton Y_α contained in X_α for α not in S. Then clearly $Y = \underset{\alpha\in}{X} Y_\alpha \subset X$. It is trivial to see that Y equipped with the relativized product σ-algebra is isomorphic to $\{0, 1\}^\omega$ equipped with the product σ-algebra. Since the latter is not a discrete space, neither is the former. Hence the product structure on X is not discrete. This contradiction shows that all but a finite numbers of X_α s are singletons. This proves the theorem in view of the remarks made at the beginning of this para.

One might be tempted to say that when we are dealing with discrete spaces it is unreasonable to look at the usual definition of product σ-algebras. One might say that the natural (for our questions) σ-algebra on X is that generated by sets of the form $\underset{\alpha\in}{X} B_\alpha$ where $B_\alpha \subset X_\alpha$. Even if one defines $\underset{\alpha\in}{X} \underline{\underline{C}}_{X_\alpha}$ to be the σ-algebra generated by such rectangles, we can prove the above theorem (Proof is also exactly the same). Leaving the details of this remark to the reader, we shall proceed to generalize theorem 6 in a different direction.

Let us say that a nonempty class $\underline{\underline{B}}$ of subsets of X is an $\aleph$ - algebra if, $\underline{\underline{B}}$ is closed under complementation and unions of $\aleph$-many sets (Here $\aleph$ is an infinite cardinal number). Thus σ-algebras are just $\aleph_0$ -algebras. If X and Y are two sets then $\underline{\underline{C}}_X \underset{\aleph}{X} \underline{\underline{C}}_Y$ denotes the smallest $\aleph$ algebra on $X \times Y$ generated by rectangles. With these concepts we have,

<u>Theorem 15:</u> Card $(X) \leq \aleph_{\alpha+1}$ implies $\underline{\underline{C}}_X \underset{\aleph_\alpha}{X} \underline{\underline{C}}_X = \underline{\underline{C}}_{X X X}$

(Here α is an ordinal number and as usual $\aleph_{\alpha+1}$ is the smallest cardinal number greater than $\aleph_\alpha$. All are infinite cardinal numbers).

<u>Proof:</u> This theorem is a direct generalization of Theorem 7 and proof is also similar. We shall just clarify an issue involved.

First take the space $Y = 2^Z$ where Card $(Z) = \aleph_\alpha$. We shall show that if f is any function on a subset of Y into Y then its graph

$$G = \left\{ (x, f(x)) : x \text{ in domain } f \right\}$$

is in $\underline{\underline{C}}_Y \underset{\aleph_\alpha}{X} \underline{\underline{C}}_Y$. To observe this, put A_z to be the set of all those points in Y whose z^{th} coordinate is 1 and which belongs to the range of f, and B_z the complement of A_z in the range of f. Put

$$G_z = [f^{-1}(A_z) \times A_z] \cup [f^{-1}(B_z) \times B_z] .$$

Each G_z contains G and belongs to $\underline{\underline{C}}_Y \underset{\aleph_\alpha}{X} \underline{\underline{C}}_Y$. Since we have $\aleph_\alpha$ many G_z s

$$G = \bigcap_{z \in Z} G_z \in \underline{\underline{C}}_Y \underset{\aleph_\alpha}{X} \underline{\underline{C}}_Y .$$

From the above para it follows that if Card $(X) \leq \aleph_{\alpha+1}$ and if f is any function defined on a subset of X into X then its graph belongs to $\underline{\underline{C}}_X \underset{\aleph_\alpha}{X} \underline{\underline{C}}_X$. From here onwards the proof of this theorem is similar to that of Theorem 6.

Analogues of Theorems 13 and 14 can also be formulated for $\aleph$ -algebras. Analogue of Theorem 6 can also be stated for $\aleph$ - algebras by assuming the generalized continuum hypothesis in view of the above Theorem. We shall leave these Corollaries to the imagination of the reader.

§ 4. An axiom.

Recall that if $\underline{\underline{F}}$ is a class of subsets of I, then the projections to I of sets in the σ-algebra on $I \times I$ generated by the rectangles with sides in $\underline{\underline{F}}$ are called $\underline{\underline{F}}$-analytic sets. Extending this idea, we define $\underline{\underline{F}}$-analytic sets of $I \times I$ to be the projections to $I \times I$ of sets in the σ-algebra on $I \times I \times I$ generated by cubes with sides in $\underline{\underline{F}}$. In what follows, we take $\underline{\underline{F}}$ to be a countable family, say $(A_n ; n \geq 1)$. Clearly, when $\underline{\underline{F}}$ happens to be the sequence of intervals with rational end points, we get the $\underline{\underline{F}}$-analytic sets to be usual analytic sets - sometimes referred to as standard analytic sets. Let f be the Marczewski function on I to I defined by

$$f(x) = \Sigma \frac{2 \chi_{A_n}(x)}{3^n}$$

The range of f will be denoted by R and its relativized Borel σ-algebra by $\underline{\underline{R}}$. Unless otherwise stated we shall fix hereafter a family $\underline{\underline{F}}$; so that the concepts depending on $\underline{\underline{F}}$ need not always be prefixed by $\underline{\underline{F}}$. We shall introduce the assumption

(H) : R contains a perfect set, that is, a homeomorph of Cantor set.

When this hypothesis is satisfied, we shall denote by P any typical perfect set in R and $Q = f^{-1}(P)$. We shall also denote, for convenience, by $\sigma \underline{\underline{F}}$ the σ-algebra generated by $\underline{\underline{F}}$ on I. We also define $\bar{f}$ from $I \times I$ to $R \times R$ by

$$\bar{f}(x, y) = (f(x), f(y))$$

Observe that $\bar{f}$ is an isomorphism between $(I \times I, \sigma \underline{\underline{F}} \times \sigma \underline{\underline{F}})$ and $(R \times R, \underline{\underline{R}} \times \underline{\underline{R}})$.

Following Ulam ([3], page 7) we shall denote by class 0, the sets in $\underline{\underline{F}}$ and their complements. Having defined classes smaller than $\alpha < \Omega$, we define sets of class α to consist of countable unions of sets in the previous classes and complements of these sets. We say that $\underline{\underline{F}}$ has sets of high Borel class number if given $\alpha < \Omega$ there is a set in $\sigma \underline{\underline{F}}$ which does not belong to any class smaller than α.

If Z is any perfect subset of I, we denote by $\underline{\underline{B}}_Z$, $\underline{\underline{A}}_Z$, $\underline{\underline{L}}_Z$ the σ-algebras on Z generated by usual Borel sets, usual analytic sets; and by sets measurable w.r.t. a fixed non-atomic probability measure on $\underline{\underline{B}}_Z$. Observe that $\underline{\underline{B}}_Z$ and $\underline{\underline{A}}_Z$ are precisely $\underline{\underline{B}}$ and $\underline{\underline{A}}$ (see section 1) restricted to $\underline{\underline{Z}}$. We immediately have the following theorem which will be needed in the next section:

<u>Theorem 16</u>: If $\underline{\underline{E}}_Z$ is any σ-algebra on Z such that $\underline{\underline{A}}_Z \subset \underline{\underline{E}}_Z \subset \underline{\underline{L}}_Z$ then $\underline{\underline{E}}_Z$ is not countably generated.

<u>Proof:</u> Either repeat the proof of theorem 1 or apply theorem 1 after identifying Z with I by any Borel isomorphism.

Jan Mycielski and H. Steinhaus [1] have introduced a set theoretic axiom - the axiom of determinateness, (A). This axiom is formulated in game theoretic terminology. It says that certain games are always determined. We shall not attempt to give a precise statement of the axiom; for it is not needed in its original form for our purposes and moreover it involves much notation too. Interesting consequences of this axiom have been studied by Jan Mycielski [1] Jan Mycielski and S. Świerczkowski [1] and by R. M. Solovoy*

Main consequences of this axiom are the following :

(A) I: 'Every subset of the real line is Lebesgue measurable' or equivalently 'for every finite denumerably additive measure μ over the field of Borel sets of a separable metric space X and every $Y \subset X$, there are

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* unpublished

Borel sets $B_1 \subset Y \subset B_2$ such that $\mu(B_1) = \mu(B_2)$'.

(A) II: 'Every subset of the Cantor discontinuum has the property of Baire, in the sense that, it is of the form $(G \cup K_1) - K_2$ where G is open and K_1, K_2 are of the first category' or equivalently 'every subset of a separable metric space has the property of Baire'.

(A) III: 'Every non-denumerable subset of the Cantor set has a perfect subset' or equivalently 'Every non-denumerable separable metric space contains a compact perfect subset'.

(A) IV: For every family of nonempty pairwise disjoint sets $\underline{\underline{F}}$ such that $|F| \leq \aleph_0$ and $\left| \bigcup_{X \subset F} X \right| \leq 2^{\aleph_0}$; there exists a choice set. The same statement remains true if 'disjoint' is omitted and accordingly 'choice set' is replaced by 'choice function' in the conclusion.

Clearly the axiom Ⓐ is inconsistent with the axiom of choice. We remind the reader that the consistency of this axiom with the usual axioms of set theory (of course, without the axiom of choice) is not known. There are, however, many interesting consequences, due especially to R.M.Solovoy* which point out towards the relative consistency of this axiom. In (Mycielski [2]) an alternative of Ⓐ is proposed which has all the four main consequences stated above. This latter axiom gives a more refined version of Ⓐ IV and also a form of the principle of dependent choices.

In concluding this section, we remind the reader that proof of the fact that 'a one to one Borel map of a Borel subset of the real line into the real line preserves Borel sets' needs no more than the separation theorem for analytic sets which in turn can be proved with the countable axiom of choice. Thus the above fact is still valid under Ⓐ. This observation will be very useful for us in Section 6.

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* unpublished

§ 5. Consequences of (H).

Throughout this section we assume that the hypothesis (H) is satisfied. Since $(R, \underline{\underline{R}})$ is isomorphic to $(I, \sigma\underline{\underline{F}})$, we can and shall suppose for our purposes that the atoms of $\sigma\underline{\underline{F}}$ are singletons.

Theorem 17: A subset X of Q is $\underline{\underline{F}}$ - analytic iff $f(X)$ is standard analytic in P.

Proof: Since f is an isomorphism between $(I, \sigma\underline{\underline{F}})$ and $(R, \underline{\underline{R}})$ we have f restricted to Q to be an isomorphism between $(Q, \sigma\underline{\underline{F}}_Q)$ and $(P, \underline{\underline{R}}_P)$. Consequently $X \subset Q$ is in $\sigma\underline{\underline{F}}$ iff $f(X) \subset P$ is Borel. Hence $\bar{X} \subset Q \times Q$ is in $\sigma\underline{\underline{F}} \times \sigma\underline{\underline{F}}$ iff $\bar{f}(\bar{X}) \subset P \times P$ is Borel. Since obviously projection $(\bar{f}(\bar{X})) =$ f(projection of $\bar{X}$) we have the desired result.

The following theorem is not needed in the sequel, but has some independent interest.

Theorem 18: $(I, \sigma\underline{\underline{F}})$ Supports a non-atomic perfect probability measure. In fact there is one such concentrated on Q.

Let us fix any nonatomic probability measure μ on $\sigma\underline{\underline{F}}$ giving strictly positive measure to Q. Let the completion of $\sigma\underline{\underline{F}}$ w.r.t. this probability measure be denoted by $\underline{\underline{L}}$.

Theorem 19: There is no countably generated σ-algebra on I containing all its $\underline{\underline{F}}$ analytic sets and contained in $\underline{\underline{L}}$.

Proof: It suffices to prove the theorem in case μ is concentrated on Q. But in that case it is a direct consequence of Theorems 11 and 12.

The above theorem is clearly a generalization of Theorem 1 (the generalization however is trivial). We have an extension of Theorem 2 in the following theorem:

Theorem 20: There are $\underline{\underline{F}}$ analytic sets of $I \times I$ not belonging to $\underline{\underline{C}} \times \underline{\underline{L}}$.

Proof: As in the previous theorem it suffices to consider the case where μ is concentrated on Q. Fix $U \subset P \times P$ any standard analytic set universal w.r.t. the standard analytic sets of P. Define $V = \bar{\bar{f}}^{-1}(U)$. Following the arguments of the proof of Theorem 17, one can show that V is an $\underline{\underline{F}}$ analytic set of $I \times I$ contained in $Q \times Q$. Using Theorem 17, it is clear that V is universal w.r.t. the $\underline{\underline{F}}$ analytic sets of I contained in Q. Now this theorem is a direct consequence of Theorem 19, just as Theorem 2 was a consequence of Theorem 1.

We now record two theorems to be used in the next section. These are analogues of their classical versions.

<u>Theorem 21</u>: There are $\underline{\underline{F}}$-analytic sets not belonging to $\sigma\underline{\underline{F}}$.

<u>Proof:</u> Obvious from Theorem 17 and the classical analogue (Sierpiński [4], page 254).

<u>Theorem 22</u>: $\sigma\underline{\underline{F}}$ has sets of high Borel class number.

<u>Proof:</u> Metrize I by $d(x,y) = |f(x) - f(y)|$. Observe that with this metric I is homeomorphic to R and in fact f is a homeomorphism. Moreover each $A_n \in \underline{\underline{F}}$ is a clopen subset of I now. Also, Q being homeomorphic to P will be a perfect subset. Now our result is a consequence of its classical version (Kuratowski [2], page 373).

Finally it should be remarked that all the theorems of this section can be stated in terms of separable metric spaces with a perfect kernel, that is, containing homeomorph of Cantor set. We have not done so in order to keep the flavour of the discussion of Ulam [3] (see Section 8 also of this Chapter).

§ 6. Consequences of Ⓐ.

Throughout this section we assume Ⓐ to be valid.

Theorem 23: There does not exist a sequence $\underline{\underline{F}} = (A_n, n \geq 1)$ of sets in I with the following properties:

(a) $\sigma\underline{\underline{F}}$ has uncountable number of atoms.

(b) $\underline{\underline{F}}$- analytic sets coincide with sets in $\sigma\underline{\underline{F}}$.

Proof: (a) implies that the range of the Marczewski function for $\underline{\underline{F}}$ is uncountable and hence by Ⓐ III contains a perfect set. Consequently we can use Theorem 21 if in its proof no 'uncountable axiom of choice' is involved. But since an effective way of obtaining an analytic non-Borel set is known (Sierpiński [4], page 254) with only the countable axiom of choice, we are through.

Theorem 24: There does not exist a sequence $\underline{\underline{F}} = (A_n, n \geq 1)$ of sets in I with the following properties:

(a) $\sigma\underline{\underline{F}}$ has sets of arbitrarily high Borel class number.

(b) $\underline{\underline{F}}$-analytic sets coincide with sets in $\sigma\underline{\underline{F}}$.

Proof: Observe that if (a) is to be satisfied then the range of the Marczewski function for $\underline{\underline{F}}$ should be uncountable and hence in view of Theorem 23, (b) can not be satisfied.

Though the above two theorems are apparently different, that they are equivalent is shown by the following:

Theorem 25: For any sequence $\underline{\underline{F}} = (A_n, n \geq 1)$ of sets in I the following are equivalent.

(a) $\sigma\underline{\underline{F}}$ has uncountable number of atoms.

(b) $\sigma\underline{\underline{F}}$ has sets of arbitrarily high Borel class number.

Proof: Since b) $\Rightarrow$ a) is obvious we show the other implication. Observe that in view of (A) III the range of the Marczewski function contains a perfect set. Consequently theorem 17 can be used if we have not used the uncountable axiom of choice in its proof. But since an effective way of exhibiting Borel sets of any class is known with only the countable axiom of choice (Kuratowski [2] page 373) we are through.

We conclude this section with a theorem which is contradictory to Theorem 6.

Theorem 26: $U \notin \underline{\underline{C}}_0 \times \underline{\underline{C}}_0$. for any separable $\underline{\underline{C}}_0 \subset \underline{\underline{C}}$.

Proof of theorem 26 can be designed following the lines of Theorem 2. For a further treatment regarding the question of whether U belongs to $\underline{\underline{C}} \times \underline{\underline{C}}$, see Chapter 3.

§ 7. Non-existence of certain structures.

This section conceptually simplifies and also unifies the proofs of the theorems in section 2, and puts them on a much more general setting.

Let $(X, \underline{\underline{B}})$ be any separable (countably generated and containing singletons) Borel space, where to avoid trivialities we assume X to be uncountable. Sets in $\underline{\underline{B}}$ are to be called as Borel subsets of X. Throughout $\underline{\underline{B}}$ is fixed.

Theorem 27: For any σ-algebra $\underline{\underline{\Sigma}}$ on X containing $\underline{\underline{B}}$ the following are equivalent:

i) Any one-one real $\underline{\underline{\Sigma}}$ - measurable function on X coincides with a $\underline{\underline{B}}$-measurable function on an uncountable Borel subset of X.

ii) Any separable σ-algebra $\underline{\underline{S}}$ on X with $\underline{\underline{B}} \subset \underline{\underline{S}} \subset \underline{\underline{\Sigma}}$ coincides with $\underline{\underline{B}}$ on an uncountable Borel subset of X; that is, on some uncountable Borel set the restrictions of $\underline{\underline{B}}$ and $\underline{\underline{S}}$ coincide.

Proof: Given (i) we can prove (ii) by looking at the Marczewski function associated with any countable generator for $\underline{\underline{S}}$. Conversely given (ii), we can prove (i) by looking at the separable σ-algebra induced by the given function and $\underline{\underline{B}}$.

