# GENERALIZED INVERSES OF SPECIAL TYPES OF MATRICES

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A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the award of the degree of

Doctor of Philosophy

Calcutta

1974

## ACKNOWLEDGEMENTS

I thank the Research and Training School of the Indian Statistical Institute for the facilities granted to me in carrying out this work.

I am highly grateful to Professor C. R. Rao, F.R.S., for supervising this work. I record my sincere thanks to him for going through this thesis and offering valuable suggestions for its improvement.

I thank Professor J. Roy for the encouragement I received throughout the period of this work. I express my deep sense of gratitude to Dr. P. Bhimasankaram for his multidimensional help and advice.

I thank Dr. A. Ramachandra Rao and Dr. K.P.S.Bhaskara Rao for the valuable discussions I had with them. I thank Dr. K.P.S.Bhaskara Rao for his permission to include our joint results in this thesis.

Finally, I thank Mr. Arun Das for his elegant typing and . Apurba Guha for his efficient cyclostyling of this thesis.

P S S N V P.Rao

C O N T E N T S

# INTRODUCTION

Inverses, in the regular sense of the term, do not exist for singular square matrices and rectangular matrices. However for such matrices there exist matrices which satisfy many important properties similar to those of inverses of nonsingular matrices and for many purposes, can be used in the same way as regular inverses. These matrices are named generalized inverses (g-inverses) to distinguish them from the inverses of nonsingular matrices. Only since 1955 this field of study of generalized inverse was investigated systematically and was explored for many beautiful and interesting results and applications though the concept of generalized inverse was first introduced by Moore in as early as 1920 as follows :

Definition (Moore) : Let $A$ be a $m \times n$ matrix over the field of complex numbers. Then $G$ is the generalized inverse of $A$ if $AG$ is the orthogonal projection operator projecting arbitrary vectors onto the column space of $A$ and $GA$ is the orthogonal projection operator projecting arbitrary vectors onto the column space of $G$.

Moore studied this concept and its properties in some details in 1935.

In 1955, unaware of the earlier work of Moore, Penrose defined generalized inverse of a matrix as follows :

Definition (Penrose) : Let $A$ be $m \times n$ matrix over the field of complex numbers. Then $G$ is a generalized inverse of $A$ if (i) $AGA = A$,

(ii) $GAG = G$, (iii) $(AG)^* = AG$ and (iv) $(GA)^* = GA$.

In 1956 Penrose showed that this generalized inverse of a matrix is unique and discussed the properties and uses of this generalized inverse of a matrix in a systematic way.

In 1956 Rado establised that the definition due to Penrose is equivalent to that of Moore. This unique generalized inverse is called Moore-Penrose inverse of a matrix.

A similar notion was also used by Bolt and Duffin in 1953 under the name constrained inverse and by Aitken with a different symbolism in 1934.

Unaware of the earlier work of Moore and contemporary work of Penrose, Rao in 1955, constructed a pseudoinverse of a singular matrix which does not satisfy all the conditions of Moore-Penrose inverse and showed that it serves the same purpose as regular inverses of a nonsingular matrix in solving normal equations and also in computing standard errors of least squares estimators. In 1962 Rao defined a generalized inverse, formally, as follows, discussed its properties in greater details and its application to the problems of Mathematical Statistics.

Definition (Rao) Let $A$ be an $m \times n$ matrix. Then an $n \times m$ matrix $G$ is a g-inverse of $A$ if $x = Gy$ is a solution of the linear system $Ax = y$ whenever it is consistent.

Rao showed that the above definition is equivalent to the following definition which is also due to him.

Definition (Rao) : Let $A$ be an $m \times n$ matrix. Then an $n \times m$ matrix $G$ is a g-inverse of $A$ if $AGA = A$.

A g-inverse of a matrix as defined by Rao, in general, is not unique and thus opens an interesting study of matrix algebra. It can be easily observed from the definitions of Penrose and Rao that the Moore-Penrose inverse is contained in the class of all g-inverses (in the sense of Rao).

In two later publications in 1965 and 1966 Rao showed that in many practical applications it is sufficient to work with g-inverse satisfying this more general definition (due to Rao). In 1967 Rao developed a alculus of generalized inverse of matrices, studied many of its important and interesting properties, classified the g-inverse based on their ses, discussed their interrelationships and their further applications Mathematical Statistics.

Following this work of Rao, Mitra in 1968, gave an equivalent finition, introduced some new class of g-inverses and suggested some rther applications to solution of matrix equations and Mathematical atistics. During 1968-1973, Rao and Mitra pursued their research generalized inverse of matrices and its applications to various entific disciplines in a series of papers and a book.

Some other principal contributors to the theory and applications generalized inverses of matrices since 1955 are Greville, Ben Israel, lyi, Odell, Bose and Khatri, Cline, Pyle, Decel, Golub, Rohde - mention only a few. References to important contributions made by these people and others will be found in the book by Rao and Mitra (1971).

Over the past two decades many interesting and important results on generalized inverses of matrices over complex field have been developed. However, till recently, not much work has been done in this regard for matrices over algebras which are not fields. Many fundamental properties of these matrices deviate from those of matrices over fields. Rao [11] explored this new field of study and suggested some applications to graph theory and network analysis. In this thesis Boolean matrices and nonnegative matrices are studied systematically with regard to the theory and computation of generalized inverses. Some other principal contributors to this part of the field are Plemmons and Cline.

Each chapter of this thesis has a detailed introduction to it. Here we just mention briefly the problems considered.

Chapter 1 mainly deals with (0, 1) Boolean matrices. Some interesting properties of Boolean matrices are proved comparing each time with the corresponding properties of real matrices. Necessary and sufficient conditions for the existence of various types of g-inverses are established along with an algorithm to compute these g-inverses. In the last section of this chapter, many results of (0, 1) Boolean matrices are extended to matrices over any arbitrary Boolean algebra.

In chapter 2, to start with, doubly stochastic matrices possessing doubly stochastic g-inverses are characterised. Then necessary and sufficient conditions for the existence of various types of nonnegative g-inverses of nonnegative matrices are proved. Also an algorithm to compute these g-inverses is given.

In chapter 3, two characterisations of merely positive subdefinite (MPSubD) matrices are proved and also a necessary and sufficient condition for an MPSubD matrix to possess an MPSubD g-inverse is given.

In chapter 4, a result of Milliken on linear estimability is extended in the first section. It is shown that his result holds in more general set up than the one considered by him. Finally, an algorithm to compute g-inverse of a matrix is given which is an extension of Goldfarb's "modified method for inverting nonsingular matrices".

The following notations are used in this thesis. Matrices are denoted by capital letters A, B, A etc., and vectors by lower case letters. I denotes the identity matrix and $e_i$ denotes the $i^{th}$ column of I. (x, y) denotes the usual Euclidean inner product of vectors x and y, i.e, y*x. $||x||$ denotes the Euclidean norm of the vector x. The symbol $\forall$ and $\in$ denote "for all" and "belongs to" respectively. $E^n$ denotes the n-dimensional unitary space. Let $A = (a_{ij})$ be an m x n matrix. Some functions of A and the symbols used are described in the following Table 1.

Table 1 (Rao and Mitra, 1971)

| Function | Symbol | Description |
|---|---|---|
| Transpose | A' | matrix with $(i,j)^{th}$ element = $a_{ji}$ |
| Conjugate transpose | A* | matrix with $(i,j)^{th}$ element = $\bar{a}_{ji}$ |
| Rank | R(A) | the number of independent columns or rows of A |
| Trace | tr A | $\Sigma a_{ii}$ |
| Column space | $\mathcal{M}(A)$ | vector space generated by columns of A |
| Orthogonal space | $\mathcal{O}(A)$ | set of all vectors x such that A'x=0 |
| Null space | $\mathcal{N}(A)$ | set of all vectors x such that Ax=0 |

Definitions of special matrices are given in Table 2.

Table 2 (Rao and Mitra, 1971)

| Type of matrix | Definition |
|---|---|
| Symmetric | A = A' |
| Hermitian | A = A* |
| Idempotent | $A^2 = A$ |
| Positive definite (p.d.) | $x^*Ax > 0$ $\forall$ nonnull x |
| Positive semi-definite (p.s.d.) | $x^*Ax \geq 0$ $\forall$ x<br>and $x^*Ax = 0$ for some nonnull x |
| Nonnegative definite (n.n.d.) | $x^*Ax \geq 0$ $\forall$ x |
| Normal | AA* = A*A |
| Orthogonal | AA' = A'A = I |
| Unitary | AA* = A*A = I |

A classification of basic types of g-inverses is given in Table 3.

Table 3 (Rao and Mitra, 1971)

| Notation | Equivalent conditions | Name of G |
|---|---|---|
| $A_L^{-1}$ | GA = I | left inverse |
| $A_R^{-1}$ | AG = I | right inverse |
| $A^-$ | AGA = A | g-inverse |
| $A_r^-$ | AGA = A; GAG = G | reflexive g-inverse |
| $A_m^-$ | AGA = A; GA = (GA)' | minimum norm g-inverse |
| $A_\ell^-$ | AGA = A; AG = (AG)' | least squares g-inverse |
| $A^+$ | AGA = A; GAG = G<br>AG = (AG)'; GA = (GA)' | Moore-Penrose inverse |

CHAPTER

# g-INVERSES OF BOOLEAN MATRICES

## 1.1 Introduction and Summary

The matrices over Boolean algebra $B$ require a completely separate treatment from that of matrices over the complex field $C$ or real field $R$ owing to the fact that these matrices do not satisfy many of the fundamental properties of matrices over $C$ or $R$. For instance a matrix over $B$ need not always possess a g-inverse in sharp contrast to the complex (real) case where every matrix has a g-inverse. Also for Boolean matrices row rank (as defined in section 1.3) and column rank need not be equal which is again a fundamental result for matrices over $C$ and $R$. Another contradiction to our basic concept for real and complex fields is, a set of n, k-tuples $(n > k)$ over $B$ may have more than $k$ independent vectors. Due to these interesting deviations from the general complex field, a few of the usual definitions for the complex (real) field need modifications to be meaningful for Boolean algebra, as for example independence of vectors, row rank and column rank of matrices etc. However the product of two matrices $A$ and $B$ of order $m \times r$ and $n \times r$ respectively and the sum of two matrices of the same order can be defined as in the case of matrices over $R$ $(C)$. Definitions which need modifications have been stated in section 1.3. Hereafter, unless otherwise stated, whenever we say a matrix $A$ we mean a matrix $A$ over $B$.

In this chapter we have considered gneral rectangular matrices. An algorithm to compute a g-inverse is also included in this chapter. These g-inverses of Boolean matrices have many applications in graph theory and network analysis.

In section 1.2 we define a few necessary and related concepts and terms and a few preliminary results are proved.

In section 1.3 through section 1.6 the underlying Boolean algebra considered is {0, 1} Boolean algebra. In section 1.3 we characterise the class of square matrices possessing an inverse. In this section a general decomposition theorem has been proved which leads to the characterisation of matrices possessing g-inverses. Later in this section it is shown that if a matrix possesses a g-inverse then one can choose a g-inverse of particular simple form, to be precise permutation matrix as defined in section 1.3. A few results on idempotent matrices are also establised.

In section 1.4, we define space decomposition of a matrix, similar to the rank factorisation of matrices over $R$ and $C$. However it is worthnoting that unlike rank factorisation, every Boolean matrix is not space decomposable. Like rank factorisation if a space decompositon of a Boolean matrix exists it is not unique. It was shown in this section that a matrix possesses a g-inverse if and only if it has a space decomposition. Later this space decomposition is used to characterise reflexive g-inverse of matrices.

In section 1.5, necessary and sufficient conditions for existence of Moore-Penrose inverse and other types of g-inverses of matrices are established.

In section 1.6, an algorithm is given which gives a g-inverse of Boolean matrix A, if A has any g-inverse. Table 4 can be used to check the existence of other types of g-inverses and to compute them.

Finally in section 1.7, the last section of the chapter, the main results of the previous sections are generalized for matrices over an arbitrary Boolean algebra.

## 1.2 Preliminaries

Let $\mathcal{B} = (\mathcal{B}, +, ., -, 0, 1)$ be a Boolean algebra with + acting as supremum, . acting as infimum, - acting as compliment, 0 acting as the zero element and 1 acting as the unit element (Halmos [3 ]). We supress the dot of a.b and simply write ab, for infimum of a and b. By a matrix over $\mathcal{B}$ we mean a matrix whose elements belong to $\mathcal{B}$. Since the order of the matrix is clear from the context, most of the time we supress the order of the matrix. Matrix addition and matrix multiplication are same as in the case of real matrices but for the concerned sums and products of elements are Boolean.

If $x_1, \ldots, x_n$ are vectors (m-tuples) over $\mathcal{B}$, the linear manifold generated by them is the set of all vectors of the form $\sum_{i=1}^{n} c_i x_i$ where $c_i \in \mathcal{B}$ and is denoted by $M(x_1 \ldots x_n)$. Other concepts such as transpose, symmetricity, idempotency, $\mathcal{M}(A)$ etc. are same as in the case of real matrices.

Definition : Let A be an $m \times n$ matrix. Then an $n \times m$ matrix G is said to be a generalized inverse (g-inverse) of A, denoted by $A^-$ if $AGA = A$.

Proposition 1 : Let A be an $m \times n$ matrix and G be an $n \times m$ matrix. Then the following statements are equivalent

(a) $AGA = A$ .

(b) Gy is a solution of the system of linear equations $Ax = y$ whenever solution exists i.e., whenever $y \in \mathcal{M}(A)$

(c) AG is idempotent and $\mathcal{M}(A) = \mathcal{M}(AG)$

(d) GA is idempotent and $\mathcal{M}(A') = \mathcal{M}(G'A')$

Proof is in the same lines as in the real case and hence we omit.

If G is a particular type of g-inverse of A, say $A_t^-$ then $Q'GP'$ is a $(PAQ)_t^-$ where P and Q are permutation matrices.

Observing the fact that if $G_1$ and $G_2$ are two g-inverses of an $m \times n$ matrix A then $(G_1 + G_2)$ is also a g-inverse of A and that number of $n \times m$ matrices is finite we define a maximum g-inverse of A.

<u>Definition</u> : A g-inverse $G$ of $A$ is said to be a maximum g-inverse of $A$ if every $A^- \leq G$.

As an immediate consequence we see that any matrix having a g-inverse has a maximum g-inverse.

## 1.3 <u>g-inverses of matrices over {0, 1} Boolean algebra</u>

<u>Definition</u> : The {0, 1} Boolean algebra denoted by $B_o$ is the set {0, 1} together with the operations +, . and - defined as follows

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| . | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$\bar{0} = 1$ and $\bar{1} = 0$

From now onwards upto section 5 we consider vectors and matrices over $B_o$ only. We say $A \leq B$ if $a_{ij} = 1 \Rightarrow b_{ij} = 1$ for all $i$ and $j$.

<u>Definition</u> : The weight of a vector $x$, denoted by $w(x)$, is the number of non-zero elements of $x$.

<u>Definition</u> : A set of vectors $\{x_1, \ldots, x_n\}$ is said to be independent if no vector is the sum of some of the remaining vectors and null vector (all elements zero) is not in the set.

<u>Definition</u> : A vector $y$ is said to be dependent on vectors $x_1, \ldots, x_n$ if $y \in M(x_1, \ldots, x_n)$. Otherwise $y$ is said to be independent of $x_1, \ldots, x_n$.

**Definition** : Let $T$ be a set of vectors. Then a set $S = \{x_1, \ldots, x_n\} \subset T$ is said to be a basis of $T$ if $S$ is independent and $M(x_1, \ldots, x_n) \supset T$.

**Proposition 2** Every set of vectors $T$, which has at least one nonnull vector has a unique basis.

**Proof** : Let $x_1 \in T$ be a minimum weight nonnull vector. Consider $T_1 = T - M(x_1)$. If $T_1$ is nonempty take $x_2 \in T_1$, a minimum weight nonnull vector and consider, $T_2 = T - M(x_1, x_2)$. Proceed likewise until for some k, $T_k = T - M(x_1, \ldots, x_k)$ is empty. Clearly $x_1, x_2, \ldots, x_k$ are independent and $T \subset M(x_1 \ldots x_k)$. Hence the set $\{x_1 \ldots x_k\}$ form a basis of $T$ and by construction $x_1, x_2, \ldots, x_k$ should be in any basis and hence it is unique.

The above proposition leads to the following definition.

**Definition** : Rank of a set of vectors $T$, denoted by $R(T)$ is the cardinality of its basis.

**Remark 1** : This rank does not satisfy the usual properties of dimension in real vector spaces, for instance $R(T_1)$ may be greater than $R(T_2)$ even though $M(T_1) \subset M(T_2)$.

Consider $T_1 = \begin{Bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{Bmatrix}$ and $T_2 = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$

re $R(T_1) = 4$ and $R(T_2) = 3$ but $M(T_1) \subset M(T_2)$

<u>Definition</u> : Row (column) rank of a matrix A is the rank of its row (column) vectors.

<u>Remark 2</u> : Row rank and column rank of a matrix A need not be equal.

Consider $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Here row rank of A is 3 and column rank is 4.

<u>Definition</u> : A matrix A is said to be of rank $r$ if row rank of A = column rank of A = $r$.

<u>Definition</u> : A matrix A of order $m \times n$ is said to be nonsingular if row rank of A is $m$ and column rank of A is $n$.

<u>Definition</u> : A permutation matrix is a square matrix such that every row and every column contain exactly one 1.

<u>Definition</u> : A matrix (need not be square) is said to be a partial permutation matrix if every row and every column of it contain atmost one 1.

<u>Remark 3</u> : Row rank and column rank of a matrix are unaltered by premultiplying and post-multiplying by permutation matrices.

Remark 4 : If $P$ is a permutation matrix then $PP' = P'P = I$ and if $Q$ is a partial permutation matrix then $Q'Q \leq I$ and $QQ' \leq I$.

Definition : A square matrix $B$ is said to be an inverse of a square matrix $A$ if $AB = BA = I$.

