THEORETICAL INPUT-OUTPUT ANALYSIS :
Three Generalisations

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Abstract

This study "generalises" input-output (IO) analysis in the sense of developing its basic theme and central idea in some specific directions. Three broad directions are explored, under the titles of "scale-dependence of IO coefficients", "structural break", and "investment and growth consistency". In each case the analysis keeps to the "logic" of IO analysis in the sense of its approach, concepts and methods. This includes in particular the adaptation of a certain basic method of IO analysis to analyse the structure of relations obtained in each case.

The study is divided into five parts where the first provides a detailed introduction to, and the last a unified review of, the generalisations reported in the three parts in between. The first part spells out the "basic theme" and "central idea" of IO analysis as seen in this study and initiates the ideas behind our generalisations in broad terms. It also provides a rather comprehensive, though purposive, overview of "theoretical IO analysis" in an attempt to make the stu conceptually and analytically self-contained. Included in this overview are accounts of endogenous treatments of consumption, investment and foreign trade from an IO standpoint.

The generalisation of Part II is essentially formal in character. It treats IO coefficients as _variables_ dependent upon the levels

of production in an IO model, and thereby removes its twin basic assumptions of "constant returns to scale" and "no externalities". The properties of the resulting non-linear IO model are analysed in details from the standpoints of both production theory and price theory, in close parallel to standard IO analysis in terms of the linear model.

Unlike Part II, Parts III and IV bring in _new_ relations within the folds of IO analysis, deploying a common conceptual frame of reference. The relations are concerned essentially with capacity, technology and investment, and the frame is designed to enable one to view these in purely ex-ante terms. IO-type models of product-balances (of varying generality depending upon the exact problem at hand) are set up for a definite future (called "terminal period"), and the required capacities are thought to be brought into existence by suitable investments over the time-span separating the "terminal period" from the present, called the "base period". In particular, these investments can bring in new technologies in pre-existing sectors of production as well as new capacities in previous "empty sectors".

The just mentioned qualitative changes are analysed in Part III under the heading of structural break. In terms of the frame, the break is from the _given structure of the base_ to a set of _open possibilities_ at the terminal. The openness comes by treating the relevant _investment decisions_ as essentially free, and the problem really is _what sorts of structural break to have, if at all_, i.e., one of _choice_. The choice issues are then resolved by a set of _secondary relations_ incorporating specific choice criteria. These criteria work back from the prospective scales of production at the terminal period, which in turn are governed by the contemporaneous final demand. A number of models are developed to focus on specific aspects of the structural break. Unlike Parts II and IV, these aspects do not constitute any systematic development of a single theme, and the models are to be seen as a sample of exercises around the notion of structural break rather than a comprehensive treatment of it.

One particular characteristic of the models of Part III is that there is no direct impact of investment for product-balances in the terminal period. The investment programme lies implicitly behind and is reflected at most in certain cost elements entering the choice criteria. Part IV, on the other hand, brings the product-use aspect of investment to the forefront. Here investment is seen exclusively as a means of expanding base capacities, relieving the initial capacity constraint on production over time. That is, the problem here is one of growth which is entailed by the given final demands of the terminal period vis-a-vis the given capacities of the base period. Now, while the requisite investment has to take place prior to the terminal period, the terminal investment in its turn is not viewed independently. Basically, it is seen as continuation of a single investment programme whose basic task is to achieve a balance between capacities and production in the terminal period. This is done by assuming a constant rate of growth of capacity for each sector. These growth rates are then treated as genuine unknowns of the problem, determined simultaneously with terminal investment and production. That is, the growth rates are then made consistent with the required terminal production levels as seen from base capacities. This part also allows for possible excess capacities at the terminal period, and for an independent restriction on growth rates with a corresponding parametric adjustment of final demands. It ends with a critical review of the technical basis of the so-called Static Multi-sectoral Planning Models which share a common problem area with this part of the study.

All the models developed in Parts II-IV are rigorously examined in the respective places. The basic method of IO analysis plays a key role in this respect. This method — which is an iterative method of solving an IO model — provides the common approach route to the detailed analysis of the models formalising the generalisations proposed. The method is suitably adapted to the formal structure of each model, and is used for both the analytical purpose of finding its basic properties and the operational purpose of computing its solution. In fact, the study can be seen as an experiment in the robustness of the method to various generalisations of the IO model.

Part V reviews the entire study from a unified standpoint, on the basis of an organisational structure of the economy common to planned or socialist economies. The standpoint is that of a particular aspect of the general planning process, viz., material balances. It is shown how the various substantive issues behind the generalisations can be tackled in a sequential process of material balancing involving different elements in the organisational structure.

Conventions and Notations

CONVENTIONS

1. Some of the symbols used in this thesis have different interpretations at different places. The appropriate interpretations are given at the beginning of each specific use of a symbol.

2. A matrix is denoted by a capital letter. The corresponding small letter with (a) subscript ij indicates the (i,j)th element, and (b) superscript j indicates the jth column of the matrix. For example, for a matrix A, a_{ij} indicates the (i,j)th element and a^j the jth column of A.

3. A matrix is called non-negative if all its elements are non-negative.

4. A vector is a matrix with either one row or one column indicating a row or a column vector respectively. Unless otherwise specified all the vectors are taken to be column vectors. For a vector x, x_i denotes the ith component of x.

5. A vector is called strictly positive if all its elements are positive.

6. For the sake of notational simplicity, the same notation '0' will be used for zero-scalar, zero-vector and zero-matrix. The context will make clear which is being used.

7. Some abbreviations have been adopted in this thesis. Although these are given in brackets when the corresponding terms are introduced for the first time, it may be helpful for the readers if a list of abbreviations is given here :

   (i) IO : Input-Output
   (ii) AA : Activity Analysis
   (iii) LP : Linear Programming
   (iv) RS : Returns to Scale
   (v) CRS : Constant Returns to Scale
   (vi) IRS : Increasing Returns to Scale
   (vii) DRS : Decreasing Returns to Scale

(viii) ACM : Arrow-Chenery Model

(ix) CLI : Cost of Living Index

(x) MB : Material Balances

(xi) SMPM : Static Multi-sectoral Planning Models

(xii) SFCF : Stock-Flow Conversion Factor

NOTATIONS

$\neq$: Not equal to

$\nless$: Not less than

$\supseteq$: Set theoretic 'contains'

$\subseteq$: Set theoretic 'contained in'

$\cup$: Set theoretic 'Union'

$(. , .)$: An open interval

$[. , .]$: A closed interval

$\in$: Belongs to

$\notin$: Does not belong to

$\forall$: For all

$\sup_{x \in X}$: Supremum over all x belonging to the set X

Δ : Forward difference, i.e., $\Delta x^t = x^{t+1} - x^t$

R^n : n-dimensional euclidean space

R^n_+ : Non-negative orthant of R^n

A^{-1} : Inverse of the matrix A

A' : Transpose of the matrix A

I : The identity matrix

e^k : The kth unit vector all of whose co-ordinates are zero except unity in the kth co ordinate.

$\hat{x}$: A diagonal matrix with x_i as its (i,i)th element where x_i is the ith co-ordinate of the vector x.

$\lambda(A)$: The dominant characteristic root of the (square) matrix A. It is defined to be the characteristic root of A which has the largest modulus.

$\min\{a,b\}$: For two n-vectors a and b, it is a vector whose ith co-ordinate is the minimum of a_i and b_i.

$\geqq$, $>>$, $>$: For two (mxn) matrices A and B, I write :

(i) $A \geqq B$ if $a_{ij} \geqq b_{ij}$ for all $i=1, \ldots, m$ and $j=1, \ldots, n$.

(ii) $A >> B$ if $a_{ij} > b_{ij}$ for all $i=1, \ldots, m$ and $j=1, \ldots, n$

(iii) $A > B$ if $A \geqq B$ and $A \neq B$.

$\{x \in X / \ldots\}$: The set of all x belonging to X and satisfying ....

PART I

THE BACKGROUND

CHAPTER 1

Scope and Approach of the Study : The Nature of Generalisations

1.1 Introduction

This study claims to provide certain 'generalisations' of Input-Output (IO) analysis. These generalisations are distinct in the sense of dealing with different substantive problems and having different formal structures as their frames of reference, i.e., as 'models' behind. A common element is provided by their starting points, for they are generalisations of the same thing. That is, each analysis starts off from the basic IO model in a certain specific direction of its own. More than that, they share a common approach to their problem formulation and a common method of analysis. Logically, these common aspects are entailed by the very idea of 'generalisation' in the sense that each claims to be a generalisation in some common sense. It seems best to set out clearly this idea of generalisation at the very outset which in turn requires us to specify precisely what is meant by 'IO analysis' itself. This discussion is taken up in the next section which is followed up, in the last section, by an outline of the substantive ideas behind our generalisations. In the remainder of this opening section I shall point out a few general characteristics of the study.

I begin with the qualification 'theoretical' in the title of the study. This is best taken as a disclaimer : IO analysis is taken up as a purely theoretical discipline without any reference to its empirical foundations or practical applications as such though the latter comes in indirectly by way of orienting the discussion. We shall now draw a rough boundary of our chosen area of study in terms of the relevant literature, leaving out Leontief's classic (1941 and 1951) — which was as much on the theory as on the empirical foundations of IO analysis. Theoretical IO analysis may be said to begin with Hawkins and Simon (1949) followed up by Georgescu-Roegen (1951), Arrow (1951 and 1954), Samuelson (1951), Solow (1952), Goodwin (1949) and Chipman (1950), among others, though the last two were in a different context. Quite a few of these pieces appeared in the collection edited by Koopmans (1951) which introduced both the conceptual frame of activity analysis (AA) in his own contribution and the technique of linear programming (LP) in that of Dantzig. Thereafter on the theoretical side all these areas — IO, AA and LP, also the area surrounding the prior fundamental analytical contributions of von Neumann, viz., the so-called von-Neumann model of production and growth and game theory — grew side by side in an overlapping fashion. The more specialised literature on IO itself similarly fused the theoretical, empirical and applied sides together, the latter receiving a particular impetus with the advent of operational planning models. The more directly discernible theoretical contributions, in their turn, are also inextricably mixed up with the pure mathematics behind the IO model, with a renewal of interest in properties of non-negative square matrices that date back to the turn of the century. In the time-span between two further studies by Leontief (1953 and 1966) there appeared, first, three important volumes of collections on IO by Morgenstern (1954), National Bureau of Economic Research (1955) and Barna (1956) and a little later, *a series of wide-ranging* theoretical treatises either on, or with substantive parts on, IO analysis, e.g., Lange (1957), Dorfman, Samuelson and Solow (1958), Karlin (1959), Chenery and Clark (1959), Gale (1960), Schwartz (1961), Morishima (1964) etc. Among the more recent contributions, mention may be made of the collections edited by Carter and Brődy (1970 and 1970a), Brődy and Carter (1972), Mathur and Bharadwaj (1967) etc.

The references above are by no means exhaustive, and the tradition is still very active. I shall later (section 6 of chapter 2) review the literature from the standpoint of our own account of IO analysis, given briefly in next section and more elaborately in chapter 2. Here I shall only state a second disclaimer which may help dispell possible misgivings. This is basically about AA which no doubt provides a general analytical frame with IO analysis as a special case. Our generalisations have nothing to do with this frame. The reasons are partly alluded to above and will be clearer later in Part I of the study. We shall return to the question in connection with the review of literature in section 2.6. Till then we keep away from references for the convenience and continuity of exposition.

Next, I may point out that excepting for one chapter (chapter 4) the study may be broadly taken to belong to so-called 'theory of economic planning'. Now, IO analysis as such is not system-specific in the economic sense. That is, no explicit reference need be made to the underlying institutional structure of the economy being studied. This is also true of this study. However, the kind of substantive problems sought to be tackled by means of IO analysis here — which in turn sets the general trend of discussion — is best seen as direct planning problems defined for the economy as a whole. This makes it convenient to assume that the economy under consideration is indeed a planned one. In our concluding chapter however we make the planning assumption in a much more substantial way. The purpose of that chapter is to review the entire study from a unified standpoint. The actual standpoint taken is that of a particular aspect of the general planning process, viz., material balances. Since the process is rooted in the organisational structure of a planned economy, it cannot really be dissociated from the assumption of a planned economy.

I shall end this section with a brief outline of the arrangement of the contents of the study. The study is divided into five parts. The "three generalisations" promised in the title are taken up separately in Parts II, III and IV. The first and the last part then take up the implied tasks of introduction and conclusion, respectively. For the sake of completeness and self-containedness, I have sought to provide all the conceptual and technical background for the rest of the study in Part I. This includes a somewhat systematic, though purposive, overview of 'theoretical IO analysis' in the next chapter. Some specific elements of this overview are pointed out in section 2 below. This chapter is to serve the purpose of a broad general introduction — to the methods as well as ideas of the study.

1.1 The logic of generalisation

To begin with, the term 'generalisation' is used in this study as a development of a basic theme, i.e., as a process of working upon some central idea at the starting point — where it is expressed in a particular context or 'frame of reference' — in somewhat wider and different contexts containing the original. The starting point can be referred to as 'standard theoretical IO analysis'. The first task we then face is to provide a clear and adequate statement of the basic theme and central idea of this standard analysis, wherefrom the development takes place. Since the "basic theme" and "central idea" of anything are really consequences of looking at it from some standpoint, the task is unavoidable in spite of the voluminous literature referred. Here it is necessary to give only a brief statement of this review which is taken up in greater details in chapter 2. In particular, a number of concepts that we shall freely use here are spelt out and properly defined in chapter 2.

To begin with, the basic theme of IO analysis is identified as interdependence of production, i.e., it conceives production as a system made up of interdependent parts, the so-called sectors of production. The sectors occur as both the origins and destinations of production flows. This immediately entails the notion of consistency of a production programme in the sense that the composition in which products are turned out has to be consistent with their use in the production programme itself. Closely related — in fact logically prior — is the concept of productivity or viability, viz., the ability of the system itself to turn out a larger volume of products than its internal product use, i.e., to throw up a positive surplus of the goods concerned. The next and crucial step in IO analysis consists of treating precisely these surpluses — so-called bill of final demands — as independent variables with output levels (production programme) as the dependent variables. This in fact can be looked upon as the basic assumption of IO analysis in the sense of its approach. With this, the basic problem is posed as that of output determination for a given bill of final demands in terms of the consistency relations posited. The outside element of final demand can then be said to provide an additional dimension of consistency itself, for the production programme has to be both internally consistent (with respect to the structure of interdependence) and externally consistent (with respect to the final demands). It need be pointed out here that the exact line of division between 'internal' and 'external' is precisely a question of the scope of consistency incorporated in "interdependence" — interdependence on account of what substantive factors ? — and this represents an entirely open area which appears very little systematically explored in the literature. Starting from the original content of interdependence as portrayed in the standard model (with its corresponding definition of final demands) we shall say that a part of final demand is internalised into the system if that part is related to the production programme by some consistency conditions initially ignored. Much of chapter 2 will be concerned precisely with this issue.

The next point within the format of standard IO analysis is one of methods of solving the basic problem. We shall identify one particular method as the IO method in view of its analytical and theoretical significance. The method referred is an iterative one which amounts to a process of successive cumulation of 'rounds' of 'derived demands' (as derived from the consistency relations) to an initial given bill of final demands. Its analytical significance derives both from the insight into the structure of interrelations of the system offered by the steps of the iteration and from the fact that, technically, the method provides a constructive approach to the formal analysis of the system itself. Theoretically, the iterations can be viewed as describing rather general economic processes, and this links up IO analysis with wider economic theory.

The IO method plays a key role in this study. It provides the common approach route to the detailed analysis of the formal structures resulting from the generalisations proposed. The bulk of the study can in fact be seen as an experiment in the robustness of the IO method to various generalisations of the model.

At this point we may briefly summarise the methodology adopted in this study as implicit in the above discussion. It should be clear that there are really two distinct levels of analysis — each requiring a certain approach — involved in any "generalisation of IO analysis" which this thesis claims to be. This is because, IO analysis can be carried out only within the frame of an IO model, and in this sense any generalisation of the former includes — rather starts out from — certain specific generalisations of the model itself. The latter generalisation of course reflects the introduction of some substantive new issues into the IO frame. The first level of analysis consists precisely of this 'model-generalisation', i.e., of formulating the new issues introduced in terms of some IO-type model which keeps to the basic logic of IO analysis. Conceptually, the logic is to follow the basic IO approach identified earlier. Technically, it means that the generalised models are amenable to analysis by means of the basic IO concepts and methods. This latter analysis — analysis, or dissection, of the model as formulated — is precisely the second level of analysis. Our approach at the first level then ensures the 'amenability' of our 'model-formulations' to the logic of IO analysis, while that at the second level consists of reliance upon the IO method for detailed 'model-analysis', as already mentioned. Since we carry out distinct generalisations of the model itself, the first level can be said to represent an area of variety — some explored, some not — while the second level provides the unifying element : a common method repeatedly applied to different 'models', with suitable modifications and adaptations.

It is to be noted here that a substantive generalisation of IO analysis need not necessarily require a corresponding generalisation of the formal structure of an IO model. This reflects precisely the open area of 'scope of consistency of an IO model' referred earlier. For a clear and specific use of language, we shall use the term "generalisation of IO analysis" more specifically to mean a <u>generalisation of the formal structure of the basic or standard IO model</u>. Correspondingly, we shall use the term <u>extension</u> rather freely to refer to any generalisation, whether it requires formal generalisation (generalisation in our sense) or not. This concludes the discussion of the logic of 'generalisation of IO analysis' promised in section 1.

1.3 <u>The substantive ideas behind generalisations</u>

As mentioned at the beginning of this chapter, this section is to give a connected account of the substantive ideas behind our generalisations. These ideas are organised around three basic concepts, viz., <u>scale-dependence</u> (of IO coefficients), <u>structural break</u> and <u>growth consistency</u> which are taken up for detailed analysis in Parts II-IV, in that order. A few clarifications appear necessary before venturing into the ideas. First, I have to point out that conceptually, there are significant overlaps between these parts. Their distinction then reflects the common fact of analysis that the same broad idea lends itself to different formulations reflecting points of emphasis and focus. This in particular is true about the relations between Parts II and III on the one hand and Parts III and IV on the other, as will be pointed out below. Second, the converse of the above is also true, viz., the same formal model can often be looked at from alternative standpoints leading to different interpretations, [illegible] to different ideas behind. On both counts, no very well defined relationship can be established between the broad ideas behind a generalisation and their specific formulations. In this section, we deal explicitly only with the former, though the latter comes in implicitly in view of the fixity provided by the very concept of 'generalisation of IO analysis' as expounded in the previous section. The explicit formulations of the three concepts noted above, in terms of some IO—or IO-type—model, are taken up in the respective parts of the study. This section can be viewed as a conceptual introduction to the more technical discussions of those parts. Inter alia, it also gives a broad outline of the contents of Parts II-IV of the study.

The first broad idea behind our generalisation is to open up IO analysis to increasing or decreasing returns to scale (IRS/DRS) in production. A straightforward and direct way of doing this is to give up the assumption of fixed IO coefficients, i.e., to treat the coefficients as variables dependent upon the levels of production in general. One then gets a general non-linear model with scale-dependent coefficients. The structure and properties of this model are investigated in details in Part II of this study. Here it may be noted that the idea of returns to scale (RS) is formulated in terms of particular inputs in this approach, i.e., one begins with input-specific RS in a sector, not with RS in a sector as such. In fact, it is not possible to define IRS/DRS in a sector in this set up excepting as a rather strong, and seemingly arbitrary, restriction on the behaviour of all the coefficients of production of that sector. The problem can be resolved to some extent by taking prices as given, so that cost of production can be defined, and one can talk of RS in terms of the behaviour of unit costs of production with respect to scale of production. This too is followed up in Part II.

Actually, the idea of fixed prices can be taken to lie implicitly behind much of this study. This appears justified in view of the fact that our direct concern is with production and its structure in physical terms, both behind (technology or input-structure) and after (structure of product-use). The area of prices as such is basically outside the scope of this study excepting at one point noted below, so that prices are implicitly taken to be determined by factors outside the area of our analysis.

The exception also occurs in Part II, reflecting largely the nature of its generalisation. Unlike our other generalisation, the non-linear model does not introduce any new relation in the IO frame : it simply formulates the same set of relations in a more general way. This makes it possible, and interesting, to develop the analysis on parallel lines. And one major line of analysis in the standard IO frame is that of price-cost relations, leading to explicit price-determination. A parallel attempt is made in this direction in terms of the non-linear model of Part II in chapter 4.

Returning to the idea of IRS/DRS, it is to be noted that the problem can be viewed, somewhat more concretely and explicitly, in the frame of alternative methods of production, representing different technologies, with specific constraints on their operation. It may be worthwhile to briefly review the basic forces behind IRS/DRS before coming to the formal aspects. The classic source of DRS is scarcity of resources, and this is perhaps a universal characteristic of any given situation. Given the technical specificities of resources, particularly those embedded in the sectoral capacities of production already installed, the choice over alternative methods of production is inextricably bound up with the complex structure of resource endowments, broadly conceived. The operation of DRS here can be portrayed as a process where the use of an inferior technology follows, so to say, that of a superior technology, after the latter has reached its maximum possible scale, as permitted by the resources specific to the latter.

In contrast, the major force behind IRS is technology itself, divested from resource constraints — in particular, installed capacities. This requires one to take a purely ex-ante view of technology where sectoral capacities are not yet decided upon. A larger capacity then entails smaller unit cost in real terms by simple mechanical laws. Since, as mentioned, alternative technologies are largely bound up with specific forms of capacities, one may simply say that the operation of a superior method of production is possible only with a larger capacity, i.e., with larger scale of operation (this expresses the basic idea of indivisibility).

It is to be noted that the depiction of IRS as above requires a distinct conceptual frame of analysis, for capacities, scales and methods of production all are thought of in purely ex-ante terms, as "prospective" ones. That is, one has to think of some definite future and consider the relation between scales and methods of production for that future. The required capacities can then be thought of as being brought into existence prior to the future, i.e., between the 'given present' and the 'postulated future'. The key to IRS then lies in investment in suitable form and direction over the intervening period which, implicitly, has to be sufficiently long to cover the so-called period of construction of the prospective capacities. It will be convenient to adopt a fixed terminology for this frame of reference. We shall refer to the 'postulated future' as the terminal period, and the present as the base period, and the total time-span covering these two and the intermediate periods as the time-horizon of analysis. Our analysis in Parts III and IV of the study draws largely upon this conceptual frame of reference.

Part III takes off directly from the kind of factors behind IRS discussed above, organised around the concept of structural break. In terms of the frame, the break is from the given structure of the base to a set of open possibilities at the terminal. The openness comes by treating the prior investment decisions as essentially free, and the problem really is what sorts of structural breaks to have, if at all, i.e., one of choice. The IO model — more properly, an

IO-type model — is defined for the terminal period, with both the base structure and the investment programme referred lying implicitly behind. The choice problem — equivalently the prior investment decisions — are then resolved by a set of secondary relations incorporating specific criteria of choice. Essentially, these criteria have to work back from the prospective scales of production at the terminal date, which in turn are governed by the contemporaneous final demand. While the substantive considerations involved in the specification of choice criteria will have to be context-specific, the approach is a general one keeping to the basic logic of IO analysis. Its independent significance is borne out in all cases where the scales of production are important determinants of the structure itself, e.g., in the case of IRS.

Now, the concept of a structural break, as introduced, is a very wide one, covering in principle all sorts of possible changes in the structure. Our own analysis under this heading is to be seen more as a sample of particular exercises than as a comprehensive treatment. This part (Part III) therefore lacks the organic unity of systematic development of a single theme which characterises the two other parts. The exercises are arranged under two broad categories. One is concerned with changes in technology in the sense of methods of production where the idea of IRS comes to the forefront. The other views the problem in terms of setting up capacities for producing commodities that were not produced earlier, i.e., imported if required (on the so-called non-competitive basis). This way, a dent is made into the so-called problem of import substitution as an aspect of structural break. We shall take off the further details of the exercises from this point in chapter 5.

Now, whether the break occurs in terms of using some technology not previously used or producing some commodity not previously produced, it requires prior investment in suitable directions, as mentioned earlier. By its very temporal priority, however, this investment does not make an explicit appearance in the IO frame of Part III. Its role has to be completely subsumed under the secondary relations, with investment-costs as elements of the choice criteria employed. Part IV on the other hand brings investment to the forefront of a terminal period IO model, using the same conceptual time-frame for a completely different set of problems. The difference between Parts III and IV is basically one of focussing on two distinct aspects of the investment-capacity relation. Looked at from the base capacities, investment over the time horizon can both change the technology incorporated in those capacities — this is the structural break problem — and expand the capacities, relieving the capacity constraint on production. The latter can be called the growth aspect, and constitutes the point of departure for Part IV.

In terms of our conceptual frame, the growth requirement is directly entailed by the given final demands of the terminal period with the given capacities of the base period. In other words, the latter is insufficient for the former, and investment comes in by way of necessary expansion of capacities in terms of given structural relations between investment and capacity expansion. Now, while this investment has to take place prior to the terminal period, the terminal investments in their turn are not viewed independently. Basically, these are seen as continuation of a single investment programme whose basic task is to achieve a balance between the terminal period capacities and terminal requirements. The simplest way of doing this is to assume a constant (unknown) rate of growth for each sector so that the terminal investments are determined directly by the terminal production levels in terms of technological relations referred earlier and the rates of growth introduced. Given the latter, this determination is nothing but a straightforward extension of the scope of IO analysis, without any change in its formal structure, that we had referred in section 1. An explicit demonstration of this is given in section 4 of chapter 2. Part IV takes off directly from this section and treats growth rates as genuine unknowns of the problem to be determined simultaneously with terminal investment and production. That is, the growth rates themselves have to be consistent with the required terminal production levels (which depend upon growth rates via investment) as seen from the base capacities.

In Part IV we explore the implications of this growth consistency requirement for terminal production and investment levels in a somewhat systematic fashion taking up issues from the basic postulate of insufficient base capacities for the terminal final demands, where the latter is obviously net of investments. In the first place (chapter 7)

~~the insufficiency is taken in the strong sense of being true sector~~ b sector. This is called the 'basic model' of Part IV. Relaxing the assumption, we then allow for excess capacities. Finally, the problem is freed from the restrictive frame of given final demands in the terminal period. This makes room for independent restrictions on growth rates themselves, with corresponding adjustment of final demands. The technique here is a parametric specification of the final demand which in turn can allow for such usual non-linearities in consumption — with respect to income and therefore implicitly production in its role of income generation — as non-unitary Engel elasticities etc. Both these extensions are taken in chapter 8.

The whole discussion of the growth consistency problem can be seen as a critique of the technical basis of the so-called static multisectoral planning models (SMM) which set out to determine a consistent terminal production programme in terms of given final demands net of investment and given base capacities without, however, having to bother about growth consistency. In fact, the very notion of rates of growth of production appear redundant in the approach taken to the construction of these models. I shall take up this critique separately at the end of Part IV (chapter 9).

To continue the outline of the study to its natural conclusion, I shall just repeat that the last chapter (chapter 10) — which also constitutes the last part of the study — stands outside. Its task is to review, not to introduce new ideas. The review is mainly in terms of the so-called method of material balances in a socialist economy, as pointed out earlier.

CHAPTER 2

An Overview of IO Analysis

This chapter gives a broad overview of IO analysis. (The qualification 'theoretical' is omitted for brevity.). The 'broadness' is to be interpreted in the sense of our requirements, not that of the literature. That is, an attempt is made to cover the entire analytical-technical background necessary for our generalisations, not to cover the entire literature. On the contrary, as an overview of the entire area, it is certainly selective and purposive.

The chapter is divided into six sections. The first section is basically an elaboration of ideas briefly presented in section 1.1. The next section provides a somewhat systematic account of the so-called IO method together with its various ramifications. The three following sections are concerned with the so-called 'scope of IO analysis' in a broad sense. Consumption, investment and foreign trade are basically looked at from the standpoint of IO analysis with a view to their endogenous treatment in the respective sections here, in that order. The concluding section provides a unified review of the literature, as promised in section 1.1.

2.1 Standard IO analysis

This section will provide a brief connected account of the basic IO system, its concepts, assumptions, properties, and problems. This will be followed up by certain standard extensions belonging to what can be called the standard IO frame.

Our first task is to clarify the basic conception of "Production

as a system made up of interdependent parts" which, as pointed out in chapter 1, is the starting point of IO analysis. These parts are the so-called 'sectors of production' (or 'sectors' for short). Each sector is conceived as both the origin and a destination of product flows; in particular, it is the origin of a distinct product specific to itself. That is, no two sectors generate the outflow of the same commodity : this is the so-called assumption of no joint production[1/]. Equivalently, a sector is defined by its production process which simply transforms a set of products as inputs into a corresponding product as output. The inputs account for destination of flows and the output for origin. The system so far can be looked upon either as a collection of interlocking sectors of product flows or as a collection of production processes transforming their products into one another. This leaves room for destination of products other than the sectors of production themselves (i.e., of process, other than production, for absorbing the products). All such destinations are brought under a single account in IO analysis, called 'final use' or 'final demand'.

Algebraically, the foregoing discussion entails the following flow or balance equations :

$$x_i = x_{i1} + x_{i2} + \dots + x_{in} + y_i \qquad \dots (2.1.0)$$
$$i = 1, 2, \dots, n$$

1/ Strictly speaking, what is required is that there is no overlap between the products of different sectors, not that each sector has a single product. IO analysis then proceeds by aggregating the products of a sector into a single "product group", using the the terms "product" and "product group " interchangeably.

where x_i and y_i are respectively the levels of production and final use of the ith product or good — to be denoted by G_i — and x_{ij} is the amount of G_i required as input to produce x_j amount of G_j. Equivalently, x_{ij} is the product flow from the i-th sector — denoted S_i — to S_j.

The basic assumption behind the construction of an IO model consists of setting the ratios :

$$a_{ij} = x_{ij}/x_j \geq 0 \qquad \ldots. \quad (2.1.1)$$

constants. These ratios are called the IO coefficients[2/]. The assumption is best seen as a composite of two assumptions. First, there is a determinate relationship between the volume of production of a sector and the amounts of inputs used there; and second, this relationship is of the proportionality type. In economic terms, the first assumption rules out alternative methods of production for any product. The second assumption itself can be seen again in a composite form, viz., the IO coefficients in a sector are independent of the scales of production both in that sector itself and in any other. The first part entails the assumption of constant returns to scale (CRS) and the second that of no externalities. In short, this assumption rules out all possibilities of internal and external scale economies or diseconomies in production. As pointed out in chapter 1, our first generalisation consists precisely of relaxing this assumption.

2/ So far all the variables are assumed to be measured in physical units. One can, however, easily transform these into value units given the prices of all commodities and redefine the IO coefficients accordingly. Formally, if $p_1, \ldots, p_n$ are the given prices of different commodities, then the IO coefficients in value units are given by $\frac{p_i a_{ij}}{{}_j}$

An IO system can now be defined from (2.1.0) and (2.1.1) as

$$x = Ax + y \qquad \ldots (2.1.2)$$

where $x = (x_i)$ and $y = (y_i) \in R^n_+$ are the vectors of production (or, production programme) and final uses respectively. $A = (a_{ij})$ is the IO matrix so that Ax is the vector of intermediate input uses of different goods.

The production process in S_j is defined by its input coefficient vector, a^j, and the system (2.1.2) can be written equivalently in terms of the production processes as :

$$x = \sum_{j=1}^{n} x_j a^j + y \qquad \ldots (2.1.2a)$$

where x_j can be interpreted as the level of operation of the production process in S_j. It is clear that the system above is defined completely by the IO matrix A and, therefore, any system characteristic is also the characteristic of A, i.e., qualifications are interchangeable.

We are now in a position to introduce the fundamental concepts of IO analysis. One group of concepts revolves around the distinction between direct and indirect inputs required for the production of any good, say G_i. Let S^o_i denote the set of products necessary for G_i (i.e., occurring as inputs in the corresponding production process). For each $G_i \in S^o_i$ one can then define a similar set, say S_{ii}, and their union $S^1_i = \bigcup_{G_i \in S^o_i} S_{ii}$, and so on. A good G_i is then said to be a direct input for G_i if $G_i \in S^o_i$ and an indirect input if $G_i \in S^k_i$ for some $k > 0$. These concepts lead to a further concept characterising structure of interdependence of the total production system, viz., that of **irreducibility**. An IO system is said to be irreducible if every product is a direct or indirect input for any other product. Algebraically, this means that there does not exist any proper, nonempty subset of sectors, K, such that $a_{ij} = 0 \; \forall i \in K$ and $j \notin K$, where indices are taken to represent corresponding sectors or products. An IO matrix which is not irreducible is said to be reducible. Clearly, the production system is interdependent in a strict sense only in the irreducible case; in the other case one will always find at least one independent (proper) subsystem consisting of sectors connected only with one another, not the rest.

A second group of ideas centres around the basic notion of consistency of a production programme. A production programme is said to be consistent — more elaborately, internally consistent — if it is able to meet its own internal requirements of goods as inputs out of its production. Formally, $x \in R^n_+$ is a consistent production programme if :

$$x \geqq Ax \qquad \ldots (2.1.3)$$

Correspondingly, one can define the set of consistent production programmes as :

$$\mathcal{H} = \left\{ x \in R^n_+ \; / \; x \geqq Ax \right\}, \qquad \ldots (2.1.4)$$

For any $x \in \mathcal{H}$, the final use vector generated — or "produced" as we shall often say — is given by

$$y = x - Ax \geqq 0 \qquad \ldots (2.1.5)$$

The production system is said to be **productive** if there is a consistent production programme producing a strictly positive final use vector and the latter is then said to be **producible**. Algebraically, the system is productive if there exists a $x \in R^n_+$ such that

$$x \gg Ax \qquad \ldots (2.1.6)$$

Obviously, productivity of the system is equivalent to the nonemptiness of the interior of X as well as to the producibility of a strictly positive vector of final use[3/].

It is to be noted that the notions of productivity or producibility as defined so far are purely local in the sense that it is only the existence of some x or y with stipulated properties that is sought for. Logically these are pure "existence" notions. Correspondingly, one can define global productivity to mean that any non-negative bill of final demands is producible. Formally, this means that equations (2.1.2) have a non-negative solution[4/] for any arbitrary final demands. This arbitrariness of course does not pertain to x, i.e., any arbitrary x is not necessarily consistent when the system is globally productive. This is precisely the force of consistency.

We can now step into the area of problems posed in IO analysis. The fundamental problem is that of output determination. To retrace a few steps back, the system formally has x and y as variables with (2.1.2)

3/ In case A is irreducible the inequality '$\gg$' in (2.1.6) is equivalent to the weaker inequality '$>$'.

4/ To avoid unnecessary repetition the qualification "non-negative" will henceforth be dropped.

(C.2.4) Each column or each row sum of A is strictly less than unity.

(C.2.5) At least one row or column sum of A is strictly less than unity.

We may note two immediate consequences of (C.2.5) : first, that local and global productivities are equivalent in the system in the sense of one implying the other; and second, that productivity itself implies the uniqueness of the system-solution. The solution is given by :

$$x = (I-A)^{-1}y = A^{*}y \qquad \ldots (2.1.7)$$

where $A^{*} = (I-A)^{-1}$

Dimensionally, the (i,j)th element a^{*}_{ij} of A^{*} represents the output of G_i required per unit final use of G_j. Since final use is part of output, one gets the input for final use by deducting it from output and thus defines new coefficients representing the input use of various products per unit of their final uses.

To bring this out, let x and y be a pair of internally consistent output and final use vector. Then the required input for y is nothing but :

$$\begin{aligned} Ax &= x - (I-A)x \\ &= (A^{*}-I)(I-A)x \qquad [\, A^{*} \text{ exists by (C.2.5)} \,] \\ &= (A^{*}-I)y \qquad [\, \text{From (2.1.2)} \,] \end{aligned}$$

so that the "total input use of G_i per unit of final use of G_j" is nothing but $(a^{*}_{ij} - \delta_{ij})$ where δ_{ij} is the so-called Kronecker delta. Clearly, for $G_i \neq G_j$, a^{*}_{ij} itself represents the required coefficient. These are sometimes called the total input coefficients, the term "input" being then meant in the sense of "input for final use", not "output" as such. We shall take up the significance of the qualification "total" in section 2 below.