A σ-algebra $\underline{\underline{\Sigma}}$ on X containing $\underline{\underline{B}}$ and satisfying any one of the above two equivalent conditions is said to be a $\underline{\underline{B}}$- Souslin σ-algebra on X (with due respect to the work done by Souslin). A σ-algebra $\underline{\underline{Z}}$ on X is said to be $\underline{\underline{B}}$ - mixing if $\underline{\underline{Z}} \supset \underline{\underline{B}}$ and any uncountable Borel subset of X contains an element of $\underline{\underline{Z}} - \underline{\underline{B}}$.

From the above definitions and Theorem 27 we have the following theorem which can be easily proved by contradiction:

Theorem 28: Let $\mathcal{Z}$ be any $\mathcal{B}$-mixing σ-algebra and Σ containing $\mathcal{Z}$ be any $\mathcal{B}$-Souslin σ-algebra for X. Then there is no separable σ-algebra on X, containing $\mathcal{Z}$ and contained in Σ. Consequently no separable σ-algebra containing $\mathcal{Z}$ can be a $\mathcal{B}$-Souslin σ-algebra.

Remark 1: If $X = I$; $\mathcal{B}$ its usual Borel σ-algebra; $\mathcal{Z} = \mathcal{A}$ (see § 1 for definition of $\mathcal{A}$); Σ is the collection Lebesgue measurable sets or sets with the Baire property; then it is trivial to verify that the conditions of the above theorem are satisfied. Consequently Theorems 1 and 3 follow from the above theorem. We believe that this theorem says something more in the following sense: Fix any analytic non-Borel set A in I. Let $\mathcal{A}_0$ be the σ-algebra on I generated by $\mathcal{B}$ and all the Borel isomorphs of A. Then $\mathcal{A}_0$ is also $\mathcal{B}$-mixing and hence Theorems 1 and 3 are still true even if $\mathcal{A}$ is replaced by $\mathcal{A}_0$. However we do not know whether $\mathcal{A}_0$ is properly contained in $\mathcal{A}$. We do not know also whether any two analytic non-Borel subsets of I are Borel isomorphic. (These problems are again discussed in Chapter 3).

The following theorem is a direct consequence of Theorem 28.

Theorem 29: Assume the hypothesis of Theorem 28. Let U be any subset of $X \times X$ such that the vertical sections of U generate $\underline{\underline{Z}}$. Then $U \notin \underline{\underline{C}} \times \underline{\underline{\Sigma}}$.

Clearly Theorems 2 and 4 are special cases of Theorem 29, in view of the remark 1 above.

Assume the set up of remark 1. If $\underline{\underline{C}}$ is a $\underline{\underline{B}}$-Souslin σ-algebra then there is no separable σ-algebra containing $\underline{\underline{A}}$. In fact there is no such containing $\underline{\underline{A}}_0$ in that case. Thus in particular if one assumes the axiom of determinateness, then there is no separable σ-algebra containing $\underline{\underline{A}}_0$ on I. However we do not know whether conversely the non-existence of a separable σ-algebra containing $\underline{\underline{A}}$ implies that $\underline{\underline{C}}$ is a $\underline{\underline{B}}$ - Souslin σ- algebra.

There are certain routine problems based on the definitions introduced above, which we shall discuss in Chapter 3.

§ 8. Historical Comments.

A part of the contents of Section 2 will be appearing in Fundamenta Mathematicae where as parts of Section 3 have already appeared in the May 1969 issue of the Bulletin of the American Mathematical Society. The contents of Sections 4-6 have been submitted to Fundamenta Mathematicae. The fact that Ulam's construction can be used to prove Theorem 10 was suggested to us by Prof. Ashok Maitra. The difference between choice set and choice function was clarified to us by Prof. J. Mycielski who also supplied to us the remark made at the end of Section 4. The contents of Section 7 have been observed by the author after writing a preliminary version of this chapter.

The fact that our Theorem 2 answers a question of Ulam (Ulam [3], page 10, lines 20- 23) was suggested to us by a referee of the Fundamenta Mathematicae. Prof. J. Mycielski has informed to us that Dr. Mansfield of the University of Manchester has also proved - unpublished - a result weaker than our Theorem 2, using the axiom of the existence of Ulam 0 - 1 measurable cardinals and some difficult and unpublished results of R. M. Solovoy. Recently Dr. Mansfield has given an alternative proof of our Theorem 1. Quite recently Dr. Ashok Maitra has remarked that yet another proof our Theorem 1 can be given by using the idea of the proof Theorem 3. Whereas we have used Baird's theorem in the proof of Theorem 3, Dr. Ashok Maitra uses in his proof of our Theorem 1, the famous theorem of Lusin - 'every Lebesgue measurable function is continuous on a set of positive measure'.

Prof. Mycielski has informed us that Theorem 6 has also been obtained by Ulam and Erdős as early as 1944 ! - unpublished - and also more recently by Dr. C. Ferens of Wroclaw and Dr. Roy.O.Davies of London - again unpublished. The proof in all the cases is same. The generalizations of Theorems 6 and 7, with which the third section is concluded, were observed by the author after writing a preliminary version of this chapter.

Our Theorem 9 answers a question of Ulam [2]. Quite recently Mr. M. N. Sastry has drawn our attention to an entirely different treatment of the same problem by F. Rothberger [1]. Rothberger uses the notion of almost equivalence of sets introduced by Hausdorff: $A \sim B$ iff $A \triangle B$ is finite.

Our definition of generalized analytic sets in Sections 3 and 4 is taken from Ulam ([3], page 9). Our Theorem 18 is motivated by a question of Ulam)[3], page 10 lines 1 - 3) and so is our Theorem 19 ([3], page 10, lines 4 - 11). We conclude this section with the following paragraph which is taken from Ulam ([3] page 10 and 11).

"The motivation for investigating the Borel operations and beyond it, the projective operations when one starts with a general sequence of sets A_n - instead of the usual one which is the sequence of rational intervals or binary intervals - lies in the following possibility. There might exist a sequence of sets such that the number of its atoms is non-countable (i.e., still non-trivial) and yet such that the projective class over this sequence is 'simpler' than the 'classical' projective class. For example a sequence such that one could define a completely additive measure function for all sets of this projective class - this is impossible, according to a result of Gödel, for the familiar projective sets: i.e., it is free from contradictions in certain systems of axioms to assume that there exist projective sets which are non-measurable in the sense of Lebesgue. Even more generally one can extend this result to show that no completely additive measure is possible for all projective sets [by a measure we understand a set function with the properties: 1. $m(E) = 1$, $m(p) = 0$ where E is whole space, (p) is a set composed of single point. 2. $m\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ if $A_i \cdot A_J = \emptyset$ for $i \neq J$].

Paradoxically enough, it is conceivable that a measure function like the above could exist, if one starts with a sufficiently 'wild' sequence of sets A_n in a class of projective sets 'over' this sequence. Possibly all such sets could have the Baire property, that is each set of the class would be of first category - or complement of such ?''

CHAPTER 2

ON BOREL STRUCTURES

§ 0. Summary.

This chapter consists of eight sections. The first section explains some of the necessary preliminaries including the notation. In section 2 we show that every countably generated Borel structure has a minimal generator. The third section exhibits two separable Borel structures on the unit interval whose intersection is not separable. In the fourth section, the existence of an atomless Borel structure on every uncountable set has been observed. In the fifth section we show that the symmetric structure on a product space is generated by the symmetric rectangles. In section six the problem treated is that of defining a Borel structure for a collection of measurable maps from one Borel space to the other in such a way that the evaluation map is jointly measurable. In particular we show that such a structure exists iff the collection is of bounded Borel class, in the separable case. After defining 'Bounded Borel class' in a suitable way we show that the same theorem is true even in the non-separable case. In section seven a start is made towards the study of the lattice of Borel structures on any set. We show in particular that the lattice of all Borel structures on I is not complemented. We also show in this section that minimal weak complements in the sense of D. Basu are really complements, though not conversely. Finally in section 8 we collect some comments and histories of the problems treated in this chapter.

As the reader proceeds, he may perhaps be led to think that we are raising more problems than actually solving. It is indeed the case. There are two reasons for this. Firstly most of the problems treated in this chapter have never been treated in the literature, even briefly (except those of section 6). Secondly, the Marczewski function - which is a useful, neat and perhaps not yet fully exploited tool for handling separable σ - algebras - has no analogue for the non-separable case. It is true that even for non-separable σ- algebras a similar function can be defined (as was done by M. H. Stone) but it does not seem to be as useful as in the separable case.

§ 1. Notation.

Our terminology for most part is that of Mackey ([1], sections 1 and 2). Let X be any non-empty set and $\underline{\underline{B}}$ a σ-algebra of subsets of X. $\underline{\underline{B}}$ is called a Borel structure for X and $(X, \underline{\underline{B}})$ a Borel space. As usual, if there is no fear of confusion we shall refer to X itself as a Borel space. A family $\underline{\underline{G}} \subset \underline{\underline{B}}$ is said to be a generator for $\underline{\underline{B}}$ if the smallest Borel structure on X containing $\underline{\underline{G}}$ coincides with $\underline{\underline{B}}$. A generator $\underline{\underline{G}}$ for $\underline{\underline{B}}$ is minimal if no proper subfamily of it is again a generator for $\underline{\underline{B}}$. $\underline{\underline{B}}$ is called countably generated if there is a countable generator for $\underline{\underline{B}}$. $\underline{\underline{B}}$ is called separable if it is countably generated and contains singletons. An atom of $\underline{\underline{B}}$ is a set A in $\underline{\underline{B}}$ which is not empty such that no non-empty proper subset of A is in $\underline{\underline{B}}$. An isomorphism between two Borel-spaces is a one to one bimeasurable map of one onto the other. As in Mackey the relativized concepts can also be defined. If $(X, \underline{\underline{B}})$ is a Borel space and $Y \subset X$, then the relativized Borel structure on Y is denoted by $\underline{\underline{B}}_Y$.

2^ω denotes the unilateral countable product of the two point space $\{0, 1\}$. The Borel structure $\underline{\underline{C}}$ on 2^ω is the product of discrete Borel structures on the component spaces. If $(G_n, n \geq 1)$ is a generator for a separable Borel space then the Marczewski function defined as

$$f(x) = \left\{ \chi_{G_n}(x) \; ; \quad n \geq 1 \right\}$$

is a Borel isomorphism between X and the range of f in 2^{ω}. $(I, \underline{\underline{B}})$ always denotes the closed unit interval with its usual Borel structure.

For any Borel space $(X, \underline{\underline{B}})$ denote the product of it with itself by $(X^2, \underline{\underline{B}}^2)$. Call a set $A \in \underline{\underline{B}}^2$ to be <u>symmetric</u> if $(x, y) \in A$ implies $(y, x) \in A$. Clearly a rectangle is symmetric iff both sides are equal. The symmetric sets form a Borel structure on X^2, to be referred to as the <u>symmetric structure</u>.

Call a separable Borel space to be analytic (co-analytic) if it is isomorphic to an analytic (respectively co-analytic) subset of I.

Some more terminology needed for sections 6 and 7 is given in those sections.

Whenever needed, we shall make use of the axiom of choice without explicit mention.

§ 2. Minimal generators.

The main theorem of this section is the following:

Theorem 1: Any countably generated Borel space has a minimal generator.

Before proving the theorem let us observe that any argument involving Zorn's Lemma fails here. It is true that the collection of generators for any Borel space do form a partially ordered set under set inclusion. But however linearly ordered sub-collections need not possess lower bounds. For example take $X = I$, $\underline{\underline{B}}$ = its Borel σ-field. Let $(I_n, n \geq 1)$ be intervals with rational end points in I. Clearly $(I_n; n \geq k)$ is a generator for $\underline{\underline{B}}$, whatever be k. However the intersection of these decreasing sequence of generators is empty.

Proof of Theorem 1: Let $(X, \underline{\underline{B}})$ be any countably generated Borel space. Since the result is obvious if X is finite, we consider the case where X is infinite.

We start with observing that the Borel structure $\underline{\underline{C}}$ on 2^{ω} has a minimal generator. Take

$$A_n = \left\{ x \in 2^{\omega} \mid n^{th} \text{ coordinate of } x = 1 \right\}$$

Clearly $(A_n ; n \geq 1)$ is a generator for $\underline{\underline{C}}$ and the removal of any A_k results in a Borel structure for 2^{ω} which can not distinguish points differing only in the k^{th} coordinate.

Define a_n for $n \geq 1$ to be that point of 2^{ω} which has only the n^{th} coordinate zero and a_0 to be the point which has zero in no coordinate place. Observe that if Z is any subset of 2^{ω} containing $(a_n ; n \geq 0)$ then $\underline{\underline{C}}_Z$ has a minimal generator. Enough to take $(B_n ; n \geq 1)$ where $B_n = A_n \cap Z$ and A_n are as described above.

Now take any separable space $(X, \underline{\underline{B}})$. Then the Marczewski function establishes an isomorphism between $(X, \underline{\underline{B}})$ and a subset Z of 2^{ω}. By suitably altering the map on a countable subset of X, we can assume that Z contains $(a_n ; n \geq 0)$. Since the property of possessing a minimal generator is an isomorphic invariant $(X, \underline{\underline{B}})$ has a minimal generator.

Finally if $(X, \underline{\underline{B}})$ is any countably generated space then it is in an obvious way structure isomorphic to a separable space (Look at the space $\bar{X}$ of atoms of $\underline{\underline{B}}$ and the natural quotient structure $\bar{\underline{\underline{B}}}$). Hence it has a minimal generator. This proves the theorem.

Remark 1: We wish to point out three facts regarding minimal generators.

Firstly, if $(X, \underline{\underline{B}})$ is any separable Borel space and $\underline{\underline{G}}$ is any generator then $\underline{\underline{G}}$ contains a countable sub-family which is also a generator. But however $\underline{\underline{G}}$ need not contain a minimal generator. For instance take $(I, \underline{\underline{B}})$ with

$$\underline{\underline{G}} = \left\{ [0, a) ; \quad 0 < a \leq 1 \right\}$$

Secondly, if $(X, \underline{\underline{B}})$ is any Borel space and $\underline{\underline{G}}$ is any generator for $\underline{\underline{B}}$ and $Y \subset X$, then $\underline{\underline{G}}_Y$ is a generator for $\underline{\underline{B}}_Y$. But however if $\underline{\underline{G}}$ is a minimal generator for $\underline{\underline{B}}$, then $\underline{\underline{G}}_Y$ need not be a minimal generator for $\underline{\underline{B}}_Y$. For instance, take

$$X = \left\{ 0, 1, 2, 3, \ldots \right\}$$

$\underline{\underline{B}}$ = class of all subsets of X

$$\underline{\underline{G}} = \left\{ \{n\} \mid n \geq 1 \right\}$$

$$Y = \left\{ 1, 2, 3, 4, \ldots \right\}$$

Thirdly, if $[(X_\alpha, B_\alpha); \ \alpha \in T]$ is a collection of separable Borel spaces then their product $(X, \underline{\underline{B}})$ is separable iff all but countable number of X_α consist of a single point. But however if each $\underline{\underline{B}}_\alpha$ has a minimal generator (though not separable) then $\underline{\underline{B}}$ also has. To see this fix any minimal generators in the coordinate spaces and look at the one dimensional cylinder subsets of X whose base lies in the fixed minimal generator on the corresponding coordinate space.

Remark 2: We raise two questions regarding minimal generators. Firstly if $(X, \underline{\underline{B}})$ is a separable Borel space, and $Y \subset X$, then $(Y, \underline{\underline{B}}_Y)$ is also a separable Borel space. We do not know if this statement is true when the term 'separable Borel space' is replaced by 'Borel space with a minimal generator'. If this were true then any Borel space would have a minimal generator. To justify this statement, observe that the product σ-algebra on $\{0, 1\}^T$ where T, is any arbitrary non-empty set, has a minimal generator and that any separated Borel space is isomorphic to a subspace of some such product space. (Separated Borel space means, given any two distinct points of the space, there is a set in the Borel structure containing exactly one of the two given points). We leave the details for the reader.

Secondly, we do not know of any Borel structure without a minimal generator. We have two possible candidates for this purpose. The first candidate is the Borel structure $\underline{\underline{A}}$ on I generated by its analytic sets. From Theorem 1 or 3 of chapter 1, $\underline{\underline{A}}$ is not countably generated. The second is a Borel structure for I obtained as follows: Fix a non-Borel set $M \subset I$, and look at the σ-algebra $\underline{\underline{B}}^M$ on I consisting of all the usual Borel subsets of I which either are disjoint with M or contain M. The next theorem points out two situations where $\underline{\underline{B}}^M$ has a minimal generator. But in the general case we do not have any answer.

Theorem 2: If either M^c is co-analytic or M^c does not contain a perfect set, then $\underline{\underline{B}}^M$ has a minimal generator.