Remark 5 : If an inverse exists it is unique.

Before proceeding further to investigate the existence and properties of g-inverses of matrices in general let us find out the conditions under which a square matrix will possess the inverse.

Theorem 1: An $n \times n$ matrix $A$ has inverse if and only if it is a permutation matrix.

Proof : 'If' part follows trivially since $AA' = A'A = I$.
'Only if' part : Let $x_i$ be the $i^{th}$ column vector of $A$ and $B$ be the inverse of $A$. Then

$AB = I \Rightarrow b_{1i}\, x_1 + b_{2i}\, x_2 + \dots + b_{ni}\, x_n = e_i$, for $i = 1,\dots,n$

$\Rightarrow e_i = x_j$ for some $j$, since $e_i$ cannot be the sum of two distinct nonnull vectors. $(i = 1,2,\dots,n)$

$\Rightarrow$ column vectors of $A$ contain $e_1,\dots,e_n$

$\Rightarrow$ $A$ is a permutation matrix since $A$ has only $n$ columns.

The following decomposition theorem is a fundamental result of our study of g-inverses of Boolean matrices.

<u>Theorem 2</u>: (Decomposition Theorem): Let $A$ be an $m \times n$ matrix with row rank $r$ and column rank $c$. Then there exist permutation matrices $P$ and $Q$ and matrices $C$ and $D$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

here $A_1$ is a nonsingular matrix of order $r \times c$.

To prove this theorem we need the following lemma :

<u>mma 1</u> : Column (row) rank of $A$ is unaltered even if a dependent (column) is deleted.

Proof of the lemma is easy and hence we omit.

<u>oof of Theorem 2</u> : Since the column rank of $A$ is $c$ the basis of ) contains $c$ column vectors of $A$. Let $Q$ be a permutation rix such that the first $c$ columns of $AQ$ form the basis of $\mathcal{M}(A)$, i.e.,

$AQ = (B : BC)$ for some $C$, where $B$ is an $m \times c$ matrix with l column rank. By the above lemma, row rank of $B$ = row rank of $A = r$. there exists a permutation matrix $P$ such that first $r$ rows of $PB$ form basis of $\mathcal{M}(B')$, i.e., $PB = \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}$ for some $D$ where $A_1$

is a $r \times c$ matrix with full row rank. Again by the above lemma, column rank of $A_1$ = column rank of $B = c$.

Therefore
$$PAQ = P(B : BC)$$
$$= (PB : PBC)$$
$$= \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} \text{ for some } C \text{ and } D,$$

ıere $A_1$ is a $r \times c$ nonsingular matrix.

**emark 6** : Observe that in the above decomposition

$$P' \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix} \text{ is a column basis of } A \text{ and } (A_1 : A_1C)\, Q'$$

a row basis of $A$.

**orem 3** : Let $PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$ where $A_1$ is nonsingular and P, are permutation matrices. Then the following statements are equivalent.

(a) $A^-$ exists

(b) $\begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}^-$ exists

(c) $(A_1 : A_1C)^-$ exists

(d) $A_1^-$ exists.

<u>Proof</u> : Given a g-inverse of any of the above four matrices, instead of ust showing the existence of g-inverses of the rest we will construct inverses for the rest. Proofs are by straightforward verifications.

(a) => (b), (c) and (d)

$A^-$ exists => $(PAQ)^-$ exists, say, $\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$ is a $(PAQ)^-$

$(G_1 + CG_3 : G_2 + CG_4)$ is a $\begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}^-$, $\begin{bmatrix} G_1 + G_2D \\ G_3 + G_4D \end{bmatrix}$ is a $(A_1 : A_1C)^-$

$(G_1 + CG_3 + G_2D + CG_4D)$ is a $A_1^-$.

(b) => (c), (d) and (a)

Let $(G_1 : G_2)$ be a $\begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}^-$, then $\begin{bmatrix} G_1 + G_2D \\ 0 \end{bmatrix}$ is a $(A_1 : A_1C)^-$, $G_1 + G_2D$ is a $A_1^-$ and $Q' \begin{bmatrix} G_1 & G_2 \\ 0 & 0 \end{bmatrix} P'$ is a $A^-$.

(c) => (d), (a) and (b)

Let $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ be a $(A_1 : A_1C)^-$. Then $(G_1 + G_2D)$ is a $A_1^-$,

$$Q' \begin{bmatrix} G_1 & 0 \\ G_2 & 0 \end{bmatrix} P' \text{ is a } A^- \text{ and } (G_1 + G_2D : 0) \text{ is a } \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}^-$$

(d) => (a), (b) and (c)

Let $G_1$ be a $A_1^-$ then $Q' \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} P'$ is a $A^-$,

$(G_1 : 0)$ is a $\begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}^-$ and $\begin{bmatrix} G_1 \\ 0 \end{bmatrix}$ is a $(A_1 : A_1C)^-$.

<u>Corollary</u> : Let $M(A) = M(B)$ then $A^-$ exists if and only if $B^-$ exists.

<u>Remark 7</u> : $\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$ is a $\begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}^-$

if and only if $(G_1 + CG_3 + G_2D + CG_4D)$ is a $A_1^-$

Hence the existence of a g-inverse of a matrix reduces to the problem of existence of g-inverses of non-singular matrices.

roposition 4 : Let a g-inverse of an $m \times n$ matrix $A$ exist. Then

(i) column rank of $A$ is $n$ implies $n \leq m$

(ii) row rank of $A$ is $m$ implies $m \leq n$.

roof : (i) Let $G$ be a g-inverse of $A$. Then $M(A) = M(AG)$, by roposition 1. This implies all independent columns, i.e., all the columns f $A$ are available among the columns of $AG$. But order of $AG$ is $\times m \Rightarrow n \leq m$.

Proof of (ii) follows in the same line.

ark 8 : Let $A$ be a nonsingular matrix then g-inverse exists $A$ is square.

ollary $A^-$ exists implies row rank of $A$ = column rank of $A$.

oof : Let $A$ be an $m \times n$ matrix of row rank $r$ and column k $c$. By the Decomposition Theorem there exist permutation matrices and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

re $A_1$ is an $r \times c$ nonsingular matrix.

by Theorem 3, $A^-$ exists $\Rightarrow A_1^-$ exists $\Rightarrow A_1$ is square, by rk 8.

$r = c$, i.e., row rank of $A$ = column rank of $A$.

**ark 9:** Every matrix need not possess a g-inverse.

**position 5 :** Let $A$ be an $m \times n$ matrix such that $A^-$ exists. Then

(i) $A$ is of full row rank implies $AG_1 = AG_2$ for all g-inverses $G_1$ and $G_2$ of $A$.

(ii) $A$ is of full column rank implies $G_1A = G_2A$ for all g-inverses $G_1$ and $G_2$ of $A$.

**f :** (i) Let $G_1$ and $G_2$ be any two g-inverses of the full row matrix $A$ By above corollary, the column rank of $A$ is $m$. $G$ be the maximum g-inverse of $A$ Then $G_1 \leq G$ and hence $\leq AG$. But $M(AG) = M(A) = M(AG_1)$ by Proposition 2, which ies that the set of column vectors of $AG_1$ is same as the set of mns of $AG$, which is again same as the column basis of $A$. together with $AG_1 \leq AG \Rightarrow AG_1 = AG$.

Hence $AG_1 = AG_2$.

Proof of (ii) follows from the above proof by taking $A'$ for $A$ serving that $A^-$ exists $\Leftrightarrow$ $(A')^-$ exists.

**lary:** If $A$ is a nonsingular matrix such that $A^-$ exists then a unique reflexive g-inverse.

By Proposition 5, for any two g-inverses $G_1$ and $G_2$, $AG_2$ and $G_1A = G_2A$.

If $G_1$ and $G_2$ are any two reflexive g-inverse of $A$ then

$$G_1 = G_1AG_1 = G_1AG_2 = G_2AG_2 = G_2$$

<u>Remark 10</u> : However, the converse of this, i.e., 'if a matrix has a unique reflexive g-inverse then it is nonsingular' is not true.

Consider $A = \begin{bmatrix} I & C \\ C' & C'C \end{bmatrix}$ where $I$ is of order $3 \times 3$

and $C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

By Remark 7, it follows that $A$ has a unique g-inverse and it is

$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ => A has a unique reflexive g-inverse $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

<u>Remark 11</u>: In case of real matrices, if the class of all g-inverses of $A$ and $B$ are same then $A = B$. But note that in case of Boolean matrices this result is not true even if $A^-$ and $B^-$ exists.

<u>Example</u>: Choose $A$ as in Remark 10 and let

$B = \begin{bmatrix} I & D \\ D' & D'D \end{bmatrix}$ where $I$ is of order $3 \times 3$

and $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Then again by Remark 7, B has a unique g-inverse and it is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
Hence class of all g-inverses of A and B are same though A is clearly not equal to B.

<u>Theorem 4</u> : Let A be a nonsingular matrix such that $A^-$ exists. Then there exists a unique permutation matrix P which is a g-inverse of A.

<u>Proof</u> : A is nonsingular and $A^-$ exists => A is square, by Remark 8. Let G be a g-inverse of A. Then $M(A) = M(AG)$ and AG is idempotent. Since A is nonsingular, columns of AG are nothing but a permutation of columns of A, i.e., there exists a permutation matrix P such that $AP = AG \Rightarrow APA = AGA = A \Rightarrow P$ is a g-inverse of A.

To show uniqueness of P, if possible let P and Q be two permutation matrices which are g-inverses of A. Then by Proposition 5, $AP_1 = AP_2$. Since columns of A are distinct because A is nonsingular, $P_1 = P_2$.

<u>Corollary</u> Let A be an m × n matrix such that $A^-$ exists. Then there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

where $A_1^2 = A_1$ and $A_1$ is nonsingular.

Proof : By Decomposition Theorem there exist permutation matrices $P$ and $Q_1$ such that

$$PAQ_1 = \begin{bmatrix} A_2 & A_2E \\ FA_2 & FA_2E \end{bmatrix} \qquad \ldots \quad (4)$$

where $A_2$ is nonsingular. Since $A$ has a g-inverse, by Theorem 3 $A_2$ has a g-inverse and by Theorem 4 there exists a permutation matrix $Q_2$ which is a g-inverse of $A_2$. Postmultiplying both sides of (4) by $\begin{bmatrix} Q_2 & 0 \\ 0 & I \end{bmatrix}$ and calling $Q_1\begin{bmatrix} Q_2 & 0 \\ 0 & I \end{bmatrix}$ as $Q$ we have

$$PAQ = \begin{bmatrix} A_2Q_2 & A_2Q_2Q_2' E \\ FA_2Q_2 & FA_2Q_2 Q_2'E \end{bmatrix} = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

where $A_1 = A_2Q_2$ is idempotent, $C = Q_2'E$ and $D = F$. Note that $Q$ is a permutation matrix.

The following is a generalisation of Theorem 4 to any matrix which has a g-inverse.

Theorem 5 : Let $A$ be an $m \times n$ matrix which has a g-inverse. Then there exists a partial permutation matrix which is a g-inverse.

Proof : By above corollary, there exist permutation matrices $P$ and $Q_1$ such that

$$P_1 A Q_1 = \begin{bmatrix} A_1 & A_1 C \\ DA_1 & DA_1 C \end{bmatrix} \quad \text{where } A_1^2 = A_1 .$$

Then partial permutation matrix $P = Q_1' \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P_1'$ is a g-inverse of $A$.

Remark 12 : This partial permutation matrix $P$ need not be unique. Before proceeding to the next section a few results on idempotent matrices are proved below.

Lemma 2 : Let $A$ be an idempotent matrix. Then $a_{ii} = 0$ implies $i^{th}$ column of A depends on the rest of the columns and $i^{th}$ row depends on the rest of the rows and also the matrix obtained by removing $i^{th}$ row and $i^{th}$ column of $A$, is idempotent.

Proof : Without loss of generality let $a_{nn} = 0$ and let

$$A = \begin{bmatrix} A_1 & \alpha \\ \beta' & 0 \end{bmatrix}$$

$$A^2 = A \Rightarrow \begin{cases} A_1^2 + \alpha\beta' = A_1 \\ A_1\alpha = \alpha \\ \beta'\alpha = 0 \\ \beta' A_1 = \beta' \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ \beta' \end{bmatrix} \alpha$$

and $(\beta' : 0) = \beta'(A_1 : \alpha)$

$A_1 = A_1^2 + \alpha\beta' = A_1^2 + A_1\alpha\beta' A_1 = A_1(I + \alpha\beta')A_1$

$\Rightarrow$ $A_1 + \alpha\beta'$ is idempotent

$\Rightarrow$ $A_1 + \alpha\beta' = A_1^2 + \alpha\beta' = A_1$

Therefore $A_1$ is idempotent.

<u>Corollary</u> : If $A$ is idempotent and nonsingular then $a_{ii} = 1$ for all $i$. However, $A$ is idempotent and $a_{ii} = 1$ for all $i$ do not imply $A$ is nonsingular.

<u>Lemma 3</u> : $i^{th}$ row of $A$ depends on the other rows implies $i^{th}$ column depends on the other columns where $A$ is idempotent and $a_{ii} = 1 \ \forall i$.

<u>Proof</u> : Without loss of generality let the last row depend on other rows

and let $A = \begin{bmatrix} A_1 & \alpha \\ \beta' & 1 \end{bmatrix}$

$A^2 = A \Rightarrow$ (i) $A_1^2 + \alpha\beta' = A_1 \Rightarrow A_1^2 \leq A_1$. But $A_1 \geq I \Rightarrow A_1^2 \geq A_1$

So $A_1^2 = A_1$.

(ii) $A_1\alpha + \alpha = \alpha \Rightarrow A_1\alpha \leq \alpha \Rightarrow A_1\alpha = \alpha$

(iii) $\beta' A_1 + \beta' = \beta' \Rightarrow \beta' A_1 \leq \beta' \Rightarrow \beta' A_1 = \beta'$

and (iv) $\beta'\alpha + 1 = 1$.

$(\beta' : 1)$ depends on the other rows implies that there exist $C$ such that $(\beta' : 1) = C'(A_1 : \alpha)$

ow $\beta' x = C' A_1 x = C' \alpha = 1$

$\Rightarrow \begin{bmatrix} \alpha \\ 1 \end{bmatrix} = \begin{bmatrix} A_1 \\ \beta' \end{bmatrix} \alpha$

the last column depends on the other columns.

w we can state the following theorem, proof of which follows from the ove two lemmas.

orem 6 : $A$ is an idempotent matrix of rank $r$ if and only if e exists a permutation matrix $P$ such that

$$PAP' = \begin{bmatrix} A_1 & A_1 C \\ DA_1 & DA_1 C \end{bmatrix}$$

e $A_1$ is $r \times r$ nonsingular idempotent matrix and $C$ and $D$ uch that $CD \leq A_1$.

lary 1 : If $A$ is nonsingular, idempotent and symmetric then

Let $A$ be an $n \times n$ matrix. If possible let $a_{ij} = 1$ for i and j such that $i \neq j$. Then there exists a $k$ such that $a_{jk}$, otherwise $i^{th}$ and $j^{th}$ rows are identical which contradicts singularity of $A$. Since $A$ is symmetric without loss of ity we assume that $a_{ik} = 0$ and $a_{jk} = 1$.

Now $A^2 = A \Rightarrow a_{ik} = \sum_{r=1}^{n} a_{ir}\, a_{rk} = 0$

$\Rightarrow a_{ir} . a_{rk} = 0$ for $r = 1,2,\ldots,n$

$\Rightarrow a_{ij} . a_{jk} = 0$

$\Rightarrow a_{jk} = 0,$

which is a contradiction and hence $a_{ij} = 0$ for $i \neq j$. A is nonsingular and idempotent implies $a_{ii} = 1$ for $i = 1,\ldots n$ from corollary of Lemma 2.

So $A = I$.

**Remark 13**: However if A is not symmetric the above result (corollary 1) is not true, for instance, consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

This is nonsingular and idempotent but clearly is not symmetric.

**Corollary 2**: A is an idempotent, symmetric matrix of rank r if and only if there exists a permutation matrix P such that

$$PAP' = \begin{bmatrix} I & C \\ C' & C'C \end{bmatrix}$$

here C is such that $CC' \leq I$.

roof follows from the above theorem and corollary.

**eorem 7** : Let A be an $m \times n$ matrix such that $a_{ii} = 1$ for all i. G be any g-inverse of A then $g_{ij} \leq a_{ij}$ for $i,j = 1,2,\ldots,r$, ere r is the minimum of m and n.

roof: Let $a_{ij} = 0$ for $i, j \leq r$ then

$$A = AGA \Rightarrow a_{ij} = \sum_{k=1}^{n} \sum_{s=1}^{m} a_{ik}\, g_{ks}\, a_{sj} = 0 \quad \text{for } i,j \leq r$$

$$\Rightarrow a_{ik} \cdot g_{ks} \cdot a_{sj} = 0 \quad \text{for } k = 1,\dots,n$$

$$s = 1,\dots,m$$

$$\text{and } i, j \leq r$$

$$\Rightarrow a_{ii}\, g_{ij}\, a_{jj} = 0 \quad \text{for } i, j \leq r$$

$$\Rightarrow g_{ij} = 0 \quad \text{for } i, j \leq r.$$

$g_{ij} \leq a_{ij}$ for $i, j \leq r$.

4 : In particular, if A is nonsingular and idempotent then g-inverse of A is A itself.

15: If A is an idempotent and nonsingular matrix then G inverse of A if and only if $I \leq G \leq A$.

ce decomposition and reflexive g-inverse

define space decomposition of Boolean matrices similar to rank factorisation for real matrices.

on : A matrix A of order $m \times n$ is said to be space decomposable, me k there exist two matrices L and R of orders $m \times k$ n respectively such that

$$A = LR \; ; \quad \mathcal{M}(A) = \mathcal{M}(L) \quad \text{and} \quad \mathcal{M}(A') = \mathcal{M}(R').$$

This decomposition we call as space decomposition of $A$.

The following theorem gives an interesting characterisation of matrices possessing g-inverses.

