CANNON-THURSTON MAPS AND RELATIVE HYPERBOLICITY

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To My Parents

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Notation:

1. For a metric space X, d_X will often denote the metric.

2. A geodesic segment joining x and y in X will be denoted by $[x, y]$.

3. For a subset $S \subset X$ and $k \geq 0$, $N_X(S, k)$ will denote the k-neighborhood of S in X.

4. A geodesic triangle with vertices x, y, z will be denoted by $\triangle xyz$.

5. $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ is the usual hyperbolic n-space with metric $ds^2 = \frac{dx_1^2 + ... + dx_n^2}{x_n^2}$.

6. $\mathbb{S}^n$ denote the usual n-sphere with center at origin and radius 1.

7. For $x, y, a \in X$, $(x, y)_a$ will denote the Gromov inner product.

8. For a proper geodesic metric space X, ∂X will denote its Gromov boundary and $\overline{X}$ will be its Gromov compactification.

9. For a geodesic segment λ in X, π_λ will denote a nearest point projection from X onto λ.

10. Let $\mathcal{H}$ denote a collection of uniformly ϵ-separated closed subsets of X. Then $\mathcal{E}(X, \mathcal{H})$ (or $\widehat{X}$ for short) will denote the coned-off space or electric space.

11. Let X be a space strongly hyperbolic relative to $\mathcal{H}$. For $H \in \mathcal{H}$, H^h will denote the hyperbolic cone constructed from H. $\mathcal{G}(X, \mathcal{H})$ (or X^h for short) will denote the hyperbolic metric space obtained from X by attaching hyperbolic cones H^h to H.

12. For an ordered quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$, $\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ (or X_{pel} for short) will denote the partially electrocuted space.

13. For a tree of spaces $\mathbb{P} : X \to T$, v a vertex in T and e an edge in T, X_v will denote the vertex space and X_e will denote the edge space.

Contents

0 Introduction **1**

1 Relative Hyperbolicity **7**

1.1 Hyperbolicity and Nearest Point Projections 7

1.2 Electric Geometry 17

1.2.1 Strongly Relatively Hyperbolic Spaces 18

1.2.2 Relatively Hyperbolic Groups 54

1.2.3 Partial Electrocution 58

1.3 Trees of Spaces: Hyperbolic and Relatively Hyperbolic 69

2 Relatively Hyperbolic Extensions of Groups **75**

2.1 Quasi-isometric Section 76

3 Cannon-Thurston Maps **83**

3.1 Preliminaries on Cannon-Thurston Maps 83

3.2 Cannon-Thurston Maps for Trees of Relatively Hyperbolic Spaces 85

3.2.1 Construction of Hyperbolic Ladder 85

3.2.2 Retraction Map 88

3.2.3 Vertical Quasigeodesic Rays 92

3.2.4 Proof of Main theorem 95

3.3 Cannon-Thurston Maps for Relatively Hyperbolic Extensions of Groups . . 97

3.3.1 Construction of Quasiconvex Sets and Retraction Map 98

3.3.2 Proof of Theorem 101

4 Examples and Applications **105**

4.1 Examples 105

4.1.1 A Combination Theorem 105

4.1.2 Examples 107

4.2 Applications 108

4.2.1 Problems 109

Bibliography **110**

Chapter 0

Introduction

Let $\mathbb{P} : Y \to T$ be a tree of strongly relatively hyperbolic spaces such that Y is also a strongly relatively hyperbolic space. Let X be a vertex space and $i : X \hookrightarrow Y$ denote the inclusion. The main aim of this thesis is to extend i to a continuous map $\bar{i} : \overline{X} \to \overline{Y}$, where $\overline{X}$ and $\overline{Y}$ are the Gromov compactifications of X and Y respectively. Such continuous extensions are called Cannon-Thurston maps. This is a generalization of [Mit98b] which proves the existence of Cannon-Thurston maps for X and Y hyperbolic. By generalizing a result of Mosher [Mos96], we will also prove the existence of a Cannon-Thurston map for the inclusion of a strongly relatively hyperbolic normal subgroup into a strongly relatively hyperbolic group. Let us first briefly sketch the genesis of this problem.

Let H be an infinite quasi-convex subgroup of a word hyperbolic group G. We choose a finite generating set of G that contains a finite generating set of H. Let Γ_H, Γ_G be their respective Cayley graphs with respect to these finite generating sets. Let $\partial\Gamma_H$ and $\partial\Gamma_G$ be hyperbolic boundaries of Γ_H and Γ_G respectively. Then it is easy to show that the inclusion $i\colon \Gamma_H \to \Gamma_G$ canonically extends to a continuous map from $\Gamma_H \cup \partial\Gamma_H$ to $\Gamma_G \cup \partial\Gamma_G$. But if H is not quasi-convex, it is not clear whether there is such an extension. It turns out that for a wide class of non-quasiconvex subgroups such an extension is possible. The first example of this sort was given by J.Cannon and W.Thurston in [CT07] (1989). They showed that if G is the fundamental group of a closed hyperbolic 3-manifold M fibering over a circle with fiber a closed surface S and if H is the fundamental group of S, then there exists a continuous extension for the embedding $i\colon \Gamma_H \to \Gamma_G$. In [Min94], Y.N.Minsky generalized Cannon-Thurston's result to bounded geometry surface Kleinian groups *without* parabolics. Later on, Mitra, in [Mit98a, Mit98b] (1998), gave a different proof of Cannon-Thurston's original result and generalized it in the following two directions:

Theorem 0.0.1. *(Mitra [Mit98a]) Let G be a hyperbolic group and let H be a hyperbolic subgroup that is normal in G. Let $i\colon \Gamma_H \to \Gamma_G$ denote the inclusion. Then i extends to a continuous map $\tilde{i}\colon \Gamma_H \cup \partial\Gamma_H \to \Gamma_G \cup \partial\Gamma_G$.*

Theorem 0.0.2. *(Mitra [Mit98b]) Let (X, d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let v be a vertex of T. If X is hyperbolic then there exists a Cannon-Thurston map for $i\colon X_v \to X$, where X_v is the vertex space corresponding to v.*

Let Σ be a compact surface of genus $g(\Sigma) \geq 1$ with a finite non-empty collection of boundary components $\{C_1, ..., C_m\}$. Subgroups of $\pi_1(\Sigma)$ corresponding to the fundamental groups of the boundary curves are called peripheral subgroups. Consider a discrete and faithful action of $\pi_1(\Sigma)$ on $\mathbb{H}^3$. The action is *strictly type preserving* if the maximal parabolic subgroups are precisely the peripheral subgroups of $\pi_1(\Sigma)$. Let N be the quotient manifold obtained from $\mathbb{H}^3$ under this action. Let $\mathrm{inj}(N)$ denote half the length of the shortest closed geodesic in N. $\mathrm{inj}(N)$ is called the injectivity radius away from cusps. B.H.Bowditch, in [Bow07], proved that if $\mathrm{inj}(N) > 0$ then there exists a Cannon-Thurston map for the induced embedding $i : \Sigma \to N$. In [Mja], Mahan Mj. gave an alternate proof of Bowditch's result and generalized it to 3-manifolds where cores are incompressible away from cusps.

M.Gromov, in [Gro87], defined the notion of relative hyperbolicity for a geodesic metric space. Let G be a finitely generated group acting properly discontinuously and cocompactly by isometries on a complete and locally compact hyperbolic space X. Then due to the Švarc Milnor Lemma (refer to [BH99]), the Cayley graph of G is quasi-isometric to X and hence G is a hyperbolic group. Now if we replace the cocompact action of G on X by an action such that the quotient space is quasi-isometric to a finite union of rays emerging from a fixed point, then we get Gromov's notion of a relatively hyperbolic group. Benson Farb, in [Far98], studied relative hyperbolicity from a different perspective. He gave an alternate definition of relative hyperbolicity.

A finitely generated group G is said to be strongly hyperbolic relative to H (in the sense of Farb) if the following two conditions hold:

1. The 'Coned-off' graph $\widehat{\Gamma}_G$, obtained from the Cayley graph Γ_G of G by coning the left cosets, is hyperbolic.

2. Two quasigeodesics in $\widehat{\Gamma}_G$ joining the same pair of points satisfy a property called 'Bounded Coset Penetration'. Roughly, it means that
   - if one quasigeodesic penetrates a left coset and the other does not then the

distance between the entry and exit points of the quasigeodesic penetrating the left coset is bounded, and

• if two quasigeodesics penetrate the same left coset then the distance between the entry points is bounded; similarly for the exit points.

If the group G satisfies only the first condition then G is said to be weakly hyperbolic relative to H. Similarly for a geodesic metric space X and a collection of uniformly separated subsets $\mathcal{H}$ of X, we have the Farb's notion of a strongly relatively hyperbolic space $(X, \mathcal{H})$ (a brief definition is given before the end of this section). As in this thesis we deal mostly with strongly relatively hyperbolic spaces, relative hyperbolicity will mean strong relative hyperbolicity.
In [Bow97], Bowditch proved the equivalence of the two notions of relative hyperbolicity. He also introduced the notion of a relative hyperbolic boundary for relatively hyperbolic metric spaces. If S is a punctured torus then its fundamental group $\pi_1(S) = \mathbb{F}(a,b)$ (free group with two generators) is hyperbolic relative to the cusp subgroup $H = < aba^{-1}b^{-1} >$. In fact, $\pi_1(S)$ acts discretely on the upper half plane $\mathbb{H}^2$ and stabilizes a point on the boundary with stabilizer subgroup H. The relative hyperbolic boundary for the Cayley graph of S is the Gromov boundary $\partial\mathbb{H}^2$ of $\mathbb{H}^2$.

In [BF92], a combination theorem for trees of hyperbolic metric spaces was proved by Bestvina and Feighn. It states that a tree of hyperbolic metric spaces is hyperbolic if it satisfy the 'quasi-isometrically embedded' condition and the 'Hallways flare' conditions. Based on their work a combination theorem for trees of (strongly) relatively hyperbolic spaces was proved by Mahan Mj. and Lawrence Reeves in [MR08]. While proving this theorem they have extended Farb's notion of strong relative hyperbolicity and construction of an electric space to that of a 'partially electrocuted space'. In a partially electrocuted space, instead of coning all of a horosphere down to a point we cone it down to a hyperbolic metric space. It is natural to ask for the existence of a Cannon-Thurston map for the inclusion of a relatively hyperbolic space as a vertex space into a tree of relatively hyperbolic spaces.

In this thesis, we prove the existence of a Cannon-Thurston map for the embedding of a vertex space into a tree of relatively hyperbolic spaces. This is a generalization of Theorem 0.0.2.

Theorem 0.0.3. *[MP][Refer to Theorem 3.2.9] Let X be a proper geodesic space and $\mathbb{P} : X \to T$ be a tree of relatively hyperbolic spaces satisfying the quasi-isometrically embedded condition. Further suppose that the inclusion of edge-spaces into vertex spaces is strictly type-preserving, and the induced tree of coned-off spaces continue* *to satisfy the quasi-isometrically embedded condition. If X is strongly hyperbolic relative to the family $\mathcal{C}$ of maximal cone-subtrees of horosphere-like sets, then a Cannon-Thurston map exists for the proper embedding $i_{v_0}\colon X_{v_0} \to X$, where v_0 is a vertex of T and (X_{v_0}, d_{v_0}) is the relatively hyperbolic metric space corresponding to v_0.*

Sketch of Proof: For a relatively hyperbolic space $(Y, \mathcal{H}_Y)$, $\widehat{Y}$ will denote the coned-off space and Y^h will denote the hyperbolic space obtained from Y by gluing 'hyperbolic cones' (brief definitions are given before the end of this section).

A Cannon-Thurston map for i_{v_0} exists (see Lemma 3.1.4) if the following holds: If the underlying relative geodesic λ (in X_{v_0}) of an electric geodesic segment $\hat{\lambda}$ in $\widehat{X}_{v_0}$ lies outside a large ball in $(X_{v_0}, d_{X_{v_0}})$ modulo horospheres then, for an electric segment $\hat{\beta}$ joining end points of λ in $\widehat{X}$, the underlying geodesic segment β lies outside a large ball in X modulo horospheres.
Let $\mathcal{TC}(X)$ be the tree of coned-off spaces obtained from the tree of relatively hyperbolic spaces, X, by coning horospheres in each vertex and edge space to a point. As in [Mit98b], the key step for proving the existence of a Cannon-Thurston map is to construct a hyperbolic ladder $\Xi_{\hat{\lambda}}$ in $\mathcal{TC}(X)$ and a large-scale Lipschitz retraction $\hat{\Pi}_{\hat{\lambda}}$ from $\mathcal{TC}(X)$ onto $\Xi_{\hat{\lambda}}$. This proves the quasiconvexity of $\Xi_{\hat{\lambda}}$. Further, we shall show that if the underlying relative geodesic λ of $\hat{\lambda}$ lies outside a large ball in $(X_{v_0}, d_{X_{v_0}})$ modulo horospheres then $\Xi_{\hat{\lambda}}$ lies outside a large ball in X modulo horospheres. Quasiconvexity of $\Xi_{\hat{\lambda}}$ ensures that geodesics joining points on $\Xi_{\hat{\lambda}}$ lie close to it modulo horospheres.

We consider here electric geodesics in the coned-off vertex and edge-spaces $\widehat{X_v}$ and $\widehat{X_e}$. In [Mit98b], it was assumed that each X_v, X_e are δ-hyperbolic metric spaces and took $\lambda = \hat{\lambda}$, hence it was necessary to find points in some C-neighborhood of λ to construct Ξ_λ. Since there is only the usual (Gromov)-hyperbolic metric in [Mit98b], this creates no confusion. But, in the present situation, we have two metrics d_{X_v} and $d_{\widehat{X}_v}$ on X_v. As electrically close (in the $d_{\widehat{X}_v}$ metric) does not imply close (in the d_{X_v} metric), we cannot take a C-neighborhood in the $d_{\widehat{X}_v}$ metric. Instead we will first construct an electroambient representative λ of $\hat{\lambda}$ in the space X_v^h and take a hyperbolic neighborhood of λ in X_v^h.
Now choose a geodesic segment with length maximal in the electric metric such that its end points lie in the intersection of a bounded neighborhood of λ and an edge space, and then 'flow' the end points to the adjacent vertex space. Join the resulting end points by geodesic segments in the corresponding vertex spaces. Repeating this process, we obtain a 'ladder' $\Xi_{\widehat{\lambda}}$. Finally we construct vertical quasigeodesic rays in $\Xi_{\hat{\lambda}}$ to show that if $\hat{\lambda} \backslash \bigcup_{H_{v\alpha} \in \mathcal{H}_v} H_{v\alpha}$ lies outside a large ball in X_v, then $(\Xi_{\hat{\lambda}} \backslash \bigcup_{C_\alpha \in \mathcal{C}} C_\alpha)$

lies outside a large ball in X. The existence of a Cannon-Thurston map follows.

Our next objective is to generalize Theorem 0.0.1 for relatively hyperbolic groups. Let K be a hyperbolic normal subgroup of a hyperbolic group G with quotient Q. The following Theorem, due to L.Mosher [Mos96], proves that Q is hyperbolic.

Theorem 0.0.4. *(Mosher [Mos96]) Let us consider the short exact sequence of finitely generated groups*

$$1 \to K \to G \to Q \to 1.$$

such that K is non-elementary word hyperbolic. If G is hyperbolic, then there exists a quasi-isometric section $s\colon Q \to G$. Hence Q is hyperbolic.

We will generalize Theorem 0.0.4 to the following :

Theorem 0.0.5. *[Pal][Theorem 2.1.6] Suppose we have a short exact sequence of finitely generated groups*

$$1 \to K \to G \xrightarrow{p} Q \to 1,$$

with K hyperbolic relative to a non-trivial proper subgroup K_1 and G preserves cusp i.e. for all $g \in G$ there exists $k \in K$ such that $gK_1g^{-1} = kK_1k^{-1}$. Then there exists a (R, ϵ)-quasi-isometric section $s\colon Q \to G$ for some constants $R \geq 1$ and $\epsilon \geq 0$.

Sketch of Proof: Let Π be the set of all parabolic end points and Π^2 denote the set of all distinct pair of parabolic end points. Let $\alpha = (\alpha_1, \alpha_2) \in \Pi^2$, then stabilizer subgroups of α_i's are $a_iK_1a_i^{-1}$ for some $a_i \in K$, where $i = 1, 2$. Due to the bounded coset penetration property, for any two relative geodesics joining left cosets a_1K_1 and a_2K_1, the diameter of the set of exit points of these relative geodesics from a_1K_1 is uniformly bounded. Let C be the set of all $(\alpha_1, \alpha_2) \in \Pi^2$ for which the identity element of K belongs to the set of exit points of relative geodesics from the left coset a_1K_1 to a_2K_1. For $g \in G$, the automorphism I_g, defined as $I_g(k) = gkg^{-1}$, acts on the relative hyperbolic boundary of K and hence acts also on Π^2. Fix an element $\eta \in \Pi^2$, let Σ be the set of all $g \in G$ for which $\eta \in I_g(C)$. Then we show that there exist constants $R \geq 1$ and $\epsilon \geq 0$ such that for all $g, g' \in \Sigma$

$$\frac{1}{R} d_Q(p(g), p(g')) - \epsilon \leq d_G(g, g') \leq R d_Q(p(g), p(g')) + \epsilon.$$

Following the scheme of the proof of 0.0.3, we will generalize Theorem 0.0.1 to the following:

Theorem 0.0.6. *[Pal][Theorem 3.3.5] Consider a short exact sequence of finitely generated groups*

$$1 \to K \to G \xrightarrow{p} Q \to 1$$

with K hyperbolic relative to a proper non-trivial subgroup K_1. Suppose that

1. *G preserves cusp,*
2. *G is (strongly) hyperbolic relative to $N_G(K_1)$ and,*
3. *G is weakly hyperbolic relative to the subgroup K_1.*

Then there exists a Cannon-Thurston map for the embedding $i\colon \Gamma_K \to \Gamma_G$, where Γ_K and Γ_G are Cayley graphs of K and G respectively.

In chapter 1, we survey some basic facts about relatively hyperbolic spaces. Here we give two definitions of a relatively hyperbolic space. For a geodesic space X and a collection of uniformly separated subsets $\mathcal{H}$ of X, we will construct a space $\mathcal{G}(X, \mathcal{H})$ (or X^h for short) from X by attaching 'hyperbolic cones H^h' (analog of horoballs) to each $H \in \mathcal{H}$. Elements of $\mathcal{H}$ will be referred to as horosphere-like sets. X is said to be hyperbolic relative to $\mathcal{H}$ in the sense of Gromov if $\mathcal{G}(X, \mathcal{H})$ is a hyperbolic metric space. Let $\mathcal{E}(X, \mathcal{H})$ (or $\widehat{X}$ for short) be the 'Coned-off' space obtained from X by coning each $H \in \mathcal{H}$ to a single point, then X is said to be hyperbolic relative to $\mathcal{H}$ in the sense of Farb if

1. $\mathcal{E}(X, \mathcal{H})$ is hyperbolic.
2. Quasi-geodesics in $\mathcal{E}(X, \mathcal{H})$ joining same pair of points satisfy 'bounded horosphere penetration' properties. It means that
   • if one quasigeodesic penetrates a horosphere-like set $H \in \mathcal{H}$ and the other does not then the distance between the entry and exit points of the quasi-geodesic penetrating H is bounded, and
   • if two quasigeodesics penetrate the same horosphere-like set then the distance between the entry points is bounded; similarly for the exit points.

In Chapter 1, we shall prove that these two definitions are equivalent. Partial electrocution and trees of relatively hyperbolic spaces are also introduced in this chapter. In chapter 2, Theorem 0.0.5 is proven. In chapter 3, we first give a criterion for the existence of a Cannon-Thurston map and then by constructing 'Hyperbolic Ladders', 'Retraction Maps' and 'Vertical Quasigeodesic Rays' in trees of relatively hyperbolic spaces, we proceed to prove Theorem 0.0.3. For a short exact sequence of relatively hyperbolic groups, we make similar constructions and prove Theorem 0.0.6. Finally, in chapter 4, we give some examples and applications.

Chapter 1

Relative Hyperbolicity

1.1 Hyperbolicity and Nearest Point Projections

Definition 1.1.1. *Let (X, d) be a metric space and $x, y \in X$. A* **geodesic path** *joining x and y is an isometric map $\gamma : [0, d(x, y)] \to X$ such that $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. X is said to be a* **geodesic metric space** *if for all $x, y \in X$ there exists a geodesic path joining x and y. A* **geodesic ray** *is a map $\gamma \colon [0, \infty) \to X$ such that $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, \infty)$.*

Definition 1.1.2. *Let (X, d) be a metric space.*

- **Geodesic Triangle:** *A geodesic triangle in X consists of three points $x, y, z \in X$ (vertices) and three geodesic segments $[x, y], [y, z], [z, x]$ (sides) joining them. A geodesic triangle with vertices x, y, z will be denoted as $\triangle xyz$.*

- **Slim Triangles:***[Aea91] Let $\delta \geq 0$. Given $x, y, z \in X$, we say that a geodesic triangle Δxyz is δ-slim if any side of the triangle Δxyz is contained in the δ- neighborhood of the union of the other two sides.*

- **Thin Triangles:***([Aea91])Let $\delta \geq 0$. Given a geodesic triangle Δxyz, let $\Delta' x'y'z'$ be a Euclidean comparison triangle with sides of the same lengths (i.e. $d_E(x', y') = d(x, y), d_E(y', z') = d(y, z), d_E(z', x') = d(z, x)$). There is a natural identification map $f : \Delta xyz \to \Delta' x'y'z'$. The maximum inscribed circle in $\Delta' x'y'z'$ meets the side $[x'y']$ (respectively $[x'z'], [y'z']$) in a point c_z (respectively c_y, c_x) such that*

  $$d(x', c_y) = d(x', c_z), d(y', c_x) = d(y', c_z), d(z', c_x) = d(z', c_y).$$

  There is a unique isometry t_Δ of the triangle $\Delta' x'y'z'$ onto a tripod T_Δ, a tree with one vertex w of degree 3, and vertices x'', y'', z'' each of degree one such *that $d(w, z'') = d(z, c_y) = d(z, c_x)$ etc. Let $f_\Delta = t_\Delta \circ f$. We say that Δxyz is δ-thin if for all $p, q \in \Delta$, $f_\Delta(p) = f_\Delta(q)$ implies $d(p, q) \leq \delta$.*

Proposition 1.1.3. *(Proposition 2.1, [Aea91]) Let X be a geodesic metric space. The following are equivalent:*
1. There exists $\delta_0 \geq 0$ such that every geodesic triangle in X is δ_0-slim.
2. There exists $\delta_1 \geq 0$ such that every geodesic triangle in X is δ_1-thin.

Definition 1.1.4. *A geodesic metric space is said to be δ-hyperbolic if it satisfies one of the equivalent conditions of Proposition 1.1.3 for that δ. A geodesic metric space is said to be hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.*

Example 1.1.5. *1. Trees are 0-hyperbolic metric spaces.*

2. It is a standard fact that $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ with metric $ds^2 = \frac{dx_1^2 + ... + dx_n^2}{x_n^2}$ is $\frac{1}{2}\log 3$-hyperbolic.

Definition 1.1.6. ***Gromov Inner Product:*** *Let (X, d) be a metric space. Choose a base point $a \in X$. The Gromov inner product on X with respect to a is defined by*

$$(x, y)_a = \frac{1}{2}(d(x, a) + d(y, a) - d(x, y)).$$

Definition 1.1.7. *Let $\delta \geq 0$. A metric space X is said to be (δ)-hyperbolic if*

$$(x, y)_a \geq \min\{(x, z)_a, (y, z)_a\} - \delta$$

for all $a, x, y, z \in X$.

Proposition 1.1.8. *[BH99] Let X be a geodesic space. X is hyperbolic in the sense of 1.1.4 if and only if there is a constant $\delta > 0$ such that X is (δ)-hyperbolic in the sense of 1.1.7.*

The following Proposition allows us to replace length spaces by metric graphs.

Proposition 1.1.9. *(Proposition 8.45, Chapter I.8, [BH99]) There exist universal constants $\mathbb{S} \geq 1$ and $\varepsilon \geq 0$ such that there is a $(\mathbb{S}, \varepsilon)$-quasi-isometry from any length space to a metric graph all of whose edges have length one.*

Let (X, d) be a geodesic metric space, we will say that two geodesic rays $c_1 : [0, \infty) \to X$ and $c_2 : [0, \infty) \to X$ are equivalent and write $c_1 \sim c_2$ if there is a $K > 0$ such that for any $t \in [0, \infty)$, $d(c_1(t), c_2(t)) \leq K$. It is easy to check that $\sim$ is an equivalence relation on the set of geodesic rays. The equivalence class of a ray c will be denoted by $[c]$.

Definition 1.1.10. *([Gro87],[BH99])* ***Geodesic boundary:*** *Let (X, d) be a δ-hyperbolic metric space. We define the geodesic boundary of X as*

$$\partial X := \{[c] |\ c : [0, \infty] \to X \textit{ is a geodesic ray } \}.$$

A metric space (X, d) is said to be proper if all closed metric balls of finite radius in X are compact.

Lemma 1.1.11. *(Visibility of ∂X)(Lemma 3.2, Chapter III.H, [BH99]) Let X be a proper, δ-hyperbolic geodesic space, then for each pair of distnct points $\xi_1, \xi_2 \in \partial X$, there exists a geodesic $c : \mathbb{R} \to X$ such that $c(\infty) = \xi_1$ and $c(-\infty) = \xi_2$.*

Notation: A *generalized ray* is a geodesic $c : I \to X$, where either $I = [0, R]$ for some $R \geq 0$ or else $I = [0, \infty)$. In case $I = [0, R]$, we define $c(t) = c(R), t \in [R, \infty)$. Thus $\overline{X} := X \cup \partial X$ is the set $\{c(\infty) \mid c \text{ a generalized ray}\}$.

Definition 1.1.12. *(The Topology on $\overline{X} = X \cup \partial X$)(Definition 3.5, Chapter III.H, [BH99]) Let X be a proper geodesic space that is δ-hyperbolic. Fix a base point $p \in X$. We define convergence in $\overline{X}$ by: $x_n \to x$ as $n \to \infty$ if and only if there exist generalized rays c_n with $c_n(0) = p$ and $c_n(\infty) = x_n$ such that every subsequence of (c_n) contains a subsequence that converges (uniformly on compact subsets) to a generalized ray c with $c(\infty) = x$. This defines a topology on $\overline{X}$: the closed subsets $B \subset \overline{X}$ are those which satisfy the condition $[x_n \in B,\ \text{for all } n > 0 \text{ and } x_n \to x] \Rightarrow x \in B$.*

Proposition 1.1.13. *(Proposition 3.7, Chapter III.H, [BH99]) Let X be a geodesic space that is δ-hyperbolic.*
(1). The topology on $\overline{X} = X \cup \partial X$ described in 1.1.12 is independent of the choice of the base point,
(2). The inclusion $X \hookrightarrow \overline{X}$ is a homeomorphism onto its image and $\partial X \subset X$ is closed,
(3). $\overline{X}$ is compact.

$\overline{X}$ will be said to be the *Gromov compactification* of X.

Let X be a δ-hyperbolic metric space and $p \in X$ be a base point. We say that a sequence $(x_n)_{n\geq 1}$ of points in X converges to infinity if $\lim_{i,j\to\infty}(x_i, x_j)_p = \infty$. Note that this definition does not depend on the choice of base point. We shall say that two sequences (x_n) and (y_n) converging to infinity are said to be equivalent and write $(x_n) \sim (y_n)$ if $\lim_{i\to\infty}(x_i, y_i)_p = \infty$. It is easy to check that $\sim$ is an equivalence relation on the set of sequences converging to infinity and that the definition of equivalence does not depend on the choice of a base point $p \in X$. The equivalence class of a sequence (x_n) converging to infinity will be denoted by $[(x_n)]$.

Definition 1.1.14. *([Gro87],[BH99],[Aea91])* ***Sequential boundary:*** *Let (X, d) be a δ-hyperbolic metric space. We define the sequential boundary of X as*

$$\partial_s X := \{[(x_n)] \mid (x_n) \text{ is a sequence converging to infinity in } X\}.$$

Lemma 1.1.15. *(Lemma 3.13, Chapter III.H, [BH99]) If X is a proper geodesic space that is δ-hyperbolic, then there is a natural bijection $\partial_s X \to \partial X$.*

Example 1.1.16.

1. *Boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ is homeomorphic to $\mathbb{S}^n$.*
2. *The boundary of a locally finite regular tree with valence of each vertex at least 3 is homeomorphic to a Cantor set.*

Definition 1.1.17. *Let $k \geq 0$. A subset S of a geodesic space X is said to be* k**-quasiconvex** *if any geodesic joining $x, y \in S$ lies in a k-neighborhood of S. A subset S is* **quasiconvex** *if it is k-quasiconvex for some k.*

Definition 1.1.18. *Let $K \geq 1$ and $\epsilon \geq 0$. A map $f : (Y, d_Y) \to (Z, d_Z)$ is said to be a* (K, ϵ)**-quasi-isometric embedding** *if*

$$\frac{1}{K} d_Y(y_1, y_2) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \epsilon$$

for all $y_1, y_2 \in Y$. If f is a (K, ϵ)-quasi-isometric embedding and every point of Z lies in a uniformly bounded distance from $f(Y)$, then f is said to be a (K, ϵ)**-quasi-isometry**.
A map $f : Y \to Z$ is said to be a **quasi-isometry** *if it is a (K, ϵ)-quasi-isometry for some $K \geq 1$ and $\epsilon \geq 0$.*

Proposition 1.1.19. *If $\phi : Y \to Z$ is a quasi-isometry then there is a quasi-isometry $\psi : Z \to Y$ such that, for all $y \in Y, z \in Z$, $d_Y(\psi(\phi(y)), y) \leq K_{1.1.19}$ and $d_Z(\phi(\psi(z)), z) \leq K_{1.1.19}$ for some number $K_{1.1.19} > 0$ depending only on constants of quasi-isometries.*

We refer to such a map ψ as a ***quasi-isometric inverse*** of ϕ. Quasi-isometric inverse of ϕ will be denoted by ϕ^{-1}.

Definition 1.1.20. *A map $f : X \to Y$ between metric spaces is said to be proper, if for all $M > 0$ there exists $N(M) > 0$ such that $d_Y(f(x), f(y)) \leq M$ implies $d_X(x, y) \leq N$.*

Lemma 1.1.21. *Let $Q \geq 0$ and suppose $i : X \to Y$ is a proper and length preserving map between two length spaces X, Y such that $i(X)$ is Q-quasiconvex in Y, then there exists $K_{1.1.21}(Q) \geq 1, \epsilon_{1.1.21}(Q) \geq 0$ such that i is an $(K_{1.1.21}, \epsilon_{1.1.21})$-quasi-isometric embedding.*

Proof. Let $x, y \in X$. As i is length preserving, for any path α in X, $l_X(\alpha) = l_Y(i(\alpha))$. Therefore, $d_Y(i(x), i(y)) \leq d_X(x, y)$. Now, as Y is a length space, for $\kappa > 0$ there exists a path $\alpha : [0,1] \to Y$ such that $l_Y(\alpha) \leq d_Y(i(x), i(y)) + \kappa$.
Let $0 = t_0 < t_1 < ... < t_n = 1$ be a partition of $[0,1]$ such that $l_Y(\alpha|_{[t_{j-1},t_j]}) = 1$ for $1 \leq j \leq n-1$ and $l_Y(\alpha|_{[t_{n-1},t_n]}) \leq 1$. For each j, there exists $p_j \in X$ such that $d_Y(\alpha(t_j), i(p_j)) \leq Q$ with $p_0 = x$ and $p_n = y$. Thus, $d_Y(i(p_j), i(p_{j+1})) \leq 2Q + 1$ for all $0 \leq j \leq n-1$. Since the map i is proper, therefore there exists $R > 0$ such that $d_X(p_j, p_{j+1}) \leq R$. Hence, by triangle inequality, we have

$$d_X(x, y) \leq nR \leq Rl_Y(\alpha) + R \leq R(d_Y(i(x), i(y)) + \kappa) + R.$$

Taking $\kappa \to 0$, we have $d_X(x, y) \leq Rd_Y(i(x), i(y)) + R$. Taking $K_{1.1.21} = \epsilon_{1.1.21} = R$, we have the required result. ☐

Definition 1.1.22. *Let $K \geq 1$ and $\epsilon \geq 0$. A (K, ϵ)-**quasigeodesic** in a metric space X is a (K, ϵ)-quasi-isometric embedding $\gamma : J \to X$, where J is an interval (bounded or unbounded) of the real line $\mathbb{R}$. A (K, K)-quasigeodesic in X will be called as K-**quasigeodesic**.*

Proposition 1.1.23. *(Taming Quasigeodesics, Lemma 1.11, Chapter III.H, [BH99]) Let X be a geodesic space. Given any (K, ϵ)-quasigeodesic $c : [a, b] \to X$, there exists a continuous $(K_{1.1.23}, \epsilon'_{1.1.23})$-quasigeodesic $c' : [a, b] \to X$ such that the following holds:*
(i) $c'(a) = c(a)$, $c'(b) = c(b)$;
(ii) $\epsilon'_{1.1.23} = 2(K + \epsilon), K_{1.1.23} = K$;
(iii) $l(c'|_{[t,t']}) \leq k^1_{1.1.23} d(c'(t), c'(t')) + k^2_{1.1.23}$ for some constants $k^1_{1.1.23} \geq 1, k^2_{1.1.23} > 0$ depending only on K, ϵ;
(iv) the Hausdorff distance between the images of c and c' is less than $K + \epsilon$.

Definition 1.1.24. *Let X be a geodesic space and $K \geq 1$ and $\epsilon \geq 0$. A path $\alpha : [0, 1] \to X$ is said to be (K, ϵ)-tamed if $l(\alpha|_{[t,t']}) \leq Kd(\alpha(t), \alpha(t')) + \epsilon$ for all $t, t' \in [0, 1]$.*

Several authors take definition of a quasigeodesic to be arc length reparametrization of a tamed path. However, for both quasigeodesics and tamed paths, the following stability property holds:

Proposition 1.1.25. *(Stability of quasigeodesics (Theorem 1.7, Chapter III.H, [BH99]), Stability of tamed path (Proposition 3.3, [Aea91])): Suppose X is a δ-hyperbolic metric space and $x, y \in X$. If α is a (K, ϵ)-quasigeodesic or a (K, ϵ)-tamed path between the points x, y then there exists $L_{1.1.25} = L_{1.1.25}(\delta, K, \epsilon) > 0$ such that if γ is any geodesic joining x and y, then $\gamma \subset N_X(\alpha, L_{1.1.25})$ and $\alpha \subset N_X(\gamma, L_{1.1.25})$.*

For a metric space Z, note that if α is a (K,ϵ)-quasigeodesic then α followed by a geodesic of length at most k is a $(K,\epsilon+k)$-quasigeodesic. Thus we have the following corollary:

Corollary 1.1.26. *Given $\delta, k, \epsilon \geq 0, K \geq 1$ there exists $L_{1.1.26} > 0$ such that the following holds:*
Suppose (X,d) is a δ-hyperbolic metric space and $x,y,z,w \in X$ such that $d(x,z) \leq k$ and $d(y,w) \leq k$. If α is a (K,ϵ)-quasigeodesic joining x,y and γ be a geodesic joining z,w then $\gamma \subset N_X(\alpha, L_{1.1.26})$ and $\alpha \subset N_X(\gamma, L_{1.1.26})$.

Definition 1.1.27. *Let $k \geq 0$. A path $\alpha : [0,1] \to X$ is said to be a* **stable k-quasiconvex path** *if for all $t,t' \in [0,1]$, the Hausdorff distance between $\alpha|_{[t,t']}$ and any geodesic joining $\alpha(t)$ and $\alpha(t')$ is at most k.*

All quasigeodesics and tamed paths in a hyperbolic metric space are stable quasiconvex paths.

Definition 1.1.28. *Suppose (X,d) is a metric space and S is a subset of X. A map π_S from X onto S is said to be a* ***nearest point projection*** *if for each $x \in X$, $d(x,\pi_S(x)) \leq d(x,y)$ for all $y \in S$.*

Suppose (X,d) is a δ-hyperbolic metric space and λ be a geodesic in X. Note that for $x \in X$ if there exist two points $a,b \in \lambda$ such that $d(x,a) \leq d(x,y)$ and $d(x,b) \leq d(x,y)$ for all $y \in \lambda$ then for the geodesic triangle $\triangle xab$, due to δ-hyperbolicity of X, there exist $w_1 \in [x,a], w_2 \in [a,b], w_3 \in [x,b]$ such that diameter of the set $\{w_1,w_2,w_3\}$ is at most δ. Now $d(w_1,a) \leq d(w_1,w_2) \leq \delta$ and $d(w_3,b) \leq d(w_3,w_2) \leq \delta$. Therefore $d(a,b) \leq d(a,w_1) + d(w_1,w_3) + d(w_3,b) \leq 3\delta$. Thus if $\pi^1_\lambda, \pi^2_\lambda$ are two nearest point projections from X onto λ, then $d(\pi^1_\lambda(x), \pi^2_\lambda(x)) \leq 3\delta$ for all $x \in X$. Similarly, for a quasiconvex set $S \subset X$, nearest point projections π_S are defined up to a bounded amount of discrepancy.

Lemma 1.1.29. *Let X be a geodesic metric space and $\lambda : [a,b] \to X$ be a geodesic. Let $x \in X$ and $s \in [a,b]$ such that $\pi_\lambda(x) = \lambda(s)$, then arc length parametrization of paths $[x,\lambda(s)] \cup [\lambda(s),\lambda(a)], [x,\lambda(s)] \cup [\lambda(s),\lambda(b)]$ are $(3,0)$-quasigeodesics.*

Proof. Let $\alpha : [0,a] \to X$ be the arc length parametrization of $[x,\lambda(s)] \cup [\lambda(s),\lambda(b)]$ such that $\alpha(0) = x, \alpha(a) = \lambda(b)$. Let $t_0 \in [0,a]$ be such that $\alpha(t_0) = \lambda(s)$. Now for $0 \leq t < t' \leq a$, if $t_0 \notin [t,t']$ then $\alpha_{[t,t']}$ is a geodesic. Now we assume $t_0 \in [t,t']$, consider the triangle $\Delta\alpha(t)\alpha(t_0)\alpha(t')$. Then

$$\begin{aligned} |t'-t| &= |t'-t_0|+|t_0-t| \\ &= d(\alpha(t'),\alpha(t_0))+d(\alpha(t_0),\alpha(t)) \\ &\leq d(\alpha(t'),\alpha(t))+d(\alpha(t),\alpha(t_0))+d(\alpha(t_0),\alpha(t)) \\ &= d(\alpha(t'),\alpha(t))+2d(\alpha(t),\pi_\lambda(x)) \\ &\leq d(\alpha(t'),\alpha(t))+3d(\alpha(t'),\alpha(t)) = 3d(\alpha(t'),\alpha(t)). \end{aligned}$$

Obviously, $d(\alpha(t'),\alpha(t)) \leq l_X(\alpha_{[t,t']}) = |t'-t|$. Hence

$$\frac{1}{3}|t-t'| \leq d(\alpha(t),\alpha(t')) \leq |t-t'| \leq 3|t-t'|.$$

Similarly, $[x,\lambda(s)]\cup[\lambda(s),\lambda(a)]$ is a $(3,0)$-quasigeodesic.

□

The following lemma is an easy consequence of δ-hyperbolicity. For the sake of completion we include the proof here.

Lemma 1.1.30. *Given $\delta>0$, there exist $D_{1.1.30}, C_{1.1.30}>0$ such that the following holds:*
1. (Lemma 3.1 of [Mit98b]) If x,y are points of a δ-hyperbolic metric space (X,d), λ is a hyperbolic geodesic in X joining x,y, and π_λ is a nearest point projection of X onto λ with $d(\pi_\lambda(x),\pi_\lambda(y))>D_{1.1.30}$, then $[x,\pi_\lambda(x)]\cup[\pi_\lambda(x),\pi_\lambda(y)]\cup[\pi_\lambda(y),y]$ lies in a $C_{1.1.30}-$neighborhood of any geodesic joining x,y.
2. Let $\alpha:[0,a]\to X$ be the arc length parametrization of $[x,\pi_\lambda(x)]\cup[\pi_\lambda(x),\pi_\lambda(y)]\cup[\pi_\lambda(y),y]$ then
(i) α is a $(K^1_{1.1.30},\epsilon^1_{1.1.30})$-tamed path for some $K^1_{1.1.30},\epsilon^1_{1.1.30}$ depending only upon δ,
(ii) α is a $(K^2_{1.1.30},\epsilon^2_{1.1.30})$-quasigeodesic for some $K^2_{1.1.30},\epsilon^2_{1.1.30}$ depending only upon δ.

Proof. 1. Let $D_{1.1.30}=6\delta$. Let $a=\pi_\lambda(x)$ and $b=\pi_\lambda(y)$. Since X is δ-hyperbolic, triangles are δ-thin, therefore there exist $w_1\in[x,a], w_2\in[a,b]$ and $w_3\in[x,b]$ such that the diameter of the set $\{w_1,w_2,w_3\}$ is bounded above by δ. Now

$$d(a,w_2)\leq d(a,w_1)+d(w_1,w_2)\leq 2d(w_1,w_2)\leq 2\delta.$$

Since $\triangle xby$ is δ-thin, $\triangle xby$ is δ-slim. Thus there exists $w_4\in[x,y]\cup[y,b]$ such that $d(w_3,w_4)\leq\delta$ and hence $d(w_2,w_4)\leq 2\delta$. If $w_4\in[y,b]$, then

$$d(a,b)\leq d(a,w_2)+d(w_2,w_4)+d(w_4,b)\leq 2\delta+2\delta+d(w_4,w_2)\leq 6\delta.$$

This contradicts $d(a,b) > D_{1.1.30} = 6\delta$. Therefore $w_4 \in [x,y]$ and $d(a,w_4) \leq 4\delta$. Similarly for b, there exists $w_5 \in [x,y]$ such that $d(b,w_5) \leq 4\delta$. Now as the triangle Δxaw_4 (resp. Δybw_5) is δ-slim, for each $p \in [x,a]$ (resp. $p \in [y,b]$) there exists $q \in [x,w_4]$ (resp. $q \in [y,w_5]$) such that $d(p,q) \leq \delta + 4\delta = 5\delta$. Now consider the quadrilateral aw_4w_5b, then for $p \in [a,b]$, due to δ-slimness of triangles Δaw_4w_5 and Δaw_5b, there exists $q \in [w_4,w_5]$ such that $d(p,q) \leq \max\{2\delta, \delta+4\delta\} = 5\delta$. Taking $C_{1.1.30} = 5\delta$, we have the required result.
2(i). Let $0 \leq s < t \leq a$ and $s_0, t_0 \in [0,a]$ such that $\alpha(s_0) = \pi_\lambda(x)$ and $\alpha(t_0) = \pi_\lambda(y)$. If $\{s_0,t_0\} \cap [s,t]$ is an empty set, then $\alpha_{[s,t]}$ is a geodesic.
If $\{s_0,t_0\} \cap [s,t]$ is a singleton set, then by Lemma 1.1.29, there exists $K_{1.1.29} \geq 1, \epsilon_{1.1.29} \geq 0$ such that $\alpha_{[s,t]}$ is a $(K_{1.1.29}, \epsilon_{1.1.29})$-tamed path.
Now let $s_0, t_0 \in [s,t]$, then by (1), $\alpha_{[s,t]}$ lies in a $C_{1.1.30}-$ neighborhood of any geodesic $[\alpha(s), \alpha(t)]$ joining $\alpha(s)$ and $\alpha(t)$. Thus for s_0, t_0, there exist $r_{s_0}, r_{t_0} \in [\alpha(s), \alpha(t)]$ such that $d(\alpha(s_0), r_{s_0}) \leq C_{1.1.30}$ and $d(\alpha(t_0), r_{t_0}) \leq C_{1.1.30}$. Therefore

$$\begin{aligned}
l(\alpha_{[s,t]}) &= l(\alpha_{[s,s_0]}) + l(\alpha_{[s_0,t_0]}) + l(\alpha_{[t_0,t]}) \\
&= d(\alpha(s), \alpha(s_0)) + d(\alpha(s_0), \alpha(t_0)) + d(\alpha(t_0), \alpha(t)) \\
&\leq d(\alpha(s), r_{s_0}) + C_{1.1.30} + d(r_{s_0}, r_{t_0}) + 2C_{1.1.30} + d(r_{t_0}, \alpha(t)) + C_{1.1.30} \\
&\leq 3d(\alpha(s), \alpha(t)) + 4C_{1.1.30}.
\end{aligned}$$

Taking $K^1_{1.1.30} = \max\{3, K_{1.1.29}\}$ and $\epsilon^1_{1.1.30} = \max\{\epsilon_{1.1.29}, 4C_{1.1.30}\}$, we have $l(\alpha_{[s,t]}) \leq K^1_{1.1.30} d(\alpha(s), \alpha(t)) + \epsilon^1_{1.1.30}$.
2(ii). Since α is the arc length parametrization of concatenation of three geodesics, therefore $l(\alpha_{[s,t]}) = |s-t|$. Hence by the above inequality, $|s-t| \leq 3d(\alpha(s), \alpha(t)) + 4C_{1.1.30}$. Hence $\frac{1}{3}|s-t| - \frac{4}{3}C_{1.1.30} \leq d(\alpha(s), \alpha(t))$. Also, $d(\alpha(s), \alpha(t)) \leq l(\alpha_{[s,t]}) = |s-t|$. Taking $K^2_{1.1.30} = 3, \epsilon^2_{1.1.30} = \frac{4}{3}C_{1.1.30}$, we have the required result. □

The following lemma states that in a hyperbolic metric space if the distance between the nearest point projection of two points onto a quasiconvex set is sufficiently large then the geodesic segment joining two points come close to the quasiconvex set.

Lemma 1.1.31. *Given $\delta, Q \geq 0$ there exist constants $D'_{1.1.31}, C'_{1.1.31} > 0$ such that the following holds: Let X be a δ-hyperbolic metric space and S be a Q-quasiconvex subset of X. For points $x, y \in X$, if $d(\pi_S(x), \pi_S(y)) > D'_{1.1.31}$ then there exist $p \in [x,y]$, $q \in S$ such that $d(p,q) \leq C'_{1.1.31}$. Further, if $\alpha : [0,a] \to X$ is an arc length parametrization of $[x, \pi_S(x)] \cup [\pi_S(x), \pi_S(y)] \cup [\pi_S(y), y]$ then α is a $K^1_{1.1.31}$-tamed path and also a $K^2_{1.1.31}$-quasigeodesic for some constants $K^1_{1.1.31}, K^2_{1.1.31} \geq 1$ depending only on δ, Q.*

Proof. Let $D_{1.1.30}, C_{1.1.30} > 0$ be constants as in Lemma 1.1.30. Let $D'_{1.1.31} = D_{1.1.30} - 2(3\delta + Q)$ and λ be a geodesic segment joining $\pi_S(x)$ and $\pi_S(y)$.
First we prove that $d(\pi_S(x), \pi_\lambda(x))$ is bounded :
Consider the triangle $\triangle x\pi_S(x)\pi_\lambda(x)$. Since triangles are δ-thin, there exist $w_1 \in [x, \pi_S(x)], w_2 \in [\pi_S(x), \pi_\lambda(x)], w_3 \in [\pi_\lambda(x), x]$ such that $diam\{w_1, w_2, w_3\} \leq \delta$. As S is Q-quasiconvex, there exists w'_2 such that $d(w_2, w'_2) \leq Q$. Thus, as π_S is a nearest point projection, $d(w_1, \pi_S(x)) \leq \delta + Q$. Also $d(w_3, \pi_\lambda(x)) \leq \delta$. Therefore $d(\pi_S(x), \pi_\lambda(x)) \leq \delta + Q + d(w_1, w_3) + \delta \leq 3\delta + Q$.
Now if $d(\pi_S(x), \pi_S(y)) > D'_{1.1.31}$, then $d(\pi_\lambda(x), \pi_\lambda(y)) > D_{1.1.30}$. By Lemma 1.1.30, for any $r \in [\pi_\lambda(x), \pi_\lambda(y)]$, we have $d(r, [x, y]) \leq C_{1.1.30}$. Therefore there exists $q \in S$ such that $d(r, q) \leq Q$ and hence $B_{Q+C_{1.1.30}}(q)$ intersects $[x, y]$. Thus there exists $p \in [x, y]$ such that $d(p, q) \leq Q + C_{1.1.30}$. Taking $C'_{1.1.31} = Q + C_{1.1.30}$, we have the required result.
The proof of α to be a tamed path or a quasigeodesic is similar to the proof of (2) in Lemma 1.1.30. □

The next Lemma states that a nearest point projection from a δ-hyperbolic metric space to a geodesic segment does not increase the distance much.