Proof: First suppose that M^c is a co-analytic set. We denote its constituents (w.r.t. any fixed sieve) by $[A'_\alpha; 1 \leq \alpha < \Omega]$ (See Kuratowski [2, p. 499). Let us define,

$$A_\alpha = A'_\alpha - \bigcup_{\beta < \alpha} A'_\beta .$$

Since M is not Borel observe that there are uncountable number of A_α s which are nonempty. By deleting the A_α s which are empty, we may suppose without loss of generality that $[A_\alpha ; 1 \leq \alpha < \Omega]$ are disjoint non-empty Borel sets whose union is M^c. Let $\underline{\underline{G}}_\alpha$ be a minimal generator for the relativized Borel σ-algebra on A_α such that the union of elements of $\underline{\underline{G}}_\alpha$ equals A_α. By a slight modification of the agruments of theorem 1 or otherwise it is not difficult to see that such a generator always exists. Let $\underline{\underline{G}} = \bigcup_{\alpha < \Omega} \underline{\underline{G}}_\alpha$. Since any Borel set contained in M^c is contained in a countable union of these A_α s (Kuratowski [2, p. 501]) it is easy to see that any Borel subset of I contained in M^c belongs to the σ-algebra generated by $\underline{\underline{G}}$. Hence $\underline{\underline{G}}$ generates $\underline{\underline{B}}^M$. To show the minimality, let, if possible, a set A from $\underline{\underline{G}}$ be omitted so that $\underline{\underline{G}}_o = \underline{\underline{G}} - \{A\}$ generates $\underline{\underline{B}}^M$. Let $A \in \underline{\underline{G}}_{\alpha_o}$. Clearly $\underline{\underline{G}}_o$ restricted to A_{α_o} does not generate $\underline{\underline{B}}^M$ restricted to A_{α_o}.

Now suppose M^c does not contain a perfect set. Then the only Borel subsets of I disjoint with M are the countable subsets of M^c. Consequently, in this case, $\underline{\underline{B}}^M$ consists of all countable subsets of M^c or co-countable subsets of I. Containing M. Hence the singletons of M^c form a generator for $\underline{\underline{B}}^M$. Clearly this is minimal. The proof is thus terminated.

We conclude this section with a generalization of Theorem 1. To start with we make a few definitions. Let X be a non-empty set and k be an infinite cardinal number. A k-algebra on X is a non-void collection of subsets of X closed under complementation and unions of k-many sets. A generator for a k-algebra $\underline{\underline{B}}$ on X is a collection $\underline{\underline{G}} \subset \underline{\underline{B}}$ such that the smallest k-algebra over $\underline{\underline{G}}$ equals $\underline{\underline{B}}$. A k-algebra is said to be λ-generated if there is a generator of cardinality less than or equal to λ - where, of course, λ is another cardinal number. The generalization of Theorem 1, that we have in mind is the following:

<u>Theorem 3</u>: Any k-generated k-algebra has a minimal generator.

The proof is a straight forward imitation of that of Theorem 1. Now we look at 2^k instead of 2^ω of that proof.

The problem of minimal generators does make sense even for arbitrary Boolean algebras.

§ 3. <u>Intersection of separable structures.</u>

We start with the following observation:

<u>Theorem 4</u>: (a) There are two countably generated σ-algebras on I whose intersection is not so.

(b) Given any separable Borel structure to I, there exists a countably generated structure whose intersection with the given one is not countably generated.

(c) There exist two countably generated substructures of the Borel structure on the real line whose intersection is not countably generated.

<u>Proof</u>:

(a): Though (a) can be deduced from (c) a simple independent proof is as follows: Let $\underline{\underline{B}}$ be the usual Borel structure on I. Fix a non-Borel set M in I and let $\underline{\underline{B}}_o$ be the structure generated by M and $\underline{\underline{B}}_{M^c}$. The intersection of these two is the $\underline{\underline{B}}^M$ of Remark 2 of § 2 which is not countably generated. However these two structures themselves are countably generated. In fact this structure is not even atomic.

(b): The same proof as above works.

(c): Let $\underline{\underline{M}}$ be the collection of all those Borel sets which are invariant under translation by 1 and $\underline{\underline{N}}$ be the collection of all those Borel sets which are invariant under translation by i where i is any fixed irrational number. It is well known that the additive subgroup of the real line generated by $\{1, i\}$ is dense in the real line and $\underline{\underline{M}} \cap \underline{\underline{N}}$ is precisely those Borel sets which are invariant under translation by elements of this group. Hence $\underline{\underline{M}} \cap \underline{\underline{N}}$ is not countably generated. This last sentence follows from a theorem of Mackey [1, 7.2], which says that such a structure which consists of Borel sets invariant under a certain sub-group - is countably generated iff the group is closed. This completes the proof.

It is interesting to observe that in all the above examples we have one or both σ-algebras under consideration to be non-separable - in the sense that they do not contain all singletons of the space. The question in which we are now interested is the possibility of finding two separable σ-algebras whose intersection is not. A simple example (better, a plethora of examples) can be given depending on rather deep facts about analytic sets. This we do in section 5. In this section we answer this question by simple and elementary methods making use of an interesting lemma of Halmos which appears in Aumann [1, Lemma 7.1]. The idea of our proof is essentially same as that of Halmos's and also similar to the related constructions treated in section 40 of Kuratowski [2]. Our proof is simple and does not use the continuum hypothesis whereas Aumann's proof is erroneous and uses the continuum hypothesis. The fact that Aumann's proof is erroneous is justified later in this section.

Lemma 1: There is a one to one map f on I onto I such that

i) $f = f^{-1}$,

ii) If A, A^c are uncountable Borel subsets of I, Then $f(A)$ is non-Borel.

Proof: Let Ω_c be the first uncountable ordinal corresponding to the cardinal c. Let A_α; $1 \leq \alpha < \Omega_c$ be an enumeration - a convenient well-ordering - of the uncountable Borel subsets of I whose complements in I are also uncountable. Since every uncountable Borel subset of I has cardinality c, we can associate with each ordinal $\alpha < \Omega_c$ three distinct points x_α, y_α, z_α in I satisfying the following two conditions:

i) $x_\alpha, y_\alpha \in A_\alpha$; $z_\alpha \in A_\alpha^c$

ii) $x_\alpha, y_\alpha, z_\alpha \notin \bigcup_{\beta < \alpha} \{x_\beta, y_\beta, z_\beta\}$.

This can be done by transfinite induction. Let f be the map which interchanges x_α with z_α and keeps every other point fixed. Obviously f is one one on I onto I and satisfies $f = f^{-1}$. Observe first that no A_β is left invariant by f. Now consider any A_α. Since any A_α contains uncountable number of A_r s and since each A_r has at least one point, namely y_r, unaltered by f; we conclude that $Z = f(A_\alpha) \cap A_\alpha$ is uncountable. In view of the fact that $f = f^{-1}$, this set Z is also invariant under f. Since the complement A_α itself was uncountable, Z^c is uncountable. Consequently if $f(A_\alpha)$ were Borel, then Z should be some A_β left invariant by f, which is impossible as already observed above. This completes the proof of the Lemma.

Now observe that if $\underline{\underline{B}}$ is the usual Borel structure on I and $\underline{\underline{B}}_o = f(\underline{\underline{B}})$ where f is any map satisfying the conditions of the Lemma, then both $\underline{\underline{B}}$ and $\underline{\underline{B}}_o$ are separable whereas $\underline{\underline{B}} \cap \underline{\underline{B}}_o$ is the countable - cocountable structure. Thus we have the following

<u>Theorem 5</u>: (a) There are two separable Borel structures on I whose intersection is not separable.

(b) Given any separable Borel structure to I such that every uncountable set in this structure has cardinality c, we can find another such structure, whose intersection with the given one is the countable - cocountable structure.

<u>Proof</u>: (a) has already been observed above. (b) follows on similar lines noting that Lemma 1, goes through even for this case.

One can now ask whether the f of Lemma 1 can be chosen so as to satisfy the further condition (iii) for every Borel B, f(B) is in $\underline{\underline{A}}$, the structure on I generated by its analytic sets. But unfortunately this can not be done. Because if this can be done, we can find two separable substructures of $\underline{\underline{A}}$ whose intersection is the countable-cocountable one. The next theorem shows that this is impossible.

Theorem 6. (a) Let $\underline{\underline{L}}$ be the Lebesgue measurable sets of I. If $(\underline{\underline{B}}_n, n \geq 1)$ are separable substructures of $\underline{\underline{L}}$ then $\cap \underline{\underline{B}}_n$ can not be the countable-cocountable structure.

(b) The above statement is valid even if $\underline{\underline{L}}$ is taken to be the collection of sets measurable w.r.t. a fixed nonatomic probability measure on $(I, \underline{\underline{B}})$.

(c) The same statement as in (a) is true even if $\underline{\underline{L}}$ is replaced by $\underline{\underline{Q}}$, the subsets of I having the Baire property.

(d) Lemma 1 is false under the Axiom of determinateness (and of course, without the axiom of choice).

Proof: (a) Let f_n be the Marczewski function associated w.r.t. any generator for $\underline{\underline{B}}_n$. Since f_n is Lebesgue measurable, it coincides on a set of measure 1 with a Borel function. Taking the intersection of these sets we have a Borel set Z of measure one - and hence uncountable - on which each $\underline{\underline{B}}_n$ coincides with the Borel σ-algebra. Thus $\cap \underline{\underline{B}}_n$ restricted to Z coincides with the Borel algebra restricted to Z. Since Z is uncountable, $\cap \underline{\underline{B}}_n$ can not be the countable cocountable structure on I.

(b) can be proved in a similar way.

(c) is also proved in the same way except the following change: Instead of using the fact that any Lebesgue function is nearly a Borel function, we make use of the fact that any function with Baire property is nearly a continuous function.

(d) Under the axiom of determinateness, observe that $\underline{\underline{L}}$ is the collection all subsets of I. If the Lemma 1 were true, then theorem 5(a) would be true which is inconsistent with 6(a). (With two σ-algebras 6(a) needs just the finite Axiom of choice).

From the above theorem we can observe an interesting property of the function f of Lemma 1.

<u>Theorem 7:</u> Let f be as given by Lemma 1 and $\underline{\underline{B}}_o =$ [illegible]$(\underline{\underline{B}})$.

(a) For every non-atomic probabi[illegible] on $\underline{\underline{B}}$ there is at least one set in $\underline{\underline{B}}_o$ which is [illegible]rable.

(b) There are sets in $\underline{\underline{B}}_0$ which do not have Baire property.

__Proof:__ In fact 7(a) and (b) are restatements of 6(b) and 6(c).

Thus here is another separable σ-algebra $\underline{\underline{B}}_0$ which contains examples of pathological sets. One such σ-algebra is already given in Theorem 9 of Chapter 1. One interesting feature of this $\underline{\underline{B}}_0$ is that it is Borel isomorphic to $\underline{\underline{B}}$ and in fact f is an isomorphism !

At this place we would like to raise two questions. First, we do not know the truth or falsity of the following statement: 'Every uncountable set X of cardinality $\leq$ c admits two separable structures whose intersection is not'. On one extreme if we assume __CH,__ then this is true in view of theorem 5(b). On the other extreme if we assume that there is an uncountable set such that its power set is the only separable structure on it, then the above statement is trivially false.

Second, we do not know if there exist two separable structures on I whose intersection is not separable, but contains a separable substructure.

We conclude this section with a comment on Aumann's proof of Lemma 1. In order to clearly see where the gap in the argument of Aumann lies we shall first prove a lemma:

<u>Lemma 2:</u> Assume <u>CH.</u> $[A_\alpha, 1 \leq \alpha < \Omega]$ be an enumeration of the uncountable Borel subsets of I whose complements are also uncountable, where Ω is the first uncountable ordinal We can associate two points x_α, y_α of I with each ordinal $\alpha < \Omega$ in such a way that the following four conditions are satisfied:

i) $x_\alpha \in A_\alpha$; $y_\alpha \in A_\alpha^c$

ii) For each α, exactly one of the two points x_α, y_α lies in $[0, \frac{1}{2})$ and the other in $[\frac{1}{2}, 1)$.

iii) For each α, $x_\alpha, y_\alpha \notin \bigcup_{\beta < \alpha} \{x_\beta, y_\beta\}$.

iv) $[0, 1) = \bigcup_{\alpha < \Omega} \{x_\alpha, y_\alpha\}$.

<u>Proof:</u> Well-order $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ say $(a_\eta; 1 \leq \eta < \Omega)$ and $(b_\eta; 1 \leq \eta < \Omega)$ respectively. We shall in a careful way select the points inductively so as to satisfy the conditions (i), (ii), (iii) and then show that (iv) is also satisfied.

If A_1 leaves uncountable number of points of $[\frac{1}{2}, 1)$ and contains uncountable number of points of $[0, \frac{1}{2})$, then define x_1 to be the first point in our well ordering of $[0, \frac{1}{2})$ which is in A_1 and y_1 to be the first point in our well ordering of $[\frac{1}{2}, 1)$ which is in A_1^c. If either A_1 does not leave uncountable number of points of $[\frac{1}{2}, 1)$ or it does not contain uncountable number of points of $[0, \frac{1}{2})$ then observe that A_1 must contain uncountable number of points of $[\frac{1}{2}, 1)$. In this case define x_1 to be the first point in our well-ordering of $[\frac{1}{2}, 1)$ which is in A_1 and y_1 to be the first point in our well-ordering of $[0, \frac{1}{2})$ which is in A_1^c.

If we have defined x_β, y_β for all $\beta < \alpha < \Omega$. Then we define x_α, y_α as follows: If A_α leaves uncountable number of points of $[\frac{1}{2}, 1)$ and contains uncountable number of points of $[0, \frac{1}{2})$, then define x_α to be the first unused point in our well-ordering of $[0, \frac{1}{2})$ which is in A_α and y_α to be the first unused point in our well-ordering of $[\frac{1}{2}, 1)$ in A_α^c. If either A_α does not leave uncountable number of points of $[\frac{1}{2}, 1)$ or it does not contain uncountable number of points of $[0, \frac{1}{2})$ we define x_α to be the first unused point in our well-ordering of $[\frac{1}{2}, 1)$ which is in A_α and y_α to be the first unused point in our well-ordering of $[0, \frac{1}{2})$ in A_α^c.

Since all the A_α s are uncountable with uncountable complements and since at each stage the number of points already selected are just countable, such a choice is always possible.

Since the verification of (i),(ii),(iii) is obvious, we shall verify (iv). We shall show that every a_η is some x_α or y_α. A similar proof holds for the set of b_η s also. If the set of unused a_η s is non-empty let a_λ be the first among them. Since $a_\lambda < \frac{1}{2}$, the set $[a_\lambda, \frac{1}{2}) - \{ a_\eta ; \eta < \lambda \} = Z$ is uncountable Borel subset of I with uncountable complement and is hence some A_α. It is obvious that a_λ should be chosen from Z. Thus the set of unused a_η s is empty completing the proof of the Lemma.

Aumann's proof of Lemma 1 goes as follows: Let $[A_\alpha, 1 \leq \alpha < \Omega]$ be the Borel sets as mentioned above. (Note that CH is assumed by Aumann). For each α choose two points x_α, y_α satisfying the conditions (i), (iii) of Lemma 2. This can be done by transfinite induction - in a simpler way than in Lemma 2. f is defined to be interchanging x_α with y_α and keeping the other points fixed. That this is insufficient for f to have the properties stated in Lemma 1 can be observed by using a choice made available by Lemma 2. In this case it is easy to see that

$$f\,[0, \tfrac{1}{2}) = [\tfrac{1}{2}, 1).$$

§ 4. Atomless Structures.

A Borel structure is atomless if it has no atoms or equivalently if every nonempty set from the structure has two disjoint nonempty subsets from the structure. It is interesting to note that this is equivalent to saying that every nonempty set from the structure contains $\aleph_1$ disjoint nonempty sets from the structure. The usual and simple example of an atomless Borel structure is on I^I. One can ask whether there is one such on I itself. The following theorem settles this question.

Theorem 8: On any uncountable set X, there is an atomless structure.

Proof: Denoting by $\aleph_1$ the first uncountable cardinal number, equip $X^{\aleph_1}$ with the product of discrete (in fact any nice) Borel structures on X. Fix $p \in X$. Let $X_0 \subset X^{\aleph_1}$ be the set of all those points which have p in all but finite number of coordinates. Then by well-known theorem (Sierpiński [5] p. 392 Theorem 1) it follows that X and X_0 have the same cardinality. Since the relativized structure on X_0 is atomless, the structure when carried over to X by any one to one map will serve as a required structure.

The above theorem says, for instance, that the real line R has an atomless structure. How nice can we choose it to be? Can we choose it to be translation invariant? Recall that a structure $\underline{\underline{Z}}$ on R is translation invariant if

$$Z \in \underline{\underline{Z}}\ ;\quad r \in R \Rightarrow Z + r \in \underline{\underline{Z}}.$$

Can we choose such a structure contained in $\underline{\underline{L}}$ - the Lebesgue measurable subsets? The following theorem considers these problems:

Theorem 9: (a) There is a translation invariant atomless structure on R.

(b) There is an atomless structure on I contained in $\underline{\underline{L}}$.

(c) There is an atomless substructure of $\underline{\underline{O}}$ on I.

Proof: (a) If X denotes the real line in the proof of theorem 8, then X and X_0 are vector spaces over rationals and have the same dimension. Consequently there exists a one to one additive map on X onto X_0. Since the relativized structure on X_0 is translation invariant, the structure on X brought by any such map is also translation invariant.

(b) Let I be decomposed as $(X_\alpha;\ \alpha \in I)$ uncountable, pairwise disjoint sets of Lebesgue measure zero. A quick way of doing this is to fix any one to one Lebesgue measure preserving bimeasurable map on I onto $I \times I$ and look at inverse images of vertical lines. Fix an atomless Borel structure $\underline{\underline{B}}_\alpha$ on X_α. The structure generated by $(\underline{\underline{B}}_\alpha,\ \alpha \in I)$ is contained in $\underline{\underline{L}}$ and is still atomless.

