Simplicial Bredon-Illman Cohomology with Local Coefficients

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September, 2010

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Thesis submitted to the Indian Statistical Institute
in partial fulfillment of the requirements
for the award of the degree of
Doctor of Philosophy.
September, 2010

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To My Parents

Acknowledgement

Looking back on the last four and half year, since I came to ISI, Kolkata, I am very grateful for all I have received throughout these years. This has been an amazing journey, encompassing both good times, that I will remember forever, because they were aplenty, and not-so-good times, during which I tried not to sway from my goal. This assiduous odyssey would not have been successful without tremendous help and support that I gratefully received from many individuals. I would like to take this opportunity to thank them.

First and foremost, I want to express my deeply-felt gratitude to my supervisor Prof. Goutam Mukherjee whose encouragement and careful supervision from the preliminary to the concluding level made this thesis possible. It was a wonderful experience working with such a caring and affectionate teacher like him. Throughout my research period, he provided encouragement, sound advice, excellent teaching, good company, and lots of good ideas. I would have been lost without him.

I would also like to express my indebtedness to Prof. Mainak Poddar, Prof. Debashish Goswami, Prof. S.M.Srivastava, Prof. Mahuya Dutta and Prof. Amitesh Dasgupta for teaching me various courses. I extend my special thanks to Mainak-da (Prof. Mainak Poddar) for his friendly words of advice that helped me stay in the right path all along.

I would like to thank Aniruddha-da (Prof. Aniruddha Naolekar) for reading a draft of this thesis and giving useful suggestions. A special thanks to Prof. Amiya Mukherjee for a very careful proofreading.

I would like to thank CSIR for providing financial support.

I greatly appreciate and wish to thank all the members of Stat-Math Unit office for their great assistance during my research project.

The road to my graduate degree has been long and winding, so I would also like to thank some people from the early days. I feel a lasting gratitude to my mentor from the school days, Mihir-sir who planted the seed of my motivation for higher studies. A special thank to R.M-sir for teaching me mathematics beautifully in the early days. It is an honor for me to thank Amal-sir and Ziten-sir from my school days.

It is a pleasure to thank my school friends Sujat, Indra, Dibyendu and Arup for helping me get through the difficult times, and for all the emotional support, entertainment, and care they provided. Life would not have been the same without them. All those people in ISI campus, with whom I have played badminton, table tennis and cricket, deserve special thank because sports is a thing I love and they have made my hostel life very enjoyable. The environment of Stat-Math department was very entertaining and supportive in the presence of Soumen, Ashis-da, Prosenjit-da, Abhijit-da, Swagata-di, Rajat, Koushik, Radhe, Subhojit, Biswarup and many others. I would like to thank all of them.

This endeavor could not have taken place without the support of my family. There can be no substitute for the unconditional love and support of my parents. My special gratitude is due to my sister and her family for their loving support.

I am always in tears whenever I think of my beloved 'dida' who was so dearly waiting for the completion of my thesis. She is no more with us. I have no words to express my heartfelt emotions for her.

Contents

Chapter 0

Introduction

0.1 Background and Motivation

The notion of cohomology with local coefficients for topological spaces arose with the work of Steenrod [Ste43, Ste99], in connection with the problem of extending sections of a fibration. This cohomology is built on the notion of fundamental groupoid of the space and can be described by the invariant cochain subcomplex of the cochain complex of the universal cover under the action of the fundamental group of the space. This later description is due to Eilenberg [Eil47]. Cohomology with local coefficients finds applications in many other situations.

We focus on one such application of this cohomology which is due to S. Gitler [Git63], where he has constructed Steenrod reduced power operations in cohomology with local coefficients. The study of cohomology operations has been one of the important areas of research in algebraic topology for a long time. They have been extensively used to compute obstructions [Ste47], to study of homotopy type of complexes [Tho56] and to show essentiality of maps of spheres [BS53]. Some of the basic operations are the reduced powers of Steenrod [Ste53b, Ste53a]. These operations are defined for cohomology with coefficients, in a fixed cyclic group of prime order $p \neq 2$. The main idea of Gitler's construction is to lift power operations in the invariant cochain subcomplex of the universal cover and reproduce the operations in cohomology with local coefficients via Eilenberg's description of the cohomology with local coefficients, where the relevant local coefficients are obtained by a fixed action of the fundamental group of the space on a fixed cyclic group of prime order $p \neq 2$.

Among many important roles played by Eilenberg-MacLane complexes, a significant one is its role in classifying cohomology. A simplicial version of this classification states that for any abelian group A and natural number q, the q^{th} Eilenberg-MacLane simplicial set $K(A,\ q)$ represents the q^{th} cohomology group functor with coefficients in A, in the sense that for every simplicial set X, there is a bijective correspondence [Dus75]

$$H^q(X; A) \cong [X, K(A,\ q)].$$

These classification results have been generalized for cohomology with local coefficients in [Hir79], [GJ99], [BFGM03], where generalized Eilenberg-MacLane complexes play the role of classifying spaces. A construction of a generalized Eilenberg-MacLane complex $L_{\pi X}(\mathcal{L}, q)$ is obtained in [BFGM03] as a homotopy colimit by using the method of Bousfield and Kan [BK72], where πX is the fundamental groupoid of X and $\mathcal{L}$ is a local coefficient system on X. The complex $L_{\pi X}(\mathcal{L}, q)$ appears as the total space of a Kan fibration $L_{\pi X}(\mathcal{L}, q) \longrightarrow N(\pi X)$, where $N(\pi X)$ denotes the nerve of the category πX. The fibration may be interpreted as an object of the slice category $\mathcal{S}/N(\pi X)$, where $\mathcal{S}$ denotes the category of simplicial sets. There is a canonical map $\eta\colon X \to N(\pi X)$ and the classification theorem states that the cohomology classes in the q^{th} cohomology with local coefficients of a Kan complex X correspond bijectively to the vertical homotopy classes of liftings of η. The proof of course depends on the usual closed model structure for the category of simplicial sets.

0.2 Outline of the thesis

The aim of the thesis is to prove equivariant versions of the results mentioned in the previous section, and are based on [MS10a, MS10b, MS11, Sen10]. The following is a chapter-wise description of the thesis.

Chapter 1, is a review of simplicial theory [May67], [GJ99]. The primary goal of this chapter is to set up notations and state results which will be used in subsequent chapters.

In Chapter 2, we deal with simplicial sets equipped with an action of a discrete group G and related objects. Let O_G denote the category of canonical orbits of G [Bre67]. We recall the notion of O_G-Eilenberg-MacLane complexes [MN98], introduce the notion of O_G-twisting function and O_G-twisted cartesian product. At the end of this chapter, we introduce an equivariant analogue of the twisted cohomology [Hir79] for a G-simplicial set.

For spaces with group actions, the analogue of cohomology with local coefficients is the Bredon-Illman cohomology with local coefficients, as introduced in [MM96], and is based on the notion of fundamental groupoid of a space equipped with a group action. We recall that for G-complexes, where G is a group, an equivariant cohomology was introduced by Bredon [Bre67]. The corresponding singular version was developed by S. Illman [Ill75] and is generally known as Bredon cohomology. The coefficients for such equivariant cohomology are contravariant functors from the category of canonical orbits O_G to the category of abelian groups, called abelian O_G-groups. When the local coefficient system is simple, in the equivariant sense, the cohomology as introduced in [MM96] reduces to the Bredon cohomology. As in the non-equivariant case, Bredon-Illman cohomology with local coefficients has been used in the study of extension problem of equivariant sections of an equivariant fibration, and an equivariant version of the Eilenberg theorem is proved in [MM96]. For some other applications of this equivariant version of cohomology with local coefficients, see [Gin04], where Steenrod squares were constructed in Bredon-Illman cohomology with suitable local coefficients and [Won05], where the cohomology is used to study fixed point properties of self maps of homogeneous spaces. In Chapter 3, we study a simplicial version of the Bredon-Illman cohomology with local coefficients. First, we introduce the notion of the fundamental groupoid of a simplicial set equipped with a given simplicial action of a discrete group G and the notion of equivariant local coefficients. Based on these notions, we introduce a simplicial version of Bredon-Illman cohomology with local coefficients [MS10a] of a G-simplicial set. Next we prove that for a suitable O_G-twisting function, induced from given equivariant local coefficients, the simplicial version of the Bredon-Illman cohomology with local coefficients of a G-simplicial set is isomorphic to its equivariant twisted cohomology. Finally in this chapter, we derive a version of the Serre spectral sequence for a G-Kan fibration, following [MS93, MP02].

In Chapter 4, we prove a classification theorem for the simplicial Bredon-Illman cohomology with local coefficients. We refer to [MN98] for a classification theorem for simplicial Bredon cohomology with coefficients in rational O_G-vector spaces, where G is a finite group. We generalize this classification result for the simplicial Bredon-Illman cohomology with local coefficients. The notion of a closed model category in the sense of Quillen [Qui67] is crucial in the proof of this classification result. Our result also retrieves the corresponding non-equivariant classification theorem [Hir79], [GJ99], [BFGM03], when the group G is a trivial group.

A well known result of Eilenberg describes the cohomology of a space with local coefficients by the cohomology of an invariant subcomplex of its universal cover equipped with the action of the fundamental group of the space [Spa81]. A simplicial version of the Eilenberg theorem is given in [Git63]. An equivariant version of the Eilenberg theorem for the Bredon-Illman cohomology with local coefficients of a G-space was proved in [MM96]. In Chapter 5, we derive Eilenberg's theorem for the simplicial Bredon-Illman cohomology with local coefficients. This is based on the notion of universal covering complexes of one vertex Kan complexes [Gug60]. In the equivariant context, the role of the universal cover is played by a contravariant functor from the category of canonical orbits to the category of one vertex Kan complexes. The main result of this chapter is deduced from a notion of an equivariant cohomology of an O_G-simplicial set equipped with an action of an O_G-group.

An important class of cohomology operations consists of the Steenrod squares and reduced power operations [Ste53b], [Ste53a], [Ara56]. The Steenrod squares are defined for cohomology with $\mathbb{Z}_2$-coefficients whereas the Steenrod reduced powers are defined in cohomology with coefficients in $\mathbb{Z}_p$, $p \neq 2$ a prime. A very general and useful method for constructing these operations is given in [May70]. A categorical approach to Steenrod operations can be found in [Eps66]. In [Git63], S. Gitler constructed reduced power operations in cohomology with local coefficients. The main idea of Gitler's construction is to lift power operations in the invariant cochain subcomplex of the universal cover of a space and reproduce the operations in cohomology with local coefficients via Eilenberg's description. The relevant local coefficient system in this context is obtained by a fixed action of the fundamental group of the space on a fixed cyclic group of prime order $p \neq 2$. In Chapter 6, we construct the Steenrod reduced power operations in simplicial Bredon-Illman cohomology with local coefficients, where the equivariant local coefficients take values in a $\mathbb{Z}_p$-algebra, for a prime $p > 2$. Throughout, our method is simplicial. Working in the simplicial category has additional advantage of proving results by combinatorial arguments. Moreover one can switch over to the topological category via geometric realization functor to get the corresponding topological results. Throughout the thesis we shall use the word complex synonymously as simplicial set.

It may be mentioned that for a space with a group action, Steenrod squares have been introduced in the Bredon-Illman cohomology with local coefficients by G. Ginot in [Gin04]. Following Gitler [Git63], we first construct the power operations in the $\underline{\pi}X$-equivariant cohomology of the 'universal O_G-covering complex' of a one vertex G-Kan complex X, where $\underline{\pi}X$ is an O_G-group defined by the fundamental groups of the fixed point subcomplexes of X. This is done by applying the algebraic description of Steenrod reduced power operations of P. May [May70]. We then use the equivariant version of Eilenberg's theorem (Theorem 5.3.4) to reproduce Steenrod reduced power operations in the present context. It may be remarked that our method also applies when $p = 2$, and hence yields Steenrod squares too.

In [Car76], H. Cartan introduced a notion of a 'Cohomology theory' to generalize Sullivan's theory of rational de Rham complexes on simplicial sets [Sul77] to cochain complexes over an arbitrary ring of coefficients. A cohomology theory A over a commutative ring Λ with identity, determines a contravariant functor from the category of simplicial sets to the category of differential graded algebras which assigns to each simplicial set X a differential graded algebra $A(X)$, and a Λ-module $\Lambda(A)$ such that the simplicial cohomology groups of X with coefficients $\Lambda(A)$ can be computed from the cohomology groups of $A(X)$. Cartan's result was generalized for cohomology with local coefficients in [Hir79]. An equivariant version of Cartan's result was proved in [MN98], where cohomology of a simplicial set is replaced by Bredon cohomology of a G-simplicial set, G being a discrete group. In Chapter 7, the final chapter of the thesis, we prove an equivariant version of Cartan's theorem for the simplicial Bredon-Illman cohomology with local coefficients.

Chapter 1

Preliminaries

1.1 Introduction

This chapter is a review of simplicial theory [May67], [GJ99]. Our primary aim in this chapter is to set up notations and state results which will be used in subsequent chapters. After reviewing some basic definitions, we recall some standard results in simplicial theory. Eilenberg-MacLane complexes and generalized Eilenberg-MacLane complexes ([Hir79], [Git63], [FG98]) play a crucial role in homotopy classification of ordinary cohomology and cohomology with local coefficients of a simplicial set, respectively. We state these classification results. We end this chapter by reviewing the notion of closed model category [Qui67] and the closed model structure on the category of simplicial sets.

1.2 Simplicial sets

Let Δ be the category whose objects are ordered sets

$$[n] = \{0 < 1 < \cdots < n\},\ n \geq 0,$$

and morphisms are non-decreasing maps $[n] \to [m]$. There are some distinguished morphisms $d^i \colon [n-1] \to [n], 0 \leq i \leq n$, called cofaces and $s^i \colon [n+1] \to [n],\ 0 \leq i \leq n$, called codegeneracies, defined as follows.

$$d^i(j) = j,\ j < i \ \text{ and } \ d^i(j) = j+1,\ j \geq i, \quad (n > 0,\ 0 \leq i \leq n);$$

$$s^i(j) = j,\ j \leq i, \ \text{ and } \ s^i(j) = j-1,\ j > i, \quad (n \geq 0,\ 0 \leq i \leq n).$$

These maps satisfy the standard cosimplicial relations,

$$d^j d^i = d^i d^{j-1}; s^j d^i = d^i s^{j-1} \text{ for } i < j,$$

$$s^j d^j = id = s^j d^{j+1},$$

$$s^j d^i = d^{i-1} s^j \text{ for } i > j+1; \; s^j s^i = s^i s^{j+1} \text{ for } i \leq j.$$

Definition 1.2.1. A simplicial object X in a category $\mathcal{C}$ is a contravariant functor $X\colon \Delta \to \mathcal{C}$. In other words, a simplicial object is a sequence $\{X_n\}_{n\geq 0}$ of objects of $\mathcal{C}$, together with $\mathcal{C}$-morphisms $\partial_i\colon X_n \to X_{n-1}$ and $s_i\colon X_n \to X_{n+1}$, $0 \leq i \leq n$, where $X_n = X([n])$, $\partial_i = X(d^i)$, and $s_i = X(s^i)$, satisfying the following simplicial identities,

$$\partial_i \partial_j = \partial_{j-1}\partial_i, \quad \partial_i s_j = s_{j-1}\partial_i, \text{ if } \; i < j,$$

$$\partial_j s_j = id = \partial_{j+1} s_j,$$

$$\partial_i s_j = s_j \partial_{i-1}, \quad i > j+1; \quad s_i s_j = s_{j+1} s_i, \quad i \leq j.$$

A simplicial map $f\colon X \to Y$ between two simplicial objects in a category $\mathcal{C}$, is a collection of $\mathcal{C}$-morphisms $f_n\colon X_n \to Y_n$, $n \geq 0$, commuting with ∂_i and s_i.

In particular, a simplicial object X in the category SETS of sets and set maps is called a simplicial set. Throughout, $\mathcal{S}$ will denote the category of simplicial sets and simplicial maps. Often we shall use the word complex (or subcomplex) synonymously with simplicial set (or subsimplicial set). A simplicial object in the category $\mathcal{G}rp$ of groups and group homomorphisms is called a simplicial group. The category of simplicial groups is denoted by $\mathcal{SG}$

For a simplicial set X, elements of X_n are called n-simplices. A simplex $x \in X_n$ is called degenerate if $x = s_i x'$ for some $x' \in X_{n-1}$, $0 \leq i \leq n-1$. Otherwise $x \in X_n$ is called non-degenerate.

For any n-simplex $x \in X_n$, in a simplicial set X, we shall use the notation $\partial_{(i_1,i_2,\cdots,i_r)}x$ to denote the $(n-1)$-simplex $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_r}x$ obtained by applying the successive face maps $\partial_{i_{r-k}}$ to x, where $0 \leq i_{r-k} \leq n-k$, $0 \leq k \leq r-1$.

Example 1.2.2. The simplicial set $\Delta[n]$, $n \geq 0$, is defined as follows. The set of q-simplices is

$$\Delta[n]_q = \{(a_0, a_1, \cdots, a_q); \;\; \text{where } a_i \in \mathbb{Z}, \; 0 \leq a_0 \leq a_1 \leq \cdots \leq a_q \leq n\}.$$

The face and degeneracy maps are defined by

$$\partial_i(a_0, \cdots, a_q) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_q),$$

$$s_i(a_0, \cdots, a_q) = (a_0, \cdots, a_i, a_i, a_{i+1}, \cdots, a_q).$$

Alternatively, $\Delta[n]$ can be viewed as the contravariant functor

$$\Delta[n] = \ \mathrm{Hom}_\Delta(\ , [n]),$$

so that $\Delta[n]_q$ is the set of Δ-morphisms from $[q]$ to $[n]$. The only non-degenerate n-simplex of $\Delta[n]$ is $id\colon [n] \to [n]$ and is denoted by Δ_n. In the earlier notation, it is simply, $\Delta_n = (0, 1, \cdots, n)$.

It is well known that if X is a simplicial set, then for any n-simplex $x \in X_n$ there is a unique simplicial map $\overline{x}\colon \Delta[n] \to X$ with $\overline{x}(\Delta_n) = x$. Often, by an n-simplex in a simplicial set X, we shall mean either an element $x \in X_n$ or the corresponding simplicial map $\overline{x}$.

We have simplicial maps

$$\delta_i\colon \Delta[n-1] \to \Delta[n], \quad \sigma_i\colon \Delta[n+1] \to \Delta[n], \quad 0 \le i \le n,$$

defined by $\delta_i(\Delta_{n-1}) = \partial_i(\Delta_n)$ and $\sigma_i(\Delta_{n+1}) = s_i(\Delta_n)$.

Definition 1.2.3. The boundary subcomplex $\partial\Delta[n]$ of $\Delta[n]$ is defined as the smallest subcomplex of $\Delta[n]$ containing the faces $\partial_i\Delta_n, \quad i = 0, 1, \cdots, n$. The k-th horn Λ^n_k of $\Delta[n]$ is the subcomplex of $\Delta[n]$ which is generated by all the faces $\partial_i\Delta_n$ except the k-th face $\partial_k\Delta_n$.

Example 1.2.4. For a topological space X, a singular n-simplex in X is a continuous map $f\colon \Delta^n \to X$, where Δ^n is the standard Euclidean n-simplex. Let $S_nX, \ n \ge 0$, be the set of all singular n-simplices of X. Define face and degeneracy operators by

$$(\partial_i f)(t_0, \cdots, t_{n-1}) = f(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}),$$

$$(s_i f)(t_0, \cdots, t_{n+1}) = f(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2} \cdots, t_{n+1}),$$

where f is a singular n-simplex in X. Then the graded set $SX = \{S_nX\}$ becomes a simplicial set.

Sometimes SX is called the total singular complex of X.

Definition 1.2.5. A simplicial set X is called a Kan complex if for every collection of $(n+1)$-tuple of n-simplices $(x_0, \cdots, x_{k-1}, \hat{x}_k, x_{k+1}, \cdots, x_{n+1})$ satisfying the compatibility conditions $\partial_i x_j = \partial_{j-1} x_i$, $i < j, \ i \neq k, \ j \neq k$, there exists an $(n+1)$-simplex x such that $\partial_i x = x_i, \ i \neq k$.

The defining condition of a Kan complex is equivalent to the following statement. Any simplicial map from the k-th horn Λ_k^{n+1} to X can be extended to $\Delta[n+1]$, where $n \geq 0$ and $0 \leq k \leq n$.

Example 1.2.6. For a topological space X, the simplicial set SX of Example 1.2.4 is a Kan complex.

Example 1.2.7. The simplicial set $\Delta[n]$ is not a Kan complex for $n > 2$ [GJ99].

Definition 1.2.8. Let $p\colon E \to B$ be a simplicial map. Then p is said to be a Kan fibration if for every $(n+1)$-tuple $(x_0, \cdots, x_{k-1}, \hat{x}_k, x_{k+1}, \cdots, x_{n+1})$ of n-simplices of E such that $\partial_i x_j = \partial_{j-1} x_i,\ i < j,\ i \neq k,\ j \neq k$ and an $(n+1)$-simplex y of B satisfying $\partial_i y = p(x_i),\ i \neq k$, there exists an $(n+1)$-simplex x of E such that $\partial_i x = x_i,\ i \neq k$, and $p(x) = y$.

The Kan condition on a simplicial map $p\colon E \to B$ is equivalent to the following fact. For every commutative diagram of simplicial maps

$$\begin{tikzcd} \Lambda_k^{n+1} \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ \Delta[n+1] \arrow[r] \arrow[ur, dotted, "\exists\theta"] & B \end{tikzcd}$$

there exists a map θ making the resulting triangles commutative, where i denotes the inclusion of the subcomplex $\Lambda_k^{n+1} \subset \Delta[n+1]$. It may be remarked that a Kan fibration yields a Serre fibration via the geometric realization functor (cf. Section 1.3).

Definition 1.2.9. A Kan complex X is said to be minimal if $\partial_i x = \partial_i y,\ i \neq k$, implies $\partial_k x = \partial_k y$.

Definition 1.2.10. A Kan fibration $p\colon E \to B$ is said to be minimal if $p(x) = p(y)$ and $\partial_i x = \partial_i y,\ i \neq k$, imply $\partial_k x = \partial_k y$. If p is minimal and B is a minimal simplicial set, then (E, p, B) is called a minimal fibre space.

Lemma 1.2.11. *[May67] Let $p\colon E \to B$ be a Kan fibration (respectively, minimal fibration).*

1. *Each fibre of p is a Kan complex (respectively, minimal complex).*
2. *If E is a Kan complex (respectively, minimal complex) and p is surjective, then B is a Kan complex (respectively, minimal complex).*
3. *If B is a Kan complex then E is a Kan complex.*

Next we briefly recall the definitions of homology and cohomology of a simplicial set. For a simplicial set X, let $C_n(X)$ denote the quotient of the free abelian group generated by the n-simplices of X by the subgroup generated by the degenerate n-simplices. Define $d\colon C_n(X) \to C_{n-1}(X)$ by $d = \sum_{i=0}^{n}(-1)^i \partial_i$. Then $\{C_*(X), d\}$ becomes a chain complex, called the normalized chain complex of X. Given an abelian group A, the normalized cochain complex $\{C^*(X;A), \delta\}$ is defined by $C^n(X;A) = Hom_{\mathcal{A}b}(C_n(X), A)$ with differential $\delta\colon C^n(X;A) \to C^{n+1}(X;A)$, given by $\delta f = (-1)^{n+1} f \circ d$, $f \in C^n(X;A)$, where $\mathcal{A}b$ denote the category of abelian groups and group homomorphisms. Then the homology and cohomology groups of X with coefficients A are defined by

$$H_n(X;A) := H_n(C_*(X) \otimes A, d \otimes id), \quad H^n(X;A) := H^n(C^*(X;A), \delta), \text{ respectively.}$$

1.3 Geometric realization

To every simplicial set X we can associate a topological space $|X|$, called the geometric realization (also called the Milnor realization) of X, as follows. Consider each X_n, $n \geq 0$, as a discrete topological space and form the disjoint union $\overline{X} = \coprod_{n\geq 0}(X_n \times \Delta^n)$ where Δ^n denotes the standard Euclidean n-simplex. Define an equivalence relation $\sim$ on $\overline{X}$ by

$$(\partial_i x_n, u_{n-1}) \sim (x_n, \eta_i u_{n-1}), \quad (s_i x_n, u_{n+1}) \sim (x_n, \zeta_i u_{n+1}),$$

where $x_n \in X_n$, $u_{n-1} \in \Delta^{n-1}$, $u_{n+1} \in \Delta^{n+1}$ and $\eta_i\colon \Delta^{n-1} \to \Delta^n$, $\zeta_i\colon \Delta^{n+1} \to \Delta^n$ are the maps given by

$$\eta_i(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}),$$

$$\zeta_i(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2} \cdots, t_{n+1}).$$

Then $|X| = \overline{X}/\sim$ is called the geometric realization of X and $|X|$ is a CW-complex with one n-cell for every non-degenerate n-simplex of X.

If $f\colon X \to Y$ is a simplicial map, then f induces a continuous map $|f|\colon |X| \to |Y|$ defined by $|f|([x_n, u_n]) = [f(x_n), u_n]$, where $[x_n, u_n]$ denotes the equivalence class of (x_n, u_n) in $|X|$. The geometric realization of a simplicial set is a functorial construction and it is left adjoint to the total singular complex functor $X \mapsto SX$.

1.4 Homotopy theory of simplicial sets

Definition 1.4.1. The cartesian product $X \times Y$ of two simplicial sets X and Y is defined by $(X \times Y)_n = X_n \times Y_n$ with the face and degeneracy maps given by

$$\partial_i(x, y) = (\partial_i x, \partial_i y) \text{ and } s_i(x, y) = (s_i x, s_i y).$$

Definition 1.4.2. Let $f, g \colon X \to Y$ be simplicial maps. Then f is said to be homotopic to g, written as $f \simeq g$, if there is a simplicial map $\mathcal{H} \colon X \times \Delta[1] \to Y$ such that

$$\mathcal{H} \circ (id \times \delta_1) = f,\ \mathcal{H} \circ (id \times \delta_0) = g,$$

where we identity $X \times \Delta[0]$ with X and $\delta_0, \delta_1 \colon \Delta[0] \to \Delta[1]$ are the simplicial maps as defined in Section 1.2.

Suppose that X' and Y' are subcomplexes of X and Y respectively such that f, g take X' into Y'. If $f|_{X'} = g|_{X'}$ (=α, say) then a homotopy $\mathcal{H} \colon f \simeq g$ is called a relative homotopy if $\mathcal{H} \circ (i \times id) = \alpha \circ pr_1$, where $pr_1 : X' \times \Delta[1] \to X'$ is the projection onto the first factor and $i \colon X' \hookrightarrow X$ is the inclusion. In this case we write $f \simeq g$ $(rel\ X')$. Intuitively, the homotopy leaves the restrictions of f to X' unchanged.

We also have the following combinatorial definition of homotopy [May67].

Definition 1.4.3. Let $f, g \colon X \to Y$ be simplicial maps. Then f is homotopic to g if there exist functions

$$h_i^n \colon X_n \to Y_{n+1},\ 0 \le i \le n,\ n \ge 0,$$

which satisfy the following identities,

$$\partial_0 h_0^n = f_n, \quad \partial_{n+1} h_n^n = g_n,$$

$$\partial_i h_j^n = h_{j-i}^{n-1} \partial_i,\ i < j,\ \partial_{j+1} h_{j+1}^n = \partial_{j+1} h_j^n,\ \partial_i h_j^n = h_j^{n-1} \partial_{i-1},\ i > j+1,$$

$$s_i h_j^n = h_{j+1}^{n+1} s_i,\ i \le j, \quad s_i h_j^n = h_j^{n+1} s_{i-1},\ i > j.$$

The homotopy relation may in general fail to be an equivalence relation on the set $Hom_{\mathcal{S}}(X, Y)$. But, homotopy (relative homotopy) is an equivalence relation on $Hom_{\mathcal{S}}(X, Y)$ $(Hom_{\mathcal{S}}((X, X'), (Y, Y')))$ if Y is a Kan complex [May67].

Thus we have the notions of homotopy equivalence, contractibility, etc., of simplicial sets. The following theorem shows that minimality of simplicial sets is a very strong condition.

Theorem 1.4.4. *[May67] A homotopy equivalence between minimal Kan complexes is an isomorphism.*

Let X be a simplicial set and $v \in X_0$. Then v generates a subcomplex of X which has exactly one simplex $s_{n-1} \cdots s_0(v)$ in dimension n. We will write v unambiguously to denote either this subcomplex or any of its simplices.

Definition 1.4.5. For a Kan complex X and $v \in X_0$, define

$$\pi_n(X, v) := Hom_{\mathcal{S}}((\Delta[n], \partial\Delta[n]), (X, v))/ \simeq (rel\ \partial\Delta[n]),\ n \geq 0.$$

For any simplicial set X, define $\pi_n(X, v) := \pi_n(S|X|, v)$.

Observe that if X is a Kan complex, then the two definitions agree. In general $\pi_0(X, v)$ is just a set. For $n \geq 1$, $\pi_n(X, v)$ is a group and it is abelian for $n > 1$. One calls $\pi_1(X, v)$ the fundamental group of X.

Definition 1.4.6. A simplicial set X is said to be connected if the set $\pi_0(X, v)$ is a singleton.

1.5 Fundamental groupoid and local coefficient system

We recall the definitions of fundamental groupoid and local coefficient system on a simplicial set [GJ99], [Ste99].

Definition 1.5.1. The fundamental groupoid πX of a Kan complex X is a category having as objects all 0-simplices of X, and a morphism $x \to y$ in πX is a homotopy class $[\omega]$ of 1-simplices $\omega \colon \Delta[1] \to X$ (rel $\partial\Delta[1]$) such that $\omega \circ \delta_0 = \overline{y}$, $\omega \circ \delta_1 = \overline{x}$.

The composition of morphisms is defined as follows. If ω_2 represents an arrow from x to y and ω_0 represents an arrow from y to z, then their composite $[\omega_0] \circ [\omega_2]$ is represented by $\Omega \circ \delta_1$, where the simplicial map $\Omega \colon \Delta[2] \to X$ corresponds to a 2-simplex, which is determined by the compatible pair of 1-simplices $(x_0 = \omega_0', \hat{x}_1, x_2 = \omega_2')$, where $\omega_i' = \omega_i(\Delta_1),\ i = 1, 2$.

For any simplicial set X the notion of fundamental groupoid is defined by $\pi X := \pi S|X|$.

Definition 1.5.2. A contravariant functor from πX to the category $\mathcal{A}b$ of abelian groups is called a local coefficient system (of abelian groups) on X.

In general, one may talk of local coefficient system of Λ-modules or Λ-algebras, where Λ is a commutative ring with unity, by replacing $\mathcal{A}b$ by Λ-mod, the category of Λ-modules and module maps, or Λ-alg, the category of commutative Λ-algebras with unity and algebra homomorphisms preserving the unity, respectively. Unless otherwise stated, by a local coefficient system we shall always mean a local coefficient system of abelian groups.

Given a local coefficient system $\mathcal{L}$ on a simplicial set X, the cohomology of X with local coefficients $\mathcal{L}$ is defined as follows [BFGM03], [Hir79].

For each n-simplex $\sigma\colon \Delta[n] \to X$ of X, we associate a 0-simplex $\sigma_\bullet\colon \Delta[0] \to X$, given by

$$\sigma_\bullet = \sigma \circ (id \times \delta_{(1,\cdots,n)}),$$

where $\delta_{(1,\cdots,n)}$ is the composition

$$\delta_{(1,\cdots,n)}\colon \Delta[0] \xrightarrow{\delta_1} \Delta[1] \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_n} \Delta[n] \text{ (cf. Section 1.2).}$$

The j-th face of σ, denoted by $\sigma^{(j)}$, is defined by

$$\sigma^{(j)} = \sigma \circ (id \times \delta_j),\ 0 \le j \le n.$$

Remark 1.5.3. Note that $\sigma_\bullet^{(j)} = \sigma_\bullet$ for $j > 0$, whereas

$$\sigma_\bullet^{(0)} = \sigma \circ \delta_{(0,2,\cdots,n)}.$$

Let $C^n(X;\mathcal{L})$ be the group of all functions f defined on n-simplices $\sigma\colon \Delta[n] \to X$ such that $f(\sigma) \in \mathcal{L}(\sigma_\bullet)$ with $f(\sigma) = 0$, if σ is degenerate. We have a morphism $\sigma_* = [\alpha]$ in πX from $\sigma_\bullet$ to $\sigma_\bullet^{(0)}$ induced by σ, where $\alpha\colon \Delta[1] \to X$ is given by $\alpha = \sigma \circ \delta_{(2,\cdots,n)}$. Define a homomorphism

$$\delta\colon C^n(X;\mathcal{L}) \to C^{n+1}(X;\mathcal{L}),\ f \mapsto \delta f,$$

by

$$(-1)^{n+1}(\delta f)(\sigma) = \mathcal{L}(\sigma_*)f(\sigma^{(0)}) + \sum_{j=1}^{n+1}(-1)^j f(\sigma^{(j)})$$

for any $(n+1)$-simplex σ of X. Then $\delta \circ \delta = 0$. Thus $\{C^*(X;\mathcal{L}), \delta\}$ is a cochain complex.

Definition 1.5.4. Let X be a simplicial set and $\mathcal{L}$ be a local coefficient system on X. Then the n-th cohomology of X with local coefficients $\mathcal{L}$ is defined by

$$H^n(X;\mathcal{L}) := H^n(\{C^*(X;\mathcal{L}), \delta\}).$$

The following discussion gives an alternative description of a local coefficient system on a connected simplicial set.

Definition 1.5.5. Let π be a group. A π-module is a pair (A, ϕ) where A is an abelian group and $\phi\colon \pi \to Aut_{\mathcal{G}rp}(A)$ a group homomorphism. A map of π-modules

$f\colon (A,\phi) \to (A',\phi')$ is a group homomorphism $f\colon A \to A'$ such that

$$f(\phi(x)a) = \phi'(x)f(a)$$

for all $x \in \pi$ and $a \in A$. The category of π-modules is denoted by π-mod. Note that a π-module is the same as a (left) module over the group ring $\mathbb{Z}\pi$.

Let X be a simplicial set and $v \in X_0$ be a 0-simplex. A local coefficient system $\mathcal{L}$ on X determines a $\pi_1(X,v)$-module (A,ϕ) in the following way. Let $A = \mathcal{L}(v)$ and for $\alpha \in \pi_1(X,v)$, define $\phi(\alpha) = \mathcal{L}(\alpha)^{-1}$. Conversely, a $\pi_1(X,v)$-module determines a local coefficient system on a connected simplicial set X [GJ99].

1.6 Twisting function

Recall that a simplicial object in the category $\mathcal{G}rp$ of groups is called a simplicial group.

Definition 1.6.1. [Moo56] Let B be a simplicial set and Γ be a simplicial group. Then a graded function

$$\kappa\colon B \to \Gamma, \quad \kappa_q\colon B_q \to \Gamma_{q-1}, \quad q \geq 1,$$

is called a twisting function if it satisfies the following identities,

$$\begin{aligned}
\partial_0(\kappa_q(b)) &= (\kappa_{q-1}(\partial_0 b))^{-1}\kappa_{q-1}(\partial_1 b), \quad b \in B_q,\\
\partial_i(\kappa_q(b)) &= \kappa_{q-1}(\partial_{i+1} b), \quad i > 0,\\
s_i(\kappa_q(b)) &= \kappa_{q+1}(s_{i+1} b), \quad i \geq 0,\\
\kappa_{q+1}(s_0 b) &= e_q, \quad e_q \text{ being the identity of the group } \Gamma_q.
\end{aligned}$$

Example 1.6.2. Let X be a connected simplicial set. Fix a vertex $v \in X_0$ and for any $x \in X_0$, let $\omega_x \in X_1$ such that $\partial_1\omega_x = v$ and $\partial_0\omega_x = x$. We choose $\omega_v = s_0 v$. Consider the group $\pi_1(X,v)$ as a simplicial group, where $\pi_1(X,v)_n = \pi_1(X,v)$ and all the face and degeneracy maps are the identity. Then the functions

$$\{\kappa(X)_n\}_{n\geq 0}\colon X \to \pi_1(X,v),$$

given by

$$\kappa(X)_n(y) = [\overline{\omega_{\partial_{(0,2,\cdots,n)}y}}]^{-1} \circ [\overline{\partial_{(2,\cdots,n)}y}] \circ [\overline{\omega_{\partial_{(1,\cdots,n)}y}}], \; y \in X_n,$$

is a twisting function. Here the composition is the composition of morphisms in the fundamental groupoid πX.

We briefly recall the definition of twisted cohomology of a simplicial set and its relation with cohomology with local coefficients [Hir79].

Let π be a group and (A, ϕ) be a π-module. Given a twisting function $\kappa\colon X \to \pi$, define the group of twisted n-cochains by

$$C^n_\phi(X;\kappa) := \{f\colon X_n \to A|\ f(x) = 0 \text{ if } x \text{ is degenerate},\ x \in X_n\}.$$

The coboundary $\delta\colon C^n_\phi(X;\kappa) \to C^{n+1}_\phi(X;\kappa)$ is given by

$$(-1)^{n+1}\delta f(x) = \kappa(x)^{-1} f(\partial_0 x) + \sum_{i=1}^{n+1}(-1)^i f(\partial_i x),\ f \in C^n_\phi(X;\kappa),\ x \in X_{n+1}.$$

Then $\delta \circ \delta = 0$. Thus $\{C^*_\phi(X;\kappa), \delta\}$ is a cochain complex. The twisted cohomology of X is then defined by

$$H^n_\phi(X;\kappa) := H^n(\{C^*_\phi(X;\kappa), \delta\}).$$

For a local coefficient system $\mathcal{L}$ on a simplicial set X, let (A, ϕ) be the $\pi_1(X, v)$-module as discussed at the end of Section 1.5. Then

$$H^*(X;\mathcal{L}) \cong H^*_\phi(X;\kappa(X)),$$

where $\kappa(X)$ is the twisting function as described in Example 1.6.2.