<u>Theorem 8</u> : $A^-$ exists if and only if $A$ is space decomposable.

<u>Proof</u> : 'If' part  Let $A = LR$ be a space decomposition of $A$

$$\left.\begin{aligned} &\Rightarrow \mathcal{M}(L) = \mathcal{M}(A) \\ &\text{and } \mathcal{M}(R') = \mathcal{M}(A') \end{aligned}\right\} \Rightarrow \quad L = AD_1 \text{ and } R = D_2 A \quad \text{for some } D_1 \text{ and } D_2$$

Now $A = LR = AD_1D_2A \Rightarrow D_1D_2$ is a $A^-$.

'Only if' part : Let $A^-$ exist. Then $A$ is of the form

$$P \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} Q$$

where $A_1$ is $r \times r$ nonsingular idempotent matrix, where $r$ is the rank of A and $P$ and $Q$ are permutation matrices .

$$\Rightarrow \quad A = P \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix} (A_1 \quad A_1C) \; Q$$

$$= LR$$

where $L = P \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}$ and $R = (A_1 \; : \; A_1C)Q$

It is easy to see that $M(A) = M(L)$ and $M(A') = M(R')$

Therefore $A = LR$ is a space decomposition.

Remark 16: Observe that $r$ the rank of the matrix is the minimum value that $k$ (in the definition of space decomposition) can take. If rank does not exist i.e., row rank $\neq$ column rank, space decomposition does not exist. In view of this, from now onwards by space decomposition of $A$ we mean $A = LR$ where $L$ is of order $m \times r$, $R$ is of order $r \times n$ where $r$ is the rank of $A$ and $M(A) = M(L)$ and $M(A') = M(R')$.

Theorem 9 : Let $A = P \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} Q$ where $P$ and $Q$ are permutation matrices, $A_1$ is nonsingular idempotent matrix. Then $A = L_1R_1$ is a space decomposition of $A$ if and only if $L_1 = LP_1$ and $R_1 = P_1'R$ for some permutation matrix $P_1$ where $L = P\begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}$ and $R = (A_1 \; : \; A_1C)Q$.

Proof: 'If' part is trivial.

'Only if' part : $A = L_1 R_1$ is spacedecomposition of $A$

$$\Rightarrow M(L_1) = M(A) = M(L) \Rightarrow L_1 = LP_1$$

$$\text{and } M(R_1') = M(A') = M(R') \Rightarrow R_1 = P_2R$$

some permutation matrices $P_1$ and $P_2$.

Now

$$A = P \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} Q = P \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix} P_1P_2(A_1 \ : \ A_1C)Q$$

$$\Rightarrow \quad A_1 = A_1 P_1P_2 A_1 \Rightarrow P_1P_2 \text{ is a } A_1^-$$

But $A_1$ being nonsingular and idempotent $I$ is the only permutation g-inverse it has, therefore $P_1P_2 = I$

$$\Rightarrow \quad P_2 = P_1'$$

Therefore $L_1 = LP_1$ and $R_1 = P_1'R$

<u>Theorem 10</u> : Let $A = LR$ be a space decomposition of $A = P \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} Q$ and $G$ be a $A^-$. Then

(a) $L^-$ and $R^-$ exist

(b) $L^-L = RR^-$

(c) $L^-A = R$ and $AR^- = L$

(d) $R^-L$ is a $A^-$

and (e) $RG$ is a $L_r^-$ and $GL$ is a $R_r^-$.

<u>**Proof**</u> : (a) Let $L = P \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix} P_1$ and $R = P_1'(A_1 \ : \ A_1C)Q$

$\mathcal{M}(A) = \mathcal{M}(L) \Rightarrow L^-$ exists and $\mathcal{M}(A') = \mathcal{M}(R') \Rightarrow R^-$ exists

(b) $L^{-}L = P_1' \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix}^{-} P'P \begin{bmatrix} A_1 \\ DA_1 \end{bmatrix} P_1 = P_1'A_1P_1$

and $RR^{-} = P_1' (A_1 : A_1C)QQ' (A_1 : A_1C)^{-}P_1 = P_1'A_1P_1$

Therefore $L^{-}L = RR^{-}$

(c) $L^{-}A = L^{-}LR = RR^{-}R = R$

and $AR^{-} = LRR^{-} = LL^{-}L = L$

(d) $AR^{-}L^{-}A = LR = A \Rightarrow R^{-}L^{-}$ is a $A^{-}$

(e) $AGA = A \Rightarrow LRGLR = LR$

$\Rightarrow LRGLRR^{-} = LRR^{-}$

i.e., $LRGLL^{-}L = LL^{-}L$

So $LRGL = L$

and therefore $RG$ is a $L^{-}$ and similarly we can show that $GL$ is a $R^{-}$. Now, for reflexivity, consider

$$RGLRG = RR^{-}RG = RG$$

Thus $RG$ is a $L_r^{-}$ and $GL$ is a $R_r^{-}$.

Remark 17 : Though $R^{-}L^{-}$ is always a $A^{-}$, every g-inverse of $A$ need not be of the form $R^{-}L^{-}$.

Ex : $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ; $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Next we turn our attention to reflexive g-inverses. In the following theorem we characterise reflexive g-inverses of A.

<u>Theorem 11</u> : G is a reflexive g-inverse of A if and only if $G = R^-L^-$ where $A = LR$ is a space decomposition of A and one of $R^-$ and $L^-$ is reflexive.

<u>Proof</u>: 'If' part : Let $A = LR$ be a space decomposition of A. We already proved that $G = R^-L^-$ is a $A^-$. For reflexivity

consider
$$\begin{aligned} GAG &= R^-L^-LRR^-L^- \\ &= R^-RR^-L^- \\ &= R^-L^- \qquad \text{assuming } R^- \text{ is reflexive} \\ &= G \end{aligned}$$

If $L^-$ is reflexive

$$\begin{aligned} GAG &= R^-L^-LRR^-L^- \\ &= R^-L^-LL \\ &= R^-L^- \\ &= G \end{aligned}$$

'Only if' part : Let G be a $A_r^-$ .

Then $G = GAG = GLRG = R_r^-L_r^-$

(from (e) of previous theorem).

Remark 18 : The condition that one of $R^-$ and $L^-$ is reflexive is necessary, for otherwise i.e., if both are not reflexive $R^-L^-$ may not be a reflexive g-inverse of A.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; $R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$R^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $L^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$G = R^-L^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a $A^-$ but not $A_r^-$.

## 1.5 Moore-Penrose inverse and other types of g-inverses

In this section we obtain necessary and sufficient conditions for a matrix A to possess various types of g-inverses, namely $A_m^-$, $A_\ell^-$ and $A^+$ where these g-inverses are as defined in Table 3.

Theorem 12 : The following statements are equivalent

(a) $A_m^-$ exists

(b) A is of the form $P \begin{bmatrix} I & C \\ D & DC \end{bmatrix} Q$ where P and Q are permutation matrices and C is such that $CC' \leq I$.

(c) $A^-$ exists and $\mathcal{M}(A) = \mathcal{M}(AA')$

and (d) there exists a G such that $GAA' = A'$.

<u>Proof</u> : (a) => (b) Let $G = A_m^-$. Then GA is symmetric and idempotent. Therefore there exists a permutation matrix Q such that

$$GA = Q' \begin{bmatrix} I & C \\ C' & C'C \end{bmatrix} Q$$

where C is such that $CC' \leq I$. Again G is a $A^-$ implies $M(A') = M((GA)')$ which implies A is of the form $P\begin{bmatrix} I & C \\ D & DC \end{bmatrix}Q$ for some D

(b) => (c) It can be easily checked that $Q' \begin{bmatrix} I & 0 \\ C' & 0 \end{bmatrix} P'$ is a $A_m^-$ and $M(A) = M(AA')$.

(c) => (d) $A^-$ exists and $M(A) = M(AA')$ => $(AA')^-$ exists and there exists a matrix D such that $A = AA'D$. Now it is easy check

$$GAA' = A' \quad \text{where} \quad G = A'(AA')^-.$$

(d) => (a) $GAA' = A'$ => GA is symmetric and hence $AGA = A$

$$\Rightarrow G \text{ is a } A_m^- .$$

This completes the proof of the theorem.

A similar theorem for $A_\ell^-$ is stated below omitting the proof as it follows on the same lines.

Theorem 13 : The following statements are equivalent

(a) $A_{\ell}^{-}$ exists

(b) A is of the form $P\begin{bmatrix} I & C \\ D & DC \end{bmatrix}Q$ where P and Q are permutation matrices and D is such that $D'D \leq I$

(c) $A^{-}$ exists and $M(A') = M(A'A)$

and (d) there exists a matrix G such that $A'AG = A'$.

Corollary1: The following statements are equivalent

(a) $A^{+}$ exists

(b) A is of the form $P\begin{bmatrix} I & C \\ D & DC \end{bmatrix}Q$ where P and Q are permutation matrices and C and D are such that $CC' \leq I$ and $D'D \leq I$

(c) $A^{-}$ exists, $M(A) = M(AA')$ and $M(A') = M(A'A)$

and (d) there exists a matrix G such that $GAA' = A'$ and $A'AG = A'$.

Corollary2 (Rao [11]) : $A^{+}$ exists $\Rightarrow A^{+} = A'$.

## 1.6 An algorithm to compute a g-inverse

In this section we develop an algorithm to compute a g-inverse if it exists. First we prove two theorems on which the algorithm is based. Before proceeding further we need the following definitions.

<u>Definition</u> : For a square matrix $A$ of order $n \times n$ we define permanent of $A$, denoted by $|A|$ as

$$|A| = \Sigma\, a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where the summation is taken over all permutations $i_1, \ldots, i_n$ of $1, 2, \ldots, n$.

Observe that if $B = PAQ$ where $P, Q$ are any two permutation matrices, then

$$|A| = |B| = |A'|$$

<u>Definition</u> : A set of vectors $T = (x_1 \ldots x_n)$ is said to be satisfying weight condition or condition $w$, if for every $i$, there are $w_i$ vectors in the set which are less than or equal to $x_i$ where $w_i$ is the weight of the vector $x_i$.

<u>Theorem 14</u>: Let $A$ be a nonsingular square matrix. Then $A^-$ exists if and only if $|A| = 1$ and columns of $A$ satisfy condition $w$.

<u>Proof</u> : 'If' part : Let $|A| = 1$, then there exists a permutation $i_1, i_2 \ldots i_n$ of $1, 2, \ldots n$ such that $a_{1i_1}, a_{2i_2} \ldots a_{ni_n} = 1$ where $n$ is the order of $A$.

Now consider the permutation matrix $P = (e_{i_1}, e_{i_2} \ldots e_{i_n})$ and let $B = AP$. Clearly $b_{ii} = 1$, for $i = 1, \ldots, n$.

$A$ is nonsingular $\Rightarrow B$ is nonsingular.

Columns of $A$ satisfy condition $w$

$\Rightarrow$ columns of $B$ satisfy condition $w$.

Next we will show that $B$ is idempotent which will imply $P$ is a $A^-$. Let $y_i$ be the $i^{th}$ column vector of $B$ and consider

$$B\, y_i = b_{1i}y_1 + b_{2i}y_2 + \dots + b_{ii}\, y_i + \dots + b_{ni}y_n \quad ..(11)$$

Let $w_i$ be weight of $y_i$. Observe that none of the $y_k$'s is null vector for $k = 1,\dots,n$. The right hand side of (11) is sum of exactly $w_i$ nonzero vectors. Now $y_k \leq y_i \Rightarrow b_{ki} = 1$ since $b_{kk} = 1$. Thus whenever $y_k \leq y_i$, $y_k$ appears as nonzero term in the right hand side of (11) and since there are $w_i$ $y_k$'s which are less than or equal to $y_i$ and only $w_i$ nonzero vectors are present in the right hand side of (11), each nonzero term is less than or equal to $y_i$ and hence their sum is also $\leq y_i$, but since $b_{ii} = 1$, $y_i$ itself is present in the right hand side of (11) and hence their sum is $y_i$.

Therefore $\qquad By_i = y_i$

So $\quad B$ is idempotent $\Rightarrow P$ is a $A^-$.

'Only if' part : $A^-$ exists => there exists a permutation matrix $P$ such that $AP$ is idempotent, by Theorem 4.

Let $B = AP$.

Then $b_{ii} = 1$, $i = 1,\ldots,n$, by Lemma 2. Hence $|A| = |B| = 1$.

Since $B$ is idempotent $By_i = y_i$, $i = 1,\ldots, n$, where $y_i$ is the $i^{th}$ column vector of $B$.

So $y_i = By_i = b_{1i}y_1 + b_{2i}y_2 + \ldots + b_{ni}\, y_n$

=> whenever $b_{ki} \neq 0$, $y_k \leq y_i$ for all $k$,

=> there are at least $w_i$, $y_k$'s $\leq y_i$, since weight of $y_i$ is $w_i$.

=> columns of $B$ satisfy the condition $w$

=> columns of $A$ satisfy the condition $w$ since columns $B$ are nothing but a permutation of columns of $A$

which completes the proof of the theorem.

<u>Corollary 1</u> : If $A$ is square, nonsingular matrix then $A^-$ exists if and only if $|A| = 1$ and the rows of $A$ satisfy condition $w$.

<u>Corollary 2</u> : If $A$ is nonsingular and $A^-$ exists then there exists column of $A$ of weight one.

Proof : Let $x$ be a column vector of $A$ with least weight say $k > 1$. But by above theorem there should be $k$ columns of $A \leq x$. Since $x$ is a column of least weight there cannot be any vector $< x$ $\Rightarrow$ there are $k$ vectors $= x \Rightarrow A$ is singular which is a contradiction. Hence $k \leq 1$. Since $k \neq 0$, $k = 1$.

Corollary 3 : Let $A$ be a square nonsingular matrix such that $|A| = 1$ and columns of $A$ satisfy condition $w$. Let $i_1, \ldots, i_n$ be such that $a_{1i_1} \ldots a_{ni_n} = 1$ then $P = [e_{i_1}\ e_{i_2} \ldots e_{i_n}]$ is a g-inverse of $A$.

In fact this is established in Theorem 14.

Let $A$ be a square nonsingular matrix such that $A^-$ exists then one and only one term in $|A|$ is nonzero.

Theorem 15 : Let $A$ be a nonsingular matrix such that $PAQ = \begin{bmatrix} 1 & \alpha' \\ 0 & A_1 \end{bmatrix}$ where $P$ and $Q$ are permutation matrices and $\alpha$ and $0$ are column vectors. Then $A^-$ exists implies $A_1^-$ exists and $A_1$ is nonsingular.

Proof : $A$ is nonsingular and $A^-$ exists

$\Rightarrow$ $A$ is square

$\Rightarrow$ $A_1$ is square

is nonsingular $\Rightarrow A$ is of full row rank

$\Rightarrow A_1$ is of full row rank.

Let $\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$ be a g-inverse of PAQ

$AGA = A \Rightarrow A_1G_4A_1 = A_1 \Rightarrow G_4$ is a $A_1^-$

$\Rightarrow$ row rank of $A_1$ = column rank $A_1$

$\Rightarrow A_1$ is nonsingular.

Corollary : If A is nonsingular and $A^-$ exists then there exist permutation matrices P and Q such that PAQ is nonsingular idempotent and upper triangular .

Algorithm

Let A be $m \times n$ matrix and A exist. For computing A we proceed as follows. First we obtain the row basis and the column basis of A. Then we compute the permutation g-inverse of the nonsingular submatrix formed by these rows and columns of A. Finally $A^-$ is constructed as in Theorem 3.

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The algorithm given in the next page is also used in chapter 2 for computing a nonnegative g-inverse of a nonnegative matrix.

Let A be the given matrix of order $m \times n$.

Step 1 : Set $p = 1$ and $B_1 = A$. Compute $w_1, w_2, \ldots, w_m$, the row weights of A. (These weights, we refer in Step 2, as original row weights). Go to step 2.

Step 2 : Compute $w_1', w_2', \ldots, w_m'$, the row weights of $B_p$. Choose a minimum weight nonnull row of $B_p$. If there are more than one row of $B_p$ with minimum weight, choose one from them with minimum original row weight. Let it be $i_p^{th}$ row of $B_p$. Form the matrix $B_{p+1}$ from $B_p$ by making $j^{th}$ column of it null if $(B_p)_{i_p j} \neq 0$ for $j = 1, 2 \ldots n$. If $B_{p+1}$ is null, go to step 3, otherwise increase the value of p by 1 and go back to step 2.

Step 3 : Let k be the value of p. Set $p = 1$. Form the matrix $C_1$ with $i_1, i_2, \ldots i_k^{th}$ rows as those of A and the rest of the rows null. Compute $v_1, v_2, \ldots v_n$ the column weights of $C_1$. (These weights are referred as original weights in step 4). Go to step 4.

Step 4 : Compute $v_1', v_2', \ldots v_n'$, the column weights of $C_p$ Choose an unmarked weight 1 column of $C_p$. (There is always one such column for $p \leq k$). If there are more than one unmarked weight 1 column of $C_p$, choose one from them with minimum original weight and mark it. Let it be $j_p^{th}$ column of $C_p$ and let $\ell_p^{th}$ element of it be nonzero. Form the matrix $C_{p+1}$ from $C_p$ by replacing all but $j_p^{th}$ element of $\ell_p^{th}$

row of $C_r$ by zeroes and keeping other elements as they are. If p=k we stop otherwise increase the value of p by 1 and go back to step 4.

Now we have

Theorem 16 : Let $A^-$ exist. Then

(a) rank of A is k

(b) $i_1, i_2, \ldots i_k$ th rows of A form the row basis of A

(c) $j_1, j_2, \ldots, j_k$ th columns of A form the column basis of A

and (d) $C_k'$ is a partial permutation g-inverse of A

where $k, i_1, i_2, \ldots i_k, j_1, j_2, \ldots, j_k$ and $C_k'$ are as in the above algorithm.

The theorem follows directly from the Theorem 3, Corollary of Theorem 15 and the above algorithm.