(c) Same as above, but now the X_αs are taken to be of the first category. A quick way of getting the decomposition is to observe that the irrationals in I and those in $I \times I$ are homeomorphic where by irrationals in $I \times I$ we meant those points with both co-ordinates irrational.

One might wonder whether a separable structure can contain an atomless structure. It can indeed happen:

__Theorem 10:__ Assume __CH.__

(a) There is a separable σ-algebra on I containing an atomless substructure.

(b) There is a subset of I whose relativized Borel structure contains an atomless substructure.

__Proof:__ (a) Taking $X = I$ in the proof of theorem 8 and starting with the usual Borel structure on X we get an atomless structure on I and the atomless structure on I obtained this way has cardinality c. Now applying theorem 6 of chapter 1 we get a separable structure containing this atomless structure.

(b) Let $\underline{\underline{C}}$ and $\underline{\underline{D}}$ be the atomless and separable structures on I obtained above. Fix any generator for $\underline{\underline{D}}$ and look at the corresponding Marczewski function g with range say R. Then the relativized Borel algebra on R which is f($\underline{\underline{D}}$) contains the atomless sub-structure f($\underline{\underline{C}}$). This completes the proof.

We do not know if the above theorem - or at least one of (a) and (b) of that theorem - could be proved without invoking CH. We also do not know if the usual Borel structure $\underline{\underline{B}}$ on I contains an atomless substructure. We feel it does not. In fact we suspect that $\underline{\underline{A}}$ - the structure generated by analytic subsets of I - does not contain any atomless substructure.

§ 5. A theorem of Blackwell-Mackey.

In this section we shall give a few applications of a theorem of Blackwell, which is also due independently to Mackey. We shall see how this theorem will be instructive in answering in the affirmative the question as to whether the symmetric structure on a product space is generated by the symmetric rectangles. The theorem we are referring to is the following:

Theorem 11 (Blackwell-Mackey): Let $(X, \underline{\underline{B}})$ be any analytic space:

(a) If $\underline{\underline{C}} \subset \underline{\underline{B}}$ is a countably generated σ-algebra then any set in $\underline{\underline{B}}$ which is a union of $\underline{\underline{C}}$-atoms belongs to $\underline{\underline{C}}$.

(b) If $\underline{\underline{C}}, \underline{\underline{D}} \subset \underline{\underline{B}}$ are two countably generated σ-algebras with the same atoms then $\underline{\underline{C}} = \underline{\underline{D}}$.

This theorem is a direct consequence of the first principle of separation for analytic sets (see Blackwell [1, section 4] and Mackey [1, section 4]).

A simple application of the above theorem is given in the following (compare with Aumann [1, p. 628, lines 3, 4]).

Theorem 12: (a) If $(X, \underline{\underline{B}})$ is an analytic space, $\underline{\underline{B}}_0$ is a proper enlargement of $\underline{\underline{B}}$ then $(X, \underline{\underline{B}})$ can not be isomorphic to $(X, \underline{\underline{B}}_0)$.

(b) There exist co-analytic spaces for which the above conclusion is false.

Proof: (a) If $(X, \underline{\underline{B}}_0)$ were isomorphic to $(X, \underline{\underline{B}})$ then it also will be an analytic space. But then $\underline{\underline{B}}$ is a proper sub σ-algebra of $\underline{\underline{B}}_0$ containing the same atoms as $\underline{\underline{B}}$, viz., singletons. Since $\underline{\underline{B}}$ is separable this is a contradiction in view of theorem 11(b).

(b) Before exhibiting an example to the effect, we remind the reader the term co-analytic space means a separable Borel space $(X, \underline{\underline{B}})$ isomorphic to a co-analytic subset of the real line.

As given by Kondô [1] fix a subset U contained in the irrationals such that any uncountable analytic subset of I is a one to one continuous image of U. Fix any one to one continuous map f on U onto I. By $\underline{\underline{B}}_1$ we denote the σ-algebra $f^{-1}(\underline{\underline{B}})$ where $\underline{\underline{B}}$ is the usual Borel structure on I. Obviously $(U, \underline{\underline{B}}_1)$ and $(I, \underline{\underline{B}})$ are Borel isomorphic and in fact f itself is an isomorphism. Let $\underline{\underline{B}}_0$ be the relativised Borel algebra on U. Observe that $(U, \underline{\underline{B}}_0)$ and $(U, \underline{\underline{B}}_1)$ are not Borel isomorphic - the former is non-Borel where as the latter is isomorphic to a Borel subset of I. Also observe that $\underline{\underline{B}}_1 \subsetneq \underline{\underline{B}}_0$. To get our example let X be the disjoint union of as many copies of U as there are integers. A σ-field $\underline{\underline{C}}$ on X is obtained as follows: For even integers equip V with $\underline{\underline{B}}_1$ and for odd integers equip U with $\underline{\underline{B}}_0$. A structure $\underline{\underline{D}}$ on X is obtained as follows: For even integers and the integer 1 equip U with $\underline{\underline{B}}_1$ and for odd integers excluding 1 equip U with $\underline{\underline{B}}_0$. Then $\underline{\underline{D}} \subsetneq \underline{\underline{C}}$. Since $(X, \underline{\underline{D}})$ and $(X, \underline{\underline{C}})$ both consist of infinitely many copies of U of which in each case infinitely many copies of U are equipped with $\underline{\underline{B}}_0$ and another infinitely many with $\underline{\underline{B}}_1$, it is easy to see that $(X, \underline{\underline{C}})$ and $(X, \underline{\underline{D}})$ are isomorphic. Obviously $(X, \underline{\underline{D}})$ is isomorphic to a co-analytic subset of the real line. Thus $(X, \underline{\underline{D}})$ is a co-analytic space and $\underline{\underline{C}}$ is an isomorphic enlargement of $\underline{\underline{D}}$. In fact one need not take Kondô's set (which Kondô called Lebesgue set), the set exhibited by A. Maitra [1] will also serve the purpose. If one does not insist on getting a co-analytic space, many examples can be constructed very easily.

As another application of theorem 11, we shall prove the following theorem:

<u>Theorem 13:</u> (a) Let $(X, \underline{\underline{B}})$ be an analytic space and f a real measurable function on X inducing the σ-algebra $\underline{\underline{B}}_f$. Then $\underline{\underline{B}}_f$ consists of all those elements of $\underline{\underline{B}}$ which are unions of f-atoms.

(b) The conclusion of (a) need not be true if $(X, \underline{\underline{B}})$ is a co-analytic space.

<u>Proof:</u> (a) is a special case of theorem 11.

(b) Either Kondô's co-analytic set or the co-analytic set exhibited by A. Maitra referred above will do the job.

Before proceeding further, we shall first give a simple and elegant proof of theorem 5(a) using theorem 11. Take any uncountable Borel set B and non-Borel analytic set A of the real line with their relativized Borel structures $\underline{\underline{B}}_B$ and $\underline{\underline{B}}_A$ respectively. Let f, g be any one to one maps of I onto B and A respectively. Let $\underline{\underline{B}}_f = \overset{-1}{f} (\underline{\underline{B}}_B)$ and $\underline{\underline{B}}_g = \overset{-1}{g} (\underline{\underline{B}}_A)$. Then $\underline{\underline{B}}_f$ and $\underline{\underline{B}}_g$ are two separable structures on I where as their intersection is not. This is obvious from theorem 11. In fact their intersection can not contain any separable structure.

<u>Theorem 14</u>: For any Borel space $(X, \underline{\underline{B}})$ the symmetric structure on X^2 is generated by the symmetric rectangles.

<u>Proof</u>: We shall give two proofs of this theorem. The first proof makes use of theorem 11. If $X = I$ and $\underline{\underline{B}}$ its Borel structure then observe that the structure generated by the symmetric rectangles is countably generated and hence contains all symmetric sets from theorem 11(a). Since the other inequality is always true, the assertion is true in this case. If $X \subset I$ and $\underline{\underline{B}}$ is its relativized Borel structure, then the symmetric structure on X^2 is the restriction of the symmetric structure on I^2 and the structure on X^2 generated by the symmetric rectangles is the restriction of the corresponding structure on I^2. Consequently in view of the Marczewski function, the result is true for separable Borel spaces. Again, in view of the structure isomorphism the result is true for any countably generated Borel space. If $(X, \underline{\underline{B}})$ is any Borel space, observe that the symmetric structure on X^2 always contains the structure generated by the symmetric rectangles. Conversely, let A be any symmetric set in $\underline{\underline{B}}^2$. To show that A is available in the structure generated by symmetric rectangles, take any countably generated $\underline{\underline{B}}_0 \subset \underline{\underline{B}}$ such that A is in $\underline{\underline{B}}_0^2$ and then apply the conclusion drawn previously.

An alternative proof is by transfinite induction. Let $\underline{\underline{C}}_0$ be the rectangles in $\underline{\underline{B}}^2$ and in general $\underline{\underline{C}}_\alpha$ $(\alpha < \Omega)$ consists of countable unions of sets in the previous classes and the complements of sets obtained in that way. Then one knows that $\underline{\underline{B}}^2 = \bigcup_{\alpha < \Omega} \underline{\underline{C}}_\alpha$. Observe that any symmetric set in $\underline{\underline{C}}_0$ is a symmetric rectangle and hence is in the structure generated by the symmetric rectangles. By transfinite induction one can show that a symmetric set in any of these classes $\underline{\underline{C}}_\alpha$ is in the structure generated by the symmetric rectangles, completing the proof.

Another interesting application of Blackwell-Mackey theorem is in the characterization of translation invariant subalgebras on the real line by D. Basu and J. K. Ghosh [1]. Take any $\partial > 0$. Identify two points $x, y \in R$ iff $x - y = n\partial$ for some integer n. This equivalence relation gives in a natural way a substructure $\underline{\underline{B}} \mid [\partial]$ of the usual Borel structure $\underline{\underline{B}}$ on R. This is countably generated and translation invariant (c.g.t.i). In fact all countably generated such structures are either trivial or can be obtained in the above fashion.

<u>Theorem 15</u> (Basu and Ghosh): Any c.g.t.i $\underline{\underline{B}}_0 \subset \underline{\underline{B}}$ is either one of the two trivial ones, viz, $\{\emptyset, R\}$ and $\underline{\underline{B}}$ or is of the form $\underline{\underline{B}}/[\partial]$ for some $\partial > 0$.

Proof: Since $\underline{\underline{B}}_0$ is countably generated, let G_0 be the atom containing 0. Suppose $\underline{\underline{B}}_0$ is not trivial. Then by translation invariance of $\underline{\underline{B}}_0$ it follows that $\underline{G}$ is a proper nontrivial subgroup of R. By theorem 11(a), $\underline{\underline{B}}_0 = \underline{\underline{B}}|G$ where $\underline{\underline{B}}|G$ is defined again as above by an equivalence relation. Now from a theorem of Mackey (Mackey [1, 7.2]) G is closed and hence is of the form $[n\delta \mid n \in \text{integers}]$, for some $\delta > 0$. This proves the theorem.

As remarked by D. Basu and J. K. Ghosh a complete characterization of the translation invariant sub σ-algebras of $\underline{\underline{B}}$ is still not known. The above theorem characterizes only those which are countably generated. It is easy to construct non-countably generated translation invariant structures not covered by the above theorem.

§ 6. Borel structures for function spaces.

The problem we treat in this section is rather different from the kind of problems we were discussing earlier in this chapter. The problem is this: Suppose $(X, \underline{\underline{B}})$ and $(Y, \underline{\underline{C}})$ are two separable Borel spaces and F is a collection of measurable maps from X to Y. There is then a natural map $\phi: F \times X \to Y$ defined by $\phi(f,x) = f(x)$. Let us say that F is admissible if there is a Borel structure $\underline{\underline{F}}$ on F such that the map ϕ from the product space $F \times X$ into Y is measurable; where of course, the product space is equipped with the product σ-algebra $\underline{\underline{F}} \times \underline{\underline{B}}$. In that case we refer to $\underline{\underline{F}}$ as an admissible structure for F. What are the admissible sets and how nice can the admissible structure be chosen? This is partially answered by the following theorem:

Theorem 16: The following conditions on a family F are equivalent:

i) F is of bounded Borel class.

ii) F is admissible.

iii) There is a separable admissible structure for F.

iv) The power set of F is an admissible structure for F.

This theorem, in essence, is not new. This was first proved, in a different form, by Aumann [1]. He has proved, to be more precise, that (i) and (ii) are equivalent and imply (iii). His proof is however quite complicated. We shall give an elegant and neat treatment of the problem. Just as the universal analytic sets played an important role in the previous chapter, we shall employ the universal functions as the key tool to prove this theorem. Before starting on the proof we have to develop a small amount of notation.

In view of the Marczewski function, we can and shall take X to be a subset of I and $\underline{\underline{B}}$ to be its relativized Borel algebra. Similar is the case with (Y, $\underline{\underline{C}}$). Without explicit mention, we shall give the relative topologies to X and Y. We shall assume the properties of the Borel classification of functions. We shall say that a collection F of measurable maps on X to Y is of bounded Borel class if there is an ordinal $\alpha < \Omega$ such that every f in F is of class $\leq \alpha$. A good source of reference is again Kuratowski [2, page 373]. We shall denote by $C_\alpha (X, Y)$ the set of all Borel functions of class less than or equal to α; for $0 \leq \alpha < \Omega$. In case no confusion arises, we shall not hesitate to write, C_α for $C_\alpha (X, Y)$. If $X = Y = I$ and $\alpha < \Omega$; then we can define a Borel function $U_\alpha(x, y)$ on $I \times I$ such that

$$C_\alpha (I, I) \subset \left\{ U_\alpha (x, \cdot); \quad x \in I \right\}.$$

This fact follows from the corresponding - and in fact stronger - result for Baire classification of functions (Natanson [1] p. 137) and the connection between the Borel and Baire classifications on the unit interval (Kuratowski [2] p. 393). Without distinguishing between additive and multiplicative classes of sets; we shall have an occasion in this section to use the Borel classification of sets in separable metric spaces (Kuratowski [2] p. 345).

We are now in a position to prove theorem 16:

<u>Proof of (i) => (ii)</u>: Since obviously subsets of admissible sets are admissible, and since any collection of functions of bounded Borel class is a subset of some $C_\alpha(X,Y)$, it suffices to show that $C_\alpha(X,Y)$ is admissible for each $\alpha < \Omega$.

In case $X = Y = I$, let us choose a function $U_\alpha(x,y)$ on $I \times I$ as mentioned above. Choose a subset Z of I (axiom of choice!) such that the map

$$T: \quad x \to U_\alpha(x, \cdot)$$

is one to one on Z onto $C_\alpha(I, I)$. Having thus identified $C_\alpha(I, I)$ with Z via T, the relativized σ-algebra on Z can be brought to $C_\alpha(I, I)$ in an obvious way. Observe that T is now an isomorphism between C_α and Z. Moreover,

$$U_\alpha(x, y) = \varphi(Tx, y) \quad \text{for } x \in Z, y \in I.$$

Since U_α is jointly measurable on $I \times I$, ϕ is measurable on $C_\alpha \times X$ as desired.

In case $X \subset I$, $Y = I$; observe that any element of $C_\alpha(X, I)$ can be regarded as the restriction to X of an element of $C_{\alpha+1}(I, I)$ - in view of the extension theorem for functions (see Kuratowski [1] p. 434, 435). Thus $C_\alpha(X,I)$ can be identified with a subset of $C_{\alpha+1}(I,I)$. Observe that this identification needs the axiom of choice, because an element of $C_\alpha(X, I)$ may be the restriction to X of more than one element of $C_{\alpha+1}(I,I)$. Since $C_{\alpha+1}(I,I)$ is admissible by the above para; it is not difficult to see that $C_\alpha(X,I)$ is also admissible.

In case $X \subset I$, $Y \subset I$; observe that $C_\alpha(X,Y)$ is a subset of $C_\alpha(X,I)$ - because any element of $C_\alpha(X,Y)$ can be thought of as an element of $C_\alpha(X,I)$ whose range is contained in Y. Since $C_\alpha(X,I)$ is admissible by the above para, we have $C_\alpha(X, Y)$ also to be admissible.

Thus our assertion is proved. Observe that in each of the above paras, the structures we have obtained for C_α is separable - a fact that is needed later.

Proof of (ii) => (i): First we show that if $(\Omega_0, \underline{\underline{B}}_0)$ is a separable metric space with its Borel σ-algebra and $(\Omega_1, \underline{\underline{B}}_1)$ is any measurable space and if $Z \subset \Omega_1 \times \Omega_0$ is any set in the product σ-algebra $\underline{\underline{B}}_1 \times \underline{\underline{B}}_0$ then there is an ordinal $\alpha < \Omega$ such that every vertical section of Z is a Borel set of class less than α in Ω_0. Since any set in the product σ-algebra is available in the σ-algebra generated by a countable number of rectangles, a moment's reflection will show that there is no loss in taking $\underline{\underline{B}}_1$ to be countably generated. If x,y are available in the same atom of $\underline{\underline{B}}_1$ then it is clear that the x- section and the y-section of Z are same (Reason: When $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_0$ are countably generated then so is $\underline{\underline{B}}_1 \times \underline{\underline{B}}_0$ and the atoms of $\underline{\underline{B}}_1 \times \underline{\underline{B}}_0$ are precisely sets of the form $A \times B$ where A and B are atoms of $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_0$ respectively. Moreover every set in $\underline{\underline{B}}_1 \times \underline{\underline{B}}_0$ is union of atoms). Since we are interested in only the vertical sections of Z, the above considerations show that there is no loss in taking $\underline{\underline{B}}_1$ to be separable. Again by using the familiar techniques involving the Marczewski function we can suppose that Ω is a separable metric space and $\underline{\underline{B}}_1$ is its Borel algebra. But in this case our assertion is a well-known easy theorem. (for instance see Kuratowski [2], p.347).