Next we recall the definition of twisted cartesian product and related facts from [May67] which will be used later in the thesis.

Definition 1.6.3. Let B, F be simplicial sets, Γ be a simplicial group which operates on F from the left, and $\kappa\colon B \to \Gamma$ be a twisting function. A twisted cartesian product (TCP), with fibre F, base B and group Γ is a simplicial set, denoted by $E(\kappa) = F \times_\kappa B$ which satisfies

$$(F \times_\kappa B)_n = F_n \times B_n$$

and has face and degeneracy operators

$$\begin{aligned} \partial_0(f, b) &= (\kappa(b)\partial_0 f, \partial_0 b),\\ \partial_i(f, b) &= (\partial_i f, \partial_i b),\ i > 0,\\ s_i(f, b) &= (s_i f, s_i b),\ i \geq 0. \end{aligned}$$

If $F = \Gamma$ with Γ acting on itself by left multiplication, then $E(\kappa)$ is called a principal twisted cartesian product (PTPC).

If B, F are Kan complexes then $E(\kappa)$ is also a Kan complex and the canonical projection $p\colon E(\kappa) \to B,\ p(f, b) = b$, is a Kan fibration.

Remark 1.6.4. The construction of twisted cartesian products is natural in the following sense. Let B', F' be simplicial sets, Γ' a simplicial group which operates on F'

from the left, and $\kappa'\colon B' \to \Gamma'$ be a twisting function. Let $\iota\colon B \to B'$, $\zeta\colon F \to F'$ and $\theta\colon \Gamma \to \Gamma'$ be simplicial maps such that

$$\theta_n \circ \kappa_{n+1} = \kappa'_{n+1} \circ \iota_{n+1} \text{ and } \zeta_n(\gamma f) = \theta_n(\gamma)\zeta_n(f),\ \gamma \in \Gamma_n,\ f \in F_n,\ n \geq 0.$$

Then the map

$$E(\kappa) \to E(\kappa'),\quad (f, b) \mapsto (\zeta(f), \iota(b)),\ f \in F,\ b \in B$$

is a simplicial map, covering ι.

Definition 1.6.5. A principal twisted cartesian product (PTPC) $E(\kappa) = \Gamma \times_\kappa B$ is said to be of type (W), if B_0 has one element b_0 and if $\partial_0\colon e_q \times B \to E(\kappa)_{q-1}$ is an isomorphism of sets for all $q \geq 1$, where e_q denote the identity of the group Γ_q.

In a subsequent chapter, we will need the following lemma [May67].

Lemma 1.6.6. *1. If $\Gamma \times_\kappa B$ is a PTPC of type (W) and Γ is a minimal complex, then $(E(\kappa), p, B)$ is a minimal fibre space.*

2. Any two PTPC's of type (W) with group complex Γ are naturally isomorphic.

Principal twisted cartesian products have an alternative description in terms of principal fibrations.

Definition 1.6.7. Let Γ be a simplicial group which operates on the right of a simplicial set E. Then Γ is said to operate principally if $xf = x$ for any $x \in E_q$, $f \in \Gamma_q$, implies $f = e_q$, the identity of the group Γ_q. If Γ operates principally on the right of E, then define a quotient complex B by identifying x and xf for all $x \in E_q$ and $f \in \Gamma_q$. The quotient map $p\colon E \to B$ is called a principal fibration with base B and structure group Γ.

On a PTPC $E(\kappa) = \Gamma \times_\kappa B$, the simplicial group Γ operates on the right by

$$(f, b)f_1 = (ff_1, b)f,\quad f_1 \in \Gamma,\ b \in B.$$

Clearly it is a principal action and B can be identified with the quotient complex of $E(\kappa)$. Thus $p\colon E(\kappa) \to B$ is a principal fibration. Conversely, we have the following.

Proposition 1.6.8. *[May67] A principal fibration with base B and structure group Γ is a PTPC with group Γ and base B, for some suitable twisting function.*

1.7 Eilenberg-MacLane complexes

Definition 1.7.1. Given a group π and a non-negative integer n, a Kan complex X is called an Eilenberg-MacLane complex of type (π, n) if $\pi_n(X, v) = \pi$ and $\pi_i(X, v) = 0$ for $i \neq n$ (cf. Definition 1.4.5). Such a complex is called a $K(\pi, n)$-complex if it is minimal.

Observe that π has to be abelian if $n > 1$. It is well known that any two $K(\pi, n)$ complexes are isomorphic and $K(\pi, n)_n = \pi$. A standard fact about Eilenberg-MacLane complexes is the following.

Proposition 1.7.2. *[May67] If π, π' are abelian groups and $f: \pi \to \pi'$ is a group homomorphism, then there exists a unique simplicial map $\phi: K(\pi, n) \to K(\pi', n)$ such that*

$$f = \phi_n : K(\pi, n)_n \to K(\pi', n)_n.$$

Note that the simplicial group $\pi_q = \pi, q \geq 0$, with all face and degeneracy maps the identity, is a $K(\pi, 0)$-complex.

Definition 1.7.3. For a group π, the group complex $\overline{W}\pi$, known as the $\overline{W}$-construction of π, is defined by setting

$$\overline{W}\pi_0 = *, \quad \overline{W}\pi_q = \pi \times \cdots \times \pi \text{ (q-factors)}, \; q > 0,$$

with face and degeneracy maps as

- $\partial_0([\gamma_1, \cdots, \gamma_q]) = [\gamma_2, \cdots, \gamma_q]$;
- $\partial_q([\gamma_1, \cdots, \gamma_q]) = [\gamma_1, \cdots, \gamma_{q-1}]$;
- $\partial_i([\gamma_1, \cdots, \gamma_q]) = [\gamma_1, \cdots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \cdots, \gamma_q] \; 1 \leq i \leq q-1$,

where $\gamma_1, \cdots, \gamma_q \in \pi$.

Note that $\overline{W}\pi$ is a $K(\pi, 1)$-complex and is the classifying space of the group π.

For an abelian group A and a non-negative integer n, we shall use the following canonical model of $K(A, n)$, for which the q-simplices are described as follows. Consider the simplicial abelian group $C(A, n)$ with q-simplices

$$C(A, n)_q = C^n(\Delta[q]; A),$$

the group of normalized n-cochains of the simplicial q-simplex $\Delta[q]$. For $\mu \in C(A, n)_q$, $\alpha \in \Delta[q-1]_n$ and $\beta \in \Delta[q+1]_n$ the face and degeneracy maps of $C(A, n)$ are defined as

$$\partial_i\mu(\alpha) = \mu(\delta_i(\alpha)), \quad s_j\mu(\beta) = \mu(\sigma_j(\beta)).$$

Here $\delta_i\colon \Delta[q-1] \to \Delta[q]$ and $\sigma_j\colon \Delta[q+1] \to \Delta[q]$ are the simplicial maps as defined in Section 1.2.

We have a simplicial group homomorphism

$$\delta^n\colon C(A,n) \to C(A,n+1)$$

such that $\delta^n c \in C(A,n+1)_q = C^{n+1}(\Delta[q];A)$ is the usual simplicial coboundary of $c \in C(A,n)_q = C^n(\Delta[q];A)$. Then

$$K(A,n)_q = Ker\ \delta^n = Z^n(\Delta[q];A)$$

the group of normalized n-cocycles.

The Eilenberg-MacLane complexes classify simplicial cohomology in the following sense.

Theorem 1.7.4. *[May67] For a simplicial set X and an abelian group A, there is natural bijection*

$$H^n(X;A) \leftrightarrow [X, K(A,n)].$$

Here $[X, K(\pi,n)]$ denote the homotopy class of simplicial maps from X to $K(A,n)$.

An analogue of the above theorem ([Hir79], [BFGM03], [Git63]) holds for local coefficients cohomology of simplicial sets, as defined in Section 1.5. The classifying complex in this context is the so called generalized Eilenberg-MacLane complex. The notion of a generalized Eilenberg-MacLane complex appears in [Git63], [Hir79], [BFGM03]. Roughly speaking, a generalized Eilenberg-MacLane complex is a one vertex minimal Kan complex having exactly two non-vanishing homotopy groups, one of them being the fundamental group. It appears as the total complex of a Kan fibration. Gitler [Git63] used it in the construction of cohomology operations in cohomology with local coefficients. It also plays a crucial role in classifying cohomology with local coefficients [Hir79], [BFGM03]. It may be remarked that a product of Eilenberg-MacLane complexes is also sometimes referred to as a generalized Eilenberg-MacLane complex.

We now describe a model for a generalized Eilenberg-MacLane complex.

Let n be a positive integer and (A,ϕ) be a π-module. Then π acts on the minimal one vertex Kan complex $K(A,n)$ in the following way,

$$\gamma\mu = \phi(\gamma) \circ \mu \text{ where } \mu \in K(A,n)_q = Z^n(\Delta[q];A), \gamma \in \pi.$$

A generalized Eilenberg-MacLane complex can be constructed as follows. Let (A,ϕ) be a π-module. Let $\overline{W}\pi$ denotes the $\overline{W}$ construction of the group π. We have a twisting function

$$\kappa(\pi)\colon \overline{W}\pi \to \pi, \quad \kappa(\pi)_q([\gamma_1, \cdots, \gamma_q]) = \gamma_1,$$

where $\gamma_1, \cdots \gamma_q \in \pi$ and π is considered as a simplicial group with each component π and all the face and the degeneracy maps being identities. For $n > 1$, let

$$L_\pi(A, n) = K(A, n) \times_{\kappa(\pi)} \overline{W}\pi,$$

where the right hand side is the twisted cartesian product (cf. Definition 1.6.3). Then it is a one vertex minimal Kan complex whose fundamental group is π, n-th homotopy group is A and all other homotopy groups are trivial. Moreover the action of the fundamental group π on the n-th homotopy group A is given by ϕ [Thu97]. We have a canonical map $p\colon L_\pi(A, n) \to \overline{W}\pi$, $p(c, x) = x$ for $c \in X, x \in \overline{W}\pi$, which is a Kan fibration.

Remark 1.7.5. Suppose (A, ϕ) is a π-module and (A', ϕ') is a π'-module. Moreover, suppose that $\alpha\colon \pi \to \pi'$ is a group homomorphism. View A' as a π-module via α. Then any π-module homomorphism $f\colon A \to A'$ induces a map

$$f_*\colon K(A, n) \times_{\kappa(\pi)} \overline{W}\pi \to K(A', n) \times_{\kappa(\pi')} \overline{W}\pi'$$

in the obvious way.

Generalized Eilenberg-MacLane complexes classify cohomology with local coefficients of simplicial sets [Git63], [Hir79], [BFGM03]. To state the result, we need to recall some standard facts about closed model category [Qui67].

1.8 Closed model categories

In [Qui67], Quillen introduced the notion of a closed model category in order to lay the foundations of what is known as 'categorical homotopy theory' or 'axiomatic homotopy theory' or 'homotopical algebra'. Broadly speaking, a closed model category is an ordinary category with three distinguished classes of morphisms which satisfy a few simple axioms that are deliberately reminiscent of properties of topological spaces. These axioms give a reasonably general context in which it is possible to set up the basic machinery of homotopy theory.

Definition 1.8.1. Let $\mathcal{C}$ be a category and $\mathcal{F}$ be a distinguished class of morphisms in

$\mathcal{C}$. Suppose we have a commutative solid arrow diagram in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

We say that i has the left lifting property (LLP) with respect to the class of morphisms $\mathcal{F}$ if the dotted arrow exists making the resulting triangles commutative for any $p \in \mathcal{F}$.

We say that p has the right lifting property (RLP) with respect to the class of morphisms $\mathcal{F}$ if the dotted arrow exists making the resulting triangles commutative for any $i \in \mathcal{F}$.

Definition 1.8.2. A category $\mathcal{C}$ with three distinguished classes of morphisms, called cofibrations, fibrations and weak equivalences (which are often denoted by $\hookrightarrow$, $\twoheadrightarrow$, $\xrightarrow{\sim}$, respectively) is called a closed model category if the following axioms are satisfied.

1. Finite limits and colimits exist in $\mathcal{C}$.
2. If $f,\ g$ are morphisms in $\mathcal{C}$ such that $f \circ g$ defined, then any two of $f,\ g$, and $f \circ g$ are weak equivalences imply the third is also so.
3. All three class of morphisms are closed under retracts.
4. (a) Every cofibration has the LLP with respect to every trivial fibration (i.e, fibration which is also a weak equivalence).
   (b) Every fibration has the RLP with respect to every trivial cofibration (i.e, cofibration which is also a weak equivalence).
5. Any morphism f in $\mathcal{C}$ admits following factorizations.
   (a) $f = qi$ where i is a cofibration and q is a trivial fibration.
   (b) $f = pj$ where p is a fibration and j is a trivial cofibration.

In a closed model category initial and terminal objects exist. To justify this, let $\mathcal{D}$ be the empty category (i.e, category with no object) and $F\colon \mathcal{D} \to \mathcal{C}$ be the unique functor. Then, by the first axiom, $colimF$ and $limF$ exist. Hence, by the definitions of colimit and limit, it follows that $\phi = colimF$ is the initial object and $\star = limF$ is the terminal object.

Definition 1.8.3. An object X of a closed model category $\mathcal{C}$ is said to be a fibrant object if the unique map $X \to \star$ is a fibration and X is called a cofibrant object if the unique map $\phi \to X$ is a cofibration.

Example 1.8.4. The category $\mathcal{S}$ of simplicial sets has a closed model structure as described below. A simplicial map $f\colon X \to Y$ is

- a weak equivalence if $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence of topological spaces.
- a fibration iff it is Kan fibration.
- a cofibration if it is dimensionwise injective.

With this closed model structure on $\mathcal{S}$, every object of $\mathcal{S}$ is cofibrant, and fibrant objects of $\mathcal{S}$ are precisely the Kan complexes.

Example 1.8.5. The category Top of topological spaces and continuous maps has a closed model structure as described below. A map $f : X \to Y$ is

- a weak equivalence if it is a weak homotopy equivalence.
- a fibration if it has homotopy lifting property with respect to all CW-complexes (called Serre fibration).
- a cofibration if it has LLP with respect to the acyclic fibrations.

Example 1.8.6. Let $\mathcal{C}$ be a closed model category and C be an object of $\mathcal{C}$. Recall that the slice category $\mathcal{C} \downarrow C$ is the category whose objects are pairs (X, u), where X is an object of $\mathcal{C}$ and $u\colon X \to C$ is a morphism in $\mathcal{C}$. A morphism in $\mathcal{C} \downarrow C$ from (X, u) to (Y, v) is a morphism $f\colon X \to Y$ in $\mathcal{C}$, such that $u = v \circ f$. Then the slice category $\mathcal{C} \downarrow C$ has a natural closed model structure, in which a morphism $f\colon (X, u) \to (Y, v)$ is is a fibration, a cofibration, a weak equivalence if the same is true for the map $f\colon X \to Y$ in the closed model category $\mathcal{C}$ (cf. page 330, [GJ99]).

Let $\mathcal{C}$ be a closed model category. Suppose that X is a cofibrant object and Y is a fibrant object of $\mathcal{C}$. Then it is possible to define 'homotopy' relation on the set $Hom_{\mathcal{C}}(X, Y)$ and it is an equivalence relation. For $C \in \mathcal{C}$, we use the notation $[X, Y]_C$ to denote the set of homotopy classes of maps in the slice category $\mathcal{C} \downarrow C$.

We are now in a position to state the homotopy classification of cohomology with local coefficients of simplicial sets.

Let X be a one vertex Kan complex with fundamental group π. Then there is a unique simplicial map $\theta\colon X \to \overline{W}\pi$ inducing the identity map on the fundamental groups [Thu97]. Let $\mathcal{L}$ be a local coefficient system on X determined by a π-module (A, ϕ). Then (X, θ) is fibrant and $L_\pi(A, n)$ is cofibrant in the closed model category $\mathcal{S} \downarrow \overline{W}\pi$.

Theorem 1.8.7. *[Hir79, Git63, BFGM03, Thu97] With the above notations, there is a natural bijection*

$$H^n(X;\mathcal{L}) \leftrightarrow [X, L_\pi(A,n)]_{\overline{W}\pi}.$$

Chapter 2

G-simplicial sets and equivariant twisted cohomology

2.1 Introduction

In this chapter we deal with simplicial sets equipped with an action of a discrete group G and related objects. Let O_G denote the category of canonical orbits of G [Bre67]. We recall the notion of O_G-Eilenberg-MacLane complexes from [MN98] and introduce the notion of O_G-twisting function and O_G-twisted cartesian product. At the end of the chapter, we introduce an equivariant analogue of the twisted cohomology for a G-simplicial set.

2.2 G-simplicial sets

Let G be a discrete group.

Definition 2.2.1. A G-simplicial set is a simplicial object in the category of G-sets. More precisely, a G-simplicial set is a simplicial set $\{X_n; \partial_i, s_i, 0 \leq i \leq n\}_{n \geq 0}$ such that each X_n is a G-set and the face maps $\partial_i \colon X_n \longrightarrow X_{n-1}$ and the degeneracy maps $s_i \colon X_n \to X_{n+1}$ commute with the G-action.

A G-simplicial map between G-simplicial sets is a simplicial map which commutes with the G-action.

We shall denote the category of G-simplicial sets and G-simplicial maps by $G\mathcal{S}$.

Let X be a G-simplicial set. For a subgroup H of G, the fixed point simplicial set X^H is a simplicial set with $X_n^H = \{x \in X_n | ax = a, \ \forall a \in H\}$ as its n-simplices and the face and degeneracy maps of X^H are the restrictions of the face maps ∂_i and the degeracy maps s_i of X to X^H.

Definition 2.2.2. A G-simplicial set X is called G-connected if the fixed point simplicial set X^H is connected for each subgroup H of G.

Let G operate trivially on $\Delta[n]$. Then for a G-simplicial set X, the product simplicial set $X \times \Delta[n]$, $n \geq 0$, is considered as a G-simplicial set with the diagonal action of G.

Definition 2.2.3. Let X, Y be G-simplicial sets and $f, g\colon X \to Y$ be G-simplicial maps. Then f and g are said to be G-homotopic if there exists a G-simplicial map $\mathcal{H}\colon X \times \Delta[1] \to Y$ such that

$$\mathcal{H} \circ (id \times \delta_1) = f, \quad \mathcal{H} \circ (id \times \delta_0) = g,$$

where $X \times \Delta[0]$ is identified with X. The map $\mathcal{H}$ is called a G-homotopy from f to g and we write $\mathcal{H}\colon f \simeq_G g$.

Suppose $i\colon X' \subseteq X$ is an inclusion of a subcomplex and $f,\ g$ agree on X'. We say that f is G-homotopic to g relative to X' if there exists a G-homotopy $\mathcal{H}\colon f \simeq_G g$ such that $\mathcal{H} \circ (i \times id) = \alpha \circ pr_1$, where $\alpha = f|_{X'} = g|_{X'}$ and $pr_1\colon X' \times \Delta[1] \to X'$ is the projection onto the first factor. In this case, we write $\mathcal{H}\colon f \simeq_G g(rel\ X')$.

Definition 2.2.4. A G-simplicial set is a G-Kan complex if the fixed point simplicial set X^H is a Kan complex for every subgroup H of G.

Example 2.2.5. Let X be a G-simplicial set. Then the geometric realization $|X|$ of X has a natural G-action, given by $a[x,u] = [ax,u]$, where $a \in G,\ x \in X_n$ and $u \in \Delta^n$. This induces a simplicial G-action on $S|X|$. Note that $(S|X|)^H = S|X|^H$, H being a subgroup of G. Hence, for a G-simplicial set X, $S|X|$ is a G-Kan complex.

For a G-simplicial map $f\colon X \to Y$, let $f^H\colon X^H \to Y^H$ denote the simplicial map $f|_{X^H}$, where $H \subseteq G$ is a subgroup.

Remark 2.2.6. Recall ([AG94], [FG98]) that the category $G\mathcal{S}$ has a closed model structure [Qui67], where a G-simplicial map $f\colon X \to Y$ is called a fibration or a weak equivalence if the simplicial map $f^H\colon X^H \to Y^H$ is so for each subgroup H of G. Therefore the fibrant objects of $G\mathcal{S}$ are the G-Kan complexes, and the cofibrant objects are the G-simplicial sets. The cofibrations are defined by the left lifting property. From this it follows that a G-homotopy on the set of G-simplicial maps $X \to Y$ is an equivalence relation, for every G-simplicial set X and G-Kan complex Y. More generally, a relative G-homotopy is an equivalence relation if the target is a G-Kan complex.

Definition 2.2.7. A G-simplicial map $f\colon X \to Y$ between G-simplicial sets X and Y is called a G-Kan fibration if the simplicial map $f^H\colon X^H \to Y^H$ is a Kan fibration for each subgroup H of G.

We consider $G/H \times \Delta[n]$ as a simplicial set with $(G/H \times \Delta[n])_q = G/H \times \Delta[n]_q$ as its q-simplices, and the face and degeneracy maps as $id \times \partial_i$ and $id \times s_i$, where ∂_i and s_i are the face and degeneracy maps of $\Delta[n]$ (cf. Example 1.2.2). Note that the group G acts on G/H by left translation. With this G-action on the first factor and the trivial action on the second factor, $G/H \times \Delta[n]$ is a G-simplicial set.

Definition 2.2.8. Let X be a G-simplicial set. A G-simplicial map $\sigma\colon G/H \times \Delta[n] \to X$ is called an equivariant n-simplex of type H in X.

Remark 2.2.9. We remark that for a G-simplicial set X, the set of equivariant n-simplices of type H in X is in a bijective correspondence with the n-simplices of X^H. For an equivariant n-simplex σ of type H, the corresponding n-simplex is $\sigma' = \sigma(eH, \Delta_n)$. The simplicial map $\Delta[n] \to X^H, \;\; \Delta_n \mapsto \sigma'$, will be denoted by $\overline{\sigma}$.

We shall call σ degenerate or non-degenerate according as the n-simplex $\sigma' \in X^H_n$ is degenerate or non-degenerate.

2.3 O_G-simplicial sets

Recall that, for a discrete group G, the category of canonical orbits, denoted by O_G, is a category whose objects are left cosets G/H, as H runs over the all subgroups of G. Note that the group G acts on the set G/H by left translation. A morphism from G/H to G/K is a G-map. Recall that such a morphism determines and is determined by a subconjugacy relation $a^{-1}Ha \subseteq K$, and is given by $\hat{a}(eH) = aK$. We denote this morphism by $\hat{a}$ [Bre67].

Definition 2.3.1. A contravariant functor from O_G to $\mathcal{S}$ is called an O_G-simplicial set. A map between O_G-simplicial sets is a natural transformation of functors.

We shall denote the category of O_G-simplicial sets by $O_G\mathcal{S}$. The notion of O_G-groups or abelian O_G-groups has the obvious meaning replacing $\mathcal{S}$ by $\mathcal{G}rp$ or $\mathcal{A}b$. Similarly, a contravariant functor from O_G to the category of chain complexes and chain maps is called an O_G-chain complex. An O_G-chain complex of abelian groups can be viewed as a sequence $\{\underline{C}_n\}_{n\geq 0}$ of abelian O_G-groups with natural transformations $\partial_n\colon \underline{C}_n \to \underline{C}_{n-1}, n \geq 1$, such that $\partial_n \circ \partial_{n+1}(G/H)$ is the zero map for each object G/H of O_G.

Definition 2.3.2. An O_G-simplicial set T is called an O_G-Kan complex if $T(G/H)$ is a Kan complex for each subgroup H of G. Similarly, a morphism $f\colon T \to S$ of O_G-simplicial sets is called an O_G-Kan fibration if the simplicial map

$$f(G/H)\colon T(G/H) \to S(G/H)$$

is a Kan fibration for each subgroup H of G.

Example 2.3.3. If X is a G-simplicial set, then we have an O_G-simplicial set ΦX defined by

$$\Phi X(G/H) := X^H,$$

for each object G/H of O_G, and $\Phi X(\hat{a})x = ax$ for $x \in X^K$, where $\hat{a}\colon G/H \to G/K$ is a morphism in O_G.

Note that, if X is a G-Kan complex then ΦX is an O_G-Kan complex.

Observe that we have a functor $\Phi\colon G\mathcal{S} \to O_G\mathcal{S}$ with ΦX as defined above, and for a G-simplicial map $f\colon X \to Y$, the natural transformation $\Phi(f)\colon \Phi X \to \Phi Y$ is defined by $\Phi(f)(G/H) = f^H\colon X^H \to Y^H$, for each object G/H of O_G.

Example 2.3.4. Let $\underline{\pi}$ be an O_G-group. We define an O_G-simplicial set $\overline{W}\underline{\pi}$ as follows. For a subgroup H of G, let

$$\overline{W}\underline{\pi}(G/H) = \overline{W}(\underline{\pi}(G/H)),$$

the $\overline{W}$-construction of the group $\underline{\pi}(G/H)$, as described in Definition 1.7.3. For a morphism $\hat{a}\colon G/H \to G/K$ in O_G, let $\overline{W}\underline{\pi}(\hat{a})\colon \overline{W}\underline{\pi}(G/K) \to \overline{W}\underline{\pi}(G/H)$ be defined by

$$\overline{W}\underline{\pi}(\hat{a})([\gamma_1, \cdots, \gamma_q]) = [\underline{\pi}(\hat{a})\gamma_1, \cdots, \underline{\pi}(\hat{a})\gamma_q],$$

where $\gamma_1, \cdots, \gamma_q \in \underline{\pi}(G/K)$. Then $\overline{W}\underline{\pi}$ is an O_G-simplicial set.

Example 2.3.5. For a G-simplicial set X, with a G-fixed 0-simplex v, we have an O_G-group $\underline{\pi}X$ defined as follows. For any subgroup H of G,

$$\underline{\pi}X(G/H) := \pi_1(X^H, v),$$

and for a morphism $\hat{a}\colon G/H \to G/K,\ \ a^{-1}Ha \subseteq K$, $\underline{\pi}X(\hat{a})$ is the homomorphism in the fundamental groups induced by the simplicial map $a\colon X^K \to X^H$.

Definition 2.3.6. The cartesian product of two O_G-simplicial sets T and S, denoted by $T \times S$, is defined by

$$(T \times S)(G/H) = T(G/H) \times S(G/H),$$

for each object G/H of O_G and $(T \times S)(\hat{a}) = (T(\hat{a}), S(\hat{a}))$ for a morphism $\hat{a}$ in O_G.

Example 2.3.7. We may consider the simplicial set $\Delta[n],\ n \geq 0$, as an O_G-simplicial set by taking $\Delta[n](G/H) = \Delta[n]$ for each object G/H of O_G and $\Delta[n](\hat{a}) = id$ for a morphism $\hat{a}$ in O_G. Therefore, for $T \in O_G\mathcal{S}$, we have the O_G-simplicial set $T \times \Delta[n],\ n \geq 0$.

The above construction of 'constant' O_G-simplicial set works with any simplicial set instead of $\Delta[n]$.

The homotopy of maps between O_G-simplicial sets is defined as follows.

Definition 2.3.8. Let $T, S \in O_G\mathcal{S}$ and $f, g\colon T \to S$ be two maps of O_G-simplicial sets. Then f is homotopic to g if there exists a map $\mathcal{H}\colon T \times \Delta[1] \to S$ of O_G-simplicial sets such that for every object G/H of O_G, $\mathcal{H}(G/H)$ is a homotopy $f(G/H) \simeq g(G/H)$ of simplicial maps. We use the notation $\mathcal{H}\colon f \simeq g$ to denote such homotopies.

Recall that $O_G\mathcal{S}$ has a closed model structure [Qui67], where a map $f\colon T \to S$ in $O_G\mathcal{S}$ is called a fibration or a weak equivalence if the simplicial map $f(G/H)$ is so, for each object G/H of O_G [DK83]. The cofibrations are defined by the left lifting property. Then the abstract notion of homotopy becomes the same as the notion of homotopy, as introduced above.

Definition 2.3.9. An O_G-group $\underline{\pi}$ is said to act on an O_G-simplicial set (O_G-group or abelian O_G-group) T if we have a natural transformation $\phi\colon \underline{\pi} \times T \to T$ such that

$$\phi(G/H)\colon \underline{\pi}(G/H) \times T(G/H) \to T(G/H)$$

is an action of the group $\underline{\pi}(G/H)$ on $T(G/H)$ for each subgroup H of G.

If $\underline{\pi}$ acts on an abelian O_G-group T, then we call T a $\underline{\pi}$-module, and denote it by (T, ϕ).

Next we recall the notion of an O_G-Eilenberg-MacLane complex from [MN98]. It may be relevant to remark that an equivariant analogue of an Eilenberg-MacLane space can be constructed from this, using generalized bar construction [Elm83].

Definition 2.3.10. Given an abelian O_G-group M_0 and an integer $n \geq 0$, an O_G-Kan complex T is called an O_G-Eilenberg-MacLane complex of type (M_0, n) if $T(G/H)$ is a $K(M_0(G/H), n)$-complex for each object G/H of O_G, and for a morphism $\hat{a}\colon G/H \to G/K$ in O_G, $T(\hat{a})\colon T(G/K) \to T(G/H)$ is the unique simplicial map induced by the linear map $M_0(\hat{a})\colon M_0(G/K) \to M_0(G/H)$ such that $T(\hat{a})_n\colon K(M_0(G/K), n)_n \to K(M_0(G/H), n)_n$ is $M_0(\hat{a})$ (cf. Proposition 1.7.2).

Using the uniqueness of $K(\pi, n)$s, the following result was proved in [MN98].

Proposition 2.3.11. *Any two O_G-Eilenberg-MacLane complexes of the same type are naturally isomorphic.*

We denote an O_G-Eilenberg-MacLane complex of type (M_0, n) by $K(M_0, n)$. Using the canonical model of an ordinary Eilenberg-MacLane complex, as described in Chapter

1, we have a canonical model of $K(M_0, n)$ given by

$$K(M_0, n)(G/H)_q = Z^n(\Delta[q]; M_0(G/H)),$$

for each object G/H of O_G with $K(M_0, n)(\hat{a})$ being induced by a coefficients change homomorphism, for a morphism $\hat{a}$ in O_G.

Remark 2.3.12. In [MN98], A. Mukherjee and A. Naolekar defined the Bredon cohomology $H^*_G(T; M_0)$ of an O_G-simplicial set T with coefficients an abelian O_G-group M_0. It has been proved in [MN98] that the Bredon cohomology $H^n_G(T; M_0)$ of T is represented by the set of homotopy classes of maps from T to the O_G-Eilenberg-MacLane complex $K(M_0, n)$.

2.4 O_G-twisting function and O_G-TCP

In the presence of an action of a discrete group G, the notion of a twisting function has the following natural generalization to an O_G-twisting function.

Definition 2.4.1. Let T be an O_G-simplicial set and Γ be a simplicial O_G-group. A natural transformation of functors $\kappa \colon T \to \Gamma$ is called an O_G-twisting function if $\kappa(G/H) \colon T(G/H) \to \Gamma(G/H)$ is an ordinary twisting function for each subgroup H of G.

Example 2.4.2. Let $\underline{\pi}$ be an O_G-group. Consider $\underline{\pi}$ as a simplicial O_G-group $\{\underline{\pi}_n\}_{n \geq 0}$, where $\underline{\pi}_n = \underline{\pi}$ for all $n \geq 0$ and all the face and degeneracy maps are the identity natural transformations. Define a natural transformation

$$\kappa(\underline{\pi}) \colon \overline{W}\underline{\pi} \to \underline{\pi} \ \text{ by } \ \kappa(\underline{\pi})_q(G/H)([\gamma_1, \cdots, \gamma_q]) = \gamma_1, \ H \text{ subgroup of } G,$$

where $\overline{W}\underline{\pi}$ is the O_G-simplicial set as introduced in Example 2.3.4 and $[\gamma_1, \cdots, \gamma_q] \in \overline{W}\underline{\pi}(G/H)_q$, $\gamma_i \in \underline{\pi}(G/H)$, $1 \leq i \leq q$. Then $\kappa(\underline{\pi})$ is an O_G-twisting function.

Example 2.4.3. Let X be a G-connected G-simplicial set and v be a G-fixed 0-simplex of X. Let $\underline{\pi}X \colon O_G \to \mathcal{G}rp$ be the O_G-group as introduced in Example 2.3.5. We regard $\underline{\pi}X$ as an O_G-group complex by $\underline{\pi}X(G/H)_n = \underline{\pi}X(G/H)$ for each object G/H of O_G and $\underline{\pi}X(\hat{a}) = id$ for a morphism $\hat{a}$ in O_G. We choose a 0-simplex x on each G-orbit of X_0 and for each such 0-simplex x, we choose a 1-simplex $\omega_x \in X_1^{G_x}$ such that $\partial_0\omega_x = x, \partial_1\omega_x = v$. Here G_x is the isotropy subgroup at $x \in X_0$, corresponding to the G-action on the set X_0. We choose $\omega_v = s_0v$. For any other 0-simplex y on the orbit of x we define $\omega_y = a\omega_x$ if $y = ax$. Then it is well-defined, for, if $y = a_1x = a_2x$ we have $a_2^{-1}a_1 \in G_x$ and then $a_2^{-1}a_1\omega_x = \omega_x$, which yields $a_1\omega_x = a_2\omega_x$. Also note that if

$y = ax$, then $G_y = aG_x a^{-1}$. Therefore $\omega_y \in X_1^{G_y}$. For a 0-simplex $x \in X^H$, we shall use the notation $\xi_H(x)$ to denote the homotopy class $[\overline{\omega}_x]$ of $\overline{\omega}_x \colon \Delta[1] \to X^H$. Here for any q-simplex σ of a simplicial set Y, $\overline{\sigma} \colon \Delta[q] \to Y$ denotes the unique simplicial map satisfying $\overline{\sigma}(\Delta_q) = \sigma$. Define

$$\{\kappa(G/H)_n\} \colon X^H \to \pi_1(X^H, v)$$

to be the twisting function $\kappa(X^H)$ as described in Example 1.6.2, that is,

$$\kappa(G/H)_n(y) = \xi_H(\partial_{(0,2,\cdots,n)}y)^{-1} \circ [\overline{\partial_{(2,\cdots,n)}y}] \circ \xi_H(\partial_{(1,\cdots,n)}y),$$

where $y \in X_n^H$.

Let ΦX be the O_G-simplicial set as defined in Example 2.3.3. We now check that

$$\kappa \colon \Phi X \to \underline{\pi} X, \quad G/H \mapsto \kappa(G/H), \quad H \subseteq G \text{ a subgroup},$$

is indeed an O_G-twisting function. Suppose H and K are subgroups of G such that $a^{-1}Ha \subseteq K$. Let $z \in X_n^K$. Then $y = az \in X_n^H$. Observe that if $x_1, x_2 \in X_1^K$ are 1-simplices such that $\overline{x}_1 \simeq \overline{x}_2$, as simplicial maps into X^K, then $\overline{y}_1 \simeq \overline{y}_2$ as simplicial maps into X^H, where $y_i = ax_i$, $\ i = 1, 2$. Thus

$$\begin{aligned}
&\kappa(G/H)_n \circ \Phi X(\hat{a})(z) \\
=&\kappa(G/H)_n(y) \\
=&\xi_H(\partial_{(0,2,\cdots,n)}y)^{-1} \circ [\overline{\partial_{(2,\cdots,n)}y}] \circ \xi_H(\partial_{(1,\cdots,n)}y) \\
=&\xi_H(a\partial_{(0,2,\cdots,n)}z)^{-1} \circ [\overline{a\partial_{(2,\cdots,n)}z}] \circ \xi_H(a\partial_{(1,\cdots,n)}z) \\
=&a\xi_K(\partial_{(0,2,\cdots,n)}z)^{-1} \circ a[\overline{\partial_{(2,\cdots,n)}z}] \circ a\xi_K(\partial_{(1,\cdots,n)}z) \\
=&\underline{\pi} X(\hat{a}) \circ \kappa(G/K)_n(z).
\end{aligned}$$

Thus $\kappa \colon \Phi X \to \underline{\pi} X$ is an O_G-twisting function.

Having defined an O_G-twisting function, we now define an O_G-twisted cartesian product. Using the naturality of twisted cartesian product, as observed in Remark 1.6.4, we have the following definition.

Definition 2.4.4. Let B, F be O_G-Kan complexes and Γ be an O_G-group complex. Suppose that Γ operates on B and $\kappa \colon B \to \Gamma$ is an O_G-twisting function. Then the O_G-Kan complex $F \times_\kappa B$, defined by

$$(F \times_\kappa B)(G/H) = F(G/H) \times_{\kappa(G/H)} B(G/H),$$

for each object G/H of O_G, and $(F \times_\kappa B)(\hat{a}) = (F(\hat{a}), B(\hat{a}))$ for a morphism $\hat{a}$ of the category O_G, is called the O_G-twisted cartesian product (O_G-TCP) with fibre F, base B, group Γ and twisting κ.

Note that the second factor projection gives an O_G-Kan fibration $p\colon (F \times_\kappa B) \to B$. We view $(F \times_\kappa B, p)$ as an object in the slice category $O_G\mathcal{S} \downarrow B$ (cf. Example 1.8.6).