Lemma 1.1.32. *(Lemma 2.2, [Mit98b]) Let (Y, d) be a δ-hyperbolic metric space and λ be a geodesic segment in Y. There exists $P_{1.1.32} > 0$ (depending only on δ) such that $d(\pi_\lambda(x), \pi_\lambda(y)) \leq P_{1.1.32}d(x, y) + P_{1.1.32}$ for all $x, y \in Y$.*

Proof. It suffices to prove that if $d(x, y) \leq 1$ then there exists $P_{1.1.32} > 0$ such that $d(\pi_\lambda(x), \pi_\lambda(y)) \leq P_{1.1.32}$. Let $D_{1.1.30}$ be the constant as in Lemma 1.1.30.
Let $d(\pi_\lambda(x), \pi_\lambda(y)) > D_{1.1.30}$, then using Lemma 1.1.30, there exist $K^1_{1.1.30} \geq 1, \epsilon^1_{1.1.30}$ such that $\beta = [x, \pi_\lambda(x)] \cup [\pi_\lambda(x), \pi_\lambda(y)] \cup [\pi_\lambda(y), y]$ is a $(K^1_{1.1.30}, \epsilon^1_{1.1.30})$-tamed path. Therefore

$$d(\pi_\lambda(x), \pi_\lambda(y)) \leq l(\beta) \leq K^1_{1.1.30}d(x, y) + \epsilon^1_{1.1.30} \leq K^1_{1.1.30} + \epsilon^1_{1.1.30}.$$

Let $P_{1.1.32} = max\{D_{1.1.30}, K^1_{1.1.30} + \epsilon^1_{1.1.30}\}$, then we have the required result. □

Corollary 1.1.33. *Let (Y, d) be a δ-hyperbolic metric space and S be a Q-quasiconvex set. There exists $P'_{1.1.33} > 0$ (depending on δ and Q) such that $d(\pi_S(x), \pi_S(y)) \leq P'_{1.1.33}d(x, y) + P'_{1.1.33}$ for all $x, y \in Y$.*

Proof. It suffices to prove that if $d(x, y) \leq 1$ then there exists $P'_{1.1.33} > 0$ such that $d(\pi_S(x), \pi_S(y)) \leq P'_{1.1.33}$. Let λ be a geodesic joining $\pi_S(x)$ and $\pi_S(y)$. Then by Lemma 1.1.32, $d(\pi_\lambda(x), \pi_\lambda(y)) \leq P_{1.1.32}$. From the proof of Lemma 1.1.31, we have $d(\pi_S(x), \pi_\lambda(x)) \leq 3\delta + Q$ and $d(\pi_S(y), \pi_\lambda(y)) \leq 3\delta + Q$. Therefore $d(\pi_S(x), \pi_S(y)) \leq$

$(3\delta+Q)+P_{1.1.32}+(3\delta+Q)=6\delta+2Q+P_{1.1.32}$. Taking $P'_{1.1.33}=6\delta+2Q+P_{1.1.32}$, we have the required result. □

Lemma 1.1.34. *Let X be a δ-hyperbolic metric space and S be a Q-quasiconvex set. Suppose $\pi_S : X \to S$ is a nearest point projection. Let $p, q \in S$ and $\lambda : [a,b] \to X$ be a (K,ϵ)-quasigeodesic in X joining p, q, then $\alpha = \pi_S(\lambda)$ is a $(K_{1.1.34}, \epsilon_{1.1.34})$ quasigeodesic, where $K_{1.1.34}, \epsilon_{1.1.34}$ depends only upon K, δ, ϵ, Q.*

Proof. For $t, t' \in [a,b]$, from corollary 1.1.33, there exists $P = P_{1.1.33} > 0$ such that $d(\alpha(t), \alpha(t')) \leq Pd(\lambda(t), \lambda(t')) + P \leq KP|t-t'| + \epsilon P + P$. Let γ be a geodesic in X joining $\lambda(a)$ and $\lambda(b)$. Then by Proposition 1.1.25, there exists $L = L_{1.1.25} > 0$ such that the Hausdorff distance between λ and γ is at most L. Thus, for $t, t' \in [a,b]$, there exist $x \in \gamma$ and $y \in \gamma$ respectively such that $d(\lambda(t), x) \leq L$ and $d(\lambda(t'), y) \leq L$. Also $d(x, \pi_S(x)) \leq Q$ and $d(y, \pi_S(y)) \leq Q$. Therefore $d(\lambda(t), \pi_S(x)) \leq L+Q$ and $d(\lambda(t'), \pi_S(y)) \leq L+Q$. Since π_S is a nearest point projection and $\alpha = \pi_S(\lambda)$, we have $d(\lambda(t), \alpha(t)) \leq L+Q$ and $d(\lambda(t'), \alpha(t')) \leq L+Q$. Therefore $d(\lambda(t), \lambda(t')) \leq d(\alpha(t), \alpha(t')) + 2(L+Q)$. Since λ is a quasigeodesic, we have $\frac{1}{K}|t-t'| - \epsilon \leq d(\lambda(t), \lambda(t'))$ and hence $\frac{1}{K}|t-t'| - \epsilon - 2(L+Q) \leq d(\alpha(t), \alpha(t'))$. Let $K_{1.1.34} = \max\{KP, K\}$ and $\epsilon_{1.1.34} = \max\{\epsilon P + P, \epsilon + 2(L+Q)\}$, then α is a $(K_{1.1.34}, \epsilon_{1.1.34})$-quasigeodesic in X. □

Lemma 1.1.35. *Suppose X is a δ-hyperbolic metric space and $p \in X$. Let μ be a stable L-quasiconvex path and λ be a geodesic in X joining end points of μ. Then $d(\pi_\lambda(p), \pi_\mu(p)) \leq L_{1.1.35}$, for some constant $L_{1.1.35} > 0$ depending only upon δ, L. In particular, this is also true for any quasigeodesic or a tamed path.*

Proof. From definition of a quasiconvex path, there exists $a \in \mu$ and $b \in \lambda$ such that $d(\pi_\lambda(p), a) \leq L$ and $d(\pi_\mu(p), b) \leq L$. Now consider the geodesic triangle $\Delta pa\pi_\mu(p)$, there exists $w \in [p, \pi_\mu(p)]$ and $w' \in [a, \pi_\mu(p)]$, with $d(w, \pi_\mu(p)) = d(w', \pi_\mu(p))$, such that $d(w, w') \leq \delta$. For w', there exists $w'' \in \mu$ such that $d(w', w'') \leq L$. Therefore $d(w, \mu) \leq \delta + L$ and hence

$$(p, a)_{\pi_\mu(p)} = d(w, \pi_\mu(p)) \leq \delta + L.$$

Thus

$$(p, \pi_\lambda(p))_{\pi_\mu(p)} \leq (p, a)_{\pi_\mu(p)} + d(\pi_\lambda(p), a) \leq \delta + 2L.$$

Similarly, $(p, \pi_\mu(p))_{\pi_\lambda(p)} \leq \delta + L$.
Therefore

$$d(\pi_\lambda(p), \pi_\mu(p)) = (p, \pi_\lambda(p))_{\pi_\mu(p)} + (p, \pi_\mu(p))_{\pi_\lambda(p)} \leq 2\delta + 3L.$$

Taking $L_{1.1.35} = 2\delta + 3L$, we have the required result. □

The following Lemma (due to Mitra [Mit98b]) says that nearest point projections and quasi-isometries in hyperbolic metric spaces 'almost commute'.

Lemma 1.1.36. *(Lemma 2.5, [Mit98b]) Suppose* (Y_1, d_1) *and* (Y_2, d_2) *are* δ*-hyperbolic metric spaces. Let* μ_1 *be a geodesic in* Y_1 *joining* a, b *and let* $p \in Y_1$*. Let* ϕ *be a* (K, ϵ)*-quasi-isometry from* Y_1 *to* Y_2*. Let* μ_2 *be a geodesic in* Y_2 *joining* $\phi(a)$ *to* $\phi(b)$*. Then* $d_{Y_2}(\pi_{\mu_2}(\phi(p)), \phi(\pi_{\mu_1}(p))) \leq P_{1.1.36}$ *for some constant* $P_{1.1.36}$ *dependent only on* K, ϵ *and* δ*.*

Due to Lemmas 1.1.35 and 1.1.36, we have the following corollary:

Corollary 1.1.37. *Suppose* (Y_1, d_1) *and* (Y_2, d_2) *are* δ*-hyperbolic metric spaces. Let* μ_1 *be a stable* L*-quasiconvex path in* Y_1 *joining* a, b *and let* $p \in Y_1$*. Let* ϕ *be a* (K, ϵ)*-quasi-isometry from* Y_1 *to* Y_2*. Let* μ_2 *be a stable* L*-quasiconvex path in* Y_2 *joining* $\phi(a)$ *to* $\phi(b)$*. Then* $d_{Y_2}(\pi_{\mu_2}(\phi(p)), \phi(\pi_{\mu_1}(p))) \leq P_{1.1.37}$ *for some constant* $P_{1.1.37}$ *dependent only on* K, ϵ, L *and* δ*.*

1.2 Electric Geometry

Let (X, d) be a path metric space. For $\nu > 0$, let $\mathcal{H}$ be a collection of closed and path connected subsets $\{H_\alpha\}_{\alpha \in \Lambda}$ of X such that each H_α is a intrinsically geodesic space with the induced path metric, denoted by d_{H_α}. The collection $\mathcal{H}$ will be said to be **uniformly** ν**-separated** if $d(H_\alpha, H_\beta) := \inf\{d(a, b) : a \in H_\alpha, b \in H_\beta\} \geq \nu$ for all distinct $H_\alpha, H_\beta \in \mathcal{H}$. We assume ν to be greater than 1. The elements of $\mathcal{H}$ are said to be uniformly properly embedded in X if for all $M > 0$ there exists $N(M) > 0$ such that for all $H_\alpha \in \mathcal{H}$ and for all $x, y \in H_\alpha$ if $d(x, y) \leq M$ then $d_{H_\alpha}(x, y) \leq N$.
Let $Z = X \bigsqcup (\sqcup_\alpha (H_\alpha \times [0, \frac{1}{2}]))$. Define a distance function as follows:

$$\begin{aligned} d_Z(x, y) &= d_X(x, y), \text{ if } x, y \in X, \\ &= d_{H_\alpha \times [0, \frac{1}{2}]}(x, y), \text{ if } x, y \in H_\alpha \text{ for some } \alpha \in \Lambda, \\ &= \infty, \text{ if } x, y \text{ does not lie on a same set of the disjoint union.} \end{aligned}$$

Let $\mathcal{E}(X, \mathcal{H})$ be the quotient space of Z obtained by identifying each $H_\alpha \times \{\frac{1}{2}\}$ to a point $v(H_\alpha)$ and for all $h \in H_\alpha$, $(h, 0)$ is identified with h. We define a metric $d_{\mathcal{E}(X,\mathcal{H})}$ on $\mathcal{E}(X, \mathcal{H})$ as follows:

$$d_{\mathcal{E}(X,\mathcal{H})}([x], [y]) = \inf \sum_{1 \leq i \leq n} d_Z(x_i, y_i),$$

where the infimum is taken over all sequences $C = \{x_1, y_1, x_2, y_2, ..., x_n, y_n\}$ of points of Z such that $x_1 \in [x], y_n \in [y]$ and $y_i \sim x_{i+1}$ for $i = 1, ..., n-1$. ($\sim$ is the equivalence relation on Z). In short, $(\mathcal{E}(X,\mathcal{H}), d_{\mathcal{E}(X,\mathcal{H})})$ will be denoted by $(\widehat{X}, d_{\widehat{X}})$. $\widehat{H}$ will denote the coned-off space obtained from $H \times [0, \frac{1}{2}]$ by coning $H \times \frac{1}{2}$ to a point.

Definition 1.2.1. *(Farb [Far98]) Let $\mathcal{H}$ be a collection of uniformly ν-separated and intrinsically geodesic closed subsets of X. The space $\mathcal{E}(X,\mathcal{H})$ constructed above corresponding to the pair $(X,\mathcal{H})$ is said to be* **electric space** *(or coned-off space). The sets $H_\alpha \in \mathcal{H}$ shall be referred to as horosphere-like sets and the points $v(H_\alpha)$'s as cone points.*

Definition 1.2.2. • *A path γ in $\mathcal{E}(X,\mathcal{H})$ is said to be an electric geodesic (resp. electric K-quasigeodesic) if it is a geodesic (resp. K-quasigeodesic) in $\mathcal{E}(X,\mathcal{H})$.*
• *γ is said to be an electric K-quasigeodesic in $\mathcal{E}(X,\mathcal{H})$* **without backtracking** *if γ is an electric K-quasigeodesic in $\mathcal{E}(X,\mathcal{H})$ and γ does not return to a* horosphere-like set *H_α after leaving it.*
• *For a path $\gamma \subset X$, there is a path $\widehat{\gamma}$ in $\mathcal{E}(X,\mathcal{H})$ obtained from γ as follows: if γ penetrates a horosphere-like set H with entry point x and exit point y, we replace the portion of the path γ lying inside H joining x, y by $[x, v_H] \cup [v_H, y]$, where v_H is the cone point over H, $[x, v_H]$ and $[v_H, y]$ are electric geodesic segments of length $\frac{1}{2}$ joining x, v_H and v_H, y respectively. If $\widehat{\gamma}$ is an electric geodesic (resp. P-quasigeodesic), γ is called a* **relative geodesic** *(resp.* **relative P-quasigeodesic***).*

Definition 1.2.3. *(Farb [Far98]) Let $\widehat{\delta} \geq 0, \nu > 0$. Let X be a geodesic metric space and $\mathcal{H}$ be a collection of uniformly ν-separated and intrinsically geodesic closed subsets of X. X is said to be* **$\widehat{\delta}$-weakly hyperbolic** *relative to the collection $\mathcal{H}$, if the electric space $\mathcal{E}(X,\mathcal{H})$ is $\widehat{\delta}$-hyperbolic.*

Example 1.2.4. *Consider the subset $X = \bigcup_{a \in \mathbb{Z}} (\{(x,y) \in \mathbb{R}^2 : x = a\} \cup \{(x,y) \in \mathbb{R}^2 : y = a\})$ of $\mathbb{R}^2$ and $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 : x = a\}$. Then X is weakly hyperbolic relative to the collection $\mathcal{H}$.*

1.2.1 Strongly Relatively Hyperbolic Spaces

Definition 1.2.5. *Relative geodesics (resp. P-quasigeodesic paths) in $(X,\mathcal{H})$ are said to satisfy* **bounded horosphere penetration** *if for any two relative geodesics (resp. P-quasigeodesic paths without backtracking) β, γ, joining $x, y \in X$ there exists $I_{1.2.1} = I_{1.2.1}(P) > 0$ such that*

Similar Intersection Patterns 1: *if precisely one of* $\{\beta, \gamma\}$ *meets a horosphere-like set* H_α*, then the distance (measured in the intrinsic path-metric on* H_α*) from the first (entry) point to the last (exit) point (of the relevant path) is at most* $I_{1.2.1}$.
Similar Intersection Patterns 2: *if both* $\{\beta, \gamma\}$ *meet some* H_α *then the distance (measured in the intrinsic path-metric on* H_α*) from the entry point of* β *to that of* γ *is at most* $I_{1.2.1}$*; similarly for exit points.*

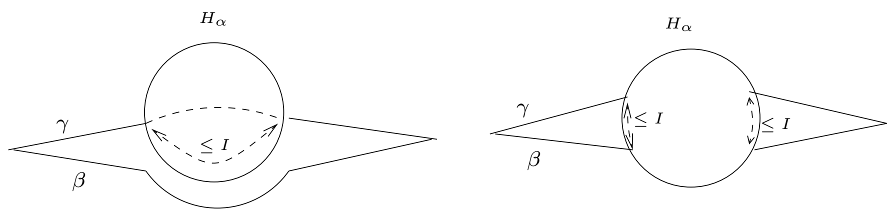

Figure 1.1: Similar Intersection Patterns.

Paths which satisfy the above properties shall be said to have *similar intersection patterns* with horospheres.

Definition 1.2.6. *(Farb [Far98]) Let* $\widehat{\delta} \geq 0$*. Let* X *be a geodesic metric space and* $\mathcal{H}$ *be a collection of uniformly* ν*-separated and intrinsically geodesic closed subsets of* X*. Then* X *is said to be* $\widehat{\delta}$*- hyperbolic relative to the collection* $\mathcal{H}$ *in the sense of Farb if*
1) X *is* $\widehat{\delta}$*-weakly hyperbolic relative to* $\mathcal{H}$*,*
2) Relative P*-quasigeodesic paths without backtracking satisfy the bounded horosphere penetration properties.*
X *is said to be hyperbolic relative to a collection* $\mathcal{H}$ *in the sense of Farb if* X *is* $\widehat{\delta}$*-hyperbolic relative to the collection* $\mathcal{H}$ *in the sense of Farb for some* $\widehat{\delta} \geq 0$*.*

<u>Warped products of metric spaces (Chen [Che99])</u>:
Suppose (X, d_X) and (Y, d_Y) are two metric spaces. Let $\gamma = (r, s) : [0,1] \to X \times Y$ be a curve and $f : Y \to R^+$ be a continuous function. Suppose $\tau :\ 0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $[0,1]$. One defines the length of γ by

$$l(\gamma) = \lim_{\tau} \sum_{1 \leq i \leq n-1} \sqrt{f^2(s(t_{i-1}))d_X^2(r(t_{i-1}), r(t_i)) + d_Y^2(s(t_{i-1}, s(t_i)))}$$

Here the limit is taken with respect to the refinement ordering of partitions over $[0,1]$. The distance between two points $x, y \in X \times Y$ is defined to be

$$d(x,y) = inf\{l(\gamma) : \gamma \text{ is a curve from } x \text{ to } y\}.$$

Proposition 1.2.7. *(Proposition 3.1, [Che99]) d is a metric on $X \times Y$.*

Definition 1.2.8. *(Definition 3.1, [Che99]) The warped product of (X, d_X) and (Y, d_Y) with respect to the warping function f is the set $X \times Y$ equipped with the metric d. We denote it by $(X \times_f Y, d)$.*

Definition 1.2.9. *(Hyperbolic Cones:) For any geodesic metric space (H, d), the hyperbolic cone (analog of a horoball), denoted by H^h, is the warped product of metric spaces $[0, \infty)$ and H with warping function $f(t) = e^{-t}$, where $t \in [0, \infty)$, i.e., $H^h := H \times_{e^{-t}} [0, \infty)$. We denote the metric on H^h by d_{H^h}.*

Note that the metric d_{H^h} is described as follows:
Let $\alpha : [0, 1] \to H \times [0, \infty) = H^h$ be a path then $\alpha = (\alpha_1, \alpha_2)$, where α_1, α_2 are coordinate functions. Suppose $\tau : \ 0 = t_0 < t_1 < ... < t_n = 1$ be a partition of $[0, 1]$. Define the length of α by

$$l_{H^h}(\alpha) = \lim_{\tau} \sum_{1 \le i \le n-1} \sqrt{e^{-2\alpha_2(t_i)} d_H(\alpha_1(t_i), \alpha_1(t_{i+1}))^2 + |\alpha_2(t_i) - \alpha_2(t_{i+1})|^2},$$

Here the limit is taken with respect to the refinement ordering of partitions over $[0, 1]$. Thus the distance between two points $x, y \in H^h$ is defined to be

$$d_{H^h}(x, y) = inf\{l_{H^h}(\alpha) : \alpha \text{ is a curve from } x \text{ to } y\}.$$

Remark 1.2.10. *The metric d_{H^h} satisfies the following two properties:*
1) $d_{H,t}((x, t), (y, t)) = e^{-t} d_H(x, y)$, where $d_{H,t}$ is the induced path metric on $H \times \{t\}$. Paths joining $(x, t), (y, t)$ and lying on $H \times \{t\}$ are called horizontal paths.
2) $d_{H^h}((x, t), (x, s)) = |t - s|$ for all $x \in H$ and for all $t, s \in [0, \infty)$, and the corresponding paths are called vertical paths. *The vertical paths are geodesics in H^h as if $\alpha = (\alpha_1, \alpha_2) : [0, 1] \to H^h$ is a path in H^h joining $(x, t), (x, s)$ then for any partition $\tau : 0 = t_0 < t_1 ... < t_n = 1$, we have*

$$\begin{aligned} & \sum_{1 \le i \le n-1} \sqrt{e^{-2\alpha_2(t_i)} d_H(\alpha_1(t_i), \alpha_1(t_{i+1}))^2 + |\alpha_2(t_i) - \alpha_2(t_{i+1})|^2} \\ \ge & \sum_{1 \le i \le n-1} (|\alpha_2(t_i) - \alpha_2(t_{i+1})|) \\ \ge & \ |t - s|. \end{aligned}$$

Hence $l_{H^h}(\alpha) \ge |t - s|$.
3) Let $(x, t) \in H^h$ and $\alpha = (\alpha_1, \alpha_2) : [0, 1] \to H^h$ be a path such that $\alpha(0) = (x, t)$ and $\alpha(1) \in H \times \{0\}$, then $t \le l_{H^h}(\alpha)$:

as for any partition $\tau : 0 = t_0 < t_1 ... < t_n = 1$, we have

$$\begin{aligned}
& \sum_{1\leq i\leq n-1} \sqrt{e^{-2\alpha_2(t_i)} d_H(\alpha_1(t_i), \alpha_1(t_{i+1}))^2 + |\alpha_2(t_i) - \alpha_2(t_{i+1})|^2} \\
\geq & \sum_{1\leq i\leq n-1} (|\alpha_2(t_i) - \alpha_2(t_{i+1})|) \\
\geq & \ t.
\end{aligned}$$

Hence $l_{H^h}(\alpha) \geq t$.

Proposition 1.2.11. *(Proposition 4.1, [Che99]) Let (Y, d_Y) be a complete, locally compact metric space and (X, d_X) be a geodesic metric space. Let function $f : Y \to R^+$ be a continuous function. Then $(X \times_f Y, d)$ is a geodesic metric space. In particular, H^h is a geodesic metric space.*

Consider the region $[0, a] \times [1, \infty)$ in $\mathbb{H}^2$, where $[0, a]$ is a horocyclic arc of length a. For $t \in [1, \infty)$, let $z_t = it, w_t = a + it \in \mathbb{H}^2$ and a_t be the length of the horocyclic arc joining z_t, w_t. Now

$$\begin{aligned}
d_{\mathbb{H}^2}(z_t, w_t) &= \log \frac{|z_t - \bar{w}_t| + |z_t - w_t|}{|z_t - \bar{w}_t| - |z_t - w_t|} \\
&= \log \frac{|-a + 2it| + a}{|-a + 2it| - a} \\
&= \log \frac{\sqrt{a^2 + 4t^2} + a}{\sqrt{a^2 + 4t^2} - a} \\
&= \log \frac{a^2 + 2t^2 + a\sqrt{a^2 + 4t^2}}{2t^2}. \qquad (1.1)
\end{aligned}$$

Therefore

$$\begin{aligned}
e^{d_{\mathbb{H}^2}(z_t, w_t)} &= \frac{a^2 + 2t^2 + a\sqrt{a^2 + 4t^2}}{2t^2} \\
&= \frac{(ae^{-t})^2 + 2(te^{-t})^2 + ae^{-t}\sqrt{(ae^{-t})^2 + 4(te^{-t})^2}}{2(te^{-t})^2} \\
& \quad \text{(multiplying numerator and denominator by } e^{-t}) \\
&= \frac{a_t^2 + a_t\sqrt{a_t^2 + 4(te^{-t})^2}}{2(te^{-t})^2} + 1 \\
&\geq \frac{a_t^2 + a_t\sqrt{a_t^2}}{2(te^{-t})^2} + 1 \\
&\geq a_t^2 + 1, \text{ since } te^{-t} \leq 1.
\end{aligned}$$

Thus

$$a_t < \sqrt{e^{d_{\mathbb{H}^2}(z_t, w_t)} - 1}. \qquad (1.2)$$

Lemma 1.2.12. *Let H be a geodesic metric space and H^h be its hyperbolic cone.*
(i). The elements of the collection $\{H \times \{t\} : t \in [0,\infty)\}$ are uniformly properly embedded in H^h, i.e, for all $M > 0$ there exists $N_{1.2.12}(M) > 0$ such that for all $t \in [0,\infty), x, y \in H \times \{t\}$ if $d_{H^h}(x,y) \leq M$ then $d_{H,t}(x,y) \leq N_{1.2.12}$, where $d_{H,t}$ is the induced path metric on $H \times \{t\}$.
(ii). If $\{x_n\}, \{y_n\}$ are two sequences in H such that $d_{H^h}(x_n, y_n) \to 0$ as $n \to \infty$, then $d_H(x_n, y_n) \to 0$ as $n \to \infty$.
(iii). Let $H_1, ..., H_n$ be geodesic spaces $x_i, y_i \in H_i$, then $\sum_{1\leq i\leq n} d_{H_i}(x_i, y_i) \leq 2((e^{\sum_{1\leq i\leq n} d_{H_i^h}(x_i,y_i)}) - 1)$.

Proof. (i) Let $x, y \in H \times \{t\}$ such that $d_{H^h}(x,y) \leq M$ and let $\alpha = (\alpha_1, \alpha_2) : [0, d_{H^h}(x,y)] \to H \times [0,\infty) = H^h$ be a geodesic in H^h joining x, y, where $\alpha_1 : [0, d_{H^h}(x,y)] \to H$, $\alpha_2 : [0, d_{H^h}(x,y)] \to [0,\infty)$ are coordinate functions. Note that image of α_1, denoted by $im(\alpha_1)$, does not contain any non-trivial loop, as if $im(\alpha_1)$ contains a non-trivial loop, then there exist distinct $s, s' \in [0, d_{H^h}(x,y)]$ such that $\alpha_1(s) = \alpha_1(s')$ and $\alpha_2(s) \neq \alpha_2(s')$. Let α' be the subsegment of α joining $(\alpha_1(s), \alpha_2(s))$ and $(\alpha_1(s'), \alpha_2(s'))$. As $im(\alpha_1)$ contains a non-trivial loop, α' is not vertical. This is a contradiction, as the vertical path is the only geodesic joining $(\alpha_1(s), \alpha_2(s))$ and $(\alpha_1(s), \alpha_2(s'))$.
Let a denote the length of α_1 in the metric space (H, d_H), then the subset $im(\alpha_1) \times [0,\infty)$ with the induced metric from H^h is isometric to a closed region bounded by two vertical asymptotic geodesic and a horocyclic arc of length a in the upper half plane, i.e., $im(\alpha_1) \times [0,\infty)$ is isometric to the region $[0,a] \times [1,\infty)$ in $\mathbb{H}^2$. Let a_t denote the length of the path $\beta_t(s) := (\alpha_1(s), t)$, where $s \in [0, d_{H^h}(x,y)]$, then $a_t = e^{-t}a$. Using equation 1.2, we have $d_{H,t}(x,y) \leq a_t \leq \sqrt{e^{d_{H^h}(x,y)} - 1} \leq \sqrt{e^M - 1}$. Taking $N_{1.2.12} = \sqrt{e^M - 1}$, we have the required result.
(ii). This follows easily from the inequality $d_H(x_n, y_n) \leq \sqrt{e^{d_{H^h}(x_n,y_n)} - 1}$.
(iii). Let $a_i = d_{H_i}(x_i, y_i)$ and $p_i = d_{H_i^h}(x_i, y_i)$ for $i \in \{1, ..., n\}$. Using equation 1.1 and putting $t = 1$, for all $i \in \{1, ..., n\}$, we have

$$p_i \geq \log \frac{a_i^2 + 2 + a_i\sqrt{a_i^2 + 4}}{2}.$$

Thus, $(a_i^2 + 2 + a_i\sqrt{a_i^2 + 4}) \leq 2e^{p_i}$ for all $1 \leq i \leq n$. Hence

$$\prod_{1\leq i\leq n} (a_i^2 + 2 + a_i\sqrt{a_i^2 + 4}) \leq 2^n e^{\sum_{1\leq i\leq n} p_i}.$$

Now,

$$2^{n-1} \sum_{1\leq i\leq n} a_i\sqrt{a_i^2 + 4} + 2^n \leq \prod_{1\leq i\leq n} (a_i^2 + 2 + a_i\sqrt{a_i^2 + 4})$$

and $a_i \leq a_i\sqrt{a_i^2+4}$ for all $1 \leq i \leq n$. Therefore

$$2^{n-1}\sum_{1\leq i\leq n} a_i + 2^n \leq 2^n e^{\sum_{1\leq i\leq n} p_i}.$$

and hence

$$\sum_{1\leq i\leq n} a_i \leq 2(e^{\sum_{1\leq i\leq n} p_i} - 1).$$

□

Lemma 1.2.13. *Let H_1, H_2 be two geodesic spaces and $\varphi : H_1 \to H_2$ be a (K,ϵ)-quasi-isometry. Let H_1^h, H_2^h be hyperbolic cones over them, then φ induces a $(K_{1.2.13}, \epsilon_{1.2.13})$-quasi-isometry $\varphi^h : H_1^h \to H_2^h$ where $K_{1.2.13} \geq 1, \epsilon_{1.2.13} \geq 0$ depends only upon K, ϵ.*

Proof. Define $\varphi^h : H_1^h \to H_2^h$ by $\varphi^h(x,t) = (\varphi(x), t)$. We will show φ^h is a quasi-isometry. First we prove that there exists $P_1 \geq 1$ such that for $(x,t),(y,s) \in H_1^h$ if $d_{H_1^h}((x,t),(y,s)) \leq 1$ then $d_{H_2^h}((\varphi(x),t),(\varphi(y),s)) \leq P_1$.
We assume $s \leq t$. Now $d_{H_1^h}((x,t),(y,s)) \leq 1$ implies that $d_{H_1^h}((y,s),(y,t)) \leq 1$. Therefore $d_{H_1^h}((x,t),(y,t)) \leq 2$. Since horosphere-like sets are properly embedded in its hyperbolic cone, there exists $N(2) > 0$ such that $d_{H_{1,t}}((x,t),(y,t)) \leq N(2)$. As φ is a (K,ϵ)-quasi-isometry, $d_{H_{2,t}}((\varphi(x),t),(\varphi(y),t)) \leq KN(2)+\epsilon$. Now $d_{H_2^h}((\varphi(y),t),(\varphi(y),s)) \leq 1$. Thus $d_{H_2^h}((\varphi(x),t),(\varphi(y),s)) \leq KN(2)+\epsilon+1 = P_1$ (say).
Now let $\alpha : [0,l] \to H_1^h$ be a geodesic in H_1^h joining (x,t) and (y,s). We partition $[0,l]$ by points $t_0, t_1, ..., t_{n-1}, t_n$ such that $\alpha(t_0) = (x,t), \alpha(t_n) = (y,s)$, for each $0 \leq i \leq n-2$, $d_{H_1^h}(\alpha(t_i), \alpha(t_{i+1})) = 1$ and $d_{H_1^h}(\alpha(t_{n-1}), \alpha(t_n)) \leq 1$. Thus, by triangle inequality, we have $d_{H_2^h}((\varphi(x),t),(\varphi(y),s)) \leq P_1 d_{H_1^h}((x,t),(y,s)) + P_1$.
Now there exists $K_1 \geq 1, \epsilon_1 \geq 0$ such that φ^{-1} is (K_1,ϵ_1)-quasi-isometry, therefore there exists $P_2 \geq 1$ such that

$$d_{H_1^h}((\varphi^{-1}(\varphi(x)),t),((\varphi^{-1}(\varphi(y)),s)) \leq P_2 d_{H_2^h}((\varphi(x),t),(\varphi(y),s)) + P_2.$$

Since φ is a quasi-isometry, there exists $r > 0$ such that for each $y \in H_2$ there exists $x \in H_1$ such that $d_{H_2}(\varphi(x), y) \leq r$ and $d_{H_1}(\varphi^{-1}(\varphi(z)), z) \leq r$ for all $z \in H_1$, therefore $d_{H_2^h}(\varphi^h(x,t),(y,t)) \leq r$ and $d_{H_1^h}((\varphi^{-1}(\varphi(z)),t),(z,t)) \leq r$. Thus

$$d_{H_1^h}((x,t),(y,s)) \leq P_2 d_{H_2^h}((\varphi(x),t),(\varphi(y),s)) + P_2 + 2r.$$

Hence there exist $K_{1.2.13} \geq 1, \epsilon_{1.2.13} \geq 0$ such that φ^h is a $(K_{1.2.13}, \epsilon_{1.2.13})$-quasi-isometry. □

For a connected graph L, Bowditch, in [Bow97], proved that the hyperbolic cone L^h is a hyperbolic metric space with Gromov boundary a singleton set. (The proof of this fact is presented on page numbers 18,19 of [Bow97], where the notation cusp(L) is used for L^h). In view of Lemma 1.2.13, Proposition 1.1.9 and the fact that hyperbolicity is a quasi-isometry invariant, we have the following proposition:

Proposition 1.2.14. *[Bow97] For any geodesic metric space (H, d), the hyperbolic cone (H^h, d_{H^h}) is a hyperbolic metric space with Gromov boundary a singleton set.*

Gromov's definition of relative hyperbolicity [Gro87] :
Let (X, d_X) be geodesic metric space and $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ be a collection of uniformly ν-separated, intrinsically geodesic, closed subsets of X.
Let $Z = X \bigsqcup (\sqcup_{\alpha\in\Lambda} H^h_\alpha)$. Define a distance function d_Z on Z as follows:

$$\begin{aligned} d_Z(x,y) &= d_X(x,y), \text{ if } x,y \in X, \\ &= d_{H^h_\alpha}(x,y), \text{ if } x,y \in H_\alpha \text{ for some } \alpha \in \Lambda, \\ &= \infty, \text{ if } x,y \text{ does not lie on a same set of the disjoint union.} \end{aligned}$$

Let $\mathcal{G}(X, \mathcal{H})$ be the quotient space of Z obtained by attaching the hyperbolic cones H^h_α to $H_\alpha \in \mathcal{H}$ by identifying $(z, 0)$ with z, for all $H_\alpha \in \mathcal{H}$ and $z \in H_\alpha$.
We define a metric $d_{\mathcal{G}(X,\mathcal{H})}$ on $\mathcal{G}(X, \mathcal{H})$ as follows:

$$d_{\mathcal{G}(X,\mathcal{H})}([x],[y]) = \inf \sum_{1\le i\le n} d_Z(x_i, y_i),$$

where the infimum is taken over all sequences $C = \{x_1, y_1, x_2, y_2, ..., x_n, y_n\}$ of points of Z such that $x_1 \in [x], y_n \in [y]$ and $y_i \sim x_{i+1}$ for $i = 1, ..., n-1$. ($\sim$ is the equivalence relation on Z). $d_{\mathcal{G}(X,\mathcal{H})}$ is a metric:
$d_{\mathcal{G}(X,\mathcal{H})}$ is indeed a pseudometric. Let $[x], [y] \in \mathcal{G}(X, \mathcal{H})$ such that $d_{\mathcal{G}(X,\mathcal{H})}([x], [y]) = 0$. If x (or y) lie in $H^h \setminus H$, then $d_{H^h}(x, H) > 0$. For any $\epsilon > 0$ there exists a sequence $\{x_1, y_1, x_2, y_2, ..., x_n, y_n\}$ such that $x_1 = x, y_n = y$, $y_i, x_{i+1} \in H_i$ ($1 \le i \le n-1$, $H_1 = H$) and $\sum_{1\le i\le n} d_X(x_i, y_i) + \sum_{1\le i\le n-1} d_{H^h_i}(y_i, x_{i+1}) \le \epsilon$. Therefore, $d_{H^h}(x, y_1) \le \epsilon$ which implies $d_{H^h}(x, H) \le \epsilon$. Taking $\epsilon \to 0$, we have $d_{H^h}(x, H) = 0$. Hence x must equals y.
Now let $x, y \in X$. For each $k \in \mathbb{N}$, there exists a sequence $\{x_1, y_1, x_2, y_2, ..., x_n, y_n\}$ such that $x_1 = x, y_n = y$, $y_i, x_{i+1} \in H_i$ and

$$\sum_{1\le i\le n} d_X(x_i, y_i) + \sum_{1\le i\le n-1} d_{H^h_i}(y_i, x_{i+1}) \le \frac{1}{k}.$$

Now, by (iii) of Lemma 1.2.12, we have

$$\begin{aligned}\sum_{1\le i\le n-1} d_{H_i}(y_i,x_{i+1}) &\le 2(e^{\sum_{1\le i\le n-1} d_{H_i^h}(y_i,x_{i+1})}-1)\\ &\le 2(e^{\frac{1}{k}}-1).\end{aligned}$$

Also, $\sum_{1\le i\le n} d_X(x_i,y_i)\le \frac{1}{k}$. Therefore, by triangle inequality, we have

$$\begin{aligned}d_X(x,y) &\le \sum_{1\le i\le n} d_X(x_i,y_i)+\sum_{1\le i\le n-1} d_{H_i^h}(y_i,x_{i+1})\\ &\le \frac{1}{k}+2(e^{\frac{1}{k}}-1)\to 0 \text{ as } k\to\infty.\end{aligned}$$

Thus, $x=y$ and hence $[x]=[y]$.

In short, $(\mathcal{G}(X,\mathcal{H}),d_{\mathcal{G}(X,\mathcal{H})})$ will be denoted by (X^h,d_{X^h}).

Observation 1.2.15. *We have the following simple observations:*
(1) The path metric induced from d_{H^h} on H is d_H,
(2) Let $\alpha:[0,1]\to X$ be a path, then $l_X(\alpha)=l_{X^h}(\alpha)$.

Proof. (1) Let α be a geodesic in H joining $x,y\in H$. As H is embedded in H^h, we can write $\alpha=(\alpha_1,\alpha_2)$ where α_2 is a constant function. For all partitions $\tau: 0=t_0<t_1...<t_n=1$, we have

$$\begin{aligned}&\sum_{1\le i\le n-1}\sqrt{e^{-2\alpha_2(t_i)}d_H(\alpha_1(t_i),\alpha_1(t_{i+1}))^2+|\alpha_2(t_i)-\alpha_2(t_{i+1})|^2}\\ =&\sum_{1\le i\le n-1}\sqrt{d_H(\alpha_1(t_i),\alpha_1(t_{i+1}))^2}\\ =&\ d_H(x,y).\end{aligned}$$

Thus $l_{H^h}(\alpha)=d_H(x,y)$.
(2) Note that α is a concatenation of paths of the form $\alpha_X:[a,b]\to X$, where $\alpha_X((a,b))\cap(\cup_{H\in\mathcal{H}}H)=\emptyset$, and $\alpha_H:[a,b]\to H$ for some $H\in\mathcal{H}$. Now α_X is a concatenation of two paths α_1,α_2 such that only one of the end points of α_i may lie on horosphere-like sets. Thus, it suffices to prove that for paths $\beta:[0,1]\to X$, with $\beta([0,1))\subset X\setminus\cup_{H\in\mathcal{H}}H$, and $\gamma:[0,1]\to H$, we have $l_X(\beta)=l_{X^h}(\beta)$ and $l_X(\gamma)=l_{X^h}(\gamma)$.
First we prove that $l_X(\beta)=l_{X^h}(\beta)$:
Let $0\le s_0<1$, then $\beta(s_0)\notin\cup_{H\in\mathcal{H}}H$. As $\mathcal{H}$ is uniformly ν-separated and horosphere-like sets are closed in X, therefore there exists $\delta>0$ such

that $B(\beta(s_0);\delta)\cap(\cup_{H\in\mathcal{H}}H)=\emptyset$. Thus, there exists $T\in(s_0,1)$ such that $l_X(\beta|_{[s,t]})\leq\frac{\delta}{10}$ for all $t\leq T$. Note that for any $s,s'\in[s_0,T]$, $d_X(\beta(s),\beta(s'))=d_{X^h}(\beta(s),\beta(s'))$. Let $\eta=\beta|_{[s_0,t]}$. Let $0<\epsilon<\frac{\delta}{10}$, then there exists a partition $\{t_0,...,t_n\}$ of $[s_0,t]$ such that $l_X(\eta)-\epsilon<\sum_{0\leq i\leq n-1}d_X(\eta(t_i),\eta(t_{i+1}))\leq l_X(\eta)$. Now for all i, $d_X(\eta(t_i),\eta(t_{i+1}))=d_{X^h}(\eta(t_i),\eta(t_{i+1}))$, therefore $l_X(\eta)-\epsilon<\sum_{0\leq i\leq n-1}d_{X^h}(\eta(t_i),\eta(t_{i+1}))$. As $\sum_{0\leq i\leq n-1}d_{X^h}(\eta(t_i),\eta(t_{i+1}))\leq l_{X^h}(\eta)$, therefore $l_X(\eta)-\epsilon<l_{X^h}(\eta)$. Taking $\epsilon\to 0$, we have $l_X(\eta)\leq l_{X^h}(\eta)$.
Now for any $\epsilon>0$ there exists a partition $\{t'_0,...,t'_m\}$ of $[s_0,t]$ such that $l_{X^h}(\eta)-\epsilon<\sum_{0\leq j\leq m-1}d_{X^h}(\eta(t'_i),\eta(t'_{i+1}))\leq l_{X^h}(\eta)$. Now $d_{X^h}(\eta(t'_i),\eta(t'_{i+1}))\leq d_X(\eta(t'_i),\eta(t'_{i+1}))$ for all i and $d_X(\eta(t'_i),\eta(t'_{i+1}))\leq l_X(\eta)$. Therefore, $l_{X^h}(\eta)-\epsilon<l_X(\eta)$. Taking $\epsilon\to 0$, we have $l_{X^h}(\eta)\leq l_X(\eta)$. Hence $l_X(\eta)=l_{X^h}(\eta)$.
Now define $F:[0,1]\to\mathbb{R}$ by $F(t)=l_X(\beta|_{[0,t]})-l_{X^h}(\beta|_{[0,t]})$.
Let $[0,s_0]$ be the maximal subinterval of $[0,1]$ for which $F(s)=0$ for all $s\in[0,s_0]$. Now from above there exist $s_0\leq T\leq 1$ such that $l_X(\beta|_{[s_0,T]})=l_{X^h}(\beta|_{[s_0,T]})$. Therefore $F(T)=0$ and hence s_0 must be equal to one. Thus $l_X(\beta)=l_{X^h}(\beta)$.
Next we prove that $l_X(\gamma)=l_{X^h}(\gamma)$:
There exists a sequence of paths $\gamma_n:[0,1]\to H^h$ such that $\gamma_n\to\gamma$ as $n\to\infty$, $\gamma_n(0)=\gamma(0)$, $\gamma_n(1)=\gamma(1)$, $\gamma_n((0,1))\cap H=\emptyset$ and $im(\gamma_n)\subset(im(\gamma)\times[0,\infty))$ for all n. Thus, $l_{X^h}(\gamma_n)\to l_{X^h}(\gamma)$ as $n\to\infty$. As $\gamma_n((0,1))\cap H=\emptyset$, similarly as above we can prove that $l_{X^h}(\gamma_n)=l_{H^h}(\gamma_n)$. Thus, $l_{X^h}(\gamma)=l_{H^h}(\gamma)$. Now the metric on H is induced from the metric d_X on X, therefore $l_H(\gamma)=l_X(\gamma)$. Also, by (1) the metric d_{H^h} on H^h induces the metric d_H on H, therefore $l_{H^h}(\gamma)=l_H(\gamma)$. Hence, $l_{X^h}(\gamma)=l_X(\gamma)$ □

Definition 1.2.16. *Let $\delta\geq 0,\nu>0$. Let X be a geodesic metric space and $\mathcal{H}$ be a collection of uniformly ν-separated, intrinsically geodesic closed subsets of X. X is said to be δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov, if the quotient space $(\mathcal{G}(X,\mathcal{H}),d_{\mathcal{G}(X,\mathcal{H})})$ is a δ-hyperbolic metric space in the sense of (1.1.7). X is said to be hyperbolic relative to $\mathcal{H}$ in the sense of Gromov if X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov for some $\delta\geq 0$.*

Note that if (X,d_X) is a proper geodesic metric space then $\mathcal{G}(X,\mathcal{H})$ is a proper path metric space. Hence $\mathcal{G}(X,\mathcal{H})$ is a geodesic space.

The following lemma proves that the vertical paths in a hyperbolic cone are geodesics in $\mathcal{G}(X,\mathcal{H})$

Lemma 1.2.17. *Let X be a geodesic metric space and $\mathcal{H}$ be a collection of uniformly ν-separated ($\nu>0$), intrinsically geodesic closed subsets of X. Let $H\in\mathcal{H}$ and $\gamma:[0,\infty)\to H^h$ be a vertical path in the hyperbolic cone H^h, where $\gamma(0)\in H$, then γ is a geodesic in X^h.*

Proof. Let $t \in [0,\infty)$ and $\alpha : [0,l] \to X^h$ be a path in X^h joining $\gamma(t)$ and $\gamma(0)$, where $\alpha(0) = \gamma(t), \alpha(l) = \gamma(0)$, then there exists $t_0 \in [0,l]$ such that $\alpha(t_0) \in H$ and $\alpha_{[0,t_0]} \subset H^h$. Now γ is a geodesic in H^h. Since $l_{H^h}(\alpha|_{[0,t_0]}) = l_{X^h}(\alpha|_{[0,t_0]})$ and $l_{H^h}(\gamma|_{[0,t]}) = l_{X^h}(\gamma|_{[0,t]})$, therefore $l_{X^h}(\gamma|_{[0,t]}) \le l_{X^h}(\alpha|_{[0,t_0]}) \le l_{X^h}(\alpha)$. Thus for all t, $\gamma|_{[0,t]}$ is a geodesic in X^h. □

Lemma 1.2.18. *(Hyperbolic Cones are uniformly properly embedded in $\mathcal{G}(X,\mathcal{H})$): Let X be a δ-hyperbolic space relative to a collection $\mathcal{H}$ of uniformly ν-separated ($\nu > 0$), intrinsically geodesic and uniformly properly embedded closed subsets of X in the sense of Gromov. Let $\mathcal{H}^h = \{H^h : H \in \mathcal{H}\}$, then elements of $\mathcal{H}^h$ are uniformly properly embedded in $\mathcal{G}(X,\mathcal{H})$, i.e., for all $M > 0$ there exists $N_{1.2.18}(M) > 0$ such that for all $H^h \in \mathcal{H}^h$ and for all $x, y \in H^h \in \mathcal{H}^h$, $d_{X^h}(x,y) \le M$ implies that $d_{H^h}(x,y) \le N_{1.2.18}$.*

Proof. Let $x, y \in H^h$ such that $d_{X^h}(x,y) \le M$. By definition of the metric d_{X^h}, there exists a path $\alpha : [0,l] \to X^h$ joining x and y such that α is a concatenation of geodesics from X and hyperbolic cones and $l_{X^h}(\alpha) \le d_{X^h}(x,y) + 1$. Therefore, $l_{X^h}(\alpha) \le M + 1$. Let $H_1^h, ..., H_{N_1}^h$ be the hyperbolic cones penetrated by α, where $H_{N_1}^h = H^h$. We partition $[0,l]$ by points $0 = s_0 \le t_0 < s_1 < t_1 < ... < s_{N_1} \le t_{N_1} = l$ such that

(i) $\alpha(0) = x$, $\alpha(t_{N_1}) = y$, $\alpha(t_0) \in H^h$, $\alpha(s_{N_1}) \in H^h$,

(ii) $\alpha_{[s_j,t_j]}$ is a geodesic in H_j^h,

(iii) $\alpha_{[t_i,s_{i+1}]}$ is a geodesic in X,

where $0 \le j \le N_1, 0 \le i \le N_1 - 1$ and $H_0^h = H^h$. Hence $\sum_{0\le j\le N_1} d_{H_j^h}(\alpha(s_j),\alpha(t_j)) \le M + 1$ and $\sum_{0\le i\le N_1-1} d_X(\alpha(t_i),\alpha(s_{i+1})) \le M + 1$. Therefore, by (iii) of Lemma 1.2.12, we have

$$\begin{aligned}\sum_{0\le j\le N_1} d_{H_j}(\alpha(s_j),\alpha(t_j)) &\le 2(e^{\sum_{0\le j\le N_1} d_{H_j^h}(\alpha(s_j),\alpha(t_j))} - 1) \\ &\le 2e^{M+1}.\end{aligned}$$

Thus $\sum_{0\le j\le N_1} d_X(\alpha(s_j),\alpha(t_j)) \le \sum_{0\le j\le N_1} d_{H_j}(\alpha(s_j),\alpha(t_j)) \le 2e^{M+1}$. Hence

$$\begin{aligned}d_X(\alpha(t_0),\alpha(s_{N_1})) &\le \sum_{1\le j\le N_1-1} d_X(\alpha(s_j),\alpha(t_j)) + \sum_{0\le i\le N_1-1} d_X(\alpha(t_i),\alpha(s_{i+1})) \\ &\le 2e^{M+1} + M + 1.\end{aligned}$$

Let $N_2 = 2e^{M+1} + M + 1$. Since elements of $\mathcal{H}$ are uniformly properly embedded in X, therefore there exists $N_3(N_2) > 0$ such that $d_H(\alpha(t_0),\alpha(s_{N_1})) \le N_3$. Hence

$$\begin{aligned}d_{H^h}(\alpha(s_0),\alpha(t_{N_1})) &\le d_{H^h}(\alpha(s_0),\alpha(t_0)) + d_{H^h}(\alpha(t_0),\alpha(s_{N_1})) + d_{H^h}(\alpha(s_{N_1}),\alpha(t_{N_1})) \\ &\le M + 1 + d_H(\alpha(t_0),\alpha(s_{N_1})) + M + 1 \\ &\le 2M + N_2 + 2.\end{aligned}$$

Taking $N_{1.2.18} = 2M + N_3 + 2$, we have the required result. $\square$

Lemma 1.2.19. *(X is properly embedded in X^h) Let X be a geodesic space δ-hyperbolic relative to a collection $\mathcal{H}$ of uniformly ν-separated ($\nu > 0$), intrinsically geodesic and uniformly properly embedded closed subsets of X in the sense of Gromov, then the inclusion $i : X \hookrightarrow X^h$ is a proper map i.e. for all $M > 0$ there exists $N_{1.2.19} > 0$ such that for all $x, y \in X$ if $d_{X^h}(x, y) \leq M$ then $d_X(x, y) \leq N_{1.2.19}$.*

Proof. Let $d_{X^h}(x, y) \leq M$. By definition of the metric d_{X^h}, there exists a path $\alpha : [0, l] \to X^h$ joining x and y such that α is a concatenation of geodesics from X and hyperbolic cones and $l_{X^h}(\alpha) \leq d_{X^h}(x, y) + 1$. Therefore, $l_{X^h}(\alpha) \leq M + 1$. Let $H_1^h, ..., H_n^h$ be the hyperbolic cones penetrated by α. We partition $[0, l]$ by points $0 = t_0 < s_1 < t_1 < ... < s_n < t_n < s_{n+1} = l$ such that
(i) $\alpha(0) = x$, $\alpha(s_{n+1}) = y$,
(ii) $\alpha_{[s_j, t_j]}$ is a geodesic in H_j^h,
(iii) $\alpha_{[t_i, s_{i+1}]}$ is a geodesic in X,
where $1 \leq j \leq n, 0 \leq i \leq n$.
Then $\sum_{1 \leq j \leq n} d_{H_j^h}(\alpha(s_j), \alpha(t_j)) + \sum_{0 \leq i \leq n} d_X(\alpha(t_i), \alpha(s_{i+1})) \leq M + 1$.
Hence $\sum_{1 \leq j \leq n} d_{H_j}(\alpha(s_j), \alpha(t_j)) \leq 2e^{M+1}$. Therefore, by triangle inequality, we have $d_X(x, y) \leq \sum_{1 \leq j \leq n} d_{H_j}(\alpha(s_j), \alpha(t_j)) + \sum_{0 \leq i \leq n} d_X(\alpha(t_i), \alpha(s_{i+1})) \leq 2e^{M+1} + M + 1$.
Taking $N_{1.2.19} = 2e^{M+1} + M + 1$, we have the required result.

$\square$

Definition 1.2.20. *(Definition 8.17, Chapter II.8, [BH99])(Busemann Function): Let (X, d) be a metric space and let $\gamma : [0, \infty) \to X$ be a geodesic ray. The function $b_\gamma : X \to \mathbb{R}$ defined by*

$$b_\gamma(x) = \lim_{t \to \infty} (d(x, \gamma(t)) - t), \ \ x \in X$$

is called the Busemann function associated to the geodesic ray γ.

