ESTIMATION OF SPECTRAL VARIATION

RAJENDRA BHATIA

Thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the award of

DOCTOR OF PHILOSOPHY

NEW DELHI
1980

CONTENTS

A C K N O W L E D G E M E N T S

While working on this thesis, I have benefitted greatly from my association with several individuals and institutions, to all of whom I am deeply indebted.

I am thankful to Professor K.R. Parthasarathy for his teaching of analysis starting from almost the first principles, for the countless hours he spent in discussion, for the effective shield he provided against troublesome bureaucrats and for the colour and spice he added to life. I am thankful to Professor Kalyan K. Mukherjea for plugging many gaps in my knowledge, for suggesting the basic methods used in this work and for actively overseeing their development, for offering practical advice on several occasions and for giving timely boosts to a periodically ebbing morale. I am thankful to Professor C.S. Seshadri for his interest in my work and for pointing out several problems in diverse areas connected with it. To all three of them I am thankful for their help, encouragement, courtesy and generosity.

Most of this work was done at the Indian Statistical Institute, New Delhi and I am thankful to its authorities for the facilities and the fellowship I was provided with. I am thankful to my friends at the Institute, and at other institutions where I spent varying amounts of time, for their help and assistance on many occasions.

INTRODUCTION

The study of spectra occupies a central place in the theory of linear operators. One part of this study consists of obtaining a detailed and complete knowledge of the spectrum of a given linear operator. The other part - perturbation theory - consists of using this knowledge to obtain information about the spectra of nearby operators.

Apart from the intrinsic mathematical interest it has, perturbation theory is of great importance in the study of several physical problems. In fact, the theory came into existence with the work of Rayleigh on sound waves and that of Schrodinger on quantum mechanics. Later, their results were put on a firm mathematical ground by Rellich and developed further by him and several other mathematicians. A detailed account of these results may be found in Kato [12].

When the underlying linear space is finite-dimensional the spectrum of an operator consists of its eighevalues. Though the finite - dimensional theory is simpler it is not trivial. (See the books by Kato [12], Rellich [17] and Baumgartel [1]). The main problems of the subject can be classified into three types of questions which have bearings upon each other. These are:

1. If $z \to A(z)$ is a holomorphic operator - valued map based on a complex domain, are the eigenvalues, eigenprojections and eigennilpotents holomorphic function of z ? This question has been discussed in detail in the three books mentioned above.

2. What are the power series expansions for the objects in Question 1 ? What are the radii of convergence of these series and what are the error - estimates when only a finite number of terms of these series are taken into account ? This question has also been considered in detail in the books refered to above.

3. If the operators A and B are close to each other how close are the eigenvalues and eigenvectors of A to those of B ?

This thesis is concerned primarily with the third question about eigenvalues. We introduce some new methods for the study of this problem and using them obtain explicit estimates for the distance between the eigenvalues of two operators.

Apart from the viewpoint of perturbation theory this problem is of interest and importance in computational linear algebra and in approximation theory. Here the problems can be stated in the following terms. If the entries of a matrix are known approximately, to what degree of approximation are the eigenvalues known ? If the entries of a matrix can be ascertained only upto a certain decimal place, to how many places should its eigenvalues be computed ? If a sequence A_n of operators converges to A, how fast do the eigenvalues of A_n converge to those of A ?

These questions have been analysed by several authors. We give below a brief summary of the prominent results germane to our study. These are given not in the chronological order in which they were obtained but in the ascending order of generality to which they pertain.

The distance between operators can be measured using several norms. For several reasons the most important ones are the Banach norm $||\cdot||_B$ and the Frobenius norm $||\cdot||_F$ defined respectively as

$$||A||_B = \sup_{||x|| = 1} ||Ax|| ,$$

$$||A||_F = (\text{trace } A^*A)^{1/2}$$

The distance between eigenvalues can also be measured in several ways. We use two such distances. Let $\text{Eig } A = \{\alpha_1, \ldots , \alpha_n\}$ and $\text{Eig } B = \{\beta_1, \ldots , \beta_n\}$ denote the eigenvalues of A and B respectively each counted as many times as its algebraic multiplicity. Define

$$d(\text{Eig } A, \text{Eig } B) = \min_{\sigma} \max_{i} |\alpha_i - \beta_{\sigma(i)}| ,$$

$$\delta(\text{Eig } A, \text{Eig } B) = \min_{\sigma} \left(\sum_i |\alpha_i - \beta_{\sigma(i)}|^2 \right)^{1/2}$$

where σ runs over all permutations of the n indices.