I shall now conclude this section with reporting two standard extensions of the standard system described above. It is basically here that our overview of the literature is somewhat particularly "purposive", the purpose being to highlight concepts that will be later significant.

First, the inputs into the various production processes so far are outputs of other sectors. However, the list of inputs can easily be extended to include such goods as are not produced within the production system under consideration. These goods are called primary goods or primary inputs; some examples being labour, imports etc. Clearly, their introduction requires a corresponding extension of the "input coefficient vector" for a sector over the set of primary goods. In keeping to the general assumption of fixed coefficients, the primary input coefficients are also treated as constants. The explicit introduction of primary goods leads to a straightforward extension of the model keeping to its basic logic. The given bill of final demands determinesthe required production levels in the manner depicted in (2.1.7), and these production levels, in turn, determine the levels or requirements of primary goods by means of the primary input coefficients of production. This obviously is one-way dependence, and it is for this reason that it is outside the strict scope of the basic model. In fact, the enumeration of 'relevant' primary goods is essentially an open-ended process from the standpoint of IO-analysis depending upon the exact purpose at hand.

The next concept surrounding the basic model is that of a constraint on production. This expresses some real factor surrounding the production system which imposes in some way a bound on the production vector. Obviously, it may not then be possible for the production system to meet any bill of final demands. The set of constraints considered then gives us a set of feasible production programmes satisfying the constraints imposed. An immediate illustration of a constraint here is provided by the case of a given availability of some primary good. The constraint then simply takes the form that its requirement defined earlier is less than or equal to this availability.

A second important case -- one that will be extensively used in this study -- is that of a capacity constraint in a sector, i.e., a constraint specifying an upper bound directly on its level of production. The idea is basically that the volume of output of a sector is conditioned not merely by the supply of inputs (both produced and primary) but also by the total existing material conditions of production there. As noted in section 1.3, these conditions restrict both the technology (and through that the structure of interdependence), as well as the volume of production of the sector concerned. This latter aspect is the capacity constraint. It is to be noted here that the concept of sectoral capacity is not directly a commodity concept. What lies behind a capacity is an ensemble of commodities attached to the sector in consideration, generating various services necessary for production. The operational significance of the ensemble lies precisely in the rate of production permitted and not in the availability of these commodities as such. One may of course attempt to make this restriction more explicit by reference to the precise structure of the services generated. This simply represents the 'open-endedness of enumeration' mentioned earlier in reference to primary inputs.

Operationally, it is clear that one can follow the IO approach in a constrained IO model only to the extent of declaring a given bill of final demands either feasible or infeasible. Further analysis obviously requires further extensions of the model itself. One extension -- again one to play an extensive role in this study -- is to bring in secondary sources of supply of the products, i.e., supply from some source other than production in the system as depicted by the IO matrix. It is to be noted that the designation 'secondary' implies that this source is called upon only when the primary source -- that is production itself -- is not possible, i.e., the final use vector is otherwise infeasible. It need be further pointed out here that the primary-secondary distinction is really a way of resolving the issue of choice thrown up by the presence of alternative sources of supply. The choice obviously has to be made in the light of some real considerations. [While] these considerations are context-specific, the approach is a [gen]eral one.

Returning to the main line of discussion, the idea of a secondary source really comes of its own in the case of capacity constraints. A secondary source here may be thought of in various ways, e.g., imports[5/], or production by a *technology different from the one represented* by the IO matrix, or expansion of capacity. This last possibility requires one to step beyond the analytical frame developed so far, for additional capacity *can be created only over time* by means of suitable investment. This requires a clear temporal specification of the model which has not so far been touched upon. This will be taken up in section 2.5.

2.2 The IO method

This section deals with what we have repeatedly called the basic IO method of analysis — in several parts. We have already mentioned about the various points of significance attached to the method (see p. 7). The points will be taken up in the following order. Part 1 gives a statement of the method and its basic properties. Part 2 interprets the method in *terms of IO* concepts. Parts 3-5 offer wider interpretations in terms of general economic theory. Finally, in the last part we shall discuss the method from the purely computational viewpoint and summarise certain refinements of the method from this angle.

2.2.1 The method and some properties

The method, as a computational procedure, is an iterative one which consists of constructing a sequence $\{x^t\}$ by the recursive formula :

$$x^t = Ax^{t-1} + y \qquad \ldots (2.2.0)$$

5/ The reference here is to so-called "competitive imports", i.e., import goods which are also produced domestically. In contrast, the import goods that are not produced domestically are called non-competitive imports. It was precisely the latter that we encountered as primary inputs earlier.

The scheme can be initiated with any arbitrary non-negative initial condition. In particular, if one starts with an 'underestimate' of the output vector — i.e., with a x^0 satisfying $x^0 \leqq Ax^0 + y$ — the sequence $\{x^t\}$ will be non-decreasing while if $x^0 \geqq Ax^0 + y$ (over-estimate), the sequence will be non-increasing. Clearly, a special case of the first situation is :

$$x^0 = y \qquad \ldots \quad (2.2.1),$$

and in this case x^t is given by :

$$x^t = (I + A + \ldots + A^t)y \qquad \ldots \quad (2.2.2)$$

It is well-known that the matrix series $(I+A+ \ldots + A^t)$ converges to $(I-A)^{-1}$, provided the system is productive. I shall refer (2.2.0) and (2.2.1) as the basic IO method.

Clearly, the method (2.2.0) requires a stopping rule or termi- n criterion. One straightforward rule is that the scheme is pped when two successive estimates become 'close' enough. Opera- y, this would require the specification of a well-defined norm , $||\cdot||$, and a positive number, ε , representing the degree of sought or the tolerance margin and the scheme is then termi- at stage T if

$$||x^{T+1} - x^T|| < \varepsilon \qquad \ldots \ (2.2.3)$$

I shall now state a well-known result which would defend on to various analytical properties of the structure (2.1.2) basic IO method. The result is stated below as a theorem.

Theorem 2.2 : The sequence $\left\{x^t\right\}$ defined in (2.2.0)-(2.2.1) converges if and only if the solution of (2.1.2) exists, and the limiting vector gives the solution.

This theorem thus establishes a complete equivalence between the workability of the iterative scheme (2.2.0)-(2.2.1) and the existence of the solution of (2.1.2).

2.2.2 A general interpretation

Here I shall give an interpretation of the basic IO method in terms of IO concepts, viz., direct and indirect requirements. Clearly, final use of any good is part of its total production, so that the vector Ay can be looked upon as the input requirements directly for the final use treated as output. This justifies the designation of a_{ij} as the direct input coefficient with respect to final use. Now, Ay similarly is also part of output and its input requirements, viz., $A(Ay)$ ($=A^2y$) can be seen as first round of indirect input requirements for the final use vector y. Continuing this way, the t-th round of indirect requirements to meet the final use vector y is given by $A^{t+1}y$. The "total indirect input vector" for y is then the sum $[A^2+A^3+ \ldots] y$. From (2.2.2) it therefore follows that the basic IO method can be interpreted as a method of finding the solution of the model by cumulation of direct and successive rounds of indirect input requirements y.

The above discussion provides the justification of the term "total input coefficient" for a^*_{ij} $(i \neq j)$, for it is the sum of the 'direct input coefficient', a_{ij}, and the 'indirect input coefficient', the latter being obtained as the infinite sum of the (i,j)th elements of the matrix series $\left\{A^2, A^3, \ldots\right\}$

2.2.3 Market adjustment process

The right hand and the left hand sides of (2.1.2) can be viewed as the demand and supply sides respectively of the production flows; the right hand side being the sum of final demands and derived intermediate input demands and the left hand side being the total supply (production) of different commodities. The equality in (2.1.2) thus reflects the state of equilibrium, viz., demand = supply.

The method (2.2.0) [with any arbitrary initial condition] in this context can be thought of as a process of adjustment of demand and supply towards reaching the equilibrium. In a general formulation, the additional supply at stage t of the process is determined by the excess demand generated at the previous stage. Formally,

$$\Delta x^{t} = F\,(d^{t} - x^{t}) \qquad \ldots (2.2.4)$$

where d^t and x^t are the demand and supply vectors at stage t of the cess and F is the so-called 'response function' which determines magnitudes by which the supplies are to be expanded in response excess demand. F is assumed to have all partial derivatives tive and F(0) = 0.

A special case of this general formulation, viz., when = x, is precisely the method (2.2.0). To bring this out clearly, is to be viewed in greater details in the last chapter. Here I shall only initiate the discussion in the simplest form.

The productive system is now assumed to be planned by a central agency, denoted by C. The actual production however is carried out sectorally in terms of n sectors, denoted by $\{S_k\}$, and the task of C is essentially one of coordinating the separate production plans of the sectors. This coordination is to achieve consistency, or balance, both external (with respect to the 'final demands' which are known to C) and internal (with respect to the 'structure of production' which, however, is known only in its separate parts to the respective sectors). The process under review is an iterative procedure used by C to achieve this balance. Each stage of the procedure is characterised by a 'provisional balance' (or plan) in the hands of C. The procedure is described below.

Let x^t be the provisional plan at stage t. C now informs each S_k of a corresponding production target, x_k^t, with the instruction to report back its input requirements. This is the 'derived demand' vector, $d^k(t)$:

$$d^k(t) = x_k^t \; a^k \qquad \ldots \qquad (3.2.5)$$

Obviously, S_k can compute it solely on the basis of its technical knowledge and the target just received from C. C now constructs a provisional plan for step (t + 1) aggregating the derived demand vectors received from $\left\{S_k\right\}$ and adding y to the aggregated derived demands. That is,

$$x^{t+1} = \sum_{k=1}^{n} d^k (t) + y \qquad \ldots \quad (2.2.6)$$

This shows how the provisional plans are revised at each step on the basis of sectoral reports. The procedure starts with an initial provisional plan :

$$x^0 = y \qquad \ldots \qquad (2.2.7)$$

It is clear that (2.2.5)-(2.2.7) together are equivalent to (2.2.0) and (2.2.1). Hence the sequence of provisional plans indeed converges to the correct plan, provided that the IO system is productive.

2.2.6 A computational procedure : some refinements

The convergence of the sequence $\left\{x^t\right\}$ as defined in (2.2.0) nd (2.2.1) can be very slow, and in this case one may rest content ith an unreasonably low estimate of the output vector or continue the omputation for a long time. I shall now discuss some possible modifi- ations of the method to speed up convergence.

The first modification is to replace the stopping rule (2.2.3) y some other rule. It is possible to estimate the limit of the sequence rom the informations obtained in the first few steps of the iterative cedure. This is done by forming the differences between the succe- ive estimates of the output vectors and then the ratios of the fferences obtained. The iterative procedure can be stopped when the average — preferably geometric average — of these ratios becomes more or less stable. Denoting the last stage of the iteration and the average of the last set of ratios by T and λ respectively, $x^{T+1} + \frac{1}{1-\lambda}(x^T - x^{T-1})$ then gives the estimate of the limit of the sequence. The mathematical justification of this procedure requires A to be <u>intrinsically positive</u> (i.e., for some $m \geq 1$, $A^m >> 0$).

A second modification is already stated in subsection 1. This is to start with some vector $\tilde{x}$ other than y as an initial estimate. In this case, the estimate of the output vector at stage t is given by :

$$x^t = (I + A + \dots + A^{t-1})\, y + A^t \tilde{x}$$

Clearly, the first term on the right hand side would converge to the solution of (2.1.2) and the second term would dwindle to zero when the system is productive, and this will hold good for any arbitrary vector $\tilde{x}$. The convergence will indeed be faster than that in the original scheme if $y \leq \tilde{x} \leq A\tilde{x} + y$. However, the extrapolation procedure described in the last paragraph need not give satisfactory results in this case.

Finally, there is another modification of the method, known as Gauss-Seidel method. Given the initial conditions, the first estimate for the first sector is obtained from the first row of the matrix A, which is then used for the first estimate for the second sector, and so on. In general, the t-th estimate for the i-th sector is obtained using (a) the i-th row of A, (b) the t-th estimate for the 1st, 2nd, ... (i-1)th sector, and (c) the (t-1)th estimate for the remaining sectors. This efficient utilisation of information makes it possible for the modified method to converge faster. Further, the use of extrapolation for termination, as stated before, is justified here also. Algebraically, the iterative scheme here can be written as follows :

$$x^o = y \qquad \ldots (2.2.8)$$

$$\left.\begin{aligned} x_1^t &= \sum_{j=1}^{n} a_{1j}\, x_j^{t-1} + y_1 \\ x_i^t &= y_i + \sum_{j=1}^{i-1} a_{ij}\, x_j^t + \sum_{j=i}^{n} a_{ij}\, x_j^{t-1} \end{aligned}\right| \qquad \ldots (2.2.9)$$

$$i = 2, \ldots, n$$

2.3 <u>Consumption from an IO standpoint</u>

In this section I shall treat consumption as an internal or endogenous part of the IO system with the help of consumption-income-production relations of a rather wide generality.

The basic idea behind internalising consumption is simply that, first, production in the different sectors generates income to households which are the basic units of consumption, and second, a major determinant of household consumption is the level of household income (in a per capita sense). In particular, in a multisector frame the latter relation can reflect both the basic Keynesian hypothesis of decreasing consumption-income ratios with increasing income by appropriate classification of income classes of households, as well as the various empirical findings in regard to the variation of commodity composition of consumption with income in terms of these size-classes.

Formally, let m income-classes of households be distinguished, say $C_1, C_2, \ldots, C_m$. C_j here represents the group of households with per capita income in a certain range. These ranges represent the different size-classes of income considered. It is taken that the ranges are arranged in an increasing order, i.e., the per capita income in C_j is greater than that of C_{j-1}.

For what follows it will be convenient to assume that all income is wage income derived from the sectors of production and that all flows are measured in a common monetary unit, so that the IO coefficients are also expressed in so-called value units. It follows that the classification of households is made on the basis of wage rates.

With this background let w_k be the total income in C_k. We now decompose the final use vector y as

$$y = u + v \qquad \ldots \quad (2.3.1)$$

where u is the vector of consumption (by commodities) for the households under consideration. Now :

$$\left.\begin{array}{ll} u_i = u_{i1} + \ldots + u_{im}, & i = 1, 2, \ldots, n \\ w_k = w_{k1} + \ldots + w_{kn}, & k = 1, 2, \ldots, m \end{array}\right\} \ldots (2.3.2)$$

where u_{ik} and w_{ki} are the amount (in value units) of G_i consumed by C_k and the income of C_k that comes from S_i respectively.

We now make the following two assumptions :

(a) $u_{ik} = c_{ik} w_k$

(b) $w_{ki} = a^o_{ki} x_i$

Here c_{ik} and a^o_{ik} are fixed and non-negative numbers c_{ik} is the <u>budget share</u> of G_i per unit income in C_k and a^o_{ki} is the <u>income-share</u> of C_k per unit value of production in S_i. Hence the matrices $A^o = (a^o_{ki})_{mxn}$ and $C = (c_{ik})_{nxm}$ can be called the <u>income distribution</u> (or <u>wage structure</u>) and <u>budget share</u> matrix respectively.

In view of assumptions (a) and (b) one can rewrite (2.3.2) as :

$$w = A^o x \qquad \ldots (2.3.3)$$

$$u = Cw \qquad \ldots (2.3.4)$$

where $w = (w_k)$ is the <u>income-vector</u> (by classes of people).

A few implicit relations in the model may now be noted. First, the total <u>employment</u> in C_k is nothing but w_k/α_k where α_k is the (average) wage-rate defining the k-th income class. This way employment is implicitly determined by income. Second, the expression $\Pi_i = (1 - \sum_j a_{ij} - \sum_k a^o_{ki})$ denotes the so-called <u>profit-margin</u> (per unit value of output) in S_i. Finally, the <u>average (= marginal) propensity to consume</u> of C_k is given by $\bar{c}_k = \sum_i c_{ik}$ and the Keynesian hypothesis referred simply boils down to the stipulation that $\bar{c}_k < \bar{c}_{k-1}$ $\forall k = 2, \ldots, m$. Similarly if G_i is thought to be a 'necessity' in the Engel's curve sense, then one would have $c_{ik} < c_{ik+1}$ $\forall k = 1, \ldots, m-1$. Also, (2.3.4) can be written as

$$u = \sum_k w_k c^k$$

Each $w_k\, c^k$ therefore reflects how income in G_k is being translated to consumption of different commodities. That is, the commodity composition of consumption for G_k is determined by the income of that class, thus reflecting the well-known Engel's law of consumption behaviour.

From (2.3.1), (2.3.3) and (2.3.4) one can write the IO system (2.1.2) equivalently as :

$$x = Ax + CA^{o}x + v$$

$$\text{or,}\quad x = (A + CA^{o})x + v \qquad \ldots\ldots \quad (2.3.5)$$

Clearly (2.3.5) has the same formal structure as (2.1.2) with $(a_{ij} + \sum_{k=1}^{m} c_{ik}\, a^{o}_{kj})$ as its (i, j)th coefficient. The formal structure (2.1.2) is thus amenable to a wider interpretation depending upon the content of the final use vector and thus upon the scope of analysis. As for the direction of determination, it is clear that with x determined from (2.3.5), w is determined by (2.3.3) and thence u from (2.3.4).

We shall conclude this section with an outline of the <u>unified multiplier process</u> referred in section 2.2.4. This is nothing but an -terpretation of the power series expansion of the matrix $(A + CA^{o})$. nsider a "rise" in investment, i.e., the vector v. In the first round, oduction (x) increases by an equal amount, whence income (w) increases determinate amounts as given by assumption (b) which in turn leads determinate extra consumption as given by assumption (a). This tra consumption then gets added to the extra input requirements as entailed by the IO relationships to generate the extra production in the second round, and so on. In the end, the composite matrix multiplier is simply given by $(I-A-CA^{c})^{-1}$. Formally, one may be said to obtain the pure "IO multiplier" by ignoring the income-consumption relations (put C=0) and the pure "consumption multiplier" by ignoring IO relation (put A=0).

It is to be noted that the convergence of the unified process hinges on the so-called "productivity" of the total matrix $(A+CA^{c})$ which, in the present context, is better referred as "viability", for the system under reference is no longer just a production system.[6/] Among the conditions stated in theorem 2.1, (C.2.4) has a direct economic interpretation here. This requires each Π_i to be positive and each $\hat{q}_k$ to be less than unity, i.e., each sector makes positive profit and the marginal (=average) propensity to consume for each class of people is less than unity. This of course is a sufficient condition. The general necessary and sufficient conditions remain as in Theorem 2.1.

2.4 Investment from an IO standpoint

In this section we shall look into the details of product-uses for investment with an eye to making it a determinate part of the basic production balance equations. The basic relation here is that between investment and the growth of production, and this cannot be depicted without an adequate recognition of the time structure of

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6/ In the literature the term "viability" is often used for both local and global productivity. Here the two concepts are operationally equivalent, but in general this need not be so.

production. Much of this section will be concerned with the specification of this time structure in the IO frame. The internalization of investment can then be accomplished, on this technical basis, by treating growth rates as parameters, like wage rates in the previous section. The vector of sectoral growth rates required for this purpose will be called a "growth programme".

The total time-structure of production is an enormously complicated subject which I can touch upon here only in its broad outlines. To begin with, one has to distinguish between a production process in the somewhat narrow sense (as hitherto) of transforming a set of products as inputs (essentially raw materials) into another as output within a set up of material conditions or capacity and the broader sense of transforming the latter as well. Each of these two transformations (IO or production proper, and capacity creation) will have their n time structures, requiring investment for growth of production. We hall refer to investment on the first account as working capital ccumulation and on the latter account as fixed capital accumulation. e now take these up one by one and then bring the two together, under implified conditions.

To begin with, the time structure of IO transformations of the rst type can, it seems, be adequately depicted by means of the so-lled periods of turn over (of raw materials etc.), i.e., the length time for which one input stays in a production process. Now choosing a time-unit of analysis greater than or equal to all turn over periods in length, it is seen that intersectoral product flows (in a given time-unit) will have to both support the production in the destination sector over the same time unit as governed by the IO coefficients as well as provide for a fraction of the use of the originating sector product in the destination sector in the next time unit, as given by the turn over period. At the same time, a part of the required flow will also come from the production of the originating sector in the previous time unit. Hence if the flows are balanced intertemporally, then one would have :

$$x_{ij}(t) = a_{ij}\, x_j(t) + \theta_{ij}\, a_{ij}\; \Delta x_j(t) \qquad \ldots \quad (2.4.1)$$

re θ_{ij} (≤ 1)[7/] is the turn over period of G_i in the production of and $\Delta x_j(t)$ is the change in the level of production of G_j between periods (t+1) and t. Technically, the second term in the right hand of (2.4.1) — denoted by v^w_{ij} below — is part of investment use in the j-th sector and $\sum_{j=1}^{n} v^w_{ij}(t)$ — denoted by $v^w_i(t)$ — is investment use of G_i in working capital, or working capital lation of G_i. One can now write,

$$v^w(t) = H\; \Delta\, (x(t) \qquad \ldots . \quad (2.4.2)$$

the (i,j)th element of the matrix H — h_{ij} — is given by ij. Now let $r = (r_i)$ be a given growth programme, where r_i is

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θ_{ij}'s were not less than unity the equation (2.4.1) would to be replaced by difference equations of higher order.

the rate of growth of production in S_i. Then (2.4.2) boils down to :

$$v^w(t) = H \hat{r} x(t) \qquad \ldots (2.4.3)$$

Coming to investment on the second account, viz., fixed capital accumulation[a], I shall begin with its simplest formulation that is very often met in the literature. In this formulation it is assumed that at any time t, unit expansion of capacity in a sector requires fixed amounts of all commodities <u>and</u> that the whole of these amounts are made available from the respective productions in the same time unit. Formally, it is assumed that the investment use of G_i on the present account for the expansion in sector j is given by :

$$v^f_{ij}(t) = b_{ij} \; \Delta x^c_j (t) \qquad \ldots (2.4.4)$$

where b_{ij} is a fixed non-negative parameter, and $\Delta x^c_j(t)$ is the expansion of capacity in S_j at time t.

The total investment use of G_i is, therefore, given by :

$$v^f_i(t) = \sum_{j=1}^{n} v^f_{ij}(t) = \sum_{j=1}^{n} b_{ij} \Delta x^c_j(t) \qquad \ldots (2.4.5)$$

$$i = 1, \ldots, n$$

Clearly, when the capacity in a sector, say S_j, grows at a given rate — r^c_j say — (2.4.5) boils down to :

$$v^f_i(t) = \sum_{j=1}^{n} b_{ij} \, r^c_j \, x^c_j$$

$$\text{or,} \quad v^f(t) = B \hat{r}^c \, x^c \qquad \ldots (2.4.6)$$

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[a] In case there is no chance of confusion, we shall often refer to "fixed capital accumulation" as "investment" without qualification.

The introduction of an explicit time structure here clearly amounts to relaxing the second assumption made above, i.e., really to breaking up b_{ij} into time specific uses of G_i for capacity creation in S_j. More specifically, one may define an investment project (or programme) in a sector as an action which once initiated is completed, with a definite lag — λ_i for S_i, say — between the initiation and completion. During this time specific amounts of different commodities are used up on this account, and at the end a definite extra capacity is installed. Stated explicitly, an investment programme at unit level in S_j initiated at time t results in one unit of additional capacity in S_j at time $(t + \lambda_j + 1)$ and uses b_{ij}^{τ} amount of G_i at time $t+\tau$, $0 \leq \tau \leq \lambda_j$. The length of λ_j which is usually taken to be greater than the unit of time under reference, is the so-called construction period of the investment project in S_j.[9/] The capacity output in S_j at time t ($> \lambda_j$) is therefore given by :

$$x_j^c(t) = x_j^c(\lambda_j) + \sum_{t'=0}^{t-\lambda_j-1} z_j(t') \qquad \ldots \; (2.4.7)$$

where $z_j(t')$ is the scale of investment programme initiated in S_j at time t'. The total investment use of G_i on fixed capital at time t is then :

$$v_i^f(t) = \sum_{j=1}^{n} \sum_{\tau=0}^{\lambda_j} b_{ij}^{\tau} \; z_j(t-\tau) \qquad \ldots \; (2.4.8)$$

9/ The λ_j's are taken to be integers.

Once again, when the rates of growth of capacities are constants, one gets from (2.4.7)

$$z_j(t-\lambda_j) = \Delta x_j^c(t) = r_j^c x_j^c(t) \qquad \forall \; t > \lambda_j$$

$$\text{or,} \quad z_j(t-\tau) = r_j^c x_j^c(t-\tau+\lambda_j) \qquad \forall \tau \leq t$$

$$\text{or,} \quad z_j(t-\tau) = r_j^c (1+r_j^c)^{\lambda_j - \tau} x_j^c(t) \qquad \ldots (2.4.9)$$

From (2.4.8) and (2.4.9) one then gets

$$v_i^f(t) = \sum_{j=1}^{n} \sum_{\tau=0}^{\lambda_j} b_{ij}^{\tau} r_j^c (1+r_j^c)^{\lambda_j - \tau} x_j^c(t)$$

$$= \sum_{j=1}^{n} g_{ij}(r_j^c) x_j^c(t)$$

where $g_{ij}(r_j^c) = \sum_{\tau=0}^{\lambda_j} b_{ij}^{\tau} r_j^c (1+r_j^c)^{\lambda_j - \tau}$ ... (2.4.10)

, $v^f(t) = G(r^c) x^c(t)$ ... (2.4.11)

where $G(r^c)$ is a matrix whose (i,j)th element is $g_{ij}(r_j^c)$ and r^c is a vector with r_i^c as its i-th component.

It is to be noted that when $\lambda_j = 0 \;\; \forall \; j$, one gets from (2.4.10):

$$g_{ij}(r_j^c) = b_{ij}^{(o)} r_j^c,$$

is back to the simplest formulation we started out with.

We now decompose the final demand vector at time t, $y(t)$, as follows :

$$y(t) = u(t) + v^W(t) + v^f(t) \qquad \ldots (2.4.12)$$

where $u(t)$ is the vector of final demands net of investment (consumption

Using (2.4.3), (2.4.11) and (2.4.12) we can now write the balance equations (2.[illegible]) at time t as :

$$x(t) = Ax(t) + H\hat{r}\,x(t) + G(r')\,x^c(t) + u(t) \qquad \ldots (2.4.13)$$

Now, if one assumes full capacity operation, i.e., $x^c(t) = x(t)$ — which in turn implies that $r_i^c = r_i$ — then (2.4.13) reduces to :

$$x(t) = Ax(t) + H\hat{r}\,x(t) + G(r)x(t) + u(t)$$

$$\text{or,} \quad x(t) = (A + B(r))\,x(t) + u(t) \qquad \ldots (2.4.14)$$

$$\text{where } B(r) = H\hat{r} + G(r) \qquad \ldots (2.4.15)$$

Clearly, when the rates of growth are treated as parameters, the formal structure of (2.4.14) is same as that of the standard IO model. This completes the discussion of the technical basis of the growth-investment relations. I shall just note that, conceptually, the treatment of fixed capital investment given here remains highly simplified, in the sense that our specifications do justice only to new vestment projects which can be defined independently of the existing pacities. The repair-maintenance-replacement-expansion on existing pacities is really a complex spectrum which has not yet found any tisfactory treatment. Properly speaking, our treatment is justified n capacity expansion is secured only by 'new' investment projects. ting this assumption, it is possible to deal with the remaining gories by assuming that all repair-maintenance-replacements due on sting capacities are in fact undertaken. The commodity requirements hese accounts can then be included in the a_{ij} coefficients, given utilization.

I shall now conclude by referring to the "viability" (productivity) condition for the system (2.4.14). It is clear from (C.2.1) that a necessary and sufficient condition for viability is :

$$\phi(A + B(r)) < 1$$

Since the elements of B(r) are <u>increasing</u> functions of r and the value of the dominant root of a non-negative matrix is also an <u>increasing</u> function of its elements, it follows that the condition is violated for high values of r. We may call this the case of an <u>infeasible growth programme</u>. Correspondingly the growth programme, r, is <u>feasible</u> — as of the consistency conditions posited — if the above condition is satisfied. In the very special case where $G(r) = B\hat{r}$ <u>and</u> r has all components equal, say of value ρ, an explicit upper bound on the feasible rates of growth can be specified, viz.,

$$\rho < \frac{1}{\phi\left[(I-A)^{-1}(H+B)\right]}$$

The matrix $(I-A)^{-1}(H+B)$ can be interpreted as the "total" (marginal) capital coefficients matrix. Its dominant root plays the same kind of role as the "capital-output ratio" of the aggregative growth models of the Harrod-Domar variety.

1.6 <u>Foreign trade from an IO standpoint</u>

The purpose of this section is to bring import and export into the picture and consider some usual treatments of these variables in an IO frame.

As for export, the standard procedure is to treat it exogenously as a part of the given bill of final demands, and I shall not go beyond this in the following discussion. Import, on the other hand, is usually treated on the basis of a dichotomy, viz., non-competitive and competitive (see footnote 5, p.29). The treatment of the former type of imports is rather straightforward, viz., the commodities are treated as primary goods. It is only the latter type of imports that I shall be dealing with in the following analysis.

From the very definition of competitive imports it is clear that these can be seen as alternative sources of supply and hence their introduction would have a direct bearing on the IO balances depicted by (2.1.2). More explicitly, the IO balances now can be written as :

$$x + z = Ax + y \qquad \ldots (2.5.1)$$

where z is the vector of imports and y is taken to include the exogenously specified vector of exports.

Clearly, the system (2.5.1) — with x and z as the endogenous variables and y the exogenous ones — has n additional degrees of [f]reedom. One simple way of closing the degree of freedom is to fix the [po]sition of alternative sources of supply, viz., to set $z_i = m_i x_i \; \forall \, i$, given m_i's. The system (2.5.1) then reduces to :

$$x = (A-M)x + y \qquad \ldots (2.5.2)$$

M is a diagonal matrix with m_i in its (i,i)th position.

One is thus back to the same formal structure of the standard but for that some of the diagonal elements of the matrix (A-M) here can be negative. Clearly, this only strengthens the concept of productivity in the sense that if x is a consistent production programme of (2.1.2) then it remains consistent for (2.5.2).

A more interesting treatment of imports starts with limitations of domestic production as the basis for imports. In particular, the limitation is brought out most clearly and sharply by specifying direct bounds (capacity constraints). As mentioned earlier, import can then be seen as secondary source of supply, i.e., it is brought in y when the domestic production reaches the specified upper bound.

y, the model can be stated as :

$$x + z = Ax + y \qquad \ldots \ (2.5.1)$$

$$x \leq \bar{x} \qquad \ldots \ (2.5.3)$$

$$(x_i - \bar{x}_i) z_i = 0 \quad \forall i \qquad \ldots \ (2.5.4)$$

Obviously, the formal structure of the standard IO model is eralised here. The formal relation between the standard model and generalisation is best seen by partitioning the set of sectors two fundamental sets entailed by relations (2.5.4). In one set — may be called the set of <u>bottleneck</u> sectors — production is at ty. In the complementary set — this may be called the set of sectors — production is less than capacity. It then follows from that there is no import of the goods in the free sectors, whereas bottleneck sectors imports become <u>residually</u> determined by demand which includes the input requirements for the products of the free sectors in the bottleneck sectors. It is evident that given the partition, the model can be reformulated as :

$$\begin{bmatrix} 0 \\ \cdots \\ z_K \end{bmatrix} + \begin{bmatrix} x_J \\ \cdots \\ \bar{x}_K \end{bmatrix} = \begin{bmatrix} A_{JJ} & A_{JK} \\ A_{KJ} & A_{KK} \end{bmatrix} \begin{bmatrix} x_J \\ \cdots \\ \bar{x}_K \end{bmatrix} + \begin{bmatrix} y_J \\ \cdots \\ y_K \end{bmatrix} \qquad \ldots(2.5.5)$$

where K denotes the set of bottleneck sectors and J its complement, i.e., the set of free sectors. The partitioning of the vectors and the matrix should be clear from the context.

(2.5.5) can be written as :

$$x_J = A_{JJ}\, x_J + (y_J + A_{JK}\, \bar{x}_K) \qquad \ldots (2.5.6)$$

$$z_K = A_{KJ}\, x_J + y_K - (I - A_{KK})\bar{x}_K \qquad \ldots (2.5.7)$$

Clearly (2.5.6) forms a reduced order IO model or a subsystem of the original system with x_J as production programme and $(y_J + A_{JK}\, \bar{x}_K)$ as final demand. The subsystem is directly solved to yield :

$$x_J = (I - A_{JJ})^{-1} (y_J + A_{JK}\, \bar{x}_K) \qquad \ldots (2.5.8),$$

and z_K is then determined residually from (2.5.7).

Formally, z_K has the same status as primary inputs in the standard IO system with primary inputs. It simply becomes the required rt vector for the given bill of final demand. Of course, which goods to be imported and which not is an internal matter; the partition is not given but has to be derived. Below we describe two methods of this derivation each of which also solves the model simultaneously.

The first algorithm consists of constructing the sequences of vectors $\left\{ x^t, z^t \right\}$ as follows :

Given x^t, z^t is computed from :

$$z^t = y - (I-A)x^t \qquad \ldots (2.5.9)$$

and a set K^t by :

$K^t = \Big\{ i$ [redacted] $/ z_i^t \geq 0 \Big\}$ ... (2.5.10)

x^{t+1} is now computed from :

$$x_{K^t}^{t+1} = \bar{x}_{K^t} \qquad \ldots (2.5.11)$$

$$x_{J^t}^{t+1} = (I-A_{J^tJ^t})^{-1} \left(y_{J^t} + A_{J^tK^t} \;\bar{x}_{K^t} \right) \qquad \ldots (2.5.12)$$

The algorithm is initiated by putting

$$x^0 = \bar{x} \qquad \ldots (2.5.13)$$

and is terminated at step T when :

$$K^{T-1} = K^T$$

In economic terms, K^t and J^t are precisely the sets of bottleneck and free sectors respectively at step (t+1). At any step, any t+1, of the algorithm, given the estimate of the output vectors of the previous step, it is first found out which of the commodities have negative imports, the corresponding sectors are then taken to be free, and their outputs are determined by solving an IO system consisting of the free-sectors, treating the input requirements of their products in the bottleneck sectors as part of their final demands, i.e., from (2.5.8) where $K = K^t$.

The second algorithm is a straightforward adaptation of the IO method (2.2.?)-(2.2.1). In words, this method equates the output of each commodity at the initial step to its capacity or its final demand whichever is smaller. Then at step t of the scheme output of each commodity is expanded to meet the derived demand given by the output vector at step (t-1) until capacity limit is reached, the remaining derived demand is met by imports. Formally, it consists of constructing two sequences $\{x^t\}$ and $\{z^t\}$ by the recursive formulae :

$$x^t = \min \left\{ y + Ax^{t-1}, \bar{x} \right\} \qquad \ldots (2.5.14)$$

$$z^t = y + Ax^{t-1} - x^t \qquad \ldots (2.5.15)$$

$$x^0 = \min \left\{ y, \bar{x} \right\} \qquad . \quad (2.5.16)$$

2.6 A review of the literature

This section is to review briefly the literature behind the account of "theoretical IO analysis" given in this chapter. The purpose is basically to cite appropriate references for points made so far in a somewhat systematic fashion. I shall begin and end the review with two broad issues, concerning respectively the scope of IO analysis and the relation between IO analysis and AA. In between, I shall attempt to trace the sources of ideas in sections 2-5 to specific contributions in the literature. Section 1 is left out, for it is based on the literature cited in chapter 1 in a general fashion, i.e., the points made in section 1 can be found in most of the references mentioned earlier in some form or other.

