SPECTRAL PROPERTIES OF LARGE DIMENSIONAL RANDOM CIRCULANT TYPE MATRICES

KOUSHIK SAHA

Indian Statistical Institute, Kolkata
September, 2010

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Thesis submitted to the Indian Statistical Institute
in partial fulfillment of the requirements
for the award of the degree of
Doctor of Philosophy.
September, 2010
Thesis Advisor: Arup Bose

STATISTICAL
INDIAN
INSTITUTE
मिन्नेष्वैक्यस्य दर्शनम्
UNITY IN DIVERSITY

Indian Statistical Institute
203, B.T. Road, Kolkata, India.

In memory of my grandparents

Acknowledgement

As the journey is more important than destination, I take this opportunity to thank all those who made my journey and realizing the destination possible.

First and foremost, Prof. Arup Bose, whose wisdom, enthusiasm, perseverance and unconditional dedication towards mathematics motivated me to begin my journey with him. It is difficult to overstate my gratitude for all his guidance, advise, inspiration, encouragement and support throughout my thesis work. Even his mere presence acted as a psychological support to an extent unknown even to him.

I thank Rajat da, my companion and witness to all the frustration and victory in the field. His enthusiasm and perseverance were a major driving force during my hard times. His valuable suggestions and ideas helped me to improve the content of this thesis. I also thank him for allowing me to include the portion of our joint work in this thesis.

I am lucky to be taught by some of the wonderful teachers of Mathematics in India, Prof. B. V. Rao, Prof. S. M. Srivastava, Prof. G. Mukherjee, Prof. D. Goswami and Dr. K. Maulik, Dr. A. Dasgupta of ISI, Kolkata and Prof. B. V. Limaye, Prof. K. D. Joshi, Prof. D. V Pai and Prof. A. Athavale of I.I.T. Bombay. I am immensely grateful to all of them. I owe special thanks to Prof. B. V. Rao for his encouragement and support. I thank Prof. B. V. Limaye for his continuous support and encouragement throughout my Ph.D work.

I take this opportunity to express my deep sense of gratitude to Milan Maharaj, S.K.C sir and M.S.R sir of Ramakrishna Mission Vidyamandira College, Belur for their help and encouragement during my study in that college.

I now come to all those who were with me in the same boat. Biswarup, Abhijit da, Debasis da, Pusti da, Prosenjit da, Subhajit, Radhe, Subhra da, Ashis da, Santanu, Krishnapada, Jyoti da, Nabin da, Suratna and Debabrata. With all their camaraderie, entertainment, cooperation, encouragement and emotional support, they made my years at SMU. The journey would have been incomplete without them.

I am indebted to my parents for all their sacrifices and their faith and confidence that made me more responsible in life. I thank Sumita for being a constant source of inspiration and support and also being so happening to me.

I am also thankful to my departmental colleagues of Bidhannagar College for their support and co-operation.

Finally I express my sincere thanks to I.S.I. and C.S.I.R. for the financial support to carry out my research work.

Contents

Notation

$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
id	The identity map
$\mathcal{R}(z)$	Real part of z
$\mathcal{I}(z)$	Imaginary part of z
$\mathrm{tr}A$	Trace of matrix A
$\mathrm{Leb}(B)$	Lebesgue measure of B in appropriate dimension
$a_n \sim b_n$	$a_n - b_n \to 0$ as $n \to \infty$
$a_n \approx b_n$	$\frac{a_n}{b_n} \to 1$ as $n \to \infty$
$\lvert x\rvert$	Euclidean norm of $x \in \mathbb{R}^d$
$\lvert z\rvert$	Modulus of $z \in \mathbb{C}$
$\lfloor \frac{a}{b} \rfloor$	Greatest integer less or equal to $\frac{a}{b}$ for $a, b \in \mathbb{N}$
$\lceil \frac{a}{b} \rceil$	Least integer greater or equal to $\frac{a}{b}$ for $a, b \in \mathbb{N}$
$\#A$	Cardinality of the set A
$F_n \xrightarrow{\mathcal{D}} F$	Distribution function F_n converges to F in distribution (weakly) as $n \to \infty$
∂A	Topological boundary of a set $A \subset \mathbb{R}^n$
A^η	Set of all points at (euclidean) distances less than η from A

Chapter 0

Introduction

Consider a sequence of matrices whose dimension increases to infinity. Suppose the entries of this sequence of matrices are random. These matrices with increasing dimension are called large dimensional random matrices (LDRM).

Practices of random matrices, more precisely the properties of their eigenvalues, has emerged first from data analysis (beginning with Wishart (1928) [132]) and then from statistical models for heavy nuclei atoms (beginning with Wigner (1955) [130]). To insist on its physical applications, a mathematical theory of the spectrum of the random matrices began to emerge with the work of E. P. Wigner, F. J. Dyson, M. L. Mehta, C. E. Porter and co-workers in the 1960's. And this established the link between various branches of mathematics including classical analysis and number theory. Slowly it appeared in other branches of sciences as well, like high dimensional data analysis, communication theory, dynamical systems, finance, diffusion process and so on. The most important papers on random matrix theory in physics from this early period are collected in the book edited by Porter (1965) [102].

Initially enumerative combinatorics was the only, though very useful, tool to analyze random matrices. Many other sophisticated and varied mathematical tools are now available in the field. These includes Fredholm determinants (in the 1960's), diffusion processes (in the 1960's), integrable systems (in the 1980's and early 1990's), and the theory of free probability (in the 1990's). Many of the mathematical elements of random matrix theory which were developed in the beginning of the 1960's has been described in the book by Mehta (2004) [90].

One of the most important objects to study in random matrix theory is the spectra of LDRM. The necessity of studying the spectra of LDRM, especially the Wigner matrices, arose in nuclear physics during the 1950's. In quantum mechanics, the energy levels of quanta are not directly observable, but can be characterized by the eigenvalues of a matrix of observations. However the empirical spectral distribution (ESD) of the eigenvalues of a random matrix has a very complicated form when the order of the matrix is high. Many conjectures, e.g., the famous circular law conjecture were made through numerical computation.

The random matrix literature is vast and evergrowing. We provide a very brief introduction restricting ourselves to areas/results which have some relevance to the problems considered in this thesis. For detailed information on these and for other developments we refer to the excellent books by Mehta (2004) [90], Bai and Silverstein (2010) [13], Anderson, Guionnet and Zeitouni (2010) [3] and the survey papers of Bai (1999) [10], Bose, Hazra and Saha (2010) [39].

In Sections 0.1 and 0.2 we provide a brief summary of existing results on the limiting spectral distribution and on the extremes of eigenvalues. In this thesis we study the circulant and related random matrices. In Section 0.3 we provide some motivation to study such matrices. In Section 0.4 we give a brief summary of the thesis.

0.1 A brief survey of existing results on limiting spectral distribution

The research on limiting spectral analysis (LSA) of LDRM has attracted considerable interest among mathematicians, probabilists and statisticians. Here we discuss some of the more common matrix models that have been dealt with in the literature.

For any square matrix A, the probability distribution which puts equal mass on each eigenvalue of A is called the Empirical Spectral Measure of A. The corresponding distribution function is called the Empirical Spectral Distribution Function (ESD) of A.

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of an $n \times n$ matrix A_n, then the Empirical Spectral Distribution Function (ESD) is given by

$$F_{A_n}(x, y) = n^{-1} \sum_{i=1}^{n} \mathbb{I}\{\mathcal{R}(\lambda_i) \leq x,\ \mathcal{I}(\lambda_i) \leq y\}.$$

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of square matrices with the corresponding ESD $\{F_{A_n}\}_{n=1}^{\infty}$. The Limiting Spectral Distribution (LSD) of the sequence is defined as the weak limit of the sequence $\{F_{A_n}\}$, if it exists.

0.1.1 Wigner matrix and the semicircular law

Wigner matrix (W_n) is a symmetric matrix and it is defined as

$$W_n = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1(n-1)} & x_{1n} \\ x_{12} & x_{22} & x_{23} & \dots & x_{2(n-1)} & x_{2n} \\ & & & \vdots & & \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{(n-1)n} & x_{nn} \end{bmatrix}.$$

Wigner (1958) [131] assumed the entries $\{x_{ij}\}$ to be i.i.d. real Gaussian, and proved that the expected ESD of $\frac{1}{\sqrt{n}}W_n$ tends to the so called semicircular law which has the density function

$$p_W(s) = \begin{cases} \frac{1}{2\pi}\sqrt{4-s^2} & \text{if } |s| \le 2, \\ \\ 0 & \text{otherwise.} \end{cases}$$

It was also noted in Wigner (1958) [131] that the semicircle law is the LSD of much more general symmetric matrix models where the entries on and above the diagonal are independent and the entries have symmetric distribution function with variance σ^2 for the nondiagonal entries and $2\sigma^2$ for the diagonal ones and all higher moments are uniformly bounded. This claim motivated the interest of relaxing the conditions on entries of the matrix to the maximum possible extent, and Grenander (1963) [72] and Arnold (1967, 1971) [4, 5] generalized this work of Wigner in various aspects. In an important review work on random matrix theory by Bai (1999) [10], two general assumptions were used on the matrix model of Wigner : let W_n be $n \times n$ Hermitian matrix whose entries above the diagonal are i.i.d. complex random variables with variance 1 and whose diagonal entries are i.i.d. real random variables (without any moment requirement) or let $W_n = [w_{ij}]$ be $n \times n$ hermitian whose entries above the diagonal are independent complex random variables with a common mean 0 and variance 1 satisfying the Lindeberg type condition, for any $\delta > 0$ as $n \to \infty$,

$$\frac{1}{\delta^2 n^2}\sum_{i,j=1}^{n} \mathrm{E}\,|w_{ij}|^2 \mathbb{I}_{(|w_{ij}|>\delta\sqrt{n})} \to 0.$$

It was shown in Bai (1999) [10] that, under either assumption, as $n \to \infty$ with probability one the ESD of $\frac{1}{\sqrt{n}}W_n$ converges weakly to the semicircular law.

Anderson and Zeitouni (2006) [2] considered an $n \times n$ symmetric random matrix with on-or-above-diagonal terms of the form $\frac{1}{\sqrt{n}}f(\frac{i}{n}, \frac{j}{n})\xi_{ij}$ where ξ_{ij} are zero mean unit variance i.i.d. random variables with all moments bounded and f is a continuous function on $[0, 1]^2$ such that $\int_0^1 f^2(x, y)dy = 1$. They show that the empirical distribution of eigenvalues converges weakly to the semicircular law. There are other extensions which we do not discuss here. See Banerjee and Bose (2010) [21].

0.1.2 Sample covariance S matrix and Marčenko and Pastur law

Suppose that $\{x_{ij};\ i,j = 1,2,\ldots\}$ is a double array of i.i.d. complex random variables with mean zero and variance 1. Write $x_k = (x_{1k}, x_{2k}, \ldots, x_{pk})'$ and $X = (x_1, x_2, \ldots, x_n)$. The sample covariance matrix is usually defined by $S = \frac{1}{n}\sum_{k=1}^{n}(x_k - \bar{x})(x_k - \bar{x})^*$. However, in spectral analysis of LDRM, the sample covariance matrix is simply defined as $S = \frac{1}{n}\sum_{k=1}^{n} x_k x_k^* = \frac{1}{n}XX^*$.

We now describe the Marčenko-Pastur law denoted by $\mathcal{L}_{MPy}$: has a positive mass $1 - \frac{1}{y}$ at the origin if $y > 1$. Elsewhere it has the density:

$$p_{MPy}(x) = \begin{cases} \frac{1}{2\pi xy}\sqrt{(b-x)(x-a)} & \text{if } a \le x \le b, \\ \\ 0 & \text{otherwise} \end{cases} \tag{0.1.1}$$

where $a = a(y) = (1-\sqrt{y})^2$ and $b = b(y) = (1+\sqrt{y})^2$.

Researchers have established the LSD of S matrix under suitable conditions on the x_{ij}'s. Here, we state the LSD result under relatively simpler conditions. Suppose $\{x_{ij}\}$ are i.i.d. with mean zero and variance 1, $p \to \infty$ and $p/n \longrightarrow y \in (0,\infty)$. Then the ESD of S converges to $\mathcal{L}_{MPy}$ a.s..

The first success in finding the LSD of S is due to Marčenko and Pastur (1967) [87]. Subsequent work was done in Bai and Yin (1988) [16], Grenander and Silverstein (1977) [73], Jonsson (1982) [78], Wachter (1978) [127] and Yin (1986) [134], Bai (1999) [10]. When the entries of X are not independent, Yin and Krishnaiah (1985) [137] investigated the LSD of S matrix when the underlying distribution is isotropic. For further developments on S matrix see Bai and Zhou (2008) [19] and Bose, Gangopadhyay and Sen (2010) [31].

0.1.3 Toeplitz, Hankel and related matrices

Toeplitz matrix T_n and Hankel matrix H_n are defined as follows:

$$T_n = \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_0 & x_1 & \ldots & x_{n-3} & x_{n-2} \\ x_2 & x_1 & x_0 & \ldots & x_{n-4} & x_{n-3} \\ & & & \vdots & & \\ x_{n-1} & x_{n-2} & x_{n-3} & \ldots & x_1 & x_0 \end{bmatrix}.$$

$$H_n = \begin{bmatrix} x_2 & x_3 & x_4 & \dots & x_n & x_{n+1} \\ x_3 & x_4 & x_5 & \dots & x_{n+1} & x_{n+2} \\ x_4 & x_5 & x_6 & \dots & x_{n+2} & x_{n+3} \\ & & & \vdots & & \\ x_{n+1} & x_{n+2} & x_{n+3} & \dots & x_{2n-1} & x_{2n} \end{bmatrix}.$$

The existence of limiting spectral distributions of Toeplitz and Hankel matrices were proved by Bryc, Dembo and Jiang (2006) [47]. Hammond and Miller (2005) [75] also proved the existence of LSD of Toeplitz matrix. Bose and Sen (2007) [43] gave a unified approach based on the work of Bryc, Dembo and Jiang (2006) [47] to prove the existence of LSD of different LDRM.

If the top right corner and the bottom left corner elements of a matrix are zeroes, we call it a band matrix. The amount of banding may change with the dimension of the matrix. The LSD of the Toeplitz, Hankel band matrices was discussed in Basak and Bose (2009) [22], Kargin (2009) [82] and Liu and Wang (2009) [85]. For other variants of Toeplitz and Hankel matrices, also see Bose and Sen (2008) [43] and Basak and Bose (2010) [23].

0.1.4 I.I.D. matrix and the circular law

The most interesting problem in LDRM literature was the so called *circular law* conjecture that the ESD of the non-symmetric random matrix with i.i.d. entries, after suitable normalization, tends to the uniform distribution over the unit disc in complex plane. This was first established for Gaussian entries by Mehta (1967) [89]. Girko (1984) [67] suggested a method of proof for the general case. Bai (1997) [9] assumed smooth densities and bounded sixth moment of the entries and showed the result to be true. Götze and Tikhomirov (2007) [68] showed the result for sub-Gaussian entries and the moment conditions were further relaxed by Pan and Zhou (2010) [95], Götze and Tikhomirov (2007) [71] and Tao and Vu (2008) [121]. The result in its final form was derived by Tao, Vu and Krishnapur (2010) [122].

0.1.5 Rate of convergence

Another important aspect arose after the LSD of an LDRM was found: the convergence rate of LSD. This is of practical interest, but had been open for decades. The first success was made in Bai (1993a, 1993b) [7,8], in which convergence rates were established for the expected ESD of a large Wigner matrix and sample covariance matrix respectively. Bai's work developed a method of discussing convergence rates of ESDs through establishing a Berry-Esséen type inequality in terms of the Stieltjes transforms. The result was later improved in Bai, Miao and Tsay (1999) [11] for Wigner matrices by assuming a slightly milder condition. Then Bai, Miao and Yao (2003) [12] improved the results of Bai (1993b) [8] on the convergence rate of LSD of the S matrix. For other development in this direction, see Götze and Tikhomirov (2004, 2005) [69, 70].

0.2 A brief survey of existing results on extreme eigenvalues

Another aspect that became the focus of research was the limiting behaviour near the "edge": of the extreme eigenvalues, spectral norm and spectral radius. This behaviour of the extreme eigenvalues and related quantities is very nontrivial for most random matrices. We now give a very brief survey of results on extreme eigenvalues.

0.2.1 Extreme of S matrix

Historically, one of the first successes in the study of the extreme eigenvalues was by Geman (1980) [63], who proved that as $n \to \infty$ and $p/n \to y$, the largest eigenvalue of S matrix converges almost surely to $(1+\sqrt{y})^2$ under certain growth conditions on the moments of the entries. Yin, Bai and Krishnaiah (1988) [136] proved the same result under the existence of the fourth moment, and Bai, Silverstein and Yin (1988) [14] proved that the existence of the fourth moment is also necessary for the existence of the limit. Silverstein (1989) [111] found a necessary and sufficient condition for the weak convergence of the largest eigenvalue of S to a nonrandom limit.

It was much harder to study the convergence of the *smallest* eigenvalue of S. The first breakthrough was obtained in Silverstein (1985) [110], who established that the smallest eigenvalue of S converges to $(1-\sqrt{y})^2$ almost surely when the entries are i.i.d. standard normal and $p/n \to y$, $n \to \infty$. Bai and Yin (1993) [18] proved the almost sure convergence of the smallest eigenvalue under finiteness of fourth moment of the underlying distribution. As a byproduct, they also established the almost sure limit of the largest eigenvalue of the S matrix.

Johansson (2000) [76] proved that the properly scaled largest eigenvalue of S converges weakly to the Tracy-Widom law as n, p (dimension of X_n) tends to ∞, $n/p \to \gamma > 0$ and the entries are i.i.d. complex Gaussian. Johnstone (2001) [77] proved a similar result when the entries are real Gaussian. Soshnikov (2002) [116] generalized these results in two directions. He proved that the joint distribution of the upper ordered eigenvalues of Wishart matrices (after proper scaling) converges to the joint Tracy-Widom distribution and also extended the results to non-Gaussian entries provided $n - p = O(p^{1/3})$. El Karoui (2003) [54] extended the result of Johnstone to the case $p/n \to 0$ or ∞. Onatski (2008) [94] showed that the joint distribution of the centered and scaled several largest eigenvalues of p-dimensional complex Wishart matrix converges to the joint Tracy-Widom law when n and p tend to infinity so that n/p remains in a compact subset of $(0, \infty)$. This result was the extension of results of Baik, Ben Arous and Péché (2005) [20] and El Karoui (2007) [55] who studied the asymptotic distribution of the largest eigenvalue of the complex Wishart matrix as n and p go to infinity so that n/p remains in a compact subset of $[1, \infty)$. Péché (2009) [98] generalized the result of Soshnikov (2002) [116] when $p/n \to \gamma$ where $\gamma \in (0, \infty]$. For results on the smallest singular values of $n \times n$ matrix with i.i.d. entries, see Tao and Vu (2010) [106]. They showed that the limiting distribution of the smallest singular value is *universal* in the sense that it does not depend on the distribution of the entries. In particular, it converges to the same limiting distribution as in the special case when the entries are i.i.d. real Gaussian, and which was explicitly calculated by Edelman (1988) [52].

0.2.2 Extreme of Wigner matrix

Juhász (1981) [79] and Füredi and Komlós (1981) [61] studied the asymptotic properties of the largest eigenvalue of W under the existence of moments of all order. Sometimes they assume the uniform boundedness of entries. Bai and Yin (1988) [17] found necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of W. Some related work can be found in Geman (1986) [64] and Bai and Yin (1986) [15]. Geman proved that the spectral radius of a square matrix of i.i.d. entries, after proper scaling tends to one almost surely under a growth condition on the moments of the underlying distribution. The same result is proved in Bai and Yin (1986) [15] under only the finiteness of the fourth moment of the entries, as a by-product of a main lemma about the limiting behaviour of the operator norm of product of random matrices.

Soshnikov (2004) [117] considered the point process based on the positive eigenvalues of appropriately scaled W with heavy tailed entries $\{x_{ij}\}$ satisfying $\mathrm{P}(|x_{ij}| > x) = h(x)x^{-\alpha}$ where h is a slowly varying function at infinity and $0 < \alpha < 2$. He showed that it converges to an inhomogeneous Poisson random point process and from there, he deduced the distributional convergence of the maximum eigenvalue of an appropriately scaled W with such heavy tailed entries. The limiting distribution is $\Phi_\alpha(x) = \exp(-x^{-\alpha})$. A similar result was proved for sample covariance matrices in Soshnikov (2006) [118]. These results were extended in Auffinger, Ben Arous and Péché (2009) [6] to $2 \le \alpha < 4$.

Another important class of matrices related to W are the Gaussian matrix ensembles, which are Gaussian measures on spaces of Hermitian matrices A, obtained by multiplying a translation-invariant measure by the Gaussian function $\exp(-\mathrm{Tr}(A^2))$. The three main examples are the Gaussian orthogonal ensemble on real Hermitian matrices, the Gaussian unitary ensemble on complex Hermitian matrices, and the Gaussian symplectic ensemble on quaternionic Hermitian matrices. These matrices are also defined by the density of their eigenvalues. The joint probability density of the eigenvalues is given by (see Mehta (2004) [90])

$$P_{n\beta}(x_1, x_2, \ldots, x_n) = C_{n\beta} e^{-\frac{1}{2}\beta \sum_{i=1}^n x_i^2} \prod_{j<k} |x_j - x_k|^\beta.$$

For $\beta = 1$ the matrices are $n \times n$ real Hermitian, for $\beta = 2$ the matrices are $n \times n$ complex Hermitian, and for $\beta = 4$ the matrices are $2n \times 2n$ self-dual Hermitian or quaternionic Hermitian matrices. For $\beta = 4$ each eigenvalue has multiplicity two.

The distributional convergence of the largest eigenvalue of Gaussian orthogonal, unitary and symplectic ensembles were studied by Tracy and Widom (1994, 1996) [123, 124] in a series of articles. See Tracy and Widom (2000) [125] for a brief survey of such results. Soshnikov (1999) [115] showed that after proper scaling, the first, second, third, etc. eigenvalues of Wigner random hermitian (respectively, real symmetric) matrix weakly converge to the distributions established by Tracy and Widom for Gaussian unitary (respectively, Gaussian orthogonal) cases. Péché and Soshnikov (2007) [99] established a probabilistic upper bound on the spectral radius of W with i.i.d. bounded centered but non-symmetrically distributed entries. Péché and Soshnikov (2008) [100], established a probabilistic lower bound on the spectral radius of W with same type of entries and combining both the results, they established a rate of convergence result for the spectral radius of W. For some recent results on extreme gaps of the eigenvalues of Gaussian unitary ensembles see Ben Arous and Bourgade (2010) [24].

0.3 Some motivation to study circulant and related matrices

In the previous section we have briefly mentioned some of the more common random matrices (Wigner, S, Toeplitz and Hankel) and the results known on their LSD and extreme eigenvalues. All these matrices are patterened random matrices.

In this thesis we concentrate on some specific type of patterned matrices, namely, circulant, symmetric circulant, reverse circulant, k-circulant and Toeplitz matrices. An $n \times n$ *k-circulant* matrix is defined as

$$A_{k,n} = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_{n-k} & x_{n-k+1} & x_{n-k+2} & \cdots & x_{n-k-2} & x_{n-k-1} \\ x_{n-2k} & x_{n-2k+1} & x_{n-2k+2} & \cdots & x_{n-2k-2} & x_{n-2k-1} \\ & & & \vdots & & \\ x_k & x_{k+1} & x_{k+2} & \cdots & x_{k-2} & x_{k-1} \end{bmatrix}_{n\times n}.$$

For $1 \leq j < n-1$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by k positions (equivalently, k mod n positions). For $k = 1$ and $k = n-1$, it is known as circulant matrix (C_n) and reverse circulant matrix (RC_n) respectively. For detailed description of all these matrices, see Section 1.1.

Why should one study such matrices?

Nonrandom Toeplitz matrices and the corresponding Toeplitz operators are of course well studied objects in mathematics. Circulant matrices play a crucial role in the study of large dimensional Toeplitz matrices with nonrandom input. See, for example, Grenander and Szegő (1984) [74] and Gray (2006) [66]. Toeplitz matrices appear as the covariance matrix of stationary processes, in shift-invariant linear filtering and in many aspects of combinatorics, time series and harmonic analysis. Bai (1999) [10] proposed the study of large Toeplitz matrix with independent inputs. So, one of the motivations to study circulant matrix is to understand the behaviour of the Toeplitz matrix.

The eigenvalues of the circulant matrices also arise crucially in time series analysis. For instance, the periodogram of a sequence $\{a_l\}_{l\geq 0}$ is defined as $n^{-1}|\sum_{l=0}^{n-1} a_l e^{2\pi i j/n}|^2$, $-\lfloor \frac{n-1}{2} \rfloor \leq j \leq \lfloor \frac{n-1}{2} \rfloor$ and is a straightforward function of the eigenvalues of the corresponding circulant matrix. The study of the properties of the periodogram is fundamental in the spectral analysis of time series. See for instance Fan and Yao (2003) [58]. The maximum of the periodogram, in particular, has been studied in Davis and Mikosch (1999) [50].

The k-circulant matrices and their block versions arise in areas such as multi-level supersaturated design of experiment (Georgiou and Koukouvinos (2006) [65]), spectra of De Bruijn graphs (Strok (1992) [109]) and $(0,1)$-matrix solutions to $A^m = J_n$ (Wu, Jia and Li (2002) [133]). See also Davis (1979) [49] and Pollock (2002) [101].

Patterned matrices have deep connection with free probability theory. Limiting spectral distribution of such patterned matrices are related to different notions of independence – classical independence, free independence and half independence. Researchers studied the arbitrary product of Wigner matrices formed from a class of independent Wigner matrices. It is well known that the trace of any such product converges and this is tied to the idea of free independence developed by Voiculescu (1991) [126]. This freeness in the limit is very special to the Wigner type matrices. Bose, Hazra and Saha (2010) [38] studied the joint convergence of symmetric patterned matrices. In particular, they show that for independent copies of the Toeplitz, Hankel, symmetric circulant and reverse circulant matrices, the tracial limits exist for any monomial formed with these independent copies. It turns out that the symmetric circulant limit is classically independent with Gaussian marginals. The reverse circulant limit is half independent with symmetrized Rayleigh marginals. The Toeplitz and Hankel limits do not seem to submit to any easy or explicit independence/dependence notions. These limits are not free, independent or half independent.

0.4 Plan of the thesis

We now present a guided tour of the thesis. The sequence $\{x_i\}$ or $\{x_{ij}\}$ which will be used to build our matrices is called the *input sequence*. The circulant, reverse circulant, symmetric circulant and k-circulant matrices will together be called "*circulant type matrices*". We will investigate the following interesting aspects of (mainly) circulant type matrices :

(i) Existence and identification of limiting spectral distribution (LSD) with independent inputs.

(ii) Existence and identification of limiting spectral distribution (LSD) with dependent inputs.

(iii) Distributional convergence of the spectral norm and spectral radius with light tail inputs.

(iv) Distributional convergence of the spectral norm and spectral radius with heavy tail inputs.

(v) Limiting behaviour of the maximum of modulus of the appropriately scaled eigenvalues with dependent inputs.

(vi) Convergence of the point process constructed from the eigenvalues of circulant type matrices.

We now give a chapterwise brief description of this thesis. In Chapter 1 we describe the structure of different circulant type matrices and their eigenvalues. We also give short descriptions of other well known matrices in random matrix literature.

In Chapter 2 we deal with the limiting spectral distribution of the above mentioned matrices. Limiting spectral distribution of the scaled eigenvalues of Toeplitz and circulant type matrices are known when the input sequence is independent and identically distributed with finite moments of suitable order. We reestablish these known limits for circulant type matrices with lesser moment assumption on the input sequence. We then derive the LSD of these matrices when the input sequence $\{x_n\}$ is a stationary, two sided moving average process of infinite order, i.e.,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}, \quad \text{where} \quad a_n \in \mathbb{R} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |a_n| < \infty. \tag{0.4.1}$$

The limits are suitable mixtures of normal, symmetric square root of the chi-square, and other mixture distributions, with the spectral density of the process involved in the mixtures. For instance, we prove that under some conditions on $\{x_n\}$, the ESD of $\frac{1}{\sqrt{n}}RC_n$ (RC_n is the reverse circulant) converges weakly to the distribution F_R, where

$$F_R(x) = \begin{cases} 1 - \int_0^{1/2} e^{-\frac{x^2}{2\pi f(2\pi t)}} dt & \text{if } x > 0 \\ \int_0^{1/2} e^{-\frac{x^2}{2\pi f(2\pi t)}} dt & \text{if } x \leq 0, \end{cases}$$

and f is the spectral density function of $\{x_n\}$. Note that f is appearing in the limiting distribution F_R.

In Chapter 3 we digress from our main flow. There we identify the tail behaviour of finite but arbitrary product of i.i.d. exponential random variables. Suppose

$$H_n(x) = \mathrm{P}[E_1 E_2 \cdots E_n > x]$$

where $\{E_i\}$ are i.i.d. standard exponentials. We prove that

$$H_n(x) = C_n x^{\alpha_n} e^{-nx^{\frac{1}{n}}} g_n(x), \quad n \geq 1,$$

where for $n \geq 1$,

$$C_n = \frac{1}{\sqrt{n}}(2\pi)^{\frac{n-1}{2}}, \quad \alpha_n = \frac{n-1}{2n} \text{ and } g_n(x) \to 1 \text{ as } x \to \infty.$$

As a consequence, it follows that this n fold product of i.i.d. exponentials lies in the maximum domain of attraction of the Gumbel distribution for any n. We use this result to derive the limiting distribution of spectral radius of k-circulant matrix in Chapter 4.

In Chapter 4 we consider the spectral norm of scaled Toeplitz, circulant, reverse circulant, symmetric circulant and spectral radius of a class of k-circulant matrices when the input sequence is independent and identically distributed with finite moments of suitable order and the dimension of the matrix tends to ∞. We first review some known results on the spectral norm of Toeplitz and Hankel matrices. Then we prove the almost sure and the distributional convergence of the spectral norm of reverse circulant and circulant matrices. For instance, suppose $\{x_i\}$ is i.i.d. with mean μ and $E|x_i|^{2+\delta} < \infty$ for some $\delta > 0$. Now consider the reverse circulant (RC_n) matrix with inputs $\{x_i\}$. Then

$$\frac{\|RC_n\| - |\mu| n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1),$$

if $\mu \neq 0$, and

$$\frac{\|\frac{1}{\sqrt{n}}RC_n\| - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

if $\mu = 0$, where

$$q = q(n) = \lfloor \frac{n-1}{2} \rfloor, \quad d_q = \sqrt{\ln q}, \quad c_q = \frac{1}{2\sqrt{\ln q}}$$

and Λ is the standard Gumbel distribution.

We then consider the joint behaviour of minimum and maximum eigenvalue of symmetric circulant matrix and from there we deduce the distributional convergence of the spectral norm. We prove the distributional convergence of the spectral radius of k-circulant matrix where $n = k^g + 1,\ g \geq 2$ and then give an idea of how to deal with the more general case, $sn = k^g + 1$ with some suitable condition on s. In most of the cases after appropriate scaling and centering the limit distribution is the standard Gumbel distribution.

In Chapter 5 we consider the distributional convergence of the spectral norm of the scaled eigenvalues of large dimensional circulant, reverse circulant and symmetric circulant matrices when the input sequence is independent and identically distributed with appropriate heavy tail. For instance, suppose $\{Z_t,\ t \in \mathbb{Z}\}$ is a sequence of i.i.d. random variables with common distribution F where F is in the *domain of attraction* of an α-*stable* random variable with $0 < \alpha < 1$. Now consider RC_n with input sequence $\{Z_t\}$. Then under some conditions on $\{Z_t\}$, we show that $\|b_n^{-1} RC_n\| \xrightarrow{\mathcal{D}} Y_\alpha$, where Y_α is distributed as $S_\alpha(C_\alpha^{-\frac{1}{\alpha}}, 1, 0)$ and $b_n \approx n^{1/\alpha} L_0(n)$ for some slowly varying function L_0. Note that in this heavy tail situation the limit distribution is different from the Gumbel distribution. We also establish the distributional convergence of the spectral norm of circulant and symmetric circulant matrices. With such heavy tail inputs we are not able to obtain the exact limit of the spectral norm of the Toeplitz matrix. But we provide good upper and lower bounds in the distributional sense.

When the input sequence is a stationary two sided moving average process of infinite order, it is difficult to derive the limiting distribution of the spectral norm. For such an input sequence, we scale the eigenvalues of circulant type matrices by the spectral density at appropriate ordinates and study the limiting behaviour of the maximum of the modulus say M, in Chapter 6. There in Section 6.1 we consider stationary two sided moving average process of infinite order based on light tail entries and show distributional convergence of M for circulant type matrices. For instance, suppose $\{x_n\}$ is the two sided moving average process as in (0.4.1) and

$$M(n^{-1/2} RC_n, f) = \max_{1 \leq k < \frac{n}{2}} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$$

where λ_k are the eigenvalues of $n^{-1/2}RC_n$ and f is the spectral density of $\{x_n\}$. Then under some assumptions on $\{x_n\}$ we show that

$$\frac{M(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where $q = q(n) = \lfloor \frac{n-1}{2} \rfloor$, $d_q = \sqrt{\ln q}$ and $c_q = \frac{1}{2\sqrt{\ln q}}$.

Then in Section 6.2, we again consider the maximum of the modulus of scaled eigenvalues (M) but with heavy tail entries and establish the weak limit of M for reverse circulant, circulant and symmetric circulant matrices.

In Chapter 7 we consider the point processes based on the eigenvalues of the reverse circulant, symmetric circulant and k-circulant matrices with i.i.d. entries and show that they converge to Poisson random measures in vague topology. For example, let

$$\eta_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_j - b_q}{a_q}\right)}(\cdot)$$

be a point process based on the points $\{(\omega_j, \frac{\lambda_j - b_q}{a_q}),\ 0 \leq j < q\}$ where $\{\lambda_j\}$ are the eigenvalues of $n^{-1/2}RC_n$ and $\{\omega_j = \frac{2\pi j}{n}\}$ are the Fourier frequencies, a_q, b_q are appropriate scaling and centering constants and $q = \lfloor \frac{n}{2} \rfloor$. Then under some conditions on the entries we showed that $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, \pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1}dt \times e^{-x}dx$.

The joint convergence of upper k-ordered eigenvalues and their spacings follow from this result of Poisson convergence. We extend these results partially to the situation where the entries come from a two sided moving average process.

In Chapter 8 we list some open problems that arise in the context of the thesis.

Chapter 1

Matrices and eigenvalues

In this chapter we give a brief description of the matrices of our interest. Although some of them have been already introduced in Chapter 0, for the sake of completeness, we again discuss them here. Then we describe their eigenvalues whenever they can be obtained in some explicit form. In Section 1.1, we first describe the circulant type matrices, and the Toeplitz and the Hankel matrix. In later chapters we deal mainly with these matrices. At the end of this section we give a brief description of two other well known matrices, namely, the Wigner matrix and the sample covariance type matrix. In Section 1.2, we describe the structure of the eigenvalues of circulant type matrices, which will be used extensively in later chapters.

1.1 Some LDRMs of interest

The sequence of variables which will be used to construct the matrices is called the *input sequence*. It shall be of the form $\{x_i; i \geq 0\}$ or $\{x_{ij};\ i, j \geq 1\}$.

Circulant matrix: The circulant matrix is defined as

$$C_n = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \dots & x_{n-3} & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \dots & x_{n-4} & x_{n-3} \\ & & & \vdots & & \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_0 \end{bmatrix}_{n \times n}.$$

For $1 \leq j < n-1$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by one positions and the (i, j)-th element of the matrix is $x_{(j-i+n) \bmod n}$.

Symmetric circulant matrix: The symmetric version of the usual circulant is defined as

$$SC_n = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_2 & x_1 \\ x_1 & x_0 & x_1 & \dots & x_3 & x_2 \\ x_2 & x_1 & x_0 & \dots & x_2 & x_3 \\ & & & \vdots & & \\ x_1 & x_2 & x_3 & \dots & x_1 & x_0 \end{bmatrix}_{n\times n}.$$

The first row $(x_0\ x_1\ x_2 \dots x_2\ x_1)$ is a palindrome and the $(j+1)$-th row is obtained by giving its j-th row a right circular shift by one position. Its (i,j)-th element is given by $x_{n/2-|n/2-|i-j||}$.

Reverse circulant matrix: The reverse circulant matrix is given by

$$RC_n = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_0 \\ x_2 & x_3 & x_4 & \dots & x_0 & x_1 \\ & & & \vdots & & \\ x_{n-1} & x_0 & x_1 & \dots & x_{n-3} & x_{n-2} \end{bmatrix}_{n\times n}.$$

For $1 \le j < n-1$, its $(j+1)$-th row is obtained by giving its j-th row a left circular shift by one position. This is a symmetric matrix and its (i,j)-th element is given by $x_{(i+j-2) \mod n}$.

k**-circulant matrix**: This is a generalization of the usual circulant matrix. For positive integers k and n, the $n \times n$ *k-circulant* matrix is defined as

$$A_{k,n} = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_{n-k} & x_{n-k+1} & x_{n-k+2} & \dots & x_{n-k-2} & x_{n-k-1} \\ x_{n-2k} & x_{n-2k+1} & x_{n-2k+2} & \dots & x_{n-2k-2} & x_{n-2k-1} \\ & & & \vdots & & \\ x_k & x_{k+1} & x_{k+2} & \dots & x_{k-2} & x_{k-1} \end{bmatrix}_{n\times n}.$$

We emphasize that all subscripts appearing above are calculated modulo n. For $1 \le j < n-1$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by k positions (equivalently, $k \mod n$ positions). Observe that the circulant and the reverse circulant are special cases of the k-circulant when we let $k = 1$ and $k = n-1$ respectively.

Toeplitz matrix and palindromic Toeplitz matrix: The Toeplitz matrix is a symmetric matrix and its (i,j)-th element is $x_{|i-j|}$. So it is given by

$$T_n = \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_0 & x_1 & \ldots & x_{n-3} & x_{n-2} \\ x_2 & x_1 & x_0 & \ldots & x_{n-4} & x_{n-3} \\ & & & \vdots & & \\ x_{n-1} & x_{n-2} & x_{n-3} & \ldots & x_1 & x_0 \end{bmatrix}_{n\times n}.$$

Nonrandom Toeplitz matrices have been around in mathematics for a long time and their properties are well understood. See for example the classic book by Grenander and Szegő (1984) [74]. Recent information on this matrix may be found in Böttcher and Silbermann (1999) [45].

The palindromic Toeplitz matrix is the palindromic version of the usual symmetric Toeplitz matrix. It is defined as (see Massey, Miller and Sinsheimer (2006) [88]),

$$PT_n = \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_2 & x_1 & x_0 \\ x_1 & x_0 & x_1 & \ldots & x_3 & x_2 & x_1 \\ x_2 & x_1 & x_0 & \ldots & x_4 & x_3 & x_2 \\ & & & \vdots & & & \\ x_1 & x_2 & x_3 & \ldots & x_1 & x_0 & x_1 \\ x_0 & x_1 & x_2 & \ldots & x_2 & x_1 & x_0 \end{bmatrix}_{n\times n}.$$

Observe that the $n \times n$ principal minor of PT_{n+1} is SC_n. So PT_n is close to SC_n.

Hankel matrix: The (i,j)-th entry of the $n \times n$ random Hankel matrix H_n is x_{i+j-2}. It is closely related to the Toeplitz matrix and is given by

$$H_n = \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & \ldots & x_n & x_{n+1} \\ & & & \vdots & & \\ x_{n-1} & x_n & x_{n+1} & \ldots & x_{2n-2} & x_{2n-1} \end{bmatrix}_{n\times n}.$$

For detailed properties of Hankel matrices see the references cited above for the Toeplitz matrices.

Now, we give brief description of a few well known matrices in random matrix literature though we shall not use them later.

Wigner matrix: A Wigner matrix (see Wigner (1955,1958) [130, 131]) of order n and scale parameter σ is a Hermitian matrix of order n, whose entries above the diagonal are independent complex random variables with zero mean and variance σ^2, and whose diagonal elements are i.i.d. real random variables. So this matrix is given by

$$W_n = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1(n-1)} & w_{1n} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2(n-1)} & w_{2n} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3(n-1)} & w_{3n} \\ & & & \vdots & & \\ w_{n1} & w_{n2} & w_{n3} & \cdots & w_{n(n-1)} & w_{nn} \end{bmatrix}_{n\times n}$$

where $w_{kj} = \overline{w}_{jk}$ for $j < k$.

Sample covariance type matrices: Suppose $\{x_{jk},\ j,k = 1,2,\ldots\}$ is a double array of i.i.d. complex random variables with mean zero and variance one. Write $\mathbf{x_k} = (x_{1k},\ldots,x_{pk})'$ and let $X_n = [\mathbf{x_1}\ \mathbf{x_2}\ \cdots\ \mathbf{x_n}]$. In LDRM literature, the matrix

$$S_n = n^{-1}X_nX_n^*$$

is called a sample covariance matrix (in short an S matrix). As a concrete example, if $\{x_{ij}\}$ are real normal random variable with mean zero and variance one, then S_n is a Wishart matrix. Note that we do not centre the matrices at the sample means as is the convention in defining the sample covariance matrix in the statistics literature. This however, does not effect the LSD.

Now let $Y_n^{1/2}$ be any $p\times p$ Hermitian matrix, independent of X_n. Define

$$B_n = n^{-1}Y_n^{1/2}X_nX_n^*Y_n^{1/2}.$$

The matrices B_n are called sample covariance type matrices.

Matrix with i.i.d. entries: The matrix with i.i.d. entries (real or complex) has also received considerable attention in the literature and has given rise to the so called circular law conjecture. It is given by

$$U_n = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_n & x_{n+1} & x_{n+2} & \cdots & x_{2n-2} & x_{2n-1} \\ x_{2n} & x_{2n+1} & x_{2n+2} & \cdots & x_{3n-2} & x_{3n-1} \\ & & & \vdots & & \\ x_{n^2-n} & x_{n^2-n+1} & x_{n^2-n+2} & \cdots & x_{n^2-2} & x_{n^2-1} \end{bmatrix}_{n\times n}.$$

1.2 Description of eigenvalues

For most of the matrices in LDRM literature, explicit expression of the eigenvalues are not known. Below we give brief descriptions of the eigenvalues of circulant type matrices and we will use these descriptions very often in the later chapters. Define

$$\omega_k = \frac{2\pi k}{n} \quad \text{for} \quad 0 \le k \le n-1. \tag{1.2.1}$$

1.2.1 Circulant matrix

Its eigenvalues $\{\lambda_k,\ 0 \le k \le n-1\}$ are given by (see, for example, Brockwell and Davis (1991) [46]):

$$\lambda_k = \sum_{j=0}^{n-1} x_j \exp(i\omega_k j) = \sum_{j=0}^{n-1} x_j \cos(\omega_k j) + i \sum_{j=0}^{n-1} x_j \sin(\omega_k j), \quad 0 \le k \le n-1.$$

1.2.2 Symmetric circulant matrix

The eigenvalues $\{\lambda_k,\ 0 \le k \le n-1\}$ of SC_n are given by:

(a) for n odd:

$$\begin{aligned}\lambda_0 &= x_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} x_j \\ \lambda_k &= x_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} x_j \cos(\omega_k j), \quad 1 \le k \le \lfloor \frac{n}{2} \rfloor\end{aligned}$$

with $\lambda_{n-k} = \lambda_k$ for $1 \le k \le \lfloor \frac{n}{2} \rfloor$.

(b) for n even:

$$\begin{aligned}\lambda_0 &= x_0 + 2\sum_{j=1}^{\frac{n}{2}-1} x_j + x_{n/2} \\ \lambda_k &= x_0 + 2\sum_{j=1}^{\frac{n}{2}-1} x_j \cos(\omega_k j) + (-1)^k x_{n/2}, \quad 1 \le k \le \frac{n}{2}\end{aligned}$$

with $\lambda_{n-k} = \lambda_k$ for $1 \le k \le \frac{n}{2}$.

1.2.3 Reverse circulant matrix

The eigenvalues $\{\lambda_k,\ 0 \le k \le n-1\}$ are given by (Bose and Mitra (2002) [41]):

$$\lambda_0 = \sum_{j=0}^{n-1} x_j$$

$$\lambda_{\frac{n}{2}} = \sum_{j=0}^{n-1} (-1)^j x_j, \quad \text{if} \quad n \quad \text{is even}$$

$$\lambda_k = -\lambda_{n-k} = |\sum_{j=0}^{n-1} x_j \exp(i\omega_k j)|, \quad 1 \le k \le \lfloor \frac{n-1}{2} \rfloor.$$

1.2.4 k-circulant matrix

Here we give a brief description of the eigenvalues of the general k-circulant matrix. A more detailed analysis of the eigenvalues, useful in deriving the limiting distribution of the spectral radius for a specific class of k-circulant matrices, has been developed in Section 4.4.2. For further information on the properties of these eigenvalues, see Bose, Mitra and Sen (2008) [44]. Let

$$\nu = \nu_n := \cos(2\pi/n) + i\sin(2\pi/n) \text{ and } \lambda_k = \sum_{l=0}^{n-1} x_l \nu^{kl}, \quad 0 \le j < n. \tag{1.2.2}$$

For any positive integers k, n, let $p_1 < p_2 < \ldots < p_c$ be all their common prime factors so that,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}. \tag{1.2.3}$$

Here $\alpha_q,\ \beta_q \ge 1$ and n', k', p_q are pairwise relatively prime. For any positive integer s, let

$$\mathbb{Z}_s = \{0, 1, 2, \ldots, s-1\}.$$

Now for fixed k and n, define the following sets

$$S(x) = \{xk^b \bmod n' : b \ge 0\}, \tag{1.2.4}$$

where $0 \le x < n'$ and n' is as in (1.2.3). For any set A, let $\#A$ denote its cardinality. Let $g_x = \#S(x)$ and

$$\upsilon_{k,n'} = \#\{x \in \mathbb{Z}_{n'} : g_x < g_1\}. \tag{1.2.5}$$

We observe the following about the sets $S(x)$.

1. $S(x) = \{xk^b \bmod n' : 0 \le b < \#S(x)\}$.

2. For $x \neq u$, either $S(x) = S(u)$ or $S(x) \cap S(u) = \emptyset$. As a consequence, the distinct sets from the collection $\{S(x) : 0 \leq x < n'\}$ forms a partition of $\mathbb{Z}_{n'}$.

We shall call $\{S(x)\}$ the *eigenvalue partition* of $\{0, 1, 2, \dots, n-1\}$ and we will denote the partitioning sets and their sizes by

$$\{\mathcal{P}_0 = \{0\}, \mathcal{P}_1, \dots, \mathcal{P}_{l-1}\}, \text{ and } \quad n_i = \#\mathcal{P}_i, \; 0 \leq i < l. \tag{1.2.6}$$

Define

$$y_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \dots, l-1 \quad \text{where} \quad y = n/n'. \tag{1.2.7}$$

The following theorem provides the formula solution for the eigenvalues of $A_{k,n}$. In what follows, $\chi(A)(\lambda)$ stands for the characteristic polynomial of the matrix A evaluated at λ but for ease of notation, we shall often suppress the argument λ and write simply $\chi(A)$.

Theorem 1.2.1 (Zhou (1996) [138])**.** *The characteristic polynomial of $A_{k,n}$ is given by*

$$\chi\left(A_{k,n}\right)(\lambda) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} \left(\lambda^{n_j} - y_j\right), \tag{1.2.8}$$

where y_j is as defined in (1.2.7).

Proof of the above theorem is available in Zhou (1996) [138](Chinese article) and also in Bose, Mitra and Sen (2008) [44]. For sake of completeness we reproduce the proof given in Bose, Mitra and Sen (2008) [44]. Here recall $\{\alpha_q\}$ and $\{\beta_q\}$ from (1.2.3) and define

$$m := \max_{1 \leq q \leq c} \lceil \beta_q / \alpha_q \rceil, \quad [t]_{m,b} := tk^m \bmod b, \; b \text{ is a positive integer.} \tag{1.2.9}$$

Let $e_{m,d}$ be a $d \times 1$ vector whose only non-zero element is 1 at (m mod d)-th position, $E_{m,d}$ be the $d \times d$ matrix with $e_{jm,d}$, $0 \leq j < d$ as its columns and for dummy symbols

$\delta_0, \delta_1, \ldots$, let $\Delta_{m,b,d}$ be a diagonal matrix as given below.

$$e_{m,d} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{d\times 1}, \tag{1.2.10}$$

$$E_{m,d} = \left[e_{0,d} \;\; e_{m,d} \;\; e_{2m,d} \ldots e_{(d-1)m,d}\right], \tag{1.2.11}$$

$$\Delta_{m,b,d} = \text{diag}\left[\delta_{[0]_{m,b}}, \; \delta_{[1]_{m,b}}, \; \ldots, \; \delta_{[j]_{m,b}}, \; \ldots, \; \delta_{[d-1]_{m,b}}\right]. \tag{1.2.12}$$

Note that

$$\Delta_{0,b,d} = \text{diag}\left[\delta_{0 \bmod b}, \; \delta_{1 \bmod b}, \; \ldots, \; \delta_{j \bmod b}, \; \ldots, \; \delta_{d-1 \bmod b}\right].$$

We need the following lemma for the main proof.

Lemma 1.2.2. *Let $\pi = (\pi(0), \; \pi(1), \; \ldots, \; \pi(b-1))$ be a permutation of $(0, 1, \ldots, b-1)$. Let*

$$P_\pi = \left[e_{\pi(0),b} \; e_{\pi(1),b} \; \ldots e_{\pi(b-1),b}\right].$$

Then, P_π is a permutation matrix and the (i,j)th element of $P_\pi^T E_{k,b}\Delta_{0,b,b}P_\pi$ is given by

$$(P_\pi^T E_{k,b}\Delta_{0,b,b}P_\pi)_{i,j} = \begin{cases} \delta_t & \text{if } (i,j) = (\pi^{-1}(kt \bmod b), \pi^{-1}(t)), \quad 0 \le t < b \\ 0 & \text{otherwise.} \end{cases}$$

The proof is easy and we omit it.

Lemma 1.2.3. *Let k and b be positive integers. Then*

$$\chi\left(A_{k,b}\right) = \chi\left(E_{k,b}\Delta_{0,b,b}\right). \tag{1.2.13}$$

where, $\delta_j = \sum_{l=0}^{b-1} a_l \omega^{jl}$, $0 \le j < b$, $\omega = \cos(2\pi/b) + i\sin(2\pi/b)$, $i^2 = -1$.

Proof. Define the $b \times b$ permutation matrix

$$P_b = \begin{bmatrix} \underline{0} & I_{b-1} \\ 1 & \underline{0}^T \end{bmatrix}.$$

Observe that for $0 \le j < b$, the j-th row of $A_{k,b}$ can be written as $a^T P_b^{jk}$ where P_b^{jk}

stands for the jk-th power of P_b. From direct calculation, it is easy to verify that $P_b = UDU^*$ is a spectral decomposition of P_b where

$$\begin{aligned} D &= \text{diag}(1, \omega, \ldots, \omega^{b-1}), && (1.2.14) \\ U &= [u_0 \ u_1 \ \cdots \ u_{b-1}] \text{ with} && (1.2.15) \\ u_j &= b^{-1/2}(1, \omega^j, \omega^{2j}, \ldots, \omega^{(b-1)j})^T, \ 0 \le j < b. \end{aligned}$$

Note that $\delta_j = a^T u_j, \ 0 \le j < b$. From easy computations, it now follows that

$$U^* A_{k,b} U = E_{k,b} \Delta_{0,b,b},$$

so that, $\chi(A_{k,b}) = \chi(E_{k,b}\Delta_{0,b,b})$, proving the lemma. □

Lemma 1.2.4. *Let k and b be positive integers and, $x = b/gcd(k,b)$. Let for dummy variables $\gamma_0, \ \gamma_1, \ \gamma_2, \ldots, \gamma_{b-1}$,*

$$\Gamma = diag(\gamma_0, \ \gamma_1, \ \gamma_2, \ldots, \gamma_{b-1}).$$

Then

$$\chi(E_{k,b} \times \Gamma) = \lambda^{b-x} \chi\left(E_{k,x} \times diag\left(\gamma_{0 \ mod \ b}, \ \gamma_{k \ mod \ b}, \ldots, \gamma_{(x-1)k \ mod \ b}\right)\right) \qquad (1.2.16)$$

Proof. Define the following matrices

$$B_{b \times x} = \left[e_{0,b} \ e_{k,b} \ e_{2k,b} \ \ldots \ e_{(x-1)k,b}\right] \ \text{and} \ \ P = [B \ B^c]$$

where B^c consists of those columns (in any order) of I_b that are not in B. This makes P a permutation matrix.

Clearly, $E_{k,b} = [B \ B \ \cdots \ B]$ which is a $b \times b$ matrix of rank x, and we have

$$\chi(E_{k,b}\Gamma) = \chi\left(P^T E_{k,b} \Gamma P\right).$$

Note that,

$$\begin{aligned}
P^T E_{k,b} \Gamma P &= \begin{bmatrix} I_x & I_x & \dots & I_x \\ 0_{(b-x)\times x} & 0_{(b-x)\times x} & \dots & 0_{(b-x)\times x} \end{bmatrix} \Gamma P \\
&= \begin{bmatrix} C \\ 0_{(b-x)\times b} \end{bmatrix} P \\
&= \begin{bmatrix} C \\ 0_{(b-x)\times b} \end{bmatrix} [B \ B^c] = \begin{bmatrix} CB & CB^c \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

where,

$$\begin{aligned}
C &= [I_x \ I_x \ \cdots \ I_x] \Gamma \\
&= [I_x \ I_x \ \cdots \ I_x] \times \text{diag}(\gamma_0, \ \gamma_1, \ \dots, \ \gamma_{b-1}).
\end{aligned}$$

Clearly, the characteristic polynomial of $P^T E_{k,b} \Gamma P$ does not depend on CB^c, explaining why we did not bother to specify the order of columns in B^c. Thus we have,

$$\chi(E_{k,b}\Gamma) = \chi\left(P^T E_{k,b}\Gamma P\right) = \lambda^{b-x} \chi(CB).$$

It now remains to show that

$$CB = E_{k,x} \times \text{diag}\left(\gamma_{0 \bmod b}, \ \gamma_{k \bmod b}, \ \gamma_{2k \bmod b}, \cdots, \gamma_{(x-1)k \bmod b}\right).$$

Note that, the j-th column of B is $e_{jk,b}$. So, j-th column of CB is actually the (jk mod b)-th column of C. Hence, (jk mod b)-th column of C is $\gamma_{jk \bmod b} \ e_{jk \bmod x}$. So,

$$CB = E_{k,x} \times \text{diag}\left(\gamma_{0 \bmod b}, \ \gamma_{k \bmod b}, \ \gamma_{2k \bmod b}, \cdots, \gamma_{(x-1)k \bmod b}\right)$$

and the Lemma is proved completely. □

Proof. of Theorem 1.2.1. We first prove the Theorem for $A_{k,n'}$. Since k and n' are relatively prime, by Lemma 1.2.3,

$$\chi(A_{k,n'}) = \chi(E_{k,n'} \Delta_{0,n',n'}).$$

Get the partitioning sets $\mathcal{P}_0$, $\mathcal{P}_1$, ... of $\{0, 1, \dots, n'-1\}$, as in (1.2.6) where $\mathcal{P}_j = \{r_j k^x \mod n', \ 0 \le x < \#\mathcal{P}_j\}$ for some integer r_j. Let $N_0 = 0$ and $N_j = \sum_{i=1}^{j} n_i$ where $n_i = \#\mathcal{P}_i$. Define a permutation π on the set $\mathbb{Z}_{n'}$ as follows:

$$\pi(0) = 0 \text{ and } \pi(N_j + t) = r_{j+1} k^{t-1} \mod n' \text{ for } 1 \le t \le n_{j+1} \text{ and } j \ge 0.$$

This permutation π automatically yields a permutation matrix P_π as in Lemma 1.2.2. Consider the positions of δ_v for $v \in \mathcal{P}_j$ in the product $P_\pi^T E_{k,n'} \Delta_{0,n',n'} P_\pi$. We know, $v = r_j k^{t-1} \mod n'$ for some $1 \le t \le n_j$. Thus,

$$\pi^{-1}\left(r_j k^{t-1} \bmod n'\right) = N_{j-1} + t, \quad 1 \le t \le n_j$$

so that, position of δ_v for $v = r_j k^{t-1} \bmod n'$, $1 \le t \le n_j$ in $P_\pi^T E_{k,n'} \Delta_{0,n'} P_\pi$ is given by

$$\left(\pi^{-1}(r_j k^t \bmod n'), \pi^{-1}(r_j k^{t-1} \bmod n')\right) = \begin{cases} (N_{j-1} + t + 1,\ N_{j-1} + t) & \text{if, } 1 \le t < n_j \\ (N_{j-1} + 1,\ N_{j-1} + n_j) & \text{if, } t = n_j \end{cases}$$

Hence,

$$P_\pi^T E_{k,n'} \Delta_{0,n',n'} P_\pi = \operatorname{diag}(L_0,\ L_1,\ \ldots)$$

where, for $j \ge 0$, if $n_j = 1$ then $L_j = \left[\delta_{r_j}\right]$ is a 1×1 matrix, and if $n_j > 1$, then,

$$L_j = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & \delta_{r_j k^{n_j - 1} \bmod n'} \\ \delta_{r_j \bmod n'} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \delta_{r_j k \bmod n'} & 0 & \ldots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & \delta_{r_j k^{n_j - 2} \bmod n'} & 0. \end{bmatrix}$$

Clearly, $\chi(L_j) = \lambda^{n_j} - y_j$. Now the result follows from the identity

$$\chi\left(E_{k,n'} \Delta_{0,n',n'}\right) = \prod_{j \ge 0} \chi(L_j) = \prod_{j \ge 0} (\lambda^{n_j} - y_j).$$

Now let us prove the results for the general case. Recall that $n = n' \times \Pi_{q=1}^{c} p_q^{\beta_q}$. Then again using Lemma 1.2.3,

$$\chi(A_{k,n}) = \chi(E_{k,n} \Delta_{0,n,n}).$$

Recalling Equation (1.2.9), Lemma 1.2.3 and using Lemma 1.2.4 repeatedly with $y = n/n'$,

$$\begin{aligned} \chi(A_{k,n}) &= \chi(E_{k,n} \Delta_{0,n,n}) \\ &= \lambda^{n-n'} \chi(E_{k,n'} \Delta_{m,n,n'}) \\ &= \lambda^{n-n'} \chi(E_{k,n'} \Delta_{m+j,n,n'}) \quad [\text{ for all } j \ge 0] \\ &= \lambda^{n-n'} \chi\left(E_{k,n'} \times \operatorname{diag}\left(\delta_{[0]_{0,n}},\ \delta_{[y]_{0,n}},\ \delta_{[2y]_{0,n}}, \ldots, \delta_{[(n'-1)y]_{0,n}}\right)\right). \end{aligned}$$

Replacing $\Delta_{0,n',n'}$ by $\operatorname{diag}\left(\delta_{[0]_{0,n}},\ \delta_{[y]_{0,n}},\ \delta_{[2y]_{0,n}}, \ldots, \delta_{[(n'-1)y]_{0,n}}\right)$, we can mimic the rest of the proof given for $A_{k,n'}$, to complete the proof in the general case. □

Chapter 2

Limiting spectral distribution of circulant type matrices

In this chapter we establish the limiting spectral distribution of circulant type random matrices with dependent inputs. Bose and Mitra (2002) [41] first established the LSD of reverse circulant matrix with i.i.d. input under finite third moment assumption and Bose and Sen (2008) [43]) relaxed the moment assumption using a different approach. Sen (2006) [108] established the LSD of the usual circulant matix under finite third moment assumption of i.i.d. inputs. Recently Bose, Mitra and Sen (2008) [44] establish LSD for some specific type of k-circulant matrices with i.i.d. entries. Thus most of the existing work on LSD of circulant type matrices assumes the input sequence $\{x_i\}$ to be independent.

It is interesting to see what happens to the LSD results of circulant type matrices if dependent inputs are allowed. There are very few works dealing with dependent inputs for random matrices. For instance, Bose and Sen (2008) [43] established LSD for some specific type of dependent entries for the Toeplitz and Hankel matrices. Bai and Zhou (2008) [19] established the LSD of large sample covariance matrices with AR(1) entries. With the current methods used to establish LSD, such as the moment method or the Stieltjes transform method, it does not appear to be easy to extend the known results on LSD to general dependent situations for circulant type matrices. So we restrict ourselves to a specific type of dependent inputs.

We assume that $\{x_i\}$ is a stationary linear process. Stationary linear process is an important class of dependent sequence. For instance the widely used stationary time series models such as AR, MA, ARMA are all linear processes. Under very modest conditions on the process, we are able to establish the LSD for circulant type matrices. These LSD are functions of the spectral density of the process.

Here is an outline of this chapter. In Section 2.1 we give a few basic definitions related to limiting spectral distribution of large dimensional random matrices. In Section 2.2 we briefly describe three methods of establishing LSD's of different LDRM. Then in Section 2.3 we deal with the LSD of circulant type matrices with independent entries and in Section 2.4 we state and prove results on the LSD for dependent entries. In Section 2.5 we report some simulation which demonstrate our theoretical results.

Throughout the chapter c and C will denote generic constants depending only on dimension, d of the corresponding Euclidean space.

The results of Bose, Hazra and Saha (2009) [33] are based on this chapter.

2.1 Basic definitions

Unless otherwise stated, the entries of all matrices are real in general.

Definition 2.1.1. *For any square matrix A, the probability distribution which puts equal mass on each eigenvalue of A is called the* **Empirical Spectral Measure** *of A. The corresponding distribution function is called the* **Empirical Spectral Distribution Function** *(ESD) of A.*

If λ is an eigenvalue of an $n \times n$ matrix A_n with multiplicity m, then the Empirical Spectral Measure of A_n puts mass m/n at λ. Note that if the entries of A are random then it is a random probability measure. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues, then the Empirical Spectral Distribution Function (ESD) of A_n is given by

$$F_{A_n}(x, y) = n^{-1} \sum_{i=1}^{n} \mathbb{I}\{\mathcal{R}(\lambda_i) \leq x,\ \mathcal{I}(\lambda_i) \leq y\}.$$

If the eigenvalues are all real then the Empirical Spectral Distribution Function (ESD) of A_n is given by

$$F_{A_n}(x) = n^{-1} \sum_{i=1}^{n} \mathbb{I}\{\lambda_i \leq x\}.$$

The *expected spectral distribution function* of A_n is defined as $\mathrm{E}(F_n(\cdot))$. This expectation always exists and is a distribution function. The corresponding probability distribution is often known as the *expected spectral measure*.