Example 2.4.5. Let M_0 be a $\underline{\pi}$-module with module structure given by ϕ. For each subgroup H of G, define an action of $\underline{\pi}(G/H)$ on $K(M_0(G/H), n)$

$$\psi(G/H)\colon \underline{\pi}(G/H) \times K(M_0(G/H), n) \to K(M_0(G/H), n)$$

as follows. For $\gamma \in \underline{\pi}(G/H)$, let $\psi(G/H)(\gamma, \text{-})$ be the unique simplicial automorphism of $K(M_0(G/H), n)$ such that

$$\psi(G/H)(\gamma, \text{-})_n\colon K(M_0(G/H), n)_n \to K(M_0(G/H), n)_n,$$

is the automorphism $\phi(G/H)(\gamma, \text{-})\colon M_0(G/H) \to M_0(G/H)$ (cf. Proposition 1.7.2). This defines an action of the O_G-group $\underline{\pi}$ on the O_G-Kan complex $K(M_0, n)$. Therefore we can form the O_G-Kan fibration $p\colon K(M_0, n) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi} \to \overline{W}\underline{\pi}$, where $\kappa(\underline{\pi})$ is the O_G-twisting function as described in Example 2.4.2. If we use the canonical model of $K(M_0, n)$, the total complex of the resulting O_G-Kan fibration is denoted by $L_\phi(M_0, n)$.

2.5 Equivariant twisted cohomology

We end this chapter by introducing the equivariant version of twisted cohomology as defined in Section 1.5. Let T be an O_G-simplicial set and M_0 be an abelian O_G-group equipped with an action $\phi\colon \underline{\pi} \times M_0 \to M_0$ of an O_G-group $\underline{\pi}$. We regard $\underline{\pi}$ as an O_G-group complex by setting $\underline{\pi}_n = \underline{\pi}$, $n \geq 0$, with all face and degeneracy maps being identity natural transformations. Let $\kappa\colon T \to \underline{\pi}$ be a given O_G-twisting function. We define equivariant twisted cohomology of T with coefficients M_0 and twisting κ as follows.

Let $\mathcal{C}_G$ denote the category of contravariant functors from O_G to the category $\mathcal{A}b$. We have a chain complex in the abelian category $\mathcal{C}_G$, defined by

$$\underline{C}_n(T)\colon O_G \to \mathcal{A}b, \quad G/H \mapsto C_n(T(G/H); \mathbb{Z}),$$

where $C_n(T(G/H); \mathbb{Z})$ denotes the free abelian group generated by the non-degenerate n-simplices of $T(G/H)$. For a morphism $\hat{a}\colon G/H \to G/K$, $a^{-1}Ha \subseteq K$ in O_G, $\underline{C}_n(T)(\hat{a})$ is given by the map $a_\#\colon C_n(T(G/K); \mathbb{Z}) \to C_n(T(G/H); \mathbb{Z})$ induced by the simplicial map $T(\hat{a})\colon T(G/K) \to T(G/H)$. The boundary map $\partial_n\colon \underline{C}_n(T) \to \underline{C}_{n-1}(T)$ is the natural transformation defined by $\partial_n(G/H)\colon C_n(T(G/H); \mathbb{Z}) \to C_{n-1}(T(G/H); \mathbb{Z})$, where

$\partial_n(G/H)$ is the ordinary boundary map of the simplicial set $T(G/H)$. Dualizing this chain complex in the abelian category $\mathcal{C}_G$, we get the cochain complex

$$\{C^*_G(T;M_0) = Hom_{\mathcal{C}_G}(\underline{C}_*(T), M_0), \delta^*\},$$

which defines the ordinary Bredon cohomology of T with coefficients M_0.

Remark 2.5.1. In particular, for a G-simplicial set X, the cohomology groups of the cochain complex $C^*_G(\Phi X; M_0)$ define the Bredon cohomology groups of X with coefficients M_0, and is denoted by $H^*_G(X; M_0)$ [Bre67].

To define the twisted cohomology of T, we modify the coboundary maps as follows.

$$\delta^n_\kappa \colon C^n_G(T;M_0) \longrightarrow C^{n+1}_G(T;M_0), \quad f \mapsto \delta^n_\kappa f,$$

where

$$\delta^n_\kappa f(G/H) \colon C_{n+1}(T(G/H);\mathbb{Z}) \to M_0(G/H)$$

is given by

$$(-1)^{n+1}\delta^n_\kappa f(G/H)(x) = (\kappa(G/H)_{n+1}(x))^{-1} f(G/H)(\partial_0 x) + \sum_{i=1}^{n+1}(-1)^i f(G/H)(\partial_i x),$$

for $x \in T(G/H)_{n+1}$. Note that the first term of the expression in the right-hand side is obtained by the given action ϕ of $\underline{\pi}$ on M_0. We denote the resulting cochain complex by $C^*_G(T;\kappa,\phi)$.

Definition 2.5.2. Let T be an O_G-simplicial set, $\underline{\pi}$ be an O_G-group, (M_0,ϕ) be a $\underline{\pi}$-module and $\kappa \colon T \to \underline{\pi}$ be an O_G-twisting function. Then the n-th equivariant twisted cohomology of T is defined by

$$H^n_G(T;\kappa,\phi) := H_n(C^*_G(T;\kappa,\phi)).$$

Definition 2.5.3. Let X be a G-simplicial set, $\underline{\pi}$ be an O_G-group, (M_0,ϕ) be a $\underline{\pi}$-module and $\kappa \colon \Phi X \to \underline{\pi}$ be an O_G-twisting function. We set $\underline{C}_*(X) = \underline{C}_*(\Phi X)$ and let $C^*_G(X;\kappa,\phi)$ denote the cochain complex $C^*_G(\Phi X;\kappa,\phi)$. Then the n-th equivariant twisted cohomology of X is defined by

$$H^n_G(X;\kappa,\phi) := H_n(C^*_G(X;\kappa,\phi)).$$

Let Y be a G-simplicial set, $\underline{\pi}'$ be an O_G-group, $\kappa' \colon \Phi Y \to \underline{\pi}'$ be an O_G-twisting function and (N_0,ϕ') be a $\underline{\pi}'$-module. Let $F \colon X \to Y$, $\nu \colon N_0 \to M_0$, $l \colon \underline{\pi} \to \underline{\pi}'$ be maps in the appropriate categories such that

$$\nu(G/H)[(l(G/H)\gamma)n_0] = \gamma[\nu(G/H)(n_0)]; \quad \kappa' \circ \Phi F = l \circ \kappa,$$

where H is a subgroup of G, $n_0 \in N_0(G/H)$, $\gamma \in \underline{\pi}(G/H)$. Then we have an induced map in the cochain complexes of equivariant twisted cohomology

$$(F, \nu, l)^* \colon C^*_G(Y; \kappa', \phi') \to C^*_G(X; \kappa, \phi),$$

defined by

$$(F, \nu, l)^*(f)(G/H) = \nu(G/H) \circ f(G/H) \circ (\Phi F)(G/H), \quad f \in C^*_G(Y; \kappa', \phi').$$

In particular, a G-simplicial map $F \colon X \to Y$ induces a cochain map

$$(F, id, id)^* \colon C^*_G(Y; \kappa', \phi) \to C^*_G(X; \kappa'\Phi F, \phi).$$

Often, we shall denote this cochain map simply by F^*.

Chapter 3

Simplicial Bredon-Illman cohomology with local coefficients

3.1 Introduction

In this chapter we introduce the fundamental groupoid of a simplicial set equipped with a given simplicial action of a discrete group G and the notion of equivariant local coefficient system. Based on these notions, we introduce simplicial version of the Bredon-Illman cohomology with local coefficients [MM96] of a G-simplicial set. Next we prove that for a suitable O_G-twisting function, induced from a given equivariant local coefficient system, the simplicial version of the Bredon-Illman cohomology with local coefficients of a G-simplicial set is isomorphic to its equivariant twisted cohomology. Finally, in this chapter we derive a version of the Serre spectral sequence for a G-Kan fibration.

3.2 Fundamental groupoid

Let X be a G-simplicial set. Recall that an equivariant n-simplex of type H in X, H being a subgroup of G, is a G-simplicial map $\sigma\colon G/H \times \Delta[n] \to X$. Each such σ corresponds to an n-simplex $\sigma' \in X^H$, and $\overline{\sigma}\colon \Delta[n] \to X^H$ is the simplicial map given by $\overline{\sigma}(\Delta_n) = \sigma' = \sigma(eH, \Delta_n)$. Suppose x_H and y_K are equivariant 0-simplices of type H and K respectively, and $\hat{a}\colon G/H \to G/K$ is a morphism in O_G, given by a subconjugacy relation $a^{-1}Ha \subseteq K$, $a \in G$, such that $\hat{a}(eH) = aK$. Moreover, suppose that we have an equivariant 1-simplex $\vartheta\colon G/H \times \Delta[1] \to X$ of type H such that

$$\vartheta \circ (id \times \delta_1) = x_H, \quad \vartheta \circ (id \times \delta_0) = y_K \circ (\hat{a} \times id).$$

Then, in particular, ϑ' is a 1-simplex in X^H such that $\partial_1\vartheta' = x'_H$ and $\partial_0\vartheta' = ay'_K$. Observe that the 0-simplex ay'_K in X^H corresponds to the composition

$$G/H \times \Delta[0] \xrightarrow{\hat{a}\times id} G/K \times \Delta[0] \xrightarrow{y_K} X$$

and ϑ is a G-homotopy $x_H \simeq_G y_K \circ (\hat{a} \times id)$.

Definition 3.2.1. Let X be a G-Kan complex. The fundamental groupoid ΠX of X is a category with objects equivariant 0-simplices

$$x_H \colon G/H \times \Delta[0] \to X$$

of type H, as H varies over all subgroups of G. Given two objects x_H and y_K in ΠX, a morphism from $x_H \to y_K$ is defined as follows. Consider the set of all pairs $(\hat{a}, \vartheta)$ where $\hat{a}\colon G/H \to G/K$ is a morphism in O_G given by a subconjugacy relation $a^{-1}Ha \subseteq K$, $a \in G$, such that $\hat{a}(eH) = aK$ and $\vartheta\colon G/H \times \Delta[1] \to X$ is an equivariant 1-simplex such that

$$\vartheta \circ (id \times \delta_1) = x_H, \quad \vartheta \circ (id \times \delta_0) = y_K \circ (\hat{a} \times id).$$

The set of morphisms in ΠX from x_H to y_K is a quotient of the set of pairs mentioned above by an equivalence relation '$\sim$', where $(\hat{a}_1, \vartheta_1) \sim (\hat{a}_2, \vartheta_2)$ if, and only if, $a_1 = a_2 = a$ (say) and there exists a G-homotopy $\mathcal{H}\colon G/H \times \Delta[1] \times \Delta[1] \to X$ of G-homotopies such that $\mathcal{H}\colon \vartheta_1 \simeq_G \vartheta_2$ (rel $G/H \times \partial\Delta[1]$). Since X is a G-Kan complex, by Remark 2.2.6, $\sim$ is an equivalence relation. We denote the equivalence class of $(\hat{a}, \vartheta)$ by $[\hat{a}, \vartheta]$. The set of equivalence classes is the set of morphisms in ΠX from x_H to y_K.

The composition of morphisms in ΠX is defined as follows. Given two morphisms

$$x_H \xrightarrow{[\hat{a}_1,\vartheta_1]} y_K \xrightarrow{[\hat{a}_2,\vartheta_2]} z_L$$

their composition $[\hat{a}_2, \vartheta_2] \circ [\hat{a}_1, \vartheta_1]$ is $[\widehat{a_1a_2}, \psi]\colon x_H \to z_L$, where the first factor $\widehat{a_1a_2}$ is the composition

$$G/H \xrightarrow{\hat{a}_1} G/K \xrightarrow{\hat{a}_2} G/L$$

and $\psi\colon G/H \times \Delta[1] \to X$ is an equivariant 1-simplex of type H as described below. Let Ω be a 2-simplex in the Kan complex X^H, determined by the compatible pair of 1-simplices $(x_0 = a_1\vartheta'_2, \hat{x}_1, x_2 = \vartheta'_1)$ so that $\partial_0\Omega = a_1\vartheta'_2$ and $\partial_2\Omega = \vartheta'_1$. Then ψ is given by $\psi(eH, \Delta_1) = \partial_1\Omega$.

For a version of the fundamental groupoid of a G-space, we refer to [MM96] and [Lüc89].

The following lemma shows that the composition is well-defined.

Lemma 3.2.2. *The equivalence class of* $(\widehat{a_1a_2}, \psi)$ *does not depend on the choice of the representatives of* $[\hat{a}_1, \vartheta_1]$ *and* $[\hat{a}_2, \vartheta_2]$.

Proof. Suppose that $[a_i, \vartheta_i] = [a_i, \lambda_i]$, $i = 1, 2$. Then there exist G-homotopies $\Theta_i \colon \vartheta_i \simeq_G \lambda_i$ (rel $G/H \times \partial\Delta[1]$) for $i = 1, 2$. Let y be a 2-simplex in X^H determined by the compatible pair of 1-simplices $(x_0 = a_1\lambda_2', \hat{x}_1, x_2 = \lambda_1')$ as described above, where $\partial_0 y = a_1\lambda_2'$ and $\partial_2 y = \lambda_1'$. Let $\xi \colon G/H \times \Delta[1] \to X$ be the equivariant 1-simplex determined by $\xi(eH, \Delta_1) = \partial_1 y$. We need to show that $(\widehat{a_1a_2}, \psi) \sim (\widehat{a_1a_2}, \xi)$. Observe that $\overline{\Theta}_i \colon \overline{\vartheta}_i \simeq \overline{\lambda}_i$ (rel $\partial\Delta[1]$), for $i = 1, 2$. Now consider the right lifting problem

$$
\begin{tikzcd}
(\Delta[2] \times \partial\Delta[1]) \cup (\Lambda^2_1 \times \Delta[1]) \arrow[r, "(\overline{y}, \overline{\Omega}, \overline{\Theta}_1, \overline{\Theta}_2)"] \arrow[d] & X^H \arrow[d] \\
\Delta[2] \times \Delta[1] \arrow[r] & *
\end{tikzcd}
$$

where in the above diagram, the right vertical arrow is a fibration and the left vertical arrow is an anodyne extension [GJ99]. Therefore, the above right lifting problem has a solution $\overline{F} \colon \Delta[2] \times \Delta[1] \to X^H$ and the composition of $\overline{F}$ with

$$\delta_1 \times id \colon \Delta[1] \times \Delta[1] \to \Delta[2] \times \Delta[1]$$

is a homotopy $\overline{\psi} \simeq \overline{\xi}$ (rel $\partial\Delta[1]$). Let $F \colon G/H \times \Delta[2] \times \Delta[1] \to X$ be the G-simplicial map determined by $F(eH, s, t) = \overline{F}(s, t)$. Then the composition

$$G/H \times \Delta[1] \times \Delta[1] \xrightarrow{id \times \delta_1 \times id} G/H \times \Delta[2] \times \Delta[1] \xrightarrow{F} X$$

is a G-homotopy $\psi \simeq_G \xi$ (rel $G/H \times \partial\Delta[1]$). Consequently,

$$[\widehat{a_1a_2}, \psi] = [\widehat{a_1a_2}, \xi].$$

□

Recall that if X is a G-simplicial set then $S|X|$ is a G-Kan complex (cf. Example 2.2.5) and the canonical map $X \to S|X|$ is a weak equivalence of G-simplicial sets.

Definition 3.2.3. For a G-simplicial set X, we define the fundamental groupoid ΠX of X by $\Pi X := \Pi S|X|$.

Note that if X, Y are G-simplicial sets and $F \colon X \to Y$ is a G-simplicial map then there exists an obvious induced functor $\Pi(F) \colon \Pi X \to \Pi Y$ which assigns to each object x_H of ΠX, the object $F \circ x_H$ of ΠY and to a morphism $[\hat{a}, \vartheta]$ in ΠX, the morphism $[\hat{a}, F \circ \vartheta]$ in ΠY.

Remark 3.2.4. If G is a trivial group, then ΠX reduces to the fundamental groupoid πX of the simplicial set X. Again, for a fixed H, the objects x_H together with the morphisms $x_H \to y_H$ with identity in the first factor, constitute a subcategory of ΠX which is precisely the fundamental groupoid πX^H of X^H. Moreover, a morphism $[\hat{a}, \vartheta]$ from x_H to y_K, corresponds to the morphism $[\overline{\vartheta}]$ in the fundamental groupoid πX^H of X^H, from x'_H to ay'_K, where $\overline{\vartheta}$ is as in Remark 2.2.9.

Suppose Υ is a morphism in πX^H from x to y given by a homotopy class $[\overline{\omega}]$, where $\overline{\omega}\colon \Delta[1] \to X^H$ is a 1-simplex in X^H such that $\partial_1\omega = x$ and $\partial_0\omega = y$. Let x_H and y_H be the objects in ΠX, defined by

$$x_H(eH, \Delta_0) = x, \quad \text{and} \quad y_H(eH, \Delta_0) = y.$$

Then we have a morphism $[id, \omega]\colon x_H \to y_H$ in ΠX, where $\omega(eH, \Delta_1) = \overline{\omega}(\Delta_1)$. We shall denote this morphism in ΠX corresponding to a morphism Υ in πX^H by $b\Upsilon$.

3.3 Equivariant local coefficient system

Definition 3.3.1. An equivariant local coefficient system (of abelian groups) on a G-simplicial set X is a contravariant functor from ΠX to the category $\mathcal{A}b$ of abelian groups.

More generally, for a commutative ring Λ with unity, an equivariant local coefficient system of Λ-algebras has the obvious meaning replacing $\mathcal{A}b$ by the category Λ-alg of commutative Λ-algebras with unity and algebra homomorphisms preserving the unity.

Example 3.3.2. Let X be a G-simplicial set and $n > 1$. For any object x_H in ΠX, define $M(x_H) = \pi_n(X^H, x_H(eH, \Delta_0))$ and, for any morphism $[\hat{a}, \vartheta]\colon x_H \to y_K$, define

$$M([\hat{a}, \vartheta]) = ([\overline{\vartheta}])^* \circ \pi_n(a),$$

where $a\colon X^K \to X^H$ is the left translation by a, and $([\overline{\vartheta}])^*$ is the isomorphism in the n-th homotopy group of X^H induced by a morphism $[\overline{\vartheta}]$ from x'_H to ay'_K. Then M is an equivariant local coefficient system on X.

The following discussion gives a generic example of an equivariant local coefficient system on a G-connected G-simplicial set X having a G-fixed 0-simplex.

Suppose that v is a G-fixed 0-simplex of X and M is an equivariant local coefficient system of Λ-algebras on X. For any subgroup H of G, let v_H be the object of type H in ΠX defined by

$$v_H\colon G/H \times \Delta[0] \to X,$$

$$(eH, \Delta_0) \longmapsto v.$$

Then for any morphism $\hat{a}\colon G/H \to G/K$ in O_G, given by a subconjugacy relation $a^{-1}Ha \subseteq K$, we have a morphism $[\hat{a}, k]\colon v_H \to v_K$ in ΠX, where $k\colon G/H \times \Delta[1] \to X$ is given by $k(eH, \Delta_1) = s_0 v$.

Define an O_G-Λ-algebra, $M_0\colon O_G \to \Lambda$-alg by

$$M_0(G/H) = M(v_H),$$

for each object G/H of O_G and $M_0(\hat{a}) = M([\hat{a}, k])$ for a morphism $\hat{a}$ in O_G. Then the O_G-Λ-algebra M_0 comes equipped with a natural action of the O_G-group $\underline{\pi}X$ (cf. Example 2.3.5) as described below.

Let $\gamma = [\overline{\vartheta}] \in \underline{\pi}X(G/H) = \pi_1(X^H, v)$. Then the morphism $[id, \vartheta]\colon v_H \to v_H$, where $\vartheta(eH, \Delta_1) = \overline{\vartheta}(\Delta_1)$, is an equivalence in the category ΠX. This yields a group homomorphism

$$b\colon \pi_1(X^H, v) \to Aut_{\Pi X}(v_H),\ \gamma = [\overline{\vartheta}] \mapsto b(\gamma) = [id, \vartheta].$$

We remark that the composition in the fundamental group $\pi_1(X^H, v)$ coincides with the morphism composition in ΠX, contrary to the usual notion of composition in the fundamental group. The composition of the map b with the group homomorphism $Aut_{\Pi X}(v_H) \to Aut_{\Lambda\text{-alg}}(M(v_H))$ which sends $\mathtt{u} \in Aut_{\Pi X}(v_H)$ to $[M(\mathtt{u})]^{-1}$ defines the action of $\pi_1(X^H, v)$ on $M_0(G/H)$. We now check that this action is natural with respect to morphisms in O_G, that is, for a subconjugacy relation $a^{-1}Ha \subseteq K$,

$$M_0(\hat{a})(\gamma z) = \underline{\pi}(\hat{a})(\gamma)(M_0(\hat{a})z),$$

where $\gamma \in \pi_1(X^K, v)$ and $z \in M_0(G/K)$. The above equality follows from the definition of M_0 and the following commutative diagram in ΠX.

$$\begin{tikzcd}
v_H \arrow[d, "{[\hat{a},k]}"'] & v_H \arrow[l, "b(a\gamma)"'] \arrow[d, "{[\hat{a},k]}"] \\
v_K & v_K \arrow[l, "b(\gamma)"']
\end{tikzcd}$$

Conversely, suppose that an O_G-Λ-algebra M_0 is equipped with an action of the O_G-group $\underline{\pi}X$, where X is a G-connected G-simplicial set and $v \in X^G$ is a fixed 0-simplex. We define an equivariant local coefficient system M on X as follows.

For every object x_H of type H, define $M(x_H) = M_0(G/H)$. To define M on morphisms, we choose a 0-simplex, say x, from each G-orbit of X_0 and an 1-simplex ω_x to fix a morphism $[\overline{\omega}_x]$ from v to x in πX^{G_x}. Here G_x is the isotropy subgroup at x.

For any other point y in the orbit of x, we fix the 1-simplex $\omega_y = a\omega_x$, where $y = ax$. Note that, if $y = a_1x = a_2x$ then $a_2^{-1}a_1 \in G_x$ and hence $a_1\omega_x = a_2\omega_x$. Thus ω_y is well-defined. Also $\omega_y \in X^{G_y}$, since $G_y = aG_xa^{-1}$. Therefore, $[\overline{\omega}_y]$ is a morphism from v to y in πX^{G_y}. Observe that if $x \in X^H$, where H is a subgroup G, then $[\overline{\omega}_x]$ is also a morphism in πX^H from v to x, as $H \subseteq G_x$.

Suppose $x_H \xrightarrow{[\hat{a},\vartheta]} y_K$ is a morphism in ΠX. Then by Remark 3.2.4, we have a morphism $[\overline{\vartheta}]$ from x'_H to ay'_K in πX^H. Define $M([\hat{a},\vartheta])$ as the following composition

$$M_0(G/K) \xrightarrow{M_0(\hat{a})} M_0(G/H) \xrightarrow{\gamma^{-1}} M_0(G/H),$$

where $\gamma \in \pi_1(X^H, v)$ is

$$\gamma := [\overline{\omega}_{ay'_K}]^{-1} \circ [\overline{\vartheta}] \circ [\overline{\omega}_{x'_H}],$$

a composition of morphisms in πX^H and the second arrow denotes the inverse of the given action of γ on $M_0(G/H)$. Then M is an equivariant local coefficient system of Λ-algebras on X. Thus we have the following proposition.

Proposition 3.3.3. *Let X be a G-simplicial set with a G-fixed 0-simplex and M be an equivariant local coefficient system of Λ-algebras on X. Then M determines an O_G-Λ-algebra M_0, equipped with an action of the O_G-group $\underline{\pi}X$. Conversely, if X is a G-connected G-simplicial set with $X^G \neq \emptyset$, then an action of the O_G-group $\underline{\pi}X$ on an O_G-Λ-algebra M_0 determines an equivariant local coefficient system of Λ-algebras on X.*

Let $F\colon X \to Y$ be a G-simplicial map between G-simplicial sets X, Y and N is an equivariant local coefficient system on Y. Then the functor $N \circ \Pi(F)$ is an equivariant local coefficient system on X, which we shall denote by F^*N. This is sometimes called the pull-back of the equivariant local coefficient system N via F.

3.4 Simplicial Bredon-Illman cohomology with local coefficients

In this section, we derive simplicial version of Bredon-Illman cohomology with local coefficients [MM96] and prove that this cohomology may be described in terms of the equivariant twisted cohomology, as introduced in Section 2.4.

Let X be a G-simplicial set and M an equivariant local coefficient system on X. For each equivariant n-simplex $\sigma\colon G/H \times \Delta[n] \to X$, we associate an equivariant 0-simplex $\sigma_H\colon G/H \times \Delta[0] \to X$, given by

$$\sigma_H = \sigma \circ (id \times \delta_{(1,2,\cdots,n)}),$$

where $\delta_{(1,2,\cdots,n)}$ is the composition

$$\delta_{(1,2,\cdots,n)}\colon \Delta[0] \xrightarrow{\delta_1} \Delta[1] \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_n} \Delta[n],$$

the maps $\delta_i,\ i = 1, \cdots, n$ being defined in Section 1.2. The j-th face of σ is an equivariant $(n-1)$-simplex of type H, denoted by $\sigma^{(j)}$, and is defined by

$$\sigma^{(j)} = \sigma \circ (id \times \delta_j),\ 0 \leq j \leq n.$$

Remark 3.4.1. Note that $\sigma_H^{(j)} = \sigma_H$ for $j > 0$, whereas

$$\sigma_H^{(0)} = \sigma \circ (id \times \delta_{(0,2,\cdots,n)}).$$

Let $C_G^n(X;M)$ be the group of all functions f defined on equivariant n-simplices $\sigma\colon G/H \times \Delta[n] \to X$ such that $f(\sigma) \in M(\sigma_H)$, with $f(\sigma) = 0$, if σ is degenerate. We have a morphism $\sigma_* = [id, \alpha]$ in ΠX from σ_H to $\sigma_H^{(0)}$ induced by σ, where the equivariant 1-simplex $\alpha\colon G/H \times \Delta[1] \to X$ is given by $\alpha = \sigma \circ (id \times \delta_{(2,\cdots,n)})$. Define a homomorphism

$$\delta\colon C_G^n(X;M) \to C_G^{n+1}(X;M),$$

$$\text{by}\ \ f \mapsto \delta f,$$

where, for any equivariant $(n+1)$-simplex σ of type H,

$$(-1)^{n+1}(\delta f)(\sigma) = M(\sigma_*)f(\sigma^{(0)}) + \sum_{j=1}^{n+1}(-1)^j f(\sigma^{(j)}).$$

Proposition 3.4.2. *The map $\delta \circ \delta\colon C_G^n(X;M) \to C_G^{n+2}(X;M)$ is the zero map.*

Proof. Let $f \in C_G^n(X;M)$ and $\sigma\colon G/H \times \Delta[n+2] \to X$ be an equivariant $(n+2)$-simplex of type H in X. Then,

$$\begin{aligned}
&(-1)^{n+1}(-1)^{n+2}\delta(\delta f)(\sigma)\\
=&(-1)^{n+1}[M(\sigma_*)(\delta f)(\sigma^{(0)}) + \sum_{j=1}^{n+2}(-1)^j(\delta f)(\sigma^{(j)})]\\
=&[M(\sigma_*)\{M(\sigma_*^{(0)})f((\sigma^{(0)})^{(0)}) + \sum_{j=1}^{n+1}(-1)^j f((\sigma^{(0)})^{(j)})\}\\
&+ \sum_{j=1}^{n+2}(-1)^j\{M(\sigma_*^{(j)})f((\sigma^{(j)})^{(0)}) + \sum_{k=1}^{n+1}(-1)^k f((\sigma^{(j)})^{(k)})\}].
\end{aligned}$$

Note that $(\sigma^{(j)})^{(k)} = (\sigma^{(k)})^{(j-1)}$ if $k < j$ and $(\sigma^{(j)})^{(k)} = (\sigma^{(k+1)})^{(j)}$ if $k \geq j$.

Also $\sigma_*^{(0)} \circ \sigma_* = \sigma_*^{(1)}$ and $\sigma_*^{(j)} = \sigma_*$ for $j > 1$. Then the first term of the above expression $M(\sigma_*)M(\sigma_*^{(0)})f((\sigma^{(0)})^{(0)})$ cancels with the first term of the summation $\sum_{j=1}^{n+2}(-1)^j M(\sigma_*^{(j)})f((\sigma^{(j)})^{(0)})$. The remaining $(n+1)$-terms of this sum cancels with $M(\sigma_*)\{\sum_{j=1}^{n+1}(-1)^j f((\sigma^{(0)})^{(j)})\}$. The double summation adds up to 0. □

Thus $\{C_G^*(X;M), \delta\}$ is a cochain complex. We are interested in a subcomplex of this cochain complex as defined below.

Let $\sigma\colon G/H \times \Delta[n] \to X$ and $\tau\colon G/K \times \Delta[n] \to X$ be two equivariant n-simplices. Suppose there exists a G-map $\hat{a}\colon G/H \to G/K,\ a^{-1}Ha \subseteq K$, such that $\tau \circ (\hat{a} \times id) = \sigma$. Then σ and τ are said to be compatible under $\hat{a}$. Observe that if σ and τ are compatible, then σ is degenerate if, and only if, τ is degenerate. Moreover, notice that in this case, we have a morphism $[\hat{a}, k]\colon \sigma_H \to \tau_K$ in ΠX, where $k = \sigma_H \circ (id \times \sigma_0)$, $\sigma_0\colon \Delta[1] \to \Delta[0]$ being the map as defined in Section 2, Chapter 1. Let us denote this induced morphism by a_*.

Definition 3.4.3. We define $S_G^n(X;M)$ to be the subgroup of $C_G^n(X;M)$ consisting of all functions f such that if σ and τ are equivariant n-simplices in X which are compatible under $\hat{a}$ then $f(\sigma) = M(a_*)(f(\tau))$.

Proposition 3.4.4. *If $f \in S_G^n(X;M)$, then $\delta f \in S_G^{n+1}(X;M)$.*

Proof. Suppose σ, τ are equivariant $(n+1)$-simplices of type H and K respectively, compatible under $\hat{a}\colon G/H \to G/K$, $a^{-1}Ha \subseteq K$. Then the faces $\sigma^{(j)}$ and $\tau^{(j)}$ are also compatible under $\hat{a}$ for all j, $0 \le j \le n+1$. Moreover, the induced morphism $a_*\colon \sigma_H^{(j)} \to \tau_K^{(j)}$ is the same as the induced morphism $a_*\colon \sigma_H \to \tau_K$ for $j \ge 1$ (cf. Remark 3.4.1) and the compositions

$$\sigma_H \xrightarrow{\sigma_*} \sigma_H^{(0)} \xrightarrow{a_*} \tau_K^{(0)} \quad \text{and} \quad \sigma_H \xrightarrow{a_*} \tau_K \xrightarrow{\tau_*} \tau_K^{(0)}$$

are the same. Thus

$$\begin{aligned}
&M(a_*)(\delta f(\tau)) \\
=&(-1)^{n+1}[M(a_*)M(\tau_*)f(\tau^{(0)}) + \sum_{j=1}^{n+1}(-1)^j M(a_*)f(\tau^{(j)})] \\
=&(-1)^{n+1}[M(\sigma_*)M(a_*)f(\tau^{(0)}) + \sum_{j=1}^{n+1}(-1)^j M(a_*)f(\tau^{(j)})] \\
=&(-1)^{n+1}[M(\sigma_*)f(\sigma^{(0)}) + \sum_{j=1}^{n+1}(-1)^j f(\sigma^{(j)})] \\
=&\delta f(\sigma).
\end{aligned}$$

□

Thus we have a cochain complex $S_G^*(X;M) = \{S_G^n(X;M),\delta\}$ which defines the simplicial Bredon-Illman cohomology.

Definition 3.4.5. Let X be a G-simplicial set with an equivariant local coefficient system M on it. Then the n-th Bredon-Illman cohomology of X with local coefficients M is defined by

$$H_G^n(X;M) := H^n(S_G^*(X;M)).$$

Suppose that X, Y are G-simplicial sets and M, N are equivariant local coefficient systems on X and Y respectively. A map from (X,M) to (Y,N) is a pair (F,λ), where $F\colon X \to Y$ is a G-simplicial map and $\lambda\colon F^*N \to M$ is a natural transformation of functors. A map $(F,\lambda)\colon (X,M) \to (Y,N)$ naturally induces a map $(F,\lambda)^{\#}\colon S_G^*(Y;N) \to S_G^*(X;M)$ as follows. For $f \in S_G^n(Y;N)$ and an equivariant n-simplex σ in X of type H, define

$$(F,\lambda)^{\#}(f)(\sigma) = \lambda(\sigma_H)f(F\circ\sigma).$$

Proposition 3.4.6. *If $f \in S_G^n(Y;N)$, then $(F,\lambda)^{\#}f \in S_G^n(X;M)$. Further, $(F,\lambda)^{\#}$ commutes with the differential δ.*

Proof. Let $\sigma\colon G/H\times\Delta[n] \to X$ and $\tau\colon G/K\times\Delta[n]\to X$ be two equivariant n-simplices in X, compatible under $\hat{a}\colon G/H \to G/K,\ a^{-1}Ha \subseteq K$. Then

$$\begin{aligned}
&M(\sigma_H \xrightarrow{a_*} \tau_K)(F,\lambda)^{\#}(f)(\tau)\\
=&M(\sigma_H \xrightarrow{a_*} \tau_K)\lambda(\tau_K)f(F\circ\tau)\\
=&\lambda(\sigma_H)\circ F^*N(a_*)f(F\circ\tau).
\end{aligned}$$

The last equality follows from the fact that $\lambda\colon F^*N \to M$ is a natural transformation. Since $f \in S_G^n(Y;N)$ and the equivariant n-simplices $F\circ\sigma$ and $F\circ\tau$ are compatible under $\hat{a}$, we have $F^*N(a_*)f(F\circ\tau) = \lambda(\sigma_H)$. Hence

$$M(a_*)(F,\lambda)^{\#}(f)(\tau) = \lambda(\sigma_H)f(F\circ\sigma) = (F,\lambda)^{\#}(f)(\sigma).$$

Next, we prove that $(F,\lambda)^{\#}$ commutes with δ. Let σ be an equivariant $(n+1)$-simplex of type H in X. Then

$$\begin{aligned}
&(F,\lambda)^{\#}(\delta f)(\sigma)\\
=&\lambda(\sigma_H)(\delta f)(F\circ\sigma)\\
=&(-1)^{n+1}[\lambda(\sigma_H)\{N((F\circ\sigma)_*)f(F\circ\sigma^{(0)}) + \sum_{j=1}^{n+1}(-1)^j f(F\circ\sigma^{(j)})\}].
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\delta((F,\lambda)^{\#}f)(\sigma)\\
=&(-1)^{n+1}[M(\sigma_*)((F,\lambda)^{\#}f)(\sigma^{(0)})+\sum_{j=1}^{n+1}(-1)^j((F,\lambda)^{\#}f)(\sigma^{(j)})]\\
=&(-1)^{n+1}[M(\sigma_*)\lambda(\sigma_H^{(0)})f(F\circ\sigma^{(0)})+\sum_{j=1}^{n+1}\lambda(\sigma_H^{(j)})f(F\circ\sigma^{(j)})].
\end{aligned}
$$

Since $N((F\circ\sigma)_*)=F^*N(\sigma_*)$ and $\lambda\colon F^*N\to M$ is a natural transformation, we have $\lambda(\sigma_H)N((F\circ\sigma)_*)=M(\sigma_*)\lambda(\sigma_H^{(0)})$. Also recall that $\sigma_H^{(j)}=\sigma_H$, for $j>1$. Hence $(F,\lambda)^{\#}\delta=\delta(F,\lambda)^{\#}$.

□

Thus $(F,\lambda)^{\#}$ is a well-defined cochain map and hence it induces a map

$$(F,\lambda)^*\colon H_G^*(Y;N)\to H_G^*(X;M)$$

in cohomology.

If M is an equivariant local coefficient system of Λ-algebras on a G-simplicial set X, then we define cup product in the simplicial Bredon-Illman cohomology with local coefficients $H_G^*(X;M)=\oplus_{n\geq 0}H_G^n(X;M)$ as follows.

Let $\sigma\colon G/H\times\Delta[n+m]\to X$ be an equivariant $(n+m)$-simplex of type H in X. Then define equivariant simplices

$$\sigma\rfloor_n=\sigma\circ(id_{G/H}\times\delta_{(n+1,\cdots,n+m)}),\ \lfloor_m\sigma=\sigma\circ(id_{G/H}\times\delta_{(0,\cdots,n)}).$$

Here the simplicial maps

$$\delta_{(n+1,\cdots,n+m)}\colon\Delta[n]\to\Delta[n+m]\text{ and }\delta_{(0,\cdots,n)}\colon\Delta[m]\to\Delta[n+m]$$

are defined as the compositions

$$\delta_{(n+1,n+2,\cdots,m+n)}\colon\Delta[n]\xrightarrow{\delta_{n+1}}\Delta[n+1]\xrightarrow{\delta_{n+2}}\Delta[n+2]\cdots\xrightarrow{\delta_{m+n}}\Delta[m+n],$$

and

$$\delta_{(0,1,\cdots,n)}\colon\Delta[m]\xrightarrow{\delta_0}\Delta[m+1]\xrightarrow{\delta_1}\Delta[m+2]\cdots\xrightarrow{\delta_n}\Delta[m+n].$$

(cf. Section 1.2 for the definitions of the maps $\delta_i,\ i=0,\cdots n+m$.)

Let $f \in C^n_G(X; M)$ and $g \in C^m_G(X; M)$. Then the cup product $f \cup g \in C^{n+m}_G(X; M)$ is the cochain whose value on an equivariant $(n+m)$-simplex σ is given by the formula

$$(f \cup g)(\sigma) = (-1)^{mn} f(\sigma\rfloor_n)\{M(\sigma_{\widehat{n+1}})g(\lfloor_m\sigma)\},$$

where $\sigma_{\widehat{n+1}} = [id, \sigma \circ (id_{G/H} \times \delta_{(1,\cdots,n,n+2,\cdots,n+m)})]$ is a morphism in ΠX from $(\sigma\rfloor_n)_H$ to $(\lfloor_m\sigma)_H$.