The scope of IO analysis can be broadly decomposed into two parts, or aspects. The first refers to the "scope of consistency" in its set of interrelations. We had repeatedly referred to the "openness" of this area in chapter 1, and in each of sections 3-5 we have explored the openness in different directions[10]. Now, the interrelations are between the 'sectors' of an IO model, and the exact coverage of these sectors (in the perspective of the whole economy) is also an open question. This may be said to represent the second aspect of the scope mentioned above.

The first point to be noted in a review of the literature is that Leontief originally started out with a "closed" version of the IO model, with an all-embracing scope vis-a-vis the economy on both counts mentioned above. In our language, this simply internalised all final demands[11]. In going over to the "open" version later (the starting point of our account), he introduced the notion of "final demands"

10/ An important difference between the extensions may be noted here. In each of sections 3 and 4, a part of the original bill of final demand is internalised into the model by a reinterpretation of the IO coefficients. This leaves the formal structure of the model intact, and extends the scope of consistency in a strict sense which is lacking in section 5, for the formal structure itself is changed in that section. However, the concepts, properties and methods of IO analysis are adapted to the changed structure in a straightforward way.

The theoretical basis of Leontief's original formulation was, however, "static" in the sense of excluding investment. It was basically household demand, or consumption, that was treated as an internal part of the set of interrelations.

basically as a formal, not a substantive, category :

"The final demand for a commodity represents that part of its total output which is treated — within the *framework* of a given analytical scheme — as an independent variable, while the derived or intermediate demand comprises the other part, which is considered to be a dependent variable" [vide Leontief (1951; p. 147)]

The distinction between "final" and "derived" demand (i.e., the first aspect of the scope question) was seen as a relative one depending upon the exact purpose of analysis. Leontief himself referred to the "practical requirements of policy-making decisions" [vide Leontief (1951; p. (ix))] as the broad purpose justifying the "open" version. This is what we had referred to as the "planning-assumption" in chapter 1. The question of validity of the relation behind a "derived" demand is of course a different one. As Leontief was to point out later :

"The usefulness of the static input-output approach in the study of an economy is conditioned by the relative invariance of its structural characteristics. The introduction of 'open system' — with the bill of final demand considered as being dependent on some 'outside forces' — serves the purpose of separating the more stable from the less stable aspects of interindustrial relationships; the input coefficients included in the structural matrix describing the former, the unaccounted for determinants of the given bill of goods represented by the latter". [vide Leontief (1953; pp. 19-20)]

We now turn to the specific contributions behind sections 2-5. IO method described in section 2 appears to have been developed two independent lines, representing economic analysis and computa- procedures respectively. The first line largely drew on the logical similarities between the Keynesian system of income determination and the IO system, and utilised the iterative scheme to depict the mechanics of the inherent multiplier process. Goodwin (1949) and Chipman (1950) can be cited as the early contributions in the field. Later economic analysis of the IO system has also significantly relied on the iterative scheme for depicting such concepts as direct and indirect inputs, total input coefficients etc. Reference may be made to Dorfman, Samuelson and Solow (1958), Gale (1960) etc. The particular economic interpretation of the method as an adjustment process and as a multiplier process are either explicit or implicit in these discussions, often under different names. The interpretation as a process of material balancing in a socialist economy is also common. A standard theoretical reference for this is Montias (1959). As a computational procedure for solving the IO system, the IO method seems to have been first proposed by Waugh (1950), followed up by Holley (1951). A very comprehensive account of the method was presented by Evans (1956). The exposition of section 2.2.6 is based largely on the last paper.

As mentioned earlier, the endogenous treatment of consumption is part of the initial development of IO analysis. Even after going over to the "open" version, Leontief referred to the possibilities of endogenous consumption on the basis of production-employment-income-expenditure relations. For example, in one of his later reviews of IO analysis, Leontief writes :

> "Households must not necessarily be considered to be part of the exogenous sectors ...... In dealing with problem of income

generation in its relation to employment, the quantities of consumers' goods and services absorbed by households can be considered (in a Keynesian manner) to be structurally dependent on the total level of employment in the same way as the quantities of coke and ore absorbed by blast furnaces are considered to be structurally related to the amount of pig iron produced by them". [*vide* Leontief (1966; pp.141-42]

The same kind of relations are quite frequently met in the literature already cited. It may nevertheless be mentioned that while the basis of endogenous consumption in the IO model seems to be agreed upon, all explicit formulations so far as known to us treat income, employment and consumption in a completely aggregative fashion. The distribution of income generated by production and its implication for the commodity pattern of consumption appears to have been mostly neglected. The formulation given in section 3 is not based on any specific contributions in this sense.

Regarding investment, mention must be made of the very large theoretical literature on so-called "Dynamic IO Analysis" which again was initiated by Leontief (1951; pp. 211-16), (1953; chapter 3). However, this literature has concerned itself with a rather different set of issues than those discussed. We shall pursue the account of section 4 later in Part IV of this study where we give a detailed review of the relevant literature (chapter 9). Here I shall only mention certain differences in the technical basis of investment-production relations between our treatment and that in the literature referred.

To recapitulate, we have based investment on two independent relations, viz., the IO relations which generate the 'working capital

ation' component of investment, and the production-capacity rela-
tions which generate the 'fixed capital accumulation' component. The two
ations are then seen to possess their own time-structures, resulting
in a general dynamic formulation. The analytical representation of this
amic system has so far been in terms of growth rates as parametres.
later analysis will be concerned essentially with turning these into
bles. In contrast, the technical basis of investment in the so-called
a IO analysis" has been sought in a set of stock-flow ratios relat-
ing stocks of inputs to output-flows, with investment as the rate of
change of stocks. This offers no clear-cut distinction either between the
two components of investment referred or between production and capacity.
The absence of the second distinction means that the formulation is
based implicitly on full capacity assumption. Regarding specific contri-
butions behind our formulation, mention may be made of the works of
ange (1959) regarding working capital accumulation and Chakravarty (1959)
ding fixed capital formation. Lange's formulation is based explicitly
turnover periods of different items of working capital as in equation
2.4.2) [p. 43]. However, he sought to include fixed capital on the same
, with 'use periods' for 'turnover periods', and this does not allow
separation between production and capacity. Chakravarty neglected
rking capital altogether, and based his relations on stock-flow ratios
Leontief. However, by distinguishing between "investment in execu-
and "finished investment" he provides the basis of our definition
an "investment programme". His concept of "gestation lags" in this
text is the same as our "construction period". However, there is again

no clear separation between production and capacity in his analysis. More recently, the literature on operational planning and projection models has reformulated the dynamic IO system in a way that is very close to our formulation. Reference may be made to Eckaus and Parikh (1968) regarding the technical basis of investment-production relations and to Stone and Brown (1962), Mathur (1964), Mukherjee (1964) etc., for the analytical representation of the dynamic system in terms of rates of growth. The basis for rate of growth determination, however, is quite different in these than that in our later discussion.

Regarding imports, the common practice in IO analysis is to treat them deterministically. Reference may be made to Chenery and Clark (1959) for the formulation given in (2.5.2) [p.49]. Similar formulations are also to be found in Leontief (1951). The model of foreign trade represented by (2.5.1), (2.5.3) and (2.5.4) [p. 50] is a special case of a more general model stated by Arrow (1954) who in turn refers to the applied works of Chenery for the formulation. The two algorithms for it reported in section 5 are due to Chander (1973) and Arrow (1954) respectively.

We shall end this section with a few comments on the AA model of production as a generalisation of the IO model. The discussion here is confined to the original exposition of AA by Koopmans (1951a), and the main purpose is to highlight the differences between the two. The main point is that AA does not represent a generalisation of IO in the conceptual sense of development of the basic theme and central idea of

IO, which is what this study purports to be. Consequently, this study has "nothing to do with the AA frame", as mentioned in chapter 1. The starting point of AA is a set of "commodities", not "products", which is then classified on a two-fold basis, viz., "available in nature" and "not available in nature", and "desired in itself" and "not desired in itself" . There is in fact no essential category of "products" as such, and the whole basis is incompatible with the "open endedness in the enumeration of primary goods" in IO analysis mentioned earlier. At the level of analysis, the fundamental assumption of AA is a given availability of some commodities, not given final demand. The fundamental concept is efficiency, not consistency, the problem is that of resource allocation, not output determination. In short, there is hardly anything in common between the approaches of IO and AA. One has to end by saying that their purposes are quite different.

PART II

SCALE DEPENDENCE OF IO COEFFICIENTS

CHAPTER 3

The Non-Linear IO Model in Physical Terms

This chapter and the next are concerned with a straightforward generalisation of the internal structure of an IO model. This generalisation consists of treating the IO coefficients as dependent upon the scales of production in a rather general fashion. This chapter will be concerned with the system at the physical level of analysis without any price-concepts. The next chapter will introduce prices and costs, and both review some of the problems of this chapter with the help of these concepts as well as take up some independent issues concerning price-formation.

This chapter is divided into two sections dealing respectively with definition (in a broad sense) and analysis. In particular, the first includes the formulation of the model, general discussion of its ideas, statement of problems and of the approach to their analysis. Section 2 takes up the detailed analysis, which is basically concerned with different aspects of the interrelated problems of productivity and output determination. By the very nature of problems tackled, the analysis is rather formal.

3.1 Basic formulation and general discussion

It will be convenient to begin straightaway with a statement of the formal structure of the model and then take up its interpretation. The formal structure is given by the following set of relations:

$$x_i = \sum_{j=1}^{n} x_{ij}(x_1, \ldots, x_n) + y_i, \qquad i = 1, 2, \ldots, n$$

$$= \sum_{j=1}^{n} f_{ij}(x_1, \ldots, x_n)\, x_j + y_i \qquad i = 1, 2, \ldots, n$$

or, in vector-matrix notation :

$$x = F(x)x + y \qquad \ldots (5.1.0)$$

where $x, y \in R^n_+$ and $F(x)$ is the (nxn) matrix with $f_{ij}(x)$ $\left[= x_{ij}/x_j \right]$[1] as its element in the (i,j)th position. In economic terms, $x_{ij}(x)$ is the amount of G_i needed as input to produce x_j amount of G_j, given that the level of production of G_k, $k \neq j$, is x_k and $f_{ij}(x)$'s are the derived IO coefficients — now variables. Obviously, $x_{ij}(x) \geq 0$, $f_{ij}(x) \geq 0$ for all $x \geq 0$ and for all i and j.

It should be obvious that the structure is very general indeed. We may now define a few general concepts regarding economies and diseconomies of scale with the help of this structure[2]. To begin with, the dependence of IO coefficients of a sector — say the kth — describing its production process upon x_k gives expression to so-called <u>internal economies or diseconomies of scale</u>. In the same way the dependence of

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1/ For $x_j = 0$, $f_{ij}(x)$ is defined to be the limit of $x_{ij}(x_1, \ldots, x_{j-1}, \bar{x}_j, x_{j+1}, \ldots, x_n)/\bar{x}_j$ as $\bar{x}_j$ goes to x_j, and this limit is assumed to exist for all i and j.

2/ Since our analysis requires only the purely formal structure of the model, it is possible to reinterpret the model more generally to include relations other than production-relations, e.g., the kind of relations discussed in sections 2.3-2.4. Obviously, the source of non-linearity may lie in any of these 'other relations'. For specificity of treatment, however, the standard interpretation of an IO model in terms of pure production interdependence will be maintained here.

$f^k(x)$ on x_j, $j \neq k$, can express direct external economies or diseconomies of scale.

On a slight digression we may note that on a priori considerations, the first direction of generalisation would appear to be more significant. However, as the structure of (3.1.0) reveals, the introduction of possible externalities does not create any new analytical complications. Basically, this reflects the fact that the production programme as a whole constitutes the basic variables of the model. Whatever effect its separate components may have for specific relations are, formally, special cases of the general effect of the production programme.

Returning to the definition of concepts, it should be obvious from (3.1.0) that any notion of IRS/DRS in the present context of purely physical specification would have to be rather strong one. To bring this out clearly, let us first of all suppose that there are no externalities, so that $f^k(x)$ is a function of x_k only, for all k. The basic notion of RS is then an input-specific-RS in each sector. That is, S_j may be said to have IRS(DRS) with respect to G_i as input if $f_{ij}(x_j)$ is a decreasing (increasing) function of x_j. As stated, this really is global IRS(DRS) in S_j with respect to G_i. The parallel local concept would require the above condition to be satisfied only in some range of the output level x_j. One can then say that there is a general IRS(DRS) in the production of G_j if each of the elements of $f^j(x_j)$ decreases (increases) with x_j. The local-global distinction can once again be made. The concept of external economies or diseconomies of scale is, however, basically a pairwise relation, viz., one can say that S_k <u>derives</u> external economies (diseconomies) of scale from S_j $(j \neq k)$ if the variation of $f_{ik}(x)$ with respect to x_j is inverse (direct), which is the same as saying that S_j <u>generates</u> external economies (diseconomies) of scale for S_k. It is to be noted that once again this notion can be input-specific (i.e., the variation mentioned above can be only for some i) and local or global. I shall now define two more concepts which will be used in the analysis of section 2.

<u>Definition 3.1</u>: There is said to be general economies (diseconomies) of scale in the production of some good, say G_j, if

$$f^j(x^1) \leqq (\geqq) \quad f^j(x^2) \qquad \forall\, x^1 \geqq x^2 \geqq 0$$

<u>Definition 3.2</u>: There is said to be general economies (diseconomies) of scale in the production system as a whole if

$$F(x^1) \leqq (\geqq) \quad F(x^2) \qquad \forall\;\; x^1 \geqq x^2 \geqq 0$$

It may be noted here that by the very generalization, certain basic properties or features of the IO model become open questions to be investigated anew in the present context. Two such properties have already been pointed out in section 2.1 (p. 25), viz., the equivalence of the notion of local and global productivity and the uniqueness of a solution.

To take up the first, it may be pointed out that the local-global equivalence need no longer be so in the present context, so that one would have to search independently for conditions guaranteeing each.

The main interest here really centres around global productivity. This is also what I shall be searching for in the first place. The next point of interest is then to search for a condition under which the property of the linear model — that local productivity implies global productivity — is obtained in this context also.

The second property of the standard model that becomes an open question here concerns, as stated before, the uniqueness of a solution of (3.1.0). In the present set up existence of a solution need not imply — as will be shown later — its uniqueness. Now, the multiplicity of solutions can also be looked upon as an opening up of a choice problem, i.e., given that multiple solutions are possible for the model, the question arises whether it is possible to 'choose' one solution — in this sense, really 'determine' the output levels for given final demands — rather than another. One obvious approach would be to minimise the cost of production on the basis of some given cost function. The problem then is to search for a method for determining the efficient solution (production programme), the one to minimise the cost satisfying all relations posited. I shall explore this aspect of the problem as well as the more formal one of looking for conditions guaranteeing the uniqueness of solutions.

One may now briefly state the strategy of our analysis. This consists of starting out with a definite method for solving the fundamental output determination problem, i.e., of determining x for a given y and establishing some basic properties of the model itself by this route. The same method — as will be seen later — is also a method for determining the efficient production programme referred above. As stated in chapter 1, this method can be seen as a straightforward generalisation of the so-called basic IO method to the present context. The 'properties of the model' referred in their turn are generalisations of the basic concept of productivity of an IO model. We shall now state the proposed method of solution and review the previous contributions in this field.

The method is an iterative one, which consists of setting :

$$x^t = F(x^{t-1})x^{t-1} + y \qquad \ldots (3.1.1)$$

$$x^0 = y \qquad \ldots (3.1.2)$$

That is, the output vector at step t of the scheme is obtained simply as the sum of the total input requirements corresponding to the output vector at step (t-1) and the given final demand vector. It is easily seen that if F(x) is a constant matrix, say A, then (3.1.1) and (3.1.2) boil down to solving the equation $(I-A)x = y$ by means of the power series expansion, $(I+A+A^2 + \ldots + A^t + \ldots)y$, i.e., to the basic IO method.

Turning to the review of literature, the basic contribution in the field as defined here is contained in the introductory part of a paper on IO computation by Evans (1956). In that paper, he introduced the method of solving an IO model by means of power series expansion precisely as a specialisation of a general iterative computational scheme for a non-linear IO model. However, he did not proceed much beyond stating the general scheme and noting some of its more elementary properties. This chapter continues with the work from the point where Evans had left it off, broadly along two directions, viz., to recount : (a) detailed and rigorous examination of the properties of the scheme proposed by Evans; and (b) application of these properties to derive certain basic properties of the model itself. I may nevertheless note here that Evans explicitly considered only internal economies or diseconomies of scale and also left the question of productivity completely unexplored. Recently Sandberg (1973) analysed mainly the global productivity aspect with much stronger continuity assumptions than what I shall require here. His approach was however completely different from the present one and in most part of his paper he also considered only internal scale economies or diseconomies.

3.2 Formal analysis : productivity and output determination

To begin with I may point out that the analysis below does not require any restriction on the formal structure of the model other than those specified hitherto. That is, the formulation, so far, is general enough, and no further general restriction on the formal structure is, in fact, involved in the analysis below. That is, there is no further assumption which is strictly logically necessary for all the results derived. It will however be convenient to record here two assumptions which are required for many results. With this, one need only refer to their logical redundancies when that is not so and economise (space) on their necessities. This is given in part 1 (subsection 3.2.1) below. The analysis proper then consists of the remaining parts. For convenience of exposition, this is divided into three subsections — 3.2.2 to 3.2.4. 3.2.2 will note a few general properties of the scheme and justify our approach to the analysis. 3.2.3 takes up the basic analytical property of viability conditions and compares the results obtained for the present model with some corresponding results for the standard model. 3.2.4 then considers the two open questions regarding uniqueness and local-global productivity mentioned earlier.

3.2.1 <u>Some frequent assumptions</u>

The assumptions are :

<u>A 3.1</u> : For all $x^1 \geq x^2 \geq 0$, $F(x^1)x^1 \geq F(x^2)x^2$

<u>A 3.2</u> : (i) if $\{x^t\}$ is a non-decreasing sequence converging to $\bar{x}$, then each $f_{ij}(x^t)x_j^t$ also converges to $f_{ij}(\bar{x})\bar{x}_j$ and,

(ii) For each i and j $\sup_{x \in R^n_+} f_{ij}(x)$ exists (to be denoted by $\bar{f}_{ij}$).

A 3.1 means that, for the production system as a whole, additional output requires additional input. That is, if there is to be a greater production of any good (and no less of any other) then the input use of <u>no</u> good can decrease. In words A 3.1 can be called a <u>monotonicity condition</u>. A 3.2 is more of a <u>regularity condition</u>. Its first part disallows certain kinds of discontinuity, and does not appear very simple to interpret in the general case. However, in the absence of externalities this only means that each $f_{ij}(x_j)x_j$

is a right continuous function of x_j at each point on R^1_+ which, obviously, enables one to handle the important case of discontinuous piece-wise linear input coefficients. The second part requires the IO coefficients to be bounded above.

3.2.2 General properties of the method

A few general properties of the sequence $\{x^t\}$ defined in (3.1.1) and (3.1.2) can now be derived. A first simple observation is stated below as a lemma, followed by a theorem, both due originally to Evans in a somewhat different form[3/].

Lemma 3.2.1 : $\{x^t\}$ is a non-negative, non-decreasing sequence.

Proof : The proof is by induction. First note that,

$$\begin{aligned} x^1 &= F(x^0)x^0 + x^0 \quad [\text{by (3.1.1) and (3.1.2)}] \\ &\geqq x^0 \quad [\because F(x^0)x^0 \geqq 0] \\ &\geqq 0 \quad [\text{by (3.1.2)}] \end{aligned}$$

Now make the induction hypothesis

$$x^t \geqq x^{t-1} \geqq 0 \qquad (H)$$

and consider,

$$\begin{aligned} x^{t+1} - x^t &= F(x^t)x^t - F(x^{t-1})x^{t-1} \quad [\text{using (3.1.1)}] \\ &\geqq 0 \quad [\text{by (H) and A.3.1}] \end{aligned}$$

This proves that

$$x^{t+1} \geqq x^t \geqq 0 \qquad \text{for } t = 0, 1, 2, \ldots \qquad \text{Q.E.D.}$$

3/ It may be noted that none of the results derived in this subsection requires the assumption A 3.2(ii).

<u>Theorem 3.2.1</u> : The sequence $\{x^t\}$ defined in (3.1.1) and (3.1.2) converges if and only if there is a solution of (3.1.0), and the limit of the sequence is a solution of (3.1.0).

<u>Proof</u> : If $\{x^t\}$ converges then its limit satisfies (3.1.0) by construction and is non-negative by lemma 3.2.1. Hence a solution of (3.1.0) exists and is given by the limit.

Conversely, suppose x^* satisfies (3.1.0), i.e.,

$$x^* = F(x^*)x^* + y \qquad \ldots \quad (3.2.1)$$

Obviously, $x^* \underset{=}{>} y$ ... (3.2.2)

Now,

$$\begin{aligned} x^1 &= F(y)y + y && \ldots \text{ [From (3.1.1) and (3.1.2)]} \\ &\underset{=}{<} F(x^*)x^* + y && \ldots \text{ [Using (3.2.2) and A.3.1]} \\ &= x^* && \ldots \text{ [From (3.2.1]} \end{aligned}$$

$\therefore \; x^1 \underset{=}{<} x^*$

Suppose, as an induction hypothesis,

$$x^t \underset{=}{<} x^* \qquad \ldots \quad (H),$$

then

$$\begin{aligned} x^{t+1} &= F(x^t)x^t + y \\ &\underset{=}{<} F(x^*)x^* + y && \ldots \text{ [Using (H) and A 3.1]} \\ &= x^* && \ldots \text{ [On account of (3.2.1)]} \end{aligned}$$

$$x^t \underset{=}{<} x^* \quad \text{for all } t = 0, 1, 2, \ldots \qquad (3.2.3)$$

By lemma 3.2.1 and (3.2.3), $\{x^t\}$ is a non-decreasing bounded sequence and hence converges. By construction, its limit satisfies (3.1.0). Q.E.D.

The theorem establishes a complete equivalence between the workability of the iterative method and the existence of a meaningful solution of the model, i.e., its so called 'productivity', and justifies the present approach to the properties of the model. Further discussions of the properties can be conducted equivalently in terms of the conditions guaranteeing the convergence of the sequence or those guaranteeing the existence of a solution of (3.1.0) directly. These are to be taken up in the next section. This section ends with reporting the following corollary, due again to Evans.

<u>Corollary 3.2.1</u> : If there are multiple solutions to the system (3.1.0) for a given $y \geq 0$, then the sequence $\{x^t\}$ defined in (3.1.1) and (3.1.2) converges to a solution which is smaller than or equal to (component by component) any other solution.

<u>Proof</u> : Let x^* be any solution of (3.1.0) and x be the limit vector of $\{x^t\}$, so that x is a solution of (3.1.0).

Then, as in the proof of Theorem 3.2.1

$$x^t \leq x^* \qquad \text{for all } t = 0, 1, 2, \ldots$$

and hence,

$$x \leq x^* \qquad \text{Q.E.D.}$$

This corollary validates the assertion made before that the method leads to the efficient production programme. The only assumption regarding the associated cost function one needs is that if there is to be a greater production of any good (and no less of any other) then the 'cost' cannot decrease. Inter alia, the corollary also proves that, if the system is productive, then it has an efficient production programme in this very general sense. From the standpoint of the method, the problem of output determination in general and that of finding an efficient production programme in particular are equivalent.

3.2.3 Viability conditions

The arrangement of the (preassigned) contents of this section is as follows : It (a) begins with a viability[4/] condition stated directly in terms of the sequence (3.1.1) and (3.1.2); (b) then establishes the connection between this condition and a standard viability condition for the linear model; and finally (c) proceeds to obtain alternative, more direct, generalisation of the latter condition to the present context.

Now, it is clear from lemma 3.2.1 that one need only impose conditions to guarantee the boundedness from above of $\{x^t\}$ to obtain the convergence of the sequence $\{x^t\}$. One sufficient condition is :

Condition 3.1 : There exists a positive scalar, λ, less than unity such that

$$|| F(x^1)x^1 - F(x^2)x^2 || \leq \lambda \quad || x^1 - x^2 || \qquad \text{for all } x^1, x^2 \in R^n_+$$

where $||\cdot||$ is any well defined norm in R^n.

Theorem 3.2.2 : If condition 3.1 holds, then there is unique solution of (3.1.0) for any $y \geq 0$.

This section is concerned only with the 'global productivity' of the non-linear IO system. For brevity, I shall often refer 'global productivity' as 'viability' in this chapter.

Proof[5/]: First observe that

$$||x^3 - x^2|| = ||F(x^2)x^2 - F(x^1)x^1|| \qquad \text{[From (3.1.1)]}$$
$$\leq \lambda \; ||x^2 - x^1|| \qquad \text{[Using condition 3.1]}$$

and, inductively, that

$$||x^{t+1} - x^t|| = ||F(x^t)x^t - F(x^{t-1})x^{t-1}||$$
$$\leq \lambda \; ||x^t - x^{t-1}||$$
$$\leq \lambda^{t-1} \; ||x^2 - x^1|| \qquad \ldots (3.2.4)$$

Repeated use of (3.2.4) yields, for $m \geq t$,

$$||x^m - x^t|| \leq ||x^m - x^{m-1}|| + ||x^{m-1} - x^{m-2}|| + \ldots + ||x^{t+1} - x^t||$$
$$\leq (\lambda^{m-2} + \lambda^{m-3} + \ldots + \lambda^{t-1}) \; ||x^2 - x^1||$$

Hence it follows that, for $m \geq t$,

$$||x^m - x^t|| \leq \frac{\lambda^{t-1}}{1-\lambda} || \; x^2 - x^1||$$

Since $0 < \lambda < 1$, the sequence $\{\lambda^{t-1}\}$ converges to zero. Therefore, $\{x^t\}$ is a cauchy sequence. The limit of the sequence, obviously, solves the system (3.1.0) by construction.

Finally, for uniqueness, suppose that there are two distinct solutions x and $\bar{x}$ of (3.1.0) for a given $y \geq 0$. Then one has,

$$||x - \bar{x}|| = ||F(x)x - F(\bar{x})\bar{x}||$$
$$\leq \lambda \; ||x - \bar{x}|| \qquad \text{[From condition 3.1]}$$

Since $x \neq \bar{x}$ then $||x - \bar{x}|| \neq 0$, so this relation implies that $1 \leq \lambda$, contrary to the hypothesis that $\lambda < 1$. Hence $x = \bar{x}$. Q.E.D.

5/ This proof is taken from Bartle (1964) [pp. 169-71] and given for the sake of completeness. It may be noted that the proof does not require A 3.1 and A 3.2(ii). Further, the sequence $\{x^t\}$ can also start from any arbitrary $x^0 \geq 0$, not necessarily satisfying (3.1.2). However, condition 3.1 implies that $F(x)x$ is uniformly continuous, and this is a much stronger assumption than A 3.2(i), ruling out, in particular, stepwise linear IO coefficient-functions.

Turning to part (b) of the analysis indicated earlier, the 'standard viability condition for the linear model' referred there is that $\rho(A) < 1$, which is necessary and sufficient for the viability of the linear model. The connection is given by the following corollary :

<u>Corollary 3.2.2</u> : If $F(x)$ is a constant non-negative matrix, say A, then condition 3.1 implies that $\rho(A) < 1$.

<u>Proof</u> : When $F(x) = A$, condition 3.1 reduces to :

$$|| Ax^1 - Ax^2 || \leq \lambda \; || x^1 - x^2 || \qquad \text{for all } x^1, x^2 \in R^n_+ \text{ and for some } \lambda,\ 0 < \lambda < 1.$$

Setting $x^2 = 0$ yields

$$|| Ax^1 || \leq \lambda \; || x^1 || \qquad \text{for all } x^1 \in R^n_+ \quad \ldots\ldots (3.2.5)$$

Now consider the following equation :

$$Ax = \rho(A)x \qquad \ldots\ (3.2.6)$$

It is well-known that (3.2.6) has a solution $x > 0$ (the characteristic vector associated with $\rho(A)$).

Now,

$$||Ax|| = \rho(A) \; || x || \qquad [\text{by } (3.2.6)] \quad \ldots\ (3.2.7)$$

and

$$||Ax|| \leq \lambda \; ||x|| \qquad [\text{by } (3.2.5) \text{ since } x > 0] \ldots (3.2.8)$$

From (3.2.7) and (3.2.8), it follows that

$$\rho(A) \; || x || \leq \lambda \; || x ||$$

i.e.,

$$\rho(A) \leq \lambda \qquad [\because \; ||x|| > 0]$$

or,

$$\rho(A) < 1 \qquad [\; 0 < \lambda < 1 \;] \qquad \text{Q. E. D.}$$

While condition 3.1 is thus seen as a generalisation of a standard viability condition for the linear model, time is now ripe to explore further whether the latter can be extended in some alternative way in a more direct, if also more mechanical, fashion. This is part (c) of the analysis. The qualification 'more direct' here refers both to the approach to the viability problem, via the iterative scheme, taken so far, and to the fact that while the condition under reference is both necessary and sufficient for viability in the linear model, its generalisation given in condition 3.1 is only sufficient, not necessary. To begin with, intuitively one would expect the value of $\phi(F(x))$ for all $x \geq 0$ to play the same kind of role in the present context as the dominant root of A in the linear model. The nearest approximation to this position appears to be the following. So far as the 'necessity' of the viability condition is concerned, one has to note that not all values of x are really admissible for the purpose. One is really interested in the value of $\phi(F(x))$ only for $x \geq 0$ satisfying (3.1.0) for some $y \geq 0$, i.e., for $x \geq 0$ satisfying $x \geq F(x)x$ — in words, for the set of output vectors consistent with some final demand vector. A straight result here is stated below as lemma 3.2.2. On the 'sufficiency' part (viability proper), something stronger than $\phi(F(x)) < 1$ for $x \geq 0$ is required, viz., $\phi(\bar{F}) < 1$. It may be noted in passing that $\bar{F}$ will not, in general, be an observable IO matrix. For it to be so, all the n^2 functions, $f_{ij}(x)$, have to attain their supremum at precisely the same point.

The results are now stated below :

Lemma 3.2.2 : $\phi(F(x)) < 1$ for all $x \geq 0$ such that $x \gg F(x)x$ [6].

6/ This will also be true for all $x \geq 0$ for which $x > F(x)x$ if the matrix $F(x)$ is irreducible for all $x \geq 0$.

Proof : Take any $x \geq 0$ such that

$$x >> F(x)x \qquad \ldots (3.2.9)$$

Now suppose $\phi(F(x)) \nless 1$

This means that there does not exist any $\bar{x} \geq 0$ such that $\bar{x} >> F(x)\bar{x}$.

But, $x >> F(x)x$ and $x \geq 0$ [From (3.2.9)]

This is a contradiction.

Hence, $\phi(F(x)) < 1$. Q. E. D.

Condition 3.2 : $\phi(\bar{F}) < 1$ where $\bar{F} = (\bar{f}_{ij})$.

Theorem 3.2.3 : If condition 3.2 holds then there is a solution of (3.1.0) for any $y \geq 0$.

Proof : Consider equation (3.1.1) :

$$x^t = F(x^{t-1})x^{t-1} + y$$
$$\leq F(x^t)x^t + y \quad \text{[Using Lemma 3.2.1 and A 3.1]}$$
$$\leq \bar{F}x^t + y \quad \text{[By construction of } \bar{F}\text{]}$$
$$x^t \leq (I - \bar{F})^{-1} y \quad \text{[In view of condition 3.2].}$$

Thus $\{x^t\}$ is bounded above and the conclusion follows at once from Lemma 3.2.1. Q. E. D.

Returning to the conjecture stated earlier, one should note that, technically, $\phi(F(x)) < 1$ for all $x \geq 0$ is neither necessary nor sufficient for viability ; the necessary condition (lemma 3.2.2) is weaker and the sufficient condition (Theorem 3.2.3), as developed here, much stronger. It is to be further pointed out that condition 3.2 guarantees only the existence of a solution of (3.1.0) for any $y \geq 0$, not its uniqueness. The following example for a two-sector case establishes this fact and hence the assertion made in section 1 :

Let $f_{ij}(x)$ be defined as follows :

$$f_{ij}(x) = \begin{cases} x_j & \text{if } x_j \leq \frac{1}{3} \\ \frac{1}{3} & \text{if } x_j > \frac{1}{3} \end{cases} \qquad i,j = 1,2. \qquad \ldots (3.2.10)$$

For a final demand vector $y' = (\frac{1}{9}, \frac{1}{9})$ it may be easily checked that both the vectors $(\frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{6}, \frac{1}{6})$ would solve (3.1.0) where $f_{ij}(x)$'s are defined as in (3.2.10). It may also be verified that the matrix $\bar{F}$ in this case is given by,

$$\bar{F} = \begin{bmatrix} \frac{1}{3}, & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

which obviously satisfies condition 3.2. We shall later look for a supplementary condition that would guarantee the uniqueness of the solution.

3.2.4 Local-global productivity and uniqueness

The purpose of this subsection is to take up the two open questions referred to in section 1. I shall first consider the 'local-global issue' and suggest a condition under which the two notions of productivity become equivalent and then show that the same condition also guarantees the uniqueness of a solution of (3.1.0).

To begin with, it should be obvious that local productivity cannot imply global productivity for the present system. Sufficient decreasing returns to scale will always ensure that (3.1.0) has no meaningful solution for y large, given that it has one for y small. Conjecturally, starting from some $(x, y) \geq 0$ satisfying (3.1.0), (a) indefinite scale expansion cannot be possible under strong decreasing returns to scale, while (b) the reverse, i.e., indefinite scale contraction, does not apparently meet any such barrier. The latter conjecture is borne out in the following theorem which again was stated by Evans.

<u>Theorem 3.2.4</u> : If there exists a solution of (3.1.0) for a $\bar{y} \geqq 0$, then there will also exist solutions of (3.1.0) for all y, $0 \leqq y \leqq \bar{y}$.

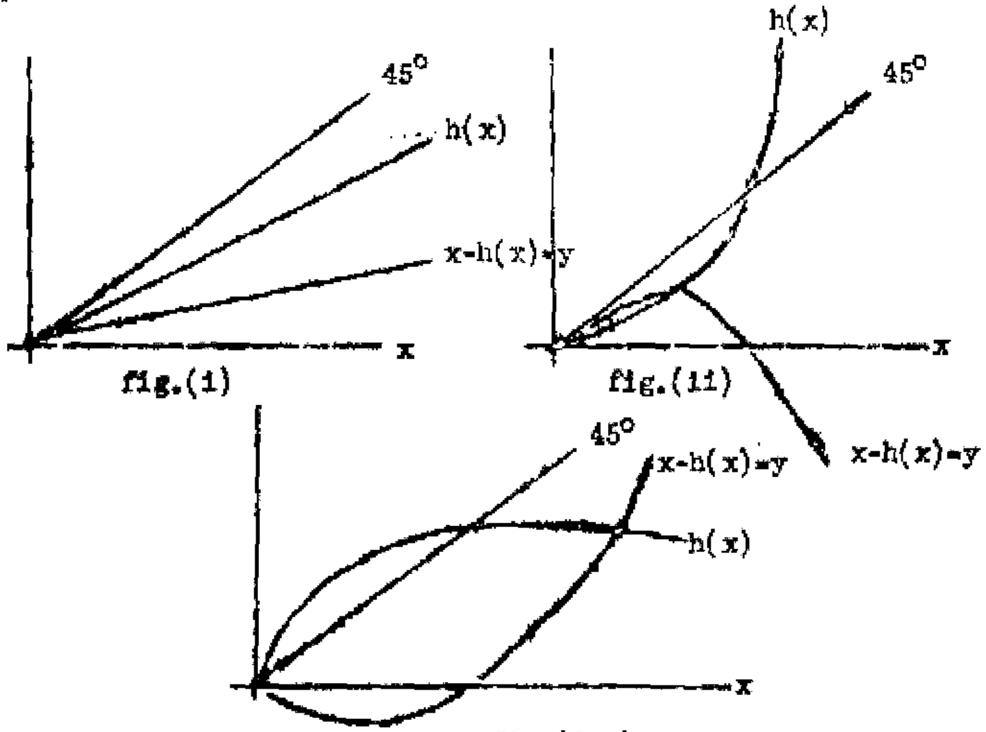
<u>Proof</u> : Let $\{x^t\}$ and $\{\bar{x}^t\}$ be the sequences generated by (3.1.1) and (3.1.2) corresponding to the exogenous vectors y and $\bar{y}$ respectively.