Next suppose that F is an admissible collection of maps from X to Y. Let I_n be intervals with rational end points in I. By the remark made above, there is an $\alpha_n < \Omega$ such that each f-section of $\phi^{-1}(I_n)$ is Borel of class $< \alpha_n$ in Y. (Remember that X and Y are subsets of I). Take $\alpha < \Omega$ which is strictly greater than $\operatorname{Sup}_n \alpha_n$. Since the f-section of $\phi^{-1}(B)$ is nothing is $f^{-1}(B)$ it is not difficult to see that F is contained in $C_{\alpha+1}(X,Y)$.

Thus (i) and (ii) are equivalent. If F has a separable admissible structure, then apriori, F is admissible and hence (iii) => (ii). Conversely if F is admissible, then as shown above F is of bounded Borel class and hence as remarked in the proof of (i) => (ii) there is a separable admissible structure for F, thus proving (ii) => (iii). Since any structure larger than an admissible structure is again admissible, obviously (iii) => (iv). To the converse, if the power set of F is admissible, then F is admissible and hence has a separable admissible structure. This completely proves the theorem.

A comment is in order. One might be wondering that in identifying an arbitrary separable Borel space with a subspace of I, via the Marczewski function; we have fixed a generator and that consequently our theorem depends on the generator. But actually it is not so. We shall now show that if under some identification a collection F of maps from X to Y is of bounded Borel class, then it remains so under any other identification. The proof of this statement depends on the composition laws for Borel functions (see Kuratowski [2] p. 376) and proceeds as follows: Suppose $(X, \underline{\underline{B}})$ $(Y, \underline{\underline{C}})$ are the two spaces to start with. Let $\underline{\underline{G}}_1$ and $\underline{\underline{G}}_2$ be two countable generators for $\underline{\underline{B}}$ and $\underline{\underline{H}}_1$, $\underline{\underline{H}}_2$ for $\underline{\underline{C}}$. Let X_1, X_2 be the subsets of I with which X is identified via the Marczewski functions g_1, g_2 associated with $\underline{\underline{G}}_1$ and $\underline{\underline{G}}_2$, respectively. Similarly Y_1, Y_2 be the subsets of I, with which Y is identified via the Marczewski functions h_1, h_2 associated with $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_2$. Let i be the map on X_2 to X_1 defined by

$$i(x) = g_1 \overset{-1}{g_2} (x)$$

and j be the map on Y_1 to Y_2 defined by

$$j(y) = h_2 \overset{-1}{h_1} (y).$$

Let i be of class β from X_2 to X_1 and j be of class

γ from Y_1 to Y_2. Suppose f is a function from X to Y which under the identification of (X, Y) with (X_1, Y_1) is of class $\leq \alpha$. Then under the identification of (X,Y) with (X_2, Y_2) it is of class $\leq \gamma + \alpha + \beta$. Thus if a collection F of functions is of bounded Borel class under some identification, it remains so under any other identification. It is however true that C_α (X, Y) depends on the particular identification.

Two remarks should be made at this point regarding Aumann's proof of this theorem. First, note that we have made use of the axiom of choice in the above proof. Aumann also makes use of it, though he does not explicitly state it (see especially the discussion following Lemma 4.1 of Aumann [1]). Second, we have made use of the already existing Borel classification of functions. Aumann defines Banach classes of functions - which is again done after fixing a generator. There is no difference between functions of Bounded Borel class and functions of Bounded Borel class and functions of Bounded Banach class. In fact, it straight away follows from the definition of Aumann, that if we start with certain types of generators, then even the class of a function will be same according to Aumann's Banach classification and the Borel classification. However we shall not enter into the details because, they are straight forward and need recalling Aumann's definitions. Moreover this will not be needed anywhere in our future discussion.

One can ask whether consistent structures can be given to C_α (X, Y) for $0 \leq \alpha < \Omega$. That is a structure $\underline{\underline{C}}_\alpha$ on C_α such that

i) $\underline{\underline{C}}_\alpha$ is separable,

ii) $0 \leq \beta < \alpha$ implies $C_\beta \in \underline{\underline{C}}_\alpha$,

iii) $0 \leq \beta < \alpha$ implies $\underline{\underline{C}}_\alpha | C_\beta = \underline{\underline{C}}_\beta$.

The answer is in the affirmative and follows from the next theorem. By C_∞ we denote the collection of all measurable functions from X to Y, that is, $\bigcup_{\alpha < \Omega} C_\alpha$.

__Theorem 17:__ There is a structure $\underline{\underline{C}}_\infty$ on C_∞ satisfying the following conditions.

i) $F \subset C_\infty$ is admissible iff $\underline{\underline{C}}_\infty | F$ is an admissible structure for F.

ii) The structures $\underline{\underline{C}}_\infty | C_\alpha$ are consistent in the above sense.

Moreover, if we assume __CH__, then $\underline{\underline{C}}_\infty$ can be chosen to be separable.

<u>Proof</u>: Put

$$C_\alpha^* = C_\alpha - \bigcup_{\beta < \alpha} C_\beta$$

Fix any separable admissible structure $\underline{\underline{C}}_\alpha^*$ for C_α^*. C_∞ is the disjoint union of $[C_\alpha^*, 1 \leq \alpha < \Omega]$. Define $\underline{\underline{C}}_\infty^*$ on C_∞ to be the structure generated by $[\underline{\underline{C}}_\alpha^*, 1 \leq \alpha < \Omega]$. We shall now show that this satisfies our requirements. First observe that any C_α is the disjoint union of the countable number of C_β^* for $\beta \leq \alpha$ and the structure $\underline{\underline{C}}_\alpha$ on C_α generated by $(\underline{\underline{C}}_\beta^*, \beta \leq \alpha)$ is admissible. In other words $\underline{\underline{C}}_\infty \mid C_\alpha$ is an admissible structure for C_α. Since if F is admissible, then F is contained in some C_α, it follows that $\underline{\underline{C}}_\infty | F$ is an admissible structure for F. Of course obviously, if $\underline{\underline{C}}_\infty | F$ is an admissible structure for F, then F is admissible. Thus (i) is proved. Since, as already observed above $\underline{\underline{C}}_\infty | C_\alpha$ is generated by the countable number of separable structures $\underline{\underline{C}}_\beta^*$ for $\beta \leq \alpha$; it is clear that $\underline{\underline{C}}_\infty | C_\alpha$ is a separable structure. Since this is observed above to be admissible (ii) is also proved.

It now remains to **prove** the last statement of the theorem. Observe that if a structure $\underline{\underline{C}}_\infty$ on C_∞ satisfies the two conditions of the theorem then any separable structure containing $\underline{\underline{C}}_\infty$ will also satisfy these conditions. Thus it suffices to exhibit a separable structure containing the $\underline{\underline{C}}_\infty$ exhibited above on C_∞. For this it suffices to verify that C_∞ has cardinality $\leq c$ (in view of the separabality of Borel spaces under consideration) and then apply theorem 6 of chapter 1.

Having settled the problem of admissibility of subsets of C_∞, through theorem 16, it remains to discuss the question as to how nice can an admissible structure be chosen. We already know one result in this direction - that is the admissible structures can be chosen to be separable. We also know that structures can be so chosen that they are consistent in a certain sense. The aspect that we now discuss is regarding minimality.

Let F be an admissible collection. Aumann defines $\underline{\underline{F}}$ to be natural if it is admissible and any admissible structure on F contains $\underline{\underline{F}}$. In otherwords an admissible structure is natural if it is the intersection of all admissible structures on F. He has given an example of a family F without a natural admissible structure. We shall explain the same example in a different - but essentially equivalent - terminology. Take any two separable σ-algebras $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$ on I whose intersection is the countable-cocountable structure. Let $\underline{\underline{B}}_2$ be the structure on I generated by $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$. Both of our spaces $(X, \underline{\underline{B}})$ and $(Y, \underline{\underline{C}})$ are $(I, \underline{\underline{B}}_2)$. Our F consists of indicators of singletons in I. Thus F and I can be identified by the map indicator of a point going to the point. Observe that with such an identification $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$ can be thought of as structures on F. The evaluation map $\varphi : F \times I \to I$ will take only two values 0 and 1. Consequently, a structure $\underline{\underline{F}}$ on F is admissible iff $\varphi^{-1}(1)$ is measurable in $\underline{\underline{F}} \times I$. In other words iff the set $[(\chi_x, x) : \chi_x$ is indicator of $x]$ is in $\underline{\underline{F}} \times \underline{\underline{B}}_2$. Since $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$ (thought of as structures on $\underline{\underline{F}}$) are separable it is easy to see that they are admissible structures. But however their intersection $\underline{\underline{B}}_0 \cap \underline{\underline{B}}_1$ which is the countable-co-countable structure can not be admissible. Thus there is no natural admissible structure for this F. However F is admissible.

Let $(I, \underline{\underline{B}})$ be as usual the unit interval with the Borel σ-field.

<u>Theorem 18</u>: $C_\alpha(I, I)$ has a natural structure if $\alpha = 0$.

<u>Proof</u>: Observe that $C_0(I, I)$ consists precisely of the continuous functions on I into I. It is complete separable metric space, when equipped with the usual supremum

metric. Its Borel σ-algebra is an admissible structure. In fact the map $\phi: C_o \times I \to I$ is continuous in either argument, having fixed the other argument, and hence will be measurable if C_o is given its topological Borel structure. As one knows the Borel structure on C_o is also generated by the evaluation maps (namely, fix a point $x \in I$ and look at the map f going to $f(x_o)$) and is consequently contained in every other admissible structure.

In passing we note that if $F_1 \subset F_2$ and if F_2 has a natural structure say $\underline{\underline{F}}_2$ then F_1 also has. In fact $\underline{\underline{F}}_2 \cap F_1$, is natural structure for F_1. To observe this note first that $\underline{\underline{F}}_2$ being admissible for F_2, $\underline{\underline{F}}_2 \cap F_1$ is admissible for F_1. Now take any other admissible structure $\underline{\underline{Z}}$ for F_1. Then the structure $\underline{\underline{Z}}^*$ on F_2 generated by $\underline{\underline{Z}}$ and $\underline{\underline{F}}_2$ restricted to $(F_2 - F_1)$ will be admissible and hence $\underline{\underline{Z}}^* \supset \underline{\underline{F}}_2$. Consequently $\underline{\underline{Z}}^* \cap F_1 \supset \underline{\underline{F}}_2 \cap F_1$. But trivially, $\underline{\underline{Z}}^* \cap F_1 = \underline{\underline{Z}}$ and hence $\underline{\underline{Z}} \supset \underline{\underline{F}}_2 \cap F_1$. Thus any admissible structure on F_1 contains $\underline{\underline{F}}_2 \cap F_1$.

This remark leads us to believe that, many subsets, even in the case of the unit interval have no natural structure in the sense of Aumann. This leads us to define naturality of a structure in a different way. A separable admissible -

structure $\underline{\underline{F}}$ for an admissible family F is natural if no proper substructure of $\underline{\underline{F}}$ is again admissible. The difference between our definition and Aumann's definition is obvious. Our definition of naturalness is weaker than that of Aumann. It is clear that if the set Z that occurs in the proof of theorem 16 is a Blackwell space then the structure we have on C_α (I, I) will be natural. Recall that a space (X, $\underline{\underline{B}}$) is a Blackwell space if it is a separable Borel space and $\underline{\underline{B}}_0$ is separable Borel structure for X contained in $\underline{\underline{B}}$ implies $\underline{\underline{B}}_0 = \underline{\underline{B}}$. For further details regarding these spaces see A. Maitra [1]. Thus the existence of natural structures (always in our sense, hereafter) seems to be connected with Blackwell selections. But however, we feel that the existence or non-existence of natural structures should better be treated directly rather than through selection theorems - as one knows that in general nice selections are difficult to obtain.

Leaving the discussion of our problem for the separable case at this point we shall now passon to the countably generated case. So let (X, $\underline{\underline{B}}$) and (Y, $\underline{\underline{C}}$) be countably generated Borel spaces. In this case either we can straight away regard these as separable pseudometric spaces with their Borel algebras or we can look at the spaces of atoms and apply theorem 16. Since any measurable function f from X to Y is constant on atoms of X and thus can be thought of as a map $\bar{f}$ on $\bar{X}$ into $\bar{Y}$, the spaces of atoms, we can define f to be of class of α if $\bar{f}$ is of class α (of course w.r.t. some fixed generators). In this case theorem 16 takes the following shape:

Theorem 19: Let F be a collection of measurable maps from $(X, \underline{\underline{B}})$ to $(Y, \underline{\underline{C}})$ where both these Borel spaces are countably generated. Then the following are equivalent:

i) F is of bounded Borel class.

ii) F is admissible.

iii) F has a countably generated admissible structure.

iv) For F, its power set is an admissible structure.

But for some fussy details regarding the quotient strucrures $\bar{X}$ and $\bar{Y}$; the proof of this theorem is straightforward by using theorem 16. For a detailed proof one can see Aumann [1].

The problem of this section in case $(X, \underline{\underline{B}})$ and $(Y, \underline{\underline{C}})$ are not countably generated is worth investigating and this is what we shall do now for the rest of this section. We shall give a complete solution of the problem. To do this we need some definitions. Hereafter we assume that $(Y, \underline{\underline{C}})$

is separable till further notice.

A collection of functions F from $(X, \underline{\underline{B}})$ to $(Y, \underline{\underline{C}})$ is said to be of bounded Borel class if there is a countably generated sub σ-algebra $\underline{\underline{B}}_0 \subset \underline{\underline{B}}$ such that

i) f in F implies f is $\underline{\underline{B}}_0$ - measurable

ii) F is of bounded Borel class w.r.t. $(X, \underline{\underline{B}}_0)$ and $(Y, \underline{\underline{C}})$.

With this definition we have the following theorem:

<u>Theorem 20</u>: For a collection F, the following are equivalent

i) F is of bounded Borel class.

ii) F is admissible.

iii) F has a countably generated admissible structure.

iv) For F, its power set is an admissible structure.

<u>Proof</u>: (i) => (ii): Suppose F is of bounded Borel class. Let $\underline{\underline{B}}_0 \subset \underline{\underline{B}}$ be fixed satisfying the definition for boundedness of F. Now we can treat F as a collection of measurable maps from X to Y. Hence by applying theorem 19, there is a structure $\underline{\underline{F}}$ on F s.t the map $\varphi : F \times X \to Y$ is measurable. Now even if we increase the structure on X from $\underline{\underline{B}}_0$ to $\underline{\underline{B}}$ - that is even if we increase the σ-field in the domain of φ ; φ still remains measurable. Thus $\underline{\underline{F}}$ is an admissible

structure.

(ii). => (i). Let $\underline{\underline{F}}$ be any admissible structure for F. We first exhibit a countably generated σ-algebra $\underline{\underline{B}}_o \subset \underline{\underline{B}}$ such that every element of F is $\underline{\underline{B}}_o$ measurable. Let (G_n; $n \geq 1$) be a generator for $\underline{\underline{G}}$. By using the familiar technique that any measurable set in a product space is in the σ-algebra generated by countable number of rectangles, we can find a countably generated σ-algebra $\underline{\underline{B}}_o \subset \underline{\underline{B}}$ such that $\phi^{-1}(G_n)$ is in $\underline{\underline{F}}_o \times \underline{\underline{B}}_o$ for all n. In other words ϕ is $\underline{\underline{F}}_o \times \underline{\underline{B}}_o$ measurable. Since for any $G \in \underline{\underline{G}}$, $\phi^{-1}(G) \in \underline{\underline{F}}_o \times \underline{\underline{B}}_o$ and since $f^{-1}(G)$ is nothing but the f-section of $\phi^{-1}(G)$ it follows that for any $f \in F$, $f^{-1}(G) \in \underline{\underline{B}}_o$. In other words any element of F is $\underline{\underline{B}}_o$ measurable. Thus F can be treated as a collection of functions from the countably generated space (X, $\underline{\underline{B}}_o$) into (Y, $\underline{\underline{G}}$). F is still admissible, since $\underline{\underline{F}}$ is still an admissible structure. Thus from theorem 19 $\underline{\underline{F}}$ is of bounded Borel class relative to $\underline{\underline{B}}_o$. In view of our definition of bounded Borel classes (i) is satisfied.

(iii) Obviously implies (ii). Conversely if $\underline{\underline{F}}$ is admissible with an admissible structure $\underline{\underline{F}}$ then ϕ: F × X→ Y is $\underline{\underline{F}} \times \underline{\underline{B}}$ measurable. By an argument analogous to that of the above para we can find a countably generated structure

$\underline{\underline{F}}_0 \subset \underline{\underline{F}}$ such that ϕ is $\underline{\underline{F}}_0 \times \underline{\underline{B}}$ measurable. This $\underline{\underline{F}}_0$ is the required countably generated admissible structure to F proving (iii).

Since structures larger than admissible structures are still admissible (iii) => (iv). Conversely (iv) a priori implies (ii) and hence implies (iii) from what has been proved above. This proves the theorem completely.