Proposition 3.4.7. *If $f \in S^n_G(X; M)$ and $g \in S^m_G(X; M)$, then $f \cup g \in S^{n+m}_G(X; M)$ and*

$$\delta(f \cup g) = \delta(f) \cup g + (-1)^{deg(f)} f \cup \delta(g).$$

Proof. We first prove that $f \cup g \in S^{m+n}_G(X; M)$. Let σ and τ be equivariant $(n+m)$-simplices of type H and K respectively. Suppose σ and τ are compatible under $\hat{a}$, $a^{-1}Ha \subseteq K$. Then the equivariant n-simplices (respectively, m-simplices) $\sigma\rfloor_n$ and $\tau\rfloor_n$ (respectively, $\lfloor_m\sigma$ and $\lfloor_m\tau$) are also compatible under $\hat{a}$. Therefore

$$\begin{aligned}
&M(\sigma_H \xrightarrow{a_*} \tau_K)(f \cup g)(\tau)\\
=&M(\sigma_H \xrightarrow{a_*} \tau_K)\{(-1)^{mn} f(\tau\rfloor_n)\{M(\tau_{\widehat{n+1}})g(\lfloor_m\tau)\}\}\\
=&(-1)^{mn}\{M(\sigma_H \xrightarrow{a_*} \tau_K)f(\tau\rfloor_n)\}\{M(\sigma_H \xrightarrow{a_*} \tau_K)M(\tau_{\widehat{n+1}})g(\lfloor_m\tau)\}.
\end{aligned}$$

Observe that $\sigma_H = (\sigma\rfloor_n)_H$ and $\tau_K = (\tau\rfloor_n)_K$. Also following equality holds in ΠX,

$$((\lfloor_m\sigma)_H \xrightarrow{a_*} (\lfloor_m\tau)_K) \circ (\sigma_H \xrightarrow{\sigma_{\widehat{n+1}}} (\lfloor_m\sigma)_H) = (\tau_K \xrightarrow{\tau_{\widehat{n+1}}} (\lfloor_m\tau)_K) \circ (\sigma_H \xrightarrow{a_*} \tau_K).$$

Combining these and using the facts that $f \in S^n_G(X; M)$ and $g \in S^m_G(X; M)$, we get $M(\sigma_H \xrightarrow{a_*} \tau_K)(f \cup g)(\tau) = (f \cup g)(\sigma)$, that is, $f \cup g \in S^{n+m}_G(X; M)$.

Next, we prove the coboundary formula for the cup product $f \cup g$. Let σ be an equivariant $(n+m+1)$-simplex of type H. Then,

$$\begin{aligned}
&\delta(f \cup g)(\sigma)\\
=&(-1)^{n+m+1}\{M(\sigma_*)(f \cup g)(\sigma^{(0)}) + \sum_{j=1}^{n+m+1}(-1)^j(f \cup g)(\sigma^{(j)})\}\\
=&(-1)^{mn+n+m+1}[M(\sigma_*)\{f(\sigma^{(0)}\rfloor_n)M(\sigma^{(0)}_{\widehat{n+1}})g(\lfloor_m(\sigma^{(0)}))\}\\
&+ \sum_{j=1}^{n+m+1}(-1)^j f(\sigma^{(j)}\rfloor_n)M(\sigma^{(j)}_{\widehat{n+1}})g(\lfloor_m(\sigma^{(j)}))].
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&(\delta f \cup g)(\sigma) + (-1)^n (f \cup \delta g)(\sigma) \\
=&(-1)^{(n+1)m}(\delta f)(\sigma\rfloor_{n+1} M(\sigma_{\widehat{n+2}})g(\lfloor_m \sigma)) \\
&+ (-1)^n(-1)^{n(m+1)} f(\sigma\rfloor_n) M(\sigma_{\widehat{n+1}})(\delta g)(\lfloor_{m+1}\sigma) \\
=&(-1)^{mn+n+m+1}[\{M((\sigma\rfloor_{n+1})_*) f(\sigma\rfloor_{n+1}^{(0)}) + \sum_{j=1}^{n+1}(-1)^j f(\sigma\rfloor_{n+1}^{(j)})\}\{M(\sigma_{\widehat{n+2}})g(\lfloor_m\sigma)\} \\
&+ (-1)^n f(\sigma\rfloor_n) M(\sigma_{\widehat{n+1}})\{M((\lfloor_{m+1}\sigma)_*)g((\lfloor_{m+1}\sigma)^{(0)}) + \sum_{j=1}^{m+1}(-1)^j g((\lfloor_{m+1}\sigma)^{(j)})\}].
\end{aligned}
$$

Note that,

$$\sigma\rfloor_n = \sigma\rfloor_{n+1}^{(n+1)},\ (\lfloor_{m+1}\sigma)^{(0)} = \lfloor_m \sigma \quad \text{and} \quad (\lfloor_{m+1}\sigma)_* \circ \sigma_{\widehat{n+1}} = \sigma_{\widehat{n+2}}.$$

Therefore, the last term of the first summation of the above expression cancels with the next term. Also

$$\sigma^{(0)}\rfloor_n = \sigma\rfloor_{n+1}^{(0)} \quad \text{and} \quad \sigma_* = (\sigma\rfloor_{n+1})_*,\ \sigma_{\widehat{n+2}} = (\sigma^{(0)})_{\widehat{n+1}} \circ \sigma_*.$$

Hence the first term of $\delta(f \cup g)(\sigma)$ agrees with the first term of $(\delta f \cup g)(\sigma) + (-1)^n(f \cup \delta g)(\sigma)$. Similar argument shows that other terms also agree. Hence $f \cup g \in S_G^{n+m}(X; M)$. □

Definition 3.4.8. Let M be an equivariant local coefficient system of Λ-algebras on X. Let $f \in S_G^n(X; M)$, $g \in S_G^m(X; M)$ and σ be an equivariant $(m+n)$-simplex of type H in X. Then the cup product $f \cup g \in S_G^{n+m}(X; M)$ is the cochain whose value on σ is given by the formula

$$(f \cup g)(\sigma) = (-1)^{mn} f(\sigma\rfloor_n)\{M(\sigma_{\widehat{n+1}})g(\lfloor_m\sigma)\},$$

where $\sigma_{\widehat{n+1}} = [id, \sigma \circ (id_{G/H} \times \delta_{(1,\cdots,n,n+2,\cdots,n+m)})]$ is a morphism in ΠX from $(\sigma\rfloor_n)_H$ to $(\lfloor_m \sigma)_H$.

The coboundary formula for the cup product shows that we have an induced cup product in cohomology. The cup product in cohomology is associative and graded commutative. Hence $H_G^*(X; M)$ is an associative graded algebra, where M is an equivariant local coefficient system of Λ-algebras.

Next we relate the simplicial Bredon-Illman cohomology with local coefficients of a G-simplicial set X to the equivariant twisted cohomology of X. Suppose X is a G-connected G-simplicial set with a G-fixed 0-simplex v. Let M be an equivariant local coefficient system of abelian groups on X and M_0 be the associated abelian O_G-group equipped with an action ϕ of the O_G-group $\underline{\pi}X$ as described in Section 3.3. Let κ be the O_G-twisting function on ΦX as introduced in Example 2.4.3.

Theorem 3.4.9. *With the above hypothesis*

$$H^n_G(X;M) \cong H^n_G(X;\kappa,\phi)$$

for all n.

Proof. Define a cochain map

$$\Psi^*\colon S^*_G(X;M) \to C^*_G(X;\kappa,\phi)$$

as follows. Let $f \in S^n_G(X;M)$ and $y \in X^H_n$ be non-degenerate. Let σ be the unique equivariant n-simplex of type H such that $\sigma(eH,\Delta_n) = y$. Then

$$\Psi^n(f)(G/H)\colon C_n(X^H) \to M_0(G/H)$$

is given by

$$\Psi^n(f)(G/H)(y) = M(b\xi_H(\partial_{(1,\cdots,n)}y))f(\sigma),$$

where $\partial_{(1,\cdots,n)}y$ is the simplex $\partial_1\cdots\partial_n y$, as introduced in Section 1.2 (cf. Example 2.4.3 and Remark 3.2.4 for other notations).

To check that $\Psi^n(f) \in C^n_G(X;\kappa,\phi)$, suppose $a^{-1}Ha \subseteq K$. Note that if $z \in X^K_n$ and $y = az$, then $y \in X^H$. Moreover, if σ is as above and τ denotes the unique equivariant n-simplex of type K such that $\tau(eK,\Delta_n) = z$, then the equivariant n-simplices σ and τ are compatible under $\hat{a}$. As $f \in S^n_G(X;M)$, we must have $f(\sigma) = M(\sigma_H \xrightarrow{a_*} \tau_K)f(\tau)$. Therefore, by the definition of Ψ^n, we have

$$\begin{aligned}
&\Psi^n(f)(G/H)(y)\\
=&M(v_H \xrightarrow{b\xi_H(\partial_{(1,\cdots,n)}y)} \sigma_H)f(\sigma)\\
=&M(v_H \xrightarrow{b\xi_H(\partial_{(1,\cdots,n)}y)} \sigma_H)M(\sigma_H \xrightarrow{a_*} \tau_K)f(\tau).
\end{aligned}$$

On the other hand,

$$M_0(\hat{a})\Psi^n(f)(G/K)(z) = M_0(\hat{a})M(v_K \xrightarrow{b\xi_K(\partial_{(1,\cdots,n)}z))} \tau_K)f(\tau).$$

Recall that $M_0(\hat{a}) = M(v_H \xrightarrow{[\hat{a},k]} v_K)$, where $k\colon G/H \times \Delta[1] \to X$ is given by

$k(eH, \Delta_1) = s_0 v$. Then note that

$$a_* \circ b\xi_H(\partial_{(1,\cdots,n)} y) = b\xi_K(\partial_{(1,\cdots,n)} z) \circ [\hat{a}, k],$$

as composition of morphisms in ΠX. Thus $\Psi^n(f) \in C_G^n(X; \kappa, \phi)$.

To check that Ψ^* is a cochain map, let $f \in S_G^n(X; M)$, $y \in X_{n+1}^H$ and let σ be the equivariant $(n+1)$-simplex of type H such that $\sigma(eH, \Delta_{n+1}) = y$. Observe that the i-th face $\sigma^{(i)}$ is such that $\sigma^{(i)}(eH, \Delta_n) = \partial_i y$. Thus by the definition of the twisted coboundary we have

$$\begin{aligned}
&(-1)^{n+1}\delta_\kappa(\Psi^n(f))(G/H)(y)\\
=&\kappa(G/H)(y)^{-1}\Psi^n(f)(G/H)(\partial_0 y) + \sum_{i=1}^{n+1}(-1)^i \Psi^n(f)(G/H)(\partial_i y)\\
=&\kappa(G/H)(y)^{-1}M(b\xi_H(\partial_{(1,\cdots,n)}\partial_0 y))f(\sigma^{(0)}) + \sum_{i=1}^{n+1}(-1)^i M(b\xi_H(\partial_{(1,\cdots,n)}\partial_i y))f(\sigma^{(i)})\\
=&\kappa(G/H)(y)^{-1}M(b\xi_H(\partial_{(1,\cdots,n)}\partial_0 y))f(\sigma^{(0)}) + \sum_{i=1}^{n+1}(-1)^i M(b\xi_H(\partial_{(1,\cdots,n+1)} y))f(\sigma^{(i)}).
\end{aligned}$$

Note that $\partial_{(1,\cdots,n+1)} y = \partial_{(1,\cdots,n)}\partial_i y$ for $i > 0$.

On the other hand,

$$\begin{aligned}
&\Psi^{n+1}((-1)^{n+1}\delta f)(G/H)(y)\\
=&M(v_H \xrightarrow{b\xi_H(\partial_{(1,\cdots,n+1)} y)} \sigma_H)((-1)^{n+1}\delta f)(y)\\
=&M(v_H \xrightarrow{b\xi_H(\partial_{(1,\cdots,n+1)} y)} \sigma_H)\{M(\sigma_*)f(\sigma^{(0)}) + \sum_{i=1}^{n+1}(-1)^i f(\sigma^{(i)})\}.
\end{aligned}$$

Therefore, we need to compare the first two terms on the left hand side of the above two expressions. By the definition of the action of $\underline{\pi}X$ on M_0 and by Example 2.4.3, we have

$$\begin{aligned}
&\kappa(G/H)(y)^{-1}M(v_H \xrightarrow{b\xi_H(\partial_{(1,\cdots,n)}\partial_0 y)} \sigma_H^{(0)})\\
=&M(b\xi_H^{-1}(\partial_{(0,2,\cdots,n+1)} y) \circ b[\overline{\partial_{(2,\cdots,n+1)} y}] \circ \xi_H(\partial_{(1,\cdots,n+1)} y))M(b\xi_H(\partial_{(1,\cdots,n)}\partial_0 y)\\
=&M(b\xi_H(\partial_{(1,\cdots,n)}\partial_0 y) \circ b\xi_H^{-1}(\partial_{(0,2,\cdots,n+1)} y) \circ b[\overline{\partial_{(2,\cdots,n+1)} y}] \circ b\xi_H(\partial_{(1,\cdots,n+1)} y))\\
=&M(b[\overline{\partial_{(2,\cdots,n+1)} y}] \circ b\xi_H(\partial_{(1,\cdots,n+1)} y))\\
=&M(b\xi_H(\partial_{(1,\cdots,n+1)} y))M(\sigma_*).
\end{aligned}$$

Observe that $\partial_{(0,2,\cdots,n+1)} y = \partial_{(1,2,\cdots,n)}\partial_0 y$. Hence Ψ^* is a cochain map.

Next we define a map

$$\Gamma^*\colon C^*_G(X;\kappa,\phi)\to S^*_G(X;M)$$

as follows. Let $f\in C^n_G(X;\kappa,\phi)$ and $\sigma\colon G/H\times\Delta[n]\to X$ be a non-degenerate equivariant n-simplex of type H. Let $y=\sigma(eH,\Delta_n)$. Define

$$\Gamma^n(f)(\sigma):=M(\sigma_H\xrightarrow{b\xi_H^{-1}(\partial_{(1,\cdots,n)}y)}v_H)f(G/H)(y).$$

To show that $\Gamma^n(f)\in S^n_G(X;M)$, suppose $a^{-1}Ha\subseteq K$, and σ, τ are non-degenerate equivariant n-simplices in X of type H and K respectively, such that σ and τ are compatible under $\hat{a}\colon G/H\to G/K$. Let $z=\tau(eK,\Delta_n)$. Then $y=az$. Note that

$$\begin{aligned}
&M(\sigma_H\xrightarrow{a_*}\tau_K)\Gamma^n(f)(\tau)\\
=&M(\sigma_H\xrightarrow{a_*}\tau_K)M(\tau_K\xrightarrow{b\xi_K^{-1}(\partial_{(1,\cdots,n)}z)}v_K)f(G/K)(z)\\
=&M(b\xi_K^{-1}(\partial_{(1,\cdots,n)}z)\circ a_*)f(G/K)(z)
\end{aligned}$$

and

$$\Gamma^n(f)(\sigma)=M(\sigma_H\xrightarrow{b\xi_H(\partial_{(1,\cdots,n)}y)}v_H)f(G/H)(y).$$

But by the naturality of f we have $f(G/H)(y)=M_0(\hat{a})f(G/K)(z)$. Moreover,

$$b\xi_K(\partial_{(1,\cdots,n)}z)\circ[\hat{a},k]=a_*\circ b\xi_H(\partial_{(1,\cdots,n)}y).$$

Hence $\Gamma^n(f)(\sigma)=M(a_*)\Gamma^n(f)(\tau)$. Thus $\Gamma^n(f)\in S^n_G(X;M)$.

It remains to show that Ψ^* and Γ^* are inverses to each other. Take $f\in S^n_G(X;M)$ and let σ be an equivariant n-simplex of type H. Then,

$$\begin{aligned}
&\Gamma^n\Psi^n(f)(\sigma)\\
=&\Gamma^n(\Psi^n(f))(\sigma)\\
=&M(\sigma_H\xrightarrow{b\xi_H^{-1}(\partial_{(1,\cdots,n)}y)}v_H)\Psi^n(f)(G/H)(y)\\
=&M(\sigma_H\xrightarrow{b\xi_H^{-1}(\partial_{(1,\cdots,n)}y)}v_H)M(v_H\xrightarrow{b\xi_H(\partial_{(1,\cdots,n)}y)}\sigma_H)f(\sigma)\\
=&f(\sigma).
\end{aligned}$$

Thus we have $\Gamma^n\Psi^n=id$. Similarly, $\Psi^n\Gamma^n=id$. This completes the proof of the theorem. $\square$

3.5 The equivariant Serre spectral sequence

The aim of this last section is to derive a version of the Serre spectral sequence for a G-Kan fibration. To do this, we give an alternative description of the simplicial Bredon-Illman cohomology with local coefficients of a G-simplicial set in terms of the cohomology of small categories, following [MS93], [MP02].

We briefly recall the definition of the cohomology of a small category, in the sense of Quillen [Qui67]. Suppose that $\mathcal{C}$ is a small category. The nerve of $\mathcal{C}$ is the simplicial set $N(\mathcal{C}) = \{N_n(\mathcal{C})\}_{n\geq 0}$ whose 0-simplices are the objects of $\mathcal{C}$ and whose n-simplices $(n > 0)$ are n-tuples $(u_1, \cdots, u_n)$ of composable maps $C_0 \xrightarrow{u_1} C_1 \xrightarrow{u_2} \cdots \xrightarrow{u_n} C_n$ in $\mathcal{C}$. The face and degeneracy maps are given as follows.

$$\partial_0(u_1, \cdots, u_n) = (u_2, \cdots, u_n),\ \partial_n(u_1, \cdots, u_n) = (u_1, \cdots, u_{n-1}),$$

$$\partial_i(u_1, \cdots, u_n) = (u_1, \cdots, u_{i-1}, u_{i+1} \circ u_i, u_{i+2}, \cdots, u_n),\quad 0 < i < n,$$

$$s_0(u_1, \cdots, u_n) = (id, u_1, \cdots, u_n),\ s_i(u_1, \cdots, u_n) = (u_1, \cdots, u_i, id, u_{i+1}, \cdots, u_n),\ i > 0.$$

Let $M\colon \mathcal{C} \to \mathcal{A}b$ be a contravariant functor. Let $C^n(\mathcal{C}; M)$ be the set of all functions on $N_n(\mathcal{C})$ which sends an n-simplex $(u_1, \cdots, u_n)$ to an element of $M(C_0)$. The differentials in the associated cochain complex $C^\bullet(\mathcal{C}; M) = \{C^n(\mathcal{C}; M)\}_{n\geq 0}$ are obtained from the face maps of $N(\mathcal{C})$ by taking alternating sums. Then the cohomology of the category $\mathcal{C}$ with coefficients M is defined by

$$H^n(\mathcal{C}; M) := H^n(C^\bullet(\mathcal{C}; M)).$$

Let G be a discrete group and X be a G-Kan complex. Then we have a category $\Delta_G(X)$ described as follows. Its objects are G-simplicial maps $\sigma\colon G/H \times \Delta[n] \to X$, where H is a subgroup of G and $n \geq 0$. A morphism from $\sigma\colon G/H \times \Delta[n] \to X$ to $\tau\colon G/K \times \Delta[m] \to X$ is a pair $(\hat{a}, \alpha)$, where $\hat{a}\colon G/H \to G/K$, $a^{-1}Ha \subseteq K$, is a G-map and $\alpha\colon \Delta[n] \to \Delta[m]$ is a simplicial map such that $\tau \circ (\hat{a} \times \alpha) = \sigma$. There is a canonical functor $v_X\colon \Delta_G(X) \to \Pi X$ which sends $\sigma\colon G/H\times\Delta[n] \to X$ to $\sigma_H = \sigma\circ(id\times\delta_{(1,2,\cdots,n)})$. For a morphism $(\hat{a}, \alpha)$ in $\Delta_G(X)$, $v_X(\hat{a}, \alpha)\colon \sigma_H \to \tau_K$ is the morphism $[\hat{a}, \vartheta]$ in ΠX where $\vartheta\colon G/H \times \Delta[1] \to X$ is an equivariant 1-simplex of type H obtained as follows. Suppose that

$$\tau \circ (id \times \delta_{(1,\cdots,\widehat{\alpha(0)}\cdots,m)})(eK, \Delta_1) = \omega \in X_1^K.$$

Let Ω be a 2-simplex in X^K determined by the compatible pair of 1-simplices $(\hat{x}_0, x_1 = s_1\partial_1\omega, x_2 = \omega)$. Then ϑ is given by $\vartheta(eH, \Delta_1) = a\partial_0\Omega$.

If X is any G-simplicial set, then we define $\Delta_G(X) = \Delta_G(S|X|)$.

For a small category $\mathcal{C}$, let $\mathcal{A}b(\mathcal{C})$ be the category of all contravariant functors from

$\mathcal{C}$ to $\mathcal{A}b$. The set of morphisms between two objects of $\mathcal{A}b(\mathcal{C})$ is the set of all natural transformations of functors.

Definition 3.5.1. A functor $M \in \mathcal{A}b(\Delta_G(X))$ is said to be G-local if

$$M = v_X^* M' = M' \circ v_X$$

for some $M' \in \mathcal{A}b(\Pi X)$. For a G-local coefficient system M, the equivariant cohomology of X with coefficients M is defined to be

$$h_G^*(X; M) := H^*(\Delta_G(X); M),$$

where the right-hand side denotes the cohomology of the category $\Delta_G(X)$.

Theorem 3.5.2. *Let X be a G-simplicial set and M be an equivariant local coefficient system on X. Then there is an isomorphism*

$$H_G^*(X; M) \cong h_G^*(X; M).$$

(On the right we identify M with $v_X^*(M)$).

Proof. Let $\tilde{X}$ be the bisimplicial set whose (p, q)-simplices are triples $(u, \alpha, \sigma) \in \tilde{X}_{p,q}$, where

$$\begin{aligned} &u = ([n_0] \xrightarrow{u_1} [n_1] \to \cdots \xrightarrow{u_p} [n_p]) \in N_p(\Delta),\\ &\alpha = (G/H_0 \xrightarrow{\alpha_1} G/H_1 \to \cdots \xrightarrow{\alpha_q} G/H_q) \in N_q(O_G),\\ &\sigma \colon G/H_q \times \Delta[n_p] \to X \text{ is a } G\text{-simplicial map.} \end{aligned}$$

The face and degeneracy maps of $\tilde{X}$ are induced from those of $N(\Delta)$ and $N(O_G)$. Then

$$\text{diagonal}(\tilde{X}) \cong N(\Delta_G(X)).$$

To every $(u, \alpha, \sigma) \in \tilde{X}_{p,q}$, associate a G-simplicial map

$$\overline{\sigma} = \sigma \circ (\alpha_q \circ \cdots \circ \alpha_1 \times u_p \circ \cdots \circ u_1) \colon G/H_0 \times \Delta[n_0] \to X.$$

Let $C^{p,q}(X; M)$ denote the set of all functions on $\tilde{X}_{p,q}$ which sends an element (u, α, σ) of $\tilde{X}_{p,q}$ to an element of $M(v_X(\overline{\sigma}))$. This yields a bicomplex $C^{\bullet,\bullet}(X; M)$ with obvious differentials d_h and d_v induced from the face maps of $\tilde{X}$. Denote the total complex of $C^{\bullet\bullet}(X; M)$ by $\operatorname{Tot} C^{\bullet\bullet}(X; M)$. Let $\operatorname{diag} C^{\bullet\bullet}(X; M)$ be the cochain complex whose p-th group is $C^{p,p}(X; M)$ and differential is $d_h d_v$. Then by a result of Dold and Puppe ([DP61]), we have

$$H^n(\operatorname{Tot} C^{\bullet\bullet}(X; M)) \cong H^n(\operatorname{diag} C^{\bullet\bullet}(X; M)).$$

Now $C^{p,p}(X;M)$ can be interpreted as the set of all functions on $N(\Delta_G(X))$ which sends a p-simplex $(\tau_0 \to \tau_1 \to \cdots \to \tau_p)$ to an element of $M(v_X(\tau_0))$, and the differential on $C^{p,p}(X;M)$ is just the differential induced from the face maps of $N_p(\Delta_G(X))$. Hence,

$$H^n(\operatorname{diag} C^{\bullet\bullet}(X;M)) \cong H^n(\Delta_G(X); v_X^* M) = h_G^n(X;M).$$

Recall that the spectral sequence associated with the p-filtration of the bicomplex $C^{\bullet\bullet}(X;M)$ converges to the cohomology of the total complex, since it is a first quadrant spectral sequence.

We now compute the E_1-term of the spectral sequence associated with the p-filtration of the bicomplex $C^{\bullet\bullet}(X,M)$.

Suppose $X_n^{(-)} \colon O_G^{op} \to SETS \subset \mathcal{C}at$ is the functor which sends G/H to X_n^H. Let

$$\mathcal{C}_n = \int_{O_G} X_n^{(-)}$$

be the category obtained by the Grothendieck construction on the functor $X_n^{(-)}$. We can identify $\mathcal{C}_n$ with the category whose objects are equivariant n-simplices of X, and a morphism from $\sigma \colon G/H \times \Delta[n] \to X$ to $\tau \colon G/K \times \Delta[n] \to X$ is a G-maps $\hat{a} \colon G/H \to G/K$ such that $\tau \circ (\hat{a} \times 1) = \sigma$.

Define a functor

$$M_n \colon \mathcal{C}_n^{op} \to \mathcal{A}b$$

which takes an object $\sigma \colon G/H \times \Delta[n] \to X$ of $\mathcal{C}_n$ to $M(v_X(\sigma))$. If $\hat{a} \colon G/H \to G/K$ is a morphism from σ to τ, then $[\hat{a}, k]$ is a morphism in ΠX from $v_X(\sigma)$ to $v_X(\tau)$ and we define $M_n(\hat{a}) = M([\hat{a}, k])$. Here $k \colon G/H \times \Delta[1] \to X$ is given by $k(eH, \Delta_1) = s_0(v_X(\sigma)(eH, \Delta_0))$.

Fix a $u = ([n_0] \xrightarrow{u_1} \cdots \xrightarrow{u_p} [n_p]) \in N_p(\Delta)$. Let us denote the composition $u_p \circ \cdots \circ u_1$ by u again. Corresponding to this u, there is a functor $F \colon \mathcal{C}_{n_p} \to \mathcal{C}_{n_0}$ which takes an object $\sigma \colon G/H \times \Delta[n_p] \to X$ of $\mathcal{C}_{n_p}$ to $\sigma \circ (id \times u) \colon G/H \times \Delta[n_0] \to X$ and a morphism $\hat{a} \colon G/H \to G/K$ between $\sigma \colon G/H \times \Delta[n] \to X$ and $\tau \colon G/K \times \Delta[n] \to X$ to $\hat{a}$. We define a functor M_u on $\mathcal{C}_{n_p}$ to be

$$M_u = M_{n_0} \circ F.$$

Then for all $p \geq 0$,

$$C^{p,q}(X;M) \cong \prod_{u \in N_p(\Delta)} C^q(\mathcal{C}_{n_p}; M_u),$$

the correspondence being given as follows. Let f be an element of $C^{p,q}(X,M)$. Then f

induces an element

$$(f_u) \in \prod_{u \in N_p(\Delta)} C^q(\mathcal{C}_{n_p}; M_u),$$

where $f_u \in C^q(\mathcal{C}_{n_p}; M_u)$ is as defined below. To a simplex

$$v = \sigma_0 \xrightarrow{\hat{a}_1 \times 1} \cdots \xrightarrow{\hat{a}_q \times 1} \sigma_q,\ \sigma_i \colon G/H_i \times \Delta[n_p] \to X,$$

of the nerve of $\mathcal{C}_{n_p}$, we associate an element $(u, \alpha, \sigma) \in \tilde{X}_{p,q}$, where u is given by the choice of the index, $\sigma = \sigma_q$ and $\alpha = (G/H_0 \xrightarrow{\hat{a}_1} G/H_1 \to \cdots \xrightarrow{\hat{a}_q} G/H_q)$. Then $f_u(v) = f(u, \alpha, \sigma)$.

Conversely, let $(f_u) \in \prod_{u \in N_p(\Delta)} C^q(\mathcal{C}_{n_p}; M_u)$. Then we get $f \in C^{p,q}(X; M)$ as follows. A (p, q)-simplex (u, α, σ) of $\tilde{X}$, given by

$$\begin{aligned}
&u = ([n_0] \xrightarrow{u_1} [n_1] \to \cdots \xrightarrow{u_p} [n_p]) \in N_p(\Delta),\\
&\alpha = (G/H_0 \xrightarrow{\hat{\alpha}_1} G/H_1 \to \cdots \xrightarrow{\hat{\alpha}_q} G/H_q) \in N_q(O_G),\\
&\sigma \colon G/H_q \times \Delta[n_p] \to X \text{ is a } G\text{-simplicial map},
\end{aligned}$$

corresponds to a q-simplex $v = \tau_0 \xrightarrow{\hat{\alpha}_1 \times 1} \tau_1 \to \cdots \xrightarrow{\hat{\alpha}_q \times 1} \tau_q$ of the nerve of $\mathcal{C}_{n_p}$, where $\tau_q = \sigma$ and $\tau_i = \tau_{i+1}(\hat{\alpha}_{i+1} \times 1)$. Set $f(u, \alpha, \sigma) = f_u(v)$.

Let us denote the differential on $C^\bullet(\mathcal{C}_{n_p}; M_u)$ by d_u. Then $C^{p,\bullet}(X; M)$ is isomorphic to the cochain complex

$$\Big(\prod_{u \in N_p(\Delta)} C^\bullet(\mathcal{C}_{n_p}; M_u), \prod_{u \in N_p(\Delta)} d_u\Big).$$

It follows that

$$H^q(C^{p,\bullet}(X; M)) \cong \prod_{u \in N_p(\Delta)} H^q(\mathcal{C}_{n_p}; M_u).$$

We now compute $H^q(\mathcal{C}_{n_p}; M_u)$. Let e_0 denote the 0-simplex $(0) \in \Delta[n_0]$. Then $u(e_0) \in \Delta[n_p]_0$. Define a G-simplicial map $\mathring{\sigma} \colon G/H \times \Delta[0] \to X$ by $\sigma(eH, u(e_0)) = \mathring{\sigma}$. Then M_u is naturally isomorphic to the functor which takes σ to $M(\mathring{\sigma})$ and hence to M_{n_p}. Thus,

$$H^*(\mathcal{C}_{n_p}; M_u) \cong H^*(\mathcal{C}_{n_p}; M_{n_p}).$$

Now, for any $n \geq 0$, $S_n(X)$ is a G-set, the G action being induced by the action on X. Recall that for the G-set $S = G/H$, the "global section" or the "inverse limit" functor,

$$\Gamma \colon \mathcal{A}b\left(\int_{O_G} (S)^{(-)}\right) \to \mathcal{A}b$$

is an exact functor ([MS93]). Also, any G-set S can be written as a union of orbits, say $S = \bigcup_H G/H$, where the union is over conjugacy classes of isotropy subgroups, one representative chosen from each class. If $\mathcal{D} = \int_{O_G} S^{(-)}$ and we let

$$\int_{O_G} (G/H)^{(-)} = \mathcal{D}_H,$$

then $\mathcal{D}$ is the union of the categories $\mathcal{D}_H$. Also, if $M \in \mathcal{A}b(\mathcal{D})$ and we denote $M|_{\mathcal{D}_H} = M_H$, then M_H is a contravariant functor on $\mathcal{D}_H$. It is clear from the definition of cohomology of categories that

$$H^q(\mathcal{D}; M) = \bigoplus_H H^q(\mathcal{D}_H; M_H).$$

Also $\Gamma(M) = \bigoplus_H \Gamma(M_H)$. Combining these facts, we get for all $n \geq 0$,

$$H^q(\mathcal{C}_n; M_n) = \begin{cases} \Gamma(M_n) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

Now recall that $\Gamma(M_n)$ consists of all functions ϕ which take an object σ of $\mathcal{C}_n$ to an element of $M_n(\sigma) = M(v_X(\sigma))$ so that if $\hat{a}\colon G/H \to G/K$ is a morphism between $\sigma\colon G/H \times \Delta[n] \to X$ and $\tau\colon G/K \times \Delta[n] \to X$ i.e. if $\tau \circ (\hat{a} \times 1) = \sigma$, then $M_n(\hat{a})(\phi(\tau)) = \phi(\sigma)$. Hence

$$\Gamma(M_n) = S^n_G(X; M).$$

Thus for each $u = ([n_0] \to \cdots \to [n_p])$ in $N_p(\Delta)$ we get a copy of $S^{n_p}_G(X; M)$ which we denote by $S^{n(u)}_G(X; M)$, and we have

$$E_1^{p,q} = H^q(C^{p,\bullet}(X; M)) = \begin{cases} \prod_{u \in N_p(\Delta)} S^{n(u)}_G(X; M) & \text{if } q = 0, \\ 0 & \text{if } q > 0; \end{cases}$$

where $S^{n(u)}_G(X; M)$ is a copy of $S^{n_p}_G(X; M)$ for every $u = ([n_0] \to \cdots \to [n_p])$. Thus,

$$\begin{aligned} H^p(\text{Tot } C^{\bullet\bullet}(X; M)) &\cong H^p(\textstyle\prod_{u \in N(\Delta)} S^{n(u)}_G(X; M)) \\ &\cong H^p(\Delta^{op}; S^\bullet_G(X; M)), \end{aligned}$$

where $S^\bullet_G(X; M)$ is the cosimplicial group which takes $[n]$ to $S^n_G(X; M)$ with obvious face and degeneracy maps induced from those of Δ. Then we know that ([MS93]),

$$H^p(\Delta^{op}; S^\bullet_G(X; M)) \cong H^p(S^\bullet_G(X; M)).$$

Hence,

$$H^p(\text{Tot } C^{\bullet\bullet}(X;M)) \cong H^p_G(X;M).$$

$\square$

We are now in a position to derive the required spectral sequence. Let X, Y be G-simplicial sets and $f\colon Y \to X$ be a G-Kan fibration. Let M be a G-local coefficient system on Y. For $q \geq 0$, we have a contravariant functor

$$h^q_G(f, M)\colon \Delta_G(X) \to \mathcal{A}b,$$

defined as follows. For an object $\sigma\colon G/H \times \Delta[n] \to X$ of $\Delta_G(X)$, let $\sigma^*(Y)$ be the total complex of the pull-back of the G-Kan fibration $f\colon Y \to X$ via σ. Then $\sigma^*(Y)$ is a G-simplicial set and define

$$h^q_G(f, M)(\sigma) := h^q_G(\sigma^*(Y); \tilde{\sigma}^*M),$$

where $\tilde{\sigma}\colon \sigma^*(Y) \to Y$ is the canonical map and $\tilde{\sigma}^*M$ is the equivariant local coefficient system on $\sigma^*(Y)$ obtained by the pull-back of M via $\tilde{\sigma}$. We claim that $h^q_G(f, M)$ factors through v_X yielding a G-local coefficient system on X. To prove this, we use the following result ([MS93], the proof of Theorem 2.3).

Theorem 3.5.3. *Let $f\colon Y \to X$ be a weak equivalence in $G\mathcal{S}$. Then for any G-local coefficient system M on X, f induces an isomorphism*

$$h^*_G(X; M) \cong h^*_G(Y; f^*M).$$

Recall that a G-simplicial map $f\colon Y \to X$ is a weak equivalence in $G\mathcal{S}$ if, and only if, $f^H\colon Y^H \to X^H$ is a weak equivalence in $\mathcal{S}$ for each subgroup H of G. Therefore, the claim will follow from the above result, provided we prove that for a simplicial map $u\colon \Delta[m] \to \Delta[n]$, the map $(\sigma(id \times u))^*(Y) \xrightarrow{\tilde{u}} \sigma^*(Y)$ covering $(id \times u)\colon G/K \times \Delta[m] \to G/K \times \Delta[n]$, is a weak equivalence in $G\mathcal{S}$. To justify this, observe that the middle and the left vertical arrows of the following pull-back diagram are G-Kan fibrations. This is because in a closed model category, the class of fibrations is closed under pull-back.

$$\begin{tikzcd}
(\sigma(id \times u))^*(Y) \arrow[r, "\tilde{u}"] \arrow[d, "f_2"'] & \sigma^*(Y) \arrow[r, "\tilde{\sigma}"] \arrow[d, "f_1"'] & Y \arrow[d, "f"] \\
G/K \times \Delta[m] \arrow[r, "id \times u"] & G/K \times \Delta[n] \arrow[r, "\sigma"] & X
\end{tikzcd}$$

Moreover, for each subgroup H of G, the simplicial map $(id \times u)^H$ is a weak equivalence in $\mathcal{S}$. Therefore, it follows from the homotopy long exact sequences of the fibrations f_1^H, f_2^H that $\tilde{u}^H$ is a weak equivalence in $\mathcal{S}$ for each subgroup H of G. Hence $\tilde{u}$ is a weak equivalence in $G\mathcal{S}$.