With these notations we can state the important results on this problem :

(A) Results for Hermitian Operators

1. Weyl's Inequalities : Let A and B be Hermitian with eigenvalues $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ respectively. Then from the minmax inequalities which can be traced back to H.Weyl [19] we have

$$\max_{1 \le j \le n} |\alpha_j - \beta_j| \le ||A-B||_B$$

In particular, this implies

$$d(\text{Eig } A, \text{Eig } B) \le ||A-B||_B \tag{0.1}$$

2. <u>Lidskii's Theorem</u> (Lidskii [14]). Let A, B be Hermitian operators with eigenvalues $\alpha_1 \le \alpha_2 \le \dots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \dots \le \beta_n$ respectively. Let the eigenvalues of $B-A$ be $\gamma_1, \gamma_2, \dots, \gamma_n$. Then the vector $(\beta_1 - \alpha_1, \dots, \beta_n - \alpha_n)$ in $\mathbb{R}^n$ lies in the convex hull generated by the vectors $(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)})$, where σ runs over all permutations of n indices. Note that this theorem puts a bound on the magnitude of $(\beta_1 - \alpha_1, \dots, \beta_n - \alpha_n)$ in terms of the magnitude of $\gamma_1, \dots, \gamma_n$.

A generalisation of this theorem for power series with Hermitian operator coefficients is given in Parthasarathy [16].

3. <u>The work of Davis and Kahan</u> (Davis [4], [5], Davis and Kahan [6], Kahan [11]). In [4] Davis studied the following problem: how much should the eigenvectors of a Hermitian operator A be rotated to obtain the eigenvectors of a neighbouring Hermitian operator B ? He obtained a lower bound for the change of eigenvalues in this case as one of his results. Further results on this problem were obtained by him and Kahan in [6]. In [11] Kahan obtained a rather curious inequality stated below. Let A be a Hermitian operator and B any other operator on an n-dimensional space. Then

$$d(\text{Eig. } A, \text{Eig } B) \le (\log_2 n + 2.038)\ ||A-B||_B.$$

(B) Results for normal operators

1. The Hoffman - Wielandt Inequality (Hoffman and Wielandt [10]). If A and B are normal operators on a finite-dimensional vector space then this inequality can be stated as

$$\delta(\text{Eig } A, \text{Eig } B) \leq ||A-B||_F \qquad (0.2)$$

Note the similarity and the differences between (0.1) and (0.2). Attempts to establish (0.1) when A and B are normal have not been successful. However for unitary operators we have the following:

2. Parthasarathy's Inequality (Parthasarathy [16]). Let U,V be unitary operators and let K be a Hermitian operator such that $UV^{-1} = \exp(iK)$. Then

$$d(\text{Eig } U, \text{Eig } V) \leq ||K||_B$$

(C) General Results

1. Ostrowski's Bound (Ostrowski [15]). Let $A = ((a_{ij}))$ and $B = ((b_{ij}))$ be two $n \times n$ matrices. Then

$$d(\text{Eig } A, \text{Eig } B) \leq 2n(n+2)\, K\, \delta^{1/n} \quad ,$$

where

$$K = \max(|a_{ij}| \, , \, |b_{ij}|) \quad ,$$

$$\delta = \frac{1}{nK} \sum_{i,j} |a_{ij} - b_{ij}|$$

2. Henrici's Bound (Henrici [9]). This bound involves a measure of non-normality of matrices and is a bit involved to state. See Section 7 of Chapter I in this thesis for the statement of this result.

One of the main results obtained in this thesis can be stated as:

Theorem Let A,B be any two linear operators on an n-dimensional space. Then

$$d(\mathrm{Eig}\ A,\ \mathrm{Eig}\ B)$$

$$\leq (2n-1)\ \{\sum_{k=1}^{n} k^{1-k/2} \binom{n}{k} (m + ||B-A||_F)^{k-1} M^{n-k}\}^{1/n}\ ||B-A||_F^{1/n}\ .$$

where

$$m = \min(||A||_F\ ,\ ||B||_F)$$

$$M = \max(||A||_F\ ,\ ||B||_F)\ .$$

This result has some theoretical and practical advantages over those of Ostrowski and Henrici which we discuss in Section 7 of Chapter I. We give two numerical examples to show that this bound compares favorably with the other two bounds. During our analysis we also obtain an estimate for the distance between the characteristic polynomials of two operators which is of independent interest. The method we develop can be adapted to give more special results in special cases some of which are investigated.

