PRICE RIGIDITIES, RATIONING SCHEMES AND ECONOMIC EFFICIENCY

By

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A thesis submitted to the Indian Statistical Institute as a partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy

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ACKNOWLEDGEMENTS

I thank Professors V.K.Chetty and D.Dasgupta for their supervision and helpful comments in preparation of this thesis. I also wish to acknowledge the benefit I received from their courses and seminars on equilibrium theory.

I thank the present and past Deans of Studies, Professors K.R.Parthasarathy and T.N.Srinivasan for encoursement received.

Finally, I thank Shri Mehar Lal and Shri V.P.Sharma for typing and Shri Surje for cyclostyling this thesis.

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1. Introduction

The prime concern of economists traditionally has been the system of markets and their role in resource allocation. Walrasian equilibrium theory, which is believed to provide an adequate framework for resource allocation through markets, has been extensively analysed in the recent past. At the same time, it has been realised, mainly through Keynesian influence, that the market mechanism is prone to failures. Downward inflexibility of wages is an obvious example of this. It is further possible that a central authority concerned with improving general welfare may tamper with the market mechanism or do away with it altogether.

There are three types of non-market distribution schemes which require varying degrees of intervention. The first one is signal mechanisms, like taxation or rationing. In taxation, taxes are used as signals. In quantity rationing, maximum tradeable or consumable quantities are used as signals. Under points rationing, prices in the different rationing currencies are used as signals. The second type of distribution schemes involves arbitration schemes like Shapley value described in [14] or a minimax rule described in [3] for coalition production economies. Third type involves lump sum transfer of resources.

The third type scheme is possible only under exceptional circumstances. The second type has not so far been applied on a large scale. Signal mechanisms like taxation and rationing have been tried on a large scale and deserve to be analysed in a general framework. In this study we analyse rationing with the help of such a framework.

The decision problem of a single consumer faced with the rationing restraints has been analysed by Samuelson [15] and Tobin [16]. Some existence theorems involving quantity restrictions have recently been proved by Drèze [6] and this study proceeds in a similar manner. A few dynamic macro-economic models can be found in the works of Barro and Grossman [2, 8].

An important case of market failure is when a price system prevails in the economy at which the supply does not equal demand. If this disequilibrium situation is more than just temporary, it becomes imperative to look for solution concepts to explain this phenomenon and possibly to help solve the problem of allocation. Dreze imposes quantity restrictions to suppress the imbalance between supply and demand. When the imbalance between supply and demand at given prices is not contrived this model provides a framework for Keynesian type of price rigidities. When the situation is contrived i.e. when the prices are purposely fixed, the same model is a suitable tool to analyse rationing.

There are instances in the real world when a commodity in excess demand gets rationed on a voluntary basis. Queues are voluntarily formed for a scarce commodity like space in a bus.

Retailers of a scarce good like cinema tickets often restrict the amount they sell per head. This kind of decentralized rationing, however, may not be widespread. Even if it is, the restrictions are not likely to be uniform over time and space. In what follows, we assume that there is a central authority which is responsible for issuing the quantity restrictions and other signals. The members of the economy are supposed to abide by these signals.

The first basic theorem in Drèze [6] is the basis for his subsequent theorems and for a portion of this study also. Here we attempt to describe it in words.

The price vectors for the n commodities are constrained to lie in a compact set. This set may turn out to be a single point also. The exchange economy consists of N consumers. The existence question is framed the following way: Is there a price vector p belonging to the given compact set P and two vectors of quantity restrictions (one denotes the maximum buying and the other maximum selling allowed) such that the following holds:

(i) Restricted demand equals restricted supply.

(ii) Buying restriction on a commodity is binding for some consumer implies that selling restriction on that commodity is not binding for any consumer and vice versa.

(iii) Restriction on buying (selling) is not binding for a commodity for any consumer unless upward (downward) movement of price of that commodity in the price set is not possible.

It can be seen that if either all buying or all selling is prohibited, then the restricted demand of each consumer is his own initial endowments. Drèze rules out this trivial equilibrium by allowing for a numeraire commodity for which no restriction can be binding on any consumer. Also, if both buying and selling are ruled out for all but one commodity, then again the same trivial equilibrium will result. Condition (ii) however, rules this out. Condition (iii) incorporates the admirable principle that prohibition of a consumer's desire by issuing quantity restrictions is used only when it is absolutely necessary, that is, when the imbalance cannot be absorbed by price movement.

Drèze shows the existence of such an equilibrium as described above. The scope of the present study is to look into different equilibrium concepts with quantity rationing for a production economy, to enquire into the efficiency aspects of equilibria with quantity restrictions and explore the alternatives to Drèze's solution concept.

In chapter 2, we show the existence of an equilibrium similar to the one in Drèze [6] with equal quantity restrictions for all the members in a production economy. Using quantity restrictions that vary with size, we define and show the existence of a few equilibrium concepts with proportional rationing.

In chapter 3, we enquire into the efficiency properties of rationing equilibria. There we find that Drèze equilibrium is in general Pareto inefficient. The implication of this to the problem of distribution as presented there and to the theory of quantity rationing are discussed in detail .

2. Production and Rationing Equilibria

2.1 Introductory remarks

As we have discussed in the introduction there exists an equilibrium with equal quantity restrictions for a price inflexible exchange economy. In this chapter we examine such equilibrium concepts for production economies. Certain aspects connected with production make our task of finding an interesting solution concept difficult.

Consider for example, an economy with some consumers, some producers and a few consumer-cum-producers. It is well known [12] that if there are no restrictions, decision making in the consumption and production activities can be considered separately. Such a facility, however, does not exist if a consumer-cum-producer is provided with a single set of quantity restrictions and providing separate restrictions for consumption and production to a single member is self-defeating.

Further, there are good reasons to treat producers and consumers differently. There is more uniformity among consumers than among producers. Consumers are treated uniformly as regards food rations and this is desirable on grounds of equity. It does not follow, however, on the same grounds that a big corporation with thousands of employees should be given the same quota of inputs as a small unit. It may make more sense to treat different production units differently allowing for size differences.

For reasons mentioned earlier, our model requires that production and consumption decisions are taken separately. In sections 2.2, 2.3 and 2.4, we consider a production economy similar to the one in [4] and [1]. We present different solution concepts for this economy and in each case prove the existence of a solution. In section 2.2, we use quantity restrictions that are equal for all members of the economy. In section 2.3, we use restrictions that are proportional to their unrestricted demands. In section 2.4, we use such proportional restrictions for the producers and equal restrictions for the consumers.

In section 2.5, we discuss the suitability of each of these solution concepts in the light of size difference between firms and the requirements of enforceability.

2.2 Production with equal quantity restrictions

Let us consider an Arrow-Debreu type of set up of a production economy with a finite number of profit maximizing firms and a finite number of consumers who own shares in these firms[1/]. Suppose that the prices are rigid and do not clear the markets by themselves. In choosing quantity restrictions to equate demand and supply, we come across a problem which does not arise in an exchange economy: The sizes of the constituents

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1/ There are a number of conceptual problems in interpreting the role of shares in a static model without uncertainty. In this work, we follow the large body of literature and ignore these problems.

may vary to a great extent. Production units range from cottage industries that employ a small number of workers to giant corporations employing thousands of workers. Consumers are in general treated alike except when it is possible to demonstrate the need for more rations for some one. Manual labourers in U.S.S.R. get more food rations than others. In production, however, it is common that the larger establishments get higher input quotas, whichever way the largeness is measured, in terms of output, capital or number of employees.

Still, there is something to be said in favour of equal quantity restrictions in production. With such restrictions, larger units are definitely worse off than their smaller counterparts. Smaller units are often favoured more in order to check concentration of economic power. Equal quantity restrictions may work against conglomeration of industries.

Further, if we do not require the production sets to be strictly convex, we shall be allowing production under constant returns to scale. It is well known that the size of a firm is indeterminate under constant returns to scale and under those conditions, equal quantity restrictions seem to be the only possibility.

In this section, we prove the existence of an equilibrium with equal quantity restrictions in a general production economy. The prices are constrained to belong to a compact set. The definition of the equilibrium and the proof is very much in the spirit of [6] which deals with an exchange economy.

The Model

The economy has N consumers, M producers and n commodities. Consumer $i \in I = \{1,2,\ldots,N\}$ is characterised by $(X_i, \succsim_i, w^i)$ representing his consumption set, preference ordering and initial endowments respectively. A producer $j \in J = \{1,2,\ldots,M\}$ is characterised by his production set Y_j. Further, a consumer i gets a fraction d_{ij} of the profits of producer j. $d_{ij} \geqq 0$ and $\sum_i d_{ij} = 1$. A commodity k belongs to the set $K = \{1,2,\ldots,n\}$. We make the following assumptions:-

Assumption (a): The consumption set $X_i \subset R^n_+$, with elements x^i, is closed, convex and satisfies $x^i \in X_i \Longrightarrow x^i + R^n_+ \subset X_i$.

Assumption (b): The preference ordering $\succsim_i$ on X_i is complete, continuous and convex. $x \geqq y \Longrightarrow x \succsim_i y$; there is an index set $A \subset K$ such that $x \geqq y$ and $x_k > y_k \; \forall k \in A \Longrightarrow x \succ_i y$.

Assumption (c): The initial endowments w^i belong to the interior of X_i.

Assumption (d): The production set $Y_j \subset R^n$, with elements y^j whose positive components denote outputs and negative components denote inputs, is closed, convex and contains 0.

Assumption (e): Let $\mathcal{Y} = \underset{j}{X} Y_j = \{(y^1, y^2, \ldots, y^M) \mid y^j \in Y_j\}$. If $y = (y^1, y^2, \ldots, y^M) \in \mathcal{Y}$ and $\sum_j y^j \geqq 0$, then $y = 0$.

Assumption (f): There is a $\bar{y} = \sum_j y^j$ for some $y^j \in Y_j$ such that $\bar{y}_k + w_k > 0 \; \forall k \in K$, where $w = \sum_i w^i$.

Let $\hat{\mathcal{Y}} = \{y \in \mathcal{Y} \mid \sum_j y^j + w \geqq 0\}$ be the set of feasible production allocations. The assumptions on producers is enough to show that $\hat{\mathcal{Y}}$ is compact and convex. A proof is given in [1].

A price system p is a vector in R^n_+. The price constraints require that p should belong to a set $P = \{p \in R^n_+, p_1 = 1 \mid \bar{p} \geqq p \geqq \underline{p}\}$. A rationing scheme is a pair of vectors $(L, \ell) \in R^n_+ \times R^n_-$. Given a price $p \in P$ and a rationing scheme (L, ℓ), the trading set of producer j is defined by

$$\lambda^j(p,L,\ell) = \{y \in Y_j \mid L \geqq -y \geqq \ell\} .$$

$$\lambda^j(p,L,\ell) \neq \phi \text{ since } 0 \in Y_j .$$

The profit function of producer j is defined by

$$\pi^j(p,L,\ell) = \max \{p\, y^j \mid y^j \in \lambda^j(p,L,\ell)\}.$$

The budget set of consumer i is defined by

$$\gamma^i(p,L,\ell) = \{x \in X^i \mid p(x-w^i) + \sum_j d_{ij}\, \pi^j(p,L,\ell) \leqq 0,\ L \geqq x-w^i \geqq \ell\}.$$

Let $\xi^i(p,L,\ell) = \{x \in \gamma^i(p,L,\ell) \mid x \succsim_i x' \ \forall x' \in \gamma^i(p,L,\ell)\}$ and

$$\hat{\xi}^i(p,L,\ell) = \{x \in \xi^i(p,L,\ell) \mid p(x-w^i) + \sum_j d_{ij}\, \pi^j(p,L,\ell) = 0\}.$$

$\hat{\xi}^i(p,L,\ell)$ we call the demand correspondence of consumer i. Let

$$\psi^j(p,L,\ell) = \{y \in \lambda^j(p,L,\ell) \mid py \geqq py' \text{ for all } y' \in \lambda^j(p,L,\ell)\}.$$

ψ^j we call the demand correspondence of producer j.

Definition: An equilibrium with price rigidities and rationing is defined as a price vector $p \in P$, and N-tuple of consumption vectors $\{x^i\}$, and an M-tuple of production vectors $\{y^j\}$ and a rationing scheme (L, ℓ) such that

(i) For all i, x^i maximizes $\succ_i$ over $Y^i(p,L,\ell)$.

(ii) For all j, $y^j \in \pi^j(p,L,\ell)$.

(iii) $\sum_i (x^i - w^i) - \sum_j y^j = 0$.

(iv) $\forall k \in K$, $L_k = x_k^i - w_k^i$ for some $i \in I$

or $L_k = -y_k^j$ for some $j \in J$

$\Longrightarrow$ $x_k^h - w_k^h > \ell_k \ \forall h \in I$

and $-y_k^h > \ell_k \ \forall h \in J$

Also, $\ell_k = x_k^i - w_k^i$ for some $i \in I$

or $\ell_k = -y_k^j$ for some $j \in J$

$\Longrightarrow$ $L_k > x_k^h - w_k^h \ \forall h \in I$

and $L_k > -y_k^h \ \forall h \in J$

(v) $\forall k \in K$, $\bar{p}_k > p_k \Longrightarrow L_k > x_k^i - w_k^i \ \forall i \in I$ and $L_k > -y_k^j \ \forall j \in J$.