Definition 1.2.21. *[CP93] Let (X, d) be a geodesic space and $k \geq 0$. A function $f : X \to \mathbb{R}$ is said to be k-quasiconvex if for each geodesic path $c : [0, 1] \to X$ parameterized proportional to arc length, we have*

$$f(c(t)) \leq (1 - t)f(c(0)) + tf(c(1)) + k \ \ \textit{for all } t \in [0, 1].$$

Lemma 1.2.22. *(Proposition 3.3, Chapter 3, [CP93]): Let $\delta \geq 0$ and X be a geodesic space which is δ-hyperbolic. Let $\gamma : [0, \infty) \to X$ be a geodesic ray, then the Busemann function $b_\gamma : X \to \mathbb{R}$ is 4δ-quasiconvex.*

Corollary 1.2.23. *Let $\delta \geq 0$ and X be a geodesic space which is δ-hyperbolic. Let $\gamma : [0, \infty) \to X$ be a geodesic ray, then the set $b_\gamma^{-1}((-\infty, 0])$ is 4δ-quasiconvex in X.*

Lemma 1.2.24. *Let $\delta \geq 0$. Let X be a geodesic metric space and $\mathcal{H}$ be a collection of uniformly ν-separated ($\nu > 0$) and intrinsically geodesic closed subsets of X. Suppose X^h is a geodesic space and X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov. Then for any $H \in \mathcal{H}$, the hyperbolic cone H^h is uniformly 4δ-quasiconvex in X^h. Moreover, for each $s \in [0, \infty)$, $(H^s)^h := H \times [s, \infty)$ is also 4δ-quasiconvex in X^h.*

Proof. Let $\gamma : [0, \infty) \to H^h$ be a vertical path in H^h, where $\gamma(0) \in H$. Then by Lemma 1.2.17, γ is a geodesic in X^h. First, we prove that $b_\gamma^{-1}(0) = H$.
Let $x \in H$, we will prove that $b_\gamma(x) = 0$. Note that

$$\begin{aligned} d_{X^h}(x, \gamma(t)) - t &\leq d_{X^h}((x, 0), (x, t)) + d_{X^h}((x, t), \gamma(t)) - t \\ &\quad (x \text{ is identified with } (x, 0)) \\ &\leq t + d_{H,t}((x, t), \gamma(t)) - t \\ &= e^{-t} d_H(x, \gamma(0)) \to 0 \text{ as } t \to \infty. \end{aligned}$$

Therefore $b_\gamma(x) \leq 0$. Also,

$$\begin{aligned} t &= d_{X^h}((x, 0), (x, t)) \\ &\leq d_{X^h}(x, \gamma(t)) + d_{X^h}(\gamma(t), (x, t)) \\ i.e. -(d_{X^h}(x, \gamma(t)) - t) &\leq e^{-t} d_H(\gamma(0), x) \text{ for all } t \in [0, \infty) \end{aligned}$$

Hence $b_\gamma(x) \geq 0$ and so $b_\gamma(x) = 0$. Thus $H \subset b_\gamma^{-1}(0)$.
Now we prove that $x \in b_\gamma^{-1}(0)$ implies $x \in H$.
Case (i): Let $x = (w, s) \in H \times [0, \infty) = H^h$. Then

$$\begin{aligned} 0 = b_\gamma(x) &= \lim_{t \to \infty} (d_{X^h}((w, s), \gamma(t)) - t) \\ &\leq \lim_{t \to \infty} (d_{X^h}((w, s), (w, t)) + d_{X^h}((w, t), \gamma(t)) - t) \\ &\leq \lim_{t \to \infty} (t - s + e^{-t} d_H(w, \gamma(0)) - t) \\ &\leq -s. \end{aligned}$$

Therefore $s = 0$ and $x = (w, 0) \in H$.
Case (ii). Let $x \in X^h \setminus int(H^h)$ and $\pi_H(x)$ be a nearest point projection of x onto H. For $t \in [0, \infty)$, let $[x, \gamma(t)]$ be a geodesic in X^h joining x and $\gamma(t)$. Let $x_t \in [x, \gamma(t)] \cap H$, then $t \leq d_{X^h}(x_t, \gamma(t))$. Now,

$$\begin{aligned} d_{X^h}(x, \pi_H(x)) &\leq d_{X^h}(x, x_t) \\ &= d_{X^h}(x, \gamma(t)) - d_{X^h}(\gamma(t), x_t) \\ &\leq d_{X^h}(x, \gamma(t)) - t. \end{aligned}$$

Taking $t \to \infty$, we have $d_{X^h}(x, \pi_H(x)) \leq b_\gamma(x)$. But $b_\gamma(x) = 0$. Thus $x = \pi_H(x) \in H$. Hence $b_\gamma^{-1}(0) = H$.
Now let $t_0 \in [0, \infty)$ and $\gamma_{t_0}(t) = \gamma(t + t_0)$, where $t \in [0, \infty)$, then similarly as above we can prove that $b_{\gamma_{t_0}}^{-1}(0) = H \times \{t_0\}$. It is easy to check that for $p \in X^h$, $b_{\gamma_{t_0}}(p) = 0 \Leftrightarrow b_\gamma(p) = -t_0$, thus $b_\gamma^{-1}((-\infty, 0]) = H^h$. Hence by corollary 1.2.23, we have that H^h is 4δ-quasiconvex.
Note that for $t_0 \in [0, \infty)$, we have $b_{\gamma_{t_0}}^{-1}((-\infty, 0]) = (H^{t_0})^h$, thus again by corollary 1.2.23, we have that $(H^{t_0})^h$ is 4δ-quasiconvex. ☐

Let X be a geodesic space and $\mathcal{H}$ be a collection of ν-separated ($\nu > 0$), intrinsically geodesic closed subsets of X. Let $\mathcal{E}(\mathcal{G}(X, \mathcal{H}), \mathcal{H}^h)$ be the space obtained from $\mathcal{G}(X, \mathcal{H})$ by coning off $H \times [0, \infty)$ for all $H \in \mathcal{H}$. In short, $\mathcal{E}(\mathcal{G}(X, \mathcal{H}), \mathcal{H}^h)$ will be denoted by $\widehat{X^h}$.
For $H \in \mathcal{H}$ and $r \in [0, \infty)$, let $H_r = H \times \{r\}$ and $\mathcal{H}_r = \{H_r : H \in \mathcal{H}\}$. Let $H_r^h = H_r \times [0, \infty)$ be the hyperbolic cone over H_r with metric $d_{H_r^h}$, then the space $H \times [r, \infty)$ with the induced metric from H^h is isometric to $(H_r^h, d_{H_r^h})$. Let $Y = \mathcal{G}(X, \mathcal{H}) \setminus \cup_{H \in \mathcal{H}} int(H \times [r, \infty))$.
Define $g : \mathcal{G}(X, \mathcal{H}) \to \mathcal{G}(Y, \mathcal{H}_r)$ as follows:
Let $x \in \mathcal{G}(X, \mathcal{H})$. If $x \in Y$, define $g(x) = x$. Now, if $x \in int(H \times [r, \infty))$ for some $H \in \mathcal{H}$, then $x = (h, t)$ for some $h \in H, t \geq r$. Define $g(x) = ((h, r), t - r)$.
Note that g is an isometry.

Lemma 1.2.25. *There exist $K_{1.2.25} \geq 1, \epsilon_{1.2.25} \geq 0$ depending on r such that $g|_X : X \to Y$ is an $(K_{1.2.25}, \epsilon_{1.2.25})$-quasi-isometric embedding.*

Proof. Note that $g(X)$ is r-quasiconvex in Y. Let $x, y \in X$ and $d_Y(g(x), g(y)) \leq M$, then $d_{X^h}(g(x), g(y)) = d_{Y^h}(g(x), g(y)) \leq M$. As X is properly embedded in X^h, there exists $N(M) > 0$ such that $d_X(g(x), g(y)) \leq N$. From definition, $g(x) = x, g(y) = y$, thus $d_X(x, y) \leq N$. Proof then follows from Lemma 1.1.21. ☐

Lemma 1.2.26. *g will induce a $(K_{1.2.26}, \epsilon_{1.2.26})$-quasi-isometry $\widehat{g} : \mathcal{E}(\mathcal{G}(X, \mathcal{H}), \mathcal{H}) \to \mathcal{E}(\mathcal{G}(Y, \mathcal{H}_r), \mathcal{H}_r)$ for some $K_{1.2.26} \geq 1$ and $\epsilon_{1.2.26} \geq 0$ depending on r, ν.*

Proof. Let $X^h = \mathcal{G}(X, \mathcal{H})$ and $Y^h = \mathcal{G}(Y, \mathcal{H}_r)$. Define $\widehat{g} : \widehat{X^h} \to \widehat{Y^h}$ as follows:
Let $\widehat{x} \in \widehat{X^h}$,
i) if $\widehat{x}$ is a cone point over H^h for some $H \in \mathcal{H}$, then define $\widehat{g}(\widehat{x})$ to be the cone point over H_r^h.
ii) Let $\widehat{x}$ lie on the interior of an edge joining some point $(h, t) \in H^h$ and the cone point $v(H^h)$ over H^h,
a) if $t \leq r$ then define $\widehat{g}(\widehat{x})$ to be the interior point on the edge joining (h, r) and

cone point $v(H_r^h)$ over H_r^h such that $d_{\widehat{X^h}}(\widehat{x},(h,t)) = d_{\widehat{Y^h}}(\widehat{g}(\widehat{x}),(h,r))$.
b) if $t > r$ then define $\widehat{g}(\widehat{x})$ to be the interior point on the edge joining (h,t) and cone point $v(H_r^h)$ over H_r^h such that $d_{\widehat{X^h}}(\widehat{x},(h,t)) = d_{\widehat{Y^h}}(\widehat{g}(\widehat{x}),(h,t))$
iii) If $\widehat{x} \in X^h$, then define $\widehat{g}(\widehat{x}) = g(\widehat{x})$.
Let $x, y \in X \subset Y$. From definition of metric $d_{\widehat{Y^h}}$, there exists a sequence $\{x_1, y_1, ..., x_n, y_n\}$ such that

- $x = x_1, y = y_n$ and for each i, $[x_i, y_i]$ is a geodesic in Y, $[y_i, x_{i+1}]$ is a geodesic (of length at most one) in the coned-off space of $H_i \times [r, \infty)$ for some $H_i \in \mathcal{H}$, and
- $\sum_{1\leq i\leq n} d_Y(x_i, y_i) + \sum_{1\leq i\leq n-1} l_{\widehat{Y^h}}([y_i, x_{i+1}]) \leq d_{\widehat{Y^h}}(x,y) + 1$.

Now, $d_X(x_1, H_1) \leq d_Y(x_1, y_1), d_X(H_i, H_{i+1}) \leq d_Y(x_{i+1}, y_{i+1})$ and $d_X(y_n, H_{n_1}) \leq d_Y(y_n, x_n)$. Therefore,

$$d_{\widehat{X^h}}(x,y) \leq \sum_{1\leq i\leq n} d_Y(x_i, y_i) + \sum_{1\leq i\leq n-1} l_{\widehat{Y^h}}([y_i, x_{i+1}]) \leq d_{\widehat{Y^h}}(x,y) + 1.$$

For the other inequality, using definition of metric $d_{\widehat{X^h}}$, there exists a sequence $\{z_1, w_1, ..., z_m, w_m\}$ such that

- $x = z_1, y = w_m$ and for each i, $[z_i, w_i]$ is a geodesic in X, $[w_i, z_{i+1}]$ is a geodesic (of length at most one) in the coned-off space $\widehat{H_i^h}$ for some $H_i \in \mathcal{H}$, and
- $\sum_{1\leq i\leq m} d_X(z_i, w_i) + \sum_{1\leq i\leq m-1} l_{\widehat{X^h}}([w_i, z_{i+1}]) \leq d_{\widehat{X^h}}(x,y) + 1$.

Let $P = K_{1.2.25}, \epsilon = \epsilon_{1.2.25}$. For each $2 \leq i \leq m-1$, $d_Y((z_i, r),(w_i, r)) \leq Pd_X(z_i, w_i) + \epsilon + 2r$, $d_Y(z_1,(w_1, r)) \leq Pd_X(z_1, w_1) + \epsilon + r$ and $d_Y((z_m, r), w_m) \leq Pd_X(z_m, w_m) + \epsilon + r$.
Now

$$\begin{aligned} d_{\widehat{Y^h}}(x,y) &\leq d_Y(z_1,(w_1,r)) + \sum_{2\leq i\leq m-1} d_Y((z_i,r),(w_i,r)) \\ &\quad + d_Y((z_m,r),w_m) + (\text{ cardinality of } \{H_1,...,H_{m-1}\}) \\ &\leq P\sum_{1\leq i\leq m} d_X(z_i,w_i) + 2r(m-1) + m\epsilon + (m-1) \\ &\leq Pd_{\widehat{X^h}}(x,y) + P + (2r+1)(m-1) + m\epsilon. \end{aligned}$$

As $\mathcal{H}$ is ν-separated, therefore $m - 1 \leq \frac{d_{\widehat{X^h}}(x,y)}{\nu} + 1$. Hence

$$d_{\widehat{Y^h}}(x,y) \leq (\frac{2r+1+\epsilon}{\nu} + P)d_{\widehat{X^h}}(x,y) + P + 2r + 1 + 2\epsilon.$$

Therefore

$$d_{\widehat{X^h}}(x,y) - 1 \leq d_{\widehat{Y^h}}(x,y) \leq (\frac{2r+1+\epsilon}{\nu} + P)d_{\widehat{X^h}}(x,y) + P + 2r + 1 + 2\epsilon.$$

Let $P_1 = \frac{2r+1+\epsilon}{\nu} + P$ and $\epsilon_1 = P + 2r + 1 + 2\epsilon$. Now for any point $\widehat{p} \in \widehat{X^h}$, there exists $p \in X$ such that $d_{\widehat{X^h}}(\widehat{p}, p) \leq 1$ and $d_{\widehat{Y^h}}(\widehat{g}(\widehat{p}), \widehat{g}(p)) \leq r + 1$. Note that $\widehat{g}(p) = p$.

Therefore, by triangle inequality, we have

$$d_{\widehat{X^h}}(\widehat{p},\widehat{q}) - (2r+4) \leq d_{\widehat{Y^h}}(\widehat{g}(\widehat{p}),\widehat{g}(p)) \leq P_1 d_{\widehat{X^h}}(\widehat{p},\widehat{q}) + 2P_1 + \epsilon_1$$

for all $\widehat{p},\widehat{q} \in \widehat{X^h}$. Taking $K_{1.2.26} = P_1$ and $\epsilon_{1.2.26} = \max\{2P_1+\epsilon_1, 2r+4\}$, we have the required result. □

Note that if $\widehat{\alpha}$ is a K-quasigeodesic path in $\mathcal{E}(\mathcal{G}(X,\mathcal{H}),\mathcal{H})$ without backtracking and $\alpha_1,...,\alpha_n$ are (consecutive) components of $\alpha^b = \widehat{\alpha} \cap X$, then $\widehat{g}(\widehat{\alpha})$ is a quasigeodesic in $\mathcal{E}(\mathcal{G}(Y,\mathcal{H}_r))$ with $g|_X(\alpha_i)$ being quasigeodesic paths in $\mathcal{G}(Y,\mathcal{H}_r)$. $g|_X$ is an identity map, therefore $g|_X(\alpha_i) = \alpha_i$ for all i. Let x_i, y_i be end points of α_i. For each $i \in \{1,...,n-1\}$, we join y_i and x_{i+1} by the path $[y_i,(y_i,r)] \cup [(y_i,r),v(H_i)] \cup [v(H_i),(x_{i+1},r)] \cup [(x_{i+1},r),x_{i+1}]$ of length $2r+1$ in $\mathcal{G}(Y,\mathcal{H}_r)$. Consequently, we obtain a path $\widehat{\mu}$ without backtracking in $\mathcal{E}(\mathcal{G}(Y,\mathcal{H}_r))$ such that

- $\widehat{\mu} \setminus \cup_{H\in\mathcal{H}} H^h = \alpha^b$,
- $\widehat{\mu}$ is a $(K+2r+1)$-quasigeodesic path.

Hence, quasigeodesics in $\mathcal{G}(X,\mathcal{H})$ have similar intersection properties with hyperbolic cones if and only if quasigeodesics in $\mathcal{G}(Y,\mathcal{H}_r)$ have similar intersection properties with hyperbolic cones. Thus, we have the following corollary,

Corollary 1.2.27. *X^h is hyperbolic relative to $\mathcal{H}^h$ in the sense of Farb if and only if Y^h is hyperbolic relative to $\mathcal{H}^h_r$ in the sense of Farb.*

Next we prove that the space $\mathcal{E}(X,\mathcal{H})$ embeds quasi-isometrically into the space $\mathcal{E}(\mathcal{G}(X,\mathcal{H}),\mathcal{H}^h)$.

Lemma 1.2.28. *Let X be a geodesic space and $\mathcal{H}$ be a collection of ν-separated ($\nu > 0$) intrinsically geodesic closed subsets of X. Then the natural inclusion $\widehat{X} \hookrightarrow \widehat{X^h}$ is a $(K_{1.2.28},\epsilon_{1.2.28})$ quasi-isometry for some numbers $K_{1.2.28} \geq 1, \epsilon_{1.2.28} \geq 0$.*

Proof. Let $j: X \hookrightarrow X^h$ denote the inclusion. Then j induces a natural inclusion $\widehat{j}: \widehat{X} \hookrightarrow \widehat{X^h}$, therefore $d_{\widehat{X^h}}(\widehat{j}(\widehat{x}),\widehat{j}(\widehat{y})) \leq d_{\widehat{X}}(\widehat{x},\widehat{y})$ for all $\widehat{x},\widehat{y} \in \widehat{X}$.
Let $x,y \in X$. By definition of the metric $d_{\widehat{X^h}}$, there exists a sequence $q_0,p_1,q_1,...,p_n,q_n,p_{n+1}$ of points in X^h such that $x = q_0, y = p_{n+1}$, $[q_i,p_{i+1}]$ $(0 \leq i \leq n)$ are geodesics in X^h with $[q_i,p_{i+1}] \subset X$, and $\sum_{0\leq i\leq n} d_{X^h}(q_i,p_{i+1}) + \sum_{1\leq j\leq n} d_{\widehat{H^h_j}}(p_j,q_j) \leq d_{\widehat{X^h}}(x,y)+1$.
Since $[q_i,p_{i+1}] \subset X$, $d_{X^h}(q_i,p_{i+1}) = d_X(q_i,p_{i+1})$ for all $0 \leq i \leq n$. Let e_j be the edge path of length one from p_j to q_j passing through the cone point $v(H_j)$, where $1 \leq j \leq n$. Then, by triangle inequality,

$$d_{\widehat{X}}(x,y) \leq \sum_{0\leq i\leq n} d_X(q_i,p_{i+1}) + \sum_{1\leq j\leq n} l_{\widehat{X}}(e_j) \leq d_{\widehat{X^h}}(x,y) + 1 + n.$$

Now, $n-1 \leq \frac{d_{\widehat{X^h}}(x,y)}{\nu}$, therefore $d_{\widehat{X}}(x,y) \leq d_{\widehat{X^h}}(x,y)(1+\frac{1}{\nu})+2$. Now j restricted on each edge over horosphere-like sets in $\widehat{X}$ is an isometry, therefore

$$d_{\widehat{X^h}}(\widehat{j}(\widehat{x}),\widehat{j}(\widehat{y})) \leq d_{\widehat{X}}(\widehat{x},\widehat{y}) \leq d_{\widehat{X^h}}(\widehat{j}(\widehat{x}),\widehat{j}(\widehat{y}))(1+\frac{1}{\nu})+2$$

for all $\widehat{x},\widehat{y} \in \widehat{X}$.
Also the Hausdorff distance between $\widehat{j}(\widehat{X})$ and $\widehat{X^h}$ is at most 1. Taking $K_{1.2.28} = 1+\frac{1}{\nu}, \epsilon_{1.2.28} = \frac{2}{1+\frac{1}{\nu}}$, we have the required result. ☐

Corollary 1.2.29. *With above notation, X is hyperbolic relative to $\mathcal{H}$ in the sense of Farb if and only if X^h is hyperbolic relative to $\mathcal{H}^h$ in the sense of Farb.*

Definition 1.2.30. *Let X_1, X_2 be two geodesic spaces and $\mathcal{H}_{X_1}, \mathcal{H}_{X_2}$ be collections of uniformly $\nu(>0)$-separated and intrinsically geodesic closed subsets of X_1, X_2 respectively. A quasi-isometry $\phi\colon X_1 \to X_2$ is said to be strictly type-preserving if $\phi(H_{X_1}) \in \mathcal{H}_{X_2}$ and $\phi^{-1}(H_{X_2}) \in \mathcal{H}_{X_1}$ for all $H_{X_1} \in \mathcal{H}_{X_1}, H_{X_2} \in \mathcal{H}_{X_2}$, where ϕ^{-1} is quasi-isometric inverse of ϕ.*

Now we prove that a strictly type-preserving quasi-isometry induces quasi-isometries between the coned-off spaces as well as between the hyperbolic spaces obtained by gluing hyperbolic cones.

Lemma 1.2.31. *Let $K \geq 1, \epsilon \geq 0, \nu > 0, r \geq 0$. Suppose X_1, X_2 be two geodesic spaces and $\mathcal{H}_{X_1}, \mathcal{H}_{X_2}$ be collections of uniformly ν-separated and intrinsically geodesic closed subsets of X_1, X_2 respectively. Let $\phi\colon X_1 \to X_2$ be a (K,ϵ)-quasi-isometry such that for each $H \in \mathcal{H}_{X_1}$ there exists $F \in \mathcal{H}_{X_2}$ such that the Hausdorff distance between $\phi(H)$ and F is at most r in X_2, and the Hausdorff distance between $\phi^{-1}(F)$ and H is at most r in X_1.*
Then $\phi\colon X_1 \to X_2$ will induce
1) a $(K^h_{1.2.31}, \epsilon^h_{1.2.31})$-quasi-isometry $\phi^h : X_1^h \to X_2^h$ for some $K^h_{1.2.31} \geq 1, \epsilon^h_{1.2.31} \geq 0$, and
2) a $(\widehat{K}_{1.2.31}, \widehat{\epsilon}_{1.2.31})$-quasi-isometry $\widehat{\phi} : \widehat{X_1} \to \widehat{X_2}$ for some $\widehat{K}_{1.2.31} \geq 1, \widehat{\epsilon}_{1.2.31} \geq 0$.
In particular, if ϕ is a strictly type-preserving quasi-isometry, then ϕ will induce strictly type preserving quasi-isometries $\phi^h : X_1^h \to X_2^h$ and $\widehat{\phi} : \widehat{X_1} \to \widehat{X_2}$.

Proof. By Lemma 1.2.26, we can assume ν to be greater than 2.
1) Define $\phi^h : X_1^h \to X_2^h$ as follows:
Let $z \in X_1^h$, define $\phi^h(z) = \phi(z)$ if $z \in X_1$, and
if $z = (w,t)$ lies inside some hyperbolic cone H^h, then there exists $f \in F$ for some

$F \in \mathcal{H}_{X_2}$ such that $d_{X_2}(\phi(w), f) \le r$, define $\phi^h(w,t) = (f,t)$.
Note that for each $(f',t) \in F^h$ there exists $(w',t) \in H^h$ such that
$d_{X_2^h}(\phi^h(w',t),(f',t)) \le 2r$.
Since ϕ is a quasi-isometry, therefore the Hausdorff distance between X_2 and $\phi(X_1)$ is uniformly bounded. Hence, the Hausdorff distance between X_2^h and $\phi^h(X_1^h)$ is uniformly bounded.
Now first we prove that there exists $P \ge 1$ such that for all $a, b \in X_1^h$ with $d_{X_1^h}(a,b) \le 1$ implies that $d_{X_2^h}(\phi^h(a),\phi^h(b)) \le P$. By definition of the metric $d_{X_1^h}$, there exists a path $\alpha : [0,1] \to X_1^h$ joining a and b such that α is a concatenation of geodesics from X_1 and hyperbolic cones and $l_{X_1^h}(\alpha) \le d_{X_1^h}(a,b) + 1$. Therefore $l_{X_1^h}(\alpha) \le 2$. Note that α can intersects at most one hyperbolic cone, say H^h. Let $a \in X_1$ and $b \in H^h$. We partition $[0,1]$ by points $0 = t_0 < s_1 < t_1 < ... < s_n \le t_n = 1$ such that
(i) $\alpha(0) = a$, $\alpha(t_n) = b$,
(ii) $\alpha_{[s_j,t_j]}$ is a geodesic in H^h,
(iii) $\alpha_{[t_i,s_{i+1}]}$ is a geodesic in X_1,
where $1 \le j \le n, 0 \le i \le n-1$.
Note that $\sum_{1\le j\le n} d_{H^h}(\alpha(s_j),\alpha(t_j)) = \sum_{1\le j\le n} l_{H^h}(\alpha_{[s_j,t_j]}) \le 2$.
Now

$$\sum_{1\le j\le n-1} d_H(\alpha(s_j),\alpha(t_j)) \le 2(e^{\sum_{1\le j\le n} d_{H^h}(\alpha(s_j),\alpha(t_j))} - 1) \le 2e^2.$$

Also $\sum_{0\le i\le n-1} d_{X_1}(\alpha(t_i),\alpha(s_{i+1})) \le 2$. Therefore by triangle inequality, we have $d_{X_1}(a,\alpha(s_n)) + d_{H^h}(\alpha(s_n),\alpha(t_n)) \le 2e^2 + 2 + 2 = 2e^2 + 4 = D$, say.
There exist $h_n \in H, l \in [0,\infty)$ such that $\alpha(t_n) = (h_n, l)$. As $d_{H^h}(\alpha(s_n),\alpha(t_n)) \le D$, we have $l \le D$. By triangle inequality, $d_{H^h}(\alpha(s_n),h_n) \le 2D$. Therefore, $d_H(\alpha(s_n),h_n) \le 2e^{2D}$ and hence

$$d_{X_1}(a,h_n) \le d_{X_1}(a,\alpha(s_n)) + d_{X_1}(\alpha(s_n),h_n) \le D + d_H(\alpha(s_n),h_n) \le D + 2e^{2D}.$$

Now there exists $F \in \mathcal{H}_{X_2}$ such that $\phi(H) \subset Nbhd_{X_2}(F;r)$. By definition, $\phi^h(\alpha(t_n)) = (f_n, l)$ for some $f_n \in F$ such that $d_{X_2}(\phi(h_n), f_n) \le r$. Thus

$$\begin{aligned}
d_{X_2^h}(\phi^h(a),\phi^h(b)) &= d_{X_2^h}(\phi^h(a),\phi^h(\alpha(t_n)))\\
&\le d_{X_2^h}(\phi(a),f_n) + d_{X_2^h}(f_n,(f_n,l))\\
&\le d_{X_2}(\phi(a),f_n) + l\\
&\le d_{X_2}(\phi(a),\phi(h_n)) + d_{X_2}(\phi(h_n),f_n) + D\\
&\le K d_{X_1}(a,h_n) + \epsilon + r + D\\
&\le K(D + 2e^{2D}) + \epsilon + r + D, \text{ where } D = e^2 + 4.
\end{aligned}$$

Taking $P = K(D + 2e^{2D}) + \epsilon + r + D$, we have $d_{X_2^h}(\phi^h(a),\phi^h(b)) \le P$. Similarly, there exists $P \ge 1$ such that the above inequality holds if both a, b lie in hyperbolic

cones or in X_1.
Now for any $x, y \in X_1^h$, by definition of metric $d_{X_1^h}$, there exists a path λ in X^h joining x, y such that
• λ is a concatenation of geodesics of X_1 and hyperbolic cones,
• $l_{X^h}(\lambda) \leq d_{X_1^h}(x, y) + 1$.
We partition λ by points $x = p_0, p_1, ..., p_n = y$ such that for $0 \leq i \leq n-1$, length of the subsegment joining p_i, p_{i+1} is equal to one and length of the subsegment joining p_{n-1}, y is at most one. Then $d_{X_1^h}(p_i, p_{i+1}) \leq 1$ and hence $d_{X_2^h}(\phi^h(p_i), \phi^h(p_{i+1})) \leq P$ for all $0 \leq i \leq n-1$. Thus, by triangle inequality, we have $d_{X_2^h}(\phi^h(x), \phi^h(y)) \leq nP \leq Pl_{X^h}(\lambda) + P \leq P(d_{X_1^h}(x, y) + 1) + P \leq Pd_{X_1^h}(x, y) + 2P$.
Let $\psi = \phi^{-1}$, then define $\psi^h : X_2^h \to X_1^h$ similarly as ϕ^h. Note that by definition of quasi-isometry, for all $z \in X_1$, $d_{X_1}(\psi(\phi(z)), z) \leq K$. Let ψ be (K_1, ϵ_1)-quasi-isometry, then by a calculation, it can be shown that for all $z \in X_1^h$, we have $d_{X_1^h}(z, \psi^h(\phi^h(z))) \leq K_1 r + r + K + \epsilon_1$. By above argument, there exists $P' \geq 1$ such that $d_{X_1^h}(\psi^h(\phi^h(x)), \psi^h(\phi^h(y))) \leq P' d_{X_2^h}(\phi^h(x), \phi^h(y)) + P'$. Therefore,

$$d_{X_1^h}(x, y) \leq P' d_{X_2^h}(\phi^h(x), \phi^h(y)) + P' + 2(K_1 r + r + K + \epsilon_1).$$

Taking $K_{1.2.31}^h = \max\{P, P'\}$, $\epsilon_{1.2.31}^h = \max\{P, P' + 2(K_1 r + r + K + \epsilon_1)\}$, we have the required result.

2) Now we define $\widehat{\phi} : \widehat{X_1} \to \widehat{X_2}$:
Let $x \in \widehat{X_1}$ and x is not a cone point, i.e., $x \in X_1$. Define $\widehat{\phi}(x) = \phi(x)$. If x is a cone point over some $H_1 \in \mathcal{H}_{X_1}$ or lies in the interior of an edge, then define $\widehat{\phi}(x)$ to be the cone point over $\phi(H_1)$. As the metric on $\widehat{X_1}$ is defined by taking infimum on chains (refer to Definition 1.2.1), proof of this fact is similar as (1).

□

Corollary 1.2.32. *With hypothesis as in above Lemma 1.2.31 , we have*
1) X_1 is hyperbolic relative to $\mathcal{H}_1$ in the sense of Gromov if and only if X_2 is hyperbolic relative to $\mathcal{H}_2$ in the sense of Gromov,
2) X_1^h is hyperbolic relative to $\mathcal{H}_1^h$ in the sense of Farb if and only if X_2^h is hyperbolic relative to $\mathcal{H}_2^h$ in the sense of Farb,
3) X_1 is hyperbolic relative to $\mathcal{H}_1$ in the sense of Farb if and only if X_2 is hyperbolic relative to $\mathcal{H}_2$ in the sense of Farb.

Proof. 1) Follows from (1) of Lemma 1.2.31.
2) Only thing we require to prove is the similar intersection properties of quasi-geodesics with hyperbolic cones. Let quasigeodesic paths in $\widehat{X_2^h}$ have similar intersection patterns with hyperbolic cones. By Lemma 1.2.31, there exists $K_1 \geq 1, \epsilon_1 \geq 0$

such that ϕ induces (K_1, ϵ_1)-quasi-isometries $\widehat{\phi^h} : \widehat{X_1^h} \to \widehat{X_2^h}$ and $\phi^h : X_1^h \to X_2^h$. Let $\widehat{\lambda} : [a,b] \to \widehat{X_1^h}$ be a P_1-quasigeodesic path, there exists $\widehat{P}_2 \geq 1$ such that $\widehat{\phi^h}(\widehat{\lambda}) : [a,b] \to \widehat{X_2^h}$ is a $\widehat{P}_2$-quasigeodesic in $\widehat{X_2^h}$.

But $\widehat{\phi^h}(\widehat{\lambda})$ may not be a continuous path, we construct a quasigeodesic path $\widehat{\alpha}$ in $\widehat{X_2^h}$ such that outside hyperbolic cones, $\widehat{\phi^h}(\widehat{\lambda})$ and $\widehat{\alpha}$ lie in a bounded neighborhood of each other in X_2^h :

Let $\lambda_1, ..., \lambda_n$ be connected components of $\widehat{\lambda} \setminus (\cup_{H^h \in \mathcal{H}_1^h} H^h)$, then each λ_j is a P_1-quasigeodesic path in X_1^h. As ϕ^h is a quasi-isometry (by Lemma 1.2.31), there exists $P_2 \geq 1$ such that each $\phi^h(\lambda_j)$ is a P_2-quasigeodesic in X_2^h.
Let $t_0 < s_1 < t_1 < ... < s_n < t_n < s_{n+1}$ be a partition of $[a,b] \cap \mathbb{Z}$ such that
• for each $j \in \{0, ..., n\}$, $\widehat{\lambda}|_{[t_j, s_{j+1}]} \subset \lambda_j$ and
• for each $i \in \{1, ..., n\}$, $\widehat{\lambda}|_{[s_i, t_i]}$ penetrates a hyperbolic cone H_i^h with $\widehat{\lambda}(s_i + 1)$, $\widehat{\lambda}(t_i - 1)$ lies in the coned-off space $\widehat{H_i^h}$.
Then $d_{X_1^h}(\widehat{\lambda}(s_i), H_i^h) \leq P_1|s_i - (s_i+1)| + P_1 = 2P_1$ and $d_{X_1^h}(\widehat{\lambda}(t_i), H_i^h) \leq P_1|(t_i - 1) - t_i| + P_1 = 2P_1$.
Now for each H_i, there exists $F_i \in \mathcal{H}_2$ such that the Hausdorff distance between $\phi(H_i)$ and F_i is at most r. Let $\widehat{\mu} = \widehat{\phi^h}(\widehat{\lambda})$, then for each i, we have $d_{X_2^h}(\widehat{\mu}(s_i), F_i^h) \leq 2K_1P_1 + \epsilon_1 + r$ and $d_{X_2^h}(\widehat{\mu}(t_i), F_i^h) \leq 2K_1P_1 + \epsilon_1 + r$.
Let $P' = 2K_1P_1 + \epsilon_1 + r$. For each i, let $\beta_i : [s_i, t_i] \to \widehat{X_2^h}$ be a reparametrization of a geodesic in $\widehat{X_2^h}$ joining $\widehat{\mu}(s_i)$ and $\widehat{\mu}(t_i)$, then $l_{\widehat{X_2^h}}(\beta_i) \leq 2P' + 1$.
As $\phi^h(\lambda_j)$ is a P_2-quasigeodesic in X_2^h and $\widehat{\phi^h}|_{X_1} = \phi^h|_{X_1}$, for $k, k+1 \in [t_j, s_{j+1}] \cap \mathbb{Z}$, we have $d_{X_2^h}(\widehat{\mu}(k), \widehat{\mu}(k+1)) \leq P_2|k - (k+1)| + P_2 = 2P_2$. Let $c_k : [k, k+1] \to X_2^h$ be a linear reparametrization of a geodesic in X_2^h joining $\widehat{\mu}(k)$ and $\widehat{\mu}(k+1)$. For each j, let $\alpha_j : [t_j, s_{j+1}] \to X_2^h$ denotes the concatenation of c_k's. Let $\widehat{\alpha} : [a,b] \to \widehat{X_2^h}$ denotes the concatenation of paths $\alpha_0, \beta_1, \alpha_1, \beta_2, ..., \beta_n, \alpha_n$.
Let $P = \max\{2P' + 1, 2P_2\}$ and $[t]$ be the integer part of t, then for all $t \in [a,b]$, we have $l_{\widehat{X_2^h}}(\widehat{\alpha}|_{[t,[t]]}) \leq P$. Thus for all $t, t' \in [a,b]$, we have

$$l_{\widehat{X_2^h}}(\widehat{\alpha}|_{[t,t']}) \leq P|t - t'| + 3P.$$

As $\widehat{\mu}$ is a $\widehat{P}_2$-quasigeodesic, we have

$$\frac{1}{\widehat{P}_2}|t - t'| - \widehat{P}_2 \leq d_{\widehat{X_2^h}}(\widehat{\mu}(t), \widehat{\mu}(t')) \leq \widehat{P}_2|t - t'| + \widehat{P}_2.$$

Therefore $d_{\widehat{X_2^h}}(\widehat{\mu}([t]), \widehat{\mu}(t)) \leq 2\widehat{P}_2$. Note that either $\widehat{\alpha}([t]) = \widehat{\mu}([t])$ or there exists $l \in \{s_j, t_j\}$ with $\widehat{\alpha}(l) = \widehat{\mu}(l)$ such that $d_{\widehat{X_2^h}}(\widehat{\alpha}(l), \widehat{\alpha}([t])) \leq P$ and $d_{\widehat{X_2^h}}(\widehat{\mu}(l), \widehat{\mu}([t])) \leq P$. Thus, by triangle inequality, $d_{\widehat{X_2^h}}(\widehat{\mu}([t]), \widehat{\alpha}([t])) \leq 2P$ and hence

$$d_{\widehat{X_2^h}}(\widehat{\mu}(t), \widehat{\alpha}(t)) \leq P + 2P + 2\widehat{P} = R, \text{ say.}$$

Thus,

$$|t-t'| \leq \widehat{P}_2\{d_{\widehat{X_2^h}}(\widehat{\alpha}(t),\widehat{\alpha}(t'))+2R\}+\widehat{P}_2^2 = \widehat{P}_2 d_{\widehat{X_2^h}}(\widehat{\alpha}(t),\widehat{\alpha}(t'))+(2R\widehat{P}_2+\widehat{P}_2^2).$$

Let $S = 2R\widehat{P}_2 + \widehat{P}_2^2$ then $S \geq P, \widehat{P}_2$, thus we have

$$\frac{1}{S}|t-t'| - S \leq d_{\widehat{X_2^h}}(\widehat{\alpha}(t),\widehat{\alpha}(t')) \leq S|t-t'|+S.$$

Thus, $\widehat{\alpha}$ is a S-quasigeodesic path in $\widehat{X_2^h}$ such that α_i lie inside $\frac{P}{2}$-neighborhood of $\phi^h(\lambda_i)$ in X_2^h and $\phi^h(\lambda_i)$ lie inside $2P_2$-neighborhood of α_i in X_2^h.
Note that α_j constructed above may intersects hyperbolic cones other than those intersected by $\widehat{\phi^h}(\widehat{\lambda})$. Let $q = \frac{P}{2}+2P_2$, then α_i and $\phi^h(\lambda_i)$ lie inside q-neighborhood of each other. Let $\mathcal{H}_{2q}^h = \{F\times[q,\infty) : F \in \mathcal{H}_2\}$ and $Y_2 = X_2^h \setminus \cup_{F\in\mathcal{H}_2} int(F\times[q,\infty))$. Then by Lemma 1.2.26, there exists a quasi-isometry $\widehat{g} : \widehat{X_2^h} \to \widehat{Y_2^h}$ and hence $\widehat{g}(\widehat{\alpha})$ is a quasigeodesic in $\widehat{Y_2^h}$. As $\phi^h(\lambda_j)$ lie outside hyperbolic cones in X_2^h and α_j lie inside q-neighborhood of $\phi^h(\lambda_j)$, therefore $\widehat{g}(\alpha_j)$ does not intersect any elements from $\mathcal{H}_{2q}^h$. Note that for all j, there exist $x_j, y_j \in F_j$ such that $d_{X_2^h}(\alpha_{j-1}(s_j), x_j) \leq P'$ and $d_{X_2^h}(\alpha_j(t_j), y_j) \leq P'$. Also note that $d_{\widehat{Y_2^h}}(x_j, y_j) \leq 2q+1$. Thus $d_{\widehat{Y_2^h}}(\alpha_{j-1}(s_j), \alpha_j(t_j)) \leq 2P'+2q+1$. We join $\alpha_{j-1}(s_j)$ and $\alpha_j(t_j))$ by a geodesic $[\alpha_{j-1}(s_j), \alpha_j(t_j))]$ in $\widehat{Y_2^h}$. Let $\widehat{\eta}$ be the concatenation of paths α_j's and $[\alpha_{j-1}(s_j), \alpha_j(t_j))]$. Thus, from $\widehat{\alpha}$ we obtain a path $\widehat{\eta}$ in $\widehat{Y_2^h}$ such that

- $\widehat{\eta}$ is a S'-quasigeodesic path in $\widehat{Y_2^h}$ for some $S' \geq 1$,
- outside hyperbolic cones in Y_2^h, $\widehat{\eta}_j$ and $\widehat{g}(\widehat{\phi^h}(\widehat{\gamma}_j))$ (j=1,2) lie in the $q+P'$-neighborhood of each other in Y_2^h, and
- horosphere-like sets intersected by $\widehat{\eta}$ are those which lie in the Hausdorff distance $r+q$ of the images of horosphere-like sets penetrated by $\widehat{\lambda}$ under the map $\widehat{g}\circ\widehat{\phi^h}$.

Now, as $(X_2^h, \mathcal{H}_2^h)$ is relatively hyperbolic in the sense of Farb, $(Y_2^h, \mathcal{H}_{2q}^h)$ is relatively hyperbolic in the sense of Farb. Suppose $\widehat{\gamma}_1, \widehat{\gamma}_2$ are two P_1-quasigeodesic paths in $\widehat{X_1^h}$ without backtracking joining same pair of points in X_1. By above there exist $S' \geq 1$ and two S'-quasigeodesic paths $\widehat{\eta}_1, \widehat{\eta}_2$ without backtracking in $\widehat{Y_2^h}$ such that outside hyperbolic cones in Y_2^h, $\widehat{\eta}_j$ and $\widehat{g}(\widehat{\phi^h}(\widehat{\gamma}_j))$ (j=1,2) lie in the $q+P'$-neighborhood of each other in Y_2^h. For each $j = 1,2$, horosphere-like sets intersected by η_j lie in a bounded Hausdorff distance from the images of hyperbolic cones intersected by γ_j under the map $\widehat{g}\circ\widehat{\phi^h}$. As $(Y_2^h, \mathcal{H}_{2q}^h)$ is relatively hyperbolic in the sense of Farb, η_1, η_2 have similar intersection properties with the sets from $\mathcal{H}_{2q}^h$. Since $\widehat{g}\circ\widehat{\phi^h}$ is a quasi-isometry, therefore γ_1, γ_2 have similar intersection properties with hyperbolic cones in X_1^h.
3) Follows from Corollary 1.2.29 and (2) of this Lemma. □

For a metric space Z, note that if α is a (P,ϵ)-quasigeodesic then α followed by a geodesic of length at most k is a $(P,\epsilon+k)$-quasigeodesic. This fact will be used in the following lemma.

Lemma 1.2.33. *Let X be hyperbolic relative to $\mathcal{H}$ in the sense of Farb. Let $x \in X$, $H \in \mathcal{H}$ and v_H be the cone point over H. Suppose λ_1 and λ_2 are two P-quasigeodesic paths in $\mathcal{E}(X,\mathcal{H})$ joining x and v_H. Let en_1 and en_2 be entry points to H of λ_1 and λ_2 respectively. Then $d_X(en_1, en_2) \leq I_{1.2.33}$, for some $I_{1.2.33}(P) > 0$ depending only on P.*

Proof. Fix some $y \in H$ and join y to v_H by a geodesic $[v_H, y]$ of length $\frac{1}{2}$ in $\mathcal{E}(X,\mathcal{H})$. Let $\lambda_1' = \lambda_1 \cup [v_H, y]$ and $\lambda_2' = \lambda_2 \cup [v_H, y]$. Then there exists $P'(P) > 0$ such that λ_1' and λ_2' are two P'-quasigeodesics in $\mathcal{E}(X,\mathcal{H})$ joining same pair of points x, y and having the same entry points as λ_1 and λ_2. By similar intersection pattern 2, there exists $I_{1.2.33} > 0$ such that $d_X(en_1, en_2) \leq I_{1.2.33}$. □

Farb's definition implies Gromov's definition

Here we prove that Farb's definition of relative hyperbolicity implies Gromov's definition. This is proved by Bowditch in [Bow97], here we propose to give another proof. To prove this we use the following criterion (due to Hamenstädt) for the hyperbolicity of a geodesic space:

Lemma 1.2.34. *([Ham05]) Let (Y,d) be a geodesic metric space. Assume that there is number $S_{1.2.34} > 0$ and for every pair of points $x, y \in Y$ there is a path $c(x,y) : [0,1] \to Y$ connecting $c(x,y)(0) = x$ to $c(x,y)(1) = y$ with the following properties:*
(1) If $d(x,y) \leq 1$ then the diameter of the set $c(x,y)[0,1]$ is at most $S_{1.2.34}$.
(2) For $x, y \in Y$ and $0 \leq s \leq t \leq 1$, the Hausdorff distance between $c(x,y)[s,t]$ and $c(c(x,y)(s), c(x,y)(t))[0,1]$ is at most $S_{1.2.34}$.
(3) For any triple (x,y,z) of points in Y, the arc $c(x,y)[0,1]$ is contained in the $S_{1.2.34}$-neighborhood of $c(x,z)[0,1] \cup c(z,y)[0,1]$.
Then the space (Y,d) is $\delta_{1.2.34}$-hyperbolic for a constant $\delta_{1.2.34} > 0$ depending only on $S_{1.2.34}$. Moreover, for all $x, y \in Y$ the Hausdorff distance between $c(x,y)$ and a geodesic connecting x to y is at most $b_{1.2.34}$, for some number $b_{1.2.34} > 0$ depending only upon $\delta_{1.2.34}$.

Theorem 1.2.35. *Given $\widehat{\delta} \geq 0, \nu > 0$ there exists $\delta_{1.2.35} \geq 0$ such that the following holds: Let X be a geodesic metric space and $\mathcal{H}$ be a collection of uniformly ν-separated and intrinsically geodesic closed subsets of X. If X is $\widehat{\delta}$-hyperbolic relative* *to the collection $\mathcal{H}$ in the sense of Farb then X is $\delta_{1.2.35}$ hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov.*

Proof. In view of Corollary 1.2.27, we can assume $\nu > 1$. By Proposition 1.1.9, we can assume X^h to be a metric graph. Hence X^h is a geodesic metric space. As X is $\widehat{\delta}$-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Farb, by Lemma 1.2.28, there exists $\widehat{\delta}_{1.2.28} \geq 0$ such that X^h is $\widehat{\delta}_{1.2.28}$ -hyperbolic relative to the collection $\mathcal{H}^h$ in the sense of Farb. Therefore triangles in $\widehat{X^h}$ are $\widehat{\delta}_{1.2.28}$-thin.

To prove X^h hyperbolic, we require to find a number $S_{1.2.34} > 0$ and a path $c(x, y)$ joining each pair of points $x, y \in X^h$ satisfying the three properties of Lemma 1.2.34. Let $x, y \in X^h$. As $X^h \subset \widehat{X^h}$, we have $x, y \in \widehat{X^h}$. Let $\widehat{c}(x, y)$ be an electric geodesic in $\widehat{X^h}$ joining x and y. Now we construct a path $c(x, y)$ from $\widehat{c}(x, y)$ in X^h joining x and y:

- If x, y lie inside a hyperbolic cone H^h, then $c(x, y)$ is a geodesic in H^h joining them.
- If x lies inside a hyperbolic cone H^h penetrated by $\widehat{c}(x, y)$, then we replace the subsegment of the geodesic $\widehat{c}(x, y)$ joining x and its exit point from H^h by a geodesic in H^h.
- If $\widehat{c}(x, y)$ penetrates a hyperbolic cone H^h with p as entry point and q as exit point, we replace the subsegment of $\widehat{c}(x, y)$ joining p and q by a geodesic in H^h joining p and q.
- If y lies inside a hyperbolic cone H^h penetrated by $\widehat{c}(x, y)$, then we replace the subsegment of the geodesic $\widehat{c}(x, y)$ joining y and its entry point to H^h by a (hyperbolic) geodesic in H^h.
- Outside hyperbolic cones, $c(x, y)$ is same as $\widehat{c}(x, y)$.

(1) <u>$c(x, y)$ satisfies property 1 of Lemma 1.2.34:</u> Let $x, y \in X^h$ such that $d_{X^h}(x, y) \leq 1$. As X^h is a graph of edge length one, therefore by construction of $c(x, y)$ and definition of the metric d_{X^h}, $c(x, y)$ is a subsegment of concatenation of at most two edges in X^h. Thus diameter of the set $c(x, y)[0, 1]$ is at most two.

(2) <u>$c(x, y)$ satisfies property 2 of Lemma 1.2.34:</u> Let $s, t \in [0, 1]$, $\mu_1 = c(x, y)[s, t]$, $\mu_2 = c(c(x, y)(s), c(x, y)(t))$. Let $\widehat{\mu}_1$ be the subsegment of $\widehat{c}(x, y)$ joining $c(x, y)(s)$ and $c(x, y)(t)$ and $\widehat{\mu}_2 = \widehat{c}(c(x, y)(s), c(x, y)(t))$. Then $\widehat{\mu}_1$ and $\widehat{\mu}_2$ are electric geodesics joining same pair of points, so they have similar intersection patterns with hyperbolic cones. We will show that there exists a number $P > 0$ such that for any $p \in \mu_1$ there exists $q \in \mu_2$ such that $d_{X^h}(p, q) \leq P$.
Let $p \in \mu_1$. If p lie in a hyperbolic cone H^h penetrated by both $\widehat{\mu}_1$ and $\widehat{\mu}_2$, then

due to similar intersection pattern 2, the distance between entry points (resp. exit points) of μ_1 and μ_2 to H^h is bounded by some number $I > 0$. Due to stability of quasigeodesics, portions of μ_1 and μ_2 lying in H_1^h are at bounded distance from each other. Thus there exists $q \in \mu_2$ such that $d_{X^h}(p,q) \leq P_1$ for some constant $P_1 > 0$. Now let $\alpha_1 : [a,b] \to X^h, \alpha_2 : [a,b] \to X^h$ be subsegments of μ_1, μ_2 respectively such that $p \in \alpha_1$ and the followings hold:
i) hyperbolic cones penetrated by $\alpha_{1|_{(a,b)}}, \alpha_{2|_{(a,b)}}$ are different,
ii) either $\alpha_1(a) = \alpha_2(a)$ or $\alpha_1(a), \alpha_2(a)$ lie on a same horosphere-like set, and
iii) either $\alpha_1(b) = \alpha_2(b)$ or $\alpha_1(b), \alpha_2(b)$ lie on a same horosphere-like set.
If end points of α_1, α_2 lie on horosphere-like sets, then due to similar intersection pattern 2, there exists $I > 0$ such that $d_{X^h}(\alpha_1(a), \alpha_2(a)) \leq I$ and $d_{X^h}(\alpha_1(b), \alpha_2(b)) \leq I$. Let $\widehat{\alpha}_i$ be the corresponding subsegment of $\widehat{\mu}_i$ in $\widehat{X^h}$ joining $\alpha_i(a), \alpha_i(b)$, where $i = 1, 2$. By stability of quasigeodesics, there exists a natural number $\widehat{P} > 0$ such that $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ lie in a $\widehat{P}$ neighborhood of each other in $\widehat{X^h}$. Let $\widehat{q}$ be a nearest point projection from p onto $\widehat{\alpha}_2$, then $d_{\widehat{X}^h}(p, \widehat{q}) \leq \widehat{P}$. Let $\widehat{\lambda}$ be a geodesic in $\widehat{X^h}$ joining p and $\widehat{q}$, then $l_{\widehat{X^h}}(\widehat{\lambda}) \leq \widehat{P}$. Thus $\widehat{\lambda}$ intersects at most $\widehat{P}$-many hyperbolic cones.

Suppose $\widehat{\gamma}_a, \widehat{\gamma}_b$ are the subsegments of $\widehat{\alpha}_2$ joining $\alpha_2(a), \widehat{q}$ and $\widehat{q}, \alpha_2(b)$ respectively. Now, by Lemma 1.1.29, $\widehat{\gamma}_a \cup \widehat{\lambda}$ and $\widehat{\gamma}_b \cup \widehat{\lambda}$ are $(3,0)$-quasigeodesics in $\widehat{X^h}$. As $\widehat{q}$ is nearest point projection, this quasigeodesic $\widehat{\gamma} \cup \widehat{\lambda}$ does not backtrack. We need to find $q \in \alpha_2$ such that $d_{X^h}(p,q)$ is bounded. If $\widehat{q} \in \widehat{\alpha}_2 \setminus \cup_{H^h \in \mathcal{H}^h} int(H^h)$, then let $q = \widehat{q}$, otherwise $\widehat{q}$ lie on an edge path of length one over some horosphere-like set H. Let $\widehat{\lambda}$ be defined on the interval $[c,d]$, then there exists $d_0 \in [c,d]$ such that $\widehat{\lambda}(d_0) \in H$. Suppose $\alpha_2(a_1)$ and $\alpha_2(b_1)$ are the entry and exit points respectively of α_2 to H. Let q be a nearest point projection from $\widehat{\lambda}(d_0)$ onto the geodesic segment joining $\alpha_2(a_1)$ and $\alpha_2(b_1)$.
Let λ_1 be the path in X^h obtained from $\widehat{\lambda}|_{[c,d_0]}$ by replacing the edge paths over the horosphere-like sets (penetrated by $\widehat{\lambda}$) by geodesics in the respective hyperbolic cones and λ_2 be a geodesic in H^h joining $\widehat{\lambda}(d_0), q$. Let $\lambda_{p,q} = \lambda_1 \cup \lambda_2$. We shall prove that $l_{X^h}(\lambda_{p,q})$ is bounded.
Suppose η_a, η_b are the subsegments of α_1 joining $\alpha_1(a), p$ and $p, \alpha_1(b)$ respectively. Let $\widehat{\eta}_a, \widehat{\eta}_b$ be the corresponding coned-off geodesic paths in $\widehat{X^h}$. Due to similar intersection patterns of
1) $\widehat{\eta}_a$ and $\widehat{\gamma}_a \cup \widehat{\lambda}$,
2) $\widehat{\eta}_b$ and $\widehat{\gamma}_b \cup \widehat{\lambda}$
with hyperbolic cones, if $\widehat{\lambda}$ penetrates a hyperbolic cone S^h with entry and exit points being x_S and y_S respectively, then $d_{S^h}(x_S, y_S) \leq I_1$ for some number $I_1 > 0$. If $\widehat{q}$ lie on an edge path of length one over some horosphere-like set H, due to similar

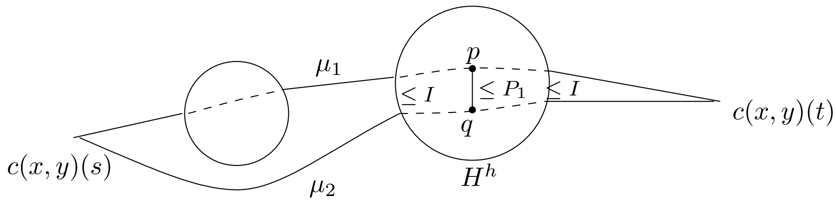

Figure 1.2: p, q lie in same hyperbolic cone.