Theorem 3.5.4. *Let X, Y be G-simplicial sets. For any G-Kan fibration $f\colon Y \to X$ and a G-local coefficient system M on Y, there is a natural spectral sequence with E_2-term $E_2^{p,q} = H_G^p(X; h_G^p(f, M))$ converging to $H_G^{p+q}(Y; M)$.*

Proof. The proof is parallel to the proof of Theorem 3.2, [MS93]. We only mention the essential steps. A G-Kan fibration $f\colon Y \to X$ induces a functor $\Delta_G(f)\colon \Delta_G(Y) \to \Delta_G(X)$ and we have a Grothendieck spectral sequence [Seg74]

$$H^p(\Delta_G(X); h^q(\Delta_G(f)/-; M)) \Rightarrow H^{p+q}(\Delta_G(Y); M).$$

It is enough to show that the two contravariant functors $h^q(\Delta_G(Y)/-; M)$ and $h_G^q(f, M)$ from $\Delta_G(X)$ to $\mathcal{A}$b are naturally isomorphic. For an object σ of $\Delta_G(X)$, let $\Delta_G(f)/\sigma$ be the comma category. This is defined as follows. Objects of $\Delta_G(f)/\sigma$ are pairs (τ, u) where $\tau \in Ob(\Delta_G(Y))$ and $u\colon \Delta_G(f)(\tau) \to \sigma$ is a map in $\Delta_G(X)$. Morphisms from (τ, u) to (τ', u') are maps $\alpha\colon \tau \to \tau'$ such that $u' \circ (\Delta_G(f)\alpha) = u$. A direct computation shows that there is a canonical equivalence of the categories

$$\Delta_G(f)/\sigma \cong \Delta_G(\sigma^*(Y)),$$

which is natural in σ. Hence we have natural isomorphism of functors

$$h^q(\Delta_G(\sigma^*(Y); \tilde{\sigma}^* M)) \cong h^q(\Delta_G(f)/\sigma; M).$$

The result now follows from Theorem 3.5.2.

□

Chapter 4

Classification of simplicial Bredon-Illman cohomology with local coefficients

4.1 Introduction

The aim of this chapter is to prove a classification theorem for simplicial Bredon-Illman cohomology with local coefficients of a G-simplicial set, as introduced in the previous chapter. We first prove a classification theorem for equivariant twisted cohomology of a G-simplicial set, generalizing the corresponding non-equivariant result [Hir79]. We then use Theorem 3.4.9 to deduce the desired result.

4.2 Generalized O_G-Eilenberg-MacLane complexes

For a group π and a non-negative integer n, let $L_\pi(A, n)$ denote the generalized Eilenberg-MacLane complex determined by a π-module (A, ϕ), as described in Section 1.7. Our aim is to derive an equivariant version of Theorem 1.8.7. The role of the classifying complex in the equivariant context is played by an O_G-Kan complex as described below.

Let $\underline{\pi}$ be an O_G-group and M_0 be an abelian O_G-group equipped with an action $\phi\colon \underline{\pi} \times M_0 \to M_0$ of $\underline{\pi}$. For an integer $n \geq 0$, we have a simplicial abelian G-group (i.e, a simplicial object in the category of abelian O_G-groups) $C(M_0, n)$, where

$$C(M_0, n)(G/H)_q = C^n(\Delta[q]; M_0(G/H)), \quad q \geq 0,$$

for every object G/H of O_G. For a morphism $\hat{a}\colon G/H \to G/K$ in O_G, the map

$C(M_0,n)(\hat{a})$ is the coefficients change homomorphism, induced by the homomorphism $M_0(\hat{a})\colon M_0(G/K) \to M_0(G/H)$. Define natural transformations

$$\delta^n \colon C(M_0,n) \to C(M_0,n+1),\ n \geq 0,$$

where $\delta^n(G/H)_q \colon C^n(\Delta[q]; M_0(G/H)) \to C^{n+1}(\Delta[q]; M_0(G/H))$ is the ordinary simplicial coboundary of the simplicial set $\Delta[q]$, $H \subseteq G$ being a subgroup and $q \geq 0$. Then it follows that $\delta^{n+1} \circ \delta^n = 0$. Note that the $\underline{\pi}$-action on M_0 induces an $\underline{\pi}$-action on $C(M_0,n)$ in the following way. For a subgroup H of G,

$$\gamma\mu = \phi(G/H)(\gamma, \text{-}) \circ \mu, \quad \mu \in C^n(\Delta[q]; M_0(G/H)),\ \gamma \in \underline{\pi}(G/H).$$

Let $\overline{W}\underline{\pi}$ denote the O_G-Kan complex, as described in Example 2.3.4. Recall that (cf. Example 2.4.2) we have an O_G-twisting function $\kappa(\underline{\pi})\colon \overline{W}\underline{\pi} \to \underline{\pi}$, given by $\kappa(\underline{\pi})(G/H)([\gamma_1, \gamma_2, \cdots, \gamma_q]) = \gamma_1$. We denote the O_G-twisted cartesian product $C(M_0,n) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}$ by $\chi_\phi(M_0,n)$. Then $\chi_\phi(M_0,n)$ is an O_G-Kan complex and for any subgroup H of G,

$$\chi_\phi(M_0,n)(G/H) = C^n(\Delta[-]; M_0(G/H)) \times_{\kappa(\underline{\pi})(G/H)} \overline{W}\underline{\pi}(G/H).$$

Let $K(M_0,n)$ denote an O_G-Eilenberg-MacLane complex of type (M_0,n). Then the canonical model of $K(M_0,n)$ is given by

$$K(M_0,n) = Ker(\delta^n \colon C(M_0,n) \to C(M_0,n+1)).$$

For the canonical model of $K(M_0,n)$, let $L_\phi(M_0,n) = K(M_0,n) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}$ be the O_G-Kan complex as introduced in Example 2.4.5. Note that for any subgroup H of G, $L_\phi(M_0,n)(G/H)$ is the generalized Eilenberg-MacLane complex

$$L_{\underline{\pi}(G/H)}(M_0(G/H),n) = Z^n(\Delta[-]; M_0(G/H)) \times_{\kappa(\underline{\pi})(G/H)} \overline{W}\underline{\pi}(G/H).$$

Since any two models of $K(M_0,n)$ are naturally isomorphic, $K(M_0,n) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}$ is isomorphic to $L_\phi(M_0,n)$ for any model of $K(M_0,n)$.

Definition 4.2.1. The O_G-Kan complex $L_\phi(M_0,n)$ is called the generalized O_G-Eilenberg-MacLane complex determined by the $\underline{\pi}$-module (M_0,ϕ).

We have the natural projections onto the second factor

$$\chi_\phi(M_0,n) \xrightarrow{p} \overline{W}\underline{\pi}, \quad L_\phi(M_0,n) \xrightarrow{p} \overline{W}\underline{\pi}$$

and we view these O_G-Kan complexes as objects in the slice category $O_G\mathcal{S} \downarrow \overline{W}\underline{\pi}$.

We shall need the following lemma.

Lemma 4.2.2. *For a subgroup H of G and an integer $q \geq 0$, consider the G-simplicial set $G/H \times \Delta[q]$. Let $\underline{\pi}$ be an O_G-group and (M_0, ϕ) be a $\underline{\pi}$-module. Suppose that $\kappa\colon \Phi(G/H \times \Delta[q]) \to \underline{\pi}$ is an O_G-twisting function. Then there is a cochain isomorphism*

$$E_H^*\colon C_G^*(G/H \times \Delta[q]; \kappa, \phi) \to C^*(\Delta[q]; M_0(G/H))$$

which is natural with respect to the morphisms in O_G.

Proof. Let $f \in C_G^n(G/H \times \Delta[q]; \kappa, \phi)$ and $\alpha \in \Delta[q]_n$ be non-degenerate. Suppose $\alpha = (\alpha_0, \cdots, \alpha_n)$, where $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_n \leq q$. Recall that (cf. Section 2.5) $f\colon \underline{C}_n(G/H \times \Delta[q]) \to M_0$ is a natural transformation. Define

$$E_H^n\colon C_G^n(G/H \times \Delta[q]; \kappa, \phi) \to C^n(\Delta[q]; M_0(G/H)),$$

by

$$E_H^n(f)(\alpha) = \kappa(G/H)(eH, (0, \alpha_0))^{-1} f(G/H)(eH, \alpha).$$

Observe that $(eH, (0, \alpha_0))$ and (eH, α) are respectively a 1-simplex and an n-simplex in $(G/H \times \Delta[q])^H$, and the right-hand side of the above equality is given by the action of $\underline{\pi}(G/H)$ on $M_0(G/H)$.

To check that E_H^* is a cochain map, let $f \in C_G^{n-1}(G/H \times \Delta[q]; \kappa, \phi)$ and $\alpha = (\alpha_0, \cdots, \alpha_n) \in \Delta[q]_n$. Then

$$\begin{aligned}
&E_H^n(\delta_\kappa f)(\alpha)\\
=&\kappa(G/H)(eH, (0, \alpha_0))^{-1}(\delta_\kappa f)(G/H)(eH, \alpha)\\
=&(-1)^n \kappa(G/H)(eH, (0, \alpha_0))^{-1}\{\kappa(G/H)(eH, \alpha)^{-1} f(G/H)(\partial_0(eH, \alpha))\\
+&\sum_{i=1}^{n}(-1)^i f(G/H)(\partial_i(eH, \alpha))\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\delta(E_H^{n-1} f)(\alpha)\\
=&(-1)^n[\sum_{i=0}^{n} E_H^{n-1}(f)(\partial_i \alpha)]\\
=&(-1)^n[\kappa(G/H)(eH, (0, \alpha_1))^{-1} f(G/H)(eH, \partial_0 \alpha)\\
&+\sum_{i=1}^{n}(-1)^i \kappa(G/H)(eH, (0, \alpha_0))^{-1} f(G/H)(eH, \partial_i \alpha)].
\end{aligned}$$

Note that $\partial_i(eH, \alpha) = (eH, \partial_i \alpha)$. Therefore, E_H^* will be a cochain map provided we show that

$$\kappa(G/H)(eH, \alpha)\kappa(G/H)(eH, (0, \alpha_0)) = \kappa(G/H)(eH, (0, \alpha_1)).$$

We may assume that $\alpha_0 \neq 0$. For, if $\alpha_0 = 0$, then by the property of a twisting function

$$\kappa(G/H)(eH, (0, \alpha_0)) = \kappa(G/H)(s_0(eH, (0))) = e_H,$$

e_H being the identity of the group $\underline{\pi}(G/H)$. Moreover,

$$\begin{aligned}
&\kappa(G/H)(eH, (0, \alpha_1)) \\
=&\kappa(G/H)(\partial_{(2,\cdots,n)}(eH, \alpha)) \\
=&\partial_{(1,\cdots,n-1)}\kappa(G/H)(eH, \alpha) \\
=&\kappa(G/H)(eH, \alpha).
\end{aligned}$$

The last equality holds because all the face maps of the group complex $\underline{\pi}(G/H)$ are identity. So suppose $\alpha_0 \neq 0$. Now observe that $\alpha = \partial_0\beta$, where $\beta = (0, \alpha_0, \cdots, \alpha_n)$ is an $(n+1)$-simplex of $\Delta[q]$. So $\kappa(G/H)(eH, \alpha) = \kappa(G/H)(\partial_0(eH, \beta))$. Furthermore,

$$\kappa(G/H)(eH, (0, \alpha_0)) = \kappa(G/H)(\partial_{(2,\cdots,n+1)}(eH, \beta)) = \kappa(G/H)(eH, \beta).$$

Therefore,

$$\begin{aligned}
&\kappa(G/H)(eH, \alpha)\kappa(G/H)(eH, (0, \alpha_0)) \\
=&\kappa(G/H)(\partial_0(eH, \beta))\kappa(G/H)(eH, \beta) \\
=&\kappa(G/H)(eH, \partial_1\beta).
\end{aligned}$$

Now note that $\partial_1\beta = (0, \alpha_1, \alpha_2, \cdots, \alpha_n)$. As a consequence,

$$\kappa(G/H)(eH, (0, \alpha_1)) = \kappa(G/H)(\partial_{(2,\cdots,n)}(eH, \partial_1\beta)) = \kappa(G/H)(eH, \partial_1\beta).$$

The inverse

$$(E_H^*)^{-1} \colon C^*(\Delta[q]; M_0(G/H)) \to C_G^*(G/H \times \Delta[q]; \kappa, \phi)$$

is defined as follows. Suppose $c \in C^n(\Delta[q]; M_0(G/H))$. Then

$$f = (E_H^*)^{-1}(c) \colon \underline{C}_n(G/H \times \Delta[q]) \to M_0$$

is given by

$$f(G/K)((\hat{a}, \alpha)) = M_0(\hat{a})(\kappa(G/H)(eH, (0, \alpha_0))c(\alpha)),$$

for any object G/K in O_G and for any n-simplex $(\hat{a}, \alpha)$ in $(G/H \times \Delta[q])^K$, where

$$\alpha = (\alpha_0, \cdots, \alpha_n) \ \text{ with } \ 0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_n \leq q.$$

Observe that

$$\hat{a} \in (G/H)^K = Hom_G(G/K, G/H) = Mor_{O_G}(G/K, G/H).$$

To prove the last part, suppose $\hat{a} \colon G/H \to G/K$, $a^{-1}Ha \subseteq K$, is a morphism in O_G. Let $\kappa' \colon \Phi(G/K \times \Delta[q]) \to \underline{\pi}$ be an O_G-twisting function. Let $\kappa = \Phi(\hat{a} \times id)\kappa'$ be the twisting function induced by the G-simplicial map $(\hat{a} \times id) \colon G/H \times \Delta[q] \to G/K \times \Delta[q]$. Let

$$(\hat{a} \times id)^* \colon C^*_G(G/K \times \Delta[q]; \kappa', \phi) \to C^*_G(G/H \times \Delta[q]; \kappa, \phi)$$

be the cochain map induced by $(\hat{a} \times id)$ (cf. Section 2.5) and let

$$M_0(\hat{a})_* \colon C^*(\Delta[q]; M_0(G/K)) \to C^*(\Delta[q]; M_0(G/H))$$

be the map induced by the coefficient homomorphism $M_0(\hat{a}) \colon M_0(G/K) \to M_0(G/H)$. We need to verify the commutativity of the following diagram.

$$\begin{tikzcd}
C^*_G(G/H \times \Delta[q]; \kappa, \phi) \arrow[r, "E^*_H"] & C^*(\Delta[q]; M_0(G/H)) \\
C^*_G(G/K \times \Delta[q]; \kappa', \phi) \arrow[r, "E^*_K"] \arrow[u, "(\hat{a}\times id)^*"] & C^*(\Delta[q]; M_0(G/K)) \arrow[u, "M_0(\hat{a})_*"']
\end{tikzcd}$$

Let $f \in C^*_G(G/K \times \Delta[q]; \kappa', \phi)$ and $\alpha = (\alpha_0, \cdots, \alpha_n)$ be a non-degenerate n-simplex of $\Delta[q]$. Then

$$M_0(\hat{a})_n \circ E^n_K(f)(\alpha) = M_0(\hat{a})(\kappa'(G/K)(eK, (0, \alpha_0))^{-1} f(G/K)(eK, \alpha)).$$

On the other hand,

$$\begin{aligned}
& E^n_H \circ (\hat{a} \times id)^n(f)(\alpha) \\
=& \kappa(G/H)(eH, (0, \alpha_0))^{-1}((\hat{a} \times id)_n(f))(G/H)(eH, \alpha) \\
=& \kappa(G/H)(eH, (0, \alpha_0))^{-1} f(G/H)(\hat{a} \times id)(eK, \alpha) \\
=& \kappa'(G/H)(aK, (0, \alpha_0))^{-1} M_0(\hat{a}) f(G/K)(eK, \alpha) \\
=& M_0(\hat{a})(\kappa'(G/K)(eK, (0, \alpha_0))^{-1} f(G/K)(eK, \alpha)),
\end{aligned}$$

by naturality of f and the definition of the twisting function κ'. Hence the above equality holds. $\square$

Suppose X is a G-simplicial set. As before, let M_0 be an abelian O_G-group with a given action ϕ of an O_G-group $\underline{\pi}$. Let $\kappa\colon \Phi X \to \underline{\pi}$ be a given O_G-twisting function. Recall that the O_G-twisting function $\kappa(\underline{\pi})\colon \overline{W}\underline{\pi} \to \underline{\pi}$ of Example 2.4.2 is given by $\kappa(\underline{\pi})(G/H)([\gamma_1, \cdots, \gamma_q]) = \gamma_1$, where $\gamma_1, \cdots, \gamma_q \in \underline{\pi}(G/H)$. We have a natural map of O_G-simplicial sets

$$\theta(\kappa)\colon \Phi X \to \overline{W}\underline{\pi},$$

defined as follows.

$$\theta(\kappa)(G/H)_q\colon X_q^H \to \overline{W}\underline{\pi}(G/H)_q, \qquad H \subseteq G \text{ a subgroup},$$

$$x \mapsto [\kappa(G/H)_q(x), \kappa(G/H)_{q-1}(\partial_0 x), \cdots, \kappa(G/H)_1(\partial_0^{q-1}x)].$$

Note that $\kappa(\underline{\pi}) \circ \theta(\kappa) = \kappa$. Let $(\Phi X, \ \chi_\phi(M_0, n))_{\overline{W}\underline{\pi}}$ denote the set of all liftings of the map $\theta(\kappa)$ with respect to $p\colon \chi_\phi(M_0, n) \to \overline{W}\underline{\pi}$. Clearly, $(\Phi X, \ \chi_\phi(M_0, n))_{\overline{W}\underline{\pi}}$ has an abelian group structure induced fibrewise from that of the cochain group. Note that we have a map

$$C(M_0, n) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi} \xrightarrow{\delta^n \times_{\kappa(\underline{\pi})} id} C(M_0, n+1) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi},$$

and the following commutative diagram.

$$\begin{tikzcd}
 & C(M_0, n) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi} \arrow[r, "\delta \times_{\kappa(\underline{\pi})} id"] \arrow[d, "p"] & C(M_0, n+1) \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi} \arrow[dl, "p"] \\
\Phi X \arrow[ur, "f"'] \arrow[r, "\theta(\kappa)"'] & \overline{W}\underline{\pi} &
\end{tikzcd}$$

Therefore, if $f \in (\Phi X, \ \chi_\phi(M_0, n))_{\overline{W}\underline{\pi}}$, then $(\delta^n \times_{\kappa(\underline{\pi})} id) \circ f \in (\Phi X, \ \chi_\phi(M_0, n+1))_{\overline{W}\underline{\pi}}$. We write

$$f(G/H)(x) = (c, \mathfrak{s}), \text{ where } x \in X_q^H,\ c \in C^n(\Delta[q]; M_0(G/H)) \text{ and } \mathfrak{s} = \theta(\kappa)(G/H)_q(x).$$

Then $(\delta^n \times_{\kappa(\underline{\pi})} id) f(G/H)(x) = (\delta^n c, \mathfrak{s})$. But $\delta^{n+1} \circ \delta^n = 0$. Thus

$$\{(\Phi X, \ \chi_\phi(M_0, *))_{\overline{W}\underline{\pi}}, \delta \times_{\kappa(\underline{\pi})} id\} = \{(\Phi X, \ \chi_\phi(M_0, n))_{\overline{W}\underline{\pi}}, \delta^n \times_{\kappa(\underline{\pi})} id\}_{n \geq 0}$$

is a cochain complex.

Theorem 4.2.3. *Let X be a G-simplicial set, $\underline{\pi}$ be an O_G-group, (M_0, ϕ) be a $\underline{\pi}$-module* *and* $\kappa\colon \Phi X \to \underline{\pi}$ *be an* O_G*-twisting function. Then there is a cochain isomorphism*

$$\Psi^*\colon \{(\Phi X, \quad \chi_\phi(M_0,*))_{\overline{W}\underline{\pi}},\ (\delta \times_{\kappa(\underline{\pi})} id)\} \cong \{C^*_G(X;\kappa,\phi),\delta_\kappa\}.$$

Proof. Suppose $f \in (\Phi X, \quad \chi_\phi(M_0,n))_{\overline{W}\underline{\pi}}$. Then $\Psi^n f\colon \underline{C}_n X \to M_0$ is a natural transformation defined as follows. Let G/H be any object in O_G and $x \in X^H_n$. Suppose

$$f(G/H)(x) = (c,\mathfrak{s}),\ c \in C^n(\Delta[n]; M_0(G/H)),\ \mathfrak{s} \in \overline{W}\underline{\pi}(G/H)_n.$$

Then $\Psi^n f(G/H)(x) = c(\Delta_n)$. The naturality of $\Psi^n f$ follows from that of f. The assignment $f \mapsto \Psi^n f$ defines the homomorphism Ψ^n.

To check that Ψ^* is a cochain map, we compute $\Psi^{n+1}(\delta^n \times_{\kappa(\underline{\pi})} id)f$. As before, for $x \in X^H_{n+1}$, if $f(G/H)(x) = (c,\mathfrak{s})$, $c \in C^n(\Delta[n+1]; M_0(G/H))$, $\mathfrak{s} = \theta(\kappa)(G/H)_{n+1}(x) \in \overline{W}\underline{\pi}(G/H)_{n+1}$, then $(\delta^n \times_{\kappa(\underline{\pi})} id)f(G/H)(x) = (\delta^n c,\mathfrak{s})$. Therefore,

$$\begin{aligned}
&\Psi^{n+1}(\delta \times_{\kappa(\underline{\pi})} id)f(G/H)(x)\\
=&\delta c(\Delta_{n+1})\\
=&(-1)^{n+1}\sum_{i=0}^{n+1}(-1)^i c(\partial_i\Delta_{n+1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\delta_\kappa(\Psi^n f)(G/H)(x)\\
=&(-1)^{n+1}[\kappa(G/H)_{n+1}(x)^{-1}\Psi^n f(G/H)(\partial_0 x) + \sum_{i=1}^{n+1}(-1)^i\Psi^n f(G/H)(\partial_i x)].
\end{aligned}$$

Since $f(G/H)$ is simplicial, we have

$$f(G/H)(\partial_0 x) = \partial_0 f(G/H)(x) = (\kappa(\underline{\pi})(G/H)(\mathfrak{s})\partial_0 c, \partial_0\mathfrak{s}),$$

by the definition of the face map ∂_0 in $\chi_\phi(M_0,n)(G/H)$. Therefore,

$$\Psi^n f(G/H)(\partial_0 x) = \kappa(\underline{\pi})(G/H)(\mathfrak{s})\partial_0 c(\Delta_n).$$

Now observe that the following diagram is commutative.

$$\begin{tikzcd} & & \chi_\phi(M_0,n) \arrow[d, "p"] \\ \Phi(G/H \times \Delta[q]) \arrow[r, "\Phi(\sigma)"] & \Phi X \arrow[ur, "f"] \arrow[r, "\theta(\kappa)"] \arrow[dr, "\kappa"'] & \overline{W}\underline{\pi} \arrow[d, "\kappa(\underline{\pi})"] \\ & & \underline{\pi} \end{tikzcd}$$

Recall that $\mathfrak{s} = \theta(\kappa)(G/H)_{n+1}(x)$ and, as a consequence,

$$\kappa(\underline{\pi})(G/H)(\mathfrak{s}) = \kappa(G/H)_{n+1}(x).$$

Thus

$$\kappa(G/H)_{n+1}(x)^{-1}\Psi^n f(G/H)(\partial_0 x) = \partial_0 c(\Delta_n) = c(\delta_0(\Delta_n)) = c(\partial_0\Delta_{n+1}).$$

Similarly, for $i > 0$,

$$\Psi^n f(G/H)(\partial_i x) = \partial_i c(\Delta_n) = c(\delta_i(\Delta_n)) = c(\partial_i\Delta_{n+1}).$$

Therefore, we have

$$\delta_\kappa(\Psi^n f) = \Psi^{n+1}(\delta^n \times_{\kappa(\underline{\pi})} id)f.$$

Hence Ψ is a chain map.

Conversely, we define a homomorphism

$$\Gamma^n \colon C^n_G(X;\kappa,\phi) \to (\Phi X, \quad \chi_\phi(M_0,n))_{\overline{W}\underline{\pi}}$$

in the following way. Let

$$f \in C^n_G(X;\kappa,\phi) = Hom_{\mathcal{C}_G}(\underline{C}_n(X), M_0).$$

To define $\Gamma^n f \colon \Phi X \to \chi_\phi(M_0,n)$, note that for any subgroup H of G and $x \in X^H_q$,

$$\Gamma^n(f)(G/H)(x) \in C^n(\Delta[q]; M_0(G/H)) \times (\overline{W}\underline{\pi}(G/H))_q$$

with $\theta(\kappa)(G/H)(x)$ as the second component, as $\Gamma^n(f)$ has to be a lift of $\theta(\kappa)$. To determine the first component of $\Gamma^n(f)(G/H)(x)$, note that the G-simplicial map

$\sigma\colon G/H \times \Delta[q] \to X,\ \ \sigma(eH, \Delta_q) = x$ induces a cochain map

$$\sigma^*\colon C^*_G(X; \kappa, \phi) \to C^*_G(G/H \times \Delta[q]; \kappa\Phi\sigma, \phi).$$

Using the isomorphism E^*_H of Lemma 4.2.2, we define

$$\Gamma^n(f)(G/H)(x) = (E^n_H\sigma^*(f), \theta(\kappa)(G/H)(x)).$$

Suppose $\hat{a}\colon G/H \to G/K,\ a^{-1}Ha \subseteq K$ is any morphism in O_G. Let $y \in X^K_q$ and $x = ay$. Suppose $\tau\colon G/K \times \Delta[q] \to X$ is the G-simplicial map with $\tau(eK, \Delta_q) = y$. Then the G-simplicial map $\sigma = \tau \circ (\hat{a} \times id)$ corresponds to x. Clearly, $(\hat{a} \times id)^* \circ \tau^* = \sigma^*$, where

$$(\hat{a} \times id)^*\colon C^*_G(G/K \times \Delta[q]; \kappa\Phi\overline{y}, \phi) \to C^*_G(G/H \times \Delta[q]; \kappa\Phi\overline{x}, \phi)$$

is induced by $\hat{a} \times id$. This observation along with the naturality of E^*_H imply that $\Gamma^n(f)$ is a natural transformation. It remains to prove that Γ^* is the inverse of Ψ^*.

Let $f \in C^n_G(X; \kappa, \phi)$. Then $\Psi^n\Gamma^n(f) = f$. For, if $H \subseteq G$ is a subgroup, $x \in X^H_n$ and σ be the equivariant n-simplex of type H with $\sigma(eH, \Delta_n) = x$, then

$$\begin{aligned}
&\Psi^n\Gamma^n(f)(G/H)(x)\\
=&E^n_H(\sigma^*f)(\Delta_n)\\
=&\{\kappa\Phi\sigma(G/H)(eH,(0,0))\}^{-1}(\sigma^*f)(eH, \Delta_n)\\
=&\{\kappa\Phi\sigma(G/H)(s_0(eH,(0)))\}^{-1}f(G/H)(x)\\
=&e_H f(G/H)(x) = f(G/H)(x).
\end{aligned}$$

The last two equalities follow from the properties of the twisting function $\kappa\Phi\sigma$. It follows that $\Psi^n\Gamma^n = id$.

Next we prove that $\Gamma^n\Psi^n(f) = f$ for $f\colon \Phi X \to \chi_\phi(M_0, n)$, a lift of $\theta(\kappa)$. Let H be a subgroup of G and $x \in X^H_q$. Let $\sigma\colon G/H \times \Delta[q] \to X$ be the G-simplicial map such that $\sigma(eH, \Delta_q) = x$. Then, by the definition of Γ^*, we have

$$\Gamma^n\Psi^n(f)(G/H)(x) = (E^n_H\sigma^*(\Psi^n f), \theta(\kappa)(G/H)(x)).$$

On the other hand, since $f\colon \Phi X \to \chi_\phi(M_0, n)$ is a lift of $\theta(\kappa)$, $f(G/H)(x) = (c, \mathfrak{s})$, where $\mathfrak{s} = \theta(\kappa)(G/H)(x)$ for some cochain $c \in C^n(\Delta[q]; M_0(G/H))$. We show that $c = E^n_H\sigma^*(\Psi^n f)$. Let $\alpha = (\alpha_0, \cdots, \alpha_n) \in \Delta[q]_n$ be a non-degenerate n-simplex. Then

$$\alpha = \partial_{(i_1, i_2, \cdots, i_{q-n})}\Delta_q,$$

where $0 \leq i_1 < i_2 < \cdots < i_{q-n} \leq q$ and

$$\{\alpha_0, \cdots \alpha_n, i_1, \cdots, i_{q-n}\} = \{0, 1, 2, \cdots, q\}.$$

Then

$$\begin{aligned}
&E^n_H(\sigma^*(\Psi^n f))(\alpha) \\
=&\kappa(G/H)_1\Phi\sigma(G/H)(eH,(0,\alpha_0))^{-1}\sigma^*(\Psi^n f)(G/H)(eH,\alpha) \\
=&\kappa(G/H)_1\Phi\sigma(G/H)(eH,(0,\alpha_0))^{-1}\Psi^n f(G/H)(\sigma(eH,\alpha)) \\
=&\kappa(G/H)_1\Phi\sigma(G/H)(eH,(0,\alpha_0))^{-1}\Psi^n f(G/H)(\partial_{(i_1,i_2,\cdots,i_{q-n})}\sigma(eH,\Delta_q)) \\
=&\kappa(G/H)_1\Phi\sigma(G/H)(eH,(0,\alpha_0))^{-1}(\Psi^n f(G/H)(\partial_{(i_1,i_2,\cdots,i_{q-n})}x).
\end{aligned}$$

Suppose $\alpha_0 = 0$. Then the properties of a twisting function imply that

$$\kappa\Phi\sigma(G/H)(eH,(0,\alpha_0)) = e_H.$$

Moreover, as $f(G/H)$ is simplicial, we have

$$\begin{aligned}
&f(G/H)\partial_{(i_1,\cdots,i_{q-n})}(x) \\
=&\partial_{(i_1,\cdots,i_{q-n})}f(G/H)(x) \\
=&\partial_{(i_1\cdots,i_{q-n})}(c,\mathfrak{s}) \\
=&(\partial_{(i_1\cdots,i_{q-n})}c,\partial_{(i_1,\cdots,i_{q-n})}\mathfrak{s}).
\end{aligned}$$

Note that since $\alpha_0 = 0$, i_1 is greater than zero. Therefore, by the definition of Ψ^*,

$$\begin{aligned}
&E^n_H(\sigma^*(\Psi^n f))(\alpha) \\
=&\Psi^n f(G/H)(\partial_{(i_1,i_2,\cdots,i_{q-n})}x) \\
=&\partial_{(i_1,i_2,\cdots,i_{q-n})}c(\Delta_n) \\
=&c(\delta_{(i_1,i_2,\cdots,i_{q-n})}\Delta_n) \\
=&c(\alpha).
\end{aligned}$$

On the other hand, if $\alpha_0 \neq 0$ then we must have $i_0 = 0$ and therefore,

$$\begin{aligned}
&f(G/H)(\partial_{(i_1,\cdots,i_{q-n})}x) \\
=&\partial_{(i_1,\cdots,i_{q-n})}f(G/H)(x) \\
=&\partial_{(0,i_2,\cdots,i_{q-n})}(c,\mathfrak{s}) \\
=&\partial_0(\partial_{(i_2,\cdots,i_{q-n})}c,\partial_{(i_2,\cdots,i_{q-n})}\mathfrak{s}) \\
=&(\kappa(\underline{\pi})(G/H)(\partial_{(i_2,\cdots,i_{q-n})}\mathfrak{s})\partial_{(0,i_2,\cdots,i_{q-n})}c,\partial_{(0,i_2,\cdots,i_{q-n})}\mathfrak{s}),
\end{aligned}$$

by the definition of the face map ∂_0 in a twisted cartesian product (cf. Definition 1.6.3). Thus, using the definition of Ψ^*, we get

$$\Psi^n(f)(G/H)(\partial_{(i_1,i_2,\cdots,i_{q-n})}x) = \kappa(\underline{\pi})(G/H)(\partial_{(i_2,\cdots,i_{q-n})}\mathfrak{s})\partial_{(0,i_2,\cdots,i_{q-n})}c(\Delta_n).$$

Now observe that

$$\begin{aligned}
&\kappa(\underline{\pi})(G/H)(\partial_{(i_2,\cdots,i_{q-n})}\mathfrak{s})\\
=&\kappa(\underline{\pi})(G/H)(\partial_{(i_2,\cdots,i_{q-n})}\theta(\kappa)(G/H)(\Phi\sigma)(G/H)(eH,\Delta_q))\\
=&\kappa(\underline{\pi})(G/H)\theta(\kappa)(G/H)\Phi\sigma(G/H)(eH,\partial_{(i_2,\cdots,i_{q-n})}\Delta_q)\\
=&\kappa(G/H)_{n+1}\Phi\sigma(G/H)(eH,(0,\alpha_0,\cdots,\alpha_n))\\
=&\kappa(G/H)_1\Phi\sigma(G/H)(eH,(0,\alpha_0)).
\end{aligned}$$

The last equality holds because $\Phi\sigma(G/H)$ is a simplicial map,

$$(0,\alpha_0) = \partial_{(2,\cdots,n+1)}(0,\alpha_0,\cdots,\alpha_n)$$

and all the face maps of the group complex $\underline{\pi}(G/H)$ are identity maps.

Therefore,

$$E^n_H(\sigma^*(\Psi^n f))(\alpha) = \partial_{(0,i_2,\cdots,i_{q-n})}c(\Delta_n) = c(\alpha).$$

□

Let $(\Phi X, \;\; L_\phi(M_0,n))_{\overline{W}\underline{\pi}}$ be the set of liftings of the map $\theta(\kappa)\colon \Phi X \to \overline{W}\underline{\pi}$ with respect to the map $p\colon L_\phi(M_0,n) \to \overline{W}\underline{\pi}$.

Corollary 4.2.4. *For every n,*

$$\Gamma^n\colon C^n_G(X;\kappa,\phi) \to (\Phi X, \quad \chi_\phi(M_0,n))_{\overline{W}\underline{\pi}}$$

restricted to cocycles induces an isomorphism

$$Z^n_G(X;\kappa,\phi) \cong (\Phi X, \quad L_\phi(M_0,n))_{\overline{W}\underline{\pi}}.$$

Proof. The n-cocycles of the cochain complex $\{(\Phi X, \;\; \chi_\phi(M_0,*))_{\overline{W}\underline{\pi}}, \; (\delta \times_{\kappa(\underline{\pi})} id)\}$ are precisely $(\Phi X, \;\; L_\phi(M_0,n))_{\overline{W}\underline{\pi}}$. Hence the corollary follows from the fact that a cochain isomorphism restricts to an isomorphism of cocycles. □

4.3 Classification

We are now ready to prove the desired classification theorem for the simplicial Bredon-Illman cohomology with local coefficients.

Composing the O_G-twisting function $\kappa(\underline{\pi})\colon \overline{W}\underline{\pi} \to \underline{\pi}$ (cf. Example 2.4.2) with the canonical projection $p\colon L_\phi(M_0, n) \to \overline{W}\underline{\pi}$, we have the O_G-twisting function

$$\kappa(\underline{\pi})p\colon L_\phi(M_0, n) \to \underline{\pi}.$$

From Section 2.5, we have the twisted cochain complex $C^*_G(L_\phi(M_0, n); \kappa(\underline{\pi})p, \phi)$. We define a cochain

$$U_0 \in C^n_G(L_\phi(M_0, n); \kappa(\underline{\pi})p, \phi) = Hom_{\mathcal{C}_G}(\underline{C}_n(L_\phi(M_0, n)), M_0)$$

as follows. For an object G/H in O_G,

$$U_0(G/H)\colon \underline{C}_n(L_\phi(M_0, n))(G/H) \to M_0(G/H)$$

is given by $U_0(G/H)(c, \mathfrak{s}) = c(\Delta_n)$, where

$$(c, \mathfrak{s}) \in L_\phi(M_0, n)(G/H)_n = Z^n(\Delta[n]; M_0(G/H)) \times_{\kappa(\underline{\pi})(G/H)} \overline{W}\underline{\pi}(G/H).$$

Then U_0 as defined above satisfies the required naturality condition and hence is a cochain.

Definition 4.3.1. We call the cochain $U_0 \in C^n_G(L_\phi(M_0, n); \kappa(\underline{\pi})p, \phi)$ the fundamental cochain.

We have the following proposition.