From lemma 3.2.1 it follows that both the sequences are non-decreasing in all components, also since $y \leqq \bar{y}$ using A 3.1 it can be easily verified that

$$x^t \leqq \bar{x}^t \qquad \text{for all } t = 0, 1, 2, \ldots \qquad \ldots (3.2.11).$$

Finally, in view of Theorem 3.2.1 and the hypothesis of this theorem it is known that the sequence $\{\bar{x}^t\}$ is bounded above. This together with (3.2.11) establishes the statement of the theorem. Q.E.D.

On a slight digression, I shall now discuss an intuitive asymmetry between the expected consequences of IRS and DRS. Sufficient DRS would imply that an arbitrarily large final demand cannot be satisfied, for the scale of production cannot be expanded that far in a self sustained manner. IRS on the other hand may require that production be carried out at certain minimum scales but this does not imply any restriction so far as the levels of final demands are concerned. The following diagrams for a one-commodity case will drive the point home :



In each of these figures, x stands for the level of production, h(x) for the input requirement function and y for the final demand. Figures (i), (ii) and (iii) correspond to the cases of CRS (linear case), DRS and IRS respectively. The assymetries are clear. In the first case, the productivity condition, $h(x)/x < 1$, is scale-independent, given which one simply gets $y = x-h(x) > 0$ for all $x > 0$. That is, y simply increases proportionately with x, there being no scale constraint on either. In neither of the two other cases, is the productivity scale-independent. With DRS (fig.(ii)), the condition is satisfied only for a certain initial range of x, with y actually rising to a maximum value for some x within this range. (It may be noted that one really requires f ent decreasingness in RS, for h(x), strictly, need not intersect the 45° diagonal.). In the case of IRS (fig.(iii)), on the other hand, the condition is satisfied for a terminal range of x. This means that as y approches zero from above, x tends to a certain positive value. There is no viable production below this limit (excepting trivially). However, once that is crossed, x and y increase monotonically in a one-to-one relation[7/].

Returning to the 'analysis' proper, it remains to suggest a condition under which local productivity implies global productivity. the above discussion it would appear that general IRS will yield equivalence. This indeed is true. In terms of our earlier defini- n 3.2 (p. 64) I have :

Condition 3.3 : There is a general economies of scale in the otion system as a whole.

Theorem 3.2.5[8/] : Under condition 3.3 if there exists a solution (3.1.0) for a $y \gg 0$, there will also exist a solution of (3.1.0) for arbitrary $y \geq 0$.

Proof : Suppose that there exists a solution of (3.1.0) for $y \gg 0$ and let x be the solution vector.

Now, consider another vector $\bar{y} \geq x$. Let $\{\bar{x}^t\}$ be the sequence erated by (3.1.1) and (3.1.2) for $y = \bar{y}$. From Lemma 3.2.1, it ws that this sequence is non-decreasing.

The above diagrammatic exposition was suggested by a discussion in Dorfman, Samuelson and Solow (1958) [pp. 273-75] on a somewhat related topic.

This theorem, theorem 3.2.4 and lemma 3.2.2 do not require A 3.2(ii).

Clearly,

$$\bar{x}^t \geqq x \qquad \forall\ t = 0, 1, 2, \ldots \qquad \ldots (3.2.12)$$

and

$$\phi(F(x)) < 1 \qquad \text{(from Theorem 3.2.2)} \qquad \ldots (3.2.13)$$

Now, from (3.1.1), I get,

$$\bar{x}^t = F(\bar{x}^{t-1})\bar{x}^{t-1} + \bar{y}$$

$$\leqq F(\bar{x}^t)\bar{x}^t + \bar{y} \qquad \text{[From Lemma 3.2.1]}$$

$$\leqq F(x)\bar{x}^t + \bar{y} \qquad \text{[On account of (3.2.12) and condition 3.3]}$$

or,

$$\bar{x}^t \leqq \left[I - F(x)\right]^{-1}\bar{y} \qquad \text{[Because of (3.2.13)]}$$

Hence it follows that the sequence $\left\{\bar{x}^t\right\}$ converges to a solution of (3.1.0) with $y = \bar{y}$. Now, letting $\bar{y}$ arbitrarily large and using Theorem 3.2.4 the conclusion follows. Q.E.D.

I shall conclude this section and hence the chapter by proving that the condition 3.3 also guarantees the uniqueness of a solution of (3.1.0).

<u>Theorem 3.2.6</u> : Under conditions 3.2 and 3.3 there exists a unique solution of (3.1.0) for any $y \geqq 0$.

<u>Proof</u> : Existence is proved in Theorem 3.2.3.

For uniqueness let there be m solution $x^1, \ldots, x^m$ of (3.1.0).

So, using (3.1.0) one gets,

$$x^i = F(x^i)x^i + y \qquad \ldots (3.2.14)$$

$$i = 1, 2, \ldots, m$$

Now making use of corollary 3.2.1 one can say that one of the x^i's will be less than or equal to all other x^j's in all components. Without loss of generality let x^1 be such that,

$$x^1 \leq x^i \qquad \text{for } i = 2, \ldots, m \qquad \ldots (3.2.15)$$

with the help of (3.2.14) one can write for any $i = 2, \ldots, m$.

$$x^i - x^1 = F(x^i)x^i - F(x^1)x^1$$

$$\leq F(x^1)x^i - F(x^1)x^1 \qquad \text{[On account of (3.2.15) and condition 3.3]}$$

$$= F(x^1)\left[x^i - x^1\right] \qquad \ldots (3.2.16)$$

$$\leq \bar{F}\left[x^i - x^1\right] \qquad \text{[By construction of } \bar{F} \text{ and using (3.2.15)]}$$

$$\therefore \ (I - \bar{F})\left[x^i - x^1\right] \leq 0$$

$$\text{or,} \quad x^i - x^1 \leq 0 \qquad \text{[Because of condition 3.2]} \qquad \ldots (3.2.17)$$

$$\text{or,} \quad x^i \leq x^1 \quad \text{for } i = 2, \ldots, m \qquad \ldots (3.2.18)$$

Comparing (3.2.15) and (3.2.18) one has,

$$x^i = x^1 \qquad \text{for } i = 2, \ldots, m \qquad \text{Q.E.D.}$$

It may be noted that in the proof above both the conditions are to prove the uniqueness. Condition 3.2 will however not be necessary [for] purpose if $y \gg 0$. This is because in this case one has $(F(x^i)) < 1$ for all $i = 1, 2, \ldots, m$ and hence from (3.2.16), one ghtaway gets (3.2.17).

CHAPTER 4

Prices and Costs in the Non-Linear IO Model

This chapter is concerned with prices and costs of production in the non-linear IO model introduced in the last chapter. The purpose, broadly, is two-fold. One is to use these notions for further analysis of the same type of problems as investigated in the last chapter. The other is to investigate directly into the structure and properties of prices. For the first purpose, the prices are taken to be given as mentioned in section 3 of chapter 1 (p. 11). On that basis unit costs of production are defined which in turn yield sharper concepts of RS, i.e., increasing/decreasing unit costs of production in place of input-specific RS and these are precisely the concepts used for the analysis referred. This, it would appear, does not require any 'theory' of prices (which is a precondition for the second purpose). This, however, is not quite true, for the prices that are to be taken as given cannot really be any arbitrary prices. The 'production of commodities by means of commodities' entails <u>consistency conditions on product prices</u> as much as on the production programme. Here also one meets the characteristic 'open scope of consistency', although the channels of interdependence would differ. This way, one is led to the problem of <u>price formation</u> in terms of the channels of interdependence considered, and that is the domain of 'price theory' proper.

In view of the above discussion, it appears best to start out with a complete statement of the consistency conditions on prices leading up to definite rules of price formation. This is given in section 1. This provides the background for sections 2 and 3 which follow up the two directions specified above, in that order. That is, section 2 takes up the further analysis of problems of the previous chapter using prices and costs as tools of analysis. Section 3 takes off independently from section 1 to conduct a somewhat basic 'comparative static' exercise in prices in terms of one of the 'price theories' defined in section 1.

Before closing this introduction I may mention that much of this chapter forms a digression from the rest of the study. As mentioned in section 1 of chapter 3, the digression is mainly a reflection of the rather general character of the non-linear model which makes it possible to go into several directions with the model only as a frame of reference. The rules of price formation, and the properties of the prices based on those rules, constitute precisely one such important direction. In terms of the literature on theoretical IO analysis cited in section 1 of chapter 1, one may particularly mention Schwartz and Morishima for is direction of enquiry. And one may add the seminal work of Sraffa 1960) which goes beyond the conceptual confines of IO analysis though the formal structure is similar.

.1 Consistency conditions and rules of price formation

Let p_i be the price per unit of G_i. The basic consistency condi- ion on the set of prices $\{p_i\}$ is that each p_i covers the respective it cost of production. These unit costs in turn depend upon the ces via the IO coefficients, and since the latter depend upon scales production, so do the unit costs. It follows that technically the sistency conditions can be defined only on the basis of *given scales* production. Further, the latter must make up a *consistent* production programme. This is the approach followed here. That is, a set of sistent prices is defined *parametrically* with respect to a consistent production programme.

Now, the costs of production definable on the basis of the IO model only the *material costs of production*. Let $m_j(p; x)$ be the unit erial cost of production in S_j. It is defined simply as :

$$m_j(p;x) = \sum_{i=1}^{n} p_i f_{ij}(x) \qquad \ldots \quad (4.1.0)$$

With these, one gets a set of *necessary* conditions for consistency, ie.,

$$p_j(x) \geq m_j(p;x) \quad \forall j$$

$$\text{or,} \quad p(x) \geq F(x)' p(x) \qquad \ldots \quad (4.1.1)$$

The only other category of costs that I shall directly deal with the *labour-costs*. This requires the introduction of labour as a input. Let $f_j(x)$ be the amount of labour services required per t output of G_j, given the production programme, x, and let w be a *wage rate*. Then the *unit labour cost of production* of S_j, $l_j(x)$, ven by :

$$l_j(x) = wf_j(x) \quad \forall j$$

$$\text{or,} \quad l(x) = wf(x) \qquad (4.1.2)$$

It will be assumed that $l(x)$ is strictly positive for all $x \in \hat{X}$, i.e.

A 4.1 $\quad l(\hat{x}) \gg 0 \quad \forall x \in \hat{X}$

The sum of labour and material cost is often called the prime cost of production[1].

With this, the set of consistent price vectors for a given production programme $x \in \hat{X}$ and a given wage rate, w can be defined. Let $P(x, w)$ be this set. Then :

$$P(x, w) = \left\{ p \in R^n_+ \,/\, p \geq F(x)' p + wf(x) \right\} \qquad \ldots (4.1.5)$$

Formally, it is to be noted that whereas consistency of a production programme was defined in a completely internal fashion, this is not so with the consistency of prices, for the wage rate enters into the conditions on an independent basis. That is, one gets a set of internal consistency conditions on prices only by neglecting labour costs. The incorporation of labour costs from that starting point has so far extended the underlying IO model in the sense of introducing exogenous elements, viz., $wf(x)$. However, just as in the case of the physical system, further account of interdependence may in fact turn the elements into endogenous ones, thereby extending the scope of consistency, as mentioned at the outset. The interdependence here is obviously between w and p. The dependence of p on w is already accounted for. We now turn to the reverse dependence, i.e., of w on p. The basic fact behind this dependence is simply that the wage rate for labour is really a means of consumption,

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1/ In case there is no confusion the terms "unit cost" and "unit prime cost" will be taken as synonyms.

es' consumption is defined in terms of the so-called wage goods ded in the set $\{G_i\}$. For any given wage-price situation (w, p), eal wage index, v, can then be defined as :

$$v = w \Big/ \sum_{i=1}^{n} p_i \; b_i \qquad (4.1.4)$$

the vector $b = (b_i)$ represents the composition of consumption erms of $\{G_i\}$. That is, b_i units of G_i is consumed with b_j units $_j$, ∀ i and j. The money wage rate w then enables the labourer to hase a real-wage basket $\{v, b_i\}$.

We may now clarify the basis of the above 'internalisation of ur costs' in terms of the mode of functioning of the economic system. ll consider separately the cases of planning (socialism) and market italism) as the system under reference. In the first case, a sector roduction may be directly required to provide for the material mea..s onsumption for the labour working there. The economic system then s without the contrivance of a money wage rate : a sector 'purchases' other sectors both the material inputs as well as the wage goods labour. The latter depends upon both the technological labour fficient as well as the level of living appropriate for labour, where ap ropriateness is really a matter of social decision.

In the second case, the system operates directly through w, and v. However, one may conceive of an independent relation between nd p, viz., that of cost-of-living-indexed wage rates. Interpreting as a vector of weights assigned to different commodities for defining a cost of living index (CLI), one can then specify a mechanism of institutionalised adjustment of the money wage rate to changes in CLI on account of price-changes. The "real wage index" in this case is provided by a historical benchmark of wage-price configuration, (w^o, p^o) say, and the mechanism ensures

$$w/w^o = \sum_{i=1}^{n} p_i b_i \Big/ \sum_{i=1}^{n} p_i^o b_i$$

which is formally equivalent to (4.1.4) with $v = w^o \Big/ \sum_{i=1}^{n} p_i^o b_i$. For the rest of the exposition I shall assume this type of wage adjustment[2].

In either case, the channel of interdependence considered has its own justification. The definition of $P(x, w)$ taking (4.1.4) into account is changed to :

$$P(x,v) = \left\{ p \in R_+^n / p \geq F(x)' p + v f(x) b' p \right\}$$

$$\text{or } P(x,v) = \left\{ p \in R_+^n / p \geq G(x,v) p \right\} \qquad \ldots (4.1.5)$$

where the (i, j)element of $G(x, v)$ — $g_{ij}(x,v)$ — is given by :

$$g_{ij}(x,v) = f_{ji}(x) + v f_i(x) b_j \qquad \ldots (4.1.6)$$

$P(x,v)$ now represents a set of internal consistency conditions on prices, entailed by the definition of consistency, technical conditions on production (IO coefficients and labour coefficients) and the conditions of employment (level of living/CLI). As earlier, one needs conditions to ensure the non-emptiness of the interior of $P(x,v)$. We shall refer to the required condition as the <u>viability condition</u> (for

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2/ Alternatively, one could stick to the assumption of a given money wage rate. We are then following the case of a <u>flexible wage system</u> in contrast to the <u>fixed wage system</u> of this alternative approach.

rices). Since $x \in \hat{X}$, the productivity condition is assumed to be satisfied. It then follows from the definition of $G(x,v)$ that there is some range of values of v say $(o, \hat{v})$ such that $\lambda(G(x,v)) < 1$ for $0 < v < \hat{v}$. For the purpose of our analysis I shall assume that the given real wage index, v, lies in the above interval. That is I assume :

A 4.2. $\lambda(G(x,v)) < 1 \qquad \forall x \in \hat{X}$

The next step takes one to 'price-theory' proper. As in the case of the (physical) IO system, two polar approaches are possible : a consistent price vector may be given, yielding value-surpluses in each sector residually (these surpluses are called profit margins). Alternatively, the margins may be given, and prices determined on that basis. The latter approach — which was identified as the IO approach earlier in the context of the physical system — can have a further refinement, viz., the (unit) profit margin in one sector stands in the same factor of proportionality to the unit cost of production as that in any other sector (this factor is called the mark-up factor, or rate of profit). In all, then, we meet three theories, which can be called the fixed price theory, the profit margin theory and the profit rate theory, respectively[3/]. There is however a rather close connection between the latter two. Analytically, the first is a closed matter so far as price determination is concerned. The second leads to the equation :

$$p = G(x,v)p + \Pi(x,v)^{4/}$$

3/ With the fixed wage system, it is possible to fuse the last two into a different theory where prices are determined conjointly by a given money wage rate and a given rate of profit. This is the version one typically meets in the IO literature referred.

4/ For notational simplicity I am denoting p(x) simply by p. Also henceforth I shall drop v from the arguments of G.

where $G(x,v)$ is defined in (4.1.6) and $\Pi(x,v) = (\Pi_i(x,v))$ is the given vector of profit margins. In tune with the general formulation, the profit margins may be taken as a function of output levels and the real wage rate[5/]. The price-solution is given by :

$$p = \left[I - G(x) \right]^{-1} \Pi(x,v).$$

Clearly, p is non-negative by A 4.2.

Before proceeding to the profit rate theory, one may note that instead of the unit profit margin, the ratio of it to the unit cost of production may be taken as given for each sector. This ratio is often called the 'degree of monopoly' of the sector concerned, and is taken to be determined by its market organisation, as a basic institutional parameter of the whole system. This approach is associated with the name of Kalecki (1971) [Chapters 5 & 6]. The profit rate theory can then be seen as a special case of this where the degree of monopoly is uniform over all sectors. Alternatively, it can be seen as based directly on a competitive mechanism which equalises the rate of return on 'capital' invested in different lines. For this, one must ignore the overhead cost elements in the profit margin, so that the entire costs of production are reflected in the prime costs. Further, the so-called 'period of production' in each sector has to be assumed to be the same, so that the return on capital is defined over the same time period

5/ $\Pi(x)$ may include elements of so-called unit overhead costs which are obviously dependent upon the levels of production.

n each sector[6]. It is basically this conceptualisation of the system that underlies the designation profit rate to the ratio of profit margin to prime costs.

Analytically, it is to be noted that the rate of profit, say ρ, cannot be given independently in this case. It has to be determined simultaneously with the prices. I now write the price system as :

$$p = (1 + \rho)\ G(x)\ p \qquad \ldots (4.1.7)$$

From A 4.2 it follows that $\phi(G(x,v))$ and the corresponding characteristic vector (which is known to be non-negative) is a meaningful solution for $\frac{1}{1+\rho}$ and p of (4.1.7) respectively. This is because $\frac{1}{1+\rho} < 1$ implies that ρ is positive. It may be noted that p here is determined uniquely upto positive scalar multiplication, i.e., only relative prices are determined uniquely. This completes our discussion of the basic formulations with respect to prices.

4.2 A reexamination of some physical properties of the model

For the discussion to be taken up in this section, I shall take f directly from the concept of 'consistent price vector' as defined the preceeding section, without going into the 'theory of prices' such. The purpose of this section is then to reconsider some of the cepts and issues discussed in the last chapter. Basically, it is the al-global productivity problem of the previous chapter that we shall

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In terms of our discussion on investment in section 4 of chapter 2, the assumption boils down to independence of θ_{ij} with respect to i and j and absence of fixed capital from the picture.

er pursue here. We have already seen that general IRS (see Definition 3.2, p. 64) is a sufficient condition for the equivalence of these two notions of productivity. Our purpose specifically is to replace the physical concept of a general IRS by a value concept and see whether the result holds. The basis of the value concepts has been indicated at the test of this chapter, and it remains to follow it up with a formal definition. The relevant definitions for our analysis are :

<u>Definition 4.1</u> : There is increasing (decreasing) unit material cost of production in S_j with respect to a price vector p^0 if

$$m_j(p^0; x^1) \geq (\leq) \; m_j(p^0; x^2) \qquad \forall\, x^1 \geq x^2 \geq 0.$$

<u>Definition 4.2</u> : There is a general increasing (decreasing) unit material cost of production in the production system as a whole with respect to a price vector p^0 if

$$m_j(p^0; x^1) \geq (\leq) \; m_j(p^0; x^2) \qquad \forall\, x^1 \geq x^2 \geq 0$$

and $\forall\, j = 1, 2, \ldots, n.$

i.e., $F(x^1)' p^0 \geq (\leq) F(x^2)' p^0 \qquad \forall\, x^1 \geq x^2 \geq 0$

The concepts defined above are generally referred as <u>value</u> <u>concepts</u>. It is clear that these enormously simplify the concept of RS seen earlier. This is because the present definitions are in terms of <u>single</u> variable rather than n variables as in definition 3.1.

As mentioned earlier, the analysis of this section is based on assumption of a <u>given consistent price vector</u> in terms of which the value concepts above are defined. Now, since the set of consistent prices, $P(x)$, itself is dependent on the production programme there is no a priori guarantee that one can find a price vector which is consistent for all $x \in \mathcal{X}$. Since to discuss RS is to vary x, I require to define the value concepts with respect to some fixed price vector which is consistent for all $x \in \mathcal{X}$. Formally, I assume :

<u>A 4.3</u> : There exists a $\bar{p}$ such that $\bar{p} \geq G(x)\bar{p} \quad \forall x \in \mathcal{X}$. (p.86),

In view of A 4.1 [it immediately follows that A 4.3 implies the existence of a $\bar{p}$ satisfying :

$$\bar{p} >> F(x)' \bar{p} \qquad \forall x \in \mathcal{X} \qquad \ldots (4.2.0)$$

Parallel to Condition 3.3 I now have the much weaker condition:

<u>Condition 4.1</u> : There is a general decreasing unit material cost of production in the production system as a whole with respect to a price vector $\bar{p}$ satisfying (4.2.0).

Also, parallel to Theorem 3.2.5 I have the following theorem.

<u>Theorem 4.2.1</u> : Under condition 4.1 if there exists a solution *of (3.1.0) for some $y >> 0$, there will also exist a solution of (3.1.0) for any arbitrary $y \geq 0$.*

<u>Proof</u> : The proof is similar to the one of theorem 3.2.5. The only difference lies in the proof of boundedness of the sequence $\{\bar{x}^t\}$ defined there. This is proved here in the following way :

From (3.1.1) with $y = \bar{y}$, one gets,

$$\bar{x}^t = F(\bar{x}^{t-1})\bar{x}^{t-1} + \bar{y}$$

or, $$\bar{x}^t \leq F(\bar{x}^t)\bar{x}^t + \bar{y} \qquad \text{[From lemma 3.2.1]} \qquad \ldots (4.2.1)$$

Now pre-multiplying both sides of (4.2.1) by $\bar{p}'$ one has,

$$\bar{p}'\bar{x}^t \leq \bar{p}' F(\bar{x}^t)\bar{x}^t + \bar{y}$$

$$\leq \bar{p}' F(x)\bar{x}^t + \bar{y} \quad \text{[On account of (3.2.12) and condition 4.1]}$$

$$\text{or, } (\bar{p}' - \bar{p}' F(x))\,\bar{x}^t \leq \bar{y} \qquad \ldots \quad (4.2.2)$$

Now, since $x \in \mathcal{X}$, using (4.2.0) it follows from (4.2.2) that the sequence $\left\{\bar{x}^t\right\}$ is bounded above. Q.E.D.

It is to be noted that condition 3.3 also yielded the uniqueness of a solution of (3.1.0). This is also true about condition 4.1. The following theorem validates this assertion.

<u>Theorem 4.2.2</u> : Under condition 4.1, there can exist at most one solution of the system (3.1.0) for a given y.

<u>Proof</u> : The proof is similar to the one of theorem 3.2.6 upto the derivation of (3.2.15). Now with the help of (3.2.14) one can write for any i = 2, ..., n :

$$x^i - x^1 = F(x^i)x^i - F(x^1)x^1$$

premultiplying both sides by $\bar{p}'$ I get,

$$\bar{p}'(x^i - x^1) = \bar{p}' F(x^i)x^i - \bar{p}' F(x^1)x^1$$

$$\leq \bar{p}' F(x^1)x^i - \bar{p}' F(x^1)x^1 \quad \text{[Using (3.2.15) and condition 4.1]}$$

$$(\bar{p}' - \bar{p}' F(x^1))\,(x^i - x^1) \leq 0 \qquad \ldots \quad (4.2.3)$$

Now suppose $x^i > x^1$. This together with (4.2.0) would mean that hand side of (4.2.3) is strictly positive thus contradicting (4.2.3).

$x^i = x^1 \quad \forall\, i = 1, 2, \ldots, m.$ Q.E.D.

4.3 *A comparative static exercise in costs and prices*

As stated earlier, the purpose of this section is to conduct a somewhat basic comparative static exercise in prices. The problem investigated is the effect of a change in the unit prime cost of production in some sector on all prices generally, including the price of the product of the same sector in particular. The change in the unit prime cost in turn can be taken as being the result of a change in the production programme, with corresponding changes in the IO coefficients.

For the purpose stated above, I shall take the prices to be determined, along with the rate of profit, by (4.1.7). The choice of this theory as the basis of the exercise is basically a reflection of its analytical superiority over the profit margin theory in that it is a completely 'closed' theory without any exogenous elements. The exogenous elements in the profit margin theory are analytically arbitrary and detract from its usability for the kind of 'pure effects' indicated above[7].

Two points are to be noted at the outset of the exercise. First, as mentioned, the endogenous variables in the profit rate theory are given by the vector (ρ , p), not p alone. Hence the problem is really one of the effect of changes in unit prime cost in some sector — I shall take this to be s_1 — on both the rate of profit and prices. Second, p is determined only upto a positive scaler multiplication.

One additional advantage of this theory is that some kind of RS to the economy as a whole is reflected in the behaviour of $\phi(G(x))$ This is because ρ in (4.1.7) is given by $\frac{1}{\phi(G(x))} - 1$ and hence an increase (decrease) in the value of $\phi(G(x))$ would imply a fall(rise) in the value of the rate of profit of the economy as a whole.

Clearly, some normalisation rule is necessary for resolving this indeterminacy. For the purpose of the analysis in this section I shall assume that p_1 is fixed, so that p_i/p_1, $i \neq 1$, is really the relative price of G_i in terms of G_1. In the language of traditional economic theory, G_1 is chosen as the numeraire. Formally, the normalisation rule can be taken to be :

$$p_1 = 1 \qquad \ldots (4.3.0)$$

Returning to the issue at hand, suppose, starting from an equilibrium solution (p, φ) of (4.1.7) and (4.3.0), the unit prime cost is lowered for only S_1. The immediate effect of this would be the emergence of a surplus in S_1. The basic result sought can be stated as a conjecture, viz., that this surplus will be eventually (i.e., in the new equilibrium) shared between (a) profits and (b) customers-in-general through, respectively, (a) a higher rate of profit, uniform over all sectors, and (b) a lower relative price of G_1 — relative to the price of each and every other G_i, i.e., a higher relative price of all G_i.

The intuitive basis of this conjecture is that cost reduction in
$_1$ would create surplus in that sector and if the rate of profit does
adjust itself upward, this would reduce p_1 contradicting the norma-
sation criterion (4.3.0). For the other sectors, a rise in the rate
profit can be secured, in the first place (i.e., on the basis of
anged unit costs), only by a rise in their prices which in turn
nges the unit costs everywhere. Clearly, the mechanism of equalisa-
of profit rates will have to work through a process. Here I abstract
the mechanics of this process[8] and concentrate only on the end
t. As mentioned, the end-result is expected to be qualitatively
ar to the immediate impact mentioned, viz., rises in the rate of
fit and in all relative prices (relative to G_1).

refer to a recent paper by Gajapathi (1975) for an analysis of
e process and its convergence.

I shall now validate the conjecture just made by the analytical results to follow. For this, let p and ρ be the solutions of (4.1.7) and (4.3.0) for some $x \in X$ and then suppose we are led to a new situation (i.e., new production programme, $\bar{x}$ say) where the unit prime cost of S_1 with respect to p has decreased. That is, the matrix $G(x)$ defined in (4.1.6) has changed to $G(\bar{x})$ such that $G^1(\bar{x})p < G^1(x)p$ and $G^i(\bar{x})p = G^i(x)p$ for $i \neq 1$, or, in other words,

$$G(x)p = G(\bar{x})p + \lambda e^1 \qquad \ldots \quad (4.3.1)$$

for some $\lambda > 0$.

Also let $(p + h)$ and $(\alpha + \theta)$ be the new solution of (4.1.7) and (4.3.0) corresponding to the 'new situation' where

$$\alpha = \frac{1}{1 + \rho} \qquad . \quad (4.3.2)$$

(clearly $\alpha > 0$, $\alpha + \theta > 0$)

So from (4.1.7) and (4.3.0) I have,

$$\alpha p = G(x)p \qquad \ldots \quad (4.3.3)$$

$$p_1 = 1 \qquad \ldots \quad (4.3.4)$$

$$(\alpha + \theta)(p+h) = G(\bar{x})(p+h) \qquad \ldots \quad (4.3.5)$$

$$p_1 + h_1 = 1 \qquad \ldots \quad (4.3.6)$$

From (4.3.2) it is clear that $\theta < 0$ implies an increase in the [rate] of profit. This is basically what is to be proved. For this, I shall [make] the following assumption :

<u>A 4.4</u> : $G(\bar{x})$ is irreducible.

Theorem 4.3.1 : The new equilibrium rate of profit is greater the old one.

Proof : First note that (4.3.4) and (4.3.6) gives

$$h_1 = 0 \qquad \ldots (4.3.7)$$

Now subtracting (4.3.3) from (4.3.5) and making use of (4.3.1) get,

$$(\alpha+\theta)h - G(\bar{x})h = -\theta p - \lambda e^1$$

Hence partitioning p, h and $G(\bar{x})$ properly one gets,

$$(\alpha+\theta)\begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} g_{11}(\bar{x}) & \bar{G}_1(\bar{x}) \\ \bar{G}^1(\bar{x}) & \bar{G}(\bar{x}) \end{bmatrix} \begin{bmatrix} h_1 \\ \cdots \\ \bar{h} \end{bmatrix} = -\theta \begin{bmatrix} p_1 \\ \cdots \\ \bar{p} \end{bmatrix} - \begin{bmatrix} \lambda \\ \cdots \\ 0 \end{bmatrix} \qquad \ldots (4.3.8)$$

Now from (4.3.8) using (4.3.7) and (4.3.4) one has :

$$-\bar{G}_1(\bar{x})\bar{h} = -\theta - \lambda \qquad \ldots (4.3.9)$$

and

$$(\alpha+\theta)\bar{h} - \bar{G}(\bar{x})\bar{h} = -\theta\bar{p} \qquad \ldots (4.3.10)$$

Now, from A 4.4 and (4.3.5) it follows first that

$$\rho\left(\frac{1}{\alpha+\theta}G(\bar{x})\right) = 1$$

and then that

$$\rho\left(\frac{1}{\alpha+\theta}\bar{G}(\bar{x})\right) < 1 \qquad \ldots (4.3.11)$$

as the dominant characteristic root of a non-negative irreducible matrix is known to be a strictly increasing function of the elements of the matrix.

Hence from (4.3.10) I get,

$$\bar{h} = -\theta \left[(\alpha+\theta)I - \bar{G}(\bar{x})\right]^{-1} \bar{p} \qquad \ldots (4.3.12)$$

Hence if θ is positive $\bar{h}$ will be negative from (4.3.12) and this will contradict (4.3.9) since $\lambda > 0$. Q.E.D.

I now have a corollary of the above theorem.

Corollary 4.3.1 : The relative price of G_i, $i \neq 1$, in terms G_1 in the new equilibrium is greater than the corresponding relative in the old equilibrium.

Proof : This follows directly from (4.3.12) since $\theta < 0$. Q.E.D.

PART III

STRUCTURAL BREAK

CHAPTER 5

Some Models of Structural Break

The concept of structural break has already been spelt out in some details in section 3 of chapter 1. To recapitulate, we presume a given structure of the economy at some initial or base period. Taking the (extended) IO model with primary inputs as the benchmark of reference, one can specify this 'structure' in terms of :

(a) a set of technological relations expressed by the IO coefficients relating quantities of products as outputs to their respective inputs; and

(b) a given classification of the totality of commodities into products and non-products or primary goods.

A structural break in this context involves a potential change either in the technological relations or in the commodity classifications (or both). We shall deal with the two pure categories defined above for the purpose of analysing problems related to the introduction of new technology (technological break) and import substitution respectively. The first relation is obvious. For the second, we need only mention that the primary goods referred in (b) are taken to represent so-called non-competitive imports, i.e., goods which are not produced domestically and hence imported. The import substitution — i.e., domestic production of the commodity concerned — then transfers the commodity from the 'non-produced' to the 'produced' group, i.e., changes the commodity classification[1].

This chapter is concerned basically with the "first stage of analysis" involved in any generalisation that we had talked about in chapter 1 (p.8). That is, the task is to formulate the issues in terms of corresponding IO — or IO-type—models and carry out some preliminary analysis. The next chapter takes up the second stage of analysis, viz., that of detailed model-analysis. More precisely the line of division is provided by the method of analysis. As elsewhere we shall rely on the IO-method(in some suitable form) as the main tool of analysis of our models. This approach is taken in the next chapter, so that in this chapter we shall carry out only such analysis as does not require the tool mentioned.

We may now repeat two observations made earlier regarding the analysis of this part of the study. First, the models developed are to be seen as a sample of exercises organised around the basic theme of structural break rather than as a comprehensive treatment of the basic theme. The models — four in all, designated Models I-IV — are arranged in two groups, each consisting of two models, dealing respectively with the two types of structural breaks referred. These are given in sections 1 and 2 below, in that order.

1/ Clearly, in broad terms, this too introduces new technology, i.e., the commodity classification itself is a reflection of technology in the broad sense. Hence our distinction between technological break and import substitution is to be taken as one of convenience rather than of definite separation.

Second, the 'time-frame of reference' for our analysis will be as spelt out in chapter 1 (p.13). In operational terms, all our models are taken to be defined for a 'reference period in the future' called the 'terminal period'. The exact time span separating the base and the terminal periods (the so-called 'time horizon of analysis'), however, does not explicitly occur in the analysis, for the problems investigated are not 'how to get from here to there' but 'whether to t there at all'[2/]. It is enough, for this purpose, to have an implicit me-horizon which makes the 'open possibilities at the terminal date' valid description. The analysis can then be conducted explicitly only reference to the terminal period. We now turn to the substantive scussions.

We shall end this introduction with a brief reference to two rtant sources of our ideas in the literature. First, on the analy- side, is the contribution of Arrow (1954) discussed in detail in on 5 of chapter 2 (pp. 48 - 53). Its broad relevance comes from explicit structure of sectors — the 'free' and 'bottleneck' rs — as an unknown of the system. As shown in the section red, the IO method there solved the output determination problem taneously with the structure. This is of key importance for us, our models and methods in this part can be seen as basically working of those ideas in different kinds of contexts. More substantively, f our models have a close connection with the above model and

chapters 7-9 we make a somewhat detailed investigation into the er type of problems.

these will be spelt out in appropriate places below.

The second source is Chakravarty (1959), on the conceptual side. The term "structural break" and the so-called "time-frame of reference" for its analysis are in fact borrowed from Chakravarty, though the specific nature of problems discussed are quite different. Again, we shall have occasion to make more specific references to his work later.

5.1 Technological break

This section is concerned with structural break of the type (a) referred at the outset of this chapter, i.e., technological break. This is divided into two subsections dealing respectively with the cases of technology-specific capacities of production and indivisibilities in the scale of production as the main factors behind such breaks.

5.1.1 Model I : Technology-specific capacity of production

The starting point of our Model I is the relation between technology and capacity that we had indicated at several places earlier, in particular chapter 1 (pp. 12-15). We shall formulate the idea in terms of two alternative technologies, represented by two IO matrices respectively, each operable on the basis of a capacity specific to itself. e of the technologies — the old one — will have an explicit capacity straint on its scale of operation while the capacity for the new hnology is assumed free. The justification is in terms of our time- of reference explained earlier.