$p_k > \underline{p}_k \Longrightarrow \ell_k < x_k^i - w_k^i \ \forall i \in I$ and $\ell_k < -y_k^j \ \forall j \in J$.

In descriptive terms, condition (iv) states that buying restriction for some commodity is binding for some agent implies selling restriction is not binding for any agent and vice versa. Condition (v) states that for any commodity, buying (selling) restriction cannot be binding unless upward (downward) movement of prices within the set P is not possible. These two conditions have already been discussed in the introduction.

We use only such quantity restrictions that satisfy $v_k \geq L_k \geq 0 \geq \ell_k \geq -v_k$. This v_k we shall define later such that the following holds: For any trade that is feasible (total excess demand is non-positive), the quantity bought and sold of commodity k by any agent is less than v_k. Such a choice of v_k is possible since the set of feasible allocations is compact. With this proviso, $\lambda^j(p,L,\ell)$ and $\gamma^i(p,L,\ell)$ are compact valued and their values are in a compact space. A few continuity properties are proved below.

Lemma 2.2.1

The correspondence λ^j is continuous in (p,L,ℓ).

Proof: Note that

$$\lambda^j(p,L,\ell) = Y_j \cap \{x \in R^n \mid -x \leqq L\} \cap \{x \in R^n \mid -x \geqq \ell\}$$

Since all the three terms are upper hemi-continuous (u.h.c.) in (p,L,ℓ), $\lambda^j(p,L,\ell)$ is u.h.c. in (p,L,ℓ). Now we shall show that $\lambda^j(p,L,\ell)$ is lower hemi-continuous (l.h.c.). Let $y \in \lambda^j(p,L,\ell)$. Let $K_1 = \{k \in K \mid y_k = 0\}$ and $K_2 = \{k \in K \mid y_k \neq 0\}$. Assume without loss of generality, $K_1 = (1,2,\ldots,m)$; $K_2 = (m+1,\ldots,n)$. We partition L, ℓ and y on same lines and obtain

$$L = (L', L''); \quad \ell = (\ell', \ell'') \text{ and } y = (0, y'').$$

Now we have $L_k > \ell_k$ for $k \in K_2$. Choose μ_1 such that $0 < \mu_1 < 1$. It is true that $L_k > -\mu_1 y_k > \ell_k$ for all $k \in K_2$ because

$$-y_k > 0 \Longrightarrow -y_k > \ell_k \Longrightarrow -\mu_1 y_k > \ell_k$$

$$\Longrightarrow -y_k \leqq L_k \Longrightarrow -\mu_1 y_k < L_k$$

also, $-y_k < 0 \Longrightarrow -y_k < L_k \Longrightarrow -\mu_1 y_k < L_k$

$$\Longrightarrow -y_k \geqq \ell_k \Longrightarrow -\mu_1 y_k > \ell_k.$$

Consider $(p^s, L^s, \ell^s) \to (p,L,\ell)$. There is positive integer s_1 such that for all $s \geqq s_1$, it holds that

$$L_k^s > \mu_1 y_k > \ell_k^s \ \forall k \in K_2.$$

Also, $\mu_1 y = (0, \mu_1 y'') \in Y_j$ since $\mu_1 y$ is the convex combination of 0 and y and Y_j is convex. So, $\mu_1 y \in \lambda^j(p^{s_1}, L^{s_1}, \ell^{s_1})$. Choose μ_n such that $0 < \mu_n < 1$, $\mu_n \to 1$ and $\mu_n > \mu_{n-1} \ldots > \mu_1$. Corresponding to μ_n, there is a positive integer $s_n > s_{n-1}$ such that

$$\mu_n y \in \lambda^j(p^{s_n}, L^{s_n}, \ell^{s_n}) \Longrightarrow \lambda^j(p,L,\ell) \subset Ls\ \lambda^j(p^s, L^s, \ell^s)$$ 2/

Lemma 2.2.2

π^j is a continuous function in (p,L,ℓ).

Proof: Suppose not. Then there is at least one sequence $(p^s, L^s, \ell^s) \to (p,L,\ell)$ and $\pi^j(p^s, L^s, \ell^s) \to \pi^1 \neq \pi^j(p,L,\ell)$. Suppose $\pi^1 > \pi^j(p,L,\ell)$. Since λ^j is compact valued there is a sequence y^s such that $p^s y^s \to \pi^1$ with $y^s \in \lambda^j(p^s, L^s, \ell^s)$. Since λ^j is u.h.c. in (p,L,ℓ), there is a subsequence of y^s, still denoted y^s, such that $y^s \to y \in \lambda^j(p,L,\ell)$. Since $p^s \to p$ and $y^s \to y$; $p^s y^s \to py \Longrightarrow py = \pi^1 > \pi^j(p,L,\ell)$. This contradicts that y is in $\lambda^j(p,L,\ell)$. So, suppose $\pi^1 < \pi^j(p,L,\ell)$. Let $y \in \lambda^j(p,L,\ell)$ and $py = \pi^j(p,L,\ell)$. Because $\lambda^j(p,L,\ell)$ is l.h.c. in (p,L,ℓ), there is a sequence $y^s \in \lambda^j(p^s, L^s, \ell^s)$ and $y^s \to y$. Now $p^s y^s \to py > \pi^1$. This implies that there is an $\epsilon > 0$ and $\bar{s}$ such that for all $s \geq \bar{s}$, $p^s y^s > \pi^1 + \epsilon$, which contradicts that $\pi^j(p^s, L^s, \ell^s) \to \pi^1$.

Lemma 2.2.3:

The correspondence $Y^i(p,L,\ell)$ from $R_+^n \times R_+^n \times R_-^n$ to X_i is continuous at every point (p,L,ℓ) where, for some k, $p_k > 0$ and $\ell_k < 0$.

2/ The notation Ls denotes the topological Limes superior of a sequence of sets. For a characterization of l.h.c. in terms of Ls, see Part I, BIII Theorem 2 in Hildenbrand [11].

Proof: Let $\alpha^i(p,L,\ell) = \{x \in X_i \mid p(x-w^i) - \sum_j d_{ij}\, \pi^j(p,L,\ell) \leq 0\}$ and $\beta^i(p,L,\ell) = \{x \in X_i \mid L \cdot (x-w^i) \geqq \ell\}$. Consider $(p^s,L^s,\ell^s) \to (p,L,\ell)$, let $x^s \in \alpha^i(p^s,L^s,\ell^s)$ and $x^s \to x$. We claim that $x \in \alpha^i(p,L,\ell)$. If not, $px > pw^i + \sum_j d_{ij}\, \pi^j(p,L,\ell)$. Or,

$$(p-p^s)\, x + p^s x > (p-p^s)\, w^i + p^s w^i + \sum_j d_{ij}\, \pi^j(p,L,\ell).$$

Let the difference be C. There is a positive integer s_1 such that for all $s \geq s_1$,

$$p^s x - p^s w^i - \sum_j d_{ij}\, \pi^j(p,L,\ell) > C/2,$$

since $p^s \to p$. This implies that there is an $s_2 > s_1$ such that for all $s > s_2$,

$$p^s x - p^s w^i - \sum_j d_{ij}\, \pi^j(p^s,L^s,\ell^s) > C/4$$

since π^j is continuous. This implies that there is a neighbourhood $N(x)$ of x and an $s_3 > s_2$ such that, for all $x' \in N(x)$,

$$p^s x' - p^s w^i - \sum_j d_{ij}\, \pi^j(p^s,L^s,\ell^s) > C/8.$$

Since $x^s \to x$, $x^s \in N(x)$ for large enough s and the last expression contradicts that $x^s \in \alpha^i(p^s,L^s,\ell^s)$. This proves that $\alpha^i(p,L,\ell)$ is a closed correspondence.

It has already been shown in [6] that $\beta^i(p,L,\ell)$ is a continuous correspondence. Now, $\gamma^i(p,L,\ell) = \alpha^i(p,L,\ell) \cap \beta^i(p,L,\ell)$, where α^i is closed and β^i is u.h.c. and compact valued. So, we obtain[3/] that

$\gamma^i(p,L,\ell)$ is u.h.c.

Consider $\alpha^{0i}(p,L,\ell) = \{x \in X^i \mid px < pw^i + \sum_j d_{ij}\, \pi^j(p,L,\ell)\}$. Since $w^i \in \alpha^i(p,L,\ell)$ and $w^i_k > 0\ \forall k$, $\alpha^{0i}(p,L,\ell)$ is non empty.

3/ Follows from Part I B.III proposition 2(b) in Hildenbrand [11].

Let $x \in \alpha^{0i}(p,L,\ell)$. Now, $(p^s,L^s,\ell^s) \to (p,L,\ell)$ implies $p^s x < p^s w^i + \sum_j d_{ij} \pi^j(p^s,L^s,\ell^s)$ for s large enough. This implies that $\alpha^{0i}(p,L,\ell)$ is l.h.c. in (p,L,ℓ). Since $\alpha^i(p,L,\ell) = Cl.\ \alpha^{0i}(p,L,\ell)$, where Cl. denotes topological closure, we obtain[4] that $\alpha^i(p,L,\ell)$ is l.h.c. in (p,L,ℓ).

To prove that $\gamma^i(p,L,\ell)$ is l.h.c., let $(p^s,L^s,\ell^s) \to (p,L,\ell)$ $p_k > 0$, $\ell_k < 0$. Let $x \in \gamma^i(p,L,\ell)$. Because β^i is l.h.c. , there is a sequence $\overline{x}^s \to x$ such that

$$\overline{x}^s \in \beta^i(p^s,\ L^s,\ \ell^s).$$

If $px < pw^i + \sum_j d_{ij} \pi^j(p,L,\ell)$, then there is s' such that for all $s \geq s'$, we have:

$$p\ \overline{x}^s < p^s w^i + \sum_j d_{ij}\ \pi^j(p^s,L^s,\ell^s)$$

and hence $\overline{x}^s \in \alpha^i(p^s,L^s,\ell^s)$ and γ^i is l.h.c. So, let $px = pw^i + \sum_j d_{ij}\ \pi^j(p,L,\ell)$,. As $p_k > 0$ and $\ell_k < 0$, there is a x^o such that $x^o \in \gamma^i(p,L,\ell)$, $px^o < pw^i + \sum_j d_{ij}\ \pi^j(p,L,\ell)$ and $p^s x^o < pw^i + \sum_j d_{ij}\ \pi^j(p,L,\ell)$ for $s > s'$. If $\overline{x}^s \notin \alpha^i(p^s,L^s,\ell^s)$, define λ^s by

$$\lambda^s\ p\ \overline{x}^s + (1-\lambda^s)\ p\ x^o = p^s w^i + \sum_j d_{ij}\ \pi^j(p^s,\ L^s,\ell^s).$$

For $s > s'$, define a sequence x^s by

$$x^s = \lambda\overline{x}^s + (1-\lambda^s)\ x^o \quad \text{if } \overline{x}^s \notin \alpha^i(p^s,L^s,\ell^s)$$

$$= \overline{x}^s, \quad \text{otherwise.}$$

When $(p^s,L^s,\ell^s) \to (p,L,\ell)$,

$$p^s\ \overline{x}^s \to px = pw^i + \sum_j d_{ij}\ \pi^j(p,L,\ell)$$

$$p^s\ x^o \to px^o < pw^i + \sum_j d_{ij}\ \pi^j(p,L,\ell).$$

4/ Follows from Part I, BIII Definition 3, Example 4, in Hildenbrand [11].

So if there are infinite s such that $\bar{x}^s \notin \alpha^i(p^s, L^s, \ell^s)$, then $p^s x^s = p^s w^i + \sum_j d_{ij} \pi^j(p^s, L^s, \ell^s)$ for these s implying $\lambda^s \to 1$. So, $x^s \to x$. It is seen that $x^s \in \gamma^i(p^s, L^s, \ell^s)$. So, γ^i is l.h.c. when $p_k > 0$ and $\ell_k < 0$.

Q.E.D.

Theorem 2.2.1

For a production economy satisfying assumptions (a) to (f) where the consumers satisfy the assumption (b) with index set $A = \{1\}$ and $P = \{p \in R^n \mid p_1 = 1, \; +\infty > \bar{p} \geq p \geq \underline{p} \geq 0\}$, there exists an equilibrium with price rigidities and rationing.

Proof Let $v_k = \max_{\hat{y}} \{ \max (\sum_j y_k^{j+}, \sum_j y_k^{j-})\} + w_j$, $\hat{y}$ is the feasible set and $(y^1, \ldots, y^N) \in \hat{y}$ and

$$y^{j+} = \max (0, y_k^j)$$

$$y^{j-} = \max (0, -y_k^j).$$

Let $M_k = \bar{p}_k + v_k$, $m_k = \underline{p}_k - v_k$.

and $Q = \{q \in R^n \mid M \geq q \geq m\}$.