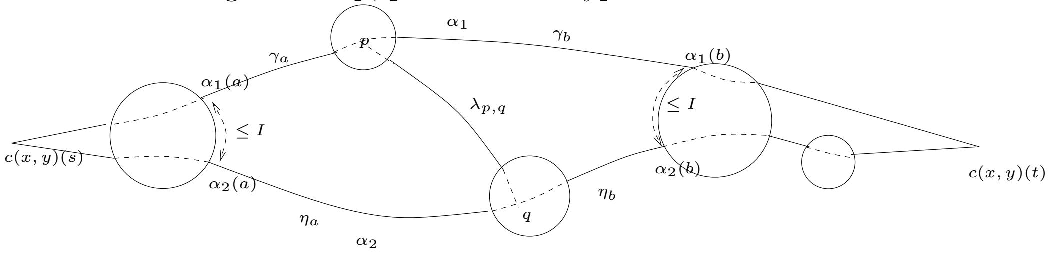

Figure 1.3: p, q lie in different hyperbolic cones.

intersection patterns of $\widehat{\eta}_a$ and $\widehat{\gamma}_a \cup \widehat{\lambda}$, $d_H(\widehat{\lambda}(d_0), \alpha_2(a_1)) \leq I_1$, where $\alpha_2(a_1)$ is the entry point of α_2 to H^h. Since q is a nearest point projection from $\widehat{\lambda}(d_0)$, we have $d_{H^h}(\widehat{\lambda}(d_0), q) \leq I_1$. Thus, $l_{X^h}(\lambda_2) \leq I_1$.
Due to similar intersection patterns of $\widehat{\alpha}_1, \widehat{\alpha}_2$ with hyperbolic cones, the lengths of the portions of α_1 lying inside hyperbolic cones are at most I. If p lie inside a hyperbolic cone K^h with $\widehat{\lambda}(c_0) \in K$ being the exit point from K of $\widehat{\lambda}$ and $\alpha_1(a_0), \alpha(b_0)$ are the entry, exit points respectively of α_1 from K^h, then $d_{K^h}(\alpha_1(a_0), \alpha(b_0)) \leq I$ and $d_K(\alpha_1(a_0), \widehat{\lambda}(c_0)) \leq I_1$. Therefore, $d_{K^h}(p, \widehat{\lambda}(c_0))) \leq I_1 + I$.
Now as $l_{\widehat{X^h}}(\widehat{\lambda}) \leq \widehat{P}$, therefore $l_{X^h}(\lambda_{p,q} \setminus \cup_{F \in \mathcal{H}} int(F^h)) \leq \widehat{P}$. $\widehat{\lambda}$ can intersects at most $\widehat{P}$-many horosphere-like sets, therefore $l_{X^h}(\lambda_1) \leq (I_1 + I) + \widehat{P} + \widehat{P}I_1$. Thus, $l_{X^h}(\lambda_{p,q}) = l_{X^h}(\lambda_1) + l_{X^h}(\lambda_2) \leq (I_1 + I) + \widehat{P} + \widehat{P}I_1 + I_1 = 2I_1 + I + \widehat{P}(1 + I_1)$. Let $P_2 = 2I_1 + I + \widehat{P}(1 + I_1)$, then $d_{X^h}(p, q) \leq P_2$.
Taking $P = \max\{P_1, P_2\}$, we have that for each $p \in \mu_1$ there exists $q \in \mu_2$ such that $d_{X^h}(p, q) \leq P$.

(3) c(x, y) satisfies property 3 of Lemma 1.2.34: Let $x, y, z \in X^h$, $\alpha = c(x, y), \beta = c(y, z), \gamma = c(z, x)$, $\overline{\alpha}(t) = \gamma(1-t)$, $\overline{\beta} = \beta(1-t)$ and $\overline{\gamma} = \gamma(1-t)$, where $0 \leq t \leq 1$.

Case (A): If α, β, γ penetrates a same hyperbolic cone H^h, then H-distance between the entry points of the pairs $(\alpha, \overline{\gamma})$, $(\overline{\beta}, \gamma)$ and $(\beta, \overline{\alpha})$ to H is at most I for some $I > 0$. Thus we get a hexagon in H^h whose length of alternate sides are bounded. H^h is hyperbolic, thus there exists $B_1' > 0$ such that the subsegment of α inside H^h lies in B_1'-neighborhood of $\beta \cup \gamma$.
Now there exist $s_0, t_0 \in (0, 1]$ such that $\alpha(s_0), \overline{\gamma}(t_0)$ lie in H, then $d_{X^h}(\alpha(s_0), \overline{\gamma}(t_0)) \leq$

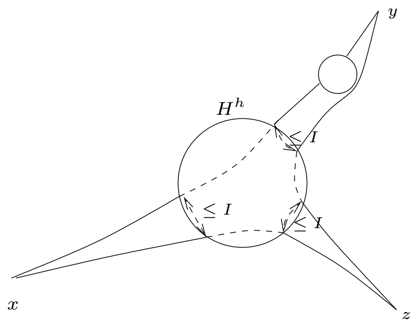

Figure 1.4: Triangle for Case (A): Three sides of $\triangle\alpha\beta\gamma$ penetrate H^h

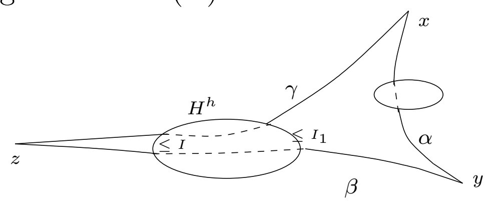

Figure 1.5: Two sides β, γ of the triangle $\triangle\alpha\beta\gamma$ penetrates H^h.

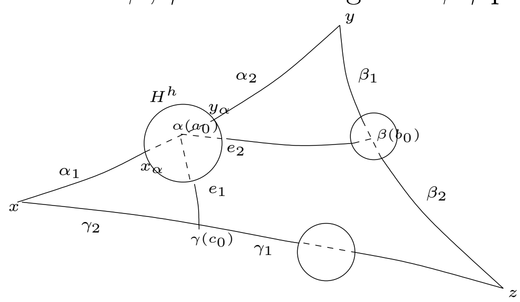

Figure 1.6: Three sides of $\triangle\alpha\beta\gamma$ penetrate distnct horosphere-like sets.

I. Now by property (2), the Hausdorff distance between $\alpha|_{[0,s_0]}$ and $\overline{\gamma}|_{[0,t_0]}$ is at most B_1'' for some $B_1'' > 0$. Similarly, there exists $s_1, t_1 \in [0,1]$ such that the Hausdorff distances between $\overline{\alpha}|_{[0,s_1]}$ and $\beta|_{[0,t_1]}$ is at most B_1''. Let $B_1 = \max\{B_1', B_1''\}$, then $\alpha \subset Nbhd_{X^h}(\beta \cup \gamma; B_1)$.

Case (B): α, β, γ does not penetrate a same hyperbolic cone:
Since triangles are $\widehat{\delta}_{1.2.28}$-thin in $\widehat{X_h}$, therefore there exists $a_0 \in [0,1]$ such that $d_{\widehat{X^h}}(\alpha(a_0), \widehat{\beta}) \leq \widehat{\delta}_{1.2.28} + 1$ and $d_{\widehat{X^h}}(\alpha(a_0), \widehat{\gamma}) \leq \widehat{\delta}_{1.2.28} + 1$. As in the proof of property (2), there exist paths $\lambda_{\alpha(a_0),\beta(b_0)}$ and $\lambda_{\alpha(a_0),\gamma(c_0)}$ in X^h joining $\alpha(a_0), \beta(b_0)$ and $\alpha(a_0), \gamma(c_0)$ respectively such that the lengths of $\widehat{\lambda}_{\alpha(a_0),\beta(b_0)}, \widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}$ in $\widehat{X^h}$ are at most $\widehat{\delta}_{1.2.28} + 2$. Let

- $\alpha_1 = \alpha|_{[0,a_0]}, \alpha_2 = \alpha|_{[a_0,1]}$,
- $\beta_1 = \beta|_{[0,b_0]}, \beta_2 = \beta|_{[b_0,1]}$,
- $\gamma_1 = \gamma|_{[0,c_0]}, \gamma_2 = \gamma|_{[c_0,1]}$.

Note that the following pairs of quasigeodesics satisfy similar intersection patterns with hyperbolic cones:
Pair (a): $\widehat{\alpha}_1$ and $\widehat{\gamma}_2 \cup \widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}$,
Pair (b): $\widehat{\alpha}_2$ and $\widehat{\beta}_1 \cup \widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$,
Pair (c): $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)} \cup \widehat{\gamma}_1$ and $\widehat{\lambda}_{\alpha(a_0),\beta(b_0)} \cup \widehat{\beta}_2$.

Now we prove that if β, γ penetrate a same hyperbolic cone H^h, then H-distance between the exit points β and γ from H is uniformly bounded:
Note that α does not penetrate H^h. By Lemma 1.2.33, the H-distance between entry points of $\overline{\beta}$, γ to H is uniformly bounded. Hence by property (2), the Hausdorff distance between the subsegments of $\overline{\beta}$, γ joining z to the respective entry points is uniformly bounded. Thus, without loss of generality, we can assume $z \in H$.
If $\beta(b_0), \gamma(c_0)$ does not lie in H^h, then due to similar intersection patterns for Pair (c), the H-distance between the exit points of β, γ from H is uniformly bounded.
If $\beta(b_0)$ or $\gamma(c_0)$ lie in H^h, then $d_{\widehat{X^h}}(H, \widehat{\alpha}) \leq \widehat{\delta}_{1.2.28}$. Let $\widehat{\tau} : [l, m] \to \widehat{X^h}$ be a shortest geodesic from z to $\widehat{\alpha} = \widehat{c}(x, y)$, then $l_{\widehat{X^h}}(\widehat{\tau}) \leq \widehat{\delta}_{1.2.28} + 1$. Let $\widehat{\alpha}_x$ and $\widehat{\alpha}_y$ be subsegments of $\widehat{\alpha}$ joining $x, \widehat{\tau}(m)$ and joining $y, \widehat{\tau}(m)$ respectively. Then $\widehat{\tau} \cup \widehat{\alpha}_x$ and $\widehat{\tau} \cup \widehat{\alpha}_y$ are quasigeodesic paths. Due to similar intersection patterns of pairs $(\widehat{\beta}, \widehat{\tau} \cup \widehat{\alpha}_y)$ and $(\widehat{\gamma}, \widehat{\tau} \cup \widehat{\alpha}_x)$, the H-distance between the exit points of $\widehat{\beta}$ and $\widehat{\gamma}$ from H is uniformly bounded.
Therefore, by stability of quasigeodesics, portions of β and γ lying inside H^h are at bounded distance from each other.

Thus, without loss of generality, we can assume that the hyperbolic cones pene-

trated by α, β and γ are different.
By construction of $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}$ and $\widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$, note that the above quasigeodesics does not backtrack. Now we have the following three situations:
(I) If $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}$ penetrates a horosphere-like set H that $\widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$ does not penetrate. Then, by Pair (c), length of the subsegment of $\lambda_{\alpha(a_0),\gamma(c_0)}$ inside H^h is at most I_1 for some $I_1 > 0$.
(II) If $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}$ and $\widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$ both penetrates a same horosphere-like set H but $\widehat{\alpha}$ does not penetrates H, then by Pair (a) and Pair (b), length of the subsegments of $\lambda_{\alpha(a_0),\gamma(c_0)}$ and $\lambda_{\alpha(a_0),\beta(b_0)}$ inside H^h is at most I_1.
(III) If $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}, \widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$ and $\widehat{\alpha}$ penetrates a same horosphere-like set H, then we have the following two cases:
Case (i): Let $\alpha(a_0) \notin H^h$. Now either α_1 intersects H^h or α_2 intersects H^h. Suppose α_1 intersects H, then by Pair (b), length of the subsegment of $\lambda_{\alpha(a_0),\beta(b_0)}$ is at most I_1. Also, due to Pair (c), H-distance between the entry points (resp. exit points) of $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}$ and $\widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$ to H is at most I_1. Thus, by triangle inequality, length of the subsegment of $\lambda_{\alpha(a_0),\gamma(c_0)}$ lying inside H^h is at most $3I_1$. Similarly, if α_2 intersects H^h, then using Pair (a), we have that the length of the subsegment of $\lambda_{\alpha(a_0),\gamma(c_0)}$ lying inside H^h is at most $3I_1$.
Case (ii): Let $\alpha(a_0) \in H^h$, x_α, y_α respectively be the entry and exit points of α to H^h. Let e_1, e_2 be the exit points of $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}, \widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$ respectively from H^h. Then,
• from Pair (a), we have $d_H(x_\alpha, e_1) \leq I_1$,
• from Pair (b), we have $d_H(e_2, y_\alpha) \leq I_1$,
• from Pair (c), we have $d_H(e_1, e_2) \leq I_1$.
Thus, $d_{H^h}(x_\alpha, y_\alpha) \leq 3I_1$. Therefore $d_{H^h}(\alpha(a_0), e_i) \leq 3I_1 + I_1 = 4I_1$ for $i = 1, 2$. Since lengths of $\widehat{\lambda}_{\alpha(a_0),\gamma(c_0)}$ and $\widehat{\lambda}_{\alpha(a_0),\beta(b_0)}$ are bounded, therefore there exists $B_2' > 0$ such that $l_{X^h}(\lambda_{\alpha(a_0),\gamma(c_0)}) \leq B_2'$ and $l_{X^h}(\lambda_{\alpha(a_0),\beta(b_0)}) \leq B_2'$. Applying property (2), there exists $B_2 > 0$ such that
• the Hausdorff distance between α_1 and γ_2 is at most B_2, and
• the Hausdorff distance between α_2 and β_1 is at most B_2.
Thus, $\alpha \subset Nbhd_{X^h}(\beta \cup \gamma; B_2)$
Taking $B = \max\{B_1, B_2\}$, we have the required result of property (3). □

Note 1.2.36. *Note that due to Lemma 1.2.34, the above paths $c(x,y)$ are stable $b_{1.2.36}$-quasiconvex paths for some number $b_{1.2.36} > 0$ depending only upon the hyperbolicity constant of $\widehat{X^h}$.*

Gromov's definition implies Farb's definition

In this subsection, we prove that Gromov's definition of relative hyperbolicity implies Farb's definition. In general, $\mathcal{G}(X,\mathcal{H})$ may not be a geodesic space, but by Lemma 1.1.9, there exists a metric graph Γ of edge length one such that $\mathcal{G}(X,\mathcal{H})$ is quasi-isometric to Γ via a map, say, Φ. By Lemma 1.2.26, we can assume ν to be large such that for all $H_1,H_2\in\mathcal{H}$, $d_\Gamma(\Phi(H_1^h),\Phi(H_2^h))\geq 1$. Let $\Phi(\mathcal{H})=\{\Phi(H):H\in\mathcal{H}\}$. Then $(X,\mathcal{H})$ is relatively hyperbolic in the sense of Farb if and only if $(\Gamma,\Phi(\mathcal{H}))$ is relatively hyperbolic in the sense of Farb. Note that since Γ is a connected graph, the coned-off space $\widehat{\Gamma}$ is a geodesic space. So, throughout this subsection, we assume that

- X is a geodesic metric space,
- $\nu>0$ and $\mathcal{H}$ is a collection of uniformly ν-separated, intrinsically geodesic and uniformly properly embedded closed subsets of X, and
- $\mathcal{G}(X,\mathcal{H}),\mathcal{E}(X,\mathcal{H})$ are geodesic spaces.

Definition 1.2.37. *(Visual Size of a horosphere-like set): Let $H\in\mathcal{H}$ be a horosphere-like set and let γ be a path in $\mathcal{G}(X,\mathcal{H})$ not intersecting H^h. Let T be the set of points $p\in H$ so that there exists some t for which a geodesic $[\gamma(t),p]$ joining $\gamma(t)$ and p intersects H in a singleton set $\{p\}$. Then the visual size of H with respect to γ is defined to be the diameter of T in the intrinsic metric of H. The visual size of the horosphere-like set H is defined to be the supremum of the visual size of H with respect to γ, where supremum is taken over all geodesics γ in $\mathcal{G}(X,\mathcal{H})$ not meeting any $H\in\mathcal{H}$.*

Lemma 1.2.38. *Let $\delta\geq 0,k\geq 1$. Suppose X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov. Then there exists $S_{1.2.38}>0$ depending on δ,k such that for a k-quasigeodesic path γ in $\mathcal{G}(X,\mathcal{H})$ lying outside a hyperbolic cone H^h, the H-diameter of the set $\pi_{H^h}(\gamma)$ is at most $S_{1.2.38}$.*

Proof. Let $H^t=H\times\{t\}$ and $H^{t^h}=H\times[t,\infty)$. By Lemma 1.2.24, H^{t^h} is 4δ-quasiconvex for all $t\in[0,\infty)$. Let $x,y\in\gamma$ and $Q=4\delta$. Using Lemma 1.1.31, there exist $D'=D'_{1.1.31}(\delta,Q)>0,C'=C'_{1.1.31}(\delta,Q)>0$ such that if $d_{X^h}(\pi_{H^{t^h}}(x),\pi_{H^{t^h}}(y))>D'$ then there exists $p\in[x,y]$ and $q\in H^{t^h}$ such that $d_{X^h}(p,q)\leq C'$. By stability of quasigeodesics, there exists $r(k)>0$ such that $[x,y]$ is contained in r-neighborhood of γ in X^h. Thus $d_{X^h}(q,\gamma)\leq C'+r$, therefore γ intersects $C'+r$-neighborhood of H^{t^h}.
Let $t=C'+r+1$, then H lies outside C' neighborhood of H^{t^h}. But γ intersects $C'+r$-neighborhood of H^{t^h}, therefore γ intersects H^h. This is a contradiction as we have assumed $\gamma\cap H^h=\emptyset$. Thus $d_{X^h}(\pi_{H^{t^h}}(x),\pi_{H^{t^h}}(y))\leq D'$

and hence $d_{X^h}(\pi_{H^h}(x), \pi_{H^h}(y)) \leq d_{X^h}(\pi_{H^h}(x), \pi_{H^{th}}(x)) + d_{X^h}(\pi_{H^{th}}(x), \pi_{H^{th}}(y)) + d_{X^h}(\pi_{H^h}(y), \pi_{H^{th}}(y)) \leq t + D' + t = D' + 2(C' + r + 1)$. Let $S' = D' + 2(C' + r + 1)$, then $d_{X^h}(\pi_{H^h}(x), \pi_{H^h}(y)) \leq S'$. As hyperbolic cones are properly embedded and horosphere like sets are properly embedded in its hyperbolic cone, there exists $S_{1.2.38}(S') > 0$ such that $d_H(\pi_{H^h}(x), \pi_{H^h}(y))) \leq S_{1.2.38}$. This holds for all $x, y \in \gamma$, therefore the H-diameter of the set $\pi_{H^h}(\gamma)$ is at most $S_{1.2.38}$. ☐

Lemma 1.2.39. *[Far98] (Horosphere-like sets are visually bounded): Let $\delta \geq 0$ and X be δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov. Then there exists $V_{1.2.39} > 0$, depending on δ, such that the visual size of each horosphere-like sets $H \in \mathcal{H}$ is at most $V_{1.2.39}$.*

Proof. Suppose $x \in X$ and let T_x be the set of all points $s \in H$ for which $[x, s] \cap H = \{s\}$. Let $s \in T_x$ and consider the triangle $\triangle x\pi_{H^h}(x)s$ in X^h, then $\triangle x\pi_{H^h}(x)s$ is δ-slim. Therefore $[\pi_{H^h}(x), s] \subset N_\delta([\pi_{H^h}(x), x] \cup [x, s])$. As $[\pi_{H^h}(x), x], [x, s]$ lie in the complement of $int(H^h)$, portions of $[\pi_{H^h}(x), s]$ lying in H^h will lie within a δ-neighborhood of H in X^h. Since H^h is 4δ-quasiconvex, the geodesic segment $[\pi_{H^h}(x), s]$ lies in the 4δ-neighborhood of H^h. Thus the geodesic ray $[\pi_{H^h}(x), s]$ lies within 4δ-neighborhood of H in X^h. Let $\alpha_s = [\pi_{H^h}(x), s]$, then the Hausdorff distance between α_s and $\pi_H(\alpha_s)$ in X^h is at most 4δ. But this Hausdorff distance approaches ∞ as $d_{X^h}(\pi_{H^h}(x), s) \to \infty$. Therefore, $d_{X^h}(\pi_{H^h}(x), s) \leq R_{1.2.39}$ for some constant $R_{1.2.39} > 0$, independent of x. Hence diameter of the set T_x is bounded by $2R_{1.2.39}$.
Now let γ be a geodesic not intersecting H and $x \in \gamma$, then the visual size of H with respect to γ is at most $2diamT_x + diam(\pi_{H^h}(\gamma)) \leq 2R_{1.2.39} + S_{1.2.38}$. Taking $V_{1.2.39} = 2R_{1.2.39} + S_{1.2.38}$, we have the required result. ☐

By replacing the geodesic γ by a quasigeodesic path in the above proof, we have the following corollary:

Corollary 1.2.40. *(Visual size of horosphere-like sets with respect to quasigeodesics is bounded): Let $\delta \geq 0, k \geq 1$ and X be δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov. Suppose $H \in \mathcal{H}$ and γ is a k-quasigeodesic path not intersecting H, then there exists $V_{1.2.40}(\delta, k) > 0$ such that visual size of H with respect to γ is at most $V_{1.2.40}$.*

Definition 1.2.41. *A collection $\mathcal{Q}$ of uniformly C-quasiconvex sets in a δ-hyperbolic metric space Z is said to be* **mutually B-cobounded** *if for all $Q_i, Q_j \in \mathcal{Q}$, $\pi_{Q_i}(Q_j)$, $i \neq j$, has diameter less than B. A collection is* **mutually cobounded** *if it is mutually B-cobounded for some $B > 0$.*

We have the following corollary of Lemma 1.1.31:

Corollary 1.2.42. *[Far98](Hyperbolic Cones are mutually cobounded): Let $\delta \geq 0$, then there exists $B_{1.2.42}(\delta) > 0$ such that the following holds: Let X be δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov, then the collection $\mathcal{H}$ is mutually $B_{1.2.42}$-cobounded.*

Proof. Let $H_1, H_2 \in \mathcal{H}$ and $x, y \in H_1$. Suppose γ is a geodesic in X^h joining x and y. From Lemma 1.2.24, the hyperbolic cone $H_2^{t^h} := H \times [t, \infty)$ is 4δ-quasiconvex for all $t \in [0, \infty)$. Let $t = 4\delta + 1$, then as H_1^h is 4δ-quasiconvex and $\mathcal{H}$ is ν (> 0)-separated, γ cannot intersects $H_2^{t^h}$. Therefore, using Lemma 1.2.38, there exists $S_{1.2.38} > 0$ such that diameter of the set $\pi_{H_2^{t^h}}(\gamma)$ is at most $S_{1.2.38}$. Hence $d_{X^h}(\pi_{H_2^{t^h}}(x), \pi_{H_2^{t^h}}(y)) \leq S_{1.2.38}$. Therefore,

$$\begin{aligned} d_{X^h}(\pi_{H_2{}^h}(x), \pi_{H_2{}^h}(y)) &\leq d_{X^h}(\pi_{H_2{}^h}(x), \pi_{H_2^{t^h}}(x)) + d_{X^h}(\pi_{H_2^{t^h}}(x), \pi_{H_2^{t^h}}(y)) \\ &\quad + d_{X^h}(\pi_{H_2{}^h}(y), \pi_{H_2^{t^h}}(y)) \\ &\leq t + S_{1.2.38} + t = 8\delta + S_{1.2.38} + 2. \end{aligned}$$

Taking $B_{1.2.42} = 8\delta + S_{1.2.38} + 2$, we have the required result. □

In [Far98], Farb proved the hyperbolicity of the electrocuted space $\mathcal{E}(X, \mathcal{H})$, where X is a pinched Hadamard manifold and $\mathcal{H}$ is the collection of uniformly separated horospheres in X. Next, we prove the general versions of Farb's theorem ensuring the hyperbolicity of electric space. Let Z be a subset of $\mathcal{G}(X, \mathcal{H})(= X^h)$. $N_{\widehat{X}}(Z, R)$ will denote the R-neighborhood about the subset Z in the electric space $(\mathcal{E}(X, \mathcal{H}), d_{\widehat{X}})$.

Lemma 1.2.43. *([Far98], [Szc98]) Let $\delta \geq 0$ and $\nu \geq 1 + 2D_{1.1.30}$, where $D_{1.1.30}(= 6\delta)$ is as in Lemma 1.1.30, then there exists $\widehat{\delta}'_{1.2.43}, \widehat{\delta}_{1.2.43} \geq 0$ such if X be δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov, then the following properties hold:*
(1). There exists $Q_{1.2.43} > 0$ with the following property:
Electric geodesics electrically track hyperbolic geodesics: Let $x, y \in X$, β be any electric geodesic from x to y in $\mathcal{E}(X^h, \mathcal{H}^h)$, and γ be a geodesic from x to y in $\mathcal{G}(X, \mathcal{H})$, then
$\beta \subset N_{\widehat{X^h}}(\gamma, Q_{1.2.43})$ and $\gamma \subset N_{\widehat{X^h}}(\beta, Q_{1.2.43})$.
(2). $\mathcal{E}(X^h, \mathcal{H}^h)$ is $\widehat{\delta}'_{1.2.43}$-hyperbolic and $\mathcal{E}(X, \mathcal{H})$ is $\widehat{\delta}_{1.2.43}$-hyperbolic.

Proof. (1). First Part: Let $D = D_{1.1.30}, C = C_{1.1.30} > 0$. Suppose β' is a maximal subsegment of β lying completely outside $N_{\widehat{X^h}}(\gamma; C)$. Let β' starts from p and ends at q, then $d_{\widehat{X^h}}(p,\gamma) = C$ and $d_{\widehat{X^h}}(q,\gamma) = C$. Let β' penetrates the hyperbolic cones $H_1^h, ..., H_N^h$. Since hyperbolic cones are uniformly ν-separated, therefore $(N-1)\nu \leq l_{\widehat{X^h}}(\beta')$ and hence $N \leq \frac{l_{\widehat{X^h}}(\beta')}{\nu} + 1$.

Let x_i be the entry point and y_i be the exit point for β' penetrating the hyperbolic cone H_i^h, where $1 \leq i \leq N$. For each i, we join x_i and y_i by a geodesic $[x_i, y_i]$ in X^h. Let $\beta_{y_j,x_{j+1}}$, $(0 \leq j \leq N)$, be the subsegment of β' joining y_j and x_{j+1}, where $y_0 = p$ and $x_{N+1} = q$. Since $\beta_{y_j,x_{j+1}}$ lies outside hyperbolic cones, it is also a geodesic in X^h. Now for each i,j, $d_{X^h}(\pi_\gamma(x_i), \pi_\gamma(y_i)) \leq D$ and $d_{X^h}(\pi_\gamma(y_j), \pi_\gamma(x_{j+1})) \leq D$ otherwise, due to Lemma 1.1.30, $[x_i, y_i]$ or $\beta_{y_j,x_{j+1}}$ would intersect C-neighborhood of γ. Therefore we have:

$$\begin{aligned}
l_{\widehat{X^h}}(\beta') &= d_{\widehat{X^h}}(p,q) \\
&\leq d_{\widehat{X^h}}(p,\pi_\gamma(p)) + d_{\widehat{X^h}}(\pi_\gamma(p),\pi_\gamma(q)) + d_{\widehat{X^h}}(\pi_\gamma(q),q) \\
&\leq 2C + d_{X^h}(\pi_\gamma(p),\pi_\gamma(q)) \\
&\leq 2C + \sum_{0\leq j\leq N} d_{X^h}(\pi_\gamma(y_j),\pi_\gamma(x_{j+1})) + \sum_{1\leq i\leq N} d_{X^h}(\pi_\gamma(x_i),\pi_\gamma(y_i)) \\
&\leq 2C + (N+1)D + ND \\
&\leq 2C + 2D(\frac{l_{\widehat{X^h}}(\beta')}{\nu} + 1) + D
\end{aligned}$$

Therefore $l_{\widehat{X^h}}(\beta') \leq \frac{(2C+3D)\nu}{\nu-2D}$ (note that $\nu > 2D$). Let $K = \frac{(2C+3D)\nu}{\nu-2D}$ and $Q^1_{1.2.43} = K + \frac{C}{2}$. Then we have $\beta \subset N_{\widehat{X^h}}(\gamma, Q^1_{1.2.43})$.

Second Part: Recall that $\gamma : [0, d_{X^h}(x,y)] \to X^h$ was a geodesic in X^h. Let $[s_0,t_0]$ be a maximal subinterval of $[0, d_{X^h}(x,y)]$ such that $\gamma|_{(s_0,t_0)}$ lie outside $Q^1_{1.2.43}$-neighborhood of β in $\widehat{X^h}$. Then there exists $\widehat{p} \in \beta$ such that $d_{\widehat{X^h}}(\widehat{p},\gamma(s_0)) \leq Q^1_{1.2.43}$ and $d_{\widehat{X^h}}(\widehat{p},\gamma(t_0)) \leq Q^1_{1.2.43}$. Then $[\gamma(s_0),\widehat{p}] \cup [\widehat{p},\gamma(t_0)]$ intersects at most $N = [2Q^1_{1.2.43}]$-many horosphere-like sets $H_1, ..., H_N$, where $[r]$ denotes the integer part of $r \in \mathbb{R}$. Let $\{p_i\} = \partial H_i^h$, where $1 \leq i \leq N$. For each $1 \leq i < N$, we join $\gamma(t_0)$ to p_i by a geodesic ray $[\gamma(t_0), p_i)$ in X^h. Let $[\gamma(s_0),p_1),(p_1,p_2),...,(p_{N-1},p_N),(p_N,\gamma(t_0)]$ be geodesics in X^h such that

$$\begin{aligned}
&([\gamma(s_0),p_1) \cup (\cup_{1\leq i\leq N-1}(p_i,p_{i+1})) \cup (p_N,\gamma(t_0)]) \setminus \cup_{1\leq i\leq N} int(H_i^h) \\
= \;&([\gamma(s_0),\widehat{p}] \cup [\widehat{p},\gamma(t_0)]) \setminus \cup_{1\leq i\leq N} int(H_i^h)
\end{aligned}$$

Since X^h is δ-hyperbolic, therefore ideal triangles are δ-thin. Thus, for $z \in \gamma|_{[s_0,t_0]}$ there exists $z' \in ([\gamma(s_0),p_1) \cup (\cup_{1\leq i\leq N-1}(p_i,p_{i+1})) \cup (p_N,\gamma(t_0)])$ such that $d_{X^h}(z,z') \leq \delta Q^1_{1.2.43}$. Hence $d_{\widehat{X^h}}(z,z') \leq d_{X^h}(z,z') \leq \delta Q^1_{1.2.43}$ and so

$$\gamma \subset N_{\widehat{X^h}}(\beta, \delta Q^1_{1.2.43} + Q^1_{1.2.43}).$$

Taking $Q^2_{1.2.43} = \delta Q^1_{1.2.43} + Q^1_{1.2.43}$, we have

$$\gamma \subset N_{\widehat{X^h}}(\beta, Q^2_{1.2.43}).$$

Taking $Q_{1.2.43} = \max\{Q^1_{1.2.43}, Q^2_{1.2.43}\}$, we have the required result.
(2). Let $x, y, z \in X^h$. Suppose $\widehat{\triangle}xyz$ is a triangle in the electric space $\widehat{X^h}$. Consider the triangle $\triangle^h xyz$ in X^h. As X^h is δ-hyperbolic, $\triangle^h xyz$ is δ-thin. From (1), $\widehat{\triangle}xyz$ is (electrically) $(2Q_{1.2.43} + \delta)$-thin. Let $\widehat{\delta}'_{1.2.43} = 2Q_{1.2.43} + \delta$ then $\widehat{X^h}$ is $\widehat{\delta}'_{1.2.43}$-thin. Now from Lemma 1.2.28, the natural inclusion $\mathcal{E}(X, \mathcal{H}) \hookrightarrow \mathcal{E}(X^h, \mathcal{H}^h)$ is a quasi-isometry. Therefore, there exists $\widehat{\delta}_{1.2.43}(\widehat{\delta}'_{1.2.43}) > 0$ such that $\mathcal{E}(X, \mathcal{H})$ is $\widehat{\delta}_{1.2.43}$-thin. □

Next we prove that the relative geodesics satisfy bounded horosphere penetration properties with horosphere-like sets.

Lemma 1.2.44. *[Far98] Let $\delta \geq 0$ and $\nu \geq 1 + 2D_{1.1.30}$, where $D_{1.1.30}$ is as in Lemma 1.1.30. Suppose X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov, where $\mathcal{H}$ is uniformly ν-separated, then the following properties hold:*
Let β be an electric geodesic in $\mathcal{E}(X, \mathcal{H})$ (resp. in $\mathcal{E}(\mathcal{G}(X, \mathcal{H}), \mathcal{H}^h)$) and γ be a geodesic in $\mathcal{G}(X, \mathcal{H})$ joining the same pair of points in X (resp. in $\mathcal{G}(X, \mathcal{H})$). Then β, γ have the following similar intersection patterns with horosphere-like sets (resp. hyperbolic cones)

1. **Similar Intersection Patterns 1:** *if precisely one of $\{\beta, \gamma\}$ meets a horosphere-like set $H \in \mathcal{H}$, then the distance (measured in the intrinsic path-metric on H) from the first (entry) point to the last (exit) point (of the relevant path) is at most $I^1_{1.2.44}$ for some $I^1_{1.2.44} > 0$.*

2. **Similar Intersection Patterns 2:** *if both $\{\beta, \gamma\}$ meet some $H \in \mathcal{H}$ then the distance (measured in the intrinsic path-metric on H) from the entry point of β to that of γ is at most $I^2_{1.2.44}$; similarly for the exit points for some $I^2_{1.2.44} > 0$.*

Proof. 1). Let us first assume that γ intersects $H \in \mathcal{H}$ and β does not intersect H. Let p be the first entry point and q be the last exit point of γ to the horosphere-like set H. We will prove that H-distance, say d_H, between p and q is bounded. Recall from Lemma 1.2.39 that the visual diameter of a horosphere-like set is at most $V_{1.2.39}$ for some $V_{1.2.39} > 0$ and from Corollary 1.2.42 that hyperbolic cones are mutually $B_{1.2.42}$-cobounded. From Lemma 1.2.43, there exists $Q_{1.2.43} > 0$ such that $\beta \subset N_{\widehat{X^h}}(\gamma; Q_{1.2.43})$. Let $V = V_{1.2.39}, B = B_{1.2.42}, Q = Q_{1.2.43}$.
Let $[a, b]$ be the domain of β. For each $t \in [a, b]$ there exist a point $p_t \in \gamma$ such that $d_{\widehat{X^h}}(\beta(t), p_t) \leq Q$. Suppose γ_1, γ_2 be two components of $\gamma \setminus int(H^h)$ containing p, q respectively. Let $[a, s_0]$ and $[t_0, b]$ be the largest subintervals of $[a, b]$ such that for

each $t \in [a, s_0) \cup (t_0, b]$, electric geodesics joining $\beta(t)$ and p_t does not intersect H. Let $\beta_1 = \beta|_{[a,s_0)}$, $\beta_2 = \beta|_{[s_0,t_0]}$ and $\beta_3 = \beta|_{(t_0,b]}$. For $s \in [a, s_0)$, suppose λ_s be an electric geodesic in $\widehat{X^h}$ joining $\beta_1(s)$ and p_s. As length of λ_s is at most Q, it penetrates at most Q-many horosphere-like sets. Let λ_s^b be the subset of λ_s lying outside horosphere-like sets and $N(\lambda_s)$ be the union of λ_s^b and horosphere-like sets penetrated by λ_s. Let γ_{p_sp} be the subsegment of γ joining p_s and p, then diameter of the set $\pi_{H^h}(\gamma_{p_sp})$ is at most V. Also diameter of $\pi_{H^h}(N(\lambda_s))$ is at most $(V+B)Q$. Therefore diameter of $\pi_{H^h}(N(\lambda_s)) \cup \gamma_{p_sp}))$ is at most $V + (V+B)Q$. This is true for each $s \in [a, s_0)$. Since $p \in \cap_{s\in[a,s_0)}\pi_{H^h}(N(\lambda_s) \cup \gamma_{p_sp}))$, we have $d_{H^h}(p, \pi_{H^h}(\beta(s_0))) \leq V + (V+B)Q$.
Using similar argument for β_3, we have $d_{H^h}(\pi_{H^h}(\beta(t_0)), q) \leq V + (V+B)Q$.
Now for β_2, $d_{\widehat{X^h}}(\beta(s_0), \pi_{H^h}(\beta(s_0))) \leq Q$ and $d_{\widehat{X^h}}(\beta(t_0), \pi_{H^h}(\beta(t_0))) \leq Q$. As β_2 is a geodesic in $\widehat{X^h}$, therefore the length of β_2 is at most $2Q$. Let $N(\beta_2)(\subset X)$ be defined similarly to $N(\lambda_s)$. Then the diameter of the set $\pi_{H^h}(N(\beta_2))$ is at most $2Q(V+B)$. Thus $d_{H^h}(\pi_{H^h}(\beta(s_0)), \pi_{H^h}(\beta(t_0))) \leq 2Q(V+B)$. Therefore $d_{H^h}(p, q) \leq 2\{V + (V+B)\} + 2Q(V+B) = I'_1(say)$. Since horosphere-like sets are properly embedded in hyperbolic cones, there exists $I^1_{1.2.44}(I'_1) > 0$ such that $d_H(p, q) \leq I^1_{1.2.44}$.
The proof of the case when β intersects H and γ does not intersect H is similar.

2). Let y, p be the first points of entry for the geodesics β, γ respectively into the horosphere-like set H. We will prove that there exists $I^2_{1.2.44} > 0$ such that $d_H(p, y) \leq I^2_{1.2.44}$. Let β' be the component of $\beta \setminus int(H^h)$ containing y and γ_1 be the component of $\gamma \setminus int(H^h)$ containing p. Let $Q = Q_{1.2.43} > 0$ be as in Lemma 1.2.43, then and let $[l, b]$ be a maximal subinterval in the domain $[a, b]$ of β' such that for all $t > l$, $d_{\widehat{X^h}}(\beta'(t), H^h) \leq Q$. For l, there exists an electric geodesic λ_l joining $\beta'(l)$ and a point $p_l \in \gamma_1$ such that the length of λ_l is at most Q and diameter of the set $\pi_{H^h}(N(\lambda_l) \cup \gamma_{p_lp})$ is at most $V + (V+B)Q$. And as above $d_{H^h}(p, \pi_{H^h}(\beta'(l))) \leq V + (V+B)Q$. Since $d_{\widehat{X^h}}(\beta'(l), H^h) = Q$, length of $\beta'|_{[l,b]}$ is at most Q. Therefore $d_{H^h}(\pi_{H^h}(\beta'(l)), y) \leq (V+B)Q$. Hence $d_{H^h}(p, y) \leq V + (V+B)Q + (V+B)Q = I'_2(say)$. Since horosphere-like sets are properly embedded in hyperbolic cones, there exists $I^2_{1.2.44}(I'_2) > 0$ such that $d_H(p, y) \leq I^2_{1.2.44}$.
The same argument works for the exit points of the geodesics β, γ from the horosphere-like set H. □

Note 1.2.45. *The above Lemma 1.2.44 is also true when geodesics are replaced by quasigeodesic paths.*

Combining Lemma 1.2.43 and Lemma 1.2.44, we have the converse of the Theorem 1.2.35 :

Theorem 1.2.46. *[Far98, Bow97] Let $\delta \geq 0$ and $\nu \geq 1 + 2D_{1.1.30}$, where $D_{1.1.30}$ is as in Lemma 1.1.30, there exists $\widehat{\delta}_{1.2.43} \geq 0$ such that the following holds: If X is δ-hyperbolic relative to a collection $\mathcal{H}$ in the sense of Gromov, where $\mathcal{H}$ is uniformly ν-separated, then X is $\widehat{\delta}_{1.2.43}$-hyperbolic relative to $\mathcal{H}$ in the sense of Farb.*

Thus, from Theorems 1.2.35, 1.2.46, we have the following equivalence of two definitions of relative hyperbolicity:

Theorem 1.2.47. *Let $\delta, \widehat{\delta} \geq 0$ and $\nu \geq 1 + 2D_{1.1.30}$, where $D_{1.1.30}(= 6\delta)$ is as in Lemma 1.1.30, then there exists $\delta_{1.2.35} \geq 0$ depending only on $\widehat{\delta}$ and there exists $\widehat{\delta}_{1.2.43} \geq 0$ depending only on δ, such that the following holds:*
(1) if X is δ-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov then X is $\widehat{\delta}_{1.2.43}$-hyperbolic relative to $\mathcal{H}$ in the sense of Farb,
(2) if X is $\widehat{\delta}$-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Farb then X is $\delta_{1.2.35}$-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov.

By Lemma 1.2.26, we have the following Theorem:

Theorem 1.2.48. *Let $\delta \geq 0, \nu > 0$ then there exist $\widehat{\delta}_{1.2.48}, \widehat{\delta}'_{1.2.48} \geq 0$ depending only on δ, ν such that the following holds:*
Let X be δ-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov, where $\mathcal{H}$ is uniformly ν-separated, then
1) X is $\widehat{\delta}_{1.2.48}$-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Farb,
2) $\mathcal{G}(X, \mathcal{H})$ is $\widehat{\delta}'_{1.2.48}$-hyperbolic relative to the collection $\mathcal{H}^h$ in the sense of Farb.

Proof. Let $D_{1.1.30} > 0$ be as in Lemma 1.1.30, $r = 1 + 2D_{1.1.30}$. For $H \in \mathcal{H}$, let $H_r = H \times \{r\}$, $\mathcal{H}_r = \{H_r : H \in \mathcal{H}\}$ and $H_r^h = H_r \times [0, \infty)$ be the hyperbolic cone over H_r with metric $d_{H_r^h}$. Let $Y = \mathcal{G}(X, \mathcal{H}) \setminus \cup_{H \in \mathcal{H}} int(H_r^h)$. Note that the collection $\mathcal{H}_r$ is $\nu + r$-separated and we have $\nu + r > 1 + 2D_{1.1.30}$. Now as X is δ-hyperbolic relative to the collection $\mathcal{H}$, Y is δ-hyperbolic relative to the collection $\mathcal{H}_r$. Thus, from Theorem 1.2.47, there exists $\widehat{\delta}_{1.2.43}, \widehat{\delta}'_{1.2.43} \geq 0$ such that Y is $\widehat{\delta}_{1.2.43}$-hyperbolic relative to $\mathcal{H}_r$ in the sense of Farb and Y^h is $\widehat{\delta}_{1.2.43}$-hyperbolic relative to $\mathcal{H}_r^h$ in the sense of Farb. By Corollary 1.2.27, there exist $\widehat{\delta}_{1.2.48}, \widehat{\delta}'_{1.2.48} \geq 0$ depending on δ, ν such that the properties (1) and (2) hold. □

Definition 1.2.49 (Electroambient Paths [Mjb])**.** *Let $\delta \geq 0$ and X be δ-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov. We start with an electric quasi-geodesic path $\widehat{\lambda} : [0, 1] \to \widehat{X^h}$ without backtracking with end points in X^h. For any*

$H^h \in \mathcal{H}^h$ penetrated by $\widehat{\lambda}$, let x_H and y_H be the first entry point and the last exit point of $\widehat{\lambda}$ respectively. We join x_H and y_H by a geodesic segment in H^h. If $\widehat{\lambda}(0)$ (resp. $\widehat{\lambda}(1)$) lies in some H^h, then we join $\widehat{\lambda}(0)$ (resp. $\widehat{\lambda}(1)$) to the exit point (resp. entry point) by a geodesic in H^h. This results in a path λ in $\mathcal{G}(X,\mathcal{H})$. The path λ will be called an **electro-ambient path**.

The following Lemma (proved in [Mjb]) proves that an electroambient path is a quasigeodesic.

Lemma 1.2.50. *(Lemmas 3.8, 3.9 of [Mjb]): Let $\delta \geq 0$ and X be a δ-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov. Suppose $\widehat{\lambda}$ is an electric (K,ϵ)-quasigeodesic path in $\widehat{X}$ without backtracking and with end points $x, y \in X$. Then an electro-ambient path representative λ of $\widehat{\lambda}$ is a $K_{1.2.50}$-tamed quasigeodesic path $\mathcal{G}(X,\mathcal{H})$ for some number $K_{1.2.50} > 0$ depending on δ, K, ϵ, ν. In particular, λ is a $Q_{1.2.50}$ quasiconvex path in $\mathcal{G}(X,\mathcal{H})$ for some $Q_{1.2.50} > 0$.*

Electric Projections

Let $\delta \geq 0$ and X be a geodesic metric space with $\mathcal{H}_X$ a collection of uniformly $\nu > 0$-separated, intrinsically geodesic and closed subsets of X. Further, assume that $\mathcal{G}(X,\mathcal{H}), \mathcal{E}(X^h,\mathcal{H}^h_X)$ are geodesic spaces. Let X be δ-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov. Let $\widehat{X} = \mathcal{E}(X,\mathcal{H}_X)$, $X^h = \mathcal{G}(X,\mathcal{H}_X)$ and $i\colon X^h \to \widehat{X^h} = \mathcal{E}(X^h,\mathcal{H}^h_X)$ denote the inclusion. Recall that from Lemma 1.2.48 that there exist $\delta_{1.2.48}, \delta'_{1.2.48} \geq 0$ such that $\widehat{X}$ is $\delta_{1.2.48}$-hyperbolic and $\widehat{X^h}$ is $\delta'_{1.2.48}$-hyperbolic.
Define $f : \widehat{X^h} \to \widehat{X}$ as follows: $f(x) = x$ if $x \in \widehat{X}$, $f(x) = v(H)$, if $x \in int(H^h)$ or x lies in an edge (of length $\frac{1}{2}$) joining the cone point over H^h to some point of $int(H^h)$ for some hyperbolic cone H^h and $v(H)$ is the cone point over H.
Recall that from Lemma 1.2.28, that the natural inclusion $\widehat{j} : \widehat{X} \hookrightarrow \widehat{X^h}$ is a $(K_{1.2.28}, \epsilon_{1.2.28})$ quasi-isometry for some $K_{1.2.28} \geq 1, \epsilon_{1.2.28} \geq 0$. It is easy to check that f is a (K_f, ϵ_f)-quasi-isometry for some $K_f \geq 1, \epsilon_f \geq 0$ and it is a quasi-isometric inverse of the natural inclusion $\widehat{j}$.
Let $\widehat{\mu}$ be a geodesic in $\widehat{X}$, μ be an electro-ambient representative of the geodesic $\widehat{\mu}$ and π_μ be a nearest point projection from X^h onto μ.

Definition 1.2.51. *(Electric Projection) Let $y \in \widehat{X}$ and $\widehat{\mu}$ be a geodesic in $\widehat{X}$. Define $\widehat{\pi}_{\widehat{\mu}} : \widehat{X} \to \widehat{\mu}$ as follows:*
For $y \in X$, define $\widehat{\pi}_{\widehat{\mu}}(y) = f(i(\pi_\mu(y)))$.
If y is a cone point over a horosphere-like set $H \in \mathcal{H}_X$, choose $z \in H$, define $\widehat{\pi}_{\widehat{\mu}}(y) = f(i(\pi_\mu(z)))$.

If y lie on the edge (of length $\frac{1}{2}$) joining the cone point over a horosphere-like set H and $z \in H$, define $\widehat{\pi}_{\widehat{\mu}}(y) = f(i(\pi_\mu(z)))$.
$\widehat{\pi}_{\widehat{\mu}}$ will be called as Electric Projection.

The next lemma, shows that $\widehat{\pi}_{\widehat{\mu}}$ is well-defined up to a bounded amount of discrepancy with respect to the metric $d_{\widehat{X}}$.

Lemma 1.2.52. *Let $\delta \geq 0$. There exists a constant $P_{1.2.52} > 0$ depending only upon δ such that the following holds:*
Let X be δ-hyperbolic relative to a collection $\mathcal{H}_X$ in the sense of Gromov. Then for any $H \in \mathcal{H}_X$ and $z, z' \in H$, if $\widehat{\mu}$ be a geodesic in $\widehat{X}$ then $d_{\widehat{X}}(\widehat{\pi}_{\widehat{\mu}}(z), \widehat{\pi}_{\widehat{\mu}}(z')) \leq P_{1.2.52}$.

Proof. Let $D' = D'_{1.1.31}, C' = C'_{1.1.31}$ be as in Lemma 1.1.31. If $d_{X^h}(\pi_\mu(z), \pi_\mu(z')) \leq D'$, then $d_{\widehat{X^h}}(i(\pi_\mu(z)), i(\pi_\mu(z'))) \leq D'$.
Let us assume $d_{X^h}(\pi_\mu(z), \pi_\mu(z')) > D'$ then, by Lemma 1.1.31, $[z, \pi_\mu(z)] \cup [\pi_\mu(z), \pi_\mu(z')] \cup [\pi_\mu(z'), z']$ is a quasi-geodesic in X^h. Thus for $\pi_\mu(z)$ there exists p_z on the geodesic $[z, z']$ in X^h such that $d_{X^h}(\pi_\mu(z), p_z) \leq C'$. Since hyperbolic cones are 4δ-quasiconvex, there exists $p \in H^h$ such that $d_{X^h}(\pi_\mu(z), p) \leq C' + 4\delta$. Similarly there exists $q \in H^h$ such that $d_{X^h}(\pi_\mu(z'), q) \leq C' + 4\delta$. Therefore $d_{\widehat{X^h}}(i(\pi_\mu(z)), i(\pi_\mu(z'))) \leq 2(C' + 4\delta) + 1$. Taking $P'_3 = max\{D', 2(C' + 4\delta) + 1\}$, we have $d_{\widehat{X^h}}(i(\pi_\mu(z)), i(\pi_\mu(z'))) \leq P'_3$. Since f is a quasi-isometry, there exists $P_{1.2.52}(P'_3) > 0$ such that $d_{\widehat{X}}(\widehat{\pi}_{\widehat{\mu}}(z), \widehat{\pi}_{\widehat{\mu}}(z')) = d_{\widehat{X}}(f(i(\pi_\mu(z)), f(i(\pi_\mu(z'))) \leq P_{1.2.52}$. ☐

Further, if $x, y \in \widehat{X}$ and $d_{\widehat{X}}(x, y) \leq 1$ then similarly we can prove that there exists a constant $R > 0$ (depending only upon δ) such that $d_{\widehat{X}}(\widehat{\pi}_{\widehat{\mu}}(x), \widehat{\pi}_{\widehat{\mu}}(y)) \leq R$. Thus we have the following lemma:

Lemma 1.2.53. *Let $\delta \geq 0$. There exists a constant $P_{1.2.53} > 0$ depending only upon δ such that the following holds:*
Let X be δ-hyperbolic relative to a collection $\mathcal{H}_X$ in the sense of Gromov and $\widehat{\mu}$ be an electric geodesic segment in $\widehat{X}$. Then for all $x, y \in \widehat{X}$, $d_{\widehat{X}}(\widehat{\pi}_{\widehat{\mu}}(x), \widehat{\pi}_{\widehat{\mu}}(y)) \leq P_{1.2.53} d_{\widehat{X}}(x, y) + P_{1.2.53}$.

Note 1.2.54. *Electric projection may not be a nearest point projection from an electric space onto an electric geodesic but analogous to Lemma 1.1.32, the above lemma says that electric projections does not increase the distance much.*

As a consequence of Lemma 1.1.36, we have the following Lemma which says that electric projections and strictly type-preserving quasi-isometries 'almost commute' in electric spaces.