In this frame, we are really looking at the terminal period the standpoint of a base period. So, the terminal period will inherit from the base a given capacity of production specific to the base technology (this is the old technology) which can still be operated to that extent in the terminal period. The new technology is brought into being, along with its matching capacity, by investments between the base and terminal periods. The scale of its operation is to be determined simultaneously with that of the old one in the light of the given final demands of the terminal period[3]. This determines investment and hence the size of new capacity as essentially 'free' variables[4], without generating any capacity constraint. This explains why there is only one capacity constraint in the model.[5]

It remains to specify the rule of operation or choice criterion regarding which technology is to be operated and to what extent. It will be assumed that the old technology is operated so long as it can be. That is, the new technology is not operated so long as the capacity constraints are ineffective. The argument is that this capacity, being pre-existent, is free in the sense that there is no cost involved in its

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3/ One may think of this final demand as being projected independently from the base period final demands and other relevant considerations.

4/ This rules out any indivisibilities in the scale of operation of the new technology. Our next model (Model II) is designed precisely to go into the question of indivisibilities. It may further be noted that the term "free variable" is used in the sense of being unconstrained, not in the sense of "costlessness". In fact, the cost of this investment is taken into account below, but this cost has nothing to do with the commodity balances in the terminal period.

5/ For this reason, we use the general term 'capacity constraint' in this section to mean 'capacity constraint specific to the old technology'.

erection. The new technology obviously entails such costs, viz., the investment costs.[6/] These costs are taken to be sufficient to make the utilisation of any existing capacity worthwhile. We shall take up the further analysis of the choice criterion from the cost-angle at the end of this subsection after a formal presentation of the ideas above and a number of clarifications.

In symbols, let A and B be the IO matrices associated with the old and new technologies (also to be called A-technology and B-technology) and $\bar{x}$ be the vector of given capacities of production associated with the former. Model I is then formally represented by :

$$x + z = Ax + Bz + y \qquad \ldots (5.1.0)$$

$$x \leq \bar{x} \qquad \ldots (5.1.1)$$

$$(x-\bar{x})' z = 0 \qquad \ldots (5.1.2)$$

where $x(\in R^n_+)$ and $z(\in R^n_+)$ are the vectors of production with the A-and B-technology respectively and y is the given bill of final demands. The equations (5.1.0), (5.1.1) and (5.1.2) represent respectively the commodity balance-equations, the capacity constraints associated with A-technology, and the choice criteria described above.

We now turn to the clarifications referred. Basically, the purpose is to clarify the relation between the model presented and certain other models that we have explicitly dealt with or referred to. There are three such models with a close relation to the above, viz.,

6/ The operational costs (so-called 'prime costs' of the last chapter) are not explicitly considered here. Implicitly, these are taken to be smaller for the new technology compared to the old. The choice criteria used in the next subsection are based on these costs.

the general AA model of production, our non-linear model of chapters 3 and 4 and the model of foreign trade discussed in section 5 of chapter 2 (pp.48-53). We shall refer to the last as the Arrow-Chenery model (ACM). Clarifications of these relations between models in turn make room for related comments on the structure of our model. These are also given below in the appropriate places.

In terms of the AA frame, our model has n commodities and 2n activities — two activities for each of the n sectors. This is clear because (5.1.0) can be equivalently written as :

$$x + z = \sum_{j=1}^{n} a^j x_j + \sum_{j=1}^{n} b^j z_j + y$$

where a^j and b^j are interpreted as the two alternative activities or production processes available to S_j, and x_j and z_j as the levels of operation of the two activities respectively. The rule (5.1.2) then states that for S_j the second process (i.e., b^j) is not operated till the first one reaches its specified upper bound. The A-and B-technology matrices here are simply formed by taking respectively a^j and b^j as their j-th column. Two points may be noted here. First, it is not necessary that $a^j \neq b^j$ for all j, i.e., there may be sectors where investment and capacity expansion does not bring about any new method of production. Second, formally Model I is generalisable to arbitrary number of technologies. This would leave our formal analysis unaffected. For this case, let there be m alternative technologies arranged hierarchically : $A_1, \ldots, A_m$, say. That is, the A_i-technology is operated only if the A_k-technologies are fully operated, $k < i$. The system will then read :

$$\sum_{i=1}^{m} x^i = \sum_{i=1}^{m} A_i x^i + y \qquad \dots (5.1.3)$$

$$x^i \leq \bar{x}^i, \quad i = 1, 2, \dots, m-1 \qquad \dots (5.1.4)$$

$$(x^i - \bar{x}^i)' x^{i+1} = 0 \quad i = 1, 2, \dots, m-1 \qquad \dots (5.1.5)$$

where x^i and $\bar{x}^i$ ($i = 1, \dots, m-1$) are the vectors of production level and capacity figures corresponding to the i-th technology and x^m is the "unconstrained" or "free" vector of production level corresponding to the m-th technology.

We now turn to the clarification regarding the relation between Model I above and the non-linear model of chapter 3. Model I in this context comes out as a special case of the non-linear model. In other words the presentation of Model I above can be thought of as an explicit and substantive depiction of a situation which gives rise to a non-linear system. This is brought out by the following equivalent formulation of Model I :

$$x^1 = F(x^1)x^1 + y \qquad \dots (5.1.6)$$

where $f_{ij}(x^1)$ — the (i,j)th element of the IO matrix $F(x^1)$ — is defined as :

$$f_{ij}(x^1) = \begin{cases} a_{ij} & \text{if } x_j^1 \leq \bar{x}_j \\ \dfrac{a_{ij}\bar{x}_j + b_{ij}(x_j^1 - \bar{x}_j)}{x_j^1} & \text{otherwise} \end{cases} \qquad \dots (5.1.7)$$

Clearly, the function $F(x)x$ as defined here satisfies the assumption A 3.1 and A 3.2(1) (p. 68). It may be noted that x^1 is really the sum of x and z, and given the first the latter two can be read off from :

$$(x_1, z_1) = \begin{cases} (x_1^1, 0) & \text{if } x_1^1 \leqq \bar{x}_1 \\ (\bar{x}_1, x_1^1 - \bar{x}_1) & \text{otherwise} \end{cases} \quad \ldots \ (5.1.8)$$

In this context, two further points are worth reporting. First, it may be noted that the non-linear model itself is a model of structural break in an implicit sense. This follows from the definition of structural break of type (a) given at the outset of this chapter since variabilities of the coefficients with respect to the scales of production can be thought of as changes in the technological relations in a broad sense. However, as illustrated in (5.1.0)-(5.1.2), the structural break proper is indicated formally by both the internal structure in the sense of coefficients and the structure of constraints. The non-linear model, on the other hand, is unconstrained.

The second point is to note a possible alternative interpretation of Model I which is polar to the one given above. This is best seen as a digression in the sense that here one is taken out of the conceptual frame of reference adopted in this chapter, viz., the problem is viewed in the light of scarcity of existing resources. The model can then be seen precisely as an upshot of the "forces behind DRS" as described in section 3 of chapter 1 (p.12) ; the A-technology is considered the superior one which however cannot be operated at any scale

PART V

A REVIEW OF THE STUDY

and as this constraint becomes binding one moves to the inferior technology, B, to meet the additional requirement.

Finally we turn to ACM. Now, Arrow in fact obtained the structure represented by (2.5.1), (2.5.3) and (2.5.4) as the 'reduced form' of a general optimising model of the AA type. Both the formal relation between Model I and the reduced form (i.e., the model as given by (2.5.1), (2.5.3) and (2.5.4)) and Arrow's derivation of the latter from the optimality criterion of his general model are worth reviewing here. The two problems are taken up below in that order.

It should be immediate that the formal structure of Model I is a generalisation of the formal structure of ACM, because putting the matrix B identically equal to zero Model I reduces to ACM.

Turning to the discussion of optimisation, I shall concern myself with a special case of Arrow's original formulation. Arrow treated imports as variables subject to upper bound constraints representing demand conditions, and distinguished between import and export prices. The special case is obtained by putting each export to its upper bound and equating export and import prices (we shall call these the international prices). Now, seen purely from the standpoint of choice there are two alternative sources of supply for each G_1 : import and (domestic) production. The former entails a cost p_1 per unit of G_1 (in any use) where p_1 is the international price of G_1. Production on the other hand has a "cost" only in terms of commodities, viz., its input vector for unit production, i.e., a^1. If all these inputs are valued at international prices then this entails a unit cost, $p'a^i$, which is dimensionally comparable to p_i. On a direct, intuitive reckoning, production is cheaper than import if and only if $p'a^i < p_i$. In economic terms, the condition requires that there is a positive value added (or surplus) in each sector at international prices (we shall call this the Arrow-condition). Arrow in fact shows that if this condition is satisfied for all i then the solution of ACM minimises the total import cost among all feasible solutions of the model defined by (2.5.1) and (2.5.3). That is, the optimality criterion or objective function used is the minimisation of import cost. Seen from this general standpoint, one can say that the condition referred to is an underlying assumption behind the formulation (2.5.1), (2.5.3) and (2.5.4). We may now briefly present the logic of this condition, for this essentially paves the path of our further analysis of the choice criterion promised earlier.

Arrow's argument is as follows :

If an optimal solution did not satisfy (2.5.4) — i.e., if an optimal solution (x, z) were such that for some i, $x_i < \bar{x}_i$ and $z_i > 0$ — one could increase the production and decrease the import of G_i by a small amount meeting the intermediate input requirements by imports and thus reduce the cost of imports. This would then clearly contradict the fact that the solution (x, z) above was an optimal one.

We now turn to a more detailed analysis of the choice criterion ed. Intuitively, given that P-technology is the superior one, one d expect the A-technology to be fully utilised only if the .

investment costs for B-technology are sufficiently high[7]. The term 'sufficiently high' here has of course to be judged in terms of the relative superiority of the B-technology over the A-technology. Hence the entire structure comes into the analysis. Below we specify a condition in respect to this total structure -- structure of investment cost vis-a-vis the structure of old and new technologies* -- which justifies our formulation of the choice criterion in the same way that the Arrow-condition justifies ACM. We shall end this section with a clarification of the intuitive basis of the condition and a formal demonstration of the justification referred. For this, let c_i be the annuitised investment cost per unit of capacity creation (for brevity, this is called 'unit investment cost' below) in S_i required for the B-technology. The condition then is :

<u>Condition S.1</u> : $c' >> c'(I - B)^{-1} A$ [8]

In economic terms, the condition requires the unit investment cost in S_i to be greater than the investment cost necessary for producing by the old method, given that all the inputs for this production are obtained by investment in the new technology (for all i).

With this, I am now in a position to state an explicit optimisation problem -- I shall call this PI -- and then justify our choice criterion on the basis of this optimisation problem.

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*It may be noted that these investment costs are not only prior but also of a once-for-all character. One may still explicitly introduce the cost element by annuitising the investment expenditure and assigning a particular value to the particular period for which the model is set up.

Here it is assumed that $\phi(B) < 1$ which -- as will be shown in the next chapter -- is a necessary and sufficient condition for the existence of solutions of Model I.

The problem, PI, is to :

minimise $c'z$

subject to (5.1.0) and (5.1.1).

The theorem below provides the 'justification'.

<u>Theorem 5.1.1</u> : Under condition 5.1, all optimal solutions of PI satisfy (5.1.2).

<u>Proof</u> : Suppose, contrary to the assertion, that (x, z) is an optimal solution of PI, <u>and</u>

$x_i < \bar{x}_i$ and $z_i > 0$ for some i.

Now increase x_i and decrease z_i by a small but same amount — so that z_i remain non-negative — meeting all the additional intermediate inputs required to increase x_i from the B-technology. From the standpoint of our purpose here, the most unfavourable situation is when all the intermediate inputs released because of the decrease of z_i are from the A-technology. Even in this situation it follows from condition 5.1 that the new solution will decrease the value of the objective function of PI contradicting the fact that we started with an optimal solution of PI. Q. E. D.

1.2 <u>Model II : Indivisibilities and alternative technologies in investment</u>

The basic idea behind the model to be set up in this subsection is the relation between <u>IRS</u> and <u>indivisibility</u>, noted in section 3 of chapter 1 (p. 12). For this purpose, we shall visualise the relevant terminal period possibilities as being described by two alternative ogies, both of which are potential in the sense of being capable operation by prior investment. The relation between IRS and indivisi-ity is then expressed by the restriction that the superior technology operable only at a minimum scale of production. This scale is of se is brought about by suitable prior investment, as already men-ed. This way, we take account of the fact that the opportunity of logical economy of scale is to be exploited really by suitable city installation which is of sufficient size for the best available ds of production to yield their results. These methods make up our or technology.

We shall formulate the model exclusively in terms of the two ative technologies referred above. This means that we abstract etely — at the level of formal analysis — from base technology capacity. Substantively, we presume — as in the previous model — t the base technology is operated so long as its capacity permits s. It is then simply assumed that the (terminal) final demands are enough to entail full utilisation of existing capacities, so that can ignore the base technology altogether along with that part of l demands which is met by its use[9]. That is, the final demand rele-for our formulation is really the net final demand substracting the referred. For convenience of exoposition however we shall call this final demand' simply 'final demand'.

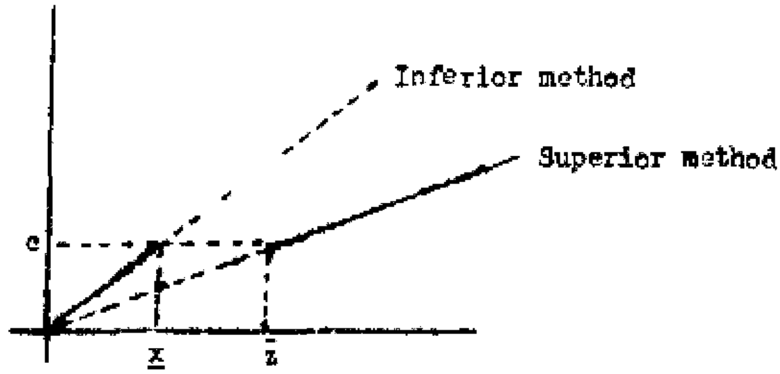
This part of final demand is of course the net output from the base technology with its production programme specified by the capacities. It is assumed that this predetermined production programme is consistent.

We can now explain the working of the model in a few words before turning to a formal representation. It is clear that because of indivisibility in the scale of operation of the superior methods, one would have to operate the inferior methods for small volumes of production. Correspondingly, as the volume of production increases, one would switch to the superior methods. This transition however will not typically be continuous. That is, the inferior method will generally be operated only upto a certain maximum efficient scale which is strictly less than the minimum technological scale of the superior method. The scale of production of a commodity would then jump from the former to the latter in a single discrete step. Clearly, the model formulation has to leave room for possible excess production, or supply, at these switch points.

We may now spell out the logic of this discrete jump in simple terms. There is a certain cost of operating the superior method at its minimum scale. At that scale obviously the cost of the inferior method is larger. It follows that the total cost of producing a smaller output by the inferior method is larger than that of producing at the minimum scale by the superior method, for some range. The total cost of production anywhere in this range is then minimised by producing more according to the superior method and having excess production. The point is illustrated in the following diagram with total operating costs[10] of

10/ These costs are all measured in money terms on the basis of fixed prices which are taken to be given for our purpose. This convention is used throughout this chapter and was justified in chapter 1 (p. 11). Further, it may be noted that these costs are the operating costs and not the 'investment costs' of last subsection. The latter ones for the two alternative technologies are assumed to be the same.

alternative methods of producing some commodity on the ordinate and the levels of their operation on the abscissa :



$\bar{z}$ here stands for the minimum scale of operation of the superior method. The total cost then is c. This same total cost is obtained for the inferior method at the level of production $\underline{x}$ which is precisely the maximum efficient scale of the inferior method referred to earlier. Clearly, for required output between $\underline{x}$ and $\bar{z}$ (this is the range referred to), one minimises cost by actually producing $\bar{z}$, with some excess production provided there is free disposal. I shall take this to be the case which in any case is a standard assumption of the so-called AA model of production[11].

Passing onto the production system as a whole, one would have, in terms of two alternative technologies[12]:

$$+ z \geq Ax + Bz + y \qquad \text{... (5.1.9)}$$

$$x \leq \underline{x} \qquad \text{... (5.1.10)}$$

$$\text{either} \quad z_i = 0 \text{ or } z_i \geq \bar{z}_i \quad \forall i \qquad \text{... (5.1.11)}$$

11/ The idea of indivisibility is thus logically linked up with that of excess production which in turn provides the operational significance of the assumption of 'free disposal'.

12/ I shall continue to denote the two technologies by their respective IO matrices — A for the inferior and B for the superior.

The lower bounds, $\bar{z}_i$'s, on the levels of production according to B-technology are taken as technological data; it is these then that determine the corresponding upper bounds on the levels of production according to A-technology, viz., $\underline{x}$, by means of the type of cost-comparison shown above. As before, this cost-comparison is assumed to be made independently on the basis of given prices, and hence does not form part of the analysis here. For the IO model, it is simply taken as given like $\bar{z}$ with the relation :

$$\bar{z} >> \underline{x} \qquad \ldots \qquad (5.1.12)$$

Formally it is also required to specify the complementary slackness relations. First,

$$x' z = 0 \qquad (5.1.13)$$

That is, production takes place according to one or the other technology available, not both[13/]. Second,

$$x_i + z_i > \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} b_{ij} z_j + y_i \qquad \ldots \qquad (5.1.14)$$

$$\text{only if} \quad z_i = \bar{z}_i$$

That is, there is excess production only at the switch points.

There is a genuine reswitching problem in this model which I should refer to here. This arises basically from the unrestricted diversity in the structure of intermediate use in the model arising from alternative combinations of processes. Thus, consider a situation

13/ This is an important point of difference between Models I and II. In the former, the superior technology was used to produce additional requirement —if any — needed after the inferior one had been fully utilised. In contrast, in Model II only one technology is operated at a time.

where G_i is produced according to the B-technology and G_j according to the A-technology; it is then possible that expansion of output of G_j justifies its switch over to the B-technology, but this in turn can so reduce the intermediate demand of G_i that its production <u>switches back</u> to A-technology. In this context we have to recall our commitment to the so-called IO method (suitably generalised) as the principal tool of analysis for our models, which works through a monotonic sequence of estimated output vectors. The switch-back to inferior methods of production then poses a genuine problem for the workability of the procedure, for it can upset this monotonicity. To avoid the problem we shall assume that the input requirements for the superior method at its minimum scale are larger, commodity by commodity, than those for the inferior method at its maximum efficient scale. Formally, I assume :

A.5.1 : $a^j x_j \leq b^j \bar{z}_j \qquad \forall\, j$

5.2 Import substitution

As in the case of technological break, we shall develop two models of import substitution below. In each model, import substitution is seen as a 'filling in' of previous 'empty sectors', i.e., sectors of production with a <u>given technology but no actual production in the base structure</u>.[14/] In the terminal period, imports can be continued or can be replaced by domestic production, the latter requiring prior

14/ This term too is borrowed from Chakravarty (1959). His definition of the concept and specification of the context provide the conceptual background of our discussion : "As a result of planned economic development, commodities which were previously imported from abroad may begin to be produced domestically .... the sectors for which imports supply the total available amount of a commodity are "empty sectors" of the economy. The above shift in the production-pattern of the economy, therefore, refers to the investment so that the import substitution problem is equivalent to the investment decision problem, as before. This is the problem tackled in both the models below, with somewhat different conceptualisation of the nature of import substituting activities with corresponding difference in the choice criterion used. In all other respects, the models are developed on the same basis.

5.2.1 Model III : Indivisibility and sectorally determined import substitution

We shall begin with the choice criterion. This is conceptually the same as in ACM, viz., no import if domestic production is possible. This possibility will be seen to be restricted essentially by the indivisibility in the size of capacity-to-be-installed. That is, a domestic capacity of production for import substitution is assumed to be of a minimum size. (As in Model II, this is basically a reflection of the nature of technology). It follows that imports will be continued unless total production is of this minimum size, so that for small final demands one will get imports, with a once-for-all switch to domestic production after a certain level, imports being put back to zero from that point onwards[15]. In formal terms, this behaviour is just

footnote 14 contd.: conversion of an "empty sector" into a producing sector ...... such a transformation leads to a change in the structural basis of the economy". /vide Chakravarty (1959), p.112/.

15/ As in Model II, general effect of indivisibilities produces this asymmetry and discreteness, viz., both inferior method (of Model II) and import (in the present case) are operated upto a point and then put back to zero, for the superior alternative can be brought in only at a minimum level.

reverse of ACM where import follows domestic production. This
eflects the "short run" nature of the problem (given capacity cons-
raint) discussed there. We may further note that although Arrow's
aper has "import substitution" in its title, the concept is used there
n a completely different sense. The import refers to competitive imports,
nd the substitution is between production and import on static grounds.
r imports are non-competitive imports, substitutable by production
ly in the future on the basis of investment.

We may now turn to a formalisation of the concept of "empty
ctors" introduced at the outset. This is best done by depicting the
tential IO matrix of the terminal period in a partitioned form as indi-
ted by the left partitioned matrix below, with the corresponding
chnology in the base period depicted on the right.

$$\begin{bmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ C & \vdots & D \end{bmatrix} \qquad \begin{bmatrix} A & \vdots & - \\ \cdots & \cdots & \cdots \\ C & \vdots & - \end{bmatrix}$$

The partitioning of the above matrices follows a corresponding
rtitioning of the set of all sectors into two, viz., the set of
roducing sectors" and "empty sectors", on the basis of whether or
t a sector takes part in domestic production at the base period.
at is, at the base period each producing sector domestically produces
e requirement of its good and, in contrast, the requirements of goods
rresponding to empty sectors are met by imports (hence the qualifi-
tion "empty"). At the terminal period, however, whereas the producing

sectors continue to produce their goods domestically, some of the empty sectors may enter into domestic production activities on the basis of the choice criterion referred. This is the 'filling in' of empty sectors mentioned at the outset of this section. The first and second set of rows or columns in the above two partitioned matrices correspond precisely to the two sets of sectors defined above. More explicitly, in the base period the IO matrix is A, with C forming the so-called "primary input coefficient matrix", the primary inputs being non-competitive imports. Since the empty sectors do not take part in domestic production at the base year, their input-coefficients — the corresponding columns in the second partitioned matrix — are left blank. As mentioned, these sectors may start producing the goods domestically in the terminal period requiring non-negative input coefficients and hence the corresponding blank columns may then be filled in by their input coefficients at the terminal date — these are e columns of the matrix $\begin{bmatrix} B \\ \cdot \\ D \end{bmatrix}$.

I shall now conclude this subsection with a formal presentation Model III. For this, let there be n producing sectors and k empty stors and the corresponding sets be denoted by N and K respectively. e Model III is then represented by the following set of equations :

$$\begin{bmatrix} x \\ \cdots \\ u \end{bmatrix} + \begin{bmatrix} o \\ \cdots \\ v \end{bmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ G & \vdots & D \end{bmatrix} \begin{bmatrix} x \\ \cdots \\ u \end{bmatrix} + \begin{bmatrix} y \\ \cdot\cdot \\ d \end{bmatrix} \qquad \ldots (5.2.1)$$

$$u_i = 0 \ \text{ or } \ u_i \geq \bar{u}_i \qquad \forall \ S_i \in K \qquad \ldots (5.2.2)$$

$$u' v = 0 \qquad \ldots (5.2.3)$$

Here $\begin{bmatrix} x \\ .. \\ u \end{bmatrix}$ and $\begin{bmatrix} y \\ .. \\ d \end{bmatrix}$ are the vectors of domestic production and demands respectively, partitioned according to the sets N and K, the vector of import levels, and $\bar{u}_i$ is the lower bound specified on level of production of the i-th commodity belonging to the set K. Equation (5.2.1) represents the basic commodity balance equations, (5.2.2) implies that any commodity belonging to the set K is either imported or domestically produced but not both. (5.2.3) then guarantees if a commodity belonging to the set K is produced domestically, level of production is higher than corresponding lower bound on it. Equations (5.2.2) and (5.2.3) together therefore take care of the criteria referred earlier. It may be noted that when $u_i = 0$, when i-th empty sector does not take part in production, the i-th of the matrix $\begin{bmatrix} B \\ .. \\ D \end{bmatrix}$ becomes ineffective.

<u>Model IV</u> : <u>Explicit cost comparison and collectively determined import substitution</u>

As mentioned earlier the model to be developed below has a ent conceptualisation of the nature of import substituting acti-. Import of commodities here are not thought to be substituted stic production separately for sectors as was the case in III. Rather the choice here is assumed to be between a <u>domestic ion programme</u> and an <u>import programme</u> for all the non-competitive together. The choice criterion is provided by an <u>explicit cost on</u>. The idea is basically that there is a <u>single investment</u> the domestic production programme, reflecting the <u>joint cost</u>

...acteristics of simultaneous capacity installation in a number of ...errelated sectors. If considered in isolation, investment in any ...tor would have a large overhead cost component (in common with others) ...ch makes individual cost comparison rather limited in significance. ...s is avoided in the present formulation, though perhaps at the cost ...a lack of flexibility. Stated in a slightly different language, one ...le of the choice before us is the scale of an <u>industrial complex</u> ...prising of several projects. The elements of the complex are the ...acities, and the scale is implicitly measured by the cost function ...ing these capacities as arguments. (I shall call this the <u>investment</u> ...<u>t function</u>.) The other side is simply <u>not having a complex at all</u>, ...t is, to import the requisite amounts of each commodity and corres-...dingly one has a composite <u>import cost function</u> with the import ...ls as arguments. It is to be noted that the investment costs are ...ght of as annuitised values as before.

We shall denote the "supply-vectors" for the sectors N and K ...x(i) and z(i) respectively, i = 1, 2. The index value i = 1 refers ...<u>no structural break</u>, so that x(1) is then a <u>production</u> vector and ...an <u>import</u> vector. The index value i = 2 stands for the case where ...<u>tural break</u> has taken place. In this case both x(2) and z(2) are ...tion vectors. With this notation for the variables the model can ...pactly stated as :

$$\begin{bmatrix} x(i) \\ \cdots \\ z(i) \end{bmatrix} = \begin{bmatrix} A & \vdots & (i-1)\,B \\ \cdots & \cdots & \cdots \\ C & \vdots & (i-1)\,D \end{bmatrix} \begin{bmatrix} x(i) \\ \cdots \\ z(i) \end{bmatrix} + \begin{bmatrix} y \\ \cdot\cdot \\ d \end{bmatrix}_{i\,=\,1,2.} \qquad \ldots (5.2.4)$$

the rest of the notation follows that of Model III.

Clearly, the choice is between the two alternative values of If the industrial complex is decided to be set up then the value of is taken to be '2' and if not then '1'. The value of i is then determed — i.e., the choice issue is settled — by comparing the compote investment and import costs, viz.,

$$i = \begin{cases} 1 & \text{if} \quad f(z(1)) < g(z(2)) \\ 2 & \text{otherwise} \end{cases} \qquad \dots (5.2.5)$$

e f and g are respectively the import cost and investment cost ctions referred above.

Finally, I assume that these cost functions are non-decreasing their elements. Formally :

<u>A 5.2</u> : $f(x^1) \geq f(x^2)$ $\quad \forall\ x^1 \geq x^2 \geq 0$

$g(x^1) \geq g(x^2)$

CHAPTER 6

Adaptations of the IO Method

The purpose of this chapter is to subject all the four models developed in the last chapter, viz., Models I-IV, to detailed analysis. Following the general approach of this study, this will be done with the help of definite methods of their solution. Since each of these models has its distinct formal structure, the proposed method for each is also going to be different from that of any other. However — as promised in chapter 1 (p. 8) — these methods will be seen to be unified by the fact that each is an appropriate generalisation of the basic IO method described in chapter 2 (pp. 29– 30). I shall take up the models one-by-one in following four sections, and propose a method for each. Some properties of the methods and thence of the models will also be rigorously examined. I should point out that by the very nature of the task faced, the presentation below is mostly algebraic.

6.1 Model I

The method that I shall propose here for solving Model I given by (5.1.0)-(5.1.2) [p. 105] is a straightforward adaptation of the iterative scheme (2.5.14)-(2.5.16) given on page 53. In economic terms, the initial estimate of production level for each commodity corresponding to the A-technology is put equal to its capacity, or the final demand, whichever is smaller, and if the final demand exceeds its city, the rest is met by the B-technology. That is, the correspond- estimate for B-technology is zero, or the excess of final demand the capacity, whichever is larger. Then at step t of the scheme, estimate of the production level for each commodity corresponding A-technology is expanded to meet the derived demand given by the mate of the production levels of all commodities and of both tech- gies at step (t-1) until capacity limit is reached, the remaining ved demand is met by B-technology. Mathematically, it consists of g two sequences $\{x^t\}$ and $\{z^t\}$ by the formulae[1]:

$$x^t = \min \left\{ Ax^{t-1} + Bz^{t-1} + y, \bar{x} \right\} \qquad \ldots (6.1.0)$$

$$z^t = Ax^{t-1} + Bz^{t-1} + y - x^t \qquad \ldots (6.1.1)$$

$$x^o = \min \left\{ y, \bar{x} \right\} \qquad \ldots (6.1.2)$$

$$z^o = y - x^o \qquad \ldots (6.1.3)$$

In this footnote I shall suggest an iterative scheme for the generalised Model I as given by (5.1.3) - (5.1.5) [p. 107]. The scheme is as follows :

$$x^{it} = \min \left\{ \sum_{j=1}^{m} A_j x^{jt-1} + y - \sum_{j=1}^{i-1} x^{jt}, \bar{x}^i \right\} \quad (i=1,\ldots,m-1)$$

$$x^{mt} = \sum_{j=1}^{m} A_j x^{jt-1} + y - \sum_{j=1}^{m-1} x^{jt}$$

$$x^{io} = \min \left\{ y - \sum_{j=1}^{i-1} x^{jo}, \bar{x}^i \right\} \qquad (i= 1, \ldots, m-1)$$

$$x^{mo} = y - \sum_{j=1}^{m-1} x^{jo}$$

It may be noted that the above iterative scheme can be equi-
ently written as :

$$(x_i^t, z_i^t) = \begin{cases} (x_i^{1t}, 0) & \text{if } x_i^{1t} \leq \bar{x}_i \\ (\bar{x}_i, x_i^{1t} - \bar{x}_i) & \text{otherwise} \end{cases} \quad \forall i \qquad \ldots (6.1.4)$$

$$x^{1t} = F(x^{1t-1})x^{1t-1} + y \qquad \ldots (6.1.5),$$

$$x^{1o} = y \qquad \ldots (6.1.6),$$

the matrix $F(x)$ is as defined in (5.1.7) [p. 107].
follows therefore that the iterative scheme (6.1.5) and (6.1.6) gene-
g the sequence $\{x^{1t}\}$ is really a special case of the iterative
(3.1.1)-(3.1.2) [p. 66] for the non-linear model of
ter 3. Also, it follows from (6.1.4) that the vector x^{1t} is
y the estimate of the "total production" vector at step t of the
tion, i.e.,

$$x^{1t} = x^t + z^t \qquad \forall t \qquad \ldots (6.1.7)$$

l now state a few properties of the scheme (6.1.0)-(6.1.3).

<u>Lemma 6.1.1</u> : The sequences $\{x^t\}$ and $\{z^t\}$ as defined in
0)-(6.1.3) are non-negative and non-decreasing.

<u>Proof</u> : It immediately follows from (6.1.5)-(6.1.6) and
3.2.1 [p. 69] that the sequence $\{x^{1t}\}$ is non-negative and
reasing. Now making use of (6.1.4) the result follows. Q.E.D.

<u>Theorem 6.1.1</u> : The sequences $\{x^t\}$ and $\{z^t\}$ as defined in
0)-(6.1.3) converge if and only if there is a solution of Model I
limiting vectors solve the system.

<u>Proof</u> : Since — as noted in the last chapter [p. 107] — Model I can also be equivalently expressed by (5.1.6)-(5.1.8), the "only if part" follows from (6.1.4)-(6.1.6).

For the "if part" suppose (x^*, z^*) solves Model I. From (6.1.5)-(6.1.7) and the proof of Theorem 3.2.1 it is evident that the sequence $\{x^t + z^t\}$ converges and

$$x^t + z^t \leq x^* + z^* \qquad \forall\ t = 0, 1, 2, \ldots \qquad \ldots (6.1.8)$$

so, from (6.1.4) one gets

$$(x^t - \bar{x})' z^t = 0 \qquad \forall\ t \qquad \ldots (6.1.9)$$

Now fix any i and consider,

Case 1 : $x_i^* < \bar{x}_i$

Clearly, $z_i^* = 0$ (from 5.1.2) and hence one gets, first,

$$x_i^t < \bar{x}_i \qquad [\text{from } (6.1.8)] \qquad \ldots (6.1.10)$$

and then,

$$z_i^t = 0 \qquad [\text{from } (6.1.10) \text{ and } (6.1.9)]$$

It then follows from (6.1.8) that

$$x_i^t \leq x_i^*$$

$$z_i^t \leq z_i^* \qquad \forall\ t = 0, 1, 2, \ldots.$$

<u>Case 2</u> : $x_i^* = \bar{x}_i$

From (6.1.4) I have,

$$x_i^t \leq \bar{x}_i \qquad \forall\ t = 0, 1, 2, \ldots$$

and hence,

$$x_i^t \leq x_i^* \qquad \forall\ t = 0, 1, 2, \ldots.$$

Now if $x_i^t < \bar{x}_i$, clearly $z_i^t = 0$ [from (6.1.9)] and I get,

$$z_i^t \leq z_i^*$$

On the contrary, if $x_i^t = \bar{x}_i$, using the hypothesis ($x_i^* = \bar{x}_i$) and(6.1.8) once again I get

$$z_i^t \leq z_i^*$$

Hence, I have,

$$x^t \leq x^*$$
$$z^t \leq z^* \qquad \forall\ t = 0,\ 1,\ 2,\ \ldots.$$

whence using Lemma 6.1.1, the conclusion follows. Q.E.D.

<u>Corollary 6.1.1</u> : The sequences $\left\{ x^t \right\}$ and $\left\{ z^t \right\}$ as defined (6.1.0)-(6.1.3) lead to the minimum solution (component by component), case there are multiple solutions of Model I for a given y.

<u>Proof</u> : The proof is exactly parallel to the one of llary 3.2.1 [p. 71]. Q. E. D.

<u>Corollary 6.1.2</u> : Under condition 5.1, the iterative scheme to an optimal solution of PI.

<u>Proof</u> : The proof follows from theorem 5.1.1 [p. 112] and above corollary. Q.E.D.

Having stated a few general properties of the iterative scheme 0)-(6.1.3), I shall now explore further properties of Model I. as stated before, will be approached via the iterative scheme 0)-(6.1.3). A neces-ary and sufficient condition for the existence lutions of Model I is stated below.

<u>Condition 6.1</u> : $\phi(B) < 1$

<u>Theorem 6.1.2</u> : Condition 6.1 is a necessary and sufficient condition for the existence of a solution of Model I for any $y \geq 0$.

<u>Proof</u> : <u>Necessity part</u> :

Let there exist solutions of Model I for any arbitrary $y \geq 0$. In particular, consider a y such that

$$y >> \bar{x} \qquad \dots (6.1.11)$$

and let (x, z) solve Model I for the y satisfying (6.1.11).

Now one gets,

$$x = \bar{x} \qquad \dots (6.1.12)$$

, if for some i, $x_i < \bar{x}_i$, one gets from (5.1.2) $z_i = 0$ and (5.1.0) be violated in view of (6.1.11).

Then from (5.1.0) one gets,

$$\begin{aligned} z &= Ax + Bz + y - x \\ &= Ax + Bz + y - \bar{x} && [\text{Using } (6.1.12)] \\ &\geq Bz + y - \bar{x} \\ &>> Bz && [\text{From } (6.1.11)] \end{aligned}$$

which implies that $\phi(B) < 1$.

<u>Sufficiency part</u> : Let condition 6.1 be satisfied.

From (6.1.0) it is clear that $x^t \leq \bar{x} \quad \forall\, t = 0, 1, 2, \dots$.