Q is compact and convex. Define $p(q)$, $L(q)$, $\ell(q)$ by

$$p_k(q) = \min (\bar{p}_k, \max (q_k, \underline{p}_k)) = \max (\underline{p}_k, \min (q_k, \bar{p}_k)).$$

$$L_k(q) = M_k - \max (\bar{p}_k, q_k).$$

$$\ell_k(q) = m_k - \min (\underline{p}_k, q_k).$$

Define $\gamma^i(q)$, $\lambda^j(q)$, $\xi^i(q)$, $\psi^j(q)$ appropriately.

Let $x^i \in \gamma^i(q)$. Then

$$q + x^i - w^i \leq q + L(q) = q + M - \max(\bar{p}, q) \leq M.$$

$$q + x^i - w^i \geq q + \ell(q) = q + m - \min(q, \underline{p}) \geq m.$$

Hence $(q + x^i - w^i) \in Q$.

Let $y^j \in \lambda^j(q)$

$q-y^j \leq q+L(q) \leq M.$

$q-y^j \geq q+\ell(q) \geq m.$

Hence $q-y^j \in Q.$

Let $Q_1 = \{q \mid q \in Q \text{ and } q_1 = 1\}$

$Q' = \{{}_1q = (q_2, q_3, \ldots, q_n) \mid q \in Q_1\}$;

$\xi^i(q)$ is non-empty, compact valued and convex since $\gamma^i(q)$ is continuous and compact. It is easily seen that $\hat{\xi}^i(q)$ is also closed and hence u.h.c. $\Psi^j(q)$ is also similarly convex, compact valued and u.h.c. Q' is compact and convex since Q_1 is compact and convex.

Define $h(q) = \{Z \mid Z \in R^n \text{ and } Z + w \in \sum_i \hat{\xi}^i(q) - \sum_j \Psi^j(q)\}$. This is the aggregate excess demand correspondance from Q_1 to R^n. Let ${}_1Z \overset{=}{\text{def}} (Z_2, \ldots, Z_n)$.

Define $\mu({}_1q) = \{{}_1q' \in R^{n-1} \mid {}_1q' = {}_1q + \frac{{}_1Z}{N+M} : Z \in h(q)\}$

$\mu({}_1q)$ is convex and compact because $h(q)$ is convex and compact. μ is u.h.c. because $h(q)$ is u.h.c. Since $Z({}_1q) = \sum_{i=1}^{N}(x_i - w_i) + \sum_{j=1}^{M} -y_j$ for some $x_i \in \gamma^i(q)$ and some $y_j \in \lambda^j(q)$, since each term $(x_i-w_i)+q$ and $-y_j+q$ for all i and j belongs to Q, and since Q is convex, taking convex combination with weights $\frac{1}{N+M}$, we get

$$q + \frac{Z(q)}{N+M} \subset Q \quad \text{and} \quad {}_1q + \frac{{}_1Z(q)}{N+M} \in Q' \implies \mu({}_1q) \subset Q'$$

Since all the conditions of Kakutani's fixed point theorem are satisfied, there is a fixed point for μ .

$\exists \; {}_1q^*$ such that $\mu({}_1q^*) = 0.$

$\implies {}_1Z({}_1q^*) = 0.$

By Walras' Law, $p(q^*) Z = 0$, where $q^* = (1, {}_1q^*)$.

Since $p_1 = 1$, $Z_1 = 0$ or $Z(q^*) = 0$.

Let $p^* = p(q^*)$; $L^* = L(q^*)$ and $\ell^* = \ell(q^*)$.

At (p^*, L^*, ℓ^*) and corresponding allocations that give $Z(q^*)$, conditions (i), (ii) and (iii) hold. It remains to check (iv) and (v). To check (v):

$v_k > w_k$ by assumption (f). Let $\bar{p}_k > p_k$. This implies

$$L_k = M_k - \bar{p}_k = v_k. \quad v_k - x_k^i + w_k^i \geq \sum_h x_k^h - x_k^i + w_k^i$$

$$> \sum_h x_k^h - x_k^i \text{ ; since } w_k^i > 0.$$

$$\geq 0$$

Also $v_k + y_k^j > 0$, by definition of v_k. So

$$\bar{p}_k > p_k \implies L_k > x_k^i - w_k^i \quad \text{all } i$$

$$L_k > -y_k^j \quad \text{all } j.$$

The reasoning is similar in the case $p_k > \bar{p}_k$. Condition (v) is alternatively stated as follows:

$L_k = x_k^i - w_k^i$ for some i

or $L_k = -y_k^j$ for some j

$\implies p_k = \bar{p}_k$.

Also, $\ell_k = x_i^k - w_k^i$ for some i or $\ell_k = -y_k^j$ for some j

$\implies p_k = \underline{p}_k$.

To show (iv): It is true that

$L_1 > x_1^i - w_1^i > \ell_1$ for all i

$L_1 > -y_1^j > \ell_1$ for all j.

So for some k, $2 \leq k \leq n$,

let $p_k = \bar{p}_k$. Then $\ell_k = m_k - \underline{p}_k = - v_k$.

$$x_k^i - w_k^i + v_k > 0 \text{ because } x_k^i \geqq 0$$

and since $v_k > w_k$ by assumption (f),

$$v_k - w_k^i \geq \sum_{h \neq i} w_k^h > 0.$$

Also $y_k^j + v_k > 0$ by definition of v_k.

Hence, if $L_k' + x_k^i - w_k^i = 0$ for some i or $L_k - y_k^j = 0$ for some j, then $p_k = \bar{p}_k$ by proof of (v) and hence $x_k^i - \ell_k > 0$ for all i and $-y_k^j - \ell_k > 0$ for all j. The reasoning is similar in the case of $p_k = \underline{p}_k$ and $\ell_k = x_k^i - w_k^i$ for some i or $\ell_k = -y_k^j$ for some j.

Q.E.D.

2.3 Production with Proportional Quantity Restrictions

In this section we examine the problem of finding an equilibrium with quantity restrictions that vary with the size of the firm. We use the same production model as in section 2.2 except for strengthening some assumptions about convexity. It has already been said that under constant returns to scale, size of a firm is indeterminate. So we assume strict convexity of preference for the consumers and strict convexity of the production sets. As a result, constant returns to scale in production is ruled out and demand correspondences of the consumers and producers become single valued.

If the quantity restrictions are to vary with the size, a natural question arises: How to define the size of a firm. We can easily rule out fuzzy concepts such as capacity. In general, we may have firms which can produce the same outputs but require different inputs and conversely. In this world of firms which produce multiple goods and differ in efficiency we may come across a firm much smaller than another in many respects and yet produces much larger amounts of a single commodity. So, it is not possible to have a single scaling factor for a firm's quantity restrictions on all the commodities. For example, it does not make sense to give a larger quota of chemical fertilizer to a large steel mill.

Further, let us take the case of two firms that produce a single commodity common to them, using a single factor of production. The relative efficiency of the firms may vary a lot with change of scale. For these reasons, it is necessary to fix one particular production level for every producer and consumption level for each consumer for the purpose of comparison. Given a rigid price, the most relevant bundle for a member of the economy is the one he would have chosen had there been no restriction. We now calculate the excess demand, positive or negative, of every member for every commodity. This we take to be the size of the member as regards that particular commodity. Quantity restrictions are issued to every member that are in proportion to this excess. It is necessary for this scheme to succeed that the members of the economy make known their unrestricted trading offers truthfully. This kind of proportional rationing is practised in the real world in isolated markets like the market for equity shares.

The working of the scheme we visualize as follows: The central authority acts as a clearing house for all transactions. It announces a price vector and invites trade offers from everyone. All the members of the economy make known their real trade offers. Now, in the case of every commodity, the authority will issue a quantity restriction to every agent who has an offer to buy, the allowed amount being a fraction of his initial buying offer. Similarly, everyone who offers to sell the commodity gets a quantity restriction that is another fraction of his initial selling offer. A prospective buyer cannot turn a seller and vice versa. On receiving such restrictions on all the commodities, every member of the economy comes up with his restricted demand. The result is that there exist a set of such fractions for which the restricted total excess demand is zero. We also require at this equilibrium that if buying offer on a commodity is slashed, then the selling offer is not and vice versa.

We now proceed to define precisely the equilibrium concept and show the existence of a solution.

Let us consider the economy with N consumers, M producers and n commodities described in section 2.2. In this section and in section 2.4, we use the index h to denote any agent, consumer or producer. We do not use any numerical value for h in order to avoid confusion. In addition to the assumptions made in section 2.2, we have two more.

Further assumptions:

(1) The preference ordering of the consumer i is strictly convex. That is, if $x \succsim_i y$ and $x \neq y$, then

$\mu x + (1-\mu) y \succ_i x$ for $0 < \mu < 1$.

(2) The production sets Y_j are strictly convex.

These two assumptions ensure that the demand correspondences of the agents are in fact single valued functions.

In this section and in section 2.4, the compact price set P of section 2.2 is specialized to consist a single point p with all the components positive. This makes the analysis simpler and the scope does not diminish significantly.

Let the unrestricted individual demands at price p be $\bar{x}^i$ for consumers and $\bar{y}^j$ for producers.

Let $\bar{Z}^i = \bar{x}^i - \omega^i$

$\bar{Z}^j = -\bar{y}^j$

be the unrestricted excess demand at price p. Define:

$\bar{Z}_k^{h+} = \max(0, \bar{Z}_k^h)$

$\bar{Z}_k^{h-} = \max(0, -\bar{Z}_k^h)$

for $h \in I \cup J$.

A proportional rationing scheme is a pair of vectors $({}_1B, {}_1b) \in [0,1]^{n-1} \times [-1,0]^{n-1}$ where ${}_1B = (B_2, B_3, \ldots, B_n)$ and ${}_1b = (b_2, b_3, \ldots, b_n)$. Given a proportional rationing scheme and the price p, define a straight rationing scheme (L^h, ℓ^h) for $h \in I \cup J$ by

$L_k^h = B_k \bar{Z}_k^{h+}$

$\ell_k^h = b_k \bar{Z}_k^{h-}$

for $k \in (2, \ldots, n)$ and

$L_1^h = v_1$ and $\ell_1^h = -v_1$

where the vector v was defined in the proof of Theorem 2.2.1.

For each producer, define his trading set by

$$\lambda^j({}_1B,{}_1b) = \{y \in Y_j \mid L^j({}_1B) \geqq -y \geqq \ell^j({}_1b)\}$$

For each producer, define his profit function by

$$\pi^j({}_1\bar{B},{}_1b) = \max\,\{p\,y^j \mid y^j \in \lambda^j({}_1B,{}_1b)\}$$

For each consumer, define his budget set by

$$\gamma^i({}_1B,{}_1b) = \{x \in X_i \mid \bar{p}(x-w^i) + \sum_j d_{ij}\;\pi^j({}_1B,{}_1b) \leq 0$$
$$L^i({}_1B) \geqq x-w^i \geqq \ell^i({}_1b)\},$$

<u>Lemma 2.3.1</u>: $\pi^j({}_1B,{}_1b)$ is a continuous function in $({}_1B,{}_1b)$.

<u>Proof</u>: Follows directly from lemma 2.2.2 since $L^j({}_1B)$ and $\ell^j({}_1b)$ are continuous in $({}_1B,{}_1b)$.

<u>Lemma 2.3.2</u>: $\lambda^j({}_1B,{}_1b)$ is a continuous correspondence in $({}_1B,{}_1b)$.

<u>Proof</u> : Follows directly from lemma 2.2.1 since $L^j({}_1B)$ $\ell^j({}_1b)$ are continuous functions in $({}_1B,{}_1b)$.

<u>Lemma 2.3.3</u>: $\gamma^i({}_1B,{}_1b)$ is a continuous correspondence in $({}_1B,{}_1b)$.

<u>Proof</u>: Follows directly from lemma 2.2.3 since $L^i({}_1B)$ and $\ell^i({}_1b)$ continuous functions in $({}_1B,{}_1b)$. The condition that there should be a k such that $p_k > 0$ and $\ell_k \neq 0$ is satisfied for commodity 1.

<u>Definition</u>: An equilibrium with proportional rationing is defined as an N-tuple of consumption vectors $\{x^i\}$, an M-tuple of production vectors $\{y^j\}$, a given price vector p with strictly positive components and a rationing scheme $({}_1B,{}_1b) \in [0,1]^{n-1} \times [-1,0]^{n-1}$ such that

(i) For all i, x^i is a maximal element for $\succsim_i$ of $\gamma^i({}_1B,{}_1b)$.

(ii) For all j, y^j is a maximal element of py over $\lambda^j({}_1B,{}_1b)$.

(iii) $\sum_i (x^i-w^i) - \sum_j y^j = 0$.

(iv) For any $k \in \{2,3,\ldots,n\}$,

$$B_k < 1 \Longrightarrow b_k = -1$$

$$b_k > -1 \Longrightarrow B_k = 1.$$

Condition (iv) says that once the agents make their trading offers known, if the prospective buyers are allowed only a proper fraction of their initial buying offer, then the prospective sellers are allowed as much as the whole of their initial selling offer and vice versa.