Lemma 1.2.55. *Let $\delta \geq 0$ and $\nu > 0$. Suppose X_1, X_2 are two geodesic spaces such that for each $i = 1, 2$, X_i is δ-hyperbolic relative to the collection $\mathcal{H}_{X_i}$ of uniformly ν-separated sets in the sense of Gromov. Let $\phi \colon X_1 \to X_2$ be a strictly type preserving (K, ϵ)-quasi-isometry, $\widehat{\phi}$ be the induced quasi-isometry from $\widehat{X}_1$ to $\widehat{X}_2$, $\widehat{\mu}_1$ be a quasigeodesic in $\widehat{X}_1$ joining a, b and $\widehat{\mu}_2$ be a quasigeodesic in $\widehat{X}_2$ joining $\widehat{\phi}(a)$ to $\widehat{\phi}(b)$. If $p \in \widehat{X}_1$ then $d_{\widehat{X}_2}(\widehat{\pi}_{\widehat{\mu}_2}(\widehat{\phi}(p)), \widehat{\phi}(\widehat{\pi}_{\widehat{\mu}_1}(p))) \leq P_{1.2.55}$, for some constant $P_{1.2.55}$ depending only on δ, K, ϵ, ν.*

Proof. Let $q \in X_1$, μ_1 and μ_2 be electroambient representatives of $\widehat{\mu}_1$ and $\widehat{\mu}_2$ respectively. There exists $P'_{1.1.37} > 0$ (by Corollary 1.1.37) such that
$d_{X_2^h}(\pi_{\mu_2}(\phi(q)), \phi(\pi_{\mu_1}(q))) \leq P'_{1.1.37}$.
Therefore $d_{\widehat{X_2^h}}(\pi_{\mu_2}(\phi(q)), \phi(\pi_{\mu_1}(q))) \leq d_{X_2^h}(\pi_{\mu_2}(\phi(q)), \phi(\pi_{\mu_1}(q))) \leq P'_{1.1.37}$.
Now the map $f : \widehat{X_2^h} \to \widehat{X_2}$ is a quasi-isometry, therefore there exists $P''_{1.1.37}(P'_{1.1.37}) > 0$ such that $d_{\widehat{X_2}}(\widehat{\pi}_{\widehat{\mu}_2}(\widehat{\phi}(q)), \widehat{\phi}(\widehat{\pi}_{\widehat{\mu}_1}(q))) \leq P''_{1.1.37}$.
If $p \in \widehat{X}_1$, there exists $p' \in X_1$ such that $d_{\widehat{X}_1}(p, p') \leq 1$. As $\widehat{\phi}$ is a quasi-isometry, $d_{\widehat{X}_2}(\widehat{\phi}(p), \widehat{\phi}(p')) \leq S_1$ for some $S_1 > 0$. From Lemma 1.2.53, there exists $P_{1.2.53}(S_1) > 0$ such that $d_{\widehat{X}_1}(\widehat{\pi}_{\widehat{\mu}_1}(p), \widehat{\pi}_{\widehat{\mu}_1}(p')) \leq P_{1.2.53}$ and $d_{\widehat{X}_2}(\widehat{\pi}_{\widehat{\mu}_2}(\widehat{\phi}(p)), \widehat{\pi}_{\widehat{\mu}_2}(\widehat{\phi}(p')) \leq P_{1.2.53}$. Since $\widehat{\phi}$ is a quasi-isometry there exists $K_2 > 0$ such that $d_{\widehat{X_2}}(\widehat{\phi}(\widehat{\pi}_{\widehat{\mu}_1}(p)), \widehat{\phi}(\widehat{\pi}_{\widehat{\mu}_1}(p'))) \leq K_2$. Therefore $d_{\widehat{X}_2}(\widehat{\pi}_{\widehat{\mu}_2}(\widehat{\phi}(p)), \widehat{\phi}(\widehat{\pi}_{\widehat{\mu}_1}(p))) \leq P_{1.2.53} + P''_{1.1.37} + K_2$. Taking $P_{1.2.55} = P_{1.2.53} + P'_{1.1.37} + K_2$ we have the required result. ☐

1.2.2 Relatively Hyperbolic Groups

Let us consider two isometries a, b of $\mathbb{H}^2$ such that it generate a free group, $\mathbb{F}(a, b)$, which acts properly discontinuously by isometries on $\mathbb{H}^2$ and the quotient space $\mathbb{H}^2/\mathbb{F}(a, b)$ is homeomorphic to a once punctured torus S. Further, $\mathbb{H}^2/\mathbb{F}(a, b)$ is quasi-isometric to the ray $[0, \infty)$. Let $K = \pi_1(S)$, then $K = \mathbb{F}(a, b)$. Let Γ_K be the Cayley graph of K with respect to the generating set $\{a, b\}$. Let $p \in \partial\mathbb{H}^2$ be the end point of a lift of this ray to $\mathbb{H}^2$ and K_1 be the stabilizer subgroup of p. For $k \in K$, let H_{kK_1} denote the closed set in Γ_K corresponding to the left coset kK_1 of K_1 in K. Then Γ_K is strongly hyperbolic relative to the collection $\mathcal{H}_{K_1} = \{H_{kK_1} : k \in K\}$. Motivated by this example we give the definition of a relatively hyperbolic group. The groups and its subgroups in the following definitions are assumed to be infinite.

Definition 1.2.56. *(Gromov [Gro87])*
(1) Let $\delta \geq 0$. A finitely generated group G is said to be δ-hyperbolic relative to the finitely generated subgroups $H_1, ..., H_n$ in the sense of Gromov (A) if it acts freely and properly discontinuously by isometries on a proper δ-hyperbolic metric space X such that the following holds:

(i) The quotient space X/G is quasi-isometric to union of n-copies of $[0, \infty)$ joined to 0.
(ii) For $i \in \{1, 2, ..., n\}$, there exists a lift $r_i : [0, \infty) \to X$ of the ith copy of $[0, \infty)$ such that H_i is the stabilizer subgroup of $r_i(\infty) \in \partial X$. The subgroups H_i's are said to be Parabolic or Cusp subgroups and the end points $r_i(\infty)$ in ∂X are said to be parabolic end points.
(2) G is said to be hyperbolic relative to finitely generated subgroups $H_1, ..., H_n$ in the sense of Gromov (A) if G is δ-hyperbolic relative to the finitely generated subgroups $H_1, ..., H_n$ in the sense of Gromov (A) for some $\delta \geq 0$.

Thus for a group G strongly hyperbolic relative to a subgroup H (in the sense of Gromov (A)) there is a natural bijective correspondence between parabolic end points and parabolic subgroups of G. In fact, a parabolic end point corresponds to a subgroup of the form aHa^{-1} for some $a \in G$.
In reference to Definition 1.2.16, we have another definition of a strongly relatively hyperbolic group.

Definition 1.2.57. *Let $\delta \geq 0$. Let G be a finitely generated group and H be a finitely generated subgroup of G such that the generating set of G contains the generating set of H. G is said to be δ- hyperbolic relative to H in the sense of Gromov (B) if the Cayley Graph Γ_G is δ-hyperbolic relative to the collection $\mathcal{H}_H = \{K_{aH} : a \in G\}$ in the sense of Gromov (refer to Definition 1.2.16), where K_{aH} is the closed set in Γ_G obtained by left translating the Cayley graph Γ_H by a in Γ_G.*
G is said to be hyperbolic relative to H in the sense of Gromov (B) if G is δ-hyperbolic relative to H in the sense of Gromov (B) for some $\delta \geq 0$.

Note that as the generating set of G contains the generating set of H, K_{aH} is an intrinsically geodesic closed subset of Γ_G and as Γ_G is locally finite and symmetric about each point, the elements of $\mathcal{H}_H$ are uniformly properly embedded.

Definition 1.2.58. *(Farb [Far98]) Let $\widehat{\delta} \geq 0$. Suppose G is a finitely generated group and H is a finitely generated subgroup of G. Let Γ_G be a Cayley graph of G. For each left coset gH of H in G, let K_{gH} be the corresponding closed set in Γ_G. We say that G is weakly $\widehat{\delta}$-hyperbolic relative to the subgroup H if Γ_G is weakly $\widehat{\delta}$-hyperbolic relative to the collection of closed sets $\mathcal{H}_H = \{K_{gH} :$ gH is a left coset of H in $G\}$ (refer to definition 1.2.3).*

The process of attaching a cone point to the closed set K_{gH} above will also be called as coning the left coset gH to a point.

Lemma 1.2.59. *(Corollary 3.2, [Far98]) Let G be a finitely generated group. Suppose Γ_1, Γ_2 are two Cayley graphs of G with respect to finite generating sets S_1, S_2 of G respectively. Let H be a subgroup of G. Then $\widehat{\Gamma}_1$ is quasi-isometric to $\widehat{\Gamma}_2$. In particular, the property of a group G being hyperbolic relative to a subgroup H is independent of the choice of generating sets for both G and H.*

Recall that geodesics in the coned-off space $\mathcal{E}(\Gamma_G, \mathcal{H}) = \widehat{\Gamma}_G$ were called as electric geodesics. For a path $\gamma \subset \Gamma_G$, there is an induced path $\widehat{\gamma}$ in $\widehat{\Gamma_G}$ obtained by replacing the portion of γ inside a left coset by edge path of length 1 passing through the cone point corresponding to that left coset. If $\widehat{\gamma}$ is an electric geodesic (resp. P-quasigeodesic), γ was called a relative geodesic (resp. relative P-quasigeodesic). If $\widehat{\gamma}$ passes through some cone point $v(gH)$, we say that $\widehat{\gamma}$ *penetrates* the coset gH. Recall that $\widehat{\gamma}$ is said to be an electric (K, ϵ)-quasigeodesic in (the electric space) $\widehat{\Gamma_G}$ without backtracking if $\widehat{\gamma}$ is an electric K-quasigeodesic in $\widehat{\Gamma_G}$ and $\widehat{\gamma}$ does not return to any left coset after leaving it.

The pair (G, H) is said to satisfy *bounded coset penetration property* if electric quasigeodesics without backtracking starting and ending at same points in Γ_G have similar intersection patterns with elements from $\mathcal{H}_H = \{K_{gH} : gH \text{ is a left coset of } H \text{ in } G\}$.

Next we recall Farb's definition of relatively hyperbolic group (in the strong sense) from [Far98]:

Definition 1.2.60. *(Farb [Far98]) Let $\widehat{\delta} \geq 0$. G is said to be $\widehat{\delta}$-hyperbolic relative to H in the sense of Farb if G is weakly $\widehat{\delta}$-hyperbolic relative to H and the pair (G, H) satisfies bounded coset penetration property.*

More generally, we can define a group hyperbolic relative to a finite set of subgroups. Let G be a finitely generated group and let $\{H_1, ..., H_m\}$ be a finite set of finitely generated subgroups of G. We form a new graph $\widehat{\Gamma}_G = \widehat{\Gamma}_G(H_1, ..., H_m)$ from the Cayley graph Γ_G of G as follows: for each left coset gH_i $(1 \leq i \leq m)$ of H_i in G, add a vertex $v(gH_i)$ to Γ_G, and add an edge of length of length $\frac{1}{2}$ from each element gh_i of gH_i to the vertex $v(gH_i)$. We call this graph $\widehat{\Gamma}_G$ the *coned-off* Cayley graph of G with respect to $\{H_1, ..., H_m\}$.

Definition 1.2.61. *[Far98] Let $\widehat{\delta} \geq 0$. A finitely generated group G is said to be $\widehat{\delta}$-hyperbolic relative to a finite set of finitely generated subgroups $\{H_1, ..., H_m\}$ in the sense of Farb if the following conditions are satisfied:*

1. *The coned-off graph $\widehat{\Gamma}_G$ is $\widehat{\delta}$-hyperbolic.*

2. Any two geodesics in $\widehat{\Gamma}_G$ with same end points satisfy bounded coset penetration properties with respect to each left coset gH_i.

G is said to be hyperbolic relative to a finite set of finitely generated subgroups $\{H_1, ..., H_m\}$ in the sense of Farb if G is $\widehat{\delta}$-hyperbolic relative to a finite set of finitely generated subgroups $\{H_1, ..., H_m\}$ in the sense of Farb for some $\widehat{\delta} \geq 0$.

In [Bow97] (Theorem 7.10 of [Bow97]), Bowditch showed the equivalence of following two definitions:
Definition C1 : We say that a group G is hyperbolic relative to a set $\mathcal{G}$ of infinite groups, if G admits a properly discontinuous isometric action on a path-metric space, X, with the following properties:
(1) X is proper and hyperbolic,
(2) every point of the boundary of X is either a conical limit point or a bounded parabolic point,
(3) the elements of $\mathcal{G}$ are precisely the maximal parabolic subgroups of G, and
(4) every element of $\mathcal{G}$ is finitely generated.
Definition C2 : We say that G is hyperbolic relative to $\mathcal{G}$, if G admits an action on a connected graph, K, with the following properties:
(1) K is hyperbolic, and each edge of K is contained in only finitely many circuits of length n for any given integer, n,
(2) there are finitely many G-orbits of edges, and each edge stabilizer is finite,
(3) the elements of $\mathcal{G}$ are precisely the infinite vertex stabilizers of K, and
(4) every element of $\mathcal{G}$ is finitely generated.
In [Dah03], (Annexe A of [Dah03]), Dahamani showed that definitions $C1$, $C2$ and Farb's definition 1.2.60 of relatively hyperbolic groups are equivalent. Farb (Proposition 4.6, Proposition 4.10 of [Far98]), Szczepański (Theorem 1 of [Szc98]), and Bumagin (Theorem 1.6 of [Bum05]) showed that if a group G is hyperbolic relative to $\{H_1, ..., H_m\}$ in the sense of Gromov (A) then G is hyperbolic relative to $\{H_1, ..., H_m\}$ in the sense of Farb. Conversely, Groves and Manning (Theorem 3.25 of [GM08]) showed that if G is hyperbolic relative to $\{H_1, ..., H_m\}$ in the sense of Farb then G is hyperbolic relative to $\{H_1, ..., H_m\}$ in the sense of Gromov (A). Thus, we have the following theorem:

Theorem 1.2.62. *([Bow97], [Szc98], [GM08], [Bum05]) G is strongly hyperbolic relative to $\{H_1, ..., H_m\}$ in the sense of Farb if and only if G is hyperbolic relative to $\{H_1, ..., H_m\}$ in the sense of Gromov (A).*

The following theorem is the group theoretic version of Theorem 1.2.35 and Theorem 1.2.48.

Theorem 1.2.63. *Let G be a finitely generated group and H be a finitely generated subgroup of G such that the generating set of G contains the generating set of H. G is hyperbolic relative to H in the sense of Farb if and only if G is hyperbolic relative to H in the sense of Gromov (B).*

Note 1.2.64. *Thus we have the following equivalence:*
G is (strongly) hyperbolic relative to H in the sense of Farb $\Leftrightarrow$ G is hyperbolic relative to H in the sense of Gromov (A) $\Leftrightarrow$ G is hyperbolic relative to H in the sense of Gromov (B).

Definition 1.2.65. *(Bowditch [Bow97])* **Relative Hyperbolic Boundary:** *Let G be a group hyperbolic relative to H, then by Definition 1.2.57, G acts properly discontinuously on the hyperbolic space Γ_G^h obtained from Γ_G by gluing hyperbolic cones. The relative hyperbolic boundary of G is the Gromov boundary, $\partial\Gamma_G^h$, of Γ_G^h. We denote the relative hyperbolic boundary of the pair (G, H) by $\partial\Gamma(G, H)$.*

Bowditch in [Bow97] showed that if G acts properly discontinuously by isometries on a proper hyperbolic space X and the action of G on ∂X is geometrically finite (i.e. every point of ∂X is either a conical limit point or a bounded parabolic point) and minimal (i.e. if the limit set $\Lambda G = \partial X$) then ∂X is homeomorphic to $\partial\Gamma(G, H)$.

1.2.3 Partial Electrocution

The notion of Partial Electrocution was introduced in [MR08]. This is a modification of Farb's [Far98] construction of an electric space described earlier. In a partially electrocuted space, instead of coning all of a horosphere down to a point we cone only it to a hyperbolic metric space.

Definition 1.2.66. *[MR08] Let $\delta \geq 0, \nu > 0$ and $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple such that the following holds:*

1. *X is a geodesic metric space and $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ is a collection of uniformly ν-separated, intrinsically geodesic and uniformly properly embedded closed subsets of X. X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov.*

2. *$\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}$ is a collection of δ-hyperbolic geodesic metric spaces and $\mathcal{G}$ is a collection of (uniformly) Lipschitz onto maps $g_\alpha : H_\alpha \to L_\alpha$ i.e. there exists a number $P_{1.2.66} > 0$ such that $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \leq P_{1.2.66} d_{H_\alpha}(x, y)$ for all $x, y \in H_\alpha$ and for all index α. Note that the indexing set for $H_\alpha, L_\alpha, g_\alpha$ is common.*

The **partially electrocuted space** *or* partially coned off space $\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ *corresponding to* $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ *is the quotient space obtained from* X *as follows:* $\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L}) = X \sqcup (\sqcup_\alpha (H_\alpha \times [0,1])) \sqcup (\sqcup_\alpha L_\alpha) / \cup_\alpha \{(x,0) \sim x, (x,1) \sim g_\alpha(x) : x \in H_\alpha\}$. *(The metric on* $H_\alpha \times [0,1]$ *is the product metric.)*
$\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ *is equipped with the quotient metric and the metric is denoted by* d_{pel}*. In short,* $\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ *will be denoted by* X_{pel}*.*

d_{pel} is a metric: d_{pel} is indeed a pseudometric (Refer to [BH99]). Now, let $x, y \in X_{pel}$ such that $d_{pel}(x,y) = 0$. If x (or y) lie outside sets in $\{H_\alpha \times \{0\}, H_\alpha \times \{1\} : H_\alpha \in \mathcal{H}\}$, then there exists $\eta > 0$ such that $d_{pel}(x, H_\alpha \times \{0\}) > \eta$ and $d_{pel}(x, H_\alpha \times \{1\}) > \eta$ for all $H_\alpha \in \mathcal{H}$. (Note that $H_\alpha \times \{1\}$ is identified with L_α in X_{pel}). Therefore, x must equals y. As $d_{H_\alpha \times [0,1]}(H_\alpha \times \{0\}, H_\alpha \times \{1\}) = 1$, therefore if $x \in H_\alpha \times \{j\}$ then $y \notin H_\alpha \times \{j+1\}$, where $j \in \{0,1\} (mod\ 2)$.
Now, let $x, y \in H_\alpha \times \{1\}$, then for $0 < \epsilon < 1$ there is a sequence of points $p_0, q_0, p_1, q_1, ..., p_n, q_n$ in $H_\alpha \times \{1\}$ such that $p_0 = x, q_n = y$; $[p_i, q_i]$ is a geodesic in $H_\alpha \times [0,1]$; $[g_\alpha(q_i), g_\alpha(p_{i+1})]$ is a geodesic in L_α and

$$\sum_{0 \le i \le n} d_{H_\alpha \times [0,1]}(p_i, q_i) + \sum_{0 \le i \le n-1} d_{L_\alpha}(g_\alpha(q_i), g_\alpha(p_{i+1})) \le \epsilon.$$

As $H_\alpha \times [0,1]$ is equipped with product metric, therefore $d_{H_\alpha \times [0,1]}(p_i, q_i) = d_{H_\alpha \times \{1\}}(p_i, q_i)$. Also, $d_{L_\alpha}(g_\alpha(p_i), g_\alpha(q_i)) \le P_{1.2.66} d_{H_\alpha \times \{1\}}(p_i, q_i)$. Therefore,

$$\begin{aligned} d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) &\le \sum_{0 \le i \le n} d_{L_\alpha}(g_\alpha(p_i), g_\alpha(q_i)) + \sum_{0 \le i \le n-1} d_{L_\alpha}(g_\alpha(q_i), g_\alpha(p_{i+1})) \\ &\le P_{1.2.66}\epsilon + \epsilon \ \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Hence $g_\alpha(x) = g_\alpha(y)$. Now $g_\alpha(x) = x$ and $g_\alpha(y) = y$ in X_{pel}. Thus, $x = y$. Similarly, if $x, y \in H_\alpha \times \{0\}$ then $x = y$.
In Farb's construction L_α is just a point. Here, in our context partial electrocution will occur in case of tree of coned-off spaces where L_α will turned around to be a tree.
In a hyperbolic metric space geodesics diverge exponentially, the following lemma 'quasi-fies' this statement:

Lemma 1.2.67. *(Proposition 4.10, [Mit98b]) Given* $\delta, A_0 \ge 0$*, there exists* $\sigma_{1.2.67} > 1, B_{1.2.67} > 0$ *such that if* $[x,y], [y,z], [z,w]$ *are geodesics in a* δ*-hyperbolic metric space* (Z, d_Z) *with* $(x,z)_y \le A_0, (y,w)_z \le A_0$ *and* $d_Z(y,z) \ge B_{1.2.67}$*, then any path joining* x, w *and lying outside a* D*-neighborhood of* $[y,z]$ *has length greater than or equal to* $\sigma_{1.2.67}^D d_Z(y,z)$*, where* $D = \min\{d_Z(x,[y,z]) - 1, d_Z(w,[y,z]) - 1\}$*.*

Corollary 1.2.68. *Given $\delta \geq 0$, there exists $\sigma_{1.2.68} > 1, B_{1.2.68} > 0$ such that the following holds:*
Let (Z, d_Z) be a δ-hyperbolic metric space, $x, y \in Z$ and λ be a geodesic segment in Z such that $d_Z(\pi_\lambda(x), \pi_\lambda(y)) \geq B_{1.2.68}$ then for any path joining x to y and lying outside a D-neighborhood of $[x, y]$ has length greater than or equal to $\sigma_{1.2.68}^D d_Z(\pi_\lambda(x), \pi_\lambda(y))$, where $D = \min\{d_Z(x, \pi_\lambda(x)) - 1, d_Z(y, \pi_\lambda(y)) - 1\}$.

Proof. Consider the triangle $\triangle x\pi_\lambda(x)\pi_\lambda(y)$. Since Z is δ-hyperbolic, triangles are δ-thin. Therefore, there exist $w_1 \in [x, \pi_\lambda(x)], w_2 \in [\pi_\lambda(x), \pi_\lambda(y)], w_3 \in [x, \pi_\lambda(y)]$ such that $diam\{w_1, w_2, w_3\} \leq \delta$. Since π_λ is a nearest point projection, $d_Z(w_1, \pi_\lambda(x)) \leq \delta$. Thus, $(x, \pi_\lambda(y))_{\pi_\lambda(x)} = d_Z(w_1, \pi_\lambda(x)) \leq \delta$. Similarly, $(y, \pi_\lambda(x))_{\pi_\lambda(y)} \leq \delta$. By Lemma 1.2.67, we have the required result. □

Corollary 1.2.69. *Given $\delta, Q \geq 0$ there exist $B_{1.2.69} > 0, \sigma_{1.2.69} > 1$ such that the following holds:*
Let (Z, d_Z) be a δ-hyperbolic metric space and S be a Q-quasiconvex set. Suppose $x, y \in Z$ and $d_Z(\pi_S(x), \pi_S(y)) \geq B_{1.2.69}$. Let β be any path in Z joining x to y such that β lie outside D-neighborhood of S, where $D = \min\{d_Z(x, \pi_S(x)), d_Z(y, \pi_S(y))\}$, then

$$l_Z(\beta) > \sigma_{1.2.69}^{D-Q-1} d_Z(\pi_S(x), \pi_S(y))$$

Proof. Let λ be a geodesic segment joining $\pi_S(x)$ and $\pi_S(y)$. It is proved in the first part of the proof of Lemma 1.1.31 that $d_Z(\pi_S(x), \pi_\lambda(x)) \leq 3\delta + Q$ and $d_Z(\pi_S(y), \pi_\lambda(y)) \leq 3\delta + Q$. Let $B_{1.2.69} = \max\{B_{1.2.68} - 2(3\delta + Q), 4(3\delta + Q)\}$. Since $d_Z(\pi_S(x), \pi_S(y)) \geq B_{1.2.69}$, we have $d_Z(\pi_\lambda(x), \pi_\lambda(y)) \geq B_{1.2.68}$ and $d_Z(\pi_\lambda(x), \pi_\lambda(y)) > \frac{1}{2} d_Z(\pi_S(x), \pi_S(y))$. Since β lie outside D-neighborhood of S, it lie outside $(D - Q)$-neighborhood of λ. Therefore by Corollary 1.2.68, we have

$$l_Z(\beta) \geq \sigma_{1.2.68}^{D-Q-1} d_Z(\pi_\lambda(x), \pi_\lambda(y)).$$

Then,

$$l_Z(\beta) > \frac{1}{2} \sigma_{1.2.68}^{D-Q-1} d_Z(\pi_S(x), \pi_S(y)).$$

Taking $\sigma_{1.2.69} = \frac{1}{2^{\frac{1}{D-Q-1}}} \sigma_{1.2.68}$, we have the required result. □

Partially Electrocuted Space is Hyperbolic

Throughout this subsection, we assume that
- X is a geodesic metric space,
- $\nu > 0$ and $\mathcal{H}$ is a collection of uniformly ν-separated, intrinsically geodesic and uniformly properly embedded closed subsets of X, and

• $\mathcal{G}(X,\mathcal{H}), \mathcal{E}(X,\mathcal{H})$ are geodesic spaces.
Recall from Lemma 1.2.24 that the hyperbolic cones H^h are 4δ-quasiconvex.

Lemma 1.2.70. *Let $\delta \geq 0$ and X be δ-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov. If λ be a geodesic in $\mathcal{G}(X,\mathcal{H})$, then the followings hold:*
1). Let $N(\lambda)$ denote the union of λ and hyperbolic cones penetrated by λ then $N(\lambda)$ is a $Q_{1.2.70}$-quasiconvex set in $\mathcal{G}(X,\mathcal{H})$ for some $Q_{1.2.70} > 0$ depending on δ.
2). Let $N_X(\lambda) = N(\lambda) \cap X$, then $N_X(\lambda)$ is a $Q^1_{1.2.70}$-quasiconvex set in X for some $Q^1_{1.2.70} > 0$.

Proof. 1). Let $x, y \in N(\lambda)$ and $[x, y]$ be a geodesic in $\mathcal{G}(X,\mathcal{H}) = X^h$ joining x and y. We assume x, y lie on different hyperbolic cones H^h_1, H^h_2 respectively. Fix $p \in N(\lambda) \cap H^h_1$ and $q \in N(\lambda) \cap H^h_2$. Let $\widehat{\mu}$ be an electric geodesic in $\widehat{X^h}$ joining x and y. Let x_1 be the exit point of $\widehat{\mu}$ from H^h_1 and y_1 be the entry point of $\widehat{\mu}$ respectively to H^h_2. We join p to x_1 by an edge path e_p of length 1 and join q to y_1 by an edge path e_q of length 1. Let $\widehat{\mu}_1$ be the concatenation of e_p, subsegment of $\widehat{\mu}$ joining x_1, y_1 and e_q; then $\widehat{\mu}_1$ is a $(1, 2)$-quasigeodesic path without backtracking. By Lemma 1.2.50, there exists $Q_{1.2.50} > 0$ such that any electroambient path representative μ_1 of $\widehat{\mu}_1$ and λ lie in $Q_{1.2.50}$-neighborhood of each other. Let μ be an electroambient path representative μ of $\widehat{\mu}$, then the subsegment of μ joining x_1, y_1 lie in the $Q_{1.2.50}$-neighborhood of λ and hence μ lie in $Q_{1.2.50}$-neighborhood of $N(\lambda)$. Also, by Lemma 1.2.50, geodesic $[x, y]$ in X^h lie in the $Q_{1.2.50}$-neighborhood of μ, therefore $[x, y]$ lie in $2Q_{1.2.50}$-neighborhood of $N(\lambda)$.
If x, y lie in same hyperbolic cone H^h, then as hyperbolic cones are 4δ-quasiconvex, $[x, y]$ lie in 4δ-neighborhood of $N(\lambda)$. Taking $Q_{1.2.70} = \max\{2Q_{1.2.50}, 4\delta\}$, we have the required result.
2). Let $Q = Q_{1.2.70}$. For $\delta, Q \geq 0$, there exists $B_{1.2.69} > 0, \sigma_{1.2.69} \geq 1$ such that the Lemma 1.2.69 holds.
Let $a, b \in N_X(\lambda)$ and $\alpha : [0, d_X(a, b)] \to X$ be a geodesic in X joining a and b. Let $[s, t]$ be a maximal subinterval of $[0, d_X(a, b)]$ such that $\alpha|_{[s,t]}$ lie outside a D-neighborhood of $N_X(\lambda)$ in X^h (D will be chosen later).
Let $B = 1 + B_{1.2.69}$. We partition $[s, t]$ by points $s = p_0 < p_1 < ... < p_{n-1} < p_n = t$ such that $d_{X^h}(\pi_{N(\lambda)}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_{i+1}))) = B$ for $1 \leq i \leq n-2$ and $d_{X^h}(\pi_{N(\lambda)}(\alpha(p_{n-1})), \pi_{N(\lambda)}(\alpha(p_n))) \leq B$. Let $d_i = d_{X^h}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_i)))$. Now consider the subsegment $\alpha|_{[p_i,p_{i+1}]}$. Let $\sigma = \sigma_{1.2.69}$. Assume $d_i \leq d_{i+1}$, then by

corollary 1.2.69, we have

$$\begin{aligned} \sigma^{d_i-Q-1} d_{X^h}(\pi_{N(\lambda)}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_{i+1}))) &\leq l_{X^h}(\alpha|_{[p_i,p_{i+1}]}) \\ &= l_X(\alpha|_{[p_i,p_{i+1}]}) \qquad (1.3) \\ &= d_X(\alpha(p_i), \alpha(p_{i+1})). \end{aligned}$$

Now $D \leq d_i$, therefore

$$\sigma^{D-Q-1} d_{X^h}(\pi_{N(\lambda)}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_{i+1}))) \leq d_X(\alpha(p_i), \alpha(p_{i+1}))$$

for $1 \leq i \leq n-2$.
Therefore by triangle inequality,
$d_{X^h}(\pi_{N(\lambda)}(\alpha(s)), \pi_{N(\lambda)}(\alpha(p_{n-1}))) \leq \frac{1}{\sigma^{D-Q-1}} d_X(\alpha(s), \alpha(p_{n-1}))$.
Now $d_{X^h}(\pi_{N(\lambda)}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_{i+1}))) = B$ for $1 \leq i \leq n-2$ and $d_{X^h}(\pi_{N(\lambda)}(\alpha(p_{n-1})), \pi_{N(\lambda)}(\alpha(p_n))) < B$. Since X is properly embedded in X^h, there exists $B_1 > 0$ such that $d_X(\pi_{N(\lambda)}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_{i+1}))) \leq B_1$ for all $1 \leq i \leq n-1$.
For $1 \leq i \leq n-2$,

$$\begin{aligned} d_X(\pi_{N(\lambda)}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_{i+1}))) &\leq B_1 \\ &\leq B_1 d_{X^h}(\pi_{N(\lambda)}(\alpha(p_i)), \pi_{N(\lambda)}(\alpha(p_{i+1}))). \end{aligned}$$

By triangle inequality,

$$\begin{aligned} d_X(\pi_{N(\lambda)}(\alpha(s)), \pi_{N(\lambda)}(\alpha(t))) &\leq B_1 d_{X^h}(\pi_{N(\lambda)}(\alpha(s)), \pi_{N(\lambda)}(\alpha(p_{n-1}))) + B_1 \\ &\leq B_1(\frac{1}{\sigma^{D-Q-1}}) d_X(\alpha(s), \alpha(p_{n-1})) + B_1 \\ &\leq B_1(\frac{1}{\sigma^{D-Q-1}}) d_X(\alpha(s), \alpha(t)) + B_1 ....(*) \end{aligned}$$

Now $d_{X^h}(\alpha(s), \pi_{N(\lambda)}(\alpha(s))) \leq D$, $d_{X^h}(\alpha(t), \pi_{N(\lambda)}(\alpha(t))) \leq D$. Since X is properly embedded in X^h, there exists $D_1(D) > 0$ such that $d_X(\alpha(s), \pi_{N(\lambda)}(\alpha(s))) \leq D_1$, $d_X(\alpha(t), \pi_{N(\lambda)}(\alpha(t))) \leq D_1$
Therefore, $d_X(\alpha(s), \alpha(t)) \leq 2D_1 + d_X(\pi_{N(\lambda)}(\alpha(s)), \pi_{N(\lambda)}(\alpha(t)))$.
Thus,

$$d_X(\alpha(s), \alpha(t)) \leq \frac{(2D_1 + B_1)\sigma^{D-Q-1}}{\sigma^{D-Q-1} - B_1}.$$

We choose D such that $\sigma^{D-Q-1} - B_1 \geq 1$. Thus $l_X(\alpha|_{[s,t]}) \leq (2D_1 + B_1)\sigma^{D-Q-1}$. Let $W = (2D_1 + B_1)\sigma^{D-Q-1}$. Taking $Q^1_{1.2.70} = D_1 + \frac{W}{2}$, we have the required result. □

Remark 1.2.71. *With the notation as Lemma 1.2.70, by equation $(*)$, we have that if $\alpha : [0,1] \to X$ is a path in X such that α lie outside a D-neighborhood of $N(\lambda)$ in*

X^h, then there exist $\sigma > 1, B_1 > 0$ (where σ, B_1 depend only upon the hyperbolicity constant δ of X^h and the quasiconvex constant $Q = Q_{1.2.70}$ of $N(\lambda)$) such that

$$d_X(\pi_{N(\lambda)}(\alpha(0)), \pi_{N(\lambda)}(\alpha(1))) \leq (\frac{B_1}{\sigma^{D-Q-1}})l_X(\alpha) + B_1.$$

Corollary 1.2.72. *Let $\delta \geq 0$ and X be δ-hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov. For a k-quasigeodesic path μ in X^h, there exists $Q_{1.2.72}(\delta, k) > 0$ such that*
(i) $N(\mu)$ is a $Q_{1.2.72}$-quasiconvex set in X^h,
(ii) $N_X(\mu) := N(\mu) \cap X$ is a $Q^1_{1.2.72}$-quasiconvex set in X.

Corollary 1.2.73. *Let $(X, \mathcal{H}.\mathcal{G}.\mathcal{L})$ be quadruple as in Definition 1.2.66, then $H_\alpha \in \mathcal{H}$, $L_\alpha \in \mathcal{L}$ are quasi-isometrically embedded in X and X_{pel} respectively.*

Proof. By Proposition 1.1.9, we can assume X_{pel} to be a connected graph of edge length one. As $(X, \mathcal{H})$ and $(X_{pel}, \mathcal{L})$ are strongly relatively hyperbolic space, by Lemma 1.2.70, H_α, L_α are quasiconvex in X, X_{pel} respectively. H_α is properly embedded in X by hypothesis, therefore H_α is quasi-isometrically embedded in X. To see L_α is properly embedded in X_{pel}, let $x, y \in L_\alpha$ such that $d_{X_{pel}}(x, y) \leq M$ and λ_p be a geodesic in X_{pel} joining x, y. Now $l_{X_{pel}}(\lambda_p) = l_{\widehat{X_{pel}}}(\lambda_p)$, therefore λ_p is a $(1, M)$-tamed path in $\widehat{X_{pel}}$. Let $\lambda_i : [0, 1] \to X_{pel}$, $i \leq i \leq N$, be N components of $\lambda_p \setminus L_\alpha$ and $\widehat{\lambda}_i$ be the coned-off path in $\widehat{X_{pel}}$ obtained from λ_i, then each $\widehat{\lambda}_i$ is a $(1, M)$-tamed path without backtracking in $\widehat{X_{pel}}$. By similar intersection pattern 2, for all i, $d_{L_\alpha}(\lambda_i(0), \lambda_i(1)) \leq I$ for some $I > 0$. Thus,

$$d_{L_\alpha}(x, y) \leq \sum_{1 \leq i \leq N} d_{L_\alpha}(\lambda_i(0), \lambda_i(1)) + \sum_{1 \leq i \leq N-1} d_{L_\alpha}(\lambda_i(1), \lambda_{i+1}(0)) \leq NI + M.$$

As X_{pel} is a graph, $N \leq M$. Therefore, $d_{L_\alpha}(x, y) \leq MI + M$. Thus, L_α is properly embedded in X_{pel} and hence quasi-isometrically embedded in X_{pel}. □

Remark 1.2.74. *Note that in the proof of the Lemma 1.2.70, if the geodesic α in X is replaced by a K-quasigeodesic for some $K \geq 1$, then also there exists $Q_{1.2.74} > 0$ such that α lies in the $Q_{1.2.74}$-neighborhood of $N_X(\lambda)$. (Without loss of generality, by Lemma 1.1.23, we can assume that α is a K-tamed path. At equation 1.3, incorporate $l_X(\alpha_{[p_i, p_{i+1}]}) \leq K d_X(\alpha(p_i), \alpha(p_{i+1})) + K$, rest of the argument is similar)*

Let $j : X \hookrightarrow X_{pel}$ denote the inclusion. We define a map $\widehat{j} : \widehat{X} \to \widehat{X_{pel}}$ as follows:
If $x \in X$, then define $\widehat{j}(x) = j(x)$.
If x is a cone point over some horosphere-like set H_α, then define $\widehat{i}(x)$ as the cone point over the hyperbolic space L_α.

Let $h \in H_\alpha$ and $e : [0, \frac{1}{2}] \to \widehat{X}$ be the edge of length $\frac{1}{2}$ joining $h \in H_\alpha \in \mathcal{H}$ and the cone point $v(H_\alpha)$ over H_α and let $e' : [0, \frac{1}{2}] \to X_{pel}$ be the edge of length $\frac{1}{2}$ joining $g(h) \in L_\alpha$ and the cone point $v(L_\alpha)$. If $x = e(t)$, where $t \in (0, \frac{1}{2}]$, then define $\widehat{j}(x) = e'(t)$.

Let $x, y \in X$, then there exists a sequence of points in $x = p_0, q_0, ..., p_n, q_n = y$ in X such that for each i, $[p_i, q_i]$ is a geodesic in X and $[q_i, p_{i+1}]$ is a geodesic in the coned-off space $\widehat{H_i}$ for some $H_i \in \mathcal{H}$; and

$$\sum_{1 \le i \le n} d_X(p_i, q_i) + \sum_{0 \le i \le n-1} d_{\widehat{H_i}}(q_i, p_{i+1}) \le d_{\widehat{X}}(x, y) + 1.$$

Now $q_i, p_{i+1} \in H_i$ and recall that $H_i \times \{1\}$ identified with L_i. Let e_i be the edge path of length one joining $(p_i, 1), (q_i, 1)$ and passing through the cone point $v(L_i)$ over L_i. Then for each i, $\alpha_i = [q_i, (q_i, 1)] \cup e_i \cup [(p_{i+1}, 1), p_{i+1}]$ is a path in $\widehat{X_{pel}}$ of length 3. Therefore,

$$\begin{aligned} d_{pel}(x, y) &\le \sum_{1 \le i \le n} d_X(p_i, q_i) + 3n \\ &\le (d_{\widehat{X}}(x, y) + 1) + 3(\frac{d_{\widehat{X}}(x, y)}{\nu} + 1) \\ &\le (1 + \frac{3}{\nu}) d_{\widehat{X}}(x, y) + 4 \end{aligned}$$

Similar to the proof of Lemma 1.2.28, we have $d_{\widehat{X}}(x, y) \le d_{\widehat{X_{pel}}}(x, y)(1 + \frac{1}{\nu}) + 2$. Thus, we have the following lemma:

Lemma 1.2.75. *There exist $K_{1.2.75} \ge 1, \epsilon_{1.2.75} \ge 0$ depending on ν such that $\widehat{j}$ is a $(K_{1.2.75}, \epsilon_{1.2.75})$-quasi-isometry.*

Thus, if $\widehat{X}$ is a hyperbolic metric space then so is $\widehat{X_{pel}}$. Also, note that if $d_{H_\alpha}(x, y) \le D$ for any $x, y \in H_\alpha \in \mathcal{H}$ then $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \le P_{1.2.66} D$. Hence, quasigeodesic paths in $\widehat{X_{pel}}$ satisfy similar intersection pattern with horosphere-like sets in $\mathcal{L}$ if quasigeodesic paths in $\widehat{X}$ satisfy similar intersection pattern with horosphere-like sets in $\mathcal{H}$. Thus, if X is hyperbolic relative to $\mathcal{H}$ in the sense of Farb, then X_{pel} is hyperbolic relative to $\mathcal{L}$ in the sense of Farb. Therefore, by Theorem 1.2.35, there exists $\delta_p(\delta, \nu, P_{1.2.66}) \ge 0$ such that the space $X^h_{pel} = \mathcal{G}(X_{pel}, \mathcal{L})$ is a δ_p-hyperbolic metric space.
Let $\widehat{\mu} = \widehat{i}(\widehat{\lambda})$ then $\widehat{\mu} \setminus \bigcup_{H_\alpha \in \mathcal{H}} (H_\alpha \times [0, 1]) = \widehat{\lambda} \setminus \bigcup_{H_\alpha \in \mathcal{H}} H_\alpha$. Let

- $N(\widehat{\lambda})$ be the union of λ^b and the hyperbolic cones from $\mathcal{H}^h$ intersected by $\widehat{\lambda}$, where $\lambda^b = \widehat{\lambda} \setminus \bigcup_{H_\alpha \in \mathcal{H}} H_\alpha$.
- $N(\widehat{\mu})$ be the union of $\widehat{\mu} \setminus \bigcup_{L_\alpha \in \mathcal{L}} L_\alpha$ and the hyperbolic cones L^h_α intersected by $\widehat{\mu}$.

• $N_X(\widehat{\lambda}) := N(\widehat{\lambda}) \cap X$ and $N_{X_{pel}}(\widehat{\mu}) := N(\widehat{\mu}) \cap X_{pel}$.
By Lemma 1.2.50 and Corollary 1.2.72, $N(\widehat{\lambda})$ is $Q_{1.2.72}$-quasiconvex in X^h and $N_X(\widehat{\lambda})$ is $Q^1_{1.2.72}$-quasiconvex in X. Since X_{pel} is δ_p-hyperbolic relative to $\mathcal{L}$, therefore there exist $Q_p(\delta_p), Q^1_p(\delta_p) > 0$ such that $N(\widehat{\mu})$ is Q_p-quasiconvex in X^h_{pel} and $N_{X_{pel}}(\widehat{\mu})$ is Q^1_p-quasiconvex in X_{pel}.

Lemma 1.2.76. *Let $K \geq 1$. Let $z, w \in N_X(\widehat{\lambda})$ and β_{pel} be a K-tamed path in X_{pel} joining z, w. Then the following holds:*
1) There exists $Q_{1.2.76} > 0$ such that $\beta_{pel} \subset Nbhd_{X_{pel}}(N_X(\widehat{\lambda}); Q_{1.2.76})$ and $\lambda^b \subset Nbhd_{X_{pel}}(\beta_{pel}; Q_{1.2.76})$.
2) Let $H \in \mathcal{H}$ be a horosphere-like set that β_{pel} intersects but $\widehat{\lambda}$ does not, then the H-distance between the first entry point and last exit point of β_{pel} to H is at most $I_{1.2.76}$ for some $I_{1.2.76} > 0$ depending on the hyperbolicity constant δ of X^h and K.

Proof. 1) Let $\widehat{\mu} = \widehat{j}(\widehat{\lambda})$ as above. From Remark 1.2.74 it follows that there exists $Q = Q_{1.2.74} > 0$ such that $\beta_{pel} \subset Nbhd_{X_{pel}}(N_{X_{pel}}(\widehat{\mu}); Q)$. Now the Hausdorff distance between $N_X(\widehat{\lambda})$ and $N_{X_{pel}}(\widehat{\mu})$ in X_{pel} is at most one. Therefore, $\beta_{pel} \subset Nbhd_{X_{pel}}(N_X(\widehat{\lambda}); Q + 1)$.
Second part: Let β_{pel} be defined on the interval $[l, m]$. Let $l = s_0 < s_1 < ... < s_n = m$ be a partition of $[l, m]$ such that for all $i < n$, $l_{X_{pel}}(\beta_{pel}|_{[s_{i-1},s_i]}) = 1$ and $l_{X_{pel}}(\beta_{pel}|_{[s_{n-1},s_n]}) \leq 1$. For each i, there exists $q_i \in N_{X_{pel}}(\widehat{\mu})$ such that $d_{X_{pel}}(\beta_{pel}(s_i), q_i) \leq Q$. Now there exist $\widehat{q_i} \in \widehat{\mu}$ such that $d_{\widehat{X_{pel}}}(q_i, \widehat{q_i}) \leq 1$. Let $\widehat{\mu}_i$ be the subsegment of $\widehat{\mu}$ joining $\widehat{q_i}$ and $\widehat{q}_{i+1}$. If q_i, q_{i+1} does not lie on the same horosphere-like set, then due to similar intersection patterns of $[p_i, \widehat{p_i}] \cup \widehat{\mu}_i \cup [\widehat{p}_{i+1}, p_{i+1}]$ and $\beta_{pel}|_{[s_i,s_{i+1}]}$ with the sets in $\mathcal{L}$, there exists $Q' > 0$ such that $\widehat{\mu} \setminus \cup_{H_\alpha \in \mathcal{H}}(H_\alpha \times [0, 1]) \subset Nbhd_{X_{pel}}(\beta_{pel}|_{[s_{i-1},s_i]}; Q')$. Now $\widehat{\mu} \setminus \cup_{H_\alpha \in \mathcal{H}}(H_\alpha \times [0, 1]) = \lambda^b$, thus for $x \in \lambda^b$ there exist i and $x' \in \beta_{pel}|_{[s_{i-1},s_i]} \subset \beta_{pel}$ such that $d_{X_{pel}}(x, x') \leq Q'$. Taking $Q_{1.2.76} = \max\{Q + 1, Q'\}$, we have the required result.
2) From above, for each i, there exists $p_i \in N_X(\widehat{\lambda})$ such that $d_{X_{pel}}(\beta_{pel}(s_i), p_i) \leq Q+1$. Then $d_{X_{pel}}(p_i, p_{i+1}) \leq 2(Q+1)+1$. Now $N_X(\widehat{\lambda})$ is properly embedded and quasiconvex in X_{pel}, hence it is quasi-isometrically embedded in X_{pel}. Thus, for each i, there exists a path $\lambda_i \subset N_X(\widehat{\lambda})$ joining p_i, p_{i+1} such that $l_{X_{pel}}(\lambda_i) \leq Q'_1$ for some $Q'_1 > 0$. Let λ_a be the concatenation of paths λ_i and $Q_1 = \frac{Q'_1}{2} + Q$, then $\lambda_a \subset N_X(\widehat{\lambda})$ with end points z, w and $\lambda_a \subset Nbhd_{X_{pel}}(\beta_{pel}; Q_1)$.
Let λ_a be defined on the interval $[0, 1]$, then for each $s \in [0, 1]$ there exists $p_s \in \beta_{pel}$ such that $d_{X_{pel}}(\lambda_a(s), p_s) \leq Q_1$. Let $[s_0, t_0]$ be a maximal subinterval of $[0, 1]$ such that $\lambda_a|_{[s_0,t_0]} \subset Nbhd_{X_{pel}}(H; Q_1)$. Due to maximality of $[s_0, t_0]$, there exist $p_{s_0}, p_{t_0} \in \beta_{pel}$ such that geodesics $[\lambda_a(s_0), p_{s_0}], [\lambda_a(t_0), p_{t_0}]$ in X_{pel} does not penetrate H but $d_{X_{pel}}(\lambda_a(s_0), H) \leq Q_1$ and $d_{X_{pel}}(\lambda_a(t_0), H) \leq Q_1$. Hence

$d_{\widehat{X_{pel}}}(\lambda_a(s_0), \lambda_a(t_0)) \leq 2Q_1 + 1$. Suppose λ'_a is the subsegment of λ_a joining $\lambda_a(s_0), \lambda_a(t_0)$, then the coned-off path $\widehat{\lambda'_a}$ intersects at most $2Q_1+1$-many horosphere-like sets. Let $N([\lambda_a(s_0), p_{s_0}])$ be the union of

- subsegments of $[\lambda_a(s_0), p_{s_0}]$ lying outside horosphere-like sets and are in X and
- horosphere-like sets in X penetrated by $[\lambda_a(s_0), p_{s_0}]$.

Note that $N([\lambda_a(s_0), p_{s_0}]) \subset X$ and similarly $N([\lambda_a(t_0), p_{t_0}])$ is defined. As lengths of subsegments of $[\lambda_a(s_0), p_{s_0}]$ lying outside horosphere-like sets are uniformly bounded in X^h, hence they are all uniform quasigeodesics in X^h.

Let $\pi_H : X \to H$ be a nearest point projection. Now horosphere-like sets are mutually co-bounded and visually bounded. Therefore there exists $I^1_{1.2.76} > 0$ such that the diameter of the set $\pi_H(N([\lambda_a(s_0), p_{s_0}]) \cup \lambda'_a \cup N([\lambda_a(t_0), p_{t_0}]))$ is at most $I^1_{1.2.76}$ (proof of this fact is same as the proof of property 1 of Lemma 1.2.44). Thus $d_H(\pi_H(p_{s_0}), \pi_H(p_{t_0}))) \leq I^1_{1.2.76}$.

Note that $d_{X_{pel}}(p_{s_0}, \pi_H(p_{s_0})) \leq 2Q_1$ and $d_{X_{pel}}(p_{t_0}, \pi_H(p_{t_0})) \leq 2Q_1$. Hence $d_{X_{pel}}(p_{s_0}, p_{t_0}) \leq 2Q_1 + I^1_{1.2.76} + 2Q_1 = 4Q_1 + I^1_{1.2.76}$. Let β'_{pel} be the subsegment of β_{pel} joining p_{s_0} and p_{t_0}, then as β_{pel} is a K-tamed path, we have $l_{X_{pel}}(\beta'_{pel}) \leq 4Q_1K + I^1_{1.2.76} + K$. Let $\widehat{\beta'_{pel}}$ be the coned-off path in $\widehat{X}$ obtained from β'_{pel} by replacing all the subpaths joining the first entry point and last exit point to horosphere-like sets in $\mathcal{H}$ by edge paths of length one. Then as length of β'_{pel} is at most $4Q_1K + I^1_{1.2.76}K + K$, hence $\widehat{\beta'_{pel}}$ is a $(1, 4Q_1K + I^1_{1.2.76}K + K)$-tamed path. Suppose x_H, y_H are the entry and exit points respectively of β_{pel} to H then x_H, y_H respectively are also the entry and exit points of $\widehat{\beta'_{pel}}$ to H. By applying Lemma 1.2.33, there exists $I_{1.2.33} > 0$ such that $d_H(x_H, \pi_H(p_{s_0})) \leq I_{1.2.33}$ and $d_H(y_H, \pi_H(p_{t_0})) \leq I_{1.2.33}$. Hence $d_H(x_H, y_H) \leq I_{1.2.33} + I^1_{1.2.76} + I_{1.2.33} = 2I_{1.2.33} + I^1_{1.2.76}$. Taking $I_{1.2.76} = 2I_{1.2.33} + I^1_{1.2.76}$, we have the required result.

□

Theorem 1.2.77. *[MR08, MP] Let $\delta \geq 0$ and suppose X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov. For $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ an ordered quadruple as in Definition 1.2.66 above, $(\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L}), d_{pel})$ is a $\delta^{pel}_{1.2.77}$-hyperbolic metric space for some $\delta^{pel}_{1.2.77} \geq 0$ depending on $\delta, P_{1.2.66}, \nu$.*

Proof. To prove hyperbolicity of (X_{pel}, d_{pel}), it suffices to prove that for all $K \geq 1$ there exists $W = W(K)$ such that for all $a, b \in X_{pel}$, K-quasigeodesics bigons in X_{pel} are W-thin, i.e, they lie in W-neighborhood of each other in X_{pel}. We assume $a, b \in X$ as Hausdorff distance (in the metric d_{pel}) between X and X_{pel} is at most one. Let β_1, β_2 be two K-quasigeodesics in X_{pel} joining a, b and $\widehat{\lambda}$ be a geodesic in $\widehat{X}$ joining a, b. In view of Lemma 1.1.23, we can assume β_1, β_2 to be K-tamed quasigeodesic path.