Hence from Lemma 6.1.1 it at once follows that the sequence $\{x^t\}$ as defined in (6.1.0)-(6.1.3) converges. Let x^* be the limit of sequence. Also, in view of Lemma 6.1.1, it is sufficient to prove the sequence $\{z^t\}$ is bounded above in order to ensure its

. Now,

$$z^t = Ax^{t-1} + Bz^{t-1} + y - x^t \qquad [\text{From } (6.1.0)]$$

$$\leq A\bar{x} + Bz^t + y \qquad [\text{Using Lemma 6.1.1 and the fact that } x^t \leq \bar{x}]$$

$$z^t \leq (I-B)^{-1} [y + A\bar{x}] \qquad [\text{On account of condition 6.1.}]$$

y it is clear that x^* and the limiting value the sequence $\{z^t\}$.

ve Model I. Q.E.D.

Finally, I shall state two alternative conditions for the unique- of a solution of Model I. The conditions are :

<u>Condition 6.2</u> : $\phi(H) < 1$ where $H = (h_{ij})$ and $h_{ij} = \max(a_{ij}, b_{ij})$ $\forall$ i and j.

<u>Condition 6.3</u> : All column sums of both the matrices A and B strictly less than unity.

<u>Theorem 6.1.3</u> : Under either condition 6.2 or condition 6.3, exists a unique solution of Model I for any $y \geq 0$.

<u>Proof</u> : Since each of conditions 6.2 and 6.3 implies condition 6.1, existence is guaranteed by theorem 6.1.2. For uniqueness, let there solutions (x^i, z^i) $(i = 1, \ldots, m)$ of Model I for a particular y.

From Corollary 6.1.1 it follows that one of the solutions is less or equal to any other solution (component by component). Let (x^1, z^1) ch a solution. Hence I can write,

$$x^1 \leq x^i$$

$$z^1 \leq z^i \qquad \text{for } i = 2, \ldots, m \qquad \ldots (6.1.13)$$

Now, first let condition 6.2 be satisfied.

From (6.1.6) I get for i = 2, ..., m

$$x^i + z^i - x^1 - z^1 = A(x^i - x^1) + B(z^i - z^1) \qquad \ldots (6.1.14)$$

$$\leqq H(x^i + z^i - x^1 - z^1) \quad \text{[By construction of H and from (6.1.13)]}$$

$$\Rightarrow \quad x^i + z^i - x^1 - z^1 \leqq 0 \quad \text{[Using condition 6.2]}$$

$$\forall\, i = 2, \ldots, m$$

which together with (6.1.13) gives

$$x^i + z^i = x^1 + z^1 \qquad \forall\, i = 1, 2, \ldots, m \quad \ldots (6.1.15)$$

Observe that if $x^i = x^1$, $\forall\, i$, one at once gets from (6.1.15) that $z^i = z^1$, $\forall\, i$ and part of the theorem is established. On the contrary, let for some i, $x^i \neq x^1$. Without loss of generality I partition x^i and x^1 into $x^i = \begin{bmatrix} x^{i1} \\ \cdots \\ x^{i2} \end{bmatrix}$ and $x^1 = \begin{bmatrix} x^{11} \\ \cdots \\ x^{12} \end{bmatrix}$ such that,

$$\left.\begin{aligned} & x^{i1} << x^{11} \\ \text{and} \quad & x^{i2} \geqq x^{12} \end{aligned}\right| \qquad \ldots (6.1.16)$$

Since all the solutions satisfy (5.1.1) and (5.1.2) I have from (6.1.16),

$$z^{11} = 0 \qquad \ldots (6.1.17)$$

where $\begin{bmatrix} z^{i1} \\ \cdots \\ z^{i2} \end{bmatrix}$ is the corresponding partitioning of z^i, i = 1, 2, ..., m.

Hence I get from (6.1.16) and (6.1.17)

$$x^{i1} + z^{i1} << x^{11} + z^{11}$$

which contradicts (6.1.15). Hence the proof.

Alternatively, suppose condition 6.3 is satisfied. Taking sum on both sides of (6.1.14) one gets

$$\sum_{j=1}^{n} (x_j^1 - x_j^1)(1 - \alpha_j) + \sum_{j=1}^{n} (z_j^1 - z_j^1)(1 - \sigma_j) = 0 \quad \dots (6.1.18)$$

where α_j and σ_j are the j-th column sums of the matrices A and B respectively. Now, using condition 6.3 and (6.1.13) it follows from (6.1.18) that

$$x^1 = x^1$$

$$\text{and} \quad z^1 = z^1 \qquad \forall\, i = 1, 2, \dots, m \qquad \text{Q.E.D.}$$

6.2 Model II

In this section I shall be concerned with the analysis of Model II given by equations (5.1.9)-(5.1.11), (5.1.13)and (5.1.14) [pp. 115-16]

In the method here two sequences $\{x^t\}$ and $\{z^t\}$ are constructed by the recursive formulae :

$$x_i^t = \begin{cases} x_i^{1t} & \text{if } x_i^{1t} \leq \underline{x}_i \\ 0 & \text{otherwise.} \end{cases} \quad \dots\dots (6.2.0)$$

$$z_i^t = \begin{cases} 0 & \text{if } x_i^{1t} \leq \underline{x}_i \\ \bar{z}_i & \text{if } \underline{x}_i < x_i^{1t} \leq \bar{z}_i \\ x_i^{1t} & \text{otherwise} \end{cases} \quad \dots\dots (6.2.1)$$

re $$x_i^{1t} = \sum_{j=1}^{n} a_{ij}\, x_j^{t-1} + \sum_{j=1}^{n} b_{ij}\, z_j^{t-1} + y_i \qquad (6.2.2)$$

$$i = 1, \dots, n$$

$$x^{10} = y \quad \dots (6.2.3)$$

In words, given the estimates of the production levels at stage t corresponding to both the technologies), the sum of the given bill of nal demands and the derived intermediate input requirements is first btained for each sector separately. Then for any sector, say S_i, this ntity is decided to be produced by A-or B-technology according as is less than $\underline{x}_i$ or greater than $\bar{z}_i$, and if that lies between $\underline{x}_i$ and $\bar{z}_i$, the B-technology is operated at the level $\bar{z}_i$. Thus the estimate of the duction levels at stage (t+1) corresponding to the two technologies obtained.

Now, I have the following property of the iterative scheme .2.0)-(6.2.3).

Theorem 6.2.1 : Under condition 6.1, the sequences $\{x^t\}$ and $\{z^t\}$ s defined in (6.2.0)-(6.2.3) converge to a solution of Model II.

Proof : From (6.2.0) and (6.2.1) it is clear that for all t and i one has,

$$x_i^t \cdot z_i^t = 0$$

$$x_i^t \leq \underline{x}_i \qquad \ldots\ldots \quad (6.2.4)$$

and either $z_i^t = 0$ or $z_i^t \geq \bar{z}_i$

Hence if the sequences converge the limiting vectors would solve equations (5.1.10), (5.1.11) and (5.1.13). Also using assumption A5.1 (5.1.12) /pp. 116-17/ it can be shown that the sequence $\{x^{1t}\}$ is non-decreasing[2/]

One need not get this monotonicity in the absence of A 5.1. The reason for this has already been given on page 117. It will be interesting to see if the iterative scheme (6.2.0)-(6.2.3) works even without A 5.1 — that is, with the possibilities of switches and reswitches. As stated before, we do not consider this case, rather rule it out in the form of assumption A 5.1.

nce from (6.2.4), (6.1.0) and (6.1.1) it follows that the sequence $\left\{z^t\right\}$ also non-decreasing and for any i the sequence $\left\{x_i^t\right\}$ is non-decreasing ll it is less than or equal to $\underline{x_i}$ and takes the constant value zero ter that.

From (6.2.2), (6.2.4) and the middle part of (6.2.1) it is o evident that the limiting vectors (if they exist) satisfy (5.1.9) l (5.1.14). Clearly $\left\{x^t\right\}$ converges. Hence it only remains to show t $\left\{z^t\right\}$ is bounded above in order to ensure its convergence and tablish the theorem. This too can be proved following almost similar guments as in the proof of theorem 6.1.2. Q.E.D.

3 Model III

The method that I shall propose for Model III — i.e., the l expressed by the equations — (5.2.1)-(5.2.3) [p. 120] — consist of constructing four sequences of estimates. Mathematically, constructs the sequences $\left\{x^t\right\}$, $\left\{u^t\right\}$, $\left\{z^t\right\}$ and $\left\{v^t\right\}$ by the following ative procedure :

$$(v_1^t,\ u_1^t) = \begin{cases} (0,\ z_1^t) & \text{if } z_1^t \geq \bar{u}_1 \\ (z_1^t,\ 0) & \text{otherwise} \end{cases} \qquad \ldots (6.3.1)$$

where z^t and x^t are given by :

$$\begin{bmatrix} x^t \\ \cdot \\ z^t \end{bmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & \cdot & \cdots \\ C & \vdots & D \end{bmatrix} \begin{bmatrix} x^{t-1} \\ \cdot \\ u^{t-1} \end{bmatrix} + \begin{bmatrix} y \\ \cdot \\ d \end{bmatrix} \qquad \ldots (6.3.2)$$

The iterative scheme is initiated by choosing such initial
as satisfy :

$$\begin{bmatrix} x^o \\ \cdot\cdot \\ z^o \end{bmatrix} \geq \begin{bmatrix} A & B \\ \cdots & \cdots \\ C & D \end{bmatrix} \begin{bmatrix} x^o \\ \cdot\cdot \\ z^o \end{bmatrix} + \begin{bmatrix} y \\ \cdot\cdot \\ d \end{bmatrix} \quad \ldots (6.3.3)$$

[3/]

$$z^o \geq \bar{u} \quad \ldots (6.3.4)$$

That is, at the initial step, the production vector $(x^o, z^o)'$ to be an <u>overestimate</u> for the given final demands as revealed 3). In addition, the production vector for the set K, i.e., the empty sectors, is taken to be greater than or equal to the f lower bounds on it — see (6.3.4). Then, at each stage of , a subset of K is formed for which each sector's production mate is higher than the corresponding lower bound. The sof production levels for the remaining sectors in K are then put and their estimated requirements are met by imports.

The result of this section is stated below as a theorem.

<u>Theorem 6.3.1</u> : The sequences $\{x^t\}$, $\{u^t\}$ and $\{v^t\}$ as in (6.3.1)-(6.3.4) converge to a solution of Model III.

<u>Proof</u> : It can be easily proved that the sequences $\{x^t\}$, $\{u^t\}$ are non-increasing and $\{v^t\}$ takes a jump from zero stage and then decreases. Since all the sequences are non-, using (6.3.1) it then at once follows that all the sequences and the limiting vectors of $\{x^t\}$, $\{u^t\}$ and $\{v^t\}$ solve I. Q.E.D.

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s assumed that $\phi \begin{bmatrix} A & B \\ C & D \end{bmatrix} < 1$ which guarantees the existence initial solution satisfying (6.3.3) and (6.3.4).

Corollary 6.3.2 : In case of multiple solutions, the solution given by the iterative scheme (6.3.1)-(6.3.4) gives the largest feasible subset of K.

Proof : Since the sequences constructed are non-increasing, it is to be noted that any sector in K which goes out of the production activity at some stage of the iteration cannot have a production level greater than the corresponding lower bound in any other situation. The conclusion therefore follows. Q. E. D.

6.4 Model IV :

Unlike the methods developed so far the present method consists of two iterative schemes — one giving a non-decreasing sequence of mates and the other a non-increasing one.

In the first scheme, the demands of all commodities belonging the "empty sectors" are taken to be met by non-competitive imports the iterations begin with an underestimate of the output and import evels. Then at each successive stage of the iteration, the estimates f the output levels are increased in the same manner as in the basic method. The estimatesof import levels are then determined with the p of the "primary input coefficient" matrix. The estimate of the rt cost is also calculated at each stage.

In the other scheme, all the commodities are taken to be ced domestically and the scheme begins with an overestimate of output levels. Then at each successive stage these estimates are sed as in the basic IO method. By the nature of initial estimates dered, this sequence becomes a non-increasing one. The estimate of investment cost is simultaneously calculated at each stage using estimate of the production levels of the "empty sectors" 4/

At any stage if the estimated investment cost becomes less than estimated import cost, the former scheme is shelved and the "domes- production programme" is chosen. Otherwise both are continued till limit and the "import programme" is chosen.

Symbolically, the first scheme is :

$$\begin{bmatrix} x^t(1) \\ \cdots \\ z^t(1) \end{bmatrix} = \left[\begin{array}{c:c} A & 0 \\ \hdashline C & 0 \end{array}\right] \begin{bmatrix} x^{t-1}(1) \\ \cdots \\ z^{t-1}(1) \end{bmatrix} + \begin{bmatrix} y \\ \cdots \\ d \end{bmatrix} \qquad \ldots (6.4.1)$$

$$\begin{bmatrix} x^o(1) \\ \cdots \\ z^o(1) \end{bmatrix} = \begin{bmatrix} y \\ \cdots \\ d \end{bmatrix} \qquad \ldots (6.4.2)$$

second one :

$$\begin{bmatrix} x^t(2) \\ \cdots \\ z^t(2) \end{bmatrix} = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \begin{bmatrix} x^{t-1}(2) \\ \cdots \\ z^{t-1}(2) \end{bmatrix} + \begin{bmatrix} y \\ \cdots \\ d \end{bmatrix} \qquad \ldots (6.4.3)$$

the initial estimate $\begin{bmatrix} x^o(2) \\ \cdots \\ z^o(2) \end{bmatrix}$ satisfying :

$$\begin{bmatrix} x^o(2) \\ \cdots \\ z^o(2) \end{bmatrix} \geq \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \begin{bmatrix} x^o(2) \\ \cdots \\ z^o(2) \end{bmatrix} + \begin{bmatrix} y \\ \cdots \\ d \end{bmatrix}^{5/} \qquad \ldots (6.4.4)$$

---

ternatively, one can have a non-increasing sequence for the er and a non-decreasing for the latter. In that case however rule 1 below has to be modified.

noted in footnote 3, I continue to assume that $\phi \begin{bmatrix} A & B \\ C & D \end{bmatrix} < 1.$

The operational rule here is :

Rule 1 : At some stage of the method if $f(z^t(1)) \geq g(z^t(2))$ [e] iterative scheme (6.4.1) and (6.4.2) is discontinued, the domestic [pro]duction programme is chosen, and the limiting vectors of (6.4.3) and [(6].4.4) are taken to be the solution. Otherwise both are continued. If [ti]ll the limit the above inequality is not satisfied the import programme [is] chosen and the solution is given by the limiting vectors of the [sc]heme (6.4.1) and (6.4.2).

I shall now close this section, this chapter and this part of [the] study with the following property of the above method.

Theorem 6.4.1 : The iterative schemes (6.4.1)-(6.4.4) together [wit]h rule 1 solve Model IV [equations (5.2.4)-(5.2.5); pp. 122-3].

Proof : First of all, it is easy to see that the sequence

$$\left.\begin{bmatrix} x^t(i) \\ \cdots \\ z^t(i) \end{bmatrix}\right\}$$ (i = 1, 2) is non-decreasing for i=1 and non-increasing

[for] i = 2 and both converge. The second one converge because it is [non-i]ncreasing and non-negative and the first one because of the condi[tion] assumed to be satisfied (see footnote 5) and that it is non-[decre]asing.

Then it is to be noted that if $f(z^t(1)) > g(z^t(2))$ for some t, [it will] also be satisfied for any (t+k), k > 0, (and hence in the limit). [This] is true because of assumption A 5.3 [p 123] and the fact that $\{z^t(1)\}$ is non-decreasing and $\{z^t(2)\}$ is non-increasing. This justifies [rule] 1 and completes the proof. Q.E.D.

PART IV

INVESTMENT AND GROWTH CONSISTENCY

CHAPTER 7

The Basic Model and Method

The nature of problems to be discussed in this part of the study [chapters 7-9] has been sufficiently explained earlier [pp. 15-17, chapter 1]. We have also laid down the technical basis of the discussion in section 4 of chapter 2 [pp. 41-48] in details. Hence we can settle down almost immediately to 'analysis' — both model formulation and model analysis. However, we need specify a few concepts somewhat more definitely than done so far.

First, the 'time-frame of reference' for our analysis remains as in the last part. Unlike that part, however, the time-horizon of the analysis plays an explicit role in the present discussion. This is taken to be given a priori, on the basis of the same considerations that specify the 'final demand' for the end-of-the-horizon, or the terminal period.

Second, again unlike the previous part, investment in the terminal period (strictly, use of terminal production for the purpose of investment in the production sectors accounted in the IO model) is treated as an endogenous element of the analysis below. Hence 'final demand' is clearly net of investment. For reasons to be clarified later, part of consumption (viz., that entailed definitely of invariant production-income-consumption relations originating from the sectors of production in the IO model) is also best considered endogenous.

basis of this endogenous treatment of consumption has been explained in details in section 3 of chapter 2 [pp.37-41]. Mainly for linguistic notational convenience, we do not make the endogeneity of consumption explicit. As indicated in the context cited, one may simply reinterpret the terms 'intermediate use' and 'IO coefficients' accordingly. We shall proceed on this implicit basis, though formally the matter is left open, i.e., one may simply take these terms as defined on their own and think of consumption as exogenously given.

In view of the openness referred, we use the term target demand instead of 'final demand'. It is simply the exogenous element in the total commodity demands of the terminal period.

This chapter formulates and analyses a 'basic model' which will be seen as a special case of a general formulation of the problems discussed in chapter 1 [pp. 15-17]. Both the general formulation and the basic model are specified in section 1 below. Section 2 takes up the corresponding 'basic method' for solving the basic model. Extensions of the basic model are taken up in the next chapter, and a review of the relevant literature is taken up in chapter 9.

.1 The basic model

We start off from equation (2.4.13) of section 4 of chapter 2 [p.47], which expresses the basic commodity-balances of an extended model where investment is treated as part of 'derived demand' on the basis of technical coefficients $\left\{a_{ij}, b_{ij}, h_{ij}, \lambda_j\right\}$, and rates of growth $\{r_j, \bar{r}_j\}$. We shall now read the equation for the terminal period which we shall denote by t=T. The same letter, T, al. stands for the length of the time-horizon, with t=0 for the base period. Clearly, production levels and capacities are time-variables and hence are defined for T. Hence adapting the notation of the equation referred to these time-specifications, we have :

$$x_i(T) = \sum_{j=1}^{n} a_{ij}x_j(T) + \sum_{j=1}^{n} h_{ij}\, r_j^T\, x_j(T) + \sum_{j=1}^{n} g_{ij}(\bar{r}_j^T)\bar{x}_j(T) + d_i(T) \quad \ldots \quad (7.1.0)$$

$$i = 1, \ldots, n.$$

where we have written $\bar{r}_i$ for r_i^o (rate of growth of capacity), $\bar{x}_i$ for x_i^o (capacity) and d_i for u_i (target demand). Otherwise the notation is the same as before. The superscript 'T' on rates of growth (r_i and $\bar{r}_i$) indicates that these too are time-variables, but not with the same order of time-dependence as production levels or capacities. That is, these are assumed to possess a certain degree of temporal stability compared to production or capacity. For example, $\bar{r}_i^T$ really stands for the rate of growth of capacity over the period $[T, T + \lambda_i]$, not $[T, T + 1]$.

To this we add the capacity constraints at T to obtain a full statement of the terminal production possibilities :

$$x_i(T) \leq \bar{x}_i(T) \quad \ldots \quad (7.1.1)$$

$$i = 1, \ldots, n.$$

Now, at t=0, we have not only the initial capacities $\bar{x}_i(0)$, but also the investment programmes initiated upto t=0, as given. By the very concept of an investment programme (see p.45, chapter 2) this means that the production capacity in S_i is predetermined for $t \leq \lambda_i$ where λ_i is the construction period of investment programme in S_i. Hence operationally, the capacity restrictions are given by $\bar{x}_i(\lambda_i)$, for all i.[1/] On the same grounds, the capacity $\bar{x}_i(T)$ depends upon this benchmark value, $\bar{x}_i(\lambda_i)$, and the investment programmes in S_i initiated between t=0 and $t = T - \lambda_i - 1$. It follows that an implicit rate of growth of capacity in S_i over the period $\left[\lambda_i, T\right]$ can be calculated from :

$$\bar{x}_i(T) = (1 + \bar{r}_i)^{T - \lambda_i} \, \bar{x}_i(\lambda_i)$$

$$\text{or} \quad \bar{r}_i = \left[\frac{\bar{x}_i(T)}{\bar{x}_i(\lambda_i)}\right]^{\frac{1}{T - \lambda_i}} - 1 \qquad \ldots (7.1.2)$$

where $\bar{r}_i$ is the rate of growth referred. For distinction we refer to r_i^T and $\bar{r}_i^T$ as post-terminal growth rates and to $\bar{r}_i$ as the preterminal or implicit growth rate. (It may be noted that the latter are defined only for capacities, not production).

This completes the background. The so-called 'general formulation' of the problems discussed in this part of the study comes from some definite relation connecting three sets of growth rates met above. This relation is the basis of the notion of growth consistency. We shall come to it in the due course of specifying the further set of relations involved in our statement of the 'basic model' as a special case of the general formulation.

1/ For convenience of language, we use the expression 'base capacity' somewhat flexibly to refer to $x_i(\lambda_i)$ in this and the following two chapters.

The basic model corresponds to the case of <u>full capacity operation at T</u> (and beyond). Since terminal capacities are made up of (a) base capacities and (b) the newly installed capacities by the relevant investment programmes over the time horizon, the assumption can be split into two : (i) there is full use of base capacities at t=T; and (ii) there is full use of newly installed capacities at t=T. The first can be seen as a reflection of the insufficiency of base capacities for the terminal production. A strongly sufficient condition for this is :

<u>Condition 7.1</u> : $\bar{x} \leqq A\bar{x} + d(T)$

where $\bar{x}$ is the vector with $\bar{x}_i(\lambda_i)$ as its i-th component. Clearly, this amounts to the insufficiency of base capacities for required terminal production ignoring investment so that some weaker condition would suffice if all investments are not zero. However, there is no direct algebraic substitute for condition 7.1 that is available to entail full use of base capacities at t=T, and we shall leave it at this. While not strictly necessary for the 'basic model', it will play a useful role in the 'basic method'.[2/]

The second assumption behind the full capacity case is simply efficient use of investment, or what can be called an <u>efficient investment rule</u>. This simply disallows any investment to generate excess capacity.

[1/] In the next chapter we shall consider an extension of the basic model where we allow for possible adequacy of base capacities for the production in some sectors. That is, in that extension we shall keep room for possible excess capacities in the terminal period.

We now come to the relations between growth rates. First, by the assumption of full capacity at t=T and beyond we have $r_i^T = \bar{r}_i^T$ for all i[3]. Second is the relation between $\bar{r}_i^T$ and $\bar{r}_i$. As mentioned in chapter 1 [p. 16], the totality of all investment programmes over the time horizon is seen as a single composite programme where the terminal investments do not appear as independent variables, i.e., these do not have any independent objectives to serve. Rather, these are taken to be continuations of the same investment programme whose basic task is to so expand base capacities that the given target demands are satisfied at the stipulated date. The 'continuation' simply means that the same rates of growth are maintained sector by sector. In all then, we assume

$$r_i^T = \bar{r}_i^T = \bar{r}_i = r_i, \quad \text{say} \qquad \forall\ i$$

This formally 'closes' the model in the sense of translating the three different sets of growth rates (parameters so far) into a single set of growth rates $\{r_i\}$ as unknowns. Formally, our basic model reads[4]:

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} h_{ij}\, r_j x_j + \sum_{j=1}^{n} g_{ij}(r_j)x_j + d_i$$

$$\text{and} \qquad r_i = \left[\frac{x_i}{\bar{x}_i}\right]^{\frac{1}{T - \lambda_i}} - 1$$

$$\forall\ i = 1, \ldots, n.$$

3/ Strictly, one requires full capacity operation only at T and (T+1).

4/ For notational convenience, we write x_i for $x_i(T)(= \bar{x}_i(T))$, $\bar{x}_i$ for $\bar{x}_i(\lambda_i)$ and d_i for $d_i(T)$. That is, we henceforth drop the period specifications unless those are explicitly necessary to avoid confusion.

n vector-matrix-scalar notation the basic model can be written as :

$$x = (A + B(r))x + d \qquad \ldots (7.1.3)$$

$$r_i = \left[\frac{x_i}{\bar{x}_i} \right]^{\frac{1}{T-\lambda_i}} - 1 \qquad \ldots (7.1.4)$$

$$i = 1, \ldots, n$$

ere $B(r)$ is as defined in (2.4.15).[5/]

Clearly, both equations above refer to consistency conditions. uation (7.1.3) is the same as (2.4.14) [p.47] and hence represents tended IO consistency, or what we can also call level consistency terms of an extended IO model internalising investment on the basis growth rates treated as parameters. These very growth rates are the nowns in (7.1.4) which require these to be consistent with respect the terminal production, as given by (7.1.3), and base capacities. shall call these growth consistency equations. Clearly (7.1.3) and .1.4) make up an interdependent system, with terminal production ending upon growth rates via investment in (7.1.3) and growth rates endent upon terminal production directly in (7.1.4). Taking the two ether, terminal production and growth rates are to be simultaneously termined for given target demands, base capacities, time-horizon and set of technical coefficients. The method of this determination the topic of our next section.

It is to be noted that by proper substitution (7.1.3) and (7.1.4) can be seen as a special case of our general non-linear model of chapter 3.

We now pass onto an important observation. It is to be recalled from section 4 of chapter 2 [p.48] that we have the case of an <u>infeasible growth programme</u> if the (extended) IO system represented in (7.1.3) is <u>not viable</u> for the vector of growth rates, r. As pointed out in the context referred, the growth programme has to be sufficiently modest, in the sense of small enough values of r_i's, to avoid the problem. Since in the present formulation, the growth programme itself is a reflection of target demands (in relation to base capacities and the time horizon), the feasibility is really that of target demands. Clearly, our formulation includes an internal examination of this question. From a substantial standpoint, the crucial issue is the implication of the magnitude of target demands for the required growth programme and thence its feasibility. The problem is thus one of both the <u>consistency and feasibility</u> of growth rates.

Formally, the internal examination referred above is easily demonstrated. This is stated below as a theorem.

<u>Theorem 7.1.1</u> : For d sufficiently large, there does not exist any solution of (7.1.3) - (7.1.4).

<u>Proof</u> : First, let (x, r) be a solution of (7.1.3)-(7.1.4) for a $d >> 0$ satisfying condition 7.1. From (7.1.3) one therefore gets,

$$x = Ax + B(r)x + d$$

$$\text{or,} \quad x >> (A + B(r))x \qquad [\because d >> 0]$$

This implies

$$\phi(A + B(r)) < 1 \qquad \ldots (7.1.5)$$

fact, (7.1.5) is a necessary condition for the existence of a solution of (7.1.3)-(7.1.4). Now, from (7.1.4) it is clear that as d increases so does r (if it exists). In fact, r is a strictly increasing function of d, so clearly, after a certain stage (7.1.5) will be vitiated . (7.1.5) being a necessary condition for the existence of a solution the conclusion follows. Q.E.D.

Thus, too ambitious target demands are rejected by the model itself in the form of growth infeasibility or non-existence of solution.

We shall end this section with a comment on the so-called 'time-path' traversed by the economy over the time horizon. Looked at from the 'time-path' standpoint, our analysis would appear to be based the assumption of constant sectoral growth rates of capacities over a time horizon (and some time beyond). This, however, is not strictly implied, for the standpoint taken is of the 'comparison of end-points type'. All that we discuss is whether or not there exist time paths of storal production and capacity with constant rates of growth which able the economy to traverse from one given endpoint to another. If e exist (i.e., there is a feasible growth programme), then these ve one set of consistent time paths. There may exist others. Nothir , s implied directly about which path to follow. That may be said to around how the so-called phasing problem is tackled, i.e., upon precise dating of particular investment programmes. This problem s outside the scope of analysis undertaken by its very approach.

In this context, we may return to the basis of the notion of owth consistency. As stated earlier /p. 144/, the basis lies in some finite relation between implicit and post-terminal growth rates. We ve taken this relation to mean simple equality, for any other relation quires additional substantive specifications. One may, for instance, troduce a definite break in the total growth programme at T, and ecify post-terminal growth rates as functions of the pre-terminal es (period of ease following that of austerity, or vice versa). The alysis carried out in this and the next chapter is immediately adapt-le to such specifications. All that one has to do is to replace the 's in (7.1.3) by $f_i(r)$ which is some increasing function expressing st-terminal growth rates as function of the pre-terminal growth ogramme. Since there is no basis for any presumption regarding these nctions $f_i(r)$, we have kept the analysis free of this unnecessary plication.

2 The basic method

As stated in the introduction, this section develops the basic thod of analysis of this chapter. Following the general approach of is study the present method is also a straightforward adaptation of IO method described in section 2 of chapter 2 to the present context. recapitulate briefly, the IO method consists of putting the estimate output levels at any stage equal to the sum of (a) the final demands, (b) the derived intermediate demands given by the estimate of the t levels of the previous stage. The initial estimate of the output vector is taken to be the final demand vector. With this as the benchmark, the generalised method that this section suggests amounts to the following:

Given the estimate of output levels at any stage, first, for each sector separately this is compared with the corresponding given base capacity to yield the estimate of the growth rate of that sector for that stage. Then the estimate of output levels for the next stage is put equal to the sum of (a) and (b) as before <u>plus</u> (c) the derived investment demands given by (i) the estimates of the output levels, and (ii) the just-computed rates of growth. Both (i) and (ii) are the current-stage estimates of the relevant variables.

Since the standard method works through an increasing sequence of output vectors, it simultaneously provides a sequence of <u>increasing growth rates</u> by comparing the output levels at each stage with the respective base capacities. Hence, when the derived investment requirements are superimposed on the demand components stage by stage, one gets a sequence of output vectors which increases at a faster rate. The road is therefore open to make growth rates endogenous to the computational procedure. It now works by first computing growth rates and then utilising these to yield the 'total' derived demand. The initial output estimates are taken to be given by the base capacities, not target demands. In view of condition 7.1 (which is explicitly assumed here), this means that the initial estimates are strict underestimates, which is all that is necessary for the 'increasingness' property referred.

Mathematically, the method consists of constructing two sequences $\{x^t\}$ and $\{r^t\}$ by the recursive formulae :

$$x^t = Ax^{t-1} + B(r^{t-1})x^{t-1} + d \qquad \ldots (7.2.0)$$

$$r_{it} = \left[\frac{x_i^t}{\bar{x}_i}\right]^{\frac{1}{T - \lambda_i}} - 1 \qquad \ldots (7.2.1)$$

where r^t is a vector with r_{it} as its i-th component.

The scheme is initiated by setting :

$$x^0 = \bar{x} \qquad \ldots (7.2.2)$$

$$r^0 = 0 \qquad \ldots (7.2.3)$$

The first simple property of the sequences $\{x^t\}$ and $\{r^t\}$ is stated below as a lemma.

<u>Lemma 7.2.1</u> : Under condition 7.1, both the sequences $\{x^t\}$ and $\{r^t\}$ as defined in (7.2.0)-(7.2.3) are non-negative and non-decreasing.

<u>Proof</u> : I shall prove this by induction.

First note that

$$\begin{aligned} x^1 &= Ax^0 + B(r^0)x^0 + d && \text{[From (7.2.0)]} \\ &= A\bar{x} + d && \text{[From (7.2.2) and (7.2.3) since } B(0) = 0\text{]} \\ &\geq \bar{x} && \text{[Using condition 7.1]} \\ &= x^0 && \text{[From (7.2.2)]} \\ &\geq 0 \end{aligned}$$

$$\therefore x^1 \geq x^0 \geq 0 \qquad \ldots (7.2.4)$$

$$\therefore r_{i1} = \left[\frac{x_i^1}{\bar{x}_i}\right]^{\frac{1}{T-\lambda_i}} - 1 \geq \left[\frac{x_i^o}{x_i^o}\right]^{\frac{1}{T-\lambda_i}} - 1 = 0 = r_{io}$$ [Using (7.2.4) and (7.2.1)]

for $i = 1, \ldots, n$

$$\therefore r^1 \geq r^o = 0$$

Now as an induction hypothesis let

$$\left.\begin{array}{l} x^t \geq x^{t-1} \geq 0 \\ r^t \geq r^{t-1} \geq 0 \end{array}\right| \quad \ldots (H),$$

r a given value of t, and consider,

$$x^{t+1} = Ax^t + B(r^t)x^t + d$$ [From (7.2.0)]

$$\geq Ax^{t-1} + B(r^{t-1})x^{t-1} + d$$ [Using (H) since B(r) increases with r]

$$= x^t$$ [From (7.2.0)]

$$\geq 0$$

$$\therefore x^{t+1} \geq x^t \geq 0 \quad \ldots \quad (7.2.5)$$

Now for any i,

$$r_{it+1} = \left[\frac{x_i^{t+1}}{\bar{x}_i}\right]^{\frac{1}{T-\lambda_i}} - 1$$ [From (7.2.1)]

$$\geq \left[\frac{x_i^t}{\bar{x}_i}\right]^{\frac{1}{T-\lambda_i}} - 1$$ [On account of (7.2.5)]

$$= r_{it}$$ [Using (7.2.1)]

$$\geq 0$$

$$\therefore r^{t+1} \geq r^t \geq 0 \quad \ldots (7.2.6)$$

and (7.2.6) prove the required result. Q.E.D.

Theorem 7.2.1 : Under condition 7.1, the sequences $\left\{x^t\right\}$ and $\left\{r^t\right\}$ defined in (7.2.0)-(7.2.3) converge if and only if there is a solu- on of (7.1.3)-(7.1.4); and the limiting values of the sequences solve system (7.1.3)-(7.1.4).

Proof : If the sequences converge, obviously the limiting ues would solve the system (7.1.3)-(7.1.4) and they are non-negative n 7.2.1].

Now suppose that there exists (x^*, r^*) non-negative satisfying .1.3)-(7.1.4).

Since $r^* \geqq 0$ it at once follows from (7.1.4) and (7.2.3)

$$x^* \geqq \bar{x} \quad \text{and} \quad r^0 \leqq r^*$$

Hence using (7.2.2) one gets

$$x^0 \leqq x^*$$

$$r^0 \leqq r^*$$

Now as an induction hypothesis suppose

$$\left.\begin{array}{l} x^t \leqq x^* \\ r^t \leqq r^* \end{array}\right. \qquad \ldots \text{(H)}$$

a given value of t, and consider,

$$x^{t+1} = Ax^t + B(r^t)x^t + d \qquad \text{[From (7.2.0)]}$$

$$\leqq Ax^* + B(r^*)x^* + d \qquad \text{[Using (H) since } B(r) \text{ increases with } r\text{]}$$

$$= x^* \qquad \text{[From (7.1.3)]}$$

$$x^{t+1} \leqq x^* \qquad \ldots (7.2.7)$$

Also for any i

$$r_{it+1} = \left[\frac{x_i^{t+1}}{\bar{x}_i}\right]^{\frac{1}{T-\lambda_i}} - 1 \leq \left[\frac{x_i^*}{\bar{x}_i}\right]^{\frac{1}{T-\lambda_i}} - 1 \quad \text{[Using (7.2.7)]}$$

$$= r_i^*$$

$$\therefore \; r^{t+1} \leq r^*$$

$$\therefore \; x^{t+1} \leq x^*$$

$$r^{t+1} \leq r^* \qquad \text{for all } t = 0, 1, 2, \ldots\ldots$$

nce using Lemma 7.2.1 the conclusion follows. Q.E.D.

Corollary 7.2.1 : In case of multiple solutions, the solution en by the scheme (7.2.0)-(7.2.3) is the minimum (component by mponent).

Proof : Let x and r be the limiting values of the sequences $\{x^t\}$ d $\{r^t\}$ respectively and $(\bar{x}, \bar{r})$ be any other solution of (7.1.3)-(7.1.