Definition 2.1.2. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of square matrices with the corresponding ESD $\{F_{A_n}\}_{n=1}^{\infty}$. The* **Limiting Spectral Distribution** *(LSD) of the sequence is defined as the weak limit of the sequence $\{F_{A_n}\}$, if it exists. If $\{A_n\}$ are random, the limit is understood to be in some probabilistic sense, such as "almost surely" or "in L_2" or "in probability".*

Suppose elements of $\{A_n\}$ are defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$, that is, $\{A_n\}$ are random. Then $\{F_{A_n}(\cdot)\}$ are random and are functions of $\omega \in \Omega$ but we suppress this dependence. Let F be a nonrandom distribution function. We say the ESD of A_n converges to F *almost surely* if for almost every $\omega \in \Omega$ and at all continuity points (x, y) of F

$$F_{A_n}(x, y) \to F(x, y) \text{ as } n \to \infty.$$

We say the ESD of A_n converges to F *in* L_2 if at all continuity points (x, y) of F,

$$\int_{\Omega} \left[F_{A_n}(x, y) - F(x, y)\right]^2 d\mathrm{P}(\omega) \to 0 \text{ as } n \to \infty.$$

The ESD of A_n converges to F *in probability* if for $\epsilon > 0$ and at all continuity points (x, y) of F,

$$\mathrm{P}(|F_{A_n}(x, y) - F(x, y)| > \epsilon) \to 0 \text{ as } n \to \infty.$$

It is easy to see that in this case,

convergence *almost surely* $\Rightarrow$ convergence *in* L_2 $\Leftrightarrow$ convergence *in probability*.

2.2 Methods used in establishing LSD

Several methods to establish the LSD of LDRM are known in the literature. Out of these, the two most common methods are the moment method and the Stieltjes transform method. Though, in this thesis we will not use any of them, for sake of completeness we briefly explain these two methods below. Then we explain the method of normal approximation which is most suited for circulant type matrices. This will be our method of choice in this thesis.

2.2.1 Moment method

Suppose $\{Y_n\}$ is a sequence of real valued random variables with distribution functions $\{F_n\}$ such that $\mathrm{E}(Y_n^h) \to \beta_h$ for every positive integer h where $\{\beta_h\}$ satisfies Carleman's condition:

$$\sum_{h=1}^{\infty} \beta_{2h}^{-1/2h} = \infty. \tag{2.2.1}$$

Then there exists a distribution function F, such that for all h

$$\beta(h) = \beta_h(F) = \int x^h dF(x).$$

and Y_n converges to F in distribution.

For any positive integer h, the h-th moment of the ESD of a real symmetric $n \times n$ matrix A, with eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_n$ has the following nice form:

$$h\text{-th moment of the ESD of } A = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^h = \frac{1}{n}\operatorname{tr}(A^h) = \beta_h(A), \text{ say.}$$

Now, suppose $\{A_n\}$ is a sequence of $n \times n$ real symmetric random matrices such that

$$\beta_h(A_n) \to \beta_h.$$

Here the convergence takes place either "in probability" or "almost surely" and $\{\beta_h\}$ are nonrandom. If $\{\beta_h\}$ satisfies Carleman's condition then we can say that the LSD of the sequence $\{A_n\}$ exists and is some distribution function F (in the corresponding "in probability" or "almost sure" sense).

If convergence of the empirical moments takes place almost surely, then

$$\omega \in \{\omega \in \Omega : \beta_h(A_n)(\omega) \to \beta_h, \text{ for all } h\} \Rightarrow F_n(\omega) \xrightarrow{\mathcal{D}} F,$$

where F_n is the ESD of A_n. That is, $F_n \xrightarrow{\mathcal{D}} F$ a.s.

Note that the computation of $\beta_h(A_n)$ involves computing the trace of A_n^h or at least its leading term. This ultimately reduces to counting the number of contributing terms in the following expansion, (a_{ij} denotes the (i, j)-th entry of A):

$$\operatorname{tr}(A^h) = \sum_{1 \le i_1, i_2, \ldots, i_h \le n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{h-1} i_h} a_{i_h i_1}.$$

The method, though straightforward, is not practically manageable in most cases. The combinatorial arguments involved become quite unwieldy and even practically impossible as h and n increase.

However, this method has been successfully applied to Wigner matrix, sample covariance matrix and F matrix and recently to symmetric Toeplitz, Hankel, Markov, reverse circulant and symmetric circulant matrices. See Bai (1999) [10] for some of the arguments in connection with Wigner, sample covariance and F matrices. For the arguments concerning Toeplitz, Hankel and Markov matrices see Bryc, Dembo and Jiang (2006) [47] and Hammond and Miller (2005) [75]. For palindromic Toeplitz and circulant matrices, see Massey, Miller and Sinsheimer (2007) [88] and for reverse circulant and symmetric circulant matrices, see Bose and Sen (2008) [43].

2.2.2 Stieltjes transform method

Stieltjes transforms play an important role in deriving LSDs. They have also been used in studying rates of convergence.

Definition 2.2.1. *For any function G of bounded variation on the real line, its Stieltjes transform s_G is defined on $\{z: \ u+iv, v \neq 0\}$ as*

$$s_G(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} G(dx).$$

We shall be concerned with cases where G is the cumulative distribution function of some probability distribution on real line. If A has real eigenvalues $\lambda_i, \ 1 \leq i \leq n$, then the Stieltjes transform of the ESD of A is

$$s_A(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i - z} = \frac{1}{n} \text{tr}[(A - zI)^{-1}].$$

Let $\{A_n\}$ be a sequence of random matrices with real eigenvalues and let the corresponding sequence of Stieltjes transform be $\{m_n\}$. If $m_n \to m$ in some suitable manner, where m is a Stieltjes transform, then the LSD of the sequence $\{A_n\}$ is the unique probability on the real line whose Stieltjes transform is the function m. The convergence of the sequence $\{m_n\}$ is verified by first showing that it satisfies some (approximate) recursion equation. Solving the limiting form of this equation identifies the Stieltjes transform of the LSD.

For detailed developments of the properties of Stieltjes transform see Silverstein (2009) [113]. The method has been successfully applied to the Wigner and the sample covariance type matrices. See Bai (1999) [10] for details on the use of this transform to derive the convergence of the ESD. For its application in the study of rate of convergence of ESD see Bai (1999) [10], Bai, Miao and Yao (2003) [12] and Götze and Tikhomirov (2004, 2005) [69, 70].

2.2.3 Method of normal approximation

This method is most suited for the circulant type matrices. To apply this method fruitfully, one has to know the explicit formula of the eigenvalues. For most of the matrices in the literature, it is very difficult to compute the eigenvalues. However, as we have seen, for circulant type matrices the eigenvalue formula is known explicitly. This makes the normal approximation method ideally suited for those matrices. Bose and Mitra (2002) [41] first used this method to find the LSD of reverse circulant and symmetric circulant matrices with i.i.d. entries. Recently Bose, Mitra and Sen (2008)

[44] used this to establish the LSD for some specific type of k-circulant matrices with i.i.d. entries.

We use this method to prove the LSD results of circulant type matrices with independent and dependent inputs. This method is explained in details later in Sections 2.3 and 2.4.

2.3 Results on LSD with independent inputs

First recall that the ESD of A_n converges to F in L_2, where F is a distribution function, if at all continuity points (x, y) of F,

$$\int_\Omega \big[F_{A_n}(x,y) - F(x,y)\big]^2 d\mathrm{P}(\omega) \to 0 \text{ as } n \to \infty. \tag{2.3.1}$$

Note that the above relation holds if

$$\mathrm{E}[F_{A_n}(x,y)] \to F(x,y) \text{ and } V[F_{A_n}(x,y)] \to 0 \tag{2.3.2}$$

at all continuity points (x, y) of F. We often write F_n for F_{A_n} when the sequence of matrices under consideration is clear from the context.

As mentioned earlier we use the method of normal approximation to prove our results. For this we first state the following result on normal approximation (Berry-Esséen bound).

Lemma 2.3.1. *Let $X_1, \ldots, X_k$ be independent random vectors with values in $\mathbb{R}^d$, having zero means and an average positive-definite covariance matrix $V_k = k^{-1}\sum_{j=1}^k Cov(X_j)$. Let G_k denote the distribution of $k^{-1/2}T_k(X_1 + \cdots + X_k)$, where T_k is the symmetric, positive-definite matrix satisfying $T_k^2 = V_k^{-1}$, $n \geq 1$. If for some $\delta > 0$, $\mathrm{E}\|X_j\|^{(2+\delta)} < \infty$, then there exists $C > 0$ (depending only on d), such that*

(i)

$$\sup_{B\in\mathcal{C}} |G_k(B) - \Phi_d(B)| \leq Ck^{-\delta/2}[\lambda_{\min}(V_k)]^{-(2+\delta)}\rho_{2+\delta}$$

(ii) for any Borel set A,

$$|G_k(A) - \Phi_d(A)| \;\leq\; Ck^{-\delta/2}[\lambda_{\min}(V_k)]^{-(2+\delta)}\rho_{2+\delta} + 2\sup_{y\in\mathbb{R}^d}\Phi_d((\partial A)^\eta - y)$$

where Φ_d is the standard d dimensional normal distribution function, $\mathcal{C}$ is the class of all Borel measurable convex subsets of $\mathbb{R}^d$, $\rho_{2+\delta} = k^{-1}\sum_{j=1}^k \mathrm{E}\|X_j\|^{(2+\delta)}$ and $\eta = C\rho_{2+\delta}n^{-\delta/2}$.

Proof of Lemma 2.3.1 follows easily from Corollary 18.1, page 181 and Corollary

18.3, page 184 of Bhattacharya and Rao (1976) [28]. We omit its proof. Now consider the following assumption.

Assumption 2.3.2. *$\{x_i\}$ are independent,* $\mathrm{E}(x_i) = 0,\ V(x_i) = 1$ *and* $\sup_i \mathrm{E}\,|x_i|^{2+\delta} < \infty$*.*

We are now ready to establish the LSD of different circulant type matrices with independent inputs satisfying Assumption 2.3.2. Most of the results in this section are known in the literature but in all cases at least finiteness of third moment of i.i.d. inputs had been assumed. Here by using a better Berry-Esséen bound we reduce the moment condition to $(2+\delta)$ for some $\delta > 0$. These results are also precursor to the new results on the LSD for dependent inputs derived in the later sections.

The first theorem is on the LSD of usual circulant matrix with independent inputs.

Theorem 2.3.3. *If Assumption 2.3.2 is satisfied then the ESD of $\frac{1}{\sqrt{n}}C_n$ converges in L_2 to the two-dimensional normal distribution given by $\mathbf{N}(0, D)$ where D is a 2×2 diagonal matrix with diagonal entries* $1/2$*.*

Remark 2.3.4. *Sen (2006) [108] proves the same result under finite third moment assumption. Meckes (2009) [91] shows similar type of result for independent complex entries. In particular, if* $\mathrm{E}(x_j) = 0$*,* $\mathrm{E}\,|x_j|^2 = 1$ *and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathrm{E}(|x_j|^2 \mathbb{I}_{|x_j| > \epsilon\sqrt{n}}) = 0$$

for every $\epsilon > 0$, then the ESD converges in L_2 to the standard complex normal distribution.

Proof of Theorem 2.3.3. First recall the eigenvalues of circulant matrices from Section 1.2.1 and then observe that we may ignore the eigenvalue λ_n and also $\lambda_{n/2}$ whenever n is even since they contribute atmost $2/n$ to the ESD $F_n(x, y)$. So for $x, y \in \mathbb{R}$,

$$\mathrm{E}[F_n(x,y)] \quad \sim \quad n^{-1} \sum_{k=1, k\neq n/2}^{n-1} \mathrm{P}(b_k \le x, c_k \le y),$$

where

$$b_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \cos(\omega_k j), \quad c_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \sin(\omega_k j), \quad \omega_k = \frac{2\pi k}{n}. \tag{2.3.3}$$

Recall from (2.3.2) that it is enough to show

$$\mathrm{E}[F_n(x,y)] \to \Phi_{0,D}(x,y) \ \text{ and } \ V[F_n(x,y)] \to 0.$$

To show $\mathrm{E}[F_n(x,y)] \to \Phi_{0,D}(x,y)$, define for $1 \le k \le n-1$, (except for $k = n/2$) and $0 \le l \le n-1$,

$$X_{l,k} = \big(\sqrt{2}x_l \cos(\omega_k l), \quad \sqrt{2}x_l \sin(\omega_k l)\big)'.$$

Note that

$$\mathrm{E}(X_{l,k}) = 0 \tag{2.3.4}$$

$$n^{-1}\sum_{l=0}^{n-1} Cov(X_{l,k}) = I \tag{2.3.5}$$

$$\sup_n \sup_{1\le k\le n} [n^{-1}\sum_{l=0}^{n-1} \mathrm{E} \parallel X_{lk} \parallel^{(2+\delta)}] \le C < \infty. \tag{2.3.6}$$

For $k \neq n/2$

$$(b_k \le x, c_k \le y) = \big\{n^{-1/2}\sum_{l=0}^{n-1} X_{l,k} \le (\sqrt{2}x, \sqrt{2}y)'\big\}.$$

Since $\{(r,s) : (r,s)' \le (\sqrt{2}x, \sqrt{2}y)'\}$ is a *convex* set in $\mathbb{R}^2$ and $\{X_{l,k},\ l = 0,1,\dots(n-1)\}$ satisfies (2.4.9)–(2.4.11), we can apply Part (i) of Lemma 2.3.1 for $k \neq n/2$ to get

$$\begin{aligned}
&\big|\mathrm{P}\big(n^{-1/2}\sum_{l=0}^{n-1} X_{l,k}) \le (\sqrt{2}x, \sqrt{2}y)'\big) - \mathrm{P}\big((N_1,\ N_2)' \le (\sqrt{2}x, \sqrt{2}y)'\big)\big| \\
&\le Cn^{-\delta/2}[n^{-1}\sum_{l=0}^{n-1} \mathrm{E}\,\|X_{lk}\|^{(2+\delta)}] \le Cn^{-\delta/2} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}[F_n(x,y)] &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1,k\neq n/2}^{n-1} \mathrm{P}\big(b_k \le x, c_k \le y\big) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1,k\neq n/2}^{n-1} (N_1,\ N_2)' \le (\sqrt{2}x, \sqrt{2}y)'\big) \\
&= \Phi_{0,D}(x,y). \qquad (2.3.7)
\end{aligned}$$

Now, to show $V[F_n(x,y)] \to 0$, it is enough to show that

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} Cov(J_k, J_{k'}) = \frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} [\mathrm{E}(J_k, J_{k'}) - \mathrm{E}(J_k)\,\mathrm{E}(J_{k'})] \to 0. \tag{2.3.8}$$

where for $1 \leq k \leq n$, J_k is the indicator that $\{b_k \leq x, c_k \leq y\}$. Now as $n \to \infty$,

$$\frac{1}{n^2} \sum_{k \neq k'; k,k'=1}^{n} \mathrm{E}(J_k)\,\mathrm{E}(J_{k'}) = \Big[\frac{1}{n}\sum_{k=1}^{n} \mathrm{E}(J_k)\Big]^2 - \frac{1}{n^2}\sum_{k=1}^{n}\big[\mathrm{E}(J_k)\big]^2 \to \big[\Phi_{0,D}(x,y)\big]^2.$$

So to show (2.3.8), it is enough to show as $n \to \infty$,

$$\frac{1}{n^2} \sum_{k \neq k'; k,k'=1}^{n} \mathrm{E}(J_k, J_{k'}) \to \big[\Phi_{0,D}(x,y)\big]^2.$$

Along the lines of the proof used to show (2.3.7) one may now extend the vectors of two coordinates defined above to ones with four coordinates and proceed exactly as above to verify this. We omit the routine details. This completes the proof of Theorem 2.3.3.

□

In the following two theorems we state the LSD results for symmetric circulant and reverse circulant matrices with independent entries. Idea of the proof is similar to the previous proof, so we skip it.

Theorem 2.3.5. *If $\{x_i\}$ satisfies Assumption 2.3.2, then the ESD of $\frac{1}{\sqrt{n}}SC_n$ converges weakly in L_2 to the standard normal distribution.*

Theorem 2.3.6. *If $\{x_i\}$ satisfies Assumption 2.3.2 then the ESD of $\frac{1}{\sqrt{n}}RC_n$ converges weakly in L_2 to F, which is the symmetric square root of the chi-square with two degrees of freedom, having density*

$$f(x) = |x| \exp(-x^2), \; -\infty < x < \infty. \tag{2.3.9}$$

This limiting distribution is also known as the symmetrized Rayleigh distribution.

Remark 2.3.7. *Bose and Mitra (2002) [41] prove similar results for symmetric circulant and reverse circulant matrices with finite third moment assumption on i.i.d. inputs. Here by using better Berry-Esséen bound we reduce the moment condition.*

Remark 2.3.8. *One can derive the LSD of the palindromic Toeplitz matrix using Theorem 2.3.5. For this, we use Cauchy's interlacing inequality (see Bhatia (1997) [27], page 59):*

Interlacing inequality: *Suppose A is an $n \times n$ symmetric real matrix with eigenvalues $\lambda_n \geq \lambda_{n-1} \geq \ldots \geq \lambda_1$. Let B be the $(n-1) \times (n-1)$ principal submatrix of A with eigenvalues $\mu_{n-1} \geq \ldots \geq \mu_1$. Then*

$$\lambda_n \geq \mu_{n-1} \geq \lambda_{n-1} \geq \mu_{n-2} \geq \ldots \geq \lambda_2 \geq \mu_1 \geq \lambda_1.$$

As a consequence

$$\|F_A - F_B\|_\infty \leq \frac{1}{n}$$

where F_A *denote the ESD of the matrix* A *and* $\|f\|_\infty = \sup_x |f(x)|$.

By using interlacing inequality, if $\{x_i\}$ *satisfies Assumption 2.3.2, then the ESD of* $\frac{1}{\sqrt{n}} PT_n$ *converges weakly in* L_2 *to the standard normal distribution.*

2.3.1 k-circulant matrix

Now we come to the k-circulant matrix. Establishing the LSD for general k-circulant matrices appears to be a difficult problem. From the formula solution of the eigenvalues of k-circulant matrix, given in Section 1.2.4 it is clear that for many combinations of k and n, a lot of eigenvalues are zero. For example, if k is prime and $n = m \times k$ where $\gcd(m, k) = 1$, then 0 is an eigenvalue with multiplicity $(n - m)$. To avoid this degeneracy and to keep our exposition simple, we primarily restrict our attention to the case when $\gcd(k, n) = 1$.

In general, the structure of the eigenvalues depend on the number theoretic relation between k and n. If we keep k (other than 1) fixed and let n tends to infinity, then LSD may not exist. For example, the LSD of usual circulant matrices $n^{-1/2}A_{1,n}$ is bivariate normal. The ESD of 2-circulant matrix $n^{-1/2}A_{2,n}$ looks like a solar ring with no mass at zero for n large odd whereas if n is even the LSD has mass at zero (see Figures 2.1 and 2.2). Similarly, if $k = 3$ then the behaviour of the ESD depends on whether n is multiple of 3 or not a multiple of 3 (see Figures 2.2 and 2.3). So, for a fixed value of $k(\neq 1)$ the LSD may exist if we let n goes to infinity only along a subsequence depending on k. LSD in a few special cases are derived in Bose, Mitra and Sen (2008) [44] for i.i.d. inputs. The next theorem of Bose, Mitra and Sen (2008) [44] tells us that the radial component of the LSD of k-circulants with $k \geq 2$ is always degenerate, at least when the input sequence is i.i.d. normal, as long as $k = n^{o(1)}$ and $\gcd(k, n) = 1$. Observe that, in this case also n tends to infinity along a subsequence and it is determined by the condition $\gcd(k, n) = 1$.

Theorem 2.3.9 (Bose, Mitra and Sen (2008) [44]). *Suppose* $\{x_i\}_{i\geq 0}$ *is an i.i.d. sequence of* $N(0, 1)$ *random variables. Let* $k \geq 2$ *be such that* $k = n^{o(1)}$ *and* $n \to \infty$ *with* $\gcd(n, k) = 1$. *Then* $F_{n^{-1/2}A_{k,n}}$ *converges weakly in probability to the uniform distribution over the circle with center at* $(0, 0)$ *and radius* $r = \exp(\mathrm{E}[\log \sqrt{E}])$, E *being an exponential random variable with mean one.*

In some special cases Bose, Mitra and Sen (2008) [44] prove the LSD with independent entries. In particular, suppose $\{x_i\}$ are independent satisfying Assumption 2.3.2. Let $\{E_i\}$ be i.i.d. $Exp(1)$, U_1 be uniformly distributed over $(2g)$-th roots of unity, U_2 be uniformly distributed over the unit circle where $\{U_i\}$, $\{E_i\}$ are mutually independent. Then the following results are established there.

Theorem 2.3.10 (Bose, Mitra and Sen (2008) [44]). *Suppose $\{x_l\}_{l\geq 0}$ satisfies Assumption 2.3.2. Fix $g \geq 1$ and let p_1 be the smallest prime divisor of g.*

(i) Suppose $k^g = -1 + sn$ where $s = 1$ if $g = 1$ and $s = o(n^{p_1-1})$ if $g > 1$. Then $F_{n^{-1/2}A_{k,n}}$ converges weakly in probability to $U_1(\prod_{j=1}^{g} E_j)^{1/2g}$ as $n \to \infty$.

(ii) Suppose $k^g = 1+sn$ where $s = 0$ if $g = 1$ and $s = o(n^{p_1-1})$ if $g > 1$. Then $F_{n^{-1/2}A_{k,n}}$ converges weakly in probability to $U_2(\prod_{j=1}^{g} E_j)^{1/2g}$ as $n \to \infty$.

2.4 Result on LSD with dependent inputs

In this section we investigate the existence of the LSD of circulant type matrices under the following dependent situation.

Assumption 2.4.1. *$\{x_n,\ n \geq 0\}$ is a two sided moving average process*

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}, \quad \text{where} \quad a_n \in \mathbb{R} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |a_n| < \infty. \tag{2.4.1}$$

Assumption 2.4.2. *$\{\epsilon_i,\ i \in \mathbb{Z}\}$ are i.i.d. random variables with mean zero, variance one and $\mathrm{E}\,|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$.*

We show that the LSD of circulant type matrices continue to exist in this dependent situation under appropriate conditions on the spectral density of the process. The LSD turn out to be appropriate mixtures of the normal distribution, the symmetrized Rayleigh distribution, and some other related distributions. Quite expectedly, the spectral density of the process is involved in these mixtures. These results also reduce to the results given in Section 2.3 when we specialize to i.i.d. inputs.

2.4.1 Spectral density and some notation

Under Assumptions 2.4.1 and 2.4.2, $\gamma_h = Cov\,(x_{t+h}, x_t)$ is finite and $\sum_{j\in\mathbb{Z}} |\gamma_j| < \infty$. The *spectral density function* f of $\{x_n\}$ exists, is continuous, and is given by

$$f(\omega) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \gamma_k \exp(ik\omega) = \frac{1}{2\pi}\big[\gamma_0 + 2\sum_{k\geq 1}\gamma_k \cos(k\omega)\big] \text{ for } \omega \in [0, 2\pi].$$

Let

$$I_n(\omega_k) = \frac{1}{n}\Big|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}\Big|^2, \quad k = 0, 1, \ldots, n-1, \tag{2.4.2}$$

denote the periodogram of $\{x_i\}$ where $\omega_k = 2\pi k/n$ are the Fourier frequencies. Define

$$\psi(e^{i\omega}) = \sum_{j=-\infty}^{\infty} a_j e^{ij\omega}, \quad \psi_1(e^{i\omega}) = \mathcal{R}[\psi(e^{i\omega})], \; \psi_2(e^{i\omega}) = \mathcal{I}[\psi(e^{i\omega})], \tag{2.4.3}$$

where a_i's are as in (2.4.1). It is easy to see that

$$|\psi(e^{i\omega})|^2 = [\psi_1(e^{i\omega})]^2 + [\psi_2(e^{i\omega})]^2 = 2\pi f(\omega).$$

Let

$$B(\omega) = \begin{pmatrix} \psi_1(e^{i\omega}) & -\psi_2(e^{i\omega}) \\ \psi_2(e^{i\omega}) & \psi_1(e^{i\omega}) \end{pmatrix} \text{ and for } g \geq 2,$$

$$B(\omega_1, ..., \omega_g) = \begin{pmatrix} \psi_1(e^{i\omega_1}) & -\psi_2(e^{i\omega_1}) & 0 & 0 & \cdots & 0 \\ \psi_2(e^{i\omega_1}) & \psi_1(e^{i\omega_1}) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \psi_1(e^{i\omega_2}) & -\psi_2(e^{i\omega_2}) & \cdots & 0 \\ 0 & 0 & \psi_2(e^{i\omega_2}) & \psi_1(e^{i\omega_2}) & \cdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \psi_1(e^{i\omega_g}) & -\psi_2(e^{i\omega_g}) \\ 0 & 0 & 0 & \cdots & \psi_2(e^{i\omega_g}) & \psi_1(e^{i\omega_g}) \end{pmatrix}.$$

The above sets, functions and matrices will play a crucial role in the statements and proofs of the main results later.

2.4.2 Circulant matrix with dependent input

Define for $(x, y) \in \mathbb{R}^2$ and $\omega \in [0, 2\pi]$,

$$H_C(\omega, x, y) = \begin{cases} \mathrm{P}\big(B(\omega)(N_1, N_2)' \leq \sqrt{2}(x, y)'\big) & \text{if } f(\omega) \neq 0, \\ \mathbb{I}(x \geq 0, y \geq 0) & \text{if } f(\omega) = 0, \end{cases}$$

where N_1 and N_2 are i.i.d. standard normal variables.
Let

$$C_0 = \{t \in [0, 1] : f(2\pi t) = 0\}.$$

Lemma 2.4.3. *(i) For fixed x, y, H_C is a bounded continuous function in ω.*

(ii) F_C defined as follows is a proper distribution function.

$$F_C(x, y) = \int_0^1 H_C(2\pi s, x, y) ds. \tag{2.4.4}$$

(iii) If $\text{Leb}(C_0) = 0$ *then* F_C *is continuous everywhere and can be expressed as*

$$F_C(x, y) = \iint \mathbb{I}_{\{(v_1,v_2)\leq(x,y)\}} \Big[\int_0^1 \frac{1}{2\pi^2 f(2\pi s)} e^{-\frac{v_1^2+v_2^2}{2\pi f(2\pi s)}} ds \Big] dv_1 dv_2. \tag{2.4.5}$$

Further, F_C *is bivariate normal if and only if* f *is constant almost everywhere (Lebesgue).*

(iv) If $\text{Leb}(C_0) \neq 0$ *then* F_C *is discontinuous only on* $D_1 = \{(x, y) : xy = 0\}$.

The proof of the Lemma is easy and we omit it. We just show how the normality claim in (iii) follows by applying Cauchy-Schwarz inequality to compare the fourth moment and square of the variance and using the fact that for the normal distribution their ratio equals 3.

Proof of 2.4.3 (iii). If f is constant it easy to see that F_C is bivariate normal. Now suppose F_C is bivariate normal. Let (X, Y) be a random vector defined on some probability space with distribution function F_C. Now it is easy to see that

$$\text{E}(X) = 0, \quad \text{E}(X^2) = \pi \int_0^1 f(2\pi s) ds \quad \text{and} \quad \text{E}(X^4) = 3\pi^2 \int_0^1 f^2(2\pi s) ds.$$

Since (X, Y) is bivariate normal, X is a normal random variable and hence

$$\begin{aligned} &\text{E}(X^4) = 3[\text{E}(X^2)]^2 \\ \Rightarrow \quad &3\pi^2 \int_0^1 f^2(2\pi s) ds = 3\pi^2 \left(\int_0^1 f(2\pi s) ds \right)^2. \end{aligned} \tag{2.4.6}$$

Now (2.4.6) holds if and only if f is constant almost everywhere. □

Theorem 2.4.4 (Bose, Hazra and Saha (2009) [33])**.** *Suppose Assumptions 2.4.1 and 2.4.2 hold. Then the ESD of* $\frac{1}{\sqrt{n}} C_n$ *converges in* L_2 *to* $F_C(\cdot)$ *given in (2.4.4)–(2.4.5).*

Remark 2.4.5. *If* $\{x_i\}$ *are i.i.d with finite* $(2 + \delta)$ *moment, then* $f(\omega) \equiv 1/2\pi$*, and* F_C *reduces to the bivariate normal distribution whose covariance matrix is diagonal with entries* $1/2$ *each. This agrees with the conclusion in Theorem 2.3.3.*

Before going into the proof of Theorem 2.4.4, we observe a general fact which will be used in the proofs.

Lemma 2.4.6. *Suppose* $\{\lambda_{n,k}\}_{1\leq k\leq n}$ *is a triangular sequence of* $\mathbb{R}^d$*-valued random variables such that* $\lambda_{n,k} = \eta_{n,k} + y_{n,k}$ *for* $1 \leq k \leq n$*. Assume the following holds:*

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P(\eta_{n,k} \leq \tilde{x}) = F(\tilde{x})$ *for* $\tilde{x} \in \mathbb{R}^d$,

(ii) $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k,l=1}^{n} P(\eta_{n,k} \le \tilde{x}, \eta_{n,l} \le \tilde{y}) = F(\tilde{x})F(\tilde{y})$, *for* $\tilde{x}, \tilde{y} \in \mathbb{R}^d$,

(iii) for any $\epsilon > 0$, $\max_{1\le k\le n} P(|y_{n,k}| > \epsilon) \to 0$ *as* $n \to \infty$.

Then,

(a) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P(\lambda_{n,k} \le \tilde{x}) = F(\tilde{x})$.

(b) $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k,l=1}^{n} P(\lambda_{n,k} \le \tilde{x}, \lambda_{n,l} \le \tilde{y}) = F(\tilde{x})F(\tilde{y})$.

Proof. We define new random variables Λ_n with $\mathrm{P}(\Lambda_n = \lambda_{n,k}) = 1/n$ for $k = 1, \ldots, n$. Then

$$\mathrm{P}(\Lambda_n \le \tilde{x}) = \frac{1}{n}\sum_{k=1}^{n} \mathrm{P}(\lambda_{n,k} \le \tilde{x}).$$

Similarly define E_n (on the same probability space) with $\mathrm{P}(E_n = \eta_{n,k}) = 1/n$ for $1 \le k \le n$ and Y_n with $\mathrm{P}(Y_n = y_{n,k}) = 1/n$ for $1 \le k \le n$. Now observe that $\Lambda_n = E_n + Y_n$ and for any $\epsilon > 0$,

$$\mathrm{P}(|Y_n| > \epsilon) = \frac{1}{n}\sum_{k=1}^{n} \mathrm{P}(|y_{n,k}| > \epsilon) \to 0, \text{ as } n \to \infty$$

by Assumption (iii). Therefore Λ_n and E_n have the same limiting distribution. Now as $n \to \infty$,

$$\mathrm{P}(E_n \le \tilde{x}) = \frac{1}{n}\sum_{k=1}^{n} \mathrm{P}(\eta_{n,k} \le \tilde{x}) \to F(\tilde{x}), \text{ (by Assumption } (i)).$$

Therefore as $n \to \infty$,

$$\frac{1}{n}\sum_{k=1}^{n} \mathrm{P}(\lambda_{n,k} \le \tilde{x}) = \mathrm{P}(\Lambda_n \le \tilde{x}) \to F(\tilde{x})$$

and this is conclusion (a).

To prove (b) we use a similar type of argument. Here we define new random variables $\tilde{\Lambda}_n$ with $\mathrm{P}(\tilde{\Lambda}_n = (\lambda_{n,k}, \lambda_{n,l})) = 1/n^2$ for $1 \le k, l \le n$. Similarly define $\tilde{E}_n$ and $\tilde{Y}_n$. Again $\tilde{\Lambda}_n = \tilde{E}_n + \tilde{Y}_n$ and

$$\mathrm{P}(\|Y_n\| > \epsilon) = \frac{1}{n^2}\sum_{k,l=1}^{n} \mathrm{P}(\|(y_{n,k}, y_{n,l})\| > \epsilon) \to 0, \text{as } n \to \infty.$$

So $\tilde{\Lambda}_n$ and $\tilde{E}_n$ will have same limiting distribution and hence conclusion (b) holds. □

Now we move to the main proof. This proof mainly depends on Lemma 2.3.1 which helps us to use normal approximation, and Lemma 2.4.7 given below which allows us to approximate the eigenvalues by appropriate partial sums of independent random variables. The latter follows easily from Fan and Yao (2003) [58] (Theorem 2.14(ii), page 63). We provide a proof for sake of completeness. For $k = 1, 2, \cdots, n$, define

$$\xi_{2k-1} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \cos(\omega_k t), \quad \xi_{2k} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \sin(\omega_k t).$$

Lemma 2.4.7. *Suppose Assumption 2.4.1 holds and $\{\epsilon_t\}$ are i.i.d random variables with mean 0, variance 1. For $k = 1, 2, \cdots, n$, write*

$$\lambda_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l e^{i\omega_k l} = \psi(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}] + Y_n(\omega_k).$$

Then

$$\max_{0 \le k < n} \mathrm{E}\,|Y_n(\omega_k)| \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof.

$$\begin{aligned}
\lambda_k &= \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} x_t e^{i\omega_k t} \\
&= \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \sum_{t=0}^{n-1} \epsilon_{t-j} e^{i\omega_k (t-j)} \\
&= \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \left(\sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t} + U_{nj} \right) \\
&= \psi(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}] + Y_n(\omega_k),
\end{aligned}$$

where

$$U_{nj} = \sum_{t=-j}^{n-1-j} \epsilon_t e^{i\omega_k t} - \sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t}, \quad Y_n(\omega_k) = n^{-1/2} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} U_{nj}.$$

Note that if $|j| < n$, U_{nj} is a sum of $2|j|$ independent random variables, whereas if $|j| \ge n$, U_{nj} is a sum of $2n$ independent random variables. Thus $\mathrm{E}\,|U_{nj}|^2 \le 2\min(|j|, n)$.

Therefore, for any fixed positive integer l and $n > l$,

$$\begin{aligned}
\mathrm{E}\,|Y_n(\omega_k)| &\leq \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} |a_j|(\mathrm{E}\,U_{nj}^2)^{1/2} \\
&\leq \sqrt{\frac{2}{n}} \sum_{j=-\infty}^{\infty} |a_j|\{\min(|j|, n)\}^{1/2} \\
&\leq \sqrt{2}\left(\frac{1}{\sqrt{n}} \sum_{|j|\leq l} |a_j||j|^{1/2} + \sum_{|j|>l} |a_j|\right).
\end{aligned}$$

The right side of the above expression is independent of k and as $n \to \infty$, it can be made smaller than any given positive constant by choosing l large enough. Hence, $\max_{1\leq k\leq n} \mathrm{E}\,|Y_n(\omega_k)| \to 0$. □

Now we are ready to prove Theorem 2.4.4. As pointed out earlier in Section 2.3, to prove that F_n converges to F (say) in L_2, it is enough to show that

$$\mathrm{E}[F_n(x, y)] \to F(x, y) \text{ and } V[F_n(x, y)] \to 0 \tag{2.4.7}$$

at all continuity points (x, y) of F. This is what we show here and in every proof later on.

Proof of Theorem 2.4.4: First assume $\mathrm{Leb}(C_0) = 0$. Recall the eigenvalues of circulant matrix from Section 1.2.1 and note that we may ignore the eigenvalue λ_n and also $\lambda_{n/2}$ whenever n is even, since they contribute atmost $2/n$ to the ESD $F_n(x, y)$. So for $x, y \in \mathbb{R}$,

$$\mathrm{E}[F_n(x, y)] \sim n^{-1} \sum_{k=1, k\neq n/2}^{n-1} \mathrm{P}(b_k \leq x, c_k \leq y),$$

where

$$b_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \cos(\omega_k j), \quad c_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \sin(\omega_k j), \quad \omega_k = \frac{2\pi k}{n}.$$

Define for $k = 1, 2, \cdots, n$,

$$\eta_k = (\xi_{2k-1}, \xi_{2k})', \quad Y_{1n}(\omega_k) = \mathcal{R}[Y_n(\omega_k)], \quad Y_{2n}(\omega_k) = \mathcal{I}[Y_n(\omega_k)],$$

where $Y_n(\omega_k)$ are same as defined in Lemma 2.4.7. Then $(b_k, c_k)' = B(\omega_k)\eta_k + (Y_{1n}(\omega_k), Y_{2n}(\omega_k))'$. Now in view of Lemma 2.4.6 and Lemma 2.4.7, to show

$\mathrm{E}[F_n(x,y)] \to F_C(x,y)$ it is sufficient to show that

$$\frac{1}{n}\sum_{k=1,k\neq n/2}^{n-1} \mathrm{P}\big[B(\omega_k)\eta_k \leq (x,y)'\big] \to F_C(x,y). \tag{2.4.8}$$

To show this, define for $1 \leq k \leq n-1$, (except for $k = n/2$) and $0 \leq l \leq n-1$,

$$X_{l,k} = \big(\sqrt{2}\epsilon_l \cos(\omega_k l), \quad \sqrt{2}\epsilon_l \sin(\omega_k l)\big)'.$$

Note that

$$\mathrm{E}(X_{l,k}) = 0, \tag{2.4.9}$$

$$n^{-1}\sum_{l=0}^{n-1} Cov(X_{l,k}) = I, \tag{2.4.10}$$

$$\sup_n \sup_{1\leq k\leq n} [n^{-1}\sum_{l=0}^{n-1} \mathrm{E} \parallel X_{l,k} \parallel^{(2+\delta)}] \leq C < \infty. \tag{2.4.11}$$

For $k \neq n/2$

$$\big\{B(\omega_k)\eta_k \leq (x,y)'\big\} = \big\{B(\omega_k)(n^{-1/2}\sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)'\big\}.$$

Since $\{(r,s) : B(\omega_k)(r,s)' \leq (\sqrt{2}x, \sqrt{2}y)'\}$ is a *convex* set in $\mathbb{R}^2$ and $\{X_{l,k},\ l = 0,1,\ldots,(n-1)\}$ satisfies (2.4.9)–(2.4.11), we can apply Part (i) of Lemma 2.3.1 for $k \neq n/2$ to get

$$\begin{aligned}
&\big|\mathrm{P}\big(B(\omega_k)(n^{-1/2}\sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)'\big) - \mathrm{P}\big(B(\omega_k)(N_1,\ N_2)' \leq (\sqrt{2}x, \sqrt{2}y)'\big)\big| \\
&\leq Cn^{-\delta/2}[n^{-1}\sum_{l=0}^{n-1} \mathrm{E}\,\|X_{lk}\|^{(2+\delta)}] \leq Cn^{-\delta/2} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore, since by Lemma 2.4.3(i), H_C is bounded continuous for every fixed (x,y),

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=1,k\neq n/2}^{n-1} \mathrm{P}\big(B(\omega_k)\eta_k \leq (x,y)'\big) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1,k\neq n/2}^{n-1} H_C(\frac{2\pi k}{n}, x, y) \\
&= \int_0^1 H_C(2\pi s, x, y)ds = F_C(x,y).
\end{aligned}$$

Hence by (2.4.8)

$$\mathrm{E}[F_n(x,y)] \to \int_0^1 H_C(2\pi s, x, y)ds = F_C(x,y). \tag{2.4.12}$$

To show $V[F_n(x,y)] \to 0$, it is enough to show that

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} Cov(J_k, J_{k'}) = \frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} [\mathrm{E}(J_k, J_{k'}) - \mathrm{E}(J_k)\,\mathrm{E}(J_{k'})] \to 0. \tag{2.4.13}$$

where for $1 \leq k \leq n$, J_k is the indicator that $\{b_k \leq x, c_k \leq y\}$. As $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} \mathrm{E}(J_k)\,\mathrm{E}(J_{k'}) = \big[\frac{1}{n}\sum_{k=1}^{n}\mathrm{E}(J_k)\big]^2 - \frac{1}{n^2}\sum_{k=1}^{n}\big[\mathrm{E}(J_k)\big]^2 \to \big[\int_0^1 H_C(2\pi s, x, y)ds\big]^2.$$

So to show (2.4.13), it is enough to show as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} \mathrm{E}(J_k, J_{k'}) \to \big[\int_0^1 H_C(2\pi s, x, y)ds\big]^2.$$

Along the lines of the proof used to show (2.4.12) one may now extend the vectors of two coordinates defined above to ones with four coordinates and proceed exactly as above to verify this. We omit the routine details. This completes the proof for the case $\mathrm{Leb}(C_0) = 0$.

When $\mathrm{Leb}(C_0) \neq 0$, we have to show (2.4.7) only on D_1^c (of Lemma 2.4.3). All the above steps in the proof will go through for all (x,y) in D_1^c. Hence if $\mathrm{Leb}(C_0) \neq 0$, we have our required LSD. This completes the proof of Theorem 2.4.4. □

2.4.3 Symmetric circulant matrix with dependent input

For $x \in \mathbb{R}$ and $\omega \in [0,\pi]$ define,

$$H_S(\omega, x) = \begin{cases} \mathrm{P}\big(\sqrt{2\pi f(\omega)}N(0,1) \leq x\big) & \text{if } f(\omega) \neq 0, \\ \mathbb{I}(x \geq 0) & \text{if } f(\omega) = 0. \end{cases} \tag{2.4.14}$$

Let

$$C_0' = \{t \in [0, 1/2] : f(2\pi t) = 0\}.$$

The following Lemma is analogous to Lemma 2.4.3. We omit the proof.

Lemma 2.4.8. *(i) For fixed x, H_S is a bounded continuous function in ω and*

$$H_S(\omega, x) + H_S(\omega, -x) = 1.$$

(ii) F_S *defined below is a proper distribution function and* $F_S(x) + F_S(-x) = 1$.

$$F_S(x) = 2\int_0^{1/2} H_S(2\pi s, x)ds. \tag{2.4.15}$$

(iii) If $\text{Leb}(C_0') = 0$ *then* F_S *is continuous everywhere and may be expressed as*

$$F_S(x) = \int_{-\infty}^{x} \Big[\int_0^{1/2} \frac{1}{\pi\sqrt{f(2\pi s)}} e^{-\frac{t^2}{4\pi f(2\pi s)}} ds\Big] dt. \tag{2.4.16}$$

Further, F_S *is normal if and only if* f *is constant almost everywhere (Lebesgue).*

(iv) If $\text{Leb}(C_0') \neq 0$ *then* F_S *is discontinuous only at* $x = 0$.

Theorem 2.4.9 (Bose, Hazra and Saha (2009) [33])**.** *Suppose Assumptions 2.4.1 and 2.4.2 hold and*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{\lfloor np/2 \rfloor} \big[f(\frac{2\pi k}{n})\big]^{-3/2} \to 0 \ \ \textit{for all} \ \ 0 < p < 1. \tag{2.4.17}$$

Then the ESD of $\frac{1}{\sqrt{n}}SC_n$ *converges in* L_2 *to* F_S *given in (2.4.15)–(2.4.16). The same limit continues to hold for* PT_n.

Remark 2.4.10. *(i) Condition (2.4.17) is satisfied if* $\inf_\omega f(\omega) > 0$. *If we do not assume (2.4.17), it is not clear whether the LSD result will be true.*
(ii) It is easy to check that the variance, μ_2 *and the fourth moment* μ_4 *of* F_S *equal* $\int_0^{1/2} 4\pi f(2\pi s)ds$ *and* $\int_0^{1/2} 24\pi^2 f^2(2\pi s)ds$ *respectively. By Cauchy-Schwarz inequality it follows that* $\frac{\mu_4}{\mu_2^2} \geq 3$ *and equal to 3 iff* $f \equiv \frac{1}{2\pi}$. *In the latter case,* F_S *is standard normal distribution function. This agrees with the conclusion of Theorem 2.3.5.*

We prove the result for symmetric circulant matrix only for odd $n = 2m + 1$. The even case follows by appropriate easy changes in the proof. First recall the eigenvalues of symmetric circulant matrix from Section 1.2.2. The partial sum approximation (Lemma 2.4.7) that has been used in the proof of Theorem 2.4.4 now takes the following form.

Lemma 2.4.11. *Suppose Assumption 2.4.1 holds and* $\{\epsilon_t\}$ *are i.i.d random variables with mean 0, variance 1. For* $n = 2m + 1$ *and* $k = 1, 2, \cdots, m$, *write*

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{m} x_t \cos\frac{2\pi kt}{n} = \psi_1(e^{i\omega_k}) \frac{1}{\sqrt{n}} \sum_{t=1}^{m} \epsilon_t \cos\frac{2\pi kt}{n} - \psi_2(e^{i\omega_k}) \frac{1}{\sqrt{n}} \sum_{t=1}^{m} \sin\frac{2\pi kt}{n} + Y_{n,k},$$

where $\psi_1(e^{i\omega_k})$, $\psi_2(e^{i\omega_k})$ *are same as defined in (2.4.3). Then* $\max_{0\leq k\leq m} \text{E}(Y_{n,k}) \to 0$ *as* $n \to \infty$.

Proof.

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{t=1}^{m} x_t \cos\frac{2\pi kt}{n}\\
= &\frac{1}{\sqrt{n}}\sum_{t=1}^{m}\sum_{j=-\infty}^{\infty} a_j\epsilon_{t-j}\big[\cos\frac{2\pi k(t-j)}{n}\cos\frac{2\pi kj}{n}-\sin\frac{2\pi k(t-j)}{n}\sin\frac{2\pi kj}{n}\big]\\
= &\frac{\psi_1(e^{i\omega_k})}{\sqrt{n}}\sum_{t=1}^{m}\epsilon_t\cos\frac{2\pi kt}{n}-\frac{\psi_2(e^{i\omega_k})}{\sqrt{n}}\sum_{t=1}^{m}\sin\frac{2\pi kt}{n}+Y_{n,k},
\end{aligned}$$

where

$$Y_{n,k}=\frac{1}{\sqrt{n}}\sum_{j=-\infty}^{\infty}a_j\big[\cos\frac{2\pi kj}{n}U_{k,j}-\sin\frac{2\pi kj}{n}V_{k,j}\big],$$

$$U_{k,j}=\sum_{t=1}^{m}\big[\epsilon_{t-j}\cos\frac{2\pi k(t-j)}{n}-\epsilon_t\cos\frac{2\pi kt}{n}\big],\ V_{k,j}=\sum_{t=1}^{m}\big[\epsilon_{t-j}\sin\frac{2\pi k(t-j)}{n}-\epsilon_t\sin\frac{2\pi kt}{n}\big].$$

Note that if $|j|<m$, $U_{k,j}, U'_{k,j}$ are sums of $2|j|$ independent random variables, whereas if $|j|\geq m$, $U_{k,j}, U'_{k,j}$ are sums of $2m$ independent random variables. Thus $\mathrm{E}\,|U_{k,j}|^2\leq 2\min(|j|,m)$. Therefore, for any fixed positive integer l and $m>l$,

$$\begin{aligned}
\mathrm{E}\,|Y_{n,k}| &\leq \frac{1}{\sqrt{n}}\Big[\sum_{j=-\infty}^{\infty}|a_j|\,\mathrm{E}(U_{k,j}^2)^{1/2}+\sum_{j=-\infty}^{\infty}|a_j|(\mathrm{E}\,V_{k,j}^2)^{1/2}\Big]\quad\Big(\because\sum_{-\infty}^{\infty}|a_j|<\infty\Big)\\
&\leq \frac{2\sqrt{2}}{\sqrt{n}}\sum_{j=-\infty}^{\infty}|a_j|\{\min(|j|,m)\}^{1/2}\\
&\leq 2\sqrt{2}\Big(\frac{1}{\sqrt{n}}\sum_{|j|\leq l}|a_j||j|^{1/2}+\sum_{|j|>l}|a_j|\Big).
\end{aligned}$$

The right side of the above expression is independent of k and as $n\to\infty$, it can be made smaller than any given positive constant by choosing l large enough. Hence, $\max_{1\leq k\leq m}\mathrm{E}(Y_{n,k})\to 0$. □

Proof of Theorem 2.4.9: Note that all eigenvalues $\{\lambda_k,\ 0\leq k\leq n-1\}$ are real in this case. As before, we provide the detailed proof only when $\mathrm{Leb}(C_0')=0$. Note that we may ignore the eigenvalue λ_0 since it contributes $1/n$ to the ESD $F_n(\cdot)$. Further, the term $\frac{x_0}{\sqrt{n}}$ can be ignored from the eigenvalue $\{\lambda_k\}$. So for $x\in\mathbb{R}$,

$$\mathrm{E}[F_n(x)]\ \sim\ \frac{2}{n}\sum_{k=1}^{m}\mathrm{P}(\frac{1}{\sqrt{n}}\lambda_k\leq x)\sim\frac{2}{n}\sum_{k=1}^{m}\mathrm{P}\Big(\frac{1}{\sqrt{n}}\sum_{t=1}^{m}2x_t\cos\frac{2\pi kt}{n}\leq x\Big).$$

Following the argument given in the circulant case and using Lemma 2.4.6 and Lemma 2.4.11, it is sufficient to show that

$$\begin{aligned} & \frac{2}{n}\sum_{k=1}^{m} \mathrm{P}\Big[\psi_1(e^{i\omega_k})\frac{2}{n}\sum_{t=1}^{m}\epsilon_t \cos\frac{2\pi kt}{n} - \psi_2(e^{i\omega_k})\frac{2}{\sqrt{n}}\sum_{t=1}^{m}\sin\frac{2\pi kt}{n} \le x\Big] \\ = & \frac{2}{n}\sum_{t=1}^{m}\mathrm{P}\Big\{m^{-1/2}\sum_{l=1}^{m}X_{l,k}\in C_k\Big\} \to F_S(x) \end{aligned}$$

where

$$\begin{aligned} X_{l,k} &= \big(2\sigma_n^{-1}\epsilon_l\cos\frac{2\pi kl}{n},\ 2\delta_n^{-1}\epsilon_l\sin\frac{2\pi kl}{n}\big),\ \sigma_n^2 = 2-1/m,\ \delta_n^2 = 2+1/m, \\ C_k &= \big\{(u,v): \sigma_n\psi_1(e^{i\omega_k})u + \delta_n\psi_2(e^{i\omega_k})v \le \sqrt{n/m}x\big\}. \end{aligned}$$

Note that

$$\mathrm{E}(X_{l,k}) = 0,\ \frac{1}{m}\sum_{l=1}^{m}Cov(X_{l,k}) = V_k \ \text{ and }\ \sup_m \sup_{1\le k\le m} m^{-1}\sum_{l=1}^{m}\mathrm{E}\,\|X_{l,k}\|^{2+\delta} \le C < \infty \tag{2.4.18}$$

where

$$V_k = \begin{pmatrix} 1 & -\frac{1}{\sqrt{4m^2-1}}\tan\frac{k\pi}{2m+1} \\ -\frac{1}{\sqrt{4m^2-1}}\tan\frac{k\pi}{2m+1} & 1 \end{pmatrix}.$$

Let α_k be the minimum eigenvalue of V_k. Then $\alpha_k \ge \alpha_m$ for $1 \le k \le m$ and

$$\alpha_m = 1 - \frac{1}{\sqrt{4m^2-1}}\tan\frac{m\pi}{2m+1} \approx 1 - \frac{2m+1}{m\pi} \approx 1 - \frac{2}{\pi} = \alpha,\ \text{say}.$$

Since $\{X_{l,k}\}$ satisfies (2.4.18) and C_k is a *convex* set in $\mathbb{R}^2$, we can apply Part (i) of Lemma 2.3.1 for $k = 1, 2, \cdots, m$ to get

$$\begin{aligned} \Big|\frac{2}{n}\sum_{k=1}^{m}\Big[\mathrm{P}\Big\{m^{-1/2}\sum_{l=1}^{m}X_{l,k}\in C_k\Big\} - \Phi_{0,V_k}(C_k)\Big]\Big| &\le Cm^{-\delta/2}\frac{2}{n}\sum_{k=1}^{m}\alpha_k^{-3/2} \\ &\le Cm^{-\delta/2}\alpha^{-3/2} \to 0. \end{aligned}$$

where Φ_{0,V_k} is a bivariate normal distribution with mean zero and covariance matrix V_k. Note that for large m, $\sigma_n^2 \approx 2$ and $\delta_n^2 \approx 2$. Hence $C_k' = \big\{(u,v): \psi_1(e^{i\omega_k})u + \psi_2(e^{i\omega_k})v \le \sqrt{x}\big\}$ serves as a good approximation to C_k and we get

$$\frac{2}{n}\sum_{k=1}^{m}\Phi_{0,V_k}(C_k) \sim \frac{2}{n}\sum_{k=1}^{m}\Phi_{0,V_k}(C_k') = \frac{2}{n}\sum_{k=1}^{m}\mathrm{P}(\mu_k N(0,1) \le x),$$

where $\mu_k^2 = \psi_1(e^{i\omega_k})^2 + \psi_2(e^{i\omega_k})^2 + 2\psi_1(e^{i\omega_k})\psi_2(e^{i\omega_k})\frac{1}{\sqrt{4m^2-1}}\tan\frac{k\pi}{2m+1}$. Define

$\nu_k^2 = \psi_1(e^{i\omega_k})^2 + \psi_2(e^{i\omega_k})^2$. Now we show that

$$\lim_{n\to\infty} \Big| \frac{2}{n} \sum_{k=1}^{m} \big[\mathrm{P}(\mu_k N(0,1) \le x) - \mathbb{P}(\nu_k N(0,1) \le x) \big] \Big| = 0. \tag{2.4.19}$$

Let $0 < p < 1$. Now as $n \to \infty$, using Assumption (2.4.17),

$$\begin{aligned}
\frac{2}{n}\Big| \sum_{k=1}^{\lfloor mp \rfloor} \big[\mathrm{P}(\mu_k N(0,1) \le x) - \mathrm{P}(\nu_k N(0,1) \le x) \big] \Big| &= \frac{2}{n} \sum_{k=1}^{\lfloor mp \rfloor} \Big| \int_{x/\nu_k}^{x/\mu_k} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \Big| \\
&\le \frac{2|x|}{n} \sum_{k=1}^{\lfloor mp \rfloor} \Big| \frac{\mu_k^2 - \nu_k^2}{\mu_k \nu_k (\mu_k + \nu_k)} \Big| \\
&\le \frac{2|x| \tan \frac{p\pi}{2}}{m^2} \sum_{k=1}^{\lfloor mp \rfloor} \frac{1}{\nu_k^3 \alpha (1+\alpha)} \to 0.
\end{aligned}$$

On the other hand, for every n,

$$\frac{2}{n}\Big| \sum_{\lfloor mp \rfloor + 1}^{m} \big[\mathrm{P}(\mu_k N(0,1) \le x) - \mathrm{P}(\nu_k N(0,1) \le x) \big] \Big| \le 4(1-p).$$

Therefore, by first letting $n \to \infty$ and then letting $p \to 1$, (2.4.19) holds. Hence

$$\begin{aligned}
\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{m} \mathrm{P}\big(\nu_k N(0,1) \le x\big) &= \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{m} \mathrm{P}\big(\sqrt{2\pi f(2\pi k/n)} N(0,1) \le x\big) \\
&\longrightarrow 2 \int_0^{1/2} H_S(2\pi s, x) ds.
\end{aligned}$$

Rest of the argument in the proof is same as in the proof of Theorem 2.4.4.

□

2.4.4 Reverse circulant matrix with dependent input

Define $H_R(\omega, x)$ on $[0, 2\pi] \times \mathbb{R}$ as

$$H_R(\omega, x) = \begin{cases} G\Big(\frac{x^2}{2\pi f(\omega)}\Big) & \text{if} \quad f(\omega) \ne 0 \\ 1 & \text{if} \quad f(\omega) = 0, \end{cases}$$

where $G(x) = 1 - e^{-x}$ for $x > 0$, is the standard exponential distribution function. The proof of the next lemma is omitted.

Lemma 2.4.12. *(i) For fixed x, $H_R(\omega, x)$ is bounded continuous on $[0, 2\pi]$.*

(ii) F_R defined below is a valid symmetric distribution function.

$$F_R(x) \quad = \quad \begin{cases} \frac{1}{2} + \int_0^{1/2} H_R(2\pi t, x)dt & \text{if } x > 0 \\ \frac{1}{2} - \int_0^{1/2} H_R(2\pi t, x)dt & \text{if } x \leq 0. \end{cases} \tag{2.4.20}$$

(iii) If $\mathrm{Leb}(C_0') = 0$ then F_R is continuous everywhere and can be expressed as

$$F_R(x) = \begin{cases} 1 - \int_0^{1/2} e^{-\frac{x^2}{2\pi f(2\pi t)}} dt & \text{if } x > 0 \\ \int_0^{1/2} e^{-\frac{x^2}{2\pi f(2\pi t)}} dt & \text{if } x \leq 0. \end{cases} \tag{2.4.21}$$

Further, F_R is the distribution of the symmetric version of the square root of chi-square variable with two degrees of freedom if and only if f is constant almost everywhere (Lebesgue).

(iv) If $\mathrm{Leb}(C_0') \neq 0$ then F_R is discontinuous only at $x = 0$.

Theorem 2.4.13 (Bose, Hazra and Saha (2009) [33])**.** *Suppose Assumptions 2.4.1 and 2.4.2 hold. Then the ESD of $\frac{1}{\sqrt{n}} RC_n$ converges in L_2 to F_R given in (2.4.20)–(2.4.21).*

Remark 2.4.14. *If $\{x_i\}$ are i.i.d, with finite $(2+\delta)$ moment, then $f(\omega) = 1/2\pi$ for all $\omega \in [0, 2\pi]$ and the LSD $F_R(\cdot)$ agrees with (2.3.9) given earlier.*

We now need the following Lemma to approximate the eigenvalues by appropriate partial sums of independent random variables. Its proof is given in Fan and Yao (2003) [58] (Theorem 2.14(ii), page 63).

Lemma 2.4.15. *Suppose Assumption 2.4.1 holds and $\{\epsilon_t\}$ are i.i.d random variables with mean 0, variance 1. For $k = 1, 2, \cdots, \lfloor \frac{n-1}{2} \rfloor$, write*

$$I_n(\omega_k) = L_n(\omega_k) + R_n(\omega_k), \ \textit{where } L_n(\omega_k) = 2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2)$$

and $I_n(\omega_k)$ is as in (2.4.2). Then $\max_{1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} \mathrm{E}\,|R_n(\omega_k)| \to 0$ as $n \to \infty$.

Proof of Theorem 2.4.13: As earlier, we give the proof only for the case $\mathrm{Leb}(C_0') = 0$. From the structure of the eigenvalues $\{\lambda_k,\ 0 \leq k \leq n-1\}$ of RC_n (see Section 1.2.3), the LSD, if it exists, is going to be a symmetric distribution. So, it is enough to concentrate on the case $x > 0$. As before we may ignore the two eigenvalues λ_0 and $\lambda_{n/2}$. Hence for $x > 0$,

$$\mathrm{E}[F_n(x)] \sim 1/2 + n^{-1} \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \mathrm{P}(\frac{1}{n}\lambda_k^2 \leq x^2) = 1/2 + n^{-1} \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \mathrm{P}(I_n(\omega_k) \leq x^2), \tag{2.4.22}$$

where $I_n(\omega_k)$ is as in (2.4.2). Along the same lines as in the proof of Theorem 2.4.4, using Lemma 2.4.6 and Lemma 2.4.15, it is sufficient to show that

$$\frac{1}{n}\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \mathrm{P}\big(L_n(\omega_k) \le x^2\big) \to \int_0^{1/2} H_R(2\pi t, x)dt$$

where $L_n(\omega_k)$ is same as in Lemma 2.4.15. Define for $k = 1, 2, \cdots, \lfloor\frac{n-1}{2}\rfloor$ and $l = 0, 1, 2, \cdots, n-1$,

$$X_{l,k} = \big(\sqrt{2}\epsilon_l \cos(l\omega_k),\ \ \sqrt{2}\epsilon_l \sin(l\omega_k)\big)',\ \ A_{kn} = \big\{(r_1, r_2) : \pi f(\omega_k)(r_1^2 + r_2^2) \le x^2\big\}.$$

Note that $\{X_{l,k}\}$ satisfies (2.4.9)–(2.4.11) and $\big\{L_n(\omega_k) \le x^2\big\} = \big\{n^{-1/2}\sum_{l=0}^{n-1} X_{l,k} \in A_{kn}\big\}$. Since A_{kn} is a *convex* set in $\mathbb{R}^2$, we can apply Part (i) of Lemma 2.3.1 to get, as $n \to \infty$

$$\frac{1}{n}\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} |\mathrm{P}(L_n(\omega_k) \le x^2) - \Phi_{0,I}(A_{kn})| \le Cn^{-\delta/2} \to 0.$$

But

$$\frac{1}{n}\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \Phi_{0,I}(A_{kn}) \ = \ \frac{1}{n}\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} H_R(\frac{2\pi k}{n}, x) \to \int_0^{1/2} H_R(2\pi t, x)dt.$$

Hence for $x \ge 0$,

$$\mathrm{E}[F_n(x)] \to \frac{1}{2} + \int_0^{1/2} H_R(2\pi t, x)dt = F_R(x).$$

Now the rest of the argument in the proof is same as in the proof of Theorem 2.4.4. □

2.4.5 k-circulant matrix with dependent input

First recall the eigenvalues of the k-circulant matrix $A_{k,n}$ and related notation from Section 1.2.4. For any positive integers k, n, let $p_1 < p_2 < \ldots < p_c$ be all their common prime factors so that,

$$n = n'\prod_{q=1}^{c} p_q^{\beta_q} \ \text{ and } \ \ k = k'\prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here $\alpha_q,\ \beta_q \ge 1$ and n', k', p_q are pairwise relatively prime. Then the charectaristic polynomial of $A_{k,n}$ is given by

$$\chi\left(A_{k,n}\right) = \lambda^{n-n'}\prod_{j=0}^{\ell-1}\left(\lambda^{n_j} - y_j\right), \tag{2.4.23}$$

where y_j, n_j are as defined in Section 1.2.4. This provides a formula solution for the eigenvalues. Also recall

$$S(x) = \{xk^b \bmod n' : b \geq 0\},\ 0 \leq x < n'; \quad g_x = \#S(x), \quad \upsilon_{k,n'} = \#\{x \in \mathbb{Z}_{n'} : g_x < g_1\}.$$

As mentioned before, it appears difficult to prove a general result on LSD for all possible pairs $(k,\ n)$. We investigate LSD for two subclasses of the k-circulant matrix, where $(k,\ n)$ satisfies either $n = k^g + 1$ or $n = k^g - 1$ and $g \geq 2$. Note that in both the cases $\gcd(n, k) = 1$ and hence, $n' = n$ in (2.4.23).

Before going into the main results we state a lemma from Bose, Mitra and Sen (2008) [44] which we shall use in the proof of the following theorems and also in Chapters 4 and 7. Here we skip the proof.