Remark 19: Whether $A^-$ exists or not the above algorithm always gives the matrix $C_k$. One way of checking whether $C_k'$ is $A^-$ or not, is checking whether $AC_k'A = A$ or not.

Once we confirm the existence of $A^-$ checking for the existence and computation of other g-inverses, viz., $A_m^-$, $A_\ell$ and $A^+$ is simple and is given in the Table 4.

Let $A^-$ exist. Let $G_1$ be the matrix with $i_1, i_2 \ldots i_r{}^{th}$ rows as those of $A$ and rest of the rows null and $G_2$ be the matrix with $j_1, j_2, \ldots j_r{}^{th}$ columns as those of $A$ and the rest of the columns null.

Table 4

| Type of $A^-$ | Condition for existence | Given by |
|---|---|---|
| $A_r^-$ | alwasys exists | $G_k' \; A \; G_k'$ |
| $A_m^-$ | weight of every column of $G_1 \leq 1$ | $G_1'$ |
| $A_\ell^-$ | weight of every row of $G_2 \leq 1$ | $G_2'$ |
| $A^+$ | weight of every column of $G_1 \leq 1$ and weight of every row of $G_2 \leq 1$ | $A'$ |

Numerical illustration :

Example 1 : $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = B_1$. Taking $i_1 = 3$

$$B_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ; \quad \text{Taking } i_2 = 5, \quad B_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here all the three nonnull rows of $B_3$ are of equal weight. But second row has minimum original weight. So we take $i_3$=2. Then $B_4$ becomes null. So k = 3. We form now

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ; \quad \text{Taking } j_1 = 1, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Taking $j_2 = 2$ we have $C_3 = C_2$ and $j_3 = 4$.

Now it can be easily checked that $C_3'$ is a $A^-$. Now $G_1 = C_1$

and $G_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ Also all the columns of $G_1$ are of

ight $\leq$ 1. So $G_1'$ is a $A_m^-$ But the first row of $G_2$ is of weight e than one and therefore $A_\ell$ does not exist.

Example 2 :

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = B_1 \text{ ; Taking } i_1 = 2$$

$$B_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \text{ Taking } i_2=1, \; B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Here both third and fifth rows are of same weight. But we take $i_3 = 5$ since fifth row has minimum original weight. Now $B_4$ becomes null so $k = 3$. We form

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{Taking } j_1 = 1, \; C_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Taking $j_2 = 2$ $\quad C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and therefore $j_3 = 5$.

$C_3'$ is not a $A^-$, $A^-$ does not exist.

1.7 g-inverses of matrices over an arbitrary Boolean algebra

In this section we deal with matrices over an arbitrary Boolean algebra $B$ not necessarily {0, 1} Boolean algebra. The case of the general Boolean algebra, not necessarily finite, we dispose of, in a remark at the end of this section. We wish to obtain results about matrices over a finite Boolean algebra $B$ anologous to the results obtained in sections 3, 4 and 5 Our main tool is the concept of homomorphism.

Definition : Let $B_0$ be the {0, 1} Boolean algebra as usual. A map $h : B \to B_0$ is called a homomorphism if

i) $h(a+b) = h(a) + h(b)$

ii) $h(a.b) = h(a).h(b)$

iii) $h(\bar{a}) = \overline{h(a)}$ for all $a,b \in B$.

For a homomorphism h and matrix $A = (a_{ij})$ over $B$, we define h(A), a matrix over $B_0$ by $(h(A))_{ij} = h(a_{ij})$. With these definitions it is easy to see $h(A.B) = h(A).h(B)$, $h(A+B) = h(A) + h(B)$ and $h(A') = [h(A)]'$. Let $H$ denote the set of all homomorphisms from $B$ to $B_0$. The following proposition is basic for our extenstions.

Proposition 6 : For matrices A and B over $B$, $h(A) = h(B)$ for all $h \in H$ if and only if $A = B$.

Proof of this proposition is an easy consequence of the fact that if $a \neq b$ then there exists $h \in H$ such that $h(a) \neq h(b)$. (Halmos, P.R.[3]). As a consequence of the above proposition, we have, G is a g-inverse of A if and only if h(G) is a g-inverse of h(A) for all $h \in H$.

Definition : A square matrix P over $B$ is called a permutation matrix if

(i) elements of any row are pairwise disjoint i.e., $a.b = 0$ if a and b are two different elements of the same row.

(ii) elements of any column are pairwise disjoint

(iii) sum of all the elements in any row is equal to 1

and (iv) sum of all the elements in any column is equal to 1.

Definition : A matrix (need not be square) over $B$ is called a partial permutation matrix if it satisfies conditions (i) and (ii).

We are justified in calling these matrices over $B$ as permutation and partial permutation matrices in view of the following lemma.

Lemma 4 : (a) P is a permutation matrix over $B$ if and only if h(P) is a permutation matrix over $B_0$ for all $h \in H$

(b) P is a partial permutation over $B$ if and only if h(P) is a partial permutation over $B_0$ for all $h \in H$.

Proof of the lemma is easy and is omitted.

We now prove

Lemma 5 : Given matrices $A_h$ over $B_o$, of the same order, indexed by $h \in H$, there exists a matrix $A$ over $B$ such that $h(A) = A_h$ for all $h \in H$.

Proof: It is sufficient to prove that, given an element $a_h \in B_o$ indexed by $h \in H$, there exists a unique element $a \in B$ such that $h(a) = a_h$ for all $h \in H$. Since $B$ is a finite Boolean algebra, we can find elements $a_1 \ldots a_n \in B$ such that, they are pairwise disjoint and any element of $B$ is a sum of some of $a_1 \ldots a_n$. For any homomorphism $h : B \to B_o$ we can find an element $a_i$ such that $h(a_i) = 1$ and $h(a_j) = 0$ if $j \neq i$. Also with the help of any $a_i$ we can define a homomorphism $h : B \to B_o$ which is an extension of $h(a_i) = 1$ and $(a_j) = 0$ for $j \neq i$. So there is one to one correspondence between and $a_1 \ldots a_n$.

Let $h_1 \ldots h_n$ be all the homomorphisms such that

$$
\begin{aligned}
h_k(a_\ell) &= 1 \quad \text{if } k = \ell \\
&= 0 \quad \text{if } k \neq \ell .
\end{aligned}
$$

the given elements be $a_{h_1}, a_{h_2} \ldots a_{h_n}$. Define $a = \Sigma a_i$ where mation is taken over all $i$ such that $a_{h_i} = 1$. It is easily ified that $h(a) = a_h$ for all $h \in H$. Uniqueness is clear.

We now ready to prove our extensions

<u>Theorem 17 :</u> For square matrices A and B over $B$, AB = I implies A is a permutation matrix over $B$ and B = A'.

<u>Proof</u> : AB = I

$\Rightarrow$ h(AB) = h(A).h(B) = h(I) = I for all $h \in H$

$\Rightarrow$ h(A) is a permutation matrix over $B_o$ and h(B) = h(A)' for all $h \in H$

$\Rightarrow$ A is a permutation matrix over $B$ and B = A'.

<u>Theorem 18 :</u> If a matrix A over $B$ has a g-inverse, then it has a partial permutation matrix over $B$ as g-inverse.

<u>Proof</u> : A has a g-inverse

$\Rightarrow$ h(A) has a g-inverse for all $h \in H$

$\Rightarrow$ h(A) has a partial permutation g-inverse, say $Q_h$ over $B_o$ for all h

$\Rightarrow$ there exists a partial permutation Q over $B$ which is a g-inverse of A.

<u>Theorem 19 :</u> If $A^+$ exists, then it is A'.

<u>Proof</u> ; Let G be $A^+$

$\Rightarrow$ $h(G) = [h(A)]^+$ for all $h \in H$

But $[h(A)]^+ = [h(A)]' = h(A')$ for all $h \in H$

$\Rightarrow$ h(G) = h(A') for all $h \in H$

$\Rightarrow$ G = A'.

Lemma 6 : Let $A$ and $B$ be two matrices over $U$. Then $M(A) = M(B)$ if and only if $M(h(A)) = M(h(B))$ for all $h \in H$.

Proof : 'if' part : $M(h(A)) = M(h(B))$ for all $h \in H$ $\Rightarrow$ $h(A) = h(B)\, C_h$ and $h(B) = h(A)\, D_h$, for all $h \in H$. Therefore we can find matrices $C$ and $D$ such that $h(C) = C_h$ and $h(D) = D_h$ for all $h \in H$ so that $h(A) = h(B).h(C) = h(BC)$ and $h(B) = h(A).h(D) = h(AD)$ for all $h \in H$ which implies $A = BC$ and $B = AD$. Hence $M(A) = M(B)$.

'Only if' part : $M(A) = M(B) \Rightarrow A = BC$ and $B = AD$ for some $C$ and $D$. Therefore $h(A) = h(B).h(C)$ and $h(B) = h(A).h(D)$ for all $h \in H$ which implies $M(h(A)) = M(h(B))$ for all $h \in H$.

Theorem 20 : An $m \times n$ matrix $A$ over $U$ has a space decomposition if and only if $A$ has a g-inverse.

Proof : 'if' part : $A$ has a g-inverse $\Rightarrow$ $h(A)$ has a g-inverse, for all $h \in H$ $\Rightarrow$ $h(A) = L_h R_h$ (space decomposition of $h(A)$) for all $h \in H$ such that $M(h(A)) = M(L_h)$ and $M(h(A)') = M(R_h')$ for all $h \in H$.

Let $r_0$ be $\max_{h \in H} r_h$ where $r_h$ is the rank of $h(A)$.

Define $L_h = (L_h : O_h)$ where $O_h$ is null matrix of order $m \times (r_0 - r_h)$ and $R_h = \begin{bmatrix} R_h \\ \cdot\cdot \\ O_h \end{bmatrix}$ where $O_h$ is null matrix of order

$(r_o - r_h) \times n$. Then $h(A) = \hat{L}_h \hat{R}_h$ for all $h \in H$, where $\hat{L}_h$ is of order $m \times r_o$ and $\hat{R}_h$ is of order $r_o \times n$ for all $h \in H$ and $M(h(A)) = M(\hat{L}_h)$ and $M(h(A)') = M(\hat{R}_h')$. So we can find matrices $L$ and $R$ over $B$ such that $h(L) = \hat{L}_h$ and $h(R) = \hat{R}_h$ for all $h \in H$.

Now it clearly follows that $A = LR$.

Other part follows trivially.

Remark 20 : All the above results of this section can be extended to matrices over any Boolean algebra $B$ (not necessarily finite). The hint is that, if one wishes to prove a result about a matrix $A$ over $B$ it is enough to consider $A$ as a matrix over a finite Boolean algebra generated by the elements of $A$. This is because of the following lemma which is easy to prove using homomorphisms.

Lemma 7 : Let $A$ be a matrix over a Boolean algebra $B$. It has a generalized inverse over $B$ if and only if it has a generalized inverse over the Boolean algebra generated by the elements of $A$.

[The author has obtained the results of this chapter jointly with Dr. K.P.S. Bhaskara Rao in January 1973, completely unaware of the earlier work of Plemmons [7]. We had belatedly known that Plemmons has obtained a version of Corollary to Proposition 4 and Theorem 4. It should be noted that our approach is entirely different than that of Plemmons. However we do acknowledge the priority of his results.]

# CHAPTER 2

## g-INVERSES OF NONNEGATIVE MATRICES

### 2.1 Introduction and Summary

In this chapter we discuss in general about nonnegative matrices having various types of nonnegative g-inverses.

To start with, in section 2.2, a simple special type of nonnegative matrices, viz., doubly stochastic matrices are considered. Doubly stochastic matrices having doubly stochastic g-inverses are characterised in this section. This result can also be obtained as a particular case of a later section. However the proofs given here are of independent interest.

In section 2.3 nonnegative matrices possessing various types of nonnegative g-inverses are characterised. It is shown that a nonnegative matrix of rank $r$ possesses nonegative g-inverse if and only if it has an $r \times r$ nonsingular diagonal submatrix. It is also established that for a nonnegative matrix to possess a nonnegative minimum norm g-inverse it is necessary and sufficient that every pair of columns are either orthogonal or one is a multiple of the other. Similar results for least squares g-inverse and Moore-Penrose inverse are also proved.

In section 4 of this chapter, stochastic matrices are considered. Necessary and sufficient conditions for a stochastic matrix to possess various types of stochastic g-inverses are established.

Finally in section 2.5 an algorithm for computing a nonnegative g-inverse of a nonnegative matrix is developed. At the end of this section computational formulas are given in a tabular form for computing minimum norm g-inverse, least squares g-inverse and Moore-Penrose inverse.

## 2.2 g-inverses of doubly stochastic matrices

**Definition :** An $n \times n$ nonnegative matrix is said to be doubly stochastic if

$$\sum_{i=1}^{n} a_{ji} = 1 \quad \text{and} \quad \sum_{i=1}^{n} a_{ij} = 1 \quad \text{for } j = 1, \ldots n.$$

**efinition :** An $n \times n$ matrix $A$ is said to be an isometry if

$||Ax|| = ||x|| \quad \forall x \in R^n$

**efinition :** An $m \times n$ matrix $A$ is said to be a partial isometry if

$$||Ax|| = ||x|| \quad \forall x \in \mathcal{M}(A')$$

rom the above definitions it immediately follows that $A$ is an isometry f and only if $A' = A^{-1}$ and $A$ is a partial isometry if and only if $' = A^{+}$.

It is well known that a nonsingular doubly stochastic matrix has doubly stochastic inverse if and only if it is a permutation matrix permutation matrices are the only doubly stochastic isometries. this section we prove that a doubly stochastic matrix has a doubly chastic g-inverse if and only if it is partial isometry.

prove a theorem on partial isometrics which is used later.

rem 1 : $A$ is a partial isometry if and only if one of the owing equivalent conditions holds.

i) $(AA')^p A$ is a partial isometry for some nonnegative integer p,

ii) $(AA')^p$ is a partial isometry for some positive integer p

f:

i) $(AA')^p A$ is partial isometry

$\iff$ $(AA')^p AA' (AA')^p (AA')^p A = (AA')^p A$

i.e., $(AA')^{3p+1} A = (AA')^p A$

$\iff$ $(AA')^{2p+1} A = A$

$\iff$ $C(AA'A-A) = 0$ where $C = [(AA')^{2p}+(AA')^{2p-1}+\ldots+(AA')+I]$

$\iff$ $AA'A = A$ Since $C$ is positive definite

$\iff$ $A$ is partial isometry

ii) $(AA')^p$ is partial isometry

$\iff$ $(AA')^{3p} = (AA')^p$

$\iff$ $(AA')^{2p+1} = AA'$

$\iff$ $C(AA'AA'-AA') = 0$ where $C = [(AA')^{2p-1}+(AA')^{2p-2}+\ldots+(AA') + I]$

$\iff$ $AA'AA' = AA'$ since $C$ is p.d.

$\iff$ $A$ is partial isometry.

This proves the theorem.

In the sequel we need a result of Sinkhorn (1968) which we state below for completeness.

Lemma 1 : If $A$ is a doubly stochastic and idempotent matrix then $A$ is symmetric.

Theorem 2 : Let $A$ be a doubly stochastic matrix possessing a doubly stochastic g-inverse. Then $A^+$ is doubly stochastic.

Proof : Let $G_1$ be a doubly stochastic g-inverse of $A$ and let $G = G_1AG_1$. Observe that $G$ is a doubly stochastic reflexive g-inverse of $A$. Further $GA$ and $AG$ are idempotent and doubly stochastic and hence by Lemma 1 are symmetric. Therefore $G = A^+$.

Remark 1 : Doubly stochastic reflexive g-inverse of a doubly stochastic matrix is unique and in fact it is the Moore-Penrose inverse.

We now prove

Theorem 3 : Let $A$ be a normal doubly stochastic matrix. Then the following statements are equivalent.

(a) $A$ has a doubly stochastic g-inverse

(b) each nonzero eigen value of $A$ is of modulus unity

(c) $A$ is a partial isometry.

<u>Proof</u> : Let $R(A) = r$. Since $A$ is normal then exists a unitary matrix $U$ such that $A = U \Lambda U^*$ where $\Lambda = \text{diag}(\lambda_1 \ldots \lambda_n)$ and $\lambda_1 \ldots \lambda_n$ are the eigen values of $A$. Without loss of generality let $\lambda_1 \ldots \lambda_r$ be nonzero and $\lambda_{r+1} \ldots \lambda_n$ be zero.

(a) => (b)

$A$ has a doubly stochastic g-inverse implies (by Theorem 2) that $A^+$ is doubly stochastic. Since $A$ is doubly stochastic $|\lambda_i| \leq 1$ for $i = 1 \ldots r$. Clearly $A^+ = U\Lambda^+U^*$ where

$\Lambda^+ = \text{diag}\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots \frac{1}{\lambda_r}\, 0 \ldots 0\right)$. Again since $A^+$ is doubly stochastic $\left|\frac{1}{\lambda_i}\right| \leq 1$ for $i = 1 \ldots r$

$$\text{Hence } |\lambda_i| = 1 \quad \text{for } i = 1 \ldots r$$

(b) => (c)

First observe that $A^* = A'$ since $A$ is real. $|\lambda_i| = 1$ for $i = 1, 2 \ldots r$ => $\Lambda\Lambda^*\Lambda = \Lambda$ and hence $AA^*A = A$.

So $A' = A^* = A^+$

(c) => (a) is trivial.

Theorem 4 : A doubly stochastic matrix A possesses a doubly stochastic g-inverse if and only if it is a partial isometry.

Proof : 'If' part is trivial.

To prove the 'Only if' part, let $G = A^+$.

If A has a doubly stochastic g-inverse, then by Theorem 2 G is doubly stochastic. So $G'G$ which is $(AA')^+$ is doubly stochastic and by Theorem 3 $AA'$ is partial isometry. So by Theorem 1, A is partial isometry.

2.3 g-inverses of non-negative matrices

Before proceeding to study nonnegative matrices possessing non-negative g-inverses we recall some results of chapter 1 on Boolean matrices which are true in general for real matrices and which are used in the sequel. Proofs are same as those of chapter 1 and hence are omitted.