A theorem similar to the above theorem can be stated and proved for the case where (Y, $\underline{\underline{C}}$) is countably generated and not necessarily separable - just as theorem 19 was formulated corresponding to theorem 16.

We shall now pass to the general case. Let (X, $\underline{\underline{B}}$) and (Y, $\underline{\underline{C}}$) be any Borel spaces. Let $\underline{\underline{C}}_0$ be a countably generated substructure of $\underline{\underline{C}}$ and F a collection of measurable maps from (X, $\underline{\underline{B}}$) to (Y, $\underline{\underline{C}}$). We say that F is bounded for $\underline{\underline{C}}_0$ if the collection F regarded as functions from (X, $\underline{\underline{B}}$) to (Y, $\underline{\underline{C}}_0$) is of bounded Borel class. Recall that this means the existence of a countably generated substructure $\underline{\underline{B}}_0 \subset \underline{\underline{B}}$ (of course depending on $\underline{\underline{C}}_0$) such that each f in F is ($\underline{\underline{B}}_0$, $\underline{\underline{C}}_0$) measurable and the collection F is of bounded Borel class between the two countably generated spaces. We now have the following theorem:

<u>Theorem 21</u>: Let $(X, \underline{\underline{B}}), (Y, \underline{\underline{C}})$ be any two Borel spaces and F a collection of measurable maps from X to Y. Let $\underline{\underline{C}}$ have a generator of cardinality $\leq N$ where N is any infinite cardinal. The following conditions on F are equivalent.

i) F is bounded for any countably generated substructure of $\underline{\underline{C}}$.

ii) F is bounded for any finitely generated substructure of $\underline{\underline{C}}$.

iii) For F, its power set is an admissible structure.

iv) F is admissible.

v) There is an N-generated admissible structure on F.

<u>Proof</u>: The sequence of the proof is (i) => (ii) => (iii) => (iv) => (v) => (i).

The implication (i) => (ii) is trivial.

(ii) => (iii) is proved as follows: Let $\underline{\underline{F}}$ denote the power set of F. Take any set $A \in \underline{\underline{C}}$. It suffices to show that $\varphi^{-1}(C)$ is in $\underline{\underline{F}} \times \underline{\underline{B}}$ where as usual φ is the map $\varphi(f,x)=f(x)$. Look at the structure on Y given by $\underline{\underline{C}}_0 = [\emptyset, A, A^c, Y]$. Since this is finitely generated, and F is bounded for $\underline{\underline{C}}_0$ now applying theorem 20 (In fact, to be more precise, the analogue of theorem 20 about which we have remarked, but not explicitly formulated) F has an admissible structure (or in other words the power set of F is admissible. Consequently $\overset{-1}{\varphi}(C) \in \underline{\underline{F}} \times \underline{\underline{B}}$.

(iii) a priori implies (iv).

(iv) => (v) can be proved as follows: Since F is admissible, let $\underline{\underline{F}}$ be any admissible structure. Fix a generator (G_α; $\alpha \in T$) for $\underline{\underline{C}}$ where cardinality of $T \leq N$. With each α, Fix a countably generated subalgebra $\underline{\underline{F}}_\alpha \subset \underline{\underline{F}}$ such that $\overset{-1}{\varphi}(G_\alpha)$ is in $\underline{\underline{F}}_\alpha \times \underline{\underline{B}}$. The structure generated by ($\underline{\underline{F}}_\alpha$; $\alpha \in T$) will now be admissible for F and is N-generated since N is infinite.

Proof that (v) => (i) is as follows: Take any admissible structure $\underline{\underline{F}}$ for F (whether N- generated or not) and any countably generated $\underline{\underline{C}}_o \subset \underline{\underline{C}}$. Since $\underline{\underline{F}}$ is still an admissible structure when F is regarded as a collection of functions from (X, $\underline{\underline{B}}$) to (Y, $\underline{\underline{C}}_o$) we have to apply theorem 20 (or its analogue as mentioned above - to be more precise) to terminate the proof of this theorem.

Though the naturality of structures can be defined even in the non-separable case; the unhappy state of affairs -

existing in the separable case have refrained us from doing so - for we have to just be content with the formulation and without even an upsilon insight into the problem.

§ 7. Lattice of Borel structures.

For any non-empty set X, we denote by L_X the collection of all σ-algebras on X. To avoid trivialities we make the blanket assumption that X has more than two points. For σ, σ' in L_X we define

$$\sigma \leq \sigma' \quad \text{iff} \quad \sigma \subset \sigma'$$

With this definition L_X is a lattice. For σ, σ' one can take

$\sigma \vee \sigma'$ = smallest structure on X generated by σ and σ'.

$\sigma \wedge \sigma'$ = set theoretic intersection of σ and σ'.

In fact L_X is a complete lattice. There is a null element in this lattice, viz. $\{\emptyset, X\}$ and a unit element, viz., $\underline{\underline{C}}_X$ the class of all subsets of X. These will be denoted, as is customary in lattice theory by 0 and 1 respectively.

Just as its cousin - the lattice of topologies, this lattice is not distributive. For instance one can take

$$X = [a, b, c]$$
$$\sigma = [\emptyset, (a, b), (c), X]$$
$$\sigma'' = [\emptyset, (b) (a, c), X]$$
$$\sigma' = [\emptyset, (a), (b, c), X]$$

to verify,

$$\sigma'' \wedge (\sigma \vee \sigma') \neq (\sigma'' \wedge \sigma) \vee (\sigma'' \wedge \sigma')$$

However there is one main difference between those two lattices - that is L_X and the lattice of topologies. The former, as we show later, is not complemented where as the latter is known to be complemented. Before proceeding to the deeper investigation of the lattice L_X, we shall recall a few definitions from lattice theory.

Let L be any abstract lattice with 0 and 1. Let $a, b \in L$. Say that an element a' in L is a <u>complement of a relative to b</u> if

$$a \vee a' = b \; ; \quad a \wedge a' = 0$$

Say that a' is a <u>weak complement</u> of a relative to b if

$$a \vee a' = b.$$

If $b = 1$, 'complement relative to b' will be simply written as <u>complement</u>. Similar remark applies for weak complements. Observe that any complement is also a weak complement, though not conversely (an example to this effect will be provided later). Also note that neither complements nor weak complements need be unique. As a matter of fact, in the example given above to show that L_X is not distributive, we find that both σ' and σ'' are complements to σ. A weak complement of a relative to b is said to be <u>minimal</u> if no element smaller than it is again a weak complement. A nonempty subset A contained in L is an <u>ideal</u> if

$$a, b \in A \quad \text{imply} \quad a \vee b \in A.$$

$$a \in A, \quad b \leq a \quad \text{imply} \quad b \in A.$$

An ideal A is said to be <u>proper</u> if it is properly contained in L. A proper ideal is <u>maximal</u> if the only proper ideal containing it is itself. A maximal ideal A is said to be principal or fixed if there is an element a_o in L (of course, depending on A) such that

$$a \in \Lambda \quad \text{iff} \quad a \leq a_o$$

A maximal ideal which is not fixed is called <u>free</u>. Dual concepts to ideals can also be defined but they are not necessary for our purposes.

Our programme now is to thoroughly discuss the complementation in L_X and then go on to maximal ideals.

<u>Theorem 22</u>: L_X is complemented iff X is countable. In case X is countable, the complement of any element other than 0 and 1 is never unique.

We shall conclude the assertion made by the theorem after observing several lemmas.

<u>Lemma 1</u>: If X is countable, then L_X is complemented.

<u>Proof</u>: Let $\sigma \in L_X$. It is easy to see that σ is countably generated. In fact it has atoms $(A_n, n \geq 1)$; either finitely many or infinitely many such that a subset of X belongs to σ if and only if it is the union of some atoms. Let A_o be any choice set for $(A_n, n \geq 1)$ - that is a set which has one and only one element in common with each A_n. Let σ' be the σ-algebra on X generated by A_o and all singletons, if any, in A_o^c. We show that σ' is a

complement of σ. Let Z be any nonempty set in $\sigma \wedge \sigma'$. Since $Z \in \sigma$ it contains some A_i and so has nonempty intersection with A_o. Since $Z \in \sigma'$ and has nonempty intersection with the atom A_o, we have $Z \supset A_o$ so that Z intersects every atom of σ. Now the fact that $Z \in \sigma$ tells us that $Z = X$. Thus $\sigma \wedge \sigma' = 0$. X being countable, in order to show that $\sigma \vee \sigma' = 1$, it suffices to verify that every singleton belongs to $\sigma \vee \sigma'$. Again by definition of σ^* it suffices to verify that every singleton in A_o belongs to $\sigma \vee \sigma'$. But this is obvious because any singleton in A_o is the intersection of A_o (belonging to σ') with some A_n (belonging to σ).

Lemma 2: If X is countable and $\sigma \in L_X$ is neither 0 nor 1 then σ has at least two complements.

Proof: By Lemma 1, σ has at least one complement.
If σ is neither 0 nor 1, then σ has an atom with at least two points and thus there exist at least two different choice sets that occur in the proof of Lemma 1. Since $\sigma \neq 0$, two different choice sets give rise to two different complements by the procedure outlined in the proof of Lemma 1.

Lemma 3: Let $\sigma < \sigma^*$ be in L_X. If σ' is a minimal weak complement of σ relative to σ^*, then σ' is also a complement relative to σ^*. However the converse need not be true.

<u>Proof</u>: We have to show, to prove the positive part of the lemma, that $\sigma \wedge \sigma' = 0$. Let A be any nonempty proper subset of X in $\sigma \wedge \sigma'$. We shall produce a weak complement of σ relative to σ^* smaller than σ', thus contradicting the minimality of σ'. To do this choose $x \in A$, $y \in A^c$. Let σ'' consist of all those elements of σ' which do not separate x and y, that is

$$\sigma'' = \left[B \in \sigma' \;\middle|\; \begin{array}{l} B \supseteq \{x, y\} \text{ or} \\ B \cap \{x, y\} = \emptyset \end{array} \right]$$

Obviously σ'' is a σ-algebra on X and hence belongs to L_X. Since $A \notin \sigma''$; $\sigma'' < \sigma'$. We shall now show that $\sigma \vee \sigma'' = \sigma^*$, thus showing that σ'' is also a weak complement. Since we already know that $\sigma \vee \sigma' = \sigma^*$ it suffices to show that $\sigma \vee \sigma'' \supseteq \sigma'$. Let $Z \in \sigma'$. If Z contains neither x nor y then by definition $Z \in \sigma''$. Also if Z contains both x and y even then $Z \in \sigma''$. So let $x \in Z$, $y \notin Z$. Since both Z and A^c belong to σ'; $Z \cap A^c \in \sigma'$ and since moreover it contains neither x nor y it belongs to σ''. By a similar argument $(Z \cap A) \cup A^c$ belongs to σ''. Since

$A \in \sigma$, it follows that

$$Z \cap A = [(Z \cap A) \cup A^c] \cap A$$

belongs to $\sigma'' \vee \sigma$. Thus both $Z \cap A$ and $Z \cap A^c$ belong to $\sigma \vee \sigma''$ and hence $Z \in \sigma \vee \sigma''$. Observe that if $Z \in \sigma'$ is such that $y \in Z$, $x \notin Z$ then by the above argument $Z^c \in \sigma \vee \sigma''$ and hence $Z \in \sigma \vee \sigma''$. Thus any $Z \in \sigma'$ also belongs to $\sigma \vee \sigma''$ as desired.

To observe that the converse need not be true take

X = Unit square.

σ^* = Borel algebra on X.

σ = Structure generated by the vertical lines, i.e., generated by the projection function to the X- coordinate.

σ'' = Structure generated by the horizontal lines, that is, generated by the projection to the Y- coordinate.

σ' = $\sigma'' \vee$ (Borel algebra on the X-axis).

Then one can easily verify that σ' and σ'' are both complements of σ relative to σ^*, $\sigma' > \sigma''$ and consequently σ^* can not be a minimal weak complement of σ relative to σ^*.

Sometimes we use w.c. for weak complement.

Lemma 4: Let $\sigma < \sigma^* \in L_X$ where σ^* is countably generated. Then any w.c. of σ relative to σ^* contains a countably generated w.c. Similar statement is true for complements.

Proof: Let σ' be a w.c. of σ relative to σ^*. Take any generators $\underline{\underline{G}}_1$ and $\underline{\underline{G}}_1'$ for σ and σ' respectively. Then $\underline{\underline{G}}^* = \underline{\underline{G}}_1 \cup \underline{\underline{G}}_1'$ is a generator for $\sigma \vee \sigma'$ and hence for σ^*. Consequently $\underline{\underline{G}}^*$ contains a countable subgenerator $(Z_n, n \geq 1)$. Let σ'' be the structure generated by all those Z_n s which are not in $\underline{\underline{G}}_1$. Obviously σ'' is countably generated. Since any Z_n which is not in $\underline{\underline{G}}_1$ should belong to $\underline{\underline{G}}_1'$ it follows that $\sigma'' \subset \sigma'$. Moreover, $\sigma \vee \sigma''$ contains all the Z_n' s and hence contains σ^*. Thus σ'' is also a w.c.

Lemma 5: Let $\sigma < \sigma^* \in L_X$ where σ^* is countably generated. Then any minimal w.c. of σ relative to σ^* - if any - is countably generated.

Proof: Obvious from minimality and Lemma 4.

Lemma 6: Let $\sigma_1, \sigma_2 \in L_X$ and N an atom of σ_2.

$$\sigma_1 \vee \sigma_2 \mid N = \sigma_1 \mid N$$

where $\sigma | N$ is the restriction of σ to N.

Proof: Obviously $\sigma_1|N \subset \sigma_1 \vee \sigma_2|N$. Since $\sigma_1 \cup \sigma_2$ is a generator for $\sigma_1 \vee \sigma_2$; inorder to show that $\sigma_1 \vee \sigma_2|N \subset \sigma_1|N$ it suffices to verify that intersection of any element in $\sigma_1 \cup \sigma_2$ with N is in $\sigma_1|N$. Again since it is obvious that the intersection of any element in σ_1 with N is in $\sigma_1|N$, it suffices to verify that $\sigma_2|N \subset \sigma_1|N$. Since N is an atom of σ_2, $\sigma_2|N = [\emptyset, N]$ and hence is contained in $\sigma_1|N$.

Lemma 7: Let X be uncountable and $\sigma^* \in L_X$ be separable. Let σ_0 be the countable-cocountable structure on X. Then σ_0 has no complement relative to σ^*.

Proof: Let σ_0' be, if possible, a complement. By Lemma 4, there is no loss in assuming that σ_0' is countably generated. Let A be an atom of σ_0'. By Lemma 6,

$$\sigma_0 \vee \sigma_0' \mid A = \sigma_0 \mid A.$$

Since $\sigma_0 \vee \sigma_0' = \sigma^*$ is separable the left side is a separable structure to A where as the right side is the countable-cocountable structure on A. Since A is uncountable (Reason: A belongs to σ_0' and hence can not be in σ_0 except when $A = \emptyset$ or X) this is a contradiction.

<u>Lemma 8:</u> With the same hypothesis as in Lemma 7, Let $\sigma \in L_X$. Then

$$\sigma \vee \sigma_0 = (Z \mid Z \triangle A \text{ is countable for some } A \text{ in } \sigma)$$

<u>Proof:</u> The right side of the above equality is a σ-algebra containing both σ and σ_0 and any structure containing σ and σ_0 should contain all such sets.

<u>Lemma 9:</u> With the same hypothesis as in Lemma 7, σ_0 has no complement in L_X.

<u>Proof:</u> Let if possible σ_0' be a complement. Put

$$\sigma_0'' = \sigma_0' \cap \sigma^*$$

We arrive at a contradiction to the conclusion of Lemma 7, if we can show that σ_0'' is a complement of σ_0 relative to σ^*. Obviously $\sigma_0 \cap \sigma_0''$ is $\{ \emptyset, X \}$. In fact

$$\{ \emptyset, X \} \subset \sigma_0 \cap \sigma_0'' \subset \sigma_0 \cap \sigma_0' = \{ \emptyset, X \}.$$

It remains to show $\sigma_0 \vee \sigma_0'' = \sigma^*$. Since both σ_0 and σ_0'' are contained in σ^* we show that

$$\sigma^* \subset \sigma_o \vee \sigma_o'' .$$

Let $Z \in \sigma^*$. Since $\sigma_o' \vee \sigma_o = 1$, we have in view of Lemma 8, sets Z_1, A_1 in σ_o' and σ_o respectively such that $Z = Z_1 \triangle A_1$. Since $Z \in \sigma^*$, $Z_1 \in \sigma^*$. Consequently $Z_1 \in \sigma_o''$. Again by Lemma 8 (or even directly) $Z \in \sigma_o \vee \sigma_o''$. This completes the proof of the Lemma.

<u>Lemma 10</u>: If $X \subset Y$ and L_X is not complemented, so is L_Y.

<u>Proof</u>: Let σ be any structure on X which has no complement in L_X. If σ_1 is the structure on Y generated by σ, then σ_1 can not have a complement in L_Y, for if it has, say, σ_1' then it is not difficult to see that $\sigma_1' | X$ is a complement to σ in L_X.

<u>Lemma 11</u>: For any uncountable X, L_X is not complemented.

<u>Proof</u>: Let X be any set with cardinality $\aleph_1$, the first uncountable cardinal. In view of Lemma 10, our assertion will be proved if, we can show that L_X is not complemented. Since we can find separable σ-algebras on X, the hypotheses of Lemma 7 is not vacuous and consequently an application of Lemma 9 completes the proof.