Lemma 2.4.16. *(i) Fix $g \geq 1$. Suppose $k^g = -1 + sn$, $n \to \infty$ with $s = 1$ if $g = 1$ and $s = o(n^{p_1 - 1})$ if $g > 1$ where p_1 is the smallest prime divisor of g. Then $g_1 = 2g$ for all but finitely many n and $\frac{\upsilon_{k,n}}{n} \to 0$.*

(ii) Suppose $k^g = 1 + sn$, $g \geq 1$ fixed, $n \to \infty$ with $s = 0$ if $g = 1$ and $s = o(n^{p_1 - 1})$ where p_1 is the smallest prime divisor of g. Then $g_1 = g$ for all but finitely many n and $\frac{\upsilon_{k,n}}{n} \to 0$.

We consider two types of k-circulant matrix, namely, k-circulant with $n = k^g + 1$ and k-circulant with $n = k^g - 1$ for some $g \geq 2$.

Type I. $n = k^g + 1$ for some fixed $g \geq 2$.

Suppose $n = k^g + 1$ and $g \geq 2$. We observe a simple but crucial property of eigenvalue partitioning $\{\mathcal{P}_j\}$ of $\mathbb{Z}_n$ (see (1.2.6)). For every integer $t \geq 0$, $tk^g = (-1 + n)t = -t$ mod n. Hence λ_t and λ_{n-t} belong to the same partition block $S(t) = S(n - t)$. Thus each $S(t)$ contains an even number of elements, except for $t = 0, \frac{n}{2}$. Hence the eigenvalue partitioning sets $\mathcal{P}_j$ are self conjugate. So, we can find sets $\mathcal{A}_j \subset \mathcal{P}_j$ such that

$$\mathcal{P}_j = \{x : x \in \mathcal{A}_j \text{ or } n - x \in \mathcal{A}_j\} \text{ and } \#\mathcal{A}_j = \frac{1}{2}\#\mathcal{P}_j. \tag{2.4.24}$$

However, it follows from Lemma 2.4.16 that for $n = k^g + 1$, $g_1 = 2g$ and $\upsilon_{k,n}/n \to 0$. For $g = 2$, it is easy to check that $S(1) = \{1, k, n-1, n-k\}$, hence, $g_1 = 4$, and

$$\upsilon_{k,n} = \{x \in \mathbb{Z}_n : g_x < g_1\} = \begin{cases} \{0, n/2\} & \text{if } n \text{ is even} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases} \tag{2.4.25}$$

As a consequence, $\upsilon_{k,n}/n \leq 2/n \to 0$.

For any $d \geq 1$, let

$$G_d(x) = \mathrm{P}\big(\prod_{i=1}^{d} E_i \leq x\big),$$

where $\{E_i\}$ are i.i.d. $Exp(1)$. Note that G_d is continuous. For any integer $d \geq 1$, define $H_d(\omega_1, \ldots, \omega_d, x)$ on $[0, 2\pi]^d \times \mathbb{R}_{\geq 0}$ as

$$H_d(\omega_1, \ldots, \omega_d, x) = \begin{cases} G_d\Big(\frac{x^{2d}}{(2\pi)^d \prod_{i=1}^{d} f(\omega_i)}\Big) & \text{if} \quad \prod_{i=1}^{d} f(\omega_i) \neq 0 \\ 1 & \text{if} \quad \prod_{i=1}^{d} f(\omega_i) = 0. \end{cases}$$

The proof of the following lemma is omitted.

Lemma 2.4.17. *(i) For fixed x, $H_d(\omega_1, \ldots, \omega_d, x)$ is bounded continuous on $[0, 2\pi]^d$.*

(ii) F_d defined below is a valid continuous distribution function.

$$F_d(x) \quad = \quad \int_0^1 \cdots \int_0^1 H_d(2\pi t_1, \ldots, 2\pi t_d, x) \prod_{i=1}^{d} dt_i \quad \text{for} \quad x \geq 0. \qquad (2.4.26)$$

Theorem 2.4.18 (Bose, Hazra and Saha (2009) [33])**.** *Suppose Assumptions 2.4.1 and 2.4.2 hold. Suppose $n = k^g + 1$ for some fixed $g \geq 2$. Then as $n \to \infty$, $F_{n^{-1/2}A_{k,n}}$ converges in L_2 to the LSD $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. with distribution function F_g given in (2.4.26) and U_1 is uniformly distributed over the $(2g)$-th roots of unity, independent of the $\{E_i\}$.*

Remark 2.4.19. *If $\{x_i\}$ are i.i.d, then $f(\omega) = 1/2\pi$ for all $\omega \in [0, 2\pi]$ and the LSD is $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. $Exp(1)$, U_1 is as in Theorem 2.4.18 and independent of $\{E_i\}$. This limit agrees with Theorem 2.3.10(i).*

Remark 2.4.20. *Using the expression (2.4.23) for the characteristic polynomial, it is then not difficult to manufacture $\{k = k(n)\}$ such that the LSD of $n^{-1/2}A_{k,n}$ has some positive mass at the origin. For example, suppose the sequences k and n satisfy $k^g = -1 + sn$ where $g \geq 1$ is fixed and $s = o(n^{1/3})$. Fix primes $p_1, p_2, \ldots, p_t$ and positive integers $\beta_1, \beta_2, \ldots, \beta_t$. Define*

$$\widetilde{n} = p_1^{\beta_1} p_2^{\beta_2} \ldots p_t^{\beta_t} n.$$

Suppose $k = p_1 p_2 \ldots p_t m \to \infty$. Then the ESD of $\widetilde{n}^{-1/2} A_{k,\widetilde{n}}$ converges weakly in probability to the LSD which has $1 - \Big(\Pi_{s=1}^{t} p_s^{\beta_s}\Big)^{-1}$ mass at zero, and rest of the probability mass is distributed as $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ where U_1 and $\{E_i\}$ are as in Theorem 2.4.18.

Proof of Theorem 2.4.18: The proof is also based on the method of normal approximation and uses the eigenvalue description given in Section 1.2.4.

For simplicity we first prove the result when $g = 2$. Note that $gcd(k, n) = 1$ and hence in this case $n' = n$ in (1.2.8). Recall that $v_{k,n}$ is the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathcal{P}_l$ and $|\mathcal{P}_l| < g_1$. In view of Lemma 2.4.16(i), we have $v_{k,n}/n \to 0$ and hence these eigenvalues do not contribute to the LSD. Hence it remains to consider only the eigenvalues corresponding to the sets $\mathcal{P}_l$ which have size *exactly equal* to g_1.

Note that $S(1) = \{1, k, n-1, n-k\}$ and hence $g_1 = 4$. Recall the quantities $n_j = \#\mathcal{P}_j$, $y_j = \prod_{t \in \mathcal{P}_l} \lambda_t$, where $\lambda_j = \sum_{l=0}^{n-1} x_l \nu^{jl}$, $0 \le j < n$ given in (1.2.2). Also, for every integer $t \ge 0$, $tk^2 = -t \mod n$, so that, λ_t and λ_{n-t} belong to the same partition block $S(t) = S(n-t)$. Thus each y_t is real. Let us define

$$I_n = \{l : \#\mathcal{P}_l = 4\}.$$

It is clear that $\frac{n}{\#I_n} \to 4$. Without any loss, let $I_n = \{1, 2, ..., \#I_n\}$.

Let $1, \omega, \omega^2, \omega^3$ be all the fourth roots of unity. Note that for every j, the eigenvalues of $A_{k,n}$ corresponding to the set $\mathcal{P}_j$ are: $y_j^{1/4}, y_j^{1/4}\omega, y_j^{1/4}\omega^2, y_j^{1/4}\omega^3$. Hence it suffices to consider only the modulus of eigenvalues $y_j^{1/4}$ as j varies: if these have an LSD F, say, then the LSD of the whole sequence will be (r, θ) in polar coordinates where r is distributed according to F and θ is distributed uniformly across all the fourth roots of unity and r and θ are independent. With this in mind and remembering the scaling $\sqrt{n}$, we consider for $x > 0$,

$$F_n(x) = \frac{1}{\#I_n} \sum_{i=1}^{\#I_n} \mathbb{I}\left(\left[\frac{y_j}{n^2}\right]^{\frac{1}{4}} \le x\right).$$

Since the set of λ values corresponding to any $\mathcal{P}_j$ is closed under conjugation, there exists a set $\mathcal{A}_i \subset \mathcal{P}_i$ of size 2 (see (2.4.24)) such that

$$\mathcal{P}_i = \{x : x \in \mathcal{A}_i \text{ or } n - x \in \mathcal{A}_i\}.$$

Combining each λ_j with its conjugate, we may write y_j in the form,

$$y_j = \prod_{t \in \mathcal{A}_j} (nb_t^2 + nc_t^2)$$

where $\{b_t\}$ and $\{c_t\}$ are given in (2.3.3). Note that for $x > 0$,

$$\mathrm{E}[F_n(x)] = \frac{1}{\#I_n} \sum_{j=1}^{\#I_n} \mathrm{P}\left(\frac{y_j}{n^2} \le x^4\right).$$

Now our aim is to show

$$\frac{1}{\#I_n}\sum_{j=1}^{\#I_n} \mathrm{P}\big(\frac{y_j}{n^2} \le x^4\big) \to F_2(x),$$

where $F_2(x)$ is as in (2.4.26) with $d = 2$. We can write $\frac{y_j}{n^2} = L_{n,j} + R_{n,j}$ for $1 \le j \le \#I_n$, where

$$\begin{aligned} L_{n,j} &= 4\pi^2 f_j \frac{\overline{y}_j}{n^2}, \quad \overline{y}_j = \prod_{t\in\mathcal{A}_j}(n\xi_{2t-1}^2 + n\xi_{2t}^2), \quad f_j = \prod_{t\in\mathcal{A}_j} f(\omega_t), \quad 1 \le j \le \#I_n, \\ R_{n,j} &= L_n(\omega_{j_1})R_n(\omega_{j_2}) + L_n(\omega_{j_2})R_n(\omega_{j_1}) + R_n(\omega_{j_1})R_n(\omega_{j_2}), \\ L_n(\omega_{j_k}) &= 2\pi f(\omega_{j_k})(\xi_{2j_k-1}^2 + \xi_{2j_k}^2), \quad k = 1, 2. \end{aligned}$$

Now using Lemma 2.4.15 it is easy to see that for any $\epsilon > 0$, $\max_{1\le j\le \#I_n} \mathrm{E}(|R_{n,j}| > \epsilon) \to 0$ as $n \to \infty$. So in view of Lemma 2.4.6 it is enough to show

$$\frac{1}{\#I_n}\sum_{j=1}^{\#I_n} \mathrm{P}\big(L_{n,j} \le x^4\big) \to F_2(x). \tag{2.4.27}$$

We show this in two steps.

Step I. Normal approximation:

$$\left|\frac{1}{\#I_n}\sum_{j=1}^{\#I_n}\Big[\mathrm{P}\Big(L_{n,j} \le x^4\Big) - \Phi_4(A_{n,j})\Big]\right| \to 0 \quad \text{as } n \to \infty, \tag{2.4.28}$$

where

$$A_{n,j} = \Big\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : \prod_{i=1}^{2}[2^{-1}(x_i^2 + y_i^2)] \le \frac{x^4}{4\pi^2 f_j}\Big\}, \quad 1 \le j \le \#I_n.$$

Step II.

$$\lim_{n\to\infty} \frac{1}{\#I_n}\sum_{j=1}^{\#I_n} \Phi_4(A_{n,j}) = F_2(x). \tag{2.4.29}$$

Proof of Step I. It is important to note that $A_{n,j}$ is *non-convex*. So we have to apply care while using the normal approximation. Define

$$X_{l,j} = 2^{1/2}\left(\epsilon_l \cos\left(\frac{2\pi tl}{n}\right),\ \epsilon_l \sin\left(\frac{2\pi tl}{n}\right),\ t \in \mathcal{A}_j\right), \quad 0 \le l < n, 1 \le j \le \#I_n,$$

Note that $\{X_{l,j}\}$ satisfies (2.4.9)–(2.4.11) and

$$\{L_{n,j} \leq x^4\} \;=\; \{n^{-1/2}\sum_{l=1}^{n-1} X_{l,j} \in A_{n,j}\}.$$

For (2.4.28), it suffices to show that for every $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all $n \geq N(\epsilon)$

$$\sup_{j\in I_n}\left|\mathrm{P}\Big(L_{n,j} \leq x^4\Big) - \Phi_4(A_{n,j})\right| \leq \epsilon.$$

Fix $\epsilon > 0$. Find $M_1 > 0$ large such that $\Phi([-M_1, M_1]^c) \leq \epsilon/16$. By Assumption 2.4.2, $\mathrm{E}(n^{-1/2}\sum_{l=0}^{n-1}\epsilon_l \cos\frac{2\pi lt}{n})^2 = \mathrm{E}(n^{-1/2}\sum_{l=0}^{n-1}\epsilon_l \sin\frac{2\pi lt}{n})^2 = 1/2$ for any $n \geq 1$ and $0 < t < n$. Now by Chebyshev bound, we can find $M_2 > 0$ such that for each $n \geq 1$ and for each $0 < t < n$,

$$\mathrm{P}\left(|n^{-1/2}\sum_{l=0}^{n-1}\epsilon_l \cos\frac{2\pi lt}{n}| \geq M_2\right) \leq \epsilon/16 \quad\text{and}\quad \mathrm{P}\left(|n^{-1/2}\sum_{l=0}^{n-1}\epsilon_l \sin\frac{2\pi lt}{n}| \geq M_2\right) \leq \epsilon/16.$$

Set $M = \max\{M_1, M_2\}$. Define the set $B := \Big\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : |x_j|, |y_j| \leq M \;\;\forall j\Big\}$. Then for all $j \in I_n$,

$$\begin{aligned}
&\left|\mathrm{P}\left(n^{-1/2}\sum_{l=0}^{n-1}X_{l,j} \in A_{n,j}\right) - \Phi_4(A_{n,j})\right| \\
&\leq \left|\mathrm{P}\left(n^{-1/2}\sum_{l=0}^{n-1}X_{l,j} \in A_{n,j}\cap B\right) - \Phi_4(A_{n,j}\cap B)\right| + \epsilon/2.
\end{aligned}$$

Since $A_{n,j}$ is a *non-convex* set, we now apply Part (ii) of Lemma 2.3.1 for $A_{n,j} \cap B$ to obtain

$$\begin{aligned}
&\sup_{j\in I_n}\left|\mathrm{P}\left(n^{-1/2}\sum_{l=0}^{n-1}X_{l,j} \in A_{n,j}\cap B\right) - \Phi_4(A_{n,j}\cap B)\right| \\
&\leq C_1 n^{-\delta/2}\rho_{2+\delta} \;+\; 2\sup_{j\in I_n}\sup_{z\in\mathbb{R}^4}\Phi_4\Big((\partial(A_{n,j}\cap B))^{\eta} - z\Big)
\end{aligned}$$

where

$$\rho_{2+\delta} = \sup_{j\in I_n} n^{-1}\sum_{l=0}^{n-1}\mathrm{E}\,\|X_{l,j}\|^{2+\delta} \quad\text{and}\quad \eta = \eta(n) = C_2\rho_{2+\delta}n^{-\delta/2}.$$

Note that $\rho_{2+\delta}$ is uniformly bounded in n by Assumption 2.4.2.
It thus remains to show that

$$\sup_{j\in I_n}\sup_{z\in\mathbb{R}^4}\Phi_4\Big((\partial(A_{n,j}\cap B))^{\eta} - z\Big) \leq \epsilon/8$$

for all sufficiently large n. Note that $\partial(A_{n,j} \cap B) \subseteq \partial A_{n,j} \cap \partial B \subseteq \partial B$ and hence

$$\begin{aligned}
& \sup_{j \in I_n} \sup_{z \in \mathbb{R}^4} \Phi_{2g}\Big((\partial(A_{n,j} \cap B))^\eta - z\Big) \\
= & \sup_{j \in I_n} \sup_{z \in \mathbb{R}^4} \int\limits_{(\partial(A_{n,j} \cap B))^\eta} \phi(x_1 - z_1) \ldots \phi(y_2 - z_4) dx_1 \ldots dy_2 \\
\leq & \sup_{z \in \mathbb{R}^4} \int\limits_{(\partial B)^\eta} \phi(x_1 - z_1) \ldots \phi(y_2 - z_4) dx_1 \ldots dy_2 \\
\leq & \int\limits_{(\partial B)^\eta} dx_1 \ldots dy_2.
\end{aligned}$$

Finally note that ∂B is a *compact* 3-dimensional manifold which has zero measure under the 4-dimensional Lebesgue measure. By compactness of ∂B, we have $(\partial B)^\eta \downarrow \partial B$ as $\eta \to 0$, and the claim follows by Dominated Convergence Theorem. Therefore

$$\mathrm{E}[F_n(x)] = \frac{1}{\# I_n} \sum_{j=1}^{\# I_n} \mathrm{P}\Big(\frac{y_j}{n^2} \leq x^4\Big) \sim \frac{1}{\# I_n} \sum_{j=1}^{\# I_n} \mathrm{P}\Big(L_{n,j} \leq x^4\Big) \sim \frac{1}{\# I_n} \sum_{j=1}^{\# I_n} \Phi_4(A_{n,j}).$$

Proof of Step II. To identify the limit, recall the structure of the sets $S(x), \mathcal{P}_j, \mathcal{A}_j$ and their properties. Since $\# I_n/n \to 1/4$, $v_{k,n} \leq 2$ and either $S(x) = S(u)$ or $S(x) \cap S(u) = \emptyset$, we have

$$\lim_{n \to \infty} \frac{1}{\# I_n} \sum_{j=1}^{\# I_n} \Phi_4(A_{n,j}) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1, |\mathcal{A}_j|=2}^{n} \Phi_4(A_{n,j}). \tag{2.4.30}$$

Also for $n = k^2 + 1$ we can write $\{1, 2, \ldots, n-1\}$ as $\{ak + b;\ 0 \leq a \leq k-1,\ 1 \leq b \leq k\}$ and using the construction of $S(x)$ we have (except for at most two values of j)

$$\mathcal{A}_j = \{ak + b, bk - a\} \text{ for } j = ak + b;\ 0 \leq a \leq k-1,\ 1 \leq b \leq k.$$

Recall that for fixed x, $H_2(\omega, \omega', x)$ is uniformly continuous on $[0, 2\pi] \times [0, 2\pi]$, . Therefore given any positive number ρ we can choose N large enough such that for all $n = k^2 + 1 > N$,

$$\sup_{0 \leq a \leq k-1,\ 1 \leq b \leq k} \Big| H_2\Big(\frac{2\pi(ak+b)}{n}, \frac{2\pi(bk-a)}{n}, x\Big) - H_2\Big(\frac{2\pi a}{\sqrt{n}}, \frac{2\pi b}{\sqrt{n}}, x\Big) \Big| < \rho. \tag{2.4.31}$$

Finally using (2.4.30), (2.4.31) we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{\#I_n}\sum_{j=1}^{\#I_n}\Phi_4(A_{n,j}) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\Phi_4(A_{n,j}) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}G_2\big(\frac{x^4}{4\pi^2 f_j}\big) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{b=1}^{\lfloor\sqrt{n}\rfloor}\sum_{a=0}^{\lfloor\sqrt{n}\rfloor}H_2\Big(\frac{2\pi(ak+b)}{n},\frac{2\pi(bk-a)}{n},x\Big) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{b=1}^{\lfloor\sqrt{n}\rfloor}\sum_{a=0}^{\lfloor\sqrt{n}\rfloor}H_2\Big(\frac{2\pi a}{\sqrt{n}},\frac{2\pi b}{\sqrt{n}},x\Big) \\
&= \int_0^1\int_0^1 H_2(2\pi s,2\pi t,x)ds\ dt = F_2(x).
\end{aligned}$$

To show that $V[F_n(x)]\to 0$, since the variables involved are all bounded, it is enough to show that

$$n^{-2}\sum_{j\neq j'}Cov\Big(\mathbb{I}\big(\frac{y_j}{n^2}\le x^4\big),\ \mathbb{I}\big(\frac{y_{j'}}{n^2}\le x^4\big)\Big)\to 0.$$

Along the lines of the proof used to show $\mathrm{E}[F_n(x)]\to F_2(x)$, one may now extend the vectors with 4 coordinates defined above to ones with 8 coordinates and proceed exactly as above to verify this. We omit the routine details. This completes the proof the Theorem for $g=2$.

The above argument can be extended to cover the general ($g>2$) case. We highlight only a few of the technicalities and omit the other details. For general g we need the following lemma.

Lemma 2.4.21. *Suppose $L_n(\omega_j)$, $R_n(\omega_j)$ are as defined in Lemma 2.4.15. Then given any $\epsilon,\eta>0$ there exist an $N\in\mathbb{N}$ such that*

$$P\big(\big|\prod_{i=1}^{s}L_n(\omega_{j_i})\prod_{i=s+1}^{g}R_n(\omega_{j_i})\big|>\epsilon\big)<\eta \textit{ for all } n\ge N.$$

Proof. Note that

$$\begin{aligned}
\mathrm{P}\big(\big|\prod_{i=1}^{s}L_n(\omega_{j_i})\prod_{i=s+1}^{g}R_n(\omega_{j_i})\big|>\epsilon\big) &\le \mathrm{P}\big(\big|L_n(\omega_{j_1})\big|\ge M\epsilon\big) \\
&\quad +\mathbb{P}\big(\big|\prod_{i=2}^{s}L_n(\omega_{j_i})\prod_{i=s+1}^{g}R_n(\omega_{j_i})\big|>1/M\big),
\end{aligned}$$

and iterating this argument,

$$\begin{aligned}
\mathrm{P}\big(\big|\prod_{i=1}^{s} L_n(\omega_{j_i})\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > \epsilon\big) &\leq \mathrm{P}\big(\big|L_n(\omega_{j_1})\big| \geq M\epsilon\big) + \sum_{i=2}^{s}\mathrm{P}\big(\big|L_n(\omega_{j_i})\big| \geq M\big)\\
&\quad +\mathrm{P}\big(\big|\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big).
\end{aligned}$$

Further note that

$$\begin{aligned}
\mathrm{P}\big(\big|\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big) &\leq \mathrm{P}\big(\big|\prod_{i=s+2}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big) + \mathrm{P}\big(\big|R_n(\omega_{j_{s+1}})\big| > 1\big)\\
&\leq \mathrm{P}\big(\big|R_n(\omega_{j_g})\big| > 1/M^s\big) + \sum_{i=1}^{g-1}\mathrm{P}\big(\big|R_n(\omega_{j_i})\big| > 1\big)\\
&\leq \big(M^s + g - s - 1\big)\max_{1\leq k\leq n}\mathrm{E}\,|R_n(\omega_k)|.
\end{aligned}$$

Combining all the above we get

$$\begin{aligned}
\mathbb{P}\big(\big|\prod_{i=1}^{s} L_n(\omega_{j_i})\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > \epsilon\big) &\leq \mathrm{P}\big(\big|L_n(\omega_{j_1})\big| \geq M\epsilon\big) + \sum_{i=2}^{s}\mathrm{P}\big(\big|L_n(\omega_{j_i})\big| \geq M\big)\\
&\quad +\big(M^s + g - s - 1\big)\max_{1\leq k\leq n}\mathrm{E}\,|R_n(\omega_k)|\\
&\leq \frac{1}{M}(s-1+1/\epsilon)4\pi \max_{\omega\in[0,2\pi]} f(\omega)\\
&\quad +\big(M^s + g - s - 1\big)\max_{1\leq k\leq n}\mathrm{E}\,|R_n(\omega_k)|.
\end{aligned}$$

First term in the right side can be made smaller than $\eta/2$ by choosing M large enough and since $\max_{1\leq k\leq n}\mathrm{E}\,|R_n(\omega_k)| \to 0$ as $n\to\infty$, we can choose $N\in\mathbb{N}$ such that the second term is less than $\eta/2$ for all $n\geq N$, proving the lemma. □

Now return to the main proof for general $g \geq 2$. As before, $n' = n$ and $v_{k,n}/n \to 0$. Hence it remains to consider only the eigenvalues corresponding to the sets $\mathcal{P}_l$ which have size exactly equal to g_1 and it follows from Lemma 2.4.16(i) that $g_1 = 2g$. We can now proceed as in $g = 2$ case. First we show

$$\frac{1}{\# I_n}\sum_{j=1}^{\# I_n}\mathrm{P}\big(\frac{y_j}{n^g} \leq x^{2g}\big) \to F_g(x). \tag{2.4.32}$$

Now write $\frac{y_j}{n^g}$ as follows

$$\frac{y_j}{n^g} = L_{n,j} + R_{n,j} \text{ for } 1 \leq j \leq \#I_n, \text{ where } L_{n,j} = \prod_{t \in \mathcal{A}_j} L_n(\omega_t) = (2\pi)^g f_j \frac{\overline{y}_j}{n^g}.$$

Using Lemma 2.4.21 it is easy show that for any $\epsilon > 0$, $\max_{1 \leq j \leq \#I_n} \mathrm{P}(|R_{n,j}| > \epsilon) \to 0$ as $n \to \infty$. So, by Lemma 2.4.6, to show (2.4.32) it is sufficient to show that

$$\frac{1}{\#I_n} \sum_{j=1}^{\#I_n} \mathrm{P}\big(L_{n,j} \leq x^{2g}\big) \to F_g(x).$$

We prove this in two steps (Step I and Step II) as we did for $g = 2$. Define

$$\bar{A}_{n,j} = \Big\{(x_i, y_i, i = 1, 2, .., g) \in \mathbb{R}^{2g} : \prod_{i=1}^{g} [2^{-1}(x_i^2 + y_i^2)] \leq \frac{x^{2g}}{(2\pi)^g f_j}\Big\}.$$

Now, in Step I, for fixed $\epsilon > 0$ we find $M_1 > 0$ large such that $\Phi([-M_1, M_1]^c) \leq \epsilon/(8g)$ and $M_2 > 0$ such that

$$\mathrm{P}\left(|n^{-1/2} \sum_{l=0}^{n-1} \epsilon_l \cos \frac{2\pi l t}{n}| \geq M_2\right) \leq \epsilon/(8g) \text{ and } \mathrm{P}\left(|n^{-1/2} \sum_{l=0}^{n-1} \epsilon_l \sin \frac{2\pi l t}{n}| \geq M_2\right) \leq \epsilon/(8g).$$

Set $M = \max\{M_1, M_2\}$ and define $B := \Big\{(x_j, y_j; 1 \leq j \leq g) \in \mathbb{R}^{2g} : |x_j|, |y_j| \leq M \ \ \forall j\Big\}$. Note that, ∂B is a *compact* $(2g-1)$-dimensional manifold which has zero measure under the $2g$-dimensional Lebesgue measure. Now proceeding as before we have

$$\left| \frac{1}{\#I_n} \sum_{l=1}^{\#I_n} \mathrm{P}\Big(L_{n,j} \leq x^4\Big) - \frac{1}{\#I_n} \sum_{l=1}^{\#I_n} \Phi_4(\bar{A}_{n,j}) \right| \to 0.$$

Now note that for $n = k^g + 1$ we can write $\{1, 2, \ldots, n-1\}$ as $\{b_1 k^{g-1} + b_2 k^{g-2} + \cdots + b_{g-1} k + b_g;\ 0 \leq b_i \leq k-1, \text{ for } 1 \leq i \leq k-1;\ 1 \leq b_g \leq k\}$. So we can write the sets $\mathcal{A}_j$ (see, 2.4.24) explicitly using this decomposition of $\{1, 2, \ldots, n-1\}$ as done in $g = 2$ case, that is, $n = k^2 + 1$ case. For example if $g = 3$, $\mathcal{A}_j = \{b_1 k^2 + b_2 k + b_3,\ b_2 k^2 + b_3 k - b_1,\ b_3 k^2 - b_1 k - b_2\}$ for $j = b_1 k^2 + b_2 k + b_3$ (except for finitely many j, bounded by $v_{k,n}$ and they do not contribute to this limit). Using this fact and proceeding as before we conclude that the LSD is now $F_g(\cdot)$, proving Theorem 2.4.18 completely. □

Type II. $n = k^g - 1$ for some $g \geq 2$.

For $z_i, w_i \in \mathbb{R}, i = 1, 2, .., g$, and with $\{N_i\}$ i.i.d. $N(0, 1)$, define

$$\mathcal{H}_g(\omega_i, z_i, w_i, i = 1, \ldots, g) = \mathrm{P}\big(B(\omega_1, \omega_2, .., \omega_g)(N_1, ..., N_{2g})' \le (z_i, w_i, i = 1, 2, .., g)'\big).$$

Proof of the following lemma is omitted.

Lemma 2.4.22. *(i) For fixed* $\{z_i, w_i, i = 1, \ldots, g\}$*,* $\mathcal{H}_g$ *is bounded continuous in* $(\omega_1, \ldots, \omega_g)$.

(ii) $\mathcal{F}_g$ *defined below is a proper distribution function.*

$$\mathcal{F}_g(z_i, w_i, i = 1, \ldots, g) = \int_0^1 \cdots \int_0^1 \mathcal{H}_g(2\pi t_i, z_i, w_i, i = 1, \ldots, g) \prod dt_i. \tag{2.4.33}$$

(iii) If $\mathrm{Leb}(C_0) = 0$ *then* $\mathcal{F}_g$ *is continuous everywhere and may be expressed as*

$$\begin{aligned}&\mathcal{F}_g(z_i, w_i, i = 1, .., g) \\ &= \int \cdot \int \mathbb{I}_{\{\boldsymbol{t} \le (z_k, w_k, k=1,.,g)\}} \Big[\int_0^1 \cdot \int_0^1 \frac{\mathbb{I}_{\{\prod f(2\pi u_i) \neq 0\}}}{(2\pi)^g \prod_{i=1}^g [\pi f(2\pi u_i)]} \prod_{i=1}^g e^{-\frac{1}{2}\frac{t_{2i-1}^2 + t_{2i}^2}{\pi f(2\pi u_i)}} \prod du_i \Big] d\boldsymbol{t}.\end{aligned}$$

where $\boldsymbol{t} = (t_1, t_2, \ldots, t_{2g-1}, t_{2g})$ *and* $d\boldsymbol{t} = \prod dt_i$*. Further* $\mathcal{F}_g$ *is multivariate normal (with independent components) if and only if* f *is constant almost everywhere (Lebesgue).*

(iv) If $\mathrm{Leb}(C_0) \neq 0$ *then* $\mathcal{F}_g$ *is discontinuous only on* $D_g = \{(z_i, w_i, i = 1, \ldots, g) : \prod_{i=1}^g z_i w_i = 0\}$.

Theorem 2.4.23 (Bose, Hazra and Saha (2009) [33])**.** *Suppose Assumptions 2.4.1 and 2.4.2 hold. Suppose* $n = k^g - 1$ *for some* $g \ge 2$*. Then as* $n \to \infty$*,* $F_{n^{-1/2}A_{k,n}}$ *converges in* L_2 *to the LSD* $(\prod_{i=1}^g G_i)^{1/g}$ *where* $(\mathcal{R}(G_i), \mathcal{I}(G_i);\ i = 1, 2, \ldots g)$ *has the distribution* $\mathcal{F}_g$ *given in (2.4.33).*

Remark 2.4.24. *If* $\{x_i\}$ *are i.i.d, with finite* $(2 + \delta)$ *moment, then* $f(\omega) \equiv 1/2\pi$ *and the LSD simplifies to* $U_2(\prod_{i=1}^g E_i)^{1/2g}$ *where* $\{E_i\}$ *are i.i.d.* $Exp(1)$ *and* U_2 *is uniformly distributed over the unit circle independent of* $\{E_i\}$*. This agrees with the conclusion in Theorem 2.3.10(ii).*

Proof of Theorem 2.4.23. First we assume $\mathrm{Leb}(C_0) = 0$. Note that $\gcd(k, n) = 1$. Since $k^g = 1 + n = 1 \mod n$, we have $g_1 | g$. If $g_1 < g$, then $g_1 \le g/\alpha$ where $\alpha = 2$ if g is even and $\alpha = 3$ if g is odd. In either case, it is easy to check that

$$k^{g_1} \le k^{g/\alpha} \le (1 + n)^{1/\alpha} = o(n).$$

Hence, $g = g_1$. By Lemma 2.4.16(ii), the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathcal{A}_l$ and $|\mathcal{A}_l| < g$ is asymptotically negligible.

Unlike the previous theorem, here the partition sets $\mathcal{A}_l$ are not necessarily self-conjugate. However, the number of indices l such that $\mathcal{A}_l$ is self-conjugate is asymptotically negligible compared to n. To show this, we need to bound the cardinality of the following set for $1 \leq l < g$:

$$D_l = \{t \in \{1, 2, \ldots, n\} : tk^l = -t \mod n\} = \{t \in \{1, 2, \ldots, n\} : n | t(k^l + 1)\}.$$

Note that $t_0 = n/\gcd(n, k^l + 1)$ is the minimum element of D_l and every other element is a multiple of t_0. Thus

$$|D_l| \leq \frac{n}{t_0} \leq \gcd(n, k^l + 1).$$

Let us now estimate $\gcd(n, k^l + 1)$. For $l > [g/2]$,

$$\gcd(n, k^l + 1) \leq \gcd(k^g - 1, k^l + 1) = \gcd\left(k^{g-l}(k^l + 1) - (k^{g-l} - 1), k^l + 1\right) \leq k^{g-l},$$

which implies $\gcd(n, k^l + 1) \leq k^{[g/2]}$ for all $1 \leq l < g$. Therefore,

$$\frac{\gcd(n, k^l + 1)}{n} = \frac{k^{[g/2]}}{(k^g - 1)} \leq \frac{2}{k^{[(g+1)/2]}} \leq \frac{2}{((n)^{1/g})^{[(g+1)/2]}} = o(1).$$

So, we can ignore the partition sets $\mathcal{P}_j$ which are self-conjugate. For other $\mathcal{P}_j$,

$$y_j = \prod_{t \in \mathcal{P}_j} (\sqrt{n} b_t + i\sqrt{n} c_t)$$

will be complex.

Now for simplicity we will provide the detailed argument assuming that $g = 2$. Then, $n = k^2 - 1$ and we can write $\{0, 1, 2, \ldots, n\}$ as $\{ak + b;\ 0 \leq a \leq k - 1,\ 0 \leq b \leq k - 1\}$ and using the construction of $S(x)$ we have $\mathcal{P}_j = \{ak + b, bk + a\}$ and $\#\mathcal{P}_j = 2$ for $j = ak + b;\ 0 \leq a \leq k - 1,\ 0 \leq b \leq k - 1$ (except for finitely many j and hence such indices do not contribute to the LSD). Let us define

$$I_n = \{j : \#\mathcal{P}_j = 2\}.$$

It is clear that $n/\#I_n \to 2$. Without any loss, let $I_n = \{1, 2, \ldots, \#I_n\}$. Suppose $\mathcal{P}_j = \{j_1, j_2\}$. We first find the limiting distribution of the empirical distribution of $\frac{1}{\sqrt{n}}(\sqrt{n} b_{j_1}, \sqrt{n} c_{j_1}, \sqrt{n} b_{j_2}, \sqrt{n} c_{j_2})$ for those j for which $\#\mathcal{P}_j = 2$ and show the convergence in L_2. Let $F_n(x, y, z, w)$ be the ESD of $\{(b_{j_1}, c_{j_1}, b_{j_2}, c_{j_2})\}$, that is

$$F_n(z_1, w_1, z_2, w_2) = \frac{1}{\#I_n} \sum_{j=1}^{\#I_n} \mathbb{I}\left(b_{j_k} \leq z_k, c_{j_k} \leq w_k,\ k = 1, 2\right).$$

We show that for $z_1, w_1, z_2, w_2 \in \mathbb{R}$,

$$\mathrm{E}[F_n(z_1, w_1, z_2, w_2)] \to \mathcal{F}_2(z_1, w_1, z_2, w_2) \text{ and } V[F_n(z_1, w_1, z_2, w_2)] \to 0. \qquad (2.4.34)$$

Define for $j = 1, 2, \cdots, n$,

$$\eta_j = (\xi_{2j_1-1}, \xi_{2j_1}, \xi_{2j_2-1}, \xi_{2j_2})',$$

and let $Y_{1n}(\omega_j) = \mathcal{R}(Y_n(\omega_j))$, $Y_{2n}(\omega_j) = \mathcal{I}(Y_n(\omega_j))$, where $Y_n(\omega_j)$ is same as defined in Lemma 2.4.7. Define

$$Y_{n,j} = \big(Y_{1n}(\omega_{j_1}), Y_{2n}(\omega_{j_1}), Y_{1n}(\omega_{j_2}), Y_{2n}(\omega_{j_2})\big).$$

Then $(b_{j_1}, c_{j_1}, b_{j_2}, c_{j_2}) = B(\omega_{j_1}, \omega_{j_2})\eta_j + Y'_{n,j}$. Note that by Lemma 2.4.7, for any $\epsilon > 0$, $\max_{1 \le j \le n} \mathrm{P}(\|Y_{n,j}\| > \epsilon) \to 0$ as $n \to \infty$. So in view of Lemma 2.4.6 to show $\mathrm{E}[F_n(z_1, w_1, z_2, w_2)] \to \mathcal{F}_2(z_1, w_1, z_2, w_2)$ it is enough to show that

$$\frac{1}{\#I_n} \sum_{j=1}^{\#I_n} \mathrm{P}(B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)') \to \mathcal{F}_2(z_1, w_1, z_2, w_2).$$

For this we use normal approximation and define

$$X_{l,j} = 2^{1/2}\left(\epsilon_l \cos\left(\frac{2\pi j_1 l}{n}\right), \epsilon_l \sin\left(\frac{2\pi j_1 l}{n}\right), \epsilon_l \cos\left(\frac{2\pi j_2 l}{n}\right), \epsilon_l \sin\left(\frac{2\pi j_2 l}{n}\right)\right)',$$

and $N = \big(N_1, N_2, N_3, N_4\big)'$, where $\{N_i\}$ are i.i.d. $N(0, 1)$. Note

$$\begin{aligned}&\big\{B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)'\big\}\\ &= \big\{B(\omega_{j_1}, \omega_{j_2})(n^{-1/2}\sum_{l=0}^{n-1} X_{l,j}) \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)'\big\}.\end{aligned}$$

Since $\big\{(r_1, r_2, r_3, r_4) : B(\omega_{j_1}, \omega_{j_2})(r_1, r_2, r_3, r_4)' \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)'\big\}$ is a *convex* set in $\mathbb{R}^4$ and $\{X_{l,j};\ l = 0, 1, \ldots, (n-1)\}$ satisfies (2.4.9)–(2.4.11), we can show using Part (i) of Lemma 2.3.1 that

$$\begin{aligned}&\frac{1}{\#I_n} \sum_{j=1}^{\#I_n} \big|\mathrm{P}(B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)')\\ &\qquad -\mathrm{P}(B(\omega_{j_1}, \omega_{j_2})N \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)')\big| \to 0,\end{aligned}$$

as $n \to \infty$. Hence

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{1}{\#I_n} \sum_{j=1}^{\#I_n} \mathrm{P}(B(\omega_{j_1}, \omega_{j_2})\eta_j \le (z_1, w_1, z_2, w_2)') \\
= & \lim_{n\to\infty} \frac{1}{\#I_n} \sum_{j=1}^{\#I_n} \mathrm{P}(B(\omega_{j_1}, \omega_{j_2})N \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)') \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{P}(B(\omega_{j_1}, \omega_{j_2})N \le (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)') \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathcal{H}_2(\omega_{j_1}, \omega_{j_2}, z_1, w_1, z_2, w_2) \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{a=0}^{\lfloor\sqrt{n}\rfloor} \sum_{b=0}^{\lfloor\sqrt{n}\rfloor} \mathcal{H}_2\big(\frac{2\pi(ak+b)}{n}, \frac{2\pi(bk+a)}{n}, z_1, w_1, z_2, w_2\big) \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{a=0}^{\lfloor\sqrt{n}\rfloor} \sum_{b=0}^{\lfloor\sqrt{n}\rfloor} \mathcal{H}_2\big(\frac{2\pi a}{\sqrt{n}}, \frac{2\pi b}{\sqrt{n}}, z_1, w_1, z_2, w_2\big) \\
= & \int_0^1 \int_0^1 \mathcal{H}_2(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\ dt = \mathcal{F}_2(z_1, w_1, z_2, w_2).
\end{aligned}
$$

Similarly we can show $V[F_n(x)] \to 0$ as $n \to \infty$.

Hence the empirical distribution of y_j for those j for which $\#\mathcal{P}_j = 2$ converges to the distribution of $\prod_{i=1}^2 G_i$ such that $(\mathcal{R}(G_i), \mathcal{I}(G_i);\ i = 1, 2)$ has distribution $\mathcal{F}_2$. Hence the LSD of $n^{-1/2}A_{k,n}$ is $\big(\prod_{i=1}^2 G_i\big)^{1/2}$, proving the result when $g = 2$ and $\mathrm{Leb}(C_0) = 0$.

When $\mathrm{Leb}(C_0) \neq 0$, we have to show (2.4.34) only on D_2^c (of Lemma 2.4.22). All the above steps in the proof will go through for all $(z_i, w_i; i = 1, 2)$ in D_2^c. Hence if $\mathrm{Leb}(C_0) \neq 0$, we have our required LSD. This proves the Theorem when $g = 2$.

For general $g > 2$, note that we can write $\{0, 1, 2, \ldots, n\}$ as $\{b_1k^{g-1} + b_2k^{g-2} + \cdots + b_{g-1}k + b_g;\ 0 \le b_i \le k-1,\ \text{for } 1 \le i \le k\}$. So we can write the sets $\mathcal{A}_j$ explicitly using this decomposition of $\{0, 1, 2, \ldots, n\}$ as done in $n = k^2 - 1$ case. For example if $g = 3$, $\mathcal{A}_j = \{b_1k^2 + b_2k + b_3,\ b_2k^2 + b_3k + b_1,\ b_3k^2 + b_1k + b_2\}$ for $j = b_1k^2 + b_2k + b_3$ (except for finitely many j, bounded by $v_{k,n}$ and they do not contribute to this limit). Using this fact and proceeding as before we will have the LSD as $\big(\prod_{i=1}^g G_i\big)^{1/g}$ such that $(\mathcal{R}(G_i), \mathcal{I}(G_i);\ i = 1, 2, \ldots g)$ has distribution $\mathcal{F}_g$. □

2.5 Simulations

2.5.1 I.I.D inputs

In Figure 2.1–2.4, we have plotted eigenvalues of k-circulant matrices for different combination of k and n when the input sequence is i.i.d.

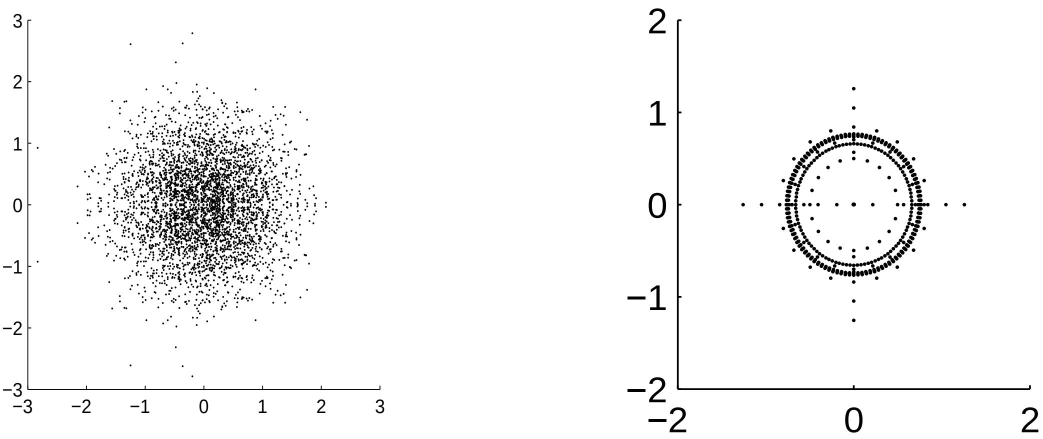

Figure 2.1: Eigenvalues of 10 realizations of $n^{-1/2}A_{k,n}$ with x_i i.i.d. $N(0,1)$ when (i) (left) $k = 1,\ n = 2000$ and (ii) (right) $k = 2,\ n = 2000$.

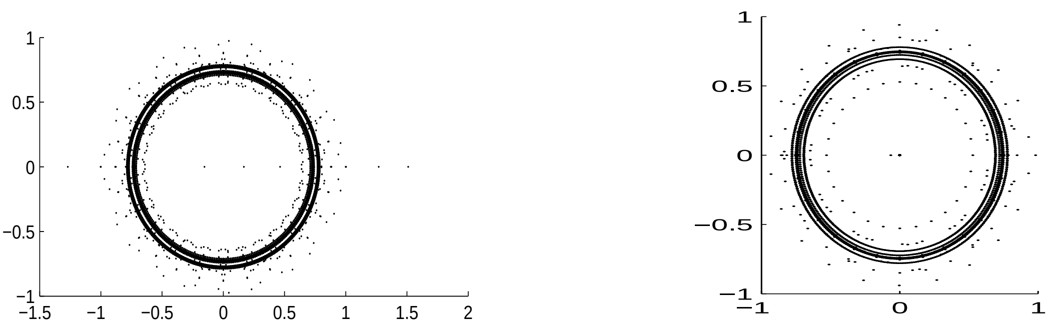

Figure 2.2: Eigenvalues of 10 realizations of $n^{-1/2}A_{k,n}$ with x_i i.i.d. $N(0,1)$ when (i) (left) $k = 2,\ n = 2001$ and (ii) (right) $k = 3,\ n = 2001$.

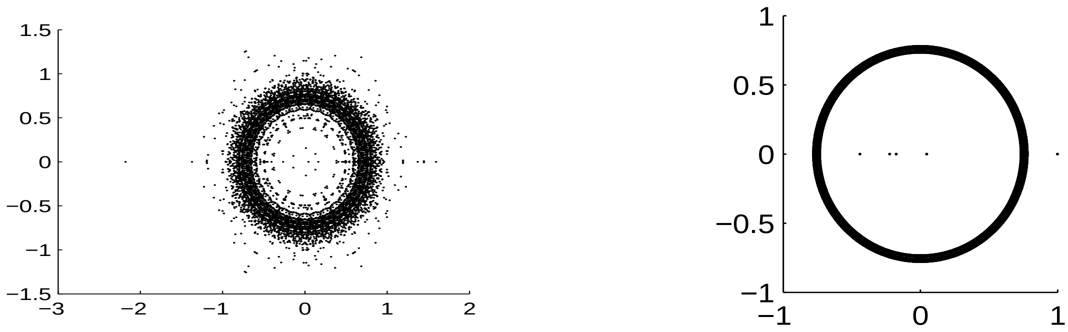

Figure 2.3: Eigenvalues of 10 realizations of $n^{-1/2}A_{k,n}$ with x_i i.i.d. $N(0,1)$ when (i) (left) $k = 3,\ n = 2002$ and (ii) (right) $k = 3,\ n = 2003$.

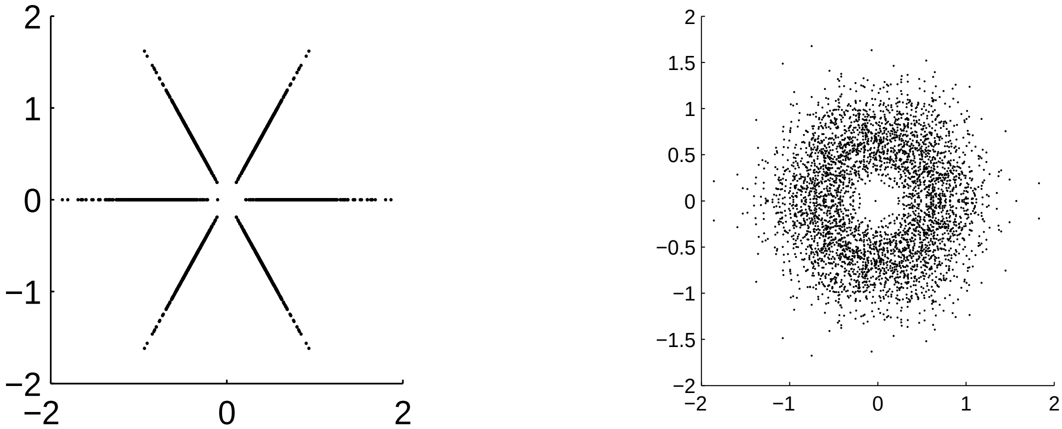

Figure 2.4: Eigenvalues of 10 realizations of $n^{-1/2}A_{k,n}$ with x_i i.i.d. $N(0,1)$ when (i) (left) $n = k^3 + 1,\ k = 10$ and (ii) (right) $n = k^3 - 1,\ k = 10$.

2.5.2 Linear process inputs

To demonstrate the limits we did some simulations with MA(1) and MA(2) processes. A process $\{X_t, t \in \mathbb{Z}\}$ is said to be a moving average process of order q (MA(q)) if

$$X_t = Z_t + a_1 Z_{t-1} + a_2 Z_{t-2} + \cdots + a_q Z_{t-q}$$

where $\{a_i\}$ is a sequence of real numbers and $\{Z_t, t \in \mathbb{Z}\}$ is a process with zero mean and covariance function $\mathrm{E}(Z_t Z_{t+h}) = \mathbb{I}\{h = 0\}\sigma^2$.

We performed numerical integration to obtain the LSD. In case of k-circulant ($n = k^2 + 1$), we have plotted the density of F_2 defined in (2.4.26).

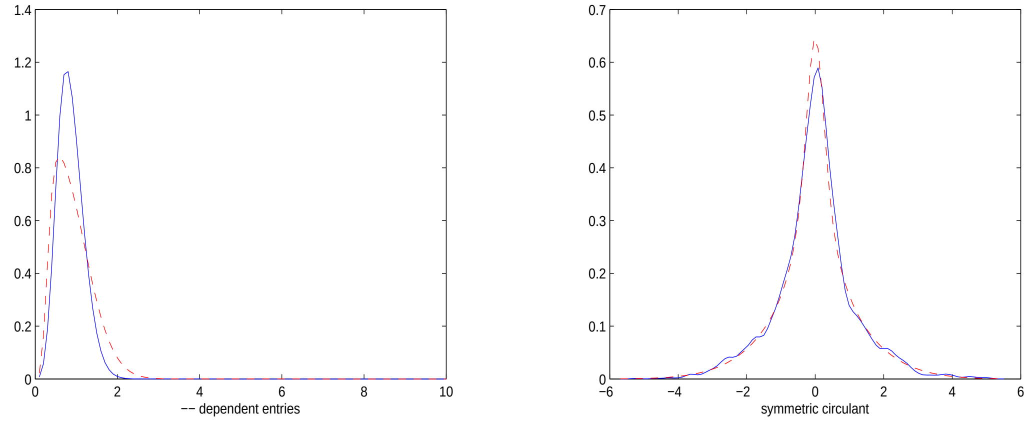

Figure 2.5: (i) (left) dashed line represents the density of F_2 when $f(\omega) = \frac{1}{2\pi}(1.25 + \cos x)$ and the continuous line represents the same with $f \equiv \frac{1}{2\pi}$. (ii) (right) dashed line represents the LSD of symmetric circulant matrix with entries $x_t = 0.3\epsilon_t + \epsilon_{t+1} + 0.5\epsilon_{t+2}$ where $\{\epsilon_i\}$ i.i.d. $N(0,1)$ and the continuous line represents the kernel density estimate of the ESD of the same matrix of order 5000×5000 and same $\{x_t\}$.

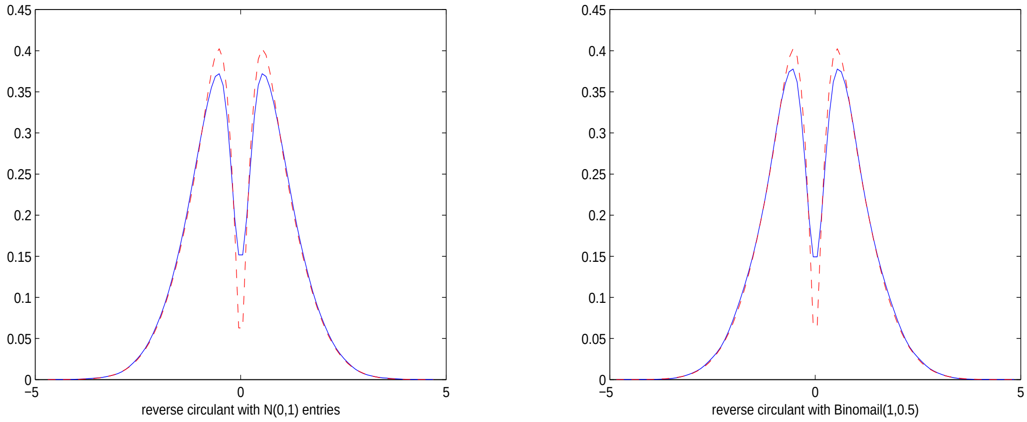

Figure 2.6: (i) (left) dashed line represents the LSD of the reverse circulant matrix with entries $x_t = 0.3\epsilon_t + \epsilon_{t+1} + 0.5\epsilon_{t+2}$ where $\{\epsilon_i\}$ i.i.d. $N(0,1)$. The continuous line represents the kernel density estimate of ESD of the same matrix of order 5000×5000 with same $\{x_t\}$. (ii) same graphs with centered and scaled Bernoulli$(1, 0.5)$.

Chapter 3

Tail of product and extreme values

In this chapter we digress from random matrix theory. Here we identify the tail behaviour of finite but arbitrary product of i.i.d. exponential random variables. As a consequence, it follows that this n fold product lies in the maximum domain of attraction of the Gumbel distribution for any n. We use this result in the next chapter to derive the limit of spectral radius of k-circulant matrices.

Several researchers have studied the distributional properties of the product of independent and identically distributed (i.i.d.) random variables. See for instance Springer and Thompson (1970) [119], Lomnicki (1967) [86] and Galambos and Simonelli (2004) [62]. However, there does not seem to be in the literature any result quantifying the nature of the tail behaviour of product beyond two or three fold product of i.i.d. exponentials.

Here is an outline of this chapter. In Section 3.1.1 we describe a few known methods for product of two exponentials. In Section 3.1.2 we derive explicitly the tail behaviour of the n fold product of exponentials (Theorem 3.1.2) by making judicious use of Laplace's asymptotic. Then in Section 3.2, using this result on tail behaviour we show that the $1/2g$-th root of product of g-many i.i.d. exponentials belongs to the max domain of attraction of the Gumbel distribution.

Some of the results of Bose, Hazra and Saha (2010) [37] are based on this chapter.

3.1 Tail of product

Let $\{E_i\}$ be i.i.d. standard exponentials. Define

$$H_n(x) = \mathrm{P}[E_1 E_2 \cdots E_n > x]. \tag{3.1.1}$$

What is the behaviour of $H_n(x)$ as $x \to \infty$? It is easy to see that this tail becomes heavier as n increases but there does not appear to be any results in the literature quantifying the nature of the tail beyond the case $n = 2$.

3.1.1 Various methods for two fold product

There are several possible approaches that come to mind to solve this problem:

(a) **Mellin Transform:** The Mellin transform of any non-negative function $f(x)$, $x \geq 0$, is defined as (see Springer and Thompson (1970) [119])

$$M\left(f(\cdot)|s\right) = \int_0^\infty x^{s-1} f(x)dx.$$

Under certain regularity conditions, this transform, considered as a function of the complex variable s, admits an inversion integral:

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M(f(\cdot)|s)ds,$$

where the path of integration is a line parallel to the imaginary axis and to the right of the origin. If X and Y are non-negative independent random variables with p.d.f. $f(\cdot)$ and $g(\cdot)$ respectively and if $h(\cdot)$ is the p.d.f. of $Z = XY$, then

$$M\left(h(\cdot)|s\right) = M\left(f(\cdot)|s\right) M\left(g(\cdot)|s\right).$$

Thus the Mellin transform for the product plays a role similar to that played by the Fourier transform for sum. It can be easily seen that

$$M(\overline{F}(\cdot)|s) = s^{-1}M(f(\cdot)|s+1), \quad \text{where} \quad \overline{F}(x) = \mathrm{P}[X > x]. \tag{3.1.2}$$

Using (3.1.2) and appropriate complex integration, Lomnicki (1967) [86] showed that,

$$M(H_n(\cdot)|s) = s^{-1}[\Gamma(s+1)]^n \quad \text{and} \quad H_n(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}s^{-1}[\Gamma(s+1)]^n ds.$$

He also derived the following series representations of the above integral for $n = 2$ and $n = 3$.

$$H_2(x) = 1 - \sum_{j=1}^{\infty} \frac{x^j}{j\{j-1!\}^2}\{-\log x + 2\psi(j) + j^{-1}\}, \quad \text{and}$$

$$H_3(x) = 1 - \frac{1}{2}\sum_{j=1}^{\infty} \frac{x^j(-1)^{j-1}}{j\{(j-1)!\}^3}\left[\{-\log x + 3\psi(j) + j^{-1}\}^2 + 3\{\psi'(1) + \sum_{k=1}^{j-1} k^{-2}\} + j^{-2}\right],$$

where $\psi(\cdot)$ is the Euler psi function (digamma function) and $\psi'(\cdot)$ it's first derivative.

For the special case of $n = 2$, comparing the above series with the series expansion of the modified Bessel function of the second kind $K_0(x)$, it can be easily shown that

$$H_2(x) \sim \sqrt{\pi} x^{\frac{1}{4}} e^{-2x^{1/2}} \quad \text{as} \quad x \to \infty.$$

However, this method does not seem to be easy to extend to other values of n.

(b) **Differential Equation:** The following differential equation can be easily derived for $H_2(\cdot)$ (see Bose, Mitra and Sen (2008) [44])

$$x \frac{d^2}{dx^2} H_2(x) - H_2(x) = 0, \quad H_2(0) = 0 \quad \text{and} \quad H_2(\infty) = 1.$$

Standard theory of second order differential equations implies that the solution can be expressed in terms of the modified Bessel function of second kind and the tail behaviour follows from that. For $n \geq 3$ we obtain higher order differential equations and their solutions appear to be intractable.

(c) **Real analysis:** Tang (2008) [120] obtained a nice formula for $H_2(x)$ using simple integral substitutions. We reproduce the result and its proof since this will be useful to motivate our result for arbitrary n.

Lemma 3.1.1.

$$H_2(x) = e^{-2x^{1/2}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \frac{z + 2x^{1/2}}{\sqrt{z^2 + 4zx^{1/2}}} dz \sim \sqrt{\pi} e^{-2x^{1/2}} x^{1/4} g_2(x),$$

where $g_2(x) \to 1$ *as* $x \to \infty$.

Proof. First note that,

$$H_2(x) = \int_0^\infty e^{-y} e^{-x/y} dy = \int_0^{x^{1/2}} e^{-(y+\frac{x}{y})} dy + \int_{x^{1/2}}^\infty e^{-(y+\frac{x}{y})} dy.$$

Let $A(y) = y + \frac{x}{y}$. Then

$$A'(y) = 1 - \frac{x}{y^2} > 0 \quad \text{if} \quad y > x^{1/2}$$
$$< 0 \quad \text{if} \quad y < x^{1/2}.$$

Hence in these two regions, consider separately, the monotone substitution $A(y) = t$.

Let the corresponding unique solutions (inverses), be $y_i(t)$, $i = 1, 2$, so that

$$y_1(t) < x^{1/2} < y_2(t).$$

Observing that both the ranges transform to $(2x^{1/2}, \ \infty)$, we obtain,

$$H_2(x) = \int_{2x^{1/2}}^{\infty} e^{-t} \left[\left(1 - \frac{x}{y_2^2(t)}\right)^{-1} + \left(\frac{x}{y_1^2(t)} - 1\right)^{-1} \right] dt.$$

Since $y_i = t, \ i = 1, 2$ are the two solutions of the quadratic equation $A(y) = t$, it is easy to see that

$$\left[\left(1 - \frac{x}{y_2^2(t)}\right)^{-1} + \left(\frac{x}{y_1^2(t)} - 1\right)^{-1} \right] = \frac{t}{\sqrt{(t^2 - 4x)}}.$$

Now, making a further substitution $t = z + 2x^{1/2}$ we get,

$$\begin{aligned} H_2(x) &= e^{-2x^{1/2}} \int_0^{\infty} \frac{e^{-z}}{\sqrt{z}} \frac{z + 2x^{1/2}}{\sqrt{z^2 + 4zx^{1/2}}} dz \\ &= \sqrt{\pi} e^{-2x^{1/2}} x^{1/4} \underbrace{\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-z}}{\sqrt{z}} \left(\frac{1 + z/2x^{1/2}}{\sqrt{1 + z/4x^{1/2}}} \right) dz.}_{g_2(x)} \end{aligned}$$

Now a straightforward application of the Dominated Convergence Theorem (DCT) implies

$$\lim_{x \to \infty} g_2(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z} z^{-1/2} dz = 1.$$

This proves the Lemma. □

3.1.2 Tail behaviour for n fold product

The following theorem provides the tail behaviour of the product of n many i.i.d. standard exponentials.

Theorem 3.1.2 (Bose, Hazra and Saha (2010) [37])**.** *There exists constants* $\{C_n, \alpha_n\}$ *such that*

$$H_n(x) = C_n x^{\alpha_n} e^{-n x^{\frac{1}{n}}} g_n(x), \quad n \geq 1, \tag{3.1.3}$$

where for $n \geq 1$,

$$C_n = \frac{1}{\sqrt{n}} (2\pi)^{\frac{n-1}{2}}, \quad \alpha_n = \frac{n-1}{2n} \quad \textit{and} \quad g_n(x) \to 1 \quad \textit{as} \quad x \to \infty.$$

We shall prove the theorem using method of induction and Laplace's method to find the integral asymptotics. First we give a brief description of Laplace's method.

Integral asymptotics: Laplace's method. Consider the following integral

$$I(x) = \int_a^b f(t) e^{-xg(t)} dt$$

where $g(t)$ is a real valued function of the real variable t, $f(t)$ is real or complex valued function and x is a large positive variable. The Laplace principle is that, the major contribution to the value of the integral $I(x)$ arises from the immediate vicinity of those points of the interval $a \le t \le b$ at which $g(t)$ assumes its minimum value.

Suppose $f(t)$ is continuous, $g(t)$ is twice continuously differentiable and $g(t)$ reaches its (strict) minumum over $[a, b]$ at an interior point c, so that $g(c) < g(t)$ for $a \le t < c$ and $c < t \le b$. Then as $x \to \infty$,

$$I(x) = e^{-xg(c)} f(c) \sqrt{\frac{2\pi}{xg''(c)}} (1 + o(1)) \tag{3.1.4}$$

and this is known as *Laplace's asymptotic*. If $g(t)$ attains its minimum at a boundary point say, at $t = a$ then Laplace's asymptotic takes the following form

$$I(x) = e^{-xg(a)} f(a) \sqrt{\frac{\pi}{2xg''(a)}} (1 + o(1)). \tag{3.1.5}$$

If $g(t)$ has finite number of minimum, we may break up the integral in a finite number of integrals so that in each integral $g(t)$ attains its minimum only at one point and no other point, and then can apply Laplace's method to each integral. For detail discussion on Laplace's method see Section 2.4 of Erdélyi (1956) [57].