Theorem 5 : Let A be an $m \times n$ matrix of rank r. Then there exist permutation matrices P and Q and matrices C and D of appropriate orders such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

where $A_1$ is $r \times r$ nonsingular matrix.

Theorem 6 : Let $A = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$ Then $G = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$

is a $A^-$ if and only if $(G_1 + CG_3 + G_2D + CG_4D)$ is a $A_1^-$.

Lemma 2 : (i) Let G be a reflexive g-inverse of A. Then

(a) $i^{th}$ row (column) of AG is null if and only if $i^{th}$ row (column) of A(G) is null.

(b) if $i^{th}$ row (column) of A is null then the matrix obtained replacing $i^{th}$ column (row) of G by null vector is also a reflexive inverse of A

(c) all the rows (columns) of A corresponding to the null umns (rows) of G are linearly dependent on the other rows (columns) A.

(ii) Let G be a least squares reflexive g-inverse of A. Then row of A is null if and only if $i^{th}$ column of G is null.

(iii) Let G be a minimum norm reflexive g-inverse of A. $i^{th}$ column of A is null if and only if $i^{th}$ row of G is null.

(iv) Let G be the Moore-Penrose inverse of A. Then $i^{th}$ row mn) of A is null if and only if $i^{th}$ column (row) if G is null.

is elementary and hence is omitted.

heorem 7: Let $A = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$ where $A_1$ is nonsingular. Then

(i) $\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$ is a reflexive g-inverse of $A$ where $B_1 = A_1^{-1}$.

(ii) $\begin{bmatrix} B_1 & B_1D' \\ 0 & 0 \end{bmatrix}$ is a least squares g-inverse of $A$ where

$B_1 = [(I + D'D)A_1]^{-1}$

(iii) $\begin{bmatrix} B_1 & 0 \\ C'B_1 & 0 \end{bmatrix}$ is a minimum norm g-inverse of $A$ where

$B_1 = [A_1(I + CC')]^{-1}$

d (iv) $\begin{bmatrix} B_1 & B_1D' \\ C'B_1 & C'B_1D' \end{bmatrix}$ is the Moore-Penrose inverse of $A$ where

$B_1 = [(I + D'D)A_1(I + CC')]^{-1}$.

e theorem follows by straightforward verification.

now prove

eorem 8 : Let $G$ be a reflexive g-inverse of an $m \times n$ matrix $A$.

en there exist permutation matrices $P$ and $Q$ such that

$$= \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} \quad \text{and} \quad Q'GP' = \begin{bmatrix} G_1 & G_1E \\ FG_1 & FG_1E \end{bmatrix}$$ where all

rows and columns of both $A_1$ and $G_1$ are non-null and $G_1$ is a

flexive g-inverse of $A_1$.

Proof : Let $i_1, i_2 \ldots i_p{}^{th}$ rows and $j_1 \ldots j_q{}^{th}$ columns of $AG$ be null and let $\{s_1, s_2 \ldots s_{m-r}\}$ be the union of $\{i_1 \ldots i_p\}$ and $\{j_1 \ldots j_q\}$. Let $P$ be a permutation matrix such that the $s_1, s_2 \ldots s_{m-r}{}^{th}$ rows and columns of $AG$ are the last $m-r$ rows and columns of $PAGP'$, i.e.,

$$PAGP' = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} = U \quad \text{(say)}$$

where $U_1$ is of order $r \times r$ and $i^{th}$ row of $[U_3 \vdots U_4]$ i.e., $(r+i)^{th}$ row of $U$ is nonnull implies $(r+i)^{th}$ column of $U$ i.e., $i^{th}$ column of $\begin{bmatrix} U_2 \\ U_4 \end{bmatrix}$ is null. Similarly $i^{th}$ column of $\begin{bmatrix} U_2 \\ U_4 \end{bmatrix}$ is nonnull implies $i^{th}$ row of $[U_3 \vdots U_4]$ is null. Also no row of $[U_1 \vdots U_2]$ is null and no column of $\begin{bmatrix} U_1 \\ U_3 \end{bmatrix}$ is null.

In a similar manner we get a permutation matrix $Q$ such that

$$Q'GAQ = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} = V \quad \text{(say)}$$

where $V_1$ is of order $t \times t$ (say) and the last $n-t$ rows and columns satisfy conditions similar to those above. Now consider

$$PAQ = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad \text{and} \quad Q'GP' = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$$

where $A_1$ is of order $r \times t$ and $G_1$ is of order $t \times r$.

First observe that $Q'GP'$ is a $(PAQ)_r^-$. We will show that $\begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}$ is also a $(PAQ)_r^-$.

Let $(r+i)^{th}$ column of $Q'GP'$ be nonnull. Then $(r+i)^{th}$ row of $U$ is null, so $(r+i)^{th}$ row of $PAQ$ is null. Therefore we can replace $(r+i)^{th}$ column of $Q'GP'$ by a null column and still have a $(PAQ)_r^-$. Thus we can replace all the nonnull columns of $\begin{bmatrix} G_2 \\ G_4 \end{bmatrix}$ by null columns and nonnull rows of $[G_3 : G_4]$ by null rows and get $\begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}$ which is a $(PAQ)_r^-$. So $G_1$ is a reflexive g-inverse of $A_1$. Now using lemma 2 we get that $PAQ$ is of the form $\begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$. Notice that if $A_1$ has a null row say, $i^{th}$ $(i \leq r)$, then $i^{th}$ row of $PAQ$ is ll, so $i^{th}$ row of $U$ is null which is a contradiction. Hence $A_1$ oes not have any null row. Similarly we can show that $A_1$ has no null lumn.

Finally by the symmetry of the argument we conclude that GP' is of the stated form and $G_1$ has no null row and null column , ce the theorem follows.

Below we state a lemma which is very easy to prove.

Lemma 3 : An $m \times n$ nonnegative matrix $A$ of rank $m$ possesses a nonnegative right inverse if and only if there exists a permutation matrix $P$ such that $AP = (A_1 : A_1C)$ where $A_1$ is a nonsingular diagonal matrix of order $m \times m$.

Corollary : A nonsingular nonnegative matrix $A$ has a nonnegative inverse if and only if there exists a permutation matrix $P$ such that $AP$ is a nonsingular diagonal matrix.

Proceeding to g-inverses, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ has a nonnegative g-inverse $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ but $A^+ = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$ is not nonnegative. So the existence of a nonnegative g-inverse does not ensure the nonnegativeness of the Moore-Penrose inverse as in the case of doubly stochastic matrices. Before proceeding further we state a theorem on nonnegative matrices (Gantmacher [1]).

Theorem 9 : To the maximal characteristic value $r$ of a nonnegative matrix $A$ there belong positive eigen vectors of $A$ and $A'$ if and only if $A$ can be represented by a permutation in quasidiagonal form $= \text{diag}(A_1 : A_2 \ldots A_s)$ where $A_1 \ldots A_s$ are irreducible matrices each of which has $r$ as its maximal characteristic value.

Corollary : Let $A$ be a nonnegative idempotent matrix such that all the columns and rows of $A$ are nonnull. Then there exists a permutation matrix $P$ such that $PAP' = \text{diag}(A_1 : A_2 \ldots A_s)$ when each $A_i$ is a rank one idempotent matrix and $s$ is the rank of $A$.

Proof: Since $A$ is idempotent with all the rows and columns nonnull its maximal eigen value is 1 and $x$, the vector of row sums is a positive characteristic eigen vector corresponding to the eigen value 1. $x$ is positive since $A$ does not have any null row. Similarly $y$, the vector of column sums is a positive eigen vector of $A'$ corresponding to the eigen value 1. Therefore by Theorem 9 it follows that there exists a permutation matrix $P$ such that $PAP' = \text{diag}(A_1 : A_2 \ldots A_s)$ where $A_1, A_2 \ldots A_s$ are irreducible and have 1 as a maximal characteristic root. Since $A_i$ is irreducible this maximal characteristic root is unique, i.e., simple. Also $A$ is idempotent implies $A_i$ is idempotent which implies $A_i$ is of rank 1. Therefore rank of $A$ is $s$. This completes the proof of the corollary.

Remark 2 : If $A \geq 0$ has a nonnegative g-inverse $G$, then it has a nonnegative namely $GAG$.

Next we proceed to prove

Theorem 10 : A nonnegative matrix $A$ of order $m \times n$, all the rows and columns of which are nonnull has a nonnegative reflexive g-inverse with all the rows and columns nonnull if and only if there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} B & BC \\ DB & DBC \end{bmatrix}$$

where $B$ is an $r \times r$ nonsingular diagonal matrix, $r$ is the rank of $A$ and $C$ and $D$ are such that $CC'$ and $D'D$ are diagonal.

<u>Proof</u>: 'If' part follows trivially once we observe that $A^+$ given by (iv) of Theorem 7 is nonnegative and has no null row or null column.

'Only if' part : Since $A$ and $G$ do not have null rows and columns, so $AG$ and $GA$ do not have null rows and columns. Therefore by corollary to Theorem 9 there exists a permutation matrix $P_1$ such that $P_1AGP_1' = \text{diag}(H_1 : H_2 \ldots H_r)$ where $H_i$'s are rank 1 idempotent matrices and $r$ is the rank of $A$. Let $H_i$ be of order $m_i \times m_i$. Partitioning $P_1A$ and $GP_1'$ as

$$P_1A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{bmatrix} \text{ and } GP_1' = [G_1 \ldots G_r]$$

here $A_i$ is of order $m_i \times n$ and $G_i$ is of order $n \times m_i$ it can be easily observed that $G_i$ is a $(A_i)^-$ and hence $A_i$ is of rank 1. Let $\alpha_i$ be a row of $A_i$.

Then $\alpha_1 \ldots \alpha_r$ are independent since $A$ is of rank $r$. Now we can rearrange the rows of $A$ by premultiplying by a permutation matrix as

$$P_2P_1A = \begin{bmatrix} B_1 \\ DB_1 \end{bmatrix} \quad \text{where} \quad B_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix}$$

and $D$ is such that each row of $D$ has exactly one nonzero element. So $D'D$ is diagonal. Thus we have

$$PA = \begin{bmatrix} B_1 \\ DB_1 \end{bmatrix}$$ where $P = P_2P_1$, $B_1$ is an $r \times n$ nonnegative matrix of rank $r$ and $D$ is a nonnegative matrix of order $(m-r) \times r$ such that $D'D$ is diagonal.

Partitioning $GP'$ as $(G_1 : G_2)$ where $G_1$ is of order $n \times r$ we can observe that $G_3 = G_1 + G_2D$ is a nonnegative reflexive g-inverse of $B_1$. Also it can be easily proved that all the rows and columns of both $B_1$ and $G_3$ are nonnull. Hence by applying the above argument we can show that there exists a permutation matrix $Q_1$ such that $B_1Q_1$ is of the form

$$B_1Q_1 = [B_2 \;:\; B_2C_1]$$

where $B_2$ is a $r \times r$ nonsingular matrix and $C_1$ is a nonnegative matrix of order $r \times (n-r)$ such that $CC'$ is diagonal. Now partitioning $Q_1'G_3$ as $(G_4' : G_5')'$ it can be seen that $G_6 = G_4 + C_1G_5$ is a nonnegative inverse of the nonnegative matrix $B_2$ which shows that there exists a permutation matrix $Q_2$ such that $B = B_2Q_2$ is diagonal.

So $\quad B_1Q_1Q_3 = (B : BC)$ where $C = Q_2'C_1 = C$

and $Q_3 = \begin{bmatrix} Q_2 & 0 \\ 0 & I \end{bmatrix}$ Thus denoting $Q_1Q_3$ by $Q$ we have

$$PAQ = \begin{bmatrix} B_1 \\ DB_1 \end{bmatrix} Q = \begin{bmatrix} B & BC \\ DB & DBC \end{bmatrix}$$

whre $B$ is an $r \times r$ nonsingular diagonal matrix and $C$ and $D$ are nonnegative matrices such that $CC'$ and $D'D$ are diagonal. This completes the proof of the theorem.

We note that in the partition of $P_1A$ as $\begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}$ in the proof of above theorem $A_i A_j' = 0$, $i \neq j$. This is because $B$ is diagonal and each column of $BC$ is just a constant multiple of a single column of so $\alpha_i' \alpha_j = 0$, $i \neq j \Rightarrow A_i A_j' = 0$. Thus we have

Remark 3: Every pair of rows(columns) of $A$ are either orthogonal or one is a multiple of the other.

Corollary : If a nonnegative matrix $A$ with nonnull rows and columns has a nonnegative g-inverse with nonnull rows and columns then $A^+$ is nonnegative and is of the form $A'D_1$ where $D_1$ is a diagonal matrix.

Proof: We have seen in the proof of Theorem 10 that there exists a permutation matrix $P$ such that

$$PA = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{bmatrix} \quad \text{where } A_i \text{ is of rank 1, for } i = 1 \ldots r.$$

From the Remark 3 above we have $A_i A_j' = 0$ for $i \neq j$.

Also observe that if $B$ is of rank 1 then $B^+ = \frac{1}{\text{tr } B'B} B'$.

Putting these facts together we have

$$A = P' \begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}$$

$$\begin{aligned} A^+ &= \begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}^+ P \\ &= (A_1^+ \ \dots \ A_r^+)\, P \\ &= (c_1 A_1' \ \dots \ c_r A_r')P \quad \text{where} \quad c_i = \frac{1}{\text{tr } A_i' A_i} \\ &= \left\{ D \begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix} \right\}' P \quad \text{where } D = \text{diag } (c_1 I : c_2 I \ \dots \ c_r I) \\ &= A'P'DP \\ &= A'D_1 \end{aligned}$$

where $D_1 = P'DP > 0$ which is diagonal. This proves the corollary.

Theorem 11 : A nonnegative matrix $G$ is a reflexive g-inverse of a nonnegative matrix $A$ of order $m \times n$ and rank $r$ if and only if there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1 C \\ DA_1 & DA_1 C \end{bmatrix} \quad \text{and} \quad Q'GP' = \begin{bmatrix} G_1 & G_1 E \\ FG & FG_1 F \end{bmatrix}$$

where $A_1$ and $G_1$ are $r \times r$ nonsingular matrices and C, D, E and are such that

$$G_1 = [(I + ED)\ A_1(I + CF)]^{-1}.$$

roof: 'If' part of the theorem follows from Theorem 6. To prove 'Only f' part, let G be a $A_r^-$. Then from Theorem 8 it follows that there xist permutation matrices $P_1$ and $Q_1$ such that

$$P_1AQ_1 = \begin{bmatrix} B & BU \\ VB & VBU \end{bmatrix} \quad \text{and} \quad Q_1'GP_1' = \begin{bmatrix} H & HX \\ YH & YHX \end{bmatrix}$$

where B and H have no null row and null column and H is a $B_r^-$. Now applying Theorem 10 we have for some permutation matrices $P_2$ and $Q_2$

$$P_2BQ_2 = \begin{bmatrix} A_1 & A_1C_1 \\ D_1A_1 & D_1A_1C_1 \end{bmatrix}$$

re $A_1$ is $r \times r$ nonsingular diagonal matrix and $C_1$ and $D_1$ are h that $C_1C_1'$ and $D_1'D_1$ are diagonal. Writing $Q_2'HP_2'$ as

$$\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{it follows from Theorem 6}$$

t $G_1 + C_1G_3 + G_2D_1 + C_1G_4D_1$ is $A_1^{-1}$ which implies $G_1$, $G_2D_1$ $_3$ are diagonal. Therefore $(G_1 + C_1G_3)$ and $(G_1 + G_2D_1)$ are diagonal. Observe that if any row of $(G_1 + G_2D_1)$ is null then a row of either $B$ or $H$ is null which is a contradiction. So $(G_1 + G_2D_1)$ is nonsingular. Similarly we can show that $(G_1 + C_1G_3)$ is nonsingular. Now $H$ is $B_r^-$ implies

$$G = (G_1 + G_2D_1)\ A_1(G_1 + C_1G_3)$$

which implies $G_1$ is nonsingular. So $Q_2'HP_2'$ is of the form

$\begin{bmatrix} G_1 & G_1E_1 \\ F_1G_1 & F_1G_1E_1 \end{bmatrix}$ where $G_1$ is $r \times r$ nonsingular diagonal matrix.

Therefore

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} \quad \text{and} \quad Q'GP' = \begin{bmatrix} G_1 & G_1E \\ FG_1 & FG_1E \end{bmatrix}$$

$$\text{where} \quad P = \begin{bmatrix} P_2 & 0 \\ 0 & I \end{bmatrix} P_1 \quad \text{and} \quad Q = Q_1 \begin{bmatrix} Q_2 & 0 \\ 0 & I \end{bmatrix}$$

Rest of the proof follows from Theorem 6.

<u>Remark 4</u> : In the above set up $ED$ and $CF$ are diagonal. This is because

$$G_1 = [(I + ED)A_1(I + CF)]^{-1}$$

$$\Rightarrow \quad (I + CF)G_1 = A_1^{-1}(I + ED)^{-1}$$

$$\Rightarrow \quad (I + ED)^{-1} \geq 0 \quad \text{since } A_1 \text{ and } G_1 \text{ are diagonal}$$

$\Rightarrow$ $(I + ED)$ is diagonal since $(I + ED) \geq 0$

$\Rightarrow$ $ED$ is diagonal.

Similarly we can show that $CF$ is diagonal.

<u>Theorem 12</u> : Let $A$ be a nonnegative matrix. Then

(i) $A$ has a nonnegative g-inverse if and only if there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

where $A_1$ is a nonsingular diagonal matrix

(ii) $A$ has a nonnegative least squares g-inverse if and only if $D'D$ is diagonal in the condition of (i)

(iii) $A$ has a nonnegative minimum norm g-inverse if and only if $CC'$ is diagonal in the condition of (i)

and (iv) $A$ has nonnegative $A^+$ if and only if both $D'D$ and $CC'$ are diagonal in condition (i).