The proof of the main theorem is thus completed in view of Lemmas 1, 2 and 11. The reader might have noticed that we have nowhere needed Lemma 5. But because it is interesting in its own right we have stated it separately - though it is included in Lemma 4. In view of our theorem and Lemma 3 (which is also not used in the proof of the theorem) it follows that even minimal weak complements need not exist. This answers a question raised by D. Basu (See. D. Basu [1]). Though our theorem theoretically answers the complementation question in L_X, still many interesting problems remain. To mention one, what are those countably generated substructures of $\underline{\underline{B}}$ on I which have complements relative to $\underline{\underline{B}}$. Trivially any $\underline{\underline{B}}_o \subset B$ which has only countable number of atoms has a relative complement. We do not have any satisfactory answer to the above question. We exhibit in the theorem below, a class of such structures. To do this we feel it convenient to work with 2^W rather than I. We denote by $\underline{\underline{B}}$ the Borel structure on 2^W. Since we do not bring in I now, there is no fear of confusion.

<u>Theorem 23</u>: Let g be a continuous function on 2^W into a polish space such that for all $v \in$ Range of g, $g^{-1}(v)$ is homeomorphic to 2^W. Let $\underline{\underline{B}}_g \subset \underline{\underline{B}}$ be the subalgebra induced

by g. Then $\underline{\underline{B}}_g$ has a minimal weak complement relative to $\underline{\underline{B}}$.

We shall break the proof into several steps. The first is a Lemma of Purves.

<u>Lemma 12:</u> There exist homeomorphisms φ_v: $g^{-1}(v)$ onto 2^W such that the combined map s: 2^W to 2^W by

$$s(x) = \varphi_v(x) \qquad \text{if } x \in g^{-1}(v)$$

is a Borel map.

For a proof see Purves [1]. Purves takes his function to be defined on a Borel subset of 2^W and demands that g be bimeasurable. His proof carries over to this case as well.

<u>Lemma 13:</u> Let 0 be any fixed point of 2^W and $R = s^{-1}(0)$. Then R is a Borel subset of 2^W and is in fact a <u>selection</u> for the partition induced by g. Here s is the map given in Lemma 1. We tacitly assume that the φ_v s stated in Lemma 1 are already fixed (Recall that a selection for a partition of a space is a subset of the space which has one and only one point in common with each element of the partition).

Proof: Since s is a Borel map, R is a Borel set. Since each φ_v is a homeomorphism between $g^{-1}(v)$ and 2^W and since s coincides with φ_v on $g^{-1}(v)$ it is obvious that R is a selection.

Lemma 14: Let r be the map on 2^W onto R given by

$$r(x) = R \cap g^{-1}(g(x))$$

Then r is a Borel map. Moreover the structure $\underline{\underline{B}}_r$ induced by r on 2^W equals $\underline{\underline{B}}_g$.

Proof: In view of Lemma 13, g restricted to R is one to one and hence by Souslin's theorem if B is a Borel subset of R then $g(B)$ is Borel. Observe that if B is a Borel subset of R, then

$$r^{-1}(B) = g^{-1}(g(B)).$$

so that r is a Borel function. It is not difficult to see that r is onto. To show $\underline{\underline{B}}_r = \underline{\underline{B}}_g$ observe that both have same atoms and are countably generated and apply the Blackwell-Mackey theorem of Section 5.

Lemma 15: Let T be the map on 2^W to $R \times 2^W$ given by

$$T(x) = (r(x); s(x))$$

Then T is a Borel isomorphism.

Proof: Since r, s are Borel functions from Lemmas 14 and 12 it is clear that T is Borel. In view of Souslin's theorem the proof is complete if T is one to one and onto. If $(\alpha,\beta) \in R \times 2^W$; since $\varphi_{g(\alpha)}$ is a homeomorphism, there is an $x \in 2^W$ such that $\varphi_{g(\alpha)}(x) = \beta$; consequently $T(x) = (\alpha,\beta)$ so that T is onto. If $x,y \in 2^W$ and $x \neq y$ then if $g(x) \neq g(y)$ then the first coordinates of $T(x)$ and $T(y)$ differ, otherwise the second coordinates differ; consequently $T(x) \neq T(y)$ so that T is one to one.

Lemma 16: Let $\underline{\underline{B}}_1$, $\underline{\underline{B}}_2$ be the σ-algebras on $R \times 2^W$ induced by the coordinate maps and $\underline{\underline{B}}_o$ its Borel algebra. Then

$$T^{-1}(\underline{\underline{B}}_1) = \underline{\underline{B}}_r .$$

Moreover $\underline{\underline{B}}_2$ is a minimal weak complement for $\underline{\underline{B}}_1$ relative to $\underline{\underline{B}}_o$.

Proof: The first sentence is again a consequence of the Blackwell-Mackey theorem and the second sentence is obvious.

Lemma 17: If T is an isomorphism on $(X, \underline{\underline{B}})$ onto $(Y, \underline{\underline{C}})$ and $\underline{\underline{C}}_2 \subset \underline{\underline{C}}$ is a minimal weak complement of $\underline{\underline{C}}_1$ then $T^{-1}(\underline{\underline{C}}_2)$ is so for $T^{-1}(\underline{\underline{C}}_1)$.

Proof is straight forward and is hence omitted.

The proof of our theorem is complete in view of the above Lemma combined with Lemmas 14, 15, 16.

Leaving aside the problem of complementation, we now embark upon the problem of characterizing the Maximal ideals in L_X. Even in this area we have only some fragmentary results. Let us say that a structure σ on X is an Ultra structure if

$$\sigma \neq \underline{\underline{C}}_X ; \text{ and } \sigma' > \sigma \text{ implies } \sigma' = \underline{\underline{C}}_X .$$

With each $\sigma \in L_X$ we can associate the ideal

$$A_\sigma = [\, \sigma' \in L_X \mid \sigma' \leq \sigma \,].$$

Lemma 18: A_σ is a maximal ideal iff σ is an Ultra structure. In that case A_σ is a fixed maximal ideal.

Proof is straight forward, by contradiction.

Thus the problem of characterizing fixed maximal ideals reduces to that of characterizing the ultra structures on X. One kind of such structures are given by the following.

Lemma 19: Fix $x \neq y \in X$. Let

$$\sigma(x,y) = \left[A \subset X \;\middle|\; \begin{array}{l} A \supset \{x,y\} \text{ or} \\ A \cap \{x,y\} \neq \emptyset \end{array} \right]$$

Then $\sigma(x,y) \in L_X$ and is an ultra structure.

Proof: It is obvious that $\sigma(x,y)$ is a σ-algebra and hence belongs to L_X. Let $\sigma' > \sigma$. Then there is a $Z \in \sigma'$ which contains only x but not y. Since $\{x,y\} \in \sigma(x,y) \subset \sigma'$ it is easy to see that $\{x\}$ and $\{y\}$ both belong to σ'. Thus σ' contains both σ and $\{x\}$ and hence it coincides with $\underline{\underline{C}}_X$ as desired. Since $\sigma(x,y) \neq \underline{\underline{C}}_X$ the lemma is proved.

Another kind of ultra structures are obtained as follows.

Lemma 20: Let $x \in X$ and $\underline{\underline{F}}_x$ be a maximal family of subsets of $X - \{x\}$ with countable intersection property. Define

$$\sigma(x, \underline{\underline{F}}_x) = \left[A \subset X \;\middle|\; \begin{array}{l} \text{Either } A \text{ or } A^c \text{ contains} \\ Z \cup \{x\} \text{ for some } Z \\ \text{in } \underline{\underline{F}}_x \end{array} \right]$$

Then σ is an ultra structure.

<u>Proof:</u> As usual, we leave it for the reader to verify that σ is a Borel structure for X. Let $\sigma' > \sigma$. If we show that $\{x\} \in \sigma'$ then we are done. Because, in that case $\underline{\underline{F}}_x \subset \sigma'$, so that by the maximality of $\underline{\underline{F}}_x$ it follows that every subset of $X - \{x\}$ belongs to σ'. Since $\{x\} \in \sigma'$ this shows that $\sigma' = \underline{\underline{C}}_X$. Since $\sigma \ (x, \underline{\underline{F}}_x) \neq \underline{\underline{C}}_X$ the lemma will be proved.

It now remains to show that $\{x\} \in \sigma'$. Take a set $Y \in \sigma' - \sigma$. By properly choosing Y or its complement we can suppose without loss of generality that $x \in Y$. Let $Y_o = Y - \{x\}$. If $Y_o \in \underline{\underline{F}}_x$ then $Y \in \sigma$, contrary to our choice so that $Y_o \notin \underline{\underline{F}}_x$. Consequently there is a set $Z_o \in \underline{\underline{F}}_x$ such that $Z_o \cap Y_o = \emptyset$. Let $Z = Z_o \cup \{x\}$. $Z \in \sigma$ implies that it belongs to σ' also. Now the proof is complete since $\{x\} = Z \cap Y$.

However in many known cases, both the above Lemmas give us the same kind of ultra structures. To make this sentence precise recall that a cardinal α is $(0 - 1)$ measurable if on the class of all subsets of a set of cardinality α we can find a measure taking only two values 0 and 1 such that every singleton has measure 0. Otherwise α is non

non (0 - 1) measurable. Regarding these cardinals we shall first observe a well-known simple lemma:

<u>Lemma 21</u>: Let card (X) = α. α is measurable iff there is a maximal family $\underline{\underline{F}}$ of subsets of X with countable intersection property such that the whole intersection of elements of $\underline{\underline{F}}$ is void.

<u>Proof</u>: If α is measurable take a 0 - 1 measure μ as available on $\underline{\underline{C}}_X$. Let

$$\underline{\underline{F}} = [Z \mid \mu(Z) = 1]$$

This satisfies all the requirements. Conversely if there is such an $\underline{\underline{F}}$ define μ on $\underline{\underline{C}}_X$ by

$$\mu(Z) = \begin{cases} 1 & \text{if } Z \in \underline{\underline{F}} \\ 0 & \text{if } Z \notin \underline{\underline{F}} \end{cases}$$

Since $\underline{\underline{F}}$ is a maximal family, for each $Z \subset X$ either Z or Z^c belongs to $\underline{\underline{F}}$ and thus μ is defined on $\underline{\underline{C}}_X$. This μ satisfies the requirements to show that α is measurable.

We shall now show that Lemmas 19 and 20 give us the same class of ultra structures in many cases.

<u>Lemma 22:</u> Every $\sigma(x, y)$ of Lemma 19 is a $\sigma(x, \underline{\underline{F}}_x)$ of Lemma 20 for some suitably chosen $\underline{\underline{F}}_x$. The converse is true iff the cardinal of X is non-(0,1) measurable.

<u>Proof:</u> The first sentence of this lemma is obvious by considering

$$\underline{\underline{F}}_x = [Z \subset X - \{x\} \mid y \in Z]$$

Suppose cardinal of X is non-measurable. Take any $\sigma(x, \underline{\underline{F}}_x)$. Since obviously cardinal of $X - \{x\}$ is also non-measurable by Lemma 21 the intersection of elements of $\underline{\underline{F}}_x$ is non void. By the maximality of $\underline{\underline{F}}_x$ this intersection is not only a singleton but also belongs to $\underline{\underline{F}}_x$. Let this be y. Then clearly

$$\sigma(x, \underline{\underline{F}}_x) = \sigma(x,y).$$

If on the other hand X is measurable, then by Lemma 21, we can fix a family of $X - \{x\}$ say $\underline{\underline{F}}_x$ with countable intersection property but the whole intersection being void. It is not difficult to see that this $\sigma(x, \underline{\underline{F}}_x)$ can not be $\sigma(x,y)$ for any y whatsoever.

We shall now collect the lemmas together and state as

<u>Theorem 24:</u> With the same notation as before, every A_σ is

a fixed maximal ideal in L_X where $\sigma = \sigma(x,y)$ or $\sigma(x, \underline{\underline{F}}_X)$. Further if cardinal of X is non (0,1) measurable each $\sigma(x, \underline{\underline{F}}_X)$ is a $\sigma(x,y)$ for some y. Moreover if X is countable every fixed maximal ideal is of this form.

Proof: First two statements of this theorem are just a combined restatement of Lemmas 19, 20 in view of Lemmas 18 and 22. Only the last sentence of the theorem needs a proof. In view of Lemma 18, it suffices to show that in the countable case every ultra structure on X is given by Lemma 19. Let σ^* be any ultra structure. Since $\sigma^* \neq \underline{\underline{C}}_X$, there is at least one atom of σ^* say A_0 which is not a singleton. Again in view of the ultraness of σ^* this A_0 should be the only atom which is not a singleton and moreover this should consist of exactly two points - say x, y . Finally by ultraness $\sigma^* = \sigma(x,y)$. This completes the proof of the theorem.

We do not know whether the above theorem characterizes all the fixed maximal ideals of L_X in case X is uncountable. Even in case $X = I$ we could not settle the question. This question is equivalent to asking whether there are ultra-structures on I containing the countable-cocountable structure as the following lemma shows:

<u>Lemma 23</u>: Let X be any set. Then all the fixed maximal *ideals in* L_X *which do not contain the* countable-conountable structure are characterized by the above theorem.

<u>Proof</u>: Let A_{σ_0} be a fixed maximal ideal not covered by the above theorem. Note that σ_0 is an ultra structure. If a singleton say x_0 does not belong to σ_0 then define

$$\underline{\underline{F}}_{x_0} = \left[Z - \{x_0\} \middle| \begin{array}{l} Z \in \sigma_0 \\ x_0 \in Z \end{array} \right]$$

$\underline{\underline{F}}_{x_0}$ is a family with countable intersection property and $\sigma_0 = \sigma(x, \underline{\underline{F}}_{x_0})$. Since σ_0 is an ultra structure it is obvious that $\underline{\underline{F}}_{x_0}$ is a maximal family with countable intersection property and consequently A_{σ_0} is covered by the previous theorem. This contradiction shows that σ_0 does contain all singletons. Thus σ_0 is an ultra structure on X containing all singletons. This proves the lemma.

The converse of the above lemma is obvious. That is if there is an ultra structure say σ_0 containing all singletons then in view of Lemma 18, A_{σ_0} is a fixed maximal ideal not covered by Theorem 24.

Since on a countable set there is no ultra structure containing all singletons the last sentence of Theorem 24 can also be deduced from Lemma 23.

It is easy to see that there do exist free maximal ideals in L_X. For example take X to be a countable infinite set. With each $x \in X$, look at $\sigma^x = [\emptyset, (x), (x)^c, X]$. Then the collection $(\sigma^x; x \in X)$ can be extended to a maximal ideal. This maximal ideal can not be fixed. For if it is fixed say A_{σ_0} then σ_0 should contain all singletons! Of course if X is finite, trivially there can not exist free maximal ideals.

As the reader might have noticed, we have only made a start of the study of the lattice L_X. Still there is much - much more - to be studied. The author is at present still working on these and related problems and hopes to consider these elsewhere.

§ 8. Historical Comments

Parts of the contents of Sections 1 to 5 will be appearing in volume 21 of the Colloquium Mathematicum . The contents of section 6 were communicated to the same journal.

Prof. D. Basu has raised (oral) the question whether the Borel algebra on the unit interval has a minimal generator. (He believed no minimal generator exists!) Theorem 1 was observed in answering this question. That the problem of minimal generators is meaningful for arbitrary Boolean algebras was suggested to us by Prof. Ashok Maitra. The example of two countably generated substructures of the Borel structure on the real line with intersection not countably generated was observed by Prof. J. K. Ghosh. The problem about symmetric Borel structures answered in Theorem 14 was raised (oral) by Mr. K. Viswanath. The contents of Section 7 have been inspired by a paper of D. Basu (see [1]). Our attention to this paper was drawn by Prof. J. K. Ghosh.

CHAPTER 3

ON ANALYTIC AND BOREL STRUCTURES

§ 0. Summary.

It is the purpose of this chapter to collect together the problems that were scattered throughout the previous two chapters. This chapter has seven sections. The first three sections deal with the problems that are raised in or have connection with the topic of chapter 1. The last three sections deal with those of chapter 2. Section 4 stands midway between both. We shall number the problems serially and the number appears in brackets. Wherever possible we shall include some discussion about the problem involved. There are no new results in this chapter except perhaps one or two in the first section. Sometimes in what follows, we shall - if necessary - invoke the axiom of choice without explicit mention. Phrases like 'one does not know' 'It is not known' do of course mean that 'the present author does not know'.

§ 1. Baire property and Lebesgue measurability.

It is always of interest - and many times useful - to know as to what properties of Lebesgue measurable sets and functions have their counterparts in sets and functions with the Baire property. To elaborate this point further, let $\underline{\underline{L}}$ denote the collection of the Lebesgue measurable sets on the real line R and $\underline{\underline{Q}}$ the collection of sets with the Baire property. At the outset both are translation invariant non-countably generated σ-algebras on R closed under the Souslin operation (See Kuratowski [2]). The counterparts of sets of measure zero in $\underline{\underline{L}}$ are those of the first category in $\underline{\underline{Q}}$. This analogy goes further. Every Lebesgue measurable function (L.M.f) is Borel measurable outside a null set (All functions, unless otherwise stated to the contrary, take their values in the real line equipped with the usual topology and Borel field). Similarly any function with the Baire property (B.P.f) is Borel outside a set of the first category. In fact more precise versions are available - every B.P.f is continuous outside a set of the first category and any L.M.f is of Baire class 1 outside a null set. This difference, however, can be explained from the fact that every set in $\underline{\underline{L}}$ is a G_δ modulo a null set whereas every set in $\underline{\underline{Q}}$ is an open set modulo a set of the first category. Theorems 1, 3, and 2, 4 of chapter 1 point out that these analogies go still deeper. Section 40 of Kuratowski [2] shows that even in the occurrence of the pathologies there is a similarity. We shall point out two more similarities later in this section - which though not surprising, are believed to be now. It would be interesting to find out many more analogies and to see the uses they can be put to (P1).