Proof of Theorem 3.1.2. We shall use the method of induction. Note, $H_1(x) = \mathrm{P}[E_1 > x] = e^{-x}$. So, $C_1 = 1$, $\alpha_1 = 0$ and $g_1(x) = 1$ for all x. Hence the result is true for $n = 1$. Now

$$\begin{aligned} H_2(x) &= \int_0^\infty e^{-y} e^{-x/y} dy \\ &= x^{1/2} \int_0^\infty e^{-x^{1/2}(t + \frac{1}{t})} dt \quad \text{(subtituting } y = tx^{1/2}) \\ &= x^{1/2} \int_0^\infty f(t) e^{-x^{1/2} g(t)} dt, \end{aligned}$$

where $f(t) = 1$ and $g(t) = t + \frac{1}{t}$. Note that g assumes a strict minimum at $t = 1$ and

$f(1) = 1 \neq 0$. So applying Laplace's asymptotic (3.1.4) we have

$$\begin{aligned} H_2(x) &= x^{1/2} e^{-x^{1/2} g(1)} f(1) \sqrt{\frac{2\pi}{x^{1/2} g''(1)}} g_2(x) \\ &= \sqrt{\pi} x^{1/4} e^{-2x^{1/2}} g_2(x) \end{aligned}$$

where $g_2(x) \to 1$ as $x \to \infty$. Hence $C_2 = \sqrt{\pi} = \frac{1}{\sqrt{2}}(2\pi)^{1/2}$ and $\alpha_k = 1$. So the result is true for $n = 2$.
Now suppose (3.1.3) is true for $n = k$. We shall prove it for $n = k + 1$.

$$\begin{aligned} & H_{k+1}(x) \\ = & \int_0^\infty e^{-y} H_k(\frac{x}{y}) dy \\ = & C_k \int_0^\infty e^{-y} (\frac{x}{y})^{\alpha_k} e^{-k(\frac{x}{y})^{1/k}} g_k(\frac{x}{y}) dy \\ = & xkC_k \int_0^\infty e^{-(ks + \frac{x}{s^k})} s^{k\alpha_k - k - 1} g_k(s^k) ds \quad \text{(substituting } x/y = s^k \text{)} \\ = & x^{\frac{k\alpha_k + 1}{k+1}} kC_k \int_0^\infty e^{-(kt + \frac{1}{t^k}) x^{\frac{1}{k+1}}} t^{k\alpha_k - k - 1} g_k(t^k x^{\frac{k}{k+1}}) dt \quad \text{(substituting } s = x^{\frac{1}{k+1}} t \text{)} \\ = & x^{\frac{k\alpha_k + 1}{k+1}} kC_k \int_0^\infty f(t) e^{-x^{\frac{1}{k+1}} g(t)} dt \end{aligned}$$

where

$$f(t) = t^{k\alpha_k - k - 1} g_k(t^k x^{\frac{k}{k+1}}) \text{ and } g(t) = kt + \frac{1}{t^k}.$$

Note that g assumes a strict minimum at $t = 1$ and $f(1) = g_k(x^{\frac{k}{k+1}}) \neq 0$, $g''(1) = k(k+1)$. Again applying Laplace's asymptotic (3.1.4) we have

$$\begin{aligned} H_{k+1}(x) &= x^{\frac{k\alpha_k + 1}{k+1}} kC_k e^{-x^{\frac{1}{k+1}} g(1)} f(1) \sqrt{\frac{2\pi}{x^{\frac{1}{k+1}} g''(1)}} h(x) \\ &= x^{\frac{k\alpha_k + 1}{k+1}} kC_k e^{-(k+1) x^{\frac{1}{k+1}}} g_k(x^{\frac{k}{k+1}}) \sqrt{\frac{2\pi}{x^{\frac{1}{k+1}} k(k+1)}} h(x) \end{aligned}$$

where $h(x) \to 1$ as $x \to \infty$. Subtituting the values of α_k and C_k we get

$$\begin{aligned} H_{k+1}(x) &= x^{\frac{k}{2(k+1)}} \frac{1}{\sqrt{k+1}} (2\pi)^{k/2} e^{-(k+1) x^{\frac{1}{k+1}}} g_k(x^{\frac{k}{k+1}}) h(x) \\ &= C_{k+1} x^{\alpha_{k+1}} e^{-(k+1) x^{\frac{1}{k+1}}} g_{k+1}(x) \end{aligned}$$

where

$$\alpha_{k+1} = \frac{k}{2(k+1)}, \quad C_{k+1} = \frac{1}{\sqrt{k+1}} (2\pi)^{k/2}, \quad g_{k+1} = g_k(x^{\frac{k}{k+1}}) h(x)$$

and $g_{k+1}(x) \to 1$ as $x \to \infty$. Hence the result is true for $n = k+1$ and this completes the proof. □

3.2 Extreme values

We start this section with the definition of the Gumbel distribution.

Definition 3.2.1. *A probability distribution is said to be Gumbel with parameter $\theta > 0$ if its cumulative distribution function is given by*

$$\Lambda_\theta(x) = \exp\{-\theta \exp(-x)\}, \quad x \in \mathbb{R}.$$

$\Lambda \equiv \Lambda_1$ is known as the (standard) Gumbel distribution.

The next theorem is an easy consequence of standard calculations in extreme value theory as found in Rootzèn (1986) [105], Embrechts, Kluppelberg and Mikosch (1997) [56].

Theorem 3.2.2 (Bose, Hazra and Saha (2010) [37])**.** *Let $\{X_n\}$ be a sequence of i.i.d. non-negative random variables with distribution F and let $F^{(n)} = \max_{1\le i\le n} X_i$. If $1 - F(x) \sim Cx^b e^{-ax^2}$ as $x \to \infty$, then*

$$\frac{F^{(n)} - d_n}{c_n} \xrightarrow{\mathcal{D}} \Lambda_1,$$

where

$$c_n = \frac{1}{2a^{1/2}(\ln n)^{1/2}} \quad \text{and} \quad d_n = \frac{\ln C - \frac{b}{2}\ln a}{2a^{1/2}(\ln n)^{1/2}} + \left(\frac{\ln n}{a}\right)^{1/2}\left[1 + \frac{b \ln\ln n}{4\ln n}\right].$$

Proof. Let $\overline{F} = 1 - F$. Then

$$\overline{F}(x) = \theta(x)\overline{F}_\#(x) \text{ where } \theta(x) \to \theta = Ce^{-a} \text{ and } \overline{F}_\#(x) = x^b \exp(-a(x^2-1)). \quad (3.2.1)$$

By invoking Proposition 1.1 given in Resnick (1987) [103] , it is now enough to show that, there exists some x_0 and a function f such that $f(y) > 0$ for $y > x_0$ and such that f has an absolute continuous density with $f'(x) \to 0$ as $x \to \infty$ so that

$$1 - F_\#(x) = \exp\Big(-\int_{x_0}^{x}(1/f(y))dy\Big), \quad x > x_0. \quad (3.2.2)$$

Further, a choice for the normalizing constants c_n and d_n is then given by

$$d_n^* = \Big(1/(\overline{F}_\#)\Big)^{-1}(n), \quad c_n^* = f(d_n^*). \quad (3.2.3)$$

Comparing the two representations of $\overline{F}_{\#}$ given in (3.2.1) and (3.2.2) implies that we may choose

$$f(x) = \frac{x}{2ax^2 - b} \sim \frac{1}{2ax} \quad \text{as} \quad x \to \infty.$$

Hence we have (noting that $d_n^* \to \infty$),

$$c_n^* = f(d_n^*) \sim \frac{1}{2ad_n^*}.$$

On the other hand, since $\overline{F}_{\#}(d_n^*) = \frac{1}{n}$, we have

$$(d_n^*)^b \exp(-a((d_n^*)^2 - 1)) = \frac{1}{n}.$$

Taking logarithms on both sides we have

$$a{d_n^*}^2 - b \ln d_n^* - a = \ln n. \tag{3.2.4}$$

Since $d_n^* \to \infty$, $d_n^* \sim \left(\frac{\ln n}{a}\right)^{1/2}$. Let $d_n^* = \left(\frac{\ln n}{a}\right)^{1/2}(1+\delta_n)$. Using this in (3.2.4) we get

$$\delta_n = \frac{\frac{b}{2}\ln\ln n + \epsilon_n}{2\ln n} + \mathrm{O}\left(\frac{(\ln\ln n)^2}{(\ln n)^2}\right),$$

where $\epsilon_n = -b\ln(1+\delta_n) - \frac{b}{2}\ln a + a$. So we get

$$\begin{aligned} d_n^* &= \left(\frac{\ln n}{a}\right)^{1/2}(1+\delta_n) \\ &= \left(\frac{\ln n}{a}\right)^{1/2}\left[1 + \frac{b}{4}\frac{\ln\ln n}{\ln n} + \frac{a - \frac{b}{2}\ln a - b\ln(1+\delta_n)}{2\ln n}\right] + \mathrm{O}\left(\frac{(\ln\ln n)^2}{(\ln n)^{3/2}}\right). \end{aligned}$$

Neglecting the lower order terms and denoting

$$\hat{d}_n = \left(\frac{\ln n}{a}\right)^{1/2}\left[1 + \frac{b}{4}\frac{\ln\ln n}{\ln n} + \frac{a - \frac{b}{2}\ln a}{2\ln n}\right] \quad \text{and} \quad \hat{c}_n = \frac{1}{2a^{1/2}(\ln n)^{1/2}}$$

we have

$$\frac{F^{(n)} - \hat{d}_n}{\hat{c}_n} \xrightarrow{\mathcal{D}} \Lambda_{Ce^{-a}}.$$

Now letting $c_n = \hat{c}_n$ and $d_n = c_n \ln(Ce^{-a}) + \hat{d}_n$ and using convergence of types result, we have

$$\frac{F^{(n)} - d_n}{c_n} \xrightarrow{\mathcal{D}} \Lambda_1.$$

□

The following corollary and lemma follow immediately using Theorem 3.1.2.

Corollary 3.2.3. *Let* $\{X_n\}$ *be a sequence of i.i.d. random variables where* $X_i \stackrel{\mathcal{D}}{=} (E_1 E_2 \dots E_k)^{1/2k}$ *and* $\{E_i\}_{1\le i\le k}$ *are i.i.d. Exp*(1) *random variables. Then*

$$\frac{\max_{1\le i\le n} X_i - d_n}{c_n} \xrightarrow{\mathcal{D}} \Lambda_1,$$

where

$$c_n = \frac{1}{2k^{1/2}(\ln n)^{1/2}}, \quad d_n = \frac{\ln C_k - \frac{k-1}{2}\ln k}{2k^{1/2}(\ln n)^{1/2}} + \left(\frac{\ln n}{k}\right)^{1/2}\left[1 + \frac{(k-1)\ln\ln n}{4\ln n}\right],$$

$$C_k = \frac{1}{\sqrt{k}}(2\pi)^{\frac{k-1}{2}}.$$

Lemma 3.2.4. *Let* $\{E_i\}$*,* c_n *and* d_n *be as in Corollary 3.2.3. Let* $\sigma_n^2 = n^{-c}$*,* $c > 0$*. Then there exists some positive constant* $K = K(x)$*, such that for all large* n *we have*

$$\mathrm{P}\left((E_1 E_2 \dots E_k)^{1/2k} > (1+\sigma_n^2)^{-1/2}(c_n x + d_n)\right) \le \frac{K}{n}, \quad x \in \mathbb{R}.$$

This lemma will be useful in the proof of Lemma 4.4.8 in Chapter 4.

Chapter 4

Spectral norm and radius of circulant type matrices with light tail

In this chapter we deal with spectral norm or spectral radius of circulant type random matrices and Toeplitz and Hankel matrices when the input sequence is independent and identically distributed. For any matrix A, its *spectral radius* $\texttt{sp}(A)$ is defined as

$$\texttt{sp}(A) := \max\Big\{|\lambda| : \lambda \text{ is an eigenvalue of } A\Big\},$$

where $|z|$ denotes the modulus of $z \in \mathbb{C}$.

A related quantity is the *spectral norm*. For any matrix A with possible complex entries, its spectral norm $\|A\|$ is the square root of the largest eigenvalue of the positive semi-definite matrix A^*A:

$$\|A\| = \sqrt{\lambda_{max}(A^*A)}$$

where A^* denotes the conjugate transpose of A. Therefore if A is an $n \times n$ real symmetric matrix or A is a normal matrix, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\|A\| = \texttt{sp}(A) = \max_{1 \le i \le n} |\lambda_i|.$$

The spectral radius and spectral norm have been important objects of study in random matrix theory. For the $n \times n$ matrix with all i.i.d. complex Gaussian entries having zero mean and variance $1/n$, Kostlan (1992) [81] gave an upper bound for the spectral radius and then Rider (2003) [104] showed that the spectral radius converges to a Gumbel distribution with appropriate scaling and centering. Silverstein (1994) [112] considered the $n \times n$ matrix with i.i.d. entries of non-zero mean and finite fourth moment, and showed that its spectral radius converges almost surely to a real value and converges weakly to the normal distribution after proper scaling and centering. For some work on spectral norm of (symmetric) Toeplitz matrices, see Meckes (2007) [92], Adamczak (2008) [1] and Bose and Sen (2007) [42]. For result on the spectral norm of symmetric circulant matrix, see Bryc and Sethuraman (2009) [48].

For symmetric or normal matrices, we prove results on spectral norm (hence spectral radius) and for other matrices we consider only the spectral radius. Since RC_n, SC_n are symmetric and C_n is normal, we consider their spectral norm. For $n = k^g + 1,\ g \geq 2$, k-circulant matrices are not normal, and hence we consider their spectral radius. Here is an outline of the chapter.

In this chapter we deal with i.i.d. light tail inputs. In Section 4.1 we review some known results on spectral norm of Toeplitz and Hankel matrices (which are close cousins of the circulant matrix and the reverse circulant matrix respectively). In Section 4.2 we prove almost sure and distributional convergence of spectral norm of reverse circulant and circulant matrices. In Section 4.3 we consider the joint behaviour of the minimum and maximum eigenvalue of the symmetric circulant matrix and from there we deduce the distributional convergence of the spectral norm. In Section 4.4 we review a known result on the spectral radius of the k-circulant matrix when $n = k^2 + 1$. Then we prove the distributional convergence of the spectral radius of the k-circulant matrix where $n = k^g + 1,\ g > 2$ and in Section 4.4.5 give an idea to deal with the more general case, $sn = k^g + 1$ with some suitable condition in s. Finally in Section 4.5 we pose some open questions. Throughout the chapter, c and C will denote a generic constant.

Some of the results of Bose, Hazra and Saha (2009, 2010) [34, 37] are based on this chapter.

4.1 Toeplitz and Hankel with light tail entries

First we state a known result for Toeplitz and Hankel matrices. Let

$$u_n = n^{-1/2}(1, 1, \ldots, 1)^T. \tag{4.1.1}$$

Theorem 4.1.1 (Bose and Sen (2007) [42])**.** *Let $\{x_i\}$ be i.i.d. with $E(x_0) = \mu > 0$ and $Var(x_0) = 1$ and let T_n be the symmetric Toeplitz matrix $((x_{|i-j|}))$. Let $T_n^0 = T_n - \mu n u_n u_n^T$. Then*

(i)

$$\frac{\|T_n\|}{n} \to \mu \quad \text{almost surely and } \|\frac{T_n^0}{\|T_n\|}\| \to 0 \quad \text{almost surely.}$$

(ii) If $E(x_0^4) < \infty$, then for $M_n = \|T_n\|$ or $M_n = \lambda_n(T_n)$, the maximum eigenvalue of

T_n,

$$\frac{M_n - \mu n}{\sqrt{n}} \to N(0, 4/3) \text{ in distribution.}$$

(iii) If T_n and T_n^0 are replaced by the corresponding symmetric Hankel matrices H_n and H_n^0, then (i) holds. Further, (ii) holds with the limiting variance being changed from $4/3$ to $2/3$.

Remark 4.1.2. *When $\{x_i\}$ are independent and centered random variables, the following results are known for the Toeplitz matrix. Meckes (2007) [92] showed that if x_i's are independent and centered uniformly subgaussian then $E\|T_n\| \leq C\sqrt{n \ln n}$. He also showed that if for all j and for some constant A, $|x_j| \leq A$ or, if $\{x_j\}$ satisfy logarithmic Sobolev inequality with constant A, that is,*

$$\mathrm{E}\left[f^2(x_j) \log f^2(x_j)\right] \leq 2A\,\mathrm{E}\left[f'(x_j)^2\right] \text{ for every smooth } f \text{ such that } \mathrm{E}\, f^2(x_j) = 1,$$

then with probability 1

$$\limsup_n \frac{\|T_n\|}{\sqrt{n \ln n}} \leq C,$$

where C depends only on A.
These results were further improved in Adamczak (2010) [1], where it was shown that for $\{x_i\}$ i.i.d. mean zero and finite variance,

$$\lim_{n \to \infty} \frac{\|T_n\|}{E\|T_n\|} = 1 \text{ a.s.}$$

Further,

$$\limsup \frac{\|T_n\|}{\sqrt{n \ln n}} < \infty \text{ a.s. if and only if } Ex_0 = 0 \text{ and } Ex_0^2 < \infty.$$

4.2 Circulant and reverse circulant with light tail entries

Results similar to Toeplitz and Hankel matrices can be established for reverse circulant, symmetric circulant and circulant matrices. In fact we shall show that in each case, the spectral norm converges in distribution when centered and scaled appropriately. Recall the eigenvalues of C_n and RC_n and, observe that $\|C_n\| = \|RC_n\|$. Hence the spectral norm for these two matrices do not have to be dealt with separately. Some results about the maximum of the singular values of circulant matrices with standard complex normal entries is known from the form of the eigenvalues. See for example Corollary 5 of Meckes (2009) [91].

We start with a result on the reverse circulant which follows easily from the existing literature.

Theorem 4.2.1 (Bose, Hazra and Saha (2009) [34]). *Suppose $\{x_i\}$ is i.i.d. with $E(x_0) = \mu$ and $Var(x_0) = 1$. Suppose RC_n is the reverse circulant matrix formed by the $\{x_i\}$. Let $RC_n^0 = RC_n - \mu n u_n u_n^T$ where u_n is as given in (4.1.1). If $\mu > 0$, then*

$$\frac{\|RC_n\|}{n} \to \mu \quad \text{almost surely and } \|\frac{RC_n^0}{\|RC_n\|}\| \to 0 \quad \text{almost surely.}$$

Similar results hold for C_n also.

Proof. The proof follows in a straightforward manner from arguments for Toeplitz and Hankel matrices given in Theorem 3 and Lemma 1(i) of Bose and Sen (2007) [42]. We omit the details. □

Remark 4.2.2. *If we assume $E(x_0^4) < \infty$, then the distributional convergence when $\mu > 0$ can also be proved following the proof of Bose and Sen (2007) [42]. However, below we establish the distributional convergence under the assumption $E|x_0|^{2+\delta} < \infty$.*

Theorem 4.2.3 (Bose, Hazra and Saha (2009) [34]). *Suppose $\{x_i\}_{i\geq 0}$ is i.i.d. with mean μ and $E|x_i|^{2+\delta} < \infty$ for some $\delta > 0$. Consider the reverse circulant (RC_n) and circulant (C_n) matrices with the input $\{x_i\}$.*

(i) If $\mu \neq 0$ then,

$$\frac{\|RC_n\| - |\mu| n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1).$$

(ii) If $\mu = 0$ then,

$$\frac{\|\frac{1}{\sqrt{n}} RC_n\| - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

where

$$q = q(n) = \lfloor \frac{n-1}{2} \rfloor, \quad d_q = \sqrt{\ln q}, \quad c_q = \frac{1}{2\sqrt{\ln q}}$$

and Λ is the standard Gumbel distribution defined in Section 3.2. The above conclusions continue to hold for C_n also.

Proof. As pointed out earlier, it is enough to deal with only RC_n. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $n^{-1/2} RC_n$. These eigenvalues are given by (see Section 1.2.3):

$$\begin{cases} \lambda_0 & = n^{-1/2} \sum_{t=0}^{n-1} x_t \\ \lambda_{n/2} & = n^{-1/2} \sum_{t=0}^{n-1} (-1)^t x_t, \quad \text{if } n \quad \text{is even} \\ \lambda_k = -\lambda_{n-k} & = \sqrt{I_{x,n}(\omega_k)}, \ 1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor \end{cases} \tag{4.2.1}$$

where

$$I_{x,n}(\omega_k) = \frac{1}{n} |\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 \text{ and } \omega_k = \frac{2\pi k}{n}.$$

Note that $\{|\lambda_k|^2;\ 1 \le k < n/2\}$ is the periodogram of $\{x_i\}$ at the frequencies $\{\frac{2\pi k}{n};\ 1 \le k < n/2\}$. If $\mu = 0$ then under the given conditions, Theorem 2.1 of Davis and Mikosch (1999) [50] yields

$$\max_{1 \le k < \frac{n}{2}} I_{x,n}(\omega_k) - \ln q \xrightarrow{\mathcal{D}} \Lambda. \tag{4.2.2}$$

Therefore

$$\max_{1 \le k < n/2} |\lambda_k|^2 - \ln q \xrightarrow{\mathcal{D}} \Lambda. \tag{4.2.3}$$

Define $g(x) = \sqrt{x}$. Then by mean value theorem,

$$g(\max_{1 \le k < n/2} |\lambda_k|^2) - g(\ln q) = g'(\xi_n)\left(\max_{1 \le k < n/2} |\lambda_k|^2 - \ln q\right)$$

where ξ_n lies between $\max_{1 \le k < n/2} |\lambda_k|^2$ and $\ln q$. From (4.2.3) we have

$$\frac{\max_{1 \le k < n/2} |\lambda_k|^2}{\ln q} \xrightarrow{\mathcal{P}} 1.$$

Therefore $\frac{\xi_n}{\ln q} \xrightarrow{\mathcal{P}} 1$. Now

$$\frac{g'(\xi_n)}{g'(\ln q)} = \left(\frac{\ln q}{\xi_n}\right)^{1/2} \xrightarrow{\mathcal{P}} 1$$

and therefore

$$\frac{g(\max_{1 \le k < n/2} |\lambda_k|^2) - g(\ln q)}{g'(\ln q)} = \frac{g'(\xi_n)}{g'(\ln q)}\left(\max_{1 \le k < n/2} |\lambda_k|^2 - \ln q\right) \xrightarrow{\mathcal{D}} \Lambda.$$

So if $\{x_i\}$ are i.i.d. with mean zero, variance 1 and $E|x_i|^{2+\delta} < \infty$, then

$$\frac{\max_{1 \le k < \frac{n}{2}} |\lambda_k| - \sqrt{\ln q}}{\frac{1}{2\sqrt{\ln q}}} \xrightarrow{\mathcal{D}} \Lambda. \tag{4.2.4}$$

Observe that we have left out λ_0 and $\lambda_{n/2}$ (if n is even) where

$$\lambda_0 = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} x_t \quad \text{and} \quad \lambda_{n/2} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} (-1)^t x_t.$$

Now suppose that mean of $\{x_i\}$ is $\mu > 0$. For $1 \le k < n/2$,

$$|\lambda_k| = \frac{1}{\sqrt{n}}\Big|\sum_{t=0}^{n-1} x_t e^{it\omega_k}\Big| = \frac{1}{\sqrt{n}}\Big|\sum_{t=0}^{n-1} (x_t - \mu) e^{it\omega_k}\Big|,$$

and $(x_t - \mu)$ has mean zero and variance 1. Therefore even when $E(x_i) > 0$, (4.2.4) holds. Note that by CLT

$$\frac{\sqrt{n}\lambda_0 - \mu n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1). \tag{4.2.5}$$

(4.2.5) implies $\lambda_0 \xrightarrow{\mathcal{P}} \infty$ and hence

$$|\lambda_0| - \mu\sqrt{n} \xrightarrow{\mathcal{D}} N(0,1).$$

Let $A_n = \max_{1\leq k<q} |\lambda_k|$. From (4.2.4) and (4.2.5)

$$\frac{A_n}{\sqrt{\ln q}} \xrightarrow{\mathcal{P}} 1 \text{ and } \frac{\lambda_0}{\mu\sqrt{n}} \xrightarrow{\mathcal{P}} 1$$

and so it follows that

$$\mathrm{P}\big[\max(A_n, |\lambda_0|) - \mu\sqrt{n} > x\big] \to \mathrm{P}\big[N(0,1) > x\big],$$

proving (i) for odd n. Since for even n,

$$\lambda_{n/2} = n^{-1/2}\sum_{t=0}^{n-1}(-1)^t x_t \xrightarrow{\mathcal{D}} N(0,1),$$

this can also be neglected as before, and hence (i) holds also for even n. Similar proof works when $\mu < 0$. This proves (i) completely.

(ii) Now assume $\mu = 0$. In contrast to the previous case, here A_n dominates $|\lambda_0|$, since $|\lambda_0|$ is tight and

$$\frac{|\lambda_0| - \sqrt{\ln q}}{(\ln q)^{-1/2}} \xrightarrow{\mathcal{P}} -\infty.$$

Hence in this case

$$\frac{\|\frac{1}{\sqrt{n}}RC_n\| - \sqrt{\ln q}}{\frac{1}{2\sqrt{\ln q}}} \xrightarrow{\mathcal{D}} \Lambda.$$

□

4.3 Symmetric circulant with light tail entries

The spectral norm of the symmetric circulant matrices behaves quite similar to reverse circulant matrices but the normalizing constants change. The following normalizing constants, well known in the context of maxima of i.i.d. normal variables, will be repeatedly used in the statements of our following results.

$$a_n = (2\ln n)^{-1/2} \quad\text{and}\quad b_n = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln 4\pi}{2(2\ln n)^{1/2}}. \tag{4.3.1}$$

We need the following Lemmata which are well known and hence we omit their proofs. The first Lemma is on the joint behaviour of maxima and minima of i.i.d normal random variables.

Lemma 4.3.1. *Let $\{N_i\}$ be i.i.d. $N(0,1)$. If $m_n = \min_{1\le i\le n} N_i$ and $M_n = \max_{1\le i\le n} N_i$, then with a_n and b_n as in (4.3.1),*

$$\Big(\frac{-m_n - b_n}{a_n}, \frac{M_n - b_n}{a_n}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda,$$

where $\Lambda \otimes \Lambda$ denotes joint distribution of two independent standard Gumbel random variables.

The statement of Lemma 4.3.2 is taken from Einmahl (1989) [53] Corollary 1(b), page 31, in combination with his Remark on page 32.

Lemma 4.3.2. *Let $\{\psi_i\}$ be independent random vectors with mean zero and values in $\mathbb{R}^d$. Assume that the moment generating functions of ψ_i, $1 \le i \le n$, exist in a neighbourhood of the origin and that*

$$Cov(\psi_1 + \psi_2 + ... + \psi_n) = B_n I_d,$$

where $B_n > 0$ and I_d denotes the d-dimensional identity matrix. Let η_k be independent $N(0, \sigma^2 Cov(\psi_k))$ random vectors, $k = 1, 2, ...n$, independent of $\{\psi_k\}$ and $\sigma^2 \in (0,1]$. Let $\psi_k^* = \psi_k + \eta_k$, $k = 1, 2, ...n$ and write p_n^* for the density of $B_n^{-1/2}\sum_{k=1}^n \psi_k^*$. Choose $\alpha \in (0, \frac{1}{2})$ such that

$$\alpha \sum_{k=1}^{n} E|\psi_k|^3 \exp(\alpha|\psi_k|) \le B_n,$$

where $|x|$ denotes the Euclidean norm in $\mathbb{R}^d$. Let

$$\beta_n = \beta_n(\alpha) = B_n^{-3/2} \sum_{k=1}^{n} E|\psi_k|^3 \exp(\alpha|\psi_k|).$$

If $|x| \le c_1 \alpha B_n^{1/2}$, $\sigma^2 \ge -c_2 \beta_n^2 \ln \beta_n$ and $B_n \ge c_3 \alpha^{-2}$, where c_1, c_2, c_3 are constants depending only on d, then

$$p_n^*(x) = \phi_{(1+\sigma^2)I_d}(x) \exp(\bar{T}_n(x)) \;\; \text{with} \;\; |\bar{T}_n(x)| \le c_4 \beta_n (|x|^3 + 1),$$

where ϕ_c is the density of a d-dimensional centered Gaussian vector with covariance matrix c and c_4 is a constant depending on d.

We shall use the above Lemma now to derive a normal approximation result which shall be used in the proof of Theorem 4.3.4 and again in Section 7.1.2. Define

$$\bar{x}_t = x_t \mathbb{I}(|x_t| \leq (1+2j)^{1/s}) - E[x_t \mathbb{I}(|x_t| \leq (1+2j)^{1/s})]. \tag{4.3.2}$$

For $1 \leq i_1 < i_2 < ... < i_d < j$ let

$$v_d(0) = \sqrt{2}(1,1,...,1), \;\; v_d(t) = 2\left(\cos\frac{2\pi i_1 t}{2j+1}, \cos\frac{2\pi i_2 t}{2j+1}, ..., \cos\frac{2\pi i_d t}{2j+1}\right) \;\; \text{for} \;\; 1 \leq t \leq j.$$

Lemma 4.3.3. *Let $n = 1+2j$ and $\sigma_j^2 = (1+2j)^{-c}$ for some $c > 0$ and let $\{x_t\}$ be i.i.d mean zero with $Ex_0^2 = 1$ and $E|x_0|^s < \infty$ for some $s > 2$. Suppose N_t's are i.i.d. $N(0,1)$ random variables independent of $\{x_t\}$ and $\tilde{p}_j(x)$ is the density of*

$$\frac{1}{\sqrt{1+2j}} \sum_{t=0}^{j} (\bar{x}_t + \sigma_j N_t) v_d(t).$$

Then for any measurable subset E of $\mathbb{R}^d$,

$$\Big| \int_E \tilde{p}_j(x)dx - \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx \Big| \;\; \leq \;\; \epsilon_j \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx + O(\exp(-(1+2j)^{\eta}))$$

where $\epsilon_j \to 0$ as $j \to \infty$, $\eta > 0$ and the above holds uniformly over d-tuples $1 \leq i_1 < i_2 < ... < i_d < j$.

Proof. Let $S_{j,\bar{x}} = \sum_{t=0}^{j} \bar{x}_t v_d(t)$ and let $s = 2+\delta$. Then observe that $Cov(S_{j,\bar{x}}) = B_j I_d$ where, $B_j = (2j+1)Var(\bar{x}_t)$ and I_d is the $d \times d$ identity matrix. Since $\{\bar{x}_t v_d(t)\}_{0 \leq t \leq j}$ is an independent collection of mean zero random vectors in $\mathbb{R}^d$, we can use Lemma 4.3.2. By choosing $\alpha = \frac{c_5(1+2j)^{-\frac{1}{s}}}{2\sqrt{d}}$, it can be easily shown that,

$$\alpha \sum_{t=0}^{j} E|\bar{x}_t v_d(t)|^3 \exp(\alpha |x_t v_d(t)|) < B_j.$$

If we define $\tilde{\beta}_j = B_j^{-3/2} \sum_{t=0}^{j} E|\bar{x}_t v_d(t)|^3 \exp(\alpha|\bar{x}_t v_d(t)|)$, then it follows that

$$\tilde{\beta}_j \leq C(1+2j)^{-(\frac{1}{2}-\frac{1-\delta}{s})}.$$

Let $c = \frac{1}{2} - \frac{1-\delta}{s} > 0$. Now choose $|x| \leq c_1 \alpha B_j^{1/2} \approx c_2(1+2j)^{\frac{1}{2}-\frac{1}{s}}$ and σ_j^2 satisfying,

$$1 \geq \sigma_j^2 \geq c_3(\ln(2j+1))(2j+1)^{-2c}.$$

Clearly $B_j \geq c_4\alpha^{-2}$ and $B_j \approx (1+2j)$. We mention here that c_1, c_2, c_3, c_4 are constants depending only on d. Then Lemma 4.3.2 implies that,

$$\tilde{p}_j(x) = \phi_{(1+\sigma_j^2)I_d}(x)\exp(|T_j(x)|)$$

with $|T_j(x)| \leq c_5\tilde{\beta}_j(|x|^3+1)$. Note that, $|T_j(x)| \to 0$ uniformly for $|x|^3 = o\{\min((1+2j)^{-c}, (1+2j)^{\frac{1}{2}-\frac{1}{s}})\}$. For the choice of $\sigma_j^2 = (1+2j)^{-c}$ the above condition can be seen to be satisfied. Now it follows from Corollary 1 of [44] that for any measurable subset E of $\mathbb{R}^d$,

$$|\int_E \tilde{p}_j(x)dx - \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx| \quad \leq \quad \epsilon_j \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx + O(\exp(-(1+2j)^{\eta}))$$

where $\epsilon_j \to 0$ as $j \to \infty$. □

For the reverse circulant, leaving out the eigenvalues λ_0 and $\lambda_{n/2}$, the maximum and minimum eigenvalues are equal in magnitude. This is not the case for symmetric circulant. Hence we now look at the joint behaviour of the maximum and minimum of the eigenvalues.

Theorem 4.3.4 (Bose, Hazra and Saha (2009) [34])**.** *Suppose* $\{\lambda_k,\ 0 \leq k \leq n-1\}$ *are the eigenvalues of* $\frac{1}{\sqrt{n}}SC_n$. *Let* $q = \lfloor \frac{n}{2} \rfloor$ *and* $M_{q,x} = \max_{1\leq k\leq q}\lambda_k$ *and* $m_{q,x} = \min_{1\leq k\leq q}\lambda_k$. *If* $\{x_i\}$ *are i.i.d. with* $Ex_0 = 0$, $Ex_0^2 = 1$ *and* $E|x_0|^s < \infty$ *for some* $s > 2$ *then we have,*

$$\Big(\frac{-m_{q,x} - b_q}{a_q}, \frac{M_{q,x} - b_q}{a_q}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda,$$

where a_q *and* b_q *are given by (4.3.1). The same limit continues to hold if the eigenvalue* λ_0 *is included in the definition of* max *and* min *above.*

Proof. First assume $n = 2j+1$, odd and let $s = 2+\delta$. The proof may be broken down into four steps. We use truncation and normal approximation (Lemma 4.3.3) along with Bonferroni Inequality.

Step 1: Truncation. Let $\bar{x}_t$ be as in (4.3.2) and

$$\tilde{x}_t = x_t\mathbb{I}(|x_t| \leq (1+2j)^{1/s}).$$

We show that it is enough to deal with the truncated random variables $\{\bar{x}_t\}$ (see (4.3.3)). If $\bar{\lambda}_k$ and $\tilde{\lambda}_k$ denote the eigenvalues of symmetric circulant matrices with entries $\bar{x}_t$ and $\tilde{x}_t$ respectively, then $\bar{\lambda}_k = \tilde{\lambda}_k$. By Borel-Cantelli lemma, $\sum_{t=1}^{\infty}|x_t|\mathbb{I}(|x_t| > (1+2j)^{1/s})$ is bounded with probability 1 and consists of only a finite number of non-zero terms.

Thus there exists a positive integer $N(\omega)$ (depending on sample point ω) such that

$$\begin{aligned}\sum_{t=0}^{j}|x_t-\tilde{x}_t| &= \sum_{t=0}^{j}|x_t|\mathbb{I}(|x_t|>(1+2j)^{1/s}) \\ &\leq \sum_{t=0}^{\infty}|x_t|\mathbb{I}(|x_t|>(1+2j)^{1/s}) \\ &= \sum_{t=0}^{N(\omega)}|x_t|\mathbb{I}(|x_t|>(1+2j)^{1/s}).\end{aligned}$$

It follows that for $2j+1 \geq \{N(\omega), |x_1|^s, \ldots, |x_{N(\omega)}|^s\}$ the left side is zero. Consequently, for all j sufficiently large, $\tilde{\lambda}_k = \lambda_k$ a.s. for all k. Therefore for all j sufficiently large,

$$\Big(\frac{-m_{j,x}-b_j}{a_j}, \frac{M_{j,x}-b_j}{a_j}\Big) \stackrel{\mathcal{D}}{=} \Big(\frac{-m_{j,\bar{x}}-b_j}{a_j}, \frac{M_{j,\bar{x}}-b_j}{a_j}\Big) \tag{4.3.3}$$

where $m_{j,\bar{x}} = \min_{1\leq k\leq j}\bar{\lambda}_k$ and $M_{j,\bar{x}} = \max_{1\leq k\leq j}\bar{\lambda}_k$.

Step 2: Application of Bonferroni Inequality.
Define for $1 \leq k \leq j$,

$$\begin{aligned}\bar{\lambda}'_k &= \frac{1}{\sqrt{2j+1}}\Big(\sqrt{2}\bar{x}_0 + 2\sum_{t=1}^{j}\bar{x}_t\cos\frac{2\pi kt}{2j+1}\Big), \\ \bar{\bar{\lambda}}'_k &= \bar{\lambda}'_k + \frac{\sigma_j}{\sqrt{1+2j}}(\sqrt{2}N_0 + 2\sum_{t=1}^{j}N_t\cos\frac{2\pi kt}{n}) \\ &= \bar{\lambda}'_k + \sigma_j N'_{j,k}.\end{aligned}$$

where $\sigma_j^2 = (1+2j)^{-c}$ for some $c>0$. Observe $N'_{j,k}$ are i.i.d. $N(0,1)$ for $k=1,2,\cdots j$. Define

$$M_{j,\bar{x}+\sigma N} = \max_{1\leq k\leq j}\bar{\bar{\lambda}}'_k \ \text{ and } \ m_{j,\bar{x}+\sigma N} = \min_{1\leq k\leq j}\bar{\bar{\lambda}}'_k.$$

Let

$$\begin{aligned}A &= \big(\frac{-m_{j,\bar{x}+\sigma N}-b_j}{a_j} > x, \frac{M_{j,\bar{x}+\sigma N}-b_j}{a_j} > y\big), \\ B &= \mathrm{P}\Big(\frac{-\min_{1\leq k\leq j}(1+\sigma_j^2)N_k - b_j}{a_j} > x, \frac{\max_{1\leq k\leq j}(1+\sigma_j^2)N_k - b_j}{a_j} > y\Big).\end{aligned}$$

Claim:

$$\lim_{j\to\infty}[\mathrm{P}(A) - \mathrm{P}(B)] = 0. \tag{4.3.4}$$

We approximate $\mathrm{P}(A)$ by $\mathrm{P}(B)$ as follows:

$$
\begin{aligned}
\mathrm{P}(A) &= \mathrm{P}\big(\frac{-m_{j,\bar{x}+\sigma N}-b_j}{a_j} > x, \frac{M_{j,\bar{x}+\sigma N}-b_j}{a_j} > y\big) \\
&= \mathrm{P}\big(m_{j,\bar{x}+\sigma N} < -a_j x - b_j, M_{j,\bar{x}+\sigma N} > a_j y + b_j\big) \\
&= \mathrm{P}\big(\cup_{k=1}^{j}\{\bar{\bar{\lambda}}'_k < -a_j x - b_j\} \cap \cup_{k=1}^{j}\{\bar{\bar{\lambda}}'_k > a_j y + b_j\}\big) \\
&= \mathrm{P}\big(\cup_{k=1}^{j}\{\bar{\bar{\lambda}}'_k \in I^j_{x,y}\}\big) = \mathbb{P}\big(\cup_{k=1}^{j} A_{k,j}\big)
\end{aligned}
$$

where, $I^j_{x,y} = (a_j y + b_j, -a_j x - b_j)$ and $A_{k,j} = \{\bar{\bar{\lambda}}'_k \in I^j_{x,y}\}$. Now by Bonferroni's inequality,

$$\sum_{t=1}^{2k}(-1)^{t-1}\tilde{A}_{t,j} \leq \mathrm{P}(A) \leq \sum_{t=1}^{2k-1}(-1)^{t-1}\tilde{A}_{t,j} \tag{4.3.5}$$

where

$$\tilde{A}_{t,j} = \sum_{1\leq i_1<i_2<\cdots<i_t\leq j} \mathrm{P}\big(A_{i_1,j}\cap ..\cap A_{i_t,j}\big).$$

$$
\begin{aligned}
\mathrm{P}(B) &= \mathrm{P}\Big(\frac{-\min_{1\leq k\leq j}(1+\sigma_j^2)N_k - b_j}{a_j} > x, \frac{\max_{1\leq k\leq j}(1+\sigma_j^2)N_k - b_j}{a_j} > y\Big) \\
&= \mathrm{P}\big(\cup_{k=1}^{j}\{(1+\sigma_j^2)^{1/2}N_k \in I^j_{x,y}\}\big) = \mathrm{P}\big(\cup_{k=1}^{j} B_{k,j}\big)
\end{aligned}
$$

where $B_{k,j} = \{(1+\sigma_j^2)^{1/2}N_k \in I^j_{x,y}\}$. Again by Bonferroni's inequality,

$$\sum_{t=1}^{2k}(-1)^{t-1}\tilde{B}_{t,j} \leq \mathrm{P}(B) \leq \sum_{t=1}^{2k-1}(-1)^{t-1}\tilde{B}_{t,j} \tag{4.3.6}$$

where

$$\tilde{B}_{t,j} = \sum_{1\leq i_1<i_2<\cdots<i_t\leq j} \mathrm{P}\big(B_{i_1,j}\cap B_{i_2,j}\cap ..\cap B_{i_t,j}\big).$$

From (4.3.5) and (4.3.6) we get

$$\sum_{t=1}^{2k}(-1)^{t-1}(\tilde{A}_{t,j}-\tilde{B}_{t,j}) - \tilde{B}_{2k+1,j} \leq \mathrm{P}(A)-\mathrm{P}(B) \leq \sum_{t=1}^{2k-1}(-1)^{t-1}(\tilde{A}_{t,j}-\tilde{B}_{t,j}) + \tilde{B}_{2k,j}. \tag{4.3.7}$$

Now note that,

$$\begin{aligned}\tilde{B}_{t,j} &= \sum_{1\le i_1<i_2\cdots<i_t\le j} \mathrm{P}\big(B_{i_1,j}\cap B_{i_2,j}\cap\ldots\cap B_{i_t,j}\big)\\ &= \sum_{1\le i_1<i_2\cdots<i_t\le j} \mathrm{P}\big((1+\sigma_j^2)^{1/2}N_{i_l}\in I_{x,y}^j; l=1,2,..,t\big)\\ &= \sum_{1\le i_1<i_2\cdots<i_t\le j} \mathrm{P}^t\big((1+\sigma_j^2)^{1/2}N_{i_l}\in I_{x,y}^j\big).\end{aligned}$$

Note here that

$$\begin{aligned}\mathrm{P}\big((1+\sigma_j^2)^{1/2}N_1\in(a_jy+b_j,-a_jx-b_j)\big) &\le \mathrm{P}\big((1+\sigma_j^2)^{1/2}N_1>a_jy+b_j\big)\\ &= \mathrm{P}\big(N_1>(a_jy+b_j)(1+\sigma_j^2)^{-1/2}\big)\\ &\le \mathrm{P}\big(N_1>(a_jy+b_j)(1-\frac{1}{2}\sigma_j^2)\big).\end{aligned}$$

Now $(a_jy+b_j)(1-\frac{\sigma_j^2}{2})\approx b_j+o(1)$ and $\mathrm{P}(N_1>b_j)\approx\frac{1}{j}$. Therefore

$$\mathrm{P}\big(N_1>(1-\frac{1}{2}\sigma_j^2)(a_jy+b_j)\big)\le\frac{K}{j}$$

and hence

$$\tilde{B}_{t,j}\le\binom{j}{t}\frac{K^t}{j^t}\le\frac{K^t}{t!}.$$

Thus

$$\lim_{t\to\infty}\overline{\lim_{j\to\infty}}\,\tilde{B}_{t,j}=0.$$

On the other hand, fixing $t\ge1$ we get,

$$\mathrm{P}(A_{i_1,j}\cap A_{i_2,j}\cap\ldots\cap A_{i_t,j})=\mathrm{P}\Big(\frac{1}{\sqrt{1+2j}}\sum_{t=0}^{j}(\bar{x}_t+\sigma_jN_t)v_d(t)\in E_t\Big),$$

where $E_t=\{(x_1,x_2,...,x_t): x_i\in I_{x,y}^j\}$. So by Lemma 4.3.3 we have that uniformly over all d-tuples $1\le i_1<i_2<\ldots<i_d\le j$,

$$\begin{aligned}&\Big|\mathrm{P}\Big(\frac{1}{\sqrt{1+2j}}\sum_{t=0}^{j}(\bar{x}_t+\sigma_jN_t)v_d(t)\in E_t\Big)-\mathrm{P}((1+\sigma_j^2)^{1/2}N_{i_l}\in I_{x,y}^j, 1\le l\le t)\Big|\\ &\qquad\le\epsilon_j\mathrm{P}((1+\sigma_j^2)^{1/2}N_{i_l}>a_jy+b_j, 1\le l\le t)+O(\exp(-(1+2j)^\eta)).\end{aligned}$$

So as $j\to\infty$ we get,

$$|\tilde{A}_{t,j}-\tilde{B}_{t,j}|\le\epsilon_j\tilde{B}_{t,j}+\binom{j}{t}O(\exp(-(1+2j)^\eta))\to0.$$

Therefore,

$$\overline{\lim_{j\to\infty}}\,|\mathrm{P}(A)-\mathrm{P}(B)| \le \overline{\lim_{j\to\infty}}\,\tilde{B}_{2k+1,j} + \overline{\lim_{j\to\infty}}\,\tilde{B}_{2k,j}$$

and letting $k\to\infty$ we get,

$$\lim_{j\to\infty}[\mathrm{P}(A)-\mathrm{P}(B)] = 0.$$

This proves the claim (4.3.4) completely.

Step 3: Claim:

$$\Big(\frac{-m_{j,\bar{x}+\sigma N}-b_j}{a_j},\frac{M_{j,\bar{x}+\sigma N}-b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda\otimes\Lambda. \tag{4.3.8}$$

As $\max_{1\le k\le j} N_k = O_p((\ln j)^{1/2})$, it follows that,

$$\Big|\frac{(1+\sigma_j^2)^{1/2}\max_{1\le k\le j}N_k - b_j}{a_j} - \frac{\max_{1\le k\le j}N_k - b_j}{a_j}\Big| \le \frac{\sigma_j \max_{1\le k\le j}|N_k|}{a_j} \xrightarrow{\mathcal{P}} 0.$$

Therefore

$$\frac{(1+\sigma_j^2)^{1/2}\max_{1\le k\le j}N_k - b_j}{a_j} \xrightarrow{\mathcal{D}} \Lambda.$$

Since $-\min_{1\le k\le j}(1+\sigma_j^2)^{1/2}N_k = \max_{1\le k\le j}\big(-(1+\sigma_j^2)^{1/2}N_k\big)$ and $-(1+\sigma_j^2)^{1/2}N_k \stackrel{\mathcal{D}}{=} (1+\sigma_j^2)^{1/2}N_k$ we get

$$\frac{\min_{1\le k\le j}-(1+\sigma_j^2)^{1/2}N_k - b_j}{a_j} \xrightarrow{\mathcal{D}} \Lambda.$$

Since $(1+\sigma_j^2)^{1/2}N_i$ are i.i.d. symmetric distributions, by Resnick (1987) [103] (Exercise 5.5.2)

$$\Big(\frac{\min_{1\le k\le j}-(1+\sigma_j^2)^{1/2}N_k - b_j}{a_j},\frac{\max_{1\le k\le j}(1+\sigma_j^2)^{1/2}N_k - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda\otimes\Lambda. \tag{4.3.9}$$

Therefore combining (4.3.4) and (4.3.9) we get,

$$\Big(\frac{-m_{j,\bar{x}+\sigma N}-b_j}{a_j},\frac{M_{j,\bar{x}+\sigma N}-b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda\otimes\Lambda.$$

This completes the proof of (4.3.8).

Step 4: Claim:

$$\Big(\frac{-m_{j,\bar{x}}-b_j}{a_j},\frac{M_{j,\bar{x}}-b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda\otimes\Lambda. \tag{4.3.10}$$

We prove this using (4.3.8). Note

$$\Big|\frac{\max(\bar{\bar{\lambda}}'_k)}{a_j} - \frac{\max(\bar{\lambda}'_k)}{a_j}\Big| \leq \frac{\sigma_j}{a_j} \max |N'_{j,k}| \xrightarrow{\mathcal{P}} 0.$$

Similarly $-\bar{\bar{\lambda}}'_k = -\bar{\lambda}'_k - \sigma N'_{j,k}$ and

$$\Big|\frac{\max(-\bar{\bar{\lambda}}'_k)}{a_j} - \frac{\max(-\bar{\lambda}'_k)}{a_j}\Big| \leq \frac{\sigma_j}{a_j} \max |N'_{j,k}| \xrightarrow{\mathcal{P}} 0.$$

Now if we denote $m'_{j,\bar{x}} = \min_{1\leq k\leq j} \bar{\lambda}'_k$ and $M'_{j,\bar{x}} = \max_{1\leq k\leq j} \bar{\lambda}'_k$ then,

$$\begin{aligned}
&\Big|\Big(\frac{-m_{j,\bar{x}+\sigma N} - b_j}{a_j}, \frac{M_{j,\bar{x}+\sigma N} - b_j}{a_j}\Big) - \Big(\frac{-m'_{j,\bar{x}} - b_j}{a_j}, \frac{M'_{j,\bar{x}} - b_j}{a_j}\Big)\Big| \\
\leq\ & C\Big[\Big|\frac{-m_{j,\bar{x}+\sigma N} - (-m'_{j,\bar{x}})}{a_j}\Big| + \Big|\frac{M_{j,\bar{x}+\sigma N} - M'_{j,\bar{x}}}{a_j}\Big|\Big] \\
\leq\ & C\Big[\Big|\frac{\max(-\bar{\bar{\lambda}}'_k) - \max(-\bar{\lambda}'_k)}{a_j}\Big| + \Big|\frac{\max(\bar{\bar{\lambda}}'_k) - \max(\bar{\lambda}'_k)}{a_j}\Big|\Big] \xrightarrow{\mathcal{P}} 0.
\end{aligned}$$

Therefore using (4.3.8), we hwve

$$\Big(\frac{-m'_{j,\bar{x}} - b_j}{a_j}, \frac{M'_{j,\bar{x}} - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda. \tag{4.3.11}$$

Again $\bar{\lambda}_k = \bar{\lambda}'_k + \frac{(1-\sqrt{2})}{\sqrt{2j+1}}\bar{x}_0$, therefore

$$\Big|\frac{M'_{j,\bar{x}} - b_j}{a_j} - \frac{M_{j,\bar{x}} - b_j}{a_j}\Big| \xrightarrow{\mathcal{P}} 0,$$

and

$$\Big|\frac{-m_{j,\bar{x}} - b_j}{a_j} - \frac{-m'_{j,\bar{x}} - b_j}{a_j}\Big| \xrightarrow{\mathcal{P}} 0.$$

Hence using (4.3.11), we have

$$\Big(\frac{-m_{j,\bar{x}} - b_j}{a_j}, \frac{M_{j,\bar{x}} - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda$$

This completes the proof of (4.3.10).

Now we get back to the proof of the main theorem. Combining (4.3.3) and (4.3.10), we can conclude that

$$\Big(\frac{-m_{j,x} - b_j}{a_j}, \frac{M_{j,x} - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda.$$

This proves the theorem when n is odd.

For the even case say $n = 2j$ it should be noted that if we work with $\lambda'_k = \sqrt{2}x_0 + \sqrt{2}(-1)^k x_j + 2\sum_{t=1}^{j-1} x_t \cos\frac{2\pi kt}{2j}$ then similar normal approximations can be done and the subsequent calculations follow after that. We omit the obvious details. This proves the theorem completely. □

The next theorem follows by calculations similar to those used in the proof of Theorem 4.2.3.

Theorem 4.3.5 (Bose, Hazra and Saha (2009) [34])**.** *Suppose $\{x_i\}_{i\geq 0}$ is an i.i.d. sequence with mean μ and $E|x_i|^{2+\delta} < \infty$ for some $\delta > 0$. Consider the symmetric circulant matrix (SC_n) with these $\{x_i\}$.*

(i) If $\mu = 0$ then,

$$\frac{\|\frac{1}{\sqrt{n}}SC_n\| - b_q - a_q \ln 2}{a_q} \xrightarrow{\mathcal{D}} \Lambda$$

where $q = q(n) \approx \frac{n}{2}$ and a_q and b_q are as in equation (4.3.1).

(ii) If $\mu \neq 0$ then,

$$\frac{\|SC_n\| - |\mu|n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, 2).$$

Proof. To prove (i), since mean $\mu = 0$, $\lambda_0 \xrightarrow{\mathcal{D}} N(0, 2)$. So we can neglect this as was done in proof of Theorem 4.2.3. Therefore, for large n with arbitrarily large probability,

$$\|\frac{1}{\sqrt{n}}SC_n\| = \max\{-\min_{1\leq i\leq \lfloor \frac{n}{2}\rfloor} \lambda_i, \max_{1\leq i\leq \lfloor \frac{n}{2}\rfloor} \lambda_i\}.$$

Hence

$$\begin{aligned} \mathrm{P}\big(\|\frac{1}{\sqrt{n}}SC_n\| \leq a_q x + b_q\big) &= \mathrm{P}\big(\frac{-\min\lambda_i - b_q}{a_q} \leq x, \frac{\max\lambda_i - b_q}{a_q} \leq x\big) \\ &\xrightarrow{\mathcal{D}} \Lambda(x)\Lambda(x) = \Lambda(x + \ln\frac{1}{2}). \end{aligned}$$

Now by convergence of types

$$\mathrm{P}\Big(\frac{\|\frac{1}{\sqrt{n}}SC_n\| - \tilde{b}_q}{\tilde{a}_q} \leq x\Big) \xrightarrow{\mathcal{D}} \Lambda(x)$$

where, $\tilde{a}_q = a_q$ and $\tilde{b}_q = b_q + a_q \ln 2$. This proves (i).

In part (ii), λ_0 dominates and the proof proceeds as in the proof given for Theorem 4.2.3. We omit the details. □

4.4 k-circulant with light tail entries

It appears difficult to establish distributional convergence of spectral norm or spectral radius for k-circulant matrix for all possible values of (k, n). A special case ($n = k^2 + 1$) was tackled in Bose, Mitra and Sen (2009) [44].

Theorem 4.4.1 (Bose, Mitra and Sen (2009))**.** *Suppose* $\{x_i\}_{i\geq 0}$ *is an i.i.d. sequence with mean zero and variance* 1 *and* $E|x_i|^{2+\delta} < \infty$ *for some* $\delta > 2$*. If* $n = k^2 + 1$ *then*

$$\frac{sp(n^{-1/2}A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

as $n \to \infty$ *where* $q = q(n) = \lfloor \frac{n}{4} \rfloor$ *and*

$$c_n = (8\ln n)^{-1/2} \quad and \quad d_n = \frac{(\ln n)^{1/2}}{\sqrt{2}}\left(1 + \frac{1}{4}\frac{\ln\ln n}{\ln n}\right) + \frac{1}{2(8\ln n)^{1/2}}\ln\frac{\pi}{2}. \tag{4.4.1}$$

We now state the following significant generalisation of the above result.

Theorem 4.4.2 (Bose, Hazra and Saha (2010) [37])**.** *Suppose* $\{x_i\}_{i\geq 0}$ *is an i.i.d. sequence of random variables with mean zero and variance* 1 *and* $\mathrm{E}\,|x_i|^{\gamma} < \infty$ *for some* $\gamma > 2$*. If* $n = k^g + 1$ *for some fixed positive integer* g*, then as* $n \to \infty$*,*

$$\frac{sp(n^{-1/2}A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

where $q = q_n = \frac{n}{2g}$ *and the normalizing constants* c_n *and* d_n *can be taken as follows*

$$c_n = \frac{1}{2g^{1/2}(\ln n)^{1/2}}, \quad d_n = \frac{\ln C_g - \frac{g-1}{2}\ln g}{2g^{1/2}(\ln n)^{1/2}} + \left(\frac{\ln n}{g}\right)^{1/2}\left[1 + \frac{(g-1)\ln\ln n}{4\ln n}\right],$$

$$and \quad C_g = \frac{1}{\sqrt{g}}(2\pi)^{\frac{g-1}{2}}.$$

The proof of the above theorem is long and is developed in the following sections. It involves some intricate study of the structure of the eigenvalues, the behaviour of the tail of product of i.i.d. standard exponential random variables $H_n(\cdot)$ from Chapter 3 and some normal approximation methods. In Section 4.4.5, we remark about the case $sn = k^g + 1$.

Here is an outline of the proof of Theorem 4.4.2. In Section 4.4.1 we discuss some distributional properties of the eigenvalues of k-circulant matrix when the input sequence is i.i.d. Gaussian. In Section 4.4.2 we provide more detailed description of the eigenvalues of k-circulant matrix for $n = k^g + 1$. Section 4.4.3 has two preparatory Lemmas on truncation and normal approximation. Drawing on the developments of

Section 3.2 of Chapter 3 and of Sections 4.4.2 and 4.4.3, we derive the limit behaviour of the spectral radius of k-circulant matrices when $n = k^g + 1 \to \infty$ (g being an integer held fixed) in Section 4.4.4. We show that the spectral radius, when scaled and centered appropriately, converges in distribution to the Gumbel distribution.

4.4.1 Properties of eigenvalues of Gaussian k-circulant for fixed n

For $1 \le t < n$, let us split λ_t into real and complex parts as $\lambda_t = a_{t,n} + ib_{t,n}$, that is,

$$a_{t,n} = \sum_{l=0}^{n-1} x_l \cos\left(\frac{2\pi tl}{n}\right), \quad b_{t,n} = \sum_{l=0}^{n-1} x_l \sin\left(\frac{2\pi tl}{n}\right). \tag{4.4.2}$$

For $z \in \mathbb{C}$, $\bar{z}$ denotes its complex conjugate. For all $0 < t, t' < n$, the following identities can easily be verified using the orthogonality relations of sine and cosine functions.

$$\mathrm{E}(a_{t,n}b_{t,n}) = 0, \quad \text{and} \quad \mathrm{E}(a_{t,n}^2) = \mathrm{E}(b_{t,n}^2) = n/2,$$

$$\bar{\lambda}_t = \lambda_{n-t}, \quad E(\lambda_t \lambda_{t'}) = n\mathbb{I}(t + t' = n), \quad E(|\lambda_t|^2) = n.$$

The following Lemma is due to Bose, Mitra and Sen (2009) [44].

Lemma 4.4.3. *(Bose, Mitra and Sen (2009) [44]) Fix k and n. Suppose that $\{x_l\}_{0 \le l < n}$ are i.i.d. standard normal random variables.*

(a) For every n, $n^{-1/2}a_{t,n}, n^{-1/2}b_{t,n}$, $0 \le t \le n/2$ are i.i.d. normal with mean zero and variance $1/2$. Consequently, any sub-collection $\{y_{j_1}, y_{j_2}, \ldots\}$ of $\{y_j\}_{0 \le j < \ell}$, so that no member of the corresponding partition blocks $\{\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots\}$ is a conjugate of any other, are mutually independent.

(b) Suppose $1 \le j < \ell$ and $\mathcal{P}_j = \{n - i : \ i \in \mathcal{P}_j\}$ and $n/2 \notin \mathcal{P}_j$. Then $n^{-n_j/2}y_j$ are distributed as $(n_j/2)$-fold product of i.i.d. exponential random variables with mean one.

4.4.2 Additional description of eigenvalues of k-circulant when $n = k^g + 1$

We need some additional facts about the eigenvalues since we are dealing with spectral radius instead of LSD. Recall the eigenvalue structure of k-circulant matrices from Section 1.2.4. Also recall that $g_x = \#S(x)$ and $S(x) = \{xk^b \mod n' : 0 \le b < g_x\}$. We call g_x the *order* of x. Note that $g_0 = 1$. It is easy to see that

$$g_x = \min\{b > 0 : b \ \text{ is an integer and } \ xk^b = x \mod n'\}. \tag{4.4.3}$$

Recall from (1.2.7)

$$y_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \ldots, l-1 \quad \text{where} \quad y = n/n'.$$

Define

$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\},$$

$$J_k := \{\mathcal{P}_i : \ \#\mathcal{P}_i = k\}, \ n_k := \#J_k, \ \ X(k) := \{x : \ x \in \mathbb{Z}_n \text{ and } x \text{ has order } k\},$$

$$\upsilon_{k,n} = \#\{x : \ x \in \mathbb{Z}_n \text{ and } g_x < g_1\}. \tag{4.4.4}$$

Lemma 4.4.4. *The eigenvalues $\{\eta_i\}$ of the k-circulant with $n = k^g + 1$, $g \geq 2$, satisfy the following:*

(a) $\eta_0 = \sum_{t=0}^{n-1} x_t$, is always an eigenvalue and if n is even, then $\eta_{\frac{n}{2}} = \sum_{t=0}^{n-1} (-1)^t x_t$, is also an eigenvalue and both have multiplicity one.

(b) For $x \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$, $g_x = g_1$ or $\frac{g_1}{b}$ for some $b \geq 2$ and $\frac{g_1}{2b}$ is an integer.

(c) For all large n, $g_1 = 2g$. Hence from (b), for $x \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$, $g_x = 2g$ or $\frac{2g}{b}$. The total number of eigenvalues corresponding to J_{2g} is

$$2g \times \#J_{2g} = \#X(2g) \approx n.$$

(d) $X(\frac{2g}{b}) = \emptyset$ for $2 \leq b < g$, b even. If g is even then $X(\frac{2g}{g}) = X(2)$ is either empty or contains exactly two elements with eigenvalues

$$\eta_l = |\lambda_l|, \ \eta_{n-l} = -|\lambda_l|, \quad \textit{for some } 1 \leq l \leq \frac{n}{2}.$$

(e) Suppose b is odd, $3 \leq b \leq g$ and $\frac{g}{b}$ is an integer. For each $\mathcal{P}_j \in J_{\frac{2g}{b}}$ there are $\frac{2g}{b}$ eigenvalues given by the $\frac{2g}{b}$-th roots of y_j. Total number of eigenvalues corresponding to the set $J_{\frac{2g}{b}}$ is

$$\frac{2g}{b} \times \#J_{\frac{2g}{b}} = \#X(\frac{2g}{b}) \approx (k^{g/b} + 1)(1 + n^{-a}) \ \ \textit{for some} \ \ a > 0.$$

There are no other eigenvalues.

Proof. Since $n = k^g + 1$, n and k are relatively prime, we have $n' = n$.
(a) $\mathcal{P}_0 = S(0) = \{0\}$ and the corresponding eigenvalue is $\eta_0 = \sum_{t=0}^{n-1} x_t$ with multiplicity one. Similarly if n is even then k is odd and hence $S(n/2) = \{\frac{n}{2}\}$, and the corresponding eigenvalue is $\eta_{\frac{n}{2}} = \sum_{t=0}^{n-1} (-1)^t x_t$ of multiplicity one.