The trivial equilibrium resulting from banning either all buying or all selling is ruled out by the choice of straight rationing on commodity 1 which cannot be binding for any agent if the allocation is feasible. Condition (iv) rules out the trivial equilibrium that could result by banning both buying and selling of all but one commodity.

A few preliminary definitions before we proceed. Let

$${}_1S = \{(s_2,s_3,\ldots,s_n) \mid 0 \le s_k \le 1,\ k=2,\ldots,n\}.$$

For ${}_1s \in {}_1S$, define ${}_1B({}_1s)$ and ${}_1b({}_1s)$ as follows. For $k \in \{2,3,\ldots,n\}$,

$$B_k({}_1s) = 2s_k \quad \text{if } 0 \le s_k \le 1/2$$

$$= 1 \quad \text{if } 1/2 \le s_k \le 1$$

$$b_k({}_1s) = -1 \quad \text{if } 0 \le s_k \le 1/2$$

$$= 2(s_k-1) \text{ if } 1/2 \le s_k \le 1.$$

It is easily seen that ${}_1B({}_1s)$ and ${}_1b({}_1s)$ are continuous in ${}_1s$ (see diagram 1). Define:

$$\pi^j({}_1s) = \pi^j({}_1B({}_1s), {}_1b({}_1s))$$

$$\lambda^i({}_1s) = \lambda^j({}_1B({}_1s), {}_1b({}_1s))$$

$$\gamma^i({}_1s) = \gamma^i({}_1B({}_1s), {}_1b({}_1s)).$$

$\pi^j({}_1s)$, $\lambda^j({}_1s)$, $\gamma^i({}_1s)$ are continuous in ${}_1s$. By standard techniques (refer to the section on mathematics in Hildenbrand [11]), continuity of $\pi^j({}_1s)$, $\lambda^j({}_1s)$ and $\gamma^i({}_1s)$ in ${}_1s$ and the continuity of $\succsim_i$ give us continuity in ${}_1s$ of the excess demand functions of the consumers and the producers. Let $Z^i({}_1s)$ be the excess demand functions of the consumer i and $Z^j({}_1s)$ be the excess demand function of producer j. Define the total excess demand function by:

$$Z({}_1s) = \sum_{i=1}^{N} Z^i({}_1s) + \sum_{j=1}^{M} Z^j({}_1s).$$

<u>Theorem 2.3.1</u>: For the economy described in this section, for a given strictly positive price vector p, there exists an equilibrium with proportional rationing.

<u>Proof</u>: Define, for $k \in \{2, \ldots, n\}$,

$$\bar{Z}_k^+ = \sum_i \bar{Z}_k^{i+} + \sum_j \bar{Z}_k^{j+}$$

$$\bar{Z}_k^- = \sum_i \bar{Z}_k^{i-} + \sum_j \bar{Z}_k^{j-} .$$

Define ${}_1t({}_1s) : {}_1S \to [-\frac{1}{2}, \frac{1}{2}]^{n-1}$ as follows.

$${}_1t({}_1s) = (t_2({}_1s), t_3({}_1s), \ldots, t_n({}_1s)) .$$

For $k = 2, \ldots, n$; $t_k({}_1s) = 0$ if $Z_k({}_1s) = 0$

$$t_k({}_1s) = \frac{Z_k({}_1s)}{2\,\overline{Z}_k^+} \qquad \text{if } Z_k({}_1s) > 0)$$

$$t_k({}_1s) = \frac{Z_k({}_1s)}{2\,\overline{Z}_k^-} \qquad \text{if } Z_k({}_1s) < 0 \;.$$

It has to be checked that ${}_1t({}_1s)$ is well defined. Suppose $Z_k({}_1s) > 0$. This means that for some agent at least, the excess demand for k-th commodity is positive. Let us say, $Z_k^i({}_1s) > 0$. This implies $\overline{Z}_k^i > 0$, because if $\overline{Z}_k^i \leq 0$, then by the definition of our budget set, $Z_k^i({}_1s) \leq 0$.

$$\overline{Z}_k^{i+} = \overline{Z}_k^i > 0$$

$$\Longrightarrow \overline{Z}_k^+ > 0.$$

Similarly, $Z_k({}_1s) < 0 \Longrightarrow \overline{Z}_k^- > 0$. So, ${}_1t({}_1s)$ is well defined.

Also, it follows from the definition of γ^i and λ^j that $\overline{Z}^{h+} \geqq Z^h({}_1s) \geqq -\overline{Z}^{h-}$. For $k \in \{2,\dots,n\}$, $0 \leq t_k({}_1s) \leq 1/2$ if $Z_k({}_1s) > 0$, $-1/2 \leq t_k({}_1s) \leq 0$ if $Z_k({}_1s) < 0$. From the above, we get $-1/2 \leq t_k({}_1s) \leq 1/2$. Now the claim is that ${}_1s - {}_1t({}_1s) \in {}_1S$. From the definition of the budget set, ${}_1B({}_1s)$, ${}_1b({}_1s)$, (L^h, ε^h) for $h \in I_{(1)}$,

(a) Given ${}_1s$, for $0 \leq s_k \leq 1/2$; $k \in \{2,\dots,n\}$, it follows that $-\overline{Z}_k^{h-} \leq Z_k^h({}_1s) \leq 2\, s_k\, \overline{Z}_k^{h+}$.

Adding over all the agents, we get

$$-\overline{Z}_k^- \leq Z_k({}_1s) \leq 2\, s_k\, \overline{Z}_k^+ .$$

If $Z_k({}_1s) = 0$, then $t_k({}_1s) = 0$.

If $Z_k({}_1s) > 0$, then $0 \leq t_k({}_1s) \leq s_k$.

If $Z_k({}_1s) < 0$, then $0 \geq t_k({}_1s) \geq -1/2$.

In all three cases, we get $0 \leq s_k - t_k({}_1s) \leq 1$.

(b) Given ${}_1s$, for $1/2 \leq s_k \leq 1$, $k \in \{2,\ldots,n\}$, it follows that: $2(s_k-1)\,\bar{Z}_k^{h-} \leq Z_k^h({}_1s) \leq \bar{Z}_k^{h+}$.

Adding over all the agents, we get

$$2(s_k-1)\,\bar{Z}_k^{-} \leq Z_k({}_1s) \leq \bar{Z}_k^{+} .$$

If $\quad Z_k({}_1s) = 0$, then $t_k({}_1s) = 0$.

If $\quad Z_k({}_1s) > 0$, then $0 \leq t_k({}_1s) \leq 1/2$

If $\quad Z_k({}_1s) < 0$, then $(s_k-1) \leq t_k({}_1s) \leq 0$.

In all the three cases, we get

$$0 \leq s_k - t_k({}_1s) \leq 1.$$

From both (a) and (b) above, we get, given ${}_1s$,

$${}_1s - {}_1t({}_1s) \in {}_1S.$$

Define $\theta : {}_1S \rightarrow {}_1S$ by $\theta({}_1s) = {}_1s - {}_1t({}_1s)$. Now $\theta({}_1s)$ is a continuous function, since ${}_1t({}_1s)$ is a continuous function. ${}_1S$ is a compact, convex set. All the conditions for the Bruower's fixed point theorem are satisfied.

Applying Bruower's fixed point theorem, there is an ${}_1s^* \in {}_1S$ such that

$$\theta({}_1s^*) = {}_1s^*.$$

$\Longrightarrow {}_1t({}_1s^*) = 0.$

$\Longrightarrow Z_k({}_1s^*) = 0$ for $k \in \{2,\ldots,n\}$.

By Walras' Law, $pZ({}_1s^*) = 0 \Longrightarrow Z_1({}_1s^*) = 0$ since $p_1 \neq 0$ and $Z_k({}_1s^*) = 0$ for $k = 2,\ldots,n$. At ${}_1B({}_1s^*)$ and ${}_1b({}_1s^*)$, conditions (ii) and (iii) for equilibrium are already satisfied and condition (iii) is satisfied since $Z({}_1s^*) = 0$. Condition (iv) also is automatically satisfied because of the definition of ${}_1B({}_1s)$ and ${}_1b({}_1s)$.

Q.E.D.

2.4 Producer Variable Rationing

In section 2.3, we considered a production economy in which the quantity restrictions are variable according to size for all the agents including consumers. In this section, we consider quantity restrictions that are equal for all the consumers but variable for all the producers. This type of rationing is useful in an economy where the consumers form a large contingent and the producers are relatively few in number. Under these conditions the cost of obtaining information about different consumers and the cost of issuing and enforcing a separate set of restrictions for every consumer may make it a wiser policy to treat all consumers equally.

We maintain the same set up of a production economy with the same assumptions as in section 2.3. For this economy, we define a new equilibrium concept. The main difference this has with an equilibrium with proportional rationing is that all the consumers face equal quantity restrictions in this while the producers face quantity restrictions that vary with the size of their demand as explained in section 2.3. We further postulate that the agents are treated similarly regarding each commodity, in spite of the difference in the nature of quantity restrictions that they face. To be more explicit, if the buying restriction is binding for a consumer or the authority does not allow the producer's full initial buying offer, then selling restriction is not binding for any consumer and the producers are allowed their full initial selling offer.

The working of this system resembles some of the decentralized planning models. At the first stage, the central authority sends a price signal to the producers. The producers, on receiving this, send back to the authority their unrestricted demands. Depending on this, there is a set of equal restrictions for consumers and demand proportional restrictions for producers at which the restricted supply equals restricted demand at the price earlier announced.

Let us consider the same production economy as we described in section 2.3 with the same assumptions. For this economy, define a proportional rationing scheme $({}_1B, {}_1b)$ as in section 2.3. Define an equal rationing scheme as a pair of vectors $(L, \ell) \in R^n_+ \times R^n_-$. We assume the prices to be a fixed vector with all positive components. Given the proportional rationing scheme $({}_1B, {}_1b)$, the profit function $\pi^j({}_1B, {}_1b)$ and the budget set $\lambda^j({}_1B, {}_1b)$ of producer j is defined as in section 2.3. Given an equal rationing scheme (L, ℓ) Tthe budget set of consumer i is defined as in section 2.2 by

$$\gamma^i(L, \ell) = \{ x \in X_i \mid p(x-w^i) + \sum_j d_{ij} \; \pi^j({}_1B, {}_1b) \leq 0, \; L \geqq x-w^i \geqq \ell \} .$$

given $({}_1B, {}_1b)$, $L^j_k({}_1B)$, $\ell^j_k({}_1b)$; $k = 2,3,\ldots,n$ are obtained as in section 2.3.

$$L^j_1 = L^i_1 = v_1 \quad \text{all } i, \text{ all } j.$$

$$\ell^j_1 = \ell^i_1 = -v_1 \quad \text{all } i, \text{ all } j.$$

where the vector v was first defined in the proof of theorem 2.2.1.

<u>Definition</u>: An equilibrium with producer variable quantity restrictions is defined as: an N-tuple of consumption vectors $\{x^i\}$, an M-tuple of production vectors $\{y^j\}$, an equal rationing scheme (L,ℓ) and a proportional rationing scheme $({}_1B,{}_1b)$ such that

(i) For all i, x^i is a maximal element for $\succsim_i$ of $Y^i(L,\ell)$.

(ii) y^j maximizes py over $Y^j({}_1B,{}_1b)$.

(iii) $\sum_i (x^i-w^i) - \sum_j y^j = 0$.

(iv) For $k \in \{2,\ldots,n\}$ $B_k < 1 \Longrightarrow b_k = -1$ and $x^i_k-w^i_k > \ell_k$; for all $i \in I$.

$b_k > -1 \Longrightarrow B_k = 1$ and $x^i_k-w^i_k < L_k$; for all $i \in I$.

(v) $x^i_k-w^i_k = L_k$ for some $k \in K$ and some $i \in I$

$\Longrightarrow x^h_k-w^h_k > \ell_k$ for all $h \in I$.

and $b_k = -1$ if $k \in \{2,3,\ldots,n\}$.

Also $x^i_k-w^i_k = \ell_k$ for some k and some $i \Longrightarrow x^h_k-w^h_k < L_k$ for all $h \in I$.

and $B_k = 1$ if $k \in \{2,3,\ldots,n\}$.

Condition (iv) states that if only a fraction of their initial buying (selling) offer for a commodity is allowed to the prospective buyers (sellers) of that commodity among the producers, then as much as the whole of their initial selling (buying) offer is allowed to the prospective sellers (buyers) of that commodity among the producers and selling (buying) restriction on the commodity cannot be binding for the consumers.

Condition (v) states that if the buying (selling) restriction is binding for some consumer for some commodity, then the selling (buying) restriction cannot be binding for any consumer for that commodity, and further, propsective sellers (buyers) of that commodity among the producers are allowed as much as the whole of their initial selling (buying) offer.

Apart from ruling out certain trivial equilibria, conditions (iv) and (v) ensure that the producers and the consumers are restricted similarly concerning each commodity.