<u>Proof</u> : 'If' part follows from Theorem 7 and the 'Only if' part of (i) follows from Theorem 11.

For 'Only if' part of (ii), let $G$ be a nonnegative least squares reflexive g-inverse of $A$. Then by Theorem 11 there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} \quad \text{and} \quad Q'GP' = \begin{bmatrix} G_1 & G_1E \\ FG_1 & FG_1E \end{bmatrix}$$

where $A_1$ and $G_1$ are nonsingular diagonal matrices and

$$G_1 = [(I + ED)A_1(I + CF)]^{-1}.$$

Now $AG$ is symmetric implies

$$A_1(I + CF)G_1E = [DA_1(I + CF)G_1]'$$

So $E = D'$ since $A_1(I + CF)G_1$ is nonsingular and diagonal. Therefore by Remark 4 we have $D'D$ is diagonal. We can prove the rest in a similar way.

<u>Corollary</u> : Let $A$ be a nonnegative matrix. Then

(i) $AA'$ has a nonnegative g-inverse if and only if $A$ has a nonnegative minimum norm g-inverse

(ii) $A'A$ has a nonnegative g-inverse if and only if $A$ has a nonnegative least squares g-inverse.

The result (i) follows trivially once we observe that $(A^-_m)'A^-_m$ is a $(AA')^-$ and $A'(AA')^-$ is a $A^-_m$. Proof of (ii) is similar.

## .4 g-inverses of stochastic matrices

Definition : A nonnegative matrix $A$ of order $m \times n$ is said to be ow stochastic if $\sum_{j=1}^{n} a_{ij} = 1$ for $i = 1,\ldots,m$.

efinition : A nonnegative matrix $A$ of order $m \times n$ is said to e column stochastic if $\sum_{i=1}^{m} a_{ij} = 1$ for $j = 1\ldots n$.

et $AB = C$. Then it is easy to check that $A$ and $B$ are row ochastic implies $C$ is row stochastic. Also $B$ and $C$ are row ochastic implies $A$ is row stochastic.

w we prove

orem 13 : Let $A$ be a row stochastic matrix of order $m \times n$ and k $r$. Suppose $A$ has no null column. Then

(a) a row stochastic $A^-$ exists if and only if there st permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

$A_1$ is $r \times r$ nonsingular diagonal matrix and $C$ is such $CC'$ is diagonal

(b) a row stochastic $A_\ell^-$ exists if and only if $D'D$ is al in the condition of (a)

(c) a row stochastic $A_m^-$ exists if and only if $C$ is column stochastic in the condition of (a)

(d) $A^+$ is row stochastic if and only if $C$ is column stochastic and $D'D$ is diagonal in the condition of (a).

Proof: For proving 'if part' of the theorem we construct the respective row stochastic g-inverses under the hypothesis. Let $X$ be an $r \times (n-r)$ matrix such that

$$x_{ij} = 1 \quad \text{if } c_{ij} \neq 0$$
$$= 0 \quad \text{otherwise.}$$

Let
$$G = Q \begin{bmatrix} G_1 & G_1 E \\ X'G_1 & X'G_1 E \end{bmatrix} P$$

Now the following statements are easy to verify under the respective hypothesis :

(a) $G$ is a row stochastic $A_r^-$ for $G_1 = I$ and $E = 0$

(b) $G$ is a row stochastic $A_{\ell r}^-$ for $G_1 = (I + D'D)^{-1}$ and $E = D'$

(c) $G$ is a row stochastic $A_{mr}^-$ for $G_1 = I$ and $E = 0$ and

(d) $G$ is row stochastic and $A^+$ for $G = (I + D'D)^{-1}$ and $E = D'$.

To prove 'only if' part of theorem, let $G$ be a row stochastic reflexive g-inverse of $A$. Then by Theorem 11 and Remark 4 it follows that there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix} \quad \text{and} \quad Q'GP' = \begin{bmatrix} G_1 & G_1E \\ FG_1 & FG_1E \end{bmatrix}$$

where $A_1$ and $G_1$ are $r \times r$ nonsingular diagonal matrices and $C$, $D$, $E$ and $F$ are such that $CF$ and $ED$ are diagonal and

$$(I + CF)\, G_1 (I + ED) = A_1^{-1}$$

$A$ and $G$ are row stochastic implies $D$ and $F$ are row stochastic.

Now $CF$ is diagonal implies every column of $C$ has atmost one nonzero element. So $CC'$ is diagonal. This completes the proof of 'only if' part of (a).

In addition to $A_r^-$ let $G$ be also a $A_\ell^-$. Then $AG$ is symmetric which implies $C = F'$. So $C$ is column stochastic. Similarly, if $G$ is a row stochastic $A_{mr}^-$ then we can show that $D = E'$ which implies $D'D$ is diagonal, since $ED$ is diagonal. Finally the 'only if' part of (d) follows from those of (b) and (c). This completes the proof of the theorem.

Remark 5 : We can relax the condition that $A$ has no null column in the case of (a) and (b). However it is necessary for the existence of row stochastic $A_m^-$ and $A^+$.

Similarly, for column stochastic matrices we have

Theorem 14 : Let $A$ be a column stochastic matrix of order $m \times n$ and rank $r$. Suppose $A$ has no null row. Then

(a) a column stochastic $A^-$ exists if and only if there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

where $A_1$ is $r \times r$ nonsingular diagonal matrix and $D$ is such that $D'D$ is diagonal

(b) a column stochastic $A_\ell^-$ exists if and only if $CC'$ is diagonal in the condition of (a).

(c) a column stochastic $A_m^-$ exists if and only if $D$ is row stochastic in the condition of (a)

(d) $A^+$ is column stochastic if and only if $D$ is row stochastic and $CC'$ is diagonal in the condition of (a).

ally we have

rem 15 : Let $A$ be a doubly stochastic matrix of order $n \times n$ rank $r$ then it has a doubly stochastic g-inverse if and only if e exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & A_1C \\ DA_1 & DA_1C \end{bmatrix}$$

where $A_1$ is $r \times r$ nonsingular diagonal matrix and $C$ and $D$ are such that both $CC'$ and $D'D$ are diagonal.

The theorem follows from the Theorems 13 and 14.

Corollary : A doubly stochastic matrix A has a doubly stochastic g-inverse if and only if there exist permutation matrices P and Q such that

$$PAQ = \text{diag}(J_1 \ldots J_r)$$

where $J_i$ is rank 1 doubly stochastic matrix i.e., all the elements of $J_i$ are equal to $\frac{1}{n_i}$ where $n_i$ is the order of $J_i$.

## 2.5 Algorithm for computing a nonnegative g-inverse

Let A be the given nonnegative matrix of order $m \times n$. To compute a nonnegative g-inverse of A we proceed as in chapter 1. We use the same algorithm of chapter 1 (Page 36) and get $k, i_1, i_2 \ldots i_k, j_1, j_2 \ldots j_k$ and $C_k$. Let G be such that

$$g_{ij} = \frac{1}{(C_k)_{ij}} \quad \text{if } (C_k)_{ij} \neq 0$$
$$= 0 \quad \text{otherwise.}$$

The following theorem is easy to prove

eorem 16 : A nonnegative $A^-$ exists if and only if G is a $A^-$.

<u>Remark 6</u> : Here again, the algorithm always gives a nonnegative matrix $G$ whether $A$ has a nonnegative g-inverse or not. So we have to check whether $G$ is $A^-$ or not, after computing $G$.

As in the case of Boolean matrices here also conditions for existence and computation of other types of g-inverses are tabulated in Table 5.

Let $A$ be an $m \times n$ matrix and $A$ exist. Let $G_1$ be the matrix with $i_p^{th}$ row as the $i_p^{th}$ row of $A$ devided by the sum of squares of elements in that row, for $p = 1,2 \dots k$ and rest of the rows are null. Let $G_2$ be the matrix with $j_p^{th}$ column as the $j_p^{th}$ column of $A$ devided by the sum of squares of elements in that column. Let $w_1, w_2, \dots, w_n$ be the column weights of $G_1$ and $w_1', w_2' \dots w_m'$ be the row weights of $G_2$. Let $G_3$ and $G_4$ be as follows :

$$(G_3)_{1j} = 1 \quad \text{if} \quad (G_1)_{ij} = 0 \quad \forall\, i$$
$$(G_3)_{ij} = 1 \quad \text{if} \quad (G_1)_{ij} \neq 0$$
$$= 0 \quad \text{otherwise}$$

and

$$(G_4)_{i1} = 1 \quad \text{if} \quad (G_2)_{ij} = 0 \quad \forall\, j$$
$$(G_4)_{ij} = 1 \quad \text{if} \quad (G_2)_{ij} \neq 0$$
$$= 0 \quad \text{otherwise}$$

A vector $x$ is said to satisfy the condition $C$ if all the nonzero elements of $x$ are equal.

Table 5

| Type of A | Type of $A^-$ | Condition for existence | Given by |
|---|---|---|---|
| Nonnegative | Nonnegative | | |
| | $A_m^-$ | $w_i \leq 1 \ \forall i$ | $G_1'$ |
| | $A_\ell^-$ | $w_i' \leq 1 \ \forall i$ | $G_2'$ |
| | $A^+$ | $w_i \leq 1$ and $w_i' \leq 1 \ \forall i$ | $G_1'C_kG_2'$ |
| Row stochastic | Row stochastic | | |
| | $A^-$ | $w_i \leq 1 \ \forall i$ | $G_3'$ |
| | $A_m^-$ | $w_i=1 \ \forall i$ and rows of $G_1$ satisfy condition C | $G_3'$ |
| | $A_\ell^-$ | $w_i \leq 1$ and $w_i' = 1 \ \forall i$ | $G_3'C_kG_2'$ |
| | $A^+$ | $w_i=1$ and $w_i'=1 \ \forall i$ and rows of $G_1$ satisfy condition C | $G_3'C_kG_2'$ |
| Column stochastic | Column stochastic | | |
| | $A^-$ | $w_i' \leq 1 \ \forall i$ | $G_4'$ |
| | $A_m^-$ | $w_i' \leq 1$ and $w_i = 1 \ \forall i$ | $G_1'C_kG_4'$ |
| | $A_\ell^-$ | $w_i'=1 \ \forall i$ and columns of $G_2$ satisfy condition C | $G_4'$ |
| | $A^+$ | $w_i'=1$ and $w_i=1 \ \forall i$ and columns of $G_2$ satisfy condition C | $G_1'C_kG_4'$ |
| Doubly stochastic | Doubly stochastic $A^-$, $A_\ell^-$, $A_m^-$, $A^+$ | $w_i=1$ and $w_i'=1 \ \forall i$ and rows and columns of A satisfy condition C | $A'$ |

[A version of Theorem 4, Corollary to Theorem 10 and parts (i) and (iv) of Theorem 12 are also independently obtained by Plemmons [8] and Plemmons and Cline [9].]

## C H A P T E R 3

## CHARACTERISATIONS OF MERELY POSITIVE SUBDEFINITE MATRICES AND RELATED RESULTS

### 3.1 Introduction and Summary

The concept of quasiconvex and pseudoconvex quadratic forms which play an important role in mathematical programming problems lead to a new subclass of real symmetric matrices, namely positive subdefinite (PSubD) matrices. Martos [ 4 ] made an interesting study of these matrices where he proves some nice properties of merely positive subdefinite (MPSubD) matrices, matrices which are PSubD but not positive semi-definite (PSemiD). He wondered whether some of the properties of these matrices, proved by him, would characterise the MPSubD matrices.

The object of this chapter is to answer his question in the affirmative, there by obtaining an interesting characterisation of MPSubD matrices. We obtain another characterisation of MPSubD matrices similar to the one of PSemiD matrices. These characterisations provide an easy recognition of quasiconvex and pseudoconvex quadratic forms. We study these matrices with respect to the generalized inverse also. It is well known that a PSemiD matrix has a PSemiD g-inverse. However as we show, barring trivial cases MPSubD matrices do not possess MPSubD g-inverses.

For completeness we give some definitions and state some theorems of Martos.

<u>Definition</u> : The real symmetric $n \times n$ matrix $A$ is positive semi-definite (PSemiD) if for any n-vector $x$

$$x'Ax \leq 0 \text{ implies } Ax = 0$$

<u>Definition</u> : The real symmetric $n \times n$ matrix $A$ is positive subdefinite (PSubD) if for any n-vector $x$

$x'Ax < 0$ implies $Ax$ is either nonnegative or nonpositive.

<u>Definition</u> : The real symmetric $n \times n$ matrix $A$ is strictly positive subdefinite if for any n-vector $x$

$x'Ax < 0$ implies $Ax$ is either strictly positive or strictly negative.

<u>Definition</u> : A PSubD matrix which is not PSemiD is called merely positive subdefinite (MPSubD).

<u>Definition</u> : A quadractic form $Q(x) = x'Ax$ is convex in the set $X$, if for all $x_1, x_2 \in X$,

$$2(x_1 - x_2)' Ax_1 \geq x_1'Ax_1 - x_2'Ax_2$$

<u>efinition</u> : $Q(x) = x'Ax$ is quasiconvex in the set $X$, if for all $, x_2 \in X$,

$$x_1'Ax_1 - x_2'Ax_2 \geq 0 \text{ implies } (x_1-x_2)'Ax_1 \geq 0$$

Definition : $Q(x) = x'Ax$ is pseudoconvex in the set $X$, if for all $x_1, x_2 \in X$

$$x_1'Ax_1 - x_2'Ax_2 > 0 \text{ implies } (x_1 - x_2)' Ax_1 > 0.$$

Theorem 1 (Martos) : An MPSubD matrix

(a) is nonpositive

(b) has exactly one (simple) negative eigen value,

and (c) has the corresponding eigen vector either nonnegative or nonpositive.

Theorem 2 (Martos) : $Q(x) = x'Ax$ is convex for every $x$ if and only if $A$ is PSemiD.

Theorem 3 (Martos) : $Q(x) = x'Ax$ is quasiconvex for every nonnegative $x(x \geq 0)$ if and only if $A$ is PSubD.

Theorem 4 (Martos) : $Q(x) = x'Ax$ is pseudoconvex for every nonnegative (nonnull) $x$ if and only if $A$ is strictly PSubD.

In the next section we prove the converse of Theorem 1 of Martos, thereby obtaining characterisation for MPSubD matrices. As the characterisations of PSemiD matrices are well known, these together characterise PSubD matrices. Thus, with the help of these characterisations, in view of Theorems 3 and 4 of Martos, quasiconvex and pseudoconvex quadratic forms are easy to identify.

Throughout this chapter $x^i$ denotes the $i^{th}$ element of the vector $x$ , and $A_k$ denotes the leading principal submatrix of order k.

## 3.2 Characterisations of MPSubD matrices

We prove

Theorem 5 : A nonpositive **symmetric** matrix, having exactly one (simple) negative eigen value, is MPSubD.

Proof : Let A be a nonpositive symmetric matrix having exactly one negative eigen value $\lambda_1$. Therefore the eigen vector of A corresponding to $\lambda_1$ is semi-unisigned* Consider the spectral decomposition of A

$$A = \lambda_1 \, p_1p_1' + \lambda_2 \, p_2p_2' + \dots + \lambda_r \, p_rp_r' + \dots + \lambda_n \, p_np_n'$$

where $\lambda_2,\dots,\lambda_r$ are positive eigen values of A are $\lambda_{r+1},\dots,\lambda_n$ are zero eigen values of A and $p_1, p_2 \dots p_n$ is the orthonormal set of eigen vectors of A corresponding to $\lambda_1,\lambda_2,\dots,\lambda_n$. Without loss of generality let $p_1 \geq 0$ because $p_1$ is semi-unisigned.

erefore

$$A = \lambda_1p_1p_1' + \lambda_2p_2p_2' + \dots + \lambda_rp_rp_r' \qquad \dots (1)$$

$x = c_1p_1 + c_2p_2 + \dots + c_np_n$ be any vector, then

$$Ax = \lambda_1c_1p_1 + \lambda_2c_2p_2 + \dots + \lambda_rc_rp_r$$

---

* A vector x is called semi-unisigned if $x \leq 0$ or $x \geq 0$ and x is called unisigned if $x < 0$ or $x > 0$

and

$$(Ax)^i = \lambda_1 c_1 p_1^i + \lambda_2 c_2 p_2^i + \ldots + \lambda_r c_r p_r^i \qquad \text{for } i = 1,2,\ldots,n$$

$$x'Ax = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \ldots + \lambda_r c_r^2.$$

To show that $x'Ax < 0 \Rightarrow Ax \gtrless 0$. Let $c_1 \geq 0$.

$$x'Ax < 0 \Rightarrow c_1 \neq 0, \text{ therefore}$$

$$-\lambda_1 > \frac{1}{c_1^2} [\lambda_2 c_2^2 + \ldots + \lambda_r c_r^2] = (u, u) \qquad \ldots(2)$$

where $u' = \frac{1}{c_1} (\sqrt{\lambda_2}\, c_2, \ldots, \sqrt{\lambda_r} c_r)$

Since $A \leq 0$

$$a_{ii} = \lambda_1 (p_1^i)^2 + \lambda_2 (p_2^i)^2 + \ldots + \lambda_r (p_r^i)^2 \leq 0$$

Now $p_1^i = 0 \Rightarrow p_2^i = p_3^i = \ldots = p_r^i = 0 \Rightarrow (Ax)^i = 0$

Otherwise if $p_1^i \neq 0$

$$-\lambda_1 \geq \frac{1}{(p_1^i)^2} [\lambda_2 (p_2^i)^2 + \ldots + \lambda_r (p_r^1)^2] = (v, v) \qquad \ldots(3)$$

Where $v' = \frac{1}{p_1^i} (\sqrt{\lambda_2}\, p_2^i, \ldots, \sqrt{\lambda_r}\, p_r^i)$

(2) and (3) $\Rightarrow$

$$-2\lambda_1 > (u,u) + (v,v) \geq 2(u,v)$$

$$\Rightarrow \quad -\lambda_1 > (u,v)$$

$$\Rightarrow \quad -\lambda_1 > \frac{1}{c_1 p_1^i} (\lambda_2 c_2 p_2^i + \dots + \lambda_r c_r p_r^i)$$

$$\Rightarrow \quad \lambda_1 c_1 p_1^i + \lambda_2 c_2 p_2^i + \dots + \lambda_r c_r p_r^i < 0 \qquad \dots (4)$$

that is $(Ax)^i < 0$ for all $i$ such that $p_1^i \neq 0$

Therefore $(Ax)^i \leq 0$ for all $i$

Hence $Ax \leq 0$.