(b) From (4.4.3) it is easy to see that g_x divides g_1 and hence $g_x = g_1$ or $g_x = \frac{g_1}{b}$ for some $b \geq 2$. Also for every integer $t \geq 0$, $tk^g = (-1+n)t = -t \mod n$. Hence λ_t and λ_{n-t} belong to same partition block $S(t) = S(n-t)$. Thus each $S(t)$ contains even number of elements, except for $t = 0, \frac{n}{2}$. Hence $\frac{g_1}{b}$ must be even, that is, $\frac{g_1}{2b}$ must be an integer.

(c) From Lemma 2.4.16(i), $g_1 = 2g$ for all but finitely many n and $\upsilon_{k,n}/n \to 0$ as $n \to \infty$. For each $\mathcal{P}_j \in J_{2g}$ we have $2g$ many eigenvalues which are $2g$-th roots of Π_j. Now the result follows from the fact that

$$n = 2g\#J_{2g} + \upsilon_{k,n}.$$

(d) Suppose $b = 2$ and $x \in X(\frac{g_1}{2}) = X(\frac{2g}{2})$. Then $xk^{\frac{g_1}{2}} = xk^g = x \mod n$. But $k^g = -1 \mod n$ and so, $xk^g = -x \mod n$. Therefore $2x = 0 \mod n$ and x can be either 0 or $n/2$. But we have already seen in part (a) that $g_0 = g_{n/2} = 1$. Hence $X(\frac{2g}{2}) = \emptyset$.

Now suppose $b > 2$, even. From Lemma 3(ii) Bose, Mitra and Sen (2008) [44], $\#X(\frac{2g}{b}) \leq \gcd(k^{2g/b} - 1, k^g + 1)$ for $b \geq 3$. Now observe that for b even,

$$\gcd(k^{2g/b} - 1, k^g + 1) = \begin{cases} 1 & \text{if } k \text{ even,} \\ 2 & \text{if } k \text{ odd.} \end{cases}$$

So we have $\#X(\frac{2g}{b}) \leq 2$ for $b > 2$ and b even.

Suppose if possible, there exist $x \in \mathbb{Z}_n$ such that $g_x = \frac{2g}{b}$. Then $\#S(x) = \frac{2g}{b}$ and for all $y \in S(x)$, $g_y = \frac{2g}{b}$. Hence

$$\#\left\{y: \ g_y = \frac{2g}{b}\right\} \geq \frac{2g}{b} > 2 \ \text{ for } \ g > b > 2, \ b \textit{ even}.$$

This contradicts the fact that $\#X(\frac{2g}{b}) \leq 2$ for $g > b > 2$, b even. Hence $X(\frac{2g}{b}) = \emptyset$ for b even and $g > b > 2$.

If $b = g$ and it is even, then from previous discussion $\#X(\frac{2g}{g}) = 0$ or 2. In the latter case there are exactly two elements in $\mathbb{Z}_n$ whose order is 2 and there will be only one partitioning set containing them. So corresponding eigenvalues will be

$$\eta_l = |\lambda_l|, \ \eta_{n-l} = -|\lambda_l|, \ \text{ for some } 1 \leq l \leq \frac{n}{2}.$$

(e) We first show that for b odd,

$$(k^{g/b} + 1) - \sum_{\substack{b_i > b, \ b_i \ odd, \\ \frac{g}{b_i} \ integer}} (k^{g/b_i} + 1) \leq \#X(\frac{2g}{b}) \leq k^{g/b} + 1.$$

Note that (e) is a simple consequence of this. Let

$$Z_{n,b} = \left\{x : \ x \in \mathbb{Z}_n \text{ and } xk^{2g/b} = x \mod (k^g+1)\right\}.$$

Then it is easy to see that

$$X(\frac{2g}{b}) \subseteq Z_{n,b}. \tag{4.4.5}$$

Let $x \in Z_{n,b}$ and $\frac{g}{b} = m$. Then

$$\begin{aligned} & k^g + 1 \mid x(k^{2g/b} - 1) \\ \Rightarrow \ & k^{bm} + 1 \mid x(k^{2m} - 1) \\ \Rightarrow \ & k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1 \mid x(k^m - 1). \end{aligned}$$

But $\gcd(k^m - 1, k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1) = 1$, and therefore x is a multiple of $(k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1)$. Hence

$$\begin{aligned} \#Z_{n,b} &= \left\lfloor \frac{k^{bm}+1}{(k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1)} \right\rfloor \\ &= k^m + 1 = k^{g/b} + 1 \end{aligned}$$

and combining with (4.4.5),

$$\#X(\frac{2g}{b}) \leq \#Z_{n,b} = k^{g/b} + 1.$$

On the other hand, if $x \in Z_{n,b}$ then either $g_x = \frac{2g}{b}$ or $g_x < \frac{2g}{b}$. For the second case $g_x = \frac{2g}{b_i}$ for some $b_i > b$, b_i odd and therefore $x \in Z_{n,b_i}$. Hence

$$\begin{aligned} \#X(\frac{2g}{b}) &\geq \#Z_{n,b} - \sum_{\substack{b_i > b, \ b_i \ odd, \\ \frac{g}{b_i} \ integer}} \#Z_{n,b_i} \\ &\geq (k^{g/b}+1) - \sum_{\substack{b_i > b, \ b_i \ odd, \\ \frac{g}{b_i} \ integer}} (k^{g/b_i}+1). \end{aligned}$$

□

4.4.3 Final preparatory lemmas: truncation and normal approximation

Truncation:

From Section 4.4.2, $n = n'$ and $S(t) = S(n-t)$ except for $t = 0,\ n/2$. So for $\mathcal{P}_j \neq S(0),\ S(n/2)$, we can define $\mathcal{A}_j$ such that

$$\mathcal{P}_j = \{x : x \in \mathcal{A}_j \text{ or } n - x \in \mathcal{A}_j\} \text{ and } \#\mathcal{A}_j = \frac{1}{2}\#\mathcal{P}_j. \tag{4.4.6}$$

For any sequence of random variables $b = \{b_l\}_{l\geq 0}$, define for $\mathcal{P}_j \in J_{2k}$

$$\beta_{b,k}(j) = \prod_{t\in\mathcal{A}_j}\left|\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} b_l\omega^{tl}\right|^2, \text{ where } \omega = \exp\left(\frac{2\pi i}{n}\right). \tag{4.4.7}$$

Suppose $\{x_l\}_{l\geq 0}$ are independent, mean zero and variance one random variables. For each $n \geq 1$, define a triangular array of centered random variables $\{\bar{x}_l^{(n)}\}_{0\leq l<n}$ by

$$\bar{x}_l = \bar{x}_l^{(n)} = x_l\mathbb{I}_{|x_l|\leq n^{1/\gamma}} - Ex_l\mathbb{I}_{|x_l|\leq n^{1/\gamma}}.$$

Now, recall from Lemma 4.4.4, $\#J_{2g} = n_{2g} \approx \frac{n}{2g}$ for $n = k^g + 1$. Without loss of generality, assuming that $\mathcal{P}_j \in J_{2g}$ for $1 \leq j \leq q = \frac{n}{2g}$, we prove the following lemma.

Lemma 4.4.5. *Assume $E|x_l|^\gamma < \infty$ for some $\gamma > 2$. Then, almost surely,*

$$\max_{1\leq j\leq q}(\beta_{x,g}(j))^{1/2g} - \max_{1\leq j\leq q}(\beta_{\bar{x},g}(j))^{1/2g} = o(1).$$

Proof. Since $\sum_{l=0}^{n-1}\omega^{tl} = 0$ for $0 < t < n$, it follows that $\beta_{\bar{x},n}(j) = \beta_{\tilde{x},n}(j)$ where

$$\tilde{x}_l = \tilde{x}_l^{(n)} = \bar{x}_l + Ex_l\mathbb{I}_{|x_l|\leq n^{1/\gamma}} = x_l\mathbb{I}_{|x_l|\leq n^{1/\gamma}}.$$

By Borel-Cantelli lemma, $\sum_{t=0}^{\infty}|x_t|\mathbb{I}_{|x_t|>t^{1/\gamma}}$ is finite a.s. and has only finitely many non-zero terms. Thus there exists a positive integer $N(\omega)$ such that

$$\sum_{t=0}^{n}|x_t - \tilde{x}_t| = \sum_{t=0}^{n}|x_t|\mathbb{I}_{|x_t|>n^{1/\gamma}} \leq \sum_{t=0}^{\infty}|x_t|\mathbb{I}_{|x_t|>t^{1/\gamma}} = \sum_{t=0}^{N(\omega)}|x_t|\mathbb{I}_{|x_t|>t^{1/\gamma}}. \tag{4.4.8}$$

It follows that for $n \geq \{N(\omega), |x_1|^\gamma, \ldots, |x_{N(\omega)}|^\gamma\}$ the left side of (4.4.8) is zero. Consequently, for all n sufficiently large,

$$\beta_{x,n}(j) = \beta_{\tilde{x},n}(j) = \beta_{\bar{x},n}(j) \text{ a.s. for all } j \tag{4.4.9}$$

and the assertion follows immediately. □

Normal approximation:

For $d \geq 1$, and any distinct integers $i_1, i_2, \ldots, i_d$, from $\left\{1, 2, \ldots, \lceil \frac{n-1}{2} \rceil\right\}$, define

$$v_{2d}(l) = \left(\cos\left(\frac{2\pi i_j l}{n}\right), \sin\left(\frac{2\pi i_j l}{n}\right) : 1 \leq j \leq d\right)^T, \quad l \in \mathbb{Z}_n.$$

Let $\phi_\Sigma(\cdot)$ denote the density of the $2d$-dimensional Gaussian vector having mean zero and covariance matrix Σ and let I_{2d} be the identity matrix of order $2d$. The following Lemma is from Davis and Mikosch (1999) [50] and it follows from strong approximation results of Einmahl (1989) [53].

Lemma 4.4.6 (Davis and Mikosch (1999) [50])**.** *Let $\{x_t\}$ be i.i.d random variables with* $\mathrm{E}[x_0] = 0$, $\mathrm{E}[x_0]^2 = 1$ *and* $\mathrm{E}[x_0]^\gamma < \infty$ *for some* $\gamma > 2$. *Let $\tilde{p}_n$ be the density function of*

$$2^{1/2} n^{-1/2} \sum_{t=1}^{n} (\bar{x}_t + \sigma_n N_t) v_d(t),$$

where $\{N_t\}$ is independent of $\{x_t\}$ and $\sigma_n^2 = \mathrm{Var}(\bar{x}_t) s_n^2$, *for some sequence $\{s_n\}$. If $n^{-2c} \ln n < s_n^2 \leq 1$ with $c = 1/2 - (1-\delta)/\gamma$ for arbitrarily small $\delta > 0$, then uniformly for $|x|^3 = o_d(\min(n^c, n^{1/2-1/s}))$,*

$$\tilde{p}_n(x) = \phi_{(1+\sigma_n^2)I_{2d}}(x)(1 + o(1)).$$

We shall use this lemma also in Section 7.1.1. Now we have the following corollary which is similar to Lemma 4.3.3.

Corollary 4.4.7. *Let $\gamma > 2$ and $\sigma_n^2 = n^{-c}$ where c is as in Lemma 4.4.6. Then for any measurable $B \subseteq \mathbb{R}^{2d}$,*

$$\left| \int_B \tilde{p}_n(x)dx - \int_B \phi_{(1+\sigma_n^2)I_{2d}}(x)dx \right| \leq \epsilon_n \int_B \phi_{(1+\sigma_n^2)I_{2d}}(x)dx + O_d(\exp(-n^\eta)),$$

where $\epsilon_n \to 0$ as $n \to \infty$ and $\eta > 0$. The above holds uniformly over all the d-tuples of distinct integers $1 \leq i_1 < i_2 < \ldots < i_d \leq \lceil \frac{n-1}{2} \rceil$.

4.4.4 Proof of Theorem 4.4.2

To establish the theorem we shall use the following lemmas whose proofs are given later. Recall that $\{\beta_{x,g}(t)^{1/2g}\}$ are the eigenvalues corresponding to the set of partitions having cardinality $2g$. We derive the behaviour of the maximum of these eigenvalues in

Lemma 4.4.8. Then using the results of Lemma 4.4.9, we show that the maximum of the remaining eigenvalues is negligible compared to the above.

Lemma 4.4.8.

$$\frac{\max_{1\le t\le q}\beta_{x,g}(t)^{1/2g}-d_q}{c_q}\xrightarrow{\mathcal{D}}\Lambda \tag{4.4.10}$$

where d_q, c_q *are as in Corollary 3.2.3,* $q=q_n=\frac{n}{2g}-k_n$ *and* $\frac{k_n}{n}\to 0$ *as* $n\to\infty$*. As a consequence,*

$$\frac{\max_{1\le t\le q}\beta_{x,g}(t)^{1/2g}-d_{n/2g}}{c_{n/2g}}\xrightarrow{\mathcal{D}}\Lambda. \tag{4.4.11}$$

The next Lemma is technical and is required in the proof of Theorem 4.4.2. Let

$$c_n(l)=\frac{1}{2l^{1/2}(\ln n)^{1/2}},\ d_n(l)=\frac{\ln C_l-\frac{l-1}{2}\ln l}{2l^{1/2}(\ln n)^{1/2}}+\left(\frac{\ln n}{l}\right)^{1/2}\left[1+\frac{(l-1)\ln\ln n}{4\ln n}\right],$$

$$C_l=\frac{1}{\sqrt{l}}(2\pi)^{\frac{l-1}{2}},\quad \text{and}$$

$$c_{n_{2j}}=c_{n_{2j}}(j),\ d_{n_{2j}}=d_{n_{2j}}(j),\ c_{n/2g}=c_{n/2g}(g)\ \text{ and }\ d_{n/2g}=d_{n/2g}(g).$$

Lemma 4.4.9. *Let* $n=k^g+1$*. If* $j<g$ *and for some* $a>0$, $2jn_{2j}=(k^j+1)(1+n^{-a})\approx n^{\frac{j}{g}}$ *or is finite, then there exists a constant* $K=K(j,g)\ge 0$ *such that,*

$$\frac{c_{n/2g}}{c_{n_{2j}}}\to K \quad \text{and} \quad \frac{d_{n/2g}-d_{n_{2j}}}{c_{n_{2j}}}\to\infty \text{ as } n\to\infty.$$

Now we prove Theorem 4.4.2 using Lemmas 4.4.8 and 4.4.9. Then we shall prove the lemmas.

Proof of Theorem 4.4.2. If $\#\mathcal{P}_i=j$, then the eigenvalues corresponding to $\mathcal{P}_i$'s are the j-th roots of Π_i and hence these eigenvalues have the same modulus. From Lemma 4.4.4, the possible values of $\#\mathcal{P}_i$ are $\{1,2,2g$ and $2g/b,\ 3\le b<g, b$ odd, $\frac{g}{b}\in\mathbb{Z}\}$. Recall from (4.4.7) that $\beta_{x,j}(i)$ is the modulus of the eigenvalue associated with the partition set $\mathcal{P}_i$, where $\#\mathcal{P}_i=2j$.

In case of Gaussian entries it easily follows that $\beta_{x,j}(i)$ is the product of j exponential random variables and they are independent as i takes n_{2j} many distinct values. So from Corollary 3.2.3, if $n_{2j}\to\infty$ then the maximum of $\beta_{x,j}(k)^{1/2j}$ has a Gumbel limit. For more general entries the method as in the proof of Lemma 4.4.8 can be adopted to get the following limit:

$$\max_{1\le k\le n_{2j}}\frac{\beta_{x,j}(k)^{1/2j}-d_{n_{2j}}}{c_{n_{2j}}}\xrightarrow{\mathcal{D}}\Lambda,\ \text{as } n_{2j}\to\infty, \tag{4.4.12}$$

where $c_{n_{2j}}$ and $d_{n_{2j}}$ are as above.
Let
$$x_n = c_n x + d_n, \ q = q(n) = \frac{n}{2g} \text{ and } \mathcal{B} = \{b : b \text{ odd}, \ 3 \le b < g, \ \frac{g}{b} \in \mathbb{Z}\}.$$
Then
$$\mathrm{P}\left(\mathtt{sp}(n^{-1/2}A_{k,n}) > x_q\right) \ge \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_{2g}} \beta_{x,g}(j)^{1/2g} > x_q\right)$$
and
$$\begin{aligned} & \mathrm{P}\left(\mathtt{sp}(n^{-1/2}A_{k,n}) > x_q\right) \\ \le \ & \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_{2g}} \beta_{x,g}(j)^{1/2g} > x_q\right) + \sum_{b \in \mathcal{B}} \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_{\frac{2g}{b}}} \beta_{x,\frac{g}{b}}(j)^{b/2g} > x_q\right) \\ & + \mathrm{P}\left(|n^{-1/2}\sum_{l=0}^{n-1} a_l| > x_q\right) + \mathrm{P}\left(|n^{-1/2}\sum_{l=0}^{n-1}(-1)^l x_l| > x_q\right) \\ & + \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_2} \beta_{x,2}(j)^{1/2} > x_q)\right) \\ =: \ & A + B + C + D + E. \end{aligned}$$

From Lemma 4.4.4, the term D appears only when $\frac{n}{2} \in \mathbb{Z}$ and the term E appears only if g is even and in that case J_2 contains only one element. It is easy to see that C, D and E tend to zero since we are taking maximum of single element.

Note that B is a sum of finitely many terms. Now suppose for $b \in \mathcal{B}$, we have some finite K_b such that

$$\frac{c_{n/2g}}{c_{n_{2g/b}}} \to K_b \text{ and } \frac{d_{n/2g} - d_{n_{2g/b}}}{c_{n_{2g/b}}} \to \infty \text{ as } n \to \infty. \tag{4.4.13}$$

Then from observation (4.4.12) and (4.4.13) we get that B goes to zero. So it remains to check that whether (4.4.13) holds for $b \in \mathcal{B}$. But (4.4.13) holds from Lemma 4.4.4(e) and Lemma 4.4.9.

Now the limit in A follows from Lemma 4.4.8, proving the theorem. □

Now we prove Lemmas 4.4.8 and 4.4.9.

Proof of Lemma 4.4.8. First assume that $\{x_l\}_{l \ge 0}$ *are i.i.d. standard normal.* Let $\{E_j\}_{j \ge 1}$ be i.i.d. standard exponentials. By Lemma 4.4.3, it easily follows that

$$\begin{aligned} & \mathrm{P}\left(\max_{1 \le t \le q} (\beta_{x,g}(t))^{1/2g} > c_q x + d_q\right) \\ = & \mathrm{P}\left(\left(E_{g(j-1)+1}E_{g(j-1)+2} \cdots E_{gj}\right)^{1/2g} > c_q x + d_q \text{ for some } 1 \le j \le q\right). \end{aligned}$$

The Lemma then follows in this special case from Corollary 3.2.3.

For the general case we break the proof into the following three steps and make use of the two results from Section 4.4.3. Fix $x \in \mathbb{R}$.

Step 1: Claim:

$$\lim_{n\to\infty}[Q_1^{(n)} - Q_2^{(n)}] = 0, \tag{4.4.14}$$

where

$$Q_1^{(n)} := \mathrm{P}\left(\max_{1\le j\le q}(\beta_{\bar{x}+\sigma_n N,g}(j))^{1/2g} > c_q x + d_q\right),$$
$$Q_2^{(n)} := \mathrm{P}\left(\max_{1\le j\le q}(1+\sigma_n^2)\left(E_{g(j-1)+1}E_{g(j-1)+2}\cdots E_{gj}\right)^{1/2g} > c_q x + d_q\right),$$

and $\{N_l\}_{l\ge 0}$ is a sequence of i.i.d. standard normal random variables.

Step 2: Claim:

$$\frac{\max_{1\le j\le q}(\beta_{\bar{x}+\sigma_n N,g}(j))^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda. \tag{4.4.15}$$

Step 3: Claim:

$$\frac{\max_{1\le t\le q}\beta_{\bar{x},g}(t)^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda. \tag{4.4.16}$$

We shall prove the above three steps later.

Now combining Lemma 4.4.5 and (4.4.16) we can conclude that

$$\frac{\max_{1\le t\le q}\beta_{\bar{x},g}(t)^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

This completes the proof of first part, (4.4.10) of the lemma. By convergence of type theorem, the second part, (4.4.11) follows since the following hold. We omit the tedious algebraic details.

$$\frac{c_q}{c_{n/2g}} \to 1 \text{ and } \frac{d_q - d_{n/2g}}{c_q} \to 0 \text{ as } n \to \infty. \tag{4.4.17}$$

Proof of Step 1: We approximate $Q_1^{(n)}$ by the simpler quantity $Q_2^{(n)}$ using Bonferroni's inequality. By Bonferroni's inequality, for all $m \ge 1$,

$$\sum_{j=1}^{2m}(-1)^{j-1}S_{j,n} \le Q_1^{(n)} \le \sum_{j=1}^{2m-1}(-1)^{j-1}S_{j,n}, \tag{4.4.18}$$

where

$$S_{j,n} = \sum_{1 \le t_1 < t_2 < \ldots < t_j \le q} \mathrm{P}\left((\beta_{\bar{x}+\sigma_n N, g}(t_i))^{1/2g} > c_q x + d_q,\ i = 1, \ldots, j \right).$$

Similarly, we have

$$\sum_{j=1}^{2m} (-1)^{j-1} T_{j,n} \le Q_2^{(n)} \le \sum_{j=1}^{2m-1} (-1)^{j-1} T_{j,n}, \tag{4.4.19}$$

where

$$T_{j,n} = \sum_{1 \le t_1 < t_2 < \ldots < t_j \le q} \mathrm{P}\left((1+\sigma_n^2)\big(E_{g(t_i-1)+1} E_{g(t_i-1)+2} \cdots E_{gt_i}\big)^{1/2g} > c_q x + d_q, i = 1, ., j \right).$$

Therefore, the difference between $Q_1^{(n)}$ and $Q_2^{(n)}$ can be bounded as follows:

$$\sum_{j=1}^{2m} (-1)^{j-1}(S_{j,n} - T_{j,n}) - T_{2m+1,n} \le Q_1^{(n)} - Q_2^{(n)} \le \sum_{j=1}^{2m-1} (-1)^{j-1}(S_{j,n} - T_{j,n}) + T_{2m,n}, \tag{4.4.20}$$

for each $m \ge 1$. By independence and Lemma 3.2.4, there exists $K = K(x)$ such that

$$T_{j,n} \le \binom{n}{j} \frac{K^j}{n^j} \le \frac{K^j}{j!} \quad \text{for all } n, j \ge 1. \tag{4.4.21}$$

Consequently, $\lim_{j\to\infty} \limsup_n T_{j,n} = 0$.

Now fix $j \ge 1$. Let us bound the difference between $S_{j,n}$ and $T_{j,n}$. Let $\mathcal{A}_t$ defined in (4.4.6) be represented as $\mathcal{A}_t = \{e_t^1, e_t^2, \ldots, e_t^g\}$. Also note $e_t^1, e_t^2, \ldots, e_t^g \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. For $1 \le t_1 < t_2 < \ldots < t_j \le q$, define

$$\begin{aligned} & v_{2gj}(l) \\ &= \left(\cos\left(\frac{2\pi l e_{t_k}^1}{n}\right), \sin\left(\frac{2\pi l e_{t_k}^1}{n}\right), \cos\left(\frac{2\pi l e_{t_k}^2}{n}\right), \ldots, , \sin\left(\frac{2\pi l e_{t_k}^g}{n}\right);\ 1 \le k \le j \right). \end{aligned}$$

Note that $\{e_{t_k}^1, \ldots, e_{t_k}^g : 1 \le k \le j\}$ is a set of distinct integers in $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\rfloor\}$. Then,

$$\mathrm{P}\left((\beta_{\bar{x}+\sigma_n N, g}(t_i))^{1/2g} > c_q x + d_q, i = 1, \ldots, j \right) = \mathrm{P}\left(2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{x}_l + \sigma_n N_l) v_{2gj}(l) \in B_n^{(j)} \right)$$

where

$$B_n^{(j)} := \left\{ y \in \mathbb{R}^{2gj} : \prod_{l=1}^{g} (y_{2gt+2l-1}^2 + y_{2gt+2l}^2)^{1/2g} > 2^{1/2}(c_q x + d_q); 0 \le t < j \right\}.$$

By Corollary 4.4.7 and the fact $N_1^2 + N_2^2 \stackrel{\mathcal{D}}{=} 2E_1$, we deduce that uniformly over all the d-tuples $1 \le t_1 < t_2 < \ldots < t_j \le q$,

$$\begin{aligned} &\Big| \mathrm{P}\Big(2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{x}_l + \sigma_n N_l) v_{2gj}(l) \in B_n^{(j)}\Big) \\ &\quad - \mathrm{P}\Big((1+\sigma_n^2)^{1/2} \big(\prod_{i=1}^{g} E_{g(t_m-1)+i}\big)^{1/2g} > c_q x + d_q, 1 \le m \le j\Big)\Big| \\ \le\ & \epsilon_n \mathrm{P}\Big((1+\sigma_n^2)^{1/2} \big(E_{g(t_m-1)+1} E_{g(t_m-1)+2} \cdots E_{gt_m}\big)^{1/2g} > c_q x + d_q, 1 \le m \le j\Big) \\ &\quad + O(\exp(-n^{\eta})). \end{aligned}$$

Therefore, as $n \to \infty$,

$$|S_{j,n} - T_{j,n}| \le \epsilon_n T_{j,n} + \binom{n}{j} O(\exp(-n^{\eta})) \le \epsilon_n \frac{K^j}{j!} + o(1) \to 0, \qquad (4.4.22)$$

where $O(\cdot)$ and $o(\cdot)$ are uniform over j. Hence using (4.4.18), (4.4.19), (4.4.21) and (4.4.22), we have

$$\limsup_n |Q_1^{(n)} - Q_2^{(n)}| \le \limsup_n T_{2m+1,n} + \limsup_n T_{2m,n} \quad \text{for each } m \ge 1.$$

Letting $m \to \infty$, we conclude

$$\lim_{n\to\infty} [Q_1^{(n)} - Q_2^{(n)}] = 0.$$

This completes the proof of Step 1.

Proof of Step 2: Since by Corollary 3.2.3,

$$\max_{1\le j\le q} \big(E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj}\big)^{1/2g} = O_p((\ln n)^{1/2}) \ \text{ and } \ \sigma_n^2 = n^{-c},$$

it follows that

$$\frac{(1+\sigma_n^2)^{1/2} \max_{1\le j\le q} \big(E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj}\big)^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

and consequently,

$$\frac{\max_{1\le j\le q}(\beta_{\bar{x}+\sigma_n N,g}(j))^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

This completes the proof of Step 2.

Proof of Step 3: In view of (4.4.15), it suffices to show that

$$\max_{1\le j\le q}(\beta_{\bar{x}+\sigma_n N,g}(j))^{1/2g} - \max_{1\le j\le q}(\beta_{\bar{x},g}(j))^{1/2g} = o_p(c_q).$$

Note that

$$\beta_{\bar{x}+\sigma_n N,g}(j) = \prod_{k=1}^{g}\left|\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}(\bar{x}_l+\sigma_n N_l)\omega^{le_j^k}\right|^2 = \prod_{k=1}^{g}\left|\alpha_{j,k}\right|^2, \text{ say,}$$

and

$$\beta_{\bar{x},g}(j) = \prod_{k=1}^{g}\left|\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\bar{x}_l\omega^{le_j^k}\right|^2 = \prod_{k=1}^{g}\left|\gamma_{j,k}\right|^2, \text{ say.}$$

Now by the inequality

$$|\prod_{i=1}^{g}a_i - \prod_{i=1}^{g}b_i| \le \sum_{j=1}^{g}(\prod_{i=1}^{j-1}b_i)|a_j-b_j|(\prod_{i=j+1}^{g}a_i) \tag{4.4.23}$$

for non-negative numbers $\{a_i\}$ and $\{b_i\}$, we have

$$|\beta_{\bar{x}+\sigma_n N,g}(j) - \beta_{\bar{x},g}(j)| \le \sum_{k=1}^{g}\left|\gamma_{j,1}\right|^2\cdots\left|\gamma_{j,k-1}\right|^2\left||\alpha_{j,k}|^2 - |\gamma_{j,k}|^2\right|\left|\alpha_{j,k+1}\right|^2\cdots\left|\alpha_{j,g}\right|^2.$$

For any sequence of random variables $\{X_n\}_{n\ge 0}$, define

$$M_n(X) := \max_{1\le t\le n}\left|n^{-1/2}\sum_{l=0}^{n-1}X_l\omega^{tl}\right|.$$

As a trivial consequence of Theorem 2.1 of Davis and Mikosch (1999) [50], we have

$$M_n^2(\sigma_n N) = O_p(\sigma_n \ln n) \text{ and } M_n^2(\bar{x}+\sigma_n N) = O_p(\ln n).$$

Therefore $\left|\alpha_{j,k}\right| = O_p(\sqrt{\ln n})$. Now,

$$\left|\gamma_{j,k}\right| \le \left|\alpha_{j,k}\right| + \sigma_n\left|\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}N_l\omega^{le_j^k}\right|$$

and therefore $\left|\gamma_{j,k}\right| = (1+\sigma_n)O_p(\sqrt{\ln n}) = O_p(\sqrt{\ln n})$. So we have

$$\begin{aligned}
\left|\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j) - \max_{1\le j\le q}\beta_{\bar{x},g}(j)\right| &\le \max_{1\le j\le q}\left|\beta_{\bar{x}+\sigma_n N,g}(j) - \beta_{\bar{x},g}(j)\right| \\
&\le \max_{1\le j\le q}\sum_{k=1}^{g}(O_p(\ln n))^{g-1}\left|\alpha_{j,k}-\gamma_{j,k}\right|(|\alpha_{j,k}|+|\gamma_{j,k}|) \\
&\le O_p(\ln n)^{g-1}O_p(\sqrt{\ln n})\max_{1\le j\le q}\sum_{k=1}^{g}\left|\alpha_{j,k}-\gamma_{j,k}\right| \\
&\le O_p(\ln n)^{g-\frac{1}{2}}g\sigma_n M_n(N) \\
&\le o_p\left(n^{-c/4}(\ln n)^g\right).
\end{aligned}$$

Hence

$$\left|\max_{1\le j\le q}(\beta_{\bar{x}+\sigma_n N,g}(j))^{1/2g} - \max_{1\le j\le q}(\beta_{\bar{x},g}(j))^{1/2g}\right| \le \left|\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j) - \max_{1\le j\le q}\beta_{\bar{x},g}(j)\right|\frac{1}{\xi^{1/2g}}$$

where ξ lies between $\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j)$ and $\max_{1\le j\le q}\beta_{\bar{x},g}(j)$. We know that

$$\frac{\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j)}{(\ln n)^g}\xrightarrow{\mathcal{P}}1 \text{ and } \frac{\left|\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j)-\max_{1\le j\le q}\beta_{\bar{x},g}(j)\right|}{(\ln n)^g}\xrightarrow{\mathcal{P}}0.$$

Therefore

$$\begin{aligned}
\frac{\max_{1\le j\le q}\beta_{\bar{x},g}}{(\ln n)^g} &= \frac{\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j)}{(\ln n)^g} + \frac{\max_{1\le j\le q}\beta_{\bar{x},g}(j)-\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j)}{(\ln n)^g} \\
&\xrightarrow{\mathcal{P}} 1.
\end{aligned}$$

Hence

$$\frac{\xi}{(\ln n)^g}\xrightarrow{\mathcal{P}}1 \Rightarrow \frac{\xi^{1-1/2g}}{(\ln n)^{g(1-1/2g)}}\xrightarrow{\mathcal{P}}1 \Rightarrow \frac{1}{\xi^{1-1/2g}} = O_p\big((\ln n)^{\frac{1}{2}-g}\big).$$

Combining all these we have

$$\begin{aligned}
\left|\max_{1\le j\le q}\beta_{\bar{x}+\sigma_n N,g}(j)^{1/2g} - \max_{1\le j\le q}\beta_{\bar{x},g}(j)^{1/2g}\right| &\le o_p\big(n^{-c/4}(\ln n)^g\big) + O_p\big((\ln n)^{\frac{1}{2}-g}\big) \\
&\le o_p(c_q).
\end{aligned}$$

This completes the proof of Step 3 and hence completes the proof of Lemma 4.4.8. □

Proof of Lemma 4.4.9. First observe that if n_j is finite then the result holds trivially. If $n_{2j} = \frac{(k^j+1)(1+n^{-a})}{2j}$ then

$$\ln n_{2j} = j\ln k + \left(\frac{1}{n^a}+\frac{1}{n^{j/g}}\right)(1+\mathrm{o}(1)) - \ln 2j$$

for some $a > 0$ and since $k = (n-1)^{\frac{1}{g}}$ we have

$$\frac{c_{n/2g}}{c_{n_{2j}}} \to \frac{j}{g} \text{ as } n \to \infty.$$

Similarly we get for some $a_0 > 0$,

$$\ln\ln n_{2j} = \ln\ln n^{\frac{j}{g}} + \left(\frac{1}{n^{a_0}\ln n}\right)(1 + \mathrm{o}(1)) - \ln 2j.$$

Now observe that $\frac{d_{n/2g} - d_{n_{2j}}}{c_{n_{2j}}}$ can be broken into the following three parts say $J_i, i = 1, 2$ or 3.

$$J_1 = 2j^{1/2}(\ln n_{2j})^{1/2}\left[\frac{\ln C_g - \frac{g-1}{2}\ln g}{2g^{1/2}(\ln\frac{n}{2g})^{1/2}} - \frac{\ln C_j - \frac{j-1}{2}\ln j}{2j^{1/2}(\ln n_{2j})^{1/2}}\right] \to m_1 \quad \text{(finite constant)}.$$

$$J_2 = 2j^{1/2}(\ln n_{2j})^{1/2}\left[\left(\frac{\ln n/2g}{g}\right)^{1/2} - \left(\frac{\ln n_{2j}}{j}\right)^{1/2}\right] \to m_2 \quad \text{(finite constant)}.$$

$$\begin{aligned}
J_3 &= 2j^{1/2}(\ln n_{2j})^{1/2}\left[\frac{(g-1)\ln\ln n/2g}{4(g\ln n/2g)^{1/2}} - \frac{(j-1)\ln\ln n_{2j}}{4(j\ln n_{2j})^{1/2}}\right] \\
&= 2j^{1/2}(\ln n_{2j})^{1/2}\left[\frac{(g-1)\ln\ln n/2g}{4(g\ln n/2g)^{1/2}} - \frac{(j-1)\sqrt{g}\ln\ln n_{2j}}{4j(\ln n/2g)^{1/2}} + \mathrm{o}(1)\right] \\
&= \frac{j^{1/2}(\ln n_{2j})^{1/2}}{2(g\ln n/2g)^{1/2}}\left[(g-1)\ln\ln n/2g - \frac{(j-1)g}{j}\ln\ln n_{2j} + \mathrm{o}(1)\right] \\
&= \frac{j^{1/2}(\ln n_{2j})^{1/2}}{2(g\ln n/2g)^{1/2}}\left[\left((g-1) - \frac{g(j-1)}{j}\right)\ln\ln n/2g + \mathrm{o}(1)\right] \to \infty \text{ (since } g > j).
\end{aligned}$$

Hence Lemma 4.4.9 is proved. □

4.4.5 Remark on k circulants with $sn = k^g + 1$

Bose, Mitra and Sen (2008) [44] show existence of the limiting spectral distribution of the k circulant matrix with $k^g = sn - 1$ assuming that $s = \mathrm{o}(n^{p_1 - 1})$ where p_1 was the smallest prime factor of g. To derive the limit of the spectral radius, we need a slightly stronger assumption that $s = \mathrm{o}(n^{p_1 - 1 - \epsilon})$ for some $0 < \epsilon < p_1$ and $s > 1$. This is essential since $s = \mathrm{o}(n^{p_1-1})$ implies $\upsilon_{k,n}/n \to 0$ which is not enough to deal with the maximum. We need the stronger result $\frac{\upsilon_{k,n}}{n} = \mathrm{o}(n^{-a_1})$ for some $a_1 > 0$, so that these terms are negligible in the log scale that we have. Note that with the above conditions $s = \mathrm{o}(n^{p_1-1})$ and $\upsilon_{k,n} = \mathrm{O}(n^{-\epsilon/p_1})$.

Since $s > 1$ it easy to see from Lemma 3 in Bose, Mitra and Sen (2008) [44] that

$$\#X(\frac{2g}{b}) \leq \gcd(k^{2g/b} - 1, \frac{k^g + 1}{s}) \leq \gcd(k^{2g/b} - 1, k^g + 1). \tag{4.4.24}$$

Also observe that,

$$\#\left\{x : \; x \in \mathbb{Z}_n \text{ and } xk^{2g/b} = x \mod (\frac{k^g+1}{s})\right\} \geq \#Z_{n,b}. \tag{4.4.25}$$

From observations (4.4.24) and (4.4.25) it easily follows that Lemma 4.4.4(d) remains valid in this case. Further, for some $\alpha > 0$ we get that

$$1 \geq \frac{\#X(\frac{2g}{b})}{k^{g/b}+1} \geq 1 - k^{-g\alpha}(1 + \mathrm{o}(1)) = 1 - (sn)^{-\alpha}(1 + \mathrm{o}(1)) \geq 1 - n^{-\alpha}(1 + \mathrm{o}(1)).$$

Hence from the above discussions we have the following Theorem.

Theorem 4.4.10 (Bose, Hazra and Saha (2010) [37])**.** *Suppose $\{x_l\}_{l\geq 0}$ is an i.i.d. sequence of random variables with mean zero and variance 1 and $\mathrm{E}\,|x_l|^\gamma < \infty$ for some $\gamma > 2$. If $s \geq 1$ and $sn = k^g + 1$ where $s = o(n^{p_1 - 1 - \epsilon})$, $0 < \epsilon < p_1$, and p_1 is the smallest prime factor of g, then as $n \to \infty$,*

$$\frac{sp(n^{-1/2}A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

where $q = q(n) = \frac{n}{2g}$ and c_n and d_n can be taken as follows

$$d_n = \frac{\ln C_g - \frac{g-1}{2}\ln g}{2g^{1/2}(\ln n)^{1/2}} + \left(\frac{\ln n}{2g}\right)^{1/2}\left[1 + \frac{(g-1)\ln\ln n}{4\ln n}\right], \quad C_g = \frac{1}{\sqrt{g}}(2\pi)^{\frac{g-1}{2}}$$

and

$$c_n = \frac{1}{2g^{1/2}(\ln n)^{1/2}}.$$

4.5 Few remarks

In Theorems 4.2.3 and 4.3.5 we saw that the nature of the limiting distribution depends on whether the input sequence has mean zero or not. Results from Adamczak (2008) [1] and Bose and Sen (2007) [42] suggest that the same should happen for the Toeplitz matrix. It would be interesting to find out the limiting distribution of the spectral norm of the Toeplitz matrix in general. Since our main focus is circulant type matrices, we have not pursued it in this thesis. Incidentally, there does not seem to be an easy answer to this question.

Theorem 4.3.4 shows that the joint distribution of the maximum and minimum of the eigenvalues of SC_n behave like the maximum and minimum of i.i.d. standard normal entries. It follows that the distribution of the range of the spectrum is the convolution of two Gumbel distributions. So one can ask a very natural question: what happens in general to the spectral gaps. We shall address this question in Chapter 7.

Chapter 5

Spectral norm of circulant type matrices with heavy tail

In this chapter we focus on spectral norm of circulant, reverse circulant, symmetric circulant and Toeplitz matrices when the input sequence is heavy tailed.

There are a few results in the literature for matrices with heavy tailed entries. Soshnikov (2004) [117] shows the distributional convergence of the maximum eigenvalue of appropriately scaled Wigner matrix with heavy tailed entries $\{x_{ij}\}$ satisfying $\mathrm{P}(|x_{ij}| > x) = h(x)x^{-\alpha}$ where h is a slowly varying function at infinity (that is, $h(tx)/h(x) \to 1$ as $n \to \infty$) and $0 < \alpha < 2$. The limiting distribution is $\Phi_\alpha(x) = \exp(-x^{-\alpha})$. A similar result was proved for sample covariance matrices in Soshnikov (2006) [118] with Cauchy entries. These results on the Wigner and sample covariance matrices were extended in Auffinger, Ben Arous and Peche (2009) [6] to $2 \leq \alpha < 4$.

Here we focus on the circulant, reverse circulant, symmetric circulant and Toeplitz matrices when the input sequence is heavy tailed with tail index $0 < \alpha < 1$. In Section 5.1 we describe the input sequence of the matrices and define a few related notions. In Sections 5.2–5.3 we establish the distributional convergence of the spectral norm and hence of the spectral radius of the three circulant matrices. Though we are unable to obtain the exact limit in the Toeplitz case, we provide upper and lower bounds in Section 5.4. Our approach is to exploit the structure of the matrices and use existing methods on the study of the maximum of periodograms for heavy tailed sequences. This approach is totally different from the methods used to derive the results in Chapter 4 with light tailed entries.

The results of Bose, Hazra and Saha (2010) [36] are based on this chapter.

5.1 Input sequence of the matrices and scaling sequence

Let $\{Z_t,\ t \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables with common distribution F where F is in the *domain of attraction* of an α-*stable* random variable with $0 < \alpha < 1$. Thus, there exist $p, q \geq 0$ with $p + q = 1$ and a *slowly varying* function $L(x)$, such that

$$\lim_{x\to\infty} \frac{\mathrm{P}(Z_1 > x)}{\mathrm{P}(|Z_1| > x)} = p, \ \lim_{x\to\infty} \frac{\mathrm{P}(Z_1 \leq -x)}{\mathrm{P}(|Z_1| > x)} = q \text{ and } \mathrm{P}(|Z_1| > x) \approx x^{-\alpha} L(x) \text{ as } x \to \infty. \tag{5.1.1}$$

A random variable Y_α is said to have a stable distribution $S_\alpha(\sigma, \beta, \mu)$ if there are parameters $0 < \alpha \leq 2, \sigma \geq 0, -1 \leq \beta \leq 1$ and μ real such that its characteristic function has the form

$$\mathrm{E}[\exp(itY_\alpha)] = \begin{cases} \exp\{i\mu t - \sigma^\alpha |t|^\alpha (1 - i\beta \operatorname{sgn}(t) \tan(\pi\alpha/2))\}, & \text{if} \quad \alpha \neq 1, \\ \exp\{i\mu t - \sigma|t|(1 + (2i\beta/\pi) \operatorname{sgn}(t) \ln|t|)\}, & \text{if} \quad \alpha = 1. \end{cases}$$

If $\beta = \mu = 0$, then Y_α is symmetric α-stable ($S\alpha S$). For details on stable processes see Samorodnitsky and Taqqu (1994) [107].

In the description of our results, we shall need the following: let $\{\Gamma_j\}$, $\{U_j\}$ and $\{B_j\}$ be three independent sequences defined on the same probability space where $\{\Gamma_j\}$ is the arrival sequence of a unit rate poisson process on $\mathbb{R}$, U_j are i.i.d. $U(0,1)$ and B_j are i.i.d. satisfying

$$\mathrm{P}(B_1 = 1) = p \ \text{ and } \ \mathrm{P}(B_1 = -1) = q, \tag{5.1.2}$$

where p and q are as defined in (5.1.1). We also define

$$Y_\alpha = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \overset{\mathcal{D}}{\simeq} S_\alpha(C_\alpha^{-\frac{1}{\alpha}}, 1, 0) \text{ where } C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx \right)^{-1}. \tag{5.1.3}$$

For a non-decreasing function f on $\mathbb{R}$, let $f^{\leftarrow}(y) = \inf\{s : f(s) > y\}$. Then the *scaling sequence* $\{b_n\}$ is defined as

$$b_n = \left(\frac{1}{\mathrm{P}[|Z_1| > \cdot]} \right)^{\leftarrow} (n) \approx n^{1/\alpha} L_0(n) \quad \text{for some slowly varying function } L_0.$$

Define

$$\omega_k = \frac{2\pi k}{n} \text{ for } 0 \leq k \leq n.$$

5.2 Reverse circulant and circulant with heavy tailed entries

Recall the eigenvalues $\{\lambda_k,\ 0 \le k \le n-1\}$ of $b_n^{-1}RC_n$ are given by (see Section 1.2.3):

$$\begin{cases} \lambda_0 & = b_n^{-1}\sum_{t=0}^{n-1} Z_t \\ \lambda_{n/2} & = b_n^{-1}\sum_{t=0}^{n-1}(-1)^t Z_t, \quad \text{if } \ n \quad \text{is even} \\ \lambda_k = -\lambda_{n-k} & = \sqrt{I_n(\omega_k)},\ 1 \le k \le \lfloor \frac{n-1}{2} \rfloor, \end{cases} \tag{5.2.1}$$

where

$$I_n(\omega_k) = \frac{1}{b_n^2}|\sum_{t=0}^{n-1} Z_t e^{-it\omega_k}|^2.$$

The eigenvalues of $b_n^{-1}C_n$ are given by

$$\lambda_j = b_n^{-1}\sum_{t=1}^{n} Z_t e^{it\omega_j}, \quad 0 \le j \le n-1.$$

Note that $\{|\lambda_k|^2;\ 1 \le k < n/2\}$ is the periodogram of $\{Z_i\}$ at the frequencies $\{\omega_k;\ 1 \le k < n/2\}$. From the eigenvalue structure of C_n and RC_n, it is clear that $\|b_n^{-1}C_n\| = \|b_n^{-1}RC_n\|$ and therefore they have identical limiting behaviour which is stated in the following result.

Theorem 5.2.1 (Bose, Hazra and Saha (2010) [36]). *Consider $\{Z_t\}$ satisfying (5.1.1). Then for $\alpha \in (0,1)$, $\|b_n^{-1}C_n\| \xrightarrow{\mathcal{D}} Y_\alpha$ and $\|b_n^{-1}RC_n\| \xrightarrow{\mathcal{D}} Y_\alpha$, where Y_α is as in* (5.1.3).

The main idea for the proof of the above result is taken from Mikosch, Resnick and Samorodnitsky (2000) [93] who show weak convergence of maximum of the periodogram based on heavy tailed sequence for $\alpha < 1$. Let $\epsilon_x(\cdot)$ denote the point measure which gives unit mass to any set containing x and let $E = [0,\ 1] \times ([-\infty, \infty]\backslash\{0\})$. Let $M_p(E)$ be the set of point measures on E, topologized by vague convergence. The following convergence result follows from Proposition 3.21 of Resnick (1987) [103]:

$$N_n := \sum_{k=1}^{n} \epsilon_{(k/n, Z_k/b_n)} \xrightarrow{\mathcal{D}} N := \sum_{j=1}^{\infty} \epsilon_{(U_j, B_j\Gamma_j^{-1/\alpha})} \quad \text{in} \quad M_p(E). \tag{5.2.2}$$

Suppose f is a bounded continuous complex valued function defined on $\mathbb{R}$ and without loss of generality assume $|f(x)| \le 1$ for all $x \in \mathbb{R}$. Now pick $\eta > 0$ and define T_η : $M_p(E) \longrightarrow C[0,\infty)$ as follows:

$$(T_\eta m)(x) = \sum_j v_j 1_{\{|v_j|>\eta\}} f(2\pi x t_j)$$

if $m = \sum_j \epsilon_{(t_j,v_j)} \in M_p(E)$ and v_j's are finite. Elsewhere, set $(T_\eta m)(x) = 0$. The following Lemma was proved by Mikosch, Resnick and Samorodnitsky (2000) [93] (Lemma 2.3) using the function $f(x) = \exp(-ix)$. The same proof works in our case. For sake of completeness we give the details.

Lemma 5.2.2. $T_\eta : M_p(E) \longrightarrow C[0,\infty)$ *is continuous a.s. with respect to the distribution of* N.

Proof. It is enough to show that if $x_n \to x \geq 0$ and $m_n \xrightarrow{v} m$ in $M_p(E)$, where

$$m\{\partial([0,1] \times \{|v| \geq \eta\}) \cap [0,1] \times \{-\infty,\infty\}\} = 0,$$

then $(T_\eta m_n)(x_n) \to (T_\eta m)(x)$. To do this denote

$$m_n = \sum_j \epsilon_{\left(t_j^{(n)},v_j^{(n)}\right)} \ \text{ and } \ m = \sum_j \epsilon_{(t_j,v_j)}.$$

Consider the set

$$K_\eta := [0,1] \times \{v : |v| \geq \eta\}.$$

K_η is compact in E with $m(\partial K_\eta) = 0$. Since $m_n \xrightarrow{v} m$, we can find an n_0 such that for $n \geq n_0$

$$m_n(K_\eta) = m(K_\eta) =: l,$$

say and there is an enumeration of the points in K_η such that

$$\left(\left(t_k^{(n)}, v_k^{(n)}\right),\ 1 \leq k \leq l\right) \to \left((t_k, v_k),\ 1 \leq k \leq l\right).$$

Without loss of generality we can assume that for given $\xi > 0$

$$\sup_{n \geq n_0} |x_n| \vee \sup_{1 \leq k \leq l} |v_k^{(n)}| \leq \xi.$$

Therefore

$$\begin{aligned}
|(T_\eta m_n)(x_n) - (T_\eta m)(x)| &= \Big|\sum_{k=1}^{l} v_k^{(n)} f(-2\pi x_n t_k^{(n)}) - \sum_{k=1}^{l} v_k f(-2\pi x t_k)\Big| \\
&\leq \sum_{k=1}^{l} \big|v_k^{(n)} f(-2\pi x_n t_k^{(n)}) - v_k f(-2\pi x t_k)\big| \\
&\leq \sum_{k=1}^{l} \big|v_k^{(n)} - v_k\big| + \sum_{k=1}^{l} |v_k| \big|f(-2\pi x_n t_k^{(n)}) - f(-2\pi x t_k)\big|,
\end{aligned}$$

so

$$\lim_{n\to\infty} |(T_\eta m_n)(x_n) - (T_\eta m)(x)| = 0.$$

This completes the proof of the Lemma. □

The proof of the following lemma is similar to the proof of Proposition 2.2 of Mikosch, Resnick and Samorodnitsky (2000) [93]. We briefly sketch the proof in our case.

Lemma 5.2.3. *For $0 < \alpha < 1$, as $n \to \infty$ the following convergence holds in $C[0,\infty)$:*

$$J_{n,Z}(x/n) := \sum_{j=1}^{n} \frac{Z_j}{b_n} f(2\pi x j/n) \xrightarrow{\mathcal{D}} J_\infty(x) := \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} f(2\pi x U_j), \quad 0 \le x < \infty.$$

Proof. Applying Lemma 5.2.2 on (5.2.2) we have

$$\begin{aligned} J_{n,Z}^{(\eta)}(x/n) &:= \sum_{j=1}^{n} \frac{Z_j}{b_n} f(2\pi x j/n) 1_{\{|Z_j| > \eta b_n\}} \\ &\xrightarrow{\mathcal{D}} \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} f(2\pi x U_j) 1_{\{\Gamma_j^{-1/\alpha} > \eta\}} := J_\infty^{(\eta)}(x) \;\text{ in }\; C[0,\infty). \end{aligned}$$

Also, as $\eta \to 0$ by dominated convergence theorem we have

$$J_\infty^{(\eta)}(x) \xrightarrow{\mathcal{D}} J_\infty(x) := \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} f(2\pi x U_j).$$

So using Theorem 3.2 of Billingsley (1999) [29], the proof will be complete if for any $\epsilon > 0$,

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \mathrm{P}\big(\|J_{n,Z}^{(\eta)} - J_{n,Z}\| > \epsilon\big) = 0, \tag{5.2.3}$$

where $\|x(\cdot) - y(\cdot)\|_\infty$ is the metric distance in $C[0,\infty)$ given by

$$\|x(\cdot) - y(\cdot)\|_\infty = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\|x(\cdot) - y(\cdot)\|_n \wedge 1\right], \;\text{ where }\; \|x(\cdot) - y(\cdot)\|_n = \sup_{t\in[0,n]} |x(t) - y(t)|.$$

Now

$$\begin{aligned} \lim_{\eta\to 0} \limsup_{n\to\infty} \mathrm{P}\big(\|J_{n,Z}^{(\eta)} - J_{n,Z}\| > \epsilon\big) &\le \lim_{\eta\to 0} \limsup_{n\to\infty} \mathrm{P}\Big(\sum_{j=1}^{n} |\frac{Z_j}{b_n}| 1_{\{|Z_j| \le \eta b_n\}} > \epsilon\Big) \\ &\le \lim_{\eta\to 0} \limsup_{n\to\infty} n\epsilon^{-1}\, \mathrm{E}\left(|\frac{Z_1}{b_n}| 1_{\{|Z_j| \le \eta b_n\}}\right). \end{aligned}$$

By an application of Karamata's theorem (see Resnick (1987) [103] Exercise 0.4.2.8) we get that

$$n\,\mathrm{E}\Big(|\frac{Z_1}{b_n}|1_{\{|Z_j|\le\eta b_n\}}\Big) \;\approx\; \frac{\alpha}{1-\alpha}n\eta\mathrm{P}(|Z_1|>\eta b_n)\approx\frac{\alpha}{1-\alpha}\eta^{1-\alpha}.$$

and $\frac{\alpha}{1-\alpha}\eta^{1-\alpha}\to 0$ as $\eta\to 0$. This completes the proof of the lemma. □

Proof of Theorem 5.2.1. We use Lemma 5.2.2 and Lemma 5.2.3 with $f(x)=\exp(-ix)$. It is immediate that

$$b_n^{-1}\|C_n\|\le b_n^{-1}\sum_{t=1}^{n}|Z_t|. \tag{5.2.4}$$

It is well known (cf. Feller (1971) [59]) that

$$b_n^{-1}\sum_{t=1}^{n}|Z_t|\stackrel{\mathcal{D}}{\to}Y_\alpha=\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\ \stackrel{\mathcal{D}}{\simeq}\ S_\alpha(C_\alpha^{-1/\alpha},1,0). \tag{5.2.5}$$

Hence it remains to show that for $\gamma>0$,

$$\liminf_{n\to\infty}\mathrm{P}(b_n^{-1}\|C_n\|>\gamma)\ge\mathrm{P}(Y_\alpha>\gamma). \tag{5.2.6}$$

Now observe that for any integer K and sufficiently large n,

$$\mathrm{P}\big(\sup_{j=1,\ldots,\lfloor\frac{n}{2}\rfloor}|J_{n,Z}(j/n)|>\gamma\big)\ge\mathrm{P}\big(\sup_{j=1,\ldots,K}|J_{n,Z}(j/n)|>\gamma\big).$$

Now from Lemma 5.2.3 we have

$$\big(J_{n,Z}(j/n),1\le j\le K\big)\stackrel{\mathcal{D}}{\to}\big(J_\infty(j),1\le j\le K\big)$$

in $\mathbb{R}^k$. Hence

$$\sup_{j=1,\ldots,K}|J_{n,Z}(j/n)|\stackrel{\mathcal{D}}{\to}\sup_{j=1,\ldots,K}|J_\infty(j)|$$

and so letting $K\to\infty$,

$$\liminf_{n\to\infty}\mathrm{P}\big(\sup_{j=1,\ldots,\lfloor\frac{n}{2}\rfloor}|J_{n,Z}(j/n)|>\gamma\big)\ge\mathrm{P}\big(\sup_{j=1,\ldots,\infty}|J_\infty(j)|>\gamma\big).$$

Now the theorem follows from Lemma 5.2.4 given below. □

Lemma 5.2.4.

$$\sup_{j=1,\ldots,\infty}|J_\infty(j)|=\sup_{j=1,\ldots,\infty}\Big|\sum_{t=1}^{\infty}B_t\Gamma_t^{-1/\alpha}\exp(-2\pi ijU_t)\Big|=Y_\alpha\quad a.s.$$

Proof. Define

$$\Omega_0 = \Big\{\omega \in \Omega : \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha}(\omega) < \infty \text{ and for all } m \geq 1, \\ (U_1(\omega), \ldots, U_m(\omega)) \text{ are rationally independent}\Big\}.$$

Then $\mathrm{P}(\Omega_0) = 1$. Let $\overline{x}$ denote the fractional part of x. For any $\omega \in \Omega_0$, by Weyl (1916) [129],

$$\big(\overline{nU_1(\omega)}, \ldots, \overline{nU_m(\omega)}\big),\ n \in \mathbb{N}$$

is dense in $[0,1]^m$. Fix any $\omega \in \Omega_0$ and $\epsilon > 0$. Then there exist an $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty} \Gamma_j^{-1/\alpha}(\omega) < \epsilon$ and from Weyl's result there exist an $N_0 \in \mathbb{N}$ such that

$$\mathrm{Real}\big(B_j \exp(-2\pi i N_0 U_j)\big) \geq 1 - \frac{\epsilon}{N\Gamma_j^{-1/\alpha}}, \quad j = 1, \ldots, N.$$

Then we have

$$\begin{aligned}
& \sup_{j=1,\ldots,\infty} \big|\sum_{t=1}^{\infty} B_t \Gamma_t^{-1/\alpha} \exp(-2\pi i j U_t)\big| \\
\geq\ & \sup_{j=1,\ldots,\infty} \big|\sum_{t=1}^{N} B_t \Gamma_t^{-1/\alpha} \exp(-2\pi i j U_t)\big| - \sum_{t=N+1}^{\infty} \Gamma_t^{-1/\alpha} \\
\geq\ & \big|\sum_{t=1}^{N} B_t \Gamma_t^{-1/\alpha} \exp(-2\pi i N_0 U_t)\big| - \epsilon \\
\geq\ & \mathrm{Real}\big(\sum_{t=1}^{N} B_t \Gamma_t^{-1/\alpha} \exp(-2\pi i N_0 U_t)\big) - \epsilon \\
\geq\ & \sum_{t=1}^{N} \big(1 - \frac{\epsilon}{N\Gamma_t^{-1/\alpha}}\big)\Gamma_t^{-1/\alpha} - \epsilon = \sum_{t=1}^{N} \Gamma_t^{-1/\alpha} - 2\epsilon.
\end{aligned}$$

Letting first $N \to \infty$ and then $\epsilon \to 0$, we get $\sup_{j=1,\ldots,\infty} |J_\infty(j)| \geq Y_\alpha$. Trivially $\sup_{j=1,\ldots,\infty} |J_\infty(j)| \leq Y_\alpha$. This completes the proof. □

5.3 Symmetric circulant with heavy tailed entries

The eigenvalues $\{\lambda_k,\ 0 \leq k \leq n-1\}$ of $b_n^{-1} SC_n$ are given by (see Section 1.2.2):
(i) for n odd:

$$\begin{cases} \lambda_0 &= b_n^{-1}\big[Z_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} Z_j\big] \\ \lambda_k &= b_n^{-1}\big[Z_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} Z_j \cos(\omega_k j)\big], \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \end{cases} \tag{5.3.1}$$

(ii) for n even:

$$\begin{cases} \lambda_0 &= b_n^{-1}\big[Z_0 + 2\sum_{j=1}^{\frac{n}{2}-1} Z_j + Z_{n/2}\big] \\ \lambda_k &= b_n^{-1}\big[Z_0 + 2\sum_{j=1}^{\frac{n}{2}-1} Z_j \cos(\omega_k j) + (-1)^k Z_{n/2}\big], \quad 1 \le k \le \frac{n}{2} \end{cases} \tag{5.3.2}$$

with $\lambda_{n-k} = \lambda_k$ in both cases.

Theorem 5.3.1 (Bose, Hazra and Saha (2010) [36]). *Assume that the input sequence is i.i.d. $\{Z_t\}$ satisfying (5.1.1). Then for $\alpha \in (0,1)$, $\|b_n^{-1} SC_n\| \xrightarrow{\mathcal{D}} 2^{1-1/\alpha} Y_\alpha$, where Y_α is as in* (5.1.3).

Proof. The proof is similar to the proof of Theorem 5.2.1. We provide the proof for n odd, and for n even, the changes needed are minor. Define

$$J_{n,Z}(x) := 2b_n^{-1} \sum_{t=1}^{q} Z_t \cos(2\pi x t) \quad \text{and} \quad M_{n,Z} := \max_{0 \le k \le q} \big|J_{n,Z}(k/n)\big|, \tag{5.3.3}$$

where $q = q(n) = \lfloor \frac{n}{2} \rfloor$. Since $\big|\|b_n^{-1} SC_n\| - M_{n,Z}\big| \to 0$ almost surely, it is enough to show $M_{n,Z} \xrightarrow{\mathcal{D}} 2^{1-1/\alpha} Y_\alpha$. Note that (5.2.2) holds with [0, 1] replaced by [0, 1/2], and letting $N_n = \sum_{k=1}^{q} \epsilon_{(k/n, Z_k/b_q)}$, $N = \sum_{j=1}^{\infty} \epsilon_{(U_j, B_j \Gamma_j^{-1/\alpha})}$ and U_j to be i.i.d. $U[0, 1/2]$. Now following the argument given in Lemma 5.2.2, Lemma 5.2.3 and taking $f(x) = \cos x$ it is easy to establish that

$$J_{n,Z}(x/n) = 2b_n^{-1} \sum_{k=1}^{q} Z_k \cos \frac{2\pi k x}{n} \xrightarrow{\mathcal{D}} 2^{1-1/\alpha} \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} \cos(2\pi x U_j) := J_\infty(x). \tag{5.3.4}$$

It is obvious that

$$M_{n,Z} \le 2b_n^{-1} \sum_{t=1}^{q} |Z_t| \xrightarrow{\mathcal{D}} 2^{1-1/\alpha} \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} = 2^{1-1/\alpha} Y_\alpha.$$

It remains to show that for $\eta > 0$,

$$\liminf_{n\to\infty} \mathrm{P}(M_{n,Z} > \eta) \ge \mathrm{P}(2^{1-1/\alpha} Y_\alpha > \eta).$$

Now following the arguments given to prove (5.2.6), we can establish this relation. This completes the proof of the theorem. □

Remark 5.3.2. *(i) Theorem 5.2.1 and 5.3.1 are rather easy to derive when $p = 1$, that is, when the left tail is negligible compared to the right tail. Let us consider $\|b_n^{-1} RC_n\|$*

and note from the eigenvalue structure that,

$$\|b_n^{-1}RC_n\| \leq b_n^{-1}\sum_{t=1}^{n}|Z_t|.$$

For the lower bound note that

$$P(\|b_n^{-1}RC_n\| > x) \geq P(\lambda_0 > x) = P(b_n^{-1}\sum_{t=1}^{n} Z_t > x).$$

Now since $P(|Z_1| > x) \approx P(Z_1 > x)$ as $x \to \infty$, the upper and lower bound converge with the same scaling constant and hence Theorem 5.2.1 holds. The details on these convergence can be found in Chapter 1 of Samorodnitsky and Taqqu (1994) [107]. Similar conclusion can be drawn for the symmetric circulant matrices too when $p = 1$.