Proposition 4.3.2. *The fundamental cochain $U_0 \in C^n_G(L_\phi(M_0, n); \kappa(\underline{\pi})p, \phi)$ is a cocycle.*

Proof. Let $(c, \mathfrak{s}) \in L_\phi(M_0, n)(G/H)_{n+1} = Z^n(\Delta[n+1]; M_0(G/H)) \times \overline{W}\underline{\pi}(G/H)_{n+1}$, where $\mathfrak{s} = [\gamma_1, \cdots, \gamma_{n+1}] \in \overline{W}\underline{\pi}(G/H)_{n+1}$, $\gamma_1 \cdots, \gamma_{n+1} \in \underline{\pi}(G/H)$. Then,

$$\begin{aligned}
&(\delta_{p\kappa(\underline{\pi})}U_0)(G/H)(c, \mathfrak{s}) \\
=&(-1)^{n+1}[(p\kappa(\underline{\pi}))(G/H)(c, \mathfrak{s})^{-1}U_0(G/H)(\partial_0(c, \mathfrak{s})) + \sum_{i=1}^{n+1}(-1)^i U_0(G/H)(\partial_i(c, \mathfrak{s}))] \\
=&(-1)^{n+1}[\gamma_1^{-1}U_0(G/H)(\kappa(\underline{\pi})(G/H)(\mathfrak{s})\partial_0 c, \partial_0\mathfrak{s})) + \sum_{i=1}^{n+1}(-1)^i U_0(G/H)(\partial_i c, \partial_i\mathfrak{s})] \\
=&(-1)^{n+1}[\gamma_1^{-1}\gamma_1\partial_0 c(\Delta_{n+1}) + \sum_{i=1}^{n+1}(-1)^i\partial_i c(\Delta_{n+1})] \\
=&\delta c(\Delta_{n+1}) \\
=&0 \ \ (\because\ c \in Z^n(\Delta[n+1]; M_0(G/H))).
\end{aligned}$$

Thus $U_0 \in C^n_G(L_\phi(M_0, n); \kappa(\underline{\pi})p, \phi)$ is a cocycle. $\square$

Remark 4.3.3. Suppose $f \in (\Phi X,\ \ L_\phi(M_0, n))_{\overline{W}\underline{\pi}}$. Then, for any object G/H in O_G,

$$f(G/H) \colon X^H \to Z^n(\Delta[-]; M_0(G/H)) \times_{\kappa(\underline{\pi})(G/H)} (\overline{W}\underline{\pi}(G/H))$$

induces a cochain map $f(G/H)^*$ from the cochain complex

$$C^*(Z^n(\Delta[-]; M_0(G/H)) \times_{\kappa(\underline{\pi})(G/H)} \overline{W}\underline{\pi}(G/H);\ M_0(G/H))$$

to the cochain complex $C^*(X^H; M_0(G/H))$ and hence

$$f(G/H)^*[U_0(G/H)] \in C^n(X^H; M_0(G/H)) = Hom_{\mathcal{A}b}(C_n(X^H), M_0(G/H)).$$

Therefore, for any $x \in X^H_n$,

$$f(G/H)^* U_0(G/H)(x) = U_0(G/H)(f(G/H)(x)) = U_0(G/H)(c, \mathfrak{s}) = c(\Delta_n).$$

Thus $\Psi^n(f)(G/H)(x) = f(G/H)^* U_0(G/H)(x)$. Hence, for $f \in (\Phi X,\ \ L_\phi(M_0, n))_{\overline{W}\underline{\pi}}$, the map $\Psi^n f$ can be described as the pull-back of the cochain U_0 by f, that is,

$$\Psi^n f = f^*(V_0).$$

Definition 4.3.4. Suppose $f,\ g \in (\Phi X,\ \ L_\phi(M_0, n))_{\overline{W}\underline{\pi}}$. Then f and g are said to be vertically homotopic if there is a homotopy $\mathcal{H} \colon f \simeq g$ of maps of the O_G-simplicial sets (cf. Definition 2.3.8) such that $p \circ \mathcal{H} = \theta(\kappa) \circ pr_1$, where $pr_1 \colon \Phi X \times \Delta[1] \to \Phi X$ is the projection onto the first factor.

Proposition 4.3.5. *Under the isomorphism*

$$Z^n_G(X; \kappa, \phi) \xrightarrow{\Gamma^n} (\Phi X,\ L_\phi(M_0, n))_{\overline{W}\underline{\pi}},$$

$f_0, f_1 \in Z^n_\phi(X; \kappa)$ *are cohomologous if and only if* $\Gamma^n f_0,\ \Gamma^n f_1$ *are vertically homotopic.*

Proof. Suppose $f_0, f_1 \in Z^n_G(X; \kappa, \phi)$ are cohomologous. Then

$$f_0 = f_1 + \delta_\kappa h$$

for some $h \in C^{n-1}_G(X; \kappa, \phi)$. Let κ_1 denote the O_G-twisting function obtained by composing κ with the projection $\Phi X \times \Delta[1] \to \Phi X$. To show that $\Gamma f_0, \Gamma f_1$ are vertically homotopic, it suffices to find $\mathcal{Y} \in Z^n_G(X \times \Delta[1]; \kappa_1, \phi)$ such that $i_0^*(\mathcal{Y}) = f_0$ and $i_1^*(\mathcal{Y}) = f_1$, where $i_0, i_1 \colon X \to X \times \Delta[1]$ are two obvious inclusions. This is because, in that case, the image of $\mathcal{Y}$ under the isomorphism

$$\Gamma\colon Z^n_G(X \times \Delta[1]; \kappa_1, \phi) \to (\Phi(X \times \Delta[1]), \quad L_\phi(M_0, n))_{\overline{W\underline{\pi}}}$$

will serve as a vertical homotopy between Γf_0 and Γf_1.

Let $\mathcal{Y}_0 = pr_1^* f_0 \in Z^n_G(X \times \Delta[1]; \kappa_1, \phi)$, where

$$pr_1^*\colon C^*_G(X; \kappa, \phi) \to C^*_G(X \times \Delta[1]; \kappa_1, \phi)$$

is the cochain map induced by the projection $X \times \Delta[1] \to X$. Clearly,

$$i_0^*(\mathcal{Y}_0) = i_1^*(\mathcal{Y}_0) = f_0,$$

where $i_0^*, i_1^*\colon C^*_G(X \times \Delta[1]; \kappa_1, \phi) \to C^*_G(X; \kappa, \phi)$ are the maps induced by i_0 and i_1 respectively. Regard $h \in C^{n-1}_G(X; \kappa, \phi)$ as a cochain defined on $i_1(X)$ and we may extend it to a cochain

$$\beta \in C^{n-1}_G(X \times \Delta[1]; \kappa_1, \phi)$$

as follows. Define $\beta(G/H)\colon X^H_{n-1} \times \Delta[1]_{n-1} \to M_0(G/H)$ by sending $(x, (1, \cdots, 1))$ to $h(G/H)(x)$, and to 0, otherwise. Clearly $\beta \in C^{n-1}_G(X \times \Delta[1]; \kappa_1, \phi)$ satisfies

$$i_0^*(\beta) = 0, \ i_1^*(\beta) = h.$$

Set $\mathcal{Y} = \mathcal{Y}_0 - \delta\beta$. Observe that

$$i_0^*(\mathcal{Y}) = i_0^*(\mathcal{Y}_0 - (\delta_{\kappa_1}\beta)) = f_0 - \delta_\kappa(i_0^*\beta) = f_0,$$

and similarly,

$$i_1^*(\mathcal{Y}) = f_0 - \delta_\kappa(i_1^*\beta) = f_0 - \delta_\kappa h = f_1.$$

Conversely, suppose $\Gamma^n f_0$ and $\Gamma^n f_1$ are vertically homotopic. Then they are homotopic in the sense of Definition 2.3.8 and so $\Gamma^n f_0(G/H)$ and $\Gamma^n f_1(G/H)$ are simplicially homotopic for any subgroup H of G. As a consequence,

$$\Gamma^n f_0(G/H)^* = \Gamma^n f_1(G/H)^*.$$

Therefore, by Remark 4.3.3, $f_0 = f_1$. □

Recall [DK83] that the category $O_G\mathcal{S}$ of O_G-simplicial sets is a closed model category in the sense of Quillen [Qui67]. Moreover, recall that if C is an object of a closed model category $\mathcal{C}$, then the slice category $\mathcal{C} \downarrow C$, the category of objects over C, has a closed model structure induced from that of $\mathcal{C}$ (cf. Example 1.8.6). In particular, the category

$O_G\mathcal{S} \downarrow \overline{W}\underline{\pi}$ of objects over $\overline{W}\underline{\pi} \in O_G\mathcal{S}$ is a closed model category. Consequently, the vertical homotopy of liftings of $\theta(\kappa)$ to $L_\phi(M_0, n)$, viewed as abstract homotopy of morphisms of $O_G\mathcal{S} \downarrow \overline{W}\underline{\pi}$, is an equivalence relation.

From Corollary 4.2.4 and Proposition 4.3.5, we obtain the following result.

Theorem 4.3.6. *Suppose X is a G-simplicial set, $\underline{\pi}$ is an O_G-group and $\kappa\colon \Phi X \to \underline{\pi}$ is an O_G-twisting function. Let M_0 be an abelian O_G-group and $\underline{\pi}$ operates on M_0. Suppose that this action is given by $\phi\colon \underline{\pi} \times M_0 \to M_0$. Then*

$$H^n_G(X; \kappa, \phi) \cong [\Phi X,\ L_\phi(M_0, n)]_{\overline{W}\underline{\pi}},$$

where the expression on the right hand side denotes the vertical homotopy classes of liftings of the map $\theta(\kappa)$.

Remark 4.3.7. Let $T \in O_G\mathcal{S}$, $\underline{\pi}$ be an O_G-group. Given a $\underline{\pi}$-module (M_0, ϕ) and an O_G-twisting function $\kappa\colon T \to \underline{\pi}$, let $H^n_G(T; \kappa, \phi)$ be the equivariant twisted cohomology of T (cf. Definition 2.5.2). As before, we have a natural map $\theta(\kappa)\colon T \to \overline{W}\underline{\pi}$,

$$\theta(\kappa)(G/H)_q\colon T(G/H)_q \longrightarrow \overline{W}\underline{\pi}(G/H)_q, \qquad H \subseteq G \text{ a subgroup},$$

$$x \mapsto [\kappa(G/H)_q(x), \kappa(G/H)_{q-1}(\partial_0 x), \cdots, \kappa(G/H)_1(\partial_0^{q-1} x)].$$

Observe that the proof of Theorem 4.3.6 can be carried through by replacing ΦX by T. Therefore, we have

$$H^n_G(T; \kappa, \phi) \cong [T,\ L_\phi(M_0, n)]_{\overline{W}\underline{\pi}}.$$

Suppose X is a G-connected G-simplicial set with a G-fixed 0-simplex v and assume that M is a given equivariant local coefficient system on X. Let M_0 be the associated abelian O_G-group equipped with an action ϕ of the O_G-group $\underline{\pi} = \underline{\pi}X$. Let κ be the O_G-twisting function as given in Example 2.4.3. Then, from Theorem 4.3.6 and Theorem 4.2.3, we obtain the following result.

Theorem 4.3.8. *Under the above hypothesis, we have*

$$H^n_G(X; M) \cong [\Phi X,\ \ L_\phi(M_0, n)]_{\overline{W}\underline{\pi}} \ \ \textit{for all} \ \ n.$$

Chapter 5

The Eilenberg theorem

5.1 Introduction

A well-known result of Eilenberg describes the cohomology of a space with local coefficients by the cohomology of an invariant subcomplex of its universal cover equipped with the action of the fundamental group of the space [Spa81], [Whi78]. A simplicial version of the Eilenberg theorem is given in [Git63]. An equivariant version of the Eilenberg theorem for the Bredon-Illman cohomology with local coefficients of a G-space was proved in [MM96]. The aim of this chapter is to derive Eilenberg's theorem for the simplicial Bredon-Illman cohomology with local coefficients, as introduced in Chapter 3. This is based on the notion of the universal covering complexes of one vertex Kan complexes [Gug60]. In the equivariant context, the role of the universal cover is played by a contravariant functor from the category of canonical orbits to the category of one vertex Kan complexes. We begin by introducing a notion of an equivariant cohomology of an O_G-simplicial set, equipped with an action of an O_G-group. This will be used to deduce the main result of this chapter.

5.2 Equivariant cohomology of O_G-simplicial sets

Let Λ be a commutative ring with unity.

Definition 5.2.1. An O_G-group ρ is said to act on an O_G-chain complex $\{\underline{C}_n\}_{n\geq 0}$ of Λ-modules if ρ acts on $\underline{C}_n$, for each $n \geq 0$, such that the differentials $\partial_n \colon \underline{C}_n \to \underline{C}_{n-1}$ satisfy

$$\partial_n(G/H)(\gamma x) = \gamma\partial_n(G/H)(x),\ x \in \underline{C}_n(G/H),\ \gamma \in \rho(G/H),\ n \geq 1,$$

for each subgroup H of G.

Let $\mathcal{A}_\Lambda$ denote the category with objects the triples (T, M_0, ρ), where T is an O_G-simplicial set, M_0 an O_G-Λ-algebra and ρ is an O_G-group which operates on both T and M_0. A morphism from (T, M_0, ρ) to (T', M_0', ρ') is a triple (f_0, f_1, f_2), where $f_0 \colon T \to T'$, $f_1 \colon M_0' \to M_0$ and $f_2 \colon \rho \to \rho'$ are maps in the appropriate categories such that

$$f_0(G/H)(\alpha x) = f_2(G/H)(\alpha) f_0(G/H)(x),$$

$$f_1(G/H)[f_2(G/H)(\alpha) m_0'] = \alpha f_1(G/H)(m_0'),$$

for each subgroup H of G, $x \in T(G/H), \alpha \in \rho(G/H), m_0' \in M_0'(G/H)$.

The ρ-equivariant cohomology of T with coefficients M_0 is defined as follows. We have an O_G-chain complex $\{\underline{C}_*(T), \partial_*\}$, defined by

$$\underline{C}_n(T) \colon O_G \to \Lambda\text{-mod}, \quad G/H \mapsto C_n(T(G/H); \Lambda),$$

where $C_n(T(G/H); \Lambda)$ is the free Λ-module generated by the non-degenerate n-simplices of $T(G/H)$. For any morphism $\hat{a} \colon G/H \to G/K$ in O_G,

$$\underline{C}_n(T)(\hat{a}) = a_\# \colon C_n(T(G/K); \Lambda) \to C_n(T(G/H); \Lambda)$$

is induced by the simplicial map $T(\hat{a})$ from $T(G/K)$ to $T(G/H)$. The boundary $\partial_n \colon \underline{C}_n(T) \to \underline{C}_{n-1}(T)$ is the natural transformation

$$\partial_n(G/H) \colon C_n(T(G/H); \Lambda) \to C_{n-1}(T(G/H); \Lambda), \quad H \subseteq G,$$

where $\partial_n(G/H)$ is the ordinary boundary map of the simplicial set $T(G/H)$. The action of ρ on T induces an action of ρ on the O_G-chain complex $\{\underline{C}_*(T), \partial_*\}$. We form the cochain complex

$$\{C^*_\rho(T; M_0) = Hom_\rho(\underline{C}_*(T), M_0), \delta^*\}$$

where $Hom_\rho(\underline{C}_n(T), M_0)$ consists of all natural transformations $\underline{C}_n(T) \xrightarrow{f} M_0$ respecting the action of ρ and the coboundary is given by $\delta^n f = (-1)^{n+1} f \circ \partial_{n+1}$.

Definition 5.2.2. Given an object $(T, M_0, \rho) \in \mathcal{A}_\Lambda$, the n-th ρ-equivariant cohomology of T with coefficients M_0 is defined as

$$H^n_\rho(X; M_0) := H_n(C^*_\rho(T; M_0)).$$

Remark 5.2.3. Observe that a morphism $(f_0, f_1, f_2) \colon (T, M_0, \rho) \to (T', M_0', \rho')$ induces a cochain map $C^*(f_0, f_1, f_2) \colon C^*_{\rho'}(T'; M_0') \to C^*_\rho(T; M_0)$ as follows. For $f \in C^*_{\rho'}(T'; M_0')$, define a natural transformation $C^*(f_0, f_1, f_2)(f) \colon \underline{C}_*(T) \to M_0$, by

$$C^*(f_0, f_1, f_2)(f)(G/H)(x) = f_1(G/H) \circ f(G/H) \circ f_0(G/H)(x),$$

for each subgroup H of G and $x \in T(G/H)$. Then $C^*(f_0, f_1, f_2)(f) \in C^*_\rho(T; M_0)$ and $C^*(f_0, f_1, f_2)$ is indeed a cochain map.

The cochain complex $C^*_\rho(T; M_0)$ is equipped with a cup product defined as follows. We have a natural transformation

$$\underline{\xi} \colon \underline{C}_*(T \times T) \to \underline{C}_*(T) \otimes \underline{C}_*(T),$$

where

$$\underline{\xi}(G/H) \colon C_*(T(G/H) \times T(G/H)) \to C_*(T(G/H)) \otimes C_*(T(G/H))$$

is the Alexander-Whitney map for the simplicial set $T(G/H)$, $H \subseteq G$ being a subgroup. The map $\underline{\xi}(G/H)$ is given by [May67],

$$\underline{\xi}(G/H)(x, y) = \sum_{i+j=n} \partial_{(i+1,\cdots,n)} x \otimes \partial_{(0,1,\cdots,n-j)} y, \quad x, y \in T(G/H)_n.$$

We have a ρ-action on $\underline{C}_*(T)$ induced by the ρ-action on T and hence, the diagonal actions of ρ on $T \times T$ and $\underline{C}_*(T) \otimes \underline{C}_*(T)$. Since the Alexander-Whitney map of simplicial sets is a natural map, $\underline{\xi}$ is equivariant with the induced actions of ρ on $\underline{C}_*(T \times T)$ and $\underline{C}_*(T) \otimes \underline{C}_*(T)$. Then the cup product is defined as the composition

$$C^*_\rho(T; M_0) \otimes C^*_\rho(T; M_0) \xrightarrow{\alpha} Hom_\rho(\underline{C}_*(T) \otimes \underline{C}_*(T), M_0) \xrightarrow{\underline{\xi}^*} C^*_\rho(T \times T; M_0) \xrightarrow{D^*} C^*_\rho(T; M_0).$$

Here $\alpha \colon C^*_\rho(T; M_0) \otimes C^*_\rho(T; M_0) \to Hom_\rho(\underline{C}_*(T \times T), M_0)$ is defined by

$$\alpha(f \otimes g)(G/H)(x \otimes y) = (-1)^{deg(x) deg(y)} f(G/H)(x) g(G/H)(y),$$

where $f, g \in C^*_\rho(T; M_0)$; $x, y \in \underline{C}_*(T)(G/H)$ and $D \colon T \to T \times T$ is the diagonal map.

Remark 5.2.4. The cochain complex $C^*_\rho(T; M_0)$, equipped with the above cup product, is an associative differential graded Λ-algebra and the induced product in the cohomology is associative and graded commutative.

5.3 The equivariant Eilenberg theorem

Let X be a one vertex Kan complex. For any $x \in X_1$, we denote by $[x]$ the element of $\pi = \pi_1(X, v)$ represented by the 1-simplex x, where v is the unique vertex of X. Recall that ([Git63], [Gug60]) the universal covering complex $\widetilde{X}$ of X is defined as follows.

$$\widetilde{X}_n = \pi \times X_n,\ n \geq 0,$$

with the face maps

$$\partial_i(\gamma, x) = (\gamma, \partial_i x),\quad 0 < i \leq n,\quad x \in X_n,\quad \gamma \in \pi,$$

$$\text{and}\quad \partial_0(\gamma, x) = ([\partial_{(2,3,\cdots,n)}x]\gamma, \partial_0 x),$$

where $\partial_{(2,3,\cdots,n)}x = \partial_2\partial_3\cdots\partial_n x$. The degeneracy maps are

$$s_i(\gamma, x) = (\gamma, s_i x)\quad 0 \leq i \leq n.$$

Then the projection onto the second factor $p\colon \widetilde{X} \to X$ has the usual properties of a universal cover. Any map $f\colon X \to Y$ of such complexes induces a map $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ by $\tilde{f}(\gamma, x) = (f_*(\gamma), f(x))$, where $f_*\colon \pi_1(X) \to \pi_1(Y)$ is the homomorphism of the fundamental groups induced by f.

Remark 5.3.1. We note that given any two 0-simplices $x_1 = (\gamma_1, v)$ and $x_2 = (\gamma_2, v)$ in $\widetilde{X}$, there is a unique homotopy class of 1-simplices ω such that $\partial_1\omega = x_1,\ \ \partial_0\omega = x_2$, as $\widetilde{X}$ is simply connected. We may represent this class by $\omega = (\gamma_1, \omega_2\omega_1^{-1})$, where ω_i represents $\gamma_i,\ i = 1,\ 2$.

For a one vertex Kan complex X, the fundamental group $\pi_1(X, v)$ operates on $\tilde{X}$ freely by

$$(\Upsilon, (\gamma, x)) \mapsto (\gamma\Upsilon^{-1}, x),\quad \Upsilon \in \pi_1(X, v).$$

This action is natural with respect to maps of complexes. The following simplicial version of the Eilenberg theorem holds.

Theorem 5.3.2. *[Git63] Let X be a one vertex Kan complex with fundamental group π and let $p\colon \tilde{X} \to X$ be the universal covering complex. Let $\mathcal{L}$ denote the local coefficient system on X determined by a π-module (A, ϕ). Then the projection p induces a natural isomorphism*

$$p^*\colon H^*(X; \mathcal{L}) \to H^*_e(\tilde{X}; A).$$

To prove an equivariant version of the above result, we define a contravariant functor from the category of canonical orbits to the category of one vertex Kan complexes as follows.

Let X be a one vertex G-Kan complex. We denote the G-fixed vertex by v. Let M be an equivariant local coefficient system of Λ-algebras on X. Let M_0 be the associated

O_G-Λ-algebra, equipped with an action of the O_G-group $\underline{\pi}X$, as described in Proposition 3.3.3. For any subgroup H of G, let

$$p_H \colon \widetilde{X^H} \to X^H$$

be the universal cover of X^H. The left translation $a \colon X^K \to X^H$, corresponding to a G-map $\hat{a} \colon G/H \to G/K$, $a^{-1}Ha \subseteq K$, induces a simplicial map

$$\tilde{a} \colon \widetilde{X^K} \to \widetilde{X^H},$$

such that $p_H \circ \tilde{a} = a \circ p_K$.

Definition 5.3.3. Let X be a one vertex G-Kan complex. Then define an O_G-Kan complex $\widetilde{X}$ by,

$$\widetilde{X}(G/H) = \widetilde{X^H},$$

for each subgroup H of G, and $\widetilde{X}(\hat{a}) = \tilde{a}$ for a morphism $\hat{a}$ in O_G. We call $\tilde{X}$ the universal O_G-covering complex of X.

The notion of the universal O_G-covering complex is the simplicial analogue of the O_G-covering space as introduced in [MM96]. A more general version, called 'universal covering functor', was introduced by W. Luck [Lüc89].

For any subgroup H, let $\tilde{v}^H \in \widetilde{X^H}$ denote the zero simplex (e_H, v), where e_H is the identity element of $\underline{\pi}X(G/H) = \pi_1(X^H, v)$. Note that the map $\tilde{a}$, induced by $a \colon X^K \to X^H$, maps $\tilde{v}^K$ to $\tilde{v}^H$. The natural actions of $\underline{\pi}X(G/H) = \pi_1(X^H, v)$ on $\widetilde{X}(G/H) = \widetilde{X^H}$ as H varies over subgroups of G, define an action of the O_G-group $\underline{\pi}X$ on $\widetilde{X}$. Thus $(\widetilde{X}, M_0, \underline{\pi}X)$ is an object of $\mathcal{A}_\Lambda$.

Theorem 5.3.4. *Let X be a one vertex G-Kan complex and M be an equivariant local coefficient system of Λ-algebras on X. Then, with the notations as above, there exists an isomorphism of graded algebras*

$$H^*_G(X; M) \cong H^*_{\underline{\pi}X}(\widetilde{X}; M_0),$$

where the right-hand side denotes the $\underline{\pi}X$-equivariant cohomology of $\tilde{X}$ as introduced in the last section.

Proof. Recall that for any two 0-simplices $x,\ y \in \widetilde{X^H}$ of the universal cover of the H-fixed point complex X^H, there is a unique homotopy class of 1-simplices ω with $\partial_1\omega = x$ and $\partial_0\omega = y$. Let us denote this class by $\widetilde{\xi}_H(x, y)$. In particular, if $x = \tilde{v}^H$, then we shall write $\widetilde{\xi}(\tilde{v}^H, y)$ simply by $\widetilde{\xi}_H(y)$. Upon projecting $\widetilde{\xi}_H(x, y)$ via p_H we get an element $\hat{\xi}_H(x, y) \in \pi_1(X^H, v)$. By Remark 3.2.4, $\hat{\xi}_H(x, y)$ corresponds to an automorphism $b\hat{\xi}_H(x,y)$ of v_H in ΠX. To simplify notation, we will denote $\hat{\xi}_H(x,y)$ by $\hat{\xi}_H(y)$ when $x = \tilde{v}^H$.

Define a map

$$\mu\colon S^n_G(X;M) \to Hom_{\underline{\pi}X}(\underline{C}_n(\widetilde{X}), M_0)$$

as follows. Let $f \in S^n_G(X;M)$ and y be a non-degenerate n-simplex in $\widetilde{X^H}$. Let σ be the equivariant n-simplex of type H in X such that $\sigma' = p_H \circ \overline{y}$, where $\overline{y}\colon \Delta[n] \to \widetilde{X^H}$ is the simplicial map with $\overline{y}(\Delta_n) = y$. Then $\mu(f) \in Hom_{\underline{\pi}X}(\underline{C}_n(\widetilde{X}), M_0)$ is given by

$$\mu(f)(G/H)(y) = M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}y))f(\sigma), \quad \text{where } H \subseteq G \text{ a subgroup.}$$

Recall that $f(\sigma) \in M(\sigma_H)$ and σ_H in this case coincides with v_H.

We check that $\mu(f)(G/H)$ is equivariant with respect to the respective actions of $\underline{\pi}X(G/H)$ on $\underline{C}_n(\widetilde{X})(G/H)$ and on $M_0(G/H)$. Let $\gamma \in \underline{\pi}X(G/H)$, $y \in \widetilde{X^H_n}$ and σ be the equivariant n-simplex determined by y as above. Then

$$\mu(f)(G/H)(\gamma y) = M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\gamma y))f(\tau),$$

where $\tau' = p_H \circ \overline{\gamma y}$. By the definition of the action of $\underline{\pi}X(G/H)$ on $C_n(\widetilde{X^H};\mathbb{Z})$, we have $p_H \circ \overline{\gamma y} = p_H \circ \overline{y}$, hence $\tau' = \sigma'$. It follows that

$$\mu(f)(G/H)(\gamma y) = M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\gamma y))f(\sigma).$$

Now write $\widetilde{\xi}_H(\partial_{(1,2,\cdots,n)}\gamma y)$ as a composition

$$\widetilde{\xi}_H(\gamma\tilde{v}^H, \partial_{(1,2,\cdots,n)}\gamma y) \circ \widetilde{\xi}_H(\gamma\tilde{v}^H)$$

of morphisms in the fundamental groupoid of $\widetilde{X^H}$. Observe that by Remark 5.3.1, $\hat{\xi}_H(\gamma\tilde{v}^H) = \gamma^{-1}$ and $\hat{\xi}_H(\gamma\tilde{v}^H, \partial_{(1,2,\cdots,n)}\gamma y) = \hat{\xi}_H(\partial_{(1,2,\cdots,n)}y)$. Therefore,

$$M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\gamma y)) = M(b\gamma)^{-1} \circ M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}y)).$$

Thus

$$\mu(f)(G/H)(\gamma y) = M(b\gamma)^{-1}\mu(f)(G/H)(y).$$

It follows from the definition of the action of $\underline{\pi}X(G/H)$ on $M_0(G/H)$ that $\mu(f)(G/H)$ is equivariant.

To check that $\mu(f)\colon \underline{C}_n(\widetilde{X}) \to M_0$ is a natural transformation, we have to show that

$$M_0(\hat{a}) \circ \mu(f)(G/K) = \mu(f)(G/H) \circ \tilde{a}_\#$$

whenever $a^{-1}Ha \subseteq K$. Recall that (cf. Section 3.3) by definition of M_0, $M_0(\hat{a}) = M(v_H \xrightarrow{[\hat{a},k]} v_K)$, where $k\colon G/H \times \Delta[1] \to X$ is given by $k(eH, \Delta_1) = s_0 v$. Let $y \in \widetilde{X_n^K}$ and $a^{-1}Ha \subseteq K$. Let τ be an equivariant n-simplex of type K in X such that $\tau' = p_K \circ \overline{y}$. Then,

$$\begin{aligned}
&M_0(\hat{a}) \circ \mu(f)(G/K)(y)\\
=&M(v_H \xrightarrow{[\hat{a},k]} v_K) \circ M(b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y))f(\tau)\\
=&M(v_H \xrightarrow{[\hat{a},k]} v_K) \circ M([id_{G/K}, \omega])f(\tau)\\
=&M([id_{G/H}, \omega] \circ [\hat{a}, k])f(\tau),
\end{aligned}$$

where as in Remark 3.2.4, ω is the equivariant 1-simplex of type K in X such that ω' represents $\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y)$. On the other hand,

$$\begin{aligned}
&\mu(f)(G/H) \circ \tilde{a}_{\#}(y)\\
=&\mu(f)(G/H)(\tilde{a}y)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{g}y))f(\sigma),
\end{aligned} \tag{5.3.1}$$

where $\sigma' = p_H \circ \overline{\tilde{a}y} = p_H \circ \tilde{a} \circ \overline{y} = a \circ p_K \circ \overline{y} = a \circ \tau'$. In particular, σ and τ are compatible n-simplices. Thus

$$\mu(f)(G/H) \circ \tilde{a}_{\#}(y) = M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y)) \circ M(a_*)f(\tau).$$

Note that v is the only vertex in X which is G-fixed and hence a_* is a morphism from v_H to v_K, and is given by $[\hat{a}, k]$, where $k = v_H \circ (id_{G/H} \times \sigma_0)$, $\sigma_0\colon \Delta[1] \to \Delta[0]$ being the simplicial map as defined in Section 1.2 . Now observe that $\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y) = \hat{\xi}_H(\tilde{a}\partial_{(1,2,\cdots,n)}y)$ can be represented by $a\omega'$. As a consequence, we may write

$$b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y) = [id_{G/H}, \omega \circ (\hat{a} \times id_{\Delta[1]})].$$

Therefore,

$$\begin{aligned}
&\mu(f)(G/H) \circ \tilde{a}_{\#}(y)\\
=&M([id_{G/H}, \omega \circ (\hat{a} \times id_{\Delta[1]})]) \circ M([\hat{a}, k])f(\tau)\\
=&M([\hat{a}, k] \circ [id_{G/H}, \omega \circ (\hat{a} \times id_{\Delta[1]})])f(\tau).
\end{aligned}$$

From the definition of composition of morphisms in ΠX, we have

$$[id_{G/K}, \omega] \circ [\hat{a}, k] = [\hat{a}, k] \circ [id_{G/H}, \omega \circ (\hat{a} \times id_{\Delta[1]})].$$

Thus $\mu(f)$ is natural.

Next we check that μ is a cochain map. Let $f \in S_G^n(X;M)$, $y \in \widetilde{X_{n+1}^H}$. Let σ denote the equivariant $(n+1)$-simplex of type H corresponding to y as described before. Then

$$\begin{aligned}
&\mu(\delta f)(G/H)(y)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n+1)}y))(\delta f)(\sigma)\\
=&(-1)^{n+1}M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n+1)}y))\{M(\sigma_*)f(\sigma^{(0)})+\sum_{j=1}^{n+1}(-1)^j f(\sigma^{(j)})\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\delta\mu(f)(G/H)(y)\\
=&(-1)^{n+1}[\sum_{i=0}^{n+1}(-1)^i\mu(f)(G/H)(\partial_i y)]\\
=&(-1)^{n+1}[\sum_{i=0}^{n+1}(-1)^i M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\partial_i y))f(\sigma^{(i)})]\\
=&(-1)^{n+1}[M(b\hat{\xi}_H(\partial_{(0,2,\cdots,n+1)}y))f(\sigma^{(0)})+\sum_{i=1}^{n+1}(-1)^i M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n+1)}y))f(\sigma^{(i)})].
\end{aligned}$$

Note that, since $\widetilde{X^H}$ is simply connected, the morphism $\hat{\xi}_H(\partial_{(0,2,\cdots,n+1)}y)$ in πX^H can be factored as

$$\hat{\xi}_H(\partial_{(1,2,\cdots,n+1)}y, \partial_{(0,2,\cdots,n+1)}y)\circ\hat{\xi}_H(\partial_{(1,2,\cdots,n+1)}y)$$

and $b\hat{\xi}_H(\partial_{(1,2,\cdots,n+1)}y, \partial_{(0,2,\cdots,n+1)}y)$ is precisely the morphism σ_*. Therefore,

$$b\hat{\xi}_H(\partial_{(0,2,\cdots,n+1)}y) = \sigma_*\circ b\hat{\xi}_H(\partial_{(1,2,\cdots,n+1)}y).$$

Hence $\mu(\delta f) = \delta\mu(f)$.

To show that μ is a cochain isomorphism define a map

$$\psi\colon Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0) \to C_G^n(X;M)$$

as follows. Let $f \in Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0)$ and σ be a non-degenerate equivariant n-simplex of type H in X. Choose an n-simplex y in $\widetilde{X^H}$ such that $p_H(y) = \sigma(eH, \Delta_n)$. Then $\psi(f)$ is given by

$$\psi(f)(\sigma) = M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}y))^{-1}f(G/H)(y).$$

Suppose z is another n-simplex in $\widetilde{X^H}$ such that $p_H(z) = \sigma(eH, \Delta_n)$. Since $\pi_1(X^H, v)$

acts transitively on each fibre of $p_H\colon \widetilde{X^H} \to X^H$, there exists an element $\gamma \in \pi_1(X^H, v)$ such that $\gamma y = z$ and hence $\gamma\partial_{(1,2,\cdots,n)}y = \partial_{(1,2,\cdots,n)}z$. Thus

$$\begin{aligned}
&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}z))^{-1}f(G/H)(z)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\gamma y))^{-1}f(G/H)(\gamma y)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}y))^{-1}M(b\gamma)^{-1}f(G/H)(y)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}y))^{-1}f(G/H)(y).
\end{aligned}$$

The last equality follows from the fact that

$$M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\gamma y)) = M(b\gamma)^{-1} \circ M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}y)),$$

which we have observed in the first part of the proof while showing that μ takes any cocycle in $S^n_G(X;M)$ into $Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0)$. Thus the map ψ is well defined.

We claim that $\psi(f) \in S^n_G(X;M)$ for any $f \in Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0)$. Let $a^{-1}Ha \subseteq K$. Let $\sigma\colon G/H \times \Delta[n] \to X$ and $\tau\colon G/K \times \Delta[n] \to X$ be equivariant n-simplices such that $\tau \circ (\hat{a} \times id) = \sigma$, so that they are compatible. We need to show that

$$\psi(f)(\sigma) = M(a_*)\psi(f)(\tau).$$

Let $y \in \widetilde{X^K}$ be such that $p_K(y) = \tau(eK, \Delta_n)$. Then the n-simplex $\tilde{a}y \in \widetilde{X^H_n}$ is such that

$$p_H(\tilde{a}y) = ap_K(y) = a\tau(eK, \Delta_n) = \tau(aK, \Delta_n) = \sigma(eH, \Delta_n).$$

By our choice, we have

$$\psi(f)(\tau) = M(b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y))^{-1}f(G/K)(y)$$

and

$$\psi(f)(\sigma) = M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y))^{-1}f(G/H)(\tilde{a}y).$$

Since $f\colon \underline{C}_n(\tilde{X}) \to M_0$ is natural, we have

$$f(G/H)(\tilde{a}y) = M_0(\hat{a})f(G/K)(y).$$

In the first part of the proof we have observed that

$$a_* \circ b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y) = b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y) \circ a_*.$$

Moreover, recall that $M_0(\hat{a}) = M(a_*)$. Therefore

$$\begin{aligned}
&M(a_*)\psi(f)(\tau)\\
=&M(a_*)M(b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y))^{-1}f(G/K)(y)\\
=&M(b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y)^{-1}\circ a_*)f(G/K)(y)\\
=&M(a_*\circ b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y))f(G/K)(y)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y))^{-1}M(a_*)f(G/K)(y)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n)}\tilde{a}y))^{-1}M_0(\hat{a})f(G/K)(y)\\
=&\psi(f)(\sigma).
\end{aligned}$$

We now check that ψ is the inverse of μ. For $f \in S^n_G(X;M)$, we have

$$\begin{aligned}
&\psi\mu(f)(\sigma)\\
=&\psi(\mu f)(\sigma)\\
=&M(b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y)^{-1}\mu(f)(G/H)(y)\\
=&M(b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y)^{-1}M(b\hat{\xi}_K(\partial_{(1,2,\cdots,n)}y)f(\sigma)\\
=&f(\sigma).
\end{aligned}\tag{5.3.2}$$

Thus $\psi\mu = id$. Similarly $\mu\psi = id$. Thus μ is indeed a cochain isomorphism.