Similarly if $c_1 < 0$ the inequality in (4) changes and $Ax \geq 0$.

Hence the theorem.

In view of Theorem 1 of Martos [4] we thus have

<u>Theorem 6</u> : A real symmetric matrix $A$ is MPSubD if and only if

(a) $A \leq 0$

and (b) has exactly one (simple) negative eigen value.

<u>Remark 1</u> : It is interesting to note that if $A$ is MPSubD and $x'Ax < 0$ then $(Ax)^i = 0$ if and only if $p_1^i = 0$, that is if and only if $i^{th}$ row and $i^{th}$ column of $A$ are null. Hence we have the following result.

An MPSubD matrix is strictly PSubD if and only if $p_1$ is unisigned that is, if and only if $A$ is irreducible (Gantmacher [1])

Also it is easy to observe that if $A$ is MPSubD then $a_{ij} = 0$ implies either $a_{ii}$ or $a_{jj}$ or both are zero, and in the case of strictly PSubD $a_{ij} = 0 \Rightarrow a_{ii} = a_{jj} = 0$.

We need the following lemma in the proof of Theorem 7. This lemma is also of independent interest.

Lemma 1 : If $A$ is an $n \times n$ MPSubD matrix and $B$ is a non-negative matrix of order $n \times p$ then $B'AB$ is also MPSubD, provided it is not null.

Proof : $x'B'ABx < 0$

$\Rightarrow y'Ay < 0$ where $y = Bx$

$\Rightarrow Ay = ABx \gtrless 0$

$\Rightarrow B'ABx \gtrless 0$ since $B \geq 0$

Thus $B'AB$ is MPSubD.

Remark 2 : The above lemma holds for $B \leq 0$ also.

It is known that a square matrix $A$ is PSemiD if and only if all its principal minors are non-negative. A similar characterisation for MPSubD matrices is proved below using a separation theorem (Wilkinson [15], pp 103).

The eigen values $\lambda_1', \lambda_2', \ldots, \lambda_{n-1}'$ of the leading principal minor matrix $A_{n-1}$ of the symmetric matrix $A_n$ separate the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A_n$.

Theorem 7 : A nonpositive symmetric matrix $A(\neq 0)$ is MPSubD if and only if all its principal minors are nonpositive.

Proof : 'If' part : The proof is by induction. Assuming $A_k = 0$ or MPSubD we will prove $A_{k+1}$ is null or MPSubD. To show $A_{k+1}$ is MPSubD it is enough to show that it has exactly one simple negative eigen value as the result follows from Theorem 5.

To start with $A_1 = a_{11} \leq 0 \Rightarrow A_1$ is 0 or MPSubD. Notice that $R(A_k) \leq R(A_{k+1}) \leq R(A_k) + 2$.

Case 1. Let $A_k = 0$.

If $A_{k+1}$ is also null we are done. Otherwise if $R(A_{k+1}) = 1$ which implies there is only one nonzero eigen value of $A_{k+1}$, which has to be negative since $tr(A_{k+1}) \leq 0$. On the other hand if $R(A_{k+1}) = 2$ then out of the nonzero eigen values of $A_{k+1}$ one is positive and the other is negative because of separation theorem.

Case 2. Let $A_k$ is MPSubD. We will show that $A_{k+1}$ is also MPSubD. Denoting by $\lambda_1', \lambda_2', \ldots, \lambda_k'$, the eigen values of $A_k$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ the eigen values of $A_{k+1}$ in increasing order we have by separation theorem $\lambda_1 < 0$, $\lambda_2 \gtreqless 0$ and $\lambda_3, \ldots, \lambda_{k+1}$ are nonnegative. Now if $\lambda_2 \geq 0$ then $\lambda_1$ is the only negative eigen value of $A_{k+1}$ and hence the result. Otherwise that is $\lambda_2 < 0$ we will show a contradiction.

Let $R(A_{k+1}) = m$. Then there exists an $m^{th}$ order nonzero principal minor of $A_{k+1}$, which can be brought to $m^{th}$ order leading principal minor of $A_{k+1}$ by using same permutation on rows and columns of $A_{k+1}$. As MPSub definiteness is undistorted by these operations (Lemma 1) without loss of generality we can assume that the m-th order leading principal minor of $A_{k+1}$ is nonzero. Therefore by hypothesis m-th order leading principal minor of $A_{k+1}$ is negative.

Considering the spectral decomposition of $A_{k+1}$ we have

$$A_{k+1} = P \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} P' = \begin{bmatrix} P_1MP_1' & P_1MP_3' \\ P_3MP_1' & P_3MP_3' \end{bmatrix}$$

where $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ is an orthogonal matrix and $M$ is diagonal matrix of $m^{th}$ order with diagonal elements as the nonzero eigen values of $A_{k+1}$. So det $M > 0$ as there are exactly two negative eigen values of $A_{k+1}$. Thus m-th order principal minor of $A_{k+1}$, that is det $P_1MP_1' \geq 0$ which is contradiction. Therefore $\lambda_2 \neq 0$. Hence $\lambda_1$ is the only negative eigen values of $A_{k+1}$. That proves the 'if' part.

'Only if' part : Given $A$ is MPSubD, to show that every principal minor of $A \leq 0$. To show this given any $r^{th}$ order principal minor there exists a permutation matrix $P$ such that the given minor is the leading principal minor of $B = PAP'$. Now consider

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} PAP' \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_r & 0 \\ 0 & 0 \end{bmatrix}$$

where $I_r$ is the identity matrix of order $r$. From Lemma 1 and corollary of Martos [ 4 ] it follows that $B_r$ is MPSubD.

Hence the theorem.

## 3.3 g-inverses of MPSubD matrices

When does a PSubD matrix have a PSubD g-inverse ? Notice that a PSubD matrix is either a PSemiD matrix or MPSubD matrix. It is known that a PSemiD matrix always possess a PSemiD g-inverse. So our main interest is towards the class of MPSubD matrices, that is, when does an MPSubD matrix have a PSubD g-inverse ? Noticing the fact that an MPSubD matrix cannot possess a PSemiD g-inverse, since a symmetric matrix $A$ have a PSemiD g-inverse if and only $A$ is PSemiD. So the only possibility is MPSubD matrix has to possess MPSubD g-inverse. A necessary and sufficient condition for an MPSubD matrix to have an MPSubD g-inverse is established in Theorem 9. Before that we prove a theorem on symmetric reflexive g-inverses of symmetric matrices.

<u>Theorem 8</u> : If $A$ is a symmetric matrix of order $n$ and rank $r$ then every symmetric reflexive g-inverse $G$ of $A$ can be written as

$$G = \frac{1}{\lambda_1} q_1 q_1' + \ldots + \frac{1}{\lambda_r} q_r q_r'$$

where $\lambda_1, \ldots, \lambda_r$ are nonzero eigen values of $A$ and $q_1, \ldots, q_r$ are independent vectors.

<u>Proof</u>: Consider spectral decomposition of $A$

$$A = P \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} P' = \lambda_1 p_1 p_1' + \lambda_2 p_2 p_2' + \ldots + \lambda_r p_r p_r'$$

where $P = [p_1 : p_2 : \ldots : p_r : \ldots p_n]$ is orthogonal matrix of eigen vectors of $A$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2 \ldots \lambda_r)$.

It is well known that $G$ is a symmetric reflexive g-inverse of $A$ if and only if $G$ is of the form

$$G = P \begin{bmatrix} \Lambda^{-1} & U \\ U' & U'\Lambda U \end{bmatrix} P'$$

where $U$ is arbitrary. Partitition $P$ as $[P_1 : P_2]$ and let $U' = (u_1 u_2 \ldots u_r)$. Consider

$$q_i = p_i + \lambda_i P_2 u_i \qquad \text{for } i = 1, 2, \ldots, r$$

Notice that $q_i$'s are independent .

Therefore $Q = [q_1 : q_2 : \ldots : q_r] = P_1 + P_2 U'\Lambda$

Now

$$G = P \begin{bmatrix} \Lambda^{-1} & U \\ U' & U'\Lambda U \end{bmatrix} P'$$

$$= [P_1 : P_2] \begin{bmatrix} \Lambda^{-1} & U \\ U' & U'\Lambda U \end{bmatrix} \begin{bmatrix} P_1' \\ \cdots \\ P_2' \end{bmatrix}$$

$$= P_1\Lambda^{-1}P_1' + P_1UP_2' + P_2U'P_1' + P_2U'\Lambda UP_2'$$

$$= (P_1 + P_2U'\Lambda) \quad \Lambda^{-1}(P_1' + \Lambda UP_2')$$

$$= Q\Lambda^{-1}Q'$$

$$= \frac{1}{\lambda_1} q_1q_1' + \frac{1}{\lambda_2} q_2q_2' + \cdots + \frac{1}{\lambda_r} q_rq_r'$$

which proves the result.

Remark 3 : Observe that $p_i'q_j = 0$ for $i \neq j$ and $p_i'q_i = 1$.

Hence $Q$ is a right inverse of $P_1'$.

Again if $Q$ is any right inverse of $P_1'$ then $Q\Lambda^{-1}Q'$ is a symmetric reflexive g-inverse of $A$. Thus we have the following result.:

If $A$ is a symmetric matrix then $G$ is a symmetric reflexive g-inverse of $A$ if and only if it is of the form

$$G = Q\Lambda^{-1}Q'$$

where $Q$ is a right inverse of $P_1'$.

Now we prove

<u>Theorem 9</u> : Let $A$ be any MPSubD matrix, then the following statements are equivalent.

(a) there exists an MPSubD g-inverse of $A$.

(b) $R(A) = 1$ or $R(A) = 2$ and the two nonzero eigen values of $A$ are of same magnitude.

(c) $A^+$ is MPSubD.

<u>Proof</u>: (a) => (b). Let $G_1$ be an MPSubD g-inverse of the MPSubD matrix $A$. Then $G = G_1AG_1$ is a reflexive g-inverse of $A$. From Lemma 1 it follows that $G$ is also MPSubD. Let $R(A) = r$

$$A = \lambda_1 p_1 p_1' + \lambda_2 p_2 p_2' + \cdots + \lambda_r p_r p_r'$$

as in Theorem 5. From Theorem 8 it follows that

$$G = \frac{1}{\lambda_1} q_1 q_1' + \frac{1}{\lambda_2} q_2 q_2' + \cdots + \frac{1}{\lambda_r} q_r q_r'$$

where $q_i = p_i + \lambda_i P_2 u_i$ for some $u_i$.

$Gp_1 = \frac{1}{\lambda_1} q_1 \Rightarrow q_1 \geq 0$ since $G \leq 0$ and $p_1 \geq 0$.

Since $G$ is MPSubD we have

$$g_{ii} = \frac{1}{\lambda_1}(q_1^i)^2 + \frac{1}{\lambda_2}(q_2^i)^2 + \dots + \frac{1}{\lambda_r}(q_r^i)^2 \leq 0 \qquad \dots(5)$$

Now $q_1^i = 0 \Rightarrow q_2^i = q_3^i = \dots = q_r^i = 0 \Rightarrow g_{ii} = 0$

Otherwise

$q_1^i \neq 0 \Rightarrow$

$$1 \geq (x,x) \quad \text{where} \quad x' = \frac{1}{q_1^i}\left(\sqrt{\frac{-\lambda_1}{\lambda_2}}\, q_2^i, \dots, \sqrt{\frac{-\lambda_1}{\lambda_r}}\, q_r^i\right)$$

Since $A$ is MPSubd

$$a_{ii} = \lambda_1(p_1^i)^2 + \lambda_2(p_2^i)^2 + \dots + \lambda_r(p_r^i)^2 \leq 0 \qquad \dots(6)$$

$$\Rightarrow 1 \geq (y,y) \text{ where } y' = \frac{1}{p_1^i}\left(\sqrt{\left(\frac{\lambda_2}{-\lambda_1}\right)}\, p_2^i, \dots, \sqrt{\left(\frac{\lambda_r}{-\lambda_1}\right)}\, p_r^i\right)$$

provided $p_1^i \neq 0$. In case $p_1^i = 0$ then $a_{ii} = 0$.

Since $2(x,y) \leq (x,x) + (y,y)$

$$\Rightarrow \frac{2}{p_1^i q_1^i}(p_2^i q_2^i + \dots + p_r^i q_r^i) \leq (x,x) + (y,y) \leq 2.$$

for all $i$ such that $p_1^i \neq 0$ and $q_1^i \neq 0$

$$\Rightarrow p_1^i q_1^i \geq p_2^i q_2^i + p_3^i q_3^i + \dots + p_r^i q_r^i \quad \text{for all } i.$$

Summing over $i$, we have

$$\Sigma\, p_1^i q_1^i \geq \Sigma\, p_2^i q_2^i + \ldots + \Sigma\, p_r^i q_r^i$$

$\Rightarrow\ 1 \geq r-1$

$\Rightarrow\ r \leq 2.$ ...(7)

Since $A \neq 0$ therefore $r = 1$, or $2$.

If $r = 2 \Rightarrow A = \lambda_1 p_1 p_1' + \lambda_2 p_2 p_2'$ and

Equality sign occurs in (7)

$\Rightarrow$ Equality sign occurs in (6)

$\Rightarrow$ tr $A = 0$

$\Rightarrow\ \lambda_1 = -\lambda_2$

that is both the nonzero eigen values are of same magnitude.

(b) => (c) If $r = 1$

$$A = \lambda_1 p_1 p_1'$$

$$\Rightarrow A^+ = \frac{1}{\lambda_1} p_1 p_1' = \left(\frac{1}{\lambda_1}\right)^2 A$$

Therefore $A^+$ is MPSubD.

If $r = 2$ and $\lambda_1 = -\lambda_2$

$$A = \lambda_1 (p_1 p_1' - p_2 p_2')$$

$$A^+ = \frac{1}{\lambda_1}(p_1 p_1' - p_2 p_2') = \left(\frac{1}{\lambda_1}\right)^2 A$$

$\Rightarrow$ $A^+$ is MPSubD.

(c) => (a) is obvious.

__Remark 4:__ When $r = 2$ and $\lambda_1 = -\lambda_2$ in the above theorem then $p_1 = |p_2|$ where $|p_2|$ is the vector obtained by replacing each element of $p_2$ by its modulus. Because

$$0 = a_{ii} = \lambda_1[(p_1^i)^2 - (p_2^i)^2]$$

$$\Rightarrow (p_1^i)^2 = (p_2^i)^2$$

$$\Rightarrow p_1 = |p_2|$$

Thus we observe that barring trivialities an MPSubO matrix does not possess an MPSubO g-inverse.

CHAPTER 4

# AN APPLICATION AND ALGORITHM

## 4.1 Introduction and Summary

In section 2 of this chapter a result of Milliken [5] on linear estimability, is extended. It is shown that his results hold in a more general set up than the one considered by him.

In section 3, a theorem is proved, based on which an algorithm for computing a g-inverse of a (real) matrix is developed. This algorithm is an extension of Goldfarbs [2] modified method for inverting nonsingular matrices.

## 4.2 A result on linear estimability

Milliken [5] gave a necessary and sufficient condition for the estimability of $A\beta$ in the linear model

$$y = X\beta + \varepsilon \qquad \ldots(1)$$

where $y$ is an $n \times 1$ random vector, $X$ is an $n \times p$ matrix of known coefficients, $\beta$ is a $p \times 1$ vector of unknown parameters and $\varepsilon$ is a random vector with $E(\varepsilon) = 0$ and $D(\varepsilon) = \sigma^2 I$. ($D(.)$ denotes the dispersion matrix). He took $A$ to be a $k \times m$ matrix of rank $k$. His Theorem 2.1 in our notation can be stated as follows :

Theorem 1 (Milliken) : Consider the model (1). $A\beta$ is estimable if and only if $R(X(I - A^{+}A)) = R(X) - R(A)$ where $A$ is a $k \times p$ matrix of rank $k$.

The object of this present section is to prove a more general theorem than Theorem 2.1 of Milliken by employing more elementary techniques and also give analogous theorems to those in section 3 of Milliken.

We state below a lemma of Mitra (1972) which is easy to prove using rank factorisation of matrices. This lemma is used in the proof of the main theorem.

Lemma 1 : $R(A+B) = R(A) + R(B)$ if and only if $\mathcal{M}(A) \cap \mathcal{M}(B) = \{\phi\}$ and $\mathcal{M}(A') \cap \mathcal{M}(B') = \{\phi\}$, where $\phi$ denotes null vector.

We now prove

Theorem 2 : Consider the model (1). $A\beta$ is estimable if and only if $R(X(I-A^{-}A)) = q-k$ where $R(X) = q$ and $R(A) = k$ (the number of rows in $A$ need not be $k$) and $A^{-}$ is any g-inverse of $A$.

Proof. 'If' part :

Write $X = XA^{-}A + X(I-A^{-}A)$. From the hypothesis it follows that

$$R(XA^{-}A) = k = R(A) \qquad \ldots(2)$$

By Lemma 1, we now have

$$\mathcal{M}(XA^{-}A) \cap \mathcal{M}(X(I-A^{-}A)) = \{\phi\} \qquad \ldots(3)$$

Let $\alpha \in \mathcal{O}(X')$. Then, $\phi = X\alpha = XA^-A\alpha + X(I - A^-A)\alpha$.
By (3) it follows that $XA^-A\alpha = \phi$. From (2), we have, $DXA^-A = A$ for some $D$. Thus $A\alpha = DXA^-A\alpha = 0$ which implies $\alpha \in \mathcal{O}(A')$. Hence $\mathcal{M}(A') \subset \mathcal{M}(X')$ and consequently $A\beta$ is estimable.