(ii) When the input sequence $\{Z_i\}$ are i.i.d. nonnegative and satisfies (5.1.1) with $\alpha \in (1, 2)$ then from above it is easy to derive the distributional behaviour of the spectral norm. In particular if $k_j = \frac{\alpha}{\alpha-1}\left(j^{\frac{\alpha-1}{\alpha}} - (j-1)^{\frac{\alpha-1}{\alpha}}\right)$ and $\widetilde{Y_\alpha} = \sum_{j=1}^{\infty}(\Gamma_j - k_j) \overset{\mathcal{D}}{\sim} S_\alpha(C_\alpha^{-\frac{1}{\alpha}}, 1, 0)$ then,

$$P\left(\frac{\|RC_n\| - n\,\mathrm{E}[Z_1]}{b_n} > x\right) \to P(\widetilde{Y_\alpha} > x) \quad \textit{as } n \to \infty,$$

and

$$P\left(\frac{\|SC_n\| - n\,\mathrm{E}[Z_1]}{b_n} > x\right) \to P(2^{1-1/\alpha}\widetilde{Y_\alpha} > x) \quad \textit{as } n \to \infty.$$

When $\alpha = 1$, and $\{Z_i\}$ are non negative

$$\mathrm{P}\left(\frac{\|RC_n\| - nb_n\int_0^\infty \sin(\frac{x}{b_n})P(Z_1 \in dx)}{b_n} > x\right) \to P(\widetilde{\widetilde{Y_\alpha}} > x),$$

where $\widetilde{\widetilde{Y_\alpha}}$ is a $S_1(2/\pi, 1, 0)$ random variable. Similar results hold for symmetric circulant matrices.

5.4 Toeplitz matrix with heavy tailed entries

Resolving the question of the exact limit of the Toeplitz spectral norm seems to be very difficult. Here we provide a good upper and lower bound in the distribution sense.

Theorem 5.4.1 (Bose, Hazra and Saha (2010) [36]). *Suppose that the input sequence* *is i.i.d.* $\{Z_t\}$ *satisfying (5.1.1). Then for* $\gamma > 0$,

$$\begin{aligned} P\big(2\sum_{j=1}^{\infty}(1-U_j)\Gamma_j^{-1/\alpha} > \gamma\big) &\leq \liminf_n P\big(b_n^{-1}\|T_n\| > \gamma\big) \\ &\leq \limsup_n P\big(b_n^{-1}\|T_n\| > \gamma\big) \leq P\big(2\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha} > \gamma\big). \end{aligned}$$

Proof. Following Meckes (2007) [92], T_n is a sub-matrix of the *infinite Laurent matrix*

$$L_n = \big[Z_{|j-k|}\mathbb{I}_{|j-k|\leq n-1}\big]_{j,k\in\mathbb{Z}}$$

so $\|T_n\| \leq \|L_n\|$, where $\|L_n\|$ denotes the operator norm of L_n acting in the standard way on $l_2(\mathbb{Z})$. If we use the Fourier basis to identify $l_2(\mathbb{Z})$ with $L_2[0,1]$, it turns out that L_n corresponds to a multiplication operator, with the multiplier

$$g(x) = \sum_{j=-(n-1)}^{n-1} Z_{|j|}e^{2\pi ijx} = Z_0 + 2\sum_{j=1}^{n-1}\cos(2\pi jx)Z_j.$$

Therefore

$$\|T_n\| \leq \|L_n\| = \|g\|_\infty = \sup_{0\leq x\leq 1}|g(x)|.$$

Hence as $n \to \infty$,

$$b_n^{-1}\|T_n\| \leq b_n^{-1}\big[|Z_0| + 2\sum_{j=0}^{n-1}|Z_j|\big] \xrightarrow{\mathcal{D}} 2\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha}$$

and we have for $\gamma > 0$

$$\limsup_n \mathrm{P}\big(b_n^{-1}\|T_n\| > \gamma\big) \leq \mathrm{P}\big(2\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha} > \gamma\big).$$

By another argument of Meckes (2007) [92], we get the following estimate

$$\|T_n\| = \sup_{v\in\mathbb{C}^n\setminus\{0\}}\frac{\langle T_n v, v\rangle}{\langle v, v\rangle} \geq \sup_{0\leq x\leq 1}\frac{1}{n}|\langle T_n v_x, v_x\rangle|,$$

where $v_x \in \mathbb{C}^n$ is defined as $(v_x)_j = e^{2\pi ixj}$ for $j = 1, 2, \ldots, n$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{C}^n$. Therefore

$$\begin{aligned}
\|T_n\| &\geq \frac{1}{n}\sup_{0\le x\le 1}\Big|\sum_{j,k=1}^{n} Z_{|j-k|}e^{2\pi i(j-k)x}\Big| \\
&= \frac{1}{n}\sup_{0\le x\le 1}\Big|\sum_{j=-(n-1)}^{n-1}(n-|j|)Z_{|j|}e^{2\pi ijx}\Big| \\
&= \sup_{0\le x\le 1}\Big|Z_0+2\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi jx)\Big|.
\end{aligned}$$

Now

$$\begin{aligned}
\liminf_n \mathrm{P}\big(b_n^{-1}\|T_n\|>\gamma\big) &\geq \liminf_n \mathrm{P}\big(b_n^{-1}\sup_{0\le x\le 1}\Big|Z_0+2\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi jx)\Big|>\gamma\big) \\
&= \lim_n \mathrm{P}\big(b_n^{-1}\sup_{0\le x\le 1}\Big|2\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi jx)\Big|>\gamma\big).
\end{aligned}$$

To find the limit in the last expression, pick $\eta>0$ and define $T_\eta : M_p(E)\longrightarrow C[0,\infty)$, as follows:

$$(T_\eta m)(x)=\begin{cases}\sum_j(1-t_j)v_j\cos(2\pi xt_j)1(|v_j|>\eta) & \text{if } m=\sum_j\epsilon_{(t_j,v_j)},\ \text{all } v_j's \text{ are finite}\\ 0 & \text{otherwise.}\end{cases}$$

Following the argument given in Lemma 5.2.2, it is easy to see that T_η is continuous a.s. with respect to the distribution of N and then using an argument from Lemma 5.2.3, we can show that for fixed x

$$2b_n^{-1}\sum_{j=1}^{n-1}(1-j/n)Z_j\cos(2\pi jx/n)\xrightarrow{\mathcal{D}}2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j). \tag{5.4.1}$$

Now for any fixed T where $n>T>0$, using (5.4.1),

$$\begin{aligned}
\sup_{0\le x\le 1}\Big|2b_n^{-1}\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi jx)\Big| &= \sup_{0\le x\le n}\Big|2b_n^{-1}\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos\frac{2\pi jx}{n}\Big| \\
&\geq \sup_{0\le x\le T}\Big|2b_n^{-1}\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos\frac{2\pi jx}{n}\Big| \\
&\xrightarrow{\mathcal{D}} \sup_{0\le x\le T}\Big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\Big|,
\end{aligned}$$

and hence

$$\liminf_n \mathrm{P}\big(b_n^{-1}\|T_n\| > \gamma\big) \quad \geq \quad \mathrm{P}\big(\sup_{0\leq x\leq T}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big| > \gamma\big).$$

Since this is true for any T, we obtain

$$\liminf_n \mathrm{P}\big(b_n^{-1}\|T_n\| > \gamma\big) \geq \mathrm{P}\big(\sup_{0\leq x<\infty}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big| > \gamma\big).$$

To identify the distribution of the random variable appearing in the right side, we follow the proof of Lemma 5.2.4. Here we use the fact $\big\{(\overline{xU_1(\omega)},\ldots,\overline{xU_m(\omega)}), x\geq 0\big\}$ is dense in $[0,1]^m$ and we get

$$\sup_{0\leq x<\infty}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big| = \sum_{j=1}^{\infty}(1-U_j)\Gamma_j^{-1/\alpha} \quad a.s.$$

This completes the proof. □

Remark 5.4.2. *The assumption of $\alpha < 1$ is crucially used only in the lower bound argument. It is clear from the above proof that the upper bound can be derived when $\alpha \in (1,2)$. Indeed, it easily follows that if $\alpha \in (1,2)$ then*

$$\limsup_{n\to\infty} P\left(\frac{\|T_n\| - 2n\,\mathrm{E}[|Z_1|]}{b_n} > x\right) \leq P(2\widetilde{Y_\alpha} > x),$$

where $\widetilde{Y_\alpha}$ is as in Remark 5.3.2 (ii).

Remark 5.4.3. *The case when $\alpha \in [1,2)$ and $p \neq 1$ and $\{Z_i\}$ are not necessarily nonnegative appears to be nontrivial. In the reverse circulant case we saw that the eigenvalue structure is similar to the square root of the periodogram and the maximum of the periodogram is not tight with the scaling $b_n^{1/\alpha}$ when $\alpha \geq 1$ (even with input sequence as i.i.d. $S\alpha S$ random variables). Instead it is tight with a different scaling (see Mikosch, Resnick and Samorodnitsky (2000) [93], Section 3 for details).*

Chapter 6

Distribution of maximum of scaled eigenvalues: dependent input

In this chapter we try to generalize the results on spectral norm of Chapters 4 and 5 when the input sequence is dependent. We take $\{x_n\}$ to be an infinite order moving average process, $x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}$, where $\{a_n; n \in \mathbb{Z}\}$ are non-random with $\sum_n |a_n| < \infty$, and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. It seems to be a nontrivial problem to derive properties of the spectral norm in this case. This is due to unequal variance of the eigenvalues. So, we resort to scaling each eigenvalue by an appropriate quantity and then consider distributional convergence of maximum of these scaled eigenvalues of different circulant matrices. Now we give an outline of this chapter.

In Section 6.1 we consider infinite order moving average process with light tail entries, that is $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. with $E(\epsilon_i) = 0$ and $V(\epsilon_i) = 1$. We scale the eigenvalues by the spectral density at the appropriate ordinate as described below and then consider their maximum. This scaling has the effect of (approximately) equalizing the variance of the eigenvalues. Similar scaling has been used in the study of periodograms (see Walker (1965) [128], Davis and Mikosch (1999) [50], Lin and Liu (2009) [84]).

For any circulant type matrix A_n we define

$$M(A_n, f) = \max_{1 \le k \le n} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$$

where f is the spectral density corresponding to $\{x_n\}$, $\{\lambda_k\}$ are eigenvalues of A_n and $\omega_k = \frac{2\pi k}{n}$ is the Fourier frequency. This rescaling by the spectral density makes the eigenvalues approximately of same variance and that makes it relatively easy to handle their maxima. We show in Theorem 6.1.3 and Theorem 6.1.16 that $M(n^{-1/2} RC_n, f)$

and $M(n^{-1/2}A_{k,n}, f)$ converge to the Gumbel distribution after proper centering and scaling. For the symmetric circulant, in Theorem 6.1.7 we show that $M(n^{-1/2}SC_n, f)$ converges to the same limit as above when we impose the extra condition $a_j = a_{-j}$ for all j. Without this condition, it is difficult to conclude the distributional convergence even if ϵ_i's are i.i.d $N(0,1)$. The convergence in probability of $M(n^{-1/2}SC_n, f)$ is discussed in Lemma 6.1.9 and Theorem 6.1.12.

In Section 6.2 we consider the infinite order moving average process based on heavy tail entries. Here also we resort to scaling each eigenvalue by the power transfer function f (defined in Section 6.2) at the appropriate ordinate and then consider their maximum. We show the distributional convergence of $M(A_n, f)$ for the three circulant matrices. These follow easily from the results on the spectral norm of their i.i.d. counterparts.

Some of the results of Bose, Hazra and Saha (2009, 2010) [34, 36] are based on this chapter.

6.1 Dependent input with light tail

Now let $\{x_n; n \geq 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i} \tag{6.1.1}$$

where $\{a_n; n \in \mathbb{Z}\}$ are non-random and $\sum_n |a_n| < \infty$, and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. random variables. Let $f(\omega)$, $\omega \in [0, 2\pi]$ be the spectral density of $\{x_n\}$. Note that if $\{x_n\}$ is i.i.d. with mean 0 and variance σ^2, then $f \equiv \frac{\sigma^2}{2\pi}$. We make the following assumption.

Assumption 6.1.1. *$\{\epsilon_i,\ i \in \mathbb{Z}\}$ are i.i.d. with $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$, $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi].$$

6.1.1 Reverse circulant and circulant: dependent input with light tail

Define $M(\cdot, f)$ for the reverse circulant matrix as follows:

$$M(n^{-1/2}RC_n, f) = \max_{1 \leq k < \frac{n}{2}} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$$

where λ_k are the eigenvalues of $n^{-1/2}RC_n$ (see Section 1.2.3). Note that $M(n^{-1/2}C_n, f)$ for the circulant matrix defined similarly satisfies $M(n^{-1/2}RC_n, f) = M(n^{-1/2}C_n, f)$. Note that λ_0 is included in the definition of $M(\cdot, f)$. When $E(\epsilon_0) = \mu = 0$, this is immaterial. However if $\mu \neq 0$, see Remark 6.1.4.

The following lemma is an approximation result which is a stronger version of Theorem 3 of Walker (1965) [128]. We will use this result in Theorem 6.1.3 but not in full force. We will again use it in Section 6.1.3.

Lemma 6.1.2. *Let $\{x_n\}$ be the two sided moving average process defined in (6.1.1) and which satisfies Assumption 6.1.1. Then*

$$\max_{1\le k<\frac{n}{2}} \Big|\frac{I_{x,n}(\omega_k)}{2\pi f(\omega_k)} - I_{\epsilon,n}(\omega_k)\Big| = o_p(n^{-1/4}\sqrt{\ln n}),$$

where

$$I_{x,n}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{it\omega_k}|^2,\ I_{\epsilon,n}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} \epsilon_t e^{it\omega_k}|^2 \text{ and } \omega_k = \frac{2\pi k}{n}.$$

Proof. First observe that $\min_{\omega\in[0,2\pi]} f(\omega) > \alpha > 0$. Now for any r,

$$|\sum_{t=1}^{r} \epsilon_t e^{i\omega t}|^2 = \sum_{s=-r}^{r} e^{i\omega s} \sum_{t=1}^{r-|s|} \epsilon_t\epsilon_{t+|s|} \le \sum_{s=-r}^{r} |\sum_{t=1}^{r-|s|} \epsilon_t\epsilon_{t+|s|}|.$$

Hence

$$\begin{aligned}
\mathrm{E}\big[\max_{0\le\omega\le\pi} |\sum_{t=1}^{r} \epsilon_t e^{i\omega t}|^2\big] &\le \mathrm{E}\big(\sum_{t=1}^{r}\epsilon_t^2\big) + 2\sum_{s=1}^{r-1}\big[\mathrm{E}(\sum_{t=1}^{r-s}\epsilon_t\epsilon_{t+s})^2\big]^{1/2} \\
&= r + 2\sum_{s=1}^{r-1}(r-s)^{1/2} \le r + 2\int_1^r x^{1/2}dx \\
&\le Kr^{3/2} \qquad (6.1.2)
\end{aligned}$$

where K is a constant independent of r. So

$$\mathrm{E}\big[\max_{0\le\omega\le\pi} |\sum_{t=1}^{r} \epsilon_t e^{i\omega t}|\big] \le K^{1/2} r^{3/4}. \qquad (6.1.3)$$

Note that (6.1.3) still holds if the limits of summation with respect to t are replaced by $1+p$ and $r+p$, where p is an arbitrary (positive or negative) integer. Let

$$J_{x,n} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1} x_t e^{i\omega t},\quad J_{\epsilon,n} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1} \epsilon_t e^{i\omega t},\ R_n(\omega) = J_{x,n}(\omega) - A(\omega)J_{\epsilon,n}(\omega),$$

$$T_n(\omega) = I_{x,n}(\omega) - |A(\omega)|^2 I_{\epsilon,n}(\omega) \text{ and } A(\omega) = \sum_{j=-\infty}^{\infty} a_j e^{i\omega j}.$$

Then it is easy to see that $2\pi f(\omega) = |A(\omega)|^2$ and

$$\begin{aligned} T_n(\omega) &= |R_n(\omega) + A(\omega)J_{\epsilon,n}(\omega)|^2 - |A(\omega)|^2 I_{\epsilon,n}(\omega) \\ &= R_n(\omega)\bar{A}(\omega)\bar{J}_{\epsilon,n}(\omega) + \bar{R}_n(\omega)A(\omega)J_{\epsilon,n}(\omega) + |R_n(\omega)|^2. \end{aligned}$$

Now

$$\begin{aligned} R_n(\omega) &= J_{x,n}(\omega) - A(\omega)J_{\epsilon,n}(\omega) \\ &= \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\Big(\sum_{t=-\infty}^{\infty} a_t\epsilon_{j-t}\Big)e^{i\omega j} - \frac{1}{\sqrt{n}}\sum_{t=-\infty}^{\infty} a_t e^{i\omega t}\sum_{j=0}^{n-1}\epsilon_j e^{i\omega j} \\ &= \frac{1}{\sqrt{n}}\sum_{t=-\infty}^{\infty} a_t e^{i\omega t}\Big[\sum_{j=0}^{n-1}\epsilon_{j-t}e^{i\omega(j-t)} - \sum_{j=0}^{n-1}\epsilon_j e^{i\omega j}\Big] \\ &= \frac{1}{\sqrt{n}}\sum_{t=-\infty}^{\infty} a_t e^{i\omega t}\Big[\Big(\sum_{j=-t}^{n-1-t} - \sum_{j=0}^{n-1}\Big)\epsilon_j e^{i\omega j}\Big] \\ &= \frac{1}{\sqrt{n}}\sum_{t=-\infty}^{\infty} a_t e^{i\omega t} Z_{n,t}(\omega), \text{ say.} \end{aligned}$$

Observe that

$$|Z_{n,t}(\omega)| \leq \begin{cases} |\sum_{j=-t}^{-1}\epsilon_l e^{i\omega j}| + |\sum_{j=n-t}^{n-1}\epsilon_j e^{i\omega j}|, & 1 \leq t < n \\ |\sum_{j=n}^{n-1-t}\epsilon_j e^{i\omega j}| + |\sum_{l=0}^{-t-1}\epsilon_j e^{i\omega j}|, & -n < t \leq -1 \\ |\sum_{-t}^{n-1-t}\epsilon_j e^{i\omega j}| + |\sum_{0}^{n-1}\epsilon_j e^{i\omega j}|, & |t| \geq n \end{cases} \tag{6.1.4}$$

and $|Z_{n,0}(\omega)| = 0$. Therefore using (6.1.3) and (6.1.4) we get

$$\begin{aligned} \mathrm{E}(\max_{0\leq\omega\leq\pi}|R_n(\omega)|) &\leq \frac{2K^{\frac{1}{2}}}{\sqrt{n}}\Big[\sum_{t=1}^{n-1}|a_t|t^{\frac{3}{4}} + \sum_{t=-n+1}^{-1}|a_t||t|^{\frac{3}{4}} + \sum_{t=n}^{\infty}|a_t|n^{\frac{3}{4}} + \sum_{t=-\infty}^{-n}|a_t|n^{\frac{3}{4}}\Big] \\ &= \frac{2K^{\frac{1}{2}}}{\sqrt{n}}\Big[\sum_{1\leq|t|\leq n-1}|a_t||t|^{\frac{3}{4}} + \sum_{|t|\geq n}|a_t|n^{\frac{3}{4}}\Big] \\ &< 2K^{\frac{1}{2}}n^{-\frac{1}{4}}\Big[\sum_{1\leq|t|\leq n-1}|a_t||t|^{\frac{1}{2}}(|t|/n)^{\frac{1}{4}} + \sum_{|t|\geq n}|a_t||t|^{\frac{1}{2}}\Big] \\ &= o(n^{-1/4}) \end{aligned} \tag{6.1.5}$$

since the second sum goes to zero as $n \to \infty$ and the first sum is not greater than

$$\sum_{k(n)<|t|\leq n-1}|t|^{1/2}|a_t| + \{k(n)/n)\}^{1/4}\sum_{1\leq|j|\leq k(n)}|j|^{1/2}|a_j|,$$

where $k(n)$ is such that $\lim_{n\to\infty}\{k(n)/n\} = 0$ and $\lim_{n\to\infty} k(n) = \infty$.

Also it is known from Davis and Mikosch (1999) [50] that under the conditions on $\{\epsilon_t\}$,

$$\max_{1\le k\le n} |I_{\epsilon,n}(\omega_k)| = O_p(\ln n)$$

and hence

$$\max_{1\le k\le n} |J_{\epsilon,n}(\omega_k)| = O_p(\sqrt{\ln n}). \tag{6.1.6}$$

Finally using (6.1.5) and (6.1.6)

$$\begin{aligned}
\max_{1\le k<\frac{n}{2}} \Big|\frac{I_{x,n}(\omega_k)}{2\pi f(\omega_k)} - I_{\epsilon,n}(\omega_k)\Big| &\le \frac{1}{2\pi\alpha}\max_{1\le k<\frac{n}{2}} |T_n(\omega_k)| \\
&\le \frac{1}{2\pi\alpha}\big[2\max_{0\le\omega\le\pi}|R_n(\omega)|\max_{0\le\omega\le\pi}|A(\omega)|\max_{1\le\omega_k<\frac{n}{2}}|J_{\epsilon,n}(\omega_k)| \\
&\quad +\{\max_{0\le\omega\le\pi}|R_n(\omega)|\}^2\big] \\
&= o_p(n^{-1/4}\sqrt{\ln n}). \qquad (6.1.7)
\end{aligned}$$

□

Theorem 6.1.3 (Bose, Hazra and Saha (2009) [34])**.** *Let $\{x_n\}$ be the two sided moving average process defined in (6.1.1) satisfying Assumption 6.1.1. Then*

$$\frac{M(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where $q = q(n) = \lfloor \frac{n-1}{2} \rfloor$, $d_q = \sqrt{\ln q}$ and $c_q = \frac{1}{2\sqrt{\ln q}}$. Same result continues to hold for $M(n^{-1/2}C_n, f)$.

Proof. From Lemma 6.1.2, we have

$$\max_{1\le k<\frac{n}{2}} \Big|\frac{I_{x,n}(\omega_k)}{2\pi f(\omega_k)} - I_{\epsilon,n}(\omega_k)\Big| = o_p(1) \tag{6.1.8}$$

where

$$I_{x,n}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 \text{ and } I_{\epsilon,n}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} \epsilon_t e^{-it\omega_k}|^2.$$

Combining this with Theorem 2.1 of Davis and Mikosch (1999) [50] we have

$$\max_{1\le k<\frac{n}{2}} \frac{I_{x,n}(\omega_k)}{2\pi f(\omega_k)} - \ln q \xrightarrow{\mathcal{D}} \Lambda.$$

Now proceeding as in the proof of Theorem 4.2.3, we can conclude that

$$\frac{M(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

□

Remark 6.1.4. *If we include λ_0 in the maximum and define $\overline{M}(n^{-1/2}RC_n, f) = \max_{0 \leq k < n/2} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$ then different limits may appear depending on mean μ' of the process $\{x_n\}$. If mean μ of ϵ_0 is 0 then by Theorem 7.1.2 of Brockwell and Davis (2002) [46] it follows that $\frac{\lambda_0}{\sqrt{2\pi f(0)}} \xrightarrow{\mathcal{D}} N(0,1)$. So by arguments similar to Theorem 4.2.3 we have*

$$\frac{\overline{M}(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

When $\mu \neq 0$ then,

$$\overline{M}(n^{-1/2}RC_n, f) - |\mu|\sqrt{n} \xrightarrow{\mathcal{D}} N(0,1).$$

Remark 6.1.5. *It appears that by the results of Lin and Liu (2009) [84], if $\{x_n\}$ is the two sided moving average process (6.1.1) where $E(\epsilon_0) = 0$, $E(\epsilon_0^2) = 1$, $E[\epsilon_0^2 \mathbb{I}\{|\epsilon| \geq n\}] = o(1/\ln n)$ and*

$$\sum_{|j| \geq n} |a_j| = o(1/\ln n) \text{ and } \min_{\omega \in [0,2\pi]} f(\omega) > 0,$$

then also

$$\frac{M(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where c_q, d_q are as in Theorem 6.1.3.

6.1.2 Symmetric circulant: dependent input with light tail

We now come to the symmetric circulant case. First we prove the following result similar to Lemma 6.1.2. It will be used in the proof of Theorem 6.1.7.

Lemma 6.1.6. *Let $\{x_n\}$ be the two sided moving average process (6.1.1) where $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$ and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi].$$

Then we have,

$$\max_{1\le k\le \lfloor \frac{n}{2}\rfloor} \Big| \frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}} - 2\frac{A_k}{\sqrt{n}}\sum_{t=1}^{\lfloor \frac{n}{2}\rfloor}\epsilon_t\cos(\frac{2\pi kt}{n}) + 2\frac{B_k}{\sqrt{n}}\sum_{t=1}^{\lfloor \frac{n}{2}\rfloor}\epsilon_t\sin(\frac{2\pi kt}{n})\Big| = o_p(n^{-1/4}) \tag{6.1.9}$$

where

$$\sqrt{2\pi f(\omega_k)}A_k = \sum_{j=-\infty}^{\infty} a_j\cos(\frac{2\pi kj}{n}) \quad \textit{and} \quad \sqrt{2\pi f(\omega_k)}B_k = \sum_{j=-\infty}^{\infty} a_j\sin(\frac{2\pi kj}{n}).$$

Proof. First observe that $\min_{\omega\in[0,2\pi]} f(\omega) > \alpha > 0$. Consider $n = 2m+1$ for simplicity. For $n = 2m$ calculations are similar.

$$\frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}} - 2\frac{A_k}{\sqrt{n}}\sum_{t=1}^{m}\epsilon_t\cos(\frac{2\pi kt}{n}) + 2\frac{B_k}{\sqrt{n}}\sum_{t=1}^{m}\epsilon_t\sin(\frac{2\pi kt}{n}) = Y_{n,k}$$

where

$$Y_{n,k} = \frac{1}{\sqrt{n}\sqrt{2\pi f(\omega_k)}}\sum_{j=-\infty}^{\infty} a_j\big[\cos\frac{2\pi kj}{n}U_{k,j} - \sin\frac{2\pi kj}{n}V_{k,j}\big],$$

$$U_{k,j} = \sum_{t=1}^{m}\big[\epsilon_{t-j}\cos\frac{2\pi k(t-j)}{n} - \epsilon_t\cos\frac{2\pi kt}{n}\big],\ V_{k,j} = \sum_{t=1}^{m}\big[\epsilon_{t-j}\sin\frac{2\pi k(t-j)}{n} - \epsilon_t\sin\frac{2\pi kt}{n}\big].$$

Note that

$$|U_{k,j}| \le \begin{cases} |\sum_{t=1-j}^{0}\epsilon_t\cos\frac{2\pi kt}{n}| + |\sum_{t=m-j+1}^{m}\epsilon_t\cos\frac{2\pi kt}{n}| & \text{if} \quad |j|<m, j\ge 0 \\ |\sum_{t=1}^{|j|}\epsilon_t\cos\frac{2\pi kt}{n}| + |\sum_{t=m+1}^{m+|j|}\epsilon_t\cos\frac{2\pi kt}{n}| & \text{if} \quad |j|<m, j<0 \\ |\sum_{t=1-j}^{m-j}\epsilon_t\cos\frac{2\pi kt}{n}| + |\sum_{t=1}^{m}\epsilon_t\cos\frac{2\pi kt}{n}| & \text{if} \quad |j|\ge m, j\ge 0 \\ |\sum_{t=|j|+1}^{|j|+m}\epsilon_t\cos\frac{2\pi kt}{n}| + |\sum_{t=1}^{m}\epsilon_t\cos\frac{2\pi kt}{n}| & \text{if} \quad |j|\ge m, j<0. \end{cases}$$

Now for any $r > 1$,

$$|\sum_{t=1}^{r}\epsilon_t\cos\frac{2\pi kt}{n}|^2 \le |\sum_{t=1}^{r}\epsilon_t e^{\frac{i2\pi kt}{n}}|^2 \le \sum_{s=-r}^{r}|\sum_{t=1}^{r-|s|}\epsilon_t\epsilon_{t+|s|}|.$$

Hence by equation (6.1.2),

$$\mathbb{E}\{\max_k|\sum_{t=1}^{r}\epsilon_t\cos\frac{2\pi kt}{n}|^2\} \le Kr^{\frac{3}{2}}.$$

Therefore

$$\mathbb{E}\{\max_k U_{k,j}^2\} \le \begin{cases} 4K|j|^{3/2} & \text{if} \quad |j|<m, \\ 4Km^{3/2} & \text{if} \quad |j|\ge m. \end{cases}$$

Similarly

$$\mathbb{E}\{\max_k V_{k,j}^2\} \leq \begin{cases} 4K|j|^{3/2} & \text{if} \quad |j| < m, \\ 4Km^{3/2} & \text{if} \quad |j| \geq m. \end{cases}$$

Now

$$\begin{aligned}
\mathbb{E}\{\max_k |Y_{n,k}|\} &\leq \frac{1}{\sqrt{2\pi\alpha}} \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} |a_j| \big[\mathbb{E}\{\max_k |U_{k,j}|\} + \mathbb{E}\{\max_k |V_{k,j}|\}\big] \\
&\leq \frac{2K^{1/2}}{\sqrt{2\pi\alpha}} \frac{1}{\sqrt{n}} \big[\sum_{|j|<m} |a_j||j|^{3/4} + \sum_{|j|\geq m} |a_j| m^{3/4} \big] \\
&\leq \frac{2K^{1/2}}{\sqrt{2\pi\alpha}} \frac{1}{n^{1/4}} \big[\sum_{|j|<m} |j|^{1/2}|a_j|(j/n)^{1/4} + \sum_{|j|\geq m} j^{1/2}|a_j| \big] \\
&= o(n^{-1/4})
\end{aligned}$$

since the second sum goes to zero as $n \to \infty$ and the first sum is not greater than

$$\sum_{k(n)<|j|<m} |j|^{1/2}|a_j| + \{k(n)/n\}^{1/4} \sum_{0\leq |j| \leq k(n)} |j|^{1/2}|a_j|,$$

where $k(n)$ is such that $\lim_{n\to\infty} \{k(n)/n\} = 0$ and $\lim_{n\to\infty} k(n) = \infty$. □

Define $M(\cdot, f)$ for the symmetric circulant matrix as was done for the reverse circulant matrix:

$$M(n^{-1/2}SC_n, f) = \max_{1\leq k<\frac{n}{2}} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$$

where λ_k are the eigenvalues of $n^{-1/2}SC_n$ as defined in Section 1.2.2. Under the additional restriction of $a_j = a_{-j}$, for all j, the following result is easy to prove.

Theorem 6.1.7 (Bose, Hazra and Saha (2009) [34]). *Let $\{x_n\}$ be the two sided moving average process defined in (6.1.1) with $a_j = a_{-j}$ and satisfies Assumption 6.1.1. Then*

$$\frac{M(n^{-1/2}SC_n, f) - b_q - a_q \ln 2}{a_q} \xrightarrow{\mathcal{D}} \Lambda$$

where $q = q(n) = \lfloor \frac{n}{2} \rfloor \approx \frac{n}{2}$ and a_q and b_q are as in equation (4.3.1).

As in Remark 6.1.4 if λ_0 is included in the definition of $M(n^{-1/2}SC_n, f)$ then the result changes only when $\mu \neq 0$.

Proof of Theorem 6.1.7. Note that if $a_j = a_{-j}$ then in Lemma 6.1.6, $B_k = 0$ and hence from the same lemma, it is easy to see that,

$$\max_{1\leq k\leq \lfloor \frac{n}{2}\rfloor}\Big|\frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}}-\lambda_{k,\epsilon}\Big|=o_p(n^{-1/4}) \tag{6.1.10}$$

where $\lambda_{k,\epsilon}$ denote eigenvalue of symmetric circulant matrix with $\{x_i\}$ replaced by $\{\epsilon_i\}$. Combining this with part (ii) of Theorem 4.3.5 we have

$$\frac{M(n^{-1/2}SC_n,f)-b_q-a_q\ln 2}{a_q}\xrightarrow{\mathcal{D}}\Lambda.$$

□

In Theorem 6.1.7, we assume that $a_j=a_{-j}$ and this condition reduces $\{x_n\}$ to a one sided moving average process. Now we focus on the case where a_j is not necessarily equal to a_{-j}. For reasons to be discussed later (see Remark 6.1.15), in this case we can deal with maximum over two different subsets L_n^1 and L_n^2 (see (6.1.12)) of $\{1\leq k\leq \lfloor \frac{n}{2}\rfloor\}$ separately. We first define some notation which will be used in our further developments. For $0<\delta_1<1/2$, define

$$p_n=(1-\frac{1}{n^{1/2+\delta_1}}) \text{ and } L_n=\{k:1\leq k\leq \lfloor np_n/2\rfloor\}, \tag{6.1.11}$$

$$L_n^1=\{k\in L_n:k\text{ is even}\} \text{ and } L_n^2=\{k\in L_n:k\text{ is odd}\}. \tag{6.1.12}$$

Let

$$\sigma_k^2=1+\frac{A_kB_k}{n}\tan(\frac{\pi k}{n}),\quad \nu_{k,k'}=\frac{D_{k,k'}}{n}\tan\frac{\pi(k+k')}{2n}+\frac{E_{k,k'}}{n}\tan\frac{\pi(k'-k)}{2n}, \tag{6.1.13}$$

$$D_{k,k'}=A_kB_{k'}+A_{k'}B_k \text{ and } E_{k,k'}=A_{k'}B_k-A_kB_{k'},$$

where A_k,B_k are as in Lemma 6.1.6.

The following lemma from Dai and Mukherjea (2001) [51] (Theorem 2.1) is an analogue of Mill's ratio in higher dimension.

Lemma 6.1.8 (Dai and Mukherjea (2001) [51])**.** *Let $(X_1,X_2,...X_n)$ be multivariate normal with zero means and a positive definite covariance matrix Σ. Let $\tilde\sigma_1\geq\tilde\sigma_2\geq...\geq\tilde\sigma_n$ denote the variances and let $I(t)=P(X_i\geq t,1\leq i\leq n)$. If $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)=\vec{1}\Sigma^{-1}$ where $\vec{1}=(1,1,...,1)$ with $\alpha_i>0$ then*

$$I(t)\approx\frac{1}{(\sqrt{2\pi})^n\sqrt{|\Sigma|}(\prod_{i=1}^n\alpha_i)t^n}\exp(-\frac{1}{2}t^2\vec{1}\Sigma^{-1}\vec{1}^T).$$

We first look at the special case where $\{\epsilon_i\}$ are standard normal random variables.

Lemma 6.1.9. *Let* $\{N_i\}$ *be i.i.d.* $N(0,1)$ *and let*

$$\lambda_{k,N} = \frac{\sqrt{2}A_k N_0}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} N_t \left(2A_k \cos(\frac{2\pi kt}{n}) - 2B_k \sin(\frac{2\pi kt}{n}) \right).$$

Then

$$\frac{\max_{k \in L_n^1} \lambda_{k,N} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \qquad (6.1.14)$$

and

$$\frac{\max_{k \in L_n^2} \lambda_{k,N} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \qquad (6.1.15)$$

where $q = q(n) = \lfloor \frac{n}{4} \rfloor$ *and* a_n *and* b_n *are as in (4.3.1).*

In particular,

$$\frac{\max_{1 \le k \le \lfloor \frac{n}{2} \rfloor} \lambda_{k,N}}{\sqrt{\ln n}} \xrightarrow{\mathcal{P}} 1. \qquad (6.1.16)$$

Proof. We shall prove (6.1.14) only. Proof of (6.1.15) is similar. Finally using these two results we shall prove (6.1.16).

Proof of (6.1.14): Consider the case $n = 2m + 1$. For $n = 2m$, calculations will be similar with minor changes. First observe that $Var(\lambda_{k,N}) = \sigma_k^2$ and for $k' > k$ we have $Cov(\lambda_{k,N}, \lambda_{k',N}) = \nu_{k,k'}$ where σ_k and $\nu_{k,k'}$ are defined in (6.1.13). Let $x_q = a_q x + b_q \approx \sqrt{2 \ln q}$. By Bonferroni inequalities we have for $j > 1$

$$\sum_{d=1}^{2j} (-1)^{d-1} \tilde{B}_d \le \mathrm{P}(\max_{k \in L_n^1} \lambda_{k,N} > x_q) \le \sum_{d=1}^{2j-1} (-1)^{d-1} \tilde{B}_d,$$

where

$$\tilde{B}_d = \sum_{i_1, i_2, \ldots, i_d \in L_n^1, \text{ all distinct}} \mathrm{P}(\lambda_{i_1,N} > x_q, \ldots \lambda_{i_d,N} > x_q)$$

Observe by the choice of p_n we have,

$$\frac{1}{n} \tan(\frac{\pi p_n}{2}) \approx \frac{2n^{1/2+\delta_1}}{\pi n} \to 0.$$

Hence for some $\epsilon > 0$, for large n we have $1 - \epsilon < \sigma_k^2 < 1 + \epsilon$ and for any $k, k' \in L_n^1$ (or L_n^2) we have $|\nu_{k,k'}| \to 0$ as $n \to \infty$. We shall use this simple observation very frequently in the proof. Next we make the following claim.

Claim:

$$\sum_{i_i, i_2, \ldots, i_d \in L_n^1, \text{ all distinct}} \mathbb{P}(\lambda_{i_1,N} > x_q, \ldots \lambda_{i_d,N} > x_q) \approx \frac{q^d \exp(-\frac{x_q^2 d}{2})}{d! x_q^d (\sqrt{2\pi})^d}, \text{ for } d \ge 1. \qquad (6.1.17)$$

To avoid notational complications we show the above claim for $d = 1$ and $d = 2$ and indicate what changes are necessary for higher dimension.

d=1: Using the fact that $\frac{\sigma_k^2}{x_q^2} \to 0$ and for $x > 0$,

$$\left(1 - \frac{1}{x^2}\right) \frac{\exp(-x^2/2)}{\sqrt{2\pi}x} \leq \mathrm{P}(N(0,1) > x) \leq \frac{\exp(-x^2/2)}{\sqrt{2\pi}x}$$

it easily follows that,

$$\sum_{k \in L_n^1} \mathrm{P}(N(0,1) > x_q/\sigma_k) \approx \sum_{k \in L_n^1} \frac{\sigma_k}{\sqrt{2\pi}x_q} \exp(-\frac{x_q^2}{2\sigma_k^2}).$$

Observe that

$$\begin{aligned}
\frac{\sum_{k \in L_n^1} \frac{\sigma_k}{\sqrt{2\pi}x_q} \exp(-\frac{x_q^2}{2\sigma_k^2})}{\frac{qp_n}{\sqrt{2\pi}x_q} \exp(-\frac{x_q^2}{2})} &= \frac{1}{qp_n} \sum_{k \in L_n^1} \sigma_k \exp(-\frac{x_q^2}{2}(\frac{1}{\sigma_k^2} - 1)) \\
&= \frac{1}{qp_n} \sum_{k \in L_n^1} \sigma_k \exp(-\frac{x_q^2}{2\sigma_k^2} \frac{A_k B_k}{n} \tan(\frac{\pi k}{n})).
\end{aligned}$$

Now using the facts that $\frac{A_k B_k x_q^2}{n\sigma_k^2} \tan(\frac{\pi p_n}{n}) \to 0$, $\sup_{k \in L_n^1} \sigma_k^2 \to 1$ and $|\{k : k \in L_n^1\}| \approx qp_n$, it is easy to see that the last term above goes to 1. Since $p_n \approx 1$ the claim is proved for $d = 1$.

d=2: We shall use Lemma 6.1.8 for this case. Without loss of generality assume that $\sigma_k^2 > \sigma_{k'}^2$. Let $\alpha = (\alpha_1, \alpha_2)$ where $\alpha = \vec{1}V^{-1}$ and

$$V = \begin{bmatrix} \sigma_k^2 & \nu_{k,k'} \\ \nu_{k,k'} & \sigma_{k'}^2 \end{bmatrix}.$$

Hence $(\alpha_1, \alpha_2) = \left(\frac{\sigma_{k'}^2 - \nu_{k,k'}}{|V|}, \frac{\sigma_k^2 - \nu_{k,k'}}{|V|}\right)$. For any $0 < \epsilon < 1$ it easily follows that $\alpha_i > \frac{1-\epsilon}{|V|}$ for large n and for $i = 1, 2$. Hence from Lemma 6.1.8 it follows that as $n \to \infty$,

$$\sum_{k,k' \in L_n^1} \mathrm{P}(\lambda_{k,N} > x_q, \lambda_{k',N} > x_q) \approx \sum_{k,k' \in L_n^1} \frac{1}{2\pi\sqrt{|V|}} \frac{\exp(-\frac{1}{2}x_q^2 \vec{1}V^{-1}\vec{1}^T)}{\alpha_1 \alpha_2 x_q^2}.$$

Now denote

$$\begin{aligned}\psi_{k,k'} &= \frac{1}{|V|}\Big[-\frac{A_kB_k}{n}\tan(\frac{\pi k}{n})-\frac{A_{k'}B_{k'}}{n}\tan(\frac{\pi k'}{n})\\&\quad+\frac{A_kB_k}{n}\tan(\frac{\pi k}{n})\frac{A_{k'}B_{k'}}{n}\tan(\frac{\pi k'}{n})-2\nu_{k,k'}+2\nu_{k,k'}^2\Big]\end{aligned}$$

and observe

$$|x_q^2\psi_{k,k'}|\le C\frac{x_q^2}{n}\tan(\frac{\pi p_n}{2})\to 0 \text{ as } n\to\infty.$$

$$\begin{aligned}&\frac{\sum_{k,k'\in L_n^1}\frac{1}{2\pi\sqrt{|V|}\alpha_1\alpha_2x_q^2}\exp(-\frac{1}{2}x_q^2\vec{1}V^{-1}\vec{1}^T)}{\frac{q^2\exp(-x_q^2)}{2!x_q^22\pi}}\\
&= \frac{2}{q^2}\sum_{k,k'\in L_n^1}\frac{1}{\sqrt{|V|}\alpha_1\alpha_2}\exp\left(-\frac{1}{2}x_q^2(\alpha_1+\alpha_2)+x_q^2\right)\\
&= \frac{2}{q^2}\sum_{k,k'\in L_n^1}\frac{|V|^{3/2}}{(\sigma_{k'}^2-\nu_{k,k'})(\sigma_k^2-\nu_{k,k'})}\exp\left(-\frac{x_q^2}{2}(\alpha_1+\alpha_2-2)\right)\\
&\le \frac{2}{q^2}\sum_{k,k'\in L_n^1}\frac{|V|^{3/2}}{(1-\epsilon)^2}\exp(-\frac{x_q^2}{2}\psi_{k,k'})\to 1 \text{ as } n\to\infty \text{ and as } \epsilon\to 0.\end{aligned}$$

Similarly the lower bound can be obtained to show that the claim is true for $d=2$.

$\mathbf{d>2}$: Now the probability inside the sum in claim (6.1.17) is $\mathrm{P}(N(0,V_n)\in E_n)$ where $E_n=\{(y_1,y_2,...,y_d): y_i>x_q, i=1,2...,d\}$, and V_n denotes covariance matrix $\{V_n(s,t)\}_{s,t=1}^d$ with $V_n(s,s)=\sigma_{i_s}^2$ and for $s\neq t$, $V_n(s,t)=\nu_{i_si_t}$, where $\sigma_{i_s},\nu_{i_si_t}$ are as in (6.1.13). Without loss of generality assume that $\sigma_{i_1}\ge\sigma_{i_2}\ge...\ge\sigma_{i_d}$, since we can always permute the original vector to achieve this and the covariance matrix changes accordingly. Note that as $n\to\infty$ we get

$$\|V_n-I_d\|_\infty\to 0,$$

where $\|A\|_\infty=\max|a_{i,j}|$. As $V_n^{-1}=\sum_{j=0}^\infty(I_d-V_n)^j$ we have $\alpha=\vec{1}+\sum_{j=1}^\infty\vec{1}(I_d-V_n)^j$. Now since $\|I_d-V_n\|_\infty\to 0$ so $\|(I_d-V_n)^j\|_\infty\to 0$ and hence elements of $(I_d-V_n)^j$ goes to zero for all j. So we get that $\alpha_i\in(1-\epsilon,1+\epsilon)$ for $i=1,2,...,d$ and $0<\epsilon<1$ and hence we can again apply Lemma 6.1.8. For further calculations it is enough to observe that for $|x|\neq 0$,

$$\frac{xV_nx^T}{|x|^2}=1+\frac{1}{|x|^2}\sum_{k=1}^d {x_k}^2A_{i_k}B_{i_k}\frac{1}{n}\tan(\frac{\pi i_k}{n})+\frac{1}{|x|^2}\sum_{1\le k\neq k'\le d}x_kx_{k'}\nu_{i_k,i_{k'}}$$

Since the last two term goes to zero in their modulus so given any $\epsilon > 0$, we get for large n

$$1 - \epsilon \leq \lambda_{min}(V_n) \leq \lambda_{max}(V_n) \leq 1 + \epsilon,$$

where $\lambda_{min}(V_n)$ and $\lambda_{max}(V_n)$ denote the minimum and maximum eigenvalue of V_n. Rest of the calculation is similar to $d = 2$ case. This proves the claim completely.

Back to the proof of (6.1.14). Using the fact that a_n and b_n are normalizing constants for maxima of standard normal it follows that,

$$\frac{q^d \exp(-\frac{x_q^2 d}{2})}{d! x_q^d (\sqrt{2\pi})^d} \approx \frac{1}{d!} \exp(-dx).$$

So from the Bonferroni inequalities and observing $\exp(-\exp(-x)) = \sum_{d=0}^{\infty} \frac{(-1)^d}{d!} \exp(-dx)$ it follows that

$$\mathrm{P}(\max_{k \in L_n^1} \lambda_{k,N} > x_q) \to \exp(-\exp(-x)),$$

proving (6.1.14) completely. For (6.1.15) calculations are similar to the proof of (6.1.14) and we omit the details.

Proof of (6.1.16): We first observe that,

$$\sum_{k=np_n/2}^{n/2} \mathrm{P}(N(0,1) > x_q/\sigma_k) \leq \frac{n}{2}(1-p_n)\mathrm{P}(N(0,1) > \frac{x_q}{\sqrt{2}}),$$

since $\sigma_k^2 \leq 2$ for $k \leq n/2$. Expanding the expressions for a_n and b_n we get,

$$\frac{x_q^2}{4} = \frac{1}{4}(a_q x + b_q)^2 = o(1) + \frac{\ln q}{2} - \frac{1}{4}\ln(4\pi \ln q) + \frac{x}{2}.$$

Now

$$\begin{aligned}
\frac{n(1-p_n)}{2}\mathrm{P}(N(0,2) > x_q) &\leq C\frac{n(1-p_n)}{2}\frac{\exp(-\frac{x_q^2}{4})}{x_q} \\
&\approx Cn^{-1/2}\frac{n(1-p_n)}{2\sqrt{\ln q}} \\
&\approx C\frac{1}{n^{\delta_1}\sqrt{\ln q}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Breaking up the set $L_1 = \{k : 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and k is even $\}$ into L_n^1 and

$\tilde{L}_n^1 = \{k : \lfloor np_n/2 \rfloor < k < \lfloor \frac{n}{2} \rfloor$ and k is even$\}$ we get,

$$\begin{aligned}
\mathrm{P}(\max_{k\in L_1} \lambda_{k,N} > x_q) &= \mathrm{P}(\max(\max_{k\in L_n^1} \lambda_{k,N}, \max_{k\in \tilde{L}_n^1} \lambda_{k,N}) > x_q) \\
&\leq \mathrm{P}(\max_{k\in L_n^1} \lambda_{k,N} > x_q) + \mathrm{P}(\max_{k\in \tilde{L}_n^1} \lambda_{k,N} > x_q) \\
&\leq \mathrm{P}(\max_{k\in L_n^1} \lambda_{k,N} > x_q) + \sum_{t=\lfloor np_n/2 \rfloor}^{\lfloor \frac{n}{2} \rfloor} \mathrm{P}(N(0,\sigma_k^2) > x_q) \\
&= \mathrm{P}(\max_{k\in L_n^1} \lambda_{k,N} > x_q) + o(1).
\end{aligned}$$

Hence the upper bound is obtained. The lower bound easily follows from (6.1.14). Similar calculations for the set $L_2 = \{k : 1 \leq k < \lfloor \frac{n}{2} \rfloor$ and k is odd$\}$ can be done. To complete the proof it is enough to observe that,

$$\mathrm{P}(\frac{\max_{1\leq k<\lfloor \frac{n}{2} \rfloor} \lambda_{k,N}}{\sqrt{\ln n}} > 1-\epsilon) \leq \mathrm{P}(\frac{\max_{k\in L_1} \lambda_{k,N}}{\sqrt{\ln n}} > 1-\epsilon) + \mathrm{P}(\frac{\max_{k\in L_2} \lambda_{k,N}}{\sqrt{\ln n}} > 1-\epsilon)$$

and the last two probabilities go to zero. This completes the proof of the Lemma. □

Remark 6.1.10. *By calculations similar to above, it can be shown that for* $\sigma^2 = n^{-c}$ *where* $c > 0$,

$$\sum_{i_i,i_2,\ldots,i_d\in L_n^1,\ \text{all distinct}} P((1+\sigma^2)^{1/2}\lambda_{i_1,N} > x_q, \ldots, (1+\sigma^2)^{1/2}\lambda_{i_d,N} > x_q) \leq \frac{K^d}{d!} \quad (6.1.18)$$

for some constant $K > 0$. *This will be used in the proof of Theorem 6.1.11.*

We now consider the symmetric circulant matrix with the general moving average process $\{x_n\}$. We shall use the result already proved for normal entries (Lemma 6.1.9).

Theorem 6.1.11 (Bose, Hazra and Saha (2009) [34])**.** *Let* SC_n *be the symmetric circulant matrix with entries from* $\{x_n\}$*, the two sided moving average process defined in (6.1.1) which satisfies Assumption 6.1.1. If* $\lambda_{k,x}$ *denote the eigenvalues of* $\frac{1}{\sqrt{n}}SC_n$ *with input* $\{x_i\}$ *then*

$$\frac{\max_{k\in L_n^1} \lambda_{k,x} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \quad (6.1.19)$$

and

$$\frac{\max_{k\in L_n^2} \lambda_{k,x} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \quad (6.1.20)$$

where $q = q(n) = \lfloor \frac{n}{4} \rfloor$ *and* a_n *and* b_n *are as in (4.3.1).*

Proof. We shall prove (8.2) only. Proof of (8.2) is similar. Again for simplicity we assume that $n = 2m + 1$. We break the proof into four steps.

Step 1: Truncation: Define

$$\tilde{\epsilon}_t = \epsilon_t \mathbb{I}(|\epsilon_t| \le n^{\frac{1}{2+\delta}}), \quad \overline{\epsilon}_t = \tilde{\epsilon}_t - E\tilde{\epsilon}_t, \quad \tilde{x}_t = \sum_{j=-\infty}^{\infty} a_j \tilde{\epsilon}_{t-j}, \quad \overline{x}_t = \sum_{j=-\infty}^{\infty} a_j \overline{\epsilon}_{t-j},$$

$$\lambda_{k,\tilde{x}} = \frac{1}{\sqrt{n}}[\tilde{x}_0 + 2\sum_{t=1}^{m} \tilde{x}_t \cos\frac{2\pi kt}{n}], \; \lambda_{k,\bar{x}} = \frac{1}{\sqrt{n}}[\overline{x}_0 + 2\sum_{t=1}^{m} \overline{x}_t \cos\frac{2\pi kt}{n}].$$

Claim: To prove (8.2) it is enough to show that,

$$\frac{\max_{k\in L_n^1} \lambda_{k,\epsilon} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda, \tag{6.1.21}$$

where

$$\lambda_{k,\epsilon} = \frac{\sqrt{2}A_k\overline{\epsilon}_0}{\sqrt{n}} + \frac{2A_k}{\sqrt{n}}\sum_{t=1}^{m} \overline{\epsilon}_t \cos(\frac{2\pi kt}{n}) - \frac{2B_k}{\sqrt{n}}\sum_{t=1}^{m} \overline{\epsilon}_t \sin(\frac{2\pi kt}{n}).$$

To prove the claim first note that

$$\begin{aligned}
\sqrt{n}\lambda_{k,\bar{x}} &= \overline{x}_0 + 2\sum_{t=1}^{m} \overline{x}_t \cos\frac{2\pi kt}{n} \\
&= \tilde{x}_0 + 2\sum_{t=1}^{m} \tilde{x}_t \cos\frac{2\pi kt}{n} + \sum_{j=-\infty}^{\infty} a_j E(\tilde{\epsilon}_{-j}) + 2\sum_{t=1}^{m}[\sum_{j=-\infty}^{\infty} a_j E(\tilde{\epsilon}_{t-j})]\cos\frac{2\pi kt}{n} \\
&= \sqrt{n}\lambda_{k,\tilde{x}} + [\sum_{j=-\infty}^{\infty} a_j E(\tilde{\epsilon}_j)][1 + 2\sum_{t=1}^{m} \cos\frac{2\pi kt}{n}] \\
&= \sqrt{n}\lambda_{k,\tilde{x}}.
\end{aligned}$$

Choose η such that $(\frac{1}{2} - \frac{1}{2+\delta} - \eta) > 0$ and observe

$$\begin{aligned}
n^{\eta}E[\max_{1\le k\le\lfloor\frac{n}{2}\rfloor} |\lambda_{k,\bar{x}} - \lambda_{k,x}|] &= n^{\eta}E[\max_{1\le k\le\lfloor\frac{n}{2}\rfloor} |\lambda_{k,\tilde{x}} - \lambda_{k,x}|] \\
&\le \frac{2}{n^{1/2-\eta}}\sum_{t=0}^{m}\sum_{j=-\infty}^{\infty} |a_j| E(|\epsilon_{t-j}|\mathbb{I}(|\epsilon_{t-j}| > n^{\frac{1}{2+\delta}})) \\
&\le \frac{2}{n^{1/2-\eta}}\sum_{t=0}^{m}\sum_{j=-\infty}^{\infty} |a_j| \big[n^{\frac{1}{2+\delta}} \mathrm{P}(|\epsilon_{t-j}| > n^{\frac{1}{2+\delta}}) \\
&\qquad + \int_{n^{\frac{1}{2+\delta}}}^{\infty} \mathrm{P}(|\epsilon_{t-j}| > u)du\big] \\
&= I_1 + I_2, \text{ say,}
\end{aligned}$$

and

$$\begin{aligned} I_1 &= \frac{2}{n^{1/2-\eta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| n^{\frac{1}{2+\delta}} \mathrm{P}(|\epsilon_{t-j}| > n^{\frac{1}{2+\delta}}) \\ &\leq \frac{2}{n^{1/2-\eta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| n^{\frac{1}{2+\delta}} \frac{1}{n} E(|\epsilon_{t-j}|^{2+\delta}) \\ &\leq \frac{E(|\epsilon_0|^{2+\delta})}{n^{\frac{1}{2}-\frac{1}{2+\delta}-\eta}} \sum_{j=-\infty}^{\infty} |a_j| \to 0, \end{aligned}$$

as $n \to \infty$ since $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Similarly

$$\begin{aligned} I_2 &= \frac{2}{n^{1/2-\eta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| \int_{n^{\frac{1}{2+\delta}}}^{\infty} \mathrm{P}(|\epsilon_{t-j}| > u) du \\ &\leq \frac{2}{n^{1/2-\eta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| \int_{n^{\frac{1}{2+\delta}}}^{\infty} \frac{E(|\epsilon_{t-j}|^{2+\delta})}{u^{2+\delta}} du \\ &\leq \frac{2E(|\epsilon_0|^{2+\delta})}{(2+\delta-1)n^{1/2-\eta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| \frac{1}{n^{1-\frac{1}{2+\delta}}} \\ &\leq \frac{E(|\epsilon_0|^{2+\delta})}{(1+\delta)n^{\frac{1}{2}-\frac{1}{2+\delta}-\eta}} \sum_{j=-\infty}^{\infty} |a_j| \to 0, \end{aligned}$$

as $n \to \infty$ for above choice of η. Hence

$$\max_{1 \leq k \leq \lfloor \frac{n}{2} \rfloor} |\lambda_{k,\bar{x}} - \lambda_{k,x}| = o_p\left(n^{-\eta}\right). \tag{6.1.22}$$

Also from Lemma 6.1.6 we have

$$\max_{k \in L_n^1} \Big| \frac{\lambda_{k,\bar{x}}}{a_q \sqrt{2\pi f(\omega_k)}} - \frac{2A_k}{\sqrt{n} a_q} \sum_{t=1}^{m} \bar{\epsilon}_t \cos(\frac{2\pi kt}{n}) + \frac{2B_k}{\sqrt{n} a_q} \sum_{t=1}^{m} \bar{\epsilon}_t \sin(\frac{2\pi kt}{n}) \Big| = o_p(\frac{\sqrt{\ln n}}{n^{\delta_1}}). \tag{6.1.23}$$

Now from (6.1.22) and (6.1.23) it follows that, to prove (8.2) it is enough to show

$$\frac{\max_{k \in L_n^1} \lambda_{k,\epsilon} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda.$$

This proves the claim in Step 1 completely.

Step 2: Normal Approximation: This is an intermediate step to approximate $\lambda_{k,\epsilon}$ by $\lambda_{k,N}$, where $\lambda_{k,N}$ is defined in Lemma 6.1.9. Define

$$\lambda_{k,\epsilon+\sigma N} = \frac{\sqrt{2}A_k}{\sqrt{n}}(\bar{\epsilon}_0+\sigma N_0)+\frac{2A_k}{\sqrt{n}}\sum_{t=1}^{m}(\bar{\epsilon}_t+\sigma N_t)\cos(\frac{2\pi kt}{n})-\frac{2B_k}{\sqrt{n}}\sum_{t=1}^{m}(\bar{\epsilon}_t+\sigma N_t)\sin(\frac{2\pi kt}{n}).$$

Claim:

$$\big|\mathrm{P}(\max_{k\in L_n^1}\lambda_{k,\epsilon+\sigma N} > x_q) - \mathrm{P}(\max_{k\in L_n^1}(1+\sigma^2)^{1/2}\lambda_{k,N} > x_q)\big| \to 0, \tag{6.1.24}$$

where $\lambda_{k,N}$ is defined in Lemma 6.1.9.
Proof of this claim is similar to the proof of Lemma 4.3.3. We use Lemma 4.3.2 to do so. Let $d \geq 1$ and $i_1, i_2, ...i_d$ be d distinct numbers from L_n^1.

$$v_d(0) = \sqrt{2}(A_{i_1}, ..., A_{i_d}) \text{ and}$$

$$v_d(t) = 2\left(A_{i_1}\cos(\frac{2\pi i_1 t}{n}) - B_{i_1}\sin(\frac{2\pi i_1 t}{n}), ..., A_{i_d}\cos(\frac{2\pi i_d t}{n}) - B_{i_d}\sin(\frac{2\pi i_d t}{n})\right).$$

Let $S_n = \sum_{t=0}^{m}\bar{\epsilon}_t v_d(t)$, and observe that $Cov(S_n) = V_n$ where V_n is the covariance matrix with diagonal entries $V_n(k,k) = B_n\sigma_{i_k}^2$ and off-diagonal entries $V_n(k,k') = B_n\nu_{i_k,i_{k'}}$ and $B_n = Var(\bar{\epsilon}_t)n \approx n$. We have in fact already seen in the proof of Lemma 6.1.9 that,

$$\|\frac{V_n}{B_n} - I_d\|_\infty \to 0.$$

To apply Lemma 4.3.2 we define

$${\epsilon'}_t = {B_n}^{1/2}V_n^{-1/2}\bar{\epsilon}_t v_d(t) \text{ for } 0 \leq t \leq \lfloor\frac{n}{2}\rfloor \text{ and } S'_n = \sum_{t=0}^{m}\epsilon'_t.$$

It is easy to see that $Cov(S'_n) = B_n I_d$. Also note the since $\|(\frac{V_n}{B_n})^{-1} - I_d\|_\infty < c'$ for some constant $c' > 0$ and hence for large n we get that $|\epsilon'_t| < 2dCn^{\frac{1}{2+\delta}}$ for some constant C . Hence $\{\epsilon'_t\}$ is a sequence of independent, mean zero random vectors with moment generating function finite in a neighborhood of zero. For verification of the other conditions choose $\tilde{\alpha} = \frac{c_1}{n^{\frac{1}{2+\delta}}2dC}$, where c_1 is a constant to be chosen later. Hence,

$$\begin{aligned}
\tilde{\alpha}\sum_{t=0}^{m}E|\epsilon'_t|^3\exp(\tilde{\alpha}|\epsilon'_t|) &\leq \tilde{\alpha}B_n^{3/2}|V_n|^{-3/2}(2d)^3\sum_{t=0}^{m}E|\bar{\epsilon}_t|^3\exp(c_1)\\
&\leq 4c_1\exp(c_1)C^2d^2n^{(1-\frac{1}{2+\delta})}E|\bar{\epsilon}_t|^3\\
&\leq 4c_1\exp(c_1)C^2d^2n^{(1-\frac{\delta_2}{2+\delta})}E|\epsilon_t|^{2+\delta_2},
\end{aligned}$$

where $\delta_2 \in (0,1)$ such that $E|\epsilon_t|^{2+\delta_2} < \infty$. Now choose c_1 such that the required condition is satisfied. Similar calculations show that

$$\beta_n = B_n^{-3/2} \sum_{t=0}^{m} E|\epsilon_t'|^3 \exp(\tilde{\alpha}|\epsilon_t'|) \leq Cn^{-c_3},$$

where $c_3 = \frac{1}{2} - \frac{1-\delta_2}{2+\delta} > 0$. Rest of the calculation is similar to the proof of Lemma 4.3.3. Let $\bar{\sigma}^2 = n^{-c_3}$ and if N_t' are i.i.d. $N(0, \bar{\sigma}^2 Cov(\epsilon_t'))$ independent of ϵ_t' and $\tilde{p}_n$ be density of $S_n^* = \frac{1}{\sqrt{B_n}} \sum_{t=0}^{m} (\epsilon_t' + N_t')$, then,

$$\tilde{p}_n(x) = \phi_{(1+\bar{\sigma}^2)I_d}(x)(1 + o(1)),$$

uniformly for all x such that $|x|^3 = o(n^{(\frac{1}{2} - \frac{1}{2+\delta})})$. Here ϕ_C denotes the d-dimensional normal density with covariance matrix C.

Let $\sigma^2 = Var(\bar{\epsilon})\bar{\sigma}^2 \approx n^{-c_3}$ and observe that $N_t' \stackrel{\mathcal{D}}{=} B_n^{1/2} V_n^{-1/2} \sigma N_t v_d(t)$, where N_t are i.i.d. $N(0,1)$ for $t = 0, 1, ..., m$.

For $x \in \mathbb{R}^d$, let $\|x\|_0 = \min_{1 \leq i \leq d} x_i$. Recall $|\cdot|$ denotes the Euclidean norm and observe that $\|x + y\|_0 \leq \|x\|_0 + |y|$. Let $\overline{S}_n = \frac{1}{\sqrt{n}} \sum_{t=0}^{m} (\bar{\epsilon}_t + N_t) v_d(t)$. Then note that $S_n^* = B_n^{1/2} V_n^{-1/2} \overline{S}_n$.