Now β_1 (resp. β_2) and $\widehat{\lambda}$ track each other outside a $Q_{1.2.76}$-neighborhood of horosphere-like sets in X_{pel}, hence β_1 and β_2 track each other outside $Q_{1.2.76}$-neighborhood of horosphere-like sets in X_{pel}.
Let β_1', β_2' be the portions of β_1, β_2 respectively, lying inside $Q_{1.2.76}$-neighborhood of a horosphere-like set L in X_{pel}. Note that the Hausdorff distance between L and $Nbhd_{X_{pel}}(L; Q_{1.2.76})$ is at most $Q_{1.2.76}$. As L is hyperbolic, $Nbhd_{X_{pel}}(L; Q_{1.2.76})$ is also a hyperbolic space. Since the end points of β_1', β_2' are at a bounded distance from each other in X_{pel}, therefore by stability of quasigeodesics, β_1', β_2' lie at a bounded distance from each other in $Nbhd_{X_{pel}}(L; Q_{1.2.76})$. Thus there exists $W = W(K) > 0$ such that the Hausdorff distance between β_1 and β_2 in X_{pel} is at most W. Hence X_{pel} is a hyperbolic metric space. □

Lemma 1.2.78. *[MR08][MP]*
Let $\delta \geq 0$ and suppose X is δ-hyperbolic relative to $\mathcal{H}$ in the sense of Gromov. Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple as in Definition 1.2.66 above.
Let $\beta_{pel} : [a, b] \to X_{pel}$ and $\widehat{\lambda} : [c, d] \to \widehat{X}$ (resp. $\lambda : [c, d] \to X^h$) denote respectively a (K_p, ϵ_p)-quasigeodesic path in (X_{pel}, d_{pel}) and a geodesic in $(\widehat{X}, d_{\widehat{X}})$ (resp. $(\mathcal{G}(X, \mathcal{H}), d_{X^h})$) joining $p, q \in X$. Then there exists $Q_{1.2.78} > 0$ depending on $\delta, \nu, K_p, \epsilon_p$ such that for $x \in \lambda^b = \widehat{\lambda} \setminus \bigcup_{H_\alpha \in \mathcal{H}} int(H_\alpha^h)$ (resp. $x \in \lambda \setminus \bigcup_{H_\alpha \in \mathcal{H}} H_\alpha$) there exists $y \in \beta_{pel}^b = \beta_{pel} \setminus \bigcup_{H_\alpha \in \mathcal{H}} (H_\alpha \times [0, 1])$ such that $d_X(x, y) \leq Q_{1.2.78}$.

Proof. Without loss of generality, by Lemma 1.1.23, we can assume β_{pel} to be (K_p, ϵ_p)-tamed quasigeodesic path. Recall that $N(\widehat{\lambda})$ is $Q_{1.2.72}$-quasiconvex in X^h. Let $Q = Q_{1.2.72}$.
First we prove that there exists $R_a > 0$ such that each component of β_{pel}^b lie in the R_a-neighborhood of $N_X(\widehat{\lambda}_a)$ in X:
Let β_{pel}' be a maximal subsegment of β_{pel} such that β_{pel}' does not intersect the sets of the form $H \times [0, 1]$ penetrated by $\widehat{\lambda}$, where $H \in \mathcal{H}$ and $H \times [0, 1]$ as in Definition (1.2.66) of partial electrocution, then the end points of β_{pel}' lie on $N_X(\widehat{\lambda})$. Let β_a' be a path in X obtained from β_{pel}' by first removing each portions of β_{pel} lying inside the sets of the form $H \times [0, 1]$ and then joining the first entry point and last exit point to H by a geodesic in H. Let β_a'' be a maximal subsegment of β_a' lying outside R-neighborhood of $N(\widehat{\lambda})$ in X^h (R will be chosen later). Suppose p'', q'' are the end points of β_a'', then $d_{X^h}(p'', N(\widehat{\lambda})) = R$ and $d_{X^h}(q'', N(\widehat{\lambda})) = R$. As X is properly embedded in X^h, therefore there exists $R_1 > 0$ such that $d_X(p'', \pi_{N(\widehat{\lambda})}(p'')) \leq R_1$ and $d_X(q'', \pi_{N(\widehat{\lambda})}(q'')) \leq R_1$. By Lemma 1.2.76, there exists $I = I_{1.2.76} > 0$ such that the length of the subsegment of β_a'' lying on a horosphere-like set in X is at most I. Thus, there exist $x'', y'' \in \beta_a''$ lying outside horosphere-like sets such that $d_X(p'', x'') \leq I$ and $d_X(q'', y'') \leq I$. Let β_{pel}'' be the subsegment of β_{pel}' joining x'' and

y''.

$$\begin{aligned} l_{X_{pel}}(\beta''_{pel}) &\leq K_p d_{X_{pel}}(x'', y'') + \epsilon_p \\ &\leq K_p d_{X_{pel}}(p'', q'') + 2K_p I + \epsilon_p \\ &\leq K_p\{2R_1 + d_{X_{pel}}(\pi_{N(\widehat{\lambda})}(p''), \pi_{N(\widehat{\lambda})}(q''))\} + 2K_p I + \epsilon_p \\ &\leq K_p\{2R_1 + d_X(\pi_{N(\widehat{\lambda})}(p''), \pi_{N(\widehat{\lambda})}(q''))\} + 2K_p I + \epsilon_p \\ &\leq K_p\{2R_1 + B_1 e^{-(R-Q-1)} l_X(\beta''_a) + B_1\} + 2K_p I + \epsilon_p, \quad \text{by Remark 1.2.71.} \end{aligned}$$

Let P be the number of horosphere-like sets penetrated by β''_{pel}, then $P \leq \frac{l_{X_{pel}}(\beta''_{pel})}{\nu} + 1$. Let $l_X(\beta''_a \setminus \cup_{H\in\mathcal{H}} H)$ be the sum of the lengths of the connected components of β''_a lying outside horosphere-like sets. Then from above, we have

$$\begin{aligned} l_{X_{pel}}(\beta''_{pel}) &\leq K_p\{2R_1 + B_1 e^{-(R-Q-1)}(l_X(\beta''_a \setminus \cup_{H\in\mathcal{H}} H) + PI)\} + 2K_p I + \epsilon_p \\ &\leq K_p\{2R_1 + B_1 e^{-(R-Q-1)}(l_{X_{pel}}(\beta''_{pel}) + I(\frac{l_{X_{pel}}(\beta''_{pel})}{\nu} + 1))\} 2K_p I + \epsilon_p. \end{aligned}$$

Thus,

$$l_{X_{pel}}(\beta''_{pel})\{1 - K_p e^{-(R-Q-1)}(B_1 + \frac{I}{\nu})\} \leq 2K_p R_1 + B_1 e^{-(R-Q-1)} K_p I + 2K_p I + \epsilon_p.$$

We choose R sufficiently large such that $1 - K_p e^{-(R-Q-1)}(B_1 + \frac{I}{\nu})) \geq \frac{1}{2}$, then for that R, we have, $l_{X_{pel}}(\beta''_{pel}) \leq 2(2K_p R_1 + B_1 e^{-(R-Q-1)} K_p I + 2K_p I + \epsilon_p) = W$ (say). Thus, $l_X(\beta''_a \setminus \cup_{H\in\mathcal{H}} H) \leq W$ and $P \leq \frac{W}{\nu} + 1$. Thus $l_X(\beta''_a) \leq W + (\frac{W}{\nu} + 1)I$.
Let $L = W + (\frac{W}{\nu} + 1)I$. Now consider the path β_a in X obtained from β_{pel} by replacing the subsegment of β_{pel} lying on sets of the form $H \times [0, 1]$ by geodesics in H joining the first entry point and last exit point to H. Let $R_a = R + \frac{L}{2}$, then $\beta_a \subset Nbhd_X(N_X(\widehat{\lambda}); R_a)$.

Let $[l, m]$ be the domain of β_a and $l = s_0 < s_1 < ... < s_n = m$ be a partition of $[l, m]$ such that for all $i < n$, $l_X(\beta_a|_{[s_{i-1}, s_i]}) = 1$ and $l_X(\beta_a|_{[s_{n-1}, s_n]}) \leq 1$. As $N_X(\widehat{\lambda})$ is quasiconvex and properly embedded in X, similar as the proof of second part in (1) of Lemma 1.2.76, by projecting $\beta_a(s_i)$ to $N_X(\widehat{\lambda})$, we obtain a path $\lambda_a \subset N_X(\widehat{\lambda})$ such that $\lambda_a \setminus \cup_{H\in\mathcal{H}} H = \lambda^b$ and $\lambda_a \subset Nbhd_X(\beta_a; R'_a)$ for some $R'_a > 0$. Now if H is a horosphere-like set that β_a intersects but λ_a does not, then the H-distance between the entry and exit points of β_a to H is at most I. Taking $Q_{1.2.78} = \max\{R_a, R'_a + \frac{I}{2}\}$, we have the required result.
As $\widehat{\lambda}$ and λ track each other outside horosphere-like sets, therefore the Lemma is also true for the geodesic λ in X^h. □

1.3 Trees of Spaces: Hyperbolic and Relatively Hyperbolic

Let S be a manifold and $\phi : S \to S$ be an orientation preserving homeomorphism. Let M be a manifold fibering over the unit circle $\mathbb{S}^1$ with fiber S, i.e.

$$M = \frac{S \times [0,1]}{\{(x,0),(\phi(x),1) : x \in S\}}.$$

Suppose $\widetilde{S}$ and $\widetilde{M}$ are universal covers of S and M respectively. Then $\widetilde{M}$ is homeomorphic to $\widetilde{S} \times \mathbb{R}$. Now $\mathbb{R}$ can be treated as a tree with vertex set as $\mathbb{Z}$ and the interval $[i, i+1]$ as an edge between i and $i+1$. For each $i \in \mathbb{Z}$, let $\widetilde{S}_i = \widetilde{S} \times \{i\}$ (called a vertex space). Let $e(i)$ be the mid point of the interval $[i, i+1]$ and let $\widetilde{S}_{e(i)} = \widetilde{S} \times \{e(i)\}$ (called a edge space). Then $\widetilde{M}$ can be viewed as a tree of spaces with vertex spaces $\widetilde{S}_i$, edge spaces $\widetilde{S}_{e(i)}$. Edge spaces $\widetilde{S}_{e(i)}$ are identified to the vertex spaces $\widetilde{S}_i$ by a lift $\tilde{\phi}_{e(i)}$ of the map ϕ to the universal cover.
Now, let S be a closed hyperbolic surface of genus greater than equal to 2, then $\widetilde{S} = \mathbb{H}^2$. ϕ induces an automorphism ϕ_* of the fundamental group $\pi_1(S)$ of S and ϕ_* induces a quasi-isometry on the Cayley graph of $\pi_1(S)$. Now $\pi_1(S)$ acts properly discontinuously and cocompactly on $\mathbb{H}^2$, therefore $\mathbb{H}^2 (= \widetilde{S})$ is quasi-isometric to the Cayley graph of $\pi_1(S)$. Hence there exists a (K,ϵ)-quasi-isometry $\widetilde{\Phi} : \widetilde{S} \to \widetilde{S}$ induced by ϕ for some $K \geq 1$ and $\epsilon \geq 0$. Thus $\widetilde{M}$ can be regarded as a tree of spaces with edge spaces identified to vertex spaces by a quasi-isometry.

Definition 1.3.1. *(Bestvina-Feighn [BF92]) Let $K \geq 1, \epsilon \geq 0$. $\mathbb{P} : X \to T$ is said to be a tree of geodesic metric spaces satisfying the (K,ϵ)-q(uasi) i(sometrically) embedded condition if the geodesic metric space (X, d_X) admits a map $\mathbb{P} : X \to T$ onto a simplicial tree T, such that there exist ϵ and $K > 0$ satisfying the following:*
1) For all $s \in T$, $X_s = \mathbb{P}^{-1}(s) \subset X$ with the induced path metric d_{X_s} is a geodesic metric space X_s. Further, the inclusions $i_s : X_s \to X$ are uniformly proper, i.e. for all $M > 0$ there exists $N > 0$ such that for all $s \in T$ and $x, y \in X_s$, $d_X(i_s(x), i_s(y)) \leq M$ implies $d_{X_s}(x,y) \leq N$.
2) For a vertex v in T, $X_v = \mathbb{P}^{-1}(v)$ will be called as vertex space for v. Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under $\mathbb{P}$ of the mid-point of e, X_e will be called as edge space for e. There exist continuous maps $f_e : X_e \times [0,1] \to X$, such that $f_e|_{X_e \times (0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric. Further, f_e is fiber-preserving, i.e. projection to the second co-ordinate in $X_e \times [0,1]$ corresponds via f_e to projection to the tree $\mathbb{P} : X \to T$.

3) Let v_1, v_2 be end points of e. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ are (K,ϵ)-quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ will occasionally be referred to as f_{e,v_1} and f_{e,v_2} respectively.

Let $\delta \geq 0$. A tree of spaces as in Definition 1.3.1 above is said to be a tree of δ-hyperbolic metric spaces, if X_v, X_e are all δ-hyperbolic for all vertices v and edges e of T.

• Define $\phi_{v,e}\colon f_{e,v_-}(X_e) \to f_{e,v}(X_e)$ as follows:
If $p \in f_{e,v_-}(X_e) \subset X_{v_-}$, choose $x \in X_e$ such that $p = f_{e,v_-}(x)$ and define $\phi_{v,e}(p) = f_{e,v}(x)$.

Note that in the above definition, x is chosen from a set of bounded diameter.

Since $f_{e,v_-}|_{X_e}$ and $f_{e,v}|_{X_e}$ are quasi-isometric embeddings into their respective vertex spaces $\phi_{v,e}$'s are uniform quasi-isometries for all vertices.

Now, let S be a hyperbolic once punctured surface with finite volume. $\pi_1(S)$ acts properly discontinuously on $\mathbb{H}^2$ and $\widetilde{S} = \mathbb{H}^2$. Let N denote S minus cusps and $\mathcal{B}$ be the collection of horodisks in $\mathbb{H}^2$ such that each element B of $\mathcal{B}$ projects down to the cusp under the quotient map $q : \mathbb{H}^2 \to \mathbb{H}^2/\pi_1(S)$, then $\widetilde{N}$ is equal to $\mathbb{H}^2$ minus horodisks in $\mathcal{B}$. $\pi_1(S)$ acts properly discontinuously and cocompactly on $\widetilde{N}$, therefore $\widetilde{N}$ is quasi-isometric to the Cayley graph of $\pi_1(S)$. Let $\phi : S \to S$ be an orientation preserving homeomorphism fixing the puncture. Then $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ preserves corresponding horodisks. Therefore $\widetilde{\phi}$ induces a (K,ϵ) quasi-isometry $\widetilde{\Phi} : \widetilde{N} \to \widetilde{N}$. Let $N = \widetilde{N}/\pi_1(S)$ and $N_\phi = \frac{N\times[0,1]}{\{(x,0),(\phi(x),1):x\in N\}}$. Then $\widetilde{N}_\phi$ can be treated as a tree of spaces with vertex spaces and edge spaces homeomorphic to $\widetilde{N}$.

Definition 1.3.2. *Let $\delta \geq 0, \nu \geq 1, \widehat{K} \geq 1, \widehat{\epsilon} \geq 0$ and X be a geodesic space. A tree $\mathbb{P} : X \to T$ of geodesic metric spaces is said to be a tree of δ-relatively hyperbolic metric spaces if in addition to above three conditions of Definition 1.3.1, we have the following conditions:*
4) for each vertex space X_v (resp. edge space X_e) there exists a collection $\mathcal{H}_v$ (resp. $\mathcal{H}_e$) of uniformly ν-separated, intrinsically geodesic and uniformly properly embedded closed subsets of X_v (resp. X_e) such that X_v (resp. X_e) is δ-hyperbolic relative to the collection $\mathcal{H}_v$ (resp. $\mathcal{H}_e$) in the sense of Gromov,
5) the maps f_{e,v_i} above $(i = 1,2)$ are **strictly type-preserving**, *i.e. $f_{e,v_i}^{-1}(H_{\alpha v_i})$, $i = 1,2$ (for any $H_{\alpha v_i} \in \mathcal{H}_{v_i}$) is either empty or some $H_{\beta e} \in \mathcal{H}_e$. Also, for all $H_{\beta e} \in \mathcal{H}_e$, there exists v and $H_{\alpha v}$, such that $f_{e,v}(H_{\beta e}) \subset H_{\alpha v}$, and*
6) the induced maps (see below) of the coned-off edge spaces into the coned-off vertex spaces $\widehat{f_{e,v_i}} : \mathcal{E}(X_e, \mathcal{H}_e) \to \mathcal{E}(X_{v_i}, \mathcal{H}_{v_i})$ $(i = 1,2)$ are uniform $(\widehat{K}, \widehat{\epsilon})$-quasi-isometric embeddings. This is called the **qi-preserving electrocution condition**

We shall denote $\mathcal{E}(X_v, \mathcal{H}_v) = \widehat{X_v}$ and $\mathcal{E}(X_e, \mathcal{H}_e) = \widehat{X_e}$.
Given the tree of relatively hyperbolic spaces with vertex spaces X_v and edge spaces X_e there exists a naturally associated tree of spaces whose vertex spaces are $\mathcal{E}(X_v, \mathcal{H}_v)$ and edge spaces are $\mathcal{E}(X_e, \mathcal{H}_e)$ obtained by simply coning off the respective horosphere-like sets. Condition (5) of the above definition ensures that we have natural inclusion maps of edge spaces $\mathcal{E}(X_e, \mathcal{H}_e)$ into adjacent vertex spaces $\mathcal{E}(X_v, \mathcal{H}_v)$. The resulting tree of coned-off spaces $\mathbb{P} : \mathcal{TC}(X) \to T$ will be called the induced tree of *coned-off spaces*. The resulting space will be denoted by $\mathcal{TC}(X)$ when thought of as tree of spaces.
Let $\nu \geq 1$. Note that by condition (4) of the above Definition 1.3.2, as each vertex space X_v (resp. edge space X_e) is δ-hyperbolic relative to the collection $\mathcal{H}_v$ (resp. $\mathcal{H}_e$) of ν-separated sets, therefore by Theorem 1.2.48, there exists $\widehat{\delta}_{1.2.48}$ such that each coned-off space $\mathcal{E}(X_v, \mathcal{H}_v)$ (resp. $\mathcal{E}(X_e, \mathcal{H}_e)$) is $\widehat{\delta}_{1.2.48}$-hyperbolic. By Lemma 1.3.4 (proven below), the spaces $\mathcal{E}(X_v, \mathcal{H}_v)$ are uniformly properly embedded in $\mathcal{TC}(X)$. Thus $\mathbb{P} : \mathcal{TC}(X) \to T$ is a tree of $\widehat{\delta}_{1.2.48}$-hyperbolic metric spaces.

The **cone locus** of $\mathcal{TC}(X)$, (the induced tree of coned-off spaces), is the graph (in fact a forest) whose vertex set $\mathcal{V}$ consists of the cone-points in the vertex set and whose edge-set $\mathcal{E}$ consists of the cone-points in the edge set. The incidence relations are dictated by the incidence relations in T. To see that the cone locus is a forest, note that a single edge space cannot have more than one horosphere-like set mapping to a common horosphere-like set in a vertex-set. Hence there are no induced loops in the cone locus, i.e. it is a forest.

Note that connected components of the cone-locus can be naturally identified with sub-trees of T. Each such connected component of the cone-locus will be called a **maximal cone-subtree**. The collection of *maximal cone-subtrees* will be denoted by $\mathcal{T}$ and elements of $\mathcal{T}$ will be denoted as T_α. Further, each maximal cone-subtree T_α naturally gives rise to a tree T_α of horosphere- like subsets Θ_α (depending on which cone-points arise as vertices and edges of T_α) as follows:
Let $x_v \in \mathcal{V}(T_\alpha)$, then x_v is a cone point over a unique horosphere-like set $H_{\alpha v}$ for some vertex space X_v and similarly for an edge $e = [w_1, w_2] \in \mathcal{E}(T_\alpha)$ there exists a unique horosphere-like set $H_{\alpha e}$ in some edge space X_e such that $f_{e,w_1}(H_{\alpha e} \times \{0\}) = H_{\alpha w_1}$ and $f_{e,w_2}(H_{\alpha e} \times \{1\}) = H_{\alpha w_2}$. Define

$$\Theta_\alpha := (\cup_{x_v \in \mathcal{V}(T_\alpha)} H_{\alpha v}) \bigcup (\cup_{e \in \mathcal{E}(T_\alpha)} f_e(H_{\alpha e} \times (0,1))).$$

Θ_α will be referred to as a **maximal cone-subtree of horosphere-like spaces**. $g_\alpha := \mathbb{P}|_{\Theta_\alpha} : \Theta_\alpha \to T_\alpha$ will denote the induced tree of horosphere-like sets. $\mathcal{G}$ will denote the collection of these maps. The collection of Θ_α's will be denoted as $\mathcal{C}$.

Lemma 1.3.3. *There exists $\zeta > 0$ such that $\mathcal{C}$ is uniformly ζ-separated.*

Proof. Let $\Theta_\alpha, \Theta_\beta$ be two distinct elements of $\mathcal{C}$. Let $x \in \Theta_\alpha$ and $y \in \Theta_\beta$. If x and y lie in different vertex spaces, then $d_X(x, y) \geq 1$.
First, let there exist two distinct edges e_1, e_2 incident on a vertex v such that $\mathbb{P}(x) \in e_1, \mathbb{P}(y) \in e_2$. Since X_v is properly embedded in X, there exists a non-negative function $\epsilon(n) \to 0$ as $n \to \infty$ such that the following holds: if $d_X(x, y) < \frac{1}{n}$, then there exist $x_v \in \Theta_\alpha \cap X_v,\ y_v \in \Theta_\beta \cap X_v$ such that $d_{X_v}(x_v, y_v) < \epsilon(n)$.
But horosphere-like sets in X_v are uniformly ν-separated, where $\nu \geq 1$, therefore there exists $\zeta_1 > 0$ such that $d_X(\Theta_\alpha, \Theta_\beta) \geq \zeta_1$.
Now, let $\mathbb{P}(x)$ and $\mathbb{P}(y)$ lie on a same edge e (resp. vertex v) of T. Similarly as above, since X_e (resp. X_v) is properly embedded in X and horosphere-like sets in X_e (resp. X_v) are uniformly ν-separated, therefore there exists $\zeta_2 > 0$ such that $d_X(\Theta_\alpha, \Theta_\beta) \geq \zeta_2$.
Let $\zeta = min\{1, \zeta_1, \zeta_2\}$, then $\mathcal{C}$ is uniformly ζ-separated.

□

Consider the partially electrocuted space $\mathcal{PE}(X, \mathcal{C}, \mathcal{G}, \mathcal{T})$. Recall that it was denoted by X_{pel}.
We define $I_{pel} : \mathcal{PE}(X, \mathcal{C}, \mathcal{G}, \mathcal{T}) \to \mathcal{TC}(X)$ as follows:
Let $x \in \mathcal{PE}(X, \mathcal{C}, \mathcal{G}, \mathcal{T})$.
i) If $x \in X$, define $I_{pel}(x) = x$.
ii) Let $x \in \Theta_\alpha \times (0, 1]$, then $x = (\theta_\alpha, t)$ for some $\theta_\alpha \in \Theta_\alpha$ and $t \in (0, 1]$,
a) if $\theta_\alpha \in H_{\alpha w}$ for some vertex w and $E_{H_{\alpha w}} : [0, \frac{1}{2}] \to \widehat{X_w}$ is the edge of length $\frac{1}{2}$ joining θ_α and the cone point $v(H_{\alpha w})$ over $H_{\alpha w}$, define $I_{pel}(x) = E_{H_{\alpha w}}(\frac{t}{2})$.
b) if $\theta_\alpha \in f_e(H_{\alpha e} \times \{s\})$ for some edge $e, s \in (0, 1)$; and $E_{f_e(H_{\alpha e} \times \{s\})} : [0, \frac{1}{2}] \to \widehat{f_e}(\widehat{X_e} \times \{s\})$ is the edge of length $\frac{1}{2}$ joining θ_α and the cone point $v(f_e(H_{\alpha e} \times \{s\}))$ over $f_e(H_{\alpha e} \times \{s\})$, define $I_{pel}(x) = E_{f_e(H_{\alpha e} \times \{s\})}(\frac{t}{2})$.
Note that I_{pel} is a bijection. Define $d_{\mathcal{TC}(X)}(x, y) := d_{X_{pel}}(I_{pel}^{-1}(x), I_{pel}^{-1}(y))$.
As $d_{X_{pel}}$ is a metric, therefore $d_{\mathcal{TC}(X)}(x, y)$ is a metric.

Lemma 1.3.4. *Let $\delta \geq 0$ and $\nu \geq 1$. Let $\mathbb{P} : X \to T$ be a tree of δ-relatively hyperbolic metric spaces such that the collections $\mathcal{H}_v, \mathcal{H}_e$ are uniformly ν-separated for each vertex v and each edge e. The induced maps $\widehat{i_v} \colon \widehat{X}_v \to \mathcal{TC}(X)$ are uniformly proper embeddings, that is, for all $M > 0$, $v \in T$ and $x, y \in \widehat{X}_v$, there exists $N > 0$ such that $d_{\mathcal{TC}(X)}(\widehat{i_v}((x), \widehat{i_v}(y)) \leq M$ implies $d_{\widehat{X}_v}(x, y) \leq N$.*

Proof. Since the natural inclusion $i_v \colon (X_v, \mathcal{H}_v) \to (X, \mathcal{C})$ takes a horosphere-like set $H_{\alpha v}$ to a horosphere-like set Θ_α and the image of no two horosphere-like sets in X_v lie in the same horosphere-like set Θ_α, i_v will induce an embedding $\widehat{i_v} \colon \widehat{X}_v \to \mathcal{TC}(X)$.

Let $x, y \in \widehat{X}_v$ such that $d_{\mathcal{TC}(X)}(\widehat{i}_v((x), \widehat{i}_v(y)) \leq M$, then there exists a path $\widehat{\lambda}$ in $\mathcal{TC}(X)$ joining $\widehat{i}_v(x)$ and $\widehat{i}_v(y)$ such that

• $\widehat{\lambda}$ concatenation of geodesics in X and geodesics in the sets of the form $\Theta \times [0,1]$, and

• $l_{\mathcal{TC}(X)} \leq M + 1$.

Since horosphere-like sets are uniformly ν-separated, $\widehat{\lambda}$ can intersects only finitely, say k, many horosphere-like sets of X, where k depends only upon M, ν. Also $\widehat{\lambda}$ intersects only finitely many, say n, vertex spaces of $\mathcal{TC}(X)$, where n depends only upon M. For $T' \subset T$, let $\mathcal{V}(T')$ denote the set of vertices of T in T'. Let $\mathcal{V}(\mathbb{P}(\widehat{\lambda})) = \{v_1, ..., v_n\}$ and e_i be the edge between v_i and v_{i+1}.

Let λ^b be the portion of $\widehat{\lambda}$ lying outside horosphere-like sets in X. Then $\lambda^b = \lambda_0 \cup \lambda_1 \cup .... \cup \lambda_k$, where λ_0 starts at x, λ_k ends at y and λ_i's are paths in X between horosphere-like sets Θ_i and Θ_{i+1} with length of λ_i at most M. As i_v's are uniformly proper embeddings, without loss of generality, we can assume each λ_i to be of the form $\lambda_{i1} \cup .... \cup \lambda_{ir_i}$, where each λ_{ij} is either a geodesic in some vertex space or of the form $[p, \phi_{v,e}(p)]$.

Let β_n be the union of those λ_{ij}'s which lie in the vertex space X_{v_n} and such that the end points of β_n lie in the edge space $f_{e_{n-1},v_n}(X_{e_{n-1}})$. Then the length of β_n in $\mathcal{TC}(X)$ is bounded above by a constant N_1 depending only upon M, k, n. Recall that $\phi_{v_{n-1},e_{n-1}}$ was the (K,ϵ)-quasi-isometry from $f_{e_{n-1},v_{n-1}}(X_{e_{n-1}})$ to $f_{e_{n-1},v_n}(X_{e_{n-1}})$. Suppose x_n and y_n are the end points of β_n, then x_n, y_n are also the end points of two paths λ_{lm}, $\lambda_{l'm'}$ respectively such that λ_{lm} and $\lambda_{l'm'}$ are of the form $[p, \phi^{-1}_{v_{n-1},e_{n-1}}(p)]$ for some $p \in f_{e_{n-1},v_n}(X_{e_{n-1}})$. Since $\widehat{f}_{e_{n-1},v_n}(\widehat{X}_{e_{n-1}})$ is quasiconvex in $\widehat{X}_{v_n}$, without loss of generality, we can take β_n in the edge space $f_{e_{n-1},v_n}(X_{e_{n-1}})$. Since length of β_n is bounded, therefore there exists a constant $N_2 > 0$ such that $d_{\widehat{X}_{v_{n-1}}}(\phi^{-1}_{v_{n-1},e_{n-1}}(x_n), \phi^{-1}_{v_{n-1},e_{n-1}}(y_n)) \leq N_2$.

Proceeding in this way, in going down from v_n to v, we get a number $N > 0$ such that $d_{\widehat{X}_v}(x, y) \leq N$. □

Thus a tree of relatively hyperbolic spaces $\mathbb{P} : X \to T$ induces a tree of coned-off spaces $\mathbb{P} : \mathcal{TC}(X) \to T$ satisfying the quasi-isometrically embedded condition.

The following corollary is a consequence of Theorem 1.2.77 and Lemma 1.2.78:

Corollary 1.3.5. *Let $\delta \geq 0$ and $\nu \geq 1$ and X be a geodesic space. Let $\mathbb{P} : X \to T$ be a tree of δ-relatively hyperbolic metric spaces such that the collections $\mathcal{H}_v, \mathcal{H}_e$ are uniformly ν-separated for each vertex v and each edge e. If X is δ-hyperbolic relative to the collection $\mathcal{C}$ in the sense of Gromov, then the followings hold:*

1) There exists $\Delta^{pel}_{1.3.5} \geq 0$ such that $\mathcal{TC}(X)$ is a $\Delta^{pel}_{1.3.5}$-hyperbolic metric space.

2) Let $\beta_p : [a,b] \to \mathcal{TC}(X)$ *and* $\widehat{\lambda} : [c,d] \to \widehat{X}$ *(resp.* $\lambda^h : [c,d] \to X^h$*) denote respectively a* (K_p, ϵ_p)*-quasigeodesic path in* $(\mathcal{TC}(X), d_{\mathcal{TC}(X)})$ *and a geodesic in* $(\widehat{X}, d_{\widehat{X}})$ *(resp.* (X^h, d_{X^h}) *) joining* p, q*. Then there exists* $Q_{1.3.5} > 0$ *such that for* $x \in \widehat{\lambda} \cap X$ *(resp.* $x \in \lambda^h \cap X$*) there exists* $y \in \beta_p \cap X$ *such that* $d_X(x,y) \leq Q_{1.3.5}$.

Chapter 2

Relatively Hyperbolic Extensions of Groups

Let us consider the short exact sequence of finitely generated groups

$$1 \to K \to G \to Q \to 1$$

such that K is non-elementary word hyperbolic. In [Mos96], Mosher proved that if G is hyperbolic, then Q is hyperbolic. To prove Q hyperbolic, Mosher (in [Mos96]) constructed a quasi-isometric section from Q to G, that is, a map $s\colon Q \to G$ satisfying

$$\frac{1}{k} d_Q(q, q') - \epsilon \leq d_G(s(q), s(q')) \leq k d_Q(q, q') + \epsilon,$$

for all $q, q' \in Q$, where d_G and d_Q are word metrics and $k \geq 1$, $\epsilon \geq 0$ are constants. In [Mit98a], existence of Cannon-Thurston map for the embedding $i\colon \Gamma_K \to \Gamma_G$ was proved, where Γ_K, Γ_G are the Cayley graphs of K, G respectively. Here in this chapter, we will generalize these results to the case where the kernel is strongly hyperbolic relative to a cusp subgroup. This is motivated by the following example: Let S be a once-punctured torus then its fundamental group $\pi_1(S) = \mathbb{F}(a, b)$ is strongly hyperbolic relative to the peripheral subgroup $H =< aba^{-1}b^{-1} >$. Let M be a 3-manifold fibering over the circle with fiber S such that the fundamental group $\pi_1(M)$ is strongly hyperbolic relative to the subgroup $H \bigoplus \mathbb{Z}$. Then we have a short exact sequence of pairs of finitely generated groups:

$$1 \to (\pi_1(S), H) \to (\pi_1(M), H \bigoplus \mathbb{Z}) \to (\mathbb{Z}, \mathbb{Z}) \to 1.$$

2.1 Quasi-isometric Section

Definition 2.1.1. *Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of finitely generated groups with K strongly hyperbolic relative to K_1. We say that G* ***preserves cusps*** *if for all $g \in G$ there exists $a_g \in K$ such that $gK_1g^{-1} = a_gK_1{a_g}^{-1}$.*

Definition 2.1.2. *(Mosher [Mos96])* **Quasi-isometric section :** *Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of finitely generated groups. A map $s\colon Q \to G$ is said to be a (R, ϵ)-quasi-isometric section if*

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_G(s(q), s(q')) \leq R d_Q(q, q') + \epsilon,$$

for all $q, q' \in Q$, where d_G and d_Q are word metrics and $R \geq 1$, $\epsilon \geq 0$ are constants.

Let K be a group strongly hyperbolic relative to a cusp subgroup K_1. For each parabolic end point $\alpha \in \partial\Gamma(K, K_1)$, there is a subgroup of the form aK_1a^{-1}. Now, Hausdorff distance between the two sets aK_1 and aK_1a^{-1} is uniformly bounded by the length of the word a. Hence α corresponds to a left coset aK_1 of K_1 in K.

Let $1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be a short exact sequence of finitely generated groups with K strongly hyperbolic relative to a subgroup K_1.

We use the following notation:

- Let Π be the set of all parabolic end points for the relatively hyperbolic group K with cusp subgroup K_1.
- Let $\Pi^2 = \{(\alpha_1, \alpha_2) : \alpha_1 \text{ and } \alpha_2 \text{ are distinct elements in } \Pi\}$.
- For $a \in K$, let $i_a\colon K \to K$ denote the inner automorphism $i_a(k) = aka^{-1}$ and $L_a\colon K \to K$ denote the left translation induced by a.
- For $g \in G$, let $I_g\colon K \to K$ be the automorphism $I_g(k) = gkg^{-1}$ and $L_g\colon G \to G$ be the left translation.

Lemma 2.1.3. *For $g \in G$, I_g will induce a quasi-isometry $\hat{I}_g : \widehat{\Gamma}_K \to \widehat{\Gamma}_K$.*

Proof. Since G preserves cusps, there exists $a_g \in K$ such that for all $a \in K$ we have $gaK_1g^{-1} = gag^{-1}a_gK_1a_g^{-1}$.
Define

$$\begin{aligned} \hat{I}_g(k) &= I_g(k), \text{ for } k \in K \\ \hat{I}_g(v(aK_1)) &= v(gag^{-1}a_gK_1), \text{ for cone points } v(aK_1) \text{ over the left cosets } aK_1. \end{aligned}$$

To prove $\hat{I}_g$ quasi-isometry it suffices to show that if $\hat{\lambda}$ is an electric geodesic between $v(K_1)$ and $v(aK_1)$ and $\hat{\lambda}$ does not penetrate any other left cosets then length of $\hat{I}_g(\hat{\lambda})$ is bounded by some constant. First note that

$$d(gK_1g^{-1}, gaK_1g^{-1}) \leq 2l(g) + d(K_1, aK_1).$$

Therefore

$$l(\hat{I}_g(\hat{\lambda})) \leq 2l(g) + l(\hat{\lambda}) \leq 2l(g) + 1.$$

□

Also note that since a left translation preserves distance between two left cosets, L_k will induce an isometry in the coned-off space.

Lemma 2.1.4. *Let G be a finitely generated group hyperbolic relative to a subgroup H. For $k \in G$, the left translation L_k and the inner automorphism i_k induce same map on the relative hyperbolic boundary.*

Proof. Let $X = \Gamma_G$ and X^h be the hyperbolic space obtained by gluing hyperbolic cones. Let $\alpha \in \partial X^h$, then there exists a sequence $\{a_n\} \subset X^h$ such that $a_n \to \alpha$. Now if $a_n \in \Gamma_G$ for some n, then $d_X(L_k(a_n), i_k(a_n)) = d_X(ka_n, ka_nk^{-1}) = d_G(1, k^{-1})$. If a_n lies in some hyperbolic cone, then $a_n = (k_n, t_n)$ for some $k_n \in \Gamma_G$ and $t_n \in [0, \infty)$ and $d_{X^h}(L_k(k_n, t_n), i_k(k_n, t_n)) = d_{X^h}((kk_n, t_n), (kk_nk^{-1}, t_n)) \leq e^{-t_n} d_X(kk_n, kk_nk^{-1}) \leq d_X(1, k^{-1})$. Therefore Hausdorff distance between two sequences $\{L_k(a_n)\}$ and $\{i_k(a_n)\}$ is bounded and hence $L_k(\alpha) = i_k(\alpha)$. □

G preserves cusps, so for each $g \in G$ there exists $a_g \in K$ such that $a_g^{-1}g \in N_G(K_1)$. If $b \in K$, then $d_K(a_gK_1, gbg^{-1}a_gK_1) \leq d_K(K_1, bK_1) + 2l_K(a_g^{-1}g)$. Since $I_g(bK_1) = g(bK_1)g^{-1} = gbg^{-1}a_gK_1a_g^{-1}$ and Hausdorff distance between $gbg^{-1}a_gK_1$ and $gbg^{-1}a_gK_1a_g^{-1}$ is bounded, I_g will induce a map $\tilde{I}_g \colon \Pi \to \Pi$ and $\tilde{I}_g$ is a bijection. Therefore, $\tilde{I}_g$ will induce a bijective map $\tilde{I}_g^2 \colon \Pi^2 \to \Pi^2$. For convenience of notation we will use I_g for $\tilde{I}_g$ and $\tilde{I}_g^2$. Similarly, for $a \in K$, i_a and L_a will induce bijective maps (with same notation) from Π to Π and Π^2 to Π^2.
Recall that for a relatively hyperbolic group (G, H), G and H are assumed to be infinite.

Lemma 2.1.5. *[Far98] Let G be a finitely generated group hyperbolic relative to a subgroup H. Then $gHg^{-1} \cap H$ is finite for all $g \in G \setminus H$.*

Proof. Let $g \in G \setminus H$. If $gHg^{-1} \cap H$ is infinite then there exists an infinite sequence $\{h_n\} \subset H$ such that $g^{-1}h_ng \in H$ and length of h_n is strictly increasing. Let λ be a relative geodesic joining 1 (the identity element) and g, then $h_n\lambda$ is a relative geodesic joining h_n and $h_n g$. This contradicts the bounded coset penetration property 2, as $h_n g \in gH$ and length of h_n is strictly increasing. $\square$

For a relatively hyperbolic group (G, H), if $gHg^{-1} = H$ for some $g \in G$, then $gHg^{-1} \cap H$ is infinite. Therefore by above Lemma 2.1.5, g must belong to H. Thus $N_G(H) = H$.

Theorem 2.1.6. *Suppose we have a short exact sequence of finitely generated groups*

$$1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1,$$

such that K is strongly hyperbolic relative to a non-trivial proper subgroup K_1 and G preserves cusps, then there exists a (R, ϵ)-quasi-isometric section $s\colon Q \to G$ for some $R \geq 1$, $\epsilon \geq 0$.

Proof. First note that if $aK_1a^{-1} = a'K_1a'^{-1}$ for some $a, a' \in K$ then, due to Lemma 2.1.5, $aK_1 = a'K_1$.
Let $\alpha = (\alpha_1, \alpha_2) \in \Pi^2$, then the stabilizer subgroups of α_i's are $a_iK_1a_i^{-1}$ for some $a_i \in K$, where $i = 1, 2$. Let B_α be the set of all exit points from a_1K_1 of relative geodesics which starts from a_1K_1 and end at some point of a_2K_1. Then, due to the bounded coset penetration property 2, B_α is a bounded set with diameter at most D for some $D > 0$.

Let $C = \{\alpha \in \Pi^2 \colon e_K \in B_\alpha\}$, where e_K is the identity element in K. We fix an element $\eta = (\eta_1, \eta_2) \in \Pi^2$. Let $\Sigma = \{g \in G : \eta \in I_g(C)\}$. Σ will be proved to be a set containing the image of a quasi-isometric section.

Step 1 For any $g \in G$, $\cup_{a\in K} I_{ga}(C) = \Pi^2$:
Let $\alpha = (\alpha_1, \alpha_2) \in \Pi^2$. Now α_i corresponds to a left coset a_iK_1, where $i = 1, 2$. Let λ be a relative geodesic in Γ_K with starting at some point of a_1K_1 and ending at some point of a_2K_1 and let x_α be its exit point from a_1K_1, then $x_\alpha \in B_\alpha$. Now there exists $k \in K$ such that $L_k(x_\alpha) = e_K$. Since L_k is an isometry, $L_k(\lambda)$ will be a relative geodesic joining points from ka_1K_1 and ka_2K_1 with e_K being the exit point of $L_k(\lambda)$ from ka_1K_1. There exists $\beta_i \in \Pi$ such that β_i corresponds to the left coset ka_iK_1, $i = 1, 2$. Then $\beta = (\beta_1, \beta_2) \in \Pi^2$ and $e_K \in B_\beta$. Therefore $L_k(\alpha) = \beta \in C$. Since L_k and i_k are same on the relative hyperbolic boundary (from Lemma 2.1.4), we have $i_k(\alpha) \in C$ and thus $\cup_{a\in K}(i_a(C)) = \Pi^2$. Consequently, for any $g \in G$, $\cup_{a\in K} I_{ga}(C) = \cup_{a\in K} I_g i_a(C) = I_g(\cup_{a\in K}(i_a(C))) = I_g(\Pi^2) = \Pi^2$.

Step 2 $p(\Sigma) = Q$:
Let $q \in Q$, then there exists $g \in G$ such that $p(g) = q$. Now $\cup_{a\in K} I_{ga}(C) = \Pi^2$ for any $g \in G$. Therefore for $\eta \in \Pi^2$ there exists $a \in K$ such that $\eta \in I_{ga}(C)$. Hence $ga \in \Sigma$ and $p(ga) = p(g) = q$.

Now we prove that there exist constants $R \geq 1, \epsilon \geq 0$ such that for all $g, g' \in \Sigma$

$$\frac{1}{R} d_Q(p(g), p(g')) - \epsilon \leq d_G(g, g') \leq R d_Q(p(g), p(g')) + \epsilon.$$

We can choose a finite symmetric generating set S of G such that $p(S)$ is also a generating set for Q. Obviously, $d_Q(p(g), p(g')) \leq d_G(g, g')$ for all $g, g' \in G$. To prove $d_G(g, g') \leq R d_Q(p(g), p(g')) + \epsilon$ for all $g, g' \in \Sigma$, it suffices to prove that there exists $R \geq 1$ such that if $d_Q(p(g), p(g')) \leq 1$ for some $g, g' \in \Sigma$, then $d_G(g, g') \leq R$.

Let $d_Q(p(g), p(g')) \leq 1$ for some $g, g' \in \Sigma$. Then $g^{-1}g' = ka$ for some $k \in K$ and a is either the identity in G or a generator of G.
Since $g, g' \in \Sigma$, $I_g(C) \cap I_{g'}(C) \neq \emptyset$. Hence $I_{ka}(C) \cap C = I_{g^{-1}g'}(C) \cap C \neq \emptyset$. Now $I_{ka} = i_k(I_a)$, therefore $i_k(I_a(C)) \cap C \neq \emptyset$.

For each $\alpha \in \Pi^2$, we choose an element $a_\alpha \in B_\alpha$. Define a map $F\colon \Pi^2 \to \Gamma_K$ by $F(\alpha) = a_\alpha$.

Since L_k is an isometry, for $k \in K$, $ka_\alpha \in B_{k\alpha}$ and hence

$$d_K(a_{k\alpha}, ka_\alpha) = d_K(F(k\alpha), kF(\alpha)) \leq D, \tag{2.1}$$

where $k\alpha$ denotes the image of α under the map $L_k\colon \Pi^2 \to \Pi^2$.

Let $B_D(e_K)$ be the closed D-neighborhood of e_K. Now $F(C)$ is contained in the union of B_α's containing the identity e_K. Therefore $F(C)$ is contained in $B_D(e_K)$. Since G preserves cusps, there exists $s \in K$ such that $F(I_a(C))$ is contained in the union of B_α's containing s and hence $F(I_a(C)) \subset B_D(s)$, where $B_D(s)$ is a closed D-neighborhood of s. From (2.1), Hausdorff distance between two sets $F(kI_a(C))$ and $kF(I_a(C))$ is bounded by D. For a set $A \subset \Gamma_K$, let $N_D(A)$ denotes the closed D-neighborhood of A. Thus

$$F(kI_a(C)) \subset N_D(kF(I_a(C))) = kN_D(F(I_a(C))) \subset kB_{2D}(s).$$

Now K acts properly discontinuously on Γ_K, therefore

$$B_D(e_K) \cap kB_{2D}(s) \neq \emptyset$$

for finitely many k's in K. This implies $F(C) \cap F(kI_a(C)) \neq \emptyset$ for finitely many k's in K. And hence $C \cap L_k(I_a(C)) = C \cap kI_a(C) \neq \emptyset$ for finitely many k's in K. L_k equals i_k on the relative hyperbolic boundary, so $C \cap (I_{ka}(C)) \neq \emptyset$ for finitely many

k's in K. Thus $g^{-1}g' = ka$ for finitely many k's. Since number of generators of G is finite, there exists a constant $R \geq 1$ such that $d_G(g, g') \leq R$.

Now we define $s\colon Q \to G$ as follows:

Let $q \in Q$ and let there exist $g, g' \in \Sigma$ such that $p(g) = p(g') = q$. Then by the above inequality $d_G(g, g') \leq R$. We choose one element $g \in p^{-1}(q) \cap \Sigma$ for each $q \in Q$ and define $s(q) = g$. Then s defines a single valued map satisfying :

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_G(s(q), s(q')) \leq R d_Q(q, q') + \epsilon.$$

for some constants $R \geq 1$, $\epsilon \geq 0$ and for all $q, q' \in Q$. □

Note that, due to bounded coset penetration properties, there exists $S_1 > 0$ such that for a group G hyperbolic relative to $\{H_1, ...H_m\}$, the diameter of the intersection of any two left cosets gH_i and $g'H_j$ is bounded above by S_1. Taking Π to be the set of all parabolic end points corresponding to the subgroups $H_1, ..., H_m$ and mimicking the proof given in Theorem 2.1.6, we have the following corollary:

Corollary 2.1.7. *Suppose we have a short exact sequence of finitely generated groups*

$$1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1,$$

such that K is strongly hyperbolic relative to $\{K_1, ..., K_m\}$ and for each $g \in G$ there exists $a_1, ..., a_m \in K$ such that $gK_1g^{-1} = a_iK_1a_i^{-1}$ for all $i = 1, ..., m$, then there exists a (R, ϵ)-quasi-isometric section $s\colon Q \to G$ for some $R \geq 1$, $\epsilon \geq 0$.

Corollary 2.1.8. *Suppose we have a short exact sequence of finitely generated groups*

$$1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1$$

Let K_1 be a finitely generated subgroup of K such that K is strongly hyperbolic relative to the subgroup K_1 and let $Q_1 = N_G(K_1)/K_1$. If G preserves cusps, then $Q_1 = Q$ and there is a quasi-isometric section $s\colon Q \to N_G(K_1)$ satisfying

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_{N_G(K_1)}(s(q), s(q')) \leq R d_Q(q, q') + \epsilon$$

where $q, q' \in Q$ and $R \geq 1$, $\epsilon \geq 0$ are constants. Further, if G is weakly hyperbolic relative to K_1, then Q is hyperbolic.

Proof. Let $q \in Q$, then there exists $g \in G$ such that $p(g) = q$. Since G preserves cusps, $gK_1g^{-1} = aK_1a^{-1}$ for some $a \in K$. Therefore $a^{-1}g \in N_G(K_1)$ and $q = p(a^{-1}g) \in Q_1$ and thus $Q_1 = Q$.

Let $\Pi^2_{K_1} = \{(\alpha_1,\alpha_2)\in\Pi^2 : \alpha_1$ corresponds to the subgroup $K_1\}$ and $C = \{\alpha \in \Pi^2_{K_1} : e_K \in B_\alpha\}$, where B_α is defined as in above theorem. We fix an element $\eta \in \Pi^2_{K_1}$ and set $\Sigma = \{g \in N_G(K_1) : \eta \in I_g(C)\}$. We choose a finite generating set S of G such that it contains a finite generating set of $K_1, K, N_G(K_1)$ and $p(S)$ is also a generating set of Q. As in the proof of Theorem 2.1.6, by replacing G with $N_G(K_1)$, we get a quasi-isometric section $s\colon Q \to N_G(K_1)$ satisfying :

$$\frac{1}{R}d_Q(q,q') - \epsilon \le d_{N_G(K_1)}(s(q),s(q')) \le Rd_Q(q,q') + \epsilon.$$

for some constants $R \ge 1$, $\epsilon \ge 0$ and for all $q,q' \in Q$.

Since $d_Q(q,q') \le d_G(s(q),s(q')) \le d_{N_G(K_1)}(s(q),s(q'))$, we can take the quasi-isometric section $s\colon Q \to N_G(K_1)$ such that

$$\frac{1}{R}d_Q(q,q') - \epsilon \le d_G(s(q),s(q')) \le Rd_Q(q,q') + \epsilon.$$

Now, let $\mathcal{E}(G,K_1)$ denotes the electrocuted space obtained from Γ_G by coning left cosets of K_1 in G. Since G is weakly hyperbolic with respect to K_1, $\mathcal{E}(G,K_1)$ is hyperbolic. We will prove that Q is hyperbolic.

Let $i : \Gamma_G \to \mathcal{E}(G,K_1)$ denote the inclusion. The quasi-isometric section $s\colon Q \to N_G(K_1)$ will induce a map $q_s : \Gamma_Q \to \Gamma_{N_G(K_1)}$. Let $\widehat{s} = q_s \circ (i|_{\Gamma_{N_G(K_1)}})$. Now for all $q,q' \in \Gamma_Q$, $d_{G^{el}}(\widehat{s}(q),\widehat{s}(q')) \le d_G(s(q),s(q')) \le R\, d_Q(q,q')+\epsilon$, where $d_{G^{el}}$ is the metric on $\mathcal{E}(G,K_1)$. For $q,q' \in Q$, let $\widehat{\alpha}$ be a geodesic in $\mathcal{E}(G,K_1)$ joining $s(q)$ and $s(q')$. Let $\widehat{\alpha}$ penetrates left cosets $g_1K_1,...,g_nK_1$ of K_1 in $N_G(K_1)$. Let x_i be the entry point and y_i be the exit point to g_iK_1 of $\widehat{\alpha}$. Since x_i, y_i lie on the same left coset, $p(x_i) = p(y_i)$. Let $\alpha = \cup_{0\le i\le n}[y_i,x_{i+1}]$, where $y_0 = s(q), x_{n+1} = s(q')$ and $[y_i,x_{i+1}]$ is a geodesic in $\Gamma_{N_G(K_1)}$ joining y_i, x_{i+1}. Note that α may not be a connected path, but $p(\alpha)$ is a connected path in Γ_Q joining q,q'. Therefore $d_Q(q,q') \le l_Q(p(\alpha)) \le \sum_{0\le i\le n} l_{[y_i,x_{i+1}]} \le d_{G^{el}}(s(q),s(q'))$.

Therefore $d_Q(q,q') \le d_{G^{el}}(\widehat{s}(q),\widehat{s}(q'))$. Hence $\widehat{s}$ is a quasi-isometric section from Q to $\mathcal{E}(G,K_1)$. Therefore $\widehat{s}(Q)$ is quasiconvex in $\mathcal{E}(G,K_1)$. Since $\mathcal{E}(G,K_1)$ is hyperbolic, Q is hyperbolic. □

Chapter 3

Cannon-Thurston Maps

3.1 Preliminaries on Cannon-Thurston Maps

For a proper hyperbolic metric space X, the Gromov compactification will be denoted by $\overline{X}$.