To complete the proof, we need to check that $\mu(f\cup g) = \mu(f)\cup\mu(g)$ for $f \in S^n_G(X;M)$ and $g \in S^m_G(X;M)$. Let $y \in \widetilde{X^H_{m+n}}$ and σ be the equivariant n-simplex of type H in X such that $\sigma' = p_H \circ \overline{y}$. Then we have,

$$\begin{aligned}
&\mu(f\cup g)(G/H)(y)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots m+n)}y))(f\cup g)(\sigma)\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots m+n)}y))(-1)^{mn}[f(\sigma\rfloor_n)\{M(\sigma_{\widehat{n+1}})g(\lfloor_m\sigma)\}] \text{ (cf. Definition 3.4.8)}\\
=&(-1)^{mn}\{M(b\hat{\xi}_H(\partial_{(1,2,\cdots m+n)}y))f(\sigma\rfloor_n)\}\{M(b\hat{\xi}_H(\partial_{(1,2,\cdots m+n)}y))M(\sigma_{\widehat{n+1}})g(\lfloor_m\sigma)\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mu f\cup\mu g)(y)\\
=&(-1)^{mn}(\mu f)(\partial_{(n+1,\cdots,n+m)}y)\{(\mu g)(\partial_{(0,1,\cdots,n)}y)\}\\
=&(-1)^{mn}\{M(b\hat{\xi}_H(\partial_{(1,\cdots,n)}\partial_{(n+1,\cdots,n+m)}y))f(\sigma\rfloor_n)\}\{M(b\hat{\xi}_H(\partial_{(1,\cdots,m)}\partial_{(0,\cdots,n)}y))g(\lfloor_m\sigma)\}.
\end{aligned}$$

Note that,

$$\partial_{(1,2,\cdots m+n)}y = \partial_{(1,\cdots,n)}\partial_{(n+1,\cdots,n+m)}y; \quad \partial_{(1,\cdots,m)}\partial_{(0,\cdots,n)}y = \partial_{(0,\cdots,\widehat{n+1},\cdots,n+m)}y.$$

Since $\widetilde{X^H}$ is simply-connected, the following composition holds in $\pi\widetilde{X^H}$,

$$\tilde{\xi}_H(\partial_{(0,\cdots,\widehat{n+1},\cdots,n+m)}y) = \widetilde{\xi}_H(\partial_{(1,2,\cdots,n+m)}y, \partial_{(0,\cdots,\widehat{n+1},\cdots,n+m)}y) \circ \widetilde{\xi}_H(\partial_{(1,2,\cdots,n+m)}).$$

Upon projecting via p_H, we get the following composition of morphisms in ΠX,

$$b\hat{\xi}_H(\partial_{(0,\cdots,\widehat{n+1},\cdots,n+m)}y) = b\hat{\xi}_H(\partial_{(1,2,\cdots,n+m)}y, \partial_{(0,\cdots,\widehat{n+1},\cdots,n+m)}y) \circ b\hat{\xi}_H(\partial_{(1,2,\cdots,n+m)}y).$$

Observe that $\sigma_{\widehat{n+1}} = b\hat{\xi}_H(\partial_{(1,2,\cdots,n+m)}y, \partial_{(0,\cdots,\widehat{n+1},\cdots,n+m)}y)$. Hence

$$\begin{aligned}
&M(b\hat{\xi}_H(\partial_{(1,\cdots,m)}\partial_{(0,\cdots,n)}y))\\
=&M(b\hat{\xi}_H(\partial_{(0,\cdots,\widehat{n+1},\cdots,n+m)}y))\\
=&M(\sigma_{\widehat{n+1}} \circ b\hat{\xi}_H(\partial_{(1,2,\cdots,n+m)}y))\\
=&M(b\hat{\xi}_H(\partial_{(1,2,\cdots,n+m)}y))M(\sigma_{\widehat{n+1}}).
\end{aligned}$$

Therefore $\mu(f \cup g) = \mu(f) \cup \mu(g)$. $\square$

Chapter 6

Steenrod reduced power operations

6.1 Introduction

An important class of cohomology operations is that of the Steenrod squares and reduced power operations [Ste53b], [Ste53a], [Ara56]. Steenrod squares are defined for cohomology with $\mathbb{Z}_2$ coefficients whereas Steenrod reduced powers are defined in cohomology with coefficients in $\mathbb{Z}_p$, $p \neq 2$ a prime. A very general and useful method of constructing these operations is given in [May70]. A categorical approach to Steenrod operations can be found in [Eps66]. In [Git63], S. Gitler constructed reduced power operations in cohomology with local coefficients. The main idea of Gitler's construction is to lift the power operations in the invariant cochain subcomplex of the universal cover of a space and reproduce the operations in cohomology with local coefficients via Eilenberg's description. The relevant local coefficient system in this context is obtained by a fixed action of the fundamental group of the space on a fixed cyclic group of prime order $p \neq 2$.

The aim of this chapter is to construct Steenrod reduced power operations in the simplicial Bredon-Illman cohomology with local coefficients, where the equivariant local coefficient system takes values in a $\mathbb{Z}_p$-algebra, for a prime $p > 2$. Throughout, our method is simplicial. It may be mentioned that for a space with a group action, the Steenrod squares have been introduced in the Bredon-Illman cohomology with local coefficients by G. Ginot in [Gin04]. Following Gitler [Git63], we first construct the power operations in the $\underline{\pi}X$-equivariant cohomology of the 'universal O_G-covering complex' of a one vertex G-Kan complex X. This is done by applying the algebraic description of the Steenrod reduced power operations of P. May [May70]. We then use the equivariant version of Eilenberg's theorem, Theorem 5.3.4, to reproduce the Steenrod reduced power operations in the present context. It may be remarked that our method also applies when $p = 2$, and hence yields Steenrod squares too.

6.2 Algebraic approach to Steenrod operations

In this section we briefly recall the relevant part of the general algebraic approach to the Steenrod operations by P. May [May70], necessary for our purpose.

Let Λ be a commutative ring. By a Λ-complex K, we will mean a $\mathbb{Z}$-graded cochain complex of Λ-modules with differential of degree 1. A morphism of Λ-complexes is a degree zero map commuting with the differential. If π is a group, we let $\Lambda\pi$ denote its group ring over Λ.

Let p be an odd prime and Σ_p denote the symmetric group on p-letters. For the rest of this chapter, unless otherwise stated, Λ will be the commutative ring $\mathbb{Z}_p$ and π will be the cyclic subgroup of Σ_p, generated by the permutation $\alpha = (p, 1, 2, \cdots, p-1)$. If not mentioned explicitly, all tensor products are over the ring Λ.

Let V, W be the free resolutions of Λ over $\Lambda\Sigma_p, \Lambda\pi$ respectively. We shall use the following canonical model of W. Let W_i be the $\Lambda\pi$-free module on one generator $e_i, i \geq 0$. Let $N = 1+\alpha+\cdots+\alpha^{p-1}$ and $T = \alpha-1$ in $\Lambda\pi$. Define the differential d, the augmentation $\epsilon\colon W_0 \to \Lambda$ and the coproduct ψ on W, respectively by the formulas

$$d(e_{2i+1}) = Te_{2i},\ d(e_{2i}) = Ne_{2i-1},\ \epsilon(\alpha^j e_0) = 1;$$

$$\psi(e_{2i+1}) = \sum_{j+k=i} e_{2j} \otimes e_{2k+1} + \sum_{j+k=i} e_{2j+1} \otimes \alpha e_{2k},$$

$$\psi(e_{2i}) = \sum_{j+k=i} e_{2j} \otimes e_{2k} + \sum_{j+k=(i-1)} \sum_{0\leq r<s<p} \alpha^r e_{2j+1} \otimes \alpha^s e_{2k}.$$

Thus W is a differential $\Lambda\pi$-coalgebra and a $\Lambda\pi$-free resolution of Λ.

We denote the p-fold tensor product $K \otimes \cdots \otimes K$ by K^p. Then K^p becomes a $\Lambda\pi$-complex by the following π operation,

$$\tau(u_1 \otimes \cdots \otimes u_p) = \gamma(\tau)u_1 \otimes \cdots u_{i-1} \otimes u_{i+1} \otimes u_i \otimes u_{i+2} \cdots \otimes u_p,$$

where $\gamma(\tau) = (-1)^{deg(u_i)deg(u_{i+1})}$ if τ is the interchange of i-th and $(i+1)$-th factor. We consider W as a non-positively graded $\Lambda\pi$-complex. The inclusion of π in Σ_p induces a morphism $j\colon W \to V$ of $\Lambda\pi$-complexes.

We have the following algebraic category $\mathfrak{C}(p)$ on which the Steenrod operations are defined. The objects of this category are pairs (K, θ), where K is a Λ-complex, equipped with a homotopy associative multiplication $K \otimes K \to K$, and $\theta\colon W \otimes K^p \to K$ is a morphism of $\Lambda\pi$-complexes satisfying the following two conditions.

1. The restriction of θ to $e_0 \otimes K^p$ is Λ-homotopic to the iterated product $K^p \to K$, associative in one fixed order.
2. The morphism θ is $\Lambda\pi$-homotopic to a composite $W \otimes K^p \xrightarrow{j\otimes 1} V \otimes K^p \xrightarrow{\varnothing} K$, where $\varnothing$ is a morphism of $\Lambda\Sigma_p$-complexes.

A morphism $f\colon (K,\theta) \to (K',\theta')$ is a morphism of Λ-complexes $f\colon K \to K'$ such that the following diagram is $\Lambda\pi$-homotopy commutative.

$$\begin{tikzcd} W \otimes K^p \arrow[r, "\theta"] \arrow[d, "id\otimes f^p"'] & K \arrow[d, "f"] \\ W \otimes (K')^p \arrow[r, "\theta'"] & K' \end{tikzcd}$$

The tensor product of two objects (K,θ) and (K',θ') is the pair $(K \otimes K', \tilde{\theta})$, where $\tilde{\theta}$ is the composition

$$W \otimes (K \otimes K')^p \xrightarrow{\psi\otimes\tilde{U}} W \otimes W \otimes K^p \otimes K'^p \xrightarrow{id\otimes\tilde{t}\otimes id} W \otimes K^p \otimes W \otimes K'^p \xrightarrow{\theta\otimes\theta'} K \otimes K'.$$

Here $\psi\colon W \to W \otimes W$ is the coproduct, $\tilde{U}\colon (K \otimes K')^p \to K^p \otimes K'^p$ is the shuffling isomorphism and $\tilde{t}(x \otimes y) = (-1)^{deg(x)deg(y)} y \otimes x$.

Definition 6.2.1. An object $(K,\theta) \in \mathfrak{C}(p)$ is said to be a Cartan object if the product $K \otimes K \to K$ is a morphism from $(K \otimes K, \tilde{\theta})$ to (K,θ).

For an object (K,θ) of $\mathfrak{C}(p)$, there are maps $D_i\colon H^q(K) \to H^{pq-i}(K), i \geq 0$, defined as follows. For $x \in H^q(K)$, $e_i \otimes x^p$ is a well-defined element of $H^{pq-i}(W \otimes_{\Lambda\pi} K^p)$ [May70] and define $D_i(x) = \theta_*(e_i \otimes x^p)$, where $\theta_*\colon H^{pq-i}(W \otimes_{\Lambda\pi} K^p) \to H^{pq-i}(K)$ is induced by θ. We make the convention that $D_i = 0$ for $i < 0$. Then the Steenrod reduced power operations

$$\mathcal{P}^s\colon H^q(K) \to H^{q+2s(p-1)}(K), \quad \beta\mathcal{P}^s\colon H^q(K) \to H^{q+2s(p-1)+1}(K),$$

are defined by the following formulas

$$\mathcal{P}^s(x) = (-1)^r (m!)^q D_{(q-2s)(p-1)}(x), \quad \beta\mathcal{P}^s(x) = (-1)^r (m!)^q D_{(q-2s)(p-1)-1}(x),$$

where $m = (p-1)/2$ and $r = s + m(q+q^2)/2$.

Proposition 6.2.2. *The power operations satisfies the following properties.*

1. *$\mathcal{P}^s$ and $\beta\mathcal{P}^s$ are natural homomorphisms.*

2. *$\mathcal{P}^s(x) = 0$ if $2s > q$, $\beta\mathcal{P}^s = 0$ if $2s \geq q$, and $\mathcal{P}^s(x) = x^p$ if $2s = q$.*

3. *If (K, θ) is a Cartan object then $\mathcal{P}^s$ and $\beta\mathcal{P}^s$ satisfy the Cartan formulas*

$$\mathcal{P}^s(xy) = \sum_{i+j=s} \mathcal{P}^i(x)\mathcal{P}^j(y),$$

$$\beta\mathcal{P}^{s+1}(xy) = \sum_{i+j=s} [\beta\mathcal{P}^{i+1}(x)\mathcal{P}^j(y) + (-1)^{deg(x)}\mathcal{P}^i(x)\beta\mathcal{P}^{j+1}(y)].$$

Remark 6.2.3. In general $\beta\mathcal{P}^s$ is single notation. But if (K, θ) is reduced mod p ([May70]) then the Bockstein homomorphism

$$\beta\colon H^n(K) \to H^{n+1}(K),$$

can be defined and $\beta\mathcal{P}^s$ is the composition of $\mathcal{P}^s$ with the Bockstein homomorphism β.

Next we recall the definition of 'Adem object' in $\mathfrak{C}(p)$ [May70]. We need the following notations for the definition.

Consider Σ_{p^2} as permutations on the p^2 symbols $\{(i, j) | 1 \leq i, j \leq p\}$. Embed $\pi =< \alpha > (\subseteq \Sigma_p)$ in Σ_{p^2} by letting $\alpha(i, j) = (i, j + 1)$. Let $\alpha_i \in \Sigma_{p^2}, 1 \leq i \leq p$, be defined by $\alpha_i(i, j) = (i, j + 1)$ and $\alpha_i(k, j) = (k, j)$ for $k \neq i$. Let

$$\beta = \alpha_1 \cdots \alpha_p,\ \nu =< \beta >,\ \sigma = \pi\nu,\ \tau =< \alpha_1, \cdots, \alpha_p, \alpha > .$$

Note that β and α_i are of order p and the following relations hold.

$$\alpha\alpha_i = \alpha_{i+1}\alpha;\ \alpha_i\alpha_j = \alpha_j\alpha_i;\ \alpha\beta = \beta\alpha.$$

Let $W_1 = W$ and $W_2 = W$ regarded as $\Lambda\pi$-free and $\Lambda\nu$-free resolutions of Λ, respectively. Let ν, π operate trivially on W_1, W_2 respectively. Then $W_1 \otimes W_2$ is a $\Lambda\sigma$-free resolution of Λ with the diagonal action of σ on $W_1 \otimes W_2$.

For any ν-module M, let τ operate on M^p by letting α operate by cyclic permutation, and by letting α_i operate on the i-th factor as does β. Let α_i operate trivially on W_1. Then τ operates on W_1 and hence τ operates diagonally on $W_1 \otimes M^p$. In particular, $W_1 \otimes W_2^p$ is then a $\Lambda\tau$-free resolution of Λ.

Let $(K, \theta) \in \mathfrak{C}(p)$. We let Σ_{p^2} operate on K^{p^2} by permutations, where we consider K^{p^2} as $\otimes_{i=1}^p(\otimes_{j=1}^p K_{i,j})$, $K_{i,j} = K$. We let ν operate on $W_2 \otimes K^p$ by letting β act as a cyclic permutation on K^p. By the previous paragraph, this fixes an action of τ on $W_1 \otimes (W_2 \otimes K^p)^p$.

Let Y be any $\Lambda\Sigma_{p^2}$-free resolution of Λ with $Y_0 = \Lambda\Sigma_{p^2}$ and let $w\colon W_1 \otimes W_2^p \to Y$ be any morphism of $\Lambda\tau$-complexes. Observe that w exists since Y is acyclic and any two choices of w are $\Lambda\tau$-equivariantly homotopic.

With these notations, we have the following definition.

Definition 6.2.4. Let $(K,\theta) \in \mathfrak{C}(p)$. We say that (K,θ) is an Adem object if there exists a morphism of the $\Lambda\Sigma_{p^2}$-complexes $\eta\colon Y \otimes K^{p^2} \to K$, such that the following diagram is $\Lambda\tau$-equivariant homotopy commutative.

$$\begin{tikzcd}
(W_1 \otimes W_2^p) \otimes K^{p^2} \arrow[r, "w\otimes id"] & Y \otimes K^{p^2} \arrow[r, "\eta"] & K \\
W_1 \otimes (W_2 \otimes K^p)^p \arrow[u, "id\times\tilde{U}"] \arrow[r, "id\otimes\theta^p"] & W_1 \otimes K^p \arrow[r, "\theta"] & K \arrow[u, "id"']
\end{tikzcd}$$

Here $\tilde{U}$ is the shuffle map and Σ_{p^2} acts trivially on K.

The following relations among the operations $\mathcal{P}^s$ and $\beta\mathcal{P}^s$ are valid on all cohomology classes of Adem objects in $\mathfrak{C}(p)$, $p > 2$ a prime, [May70]

- If $a < pb$ then $\beta^e\mathcal{P}^a\mathcal{P}^b = \sum_i(-1)^{a+i}(a-pi,(p-1)b-a+i-1)\beta^e\mathcal{P}^{a+b-i}\mathcal{P}^i$.
- If $a \leq pb$ then $\beta^e\mathcal{P}^a\beta\mathcal{P}^b = (1-e)\sum_i(-1)^{a+i}(a-pi,(p-1)b-a+i-1)\beta\mathcal{P}^{a+b-i}\mathcal{P}^i$
$- \sum_i(-1)^{a+i}(a-pi-1,(p-1)b-a+i)\beta^e\mathcal{P}^{a+b-i}\beta\mathcal{P}^i$,

where $e = 0,1$ and $\beta^0\mathcal{P}^s = \mathcal{P}^s$ and $\beta^1\mathcal{P}^s = \beta\mathcal{P}^s$.

6.3 Steenrod reduced power operations in simplicial Bredon-Illman cohomology with local coefficients

We apply the general method of the previous section to construct the Steenrod reduced power operations in the equivariant cohomology of O_G-simplicial sets, as defined in Section 6.3.4. In particular, for a one vertex G-Kan complex X, we have the reduced power operations defined for the $\underline{\pi}X$-equivariant cohomology of the universal O_G-covering complex $\tilde{X}$ of X (cf. Definition 5.3.3). We then apply Theorem 5.3.4 to deduce the Steenrod reduced power operations in the simplicial Bredon-Illman cohomology with local coefficients.

Definition 6.3.1. Let ρ be an O_G-group and T, T' be O_G-simplicial sets. Assume that ρ acts on both T and T'. A map $f\colon T \to T'$ is called ρ-equivariant if

$$f(G/H)(ax) = af(G/H)(x),\ a \in \rho(G/H),\ x \in T(G/H),$$

for each subgroup H of G.

Definition 6.3.2. Let E, E' be O_G-chain complexes. Two natural transformations $v = \{v_n\}, w = \{w_n\}\colon E \to E'$ are said to be homotopic if there exist natural transformations

$$\mathcal{H}_n : v_n \to w_{n+1},\ n \geq 0,$$

such that $\{\mathcal{H}_n(G/H)\}_{n\geq 0}$ is a chain homotopy of the chain maps $v(G/H), w(G/H)$ for each subgroup H of G. Symbolically, we write $\mathcal{H} : v \simeq w$.

If an O_G-group ρ acts on E, E' and v, w are ρ-equivariant, then v, w are said to be ρ-equivariantly homotopic if there exists a homotopy $\mathcal{H}\colon v \simeq w$ which satisfies

$$\mathcal{H}_n(G/H)(ax) = a\mathcal{H}_n(G/H)(x) \text{ for } a \in \rho(G/H),\ x \in E(G/H)_n,$$

for each subgroup H of G.

Definition 6.3.3. Let E, E' be two O_G-chain complexes. Then their tensor product is the O_G-chain complex $E \otimes E'$, defined by

$$(E \otimes E')(G/H) = E(G/H) \otimes E'(G/H),$$

for each object G/H of O_G and $(E \otimes E')(\hat{a}) = E(\hat{a}) \otimes E'(\hat{a})$ for a morphism $\hat{a}$ in O_G.

Note that a chain complex W can be considered as an O_G-chain complex in the obvious way, that is, $W(G/H) = W$ for an object G/H of O_G and $W(\hat{a}) = id$ for a morphism $\hat{a}$ in O_G. So the tensor product of W with an O_G-chain complex is defined.

Let (T, M_0, ρ) be an object of $\mathcal{A}_\Lambda$ (cf. Section 5.2). Recall that the cochain complex $C^*_\rho(T; M_0)$, equipped with the cup product, is an associative differential graded Λ-algebra (cf. Remark 5.2.4). We now construct a morphism of $\Lambda\pi$-complexes

$$\theta\colon W \otimes C^*_\rho(T; M_0)^p \to C^*_\rho(T; M_0),$$

so that $(C^*_\rho(T; M_0), \theta)$ becomes an object of the category $\mathfrak{C}(p)$.

For a simplicial set L, let $C_*(L)$ denote the normalized chain complex of L with coefficients Λ. We recall the following lemma from [May70].

Lemma 6.3.4. *Let π be a subgroup of Σ_p (π not necessarily cyclic of order p) and W be a $\Lambda\pi$-free resolution of Λ such that $W_0 = \Lambda\pi$ with generator e_0. For simplicial sets $L_1, \cdots, L_p$, there exists a chain map*

$$\Phi\colon W \otimes C_*(L_1 \times \cdots \times L_p) \to W \otimes C_*(L_1) \otimes \cdots \otimes C_*(L_p),$$

which is natural in the L_i and satisfies the following properties.

1. *For $\sigma \in \pi$, the following diagram is commutative.*

$$\begin{tikzcd} W \otimes C_*(L_1 \times \cdots \times L_p) \arrow[r, "\Phi"] \arrow[d, "\sigma"] & W \otimes C_*(L_1) \otimes \cdots \otimes C_*(L_p) \arrow[d, "\sigma"] \\ W \otimes C_*(L_{\sigma(1)} \times \cdots \times L_{\sigma(p)}) \arrow[r, "\Phi"] & W \otimes C_*(L_{\sigma(1)}) \otimes \cdots \otimes C_*(L_{\sigma(p)}) \end{tikzcd}$$

2. *Φ is the identity homomorphism on $W \otimes C_0(L_1 \times \cdots \times L_p)$.*

3. *$\Phi(e_0 \otimes (x_1, \cdots, x_p)) = e_0 \otimes \xi(x_1, \cdots, x_p)$, where $x_i \in L_j$ for $1 \leq i \leq p$ and*

$$\xi\colon C_*(L_1 \times \cdots \times L_p) \to C_*(L_1) \otimes \cdots \otimes C_*(L_p)$$

is the Alexander-Whitney map.

4. *$\Phi(W \otimes C_j(L_1 \times \cdots \times L_p)) \subseteq \sum_{k \leq pj} W \otimes [C_*(L_1) \otimes \cdots \otimes C_*(L_p)]_k$.*

5. *Any two such Φ are naturally equivariantly homotopic.*

In the special case $L_1 = \cdots = L_p = L$, we obtain a natural morphism of chain complexes of $\Lambda\pi$-modules

$$\Phi\colon W \otimes C_*(L^p) \to W \otimes C_*(L)^p,$$

which satisfies the last four conditions of Lemma 6.3.4.

Let $T \in O_G\mathcal{S}$. Applying the above special case of Lemma 6.3.4 to each simplicial set $T(G/H)$, we obtain chain maps $\Phi_H\colon W \otimes C_*(T(G/H)^p) \to W \otimes C_*(T(G/H))^p$ which is π-equivariant. Since Φ_H is natural with respect to maps of simplicial sets, we see that $\Phi_H \circ (id_W \otimes \underline{C}_*(T(\hat{a})^p)) = (id_W \otimes \underline{C}_*(T(\hat{a}))^p) \circ \Phi_K$, where $a^{-1}Ha \subseteq K$. Thus we have a morphism $\underline{\Phi}$ of O_G-chain complexes

$$\underline{\Phi}\colon W \otimes \underline{C}_*(T^p) \to W \otimes \underline{C}_*(T)^p, \text{ defined by } \underline{\Phi}(G/H) = \Phi_H,\ G/H \in O_G.$$

Now suppose that an O_G-group ρ operates on T. The diagonal action of ρ on T^p induces a ρ-action on $\underline{C}_*(T^p)$. Also, we have an induced ρ-action on $\underline{C}_*(T)$. We let ρ operate diagonally on $\underline{C}_*(T)^p$ and trivially on W. The naturality of Φ_H with respect to maps from $T(G/H)$ into itself shows that Φ_H is $\rho(G/H)$-equivariant. Thus the map $\underline{\Phi}$ is $(\pi \times \rho)$-equivariant. Hence we obtain the following corollary.

Corollary 6.3.5. *Let $T \in O_G\mathcal{S}$ and suppose an O_G-group ρ operates on T. For a subgroup π of Σ_p (π not necessarily cyclic of order p), let W be a $\Lambda\pi$-free resolution of Λ such that $W_0 = \Lambda\pi$ with generator e_0. Then there is a natural transformation*

$$\underline{\Phi}\colon W \otimes \underline{C}_*(T^p) \to W \otimes \underline{C}_*(T)^p$$

such that

1. *$\underline{\Phi}$ is $(\pi \times \rho)$-equivariant.*
2. *$\underline{\Phi}$ is the identity homomorphism on $W \otimes \underline{C}_0(T^p)$.*
3. *$\underline{\Phi}(G/H)(e_0 \otimes (x_1, \cdots, x_p)) = e_0 \otimes \underline{\xi}(G/H)(x_1, \cdots, x_p)$, where $x_i \in T(G/H)$ for $1 \leq i \leq p$ and $\underline{\xi}(G/H)\colon C_*(T(G/H)^p) \to C_*(T(G/H))^p$ is the Alexander-Whitney map of the simplicial set $T(G/H)$.*
4. *$\underline{\Phi}(G/H)(W \otimes C_j(T(G/H)^p)) \subseteq \sum_{k \leq pj} W \otimes (C_*(T(G/H))^p)_k$.*
5. *The map $\underline{\Phi}$ is natural with respect to the equivariant maps of O_G-simplicial sets and any two such $\underline{\Phi}$ are naturally equivariantly homotopic.*

Next we construct the map $\theta\colon W \otimes C^*_\rho(T; M_0)^p \to C^*_\rho(T; M_0)$.

For an object $(T, M_0, \rho) \in \mathcal{A}_\Lambda$, let $D\colon T \to T^p$ be the diagonal map

$$D(G/H)(x) = (x, \cdots, x),\ x \in T(G/H),$$

which induces a map $D_*\colon \underline{C}_*(T) \to \underline{C}_*(T^p)$. Define $\underline{\Delta}\colon W \otimes \underline{C}_*(T) \to \underline{C}_*(T)^p$ to be the composite

$$\underline{\Delta}\colon W \otimes \underline{C}_*(T) \xrightarrow{id \otimes D_*} W \otimes \underline{C}_*(T^p) \xrightarrow{\underline{\Phi}} W \otimes \underline{C}_*(T)^p \to \underline{C}_*(T)^p,$$

where the last map is the augmentation. Observe that the map $\underline{\Delta}$ is $(\pi \times \rho)$-equivariant. Moreover, we have a natural map

$$\alpha\colon [C^*_\rho(T; M_0)]^p \to Hom_\rho(\underline{C}_*(T)^p, M_0)$$

defined by

$$\alpha(f_1 \otimes \cdots \otimes f_p)(G/H)(x_1 \otimes \cdots \otimes x_p) = (-1)^a f_1(G/H)(x_1) \cdots f_p(G/H)(x_p),$$

where $f_i \in C^*_\rho(T; M_0),\ x_i \in \underline{C}_*(T)(G/H),\ i = 1, \cdots, p$ and $a = \prod_{k=1}^p deg(x_k)$. Hence dualising $\underline{\Delta}$, we get a natural morphism of $\Lambda\pi$-complexes,

$$\theta\colon W \otimes C^*_\rho(T; M_0)^p \to C^*_\rho(T; M_0),$$

given by

$$\theta(w \otimes f)(G/H)(x) = (-1)^{deg(w)deg(x)} \alpha(f)(G/H)(\underline{\Delta}(G/H)(w \otimes x)),$$

where $w \in W,\ f \in C^*_\rho(T; M_0)^p,\ x \in C_*(T(G/H))$.

Remark 6.3.6. Note that $\theta(e_0 \otimes f) = D^* \underline{\xi}^* \alpha(f)$ for any $f \in C^*_\rho(T; M_0)^p$. As before let V denote a $\Lambda\Sigma_p$-free resolution of Λ and $j \colon W \to V$ be the map induced by the inclusion $\pi \hookrightarrow \Sigma_p$. We apply Corollary 6.3.5 for the (sub)group Σ_p to get $\tilde{\Phi} \colon V \otimes \underline{C}_*(T^p) \to W \otimes \underline{C}_*(T)^p$. Then $\tilde{\Phi} \circ (j \otimes id)$ satisfies the first four conditions of Corollary 6.3.5 for the subgroup π and hence must be equivariantly homotopic to $\underline{\Phi}$. Therefore, $\tilde{\theta} \colon V \otimes C^*_\rho(T; M_0)^p \to C^*_\rho(T; M_0)$ can be defined such that $\tilde{\theta} \circ (j \otimes id)$ is $\Lambda\pi$-equivariantly homotopic to θ. Therefore $(C^*_\rho(T; M_0), \theta)$ is an object of the category $\mathfrak{C}(p)$. Thus we obtain a contravariant functor $\Gamma \colon \mathcal{A}_\Lambda \to \mathfrak{C}(p)$ by letting $\Gamma(T, M_0, \rho) = (C^*_\rho(T; M_0), \theta)$ and $\Gamma(f_0, f_1, f_2) = C^*(f_0, f_1, f_2)$ on morphisms (cf. Remark 5.2.3).

The next lemma is the key to show that $(C^*_\rho(T; M_0), \theta)$ is a Cartan object of $\mathfrak{C}(p)$. Let $\phi = (\epsilon \otimes id)\Phi$ where Φ is obtained from Lemma 6.3.4 and $\epsilon \colon W \to \Lambda$ is the augmentation.

Lemma 6.3.7. *Let L_i, S_i $i = 1, \cdots, p$ be simplicial sets. Let $u \colon (\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i) \to \prod_{i=1}^p (L_i \times S_i)$ and $U \colon (\otimes_{i=1}^p C_*(L_i)) \otimes (\otimes_{i=1}^p C_*(S_i)) \to \otimes_{i=1}^p [C_*(L_i) \otimes C_*(S_i)]$ be shuffle maps. Let t denote the flip map, that is $t(x \otimes y) = y \otimes x$. Then there exists a homotopy*

$$\mathcal{H} \colon W \otimes C_*(\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i) \to \bigotimes_{i=1}^p [C_*(L_i) \otimes C_*(S_i)]$$

of the chain maps $\xi^p \phi(id \otimes u)$ and $U(\phi \otimes \phi)(id \otimes t \otimes id)(\psi \otimes id \otimes id)(id \times \xi)$, so that the following diagram is homotopy commutative.

$$\begin{tikzcd}
W \otimes C_*(\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i) \arrow[r, "id \times u"] \arrow[d, "id \otimes \xi"'] & W \otimes C_*(\prod_{i=1}^p (L_i \times S_i)) \arrow[r, "\phi"] & \bigotimes_{i=1}^p [C_*(L_i \times S_i)] \arrow[d, "\xi^p"] \\
W \otimes C_*(\prod_{i=1}^p L_i) \otimes C_*(\prod_{i=1}^p S_i) \arrow[rr, "U(\phi\otimes\phi)(id\otimes t\otimes id)(\psi\otimes id\otimes id)"'] & & \bigotimes_{i=1}^p [C_*(L_i) \otimes C_*(S_i)]
\end{tikzcd}$$

Moreover, the homotopy $\mathcal{H}$ is natural in the L_i, S_i and the following diagram com-

mutes for $\sigma \in \pi$.

$$\begin{tikzcd} W \otimes C_*(\prod_{i=1}^{p} L_i \times \prod_{i=1}^{p} S_i) \arrow[r, "\mathcal{H}"] \arrow[d, "\sigma\otimes\sigma"'] & \bigotimes_{i=1}^{p}[C_*(L_i) \otimes C_*(S_i)] \arrow[d, "\sigma"] \\ W \otimes C_*(\prod_{i=1}^{p} L_{\sigma(i)} \times \prod_{i=1}^{p} S_{\sigma(i)}) \arrow[r, "\mathcal{H}"'] & \bigotimes_{i=1}^{p}[C_*(L_{\sigma(i)}) \otimes C_*(S_{\sigma(i)})] \end{tikzcd}$$

Proof. The proof is similar to the proof of Lemma 7.1 of [May70]. Let $A_j = C_j(\prod_{i=1}^{p} L_i \times \prod_{i=1}^{p} S_i)$ and $B_j = [\otimes_{i=1}^{p} C_*(L_i) \otimes C_*(S_i)]_j$. We construct $\mathcal{H}$ on $W_i \otimes A_j$ by induction on i and for fixed i by induction on j. Note that the two maps agree on $W \otimes A_0$. So H is the zero map on $W \otimes A_0$. To define $\mathcal{H}$ on $W_0 \otimes A_j,\ j \geq 0$, it suffices to define on $e_0 \otimes A_j$, since $\mathcal{H}$ can then be uniquely extended to all of $W_0 \otimes A_j$ using the commutativity of the second diagram. The functor $e_0 \otimes A_j$ is represented by the model $\Delta[j]^p \times \Delta[j]^p$ and $W \otimes B_j$ is acyclic on this model. Therefore, by acyclic model argument, $\mathcal{H}$ can be defined on $e_0 \otimes A_j$, provided $\mathcal{H}$ is known on $e_0 \otimes A_{j-1}$. But $\mathcal{H}$ has already been defined on $W_0 \otimes A_0$. Hence by induction on j, we can define $\mathcal{H}$ on $e_0 \otimes A_j,\ j \geq 0$. To define $\mathcal{H}$ on $W_i \otimes A_j$, assume that it has already been defined on $W_{i'} \otimes A_j,\ i' < i,\ j \geq 0$ and on $W_i \otimes A_{j'},\ j' < j$. Choose a $\Lambda\pi$-basis $\{w_k\}$ for W_i. As before, it suffices to define $\mathcal{H}$ on $w \otimes A_j,\ w \in \{w_k\}$. We can repeat the acyclic model argument replacing e_0 by w, and hence we are through by induction. □

In the special case $L_1 = \cdots = L_p = L,\ S_1 = \cdots = S_p = S$, we obtain the following corollary.

Corollary 6.3.8. *For simplicial sets* L, S *the two chain maps* $\xi^p\phi(id \otimes u)$ *and* $U(\phi \otimes \phi)(id \otimes t \otimes id)(\psi \otimes id \otimes id)(id \times \xi)$ *from* $W \otimes C_*(L^p \times S^p)$ *to* $[C_*(L) \otimes C_*(S)]^p$ *are* $\Lambda\pi$*-equivariantly homotopic and the homotopy is natural in* L *and* S.

Suppose (T, M_0, ρ) and (T', M_0', ρ') are objects of $\mathcal{A}_\Lambda$. With the product actions of $\rho \times \rho'$ on $T \times T'$ and $M_0 \otimes M_0'$, we have an object $(T \times T', M_0 \otimes M_0', \rho \times \rho') \in \mathcal{A}_\Lambda$. The following lemma relates $\Gamma(T \times T', M_0 \otimes M_0', \rho \times \rho') = (C^*_{\rho\times\rho'}(T \times T'; M_0 \otimes M_0'), \theta)$ to $\Gamma(T, M_0, \rho) \otimes \Gamma(T', M_0', \rho') = (C^*_\rho(T; M_0) \otimes C^*_{\rho'}(T'; M_0'), \tilde{\theta})$.

Let

$$\tilde{\alpha} \colon C^*_\rho(T; M_0) \otimes C^*_{\rho'}(T'; M_0') \to Hom_{\rho\times\rho'}(\underline{C}_*(T) \otimes \underline{C}_*(T'), M_0 \otimes M_0')$$

be defined by

$$\tilde{\alpha}(f \otimes g)(G/H)(x \otimes y) = (-1)^{deg(x)deg(y)} f(G/H)(x) \otimes g(G/H)(y),\ H \subseteq G,$$

where $f \in C^*_\rho(T; M_0),\ g \in C^*_{\rho'}(T'; M'_0),\ x \in \underline{C}_*(T)(G/H),\ y \in \underline{C}_*(T')(G/H)$.

Lemma 6.3.9. *With the notations as above, the following diagram is $\Lambda\pi$-homotopy commutative.*

$$\begin{tikzcd}
W \otimes C^*_{\rho\times\rho'}(T \times T'; M_0 \otimes M'_0)^p \arrow[r, "\theta"] & C^*_{\rho\times\rho'}(T \times T'; M_0 \otimes M'_0) \\
W \otimes [C^*_\rho(T; M_0) \otimes C^*_{\rho'}(T'; M'_0)]^p \arrow[r, "\tilde{\theta}"] \arrow[u, "id\otimes(\underline{\xi}^*\tilde{\alpha})^p"] & C^*_\rho(T; M_0) \otimes C^*_{\rho'}(T'; M'_0) \arrow[u, "\underline{\xi}^*\tilde{\alpha}"']
\end{tikzcd}$$

Proof. Let $D,\ D',\ \tilde{D}$ be the diagonals for $T,\ T',\ T \times T'$ respectively. Let

$$\underline{u}\colon T^p \times T'^p \to (T \times T')^p \quad \text{and} \quad \underline{U}\colon \underline{C}_*(T)^p \otimes \underline{C}_*(T')^p \to [\underline{C}_*(T) \otimes \underline{C}_*(T')]^p$$

be the shuffle maps. Let t be the switch map.

By the definitions of θ and $\tilde{\theta}$, it suffices to prove that the following diagram of O_G-chain complexes is $\Lambda(\pi \times \rho \times \rho')$-equivariant homotopy commutative.

$$(1)\cdots \quad \begin{tikzcd}
W \otimes \underline{C}_*(T \times T') \arrow[r, "\underline{\Delta}"] \arrow[d, "id\times\underline{\xi}"'] & \underline{C}_*(T \times T')^p \arrow[d, "\underline{\xi}^p"] \\
W \otimes \underline{C}_*(T) \otimes \underline{C}_*(T') \arrow[r, "\zeta"] & [\underline{C}_*(T) \otimes \underline{C}_*(T')]^p
\end{tikzcd}$$

Here

$$\underline{\Delta} = (\epsilon \otimes id)\underline{\Phi}(id \otimes \tilde{D}),\ \zeta = \underline{U}(\underline{\Delta} \otimes \underline{\Delta})(id \otimes t \otimes id)(\psi \otimes id \otimes id).$$

Let $\underline{\phi} = (\epsilon \otimes id)\underline{\Phi}$. Observe that $\tilde{D} = \underline{u}(D \times D')$ and

$$(id \otimes D \otimes id \otimes D')(id \otimes t \otimes id)(\psi \otimes id \otimes id) = (id \otimes t \otimes id)(\psi \otimes id \otimes id)(id \otimes D \otimes D').$$

Observe that the following diagram commutes by the naturality of $\underline{\xi}$.

$$(2)\cdots\quad \begin{array}{ccc} W\otimes \underline{C}_*(T\times T') & \xrightarrow{id\otimes(D\times D')} & W\otimes \underline{C}_*(T^p\times T'^p) \\ \Big\downarrow{\scriptstyle id\otimes\underline{\xi}} & & \Big\downarrow{\scriptstyle id\otimes\underline{\xi}} \\ W\otimes \underline{C}_*(T)\otimes \underline{C}_*(T') & \xrightarrow{id\otimes D\otimes D'} & W\otimes \underline{C}_*(T^p)\otimes \underline{C}_*(T'^p) \end{array}$$

Let $\mathcal{F}$ denote the following diagram of O_G-chain complexes of Λ-modules.

$$(3)\cdots\quad \begin{array}{ccccc} W\otimes \underline{C}_*(T^p\times T'^p) & \xrightarrow{id\otimes\underline{u}} & W\otimes \underline{C}_*([T\times T']^p) & \xrightarrow{\underline{\phi}} & \underline{C}_*(T\times T')^p \\ \Big\downarrow{\scriptstyle id\otimes\underline{\xi}} & & & & \Big\downarrow{\scriptstyle \underline{\xi}^p} \\ W\otimes \underline{C}_*(T^p)\otimes \underline{C}_*(T'^p) & & \xrightarrow{\underline{U}(\underline{\phi}\otimes\underline{\phi})(id\otimes t\otimes id)(\psi\otimes id\otimes id)} & & [\underline{C}_*(T)\otimes \underline{C}_*(T')]^p \end{array}$$

Then $\mathcal{F}(G/H)$ is $\Lambda\pi$-equivariant homotopy commutative, by Corollary 6.3.8. The naturality of this homotopy with respect to maps from $T(G/H)$ into itself implies that the homotopy is equivariant for the $\rho(G/H)$-action on $T(G/H)$. Similarly, the homotopy is $\rho'(G/H)$-equivariant. These natural equivariant homotopies of chain complexes combine together to form $\Lambda(\pi\times\rho\times\rho')$-equivariant homotopy, which makes the diagram (3) $\Lambda(\pi\times\rho\times\rho')$-equivariant homotopy commutative.