Let $r_n = o(n^{(\frac{1}{2} - \frac{1}{2+\delta})})$ and denote $K_n = \{y \in \mathbb{R}^d : \|B_n^{-1/2} V_n^{1/2} y\|_0 > x_q\}$ and break it into the following two sets $K_{1,n} = \{y \in \mathbb{R}^d : \|B_n^{-1/2} V_n^{1/2} y\|_0 > x_q, |y| > r_n\}$ and $K_{2,n} = \{y \in \mathbb{R}^d : \|B_n^{-1/2} V_n^{1/2} y\|_0 > x_q, |y| \leq r_n\}$. Then

$$\begin{aligned}
& \mathrm{P}(\|\overline{S}_n\|_0 > x_q) \\
\leq\ & \mathrm{P}(\|B_n^{-1/2} V_n^{1/2} S_n^*\|_0 > x_q) \\
=\ & \int_{K_n} \tilde{p}_n(y) dy \\
=\ & \int_{K_{2,n}} \tilde{p}_n(y) dy + \int_{K_{1,n}} \tilde{p}_n(y) dy \\
=\ & (1 + o(1)) \int_{K_{2,n}} \phi_{(1+\sigma^2)I_d}(y) dy + \int_{K_{1,n}} \tilde{p}_n(y) dy \\
=\ & (1 + o(1)) \int_{K_n} \phi_{(1+\sigma^2)I_d}(y) dy - (1 + o(1)) \int_{K_{1,n}} \phi_{(1+\sigma^2)I_d}(y) dy + \int_{K_{1,n}} \tilde{p}_n(y) dy \\
=\ & (1 + o(1)) \mathrm{P}(\|(1 + \sigma^2)^{1/2} \frac{1}{\sqrt{n}} \sum_{t=0}^{m} N_t v_d(t)\|_0 > x_q) \\
& -(1 + o(1)) \int_{K_{1,n}} \phi_{(1+\sigma^2)I_d}(y) dy + \int_{K_{1,n}} \tilde{p}_n(y) dy.
\end{aligned}$$

The third integral is less than

$$\mathrm{P}\left(|\frac{1}{\sqrt{n}}\sum_{t=0}^{m} B_n^{1/2}V_n^{-1/2}(\epsilon_t+\sigma N_t)v_d(t)| > r_n\right).$$

Now using the fact that $\|(\frac{V_n}{B_n})^{-1/2}\|_\infty \leq C_5$ for some constant $C_5 > 0$ and using calculations similar to Corollary 1 of Bose, Mitra and Sen (2009) [44] we conclude that the third integral is bounded by $K_1\exp(-K_2 n^{\delta_3})$ for some constant $K_1, K_2 > 0$ and depending only on d and $\delta_3 > 0$. Similarly the integral in the second term is bounded by

$$\int_{|y|>r_n} \phi_{(1+\sigma^2)I_d}(y)dy \leq 2d\exp(-\frac{r_n}{2d}).$$

From all the above observations it is easy to conclude that, for $\epsilon_n \to 0$ we get uniformly over d distinct tuples $i_1, i_2, ...i_d \in L_n^1$ that

$$\begin{aligned} &\big|\mathrm{P}(\|\overline{S}_n\|_0 > x_q) - \mathrm{P}(\|(1+\sigma^2)\frac{1}{\sqrt{n}}\sum_{t=0}^{m} N_t v_d(t)\|_0 > x_q)\big| \\ \leq\ & \epsilon_n P(\|(1+\sigma^2)\frac{1}{\sqrt{n}}\sum_{t=0}^{m} N_t v_d(t)\|_0 > x_q) + K_3\exp(-K_4 n^{\delta_3}), \end{aligned} \tag{6.1.25}$$

where K_3, K_4 are constants depending on d.
Now by arguments similar to Step 2 of the proof of Theorem 4.3.4 and using (6.1.18) and (6.1.25) it follows that,

$$\big|\mathrm{P}(\max_{k\in L_n^1}\lambda_{k,\epsilon+\sigma N} > x_q) - \mathrm{P}(\max_{k\in L_n^1}(1+\sigma^2)^{1/2}\lambda_{k,N} > x_q)\big| \to 0.$$

This proves the claim (6.1.24) in Step 2 completely.

Step 3: Claim:

$$\lim_{n\to\infty}\mathrm{P}(\max_{k\in L_n^1}\lambda_{k,\epsilon+\sigma N} > x_q) = \Lambda(x). \tag{6.1.26}$$

Proof of this step is similar to Step 3 of the proof of Theorem 4.3.4. Now since $\max_{k\in L_n^1}\lambda_{k,N} = O_P(\sqrt{\ln n})$ (see Lemma 6.1.9) and $\sigma^2 = n^{-c_3}$ we get as $n\to\infty$,

$$\mathrm{P}\left(\max_{k\in L_n^1}(1+\sigma^2)^{1/2}\lambda_{k,N} > x_q\right) \to \Lambda(x).$$

Combining this with (6.1.24) we get,

$$\lim_{n\to\infty}\mathrm{P}(\max_{k\in L_n^1}\lambda_{k,\epsilon+\sigma N} > x_q) = \Lambda(x).$$

This completes the proof of Step 3.

Step 4: In this step we shall prove (6.1.21). Observe that

$$\Big|\frac{\max_{k\in L_n^1}\lambda_{k,\epsilon+\sigma N}-b_q}{a_q}-\frac{\max_{k\in L_n^1}\lambda_{k,\epsilon}-b_q}{a_q}\Big|\leq\frac{\sigma\max_{k\in L_n^1}\lambda_{k,N}}{a_q}\xrightarrow{\mathcal{P}}0.$$

Now using (6.1.26) it follows that

$$\frac{\max_{k\in L_n^1}\lambda_{k,\epsilon}-b_q}{a_q}\xrightarrow{\mathcal{D}}\Lambda.$$

This completes the proof of Step 4.

Hence from the claim in Step 1 it follows that

$$\frac{\max_{k\in L_n^1}\lambda_{k,x}-b_q}{a_q}\xrightarrow{\mathcal{D}}\Lambda$$

proving (8.2) completely. For (8.2) calculations are similar to the proof of (8.2) and we omit the details. This completes the proof of the theorem. □

Theorem 6.1.12 (Bose, Hazra and Saha (2009) [34])**.** *If $\{\lambda_{k,x}\}$ are the eigenvalues of $\frac{1}{\sqrt{n}}SC_n$ then under the assumptions of Theorem 6.1.11,*

$$\frac{\max_{1\leq k\leq\lfloor\frac{n}{2}\rfloor}\frac{\lambda_{k,x}}{\sqrt{2\pi f(w_k)}}}{\sqrt{\ln n}}\xrightarrow{\mathcal{P}}1\quad where\quad\omega_k=\frac{2\pi k}{n}.$$

Proof. As before we assume $n=2m+1$. It is now easy to see from the truncation part of Theorem 6.1.11 and Lemma 6.1.6 that it is enough to show that,

$$\frac{\max_{1\leq k\leq\lfloor\frac{n}{2}\rfloor}\lambda_{k,\epsilon}}{\sqrt{\ln n}}\xrightarrow{\mathcal{P}}1,$$

where,

$$\lambda_{k,\epsilon}=\frac{\sqrt{2}A_k\bar{\epsilon}_0}{\sqrt{n}}+\frac{2A_k}{\sqrt{n}}\sum_{t=1}^{m}\bar{\epsilon}_t\cos(\frac{2\pi kt}{n})-\frac{2B_k}{\sqrt{n}}\sum_{t=1}^{m}\bar{\epsilon}_t\sin(\frac{2\pi kt}{n}),$$

and $\bar{\epsilon}_t=\epsilon_t\mathbb{I}(|\epsilon_t|\leq n^{1/s})-E\epsilon_t\mathbb{I}(|\epsilon_t|\leq n^{1/s})$. The steps are same as the steps required to prove (6.1.16) in Lemma 6.1.9 and observe from there that to complete the proof it is enough to show,

$$\sum_{k=\lfloor np_n/2\rfloor+1}^{\lfloor\frac{n}{2}\rfloor}\mathrm{P}(\lambda_{k,\epsilon}>x_q)\to0\text{ as }n\to\infty.\tag{6.1.27}$$

Denote

$$m = \lfloor \frac{n}{2} \rfloor, \; v_1(0) = \sqrt{2} A_k \text{ and } v_1(t) = 2A_k \cos(\frac{2\pi kt}{n}) - 2B_k \sin(\frac{2\pi kt}{n}).$$

Since $\{\overline{\epsilon}_t v_1(t)\}$ is a sequence of bounded independent mean zero random variable, by applying Bernstein's inequality we get

$$\begin{aligned}
P(\frac{1}{\sqrt{m}} \sum_{t=0}^{m} \overline{\epsilon}_t v_1(t) > x_q) &\leq P(|\sum_{t=0}^{m} \overline{\epsilon}_t v_1(t)| > \sqrt{m} x_q) \\
&= P(|\sum_{t=0}^{m} \overline{\epsilon}_t v_1(t)| > m \frac{x_q}{\sqrt{m}}) \\
&\leq 2 \exp\big(- \frac{m x_q^2}{2 \sum_{t=0}^{m} Var(\epsilon_t v_1(t)) + \frac{2}{3} C n^{1/s} m \frac{x_q}{\sqrt{m}}} \big).
\end{aligned}$$

Denote by $C_k = A_k B_k$ and observe

$$\begin{aligned}
D &:= \frac{m x_q^2}{2 \sum_{t=0}^{m} Var(\epsilon_t v_1(t)) + \frac{2}{3} C n^{1/s} m \frac{x_q}{\sqrt{m}}} \\
&\geq \frac{x_q^2}{4 \frac{1}{n} \sum_{t=0}^{m} Var(\epsilon_t v_1(t)) + \frac{4}{3} C n^{1/s - 1/2} x_q} \\
&= \frac{x_q^2}{4(1 + \frac{C_k}{n} \tan \frac{\pi k}{n}) + \frac{4}{3} \frac{C x_q}{n^{1/2 - 1/s}}} \\
&\geq \frac{x_q^2}{4(1 + \frac{2}{\pi}) + o(1)} \geq \frac{x_q^2}{8}.
\end{aligned}$$

Therefore

$$P(|\sum_{t=0}^{m} \overline{\epsilon}_t v_1(t)| > \sqrt{m} x_q) \leq 2 \exp(-\frac{x_q^2}{8}),$$

and hence

$$\sum_{t=\lfloor n p_n / 2 \rfloor}^{\lfloor \frac{n}{2} \rfloor} P(\frac{1}{\sqrt{n}} |\sum_{t=0}^{m} \overline{\epsilon}_t v_1(t)| > x_q) \quad \leq \quad n(1 - p_n) \exp(-\frac{x_q^2}{4}) \leq \frac{C}{n^{\delta_1} (\ln n)^{1/4}} \longrightarrow 0.$$

This completes the proof of (6.1.27) and hence the proof of the theorem. □

Remark 6.1.13. *Note that the above calculation can be imitated with ease to conclude that under the conditions of Theorem 6.1.12,*

$$\frac{\max_{1 \leq k \leq \lfloor \frac{n}{2} \rfloor} \frac{|\lambda_{k,x}|}{\sqrt{2\pi f(\omega_k)}}}{\sqrt{\ln n}} \xrightarrow{\mathcal{P}} 1.$$

The proof is same, with only the normalizing constants changed suitably.

Remark 6.1.14. *If we include λ_0 in the definition $M(n^{-1/2}SC_n, f)$ that is, if $\overline{M}(n^{-1/2}SC_n, f) = \max_{0\leq k\leq \lfloor \frac{n}{2}\rfloor} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$ then it is easy to see that under Assumption 6.1.1 except that mean μ of $\{\epsilon_i\}$ is now non-zero,*

$$\overline{M}(n^{-1/2}SC_n, f) - |\mu|\sqrt{n} \xrightarrow{\mathcal{D}} N(0,2).$$

Remark 6.1.15. *In Theorem 6.1.11 we were unable to consider the convergence over $L_n^1 \cup L_n^2$. It is not clear if the maximum over the two subsets are asymptotically independent and hence it is not clear if we would continue to obtain the same limit. Observe that for example, if k is odd and k' is even then*

$$Cov(\lambda_{k,x}, \lambda_{k',x}) = \frac{-D_{k,k'}}{n}\cot\frac{\pi(k+k')}{2n} - \frac{E_{k,kk'}}{n}\cot\frac{\pi(k'-k)}{2n}.$$

So for this covariance terms to tend to zero, we have to truncate the index set from below appropriately. For instance, in the Gaussian case we may consider the set $L' = \{(k,k') : 1 < k < \lfloor np_n/2\rfloor,\ k + \lfloor nq_n/2\rfloor < k' < \lfloor np_n/2\rfloor\}$ with $q_n \to 0$, and can approximate it by the i.i.d. counterparts since $\sup_{k,k'\in L'} |Cov(\lambda_{k,x}, \lambda_{k',x})| \to 0$ as $n \to \infty$. The complication comes when dealing with the complement of L' since it has no longer small cardinality.

6.1.3 k-circulant: dependent input with light tail

First recall the eigenvalues of k-circulant matrix $A_{k,n}$ from Section 1.2.4. For any positive integers k, n, let $p_1 < p_2 < \ldots < p_c$ be all their common prime factors so that,

$$n = n'\prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k'\prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here α_q, $\beta_q \geq 1$ and n', k', p_q are pairwise relatively prime. Then the characteristic polynomial of $A_{k,n}$ is given by

$$\chi\left(A_{k,n}\right) = \lambda^{n-n'}\prod_{j=0}^{\ell-1}\left(\lambda^{n_j} - y_j\right), \tag{6.1.28}$$

where y_j, n_j are as defined in Section 1.2.4.

k-circulant for $n = k^2 + 1$.

We first consider k-circulant matrix with $n = k^2 + 1$. In this case, clearly $n' = n$ and $k' = k$. From Lemma 2.4.16(i), $g_1 = 4$ and the eigenvalue partition of $\{0, 1, 2, \ldots, n-1\}$

contains exactly $q = \lfloor \frac{n}{4} \rfloor$ sets of size 4, say $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{\lfloor \frac{n}{4} \rfloor}\}$. Since each $\mathcal{P}_i$ is self-conjugate, we can find a set $\mathcal{A}_i \subset \mathcal{P}_i$ of size 2 such that

$$\mathcal{P}_j = \{x : x \in \mathcal{A}_j \text{ or } n - x \in \mathcal{A}_j\}.$$

Since we shall be using the bounds from Lemma 6.1.2 we define a few relevant notation for convenience. Define,

$$I_{x,n}(\omega_j) = \frac{1}{n}\Big|\sum_{l=1}^{n} x_l e^{i\omega_j l}\Big|^2, \quad I_{\epsilon,n}(\omega_j) = \frac{1}{n}\Big|\sum_{l=1}^{n} \epsilon_l e^{i\omega_j l}\Big|^2,$$

$$J_{x,n}(\omega) = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} x_l e^{i\omega_j l}, \quad J_{\epsilon,n}(\omega) = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} \epsilon_l e^{i\omega_j l},$$

$$\beta_{x,n}(t) = \prod_{j \in A_t} I_{x,n}(\omega_j), \quad \beta_{\epsilon,n}(t) = \prod_{j \in \mathcal{A}_t} I_{\epsilon,n}(\omega_j),$$

$$A(\omega_j) = \sum_{t=-\infty}^{\infty} a_t e^{i\omega_j t}, \quad T_n(\omega_j) = I_{x,n}(\omega_j) - |A(\omega_j)|^2 I_{\epsilon,n}(\omega_j),$$

$$\widetilde{\beta}_{x,n}(t) := \frac{\beta_{x,n}(t)}{\prod_{j \in \mathcal{A}_t} 2\pi f(\omega_j)} \quad \text{and} \quad M(n^{-1/2}A_{k,n}, f) = \max_{1 \le t \le q} \big(\widetilde{\beta}_{x,n}(t)\big)^{1/4}.$$

Theorem 6.1.16 (Bose, Hazra and Saha (2009) [34]). *Let $\{x_n\}$ be the two sided moving average process defined in (6.1.1) and satisfies Assumption 6.1.1. Then for $n = k^2 + 1$,*

$$\frac{M(n^{-1/2}A_{k,n}, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

as $n \to \infty$ where $q = q(n) = \lfloor \frac{n}{4} \rfloor$ and c_q, d_q are same as defined in Theorem 4.4.1.

Proof. Observe that,

$$\widetilde{\beta}_{x,n}(t) := \frac{\beta_{x,n}(t)}{\prod_{j \in \mathcal{A}_t} 2\pi f(\omega_j)} = \beta_{\epsilon,n}(t) + R_n(t),$$

where

$$R_n(t) = I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})} + I_{\epsilon,n}(\omega_{t_2})\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})} + \frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}.$$

Let $q = \lfloor \frac{n}{4} \rfloor$. Recall that,

$$\|n^{-1/2}A_{k,n}\| = \max_{1 \le t \le q} \big(\beta_{x,n}(t)\big)^{1/4} \text{ and } M(n^{-1/2}A_{k,n}, f) = \max_{1 \le t \le q} \big(\widetilde{\beta}_{x,n}(t)\big)^{1/4}. \quad (6.1.29)$$

We shall show $\max_{1\le t\le q}|\widetilde{\beta}_{x,n}(t)-\beta_{\epsilon,n}(t)|\to 0$ in probability.
Now

$$|\widetilde{\beta}_{x,n}(t)-\beta_{\epsilon,n}(t)|\le|I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}|+|I_{\epsilon,n}(\omega_{t_2})\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}|+|\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}|,$$

Note that

$$\max_{1\le t\le q}|I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}|\le\frac{1}{2\pi\alpha}\max_{1\le t<\frac{n}{2}}|I_{\epsilon,n}(\omega_t)|\max_{1\le t<\frac{n}{2}}|T_n(\omega_t)|.$$

From (6.1.7) we get

$$\max_{1\le t\le n}|T_n(\omega_t)|=O_p(n^{-1/4}(\ln n)^{1/2}).$$

Therefore

$$\max_{1\le t\le q}|I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}|=O_p(n^{-1/4}(\ln n)^{3/2})$$

and

$$\max_{1\le t\le q}|\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}|=O_p(n^{-1/2}\ln n).$$

Combining all this we have

$$\max_{1\le t\le q}|R_n(t)|=\max_{1\le t\le q}|\widetilde{\beta}_{x,n}(t)-\beta_{\epsilon,n}(t)|=O_p(n^{-1/4}(\ln n)^{3/2}).$$

Note that

$$\big(\beta_{\epsilon,n}(t)\big)^{1/4}-|R_n(t)|^{1/4}\le\big(\widetilde{\beta}_{x,n}(t)\big)^{1/4}\le\big(\beta_{\epsilon,n}(t)\big)^{1/4}+|R_n(t)|^{1/4}$$

and hence

$$\Big|\max_{1\le t\le q}\big(\widetilde{\beta}_{x,n}(t)\big)^{1/4}-\max_{1\le t\le q}\big(\beta_{\epsilon,n}(t)\big)^{1/4}\Big|=O_p(n^{-1/16}(\ln n)^{3/8}).\tag{6.1.30}$$

From Theorem 4.4.1 we know

$$\frac{\max_{1\le t\le q}\big(\beta_{\epsilon,n}(t)\big)^{1/4}-d_q}{c_q}\xrightarrow{\mathcal{D}}\Lambda.\tag{6.1.31}$$

Hence from (6.1.29), (6.1.30) and (6.1.31) it follows that,

$$\frac{M(n^{-1/2}A_{k,n},f)-d_q}{c_q}\xrightarrow{\mathcal{D}}\Lambda.$$

□

k-circulant with $n = k^g + 1$, $g > 2$.

Now we extend Theorem 6.1.16 for $n = k^g + 1$ where $g > 2$. Here, we use slightly different notation to use the developments of Sections 4.4.2–4.4.4. Define,

$$\widetilde{\beta}_{x,j}(t) := \frac{\beta_{x,j}(t)}{\prod_{l \in \mathcal{A}_t} 2\pi f(\omega_l)} \text{ and } M(n^{-1/2}A_{k,n}, f) = \max_{l} \max_{j:\mathcal{P}_j \in J_l} \left(\widetilde{\beta}_{x,l}(j)\right)^{1/2l}.$$

Theorem 6.1.17 (Bose, Hazra and Saha (2009) [34])**.** *Let $\{x_n\}$ be the two sided moving average process defined in (6.1.1) and satisfies Assumption 6.1.1. Then for $n = k^g + 1,\ g > 2$,*

$$\frac{M(n^{-1/2}A_{k,n}, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

as $n \to \infty$ where $q = q(n) = \frac{n}{2g}$ and c_q, d_q are as defined in Theorem 4.4.2.

Proof. The line of argument is similar as $g = 2$ case. To prove the result we use following two facts:
(i) From (6.1.7),

$$\max_{1 \le t < \frac{n}{2}} |T_n(\omega_t)| = o_p(n^{-1/4}(\ln n)^{1/2}).$$

(ii) From Davis and Mikosch (1999) [50],

$$\max_{1 \le t < \frac{n}{2}} |I_{\epsilon,n}(\omega_t)| = O_p(\ln n) \quad \text{and} \quad \max_{1 \le t < \frac{n}{2}} |I_{x,n}(\omega_t)| = O_p(\ln n).$$

Using these and inequality (4.4.23), it is easy to see that, for some $\delta_0 > 0$

$$\max_{l} \max_{j:\mathcal{P}_j \in J_l} \left|\widetilde{\beta}_{x,l}(t) - \beta_{\epsilon,l}(t)\right| = o_p(n^{-\delta_0}). \tag{6.1.32}$$

Now the results follows from Theorem 4.4.2 and (6.1.32). □

6.2 Dependent input with heavy tail

Now suppose that the input sequence is a *linear process* $\{X_t, t \in \mathbb{Z}\}$ given by

$$X_t = \sum_{j=-\infty}^{\infty} a_j Z_{t-j}, \quad t \in \mathbb{Z}, \text{ where } \sum_{j=-\infty}^{\infty} |a_j|^{\alpha-\epsilon} < \infty \text{ for some } 0 < \epsilon < \alpha. \tag{6.2.1}$$

Suppose that $\{Z_i\}$ are i.i.d random variables satisfying (5.1.1) with $0 < \alpha < 1$. Using $\mathrm{E}\,|Z|^{\alpha-\epsilon} < \infty$ and the assumption on the $\{a_j\}$ we have,

$$\mathrm{E}\,|X_t|^{\alpha-\epsilon} \leq \sum_{j=-\infty}^{\infty} |a_j|^{\alpha-\epsilon}\, \mathrm{E}\,|Z_{t-j}|^{\alpha-\epsilon} = \mathrm{E}\,|Z_1|^{\alpha-\epsilon} \sum_{j=-\infty}^{\infty} |a_j|^{\alpha-\epsilon} < \infty.$$

Hence X_t is finite a.s. Let

$$\psi(x) = \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi x j), \quad x \in [0,1]$$

be the *transfer function* of the linear filter $\{a_j\}$ and $f_X(x)$ be the *power transfer function* of $\{X_t\}$. Then

$$f_X(x) = |\psi(x)|^2.$$

Define

$$M(RC_n, f_X) = \max_{0 \leq k < \frac{n}{2}} \frac{|\lambda_k|}{\sqrt{f_X(k/n)}}, \quad M(C_n, f_X) = \max_{0 \leq k < \frac{n}{2}} \frac{|\lambda_k|}{\sqrt{f_X(k/n)}},$$

$$M(SC_n, f_X) = \max_{0 \leq k < \frac{n}{2}} \frac{|\lambda_k|}{\sqrt{f_X(k/n)}},$$

where in each case $\{\lambda_k\}$ are the eigenvalues of the corresponding matrix. From the eigenvalue structure of C_n and RC_n, $M(C_n, f_X) = M(RC_n, f_X)$.

Theorem 6.2.1 (Bose, Hazra and Saha (2010) [36]). *Assume that $\{X_n\}$ and $\{a_j\}$ satisfy (6.2.1) and $\{Z_t\}$ is i.i.d satisfying (5.1.1). Suppose f_X is strictly positive on $[0, 1/2]$. Then*

(*a*) $M(b_n^{-1}C_n, f_X) \xrightarrow{\mathcal{D}} Y_\alpha$ *and* $M(b_n^{-1}RC_n, f_X) \xrightarrow{\mathcal{D}} Y_\alpha$.

(*b*) *Further, if* $a_j = a_{-j}$, *then* $M(b_n^{-1}SC_n, f_X) \xrightarrow{\mathcal{D}} 2^{1-1/\alpha} Y_\alpha$.

Proof. (a) The proof is along the lines of the proof of Lemma 2.6 in Mikosch, Resnick and Samorodnitsky (2000) [93]. Let $\widehat{C_n}$ be the circulant matrix formed with independent entries $\{Z_i\}$. To prove the result it is enough to show that

$$\big|M(b_n^{-1}C_n, f_X) - \|b_n^{-1}\widehat{C_n}\|\big| \xrightarrow{\mathcal{P}} 0.$$

Let $J_{n,Z}(x) = b_n^{-1}\sum_{t=1}^{n} Z_t \exp(-i2\pi xt)$. Note

$$\begin{aligned}\big|M(b_n^{-1}C_n, f_X) - \|b_n^{-1}\widehat{C_n}\|\big| &= \big|\sup_{1\le k\le n}(f_X(k/n))^{-1/2}|J_{n,X}(k/n)| - \sup_{1\le k\le n}|J_{n,Z}(k/n)|\big| \\ &\le \sup_{1\le k\le n}\big||\psi(k/n)^{-1}J_{n,X}(k/n)| - |J_{n,Z}(k/n)|\big| \\ &\le \sup_{1\le k\le n}\big|\psi(k/n)^{-1}J_{n,X}(k/n) - J_{n,Z}(k/n)\big|\end{aligned}$$

and

$$\begin{aligned}J_{n,X}(x) &= b_n^{-1}\sum_{t=1}^{n} X_t\exp(-i2\pi xt) \\ &= b_n^{-1}\sum_{j=-\infty}^{\infty} a_j\exp(-i2\pi xj)\big(\sum_{t=1}^{n} Z_t\exp(-i2\pi xt) + V_{n,j}\big) \\ &= \psi(x)J_{n,Z}(x) + Y_n(x), \qquad (6.2.2)\end{aligned}$$

where

$$V_{n,j} = \sum_{t=1-j}^{n-j} Z_t\exp(-i2\pi xt) - \sum_{t=1}^{n} Z_t\exp(-i2\pi xt),\ Y_n(x) = b_n^{-1}\sum_{j=-\infty}^{\infty} a_j\exp(-i2\pi xj)V_{n,j}.$$

Since f_X is bounded away from 0 and (6.2.2) holds, it is enough to show that $\max_{1\le k\le n}|Y_n(k/n)| \xrightarrow{\mathcal{P}} 0$. Now

$$\begin{aligned}Y_n(x) &= b_n^{-1}\sum_{j=n+1}^{\infty} a_j\exp(-i2\pi xj)V_{n,j} + b_n^{-1}\sum_{j=1}^{n} a_j\exp(-i2\pi xj)V_{n,j} \\ &\quad + b_n^{-1}\sum_{j=-\infty}^{-n-1} a_j\exp(-i2\pi xj)V_{n,j} + b_n^{-1}\sum_{j=-n}^{-1} a_j\exp(-i2\pi xj)V_{n,j} \\ &= S_1(x) + S_2(x) + S_3(x) + S_4(x).\end{aligned}$$

Now following an argument similar to that given in the proof of Lemma 2.6 in Mikosch, Resnick and Samorodnitsky (2000) [93], we can show that

$$\max_{1\le k\le n}|S_i(k/n)| \xrightarrow{\mathcal{P}} 0 \ \text{ for } \ i = 1, 2.$$

The behaviour of $S_3(x)$ and $S_4(x)$ are similar to $S_1(x)$ and $S_2(x)$ respectively. Therefore, following similar argument we can show that $\max_{1\le k\le n}|S_j(k/n)| \xrightarrow{\mathcal{P}} 0$ for $j = 3, 4$. This completes the proof of part (a).

(b) Let $\widehat{SC_n}$ be the symmetric circulant matrix formed with independent entries $\{Z_i\}$.

In view of Theorem 5.3.1, it is enough to show that

$$\big|M(b_n^{-1}SC_n, f_X) - \|b_n^{-1}\widehat{SC_n}\|\big| \xrightarrow{\mathcal{P}} 0.$$

Let $q = q(n) = \lfloor \frac{n}{2} \rfloor$ and

$$\begin{aligned} J_{n,Z}(x) &:= 2b_n^{-1}\sum_{t=1}^{q} Z_t \cos(2\pi xt) \\ &= b_n^{-1}\sum_{t=1}^{q} Z_t \exp(i2\pi xt) + b_n^{-1}\sum_{t=1}^{q} Z_t \exp(-i2\pi xt). \end{aligned}$$

Then using $a_j = a_{-j}$ we have

$$\begin{aligned} J_{n,X}(x) &:= b_n^{-1}\sum_{t=1}^{q} X_t \exp(i2\pi xt) + b_n^{-1}\sum_{t=1}^{q} X_t \exp(-i2\pi xt) \\ &= b_n^{-1}\sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj)\big(\sum_{t=1}^{q} Z_t \exp(i2\pi xt) + U_{n,j}\big) \\ &\quad + b_n^{-1}\sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj)\big(\sum_{t=1}^{q} Z_t \exp(-i2\pi xt) + V_{n,j}\big) \\ &= \psi(x)J_{n,Z}(x) + Y_{1n}(x) + Y_{2n}(x), \end{aligned}$$

where

$$U_{n,j} = \sum_{t=1+j}^{q+j} Z_t \exp(i2\pi xt) - \sum_{t=1}^{q} Z_t \exp(i2\pi xt),$$

$$V_{n,j} = \sum_{t=1-j}^{q-j} Z_t \exp(-i2\pi xt) - \sum_{t=1}^{q} Z_t \exp(-i2\pi xt),$$

$$Y_{1n} = b_n^{-1}\sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj)U_{n,j}, \quad Y_{2n} = b_n^{-1}\sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj)V_{n,j}.$$

Since f_X is bounded away from 0, it is enough to show that

$$\sup_{1\le k\le q}\big|J_{n,X}(k/n) - \psi(k/n)J_{n,Z}(k/n)\big| \;\le\; \sup_{1\le k\le q}\big|Y_{1n}(k/n)\big| + \sup_{1\le k\le q}\big|Y_{2n}(k/n)\big| \xrightarrow{\mathcal{P}} 0.$$

Now

$$\begin{aligned}
Y_{1n}(x) &= b_n^{-1}\sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj)U_{n,j} \\
&= b_n^{-1}\sum_{j=q+1}^{\infty} a_j \exp(-i2\pi xj)U_{n,j} + b_n^{-1}\sum_{j=1}^{q} a_j \exp(-i2\pi xj)U_{n,j} \\
&\quad + b_n^{-1}\sum_{j=-\infty}^{-q-1} a_j \exp(-i2\pi xj)U_{n,j} + b_n^{-1}\sum_{j=-q}^{-1} a_j \exp(-i2\pi xj)U_{n,j} \\
&= S_1(x) + S_2(x) + S_3(x) + S_4(x).
\end{aligned}$$

Again following an argument similar to that in the proof of Lemma 2.6 in [93], we can show that $\sup_{1\le k\le q}\left|S_i(k/n)\right| \xrightarrow{\mathcal{P}} 0$ for $1 \le i \le 4$. Hence $\sup_{1\le k\le q}\left|Y_{1n}(k/n)\right| \xrightarrow{\mathcal{P}} 0$. Similarly $\sup_{1\le k\le q}\left|Y_{2n}(k/n)\right| \xrightarrow{\mathcal{P}} 0$. This completes the proof of part (b). □

Chapter 7

Poisson convergence of eigenvalues of circulant type matrices

In this chapter we deal with weak convergence of point process based on the eigenvalues of circulant type random matrices. There appears to have been only limited studies on the weak convergence of point process based on the eigenvalues of random matrices. Soshnikov (2004) [117] considers the point process based on the positive eigenvalues of an appropriately scaled Wigner matrix with heavy tailed entries $\{x_{ij}\}$ satisfying $\mathrm{P}(|x_{ij}| > x) = h(x)x^{-\alpha}$ where h is a slowly varying function at infinity and $0 < \alpha < 2$. He showed that it converges to an inhomogeneous Poisson random point process. A similar result was proved for sample covariance matrices with Cauchy entries in Soshnikov (2006) [118]. These results on Wigner and sample covariance matrices were extended in Auffinger, Ben Arous and Peche (2009) [6] to $2 \leq \alpha < 4$.

On the other hand, in Chapter 4 we have established the distributional convergence of the maximum of the modulus of the eigenvalues of circulant type matrices. The same result for k-circulant matrix for $n = k^2 + 1$ was also derived in Bose, Mitra and Sen (2008) [44]. The main tool for proving such a result was the strong approximation theorem of Einmahl (1989) [53] for i.i.d random vectors. It seems then natural to study the joint distribution of the ordered eigenvalues of circulant type matrices.

Here is an outline of this chapter. In Section 7.1 we deal with circulant type matrices with i.i.d. light tailed entries and consider the point process based on the points $\{(\omega_k, \frac{\lambda_k - b_q}{a_q}),\ 0 \leq k < n\}$ where $\{\lambda_k\}$ are the eigenvalues as given in Section 1.2 and $\{\omega_k = \frac{2\pi k}{n}\}$ are the Fourier frequencies, and a_q, b_q are appropriate scaling and centering constants appearing in the weak convergence of the spectral radius in Chapter 4. We show that the limit measure is Poisson. In particular this yields the distributional convergence of any *k-upper* ordered eigenvalues of these matrices and also yields the joint distributional convergence of any k spacings of the upper ordered eigenvalues. Then in Section 7.2 we extend these results partially to two sided moving average process entries under certain restriction on the process.

The results of Bose, Hazra and Saha (2010) [35] are based on this chapter.

7.1 Results for i.i.d. input

We first recall the definition of *point process* and *simple point process*. Let $M_p(E)$ be the space of all point measures on E equipped with an appropriate sigma algebra $\mathcal{M}_p(E)$.

Definition 7.1.1. *A point process on E is a measurable map*

$$N : (\Omega, \mathcal{F}, P) \to (M_p(E), \mathcal{M}_p(E)).$$

A point process N is simple if

$$P(N(\{x\}) \leq 1, x \in E) = 1.$$

We initially assume that the input sequence $\{x_i\}$ is a sequence of i.i.d. random variables. Recall the eigenvalues of circulant type matrices from Section 1.2. We use a little different notation for their eigenvalues in this chapter. Necessity of using this will be clear as we go along.

7.1.1 Reverse circulant

Let $\lambda_{n,x}(\omega_0), \lambda_{n,x}(\omega_1), \ldots, \lambda_{n,x}(\omega_{n-1})$ be the eigenvalues of $n^{-1/2}RC_n$. The subscript x in the eigenvalues keeps track of the fact that the input sequence is $\{x_i\}$. This notation will come in handy later on when we have the need to consider matrices with different input sequences. These eigenvalues are given by (see Section 1.2.3):

$$\begin{cases} \lambda_{n,x}(\omega_0) & = n^{-1/2}\sum_{t=0}^{n-1} x_t \\ \lambda_{n,x}(\omega_{n/2}) & = n^{-1/2}\sum_{t=0}^{n-1}(-1)^t x_t, \quad \text{if } n \quad \text{is even} \\ \lambda_{n,x}(\omega_k) = -\lambda_{n,x}(\omega_{n-k}) & = \sqrt{I_{n,x}(\omega_k)}, \ 1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor. \end{cases} \tag{7.1.1}$$

where

$$I_{n,x}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 \text{ and } \omega_k = \frac{2\pi k}{n}.$$

Note that $\{|\lambda_{n,x}(\omega_k)|^2;\ 1 \leq k < n/2\}$ is the periodogram of $\{x_i\}$ at the frequencies $\{\omega_k = \frac{2\pi k}{n};\ 1 \leq k < n/2\}$. This explains our notation of using ω_k as an argument of the eigenvalues $\lambda_{n,x}$. Since the eigenvalues occur in pairs with opposite signs (except for perhaps one eigenvalue), it suffices for our purposes to define our point process based on the points $(\omega_k, \frac{\lambda_{n,x}(\omega_k)-b_q}{a_q})$ for $k = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. Let $\epsilon_x(\cdot)$ denote the point measure which gives unit mass to any set containing x. With $q = q(n) = \lfloor \frac{n}{2} \rfloor$, $a_q = \frac{1}{2\sqrt{\ln q}}$ and $b_q = \sqrt{\ln q}$, define

$$\eta_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot). \tag{7.1.2}$$

Let $M_p([0,\pi]\times(-\infty,\infty])$ denote the set of all point measures on the set $[0,\pi]\times(-\infty,\infty]$ endowed with the topology of vague convergence. Let $\xrightarrow{\mathcal{D}}$ denote the convergence in distribution relative to the vague topology. Now consider the following assumption.

Assumption 7.1.2. $\{x_i\}$ *are i.i.d.,* $\mathrm{E}[x_0] = 0$, $\mathrm{E}[x_0]^2 = 1$ *and* $\mathrm{E}\,|x_0|^s < \infty$ *for some* $s > 2$.

We then have the following Theorem.

Theorem 7.1.3 (Bose, Hazra and Saha (2010) [35])**.** *Let* $\{x_i\}$ *be i.i.d. random variables which satisfy Assumption 7.1.2. Then for the sequence of point processes* η_n *defined in (7.1.2), we have* $\eta_n \xrightarrow{\mathcal{D}} \eta$, *where* η *is a Poisson process on* $[0,\pi]\times(-\infty,\infty]$ *with intensity measure* $\pi^{-1}dt \times e^{-x}dx$.

Before going into the proof of Theorem 7.1.3 we state a result which plays a key role in the proof and which will also be used later.

Lemma 7.1.4 provides a criterion for convergence. Its proof is available in Kallenberg (1983) [80], Resnick (1987) [103] and Embrechts, Kluppelberg and Mikosch (1997) [56].

Lemma 7.1.4. *Let* $\{N_n\}$ *be a sequence of point process and* N *be a simple point process on a complete separable metric space* E. *Let* $\mathcal{T}$ *be a basis of relatively compact open sets such that* $\mathcal{T}$ *is closed under finite unions and intersections and for* $I \in \mathcal{T}$, $\mathrm{P}[N(\partial I) = 0] = 1$. *If* $\lim_{n\to\infty} \mathrm{P}[N_n(I) = 0] = \mathrm{P}[N(I) = 0]$ *and* $\lim_{n\to\infty} \mathrm{E}[N_n(I)] = \mathrm{E}[N(I)] < \infty$ *then* $N_n \xrightarrow{\mathcal{D}} N$ *in* $M_p(E)$.

Now first suppose that the input sequence is i.i.d. normal. Then the eigenvalues (apart from negative eigenvalues) are independent square root of exponentials (see Lemma 4.4.3) and in this case the Poisson process result is easy to derive. As we have already seen when the entries are not normal, the eigenvalues are asymptotically uncorrelated and asymptotically distributed as square root of exponential. This is also easy to see using central limit theorem for independent random variables. The rate of convergence is sharp provided the $(2+\delta)$ moment is finite.

Recall the sophisticated normal approximation result given in Lemma 4.4.6 which allows us to replace the variables by appropriate normal variables. But it is available only after appropriate truncations. Thus let the truncated and centered i.i.d. random variables be

$$\overline{x}_t = x_t \mathbb{I}(|x_t| < n^{1/s}) - \mathrm{E}[x_t \mathbb{I}(|x_t| < n^{1/s})].$$

Let $\{N_t\}$ be a sequence of i.i.d $N(0,1)$ random variables and ϕ_{C_d} be the density of d dimensional centered Gaussian vector with covariance matrix C_d. Define for $d \geq 1$,

$$v_d(t) = (\cos(\omega_{i_1} t), \sin(\omega_{i_2} t), ..., \cos(\omega_{i_d} t), \sin(\omega_{i_d} t))' \tag{7.1.3}$$

where $\omega_{i_1}, ..., \omega_{i_d}$ are any distinct Fourier frequencies.

A sketch of the proof: Suppose for a moment that the entries are i.i.d. standard normal random variables. Then it is easy to see from Lemma 4.4.3 of Chapter 4 that the eigenvalues are independent and distributed as symmetric square root $\sqrt{E_1}$ where E_1 is a standard exponential random variable. In this case the Poisson convergence result is immediate.

Now consider the reverse circulant matrices with the input sequences $\{\bar{x}_t + \sigma_n N_t\}$ and $\{\bar{x}_t\}$. Let η_n^* and $\bar{\eta}_n$ be the respective point processes η_n but with the above input sequences.

In Step 1 we show that η_n^* converges to the required Poisson process. For technical convenience, to define η_n^*, we just consider the distinct eigenvalues and also leave out λ_0. In Step 2 we show that η_n^* and $\bar{\eta}_n$ are close in probability. Finally using some inequalities we show that the original point process η_n and $\bar{\eta}_n$ are close.

This is essentially the programme that is carried out for other matrices also. Finally, the dependent case is reduced to the independent case by an appropriate approximation result (such as Lemma 6.1.2 for reverse circulant matrix).

Proof of Theorem 7.1.3. **Step 1:** We first show that $\eta_n^* \xrightarrow{\mathcal{D}} \eta$ where

$$\eta_n^*(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j) - b_q}{a_q}\right)}(\cdot)$$

and $\lambda_{n,\bar{x}+\sigma_n N}(\omega_k)$ are the eigenvalues of $n^{-1/2} RC_n$ with entries $\{\bar{x}_t + \sigma_n N_t\}$ with $\sigma_n^2 = n^{-c}$ and c is as in Lemma 4.4.6. First note that if we define the set

$$A_q^d = \{(x_1, y_1, ..., x_d, y_d)' : \sqrt{x_i^2 + y_i^2} > 2z_q\}$$

where $z_q = a_q x + b_q$, it easily follows that

$$\begin{aligned} & \mathrm{P}\left(\lambda_{n,\bar{x}+\sigma_n N}(\omega_{i_1}) > z_q, ..., \lambda_{n,\bar{x}+\sigma_n N}(\omega_{i_d}) > z_q\right) \\ = \;& \mathrm{P}\left(2^{1/2} n^{-1/2} \sum_{t=1}^{n} (\overline{x}_t + \sigma_n N_t) v_d(t) \in A_q^d\right) \\ = \;& \int\limits_{A_q^d} \phi_{(1+\sigma_n^2) I_{2d}}(x)(1 + o(1)) dx \\ = \;& q^{-d} \exp(-dx)(1 + o(1)). \end{aligned} \tag{7.1.4}$$

Since the limit process η is simple, to show $\eta_n^* \xrightarrow{\mathcal{D}} \eta$ it suffices to show (see Lemma 7.1.4) that

$$\mathrm{E}\,\eta_n^*((a,b] \times (x,y]) \to \mathrm{E}\,\eta((a,b] \times (x,y]) = \frac{b-a}{\pi}(e^{-x} - e^{-y}) \tag{7.1.5}$$

for all $0 \le a < b \le \pi$ and $x < y$ and for all $k \ge 1$,

$$\begin{aligned} & \mathrm{P}(\eta_n^*((a_1,b_1] \times R_1) = 0, \ldots, \eta_n^*((a_k,b_k] \times R_k) = 0) \\ & \quad \to \mathrm{P}(\eta((a_1,b_1] \times R_1) = 0, \ldots, \eta((a_k,b_k] \times R_k) = 0), \end{aligned} \tag{7.1.6}$$

where $0 \le a_1 < b_1 < \cdots < a_k < b_k \le \pi$ and $R_1, \ldots, R_k$ are bounded Borel sets, each consisting of a finite union of intervals on $(-\infty, \infty]$.

Proof of (7.1.5): It is established as follows:

$$\begin{aligned} \mathrm{E}\,\eta_n^*((a,b] \times (x,y]) \;&=\; \sum_{\omega_j \in (a,b]} \mathrm{P}(a_q x + b_q < \lambda_{n,\bar{x}+\sigma_n N}(\omega_j) \le a_q y + b_q) \\ \text{(by (7.1.4))} \quad &\sim\; \frac{(b-a)n}{2\pi} q^{-1}(e^{-x} - e^{-y}) \to \frac{(b-a)}{\pi}(e^{-x} - e^{-y}). \end{aligned}$$

Proof of (7.1.6): Set $n_j := \#\{i : \omega_i \in (a_j, b_j]\} \sim n(b_j - a_j)$. Then the complement of the event in (7.1.6) is the union of $m = n_1 + \ldots + n_k$ events, that is,

$$\begin{aligned} & 1 - \mathrm{P}(\eta_n^*((a_1,b_1] \times R_1) = 0, \ldots, \eta_n^*((a_k,b_k] \times R_k) = 0) \\ & = \mathrm{P}\left(\cup_{j=1}^k \cup_{\omega_i \in (a_j,b_j]} \{\frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_i) - b_q}{a_q} \in R_j\}\right). \end{aligned} \tag{7.1.7}$$

Now for any choice of d distinct integers $i_1, \ldots, i_d \in \{1, \ldots, q\}$ and integers $j_1, \ldots, j_d \in \{1, \ldots, k\}$ we have from (7.1.4) that

$$\mathrm{P}\left(\cap_{r=1}^d \{\frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_{i_r}) - b_q}{a_q} \in R_{j_r}\}\right) = q^{-d} \prod_{r=1}^d \lambda(R_{j_r})(1 + o(1)), \tag{7.1.8}$$

where $\lambda(\cdot)$ is the measure on $(-\infty, \infty]$ given by $e^{-x}dx$ and the relation is uniform over all d-tuples $i_1, \ldots, i_d$. Using elementary counting argument and (7.1.8), the sum of the probabilities of all collections of d distinct sets from the m sets that comprise the union in (7.1.7) is given by

$$\begin{aligned} S_d &= \sum_{\substack{(u_1,\ldots,u_k),\\ u_1+\cdots+u_k=d}} \binom{n_1}{u_1}\cdots\binom{n_k}{u_k} q^{-u_1}\lambda^{u_1}(R_1)\cdots q^{-u_k}\lambda^{u_k}(R_k)(1+o(1)) \\ &= \sum_{\substack{(u_1,\ldots,u_k),\\ u_1+\cdots+u_k=d}} \frac{1}{u_1!u_2!\cdots u_k!\pi^d}((b_1-a_1)\lambda(R_1))^{u_1}\cdots((b_k-a_k)\lambda(R_k))^{u_k}(1+o(1)) \\ &\to (d!)^{-1}\pi^{-d}((b_1-a_1)\lambda(R_1)+\cdots+(b_k-a_k)\lambda(R_k))^d. \end{aligned}$$

Now it follows that,

$$\begin{aligned} \sum_{j=1}^{2s}(-1)^{j-1}S_j &\stackrel{n\to\infty}{\longrightarrow} \sum_{j=1}^{2s}\frac{(-1)^{j-1}}{j!\pi^j}((b_1-a_1)\lambda(R_1)+\ldots+(b_k-a_k)\lambda(R_k))^j \\ &\stackrel{s\to\infty}{\longrightarrow} 1-\exp\big(-\sum_{j=1}^{k}(b_j-a_j)\pi^{-1}\lambda(R_j)\big), \end{aligned}$$

which by Bonferroni inequality and (7.1.7), proves (7.1.6).

Step 2: It remains to transfer the convergence of η_n^* onto η_n. First define the point process

$$\bar{\eta}_n(\cdot) = \sum_{j=1}^{q}\epsilon_{\left(\omega_j, \frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q}\right)}(\cdot) \quad\text{and}\quad \eta'_n(\cdot) = \sum_{j=1}^{q}\epsilon_{\left(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot).$$

It then suffices to show that (see Theorem 4.2 of Kallenberg (1983) [80])

$$\bar{\eta}_n - \eta_n^* \xrightarrow{\mathcal{P}} 0, \tag{7.1.9}$$

$$\bar{\eta}_n - \eta'_n \xrightarrow{\mathcal{P}} 0 \tag{7.1.10}$$

and

$$\eta'_n - \eta_n \xrightarrow{\mathcal{P}} 0. \tag{7.1.11}$$

Equivalently, that for any continuous function f on $[0, \pi] \times (-\infty, \infty]$ with compact support,

$$\bar{\eta}_n(f) - \eta_n^*(f) \xrightarrow{\mathcal{P}} 0\ ,\ \bar{\eta}_n(f) - \eta'_n(f) \xrightarrow{\mathcal{P}} 0,\ \text{and}\ \eta'_n(f) - \eta_n(f) \xrightarrow{\mathcal{P}} 0$$

where the notation $\eta(f)$ denotes $\int f d\eta$. Suppose the compact support of f is contained in the set $[0,\pi]\times[K+\gamma_0,\infty)$ for some $\gamma_0>0$ and $K\in\mathbb{R}$. Since f is uniformly continuous , $\omega(\gamma):=\sup\{|f(t,x)-f(t,y)|;\ t\in[0,\pi],\ |x-y|\le\gamma\}\to 0$ as $\gamma\to 0$.

Proof of (7.1.9): On the set $A_n=\{\max_{j=1,\ldots,q}|\frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j)}{a_q}-\frac{\lambda_{n,\bar{x}}(\omega_j)}{a_q}|\le\gamma\}$, we have for $\gamma<\gamma_0$,

$$\big|f(\omega_j,\frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q})-f(\omega_j,\frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q})\big|\le\begin{cases}\omega(\gamma) & \text{if } \frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q}>K\\ 0 & \text{if } \frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q}\le K.\end{cases} \tag{7.1.12}$$

Also note

$$\begin{aligned}
&\frac{1}{a_q}\max_{1\le j\le q}|\lambda_{n,\bar{x}+\sigma_n N}(\omega_j)-\lambda_{n,\bar{x}}(\omega_j)|\\
\le\ &\frac{1}{a_q}\max_{1\le j\le q}|\frac{\sigma_n}{\sqrt{n}}\sum_{t=1}^{n}N_t e^{i\omega_j t}|\\
\le\ &\frac{\sigma_n}{a_q}\max_{1\le j\le q}\sqrt{\frac{1}{n}(\sum_{t=1}^{n}N_t\cos\frac{2\pi kt}{n})^2+\frac{1}{n}(\sum_{t=1}^{n}N_t\sin\frac{2\pi kt}{n})^2}\\
\le\ &\frac{\sigma_n}{a_q}\max_{1\le j\le q}\sqrt{X_{1j}^2+X_{2j}^2}
\end{aligned}$$

where $\{X_{1j},X_{2j};1\le j\le q\}$ are i.i.d. $N(0,1)$. Now $\frac{\sigma_n}{a_q}\max_{1\le j\le q}\sqrt{X_{1j}^2+X_{2j}^2}=O_P(\sigma_n\ln n)$. Therefore $\lim_{n\to\infty}\mathrm{P}(A_n^c)=0$. For any $\epsilon>0$, choose γ sufficiently small that $\gamma<\gamma_0$. Define $B_n=\{|\bar{\eta}_n(f)-\eta_n^*(f)|>\epsilon\}$. Then

$$\begin{aligned}
\limsup_{n\to\infty}\mathrm{P}(B_n) &\le \limsup_{n\to\infty}(\mathrm{P}(B_n\cap A_n)+\mathrm{P}(A_n^c))\\
&\le \limsup_{n\to\infty}\mathrm{P}(\omega(\gamma)\eta_n^*([0,\pi]\times[K,\infty))>\epsilon)+\limsup_{n\to\infty}\mathrm{P}(A_n^c)\\
&\le \limsup_{n\to\infty}\mathrm{E}\,\eta_n^*([0,\pi]\times[K,\infty))\omega(\gamma)/\epsilon\\
&\le e^{-K}\omega(\gamma)/\epsilon.
\end{aligned}$$

Since $\omega(\gamma)\to 0$ as $\gamma\to 0$, (7.1.9) follows.

Proof of (7.1.10): This is essentially identical to the argument given for (7.1.9). For completeness we give the details. Define $C_n=\{\max_{1\le j\le q}|\frac{\lambda_{n,x}(\omega_j)}{a_q}-\frac{\lambda_{n,\bar{x}}(\omega_j)}{a_q}|<\gamma\}$. Again on the set C_n, we have for $\gamma<\gamma_0$

$$\big|f(\omega_j,\frac{\lambda_{n,x}(\omega_j)-b_q}{a_q})-f(\omega_j,\frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q})\big|\le\begin{cases}\omega(\gamma) & \text{if } \frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q}>K\\ 0 & \text{if } \frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q}\le K.\end{cases} \tag{7.1.13}$$

Now

$$|\lambda_{n,x}(\omega_j) - \lambda_{n,\bar{x}}(\omega_j)| = |\frac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t \mathbb{I}(|x_t| > n^{1/s}) e^{i\omega_j t}|$$

and hence

$$\begin{aligned}
\frac{1}{a_q}\,\mathrm{E}\{\max_{1\le j\le q} |\lambda_{n,x}(\omega_j) - \lambda_{n,\bar{x}}(\omega_j)|\} &\le \frac{1}{a_q}\,\mathrm{E}\{\max_{1\le j\le q} |\frac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t \mathbb{I}(|x_t| > n^{1/s}) e^{i\omega_j t}|\} \\
&\le \frac{\sqrt{\ln n}}{\sqrt{n}}\,\mathrm{E}\{\sum_{t=1}^{n} |x_t| \mathbb{I}(|x_t| > n^{1/s})\} \\
&\le \sqrt{n \ln n}\,\mathrm{E}\,|x_1| \mathbb{I}(|x_1| > n^{1/s}) \\
&= \sqrt{n \ln n}\big[n^{1/s}\,\mathrm{P}(|x_1| > n^{1/s}) + \int_{n^{1/s}}^{\infty} \mathrm{P}(X_1 > x)dx\big] \\
&\le \sqrt{n \ln n}\big[n^{1/s}\frac{\mathrm{E}\,|x_1|^s}{n} + \frac{\mathrm{E}\,|x_1|^s}{n^{1-1/s}}\big] \\
&\le 2\frac{\ln n}{n^{1/2-1/s}}\,\mathrm{E}\,|x_1|^s \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore $\mathrm{P}(C_n^c) \to 0$. Now for any $\epsilon > 0$, choose γ sufficiently small that $\gamma < \gamma_0$. Then by intersecting the event $\{|\eta_n(f) - \bar{\eta}_n(f)| > \epsilon\}$ with C_n and C_n^c, respectively and using (7.1.13) and (7.1.5), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \mathrm{P}(|\bar{\eta}_n(f) - \eta_n'(f)| > \epsilon) &\le \limsup_{n\to\infty} (\mathrm{P}(\omega(\gamma)\bar{\eta}_n([0,\pi]\times[K,\infty)) > \epsilon) + \mathrm{P}(C_n^c)) \\
&\le \limsup_{n\to\infty} \mathrm{E}\,\bar{\eta}_n([0,\pi]\times[K,\infty))\omega(\gamma)/\epsilon \\
&\le e^{-K}\omega(\gamma)/\epsilon.
\end{aligned}$$

Since $\omega(\gamma) \to 0$ as $\gamma \to 0$, (7.1.10) follows.

Proof of (7.1.11): Finally for any $\epsilon > 0$

$$\begin{aligned}
\mathrm{P}(|\eta_n'(f) - \eta_n(f)| > \epsilon) &= \mathrm{P}(|f(0, \frac{\lambda_{n,x}(\omega_0) - b_q}{a_q})| > \epsilon) \\
&\le \mathrm{P}(\frac{\lambda_{n,x}(\omega_0) - b_q}{a_q} \ge K) \\
&= \mathrm{P}(\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} x_l > K a_q + b_q) \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore $\eta_n - \eta_n' \xrightarrow{\mathcal{P}} 0$, that is (7.1.11) holds.

Since Step 1 and Step 2 are completely proved, the proof of Theorem 7.1.3 is over.

□

The relation $\eta_n \xrightarrow{\mathcal{D}} \eta$ immediately yields the joint weak convergence of a finite vector of k upper ordered eigenvalues. To be precise, we introduce for every n the ordered version of the eigenvalues $\lambda_{n,x}(\omega_j), j = 0, 1, \ldots, n-1$,

$$\lambda_{n,(q)} \leq \cdots \leq \lambda_{n,(2)} \leq \lambda_{n,(1)}.$$

Let $x_k < \cdots < x_1$ be any real numbers, and write $N_{i,n} = \eta_n([0,\pi] \times (x_i, \infty))$ for the number of exceedances of x_i by $\frac{\lambda_{n,x}(\omega_j) - b_q}{a_q}$, $j = 1, \ldots, q$. Then

$$\{\frac{\lambda_{n,(1)} - b_q}{a_q} \leq x_1, \ldots, \frac{\lambda_{n,(k)} - b_q}{a_q} \leq x_k\} = \{N_{1,n} = 0, N_{2,n} \leq 1, \ldots, N_{k,n} \leq k-1\}.$$

Then the joint limit distribution of the vector of the k upper ordered eigenvalues $\lambda_{n,x}(\omega_j)$ as well as their spacings can be derived from Theorem 7.1.3.

Corollary 7.1.5. *Under the assumption of Theorem 7.1.3,*

(i) for any real numbers $x_k < \cdots < x_2 < x_1$,

$$\mathrm{P}\left(\frac{\lambda_{n,(1)} - b_q}{a_q} \leq x_1, \cdots, \frac{\lambda_{n,(k)} - b_q}{a_q} \leq x_k\right) \to \mathrm{P}(Y_{(1)} \leq x_1, \cdots, Y_{(k)} \leq x_k),$$

where $(Y_{(1)}, \cdots, Y_{(k)})$ *has the density* $\exp(-\exp(-x_k) - (x_1 + \cdots + x_{k-1}))$.

(ii) $\left(\frac{\lambda_{n,(i)} - \lambda_{n,(i-1)}}{a_q}\right)_{i=1,\ldots,k} \xrightarrow{\mathcal{D}} (i^{-1}E_i)_{i=1,\ldots,k}$ *where* $\{E_i\}$ *is an i.i.d standard exponential sequence.*

Proof. The proof is similar to the proof of Theorem 4.2.8 of Embrechts, Kluppelberg and Mikosch (1997) [56]. We just briefly sketch the steps. We have already seen that for finite k,

$$\begin{aligned} \mathrm{P}(\frac{\lambda_{n,(1)} - b_q}{a_q} \leq x_1, \ldots, \frac{\lambda_{n,(k)} - b_q}{a_q} \leq x_k) &= \mathrm{P}(N_{1,n} = 0, N_{2,n} \leq 1, \ldots, N_{k,n} \leq k-1) \\ &\to \mathrm{P}(N_1 = 0, N_2 \leq 1, \cdots, N_k \leq k-1), \end{aligned}$$

where $N_i = \eta([0,\pi] \times (x_i, \infty])$. Let us denote $Z_i = \eta([0,\pi] \times (x_i, x_{i-1}])$ with $x_0 = \infty$. Now observe that to calculate $\mathrm{P}(N_1 = 0, N_2 \leq 1, \cdots, N_k \leq k-1)$, it is enough to consider $\mathrm{P}(N_1 = a_1, N_2 = a_1 + a_2, \cdots, N_k = a_1 + \cdots + a_k)$, where $a_i \geq 0$ and

$$\begin{aligned} &\mathrm{P}(N_1 = a_1, N_2 = a_1 + a_2, \cdots, N_k = a_1 + \cdots + a_k) \\ &= \mathrm{P}(Z_1 = a_1, Z_2 = a_2, \cdots, Z_k = a_k) \\ &= \frac{(e^{-x_1})^{a_1}}{a_1!} \frac{(e^{-x_2} - e^{-x_1})^{a_2}}{a_2!} \cdots \frac{(e^{-x_k} - e^{-x_{k-1}})^{a_k}}{a_k!} e^{-e^{-x_k}}. \end{aligned}$$

This proves Part (i). Part (ii) is an easy consequence of Part (i). □

7.1.2 Symmetric circulant

Let $\lambda_{n,x}(\omega_0), \lambda_{n,x}(\omega_1), \ldots, \lambda_{n,x}(\omega_{n-1})$ be the eigenvalues of $n^{-1/2}SC_n$. These eigenvalues are given by (see Section 1.2.2):
(i) for n odd:

$$\begin{cases} \lambda_{n,x}(\omega_0) & = \frac{1}{\sqrt{n}}\left[x_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} x_j\right] \\ \lambda_{n,x}(\omega_k) & = \frac{1}{\sqrt{n}}\left[x_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} x_j \cos(\omega_k j)\right], \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \end{cases} \tag{7.1.14}$$

(ii) for n even:

$$\begin{cases} \lambda_{n,x}(\omega_0) & = \frac{1}{\sqrt{n}}\left[x_0 + 2\sum_{j=1}^{\frac{n}{2}-1} x_j + x_{n/2}\right] \\ \lambda_{n,x}(\omega_k) & = \frac{1}{\sqrt{n}}\left[x_0 + 2\sum_{j=1}^{\frac{n}{2}-1} x_j \cos(\omega_k j) + (-1)^k x_{n/2}\right], \quad 1 \leq k \leq \frac{n}{2} \end{cases} \tag{7.1.15}$$

with $\lambda_{n,x}(\omega_{n-k}) = \lambda_{n,x}(\omega_k)$ in both cases.

Now define a sequence of point processes based on the points $(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q})$ for $k = 0, 1, \ldots, q(= \lfloor \frac{n}{2} \rfloor)$, where $\lambda_{n,x}$ are as in (7.1.14). Note that we have not considered the eigenvalues λ_{n-k} for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ to define the point process since $\lambda_{n,x}(\omega_{n-k}) = \lambda_{n,x}(\omega_k)$ for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ and it does not affect our goal of finding the limit distribution of upper order eigenvalues. Define

$$\eta_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot) \tag{7.1.16}$$

where

$$b_n = c_n + a_n \ln 2, \; a_n = (2\ln n)^{-1/2} \text{ and } c_n = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln 4\pi}{2(2\ln n)^{1/2}}. \tag{7.1.17}$$

Theorem 7.1.6 (Bose, Hazra and Saha (2010) [35])**.** *Let $\{x_t\}$ be i.i.d random variables which satisfy Assumption 7.1.2. Then for the sequence of point processes η_n defined in (7.1.16), we have $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, \pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1}dt \times e^{-x}dx$.*

We shall use Lemma 4.3.3, a strong approximation result similar to Lemma 4.4.6, in the proof of Theorem 7.1.6.

Proof of Theorem 7.1.6. The idea of the proof is similar to that of Theorem 7.1.3. So we mention only the main steps and a few technical details. We first establish convergence in distribution for the point process based on the points $(\omega_j, \frac{\lambda'_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q})$ for $j = 1, 2, \ldots, q$, where

$$\lambda'_{n,\bar{x}+\sigma_n N}(\omega_j) = \frac{1}{\sqrt{n}}\big[\sqrt{2}(\bar{x}_0 + \sigma_n N_0) + 2\sum_{t=1}^{\lfloor \frac{n}{2} \rfloor}(\bar{x}_t + \sigma_n N_t)\cos\frac{2\pi jt}{n}\big], \quad 0 \le j \le \lfloor \frac{n}{2} \rfloor.$$

Define

$$\eta_n^*(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\lambda'_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q}\right)}(\cdot).$$

We first show $\eta_n^* \xrightarrow{\mathcal{D}} \eta$. Since the limit process η is simple, it suffices to show (7.1.5) and (7.1.6) for above η_n^*. We can establish them following arguments similar to those given in the proof of Theorem 7.1.3 and using Lemma 4.3.3.