It is a striking - but unfortunately not so well known - theorem of Sierpiński [1] that there exists a null set A, in L such that the sum

$$A + A = [\, x + y \mid x \in A,\ y \in A \,]$$

is not in L. This can be formulated in Q as follows:

Theorem 1: There exists a set A of the first category in Q such that A + A is not in Q.

Though the proof of this theorem is just similar to that of the above quoted theorem of Sierpiński, we give it in detail here since it is simple and elegant (since moreover the proof of Sierpiński's theorem is less well known). We shall first observe two simple facts.

Lemma 1: Let $Z = X \triangle Y$ where X is a nonempty open set and Y is of the first category. Then the difference set of Z contains an interval.

Proof: No loss in assuming that X = (a, b), a finite interval.

Take ∂ such that $0 < \partial < \frac{b-a}{4}$. We exhibit a point x in Z such that $x+\partial \in Z$, which will complete the proof of the theorem. For this it suffices to observe that $Y \cup (Y-\partial)$ is of the first category and hence there exists at least one x in $(a, a+\frac{b-a}{4})$ not in $Y \cup (Y-\partial)$.

Lemma 2: There exists a Hamel basis in $\underline{\underline{Q}}$. There exists a Hamel basis not in $\underline{\underline{Q}}$. Any Hamel basis in $\underline{\underline{Q}}$ is of the first category.

Proof: To prove the first sentence it is not difficult to observe that a Hamel basis lives in the Cantor set. For the third sentence, observe that if a Hamel basis H in $\underline{\underline{Q}}$ is not of the first category then by Lemma 1, the difference set of H contains an interval and hence any real number is a rational linear combination of two elements from H. For the second sentence of theorem choose by transfinite induction (Kuratowski [2] or Lévy*) a Hamel basis H which has non-empty intersection with every homeomorph of the Cantor set on R. If H is in $\underline{\underline{Q}}$ then it should be of the first category and consequently H^c contains a homeomorph of the Cantor set, which can not happen by the construction of H.

* See Proc. IV Berk. Symp. Vol. II p. 273-2[illegible]
'An extension of the Lebesgue measure of [illegible]

<u>Proof of theorem 1:</u> By Lemma 2 choose a Hamel basis H of the first category. Let

$$H_0 = \bigcup_{r:\ \text{rational}} rH$$

$$H_n = H_{n-1} + H_{n-1} \qquad \text{for} \quad n \geq 1$$

Observe that,

$$R = \bigcup_{n \geq 0} H_n$$

By using arguments similar to those of Lemma 2 one can show that if H_n is in $\underline{\underline{Q}}$ then it should be of the first category Since R is not of the first category it follows that one H_n is not in $\underline{\underline{Q}}$. Let p be the smallest integer such that H_p is not in $\underline{\underline{Q}}$. By the choice of H, it follows that $p \geq 1$. Now clearly H_{p-1} is of the first category such that $H_{p-1} + H_{p-1}$ is not in $\underline{\underline{Q}}$.

Just as from Sierpiński's theorem it has been observed by L. A. Rubel [1] that there exist pathological L.M.fs, one can observe the existence of pathological L.P.fs.

It is a well known fact that any additive function into itself is continuous, if it is either L.M.f. But strangely enough one knows a stronger and r result for L.M.f - again due to Sierpiński [2].

is continuous. But however no similar result is available for B.P.f (P2).

By assuming CH it is possible to exhibit an uncountable subset of R such that all its continuous images in R are null sets (Sierpiński [3]). But one does not know whether there is an uncountable subset of R such that all its continuous images in R are of the first category (P3).

An interesting theorem due to H. Cartan and G. Choquet (which appears in P. Lévy: ibid) says that R can be written as the direct sum of a countable number of Lebesgue full subgroups of R, that is, each has full Lebesgue outer measure. A similar statement is true with the Baire property. Let us say that a subset A of R is Baire full if any subset of A^c which has Baire property is of the first category. Then one can state that the real line is the direct sum of a countable number of Baire full sub-groups of R. The sub-groups exhibited by Cartan and Choquet will serve the purpose.

Paul Lévy defines a Hamel function as follows: Take two Hamel basis $H_1 = (W_\alpha \; ; \; \alpha \in T)$ and $H_2 = (W'_\alpha \; ; \; \alpha \in T)$ indexed by the same set T. Define a function on R to I as follows:

$$\text{If} \quad x = \Sigma\, r_i\, w_i \qquad w_i \in H_1$$

$$\text{then} \quad f(x) = \Sigma\, r_i\, w_i' \qquad w_i' \in H_2$$

Any function obtained in this way is called a Hamel function. Then an interesting theorem proved by Lévy says that the graph of any Hamel function - as a subset of R^2 - is either null or full. We do not know whether a similar statement holds in terms of the Baire property (P4). One feels it should be true - that is the graph of any Hamel function should be either of the first category in the plane or it should be full in the sense described previously.

Though P2, P3, P4 are included in P1, they are mentioned separately because they are interesting and rather more concretely posed. Well, of course, there are many more.

§ 2. Consistency problems (P5).

We have proved in chapter 1 that the product of discrete σ-algebras on I is the discrete σ-algebra on $I \times I$ - by assuming CH besides the axiom of choice. Let us, for ease of reference denote by D, D' and D'' the following statements:

D : $\underline{\underline{C}}_I \times \underline{\underline{C}}_I \neq \underline{\underline{C}}_{I \times I}$

D': There is a $U \subset I \times I$ such that for any separable $\underline{\underline{C}}_o \subset \underline{\underline{C}}_I$

$$U \notin \underline{\underline{C}}_o \times \underline{\underline{C}}_o$$

D'': $\underline{\underline{C}}_R \times \underline{\underline{C}}_R \neq \underline{\underline{C}}_{R \times R}$.

Let ZF_o be the Zermelo-Fraenkel theory without the axiom of choice and ZF be ZF_o + Axiom of choice. Then negation of D is a theorem of ZF + <u>CH</u>. But however we do not know whether ZF_o + D and ZF + D are consistent. A statement like ZF_o + D is consistent is used in the usual sense that if ZF_o does not lead to a contradiction then nor does ZF_o + D. In ZF the two statements D and D' are equivalent. We do not know whether they are equivalent in ZF_o. Of course it is always true that D => D'. But the statement D' => D can be proved with the axiom of choice. To be more clear let us recall the proof of D' => D. To see this what one does is that one shows that the class $\underline{\underline{S}}$ of all $U \in \underline{\underline{C}}_I \times \underline{\underline{C}}_I$ for which there is a separable $\underline{\underline{C}}_o \subset \underline{\underline{C}}_I$ with U in $\underline{\underline{C}}_o \times \underline{\underline{C}}_o$ forms a σ-algebra. It is easy to see that $\underline{\underline{S}}$ is closed under complementation but to show that $\underline{\underline{S}}$ is closed under countable unions one needs the axiom of choice. One can not show this with just the form $(\bar{A})$ IV of the countable axiom of choice as assured by the axiom of determinateness. Because $(\bar{A})$ IV only assures that if we have a countable number of sets with cardinality of their union smaller than c, then there is a choice function. Thus though D' is a theorem of $ZF_0 + (\bar{A})$ (in view of section 6 of chapter 1) we do not know whether D is a theorem in $ZF_0 + (\bar{A})$. Though there are many results pointing out towards the consistency of $ZF_0 + (\bar{A})$ (remarked to us by J. Mycielski) it is not known whether it is consistent. Observe that for D' to be consistent with ZF_0 it is not necessary that $(\bar{A})$ be consistent with ZF_0. In fact looking more closely to sections 2 and 6 of chapter 1, one can see that if $(\bar{A})$IV and $(\bar{\bar{A}})$I are consistent with ZF_0 then D' is consistent with ZF_0. Recall that $(\bar{A})$ IV states that every set of real numbers is Lebesgue measurable. The rest of this section is devoted to discuss about the consistency of D with ZF.

As observed in chapter 1, the problem about discrete spaces is one about cardinalities and consequently consistency of D with ZF is same as that of D'' with ZF. Since we know that D'' is true for a set R of cardinality > c, it is quite possible that D'' is a theorem when one starts with a suitable model for the real number system - more specifically with those models where R has a very large number of elements (of course, the word 'large' here is for a person outside the model). For instance the Scott- Solovoy model (See Dana Scott [1]) is one such. This model was used by Scott and Solovoy to show that the negation of CH is consistent with ZF.

We feel that for some suitably chosen Scott- Solovoy model one can show that D is consistent with ZF. It should be mentioned that Prof. Jan Mycielski, in a letter, expressed that he also arrived independently at these ideas.

What makes us worried is that though we know D' to be a theorem in $ZF_0 + (\bar{A})$ we know nothing about the status of D in $ZF_0 + (\bar{A})$.

If $\underline{B}$ denotes the Borel algebra of I, then we do not know the status of the proposition '$\underline{C}_I$ is a $\underline{B}$ - Souslin σ-algebra'. For definition of $\underline{B}$ - Souslin σ-algebra see section 7 of chapter 1.

§ 3. An isomorphism problem.

We know that any two uncountable Borel sets situated in any two Polish spaces are Borel isomorphic. Thus roughly speaking there is only one uncountable standard Borel set.

all the others being isomorphic to it. Is the same true for non-Borel analytic sets. Is there essentially one non-Borel analytic set? More precisely, given any two analytic non-Borel subsets of Polish spaces are they isomorphic? (P6). It would be very interesting if the answer turns out to be in the negative. Then the analytic sets can be classified into various isomorphism classes. There are two interesting special cases of the above problem which are worth mentioning. Are any two Universal analytic subsets of the plane isomorphic? (P7). The next one raised by Dr. Ashok Maitra is the following: Take any Universal analytic set U in the plane. Are U and $U \cup D$ isomorphic where D is the diagonal in the plane? (P8).

One way of looking at P6 may be as follows: Take two analytic non-Borel sets A and B on the Real line. After fixing sieves, which when sifted give A and B, one can talk about the constituents for A and B (See Kuratowski [2]) which are Borel sets. By using the isomorphism properties for Borel sets one can attempt to show that, under certain conditions on the constituents, A, B are isomorphic.

Another way of looking at the problem - which is not likely to succeed, but may give insight into the problem - is to try for Sieves V_r and W_r which when sifted give A and

B such that the following condition is satisfied: There is an order preserving map $\emptyset : Q \to Q$ where Q is the positive rationals in (0, 1) such that $V_r = W_{\emptyset(r)}$. Such a method has been used by Kondô [1] in a specific situation.

Yet another way of looking at P6 may be as follows: Fix an analytic non-Borel set A on the real line R. Look at all those subsets of R which are isomorphic to A. Denote this class of sets by Iso (A). Perhaps a study of the properties of the class Iso (A) may eventually lead to a clue for the solution to our problem. As a matter of fact this itself seems to be an independent interesting problem. For instance it is trivial to see that Iso (A) is translation invariant. Since any isomorph of A is again analytic non-Borel it is trivial to see that Iso (A) is anti-complemented in the sense that complement of no set in Iso (A) can again be in Iso (A).

It is however worth noting that the problem P6 formulated for coanalytic sets can is not so interesting as above. One knows that there could exist an uncountable coanalytic set without containing a homeomorph of the Cantor set. Take any such set say X and take any other non-Borel coanalytic set Y containing a perfect set. Then X and Y can not be isomorphic.

Unfortunately, from this fact one can not conclude that the two analytic sets X^c and Y^c are non-isomorphic. Of course P6 can be modified carefully for coanalytic sets. For instance, one can ask whether two coanalytic (non-Borel) sets both of which contain perfect sets are isomorphic.

The collection of non-empty Borel subsets of the real line can be decomposed into countable number of isomorphism classes. How nice it would be if the collection of analytic non-Borel sets can also be decomposed in an elegant way into isomorphism classes!

§ 4. Blackwell spaces and Souslin spaces.

Recall that a separable Borel space (X, $\underline{\underline{B}}$) is a **Blackwell space** if every separable sub-σ-algebra contained in $\underline{\underline{B}}$ equals $\underline{\underline{B}}$. Every analytic space is a Blackwell space. Blackwell [1] has raised the interesting question (P 9) as to whether every such space is isomorphic to an analytic space. Nothing much is known about this problem to this day. Maitra [1] has observed that there exist coanalytic spaces that are not Blackwell spaces. Again one does not know whether any coanalytic space is a Blackwell space. The interest in coanalytic spaces in this connection is due to the following reason.

the analytical heirarchy of sets the coanalytic sets are easy to handle when one is interested in finding out whether a nonanalytic space can be a Blackwell space.

We feel that (P9) is connected with the foundations of set theory. For instance, as remarked in section 3 of chapter 2 suppose there exists an uncountable set Z such that the only separable σ-algebra on Z is the class of all subsets of Z. Then Z with its power set is a Blackwell space. Since in this case cardinality of $Z < c$, it can not be an analytic space. We do not know whether the existence of such a set Z is consistent with ZF_0 or ZF. (P10). Of course if one accepts CH then such a set can not exist. As a matter of fact even if one accepts the hypothesis that

$$N \text{ is an uncountable cardinal} \Longrightarrow 2^N > c$$

then such a set Z can not exist.

Suppose $(X, \underline{\underline{B}})$ $(Y, \underline{\underline{C}})$ are two Blackwell spaces. We do not know whether $(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}})$ can be a non-Blackwell space (P 11). We do not know whether $(X \underset{d}{\cup} Y, \underline{\underline{B}} \underset{d}{\cup} \underline{\underline{C}})$ is a Blackwell space (P 12). Here $X \underset{d}{\cup} Y$ denotes the disjoint union. We do not know the answer to P 12 even if both the spaces are the same. If $(X, \underline{\underline{B}})$ is an uncountable Blackwell space, we do not know whether X contains a set Z such that $(Z, \underline{\underline{B}}_Z)$ is isomorphic to the unit interval (P 13). The interest in the problems P 11, P 12, P 13 is because of the following reason: Perhaps after studying the theory of Blackwell spaces in detail one can compare their properties with the - rich and abundant - properties of the analytic spaces so that any clues to the tough problem P 9 can be obtained.

Leaving the discussion on Blackwell spaces we shall mention a problem about Souslin spaces. Fix any separable Borel space $(X, \underline{\underline{B}})$. Recall that a σ-algebra $\underline{\underline{S}}$ on X containing $\underline{\underline{B}}$ is said to be $\underline{\underline{B}}$ - Souslin if any $\underline{\underline{S}}$ measurable real function on X is $\underline{\underline{B}}$ measurable when restricted to some uncountable set in $\underline{\underline{B}}$. If $\underline{\underline{S}}$ is $\underline{\underline{B}}$ - Souslin we do not know whether $\underline{\underline{S}} \times \underline{\underline{S}}$ is $\underline{\underline{B}} \times \underline{\underline{B}}$ - Souslin (P 14).

§ 5. Miscellany.

One of the interesting problems is to exhibit a Borel structure without a minimal generator (P 15). We do not know whether there exist two separable σ-algebras on I whose intersection is itself not separable but contains a [illegible] sub-σ-algebra (P 16). If the last conditi[illegible] following chapter 2 one can give plenty [illegible] do not know how to prove that the usual Borel structure on I has no atomless substructure (P 17). Without CH we do not know whether a separable σ-algebra can contain an atomless substructure (P 18). If CH is assumed we have answered this question in the affirmative in section 4 of chapter 2. An interesting problem raised by J. K. Ghosh and D. Basu is to characterize all the translation invariant sub-σ-algebras of the Borel σ-algebra or the real line (P 19). Observe that their theorem characterizes such structures which are countably generated (Section 5 of Chapter 2). In fact we feel that th their theorem characterizes all those countably generated translation invariant structures contained in L or Q. We do not know how to prove it (P 20). In this connection we also do not know the conditions under which a translation invariant extension of a countably generated translation invariant σ-algebra on R by adding one set will again be countably generated (P 21).

§ 6. Natural structures.

For the terminology to be used in this and next sections, the reader is referred to Chapter 2.

Suppose F is an admissible collection of measurable maps from one separable space $(X, \underline{\underline{B}})$ to another separable space $(Y, \underline{\underline{C}})$. A problem of Aumann is to find conditions under which a natural structure for F is available (P 22). Even if $X = I = Y$ we suspect that the Baire classes C_α for $\alpha > 1$ have no natural structure. It would be interesting to find out conditions under which F has a natural structure in our sense (P 23). This may be difficult but perhaps a study of this might reveal some properties about Blackwell spaces, as mentioned in section 6 of chapter 2.

§ 7. The Lattice L_X.

But for the few observations made in section 7 of chapter 2 the study of L_X seems to be completely blank. Thus the whole topic itself is wide open. We shall however be content with formulating two concrete problems. What are those sub-algebras of the usual Borel structure on I which are complemented relative to the Borel structure? (P 24). The same question can be asked for minimal weak complements of D. Basu. Next problem is to characterize all the maximal ideals in L_X. (P 25). The partial solutions of these problems we have given in chapter 2 are far from complete.

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