Definition 3.1.1. *Let X and Y be proper hyperbolic metric spaces and $i : Y \to X$ be an embedding. A* **Cannon-Thurston map** *$\overline{i}$ from $\overline{Y}$ to $\overline{X}$ is a continuous extension of i to the Gromov compactifications $\overline{X}$ and $\overline{Y}$.*

An embedding $i : Z \to W$ is said to be a proper embedding if for all $P > 0$ there exists $Q > 0$ such that for $x, y \in Z$, $d_W(i(x), i(y)) \leq P$ implies that $d_Z(x, y) \leq Q$. The following lemma, given in [Mit98b], gives a necessary and sufficient condition for the existence of Cannon-Thurston maps.

Lemma 3.1.2. *[Mit98b] Let X and Y be proper hyperbolic metric spaces and i : $Y \to X$ be a proper embedding. A Cannon-Thurston map $\overline{i}$ from $\overline{Y}$ to $\overline{X}$ exists for the proper embedding $i\colon Y \to X$ if and only if there exists a non-negative function $m(n)$ with $m(n) \to \infty$ as $n \to \infty$ such that the following holds:*
Given $y_0 \in Y$, for all geodesic segments λ in Y lying outside an n-ball around $y_0 \in Y$ any geodesic segment in X joining the end points of $i(\lambda)$ lies outside the $m(n)$-ball around $i(y_0) \in X$.

Note that the above statement is also true if geodesics are replaced by stable quasiconvex paths.

Let $\delta \geq 0$. Let X (resp. Y) be δ-hyperbolic relative to the collections $\mathcal{H}_X$ (resp. $\mathcal{H}_Y$) of uniformly ν (≥ 1)-separated subsets of X (resp. Y) in the sense of Gromov. Let $i\colon Y \to X$ be a strictly type-preserving proper embedding, i.e. for $H_Y \in \mathcal{H}_Y$

there exists $H_X \in \mathcal{H}_X$ such that $i(H_Y) \subset H_X$ and images of distinct horospheres-like sets in Y lie in distinct horosphere-like sets in X. As X is uniformly properly embedded in $\mathcal{G}(X, \mathcal{H}_X)$, the proper embedding $i\colon Y \to X$ will induce a proper embedding $i_h\colon \mathcal{G}(Y, \mathcal{H}_Y) \to \mathcal{G}(X, \mathcal{H}_X)$.

Definition 3.1.3. *A Cannon-Thurston map is said to exist for the pair (Y, X) of relatively hyperbolic metric spaces and a strictly type-preserving inclusion $i : Y \to X$ if a Cannon-Thurston map exists for the induced map $i_h\colon \mathcal{G}(Y, \mathcal{H}_Y) \to \mathcal{G}(X, \mathcal{H}_X)$ between the respective hyperbolic cones.*

We now give a criterion for the existence of Cannon-Thurston maps for relatively hyperbolic spaces. Let $Y^h = \mathcal{G}(Y, \mathcal{H}_Y), \widehat{Y} = \mathcal{E}(Y, \mathcal{H}_Y)$, $X^h = \mathcal{G}(X, \mathcal{H}_X), \widehat{X} = \mathcal{E}(X, \mathcal{H}_X)$ and $\widehat{X^h} = \mathcal{G}(X^h, \mathcal{H}^h_X)$. Recall from Theorem 1.2.48 that there exist $\widehat{\delta}_{1.2.48}, \widehat{\delta}'_{1.2.48} \geq 0$ such that $\widehat{X}$ is $\widehat{\delta}_{1.2.48}$-hyperbolic and $\widehat{X^h}$ is $\widehat{\delta}'_{1.2.48}$-hyperbolic. Let $B^h_R(Z) \subset X^h$ denotes the R-neighborhood of Z in (X^h, d_{X^h}).

Lemma 3.1.4. *Let $\delta \geq 0$ and X, Y be proper geodesic spaces. Let X and Y be δ-hyperbolic relative to the collections $\mathcal{H}_X$ and $\mathcal{H}_Y$ respectively in the sense of Gromov. Let $i\colon Y \to X$ be a strictly type-preserving proper embedding. A Cannon-Thurston map for $i\colon Y \to X$ exists if and only if there exists a non-negative function $m(n)$ with $m(n) \to \infty$ as $n \to \infty$ such that the following holds:*
Suppose $y_0 \in Y$, and $\widehat{\lambda}$ in $\widehat{Y}$ is an electric geodesic segment starting and ending outside horospheres. If $\lambda^b = \widehat{\lambda} \setminus \bigcup_{K \in \mathcal{H}_Y} K$ lies outside an $B_n(y_0) = \{y \in Y : d_Y(y, y_0) \leq n\}$, then for any electric geodesic $\widehat{\beta}$ joining the end points of $\widehat{i}(\widehat{\lambda})$ in $\widehat{X}$, $\beta^b = \widehat{\beta} \setminus \bigcup_{H \in \mathcal{H}_X} H$ lies outside $B_{m(n)}(i(y_0)) = \{x \in X : d_X(x, i(y_0)) \leq m(n)\}$.

Proof. Let λ be the electroambient representative of $\widehat{\lambda}$. Since λ^b lies outside the ball $B_n(y_0)$, there exists $n_1(n) > 0$ such that a geodesic λ^h in Y^h joining end points of $\widehat{\lambda}$ lies outside the ball $B^h_{n_1}(y_0)$ in Y^h. By Lemma 3.1.2, we note that a Cannon-Thurston map exists for the pair (X^h, Y^h) if and only if there exists a non-negative function $m_1(n_1)$ with $m_1(n_1) \to \infty$ as $n_1 \to \infty$ such that the following holds:
If a geodesic $\lambda^h \subset Y^h$ lies outside $B^h_{n_1}(y_0) \subset Y^h$, then any geodesic $\beta^h \subset X^h$ joining the end-points of $i(\lambda^h)$ lies outside $B^h_{m_1(n_1)}(i(y_0)) \subset X^h$.
Let β be an electroambient representative of $\widehat{\beta}$, then β is a stable quasiconvex path. Since β^h lies outside $B^h_{m_1(n_1)}(i(y_0))$, there exists $m_2 > 0$ such that β lie outside $B^h_{m_2}(i(y_0))$. X is properly embedded in X^h, therefore there exists $m(m_2) > 0$ such that β^b lie outside $B_m(i(y_0))$. □

3.2 Cannon-Thurston Maps for Trees of Relatively Hyperbolic Spaces

Throughout this section, we will assume that trees of relatively hyperbolic spaces are as in Definition 1.3.1 and horosphere-like sets are uniformly ν-separated, where $\nu \geq 1$. In view of Lemma 1.1.9, all metric spaces in this section are assumed to be connected graphs of edge length one.

3.2.1 Construction of Hyperbolic Ladder

Let $\mathbb{P} : X \to T$ be a tree of δ-relatively hyperbolic metric spaces. Given a geodesic segment $\widehat{\lambda} \subset \widehat{X}_{v_0}$ with end points lying outside horospheres-like sets, we now construct a quasiconvex set $\Xi_{\widehat{\lambda}} \subset \widehat{X}$ containing $\widehat{\lambda}$.

The (K, ϵ)-quasi-isometric embedding $f_{e,v} \colon X_e \to X_v$ will induce a map $f^h_{e,v} \colon X^h_e \to X^h_v$ in the following way:
Let $x \in X^h_e$. If $x \in X_e$, then define $f^h_{e,v}(x) = f_{e,v}(x)$. If x lies in the hyperbolic cone H^h_e of the edge space X_e, then $x = (x_e, t)$ for some $x_e \in H_e, t \in [0, \infty)$. Define $f^h_{e,v}((x_e, t)) = (f_{e,v}(x_e), t)$.
Recall from Definition 1.3.2 that $\widehat{f_{e,v}} : \widehat{X_e} \to \widehat{X_v}$ are $(\widehat{K}, \widehat{\epsilon})$-quasi-isometric embeddings.

Lemma 3.2.1. *For $f^h_{e,v}$ defined above, there exist $C_{3.2.1} \geq 0, K^h_{3.2.1} \geq 1, \epsilon^h_{3.2.1} \geq 0$ depending on $\delta, K, \epsilon, \widehat{K}, \widehat{\epsilon}$ such that $f^h_{e,v}(X^h_e)$ is a $C_{3.2.1}$-quasiconvex set and $f^h_{e,v}$ is a $(K^h_{3.2.1}, \epsilon^h_{3.2.1})$-quasi-isometric embedding.*

Proof. Since $\widehat{f_{e,v}} : \widehat{X_e} \to \widehat{X_v}$ is a quasi-isometric embedding, therefore $\widehat{f_{e,v}}(\widehat{X_e})$ will be a quasiconvex subset of $\widehat{X_v}$ (as $\widehat{X_v}$ is a hyperbolic space). Let $x, y \in f_{e,v}(X_e)$, γ be a geodesic in X^h_v joining x and y and $\widehat{\lambda}$ be a geodesic in $\widehat{X}_v$ joining x and y. Suppose $Pr_v : \widehat{X}_v \to \widehat{f_{e,v}}(\widehat{X_e})$ is a nearest point projection, then by Lemma 1.1.34, $Pr_v(\widehat{\lambda})$ is a quasigeodesic in $\widehat{f_{e,v}}(\widehat{X_e})$ joining x and y. Thus $\widehat{\lambda}$ and $Pr_v(\widehat{\lambda})$ are quasigeodesics in $\widehat{X}_v$ joining same pair of points . Note that $Pr_v(\widehat{\lambda})$ may not be a continuous path, but in view of Lemma 1.1.23, we can assume $Pr_v(\widehat{\lambda})$ to be a continuous quasigeodesic path. Also, we can modify $Pr_v(\widehat{\lambda})$ to a quasigeodesic path such that it does not backtrack, so we assume $Pr_v(\widehat{\lambda})$ does not backtrack. Since $\widehat{f_{e,v}}$ is a quasi-isometric embedding, we can assume $Pr_v(\widehat{\lambda}) \subset \widehat{f_{e,v}}(\widehat{X_e})$. Due to similar intersection patterns, electroambient representatives $\lambda, Pr_v(\lambda)$ of $\widehat{\lambda}_v, Pr_v(\widehat{\lambda})$, respectively, lie in a bounded neighborhood of each other. Also, the Hausdorff distance between λ and γ is bounded. As $Pr_v(\lambda) \subset f^h_{e,v}(X^h_e)$, therefore $f^h_{e,v}(X^h_e)$ is $C_{3.2.1}$-quasiconvex for some $C_{3.2.1} > 0$.

Next we prove that the map $f^h_{e,v}$ is proper, i.e., for $M > 0$ there exists $N(M) > 0$ such that for $x, y \in X^h_e$ if $d_{X^h_v}(f^h_{e,v}(x), f^h_{e,v}(y)) \leq M$ then $d_{X^h_e}(x, y) \leq N$.
Let γ be a geodesic in X^h_v joining $f^h_{e,v}(x)$ and $f^h_{e,v}(y)$, then its length is at most M. Let $\widehat{\lambda}$ be a geodesic in $\widehat{X^h_v}$ joining $f^h_{e,v}(x)$ and $f^h_{e,v}(y)$, then due to similar intersection patterns of γ and $\widehat{\lambda}$ with hyperbolic cones there exists $N_1(M) > 0$ such that the length of an electroambient representative λ of $\widehat{\lambda}$ is at most N_1. Let $Pr_v(\lambda)$ be an electroambient representative of $Pr_v(\widehat{\lambda})$ in X^h_v, then due to similar intersection patterns of $\widehat{\lambda}$ and $Pr_v(\widehat{\lambda})$ with hyperbolic cones, length of $Pr_v(\lambda)$ is at most N_2 for some $N_2 > 0$. Now as $f_{e,v} : X_e \to X_v$ is a quasi-isometric embedding and $f^h_{e,v}|_{H^h_e} : H^h_e \to f^h_{e,v}(H^h_e)$ is a quasi-isometry (by Lemma 1.2.13), there exists $N(N_2) > 0$ such that $d_{X^h_e}(x, y) \leq N$.
Now we show that $f^h_{e,v}$ is a quasi-isometric embedding:
By the first part of the proof of Lemma 1.2.31, there exists $R \geq 1, \varepsilon \geq 0$ such that $d_{X^h_v}(f^h_{e,v}(x), f^h_{e,v}(y)) \leq R d_{X^h_e}(x, y) + \varepsilon$ for all $x, y \in X^h_e$. Now to prove the other inequality let $x, y \in X^h_e$ and α be a geodesic in X^h_v joining $f^h_{e,v}(x)$ and $f^h_{e,v}(y)$. We partition α by points $a_0, a_1, ..., a_n$ such that $d_{X^h_v}(a_{i-1}, a_i) = 1$ $(0 \leq i \leq n-1)$ and $d_{X^h_v}(a_{n-1}, a_n) \leq 1$ with $a_0 = f^h_{e,v}(x)$ and $a_n = f^h_{e,v}(y)$. Since $f^h_{e,v}(X^h_e)$ is $C_{3.2.1}$-quasiconvex, there exists $b_i \in f^h_{e,v}(X^h_e)$ such that $d_{X^h_v}(a_i, b_i) \leq C_{3.2.1}$. Now for each i, we have $d_{X^h_v}(b_{i-1}, b_i) \leq 2C_{3.2.1} + 1$. There exists $c_i \in X^h_e$ such that $b_i = f^h_{e,v}(c_i)$, then as the map $f^h_{e,v}$ is proper, there exists $R'(C_{3.2.1}) > 0$ such that $d_{X^h_e}(c_{i-1}, c_i) \leq R'$. Therefore, by triangle inequality, we have $d_{X^h_e}(x, y) \leq R' d_{X^h_v}(f^h_{e,v}(x), f^h_{e,v}(y)) + R'$. Taking $K_{3.2.1} = \max\{R, R'\}, \epsilon_{3.2.1} = \max\{R', \varepsilon\}$, we have the required result. □

Let e denote the directed edge from v_- to v.
• Define $\phi^h_{v,e} \colon f^h_{e,v_-}(X^h_e) \to f^h_{e,v}(X^h_e)$ as follows:
If $p \in f^h_{e,v_-}(X^h_e) \subset X^h_{v_-}$, choose $x \in X^h_e$ such that $p = f^h_{e,v_-}(x)$ and define $\phi^h_{v,e}(p) = f^h_{e,v}(x)$.
Note that $\phi^h_{v,e}$ are all uniform quasi-isometries. Let $C_{1.1.30}$ be as in Lemma 1.1.30 and $f^h_{e,v}(X^h_e)$ be $C_{3.2.1}$-quasiconvex. By Lemma 1.2.50, there exists $Q_{1.2.50} > 0$ such that electroambient path representatives of electric geodesics are stable $Q_{1.2.50}$-quasiconvex path, we assume them to be stable $Q_{1.2.50}$-quasiconvex path for some $Q_{1.2.50} > 0$. Let $C = C_{1.1.30} + C_{3.2.1} + Q_{1.2.50}$.

For $Z \subset X^h_v$, let $N_C(Z)$ denote the C-neighborhood of Z in X^h_v, where C is as above.

Hyperbolic Ladder $\Xi_{\widehat{\lambda}}$

Recall that $\mathbb{P} : \mathcal{TC}(X) \to T$ is the usual projection to the base tree.

For convenience of exposition, T shall be assumed to be rooted, i.e. equipped with a base vertex v_0. Let $v \neq v_0$ be a vertex of T. Let v_- be the penultimate vertex on the geodesic edge path from v_0 to v. Let e denote the directed edge from v_- to v.

Recall that we have defined $\phi_{v,e}\colon f_{e,v_-}(X_e) \to f_{e,v}(X_e)$ in the following way:
If $p \in f_{e,v_-}(X_e) \subset X_{v_-}$, choose $x \in X_e$ such that $p = f_{e,v_-}(x)$ and define $\phi_{v,e}(p) = f_{e,v}(x)$.
Since f_{e,v_-} and $f_{e,v}$ are quasi-isometric embeddings into their respective vertex spaces $\phi_{v,e}$'s are uniform quasi-isometries for all vertices. We shall denote $\mathcal{E}(X_v, \mathcal{H}_v) = \widehat{X_v}$ and $\mathcal{E}(X_e, \mathcal{H}_e) = \widehat{X_e}$.

Step 1

Let $\widehat{\mu} \subset \widehat{X_v}$ be a geodesic segment in $(\widehat{X_v}, d_{\widehat{X_v}})$ with starting and ending points lying outside horoballs and μ be the corresponding electroambient path representative in X_v^h (cf Lemma 1.2.50). Then $\mathbb{P}(\widehat{\mu}) = v$. For the collection of edges e' incident on v, but not lying on the geodesic (in T) from v_0 to v, consider the subcollection of edges $\{e\}$ for which $N_C^h(\mu) \cap f_{e,v}(X_e) \neq \emptyset$ and for each such e, choose $p_e, q_e \in N_C^h(\mu) \cap f_{e,v}(X_e)$ such that $d_{X_v^h}(p_e, q_e)$ is maximal. Let $\{e_i\}_{i \in I_v}$ be the further subcollection of $\{e\}$ for which $d_{\widehat{X}_v}(p_{e_i}, q_{e_i}) > D_{1.1.30}$ where $D_{1.1.30}$ is as in Lemma 1.1.30. Let v_i be the terminal vertex of the edge e_i. Let $\widehat{\mu}_{v,e_i}$ be a geodesic in $\widehat{X}_v$ joining p_{e_i} and q_{e_i}. Define

$$\Xi^1(\widehat{\mu}) = i_v(\widehat{\mu}) \cup \bigcup_{i \in I_v} \widehat{\Phi}_{v,e_i}(\widehat{\mu}_{v,e_i})$$

where $\widehat{\Phi}_{v,e_i}(\widehat{\mu}_{v,e_i})$ is an electric geodesic in $\widehat{X}_{v_i}$ joining $\phi_{v,e_i}(p_{e_i})$ and $\phi_{v,e_i}(q_{e_i})$.

Step 2

Step 1 above constructs $\Xi^1(\widehat{\lambda})$ in particular. We proceed inductively on $m \in \mathbb{N}$.
Suppose that $\Xi^m(\widehat{\lambda})$ has been constructed such that the vertices in $\mathbb{P}(\Xi^m(\widehat{\lambda})) \subset T$ are the vertices of a subtree. Let $\{w_i\}_i = \mathbb{P}(\Xi^m(\widehat{\lambda})) \setminus \mathbb{P}(\Xi^{m-1}(\widehat{\lambda}))$.
Assume further that $\mathbb{P}^{-1}(w_k) \cap \Xi^m(\widehat{\lambda})$ is a path of the form $i_{w_k}(\widehat{\lambda}_{w_k})$, where $\widehat{\lambda}_{w_k}$ is a geodesic in $(\widehat{X}_{w_k}, d_{\widehat{X}_{w_k}})$ and note that $\mathbb{P}^{-1}(w_k) = \widehat{X}_{w_k}$.
Define

$$\Xi^{m+1}(\widehat{\lambda}) = \Xi^m(\widehat{\lambda}) \cup \bigcup_k (\Xi^1(\widehat{\lambda}_{w_k}))$$

where $\Xi^1(\widehat{\lambda}_{w_k})$ is defined in step 1 above.
Define

$$\Xi_{\widehat{\lambda}} = \cup_{m \geq 1} \Xi^m(\widehat{\lambda})$$

Observe that the vertices comprising $\mathbb{P}(\Xi_{\widehat{\lambda}})$ in T are the vertices of a subtree, say, T_1.

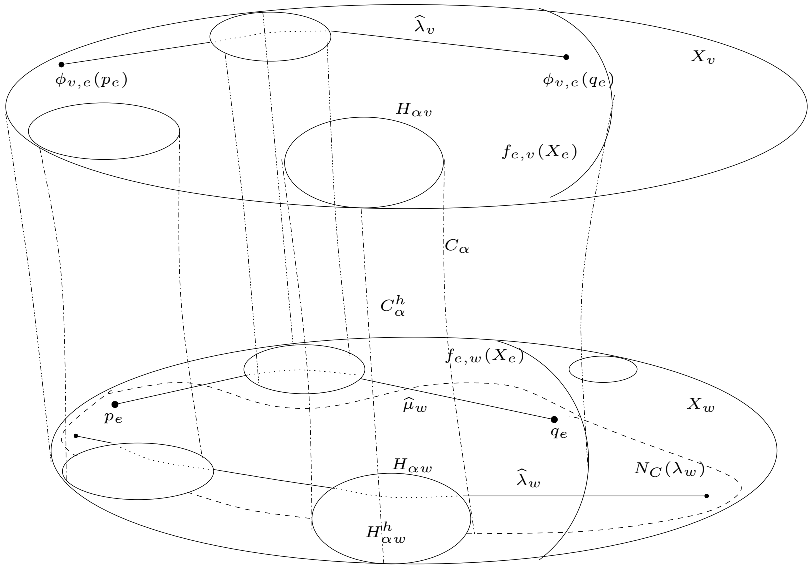

Figure 3.1: Hyperbolic Ladder for an edge e with vertices w and v.

Roughly speaking, what we have done is that at each stage we take a geodesic $\widehat{\lambda}_v$ look at all edge spaces which hit $\widehat{X}_v$ near $\widehat{\lambda}_v$, 'break' $\widehat{\lambda}_v$ into maximal subpieces coarsely contained in the images of these edge spaces and then 'flow' them (via the $[0,1]$ direction in $\widehat{X}_e \times [0,1]$) into adjacent vertex spaces. The maximal subpieces are the $\widehat{\mu}$'s.

3.2.2 Retraction Map

In order to prove $\Xi_{\widehat{\lambda}}$ is quasiconvex in $\mathcal{TC}(X)$, we will construct a *retraction map* $\widehat{\Pi}_{\widehat{\lambda}}$ from $\mathcal{TC}(X)$ to $\Xi_{\widehat{\lambda}}$ which is coarsely Lipschitz. For convenience of exposition, we shall define $\widehat{\Pi}_{\widehat{\lambda}}$ only on the union of vertex spaces of $\mathcal{TC}(X)$.

For a tree T, let $\mathcal{V}(T)$ denote the vertex set of T.

Definition 3.2.2. *(Retraction Map) Let $\widehat{\pi}_{\widehat{\lambda}_v} : \widehat{X}_v \to \widehat{\lambda}_v$ be an electric projection from $\widehat{X}_v$ onto $\widehat{\lambda}_v$ (See Definition 1.2.51).*

Define $\widehat{\Pi}_{\widehat{\lambda}} : \bigcup_{v \in \mathcal{V}(T_1)} \widehat{X}_v \to \Xi_{\widehat{\lambda}}$ by

$$\widehat{\Pi}_{\widehat{\lambda}}(x) = \widehat{i}_v(\widehat{\pi}_{\widehat{\lambda}_v}(x)) \text{ for } x \in \widehat{X}_v.$$

If $x \in \mathbb{P}^{-1}((\mathcal{V}(T) \setminus \mathcal{V}(T_1)))$, choose $x_1 \in \mathbb{P}^{-1}(\mathcal{V}(T_1))$ such that $d_X(x, x_1) = d_X(x, \mathbb{P}^{-1}(\mathcal{V}(T_1)))$ and define $\widehat{\Pi}'_{\widehat{\lambda}}(x) = x_1$, d_X is the metric on X. Next define $\widehat{\Pi}_{\widehat{\lambda}}(x) = \widehat{\Pi}_{\widehat{\lambda}}(\widehat{\Pi}'_{\widehat{\lambda}}(x))$.

The following is the main theorem of this subsection.

Theorem 3.2.3. *There exists* $P_{3.2.3} \geq 0$ *such that*

$$d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) \leq P_{3.2.3} d_{\mathcal{TC}(X)}(x, y) + P_{3.2.3} \text{ for } x, y \in \mathcal{TC}(X).$$

In particular, if $\mathcal{TC}(X)$ *is hyperbolic, then* $\Xi_{\widehat{\lambda}}$ *is uniformly (independent of* $\widehat{\lambda}$*) quasiconvex.*

Recall that $\widehat{\lambda}_v = \widehat{X}_v \cap \Xi_{\widehat{\lambda}}$ and λ_v is the electroambient representative of $\widehat{\lambda}_v$. The following lemma says that points in the corresponding edge space and which are at bounded distance from λ_v's are also at a bounded distance from 'maximal subpieces'. Recall that $f^h_{e,v}(X^h_e)$ are uniformly $C_{3.2.1}$-quasiconvex and $C = C_{1.1.30}+C_{3.2.1}+Q_{1.2.50}$.

Lemma 3.2.4. *([Mit98b]) Let* $\widehat{\mu}_1 \subset \widehat{X}_v$ *be an electric geodesic with end points* a *and* b *lying outside a horosphere-like set. Let* μ_1 *be the corresponding electroambient representative in* X^h_v*. Let* $p, q \in N_C(\mu_1) \cap f^h_{e,v}(X^h_e)$ *be such that* $d_{X^h_v}(p, q)$ *is maximal. Let* $\widehat{\mu}_2$ *be a geodesic in* $\widehat{X}_v$ *joining* p *and* q *and* μ_2 *be its electro-ambient representative. If* $r \in N_C(\mu_1) \cap f^h_{e,v}(X^h_e)$ *then* $d_{X^h_v}(r, \mu_2) \leq P_{3.2.4}$ *for some constant* $P_{3.2.4}$ *depending only on* $C, D_{1.1.30}, \delta$*.*

Proof. Let $[a, b]$ and $[p, q]$ be geodesics in X^h joining a, b and p, q respectively. Then μ_1, $[a, b]$ lie in an $Q_{1.2.50}$-neighborhood of each other and $[p, q]$, μ_2 lie in an $Q_{1.2.50}$-neighborhood of each other. Let π_1 be a nearest point projection from X^h_v onto $[a, b]$. If $\pi_1(r) \in [\pi_1(p), \pi_1(q)] \subset [a, b]$, then there exists $C' > 0$ such that $d_{X^h_v}(r, \mu_2) \leq C + C'$.
Let $\pi_1(r) \notin [\pi_1(p), \pi_1(q)]$ and we assume that $\pi_1(r) \in [a, \pi_1(p)] \subset [a, \pi_1(q)]$. Then we have

$$\begin{aligned} d_{X^h_v}(\pi_1(r), \pi_1(q)) &\leq d_{X^h_v}(r, q) + 2C \\ &\leq d_{X^h_v}(p, q) + 2C, \quad \text{as } d_{X^h_v}(p, q) \text{ is maximal.} \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} d_{X^h_v}(\pi_1(r), \pi_1(q)) &= d_{X^h_v}(\pi_1(r), \pi_1(p)) + d_{X^h_v}(\pi_1(p), \pi_1(q)) \\ &\geq d_{X^h_v}(\pi_1(r), \pi_1(p)) + d_{X^h_v}(p, q) - 2C \end{aligned} \tag{3.2}$$

Thus from equations 3.1 and 3.2, we have $d_{X^h_v}(\pi_1(r), \pi_1(p)) \leq 4C$. This implies $d_{X^h_v}(r, p) \leq 6C$ and therefore $d_{X^h_v}(r, \mu_2) \leq 6C$.
Taking $P_{3.2.4} = \max\{C + C', 6C\}$, we have the required result. ☐

The following lemma says that the images of the edge space onto λ_v's and 'maximal subpieces' under nearest point projections are at bounded Hausdorff distance.

Lemma 3.2.5. *([Mit98b]) Let* $\widehat{\mu}_1$ *and* $\widehat{\mu}_2$ *be as in Lemma 3.2.4. If* $s \in f_{e,v}(X_e)$*, then* $d_{X_v^h}(\pi_{\mu_1}(s), \pi_{\mu_2}(s)) \leq P'_{3.2.5}$ *for some constant* $P'_{3.2.5} > 0$ *depending only on* $\delta, C, D_{1.1.30}$*. Moreover, if* $s \in \widehat{f}_{e,v}(\widehat{X}_e)$*, there exists* $P_{3.2.5}(P'_{3.2.5}) > 0$ *such that* $d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\mu}_1}(s), \widehat{\pi}_{\widehat{\mu}_2}(s)) \leq P_{3.2.5}$*.*

Proof. Recall that $[a, b]$ is a geodesic joining end points of μ_1 and $[p, q]$ is a geodesic joining end points of μ_2. Let π_1, π_2 be nearest point projections from X_v^h onto $[a, b], [p, q]$ respectively. Let $D_{1.1.30}$ be as in Lemma 1.1.30. If $d_{X_v^h}(\pi_1(s), \pi_1\pi_2(s)) \leq D_{1.1.30}$, then $d_{X_v^h}(\pi_1(s), \pi_2(s)) \leq D_{1.1.30} + C$.
Let $d_{X_v^h}(\pi_1(s), \pi_1\pi_2(s)) > D_{1.1.30}$, then by Lemma 1.1.30, $[s, \pi_1(s)] \cup [\pi_1(s), \pi_1\pi_2(s)] \cup [\pi_1\pi_2(s), \pi_2(s)]$ is a quasigeodesic. Since $f_{e,v}^h(X_e^h)$ is $C_{3.2.1}$- quasiconvex, there exists $r \in f_{e,v}^h(X_e^h)$ such that $d_{X_v^h}(r, \pi_1(s)) \leq C_{1.1.30} + C_{3.2.1}$. This implies $d_{X_v^h}(r, \mu_1) \leq C_{1.1.30} + C_{3.2.1} + Q_{1.2.50} = C$ and therefore $r \in N_C(\mu_1) \cap f_{e,v}^h(X_e^h)$. By using Lemma 3.2.4, there exists $r' \in [p, q]$ such that $d(r, r') \leq P_{3.2.4} + Q_{1.2.50}$ and hence $d_{X_v^h}(r', \pi_1(s)) \leq C_{1.1.30} + C_{3.2.1} + P_{3.2.4} + Q_{1.2.50} = P_{3.2.4} + C$.
Since π_2 is a nearest point projection, $(s, r')_{\pi_2(s)} \leq 2\delta$. Therefore

$$(s, \pi_1(s))_{\pi_2(s)} \leq (s, r')_{\pi_2(s)} + d_{X_v^h}(r', \pi_1(s)) \leq 2\delta + C + P_{3.2.4}.$$

Since π_1 is a nearest point projection, $(s, \pi_1\pi_2(s))_{\pi_1(s)} \leq 2\delta$. Thus

$$(s, \pi_2(s))_{\pi_1(s)} \leq (s, \pi_1\pi_2(s))_{\pi_1(s)} + d_{X_v^h}(\pi_1\pi_2(s), \pi_2(s)) \leq 2\delta + C_{1.1.30} + C_{3.2.1}.$$

Now $d_{X_v^h}(\pi_1(s), \pi_2(s)) = (s, \pi_1(s))_{\pi_2(s)} + (s, \pi_2(s))_{\pi_1(s)}$, therefore $d_{X_v^h}(\pi_1(s), \pi_2(s)) \leq 2\delta + C + P_{3.2.4} + 2\delta + C_{1.1.30} + C_{3.2.1} = K(say)$. Thus from Lemma 1.1.35, we have $d_{X_v^h}(\pi_{\mu_1}(s), \pi_{\mu_2}(s)) \leq K + 2L_1$.
Taking $P'_{3.2.5} = \max\{D_{1.1.30} + C, K + 2L_1\}$, we have the required result. □

Suppose $x, y \in \mathcal{TC}(X)$ and $d_{\mathcal{TC}(X)}(x, y) \leq 1$. Since $\widehat{i}_v$'s are uniformly proper embedding from $\widehat{X}_v$ to $\mathcal{TC}(X)$, there exists a constant $M > 0$ such that $d_{\widehat{X}_v}(x, y) \leq M$.
Let $\mathbf{P} = max\{P_{1.1.32}, P_{1.1.36}, P_{1.2.52}, P_{1.2.53}, P_{1.2.55}, P_{3.2.4}, P_{3.2.5}, M\}$.

Proof of theorem 3.2.3

It suffices to prove that if $d_{\mathcal{TC}(X)}(x, y) \leq 1$ then $d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) \leq P_{3.2.3}$.

Let $d_{\mathcal{TC}(X)}(x, y) \leq 1$.

Case 1: Let $x, y \in \mathbb{P}^{-1}(v)$ for some $v \in T_1$. Using Lemma 1.2.53, there exists a constant $K_0(\mathbf{P}) > 0$ such that

$$d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}_v}(x), \widehat{\Pi}_{\widehat{\lambda}_v}(y)) \leq d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\lambda}_v}(x), \widehat{\pi}_{\widehat{\lambda}_v}(y)) \leq K_0.$$

Case 2: Let $x \in \mathbb{P}^{-1}(w)$ and $y \in \mathbb{P}^{-1}(v)$ for some $v, w \in T_1$ such that $v \neq w$. Now v and w are adjacent in T_1 and $x \in f_e(X_w)$ since $d_{\mathcal{TC}(X)}(x,y) \leq 1$. Without loss of generality we can assume that $w = v_-$. Let e be the edge between v and w. Recall that $\Xi_{\widehat{\lambda}} \cap \mathbb{P}^{-1}(v) = \widehat{\lambda}_v$, $\Xi_{\widehat{\lambda}} \cap \mathbb{P}^{-1}(w) = \widehat{\lambda}_w$, $\widehat{\lambda}_v = \widehat{\Phi}_{v,e}(\widehat{\mu}_{w,e})$, where $\widehat{\mu}_{w,e}$ is the geodesic in $\widehat{X}_w$ joining $p_e, q_e \in X_w$, p_e, q_e lie in a C-neighborhood of λ_w, $d_{\widehat{X}_w}(p_e, q_e) > D_{1.1.30}$ and $\widehat{\Phi}_{v,e}(\widehat{\mu}_{w,e})$ is the geodesic in $\widehat{X}_v$ joining $\phi_{v,e}(p_e)$ and $\phi_{v,e}(p_e)$. For simplicity, we denote $\widehat{\mu}_{w,e}$ by $\widehat{\mu}_w$ and the quasi-isometry $\phi_{v,e}$ by ϕ_v.
Step 1: From lemma 3.2.5,

$$d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}_w}(x), \widehat{\Pi}_{\widehat{\mu}_w}(x)) \leq d_{\widehat{X}_w}(\widehat{\pi}_{\widehat{\lambda}_w}(x), \widehat{\pi}_{\widehat{\mu}_w}(x)) \leq \mathbf{P}.$$

Step 2: $\widehat{f}_{e,v}(\widehat{X}_v)$ and $\widehat{f}_{e,w}(\widehat{X}_w)$ are uniformly quasiconvex in $\widehat{X}_v$ and $\widehat{X}_w$ respectively. Let $Pr_w : \widehat{X}_w \to \widehat{f}_{e,w}(\widehat{X}_w)$ be a nearest point projection, then from Corollary 1.1.33, we have Pr_w is coarsely Lipschitz. Therefore, by Lemma 1.1.34, $Pr_w(\widehat{\mu}_w)$ is a quasigeodesic in $\widehat{f}_{e,w}(\widehat{X}_w)$. Let $\widehat{\mu}'_w = Pr_w(\widehat{\mu}_w)$. Using Lemma 1.1.35, we have

$$d_{\widehat{X}_w}(\widehat{\pi}_{\widehat{\mu}_w}(x), \widehat{\pi}_{\widehat{\mu}'_w}(x)) \leq K' \text{ for some constant } K' > 0. \tag{3.3}$$

By using Lemma 1.2.55, there exists a constant $R > 0$ such that

$$d_{\widehat{X}_v}(\widehat{\phi}_v(\widehat{\pi}_{\widehat{\mu}'_w}(x)), \widehat{\pi}_{\widehat{\lambda}_v}(\widehat{\phi}_v(x))) \leq R. \tag{3.4}$$

$$\begin{aligned} d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\mu}_w}(x), \widehat{\Pi}_{\widehat{\lambda}_v}(\widehat{\phi}_v(x))) &= d_{\mathcal{TC}(X)}(\widehat{\pi}_{\widehat{\mu}_w}(x), \widehat{\pi}_{\widehat{\lambda}_v}(\widehat{\phi}_v(x))) \\ &\leq d_{\mathcal{TC}(X)}(\widehat{\pi}_{\widehat{\mu}_w}(x), \widehat{\pi}_{\widehat{\mu}'_w}(x)) + d_{\mathcal{TC}(X)}(\widehat{\pi}_{\widehat{\mu}'_w}(x), \widehat{\phi}_v(\widehat{\pi}_{\widehat{\mu}'_w}(x)) \\ &\quad + d_{\mathcal{TC}(X)}(\widehat{\phi}_v(\widehat{\pi}_{\widehat{\mu}'_w}(x)), \widehat{\pi}_{\widehat{\lambda}_v}(\widehat{\phi}_v(x))) \\ &\leq K' + 1 + R. \end{aligned}$$

Step 3: $d_{\mathcal{TC}(X)}(x,y) = 1 = d_{\mathcal{TC}(X)}(x, \widehat{\phi}_v(x))$. Then $d_{\widehat{X}_v}(\widehat{\phi}_v(x), y) \leq 2M$. Thus using lemma 1.2.53, we have
$d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}_v}(\widehat{\phi}_v(x)), \widehat{\Pi}_{\widehat{\lambda}_v}(y)) \leq d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\lambda}_v}(\widehat{\phi}_v(x)), \widehat{\pi}_{\widehat{\lambda}_v}(y)) \leq 2\mathbf{P}M + \mathbf{P}$.
Thus from above three steps, there exists a constant $K_1(\mathbf{P}) > 0$ such that $d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) \leq K_1$.

Case 3: Let $\mathbb{P}([x,y])$ be not contained in T_1. Then $\mathbb{P}(x)$ and $\mathbb{P}(y)$ belong to the closure of the same component of $T \setminus T_1$. Then $\mathbb{P}(\widehat{\Pi}'_{\widehat{\lambda}}(x)) = \mathbb{P}(\widehat{\Pi}'_{\widehat{\lambda}}(y)) = v$ for some $v \in \mathcal{V}(T_1)$ by the second part of Definition 3.2.2. Let $x_1 = \widehat{\Pi}'_{\widehat{\lambda}}(x)$ and $y_1 = \widehat{\Pi}'_{\widehat{\lambda}}(y)$. Now $x_1, y_1 \in f^h_{e,v}(X^h_e)$ for some edge e with initial vertex v.
If $d_{X^h_v}(\pi_{\lambda_v}(x_1), \pi_{\lambda_v}(y_1)) < D_{1.1.30}$ then

$$d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) \leq d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\lambda}_v}(x), \widehat{\pi}_{\widehat{\lambda}_v}(y)) \leq D_{1.1.30} + 2.$$

Let us assume $d_{X_v^h}(\pi_{\lambda_v}(x_1),\pi_{\lambda_v}(y_1)) > D_{1.1.30}$, then by Lemma 1.1.30 $[x_1,\pi_{\lambda_v}(x_1)]\cup[\pi_{\lambda_v}(x_1),\pi_{\lambda_v}(y_1)]\cup[\pi_{\lambda_v}(y_1),y_1]$ is a quasi-geodesic lying in a $C_{1.1.30}$-neighborhood of $[x_1,y_1]$.
Since $f^h_{e,v}(X^h_e)$ is $C_{3.2.1}$-quasiconvex in X^h_v, there exist $x_2,y_2\in f^h_{e,v}(X^h_e)$ such that $d_{X_v^h}(\pi_{\lambda_v}(x_1),x_2)\le C_{1.1.30}+C_{3.2.1}\le C$ and $d_{X_v^h}(\pi_{\lambda_v}(y_1),y_2)\le C_{1.1.30}+C_{3.2.1}\le C$ and thus $x_2,y_2\in N_C(\lambda_v)\cap f^h_{e,v}(X^h_e)$. Now there exist $x'_2,y'_2\in N_C(\lambda_v)\cap f_{e,v}(X_e)$ such that $d_{\widehat{X_v^h}}(x_2,x'_2)\le 1$ and $d_{\widehat{X_v^h}}(y_2,y'_2)\le 1$. Therefore $d_{\widehat{X}_v}(x'_2,y'_2)\le d_{\widehat{X_v^h}}(x'_2,y'_2)\le 2$.
Let $D_1 > P_{1.2.53}D_{1.1.30}+P_{1.2.53}$. If $D_1 < d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\lambda}_v}(x'_2),\widehat{\pi}_{\widehat{\lambda}_v}(y'_2))$, then by Lemma 1.2.53 $d_{\widehat{X}_v}(x'_2,y'_2) > D_{1.1.30}$. This implies that the edge $\mathbb{P}([x,y])$ of T would be in T_1, (because we would be able to to continue the construction of the ladder $\Xi_{\widehat{\lambda}}$ beyond the vertex v) which is a contradiction. Therefore $d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\lambda}_v}(x'_2),\widehat{\pi}_{\widehat{\lambda}_v}(y'_2))\le D_1$.

$$\begin{aligned}
d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\lambda}}(x),\widehat{\Pi}_{\widehat{\lambda}}(y)) &\le d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\lambda}_v}(x_1),\widehat{\pi}_{\widehat{\lambda}_v}(y_1))\\
&\le d_{\widehat{X}_v^h}(\pi_{\lambda_v}(x_1),\pi_{\lambda_v}(y_1))+2\\
&\le d_{\widehat{X}_v^h}(\pi_{\lambda_v}(x_1),x_2)+d_{\widehat{X}_v^h}(x_2,y_2)+d_{\widehat{X}_v^h}(y_2,\pi_{\lambda_v}(y_2))+2\\
&\le d_{X_v^h}(\pi_{\lambda_v}(x_1),x_2)+d_{X_v^h}(\pi_{\lambda_v}(y_1),y_2)\\
&\quad +d_{\widehat{X}_v}(x'_2,y'_2)+2+2\\
&\le 2C+d_{\widehat{X}_v}(x'_2,\widehat{\pi}_{\widehat{\lambda}_v}(x'_2))+d_{\widehat{X}_v}(y'_2,\widehat{\pi}_{\widehat{\lambda}_v}(y'_2))\\
&\quad +d_{\widehat{X}_v}(\widehat{\pi}_{\widehat{\lambda}_v}(x'_2),\widehat{\pi}_{\widehat{\lambda}_v}(y'_2))+4\\
&\le 4C+D_1+4=K_2(say).
\end{aligned}$$

Taking $P_{3.2.3}=\max\{K_0,K_1,K_2,D_{1.1.30}+2\}$, we have the required result. □

3.2.3 Vertical Quasigeodesic Rays

Let $\delta\ge 0$. Let $\mathbb{P}:X\to T$ be a tree of δ-relatively hyperbolic metric spaces with (X,d_X) is δ-hyperbolic relative to the collection $\mathcal{C}$ maximal cone subtree of ν (≥ 1)-separated horosphere-like spaces. Then the tree of coned-off spaces $\mathcal{TC}(X)$ is hyperbolic. Let $\widehat{\lambda}_{v_0}$ be an electric geodesic segment from a to b in $\widehat{X}_{v_0}$ with a and b lying outside horosphere-like sets and λ_{v_0} denotes its electroambient path representative in $X^h_{v_0}$. Recall that we have constructed a set $\Xi_{\widehat{\lambda}_{v_0}}=\bigcup_{v\in\mathcal{V}(T_1)}\widehat{i}_v(\widehat{\lambda}_v)$, called as hyperbolic ladder, in $\mathcal{TC}(X)$ satisfying the following properties:
(i) $\Xi_{\widehat{\lambda}_{v_0}}$ is a quasiconvex set in $\mathcal{TC}(X)$ containing $\widehat{\lambda}_{v_0}$.
(ii) $\mathbb{P}(\Xi_{\widehat{\lambda}_{v_0}})=\mathcal{V}(T_1)$ and for $v\in\mathcal{V}(T_1)$, $\widehat{X}_v\cap\Xi_{\widehat{\lambda}_{v_0}}=\widehat{\lambda}_v$, where $\widehat{\lambda}_v$ is a geodesic in $\widehat{X}_v$.
(iii) Let $v,w\in\mathcal{V}(T_1)$ be adjacent vertices with e being the edge between them and $w=v\_$. There exists a geodesic $\widehat{\mu}_w$ with its end point lying in X_w and its elec-

troambient representative μ_w lies in a bounded neighborhood of the electroambient representative λ_w of $\widehat{\lambda}_w$. $\widehat{\lambda}_v$ is a geodesic in $\widehat{X}_v$ joining the end points of $\widehat{\phi}_{v,e}(\widehat{\mu}_w)$.

Let

• λ_v^c be the union of geodesic subsegments of the electroambient path representative λ_v lying inside the hyperbolic cones penetrated by λ_v.

• $\lambda_v^b = \lambda_v \setminus \lambda_v^c$. (Note that $\lambda_v^b \subset \widehat{\lambda}_v$).

• $\Xi_{\lambda_{v_0}^b} = \bigcup_{v \in \mathcal{V}(T_1)} i_v(\lambda_v^b)$. (Then $\Xi_{\lambda_{v_0}^b} \subset \Xi_{\widehat{\lambda}_{v_0}}$).

If $x \in \Xi_{\lambda_{v_0}^b}$, then there exists $v \in T_1$ such that $x \in \lambda_v^b$. Let $S = [v_n, v_{n-1}] \cup ... \cup [v_1, v_0]$ be the geodesic edge path in T_1 joining v and v_0.

We will construct a map $r_x \colon S \to \Xi_{\lambda_{v_0}^b}$ satisfying

• $d_S(w, w') \leq d_X(r_x(w), r_x(w') \leq R_{3.2.3} d_S(w, w')$ for all $w, w' \in S$.

• $r_x(v_i) \in X_{v_i}$.

r_x will be called a $R_{3.2.3}$-*vertical quasigeodesic ray.*

Recall that $\phi_{u,e} \colon f_{e,u_-}(X_e) \to f_{e,u}(X_e), \phi_{u,e}^h \colon f_{e,u_-}^h(X_e^h) \to f_{e,u}^h(X_e^h)$ are quasi-isometries, for the sake of simplicity of notation, here we will denote $\phi_{u,e}$ by ϕ_u and $\phi_{u,e}^h$ by ϕ_u^h.

<u>Construction of r_x:</u>

• For $v_n \in S$, define $r_x(v_n) = x$

Let $v = v_n, w = v_{n-1}$, $e_i = [v_i, v_{i-1}]$, $\psi_{v_i} = \phi_{v_i}^{-1} \colon f_{e_i,v_i}(X_{e_i}) \to f_{e_i,v_{i-1}}(X_{e_i})$ and $\psi_{v_i}^h = (\phi_{v_i}^h)^{-1} \colon f_{e_i,v_i}^h(X_{e_i}^h) \to f_{e_i,v_{i-1}}^h(X_{e_i}^h)$ for all $i = 1, ..., n$. Then $\psi_{v_i}, \psi_{v_i}^h$ are quasi-isometries.

Since x lies outside horosphere-like sets and ψ_v preserves horosphere-like sets (by the strictly type-preserving condition), $\psi_v(x)$ will lie outside horosphere-like sets.

Let $[a, b]$ be the maximal connected component of λ_v^b on which x lies. Then there exist two horosphere-like sets H_1 and H_2 such that $a \in H_1$ (or is a initial point of λ_v) and $b \in H_2$ (or is a terminal point of λ_v). Since ψ_v preserves horosphere-like sets, $\psi_v([a, b]) \setminus \{\psi(a), \psi(b)\}$ will lie outside horosphere-like sets.

As $[a, b]$ lies outside horosphere-like sets, $\psi_v^h([a, b]) = \psi_v([a, b]) \subset X_w$ and $\psi_v([a, b])$ is a quasigeodesic in X_w^h. Let $\Psi_v^h([a, b])$ be a hyperbolic geodesic in X_w^h joining $\psi_v(a)$ and $\psi_v(b)$. Then $\psi_v([a, b])$ will lie in a bounded neighborhood of $\Psi_v^h([a, b])$ and therefore there exists $R_1 > 0$ such that $d_X(\psi_v(x), \Psi_v^h([a, b])) \leq R_1$. By Lemma 1.2.44, there exists an upper bound on how much $\Psi_v^h([a, b])$ can penetrate hyperbolic cones, that is, for all $z \in \Psi_v^h([a, b])$ there exists $z' \in \Psi_v^h([a, b])$ lying outside hyperbolic cones such that $d_X(z, z') \leq I$. Hence there exists $y_1 \in \Psi_v^h([a, b])$ such that $d_X(\psi_v(x), y_1) \leq I + R_1$ and y_1 lies outside horosphere-like sets.

Again, $\Psi_v^h([a, b])$ lies at a uniformly bounded distance $\leq R_2$ from μ_v (the electroambient path representative of $\widehat{\mu}_v$ in the construction of $\Xi_{\widehat{\lambda}_{v_0}}$). Let $c, d \in \mu_v$ such that $d_X(a,c) \leq R_2$ and $d_X(b,d) \leq R_2$. Then $\Psi^h_v([a,b])$ and the quasigeodesic segment $[c,d] \subset \mu_v$ have similar intersection patterns (Lemma 1.2.44) with hyperbolic cones. Therefore $[c,d]$ can penetrate only a bounded distance $\leq I$ into any hyperbolic cone. Hence there exists $y_2 \in \mu_v$ and y_2 lies outside horosphere-like sets such that $d_X(y_1, y_2) \leq R_2 + I$.

Since end points of μ_v lie at a bounded neighborhood of λ_w, there exists $R_3 > 0$ such that μ_v will lie at a R_3 neighborhood of λ_w. Therefore there exists $y_3 \in \lambda_w$ such that $d_X(y_2, y_3) \leq R_3$. Now y_3 may lie inside a hyperbolic cone. Since μ_v and $\pi_{\lambda_w}(\mu_v)$ lies in a bounded neighborhood of each other, by Lemma 1.2.44 they have similar intersection patterns with hyperbolic cones. Therefore there exists $I > 0$ and $y \in \lambda_w$ such that y lies outside horosphere-like sets and $d_X(y_3, y) \leq I$.

Hence $d_X(x,y) \leq 1 + R_1 + R_2 + R_3 + 3I = R_{3.2.3}$(say).

- Recall that $w = v_{n-1}$. Define $r_x(v_{n-1}) = y$.

Thus we have $1 \leq d_X(r_x(v_n), r_x(v_{n-1})) \leq R_{3.2.3}$.

Using the above argument repeatedly, inductively replacing x with $r_x(v_i)$ in each step, we get the following. Since $r_x(v) \in X_v$, we have $d_S(v,w) \leq d_X(r_x(v), r_x(w))$.

Lemma 3.2.6. *There exists $R_{3.2.3} \geq 0$ such that the following holds:*
For all $x \in \lambda^b_v \subset \Xi^b_{\lambda_{v_0}}$, there exists a $R_{3.2.3}$-vertical quasigeodesic ray $r_x \colon S \to \Xi^b_{\lambda_{v_0}}$ such that $r_x(v) = x$ and $d_S(v,w) \leq d_X(r_x(v), r_x(w)) \leq R_{3.2.3} d_S(v,w)$, where S is the geodesic edge path in T_1 joining v and v_0 and $w \in S$.

The following is the concluding Lemma of this subsection.

Lemma 3.2.7. *Let $R_{3.2.3} > 0$ be as above:*
Fix a reference point p lying outside the horosphere-like sets in X_{v_0}. $B_n(p)$ denotes the n-ball around p in $(X_{v_0}, d_{X_{v_0}})$. Let $\lambda^b_{v_0}$ lies outside $B_n(p)$ (and hence entry and exit points of $\widehat{\lambda}$ to a horosphere-like set lie outside $B_n(p)$). Then for any $x \in \lambda^b_v (\subset \Xi^b_{\lambda_{v_0}} \subset \Xi_{\widehat{\lambda}_{v_0}})$, x lies outside an $n/(R_{3.2.3}+1)$-ball about p in X.