As in the proof of theorem 7.2.1, it can be shown that

$$x^t \leq \bar{x}$$

$$r^t \leq \bar{r}$$

d hence

$$x \leq \bar{x}$$

$$r \leq \bar{r}$$

Q.E.D.

I now conclude this chapter with the following observation on the sults derived in this section. If one looks at the basic model as a ecial case of the non-linear model of chapter 3 [see footnote 5, p. 145], e parallelism of the results here with the ones stated in section 3.2.2 p.69 - 72] becomes obvious. In fact, the basic method can also be seen a special case of the iterative method (3.1.1) [p.66].

CHAPTER 8

Two Extensions

This chapter consists of two extensions of the basic model of previous chapter, given in the two respective sections below. Each ion contains the appropriate 'generalisation of the IO method' for analysis.

System with excess capacity

Our first generalisation consists of opening up the basic model last chapter to possible excess capacities (at the terminal period). the 'general formulation' of section 1 of last chapter kept room this extension, we can start off directly from there, viz., from ations (7.1.0)-(7.1.2) /pp. 141-42/. We shall keep both to the ity between post-and pre-terminal growth rates as the basis for th consistency, and to the so-called 'investment rule' of the ous chapter. In fact, the model developed below differs from the c model only in respect of giving up the full capacity assumption hence condition 7.1 /p. 143/.

We may begin by restating the <u>investment rule</u>. As mentioned chapter 7 /p. 143/, no investment is allowed to generate excess ities. That is, there is full utilisation of capacity at the al period in any sector where investment takes place. Conversely, is excess capacity only if investment is zero, in which case the al capacity is clearly equal to the base capacity. In other words excess terminal capacity is really a reflection of excess base capacity. What it expresses is the fact that the base capacity of some sector may be redundant even for the requirements of terminal production. No investment is then made in such sectors, no new capacities come into being and there is no avoidable wastage in this sense[1].

Translated into 'growth rates', the above argument leads to the conclusion that sectors with excess capacity have zero growth rates. Conversely, positive growth rates are possible only for the full capacity sectors. We shall call the full capacity sectors the bottleneck sectors and denote the set of bottleneck sectors by J. Clearly, the growth consistency specification remains the same as before, viz., equation (7.1.4) [p. 145], with the domain of sectors restricted to J. For $i \notin J$, $r_i = 0$.[2]

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[1] This is really a question of the objectives of growth vis-a-vis the initial conditions. There is no reason to believe that all previously built-in capacities resulting from past investments (prior to t=0) have commensurate measures of significance under the conditions reflected in the target demands. Instances would be pre-existent capacities in sectors producing mainly luxury goods, vulnerable exports etc. Indeed, a definite 'historical break' between the pre-base-period past and the time-horizon is implicitly presumed in the total frame of analysis. Investments initiated after t=0 are governed by the internal forces represented, those initiated earlier were governed by some different set of forces.

[2] I should note here that the formulation below does not put any prior restriction on J. If J is the empty set one is back to the standard IO model and if it is the set of all sectors one gets the basic model of chapter 7. Both the cases are possible. Clearly, for there to be investment it is only necessary that J is non-empty.

There remains one point to clear up before we can formalise the ations. This refers to the relations between the growth rates appear- in the general formulation (7.1.0)-(7.1.2). As noted in the previous ter [p. 144], there were two specific relations involved in the o model, viz., $r_i^T = \bar{r}_i^T$ on the basis of full capacity, and $\bar{r}_i^T = \bar{r}_i$ the basis of growth consistency. Clearly, we shall now have the st equation only for $S_i \in J$. For $S_i \notin J$, on the other hand, we have eady seen that $\bar{r}_i = 0$; hence $\bar{r}_i^T = 0$. This leaves r_i^T for $S_i \notin J$ eterminate. We shall assume that $r_i^T = 0$ for $S_i \notin J$, i.e., there is growth of production in the terminal period in any sector with ss capacity. There is really no *a priori* justification of this tion. It is believed that the degree of error introduced on unt of the assumption is not very significant (i.e., the invest- t in working capital is taken to account for a relatively small of total production).

Having done the groundwork we shall now go on to the formal ysis of this section. First, we shall give the formal representa- of the model developed so far and then pass on to analysing it.

To take up the first task, we have

$$J = \left\{ S_i / \; x_i(T) = \bar{x}_i(T) \right\} \qquad \dots (8.1.0)$$

hence the growth consistency relations and capacities are given ectively by :

$$r_i = \begin{cases} \left[\dfrac{\bar{x}_i(T)}{\bar{x}_i(\lambda_i)} \right]^{\frac{1}{T - \lambda_i}} - 1 & \text{if } S_i \in J \\ 0 & \text{otherwise} \end{cases} \qquad \dots (8.1.1)$$

$$\bar{x}_i(T) = \begin{cases} x_i(T) & \text{if } S_i \in J \\ \bar{x}_i(\lambda_i) & \text{otherwise} \end{cases} \qquad \ldots (8.1.2)$$

The level *consistency equations*, however, remain the same as n the basic model, viz.,

$$x(T) = \left[A + B(r)\right] x(T) + d(T) \qquad \ldots (8.1.3)$$

This follows from (8.1.0) and (8.1.1), for if $S_i \notin J$ then $_i = 0$, which reduces the i-th column of B(r) to the null vector and se does not call forth any commodity use for investment in S_i.

Now using the simpler notations introduced in the last chapter see footnote 4, p. 144] and by proper substitution the model formula above reduces to :

$$x = (A + B(r))x + d \qquad \ldots (8.1.4)$$

$$J = \left\{ S_i / x_i > \bar{x}_i \right\} \qquad \ldots (8.1.5)$$

$$r_i = \begin{cases} \left[\dfrac{x_i}{\bar{x}_i}\right]^{\frac{1}{T - \lambda_i}} - 1 & \text{if } S_i \in J \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (8.1.6)$$

Having spelt out the model I am now in a position to describe e iterative method of its solution. This is an adaptation of the sic method (7.2.0)-(7.2.3) [p. 150] to the present context. In words, the method is as follows :

Given the estimate of output levels at any stage t, first, 1.5) is used for an estimate of the set J for that stage; then for sector belonging to this estimate of the set J, the output esti- is compared with the corresponding given base capacity to yield the mate of the growth rate of that sector for that stage, and for other s the estimate of growth rates is put equal to zero for that age. Then the estimate of the output levels for the next stage is put to the sum of (a) the target demand vector, (b) the derived inter- ate input demand given by the estimate of the output vector at stage t, (c) the derived investment demand given by the estimate of output els and the foregoing growth rates at stage t. The procedure starts th the target demand vector as the estimate of the output levels. thematically, it consists of constructing the sequences $\{x^t\}$, $\{r^t\}$ $\{J^t\}$ by the recursive formulae :

$$x^t = Ax^{t-1} + B(r^{t-1})x^{t-1} + d \qquad \ldots (8.1.7)$$

$$J^t = \left\{ S_i \quad / x_i^t > \bar{x}_i \right\} \qquad \ldots (8.1.8)$$

$$r_{it} = \begin{cases} \left[\dfrac{x_i^t}{\bar{x}_i} \right]^{\frac{1}{T - \lambda_i}} - 1 & \text{if } S_i \in J^t \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (8.1.9)$$

$$x^0 = d \qquad \ldots (8.1.10)$$

$$J^0 = \left\{ S_i \quad / x_i^0 > \bar{x}_i \right\} \qquad \ldots (8.1.11)$$

$$r_{io} = \begin{cases} \left[\dfrac{x_i^0}{\bar{x}_i} \right]^{\frac{1}{T - \lambda_i}} - 1 & \text{if } S_i \in J^0 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (8.1.12)$$

I shall now close this section after stating some properties the scheme (8.1.7)-(8.1.12), which can be proved following almost al arguments as in the corresponding proofs of chapter 7.

Lemma 8.1.1 : The sequences $\{x^t\}$, $\{r^t\}$ and $\{J^t\}$ as defined (8.1.7)-(8.1.12) satisfy : $x^t \geq x^{t-1} \geq 0$, $r^t \geq r^{t-1} \geq 0$ and $\geq J^{t-1}$ for all t = 1, 2, 3, ...

Theorem 8.1.1 : The sequences $\{x^t\}$ and $\{r^t\}$ as defined in 1.7)-(8.1.12) converge if and only if there exists solution of the (8.1.4)-(8.1.6) and the limiting values of the sequences solve system (8.1.4)-(8.1.6).

Corollary 8.1.1 : In case of multiple solutions the iterative (8.1.7)-(8.1.12) leadsto the minimum solution (component by nent).

We may finally note that the set of bottleneck sectors is an genous element of the method described above. In this context, must acknowledge that this idea has been drawn from Arrow's method solving ACM discussed in section 5 of chapter 2 [p. 53].

Size of target demand Vs. rate of growth

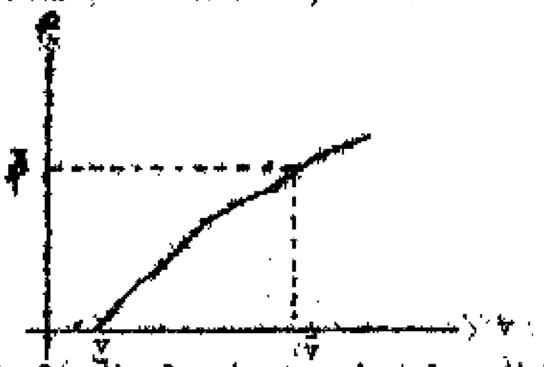
The idea behind the extension of this section has been touched n in chapter 1 [p. 17]. As mentioned there, the extension consists ally of freeing the analysis of chapter 7 from the "restrictive e of given target demands". This clearly is a step beyond the eral formulation of chapter 7. We shall also allow for excess capa-ties. Hence the generalisation is really of the model of the previous tion.

It will be convenient to begin with a recapitulation of the positive contributions of the foregoing analysis. We have shown that if one starts from some given base capacities and has a set of given target demands for a certain future time (the terminal period) in mind, then it is possible to adapt both the standard IO model as well as the standard method of its solution to finding the required output and investment levels in the terminal period, where the investment levels depend upon a consistent set of growth rates. For the substantive significance of these results we may refer to what was earlier pointed out as a "crucial issue", viz., "the implication of the magnitude of target demands for the required growth programme and thence its feasibility" [p. 146].

Clearly, the larger the target demand the higher the growth rates. To abstract from the technicalities of the vector relations, we may conduct the analysis in terms of scalar measures of the target demands and growth rates, say v and ρ respectively. The simplest case will be to value the output levels at some given "reference prices". Actually, without any loss of generality, we may simply assume that this is already done, i.e., the variables are all measured in a common monetary unit and IO coefficients are also specified in corresponding value units. The scalar measures referred are then defined as :

$$v = \Sigma d_i, \quad \rho = \Sigma r_i \frac{\bar{x}_i(\lambda_i)}{\Sigma \bar{x}_i(\lambda_i)} \qquad \ldots \quad (8.2.0)$$

To systematically, one can specify the target demand vector, d, as a function of the 'size', v , and treat the latter as an independent ter, chalking out the corresponding 'size' of the growth programme or vice versa. We shall refer to v and ρ simply as the size-index target demand) and the (overall) growth rate respectively. The re class of models, therefore, can be characterised by the relation up between v and ρ — with the base capacities and the tech- coefficients as the only 'givens' of the analysis — which is early positive in all cases, as illustrated below.:



here stands for the least upper bound on the overall growth rate sed by growth feasibility; and $\bar{v}$ the corresponding (unattainable) ex of target demands. More precisely, $\bar{v}$ is the limit of v as from below. $\underline{v}$ is to be taken as the maximum size-index permitted base capacities, so that $\rho > 0$ only for $v > \underline{v}$. Our analysis has been to stipulate v from outside — consequently d has a datum of our analysis — and determine accordingly. In section we really try out the alternative approach. Before doing we have to formalise the relation between d and v mentioned above. aightforward formulation here would be

$$d = v.y$$

y fixes the commodity composition of target demands. More gene-
y, we may simply write

$$d = f(v)$$

e f(v) is an increasing function of v in all components. Substan-
y, this would allow for variations in the composition of target
ds with respect to its size (as indicated by v) reflecting, e.g.,
's law of consumption.

We may now state explicitly the alternative approach. This
ld stipulate ρ as an outside parameter and determine v. As we shall
the method of the last section can in fact be suitably generalised
a built-in feedback mechanism to solve the present system. Also, the
would give — as an efficiency property — the maximum v that
consistent with the stipulated ρ and other consistency requirements.
.., level and growth consistencies). This is really a reflection of
efficiency property of the method met earlier (corollary 8.1.1).

Before settling down to analysis it only remains to give a
presentation of the system which is as follows (with $\bar{\bar{x}}$ and ρ^*
the exogenous variables) :

$$x = Ax + B(x)x + f(v) \quad \ldots \quad (8.2.1)$$

$$J = \left\{ S_i / x_i > \tilde{x}_i \right\} \quad \ldots \quad (8.2.2)$$

$$r_{\lambda_i} = \begin{cases} \left[\dfrac{x_i^*}{x_i}\right]^{\frac{1}{T-\lambda_i}} - 1 & \text{if } S_i \in J \\ 0 & \text{otherwise} \end{cases} \quad \ldots \quad (8.2.3)$$

$$\rho = \rho^* \quad \ldots \quad (8.2.4)$$

ρ is given by (8.2.0). It is assumed that $0 < \rho^* < \bar{\rho}$.

Clearly, for a fixed value of v (8.2.1)-(8.2.5) is same as the 8.1.4)-(8.1.6) of last section and I shall denote such a model by
uely, by corollary 8.1.1 [p. 159], the solution of M_v obtained method (8.1.7)-(8.1.12) will give the minimum possible value of ρ, minimum value of ρ will be a function of v. I shall denote such as by $g(v)$. It is known to be increasing. I also assume it to be
us function with a finite slope in the range $0 \leq v \leq \bar{v}$.
the assumption is :

A 8.1 : There exists a scalar $\lambda > 0$ such that

$$g(v_1) - g(v_2) \geq \lambda(v_1 - v_2) \quad \text{for all } 0 < v_1 < v_2 < \bar{v}$$

The assumption plays an important role in the method described Here I shall note that it also guarantees the existence of a solution of the equation $g(v) = \rho$ for all $0 < \rho < \bar{\rho}$. I shall the solution for $\rho = \rho^*$ by v^* i.e.,

$$g(v^*) = \rho^* \qquad \ldots\ldots (8.2.5)$$

(Clearly, $\underline{v} < v^* < \bar{v}$)

At this stage I shall state a few properties of the model which useful later on.

Lemma 8.2.1 : For any $v \leq \bar{v}$, there exists a solution of M_v.

Lemma 8.2.2 : $g(v_1) \geq g(v_2)$ for all $\bar{v} \geq v_1 > v_2 \geq 0$. The ty will be strict if either of v_1 or v_2 lies to the right of $\underline{v}$.

The proof of the above results are straightforward.

As stated before, my job now is to propose a method for solving (8.2.1)-(8.2.4). I will first state the method in its broad o and then spell out a certain refinement which increases its ciency. In broad terms, the method approaches the solution-value, f (8.2.5), monotonically from below by constructing an increasing ence $\left\{ v_n \right\}$. Formally, the method consists of the iterations :

$$v_{n+1} = v_n + \frac{1}{\lambda}\left[\rho^* - g(v_n)\right] \qquad (8.2.6)$$

the initial condition satisfying

$$g(v_0) < \rho \qquad \ldots \quad (8.2.7)$$

From (8.2.5), (8.2.7) and the properties of the function $g(v)$ llows that

$$v_0 < v^* < \bar{v} \qquad \ldots \quad (8.2.8)$$

It may be noted that $g(v_n)$ in (8.2.6) is obtained on the basis solution of M_{v_n}. Hence the sequence above is based on the solu- of a sequence of models $\left\{ M_{v_n} \right\}$. For each M_{v_n}, the method of the section (with $d = f(v_n)$) remains applicable. However, one significant ation of the latter is possible in the present context; and this to the "initial conditions", equation (8.1.10)-(8.1.12). Since an ing sequence $\left\{ v_n \right\}$ is now being worked out, it is no longer to begin the iterations for M_{v_n} as in (8.1.10) [with $d = f(v_n)$]. y, it is necessary to use this initial condition only for solving at is, M_{v_0} is solved by the method (8.1.7)-(8.1.12) [with $d=f(v_0)$]. stage t, M_{v_t} is solved by the method (8.1.7)-(8.1.9) where the estimates are given by the solutions of $M_{v_{t-1}}$. It can be easily that the properties of the method (8.1.7)-(8.1.12) [Lemma 8.1.1, 8.1.1 and Corollary 8.1.1] would also hold good for this modified thm.

I now state the all important result of this section.

<u>Theorem 8.2.1</u> : The sequence $\{v_n\}$ defined in (8.2.6) and (8.2.7) verges to a solution of the system (8.2.1)-(8.2.4).

<u>Proof</u> : First note that because of (8.2.8) and Lemma 8.2.1, has a solution. Now from (8.2.6) one has
o

$$v_1 = v_o + \frac{1}{\lambda}\left[\rho^* - g(v_o)\right]$$

or, $v_1 > v_o$ [From (8.2.7)]

Now, if $v_1 \geq v^*$ I get from (8.2.6)

$$v_o + \frac{1}{\lambda}(\rho^* - g(v_o)) \geq v^*$$

or $\rho^* - g(v_o) \geq \lambda(v^* - v_o)$

or $g(v^*) - g(v_o) \geq \lambda(v^* - v_o)$ [From (8.2.5)]

This clearly contradicts A 8.1 because of (8.2.8). Hence $v_1 < v^*$.

Now as an induction hypothesis suppose for m = t,

$$v_{t-1} < v_t < v^* \qquad \ldots \text{ (H)}$$

This, with the help of Lemma 8.2.1 implies that both M_{v_t} and $M_{v_{t-1}}$ have solutions.

Now consider,

$$v_{t+1} - v_t = (v_t - v_{t-1}) - \frac{1}{\lambda}(g(v_t) - g(v_{t-1})) \text{ [From (8.2.6)]}$$

$$> 0 \qquad \text{[From (H) and A 8.1]}$$

$\therefore$ $v_{t+1} > v_t$, and hence

$$v_t > v_{t-1} \qquad \forall t = 1, 2, \ldots\ldots \quad (8.2.9)$$

Now from (8.2.6) and (8.2.9) it at once follows that,

$$g(v_t) < \rho^* \qquad \forall\ t = 0, 1, 2, \ldots.$$

e using Lemma 8.2.2 and (8.2.5) one gets,

$$v_t < v^* \qquad \forall\ t = 0, 1, 2, \ldots. \qquad (8.2.10)$$

.9) and (8.2.10) together, thus, guarantees the convergence.
(8.2.6) and A.8.1, it then follows that the limiting value of the
ence $\{v_t\}$ - v, say — satisfies $g(v) = \rho^*$, and thus the solu-
of M_v and v solves the system (8.2.1)-(8.2.4). Q.E.D.

The efficiency property of the above method is now stated
w as a corollary.

<u>Corollary 8.2.2</u> : The method (8.2.6)-(8.2.7) gives the largest
on of v in case there are multiple solutions of (8.2.1)-(8.2.4).

<u>Proof</u> : Let v be the limiting value of the sequence $\{v_n\}$ as
ed in (8.2.6)-(8.2.7). Since g(v) is the minimum value of ρ one
got by solving M_v, it follows from Lemma 8.2.2 that for any
$v_1 > v$, $g(v_1) > \rho^*$. Hence the proof follows. Q.E.D.

Before leaving the technical area I have to mention that the
of the present extension has been borrowed from Chander (1973)
suggested a similar extension to the model of foreign trade dis-
ed in section 5 of chapter 2. Chander, however, did not suggest
rigorous solution of his problem.

I shall close this chapter with a few observations on the
se carried out in relation to the total conception of the problem
sistent and feasible growth. First, the question of prior

...ation of the size-index of target demand vs. overall growth
e can be seen as one of starting directly from objectives vs. from
e implicit notion of viability. Such a notion is rather difficult to
ing on the size index but is quite natural for the rate of growth,
r there may be independent forces restraining the latter as part of
g run historical conditions. One may mention the idea of so-called
absorptive capacity' in this context.

It is to be noted that this implicit notion of growth feasibility
nothing to do with the bounds on feasible growth rates that emerge
the model itself. The problem was discussed in section 4 of
ter 2 [p. 48] and is reflected in $\bar{\rho}$ of the diagram on
ge 161 . In particular, one may refer here to the so-called
savings constraint' on growth rates. Since this constraint operates
rough the technology, it is really part of the analysis given. One
s to recall the qualifications made at the outset of chapter 7
garding the treatment of consumption out of income generated in the
ctors of production. Clearly, when this is treated as an endogenous
ement of demand (formally, as explained by the reinterpreted IO
efficients — see pp. 39-40, section 3 of chapter 2), the
-called 'savings constraint' becomes an internal part of the set of
ations in the model. Operationally, this would be reflected in a
ler 'feasibility range' of the overall growth rate (0, $\bar{\rho}$). With
s interpretation, the target demand would strictly consist of
ds generated by the "rest of the economy", i.e., by the economy

~~not covered by the sectors of production accounted for. A little more~~ generally, one may adopt a somewhat 'flexible' approach to consumption, and split it between a "necessary" and a "surplus" part. The "necessary" part can then be related to production (via labour requirements, wage rates and budget relations) in the manner depicted earlier (pp. 37 - 41). While the "surplus" part would form part of the target demand. This would reflect both the idea of 'savings constraint' and of 'consumption target'.

CHAPTER 9

The Approach of Static Multi-sectoral Planning Models : A Critical Review

The purpose of this chapter is to review the contributions of chapters 7 and 8 in the light of the recent, and growing literature on what can be called static multi-sectoral planning models (SMPM). The reference here is to a class of models all of which are of a mixed analytical-empirical (numerical) character, being studies on planning in a large number of countries in the last decade-and-a-half. The earliest instance appears to be Sandee (1960), while the Technical Note on the Approach to the Fifth Five Year Plan of India — Government of India (1973) — can be cited as a recent example. A representative sample of the models developed in between would be : Manne (1966), Bruno (1966), Manne-Rudra (1965), Tendulkar (1971), Weisscopf (1967) etc. Review of this literature is available in Bhagwati and Chakravarty (1971, pp. 16-21).

As should be expected from the variety of references, individual members in the class differ significantly from one another in respect of number of details, and have a wider or narrower set of problems before them. We shall be concerned with a certain basic and common element in their set of problems, and this concerns the problem of consistency with regard to both production and investment. This problem is sought to be tackled within the same analytical frame in all the models following common approach. For the connection between our analysis and this literature, we may immediately point out that the nature of problems and the analytical frame in TPM are exactly the same as ours, but the approaches taken are completely different. Hence our 'critique' of the literature is essentially one of its approach[1]. There are many matters of detail where the literature is based on rather simplified assumptions compared to the analysis we have given, but these are to be regarded as secondary matters. For a clear development of the differences, we ignore these details and base the arguments on the simplest formulation of endogenous investment in the IO model. In fact, the bulk of the literature referred uses explicitly this formulation which is given by the so-called dynamic IO model as stated by Leontief (1953, pp. 53-90). The formulation is as follows :

$$x(t) = Ax(t) + B(x(t+1) - x(t)) + d(t) \qquad \ldots \quad (9.1)$$

where B is the so-called (marginal) capital coefficient matrix[2].

This completes a statement of the background necessary for the critique. I shall now first develop the approach of SMPM and then base the critique on this approach, as mentioned. Before venturing into it, I should mention that most of the ideas developed in the previous two chapters are in fact derived from the literature mentioned. In fact, our analysis can be said to have grown out of a feeling of inadequacy of

1/ In fact, the approach is shared by a wider class of planning models, e.g., the aggregative model of Chenery and Bruno (1962).

2/ The 'simplified assumptions' mentioned above are of full capacity operation, absence of working capital from the picture and finally of no construction period. In terms of our analysis in section 4 of chapter 2, these mean $x_i^c(t) = x_i(t)$, $\theta_{ij} = 0$ and $\lambda_i = 0$ for all i and j.

treatment of not only the production-investment-consistency problem but also a number of related issues mentioned in the literature. We shall try to give a somewhat connected account of the sources of our ideas in the literature at the end of the chapter.

I have already mentioned that the analytical frame used by SMPM is the same as ours, i.e., the consistency problems are posed only for a terminal year of planning with a benchmark value of capacity (= production) for the base as given. That is, equation (9.1) is discussed for t=T where T refers to both the terminal period and the time-horizon, or planning horizon as it is called in the literature (t=0 is the base as before). The problem is essentially one of finding the terminal investment, say v(T). Obviously,

$$v(T) = B(x(T+1) - x(T)).$$

Clearly, once v(T) is expressed uniquely in terms of x(T), both can be solved from (9.1).

The approach in SMPM simply consists of viewing this problem as one of assigning a part of the 'total' investment (i.e., the total over the planning horizon) to the terminal date, i.e., essentially as one of time allocation or phasing of investment. Conceptually, in this approach, investment over the planning horizon and its part to be executed in the terminal year constitute two separate unknowns determined on independent considerations. In analytical terms, the approach is as follows. For any stipulated value of x(T), one gets a otal investment over the time horizon equal to $B(x(T) - \bar{x})$ where $\bar{x}$ is the vector of base capacities. The next problem is then to allocate a definite part of this total investment to the terminal period. This is sought to be tackled by a coefficient, say α, representing the fraction of the total investment carried out in the terminal period. This coefficient is called the stock-flow conversion factor (SFCF) in the literature in view of the fact that dimensionally it represents the ratio of a flow to its integral over a certain time. That is, the denominator of this ratio can be looked upon as the difference between two stocks and hence itself a stock.

Postponing a discussion of the derivation of SFCF, it is easily seen that given its magnitude, the approach leads to the following rmulation of terminal production levels :

$$x(T) = Ax(T) + \alpha B(x(T) - \bar{x}) + d(T) \qquad \ldots (9.2)$$

is is the form in which the model is formulated explicitly in ahne-Rudra (1965). As mentioned earlier the same formulation is also tainable from the other models referred after suitable simplifications of other relations in those models.

We now take up the derivation of SFCF. As mentioned earlier, e basis of SFCF is some stipulated time-distribution of investment ver the horizon. The simplest case here would be that of a uniform stribution which immediately leads to :

$$\alpha = \frac{1}{T}$$

ere T is the length of the planning horizon.

Next, the distribution can be taken to be given by an *increasing investment path*. Sandee (1960) assumed the path to be of the linear type. That is, he assumed investment at time t, v(t) say, to be of the form $v(t) = a + bt$. The SFCF in this case is then given by :

$$\alpha = \frac{a + bT}{aT + \frac{1}{2} bT^2}$$

Manne (1966) considered an exponential growth of investment and derived the SFCF as :

$$\alpha = \frac{e^{\rho T}}{\int_0^T e^{\rho t}\, dt} = \frac{\rho}{1 - e^{-\rho T}} \qquad \ldots (9.3)$$

where ρ is the stipulated uniform rate of growth of investment.

Modifications of the last derivation of SFCF were later introduced by Weisscopf (1967), Manne-Rudra (1965). Finally, I may point out that instead of assuming a *single growth rate* for all investments, one may treat the rate of growth of investment in each sector as a constant and replace the single SFCF by a vector of sectoral SFCF's. For this, one need only write ρ_i for the rate of growth of investment in S_i and get the SFCF for S_i — α_i say — from (9.3). In this case the model (9.2) has to be replaced by the model:

$$x(T) = Ax(T) + \hat{\alpha} B (x(T) - \bar{x}) + d(T)$$

where $\hat{\alpha}$ is a diagonal matrix with α_i in its (i, i)th position.

This completes the statement of the approach to the specific method of endogenous treatment of investment in the models referred. We can now develop the critique in terms of our own analysis. First and foremost, in SMPM the problem of consistency is viewed exclusively in terms of (9.2), with the SFCF given. And the determination of SFCF has nothing 'endogenous' about it; it is simply determined by outside constants as demonstrated above. Conceptually, it is difficult to pinpoint the exact nature of consistency in this approach, for properly speaking (9.2) represents neither level consistency, nor growth consistency. It may be convenient here to restate the content of these two separate aspects of consistency in slightly different terms. There is, first, a problem of purely contemporaneous consistency among the product flows and their uses in the period under consideration, i.e., the terminal period. One may therefore call this the intraperiod (terminal) consistency. Then, because a benchmark value of production for an earlier period (the base eriod) is given and this implicitly is seen to be responsible for restment and capacity expansion, there is a further problem of inter-riod (terminal-base) consistency. In our language, these correspond ecisely to level consistency and growth consistency respectively. ated in the new language, (9.2) is an attempt at composite intraperiod-terperiod consistency, for both terminal production and base capacity e involved in (9.2). As claimed earlier, neither is properly reflected. early, there is no interperiod consistency, for the SFCF is a cons-nt of the analysis, independent of the values of $x(T)$ and $\bar{x}$. Hence in way can it represent the required growth for going from $\bar{x}$ to $x(T)$. t then, neglecting this aspect, one cannot interpret (9.2) as exhibiting intraperiod consistency since the base period is directly involved in it. Formally, one may try to interpret α as the rate of growth in the terminal period, which would define $(A + \alpha B)\ x(T)$ as the 'derived' part of the production vector. But this requires $(d(T) - \alpha B\bar{x})$ to be interpreted as a final demand vector, and this is without justification. For one thing, there is no guarantee that this vector is non-negative. For another, this whole interpretation breaks own once one leaves the case of uniform growth, for the vector 'α' annot be even formally interpreted as a vector of growth rates.

Cutting this methodological critique short, we may point out e operational implications of (9.2). First, it is clear that there o no notion of growth feasibility on internal grounds in the approach tated. No matter how large d(T) is compared to $\bar{x}$, there is always a asible plan. One simply chooses α reasonably low so that $(A + \alpha B)$ viable (and $\left[d(T) - \alpha B\bar{x}\right]$ is non-negative). Conversely, no matte small d(T) is, it can be rendered infeasible just by a choice of T, α is critically inversely sensitive to the value of T. That is, e exists a certain time-span, say T*, such that $(A + \alpha B)$ is ble or not as $T \gtrless T^*$. On one side of this critical value, all targets d consistent solution, on the other none ! This has nothing to do the real life phenomenon that more ambitious targets can be eved only over longer horizons.

Viewed analytically, SFCF is the all important coefficient in differentiating it from an ordinary IO model with only technical

fficients (A and B matrices) We have indicated its conceptual basis
ve, We now note some analytical discussions of its use in the lite-
re mentioned. First, it is clear that (9.2) has a solution if and
y if the matrix ($A + \alpha B$) is viable. This is usually dealt with
functorily in the literature. The problem is viewed as a mere
icality and more or less forthwith dispensed with. For example,
e-Rudra (1965, p. 61) simply assumed that the so-called Solow
tions were satisfied by their matrix ($A + \alpha B$). In a slightly
rent context, Manne (1974, p.58) writes,

"All that is required is that the geometric growth rate r be set sufficiently low so that there is some feasible terminal period solution x_T".

Second, there is some discussion of parametric variations of α
9.2), based on variations of ρ in (9.3). There is a much-repeated
sitivity result" here. The insensitivity referred is that of α to ρ.
e (1966) reported that as ρ was varied from 5 per cent to 12 per
, α varied from 12.7 per cent to 17.2 per cent. This is the insensi-
ty result. While the claim is not clear -- in fact, if change in ρ
the corresponding change in α is 64 -- the result has been used
stify the procedure, and this appears still less clear. For, the
tivity one is interested in is that of production to targets --
-vis capacities, and neither of the two latter are present in the
se referred. All that one can get from it is at most a 'relative
tivity of production' to the <u>rate of growth of investment</u>. But

the latter is neither here nor there. It comes in only because of the approach of SMPM. It is nothing 'real' in the sense that 'rates of growth of production' are, for these directly connect terminal production (reflecting targets) and base capacities.

Finally, I have to mention that there have been occasional talks bout "revising growth rates" on the basis of the results thrown up by (9.2)[5]. As mentioned above, the growth rates here being those of investment, one looks the proper basis of any such revision, for (9.2) can throw up only a growth rate of production on comparison of $x(T)$ and $\bar{x}$. Even granting some definite connection between the two growth rates investment and production), the literature offers no definite method of revision.

This completes the critique. I now turn to the ideas often expressed both in the literature/which I have sought to exploit in the two previous chapters. This source is in fact already implicitly acknowledged at the beginning of this chapter when it was pointed out that both the nature of the overall problem and the analytical frame are common between SMPM and our discussions undertaken in chapters 7 and 8. he two differ basically in respect of approach (and secondarily in espect of details) which lead to very different formalisations of the ame set of issues. I shall now give a somewhat more systematic account of specific ideas and their use in our analysis in the two previous chapters.

referred and in some related literature

To quote Weisscopf (1967, p.279),

"If the initial estimate of r^0 proves to be inconsistent with the results of the programming run, it is always possible to revise it for a second run and thus proceed by iteration to a consistent solution ....".

To begin with, the very idea of simultaneous production-investment consistency is repeatedly stressed in the literature. In fact, this as seen as the major point of departure of the class of models from the static IO models concerned solely with intraperiod consistency. In the words of Manne (1966, p. 269) :

> "The principal technical difference consists of the treatment of investment within the key sectors. Here this source of inter-industry demand is treated as an endogenous element ....."

As we have seen, the basis of this endogenous treatment was provided by NFCF. The rate of growth of production in that approach is obtainable only as reflected in the rate of growth of investment, and this, as mentioned, has no definite basis in the analytical frame presumed.

However, later writings — not strictly belonging to the literature referred — have sought to bring the rate of growth of production to the forefront. In setting up a terminal-period consistency model based on (9.1) Bruno (1971, p.176) explicitly stated :

> "The particular form in which we choose to take care of the capital coefficients has some clear analytical drawbacks, as it ignores problems of excess capacity and of intertemporal choice within the planning horizon. At the same time, if we take a pragmatic view, with a long enough planning period (5 years at least), problems of excess capacity become relatively less important. Also successive approximation can be used to make a sensible choice of the terminal growth rates so that they are not too far off from the implied intra-period exponential growth rates". (emphasis added).

"sensible choice" problem is precisely our "growth consistency" blem tackled in the two foregoing chapters. The issue of "excess capacity" was taken up in section 1 of chapter 8 where we had also argued against Bruno's views about its "importance" [see, in particular, footnote 1, p. 155]. Bruno does not specify the content of "intertemporal choice". Strictly, the question cannot be properly tackled in the analytical frame posited, as mentioned earlier in connection with the problem of "phasing" (see p. 147, chapter 7). However, our formulation of "efficient investment rule" resolves one kind of choice problems with regard to investment, and hence one aspect of intertemporal choice. The last point — that of "successive approximation" — actually recurs a number of times in the literature (surrounding SMPM) in the form of revision of rates of growth. For example Bergsman and Manne (1966) state explicitly that an iterative method for revising growth rates was adopted in their numerical calculations. However, the method is not explicitly stated, and in their algebraic formulation, the growth rates are taken as exogenously specified. To quote them (p. 255) :

> "The computer was programmed to calculate the 1970-71 output levels, then revise the fifth plan growth rates, then revise 1970-71 output levels and so on . ... "

In the light of this discussion I have to point out that a proper formulation really obviates the need for any separate revision of growth rates. As the methods of chapters 7 and 8 show, rates of growth are calculated <u>on the way</u> towards a solution of consistent production.

Finally, I may quote a broader overview of the nature and scope of consistency in the class of models :

> "The consistency model is of the 'open' rather than 'closed' type. In order to apply it, the first step is to project the principal components of gross domestic expenditure, and to <u>translate these into final demands for individual commodities</u>. The model's job is then to deduce an internally consistent set of sectoral output levels, imports and investment requirements. Unlike a closed model, no <u>explicit feedback link</u> is provided here from the process of production back to generation of income and in turn back to the principal components of gross domestic expenditure" [<u>vide</u> Manne-Rudra (1965, p. 67)].