Now observe that the diagram (1) is the juxtaposition of the diagrams (2) and (3). Hence the diagram (1) is $\Lambda(\pi\times\rho\times\rho')$-equivariant homotopy commutative. $\square$

Proposition 6.3.10. *For an object (T, M_0, ρ) of $\mathcal{A}_\Lambda$, $\Gamma(T, M_0, \rho) = (C^*_\rho(T; M_0), \theta)$ is a Cartan object of $\mathfrak{C}(p)$.*

Proof. Recall that $(C^*_\rho(T; M_0), \theta)$ is called a Cartan object if the cup product is a morphism of $\mathfrak{C}(p)$. Now observe that

$$(T, M_0, \rho) \xrightarrow{(D,id,id)} (T\times T, M_0, \rho) \xrightarrow{(id,m,D)} (T\times T, M_0\otimes M_0, \rho\times\rho)$$

are morphisms in $\mathcal{A}_\Lambda$, where $m\colon M_0\otimes M_0\to M_0$ is the multiplication, D denotes the diagonal map, and we let ρ to operate diagonally on $T\times T$.

Applying Lemma 6.3.9 with $(T, M_0, \rho) = (T', M_0', \rho')$, and composing with the morphism $C^*(id, m, D)$, we see that the composite $\underline{\xi}^*\alpha$

$$C^*_\rho(T; M_0)\otimes C^*_\rho(T; M_0) \xrightarrow{\alpha} Hom_\rho(\underline{C}_*(T)\otimes \underline{C}_*(T), M_0) \xrightarrow{\underline{\xi}^*} C^*_\rho(T\times T; M_0)$$

is a morphism in $\mathfrak{C}(p)$. Also note that $C^*(D, id, id)\colon C^*_\rho(T\times T; M_0)\to C^*_\rho(T; M_0)$ is a morphism in $\mathfrak{C}(p)$. Hence the cup product is a morphism in $\mathfrak{C}(p)$. □

Next we show that $C^*_\rho(T; M_0)$ is an 'Adem object' in $\mathfrak{C}(p)$.

Proposition 6.3.11. *For $(T, M_0, \rho) \in \mathcal{A}_\Lambda$, $\Gamma(T, M_0, \rho) = (C^*_\rho(T; M_0), \theta)$ is an Adem object in $\mathfrak{C}(p)$.*

Proof. With the notations of Definition 6.2.4, we first construct the map

$$\eta\colon Y \otimes C^*_\rho(T; M_0)^{p^2} \to C^*_\rho(T; M_0).$$

The procedure is similar to the construction of θ. We remark that the proof of Lemma 6.3.4 works for any subgroup π of Σ_r, r being any positive integer. Thus we have a chain map

$$\Phi\colon Y \otimes C_*(L_1 \times \cdots \times L_r) \to Y \otimes C_*(L_1) \otimes \cdots \otimes C_*(L_r),$$

satisfying the properties of Lemma 6.3.4. As before, we specialize to $L_1 = \cdots = L_r = L$ and take $\pi = \Sigma_r$. The naturality of Φ with respect to maps of a simplicial set into itself allows us to pass to an O_G-simplicial set T, equipped with an action of an O_G-group ρ, so that we get $\Lambda(\Sigma_r \times \rho)$-equivariant map of O_G-chain complexes $\underline{\Phi}\colon Y \otimes \underline{C}_*(T^r) \to Y \otimes \underline{C}_*(T)^r$. As a consequence, we obtain a map of O_G-chain complexes $\underline{\Delta}\colon Y \otimes \underline{C}(T) \to \underline{C}(T)^{p^2}$ which is $(\Sigma_{p^2} \times \rho)$-equivariant. Next, following the construction of the map θ, we obtain η.

Note that, dualizing the diagram in Definition 6.2.4, it suffices to prove that the following diagram is $\Lambda(\tau \times \rho)$-homotopy commutative.

$$\begin{tikzcd}
W_1 \otimes W_2^p \otimes \underline{C}_*(T) \arrow[r, "w\otimes id"] \arrow[d, "t\otimes id"'] & Y \otimes \underline{C}_*(T) \arrow[r, "\underline{\Delta}"] & \underline{C}_*(T)^{p^2} \\
W_2^p \otimes W_1 \otimes \underline{C}_*(T) \arrow[r, "id\otimes\underline{\Delta}"] & W_2^p \otimes \underline{C}_*(T)^p \arrow[r, "\underline{U}"] & {[W_2 \otimes \underline{C}_*(T)]^p} \arrow[u, "\underline{\Delta}^p"']
\end{tikzcd}$$

Here the notations are as in Lemma 6.3.9. Define the maps of O_G-chain complexes $\chi, \Omega\colon W_1 \otimes W_2^p \otimes \underline{C}_*(T^{p^2}) \to \underline{C}_*(T)^{p^2}$ by

$$\chi = \underline{\phi}(w \otimes id_{\underline{C}_*(T^{p^2})}) \text{ and } \Omega = \underline{\phi}^p \underline{U}(id_{W_1 \otimes W_2^p} \otimes \underline{\phi})(t \otimes id_{\underline{C}_*(T^{p^2})}).$$

Let $D\colon \underline{C}_*(T) \to \underline{C}_*(T^{p^2})$ be induced by the diagonal. Following [May67], we observe that,

$$\underline{\Delta}(w \otimes id) = \chi(id \otimes id \otimes D),$$

and

$$\underline{\Delta}^p \underline{U}(id \otimes \underline{\Delta})(t \otimes id) = \Omega(id \otimes id \otimes D).$$

Therefore it suffices to show that the maps of O_G-chain complexes χ, Ω are $\Lambda(\tau \times \rho)$-equivariantly homotopic. Here τ operates by permutation of factors and the O_G-group ρ operates diagonally on T^{p^2} and on $\underline{C}_*(T)^{p^2}$. We replace $\underline{C}_*(T^{p^2})$ by $C_*(\prod_{i,j=1}^{p} L_{i,j})$ and $\underline{C}_*(T)^{p^2}$ by $\bigotimes_{i,j=1}^{p} C_*(L_{i,j})$ in the definitions of the maps χ and Ω, where $L_{i,j}$s are simplicial sets. Then the chain maps, corresponding to χ and Ω can be shown to be τ-equivariantly homotopic, and the homotopy is natural with respect to maps of simplicial sets. In the special case $L_{i,j} = L,\ 1 \leq i, j \leq p$, the naturality of this homotopy for maps of a simplicial set into itself implies that the chain maps $\chi(G/H)$ and $\Omega(G/H)$ are $\Lambda(\tau \times \rho(G/H))$-equivariantly homotopic, $H \subseteq G$ being a subgroup. Again the naturality of homotopy shows that the maps of O_G-chain complexes χ, Ω are $\Lambda(\tau \times \rho)$-equivariantly homotopic.

□

Thus we have the following theorem.

Theorem 6.3.12. *Let $(T, M_0, \rho) \in \mathcal{A}_\Lambda$, $\Lambda = \mathbb{Z}_p$, $p > 2$ a prime. Then there exist functions*

$$\mathcal{P}^s \colon H^q_\rho(T; M_0) \to H^{q+2s(p-1)}_\rho(T; M_0),$$

$$\beta\mathcal{P}^s \colon H^q_\rho(T; M_0) \to H^{q+2s(p-1)+1}_\rho(T; M_0),$$

which satisfy the following properties.

1. *$\mathcal{P}^s$ and $\beta\mathcal{P}^s$ are natural homomorphisms.*

2. *$\mathcal{P}^s = \beta\mathcal{P}^s = 0$ if $s < 0$. Also $\mathcal{P}^s(x) = 0$ if $2s > q$, $\beta\mathcal{P}^s = 0$ if $2s \geq q$.*

3. *$\mathcal{P}^s(x) = x^p$ if $2s = q$.*

4. *(Cartan formula) For $x, y \in H^q_\rho(T; M_0)$,*

$$\mathcal{P}^s(x \cup y) = \sum_{i+j=s} \mathcal{P}^i(x) \cup \mathcal{P}^j(y),$$

$$\beta\mathcal{P}^{s+1}(x \cup y) = \sum_{i+j=s} [\beta\mathcal{P}^{i+1}(x) \cup \mathcal{P}^j(y) + (-1)^{deg(x)}\mathcal{P}^i(x) \cup \beta\mathcal{P}^{j+1}(y)].$$

5. *(Adem relation) If $a < pb$ then*

$$\beta^e\mathcal{P}^a\mathcal{P}^b = \sum_i (-1)^{a+i}(a - pi, (p-1)b - a + i - 1)\beta^e\mathcal{P}^{a+b-i}\mathcal{P}^i.$$

If $a \leq pb$ then

$$\beta^e\mathcal{P}^a\beta\mathcal{P}^b = (1-e)\sum_i (-1)^{a+i}(a - pi, (p-1)b - a + i - 1)\beta\mathcal{P}^{a+b-i}\mathcal{P}^i$$

$$-\sum_i (-1)^{a+i}(a-pi-1,(p-1)b-a+i)\beta^e\mathcal{P}^{a+b-i}\beta\mathcal{P}^i,$$

where $e=0,1$ *and* $\beta^0\mathcal{P}^s=\mathcal{P}^s$ *and* $\beta^1\mathcal{P}^s=\beta\mathcal{P}^s$.

Proof. We only need to prove that $\mathcal{P}^s=\beta\mathcal{P}^s=0$ for $s<0$. By definition of the power operations, it suffices to show that $D_i(x)=0$ for $i>pq-q$, $deg(x)=q$ (cf. Section 6.2). Recall that $\underline{\Delta}=(\epsilon\otimes id)\underline{\Phi}(id\times D)$ and

$$\underline{\Phi}(e_i\otimes D(x))\in\sum_{j<pq} W_{pq-j}\otimes[\underline{C}_*(T)]^p_j\subseteq Ker(\epsilon\otimes id) \text{ for } i>pq-q.$$

Hence $\underline{\Delta}(e_i\otimes x)=0$ for $x\in\underline{C}_{pq-i}(T)$. □

Let X be a one vertex G-Kan complex and M be an equivariant local coefficient system of Λ-algebras on X, where $\Lambda=\mathbb{Z}_p$, $p>2$ a prime. We define the Steenrod reduced power operations in the simplicial Bredon-Illman cohomology with local coefficients by

$$\mathcal{P}^s=\mu^{*-1}\mathcal{P}^s\mu^* \text{ and } \beta\mathcal{P}^s=\mu^{*-1}(\beta\mathcal{P}^s)\mu^*,$$

where the symbols $\mathcal{P}^s$ and $\beta\mathcal{P}^s$ on the right side of the above equalities denote the power operations as constructed in the category $\mathcal{A}_\Lambda$ and $\mu^*\colon H^*_G(X;M)\cong H^*_{\underline{\pi}X}(\widetilde{X};M_0)$ is the isomorphism as obtained in Theorem 5.3.4. Thus we have the following theorem.

Theorem 6.3.13. *Let X be a one vertex G-Kan complex and M be an equivariant local coefficient system of Λ-algebras on X, $\Lambda=\mathbb{Z}_p$, $p>2$ a prime. Then there exist natural homomorphisms*

$$\mathcal{P}^s\colon H^q_G(X;M)\to H^{q+2s(p-1)}_G(X;M),$$
$$\beta\mathcal{P}^s\colon H^q_G(X;M)\to H^{q+2s(p-1)+1}_G(X;M),$$

which satisfy the properties $(1)-(5)$ *of Theorem 6.3.12.*

If G is a trivial group, then $\mathcal{P}^s$ can be naturally identified with the reduced power operations in cohomology with local coefficients [Git63].

Proof. Since the isomorphism μ^* of the Eilenberg theorem, Theorem 5.3.4, is natural and respects the cup product, the first part follows from Theorem 6.3.12.

For the second part, we just remark that when G is trivial, the map

$$\underline{\Delta}\colon W\otimes\underline{C}_*(T)\to\underline{C}_*(T)^p$$

reduces to the $(\pi\times\rho)$-equivariant chain mapping $\phi'\colon W\otimes C_*(X)\to C_*(X)^p$, as constructed by Gitler in Section 4.2 of [Git63]. □

Chapter 7

Equivariant twisted Cartan cohomology theory

7.1 Introduction

In this final chapter of the thesis, we prove an equivariant version of a result of Cartan ([Car76], [McC82]) for the simplicial Bredon-Illman cohomology with local coefficients. In Section 7.2, we recall the statement of Cartan's theorem. In Section 7.3 we introduce the notion of an equivariant twisted Cartan cohomology theory and prove the main result.

7.2 Cartan cohomology theory

To generalize Sullivan's theory of rational de Rham complexes of simplicial sets [Sul77] to cochain complexes over arbitrary ring of coefficients, Cartan [Car76] introduced the notion of a 'Cohomology theory'. Let Λ be a commutative ring with identity.

Definition 7.2.1. A differential graded algebra over Λ is a graded Λ-modules $A^* = \oplus_{n\geq 0}A^n$ with an associative Λ-linear multiplication $A^n \otimes_\Lambda A^m \to A^{n+m}$ and a degree 1 Λ-linear map $\delta\colon A^* \to A^*$ such that

$$\delta^2 = 0; \quad \delta(xy) = (\delta x)y + (-1)^{deg(x)}x(\delta y).$$

Let $\mathbf{DGA}/\Lambda$ be the category whose objects are differential graded algebras over Λ, and morphisms are degree zero maps commuting with the differentials.

Definition 7.2.2. A simplicial differential graded algebra over Λ is a simplicial object in the category $\mathbf{DGA}/\Lambda$, so that for each $p \geq 0$ we have a differential graded algebra

$$(A_p^*, \delta) \colon A_p^0 \xrightarrow{\delta} A_p^1 \xrightarrow{\delta} A_p^2 \to \cdots$$

together with face and degeneracy maps $\partial_i \colon A_{p+1}^* \to A_p^*$ and $s_i \colon A_p^* \to A_{p+1}^*$ which are homomorphisms of differential graded algebras satisfying the usual simplicial and differential identities.

A map of simplicial differential graded algebras over Λ is a natural transformation of functors. The category of simplicial differential graded algebras over Λ is denoted by $\mathcal{S}\mathbf{DGA}/\Lambda$.

With these definitions, Cartan's result can be described as follows.

Definition 7.2.3. A cohomology theory in the sense of Cartan over a commutative ring Λ is a simplicial differential graded algebra A over Λ which satisfies the following conditions.

1. For each $p \geq 0$, the cochain complex (A_p^*, δ) is exact and $Z^0A = Ker(A_*^0 \xrightarrow{\delta} A_*^1)$ is a simplicially trivial algebra over Λ (A simplicial object is said to be simplicially trivial if all the face and degeneracy maps are isomorphisms).

2. The homotopy groups $\pi_i(A_*^n)$ of the simplicial set $A_*^n = \{A_p^n\}_{p \geq 0}$ are trivial for all $i, n \geq 0$.

Example 7.2.4. Let $\Lambda = \mathbb{R}$, the field of real numbers and $\Omega_p^* = \Omega^*(\Delta^p)$, the differential graded algebra of smooth differential forms on the standard Euclidean p-simplex Δ^p. Then Ω^* is a is a cohomology theory in the sense of Cartan with $(Z^0\Omega)_0 = \mathbb{R}$.

A cohomology theory A determines a contravariant functor from the category of simplicial sets to $\mathbf{DGA}/\Lambda$ which assigns to each simplicial set X the differential graded algebra $A(X) = \{Hom(X, A_*^n)\}_{n \geq 0}$, where $Hom(X, A_*^n)$ is the Λ-module of simplicial maps $X \to A_*^n$ and the differential on $A(X)$ is induced from that of A. Then Cartan's theorem can be stated as follows.

Theorem 7.2.5. *([Car76]) Let A be a cohomology theory. Then there is a natural isomorphism of graded Λ-modules*

$$H^*(A(X)) \cong H^*(X; \Lambda(A)),$$

for every simplicial set X, where $\Lambda(A)$ is the Λ-module $(Z^0A)_0$.

In [Hir79], Hirashima generalized Cartan's result for cohomology with local coefficients. Moreover, for a discrete group G, Cartan's theorem was generalized in [MN98] for G-simplicial sets, where in the equivariant setting, the ordinary cohomology of simplicial sets is replaced by the Bredon cohomology of G-simplicial sets (cf. Remark 2.5.1). Thus Cartan's theorem as generalized in [MN98] may be described as follows.

Definition 7.2.6. Let G be a discrete group. Then a G-cohomology theory over Λ is a contravariant functor $A\colon O_G \to \mathcal{S}\mathbf{DGA}/\Lambda$ such that $A(G/H)$ is a cohomology theory over Λ for each subgroup H of G.

For a G-cohomology theory A, let $A^n \in O_G\mathcal{S}$ be defined by $A^n(G/H)_q = A(G/H)^n_q$, for an object G/H of O_G and $A^n(\hat{a})_q = A(\hat{a})|_{A(G/H)^n_q}$ for a morphism $\hat{a}$ in O_G. Note that, A determines an O_G-Λ-module λ_A, defined by

$$\lambda_A(G/H) = \Lambda(A(G/H)),$$

for each object G/H of O_G and $\lambda_A(\hat{a}) = A(\hat{a})|_{Z^0(A(G/K))_0}$ for a morphism $\hat{a}\colon G/H \to G/K$ in O_G. Moreover, A determines a differential graded algebra of Λ-modules $A_G(X)$ for any G-simplicial set X, where

$$A_G(X) := \bigoplus_{n\geq 0} Hom_{O_G\mathcal{S}}(\Phi X, A^n).$$

The differential and the algebra structures on $A_G(X)$ are induced from those of the G-cohomology theory A.

Theorem 7.2.7. *[MN98] Let A be a G-cohomology theory. Then for a G-simplicial set X, there is a natural isomorphism of graded Λ-modules*

$$H^*_G(X;\lambda_A) \cong H^*(A_G(X)),$$

where $H^*_G(X;\lambda_A)$ is the Bredon cohomology of the G-simplicial set X with coefficients λ_A (cf. Remark 2.5.1).

7.3 Equivariant twisted Cartan cohomology theory

In this section, we formulate an equivariant twisted version of Cartan Cohomology theory [Car76] and prove that the simplicial Bredon-Illman cohomology with local coefficients of a G-simplicial set can be computed by the cohomology of a differential graded algebra determined by a given cohomology theory.

Let G be a discrete group. Let O_G-Λ-mod denote the category of contravariant functors from O_G to the category Λ-mod of Λ-modules and module maps. An object of O_G-Λ-mod is called an O_G-Λ-module and a simplicial object in the category O_G-Λ-mod is called a simplicial O_G-Λ-module. The category of simplicial objects in the category Λ-mod is denoted by $\mathcal{S}\Lambda$-mod.

We formulate the following equivariant generalization of Cartan Cohomology theory suitable for our purpose.

Definition 7.3.1. An equivariant twisted Cartan cohomology theory over Λ is a sequence $\mathcal{A} = \{A^i\}_{i\geq 0}$ of simplicial O_G-Λ-modules A^i, together with simplicial differentials $\delta^i \colon A^i \to A^{i+1}$ such that the following axioms are satisfied.

1. For each subgroup $H \subseteq G$, $\mathcal{A}(G/H) = (A^*(G/H)_*, \delta^*(G/H))$ is a simplicial differential graded algebra over Λ.

2. For each $p \geq 0$,
$$A^0_p \xrightarrow{\delta^0_p} A^1_p \xrightarrow{\delta^1_p} A^2_p \to \cdots$$
is an exact sequence in the abelian category of O_G-Λ-modules.

3. The O_G-group $\pi_n \circ A^i$ is the zero O_G-group, for all $n, i \geq 0$.

4. The simplicial O_G-Λ-module $Z^0\mathcal{A} = ker(A^0 \xrightarrow{\delta^0} A^1)$ is simplicially trivial.

5. For each subgroup $H \subseteq G$ and an integer $i \geq 0$ there is a group homomorphism
$$\psi^i_H \colon Aut_{\Lambda\text{-mod}}((Z^0\mathcal{A})_0(G/H)) \to Aut_{\mathcal{S}\Lambda\text{-mod}}(A^i(G/H))$$
satisfying
   - $\delta^i \circ \psi^i_H(\alpha) = \psi^{i+1}_H(\alpha) \circ \delta^i, \alpha \in Aut((Z^0\mathcal{A})_0(G/H)),\ i \geq 0.$
   - If $\alpha \in Aut_{\Lambda\text{-mod}}((Z^0\mathcal{A})_0(G/H))$ and $\beta \in Aut_{\Lambda\text{-mod}}((Z^0\mathcal{A})_0(G/K))$ be such that
   $$\alpha \circ (Z^0\mathcal{A})_0(\hat{a}) = (Z^0\mathcal{A})_0(\hat{a}) \circ \beta,\ \text{where } a^{-1}Ha \subseteq K,$$
   then the following diagram commutes.
$$\begin{tikzcd}
A^i(G/H) \arrow[r, "\psi^i_H(\alpha)"] & A^i(G/H) \\
A^i(G/K) \arrow[u, "A^i(\hat{a})"] \arrow[r, "\psi^i_K(\beta)"] & A^i(G/K) \arrow[u, "A^i(\hat{a})"']
\end{tikzcd}$$

Example 7.3.2. Let M_0 be an O_G-Λ-module. Define
$$A^n = C(M_0, n),\ n \geq 0,$$

with differentials δ^n as described in Section 4.2. Then $\mathcal{A} = \{A^n\}_{n\geq 0}$ is an equivariant twisted Cartan cohomology theory with $(Z^0\mathcal{A})_0 = M_0$.

Given an equivariant twisted Cartan Cohomology theory $\mathcal{A}$, we have simplicial O_G-Λ-modules $Z^n\mathcal{A}$, $n \geq 0$, defined by

$$Z^n\mathcal{A}(G/H) = \{Ker(\delta^n(G/H)\colon A^n(G/H) \to A^{n+1}(G/H))\},$$

for each object G/H of O_G and $Z^n\mathcal{A}(\hat{a}) = A^n(\hat{a})|_{Z^n\mathcal{A}(G/H)}$ for a morphism $\hat{a}$ in O_G.

Lemma 7.3.3. *Let $\mathcal{A}\colon A^0 \xrightarrow{\delta} A^1 \xrightarrow{\delta} \cdots$ be an equivariant twisted Cartan cohomology theory. Then each A^n is contractible as an object of $O_G\mathcal{S}$.*

Proof. For an integer $n \geq 0$ and a subgroup H of G, we have a short exact sequence

$$0 \to Z^n\mathcal{A}(G/H) \hookrightarrow A^n(G/H) \xrightarrow{\delta} Z^{n+1}\mathcal{A}(G/H) \to 0$$

of simplicial abelian groups. Therefore $A^n(G/H) \to Z^{n+1}\mathcal{A}(G/H)$ is a principal fibration with fibre $Z^n\mathcal{A}(G/H)$ in the category of simplicial sets, and hence a principal twisted cartesian product (PTCP) of type (W) with group complex $Z^n\mathcal{A}(G/H)$ (cf. Proposition 1.6.8). This PTCP of type (W) is naturally isomorphic to the universal PTCP of type (W), $W(Z^n\mathcal{A}(G/H)) \to \overline{W}(Z^n\mathcal{A}(G/H))$. But $W(Z^n\mathcal{A}(G/H))$ is contractible. The functions

$$h^H_{q-i}\colon W(Z^n\mathcal{A}(G/H))_q \to W(Z^n\mathcal{A}(G/H))_{q+1},\ 0 \leq i \leq q,\ q \geq 0,$$

$$h^H_{q-i}(x_q, \cdots, x_0) = (0^H_{q+1}, \cdots, 0^H_{i+1}, \partial_0^{q-i}x_q \cdots \partial_0 x_{i+1} \cdot x_i, x_{i-1}, \cdots x_0),$$

where $x_j \in Z^n\mathcal{A}(G/H)_j,\ 0 \leq j \leq q$ and 0^H_{q+1-r} is the zero element of the abelian group $Z^n\mathcal{A}(G/H)_{q+1-r},\ 0 \leq r \leq q-i$, define a contraction of $W(Z^n\mathcal{A}(G/H))$ which is natural with respect to morphisms in O_G. Hence $A^n(G/H)$ is also contractible and the contraction is natural. Consequently, A^n is contractible as an object of $O_G\mathcal{S}$. □

Consider an equivariant twisted Cartan cohomology theory $\mathcal{A} = \{A^i\}_{i\geq 0}$. Let M_0 denote the O_G-Λ-module $(Z^0\mathcal{A})_0$. Given a G-simplicial set X, an O_G-group $\underline{\pi}$, an O_G-twisting function $\kappa\colon \Phi X \to \underline{\pi}$ and a $\underline{\pi}$-module structure ϕ on M_0, we shall construct a differential graded algebra over Λ whose cohomology will compute the equivariant twisted cohomology of (X, ϕ, κ).

Observe that a $\underline{\pi}$-module structure ϕ on the O_G-Λ-module M_0 determines and is determined by the group homomorphisms $\phi_H\colon \underline{\pi}(G/H) \to Aut_{\Lambda\text{-mod}}(M_0(G/H))$ for each subgroup H of G, such that

$$\phi_H(\underline{\pi}(\hat{a})\gamma) \circ M_0(\hat{a}) = M_0(\hat{a}) \circ \phi_K(\gamma),$$

for a morphism $\hat{a}\colon G/H \to G/K$ in O_G. Then $(A^n, \psi^n\phi)$ is a $\underline{\pi}$-module, where $(\psi^n\phi)_H = \psi^n_H \circ \phi_H \colon \underline{\pi}(G/H) \to Aut_{\mathcal{S}\Lambda\text{-mod}}(A^n(G/H))$. To justify this, we take $\alpha = \phi_H(\underline{\pi}(\hat{a})\gamma)$ and $\beta = \phi_K(\gamma)$ in the second condition of the fifth axiom in Definition 7.3.1 and hence we obtain the relation

$$\psi^n_H \circ \phi_H(\underline{\pi}(\hat{a})\gamma) \circ A^n(\hat{a}) = A^n(\hat{a}) \circ \psi^n_K \circ \phi_K(\gamma), \text{ where } a^{-1}Ha \subseteq K.$$

Therefore, in view of the observation at the beginning of this paragraph, $(A^n, \psi^n\phi)$ is a $\underline{\pi}$-module.

Given an O_G-group $\underline{\pi}$, consider the O_G-twisting function $\kappa(\underline{\pi})\colon \overline{W}\underline{\pi} \to \underline{\pi}$, as introduced in Example 2.4.2. We form the O_G-Kan fibration $p\colon A^n \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi} \to \overline{W}\underline{\pi}$ by taking the O_G-twisted cartesian product as described in Section 2.4.

The given O_G-twisting function $\kappa\colon \Phi X \to \underline{\pi}$ determines a map of the O_G-simplicial sets $\theta(\kappa)\colon \Phi X \to \overline{W}\underline{\pi}$, defined by

$$\theta(\kappa)(G/H)_q(x) = [\kappa(G/H)(x), \kappa(G/H)(\partial_0 x), \cdots, \kappa(G/H)(\partial_0^{q-1}x)],\ x \in X^H_q.$$

We define a differential graded algebra $\mathcal{A}_\phi(X;\kappa)$ as follows.

Definition 7.3.4. Let

$$\mathcal{A}^n_\phi(X;\kappa) = \{f\colon \Phi X \to A^n \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi} \mid p \circ f = \theta(\kappa)\}.$$

This set has a Λ-module structure by fibrewise addition, scalar multiplication and the zero section. We define a differential $\overline{\delta}^n\colon \mathcal{A}^n_\phi(X;\kappa) \to \mathcal{A}^{n+1}_\phi(X;\kappa)$ by

$$(\overline{\delta}^n f)(G/H)(x) = (\delta^n(G/H)c, b),$$

where $f \in A^n_\phi(X;\kappa),\ x \in X^H,\ f(G/H)(x) = (c, b)$. Then $\mathcal{A}_\phi(X;\kappa) = \{\mathcal{A}^*_\phi(X;\kappa), \overline{\delta}\}$ is a cochain complex of Λ-modules. Furthermore, $\mathcal{A}_\phi(X;\kappa)$ admits a graded algebra structure induced from the differential graded algebra $\mathcal{A}$. The zero element of this algebra is given by the trivial lift $\mathbf{0}$, defined by

$$\mathbf{0}(G/H)_q(x) = (0^H_q, \theta(\kappa)(G/H)_q(x)),$$

where $x \in X^H_q$ and 0^H_q is the zero of the abelian group $A(G/H)_q$.

As before we use the notation $[\Phi X,\ Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}]_{\overline{W}\underline{\pi}}$ to denote the set of vertical homotopy classes of liftings of $\theta(\kappa)$.

Proposition 7.3.5. *With the above notations, we have*

$$H^n(\mathcal{A}_\phi(X;\kappa)) = [\Phi X,\ Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}]_{\overline{W}\underline{\pi}}.$$

Proof. From the definition of $\overline{\delta}$, it follows that $Ker(\overline{\delta}^n) = (\Phi X,\ Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi})_{\overline{W}\underline{\pi}}$. We now show that

$$\mathrm{Im}(\overline{\delta}^{n-1}) = \{f \in (\Phi X,\ Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi})_{\overline{W}\underline{\pi}} | f \sim_v \mathbf{0}\}.$$

Let $F\colon f \sim_v \mathbf{0}$. Consider the following left lifting problem in the closed model category $O_G\mathcal{S} \downarrow \overline{W}\underline{\pi}$ (cf. Example 1.8.6).

$$\begin{tikzcd}
\Phi X \arrow[r, "(0,\theta(\kappa)x)"] \arrow[d, "i_1"'] & A^{n-1} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi} \arrow[d, "\overline{\delta}^{n-1}"] \\
\Phi X \times \Delta[1] \arrow[r, "F"'] \arrow[ur, dotted, "\tilde{F}"] & Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}
\end{tikzcd}$$

We identify ΦX with $\Phi X \times \Delta[0]$. The canonical inclusions $\delta_0, \delta_1\colon \Delta[0] \to \Delta[1]$ induce natural inclusions $i_0, i_1\colon \Phi X \to \Phi X \times \Delta[1]$ where we identify $X \times \Delta[0]$ with X. Note that i_1 is a trivial cofibration and $\overline{\delta}^{n-1}$ is a fibration in $O_G\mathcal{S} \downarrow \overline{W}\underline{\pi}$. Hence the above left lifting problem has a solution $\tilde{F}$. Then $\tilde{F}i_0 \in A_\phi^{n-1}(X;\kappa)$ such that $\overline{\delta}^{n-1}(\tilde{F}i_0) = f$. Therefore $f \in \mathrm{Im}(\overline{\delta}^{n-1})$.

On the other hand, suppose that $f = \overline{\delta}^{n-1}h$, where $f \in \mathcal{A}_\phi^n(X;\kappa)$ and $h \in \mathcal{A}_\phi^{n-1}(X;\kappa)$. Then clearly $f \in (\Phi X,\ Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi})_{\overline{W}\underline{\pi}}$. Composing h with first factor projection map, we get a map $h'\colon \Phi X \to A^{n-1}$ of $O_G\mathcal{S}$. But by Lemma 7.3.3 A^{n-1} is contractible. Let $\mathcal{H}\colon \Phi X \times \Delta[1] \to A^{n-1}$ be a contracting homotopy for the O_G-simplicial set A^{n-1}. Then define

$$\tilde{\mathcal{H}}\colon \Phi X \times \Delta[1] \to \mathcal{A}_\phi^{n-1}(X;\kappa)$$

by $\tilde{\mathcal{H}}(x,t) = (\mathcal{H}(x,t), \theta(\kappa)x)$. Clearly $\tilde{\mathcal{H}}\colon h \sim_v \mathbf{0}$ in $O_G\mathcal{S} \downarrow \overline{W}\underline{\pi}$. Hence $\overline{\delta}^{n-1} \circ \tilde{\mathcal{H}}\colon f \sim_v \mathbf{0}$. This proves the proposition for $n > 0$.

For $n = 0$, we note that $H^0(\mathcal{A}_\phi(X;\kappa)) = (\Phi X,\ Z^0\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi})_{\overline{W}\underline{\pi}}$ and two elements in the right hand side are homotopic if and only of they are equal. □

Observe that the fourth axiom of Definition 7.3.1 implies that $Z^0\mathcal{A}$ is an O_G-Eilenberg-MacLane complex of type $(M_0, 0)$ and hence by induction $Z^n\mathcal{A}$ is an O_G-Eilenberg-MacLane complex of type (M_0, n). To justify this, consider the fibration

$$A^n(G/H) \to Z^{n+1}\mathcal{A}(G/H)$$

with fiber $Z^n\mathcal{A}(G/H)$, $H \leq G$. As noted in Lemma 7.3.3, this is a PTCP with fibre $Z^n\mathcal{A}(G/H)$. Therefore, if $Z^n\mathcal{A}(G/H)$ is minimal then the above fibration is a minimal fibre space by Lemma 1.6.6 and so $Z^{n+1}\mathcal{A}(G/H)$ is a minimal complex. But $Z^0\mathcal{A}(G/H)$, being simplicially trivial, is minimal. Hence by induction on n, it follows that $Z^n\mathcal{A}(G/H)$ is minimal for all n.

Now applying the homotopy long exact sequence to the above fibration, we get the following long exact sequence.

$$\to \pi_i(Z^n\mathcal{A}(G/H)) \to \pi_i(A^n(G/H)) \to \pi_i(Z^{n+1}\mathcal{A}(G/H)) \to \pi_{i-1}(Z^n\mathcal{A}(G/H)) \to .$$

In view of the third axiom of Definition 7.3.1, we see that if $Z^n\mathcal{A}$ is an O_G-Eilenberg-MacLane complex of type (M_0, n), then $Z^{n+1}\mathcal{A}$ is an O_G-Eilenberg-MacLane complex of type $(M_0, n+1)$. But we have already observed that $Z^0\mathcal{A}$ is an O_G-Eilenberg-MacLane complex of type $(M_0, 0)$. Therefore by induction on n, it follows that $Z^n\mathcal{A}$ is an O_G-Eilenberg-MacLane complex of type (M_0, n) and hence it is isomorphic to the canonical model of $K(M_0, n)$ by Proposition 2.3.11.

As a result $(Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}, p)$ is isomorphic to $(L_\phi(M_0, n), p)$ as objects in the slice category $O_G\mathcal{S} \downarrow \overline{W}\underline{\pi}$. So we have,

$$\begin{aligned} H^n(\mathcal{A}_\phi(X;\kappa)) &= [\Phi X,\ Z^n\mathcal{A} \times_{\kappa(\underline{\pi})} \overline{W}\underline{\pi}]_{\overline{W}\underline{\pi}} \\ &\cong [\Phi X,\ L_\phi(M_0, n)]_{\overline{W}\underline{\pi}}. \end{aligned}$$

It follows from Theorem 4.3.6 that

$$H^n(\mathcal{A}_\phi(X;\kappa)) \cong H^n_G(X;\phi,\kappa).$$

Thus we have proved the following theorem.

Theorem 7.3.6. *Suppose $\mathcal{A}$ is an equivariant twisted Cartan cohomology theory. Then for every G-simplicial set X together with an O_G-group $\underline{\pi}$, an O_G-twisting function $\kappa\colon \Phi X \to \underline{\pi}$ and an action ϕ of $\underline{\pi}$ on the abelian O_G-group $(Z^0\mathcal{A})_0$ there is a natural isomorphism of graded Λ-modules*

$$H^*_G(X;\kappa,\phi) \cong H^*(\mathcal{A}_\phi(X;\kappa)),$$

where $\mathcal{A}_\phi(X;\kappa)$ is the differential graded algebra as defined in Definition 7.3.4.

Combining Theorem 3.4.9 with Theorem 7.3.6 we have the following result.

Theorem 7.3.7. *Suppose $\mathcal{A}$ is an equivariant twisted Cartan cohomology theory. Given any G-connected G-simplicial set X with a G-fixed 0-simplex and an action ϕ of $\underline{\pi}X$ on $(Z^0\mathcal{A})_0$, let M be the equivariant local coefficient system of Λ-modules determined by the*

$\underline{\pi}X$-module $(Z^0\mathcal{A})_0$ on X. Then there is a natural isomorphism of graded Λ-modules

$$H^*_G(X;M) \cong H^*(\mathcal{A}_\phi(X;\kappa)),$$

where $\mathcal{A}_\phi(X;\kappa)$ is the differential graded algebra as defined in Definition 7.3.4.

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