Before we proceed, we have some more definitions. Define the set ${}_1S$ exactly as we did in section 2.3. Given ${}_1s \in {}_1S$, define ${}_1B({}_1s)$ and ${}_1b({}_1s)$ as we did in 2.3. Given ${}_1s \in {}_1S$, define an equal rationing scheme by:

For $k \in \{2,3,\ldots,n\}$, $L_k({}_1s) = v_k\ B_k({}_1s)$.

$$\ell_k({}_1s) = v_k\ b_k({}_1s).$$

$$L_1({}_1s) = v_k \text{ and } \ell_1({}_1s) = -v_k.$$

where v_k is as defined in section 2.2. Define $\gamma^i({}_1s)$, $\lambda^j({}_1s)$ and $\pi^j({}_1s)$ suitably as in section 2.3. Now again $\gamma^i({}_1s)$ and $\lambda^j({}_1s)$ are continuous correspondences. Since $\succsim_i$ and $\pi^j({}_1s)$ are continuous, it follows from standard techniques that the excess demand functions are continuous for producers as well as consumers.

<u>Theorem 2.4.1</u>: For the economy detailed in this section 2.4 given a price vector p with strictly positive components, an equilibrium with producer variable quantity restrictions exists.

<u>**Proof**</u>: Define

$$G_k^+ = N\ v_k + \sum_j \overline{Z}_k^{j+}$$

$$G_k^- = N\ v_k + \sum_j \overline{Z}_k^{j-} .$$

Define a function, ${}_1t'({}_1s)$ by:

For $k \in \{2,\ldots,n\}$,

$$t'_k({}_1s) = 0 \qquad \text{if } Z_k({}_1s) = 0$$

$$t'_k({}_1s) = \frac{Z_k({}_1s)}{2\, G_k^+} \text{ if } Z_k({}_1s) > 0$$

$$t'_k({}_1s) = \frac{Z_k({}_1s)}{2\, G_k^-} \text{ if } Z_k({}_1s) < 0.$$

As we showed in section 2.3 that ${}_1t$ is well defined, it can be shown that ${}_1t'$ is well defined.

From the definition of the budget sets,

$${}_1B({}_1s),\ {}_1b({}_1s),\ L^j({}_1s),\ \ell^j({}_1s),\ L({}_1s),\ \ell({}_1s),$$

(a) Given ${}_1s$, for $0 \leq s_k \leq 1/2$, $k \in \{2,\ldots,n\}$ it follows that

$$-v_k \leq Z_k^i({}_1s) \leq 2\, s_k\, v_k \quad \text{for all } i,$$

$$-\bar{Z}_k^{j-} \leq Z_k^j({}_1s) \leq 2\, s_k\, \bar{Z}_k^{j+} \quad \text{for all } j.$$

Adding over all the agents, we get

$$-G_k^- \leq Z_k({}_1s) \leq 2\, s_k\, G_k^+ .$$

If $Z_k({}_1s) = 0$, then $t'_k({}_1s) = 0$.

If $Z_k({}_1s) > 0$, then $0 \leq t'_k({}_1s) \leq s_k$.

If $Z_k({}_1s) < 0$, then $-1/2 \leq t'_k({}_1s) \leq 0$.

In all three cases, $0 \leq s_k - t'_k({}_1s) \leq 1$.

(b) Given ${}_1s$, for $1/2 \leq s_k \leq 1$; $k \in \{2,\ldots,n\}$, it follows that

$$2(s_k-1)\, v_k \leq Z_k^i({}_1s) \leq v_k \quad \text{for all } i$$

$$2(s_k-1)\, \bar{Z}_k^{j-} \leq Z_k^j({}_1s) \leq \bar{Z}_k^{j+} \quad \text{for all } j.$$

Adding over all the agents, we get

$$2(s_k-1)\, G_k^- \leq Z_k({}_1s) \leq G_k^+ .$$

If $\quad Z_k({}_1s) = 0$, then $t_k'({}_1s) = 0$.

If $\quad Z_k({}_1s) > 0$, then $0 \leq t_k'({}_1s) \leq 1/2$.

If $\quad Z_k({}_1s) < 0$, then $(s_k-1) \leq t_k'({}_1s) \leq 0$.

In all three cases, we get $0 \leq s_k - t_k'({}_1s) \leq 1$. From both (a) and (b), we get, given ${}_1s$,

$${}_1s - {}_1t'({}_1s) \in {}_1S.$$

Define $\theta' : {}_1S \rightarrow {}_1S$ by $\theta'({}_1s) = {}_1s - {}_1t'({}_1s)$. Now $\theta'({}_1s)$ is continuous since ${}_1t'({}_1s)$ is continuous. ${}_1S$ is a compact, convex set. All the conditions for Bruower's fixed point theorem are satisfied. So, there is an ${}_1s^* \in {}_1S$ such that $\theta'({}_1s^*) = {}_1s^*$.

$\Longrightarrow \quad {}_1t({}_1s^*) = 0.$

$\Longrightarrow \quad Z_k({}_1s^*) = 0 \quad$ for $k \in \{2, \ldots, n\}$.

By Walras' Law, $p\, Z({}_1s^*) = 0$. $\Longrightarrow Z_1({}_1s^*) = 0$ since $p_1 > 0$. Conditions (i), (ii), (iii) for the equilibrium are already satisfied. To check (iv):

It easily follows from the definition of ${}_1B({}_1s)$, ${}_1b({}_1s)$, $L({}_1s)$, $\ell({}_1s)$ that

$$B_k < 1 \Longrightarrow b_k = -1 \text{ and } \ell_k = -v_k$$

and $\quad b_k > -1 \Longrightarrow B_k = 1$ and $L_k = v_k$.

Further, it can be checked as in section 2.2 that for any feasible allocation.

$$x_k^i - w_k^i > -v_k$$

and $\quad x_k^i - w_k^i < v_k$ for all i.

To check (v):

$$x_k^i - w_k^i = L_k \text{ for some } k \text{ and some } i$$

$$\Longrightarrow \quad 0 \leq L_k < v_k$$

$$\Longrightarrow \quad 0 \leq a_k^* \leq 1/2 \text{ by definition of } L({}_1a).$$

$$\Longrightarrow \quad \ell_k = -v_k \text{ and } b_k = -1.$$

Also, $\ell_k = -v_k \Longrightarrow x_k^i - w_k^i > \ell_k$ all i. Similarly for the case

$$x_k^i - w_k^i = \ell_k \text{ for some } k \text{ and some } i.$$

Q.E.D.

2.5 Concluding remarks on production and rationing equilibria

As we have stated, the equilibrium with equal rationing we defined in section 2.2 is the only one we have to offer in the general case where production is possible with constant returns to scale. This is because the size of the firm is then indeterminate.

The equilibrium with proportional rationing we defined in section 2.3 is ideal for the case where the number of consumers is small. In a typical peasant economy with a large chunk of the population being peasants who take their consumption and production decisions together, proportional rationing may find applications. Without going to the fundamentals of joint consumption and production decision making, if we assume that each agent has a restricted demand function that is continuous in the restrictions, then the results of section 2.3 are still valid.

The equilibrium with producer variable rationing as defined in section 2.4, we hope, will be the most widely applicable equilibrium when prices are rigid. The essential difference in enforcing equal restrictions and this one is that the former does not require information about individual demand while the latter does. But most governments do keep a tab on the producers. In a planned economy in particular, when a good portion of the production units are state enterprises, the planning authority should have enough information to practise commodity rationing as outlined in section 2.4.

3. Efficiency of Non-Walrasian Equilibria

3.1 Rationing and Distribution

In many developing countries with a large portion of the population below the poverty line, there is a lot of concern over the distribution of income. It is recognized that the set of Walras equilibria, the solutions given raise to by the market mechanism which operates with the help of price signals, may not contain any solution with an equitable distribution of income. So, policy makers frequently resort to quantity rationing which uses prices as well as quantity restrictions as signals, as a practical alternative to the market mechanism. In this chapter we enquire into the efficiency of such rationing. Pareto efficiency and its relationship to Walras equilibrium has been discussed extensively in [1]. We make use of the same concepts as presented there, except that we confine ourselves to an exchange economy for the sake of simplicity.

Let us consider an exchange economy with N consumers (indexed i) and n commodities (indexed k). The utility maximizing consumer i is characterized by (X_i, U_i, w_i) representing his consumption set, utility function and initial endowments in that order.

A feasible consumption allocation is defined as N-tuple of consumption vectors

$$x^i \in X_i \text{ such that } \sum_i x^i \leqq \sum_i w^i.$$

Define the set of feasible utility allocations U by

$$U = \{u \in R^N \mid u_i = U_i(x^i) \text{ and } (x^1, x^2, \ldots, x^N) \text{ is feasible}\}.$$

E_1, the set of feasible Pareto efficient utility allocations or the Pareto frontier, is defined by

$$E_1 = \{u \in U \mid \text{There is no } u' \in U \text{ and } u' \geqq u \text{ and } u'_i > u_i \text{ for some } i\}$$

E_2, the set of individually rational Pareto efficient utility allocations is defined by

$$E_2 = \{u \in E_1 \mid u_i \geqq U_i(w^i)\} .$$

It is easily seen that if $X_i = R^n_+$ and U_i is strictly monotone for every i, then E_1 coincides with the set

$$\{u \in U \mid \text{There is no } u' \in U \text{ with } u'_i > u_i \;\; \forall\, i\} .$$

Conforming to the earlier literature, we define Pareto efficiency using utility allocations and not consumption allocations. This way it so happens that if inter-personal comparisons are ruled out, any two Pareto efficient points become non-comparable to each other.

In sections 3.2, 3.3, 3.4 we find that although we may be able to generate a large number of allocations using rationing, the chances of achieving a Pareto efficient allocation remain as slim as with the market mechanism. Rationing is in general Pareto inefficient. The implication of this is discussed in section ~~3.5~~ 3.4.

3.2 Efficiency of Drèze Equilibria

It is well known that Walras equilibrium in general is Pareto efficient. When a different solution concept is proposed for the problem of allocation, as has been done by J.H.Drèze [6], it is natural for one to enquire whether Drèze equilibrium is Pareto efficient. The set of Drèze equilibrium consumption allocations as we define here will be the set of all the allocations that can be obtained through commodity rationing with equal quantity restrictions. This adds more significance to the question of efficiency of Drèze equilibria. Herein we show that though Drèze equilibrium can give raise to a large number of allocations, the solutions are in general Pareto inefficient. We make the following assumptions:

1) $X_i = R^n_+$ and $w^i \in$ interior X_i.

2) (Monotonicity) The utility function U_i are monotone in their arguments.

3) (Smooth indifference curves) The utility functions U_i are differentiable.

4) (Strictly convex indifference curves) For any $x \in X_i$ with strictly positive components, the set $R(x)$ defined by
$R(x) = \{y \in X_i \mid U_i(y) \geq U_i(x)\}$
is strictly convex.

The assumptions have a long history and were used by Hicks [10] among others. They ensure that the preference orderings are strongly convex, i.e. if $U_i(x) = U_i(y)$ and $x \neq y$, then

$$U_i(\lambda x + (1-\lambda)y) > U_i(x) \quad \text{for } 0 < \lambda < 1.$$

One more implication of assumption 4 is that the indifference curves do not touch the axes and the utility maximization problem has an interior solution with all components positive.

In what follows, we make good use of this well known result: under the assumptions made, any Pareto efficient allocation can be obtained as a Walras equilibrium. This result is found in (1) and requires some clarification. Given the initial endowments, if we make a certain lump sum transfer of these (for example, a transfer that gives the Pareto efficient allocation itself as the new endowments), then there is a price vector at which the given Pareto efficient allocation is a Walras equilibrium.

Consider a rationing scheme $(L,\ell) \varepsilon R^n_+ \times R^n_-$. Given any price vector $p \varepsilon R^n_+$, with the first component $p_1 = 1$, define the budget set of consumer i by

$$\gamma^i(p,L,\ell) = \{x \varepsilon X_i \mid p(x-w^i) \leqq 0,\ L \geqq x-w^i \geqq \ell\} .$$

An N-tuple of consumption allocation $\{x^i\}$ is said to be obtained as a Drèze equilibrium if there is a strictly positive $p\varepsilon R^n_+$ with $p_1 = 1$ and a rationing scheme $(L,\ell)\varepsilon R^n_+ \times R^n_-$ such that these conditions hold:

(i) x^i is maximal for U_i over $\gamma^i(p,L,\ell)$.

(ii) $\sum_i x^i - \sum_i w^i = 0$.

(iii) $L_k = x^i_k - w^i_k$ for some i and some k $\Longrightarrow x^i_k - w^i_k > \ell_k$ for all i.

$\ell_k = x^i_k - w^i_k$ for some i and some k $\Longrightarrow L_k > x^i_k - w^i_k$ for all i.

If the following condition holds in addition to the above three, we say that $\{x^i\}$ is obtained as a numeraire exempted Drèze equilibrium.

(iv) $L_1 > x^i_1 - w^i_1 > \ell_1$ for all i.

Condition (iii) states that if buying restriction is binding for some agent on a commodity, then selling restriction is not binding for all agents on the same commodity and vice versa. Condition (iv) states that no restriction can be binding for any agent on the numeraire commodity.