Proof. Since $r_x(v_0) \in \lambda^b_{v_0}$, $r_x(v_0)$ lies outside $B_n(p)$. Let m be the first non-negative integer such that $v \in \mathbb{P}(\Xi^m(\widehat{\lambda}_{v_0})) \setminus \mathbb{P}(\Xi^{m-1}(\widehat{\lambda}_{v_0}))$. Then $d_{T_1}(v_0, v) = m$, and $d_X(x,p) \geq m$ (since $r_x(v) = x \in \lambda^b_v$).
From Lemma 3.2.6, $m \leq d_X(r_x(v), r_x(v_0)) \leq R_{3.2.3} m$.
Since $r_x(v_0)$ lies outside $B_n(p)$, $d_X(r_x(v_0), p) \geq n$.
$n \leq d_X(r_x(v_0), p) \leq d_X(r_x(v_0), r_x(v)) + d_X(r_x(v), p) \leq mR_{3.2.3} + d_X(r_x(v), p)$.
Therefore, $d_X(r_x(v), p) \geq n - mR_{3.2.3}$ and $d_X(r_x(v), p) \geq m$.
Hence $d_X(x,p) = d_X(r_x(v), p) \geq \frac{n}{R_{3.2.3}+1}$. ☐

3.2.4 Proof of Main theorem

Let $\mathbb{P} : X \to T$ be a tree of relatively hyperbolic metric spaces with (X, d_X) is strongly hyperbolic relative to the collection $\mathcal{C}$ maximal cone subtree of horosphere-like spaces.
Let
• $\widehat{\lambda}_{v_0}$ = electric geodesic in $\widehat{X}_{v_0}$ joining $a, b \in X_{v_0}$ with $\lambda^b_{v_0}$ lying outside an n-ball $B_n(p)$ around p in X_{v_0}, for a fixed reference point $p \in X_{v_0}$ lying outside the horosphere-like sets.
• λ_{v_0} = electroambient path representative of $\widehat{\lambda}_{v_0}$ in $X^h_{v_0}$ constructed from $\widehat{\lambda}_{v_0}$.
• $\widehat{\mu}$ be a geodesic in the electric space $\widehat{X}$ joining a, b.
• μ be an electroambient representative of $\widehat{\mu}$.
• β_p =quasi-geodesic in the space $\mathcal{TC}(X)$ joining a, b.
• $\beta'_p = \widehat{\Pi}_{\Xi_{\widehat{\lambda}_{v_0}}}(\beta_p)$, where $\widehat{\Pi}_{\Xi_{\widehat{\lambda}_{v_0}}}$ is a nearest point projection map from $\mathcal{TC}(X)$ to the quasi-convex set $\Xi_{\widehat{\lambda}_{v_0}}$.

Recall that by Corollary 1.3.5, $\mathcal{TC}(X)$ is hyperbolic. By Lemma 1.1.34, β'_p is a $K_{1.1.34}$-quasigeodesic for some $K_{1.1.34} \geq 1$ in the space $\mathcal{TC}(X)$ joining a, b and lying on $\Xi_{\widehat{\lambda}_{v_0}}$. We will construct a tamed quasigeodesic path γ_p from β'_p in $\mathcal{TC}(X)$ joining a, b such that $\gamma_p \cap X$ lie in a C_2-neighborhood of $\Xi_{\lambda^b_{v_0}}$. Let $[l, m]$ be the domain of β'_p and $\mathcal{P} = \{l, m\} \cup (\mathbb{Z} \cap (l, m))$. For two successive points $t_i, t_{i+1} \in \mathcal{P}$, we have $d_{\mathcal{TC}(X)}(\beta'_p(t_i), \beta'_p(t_{i+1})) \leq 2K_{1.1.34}$. Let $\beta'_p(t_i) \in \widehat{X}_{u_i}, \beta'_p(t_{i+1}) \in \widehat{X}_{u_{i+1}}$, then $d_T(u_i, u_{i+1}) \leq 2K_{1.1.34}$. Recall that for $w \in \mathbb{P}(\Xi_{\widehat{\lambda}_{v_0}})$, $\widehat{X}_w \cap \Xi_{\widehat{\lambda}_{v_0}} = \widehat{\lambda}_w$, where $\widehat{\lambda}_w$ is a geodesic in $\widehat{X}_w$. By construction of $\Xi_{\widehat{\lambda}_{v_0}}$, there exists a vertex v in the geodesic $[u_i, u_{i+1}]$ such that $\widehat{\lambda}_{u_i} = \Xi_{\widehat{\lambda}_v} \cap \widehat{X}_{u_i}$ and the geodesic joining $\beta'_p(t_i), \beta'_p(t_{i+1})$ in $\mathcal{TC}(X)$ intersects $\widehat{X}_v$. Now for each i there exists $a_i \in \lambda^b_{u_i}$ such that $d_{\mathcal{TC}(X)}(\beta'_p(t_i), a_i) \leq 1$. By Lemma 3.2.6, there exist $x_v, y_v \in \lambda^b_v$ such that $d_X(a_i, x_v) \leq 2K_{1.1.34}R_{3.2.6}$ and $d_X(a_{i+1}, y_v) \leq 2K_{1.1.34}R_{3.2.6}$. Therefore, by triangle inequality,

$$\begin{aligned} d_{\mathcal{TC}(X)}(x_v, y_v) &\leq d_X(x_v, a_i) + d_{\mathcal{TC}(X)}(a_i, a_{i+1}) + d_X(a_{i+1}, y_v) \\ &\leq 2K_{1.1.34}R_{3.2.6} + (2K_{1.1.34} + 2) + 2K_{1.1.34}R_{3.2.6} = K_1 \text{ say.} \end{aligned}$$

Now $\widehat{X_v}$ is uniformly properly embedded in $\mathcal{TC}(X)$, therefore there exists $P = P(K_1) > 0$ such that $d_{\widehat{X_v}}(x_v, y_v) \leq P$. Let $\widehat{\lambda}'_v$ be the subsegment of $\widehat{\lambda}_v$ joining x_v, y_v; $[\beta'_p(t_i), a_i]$ be a geodesic in $\mathcal{TC}(X)$ and $[a_i, x_v]_X, [b_i, y_v]_X$ be geodesics in X. Let $\beta_i = [\beta'_p(t_i), a_i] \cup [a_i, x_v]_X \cup \widehat{\lambda}'_v \cup [y_v, b_i]_X \cup [b_i, \beta'_p(t_{i+1})]$, then the length of β_i in $\mathcal{TC}(X)$ is at most $2 + 4K_{1.1.34}R_{3.2.6} + P$. Let $\gamma_p = \cup_i \beta_i$, then γ_p is a tamed quasi-geodesic path in $\mathcal{TC}(X)$. Let $C_2 = 2K_{1.1.34}R_{3.2.6}$ then for $z \in \gamma_p \cap X$ there exists $w \in \lambda^b_v \subset \Xi_{\lambda^b_{v_0}}$ such that $d_X(z, w) \leq C_2$.

Now a, b are end points of $\widehat{\lambda}_{v_0}$, therefore $a, b \in \Xi_{\widehat{\lambda}_{v_0}}$ and end points of γ_p are a, b. By Lemma 1.3.5, there exists $C_1 \geq 0$ such that if $x \in \mu^b = \widehat{\mu} \cap X$, then there exists $y \in \gamma_p^b = \gamma_p \cap X$ such that $d_X(x, y) \leq C_1$. For $y \in \gamma_p^b$, there exists $y_1 \in \Xi_{\lambda_{v_0}^b}$ such that $d_X(y, y_1) \leq C_2$.

It follows from lemma 3.2.7 that $d_X(y_1, p) \geq \frac{n}{R_{3.2.3}+1}$.

So, $\frac{n}{R_{3.2.3}+1} \leq d_X(y_1, y) + d_X(y, x) + d_X(x, p) \leq C_2 + C_1 + d_X(x, p)$,

i.e. $d_X(x, p) \geq \frac{n}{R_{3.2.3}+1} - C_1 - C_2$ (=$M(n)$, say).

Thus we have the following proposition :

Proposition 3.2.8. *Let $\delta \geq 0, \nu \geq 1$ and X be a proper geodesic space. Let $\mathbb{P} : X \to T$ be a tree of δ-relatively hyperbolic metric spaces such that the collections $\mathcal{H}_v, \mathcal{H}_e$ are uniformly ν-separated for each vertex v and each edge e. Let X be δ-hyperbolic relative to the collection $\mathcal{C}$ in the sense of Gromov. For a vertex v_0 of T, let $\widehat{\lambda}_{v_0}$ be an electric geodesic in $\widehat{X_{v_0}}$ joining $a, b \in X_{v_0}$ with $\lambda_{v_0}^b$ lying outside an n-ball around p in X_{v_0}, for a fixed reference point $p \in X_{v_0}$ lying outside the horosphere-like sets. Let $\widehat{\mu}$ be a geodesic in $\widehat{X}$ joining a, b and $\mu^b = \widehat{\mu} \cap X$. Then for every point x on μ^b, x lies outside an $M(n)$-ball around p in X, such that $M(n) \to \infty$ as $n \to \infty$.*

It is now easy to assemble the pieces to deduce the existence of Cannon-Thurston maps.

Theorem 3.2.9. *Let $\delta \geq 0, \nu \geq 1$ and X be a proper geodesic space. Let $\mathbb{P} : X \to T$ be a tree of δ-relatively hyperbolic metric spaces such that the collections $\mathcal{H}_v, \mathcal{H}_e$ are uniformly ν-separated for each vertex v and each edge e. If X is δ-hyperbolic relative to the collection $\mathcal{C}$ in the sense of Gromov, then a Cannon-Thurston map exists for the proper embedding $i_{v_0} \colon X_{v_0} \to X$.*

Proof. A Cannon-Thurston map exists if it satisfies the condition of Lemma 3.1.4. So for a fixed reference point $p \in X_{v_0}$ with p lying outside horosphere-like sets, we assume that $\widehat{\lambda}_{v_0}$ is an electric geodesic in $\widehat{X}_{v_0}$ such that $\lambda_{v_0}^b = \widehat{\lambda}_{v_0} \cap X \subset X_{v_0}$ lies outside an n-ball $B_n(p)$ around p in X_{v_0}. Since i_{v_0} is a proper embedding, $\lambda_{v_0}^b$ lies outside a $f(n)$-ball around p in X such that $f(n) \to \infty$ as $n \to \infty$. From the Proposition 3.2.8, if $\widehat{\mu}$ is a geodesic in $\widehat{X}$ joining the end points of λ_{v_0}, then μ^b lies outside an $M(f(n))$-ball around p in X such that $M(f(n)) \to \infty$ as $n \to \infty$. From Lemma 3.1.4, a Cannon-Thurston map for $i \colon X_{v_0} \to X$ exists. ∎

3.3 Cannon-Thurston Maps for Relatively Hyperbolic Extensions of Groups

Here, we will work with the following short exact sequence of finitely generated groups :

$$1 \to K \to G \xrightarrow{p} Q \to 1$$

Let K_1 be a finitely generated non-trivial proper subgroup of K such that K is strongly hyperbolic relative to the subgroup K_1 and G preserves cusps. Since all groups are finitely generated, we can choose a finite generating set S of G such that S contains finite generating set of $K, K_1, N_G(K_1)$ and $p(S)$ is also a finite generating set of Q. There exists a (R, ϵ)-quasi-isometric section $s\colon Q \to G$ such that

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_G(s(q), s(q')) \leq R d_Q(q, q') + \epsilon.$$

From Corollary 2.1.8 and using a left translation L_k by an element $k \in K_1$, we can assume that $s(Q)$ contains the identity element e_K of K and $s(Q) \subset N_G(K_1)$. Further, we assume that G is strongly hyperbolic relative to the subgroup $N_G(K_1)$ and G is weakly hyperbolic relative to the subgroup K_1. We have assumed that generating set of G contains the generating set of K, K_1, thus the Cayley graphs of Γ_K, Γ_{K_1} are connected subgraphs of Γ_G. For $a \in G$, let L_a denotes the left translation by a. L_a acts by isometry on Γ_G. Let $H_{aK_1} = L_a(\Gamma_{K_1})$, $\mathcal{H}_{K_1} = \{H_{aK_1} : a \in G\}$ and $\mathcal{E}(G, K_1) = \mathcal{E}(\Gamma_G, \mathcal{H}_{K_1})$. Similarly we have $H_{gN_G(K_1)}$ and $\mathcal{H}_{N_G(K_1)}$. As G is weakly hyperbolic to K_1, $\mathcal{E}(G, K_1)$ is a hyperbolic metric space. Since G is strongly hyperbolic relative to $N_G(K_1)$ (Here the definition of a relatively hyperbolic group will taken in the sense of Gromov (B), 1.2.57), $(\Gamma_G, \mathcal{H}_{N_G(K_1)})$ is a strongly relatively hyperbolic space.

Let Λ be a subset of G such that the identity element $e_G \in \Lambda$ and for two distinct elements $g, g' \in \Lambda$ we demand $gN_G(K_1) \neq g'N_G(K_1)$. Let $(Q_g, d_{Q_g}) = (Q, d_Q)$ and $\mathcal{Q} = \{Q_g : g \in \Lambda\}$.

For each $g \in \Lambda$, define $F_g : H_{gN_G(K_1)} \to Q_g$ by $F_g(gb) = p(gb)$ for all $b \in N_G(K_1)$.

As $d_Q(p(a), p(a')) \leq d_{N_G(K_1)}(a, a')$ for all $a, a' \in N_G(K_1)$, therefore $d_{Q_g}(F_g(x), F_g(x')) \leq d_{H_{gN_G(K_1)}}(x, x')$ for all $x, x' \in H_{gN_G(K_1)}$.

Let $\mathcal{F} = \{F_g : g \in \Lambda\}$. Since Γ_Q is quasi-isometrically embedded in $\mathcal{E}(G, K_1)$ (from Corollary 2.1.8), Γ_Q is hyperbolic and hence Q_g is hyperbolic for all $g \in \Lambda$. Then, according to Definition 1.2.66, $\mathcal{PE}(\Gamma_G, \mathcal{H}_{N_G(K_1)}, \mathcal{F}, \mathcal{Q})$ is a partially electrocuted space. Let $\Gamma_G^{pel} = \mathcal{PE}(\Gamma_G, \mathcal{H}_{N_G(K_1)}, \mathcal{F}, \mathcal{Q})$, then from Theorem 1.2.77 is a hyperbolic metric space.

Since (K, K_1) and (G, G_1) are relatively hyperbolic groups, due to Theorem 1.2.63, $(\Gamma_K, \mathcal{H}_{K_1})$ and $(\Gamma_G, \mathcal{H}_{G_1})$ are relatively hyperbolic metric spaces both in the sense of Gromov (B) (Defn. 1.2.57) and Farb (Defn. 1.2.6). Let $\lambda^b = \widehat{\lambda} \backslash \mathcal{H}_{K_1}$ denote the portions of $\widehat{\lambda}$ that do not penetrate horosphere-like sets in $\mathcal{H}_{K_1}$. The following Lemma gives a sufficient condition for the existence of a Cannon-Thurston map for the inclusion $i\colon (\Gamma_K, \mathcal{H}_{K_1}) \to (\Gamma_G, \mathcal{H}_{G_1})$. For proof refer to Lemma 3.1.4.

Lemma 3.3.1. *A Cannon-Thurston map for* $i\colon (\Gamma_K, \mathcal{H}_{K_1}) \to (\Gamma_G, \mathcal{H}_{G_1})$ *exists if there exists a non-negative function* $M(N)$ *with* $M(N) \to \infty$ *as* $N \to \infty$ *such that the following holds:*

Given $y_0 \in \Gamma_K$ *and an electric quasigeodesic segment* $\widehat{\lambda}$ *in* $\widehat{\Gamma}_K$ *if* $\lambda^b = \widehat{\lambda} \setminus \mathcal{H}_{K_1}$ *lies outside an* N*-ball around* $y_0 \in \Gamma_K$*, then for any geodesic* $\widehat{\mu}$ *in* $\widehat{\Gamma}_G$ *joining end points of* $\widehat{\lambda}$*,* $\mu^b = \widehat{\mu} \setminus \mathcal{H}_{G_1}$ *lies outside an* $M(N)$*-ball around* $i(y_0)$ *in* Γ_G*.*

3.3.1 Construction of Quasiconvex Sets and Retraction Map

Recall that for $g \in G$, $L_g\colon G \to G$ denotes the left translation by g and $I_g\colon K \to K$ denotes the automorphism $I_g(k) = gkg^{-1}$. Let $\phi_g = I_{g^{-1}}$ then $\phi_g(a) = g^{-1}ag$. Since L_g is an isometry, L_g preserves distance between left cosets of G_1 in G. Hence L_g induces an isometry $\widehat{L}_g\colon \Gamma_G^{pel} \to \Gamma_G^{pel}$. The embedding $i\colon \Gamma_K \to \Gamma_G$ will induce an embedding $\widehat{i}\colon \widehat{\Gamma}_K \to \Gamma_G^{pel}$.

Let $\widehat{\lambda}$ be an electric geodesic segment in $\widehat{\Gamma}_K$ with end points a and b in Γ_K. Let $\widehat{\lambda}_g$ be an electric geodesic in $\widehat{\Gamma}_K$ joining $\phi_g(a)$ and $\phi_g(b)$.

Define

$$\Xi_{\widehat{\lambda}} = \bigcup_{g \in s(Q)} \widehat{L}_g.\widehat{i}(\widehat{\lambda}_g).$$

Recall from definition 1.2.16 that $\Gamma_K^h = \mathcal{G}(\Gamma_K, \mathcal{H}_{K_1})$ is the hyperbolic metric space obtained from Γ_K by hyperbolic cone construction. For $g \in G$, let $\widehat{\pi}_{\widehat{\lambda}_g}\colon \widehat{\Gamma}_K \to \widehat{\lambda}_g$ be the electric projection. (Refer to Definition 1.2.51).
From Lemma 1.2.53, there exists $P_{1.2.53} > 0$ such that

$$d_{\widehat{\Gamma}_K}(\widehat{\pi}_{\widehat{\lambda}_g}(k), \widehat{\pi}_{\widehat{\lambda}_g}(k')) \leq P_{1.2.53} d_{\widehat{\Gamma}_K}(k, k') + P_{1.2.53}$$

for all $k, k' \in \widehat{\Gamma}_K$, where $P_{1.2.53}$ depends only on the hyperbolic constant of $\widehat{\Gamma}_K$.
For each $g \in G$, $\phi_g\colon \Gamma_K \to \Gamma_K$ is a quasi-isometry and it induces a quasi-isometry $\widehat{\phi}_g\colon \widehat{\Gamma}_K \to \widehat{\Gamma}_K$. Thus from Lemma 1.2.55, there exists a constant $P_{1.2.55} > 0$ such that if $x \in \widehat{\Gamma}_K$ and $\widehat{\lambda}$ is a geodesic in $\widehat{\Gamma}_K$ joining a and b then $d_{\widehat{\Gamma}_K}(\widehat{\phi}_g(\widehat{\pi}_{\widehat{\lambda}}(x)), \widehat{\pi}_{\widehat{\lambda}_g}(\widehat{\phi}_g(x))) \leq P_{1.2.55}$.

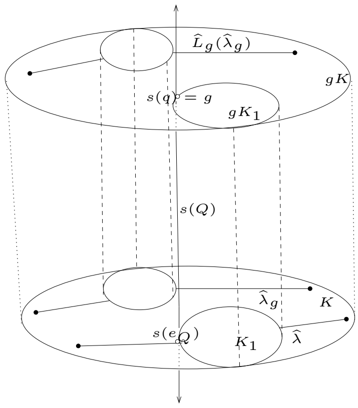

Figure 3.2: Quasiconvex set $\Xi_{\widehat{\lambda}}$.

Analogous to the Definition 3.2.2, we define the Retraction Map in a group theoretic setting.

Definition 3.3.2. *(Retraction Map) Define* $\widehat{\Pi}_{\widehat{\lambda}}\colon \Gamma_G^{pel} \to \Xi_{\widehat{\lambda}}$ *as follows:*

Let $x \in \Gamma_G^{pel}$*. There exists a unique* $g \in s(Q)$ *such that* $\widehat{L_g}(\widehat{i}(k)) = x$ *for some unique* $k \in K$*, define* $\widehat{\Pi}_{\widehat{\lambda}}(x) = \widehat{L_g}(\widehat{i}(\widehat{\pi}_{\widehat{\lambda}_g}(k)))$*.* $\widehat{\Pi}_{\widehat{\lambda}}$ *will be called a Retraction Map.*

The following theorem says that the Retraction Map $\widehat{\Pi}_{\widehat{\lambda}}$ is coarsely Lipschitz.

Theorem 3.3.3. *There exists a constant* $P_{3.3.3} > 0$ *such that for the short exact sequence of pair of finitely generated groups*

$$1 \to (K, K_1) \xrightarrow{i} (G, N_G(K_1)) \xrightarrow{p} (Q, Q) \to 1$$

with G *preserving cusps;* K, G *strongly hyperbolic relative to the subgroups* $K_1, N_G(K_1)$ *respectively, and* G *weakly hyperbolic relative to the collection* K_1*, the following inequality holds:*

$$d_{pel}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(x')) \leq P_{3.3.3} d_{pel}(x, x') + P_{3.3.3}$$

for all $x, x' \in \Gamma_G^{pel}$*, where* $(\Gamma_G^{pel}, d_{pel})$ *is the coned-off space corresponding to the pair* $(G, i(K_1))$*. In particular, if* Γ_G^{pel} *is hyperbolic then* $\Xi_{\widehat{\lambda}}$ *is uniformly (independent of* $\widehat{\lambda}$*) quasiconvex.*

Proof. Since cone points in Γ_G^{pel} lie within a unit distance from the points of Γ_G, it suffices to prove the theorem for points lying in Γ_G. Also, it suffices to prove that there exists $P_{3.3.3} > 0$ such that for $x, y \in \Gamma_G$ if $d_{pel}(x, y) \leq 1$ then $d_{pel}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) \leq P_{3.3.3}$. The embedding $i\colon \Gamma_K \to \Gamma_G$ induces an embedding $\widehat{i}\colon \widehat{\Gamma}_K \to \Gamma_G^{pel}$, so we identify x with its image $\widehat{i}(x)$.

Case i: Let x and y lie in the same left coset gK of K in G.
Then using Lemma 1.2.53, there exists a constant $P_{1.2.53} > 0$ such that

$$d_{pel}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) \leq d_{\widehat{\Gamma}_K}(\widehat{\pi}_{\widehat{\lambda}_g}(x), \widehat{\pi}_{\widehat{\lambda}_g}(y)) \leq P_{1.2.53} d_{\widehat{\Gamma}_K}(x, y) + P_{1.2.53} = 2P_{1.2.53}.$$

Case ii: Let x and y lie in different left cosets. Therefore $d_{pel}(x, y) = 1$ and hence $d_Q(p(x), p(y)) = 1$. Recall that $s\colon Q \to \Gamma_G$ is a quasi-isometric section. Therefore there exists $k_1 > 0$ such that $d_G(s(p(x)), s(p(y))) \leq k_1$.
Now there exists $g_0 \in s(Q)$ and $g \in G$ with length of g is bounded above by the constant k_1 such that $x \in L_{g_0}(\Gamma_K)$ and $y \in L_{g_0 g}(\Gamma_K)$. Therefore $x = L_{g_0}(x_1), y = L_{g_0 g}(y_1)$ for some $x_1, y_1 \in \Gamma_K$. By definition, $\widehat{\Pi}_{\widehat{\lambda}}(x) = \widehat{L}_{g_0} \widehat{\pi}_{g_0}(x_1)$ and $\widehat{\Pi}_{\widehat{\lambda}}(y) = \widehat{L}_{g_0} \widehat{\pi}_{g_0}(y_1)$.
Now for all $k \in K$, $d(L_{g_0}(k), L_{g_0 g}\phi_g(k)) = d(k, kg) \leq k_1$. Therefore

$$d_{pel}(\widehat{L}_{g_0}(\widehat{\pi}_{\widehat{\lambda}_{g_0}}(x_1)), \widehat{L}_{g_0 g}\widehat{\phi}_g(\widehat{\pi}_{\widehat{\lambda}_{g_0}}(x_1))) \leq k_1.$$

Also we have,

$$\begin{aligned}
d_{pel}(\widehat{L}_{g_0 g}(\widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(\widehat{\phi}_g(x_1))), \widehat{L}_{g_0 g}\widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(y_1))) &= d_{pel}(\widehat{\phi}_g(\widehat{\pi}_{\widehat{\lambda}}(x)), \widehat{\pi}_{\widehat{\lambda}_g}(\widehat{\phi}_g(x))) \\
&\leq d_{\widehat{\Gamma}_K}(\widehat{\phi}_g(\widehat{\pi}_{\widehat{\lambda}}(x)), \widehat{\pi}_{\widehat{\lambda}_g}(\widehat{\phi}_g(x))) \\
&\leq P_{1.2.55}
\end{aligned}$$

Since $x_1 \in \Gamma_K$, $\widehat{\phi}_g(x_1) = \phi_g(x_1)$. Thus

$$\begin{aligned}
d_{pel}(\widehat{\phi}_g(x_1), y_1) &= d_{pel}(\phi_g(x_1), y_1) \\
&\leq d_{pel}(xg, y) \\
&\leq d_{pel}(xg, x) + d_{pel}(x, y) \\
&\leq k_1 + 1.
\end{aligned}$$

Now $\widehat{\Gamma}_K$ is properly embedded in Γ_G^{pel} therefore there exists a constant $M_1(k_1) > 0$ such that $d_{\widehat{\Gamma}_K}(\widehat{\phi}_g(x_1), y_1) \leq M_1$. Since $\widehat{L}_{g_0 g}$ is an isometry, we have
$d_{pel}(\widehat{L}_{g_0 g}(\widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(\widehat{\phi}_g(x_1))), \widehat{L}_{g_0 g}(\widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(y_1))) = d_{pel}(\widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(\widehat{\phi}_g(x_1)), \widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(y_1))$.
From Lemma 1.2.53, there exists a constant $P_{1.2.53} > 0$ such that

$$\begin{aligned}
d_{pel}(\widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(\widehat{\phi}_g(x_1)), \widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(y_1)) &\leq d_{\widehat{\Gamma}_K}(\widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(\widehat{\phi}_g(x_1)), \widehat{\pi}_{\widehat{\lambda}_{g_0 g}}(y_1)) \\
&\leq P_{1.2.53} M_1 + P_{1.2.53}.
\end{aligned}$$

Thus, finally we have

$$\begin{aligned} d_{pel}(\widehat{\Pi}_{\widehat{\lambda}}(x), \widehat{\Pi}_{\widehat{\lambda}}(y)) &= d_{pel}(\widehat{L}_{g_0}(\widehat{\pi}_{\widehat{\lambda}_{g_0}}(x_1)), \widehat{L}_{g_0g}(\widehat{\pi}_{\widehat{\lambda}_{g_0g}}(y_1))) \\ &\leq d_{pel}(\widehat{L}_{g_0}(\widehat{\pi}_{\widehat{\lambda}_{g_0}}(x_1)), \widehat{L}_{g_0g}\widehat{\phi}_g(\widehat{\pi}_{\widehat{\lambda}_{g_0}}(x_1))) + \\ &\quad d_{pel}(\widehat{L}_{g_0g}\widehat{\phi}_g(\widehat{\pi}_{\widehat{\lambda}_{g_0}}(x_1)), \widehat{L}_{g_0g}(\widehat{\pi}_{\widehat{\lambda}_{g_0g}}(\widehat{\phi}_g(x_1)))) + \\ &\quad d_{pel}(\widehat{L}_{g_0g}(\widehat{\pi}_{\widehat{\lambda}_{g_0g}}(\widehat{\phi}_g(x_1))), \widehat{L}_{g_0g}(\widehat{\pi}_{\widehat{\lambda}_{g_0g}}(y_1))) \\ &\leq k_1 + P_{1.2.55} + P_{1.2.53}M_1 + P_{1.2.53}. \end{aligned}$$

Taking $P_{3.3.3} = max\{2P_{1.2.53}, k_1+P_{1.2.55}+P_{1.2.53}M_1+P_{1.2.53}\}$, we have the required result. □

3.3.2 Proof of Theorem

Since $i\colon \Gamma_K \to \Gamma_G$ is an embedding we identify $k \in K$ with its image $i(k)$. Let

- $\widehat{\mu}_g = \widehat{L}_g(\widehat{\lambda}_g)$, where $g \in s(Q)$.
- $\mu_g^b = \widehat{\mu}_g \setminus \mathcal{H}_{N_G(K_1)}$.
- $\lambda_g^b = \widehat{\lambda}_g \setminus \mathcal{H}_{K_1}$.
- $\Xi_{\lambda^b} = \bigcup_{g\in s(Q)} \mu_g^b$.
- $Y = \Gamma_K$ and $X = \Gamma_G$.

Lemma 3.3.4. *There exists $A > 0$ such that if λ^b lies outside $B_N(p)$ for a fixed reference point $p \in \Gamma_K$, then for all $x \in \mu_g^b \subset \Xi_{\lambda^b} \subset \Xi_{\widehat{\lambda}}$, x lies outside a $\frac{f(N)}{A+1}$ ball about p in Γ_G, where $f(N) \to \infty$ as $N \to \infty$.*

Proof. Let $x \in \mu_g^b$ for some $g \in s(Q)$. Let γ be a geodesic path in Γ_Q joining the identity element e_Q of Γ_Q and $p(x) \in \Gamma_Q$. Order the vertices on γ so that we have a finite sequence $e_Q = q_0, q_1, ..., q_n = p(x) = p(g)$ such that $d_Q(q_i, q_{i+1}) = 1$ and $d_Q(e_Q, p(x)) = n$. Since s is a quasi-isometric section, this gives a sequence $s(q_i) = g_i$ such that $d_G(g_i, g_{i+1}) \leq R + \epsilon = R_1$ (say). Observe that $g_n = g$ and $g_0 = e_G$. Let $B_{R_1}(e_G)$ be a closed ball around e_G of radius R_1, then $B_{R_1}(e_G)$ is finite. Now for each $g \in G$, the automorphism ϕ_g is a quasi-isometry. Thus there exists $K \geq 1$ and $\epsilon \geq 0$ such that for all $g \in B_{R_1}(e_G)$, ϕ_g is a (K, ϵ) quasi-isometry and K, ϵ are independent of elements of G. Let $s_i = g_{i+1}^{-1}g_i$, then $s_i \in B_{R_1}(e_G)$, where $i = 0, ..., n-1$. Hence ϕ_{s_i} is a (K, ϵ) quasi-isometry.

Since $s(Q) \subset N_G(K_1)$ so we have $s_i \in N_G(K_1)$ for all i. ϕ_{s_i} will induce a $(\widehat{K}, \widehat{\epsilon})$ quasi-isometry $\widehat{\phi}_{s_i}$ from $\widehat{\Gamma}_K$ to $\widehat{\Gamma}_K$, where $\widehat{K}, \widehat{\epsilon}$ depends only K and ϵ.

Now $x \in \mu^b_{g_n}$ and L_g preserves distance between left cosets for all $g \in G$, hence there exists $x_1 \in \lambda^b_{g_n}$ such that $x = L_{g_n}(x_1)$.

Let $[p,q]_{g_n} \subset \lambda^b_{g_n}$ be the connected portion of $\lambda^b_{g_n}$ on which x_1 lies. Note that $[p,q]_{g_n}$ is a geodesic in Y^h. Since $\phi_{s_{n-1}}$ is a strictly type preserving quasi-isometry, it induces a quasi-isometry $\phi^h_{s_{n-1}} : Y^h \to Y^h$ and as $\lambda^b_{g_n}$ lies outside horosphere-like sets, $\phi_{s_{n-1}}([p,q]_{g_n})(= \phi^h_{s_{n-1}}([p,q]_{g_n}))$ is a quasigeodesic in Y^h lying at a uniformly bounded distance $\leq C_1$ from $\lambda_{g_{n-1}}$ in Y^h, where $\lambda_{g_{n-1}}$ is an electroambient representative of $\widehat{\lambda}_{g_{n-1}}$. Thus there exist $x_2 \in \lambda_{g_{n-1}}$ such that $d_{X^h}(\phi_{s_{n-1}}(x_1), x_2) \leq d_{Y^h}(\phi_{s_{n-1}}(x_1), x_2) \leq C_1$. But x_2 may lie inside a hyperbolic cone penetrated by $\widehat{\lambda}_{g_{n-1}}$. Due to bounded coset (horosphere) penetration properties there exists $y \in \lambda^b_{g_{n-1}}$ such that $d_{X^h}(x_2, y) \leq I$ for some $I > 0$.

Thus $d_{X^h}(\phi_{s_{n-1}}(x_1), y) \leq C_1 + I$. Since $X = \Gamma_G$ is properly embedded in X^h, there exists $M > 0$ depending only upon C_1, I such that $d_G(\phi_{s_{n-1}}(x_1), y) \leq M$.

Hence $d_G(L_{g_{n-1}}(\phi_{s_{n-1}}(x_1)), L_{g_{n-1}}(y)) = d_G(\phi_{s_{n-1}}(x_1), y) \leq M$ and $L_{g_{n-1}}(y) \in \mu^b_{g_{n-1}}$.

Let $z = L_{g_{n-1}}(y)$, then

$$\begin{aligned} d_G(x,z) &\leq d_G(x, L_{g_{n-1}}(\phi_{s_{n-1}}(x_1))) + d_G(L_{g_{n-1}}(\phi_{s_{n-1}}(x_1)), L_{g_{n-1}}(y)) \\ &\leq d_G(x, xs_{n-1}) + M \\ &\leq R_1 + M = A(say). \end{aligned}$$

Thus, we have shown that for $x \in \mu^b_{g_n}$ there exists $z \in \mu^b_{g_{n-1}}$ such that $d_G(x,z) \leq A$. Proceeding in this way, for each $y \in \mu^b_{g_i}$ there exists $y' \in \mu^b_{g_{i-1}}$ such that $d_G(y,y') \leq A$.

Hence there exists $x' \in \lambda^b$ such that $d_G(x,x') \leq An$.

Since Γ_K is properly embedded in Γ_G there exists $f(N)$ such that λ^b lies outside $f(N)$-ball about p in Γ_G and $f(N) \to \infty$ as $N \to \infty$.

Therefore $d_G(x',p) \geq f(N)$, thus

$$d_G(x,p) \geq f(N) - d_G(x,x') \geq f(N) - An.$$

Also we know that $d_G(x,p) \geq n$, therefore $d_G(x,p) \geq \frac{f(N)}{A+1}$, i.e., x lies outside $\frac{f(N)}{A+1}$-ball about p in Γ_G. □

Theorem 3.3.5. *Consider a short exact sequence of finitely generated groups*

$$1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1$$

with K hyperbolic relative to a proper non-trivial subgroup K_1. Suppose that

1. *G preserves cusp,*

2. G is (strongly) hyperbolic relative to $N_G(K_1)$ and,

3. G is weakly hyperbolic relative to the subgroup K_1.

Then there exists a Cannon-Thurston map for the embedding $i\colon \Gamma_K \to \Gamma_G$, where Γ_K and Γ_G are Cayley graphs of K and G respectively.

Proof. It suffices to prove the condition of Lemma 3.3.1.

So for a fixed reference point $p \in \Gamma_K$, we assume that $\widehat{\lambda}$ is an electric geodesic segment in $\widehat{\Gamma}_K$ such that $\lambda^b(\subset \Gamma_K)$ lies outside an N-ball $B_N(p)$ around p. Let β_{pel} be a quasigeodesic in the partially electrocuted space Γ_G^{pel} joining the end points of $\widehat{\lambda}$. Let $\widehat{\Pi}_{\Xi_{\widehat{\lambda}}}$ be a nearest point projection from Γ_G^{pel} onto the quasiconvex set $\Xi_{\widehat{\lambda}}$ which satisfies the Lipschitz's condition. Let $\beta'_{pel} = \widehat{\Pi}_{\Xi_{\widehat{\lambda}}}(\beta_{pel})$, then β'_{pel} is a quasigeodesic in Γ_G^{pel} lying on $\Xi_{\widehat{\lambda}}$. So β'_{pel} lies in a P-neighborhood of β_{pel} in Γ_G^{pel}. Let $\mathcal{C} = \{C_{gN_G(K_1)} : g \in G\}$. As in proof of Proposition 3.2.8, there exists a tamed quasigeodesic path γ_{pel} obtained from β'_{pel} in Γ_G^{pel} joining a, b such that $\gamma_{pel} \cap \Gamma_G$ lie in a R_1-neighborhood of Ξ_{λ^b} in Γ_G for some $R_1 > 0$.
Let $\widehat{\mu}$ be an electric geodesic in $\widehat{\Gamma}_G$, there exists $R_2 \geq 0$ such that if $x \in \mu^b = \widehat{\mu} \setminus \cup\mathcal{C}$, then there exists $y \in \gamma_{pel}^b = \gamma_{pel} \cap \Gamma_G$ such that $d_G(x, y) \leq R_2$. For $y \in \gamma_{pel}^b$ there exists $y_1 \in \Xi_{\lambda_v^b}$ such that $d_G(y, y_1) \leq R_1$.

Since $y_1 \in \Xi_{\lambda^b}$, by Lemma 3.3.4, $d_G(y_1, p) \geq \frac{f(N)}{A+1}$.

Therefore, $d_G(x, p) \geq \frac{f(N)}{A+1} - R_2 - R_1$ $(= M(N)$, say) and $M(N) \to \infty$ as $N \to \infty$. By Lemma 3.3.1, a Cannon-Thurston map for $i\colon \Gamma_K \to \Gamma_G$ exists. □

Chapter 4

Examples and Applications

We first recall a combination theorem for trees of hyperbolic spaces (due to Bestvina and Feighn [BF92]), which ensures its hyperbolicity. Using this, in [Mos97], Mosher extended a closed hyperbolic surface group by a free group generated by some sufficiently large powers of hyperbolic automorphisms, i.e. pseudo-Anosov mapping classes. In [MR08], the combination theorem was generalized for trees of relatively hyperbolic spaces. Analogously, in [MR08], Mosher's result was generalized to punctured surface groups.

4.1 Examples

4.1.1 A Combination Theorem

Let X be a tree of hyperbolic metric spaces.

Definition 4.1.1. *(Bestvina and Feighn [BF92])*
A disk $\triangle\colon [-m-\frac{1}{4}, m+\frac{1}{4}] \times I \to X$ *is a hallway of length* $2m$ *if it satisfies :*

1. $\triangle^{-1}(\bigcup X_e \colon e \in \mathcal{E}(T)) = \{-m, ..., m\} \times I$.

2. *(a)* $\triangle$ *maps* $i \times I$ *to a geodesic in* X_e *for some edge* e*;*
   (b) image of $(i-\frac{1}{4}, i+\frac{1}{4}) \times I$ *under* $\triangle$ *lies in* $f_e|_{X_e \times (0,1)}(X_e \times (0,1))$ *such that* $d_X(\triangle(t,s), \triangle(t',s)) = 2|t-t'|$ *for all* $t, t' \in (i-\frac{1}{4}, i+\frac{1}{4}), s \in I$*, and*
   (c) if $\triangle(i \times I) \subset X_e$ *with* $e = [v_-, v_+]$*, then* $\triangle((i-\frac{1}{4}) \times I) \subset X_{v_-}$ *and* $\triangle((i+\frac{1}{4}) \times I) \subset X_{v_+}$.

Definition 4.1.2. *(Bestvina and Feighn [BF92])*

1. *A hallway is* ρ *-thin if* $d_{X_v}(\triangle(i+\frac{1}{4}, t), \triangle((i+1)-\frac{1}{4}, t)) \leq \rho$ *for all* i, t*, where* X_v *is the vertex space for which* $\triangle(((i+1)-\frac{1}{4}) \times I), \triangle((i+\frac{1}{4}) \times I) \subset X_v$

2. *A hallway is* λ *-hyperbolic if*

   $\lambda l(\triangle(\{0\} \times I)) \leq max(l(\triangle(\{-m\} \times I), l(\triangle\{m\} \times I))$

3. *A hallway is essential if the edge path in* T *resulting from projecting* X *onto* T *does not back track (and is therefore a geodesic segment in the tree* T*).*

4. *The girth of the hallway* $\triangle$ *is length of* $\triangle(\{0\} \times I)$*.*

Definition 4.1.3. *[MR08] An essential hallway of length* $2m$ *is* **cone-bounded** *if* $\triangle(i \times \partial I)$ *lies in the cone-locus for* $i = \{-m, ..., m\}$*.*

Definition 4.1.4. *(Bestvina and Feighn [BF92]) The tree of spaces ,* X*, is said to satisfy the hallways flare condition if there are numbers* $\lambda > 1$ *and* $m \geq 1$ *such that for all* ρ *there is a constant* $H(\rho)$ *such that any* $\rho-$ *thin essential hallway of length* $2m$ *and girth at least* H *is* $\lambda-$*hyperbolic.*

The main theorem of Bestvina and Feighn which ensures the hyperbolicity of trees of hyperbolic spaces is as follows:

Theorem 4.1.5. *(Bestvina and Feighn [BF92]) Let* X *be a tree of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition and the hallways flare condition. Then* X *is hyperbolic.*

Using the theorem of Bestvina and Feighn, in [MR08], a combination theorem for trees of strongly relatively hyperbolic spaces was proved, where the following new condition was introduced :

Definition 4.1.6. Cone-bounded hallways strictly flare condition *[MR08]:*

The tree of spaces, X*, is said to satisfy the Cone-bounded hallways strictly flare condition if there are numbers* $\lambda > 1$ *and* $m \geq 1$ *such that any cone-bounded hallway of length* $2m$ *is* λ*-hyperbolic.*

Theorem 4.1.7. *(Mj-Reeves) [MR08]*

Combination Theorem for trees of strongly relatively hyperbolic spaces:

Let $\mathbb{P} : X \to T$ *be a tree of strongly relatively hyperbolic spaces satisfying*

1. *the qi-embedded condition (refer to Definition 1.3.1)*

2. *the strictly type-preserving condition*

3. *the qi-preserving electrocution condition*

4. the induced tree of coned-off spaces satisfies the hallways flare condition

5. the cone-bounded hallways strictly flare condition

Then X is strongly hyperbolic relative to the family $\mathcal{C}$ of maximal cone-subtrees of horosphere-like spaces.

4.1.2 Examples

Let S be a hyperbolic surface of finite volume with finitely punctures and $\mathcal{MCG}(S)$ be the mapping class group of S. Let $\Phi_1, ..., \Phi_m \in \mathcal{MCG}(S)$ be m orientation preserving pseudo-Anosov homeomorphisms of S preserving punctures with different sets of stable and unstable singular foliations. By taking a suitable power, we can assume that each homeomorphism Φ_i fixes punctures. For each puncture $p \in S$, there exists an open region $N(p)$ around p such that $N(p)$ is homeomorphic to interior of $\mathbb{D}^2 \setminus (0, 0)$. Let N be the compact surface with boundary obtained from S by deleting $N(p)$ from S for each puncture p, then $int(N)$ admits a hyperbolic structure. Each Φ_i induces a homeomorphism (with same notation) from N to N fixing the boundary components.

Construct a complex M as follows: Let $N_1, ..., N_m$ be m homeomorphic copies of N via homeomorphisms $f_i : N_i \to N$, where $i = 1, ..., m$. Let M be the quotient space obtained from $(\bigcup_{i=1}^{m} N_i \times [0, 1]) \cup N$ by identifying $(x_i, 0) \sim f_i(x_i), (x_i, 1) \sim \Phi_i(f_i(x_i))$, for $x_i \in N_i$, $i = 1, ..., m$.

For a puncture $p \in S$, let $K_p = \pi_1(N(p))$. Then $\pi_1(S)$ is hyperbolic relative to the finite collection of subgroups $\{K_p : p \text{ is a puncture of } S\}$. $K(p)$'s are often called peripheral subgroups. Let $\mathcal{N} = \{N(p) : p \text{ is a puncture}\}$ and $\mathcal{E}(N, \mathcal{N})$ be the electric space (or coned-off space) obtained from N by coning each $N(p)$ to a single point. Let α be a geodesic in $\mathcal{E}(N, \mathcal{N})$ and $\alpha^b = \alpha \setminus \cup_{N(p) \in \mathcal{N})} N(p)$ then components of α^b are geodesics in N. Let $\Phi \in \mathcal{MCG}(S)$ be such that Φ is an orientation preserving pseudo-Anosov homeomorphism fixing the punctures. We say Φ stretch α by a factor of k if the length of each component of $\Phi(\alpha^b)$ is greater than k-times the length of that component of α^b. The following lemma plays a crucial role in proving the relative hyperbolicity of the fundamental group $\pi_1(M)$ (viz. Theorem 4.1.9). This is essentially the generalization of Mosher's 'three out of four stretch' ([Mos97]) lemma.

Lemma 4.1.8. *[MR08] For any $k > 1$, there exists positive integers $r_1, ..., r_m$ such that for any geodesic α in $\mathcal{E}(N, \mathcal{N})$, at least $2m - 1$ elements of the set $\{\Phi_1^{r_1}, \Phi_1^{-r_1}, ..., \Phi_m^{r_m}, \Phi_m^{-r_m}\}$ stretch α by a factor of k.*

As an application of the Combination Theorem 4.1.7, we have:

Theorem 4.1.9. *[MR08] Let S be a hyperbolic surface of finite volume with finitely many punctures. Let $\Phi_1, ..., \Phi_m \in \mathcal{MCG}(S)$ be m orientation preserving pseudo-Anosov homeomorphisms of S with different sets of stable and unstable foliations. Then there are positive integers $n_1, ..., n_m$ such that the homeomorphisms $\Phi_1^{n_1}, ..., \Phi_m^{n_m}$ generate a free group F and the group $\pi_1(M)$ is given by the short exact sequence:*

$$1 \to \pi_1(S) \xrightarrow{i} \pi_1(M) \xrightarrow{p} F \to 1$$

and $\pi_1(M)$ is (strongly) hyperbolic relative to the finite collection of parabolic subgroups $\{N_G(K_p) : p \text{ is a puncture of } S\}$.

4.2 Applications

1) Let S be a hyperbolic surface of finite volume with finitely many punctures and let $\Phi_1, ..., \Phi_m$ be m orientation preserving pseudo-Anosov homeomorphisms of S fixing punctures. Recall from the above example 4.1.2 that N is a compact surface with boundary obtained from S by deleting a disc around a puncture. Let M be the complex as constructed in the above example 4.1.2. Let $\widetilde{S}, \widetilde{N}$ and $\widetilde{M}$ be the universal covers of S, N and M respectively. Then $\widetilde{N}$ is obtained from $\widetilde{S}$ by deleting horoballs corresponding to the punctures.

From above we have a short exact sequence of relatively hyperbolic groups

$$1 \to \pi_1(N) \xrightarrow{i} \pi_1(M) \xrightarrow{p} F \to 1,$$

where F is a free group generated by $\Phi_1^n, ..., \Phi_m^n$ for some large n. Let $\Gamma_N, \Gamma_M, \Gamma_F$ be the Cayley Graphs of $\pi_1(N), \pi_1(M), F$ respectively. Since F is free, Γ_F is a tree. Γ_M can be treated as a tree of spaces with vertex and edge spaces homeomorphic to Γ_N and the tree as Γ_F . Each Φ_i $(1 \leq i \leq m)$ induces an automorphism Φ_i^* of the fundamental group of N. Hence each Φ_i^* induces a (K_i, ϵ_i)-quasi-isometry from Γ_N to Γ_N. Thus edge spaces in the tree of spaces are quasi-isometrically embedded in the vertex spaces. Since Φ_i fixes punctures of N, $\pi_1(M)$ preserves punctures of N. Now $\widetilde{N}, \widetilde{M}$ are quasi-isometric to the Cayley graphs Γ_N, Γ_M respectively. Therefore $\widetilde{M}$ can be treated as a tree of spaces with vertex and edge spaces homeomorphic to $\widetilde{N}$.

By Theorem 3.3.5, a Cannon-Thurston map exists for the inclusion $i : \widetilde{N} \to \widetilde{M}$ i.e. the inclusion i can be extended continuously to a map $\tilde{i} : \partial_{rel}\widetilde{N} \to \partial_{rel}\widetilde{M}$, where $\partial_{rel}\widetilde{N}, \partial_{rel}\widetilde{M}$ are the relative hyperbolic boundaries of relatively hyperbolic groups $\pi_1(N), \pi_1(M)$ respectively. Now $\partial_{rel}\widetilde{N}$ is homeomorphic to the unit circle $\mathbb{S}^1$.

Let $H = \pi_1(N)$, then H acts on the relative hyperbolic boundary $\partial_{rel}\widetilde{M}$. Let ΛH be the limit set of H in $\partial_{rel}\widetilde{M}$. As H is a normal subgroup of $\pi_1(M)$, $\Lambda H = \partial_{rel}\widetilde{M}$ and therefore $\tilde{i}(\partial_{rel}(N)) = \Lambda H = \partial_{rel}\widetilde{M}$. Since $\partial_{rel}(N)$ is homeomorphic to $\mathbb{S}^1$, we have an example of space filling curve.

2) [Bow07, Mja] Let S^h be a punctured hyperbolic surface with finite volume. Let $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ be a discrete and faithful representation such that the 3-manifold $N^h = \mathbb{H}^3/\rho(\pi_1(S))$ has injectivity radius bounded below by some $\epsilon_0 > 0$ and maximal parabolic subgroups of $\pi_1(N^h)$ are precisely the parabolic subgroups of $\pi_1(S)$. (Injectivity radius is the half of the length of the shortest closed geodesic in N). Let N denote N^h minus cusps, S denote S^h minus cusps and $\widetilde{S}$ be the universal cover of S. Fix a base surface in N and identify it with S. In [Mja] it is shown that the universal cover $\widetilde{N}$ of N is quasi-isometric to a tree T of relatively hyperbolic metric spaces where each vertex and edge spaces are copies of $\widetilde{S}$. And T is a semi-finite interval or a bi-infinite interval in $\mathbb{R}$ according as N is one or two ended.
Let $i^h \colon S^h \to N^h$ be a proper homotopy equivalence then i^h induces a proper homotopy equivalence $i : S \to N$. Let $\tilde{i}$ be a lift of i to their universal covers. Then by Theorem 3.2.9, $\tilde{i} \colon \widetilde{S} \to \widetilde{N}$ extends continuously to the relative hyperbolic boundary $\bar{i} \colon \partial_{rel}\widetilde{S} \to \partial_{rel}\widetilde{N}$. Let Λ denotes the limit set of $\pi_1(S)$ in $\partial_{rel}\widetilde{N}(= \mathbb{S}^2)$, then $\bar{i}(\partial_{rel}\widetilde{S}) = \Lambda$. Now $\partial_{rel}\widetilde{S}$ is homeomorphic to $\mathbb{S}^1$ and continuous image of a compact locally connected space is locally connected ([HY61]). Therefore Λ is locally connected.

4.2.1 Problems

1. Let us consider the short exact sequence of hyperbolic groups

$$1 \to K \xrightarrow{i} G \to Q \to 1$$

with K non-elementary. It was shown by Mitra in [Mit98a] that a Cannon-Thurston $\tilde{i}$ map exists for i. In [Mit97], Mitra gave an explicit description of the Cannon-Thurston map $\tilde{i}$. It was proved (in [Mit97]) that the end points (in ∂K) of a "leaf" of an "ending lamination" are precisely the points which are identified to a single point in ∂G under $\tilde{i}$. In reference to Theorem 3.3.5, analogously, for the short exact sequence of relatively hyperbolic groups

$$1 \to (K, K_1) \xrightarrow{i} (G, N_G(K_1)) \xrightarrow{p} (Q, Q) \to 1,$$

give an explicit description of the Cannon-Thurston map $\tilde{i}$.

2. In [Bow02], Bowditch constructed a “Stack” of hyperbolic spaces which roughly consists of a path metric space decomposed into “sheets” of uniformly hyperbolic spaces. We assume that the stack is hyperbolic. For a closed hyperbolic surface S, a *hyperbolic surface stack* consists of a proper hyperbolic stack of hyperbolic planes isometric to $\mathbb{H}^2$, together with a sheet preserving isometric action of $\pi_1(S)$, such that the induced action on each sheet is properly discontinuous and cocompact. The main theorem of [Bow02] states that if there are two stacks arising from a surface and having the same ending lamination then there is an equivariant sheet-preserving quasi-isometry between them. In view of relative hyperbolicity, the whole theory should have generalizations for non-compact hyperbolic surfaces.

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