'Only if' part : $A$ is estimable $\Rightarrow A = DX$ for some $D$. Write

$X = XA^-A + X(I - A^-A)$. Let $\alpha \in \mathcal{M}(XA^-A) \cap \mathcal{M}(X(I-A^-A))$. Then

$$\alpha = XA^-A\theta_1 = X(I-A^-A)\theta_2 \quad \text{for some } \theta_1 \text{ and } \theta_2$$

$$\Rightarrow \quad D\alpha = A\theta_1 = \phi$$

$$\Rightarrow \quad \alpha = \phi$$

Hence $\mathcal{M}(XA^-A) \cap \mathcal{M}(X(I-A^-A)) = \{\phi\}$. Clearly $\mathcal{M}((XA^-A)') \cap \mathcal{M}((X(I-A^-A))') = \{\phi\}$. Therefore by Lemma 1 $R(X) = R(XA^-A) + R(X(I-A^-A))$. Further

$k = R(A) = R(DXA^-A) \leq R(XA^-A) \leq R(A) = k$

Hence $R(XA^-A) = k$ and $R(X(I-A^-A)) = q - k$

This completes the proof of Theorem 2.

Martos also proved a few theorems on testing of hypothesis about estimable linear combinations. We now state a theorem which is analogous to Theorems 3.1 and 3.2 of Milliken.

For the linear model (1) consider testing of hypothesis

$$H_0 : A\beta = \phi \quad \text{against} \quad A\beta \neq \phi \qquad \ldots(4)$$

where $A\beta$ is estimable.

Theorem 3 : The restricted model used to obtain the sum of squares due to the null hypothesis (4) is

$$Y = X(I - A^{-}A)\beta + \varepsilon \quad \text{where} \quad E(\varepsilon) = 0 \quad \text{and} \quad D(\varepsilon) = \sigma^2 I.$$

The sum of squares due to the hypothesis is

$$Q = Y'[XX_{\ell}^{-} - (X(I-A^{-}A))\ (X(I-A^{-}A))_{\ell}^{-}]Y.$$

$Q.\sigma^{-2}$ is distributed as a noncentral chi-square with $k$ degrees of freedom.

Proof is easy and hence we omit.

Remark 1 : Theorem 2 remains valid even if $D(\varepsilon) = \sigma^2\Lambda$ where $\Lambda$ is any positive definite matrix and Theorem 2 holds (when $D(\varepsilon) = \sigma^2\Lambda$) with obvious modifications.

The following theorem is worthnoting.

Theorem 4 : Let $X$ be any $n \times p$ matrix. If $R(X(I-GA)) = R(X) - R(A)$ then $G$ is a g-inverse of $A$.

Proof : By Frobenius inequality, we have

$$R(X) + R(I-GA) \leq p + R(X(I-GA)) = p + R(X) - R(A)$$

$\Rightarrow R(I-GA) \leq p - R(A)$

Also $R(I-GA) \geq p - R(GA) \geq p - R(A)$

Therefore $R(I - GA) = p - R(A)$ and hence by Theorem 2.2.1 of Rao and Mitra (1971) it follows that $G$ is a $A^-$.

Remark 2 : Combining Theorems 2 and 4 we have the following: Consider the set up (I). $A\beta$ is estimable if and only if there exists a matrix $G$ such that $R(X(I-GA)) = R(X) - R(A)$ in which case $G$ is a g-inverse of $A$.

## 4.3 An algorithm to compute a generalized inverse of a matrix

In this section we present an algorithm to compute a g-inverse of a matrix. This algorithm is a generalisation of Goldfarbs modified method of computing inverse of a nonsingular matrix.

Let $A$ be an $m \times n$ matrix. Without loss of generality let $m \leq n$ otherwise we can take $A'$ and compute $(A')^-$.

Choose an $n \times m$ matrix $B_o$ such that $R(B_oA) = R(A)$ as an initial approximation. Let $Z$ be an orthonormal basis of $E^n$. Compute

$$B_k = B_{k-1} - \frac{(B_{k-1}A\, y_k - x_k)\, x_k'\, B_{k-1}}{x_k'\, B_{k-1}\, A\, y_k} \qquad \ldots(5)$$

for some $x_k \in Z - \{x_1 \ldots x_{k-1}\}$ and $y_k \in Z - \{y_1 \ldots y_{k-1}\}$ such that $x_k' B_{k-1} A y_k \neq 0$ for $k = 1,2 \ldots$. Then we will show that $Y X' B_r$ is a $A^-$ where $Y = [y_1 \ldots y_n]$ and $X = [x_1 \ldots x_n]$, for some $r \leq m$.

Assuming $B_1 \ldots B_{k-1}$ are already computed choosing $x_1 \ldots x_{k-1}$ and $y_1 \ldots y_{k-1}$ we choose $x_k$ and $y_k$ as follows for computing $B_k$. Choose $x \in Z - \{x_1 \ldots x_{k-1}\}$ and $y \in Z - \{y_1 \ldots y_{k-1}\}$ such that $x' B_{k-1} A y \neq 0$ and call them $x_k$ and $y_k$ respectively and compute $B_k$. If no such $x$ and $y$ exist i.e., if $x' A B_{k-1} y = 0$ for all $x \in Z - \{x_1 \ldots x_{k-1}\}$ and for all $y \in Z - \{y_1 \ldots y_{k-1}\}$ we stop at this stage.

Let the procedure stops at $(r+1)^{th}$ stage i.e., after computing $B_r$. Then denoting $X_1 = (x_1 \ldots x_r)$ and $X_2$ the matrix with the rest of the vectors of $Z$ as columns and $Y_1 = (y_1 \ldots y_r)$ and $Y_2$ the rest of the vectors of $Z$ as columns, we have

$$X_2' B_r A Y_2 = 0.$$

Let $X = [X_1 : X_2]$ and $Y = [Y_1 : Y_2]$.

$$\text{Consider} \quad X' B_r A Y = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} B_r A [Y_1 : Y_2] = \begin{bmatrix} X_1' B_r A Y_1 & X_1' B_r A Y_2 \\ X_2' B_r A Y_1 & X_2' B_r A Y_2 \end{bmatrix}$$

$$= \begin{bmatrix} X_1' B_r A Y_1 & X_1' B_r A Y_2 \\ X_2' B_r A Y_1 & 0 \end{bmatrix} \qquad \ldots(6)$$

At this stage we need the following lemmas.

Lemma 2 : $B_k A y_p = x_p$ for $p = 1,2 \dots k$ and $k = 1,2 \dots r$.

Proof : We will prove this by induction.

First we show that the result is true for $p = k$ and using it we will prove the result for $p = k-1$ and so on.

Let $p = k$. Then

$$B_k A y_p = B_k A y_k = (I - \frac{(B_{k-1} A y_k - x_k)x_k'}{x_k' B_{k-1} A y_k}) B_{k-1} A y_k$$

$$= x_k$$

So the result is true for all $p = k \leq r$.

Now assuming $B_k A y_p = x_p$ for $p = k, k-1, \dots, k-j+1$, $k=1 \dots r$ we show that $B_k A y_{k-j} = x_{k-j}$

$$B_k A y_{k-j} = [ I - \frac{(B_{k-1} A y_k - x_k)x_k'}{x_k' B_{k-1} A y_k}] B_{k-1} A y_{k-j}$$

$$= [ I - \frac{(B_{k-1} A y_k - x_k)x_k'}{x_k' B_{k-1} A y_k}] B_{k-1} A y_{(k-1)-(j-1)}$$

$$= [ I - \frac{(B_{k-1} A y_k - x_k)x_k'}{x_k' B_{k-1} A y_k}] x_{k-j}$$

$$= x_{k-j} \qquad \text{since} \quad x_k' x_{k-j} = 0$$

Rest of the proof follows by induction.

<u>Lemma 3</u> : $R(B_k A) = R(B_{k-1} A)$ for $k = 1,\ldots,r.$

<u>Proof:</u> By construction it is obvious that $N(B_k A) \supset N(B_{k-1} A)$.

To show the other inequality let $\alpha \in N(B_k A)$

$$\Rightarrow B_k A \alpha = 0$$

$$\Rightarrow x_k' B_k A \alpha = 0$$

$$\Rightarrow x_k' B_{k-1} A \alpha - \frac{x_k'(B_{k-1} A y_k - x_k) x_k' B_{k-1} A\alpha}{x_k' B_{k-1} A y_k} = 0$$

$$\Rightarrow x_k' B_{k-1} A\alpha - x_k' B_{k-1} A\alpha + \frac{x_k' B_{k-1} A\alpha}{x_k' B_{k-1} A y_k} = 0$$

$$\Rightarrow \frac{x_k' B_{k-1} A \alpha}{x_k' B_{k-1} A y_k} = 0$$

$$\Rightarrow x_k' B_{k-1} A \alpha = 0$$

$$\Rightarrow B_k A \alpha = B_{k-1} A \alpha = \phi$$

$$\Rightarrow \alpha \in N(B_{k-1} A)$$

$$\Rightarrow N(B_k A) \subset N(B_{k-1} A)$$

Hence $N(B_k A) = N(B_{k-1} A)$

Therefore $R(B_k A) = R(B_{k-1} A)$

which completes the proof of Lemma 3.

Now from (6) we have

$$X'B_rAY = \begin{bmatrix} X_1'X_1 & X_1'B_rA\ Y_2 \\ X_2'X_1 & 0 \end{bmatrix} = \begin{bmatrix} I & X_1'B_rA\ Y_2 \\ 0 & 0 \end{bmatrix}$$

which is idempotent and of rank $r$ which implies $R(B_rA) = r$ since $X$ and $Y$ are nonsingular matrices. As $B_o$ is chosen initially such that $R(B_oA) = R(A)$ in view of Lemma 3 we have $R(B_rA) = R(A)$. Also $R(AY) = R(A) = R(B_rA) = R(X'B_rAY)$ and $X'B_rA\ Y$ is idempotent implies $X'B_r$ is a g-inverse of $AY$ and hence $YX'B_r$ is a $A^-$. Thus we have

Theorem 5 : Let $A$ be an $m \times n$ matrix. Choose an $n \times m$ matrix $B_o$ such that $R(B_oA) = R(A)$ as an initial approximation and let $Z$ be an orthonormal basis of $E^n$. Compute $B_k$ as in (5) for $k = 1,2...r$ where $r = R(A)$ and $\{x_i\}$ and $\{y_i\}$ are some permutation of vectors of $Z$. Then $YX'B_r$ is $A^-$ where $X$ and $Y$ are as above. Observe that $YX'$ is a permutation matrix.

Let $A$ be an $m \times n$ matrix such that $n \geqslant m$. The most economical choice of $B_o$ and $Z$ in terms of computer storage and number of operations performed is

$$B_o = \begin{bmatrix} I \\ 0 \end{bmatrix} \text{ and } Z = \{e_1, e_2, \ldots, e_n\}$$

This leads to the following algorithm. Let $b_{ij}$ be the $(i,j)^{th}$ element of $B_{k-1}$ and $c_{ij}$ be the $(i,j)^{th}$ element of $B_k$.

At $k^{th}$ step, for $k = 1,2,\ldots,$ using $k^{th}$ column of $A$ compute

$$u_i = \sum_{j=1}^{k-1} b_{ij}\, a_{jk} + a_{ik}, \quad i = k,\ldots,m.$$

If $u_k \neq 0$, compute

$$u_i = \sum_{j=1}^{k-1} b_{ij}\, a_{jk}, \quad i = 1,2,\ldots,(k-1)$$

$$c_{kk} = 1/u_k$$

$$c_{kj} = c_{kk}\, b_{kj}, \qquad j = 1,2,\ldots,(k-1)$$

$$c_{ij} = b_{ij} - u_i\, c_{kj}, \quad i = 1,2,\ldots,m,\ i \neq k;\ \ j = 1,2,\ldots,(k-1)$$

and $$c_{ik} = -u_i\, c_{kk}, \qquad i = 1,2,\ldots,m,\ i \neq k.$$

If $u_k = 0$, take a nonzero $u_i$ and interchange $i^{th}$ row and $k^{th}$ row of $B_{k-1}$ and proceed. Thus a row interchange may be necessary. However, if $u_i = 0$ for $i = k,\ldots,m$, compute $u_i$ for $i = k,\ldots,m$ as above using $p^{th}$ $(p > k)$ column of $A$ and interchange $p^{th}$ and $k^{th}$ column of $A$. Thus a column interchange may be necessary. In practice, the actual interchange of rows and columns may be avoided by using permutation vectors. We stop at $(r+1)^{th}$ step where $u_i = 0$ for $i = (r+1),\ldots,m$ and for the rest of the columns of $A$.

Remark 3 : If $u_i = 0$ for $i = k,\dots,m$ for $p^{th}$ column of $A$ at $k^{th}$ step then $u_i = 0$ for $i = k,\dots,m$ for $p^{th}$ column of $A$ at $(k+1)^{th}$ step also.

Computational aspects

To reduce the computational error it is advisable to use the maximum absolute $u_i$ as $u_k$, the pivotal element. At $k^{th}$ step $2(m-1)(k-1)$ additions and subtractions, $2m(k-1) + (m-1)$ multiplications and one division are needed. So if the matrix is of rank $r$ we need $r(m-1)(r-1)$ additions and subtractions, $(mr^2-r)$ multiplications and $r$ divisions, without taking into account the extra computations needed when a column permutation has occured. Totally $(n-r)$ column permutations are necessary and each column permutation at $k^{th}$ step needs $(m-k+1)(k-1)$ multiplications and the same number of additions, which attain maximum at $k = [\frac{m}{2}] + 1$ where $[\frac{m}{2}]$ is the integral part of $\frac{m}{2}$ So the maximum number of computations that are needed in these $(n-r)$ column permutations is $(n-r)[\frac{m^2}{4}]$. Thus the total number of additions and subtractions needed $\leq r(m-1)(r-1) + (n-r)[\frac{m^2}{4}]$, total number of multiplications needed $\leq mr^2-r+(n-r)[\frac{m^2}{4}]$ and $r$ divisions are needed.

It is easy to observe that the number of computations is minimum when the first $(n-r)$ columns are null and in that case we need $r(m-1)(r-1)$ additions and subtractions, $(mr^2-r)$ multiplications and $r$ divisions. The number of computations is maximum when $[\frac{m}{2}] + i$, for $i = 1,\dots,n-r$, columns depend on the first $[\frac{m}{2}]$ columns.

## 4.4 Numerical illustration

In this section we consider three matrices and compute their g-inverses using the algorithm given in the previous section.

Example 1 :

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 3 & 1 & 1 & 0 \\ 2 & 2 & -2 & 6 & 4 & 0 & 2 \\ 3 & 3 & -3 & 9 & 6 & 0 & -3 \\ 4 & 4 & -4 & 12 & 8 & 0 & 4 \end{bmatrix}$$

$$A^{-} = \begin{bmatrix} 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.50 & -0.50 & 0.25 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ -3.00 & -3.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -2.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ -0.50 & -0.50 & 0.25 & 0.00 & 0.00 \end{bmatrix}$$

This is an example where the number of computations attains the maximum.

Example 2 :

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} -1.50 & 2.00 & 0.50 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ -1.00 & -1.00 & 0.00 & 1.00 & 0.00 \\ -0.50 & 1.00 & -0.50 & 0.00 & 0.00 \\ 1.50 & -2.00 & -0.50 & 0.00 & 0.00 \\ -0.50 & 1.00 & 0.50 & 0.00 & 0.00 \end{bmatrix}$$

This is an example where the number of computations attains the minimum.

Example 3 :

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ 2 & 1 & 3 & -3 & 1 & -2 \end{bmatrix}$$

$$A^{-} = \begin{bmatrix} -2.00 & -1.50 & 0.00 & -0.50 \\ 1.00 & 1.50 & 0.00 & 0.50 \\ 1.00 & 0.50 & 0.00 & 0.50 \\ 1.00 & 1.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

## REFERENCES


[1] Gantmacher, F.R. (1960) : The Theory of Matrices, Vol.2. Chelsea Publishing Company, New York.

[2] Goldfarb, D. (1972) : Modification Methods for Inverting Matrices and Solving Systems of Linear Algebraic Equations. Math. Comp., 26, 829-852.

[3] Halmos, P.R. (1963) : Lectures on Boolean Algebras. D. Van Nostrand Company.

[4] Martos, B. (1969) : Subdefinite Matrices and Qudratic Forms. SIAM J. Appl. Math., 17, 1215-1223.

[5] Milliken, G.A. (1971) : New Criteria for Estimability in Linear Models. Ann. Math. Stat., 42, 1588-1594.

[6] Mitra, S.K. (1972) : Fixed Rank Solutions of Linear Matrix Equations. Sankhyā, Ser. A, 34, 387-392.

[7] Plemmons, R.J. (1971) : Generalized Inverses of Boolean Relation Matrices. SIAM J. Appl. Math., 20, 426-433.

[8] ________ (1973) : Regular Nonnegative Matrices. Proc. Amer. Math. Soc., 39, 26-32.

[9] Plemmons, R.J. and Cline, R.E. (1972) : The Generalized Inverse of a Nonnegative Matrix. Proc. Amer. Math. Soc., 31, 46-50.

[10] Rao, C.R. (1965) : Linear Statistical Inference and Its Applications. John Wiley and Sons, New York.

[11] ________ (1972) : A talk on Generalized Inverses of Boolean Valued Matrices - delivered at the Conference on Combinatorial Mathematics held at Delhi, India.

[12] Rao, C.R. and Mitra, S.K. (1971) : Generalized Inverse of Matrices and Its Applications. John Wiley and Sons, New York.

[13] Rao, P.S.S.N.V.P. (1973) : On Generalized Inverses of Doubly Stochastic Matrices. Sankhyā, Ser. A, 35, 103-105.

[14] Sinkhorn, R. (1968) : Two Results Concerning Doubly Stochastic Matrices. Amer. Math. Monthly, 632-634.

[15] Wilkinsen, J.H. (1965) : The Algebraic Eigen Value Problem. Clarenden Press, Oxford.