As a particular case of Drèze's first theorem in [6], we get that given any $p \in R^n_+$ with $p_1 = 1$, there is an N-tuple $\{x^i\}$ which is a Drèze equilibrium at price p. Moreover, $\{x^i\}$ turn out to be a numeraire exempted Drèze equilibrium as well, though Drèze does not mention (iv) as a condition. It is stated in the introduction how conditions (iii) and (iv) rule out certain trivial allocations.

Define:

$D_1 = \{u \in U \mid u_i = U_i(x^i)$ and $\{x^i\}$ is a Drèze equilibrium $\}$.

$D_2 = \{u \in U \mid u_i = U_i(x^i)$ and $\{x^i\}$ is a numeraire exempted Drèze equilibrium $\}$.

$W = \{u \in U \mid u_i = U_i(x^i)$ and $\{x^i\}$ is a Walras equilibrium $\}$.

Drèze shows in [6] the existence of an equilibrium with price rigidities that satisfies conditions (i), (ii), (iii), (iv) and one more boundary condition when the prices are constrained to lie in a certain set $P \subset R^n_+$. For any such equilibrium with price rigidities, it is seen that the corresponding utility allocation will lie in D_1 and D_2 ($D_2 \subset D_1$) for whatever price set P, since $w^i \in Y^i(p,L,\ell)\ \forall i$, $\forall (p,L,\ell)$.

The following lemma plays a fundamental role in all that follows.

Lemma 3.2.1: Let X_i be R^2_+. Let $(\bar{x}_1, \bar{x}_2)$ be the preferred bundle for consumer i at prices $(1, p_2)$ and income $\bar{x}_1 + p_2 \bar{x}_2$. $\bar{x}_1 + p_2 \bar{x}_2 = x_1 + p_2 x_2$ and (x_1, x_2) is the preferred bundle at prices $(1, p_2')$ and income $x_1 + p_2' x_2$. Both p_2 and p_2' are positive. Then

$$x_2 > \bar{x}_2 \iff p_2' < p_2 .$$
$$\bar{x}_2 > x_2 \iff p_2 < p_2' .$$

Proof: Let $x_2 > \bar{x}_2$. We have to prove that $p_2' < p_2$. Let us suppose that $p_2' \geq p_2$.

Now, $\bar{x}_1 + p_2 \bar{x}_2 = x_1 + p_2 x_2$ (by assumption)

or $(\bar{x}_1 - x_1) = p_2(x_2 - \bar{x}_2)$

$\leqq p_2'(x_2 - \bar{x}_2)$,

since $(x_2 - \bar{x}_2) > 0$ and $p_2' \geqq p_2 > 0$.

Rearranging terms, we get

$$\bar{x}_1 + p_2' \bar{x}_2 \leqq x_1 + p_2' x_2 \qquad (1)$$

Due to strong convexity of preferences, we know that the preferred bundle is unique at any income. So, $U_i(\bar{x}_1, \bar{x}_2) > U_i(x_1, x_2)$ since at income $\bar{x}_1 + p_2 \bar{x}_2$ and prices $(1, p_2)$, $(\bar{x}_1, \bar{x}_2)$ is revealed preferred to (x_1, x_2). By (1), $(\bar{x}_1, \bar{x}_2)$ is in the budget set with income $(x_1 + p_2' x_2)$ and prices $(1, p_2')$. This contradicts that (x_1, x_2) is the preferred bundle in that budget set. Hence $p_2' < p_2$.

Now let $p_2' < p_2$. We first prove that $x_2 \geqq \bar{x}_2$.

Let $x_2 < \bar{x}_2$.

$$\bar{x}_1 + p_2 \bar{x}_2 = x_1 + p_2 x_2 .$$
$$(x_1 - \bar{x}_1) = p_2(\bar{x}_2 - x_2) > p_2'(\bar{x}_2 - x_2).$$

Rearranging terms,

$$\bar{x}_1 + p_2'\bar{x}_2 < x_1 + p_2x_2.$$

This expression again contradicts that (x_1,x_2) is the preferred bundle at prices $(1,p_2')$ and income $x_1+p_2'x_2$. Hence, $x_2 \geqq \bar{x}_2$.

If $\bar{x}_2 = x_2$, then $x_1 = \bar{x}_1$, since $\bar{x}_1 + p_2\bar{x}_2 = x_1 + p_2x_2$. At $(x_1,x_2) = (\bar{x}_1,\bar{x}_2)$, the marginal rate of substitution (m.r.s.) of the first commodity for the second is p_2 since $(\bar{x}_1,\bar{x}_2)$ is obtained as a preferred bundle at prices $(1,p_2)$. This m.r.s. is also p_2' since (x_1,x_2) is obtained as a preferred bundle at prices $(1,p_2')$. Since m.r.s. is unique at every point by assumption 3, this is contradiction. So, $\bar{x}_2 \neq x_2$ and $x_2 > \bar{x}_2$.

The proof for $\bar{x}_2 > x_2 \Longleftrightarrow p_2 > p_2'$ is similar to the above.

Q.E.D.

The reason why the lemma is true can be seen with the help of a diagram. In diagram 2, if $x_2 > \bar{x}_2$ and $p_2' \geqq p_2$, then (x_1,x_2) would have to be a point like H which is in the original budget set DOE. Hence $(\bar{x}_1,\bar{x}_2)$ is preferred to (x_1,x_2). We also have A in the budget set JOG in which H is preferred. So (x_1,x_2) is preferred to $(\bar{x}_1,\bar{x}_2)$ which is a contradiction.

Suppose on the other hand $p_2' < p_2$ and $x_2 \leqq \bar{x}_2$. Now x_2 cannot be equal to $\bar{x}_2$ since our assumption of smoothness of indifference curves rules out two supporting prices at the same point. So, let (x_1,x_2) be a point like I. Since I is in the budget set DOE, $(\bar{x}_1,\bar{x}_2)$ is preferred to (x_1,x_2). Since A is in the budget set BOC, I is preferred to A, or (x_1,x_2) is preferred to $(\bar{x}_1, \bar{x}_2)$ which is a contradiction.

Now we prove a result which shows that the set of Drèze equilibrium allocations can be very large.

Proposition 3.2.1: Let there be two consumers A and B with initial endowments (w_1^A, w_2^A) and (w_1^B, w_2^B) of the two commodities 1 and 2 in the economy. If the initial endowments do not constitute a Walras equilibrium, then the set D_2 is uncountable.

Proof: Let the marginal rates of substitution at the initial endowments $[(\delta U_i/\delta x_2^i) \,/\, (\delta U_i/\delta x_1^i)|\,(x_1^i, x_2^i) = (w_1^i, w_2^i)$ for $i = A, B]$ be p_1 for consumer A and p_2 for consumer B. If $p_1 = p_2$, then the initial endowments constitute a Walras equilibrium. Therefore, $p_1 \neq p_2$. Without loss of generality, let $p_1 < p_2$. Consider any p_3 such that $p_1 < p_3 < p_2$. At prices $(1, p_3)$ for the first and the second commodity, we know from [6] that there is a single restriction equal for both the consumers on either buying or selling of the second commodity such that there is an equilibrium with that restriction. Commodity 1 is taken as numeraire and does not have any binding restriction. Let $(\bar{x}_1^A, \bar{x}_2^A)$ and $(\bar{x}_1^B, \bar{x}_2^B)$ be the unrestricted demands of A and B. By an application of lemma 3.2.1, it is true that $w_2^A > \bar{x}_2^A$ since $p_3 > p_1$ and $w_2^B < \bar{x}_2^B$ since $p_2 > p_3$.

Since there can be only one restriction on either buying or selling of commodity 2, either A or B is not restricted in his trade at the corresponding Dreze equilibrium. This implies that something is definitely traded at the Drèze equilibrium with prices $(1, p_3)$. Let (x_1^A, x_2^A), (x_1^B, x_2^B) be this Drèze equilibrium consumption allocation.

Now, $$x_1^A + p_3\, x_2^A = w_1^A + p_3\, w_2^A$$

Or, $$(x_1^A - w_1^A) = p_3(w_2^A - x_2^A)$$

$$\neq p_4(w_2^A - x_2^A)$$

for any $p_4 \neq p_3$ since $x_1^A \neq w_1^A$, as we have shown.

So $$x_1^A + p_4\, x_2^A \neq w_1^A + p_4\, w_2^A$$

for any $p_4 \neq p_3$, ruling out the possibility that a Drèze equilibrium with prices $(1, p_3)$ can be obtained by any other price. Therefore, the set of Drèze equilibrium consumption allocations is uncountable since p_3 satisfying $p_1 < p_3 < p_2$ are uncountable.

It remains to show that the set of corresponding utility allocations is uncountable. Let us suppose, without loss of generality, that there are uncountable p_3 with $p_1 < p_3 < p_2$ and only selling of commodity 2 prohibited affecting the trade of consumer A only at the corresponding Drèze equilibria. Consider two such allocations given by the prices $(1, p_4)$ and $(1, p_5)$ with $p_1 < p_4 < p_5 < p_2$. Let (x_1^B, x_2^B) be the Drèze equilibrium allocation of B at prices $(1, p_5)$.

Now, $$x_1^B + p_5\, x_2^B = w_1^B + p_5\, w_2^B$$

or $$(w_1^B - x_1^B) = p_5\,(x_2^B - w_2^B)$$

$$\geq p_4(x_2^B - w_2^B),$$

since $x_2^B > w_2^B$.

This implies, $w_1^B + p_4\, w_2^B \geq x_1^B + p_4\, x_2^B$.

Since the preferred bundle is unique in any budget set, utility for B at Drèze equilibrium with prices $(1, p_4)$ will be higher than at prices $(1, p_5)$. This proves that the set D_2 is uncountable.

Let us now restrict ourselves to an economy with two agents A and B and two commodities, 1 and 2. (w_1^A, w_2^A) and (w_1^B, w_2^B) are the initial endowments and the economy satisfies the assumptions made earlier.

<u>Theorem 3.2.1:</u> When $N = n = 2$, $E_2 \cap D_1 = W$.

<u>Proof:</u> We first show that $W \not\supset E_2 \cap D_1$. Let $(u_A, u_B) \in E_2 \cap D_1$. Let $\{x\} = \{(x_1^A, x_2^A), (x_1^B, x_2^B)\}$ be a Drèze equilibrium allocation such that $U_A(x_1^A, x_2^A) = u_A$ and $U_B(x_1^B, x_2^B) = u_B$. We consider two cases:

<u>Case (a):</u> Let $(x_1^A, x_2^A) \neq (w_1^A, w_2^A)$.

Now, the only price that makes x a Drèze equilibrium is $[1, (w_2^A - x_2^A)/(x_1^A - w_1^A)]$ which we call $(1,p)$. Since x is Pareto efficient, there is a price which makes it a Walras equilibrium for an appropriate distribution of endowments, and in particular when x itself is the endowments. Let this price be $(1,p')$. If $p \neq p'$, then let $p' > p$. Also, let $(\bar{x}_1^A, \bar{x}_2^A)$, $(\bar{x}_1^B, \bar{x}_2^B)$ be the preferred bundles without restrictions at price $(1,p)$ and incomes $x_1^A + px_2^A$ for A and $x_1^B + px_2^B$ for B. It follows from lemma 3.2.1 that $\bar{x}_2^A > x_2^A$ and $\bar{x}_2^B > x_2^B$. Now, either $w_2^A > x_2^A$ or $w_2^B > x_2^B$. Without loss of generality, $w_2^A > x_2^A$. Now, since $\bar{x}_2^A > x_2^A$, either $w_2^A \geq \bar{x}_2^A > x_2^A$ or $\bar{x}_2^A > w_2^A > x_2^A$.

<u>Case (i):</u> Let $w_2^A \geq \bar{x}_2^A > x_2^A$.

Since

$$w_1^A + pw_2^A = \bar{x}_1^A + p\bar{x}_2^A ,$$

$$w_2^A \geq \bar{x}_2^A \iff w_1^A \leq \bar{x}_1^A$$

Since $\quad \overline{x}_1^A + p\overline{x}_2^A = x_1^A + px_2^A$,

$$\overline{x}_2^A > x_2^A \Longleftrightarrow \overline{x}_1^A < x_1^A .$$

Or $\quad w_2^A \geq \overline{x}_2^A > x_2^A \Longleftrightarrow w_1^A \leq \overline{x}_1^A < x_1^A.$

So, a choice of $(\overline{x}_1^A, \overline{x}_2^A)$ involves less of buying of commodity 1 and less of selling of commodity 2 than a choice of (x_1^A, x_2^A). This contradicts that x is a Drèze equilibrium.

<u>Case (ii)</u>: If $\overline{x}_2^A > w_2^A > x_2^A$.

Let $w_2^A = \lambda\overline{x}_2^A + (1-\lambda)\ x_2^A$ with $0 < \lambda < 1$. $\qquad$ (1)

Because of the budget equation,

$$x_1^A = w_1^A + p(w_2^A - x_2^A) \qquad \text{and}$$

$$\overline{x}_1^A = w_1^A + p(w_2^A - \overline{x}_2^A)$$

$$\lambda\overline{x}_1^A+(1-\lambda)x_1^A = w_1^A + p\lambda(w_2^A-\overline{x}_2^A) + (1-\lambda)\ (w_2^A - x_2^A)$$

$$= w_1^A, \text{ using } (1)$$

So, $(w_1^A,w_2^A)= \lambda(\overline{x}_1^A,\overline{x}_2^A) + (1-\lambda)\ (x_1^A,x_2^A)$.