The above excerpt has two clear points relevant for our discussion. The "first step" mentioned really conforms to our <u>parametric specification</u> of target demands in section 2 of chapter 8. Second, the reference to "feed back link" precisely relates to what we have called "internalisation of consumption" in section 5 of chapter 2 [pp. 37-41]. However, as we have argued in the concluding paragraph of the last chapter, from the standpoint of planning only a part of the consumption (the "necessary" part) can properly be accounted for in the 'derived' part of the production vector via production-income-consumption relations. The rest can then be seen as part of the parametric target demands.

CHAPTER 10

The Method of Material Balances

This, the concluding, chapter of the study is to review the entire work done so far from a unified standpoint. As mentioned in chapter 1, the standpoint represents a particular aspect of general planning processes which in turn is rooted in the organisational structure of a planned, or socialist, economy. We have already given a brief glimpse of the process and the structure in section 2 of chapter 2 (pp. 33-35), and our discussion in this chapter can be seen as a generalisation of the discussion, conducted there in terms of the standard IO model, to various generalisations of the model discussed. However, for the purpose f an overall review, the conceptual basis of the discussion itself nee further clarified. This is undertaken in section 1 below. The follow- g four sections then take up the task of 'generalisation' referred bove in an order which will be clarified below. I should just mention at this order has its basis in the discussion of section 1 and is fferent from the order of our 'model-generalisations' reported in the udy. Finally, section 6 offers some concluding remarks on the area of r work, going beyond the scope indicated by the title of this chapter. rictly, the title is to be taken in the sense of the major theme, not a mplete coverage, of this chapter.

.1 The conceptual basis of the review

We may begin by distinguishing between two separate kinds of nceptual clarifications that we have to provide in this section. The first is in terms of our work : its approach and methodology. This amounts basically to reinterpreting our concepts and problems in the light of the standpoint proposed for this chapter. The second is to clarify the nature of planning process and organisational structure of planning referred earlier.

Regarding the first, we begin by observing that for the purpose of this chapter, all the 'models' discussed so far have to be interpreted as 'planning models'. Formally, a planning model is distinguished from a general economic model by a classification of all variables into the sets of target variables (ends of planning), instrument variables (means of planning) and irrelevant variables (irrelevant for planning ut not for economic description). We shall presently count upon this thodology vis-a-vis the IO frame. Here it need be pointed out that r discussion will be concerned not really with the nature of planning dels as such, but with a procedure of planning. This procedure in rn is really an interpretation of the basic IO method that we have peatedly modified and adopted in the context of various models. This precisely the way the procedure was introduced in section 2 of ter 2. As we had pointed out in chapter 1, the area of 'models' resents 'variety', and that of 'methods', 'unity' for this study. In t light, the task of this chapter is to see the 'unity' at the l of a substantive process in the frame of an 'economic system' ined by planning, not just as a formal method of analysis.

Returning to the methodology of planning models, it is usual to treat the dependent-independent classification of variables of IO analysis as one-to-one with the instrument-target variable classification of planning models, there being no 'irrelevant variables'. The nature of planning models is then said to be of the 'fixed target' variety on this basis. This, however, is too restrictive a view. The IO model as a tool of planning is really far more flexible. For example, consumption may perhaps always be thought of as belonging to the domain of plan objectives. Clearly, the endogenous treatment of consumption (or some part of it) allows objectives a wider scope than 'final demand' in the formal sense of the term. Similarly, as we have also shown, the final demand vector itself may be suitably parametrised and internally adjusted with the endogenous variables of the model. On both counts, there is no necessary rigid adherence to the 'fixed target' variety of planning for the operational task of plan formulation.

We now come to the second type of conceptual basis needed for our sion. Now, the really distinguishing feature of a planning model our purpose is an organisational structure of the economy in terms f which all the specific calculations required for plan formulation are be carried out. We shall distinguish between two types of elements in s structure : the essentially coordinating elements on the one hand the operative elements on the other. For our models, the operative ts consist simply of the so-called sectors of production : those places where production is actually carried out in separation from another (this gives rise to the problem of <u>coordination</u>) and which therefore also the unique repositories of the corresponding technical edge regarding methods of production, specific capacities, methods eir expansion etc. (this gives rise to the problem of <u>information</u> rements for making an overall plan). In other words, production is sed in distinct sectors or units, to be denoted by S_k, $k=1,2,\ldots, n$. arly, there may be a <u>structure of coordination</u> itself. For example, of production, consumption, investment, import, export may be based ate coordinations of the relevant activities. For our purpose, duction coordination is the basic one, with the others being nt only in so far as these have some implication for the former. implications, however, may be separately worked out, and then into the procedure for production coordination. It follows that tructure of coordination *must have an apex which is responsible* <u>al coordination</u>, with <u>subsidiary</u> coordinating agencies respon- for various 'sub-plans' so to say. We shall call this apex the (C). The plan objectives are also taken to be formulated by C. up, the overall structure of the system consists of C on the d and $\{S_k, W_i\}$ on the other where $\{S_k\}$ represent operative ts and $\{W_i\}$ the (subsidiary) coordinating elements.

With this background, one may define a general planning process ghting one aspect of the organisational structure described This is the information aspect. For any kind of a general ot planning model in the background, we may refer to the total of known elements in it as its <u>information content</u>. This information ... t will be distributed among the various participants in the process $\{C, W_i, S_k\}$ — in a way dependent upon the substantive nature of element in the context of the model. We shall refer to this distribution as the <u>initial information distribution</u>. A general planning ... in this context is an iterative procedure of information exchange $\{S_k\}$ to $\{W_i\}$ and/or C, also from $\{W_i\}$ to C, and in the ... direction again. Roughly, one can say that the process is one of ...ssive plan formulations (sequence of provisional plans) by C on the ... of informations (proposals) received by it from $\{S_k\}$ and $\{W_i\}$ ...sponse to informations (instructions, directions etc.,) sent out by ...

For convenience all informations transacted are often referred ... as messages.

As mentioned earlier, the process of material balances (MB) can ...ined as a specific procedure or method of planning belonging to the class[1] defined above. Its specificity is really a reflection of ...pose it seeks to serve which is to <u>balance</u> the requirements and ...abilities of various commodities in the form of a set of MB (one

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...should mention that the class is "broad" only relative to the ...ocess of MB, not in general. I have already mentioned that it ...lights only the information aspect of the organisational ...cture. The interrelated aspects of "incentives", "lack of full ...tral authority", "conflict of interests" etc., are totally ...ored. It may also be mentioned that there is a significant over... between our area of enquiry and the so-called "decentralisa...n procedures in planning", though the latter has a somewhat ...ferent focus on issues. Our definition of a "general planning ...ocess" is roughly similar to the definition of a "decentralisation ...cedure" in Malinvaud (1960). A broader definition is given in ...ndar (1973).

for each commodity group) for the economy as a whole. The focus, as before, lies on the so-called products, i.e., availability is thought of primarily as production. We have already illustrated the working of this method in chapter 2 (pp.33-35) which brings out its essential features. Starting from the presentation of the method in chapter 2 as the benchmark, our purpose is to find precisely in what ways the method can be suitably generalised to tackle various issues. Methodologically, this ties up with the logic of generalisations underlying this study, as explained in chapter 1. The initial benchmark — standard IO analysis for the entire study, the method of MB in terms of the standard IO model for this chapter — provides us with some 'central ideas' or 'basic themes' or 'essential features' which we pursue under wider and/or different contexts. Thus the method of MB in our analytical scheme stands in the same relation to the planning models as the standard IO method does to the various extensions of the IO model, whether formal or interpretive. In substantive terms, this means that we open up the method of MB to issues of production-income-consumption relations, internal and external economies of scale, structural break, investment and growth etc. The theoretical literature on the method itself appears to be yet rather limited. There does not appear to be any analytical treatment of the set of issues referred in the literature, although the problems have been mentioned. We may refer here to the "concluding remarks" of a standard theoretical reference in the field :

"This paper confines itself to the purely technical aspects of drawing up a consistent set of interlocking balances of material resources. We have not related these balance sheets to the composite ("synthetic") balance sheets of money flows which are used, among other purposes, to equilibrate the aggregate supply and demand for consumer goods. Neither have we related the short-term balances to investment planning or to plant capacity. We have thus by-passed one essential function of the material balances : the detection of bottlenecks and their eventual elimination by suitable investments". [vide Montias (1959,p.981)]

As will be seen later, all these issues are in fact amenable to the process by straightforward adaptation of its benchmark exposition of chapter 2. Such possibilities are mentioned in the literature, but there do not appear to be any systematic formal representations.One may, e.g., refer to Wellisz (1964, chapter 5) which quotes, inter alia, the following passage from a document entitled Economic Planning in Poland (Polish Planning Commission) :

> "The main advantage of the approach used in practice [i.e., of material balancing] is that we are not obliged to make any aprioristic assumptions about functional (in fact proportional) relationship betwen variables. We can (in theory, at any rate) take into account in every step of sectoral analysis different relationships between changes in variables".

We shall now briefly recapitulate the working of the basic method, i.e., of the benchmark. The method can be specified in terms of three 'rules' : (a) at each stage, C sends a production target to each S_k; (b) each S_k reports back its material requirements on the basis of its targets; and (c) C revises the production targets on the basis of these rts. These may be called the 'rules of the game' at the benchmark. We may note in passing that the operational crux of step (b) lies in the assumption that the sectors are able to translate directly the targets into requirements without further information from C. That is, whatever further informations may be required for this purpose are provided on an independent basis. The definition of general planning process keeps enough room for this. The method of MB, as mentioned, is just one aspect of this general process. From the standpoint of economic analysis, this links up with the idea of fixed prices lying implicitly behind our analysis of production and its structure (chapter 1, p.11). In terms of planning, this means that price-setting is done independently — possibly as an independent part of the general planning process defined earlier[2].

We shall now clarify a few points in the above discussion which will save considerable time later. First, the distinction between operative and coordinating elements in the organisational structure is relative to the degree of aggregation at which the overall planning problems are considered. Clearly, to use the sectors of an IO model as the operative elements is to aggregate over the individual firms or plants making up a particular sector. In a more detailed analysis, the plants may be the operative elements with sectors (ministries, departments etc.) as coordinating elements. Second, for a formal analysis of the planning process, the subsidiary coordinating agencies $\{W_1\}$ really have a purely intermediate role to play. They simply provide

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2/ One may refer to the work of Taylor (1929) in this regard. A recent formalisation of Taylor's work is given in Chander (1973).

possible intermediate links between the operative elements on the one hand and the final coordinating element on the other. For convenience of exposition etc., one may simply subsume these appropriately within C or $\{a_k\}$. The explicit form of the organisational structure is therefore open to an analytical choice. We shall exploit this choice according to convenience.

Finally, I may point out that since the method of MB describes an iterative process, it would require an initiation and a termination. For the former, I shall generally assume that C initiates the process with 'final demands' as 'production targets'. Hence only departures from this rule will be mentioned in appropriate places below. The decision to terminate has clearly to be based on some definite criterion which has to ensure that the plan at the terminating stage is an "acceptable" one in some sense. This includes the 'degree of approximation sought', or 'tolerable margin of error' in the plan. Sometimes, after a few stages of the iteration, an extrapolation method is made use of to arrive at the solution (see chapter 2, pp. 35-36). For our purpose, we shall assume that C has some such termination criterion and make no explicit reference to it in our analysis.

We shall now end this section with an ordering of issues taken up in the following sections, each extending our analysis of the method of MB of chapter 2 in an appropriate direction. The ordering is in terms of the substantive scope of MB, not its formal structure, and resembles the ordering of issues in chapter 3. We begin with pure production coordination (section 2) and follow up by successively incorporating consumption (section 3), investment (section 4) and foreign trade (section 5) as separate and explicit elements in the overall coordination. The 'models' covered are of course drawn from all over the study. Section 2 includes the general non-linear model of chapter 3 and Models I and II of chapter 5, section 3 covers only the model of section 3 of chapter 2, section 4 the models dealt with in chapter 8 and section 5 consists of the model of section 5 of chapter 2 and Models III and IV (on import substitution) of chapter 5. Since the conceptual background for the method of MB is provided by the organisational structure and the 'initial information distribution', we shall begin each discussion clarifying these and then take up the specific form of the method.

Finally, let me note a point of caution : the same notation will used in different contexts for notational simplicity, e.g., matrices and C are to be interpreted differently in different contexts. The tations used in the preceding chapters are, however, maintained and case of confusion one is requested to refer back to the relevant pages.

.2 Pure production coordination

The 'model-content' of this section has already been specified. will be convenient to take up Model I of Chapter 5 first, followed Model II of the same chapter, and finally the general non-linear lel of Chapter 3.

The organisational structure for all the models of this section taken to be the same as in the basic method of chapter 2, i.e., it consists of final coordinating agency, C, and the n operative elements, $\{S_k\}$. The initial information distribution is taken to correspond exactly to the organisational structure in the sense that C is assumed to posses prior information only of the plan objectives and S_k of everything that relates exclusively to production in that sector (e.g., methods of production, capacities etc.). Since there is no explicit reference to consumption etc. in these models, the plan objectives are taken to correspond to final demands. Clearly, the formal expression of the information content of S_k will vary a lot from model to model, and we need only specify this to begin the discussion for each model. Regarding the specifics of the method of MB, the only departure from its benchmark of reference is in regard to the way a sector calculates its input requirements at each stage. The rest is exactly the same as before, for the formal task, we need only specify (a) the initial distribution of information and (b) the method of sectoral calculations at each stage. As in section 2 of chapter 2 we shall denote the input requirement vector of S_k at stage t by $d^k(t)$.

We now take up Model I of chapter 5 given by the equations (5.1.0)–(5.1.2) /p. 105 _7. For the purpose of the present analysis, however, it will be convenient to refer to its equivalent formulation as given in equations (5.1.6)-(5.1.8) /pp. 107-108 _7 and to denote x^1 there by mply x. The information content of the model is defined by $(A, B, \bar{x}, y)$. e process of MB begins with S_k possessing prior knowledge of a^k, b^k d $\bar{x}_k$, and C of y ; this is the initial distribution of information

The calculation of $d^k(t)$ is given by :

$$d^k(t) = \begin{cases} a^k x_k^t & \text{if } x_k^t \leq \bar{x}_k \\ a^k \bar{x}_k + b^k (x_k^t - \bar{x}_k) & \text{otherwise} \end{cases} \quad \ldots (10.3.0)$$

Passing on to Model II of chapter 5 [equations (5.1.9)-(5.1.11), (5.1.13) and (5.1.14); pp. 115-116], its information content is given by $(A, B, \underline{x}, \bar{x}, y)$. Regarding the initial distribution of information it is taken that each S_k has prior knowledge about a^k, b^k, $\underline{x}_k$ and $\bar{x}_k$, and C about y. It is to be recalled that there is room for excess production in this model. It is assumed here that any decision for excess production over the target is taken by the corresponding sector. In fact, the centre is not even informed about it. The calculation of $d^k(t)$ here is modified as :

$$d^k(t) = \begin{cases} a^k x_k^t & \text{if } x_k^t \leq \underline{x}_k \\ b^k \bar{x}_k & \text{if } \underline{x}_k < x_k^t \leq \bar{x}_k \\ b^k x_k^t & \text{otherwise} \end{cases}$$

We now come to the non-linear model of chapter 5 [equation (5.1.0); p. 63]. We shall begin with the assumption of no externalities, i.e., all arguments in the function $f_{ij}(x)$ are assumed irrelevant excepting x_j, for all i and j. Notationally, we express this by writing

$$f_{ij}(x) = f_{ij}(x_j) \quad \forall \; i \text{ and } j.$$

The information content of the model now is simply $(F(x), y)$ where elements of $F(x)$ satisfy the restriction specified. S_k has prior knowledge of $f^k(x_k)$, and C of y. The calculation of $d^k(t)$ is straightforward :

$$d^k(t) = x_k^t \quad f^k(x_k^t) \qquad \ldots (10.2.1)$$

So far, the 'rules of the game' are exactly the same as before. With externalities, this is no longer so, for a sector cannot then find its input requirements just on the basis of <u>its</u> production target. It has to have information about the general production plan. This means that C has to make a <u>general announcement</u> of its provisional plan at each stage; it cannot carry out separate dialogues with the sectors. If this is done — C informs each S_k of x^t, not just x_k^t — then one need only write (10.2.1) as

$$d^k(t) = x_k^t \quad f^k(x^t)$$

and the story is the same as before.

It is to be noted here that given certain 'specificities' in e matter of externalities, the lack of separation can in fact be ercome by appropriate change in the organisational structure. The ecificities referred require 'limited externalities' in the sense at a set of sectors generate external effects only for sectors longing to the set. If all externalities are of this type, then the duction system can be <u>partitioned</u> into a number of <u>sets of sectors</u> out any externalities between sets. C can then carry out separate logues with <u>one agency representing each set</u>. The organisational ucture is then defined by these agencies, not the sectors. Any such agency would receive the production targets of all its constituent sectors; and pass on the corresponding input requirements to C. It may be noted that with production actually carried out on a sectoral basis, these agencies can also be looked upon as subsidiary coordinating elements. Their tasks boil down to coordination of the corresponding constituent production sectors, and they become necessary elements in the organisational structure between $\{S_k\}$ and C if the element of separation is to be maintained.

We shall end this section with a point regarding the (additional) information content of the successive steps of the method of MB. The point was raised by Chander (1973). His point can be restated as follows: If the IO matrix is a constant one, say A, then C comes to possess effective information of the full IO matrix at the very first step of the iteration. This is evident from (10.2.1), since x^o (=y) is known to C. More explicitly, at the initial step, C gets to know a collection of n vectors, $\{d^k(o)\} = \{x^o_k \; a^k\}$ Since x^o is known, C obtains a^k by simply multiplying $d^k(o)$ by $(\frac{1}{x^o_k})$. Thus, the subsequent steps of the process do not add anything to the information possessed by C and in this sense, can be said to be <u>informationally superfluous</u>.

In the non-linear case however C never comes to possess the full information regarding production processes. In the end, S_k had to compute the value of $f^k(x)$ only for a finite number of discrete points which are implicitly passed on to C, and which in general are different from one another. Thus the process of exchange of <u>relevant</u> informations can be said to continue till the termination stage. There is thus a considerable gain for C in terms of its information-requirement for making a plan. In general, therefore, the process of MB described for the non-linear system saves the centre the trouble both of carrying out extensive computations and of acquiring the full technological details of unit production processes. In the standard case only the first, computational, advantage is present; not the second, informational, one.

10.3 Material balances with endogenous consumption

This section is based on the extended IO model of section 3 of chapter 2 [equations (2.3.3)-(2.3.5); pp. 39-40]. I have to begin by pointing out that the set of technical relations relating consumption, income and production in that model can be obtained on the basis of alternative institutional arrangements even in a planned economy. The method of MB will be dependent upon the nature of these arrangements. Below we describe two alternative arrangements.

In the first alternative, the sectors have to provide for consumption for the labour required in its production. The commodities on this account then form just a part of its own requirements. Clearly, in this case, no further extension is necessary regarding the structure of MB stated in chapter 2. One has only to treat $d^k(t)$ as a vector of commodity requirements in S_k at stage t, both as material inputs and as consumption for labour.

In the second arrangement, the task of a sector ends with wage payments. The actual consumption demands are then made by labourers on the basis of their income. This requires a separate agency for consumption planning which has to be reflected in the organisational structure. Clearly, this agency is a subsidiary coordinating element in the structure. We shall denote it by W_1. Informationally, it is assumed that W_1 possesses all the necessary knowledge for relating consumption to income. Its task then is basically to receive income estimates from sectors of production, translate these into consumption estimates, and forward these to the centre for incorporation in the overall production plan.

In operational terms, the information content of the planning model is now described by (A, A^o, C, v). W_1 has prior knowledge about the matrix C, each S_k about a^k and a^{ok}, and the centre about v.

At stage t of the process C informs each S_k of its provisional production target, x_k^t. Each S_k, in turn, reports its input requirements, $d^k(t)$, to C directly, and the vector of income generated, say $w^k(t)$, to W_1. These are given respectively by :

$$d^k(t) = x_k^t \; a^k \qquad (10.3.1)$$

$$w^k(t) = x_k^t \; a^{ok} \qquad (10.3.2)$$

W_1 adds up the $w^k(t)$'s to obtain the corresponding income vector w^t :

$$w^t = \sum_{t=1}^{n} w^k(t) = A^o x^t \qquad \ldots \quad (10.3.3)$$

It then finds the corresponding consumption requirements, u^t, by postmultiplying C by w^t, i.e.,

$$u^t = Cw^t \qquad (10.3.4)$$

This vector u^t then forms the message of W_1 to C at that stage. C now constructs a provisional plan for stage (t+1) by aggregating $d^k(t)$'s and adding u^t and v to the aggregated demand vector, i.e.,

$$x^{t+1} = \sum_{k=1}^{n} d^k(t) + u^t + v \qquad \ldots (10.3.5)$$

From (10.3.1)-(10.3.5) it is evident that

$$x^{t+1} = Ax^t + CA^o x^t + v$$

This establishes the formal equivalence of the IO method and the method of MB in the present case.

10.4 Material balances with endogenous investment and growth

The planning model here is defined by the equations (8.1.4)-(8.1.6) of chapter 8 (p. 157). The organisational structure here is taken to be the same as in the benchmark case and the information content of the model is given by $(A, B(r), \lambda_1, \ldots, \lambda_n, T, \bar{x}, d)$ The initial information distribution is assumed as follows : C has prior knowledge about d, and each S_k about λ_k, a^k, $b^k(r_k)$ and $\bar{x}_k$.[3/] T is assumed to be known by all the parties.

The material requirements of a sector will now consist of requirements both for production and investment. The former ($d^k(t)$) is calculated exactly as before. For the latter, the sector first computes growth rates as :

3/ From the construction of the matrix B(r) [see section 4 of chapter 2, pp. 41 - 47] it is clear that the kth column of it does not depend on any r_i, $i \neq k$ ($\forall$ k)

$$r_{kt} = \begin{cases} 0 & \text{if } x_k^t \leq \bar{x}_k \\ \left[\dfrac{x_k^t}{\bar{x}_k}\right]^{\frac{1}{T-\tau_1}} - 1 & \text{otherwise} \end{cases} \qquad \ldots (10.4.1)$$

and then the derived investment requirements, $I^k(t)$ as :

$$I^k(t) = b^k(r_{kt}) \; x_k^t \qquad \ldots (10.4.2)$$

The sum of these requirement vectors, i.e., $d^k(t) + I^k(t)$, is reported back to C [4/] who then constructs the provisional plan for the next stage by simply aggregating these vectors and adding d to it, i.e.,

$$x^{t+1} = \sum_{k=1}^{m} (d^k(t) + I^k(t)) + d \qquad \ldots (10.4.3)$$

From (10.4.1)-(10.4.3) it is clear that

$$x^{t+1} = Ax^t + B(r^t)x^t + d$$

Thus the formal similarity between the method of MB in this case and the method (8.1.7)-(8.1.12) /p. 158/ is clear-cut.

I shall now point out a modification of the method described above when the planning model is defined by (8.2.1)-(8.2.4) /p. 162/, that is, when an overall growth rate is stipulated from outside a... a flexibility of adjustment in the target demand vector d. The overall ...wth rate, ρ, is treated as a function of sectoral growth rates, $\rho = h(r)$, and the target demand vector is a function of a size-of target demands, say $d = f(v)$, where v is the size-index ex/variable. It is presumed here that C knows both the functions $h(r)$ $f(v)$. It is also taken that a v_0 satisfying (8.2.7) /p. 164/ and

---

It may be noted that one may consider a separate investment agency whose role would be to calculate the investment demands. Here implicitly the role of this agency is subsumed in the S_k's.

a λ satisfying A B.1 /p. 163_7 can be found independently by C.

Here, first of all, the process of MB described above is carried out with d replaced by $f(v_o)$. When C decides to terminate the process, it instructs S_k's to report their estimates of the rates of growth. C then calculates the scaler measure of these rates of growth and on the basis of this revises the value of v up from v_o and thus the target demands with the help of (8.3.6) /p. 164_7. It then re-initiates the above process with the provisional production target for any sector put equal to the estimate of its production level arrived at the end of the initial process. This continues till the solution of v is reached.

On the face of it, the method of MB may appear discontinuous in the sense that the procedure is started anew after each revision of v. This however is not quite so. Since the revision of target demands is done by C, the sectors simply continue to receive increasing targets for their production from C all along. To the sectors, the process works exactly in the same way as in other cases. The only difference is that instead of just aggregating material requirements with an unchanged bill of target demands, C actually revises the latter periodically in response to the different sets of estimates from the sectors.

10.5 Material balances with foreign trade

This section deals with foreign trade only in the sense of endogeneous imports estimated as part of the method of MB. Exports are taken to be exogeneus as part of the given 'final demand'. As stated earlier, this section covers the model of foreign trade of section 5 of chapter 2, and Models III and IV of chapter 5 on import substitution.

The method of MB for the first model [equations (2.5.1)-(2.5.4); p. 50] is dispensed with in a few words. As we had shown earlier, in purely formal terms the model is a special case of Model I of Chapter 5. So is the case with the method of MB. More explicitly, till the terminating stage everything here remains the same as in the method of MB described for Model I of chapter 5 in the section 2 above, only the vector b^k is taken to be identically equal to zero for all k. This would only affect the calculation of $d^k(t)$ which is now given by :

$$d^k(t) = \begin{cases} a^k x_k^t & \text{if } x_k^t \leqq \bar{x}_k \\ a^k \bar{x}_k & \text{otherwise} \end{cases}$$

At the terminating stage, however, C instructs each S_k to report back the shortfall of production from targets. These shortfalls, z_k^t say, make up the required import bill and are calculated by the S_k's from:

$$z_k^t = \begin{cases} x_k^t - \bar{x}_k^* & \text{if } x_k^t > \bar{x}_k \\ 0 & \text{otherwise} \end{cases}$$

It may be noted that by suitable adjustments of language one can specify the method with a separate import agency in charge of the import plan. As explained above, such an agency will have a role only at the terminal stage of the process.

We now move on to the Model III of chapter 5 [equations (5.2.1)-(5.2.3); p. 120]. The organisational structure here consists of C and (n+k) operative elements $\left\{ S_i, \ i = 1, \dots, n+k \right\}$. There are two groups of commodities, $\left\{G_i, \quad i = 1, \ldots, n\right\}$ and $\left\{G_i, \quad i = n+1, \ldots, n+k\right\}$ — to be called first group and second group of commodities respectively. The first group of commodities are always produced domestically — G_i by S_i —, and any commodity in the second group is either domestically produced (with a given lower bound on its production) — or imported. Correspondingly, the groups of sectors $\left\{S_i, \quad i = 1, \ldots, n\right\}$ and $\left\{S_i, \quad i = n+1, \ldots, n+k\right\}$ will be called first group and second group of sectors and denoted by N and K respectively. The information content of the model is given by $(A, B, C, D, \bar{u}, y, d)$, and the distribution of this information is as follows : each $S_i \in N$ knows about a^i and c^i, each $S_i \in K$ about b^i, d^i and $\bar{u}_i$, and C about y and d. It is also presumed that a vector $(x^{o\prime} : z^{o\prime})$ satisfying (6.3.3)-(6.3.4) [p. 135, chapter 6] can be found independently by C.

The process starts with C informing each S_i of its provisional production target which is x_i^o for $S_i \in N$ and z_i^o for $S_i \in K$. Each S_i then calculates the input requirement vector as :

$$d^{i\prime}(o) = \begin{cases} (a^{i\prime} : c^{i\prime})\, x_i^o & \text{if } S_i \in N \\ (b^{i\prime} : d^{i\prime})\, z_i^o & \text{if } S_i \in K \end{cases} \qquad \ldots (10.5.1)$$

and reports back to C this vector as its message. C then aggregates all these vectors and adds $\begin{bmatrix} y \\ d \end{bmatrix}$ to it, and this becomes C's message to the S_k's for the next stage. That is :

$$\begin{bmatrix} x^1 \\ \cdots \\ z^1 \end{bmatrix} = \sum_{i=1}^{n+k} d^i(0) + \begin{bmatrix} y \\ \cdots \\ d \end{bmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ C & \vdots & D \end{bmatrix} \begin{bmatrix} x^o \\ \cdots \\ z^o \end{bmatrix} + \begin{bmatrix} y \\ \cdots \\ d \end{bmatrix} \qquad \ldots (10.5.2)$$

Now, a $S_i \epsilon K$ may find that its production target is less than the corresponding lower bound on its production. In this case, S_i is assumed to withdraw from production and inform C of this decision. The rest report back to C their input requirements as before. C now faces two kinds of tasks. One is to find new production targets. This is confined to sectors which do not withdraw from production, and is done in the same way as before (viz., by adding final demand to aggregated input requirements for the relevant commodities). The other task of C is to find import requirements. Imports become necessary on account of the withdrawal of sectors from production. The amount of import of the corresponding commodity is given simply by the requirement of that commodity as final demand and as inputs in those sectors which still continue production. Obviously, C can perform both tasks on the basis of its prior informations and messages just received from sectors. Further, since production targets in the present case form a decreasing sequence, withdrawal of a sector at a stage does not signify any loss of information. Later production targets could not have justified its reentry anyway.

We now pass on to Model IV of chapter 5 [equations (5.2.4)-(5.2.5); pp. 122 - 123]. The organisational structure here is as just described. The information content of the model is given by $(A, B, C, D, y, d, f(.), g(.))$. Each $S_i \epsilon N$ possesses information about a^i and c^i, each $S_i \epsilon K$ about b^i and d^i, and C about y, d, f(.) and g(.). It is also taken that a $(x^{o\prime} : z^{o\prime})$ satisfying (6.4.4) [p. 137] is known to C.

There is a 'two-tier' plan formulation for this model. For the set of sectors, K, there is both a provisional production plan and a provisional import plan at each stage with corresponding alternative production targets for the set of sectors, N. Formally, C sends (a) two alternative production targets, $x_i^t(1)$, $x_i^t(2)$ for all $S_i \in N$; (b) one target, $z_i^t(2)$ for all $S_i \in K$, with the instruction to report back the input requirements for all the targets.5/

Each $S_i \in N$ then calculates two alternative vectors of requirements corresponding to the two targets as :

$$d^i(t, j) = \begin{bmatrix} a^i \\ \cdot\cdot \\ c^i \end{bmatrix} x_i^t(j) \qquad (10.5.3)$$

$$j = 1, 2; \quad i = 1, \ldots, n.$$

and each $S_i \in K$ calculates its requirements as :

$$d^i(t) = \begin{bmatrix} b^i \\ \cdot\cdot \\ d^i \end{bmatrix} z_i^t(2) \qquad \ldots (10.5.4)$$

$$i = n + 1, \ldots, n+k.$$

All these requirements are then reported back to C. On this basis, C makes two alternative provisional plans as follows :

$$\begin{bmatrix} x^{t+1}(1) \\ \ldots \\ z^{t+1}(1) \end{bmatrix} = \sum_{i=1}^{n} d^i(t, 1) + \begin{bmatrix} y \\ \ldots \\ d \end{bmatrix} \qquad \ldots (10.5.5)$$

$$\begin{bmatrix} x^{t+1}(2) \\ \ldots \\ z^{t+1}(2) \end{bmatrix} = \sum_{i=1}^{n} d^i(t,2) + \sum_{i=n+1}^{n+k} d^i(t) + \begin{bmatrix} y \\ \ldots \\ d \end{bmatrix} \qquad \ldots (10.5.6)$$

The process initiates with $x^o(1) = y$, $x^o(2) = x^o$ and $z^o(2) = z^o$

Two alternative cases are now possible ; $f(z^{t+1}(1)) < g(z^{t+1}(2))$ for all t (case A); and $f(z^{t+1}(1)) \geq g(z^{t+1}(2))$ at some $t = t_o$, say (case B). In case A, the process is continued as above. After termination, the terminal values of the first alternative, i.e., of equation (10.5.5), are accepted as the plan. This involves a domestic production plan $x^T(1)$ and an import plan $z^T(1)$ where T is the terminal step. There is no production of the second group of commodities. In case B on the other hand, the first alternative is discontinued at stage t_o, and the process boils down to the standard case of chapter 2. Only $x_i^t(2)$ is sent out a target to $S_i \in N$ for $t \geq t_o$, and C recomputes targets as per equation (10.5.6).

10.6 Concluding remarks

In bringing this study to a conclusion it is worth returning to the opening points. As stated at the very outset, the only common element behind our generalisations has been their starting point in IO analysis. As we have viewed it, a generalisation is essentially the further development of some central idea or basic theme at the starting point, and the process is essentially open ended. This also means that there is no prior fixity of the generalisations in any given conceptual frame of reference. The latter is relative to the substantive content of the generalisations, subject only to the requirement that it does not constrain or bind the expression of the central idea or basic theme sought to be developed. Stated in a somewhat different language, our concern has been with IO analysis, not IO models as such. For this reason, we have often called the models analysed in this study 'IO-type', rather than IO, models.

At the cost of some repetition, we may spell out the 'logic' of IO analysis as we have seen it in this study. At one level, the logic simply consists of working out the structure of production, and all that is entailed by this structure, from some components of total demand treated as autonomous elements. This does not imply any particular restriction on the formal structure of relations embodied in the 'model'. Our generalisations can be seen as providing some proof of the flexibility of the formal structure for this purpose. Each of the specific models developed in this study has its own particular structure varying significantly from one another. In formal terms, the range covered has varied from very general sort of nonlinearities to rather complex, though particular, structures of inequalities and constraints. In substantive terms, we have touched upon issues of IRS/DRS, technological alternatives and growth, all within the basic consistency frame of IO analysis.

We may return here briefly to the 'scope' of IO analysis, which provides the substantive underpinning to any model amenable to IO analysis. We have dealt at length with the scope of consistency at various parts of the study. As we have seen, this takes one beyond 'production' in the technological sense. IO consistency can be based on behavioural-institutional relations as much as on technological relations. The question of validity in this respect is ultimately one of stability of the relations on some objective basis.[6/] Once this is recognised, no

6/ Cf. the views expressed by Leontief in our quotation on page 55 (chapter 2).

special significance attaches to technological relations per se. On the contrary, the technological relations themselves can be the weakest links for some 'sectors of production' depending upon the level of its technological-institutional development in a broad sense. This represents precisely the openness of the 'proper' sector-coverage of IO analysis that we had earlier referred to (section 2.6, p. 54). Just as recognition of various aspects of consistency, or channels of interdependence deepens IO analysis so to say, so also, the recognition of weak or uncertain links emerging from some sectors restricts its width. The proper scope can be settled only by "hard theory".

So much for the 'model' aspect of our generalisations. As repeatedly pointed out, the basic unifying element of the study comes from its repeated reliance upon the IO method in some form or other for the detailed analysis of each model. The method in fact is an expression of the logic of IO analysis at a second level. In other words, the 'proof' of keeping to the logic of IO analysis in some particular context is seen to be the use of IO method in that context, with suitable modifications, if necessary. This sums up our position.

We conclude on a historical note. As our brief review of the origins of 'theoretical IO analysis' /pp. 2-3_7 shows, the period of intense developments in the field was early fifties (1949-56, to be precise, according to our references). This was also the period that brought the new fields of AA and LP into existence. Later, all these fields developed in an overlapping fashion, gradually fusing into a single area that is often called 'linear economic models'. The 'speciality' of IO analysis — its approach and logic — however got lost in this fusion. It is interesting to note in this connection that the two analytical contributions that lie most significantly behind this study date back to the period mentioned, viz., Arrow (1954) and Evans (1956). I end by recalling that Wood and Dantzig (1951) saw the IO method and LP in coordinate terms, each providing an operational technique for a large class of quantitative economic problems. The period since then has seen huge developments in the latter, hardly any in the former. Ours can be taken to be a contribution in what appears to be an almost abandoned area.

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ON A GENERAL CLASS OF UNBIASED RATIO, PRODUCT, RATIO-CUM-PRODUCT AND REGRESSION TYPE ESTIMATORS

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