Now define the following point processes

$$\bar{\eta}'_n(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\lambda'_{n,\bar{x}}(\omega_j)-b_q}{a_q}\right)}(\cdot), \quad \bar{\eta}_n(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q}\right)}(\cdot),$$

$$\eta'_n(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot),$$

where

$$\lambda'_{n,\bar{x}}(\omega_j) = \frac{1}{\sqrt{n}}\big[\sqrt{2}\bar{x}_0 + 2\sum_{t=1}^{\lfloor \frac{n}{2} \rfloor}\bar{x}_t\cos\frac{2\pi jt}{n}\big], \quad 0 \le j \le \lfloor \frac{n}{2} \rfloor,$$

and $\{\lambda_{n,\bar{x}}(\omega_j)\}$ are given in (7.1.14) with x_t replaced by $\bar{x}_t$. As before it now suffices to show that (see Theorem 4.2 of Kallenberg (1983) [80])

$$\bar{\eta}'_n - \eta_n^* \xrightarrow{\mathcal{P}} 0, \quad \bar{\eta}_n - \bar{\eta}'_n \xrightarrow{\mathcal{P}} 0, \quad \bar{\eta}_n - \eta'_n \xrightarrow{\mathcal{P}} 0 \text{ and } \eta'_n - \eta_n \xrightarrow{\mathcal{P}} 0. \tag{7.1.18}$$

For the first relation in (7.1.18) define $A_n = \{\max_{1\le j\le q}|\lambda'_{n,\bar{x}}(\omega_j) - \lambda_{n,\bar{x}+\sigma_n N}(\omega_j)| \le \gamma\}$ and observe that

$$\max_{1\le j\le q}|\lambda'_{n,\bar{x}}(\omega_j) - \lambda_{n,\bar{x}+\sigma_n N}(\omega_j)| = \frac{\sigma_n}{\sqrt{n}}\max_{1\le j\le q}|\sqrt{2}N_0 + 2\sum_{t=1}^{\lfloor n/2 \rfloor} N_t\cos\frac{2\pi jt}{n}| = O_p(\sigma_n \ln n).$$

Hence $\mathrm{P}(A_n^c) \to 0$. The remaining argument is similar to the proof of (7.1.9). For the second relation note that

$$\mathrm{P}\big(\max_{1\le j\le q}|\lambda_{n,\bar{x}}(\omega_j) - \lambda'_{n,\bar{x}}(\omega_j)| > \epsilon\big) \le \mathrm{P}\big(\frac{(\sqrt{2}-1)|x_0|}{\sqrt{n}} > \epsilon\big) \to 0.$$

Proof of the third and fourth relations are similar to the proofs of (7.1.10) and (7.1.11) in the proof of Theorem 7.1.3. □

Note that a result similar to Corollary 7.1.5 holds with $\{\lambda_{n,(i)}\}_{1\le i\le k}$ as the ordered eigenvalues of the symmetric circulant matrix. Here we skip the proof.

Corollary 7.1.7. *Under the assumption of Theorem 7.1.6,*

(i) for any real numbers $x_k < \cdots < x_2 < x_1$*,*

$$\mathrm{P}\left(\frac{\lambda_{n,(1)}-b_q}{a_q} \le x_1, \cdots, \frac{\lambda_{n,(k)}-b_q}{a_q} \le x_k\right) \to \mathrm{P}(Y_{(1)} \le x_1, \cdots, Y_{(k)} \le x_k),$$

where $(Y_{(1)}, \cdots, Y_{(k)})$ *has the density* $\exp(-\exp(-x_k) - (x_1 + \cdots + x_{k-1}))$.

(ii) $\left(\frac{\lambda_{n,(i)}-\lambda_{n,(i-1)}}{a_q}\right)_{i=1,\ldots,k} \xrightarrow{\mathcal{D}} (i^{-1}E_i)_{i=1,\ldots,k}$ *where* $\{E_i\}$ *is an i.i.d standard exponential sequence.*

7.1.3 k-circulant, $n = k^2 + 1$.

First recall the eigenvalues of the k-circulant matrix $A_{k,n}$. For any positive integers k, n, let $p_1 < p_2 < \ldots < p_c$ be all their common prime factors so that,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here α_q, $\beta_q \ge 1$ and n', k', p_q are pairwise relatively prime. Then the characteristic polynomial of $A_{k,n}$ (whence its eigenvalues follow) is given by

$$\chi(A_{k,n}) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} (\lambda^{n_j} - y_j), \qquad (7.1.19)$$

where y_j, n_j are as defined in Section 1.2.4.

For simplicity, here we consider k-circulant matrix only for $n = k^2 + 1$. One can consider point process based on eigenvalues of k-circulant matrix for $n = k^g + 1$ where $g > 2$ and can prove result similar to Theorem 7.1.9. But for general $g > 2$ algebraic details will be much more complicated.

In the present case, clearly $n' = n$ and $k' = k$. From Lemma 2.4.16 and (2.4.24) of Chapter 2, $g_1 = 4$ and the eigenvalue partition of $\{0, 1, 2, \ldots, n-1\}$ contains exactly $q = \lfloor \frac{n}{4} \rfloor$ sets of size 4 and each set is self-conjugate. Moreover, if k is even then there is only one more partition set containing only 0, and if k is odd then there are two more partition sets containing only 0 and only $n/2$ respectively.

For the development of the point process we need a clear picture of the eigenvalue partition of $\{0, 1, 2, \ldots, n-1\}$. For this we represent the set $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ in the following form

$$\mathbb{Z}_n = \{ak + b; 0 \leq a \leq k-1, 1 \leq b \leq k\} \cup \{0\}. \tag{7.1.20}$$

Then we can write $S(x)$ defined in (1.2.4) as follows

$$S(ak+b) = \{ak+b, bk-a, n-ak-b, n-bk+a\};\ 0 \leq a \leq k-1,\ 1 \leq b \leq k.$$

Lemma 7.1.8. *For $n = k^2 + 1$,*

$$\mathbb{Z}_n = \bigcup_{0 \leq a \leq \lfloor \frac{k-2}{2} \rfloor, a+1 \leq b \leq k-a-1} S(ak+b) \bigcup S(0),\ \text{if } k \text{ is even} \tag{7.1.21}$$

and

$$\mathbb{Z}_n = \bigcup_{0 \leq a \leq \lfloor \frac{k-2}{2} \rfloor, a+1 \leq b \leq k-a-1} S(ak+b) \bigcup S(0) \bigcup S(n/2),\ \text{if } k \text{ is odd} \tag{7.1.22}$$

where all $S(ak+b)$ are mutually disjoint and hence form an eigenvalue partition of $\mathbb{Z}_n$.

Proof. First observe that $S(0) = \{0\}$ and $S(n/2) = \{n/2\}$ if k is odd and

$$\#\{x :\ x \in S(ak+b); 0 \leq a \leq \lfloor \frac{k-2}{2} \rfloor, a+1 \leq b \leq k-a-1\} = \begin{cases} n-1 & \text{if} \quad k \text{ even} \\ n-2 & \text{if} \quad k \text{ odd.} \end{cases}$$

So if we can show that $S(ak+b); 0 \leq a \leq \lfloor \frac{k-2}{2} \rfloor, a+1 \leq b \leq k-a-1$ are mutually disjoint then we are done. We shall show $S(a_1k+b_1) \cap S(a_2k+b_2) = \emptyset$ for $a_1 \neq a_2$ or $b_1 \neq b_2$. We divide the proof into four different cases.

Case (i) $(a_1 < a_2, b_1 > b_2)$ Note that

$$a_1 + 1 < a_2 + 1 \leq b_2 < b_1 \leq k - (a_1 + 1).$$

Since $\{S(x); 0 \leq x \leq n-1\}$ forms a partition of $\mathbb{Z}_n$, it is enough to show that $a_1k+b_1 \notin S(a_2k+b_2)$. As $(a_2-a_1)k > k$ and $(b_1-b_2) < k$, we have $a_1k+b_1 \neq a_2k+b_2$. Also $(b_2-a_1)k \geq 2k$ and $a_2+b_1 \leq \lfloor \frac{k-2}{2} \rfloor + k - (a_1+1) \leq \frac{3k}{2}$, therefore $a_1k+b_1 \neq b_2k-a_2$.

Note that

$$\begin{aligned} a_1k + b_1 + a_2k + b_2 &\le (a_1 + a_2)k + 2k - 2(a_1 + 1) \\ &\le 2\lfloor \frac{k-2}{2} \rfloor k + 2k - 2(a_1 + 1) \\ &\le k^2 - 2k + 2k - 2(a_1 + 1) \\ &< k^2 + 1 = n. \end{aligned}$$

Therefore $a_1k + b_1 \neq n - (a_2k + b_2)$. Similarly,

$$a_1k + b_1 + b_2k - a_2 \le a_1k + k - (a_1 + 1) + (k - (a_2 + 1))k - a_2 < k^2 + 1 = n$$

and therefore $a_1k+b_1 \neq n-(b_2k-a_2)$. Hence in this case $S(a_1k+b_1) \cap S(a_2k+b_2) = \emptyset$.

Case (ii) $(a_1 < a_2, b_1 < b_2)$ In this case it is very easy to see that $a_1k+b_1 \notin S(a_2k+b_2)$ and hence $S(a_1k + b_1) \cap S(a_2k + b_2) = \emptyset$.

Case (iii) $(a_1 = a_2, b_1 < b_2)$ Let $a_1 = a_2 = a$. Obviously $ak + b_1 \neq ak + b_2$. Since $0 \le a \le \lfloor \frac{k-2}{2} \rfloor$ and $a+1 \le b_1 < b_2 \le k-(a+1)$, we have $(b_2-a)k \ge 2k > (a+b_1)$. Hence $ak+b_1 \neq b_2k-a$. Also $2ak+b_1+b_2 \le k(k-2)+2k = k^2 < n$, so $ak+b_1 \neq n-(ak+b_2)$. Finally,

$$b_1 + b_2k + ak - a \le [k - (a + 1)](k + 1) + ak - a = k^2 - 2a - 1 < k^2 + 1 = n,$$

implies $ak+b_1 \neq n-(b_2k-a)$. Hence $ak+b_1 \notin S(ak+b_2)$ and $S(a_1k+b_1) \cap S(a_2k+b_2) = \emptyset$.

Case (iv) $(a_1 < a_2, b_1 = b_2)$ In this case also it is very easy to show that $S(a_1k + b_1) \cap S(a_2k + b_2) = \emptyset$. This completes the proof. □

Now we are ready to define our point process based on the eigenvalues of the k-circulant matrix. For our purpose we neglect $\{0, n/2\}$ if n is even and $\{0\}$ if n is odd. Denote

$$S = \mathbb{Z}_n - \{0, n/2\},\ T_n = \{(a, b) : 0 \le a \le \lfloor \frac{k-2}{2} \rfloor, a + 1 \le b \le k - (a + 1)\},$$

$$\lambda_t(x) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp(\frac{2\pi jt}{n}),\ \beta_{x,n}(a, b) = \prod_{t \in S(ak+b)} \lambda_t(x) \text{ and } \lambda_x(a, b) = (\beta_{x,n}(a, b))^{1/4}.$$

Now define a sequence of point process based on the points $\{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_x(a,b)-d_q}{c_q}) : (a, b) \in$

$T_n\}$. Define

$$\eta_n(\cdot) = \sum_{(a,b)\in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_x(a,b)-d_q}{c_q}\right)}(\cdot) \tag{7.1.23}$$

where $q = q(n) = \lfloor \frac{n}{4} \rfloor$ and

$$c_n = (8\ln n)^{-1/2} \text{ and } d_n = \frac{(\ln n)^{1/2}}{\sqrt{2}}\left(1 + \frac{1}{4}\frac{\ln\ln n}{\ln n}\right) + \frac{1}{2(8\ln n)^{1/2}} \ln\frac{\pi}{2}. \tag{7.1.24}$$

Theorem 7.1.9 (Bose, Hazra and Saha (2010) [35])**.** *Let $\{x_t\}$ be i.i.d random variables which satisfy Assumption 7.1.2. Then for the sequence of point processes η_n defined in (7.1.23), we have $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, 1/2] \times [0, 1] \times [0, \infty]$ with intensity measure $4\mathbb{I}_{\{s\le t\le 1-s\}}e^{-x}dsdtdx$.*

Proof. Though the main idea of the proof is similar to the proof of Theorem 7.1.3, the details are more complicated. We do it in two steps.

Step 1: We first establish convergence in distribution for the point process based on the points $\{\left(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\right) : (a,b) \in T_n\}$ where $\lambda_{\bar{x}+\sigma_n N}(a,b)$ is obtained from $\lambda_x(a,b)$ replacing $\{x_i\}$ by $\{\bar{x}_i + \sigma_n N_i\}$. Define

$$\eta_n^*(\cdot) = \sum_{(a,b)\in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\right)}(\cdot).$$

We show $\eta_n^* \xrightarrow{\mathcal{D}} \eta$. Observe that first two components of the limit is uniformly distributed over a triangle whose vertices are $(0,0), (1/2,1/2), (0,1)$. Denote this triangle by $\triangle$. Since the limit process is simple it suffices to show that

$$\mathrm{E}\,\eta_n^*((a_1,b_1]\times(a_2,b_2]\times(x,y]) \to \mathrm{E}\,\eta((a_1,b_1]\times(a_2,b_2]\times(x,y]) \tag{7.1.25}$$

for all $0 \le a_1 < b_1 \le 1/2$, $0 \le a_2 < b_2 \le 1$ and $x < y$, and for all $l \ge 1$,

$$\mathrm{P}(\eta_n^*((a_1,b_1]\times(c_1,d_1]\times R_1) = 0,\ldots,\eta_n^*((a_l,b_l]\times(c_l,d_l]\times R_l) = 0) \tag{7.1.26}$$

$$\longrightarrow \mathrm{P}(\eta((a_1,b_1]\times(c_1,d_1]\times R_1) = 0,\ldots,\eta((a_l,b_l]\times(c_l,d_l]\times R_l) = 0),$$

where $\cap_{i=1}^{l}(a_i,b_i]\times(c_i,d_i] = \emptyset$ and $R_1,\ldots,R_l$ are bounded Borel sets, each consisting of a finite union of intervals on $[0,\infty]$.

Proof of (7.1.25): We shall first prove condition (7.1.25) for the following type of sets:

(i) $(a_1,b_1]\times(a_2,b_2]$ lies entirely inside the triangle $\triangle$.

(ii) $(a_1, b_1] \times (a_1, b_1]$ where $0 \le a_1 < b_1 \le 1/2$.
(iii) $(a_1, b_1] \times (1 - b_1, 1 - a_1]$ where $0 \le a_1 < b_1 \le 1/2$.
(iv) $(a_1, b_1] \times (a_2, b_2]$ lies entirely outside of the triangle $\triangle$.

Graphically the mentioned boxes are as in Figure 1.

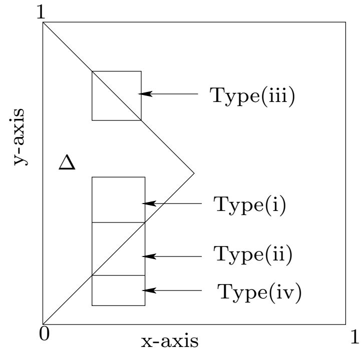

Figure 1

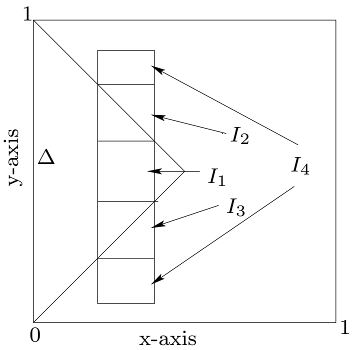

Figure 2

Figure 1 shows four types of basic sets and Figure 2 shows the decomposition of a rectangle into these four types of sets.

Since any rectangles in $[0, 1/2] \times [0, 1]$ can be expressed as disjoint union of these four kinds of sets (see Figure 2), it is sufficient to prove (7.1.25) and (7.1.26) for the above four kind of boxes only. Let I_i denote i-th type of set. Enough to prove that for each i, as $n \to \infty$, $\mathrm{E}\,\eta_n^*(I_i \times (x, y]) \to \mathrm{E}\,\eta(I_i \times (x, y])$.

(a) Proof of (7.1.25) for Type (i) sets:

$$\begin{aligned}
& \mathrm{E}\,\eta_n^*((a_1, b_1] \times (a_2, b_2] \times (x, y]) \\
= \;& \mathrm{E}\Big(\sum_{(a,b) \in T_n} \epsilon_{\big(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}+\sigma_n N(a,b)} - d_q}{c_q}\big)}((a_1, b_1] \times (a_2, b_2] \times (x, y])\Big) \\
= \;& \sum_{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}) \in (a_1, b_1] \times (a_2, b_2]} \mathrm{P}\Big(\frac{\lambda_{\bar{x}+\sigma_n N}(a, b) - d_q}{c_q}) \in (x, y]\Big) \\
\sim \;& (b_1 - a_1)(b_2 - a_2) n \frac{1}{q}(e^{-x} - e^{-y})(1 + o(1)) \\
\to \;& 4(b_1 - a_1)(b_2 - a_2)(e^{-x} - e^{-y}) \\
= \;& \mathrm{E}\,\eta((a_1, b_1] \times (a_2, b_2] \times (x, y]).
\end{aligned}$$

(b) Proof of (7.1.25) for Type (ii) sets:

$$\begin{aligned}
& \mathrm{E}\,\eta_n^*((a_1,b_1]\times(a_1,b_1]\times(x,y]) \\
= \;& \mathrm{E}\,\Big(\sum_{(a,b)\in T_n} \epsilon_{\big(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\big)}((a_1,b_1]\times(a_1,b_1]\times(x,y])\Big) \\
= \;& \sum_{(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_1,b_1]\times(a_1,b_1]} \mathrm{P}\,\Big(\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q})\in(x,y]\Big) \\
\sim \;& \frac{1}{2}(b_1-a_1)(b_1-a_1)n\frac{1}{q}(e^{-x}-e^{-y})(1+o(1)) \\
\to \;& \frac{1}{2}(b_1-a_1)^2 4(e^{-x}-e^{-y}) \\
= \;& \mathrm{E}\,\eta((a_1,b_1]\times(a_1,b_1]\times(x,y]).
\end{aligned}$$

(c) Proof of (7.1.25) for Type (iii) sets is exactly similar as Type (ii) sets.

(d) Proof of (7.1.25) for Type (iv) sets:

$$\begin{aligned}
& \mathrm{E}\,\eta_n^*((a_1,b_1]\times(a_2,b_2]\times(x,y]) \\
= \;& \mathrm{E}\,\Big(\sum_{(a,b)\in T_n} \epsilon_{\big(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\big)}((a_1,b_1]\times(a_2,b_2]\times(x,y])\Big) \\
= \;& \sum_{(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_1,b_1]\times(a_2,b_2]} \mathrm{P}\,\Big(\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q})\in(x,y]\Big) \\
= \;& 0 = \mathrm{E}\,\eta((a_1,b_1]\times(a_2,b_2]\times(x,y]),
\end{aligned}$$

since $\{(a,b)\in T_n : (\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_1,b_1]\times(a_2,b_2]\}=\emptyset$. This completes the proof of (7.1.25).

Proof of (7.1.26): We prove (7.1.26) for the four types of sets separately.

(a) Proof of (7.1.26) for Type (i) sets: $(a_i,b_i]\times(c_i,d_i]$ lies completely inside the triangle $\triangle$ for all $i=1,2,\ldots,l$. Let

$$\begin{aligned}
n_j &= \#\{(a,b):(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_j,b_j]\times(c_j,d_j]\} \\
&\sim \sqrt{n}(b_j-a_j)\sqrt{n}(d_j-c_j)=n(b_j-a_j)(d_j-c_j).
\end{aligned}$$

Then the complement of the event in (7.1.26) is the union of $m=n_1+\ldots+n_l$ events, that is

$$\begin{aligned}
& 1-\mathrm{P}\left(\eta_n^*((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(c_l,d_l]\times R_l)=0\right) \\
= \;& \mathrm{P}\left(\cup_{j=1}^{l}\cup_{(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_j,b_j]\times(c_j,d_j]}\{\frac{\lambda_{\bar{x}+\sigma_n N-d_q}}{c_q}\in R_j\}\right).
\end{aligned}$$

Now following the argument to prove (7.1.6) given in Theorem 7.1.3, we get

$$\begin{aligned}
& \mathrm{P}\left(\eta_n^*((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(c_l,d_l]\times R_l)=0\right) \\
& \stackrel{n\to\infty}{\longrightarrow} \exp\{-\sum_{j=1}^{l}(b_j-a_j)(d_j-c_j)4\lambda(R_j)\} \\
& = \mathrm{P}(\eta((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta((a_l,b_l]\times(c_l,d_l]\times R_l)=0).
\end{aligned}$$

This proves (7.1.26) for Type (i) sets.

(b) Proof of (7.1.26) for Type (ii) sets: Here $c_i=a_i$, $d_i=b_i$ and

$$\begin{aligned}
n_j &= \#\{(a,b):(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_j,b_j]\times(a_j,b_j]\} \\
&\sim \frac{1}{2}\sqrt{n}(b_j-a_j)\sqrt{n}(b_j-a_j)=\frac{n}{2}(b_j-a_j)^2.
\end{aligned}$$

Remaining part of the proof is as in the previous case. Finally we get

$$\begin{aligned}
& \mathrm{P}\left(\eta_n^*((a_1,b_1]\times(a_1,b_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(a_l,b_l]\times R_l)=0\right) \\
& \stackrel{n\to\infty}{\longrightarrow} \exp\{-\sum_{j=1}^{l}\frac{1}{2}(b_j-a_j)^2 4\lambda(R_j)\} \\
& = \mathrm{P}(\eta((a_1,b_1]\times(a_1,b_1]\times R_1)=0,\ldots,\eta((a_l,b_l]\times(a_l,b_l]\times R_l)=0).
\end{aligned}$$

(c) Proof of (7.1.26) for Type (iii) sets is same as Type (ii) sets.

(d) Finally we prove it for Type (iv) sets. In this case $(a_i,b_i]\times(c_i,d_i]\bigcap\triangle=\emptyset$ for all $i=1,\ldots,l$. Note that for all i, $\#\{(a,b)\in T_n:(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_i,b_i]\times(c_i,d_i]\}=0$ and therefore

$$\mathrm{P}\left(\eta_n^*((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(c_l,d_l]\times R_l)=0\right)=1.$$

Also from intensity measure of η,

$$\mathrm{P}(\eta((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta((a_l,b_l]\times(c_l,d_l]\times R_l)=0)=1.$$

Hence (7.1.26) is proved for all four types of sets separately. Consequently, the proof of Step 1 is complete.

Step 2: It remains to transfer the convergence of η_n^* onto η_n. First define the following process

$$\bar{\eta}_n(\cdot) = \sum_{(a,b)\in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}}(a,b)-d_q}{c_q}\right)}(\cdot).$$

Then it suffices to show that for any continuous function f on $[0,1/2]\times[0,1]\times[0,\infty)$ with compact support,

$$\bar{\eta}_n(f) - \eta_n^*(f) \xrightarrow{\mathcal{P}} 0 \text{ and } \bar{\eta}_n(f) - \eta_n(f) \xrightarrow{\mathcal{P}} 0. \tag{7.1.27}$$

Suppose the compact support of f is contained in the set $[0,1/2]\times[0,1]\times[K+\gamma_0,\infty)$ for some $\gamma_0 > 0$ and $K \in \mathbb{R}$. Since f is uniformly continuous, $\omega(\gamma) := \sup\{|f(s,t,x) - f(s,t,y)|;\ s\in[0,1/2],\ t\in[0,1],\ |x-y|\le\gamma\} \to 0$ as $\gamma \to 0$.

Proof of $\bar{\eta}_n(f) - \eta_n^*(f) \xrightarrow{\mathcal{P}} 0$**:**
On the set $A_n = \{\max_{(a,b)\in T_n} |\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)}{c_q} - \frac{\lambda_{\bar{x}}(a,b)}{c_q}| \le \gamma\}$, we have for $\gamma < \gamma_0$,

$$\begin{aligned} &\left|f\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}+\sigma_n N}(a,b) - d_q}{c_q}\right) - f\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}}(a,b) - d_q}{c_q}\right)\right| \\ &\le \begin{cases} \omega(\gamma) & \text{if } \frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j) - b_q}{a_q} > K \\ 0 & \text{if } \frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j) - b_q}{a_q} \le K. \end{cases} \end{aligned} \tag{7.1.28}$$

Now if $\mathrm{P}(A_n^c) \to 0$, then using (7.1.28)

$$\limsup_{n\to\infty} \mathrm{P}\left(|\eta_n^*(f) - \bar{\eta}_n(f)| > \epsilon\right) \le \frac{\omega(\gamma)}{\epsilon} 4e^{-K} \to 0, \quad \text{as } \gamma \to 0.$$

Now we show $\mathrm{P}(A_n^c) \to 0$. For any sequence of random variables $(X_i)_{0\le i<n}$, define

$$M_n(X) = \max_{1\le t\le n} \left| n^{-1/2} \sum_{l=0}^{n-1} X_l \exp(i2\pi t l/n) \right|.$$

We can use the basic inequalities

$$\big||z_1 z_2| - |w_1 w_2|\big| \le (|z_1| + |w_2|) \max\{|z_1 - w_1|, |z_2 - w_2|\}, \tag{7.1.29}$$

and

$$||w_1|^{1/2} - |w_2|^{1/2}| \le |w_1 - w_2|^{1/2}, \quad z_i, w_i \in \mathbb{C},\ 1 \le i \le 2, \tag{7.1.30}$$

to obtain

$$\begin{aligned}
&\max_{a,b} \big|\lambda_{\bar{x}+\sigma_n N}(a,b) - \lambda_{\bar{x}}(a,b)\big| \\
&\leq \big[(M_n(\bar{x}+\sigma_n N))^{1/2} + (M_n(\bar{x}))^{1/2}\big](M_n(\sigma_n N))^{1/2} \quad \text{(by (7.1.29))} \\
&\leq \big[2(M_n(\bar{x}+\sigma_n N))^{1/2} + (M_n(\sigma_n N))^{1/2}\big](M_n(\sigma_n N))^{1/2} \quad \text{(by (7.1.30))}.
\end{aligned}$$

By Davis and Mikosch (1999) [50], we have

$$M_n^2(\sigma_n N) = O_p(\sigma_n^2 \ln n) \text{ and } M_n^2(\bar{x}+\sigma_n N) = O_p(\ln n),$$

with $\sigma_n^2 = n^{-c}$. Therefore

$$\max_{a,b} \frac{1}{c_q} \big|\lambda_{\bar{x}+\sigma_n N}(a,b) - \lambda_{\bar{x}}(a,b)\big| = O_p((\ln n) n^{-c/2}).$$

Hence

$$\begin{aligned}
\mathrm{P}(A_n^c) &= \mathrm{P}\big(\max_{a,b} \big|\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)}{c_q} - \frac{\lambda_{\bar{x}}(a,b)}{c_q}\big| > \epsilon\big) \\
&= \mathrm{P}\big(\frac{n^{c/4}}{\ln n} \max_{a,b} \frac{1}{c_q} \big|\lambda_{\bar{x}+\sigma_n N}(a,b) - \lambda_{\bar{x}}(a,b)\big| > \frac{\epsilon n^{c/4}}{\ln n}\big) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This completes the proof of $\bar{\eta}_n(f) - \eta_n^*(f) \xrightarrow{\mathcal{P}} 0$.
The other part of (7.1.27) follows from the fact (4.4.9) of Chapter 4. This completes the proof of Step 2 and hence the theorem is completely proved. □

Let $x_k < \cdots < x_1$ be real numbers, and let $N_{i,n} = \eta_n([0, \frac{1}{2}] \times [0,1] \times (x_i, \infty))$ be the number of exceedances of x_i by $\frac{\lambda_x(a,b) - d_q}{c_q}$. Then the joint distribution of the k upper order eigenvalues can be written in terms of $\{N_{i,n}\}_{1 \leq i \leq k}$. From this it is easy to derive distributional convergence of the k upper order eigenvalues. Hence a similar result as Corollary 7.1.5 holds with $\{\lambda_{n,(i)}\}_{1 \leq i \leq k}$ representing the ordered eigenvalues of k-circulant matrix.

7.2 Results for dependent input

Let $\{x_n; n \geq 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i} \tag{7.2.1}$$

where $\{a_n; n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |a_n| < \infty$, are non-random and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. random variables. Let $f(\omega),\ \omega \in [0, 2\pi]$ be the spectral density of $\{x_n\}$. Note that if $\{x_n\}$ is i.i.d. with mean 0 and variance σ^2, then $f \equiv \frac{\sigma^2}{2\pi}$.

It seems to be a non-trivial problem to derive Poisson convergence of the point processes based on eigenvalues of the matrices with such dependent entries. As seen in Chapter 6, an individual scaling of each eigenvalue is needed. We resort to scaling each eigenvalue by the spectral density at the appropriate ordinate, as done in Chapter 6 and then consider their point process.

We shall prove our next theorems under the following assumption on the two sided moving average process $\{x_n\}$ defined in (7.2.1).

Assumption 7.2.1. $\{\epsilon_i; i \in \mathbb{Z}\}$ *are i.i.d. random variables with* $\mathrm{E}(\epsilon_0) = 0$, $\mathrm{E}(\epsilon_0^2) = 1$ *and* $\mathrm{E}\,|\epsilon_0|^s < \infty$ *for some* $s > 2$ *and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi].$$

7.2.1 Reverse circulant

Let $\lambda_{n,x}(\omega_k)$ be the eigenvalues of $n^{-1/2}RC_n$ defined in (7.1.1). Define the sequence of point processes based on the points $\tilde{\lambda}_{n,x}(\omega_k) = \frac{\lambda_{n,x}(\omega_k)}{\sqrt{2\pi f(\omega_k)}}$ as

$$\tilde{\eta}_n(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\tilde{\lambda}_{n,x}(\omega_k) - b_q}{a_q}\right)}(\cdot) \tag{7.2.2}$$

where $a_q = \frac{1}{2\sqrt{\ln q}}$, $b_q = \sqrt{\ln q}$ and $q = q(n) = \lfloor \frac{n}{2} \rfloor$.

Theorem 7.2.2 (Bose, Hazra and Saha (2010) [35]). *Let $\{x_n\}$ be the two sided moving average process defined in (7.2.1) and which satisfies Assumption 7.2.1. Then for the sequence of point processes $\tilde{\eta}_n$ defined in (7.2.2), we have $\tilde{\eta}_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, \pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1}dt \times e^{-x}dx$.*

Proof. First observe that $\min_{\omega \in [0,2\pi]} f(\omega) > \alpha > 0$. We define another sequence of point process based on the points $\left(\omega_k, \frac{\lambda_{n,\epsilon}(\omega_k) - b_q}{a_q}\right)$ for $k = 1, 2, \ldots, q$ where $\lambda_{n,x}(\omega_k)$ are the eigenvalues of $n^{-1/2}RC_n$ with x_i replaced by ϵ_i. Define

$$\eta_n(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,\epsilon}(\omega_j) - b_q}{a_q}\right)}(\cdot). \tag{7.2.3}$$

In Theorem 7.1.3, we have shown that $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, \pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1}dt \times e^{-x}dx$. Now if we can show that $\tilde{\eta}_n - \eta_n \xrightarrow{\mathcal{P}} 0$, then we will be through. Equivalently, we have to show that for any continuous function g on E with compact support,

$$\tilde{\eta}_n(g) - \eta_n(g) \xrightarrow{\mathcal{P}} 0$$

as $n \to \infty$. Suppose the compact support of g is contained in the set $[0,\pi] \times [K+\gamma_0, \infty)$ for some $\gamma_0 > 0$ and $K \in \mathbb{R}$. Since g is uniformly continuous , $\omega(\gamma) := \sup\{|g(t,x) - g(t,y)|;\ t \in [0,1],\ |x-y| \le \gamma\} \to 0$ as $\gamma \to 0$. On the set $A_n = \{\max_{j=1,\ldots,q} |\frac{\lambda_{n,x}(\omega_j)}{a_q\sqrt{2\pi f(\omega_j)}} - \frac{\lambda_{n,\epsilon}(\omega_j)}{a_q}| \le \gamma\}$, we have for $\gamma < \gamma_0$,

$$\left|g(\omega_j, \frac{\tilde{\lambda}_{n,x}(\omega_j) - b_q}{a_q}) - g(\omega_j, \frac{\lambda_{n,\epsilon}(\omega_j) - b_q}{a_q})\right| \le \begin{cases} \omega(\gamma) & \text{if } \frac{\lambda_{n,\epsilon}(\omega_j) - b_q}{a_q} > K \\ 0 & \text{if } \frac{\lambda_{n,\epsilon}(\omega_j) - b_q}{a_q} \le K. \end{cases} \tag{7.2.4}$$

Observe

$$\begin{aligned} & \frac{1}{a_q} \max_{1\le j\le q} |\frac{\lambda_{n,x}(\omega_j)}{\sqrt{2\pi f(\omega_j)}} - \lambda_{n,\epsilon}(\omega_j)| \\ \le\ & \frac{1}{\alpha a_q} \max_{1\le j\le q} |\lambda_{n,x}(\omega_j) - \sqrt{2\pi f(\omega_j)}\lambda_{n,\epsilon}(\omega_j)| \\ \le\ & \frac{1}{\alpha a_q} \max_{1\le j\le q} |\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l e^{i\omega_j l} - (\sum_{t=-\infty}^{\infty} a_t e^{i\omega_j t}) \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \epsilon_l e^{i\omega_j l}| \end{aligned}$$

and from (6.1.5), we have

$$\max_{1\le j\le q} |\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l e^{i\omega_j l} - (\sum_{t=-\infty}^{\infty} a_t e^{i\omega_j t}) \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \epsilon_l e^{i\omega_j l}| = o_p(n^{-1/4}).$$

Therefore $\lim_{n\to\infty} \mathrm{P}(A_n^c) = 0$. Now, for any $\delta > 0$, choose γ sufficiently small that $\gamma < \gamma_0$. Then, by intersecting the event $\{|\tilde{\eta}_n(g) - \eta_n(g)| > \delta\}$ with A_n and A_n^c and using (7.2.4), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \mathrm{P}(|\tilde{\eta}_n(g) - \eta_n(g)| > \delta) &\le \limsup_{n\to\infty} (\mathrm{P}(\{|\tilde{\eta}_n(g) - \eta_n(g)| > \delta\} \cap A_n) + \mathrm{P}(A_n^c)) \\ &\le \limsup_{n\to\infty} \mathrm{P}(\omega(\gamma)\eta_n([0,\pi] \times [K,\infty)) > \epsilon) + \limsup_{n\to\infty} \mathrm{P}(A_n^c) \\ &\le \limsup_{n\to\infty} \mathrm{E}\,\eta_n([0,\pi] \times [K,\infty))\omega(\gamma)/\epsilon \le e^{-K}\omega(\gamma)/\epsilon. \end{aligned}$$

Since $\omega(\gamma) \to 0$ as $\gamma \to 0$, $\tilde{\eta}_n - \eta_n \xrightarrow{\mathcal{P}} 0$. □

7.2.2 Symmetric circulant

Here we consider the two sided moving average process defined in (7.2.1) with an extra assumption that $a_j = a_{-j}$ for all $j \in \mathbb{N}$. Define

$$\tilde{\eta}_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\omega_j, \frac{\tilde{\lambda}_{n,x}(\omega_j) - b_q}{a_q}\right)}(\cdot) \tag{7.2.5}$$

where $q = q(n) \sim \frac{n}{2}$, $\tilde{\lambda}_{n,x}(\omega_j) = \frac{\lambda_{n,x}(\omega_j)}{\sqrt{2\pi f(\omega_j)}}$ and $\lambda_{n,x}(\omega_j)$ are the eigenvalues of symmetric circulant matrix given in (7.1.14) and a_q, b_q are as in (7.1.17).

Theorem 7.2.3 (Bose, Hazra and Saha (2010) [35]). *Let $\{x_n\}$ be the two sided moving average process defined in (7.2.1) with $a_j = a_{-j}$ and satisfies Assumption 7.2.1. Then for the sequence of point processes $\tilde{\eta}_n$ defined in (7.2.5), we have $\tilde{\eta}_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, \pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1}dt \times e^{-x}dx$.*

Proof. The line of argument is similar as in Theorem 7.2.2. We omit the details but mention that to show $\lim_{n\to\infty} \mathrm{P}(A_n^c) = 0$, we use the following fact from (6.1.10) of Chapter 6:

$$\max_{1 \le k \le \lfloor \frac{n}{2} \rfloor} \left| \frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}} - \lambda_{k,\epsilon} \right| = o_p(n^{-1/4}).$$

□

7.2.3 k-circulant, $n = k^2 + 1$.

First recall the eigenvalues of k-circulant matrix for $n = k^2 + 1$ given in Section (7.1.3) and define following notation based on that

$$\beta_{\epsilon,n}(a,b) = \prod_{t \in S(ak+b)} \lambda_t(\epsilon), \quad \lambda_\epsilon(a,b) = (\beta_{\epsilon,n}(a,b))^{1/4},$$

$$\tilde{\beta}_{x,n}(a,b) = \frac{\prod_{t \in S(ak+b)} \lambda_t(x)}{4\pi^2 f(\omega_{ak+b}) f(\omega_{bk-a})} \quad \text{and} \quad \tilde{\lambda}_x(a,b) = (\tilde{\beta}_{x,n}(a,b))^{1/4}.$$

Now with $q = q(n) = \lfloor \frac{n}{4} \rfloor$ and $d_q,\ c_q$ as in (7.1.24), define our point process based on points $\{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\tilde{\lambda}_x(a,b) - d_q}{c_q}) : (a,b) \in T_n\}$ as:

$$\tilde{\eta}_n(\cdot) = \sum_{(a,b) \in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\tilde{\lambda}_x(a,b) - d_q}{c_q}\right)}(\cdot). \tag{7.2.6}$$

Theorem 7.2.4 (Bose, Hazra and Saha (2010) [35]). *Let $\{x_n\}$ be the two sided moving average process defined in (7.2.1) and which satisfies Assumption 7.2.1. Then for the* *sequence of point processes* $\tilde{\eta}_n$ *defined in (7.2.5), we have* $\tilde{\eta}_n \stackrel{\mathcal{D}}{\to} \eta$*, where* η *is a Poisson process on* $[0,1/2] \times [0,1] \times [0,\infty]$ *with intensity measure* $4\mathbb{I}_{\{s \le t \le 1-s\}} e^{-x} dsdtdx$*.*

Proof. First define a point process based on $\{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_\epsilon(a,b)-d_q}{c_q}) : (a,b) \in T_n\}$,

$$\eta_n(\cdot) = \sum_{(a,b)\in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_\epsilon(a,b)-d_q}{c_q}\right)}(\cdot).$$

First note that in Theorem 7.1.9, we have shown that $\eta_n \stackrel{\mathcal{D}}{\to} \eta$, where η is a Poisson process on $[0,1/2] \times [0,1] \times (-\infty,\infty]$ with intensity measure $4\mathbb{I}_{\{s \le t \le 1-s\}} e^{-x} dsdtdx$. Rest of the argument is similar to the proof of Theorem 7.2.2. The additional point that needs to be noted is that $\mathrm{P}(\max_{(a,b)\in T_n} \left|\tilde{\lambda}_x(a,b) - \lambda_\epsilon(a,b)\right| > \gamma) \to 0$ follows from the proof of Theorem 6.1.16 of Chapter 6. □

Chapter 8

Few remarks and further research

In this chapter we indicate a list of problem that arise in the context of this thesis for circulant and related matrices. We list the problems topicwise for future study and hope that our article will generate activity in this interesting area of research.

8.1 Limiting spectral distribution

As discussed in Chapter 0, the study of the limit of the empirical spectral distribution of random matrices when the dimension tends to infinity has a long history, specially where the input sequence has light tails. But there have been very few works where the input sequence has heavy tails. This thesis also concentrated on the light tailed case.

For the Wigner matrix, when the input sequence belongs to the domain of attraction of an α stable law with $\alpha \in (0, 2)$, Ben Arous and Guionnet (2008) [26] showed that the LSD exists in probability and it has heavy tails. Later Belinschi, Dembo and Guionnet (2009) [25] studied some symmetric band matrices and the sample variance covariance matrices with heavy tailed inputs. In both these articles the LSD was shown to be nonrandom.

In Chapter 2 we have discussed the LSD of the scaled eigenvalues of circulant type matrices when the input sequence is i.i.d. with finite moments of suitable order. We then also derived the LSD of these matrices when the input sequence is a stationary, two sided moving average process of infinite order.

What is the limiting behaviour of the ESD of circulant type matrices when the input sequence has heavy tails?

The LSD of the reverse circulant and circulant matrices with the (i.i.d.) input sequence belonging to the domain of attraction of an α stable law with $\alpha \in (0,\ 2)$ was shown to exist in Bose, Chatterjee and Gangopadhyay (2003) [30] using the methods of Freedman and Lane (1981) [60]. Knight (1991) [83] has been able to obtain some very nice representation of the empirical distribution of periodogram entries of $\{X_i, 1 \leq i \leq n\}$ and provide its limiting distribution including a representation for the limit. If the empirical measure of $\{X_j\}$ converges in distribution, then for any continuous function f, the empirical measure of $\{f(X_j)\}$ also converges in distribution. Since the eigenvalues of the reverse circulant matrix are the square root of the periodogram entries of $\{X_i, 1 \leq i \leq n\}$, the LSD of the reverse circulant matrix follows from Theorem 5 of Knight (1991) [83] with an appropriate choice of f.

Very recently, Bose et al. (2011) [32] have extended the above works and considered the k-circulant matrix for $k^g = n \pm 1$. Assuming that the input sequence belongs to the domain of attraction of an α stable law with $\alpha \in (0,\ 2)$, they have shown that the LSD exists. They also determined explicit representations of the limits. We now briefly describe their results.

Assumption on the input sequence : Suppose that the input sequence $\{X_i\}$ is defined on a probability space $(\Omega, \mathcal{A}, P)$. Suppose it is i.i.d. in the domain of attraction of a stable law with index $\alpha \in (0,2)$, that is, there exists $a_n \to \infty$ such that

$$a_n^{-1} \sum_{k=1}^{n} (X_k - c_n) \xrightarrow{\mathcal{D}} S_\alpha,$$

where S_α is a stable random variable and $c_n = \mathrm{E}[X_1 I(|X_1| \leq a_n)]$.

It is well known that a random variable X is in the domain of attraction of a (nonnormal) stable law with index $\alpha \in (0,2)$ if and only if $\mathrm{P}[|X| > t] = t^{-\alpha} l(t)$, for some slowly varying l and

$$\lim_{t \to \infty} \frac{\mathrm{P}[X > t]}{\mathrm{P}[|X| > t]} = p \in [0,1]. \tag{8.1.1}$$

Also the normalizing constants a_n are such that

$$n\,\mathrm{P}[|X| > a_n x] \to x^{-\alpha}.$$

8.1.1 k-circulant with $n = k^g + 1$ (heavy tailed input)

We now analyze the eigenvalues for this particular case in more detail. First suppose $n = k^2 + 1$. Then from Bose, Mitra and Sen (2008) [44], if k is even then there is one singleton partition set $\{0\}$ and if k is odd then there are two singleton partition sets $\{0\}$ and $\{n/2\}$ respectively; all the remaining partitions have four elements each. Thus apart from these finitely many (hence negligible) singleton partitions, all others are of equal size of four.

In general for $n = k^g + 1,\ g \geq 1$, the eigenvalue partition (see Section 4.1 of Bose, Mitra and Sen (2008) [44]) of $\{0, 1, 2, \ldots, n-1\}$ contains approximately $q = [\frac{n}{2g}]$ sets each of size $(2g)$ and each set is self-conjugate; in addition, the remaining sets do not contribute to the LSD. We shall call the partition sets of size $(2g)$ as *major partition sets*.

We shall now use these major partition sets and express the eigenvalues in a convenient form. This is given in the following Lemma (see Bose et al. (2011) [32]) for easy reference. To do this, observe that a typical $S(x)$ may be written as

$$S(b_1 k^{g-1} + b_2 k^{g-2} + \cdots + b_g)$$

which in turn is the union of the following two sets

$$\left\{b_1 k^{g-1} + b_2 k^{g-2} + \cdots + b_g, b_2 k^{g-1} + b_3 k^{g-2} + \cdots + b_g k - b_1, \ldots, b_g k^{g-1} - b_1 k^{g-2} - \cdots - b_{g-1}\right\}$$

and its conjugate i.e.

$$\left\{n - (b_1 k^{g-1} + b_2 k^{g-2} + \cdots + b_g), \ldots, n - (b_g k^{g-1} - b_1 k^{g-2} - \cdots - b_{g-1})\right\}$$

where

$$0 \le b_1 \le k-1, \ \ldots, 0 \le b_{g-1} \le k-1 \ \text{ and } \ 1 \le b_g \le k.$$

Define

$$T_n = \{(b_1, b_2, \ldots b_g) : \ 0 \le b_1 \le k-1, \ldots, 1 \le b_g \le k\},$$

$$C_t = \sum_{j=1}^{n} X_j \cos(\frac{2\pi j t}{n}) \quad \text{and} \quad S_t = \sum_{j=1}^{n} X_j \sin(\frac{2\pi j t}{n}) \ \text{ for } \ t \in \mathbb{N}.$$

So far the scaling constant $\{a_n\}$ has been taken to be $n^{1/2}$ (see Chapter 2) but now since the entries are heavy tailed, the square root scaling is not the appropriate scaling any longer.

Lemma 8.1.1. *The eigenvalues of the k-circulant $a_n^{-1} A_{k,n}$ with $n = k^g + 1$ corresponding to the major partition sets may be written as*

$$\left\{\lambda_{(b_1, b_2, \ldots, b_g)}, \lambda_{(b_1, b_2, \ldots, b_g)} \omega_{2g}, \ldots, \lambda_{(b_1, b_2, \ldots, b_g)} \omega_{2g}^{2g-1} : (b_1, b_2, \ldots, b_g) \in T_n\right\}$$

where ω_{2g} is the primitive $(2g)$-th root of unity and

$$\lambda_{(b_1, b_2, \ldots, b_g)} = a_n^{-1} \left(C^2_{b_1 k^{g-1} + \cdots + b_g} + S^2_{b_1 k^{g-1} + \cdots + b_g}\right)^{1/2g} \cdots \left(C^2_{b_g k^{g-1} - \cdots - b_{g-1}} + S^2_{b_g k^{g-1} - \cdots - b_{g-1}}\right)^{1/2g}.$$

In view of Lemma 8.1.1, to find the LSD of the k-circulant $a_n^{-1} A_{k,n}$ where $n = k^g + 1$, it suffices to consider the ESD of $\left\{\lambda_{(b_1, b_2, \ldots, b_g)} : \ (b_1, \ldots, b_g) \in T_n\right\}$: if these have an LSD

F, then the LSD of $a_n^{-1}A_{k,n}$ will be (r,θ) in polar coordinates where r is distributed according to F, and θ is distributed uniformly across all the $(2g)$-th roots of unity and r and θ are independent. With this in mind, define

$$L_{A_{k,n}}(A,\omega) \quad = \quad \frac{1}{|T_n|}\sum_{(b_1,\ldots,b_g)\in T_n} I(\lambda_{(b_1,\ldots,b_g)}\in A).$$

Further, let $\{\Gamma_j\}$, $\{B_j\}$, $\{U_j\}$, $\{U_j^*\}$ and $\{U_{t,j}^*\}$, be independent random sequences defined on the same probability space where $\Gamma_j = \sum_{i=1}^{j} E_i$ and $\{E_i\}$ is a sequence of i.i.d. exponential with mean 1, and B_j are i.i.d. satisfying $\mathrm{P}[B_1 = 1] = p = 1 - \mathrm{P}[B_1 = -1]$ where p is defined by equation (8.1.1) and the rest of the variables are i.i.d $U(0,1)$. Finally, let

$$Z_j = \Gamma_j^{-1/\alpha} = \left(\sum_{t=1}^{j} E_t\right)^{-1/\alpha} \quad \text{and} \quad \mu_t = \mathrm{E}[B_t Z_t I(Z_t \le 1)].$$

We now state the following theorem. A typical element of Ω will be denoted by ω.

Theorem 8.1.2 (Bose et al. (2011) [32])**.** *Suppose g is fixed and $n = k^g + 1$. Then $L_{A_{k,n}} \xrightarrow{\mathcal{D}} L_{A_k}$ as $n\to\infty$, $L_{A_k}(\cdot,\omega)$ being the random distribution induced by $L_1(\omega)^{\frac{1}{2g}}L_2(\omega)^{\frac{1}{2g}}\cdots L_g(\omega)^{\frac{1}{2g}}$ and*

$$L_j(\omega) = \left(\sum_{t=1}^{\infty}\sin(2\pi U_{t,j}^*)B_t(\omega)Z_t(\omega)\right)^2 + \left(\sum_{t=1}^{\infty}\cos(2\pi U_{t,j}^*)B_t(\omega)Z_t(\omega)\right)^2, \; 1\le j\le g.$$

8.1.2 k-circulant with $n = k^g - 1$ (heavy tailed input)

Now there are approximately $q = [\frac{n}{g}]$ major partition sets, each of size g. For detailed illustration see Bose, Mitra and Sen (2008) [44]. The major partition sets $\{S(x)\}$ may now be listed as

$$\{b_1k^{g-1}+b_2k^{g-2}+\cdots+b_g,\, b_2k^{g-1}+b_3k^{g-2}+\cdots+b_gk+b_1,\ldots,b_gk^{g-1}+b_1k^{g-2}+\cdots+b_{g-1}\}$$

where $0\le b_1\le k-1,\ldots,0\le b_{g-1}\le k-1, 1\le b_g\le k.$, with

$$(b_1,b_2,...,b_g)\neq(k-1,k-1,...,k-1) \text{ and } (b_1,b_2,...,b_g)\neq(k-1,k-1,\ldots,k-1,k).$$

Now define

$$T_n' = \big\{(b_1,b_2,...b_g) \;:\; 0\le b_1\le k-1,...,1\le b_g\le k,\; (b_1,b_2,...,b_g)\neq(k-1,k-1,...,k-1) \\ \text{and } (b_1,b_2,...,b_g)\neq(k-1,k-1,\ldots,k-1,k)\big\},$$

$$\gamma_{(b_1,b_2,\ldots,b_g)} = a_n^{-1}\left(C_{b_1k^{g-1}+\cdots+b_g} + iS_{b_1k^{g-1}+\cdots+b_g}\right)\ldots\left(C_{b_gk^{g-1}+\cdots+b_{g-1}} + iS_{b_gk^{g-1}+\cdots+b_{g-1}}\right),$$
$$\eta_{(b_1,b_2,\ldots,b_g)} = |\gamma_{(b_1,b_2,\ldots,b_g)}|^{1/g}\exp\{\frac{i\arg\left(\gamma_{(b_1,b_2,\ldots,b_g)}\right)}{g}\}.$$

Then the eigenvalues of the k-circulant $a_n^{-1}A_{k,n}$ with $n = k^g - 1$ corresponding to the partition set $S(b_1k^{g-1} + b_2k^{g-2} + \cdots + b_g)$ are

$$\eta_{(b_1,b_2,\ldots,b_g)}, \eta_{(b_1,b_2,\ldots,b_g)}\omega_g, \eta_{(b_1,b_2,\ldots,b_g)}\omega_g^2, \ldots, \eta_{(b_1,b_2,\ldots,b_g)}\omega_g^{g-1}$$

where ω_g is g-th root of unity. So, to find the LSD, it suffices to consider the ESD of $\{\gamma_{(b_1,b_2,\ldots,b_g)} : (b_1, \ldots, b_g) \in T_n'\}$: if these have an LSD F, then the LSD of $a_n^{-1}A_{k,n}$ will be (r', θ) where r' is distributed according to $h(F)$ where $h(z) = |z|^{1/g}e^{\frac{i\arg(z)}{g}}$ and θ is distributed uniformly across all the g-th roots of unity, and r' and θ are independent. Hence define

$$L_{A_{k,n}}(A, \omega) = \frac{1}{|T_n'|}\sum_{(b_1,\ldots,b_g)\in T_n'} I(\gamma_{(b_1,\ldots,b_g)} \in A).$$

Theorem 8.1.3 (Bose et al. (2011) [32]). *Suppose g is fixed and $n = k^g - 1$. Then $L_{A_{k,n}} \xrightarrow{\mathcal{D}} L_{A_k}$ as $n \to \infty$, $L_{A_k}(., \omega)$ being the random distribution induced by $L_1(\omega)L_2(\omega)\ldots L_g(\omega)$, and*

$$L_j(\omega) = \left(\sum_{t=1}^{\infty}\cos(2\pi U_{t,j}^*)B_t(\omega)Z_t(\omega)\right) + i\left(\sum_{t=1}^{\infty}\sin(2\pi U_{t,j}^*)B_t(\omega)Z_t(\omega)\right), 1 \le j \le g.$$

8.1.3 Symmetric circulant matrix with heavy tailed input

The (i, j)-th element of the symmetric circulant, SC_n is given by $X_{n/2+1-|n/2-|i-j||}$. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $a_n^{-1}SC_n$. Then the ESD of $a_n^{-1}SC_n$ is given by

$$L_{SC_n}(A, \omega) = \frac{1}{n}\sum_{j=0}^{n-1} I(\lambda_j \in A).$$

Theorem 8.1.4 (Bose et al. (2011) [32]). *As $n \to \infty$, $L_{SC_n} \xrightarrow{\mathcal{D}} L_{SC}$, where $L_{SC}(\cdot, \omega)$ is the distribution of* $2\sum_{t=1}^{\infty}\cos(2\pi U_t^*)B_t(\omega)Z_t(\omega)$.

The method of proof of the above results (Theorem 8.1.2, 8.1.3, 8.1.4) heavily relies on the extension of the results of Freedman and Lane (1981) [60] accomplished in Knight (1991) [83] together with some intricate study of the eigenvalue structure of the k-circulant with $n = k^g \pm 1$.

8.1.4 k-circulant with $n \neq k^g \pm 1$

We have already discussed in Section 2.3.1 that establishing the LSD for general k-circulant matrices is a difficult problem even with light tailed entries. For general k and n, the eigenvalue partitions have different sizes and varied compositions, and hence establishing the LSD is much more difficult for both light tailed and heavy tailed entries. It is an open problem.

Bose, Mitra and Sen (2008) [44] showed that the radial component of the LSD of k-circulants with $k \geq 2$ is always degenerate, at least when the input sequence is i.i.d. normal, as long as $k = n^{o(1)}$ and $\gcd(k, n) = 1$.

Theorem 8.1.5 (Bose, Mitra and Sen (2008) [44])**.** *Suppose the input sequence is i.i.d.* $N(0, 1)$ *random variables. Let* $k \geq 2$ *be such that* $k = n^{o(1)}$ *and* $n \to \infty$ *with* $\gcd(n, k) = 1$*. Then* $F_{n^{-1/2}A_{k,n}}$ *converges weakly in probability to the uniform distribution over the circle with center at* $(0, 0)$ *and radius* $r = \exp(\mathrm{E}[\log \sqrt{E}])$*,* E *being an exponential random variable with mean one.*

The proof of this result uses the normality of the variables very crucially. It would be interesting to establish this result when the normality assumption is dropped. It would also be interesting to find the other possible choices of (k, n) for which the LSD has degenerate radial component.

8.2 Spectral radius and spectral norm

As we have seen, the behaviour of the extreme eignevalues of general large dimensional random matrices is a very nontrivial issue. The class of k-circulants admit a formula solution for its eigenvalues. This helped in the study of the extreme values but the issue of non Gaussianity of the entries was taken care of after considerable amount of approximation by the Gaussian case. Even then this required the finiteness of a moment of order larger than two. Moreover, these results were proved only for certain subclasses of the k-circulants. There are some results (see Section 4.1, 5.4) known for the related Toeplitz and Hankel matrices but even there, a host of unanswered questions remain.

For the Toeplitz matrices with mean zero entries nothing is known about the limiting distribution of the spectral norm (after centering and scaling). As seen in Section 4.1 only the almost sure and in probability convergence (see Remark 4.1.2) of spectral norm is known. It would be nice to find appropriate centering and scaling in such a case. Similar questions can be asked about the Hankel matrices. Moreover, even for the almost sure convergence results, the results are not completely sharp and the exact limits if any, are not known. It would also be interesting to study the limiting behaviour of the extreme eigenvalues of palindromic Toeplitz matrices and Toeplitz matrices with band structure.

Results on spectral radius of k-circulant matrices were proved in Section 4.4 for the case when $n = k^g \pm 1$ with $g \geq 1$. It would be interesting to find out what happens for other combinations of k and n.

As mentioned in the beginning of Chapter 4, k-circulant matrices with $k = 1, n-1$ (circulant matrix and reverse circulant matrix respectively) are normal matrices and hence their spectral norm and spectral radius are same. The limiting behaviour of spectral norm has been derived from the behaviour of the spectral radius (see Section 4.2). The behaviour of the spectral norm for other k-circulant matrices is not known as the matrices in such cases are non-normal matrices.

In Chapter 5 we derived the behaviour of the spectral norm of circulant and reverse circulant matrices when the input sequence is in the domain of attraction of α stable law with $0 < \alpha < 1$. Results for the case $1 \leq \alpha \leq 2$ are not known. In the heavy tailed case no results for the spectral norm and spectral radius of k-circulant matrices is known even for the case when $n = k^2 + 1$.

It is interesting to study the spectral norm and spectral radius when one goes out of the independent regime. Suppose the input sequence $\{x_n\}$ is an infinite order moving average process, $x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}$, where $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. with $E(\epsilon_i) = 0$ and $V(\epsilon_i) = 1$. It seems to be a nontrivial problem to derive properties of the spectral norm and spectral radius in this case. The spectral density is expected to appear in some form in the limit. This seems to be a difficult problem.

We obtained some results (see Chapter 6) when one resorts to scaling each eigenvalue by the spectral density at the appropriate ordinate and then considering their maximum. This scaling has the effect of equalizing the variance of the eigenvalues. However, it is not known what happens if we consider the maximum without such scaling.

In Section 6.1.2, for SC_n with inputs from a linear process we have shown that the maximum of the eigenvalues over certain subsets converges in distribution to the Gumbel distribution. For instance, in Theorem 6.1.11, we have shown that if $\lambda_{k,x}$ denote the eigenvalues of $\frac{1}{\sqrt{n}} SC_n$ with input $\{x_i\}$ then

$$\frac{\max_{k \in L_n^1} \lambda_{k,x} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda$$

and

$$\frac{\max_{k \in L_n^2} \lambda_{k,x} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where

$$L_n^1 = \{k \in L_n : k \text{ is even}\} \text{ and } L_n^2 = \{k \in L_n : k \text{ is odd}\},$$

$$L_n = \{k : 1 \le k \le \lfloor np_n/2 \rfloor\} \text{ and } p_n = (1 - \frac{1}{n^{1/2+\delta_1}}), \quad 0 < \delta_1 < 1/2.$$

However, it is not clear what will happen to this limiting distribution when maximum is taken over all the eigenvalues. This is an interesting open problem.

8.3 Poisson convergence

In Section 7.1.3 we saw that a detailed study of the eigenvalues (see Lemma 7.1.8) was used to exhibit the point process convergence for k-circulant matrices with $n = k^2 + 1$. This explicit study for the eigenvalue partition is not known for $n = k^g + 1$ when $g > 2$. If this study is accomplished then one can expect a point process convergence result similar to Theorem 7.1.9 in this case.

In Section 7.2 we have considered the point processes based on the eigenvalues scaled by the spectral density. The above Poisson convergence results immediately imply that results similar to Corollary 7.1.5 hold for the corresponding ordered values in each case. However, it is not at all obvious how to derive the joint distributional convergence of k upper ordered eigenvalues in this dependent situation.

In Theorem 7.2.2, for the point process convergence of the eigenvalues of SC_n we needed an extra assumption $a_j = a_{-j}$ on the process $\{x_t\}$. We have also seen in Theorem 6.1.7 of Chapter 6 that without this extra assumption the distributional convergence of the maximum of properly scaled eigenvalues of SC_n is not known. Similarly, here also it is not clear whether Theorem 7.2.2 will hold without this assumption.

8.4 Minimum of the eigenvalues

As discussed in Chapter 0 it is much harder to study the convergence of the *smallest* eigenvalue of random matrices. For S matrix it was studied by Silverstein (1985) [110], Bai and Yin (1993) [18]. For circulant type matrices also it is not at all obvious how the minimum of the absolute eigenvalues behave, and the answer is not known in general. Recently Bose, Hazra and Saha (2011) [40] shed some light for a specific subclass of k-circulant matrices when the input sequence is Gaussian. They established the following result on distributional convergence of the minimum of modulus of eigenvalues of circulant, reverse circulant, symmetric circulant and k-circulant matrices for $n = k^g + 1$ with Gaussian entries.

Theorem 8.4.1 (Bose, Hazra and Saha (2011) [40]). *Suppose $\{x_i\}_{i \ge 0}$ are i.i.d. standard normal random variables. Consider any one of the circulant type matrices $\{B_n\}$ with* *the input* $\{x_i\}$. *Then as* $n \to \infty$,

$$\frac{\min_{1 \le i \le n} |\lambda_i|}{c_q} \xrightarrow{\mathcal{D}} F, \tag{8.4.1}$$

where $\{\lambda_i, 1 \le i \le n\}$ *are the eigenvalues of* $n^{-1/2} B_n$ *and for*

(i) $B_n = RC_n$ *or* $B_n = C_n$,

$$q = q(n) = [\frac{n-1}{2}], \quad c_q = q^{-1/2} \quad \text{and} \quad F(x) = 1 - \exp(-x^2),$$

(ii) $B_n = SC_n$,

$$q = q(n) = [\frac{n}{2}], \quad c_q = \sqrt{\frac{2}{\pi}} q^{-1} \quad \text{and} \quad F(x) = 1 - \exp(-x),$$

(iii) $B_n = A_{k,n}$ *with* $n = k^g + 1$,

$$q = q(n) = \frac{n}{2g}, \quad c_q = q^{-1/2g} (\log q)^{-\frac{g-1}{2g}} \quad \text{and} \quad F(x) = 1 - \exp(-x^{2g}).$$

A similar result for non-Gaussian entries is not known. The normal approximation results used in Chapter 4 for the spectral radius do not seem to be able to salvage the situation for the minimum. Similarly, the behaviour when the input sequence is heavy tailed is also not known. Also, no results for the minimum of modulus of eigenvalues of k-circulant matrices is known when $n = k^g - 1$.

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