Since both $(\overline{x}_1^A,\overline{x}_2^A)$ and (x_1^A,x_2^A) belong to $R\ \{(x_1^A,x_2^A)\}$, by assumption 4 it follows that $U_A(w_1^A,w_2^A) < U_A(x_1^A,x_2^A)$. This again contradicts that x is a Drèze equilibrium since (w_1^A,w_2^A) is contained in the budget set for A under any restrictions.

Similar contradictions are obtained in the case when $p' < p$. Hence $p = p'$. Also, we have $x_1^A + p'x_2^A = x_1^A+px_2^A = w_1^A+pw_2^A = w_1^A + p'w_2^A$ since $p=p'$. This implies that (x_1^A,x_2^A) is maximal for U_A with initial endowments (w_1^A,w_2^A) and prices $(1,p)$ with no restrictions. Similarly, (x_1^B,x_2^B) is maximal for U_B with endowments (w_1^B,w_2^B) and prices $(1,p)$ with no restrictions. This implies $u \in W$.

Case (b): $(x_1^A, x_2^A) = (w_1^A, w_2^A)$.

Now, x is Pareto efficient and hence is a Walras equilibrium at the supporting prices.

This proves that $E_2 \cap D_1 \subset W$. Any Walras equilibrium is Pareto efficient and is a Drèze equilibrium as well for large enough restrictions. So, $W \subset E_2 \cap D_1$ or, $E_2 \cap D_1 = W$.

Q.E.D.

It can be seen that the above result does not hold if the assumptions 2 and 4 are violated.

Example: Let us consider an economy in which there are two consumers A and B both having initial endowments (1,1) of the two commodities. Let A's preference ordering be given by the function $U_A(x_1^A, x_2^A) = x_1^A + x_2^A$ and B's préference ordering by the function $U_B(x_1^B, x_2^B) = x_1^B$. Now it is easily seen that (1,1) is the only possible price vector for the two commodities as a Walras equilibrium. The corresponding allocation is consumptions of (0,2) for A and (2,0) for B with utility 2 for both.

Consider (1,p) as the price with $0 < p < 1$. The unrestricted demand will be $(0, 1 + \frac{1}{p})$ for A and (1+p,0) for B. Since these do not balance, we restrict A's buying of commodity 2 to one unit. In that case, (1,p) is obtained as a Drèze equilibrium with consumption allocation {(1-p,2), (1+p,0)} and utility allocation {3-p, 1+p} for A and B. This utility allocation is Pareto efficient.

A similar, though slightly weaker result can be shown in the general case of the same exchange economy with N consumers and n commodities. We need a few definitions and a lemma to prove the result in the general case.

Given a strictly positive $x^i \in R^n_+$, define $U^h_i : R^2_+ \to R$ as follows:

$$U_i^h(x_1'^i, x_h'^i) = U_i(x_1'^i, x_2^i, \ldots, x_{h-1}^i, x_h'^i, x_{h+1}^i, \ldots, x_n^i)$$

It is easily verified that U_i^h satisfies the assumptions 2,3 and 4 if U_i does. It is obvious that if U_i is monotone and differentiable, U_i^h is also monotone and differentiable. It remains to check assumption 4.

Let $\quad U_i^h(y_1, y_h) \geqq U_i^h(x_1'^i, x_h'^i)$

and $\quad U_i^h(y_1', y_h') \geqq U_i^h(x_1'^i, x_h'^i)$.

where $\quad (y_1, y_h) \neq (y_1', y_h')$.

Let $\quad (y_1'', y_h'') = \lambda(y_1, y_h) + (1-\lambda)(y_1', y_h')$ where $0 < \lambda < 1$.

Now, if U_i satisfies assumption 4, it is true that

$$(y_1'', x_2^i, \ldots, x_{h-1}^i, y_h'', x_{h+1}^i, \ldots, x_h^i)$$

is in the interior of

$$R\{(x_1'^i, x_2^i, \ldots x_{h-1}^i, x_h'^i, x_{h+1}^i, \ldots, x_n^i)\}$$

$\Longrightarrow$ there is a vector in

$$R\{(x_1'^i, x_2^i, \ldots x_{h-1}^i, x_h'^i, x_{h+1}^i, \ldots x_n^i)\}$$

that is strictly less in every component than

$$(y_1'', x_2^i, \ldots x_{h-1}^i, y_h'', x_{h+1}^i, \ldots, x_n^i).$$

By monotonicity of U_i

$$U_i(y_1'', x_2^i, \ldots, x_{h-1}^i, y_h'', x_{h+1}^i, \ldots, x_n^i) >$$
$$U_i(x_1'^i, x_2^i, \ldots, x_{h-1}^i, x_h'^i, x_h^i, \ldots, x_n^i).$$

Or $\quad U_i^h(y_1'', y_h'') > U_i^h(x_1'^i, x_h'^i)$.

Or $\quad U_i^h$ satisfies assumption 4.

Definition: Let $\{x^i\}$ be a Drèze equilibrium with numeraire exemption. Let p be an associated price vector. A commodity k is said to have a binding quantity restriction on it if for some i_o,

$$(x_1^{i_o}, x_k^{i_o}) \neq (\bar{x}_1^{i_o}, \bar{x}_k^{i_o})$$

where $(\bar{x}_1^{i_o}, \bar{x}_k^{i_o})$ is maximal on R_+^2 for $U_{i_o}^k$ given x^{i_o} with an income $x_1^{i_o} + p_k x_k^{i_o}$ and prices $(1, p_k)$.

Lemma 3.2.2: For any $u \in E_2 \cap D_2$ with the corresponding Drèze equilibrium $\{x^i\}$ and an associated price vector p, there is no h such that both (i) and (ii) hold.

(i) $x_h^{i_o} \neq w_h^{i_o}$ for some i_o.

(ii) A binding restriction is there on h.

Proof: Suppose there is a $u \in E_2 \cap D_2$ and a commodity h that satisfies conditions (i) and (ii). Let us suppose $\bar{x}_h^{i_o} > x_h^{i_o}$ where $(\bar{x}_1^{i_o}, \bar{x}_h^{i_o})$ maximizes $U_{i_o}^h$ given x^{i_o} on R_+^2 with income $x_1^{i_o} + p_h x_h^{i_o}$ and prices $(1, p_h)$. As $x_h^{i_o} \neq w_h^{i_o}$, there is a consumer g such that $x_h^g < w_h^g$.

Since $\{x^i\}$ is Pareto efficient, there is a price vector p' at which $\{x^i\}$ is a Walras equilibrium for an appropriate distribution of initial endowments, in particular with $\{x^i\}$ itself as the initial endowments. The first components of both p and p' is 1 by convention.

U_{i_o} is maximized on R^n_+ at x^{i_o} with income $p'x^{i_o}$ and prices p'. This implies that $U^h_{i_o}$ given x^{i_o} is maximized on R^2_+ at $(x_1^{i_o}, x_h^{i_o})$ with income $x_1^{i_o} + p_h' x_h^{i_o}$ and prices $(1, p_h')$. Also, $\bar{x}_h^{i_o} > x_h^{i_o}$ by asupposition. By lemma 3.2.1 it follows that $p_h < p_h'$.

Let $(\overline{x}_1^g, \overline{x}_h^g)$ be the bundle that maximizes U^h_g given x^g on R^2_+ with income $(x_1^g + p_h x_h^g)$. Since (x_1^g, x_h^g) maximizes U^h_g given x^g on R^2_+ with income $x_1^g + p_h' x_h^g$ and prices $(1, p_h')$, and since $p_h < p_h'$, it follows from lemma 3.2.1 that $\overline{x}_h^g > x_h^g$. Now there are two possibilities. Either $w_h^g \geq \overline{x}_h^g > x_h^g$ or $\overline{x}_h^g > w_h^g > x_h^g$.

(a): If $w_h^g \geq \overline{x}_h^g > x_h^g$,

then $(\overline{x}_1^g, x_2^g, \ldots x_{h-1}^g, \overline{x}_h^g, x_{h+1}^g, \ldots, x_n^g)$

is in the restricted budget set. Further, since

$U^h_g(\overline{x}_1^g, \overline{x}_h^g) > U^h_g(x_1^g, x_h^g)$, it follows that

$U_g(\overline{x}_1^g, x_2^g, \ldots, x_{h-1}^g, \overline{x}_h^g, x_{h+1}^g, \ldots, x_n^g) > U_g(x^g)$.

This contradicts that $\{x^i\}$ is a Drèze equilibrium.

(b): If $\overline{x}_h^g > w_h^g > x_h^g$,

let $w_h^g = \lambda \overline{x}_h^g + (1-\lambda)\, x_h^g$ with $0 < \lambda < 1$.

Let $y_1^g = \lambda\, \overline{x}_1^g + (1-\lambda)\, x_1^g$

Since $x_1^g + p_h x_h^g = \overline{x}_1^g + p_h \overline{x}_h^g$, it follows that

$y_1^g + p_h w_h^g = x_1^g + p_h x_h^g$.

We have $U^h_g(\overline{x}_1^g, \overline{x}_h^g) \geq U^h_g(x_1^g, x_h^g)$.

Also, $(\overline{x}_1^g, \overline{x}_h^g) \neq (x_1^g, x_h^g)$

and $(y_1^g, w_h^g) = \lambda(\overline{x}_1^g, \overline{x}_h^g) + (1-\lambda)(x_1^g, x_h^g)$

with $0 < \lambda < 1$. As U_g^h satisfies assumption 4, we have

$$U_g^h(y_1^g, w_h^g) > U_g^h(x_1^g, x_h^g).$$

Further, $(y_1^g, x_2^g, \ldots, x_{h-1}^g, w_h^g, x_{h+1}^g, \ldots, x_n^g)$ belongs to the restricted budget set and

$$U_g(y_1^g, x_2^g, \ldots, x_{h-1}^g, w_h^g, x_{h+1}^g, \ldots, x_n^g) > U_g(x^g).$$

The above expression contradicts that $\{x^i\}$ is a Drèze equilibrium.

We get similar contradictions in the case when $\overline{x}_h^{i_o} < x_h^{i_o}$. So, we cannot find an u satisfying the conditions of the lemma.

Q.E.D.

<u>Theorem 3.2.2</u>: $E_2 \cap D_2 = W$.

<u>Proof</u>: Let $u \in E_2 \cap D_2$. Let $\{x^i\}$ be the corresponding Drèze equilibrium with associated price $p = (1, p_2, \ldots, p_n)$. Let $p' = (1, p_2', \ldots, p_n')$ be the price vector at which $\{x^i\}$ is a Walras equilibrium with $\{x^i\}$ itself as the initial endowments.

Let $K_1 = \{k \in \{2,3,\ldots,n\} \mid x_k^i \neq w_k^i$, some i$\}$.

and $K_2 = \{k \in \{2,3,\ldots,n\} \mid x_k^i = w_k^i$, all i $\}$.

K_1 or K_2 can be null.

Take any $k \in K_1$. Let $(\overline{x}_1^i, \overline{x}_k^i)$ be the bundle that maximizes U_i^k given x^i with an income $x_1^i + p_k x_k^i$ and price $(1, p_k)$. It is clear as explained in the proof of lemma 3.2.2 that (x_1^i, x_k^i) maximizes U_i^k given x^i with an income $x_1^i + p_k' x_k^i$ and prices $(1, p_k')$. From lemma 3.2.1, if $p_k > p'_k$, then $\bar{x}^i_k < x^i_k$, all i and

if $p_k < p'_k$, then $\bar{x}^i_k > x^i_k$, all i.

We obtain from lemma 3.2.2 that there cannot be a binding quantity restriction on k. So, $p_k = p'_k$ for all $k \in K_1$.

By monotonicity of the utility functions U_i, we have $pw^i = px^i$ for all i.

Since $p_k = p'_k$ for $k \in K_1 \cup \{1\}$

and $w^i_k = x^i_k$ for $k \in K_2$,

we have $p'w^i = p'x^i$.

So, x^i is in the budget set for every i with initial endowments w^i and prices p'. We know by the definition of p' that x^i maximizes U_i at income $p'x^i = p'w^i$ and prices p'. So, $\{x^i\}$ is obtained as a Walras equilibrium with initial endowments $\{w^i\}$ and prices p'. So, $u \in W$.

We have just shown that $u \in E_2 \cap D_2 \Longrightarrow u \in W$. The reverse inclusion is also true since any Walras equilibrium is Pareto efficient and for a choice of large enough quantity restrictions, a Drèze equilibrium as well.

Q.E.D.

3.3 Efficiency of other non-Walrasian Equilibria

Non-Walrasian economies can be defined, following Hahn [9], to be those economies where the individual budget set is not dependent only on a price and initial endowments. Drèze equilibrium is a special case of non-Walrasian equilibria, where all the individuals face equal quantity restrictions. A possible generalization of Drèze equilibrium is one in which we do not insist that all individuals face the same quantity restrictions. This equilibrium has been analysed by Hahn [9]. In what follows, we have the same exchange economy as described in section 3.2.

Definition: $\{x^i\}$ is said to be obtained as a Hahn equilibrium if there is a strictly positive vector $p \in R^n_+$ with $p_1 = 1$ and N rationing schemes $(L^i, \ell^i) \in R^n_+ \times R^n_-$ such that these conditions hold.

(i) x^i is maximal for U_i over $\gamma^i(p, L^i, \ell^i)$.

(ii) $\sum_i x^i - \sum_i w^i = 0$

(iii) $L^i_k = x^i_k - w^i_k$ for some i and some k

$\Longrightarrow$ $x^i_k - w^i_k > \ell^i_k$ for all i.

$\ell^j_k = x^i_k - w^i_k$ for some i and some k

$\Longrightarrow$ $L^i_k > x^i_k - w^i_k$ for all i.

(iv) $L^i_1 > x^i_1 - w^i_1 > \ell^i_1$ for all i.

Define:

$H = \{u \in U \mid$ There is $\{x^i\}$ such that $U_i(x^i) = u_i$ and $\{x^i\}$ is a Hahn equilibrium$\}$.

Theorem 3.3.1: $E_2 \cap H = W$.

Proof : The same proof as for theorem 3.2.2 goes through since it was never needed there that the quantity restrictions are equal for all the consumers.

A proportional rationing scheme (see section 2.3) is a pair of vectors

$$({}_1B, {}_1b) \in [0,1]^{n-1} \times [-1,0]^{n-1},$$

where ${}_1B = (B_2, B_3, \ldots, B_n)$ and ${}_1b = (b_2, b_3, \ldots, b_n)$.

Given a proportional rationing scheme and a price $p \in R^n_{+}$, a straight rationing scheme $(L^i, \ell^i) \in R^n_+ \times R^n_-$ is defined by

$$L^i_k = B_k \, \overline{Z}^{i+}_k$$

$$\ell^i_k = b_k \, \overline{Z}^{i-}_k$$

for $k \in \{2,\ldots,n\}$ and

$$L^i_1 = \sum_i w^i_1 \text{ and } \ell^i_1 = -\sum_i w^i_1$$

where $\overline{Z}^{i+}_k = \max(0, \overline{Z}^i_k)$

$$\overline{Z}^{i-}_k = \max(0, -\overline{Z}^i_k)$$

and $\overline{Z}^i_k$ is the unrestricted demand of consumer i at prices p. Consumers i's budget set under proportional rationing is defined by

$$\gamma^i(p, {}_1B, {}_1b) = \{x \in X_i \mid p(x-w^i) \leq 0,\ L^i({}_1B) \geq x-w^i \geq \ell^i({}_1b)\}$$

<u>Definition</u>: An N-tuple of consumption vectors $\{x^i\}$ is obtained as a proportional rationing equilibrium if there exists a strictly positive vector $p \in R^n_+$ and a proportional rationing scheme $({}_1B, {}_1b) \in [0, 1]^{n-1} \times [-1, 0]^{n-1}$ such that the following holds.

(i) For all i, x^i is a maximal element for U_i of $\gamma^i({}_1B, {}_1b)$.

(ii) $\sum_i (x^i - w^i) = 0$.

(iii) For any $k \in \{2, 3, \ldots, n\}$,

$$B_k < 1 \implies b_k = -1$$

$$b_k > -1 \implies B_k = 1.$$

Let $G = \{u \in U \mid$ There is a proportional rationing equilibrium $\{x^i\}$ with $U_i(x^i) = u_i\}$

Theorem 3.3.2: $E_2 \cap G = W$.

The proof for the above theorem is on the same lines as that of lemma 3.2.2 and Theorem 3.2.2 and hence is omitted.

3.4 Conclusions on efficiency of rationing equilibria

The proposition in section 3.2 indicates that the set of Drèze equilibria can be really large. This is a distinct advantage a policy of rationing has over the market mechanism which is rather restrictive. Indeed, if all the goods are gross substitutes at all the prices, the Walras equilibrium is unique.

Given a welfare function that is monotone in the individual utilities, a Pareto efficient allocation is not necessarily good (equitable) but a Pareto inefficient allocation is necessarily not good since it can be improved upon. The results of section 3.2 and 3.3 indicate that rationing equilibria are in general Pareto inefficient except in the trivial case when the allocation is a Walras equilibrium. Since a Pareto inefficient allocation does not belong to the core, there is every incentive for private redistribution under rationing. This may explain why sometimes black markets do come up under rationing. Even if these are restricted, there is no reason for the central authority to remain satisfied with an inferior allocation.

The only policy known that can generate all the Pareto efficient points is a policy of lump sum transfer of initial endowments. This policy violates the property rights of individuals in private ownership economies. Further, since some of these endowments may include types of labour, a lump sum transfer of these amounts to allowing slavery.

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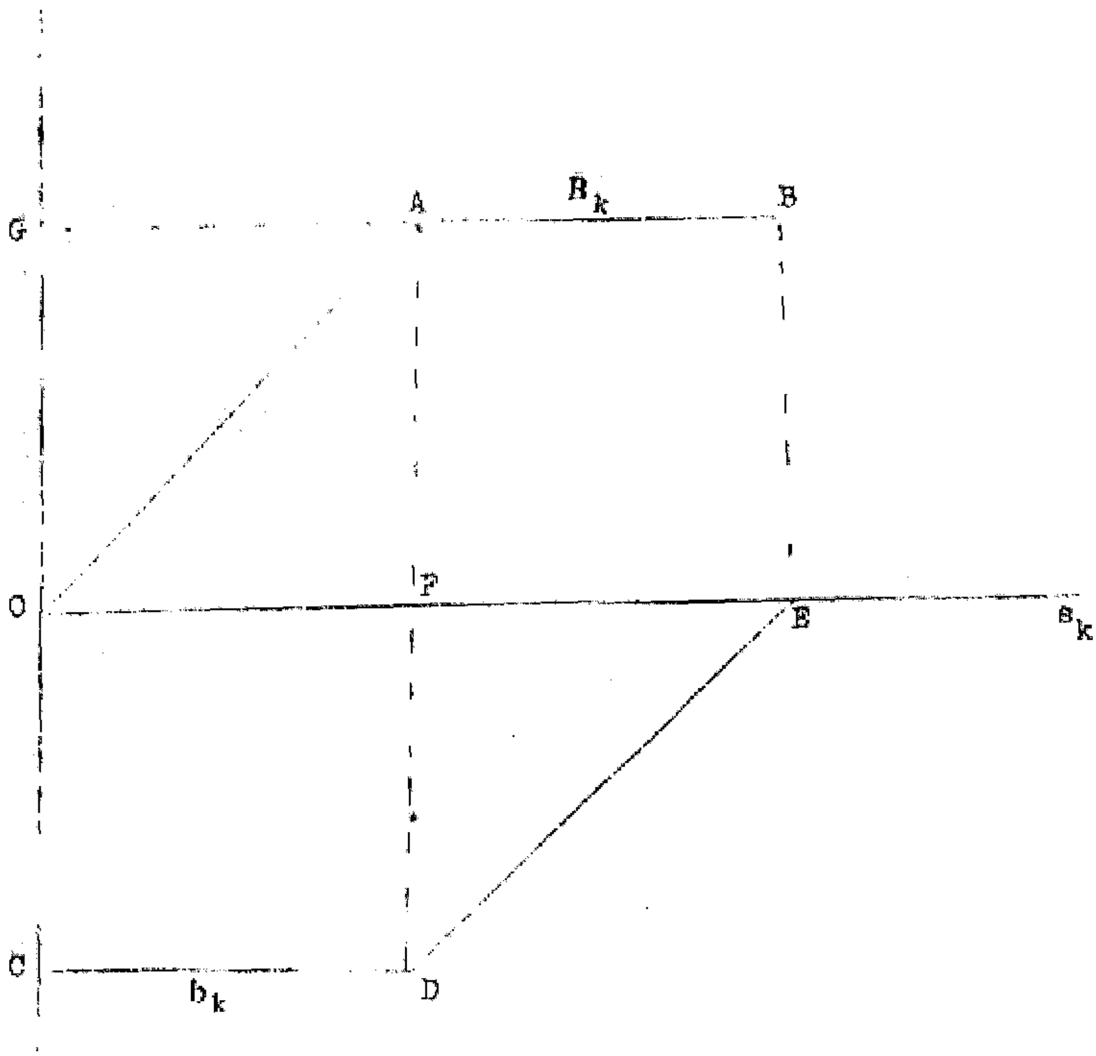

DIAGRAM 1

OAB represents B_k and CDE represents b_k as functions of s_k

$OF = \frac{1}{2}$ · $OE = 1$ · $OG = \frac{1}{2}$ · $OC = -\frac{1}{2}$

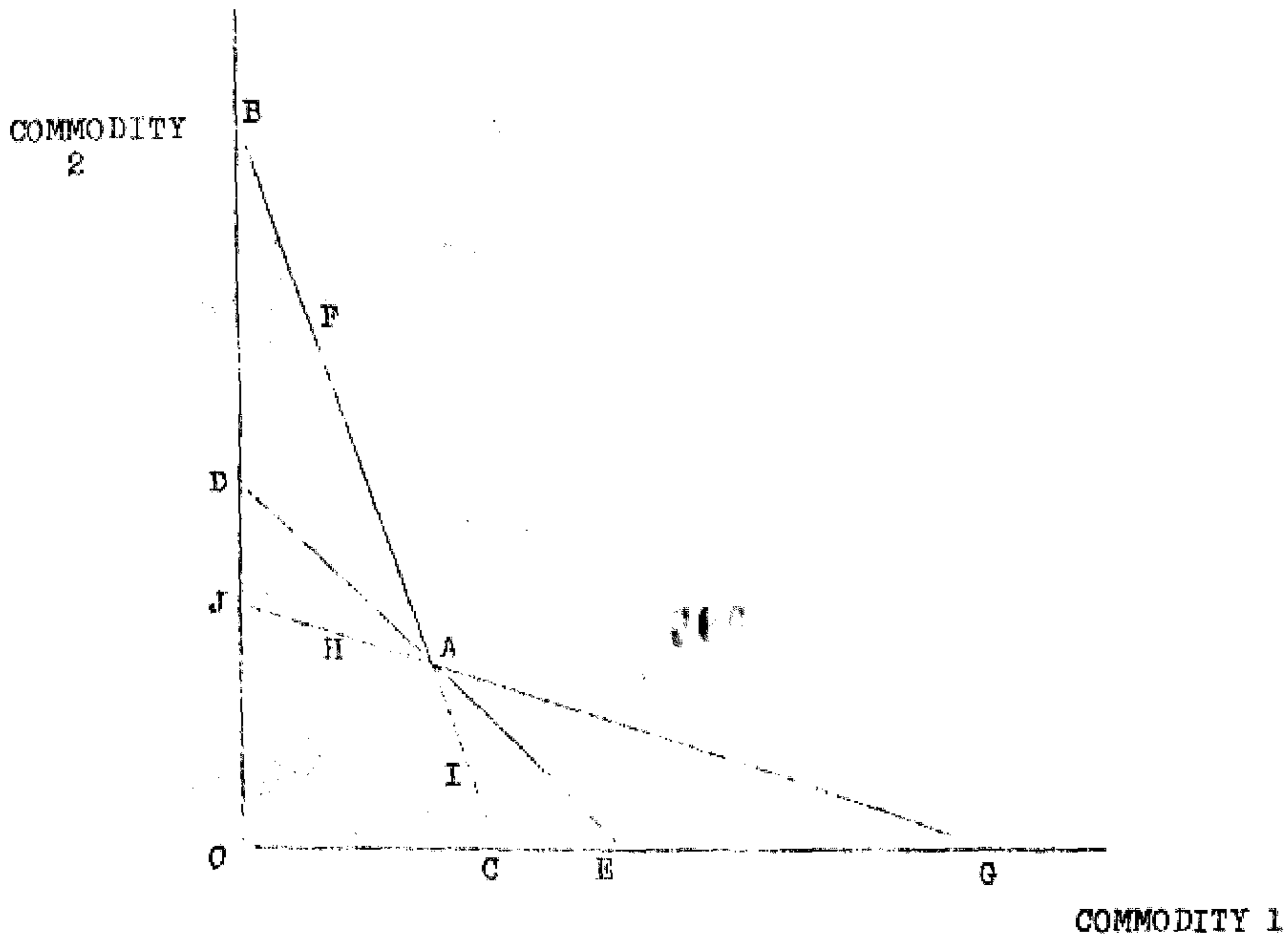

DIAGRAM 2

$A = (\bar{\bar{x}}_1, \bar{\bar{x}}_2)$

DAE denotes the prices $(1, p_2)$.

BAC denotes the prices $(1, p_2')$ with $p_2' < p_2$.

JAG denotes the prices $(1, p_2')$ with $p_2 < p_2'$.

The points F, H and I denote different possibilities of (x_1, x_2)