An important feature of the bounds for d(Eig A, Eig B) in the general case is the exponent 1/n. This makes the general result rather weak, in the sense that when $||A-B||$ is small the bound for d(Eig A, Eig B) is large in comparison. There are examples to show that in general, this order is best possible. However, in some special cases it can be improved. We identify some of these cases. We show that when A and B lie in the Lie algebra of complex skew-symmetric matrices or in the symplectic Lie algebra then d(Eig A^2, Eig B^2) is of order $||A-B||^{1/r}$ where r is the rank of these Lie algebras.

Finally, using a characterisation of similarity orbits and their tangent spaces, we introduce a geometric method by which it might become possible to obtain better inequalities. We show how such inequalities may be obtained in some special cases.

The organisation of the thesis is as follows. In Chapter I, after a few preliminary sections, we introduce our approach to the problem. We look upon the map $A \to$ Eig (A) as a map from the space of linear operators into the space $\mathbb{C}^n_{sym}$ of unordered n-tuples of complex numbers. There is a homeomorphism $\bar{S} : \mathbb{C}^n_{sym} \to \mathbb{C}^n$ which takes the (unordered) roots of a polynomial to the (ordered) n-tuple of the coefficients of the polynomial. The composite map $\bar{S} \circ$ Eig is then a holomorphic map from the Banach space of linear operators to the Banach space $\mathbb{C}^n$. In Section 5 we obtain estimates for the derivative of this map. Using these estimates and the mean value theorem for calculus in Banach spaces we can then estimate the distance between the coefficients of the characteristic polynomials of two linear operators. In Section 7 we combine these results with a theorem of Ostrowski on roots of polynomials to obtain estimates for the distance between the eigenvalues of two operators. Chapter II is divided into two independent parts.

In Part A we show how the order of these estimates can be improved in some special cases. In Part B we show how to connect two operators by a suitable path so that a "bad part" of spectral variation may be isolated. Some topics arising out of these considerations are dealt with in two appendices.

CHAPTER I

THE RATE OF CHANGE OF SPECTRA

In this chapter, our principal aim is, first, to put the study of spectral variation in its proper conceptual setting and then, using the general results obtained, to derive an estimate of the distance between the eigenvalues of two operators. We analyse first the following problem in a general setting : if f_t is a continuously varying family of endomorphisms of a finite-dimensional vector space, how does the spectrum - the set of eigenvalues - of f_t change with t ? In most of the work on this problem a recurring feature has been the problem of coherently ordering the eigenvalues of f_t as t varies. Since no globally coherent ordering exists in general, a lot of difficulties arise. Here, we regard the set of eigenvalues of an endomorphism as an <u>unordered n-tuple</u> i.e., as a point in the nth symmetric power of $\mathbb{C}$. This simplifies matters considerably and using some elementary methods of multilinear algebra and calculus in Banach spaces we can obtain an estimate for the rate of change of eigenvalues of f_t. We then use these estimates to obtain a bound for the distance between the eigenvalues of any two given operators.

1. The spaces $\mathcal{E}_\sigma(V)$ and $M(n)$

Let V be an n-dimensional complex vector space. We denote by $\langle \cdot,\cdot \rangle$ the usual Euclidean inner product on V and by $||\cdot||$ the associated Euclidean norm on V. The set of all linear operators on V (i.e. the set of all endomorphisms of V) will be denoted by $\mathcal{E}(V)$. The set of all $n \times n$ complex matrices will be denoted by $M(n, \mathbb{C})$ or simply by $M(n)$. When a basis has been chosen for V every element A of $\mathcal{E}(V)$ has a unique element of $M(n)$ associated with it, viz., the matrix of A in this basis. We denote this matrix also by A and we denote the entries of this matrix by $(a_{ij})_{1 \leq i, j \leq n}$.

The spaces $\mathcal{E}(V)$ and $M(n)$ are complex vector spaces of dimension n^2. Among the several norms that can be defined on these spaces, particularly significant are those that are unitary invariant.

<u>Definition 1.1.</u> A norm $||.||$ on $\mathcal{E}(V)$ or $M(n)$ is called <u>unitary invariant</u> if for all A we have

$$||UAU^{-1}|| = ||A|| ,$$

for every unitary operator (matrix) U.

For such a norm the matrix representation of an operator A in every orthonormal basis for V has the same norm.

Two most frequently used unitary invariant norms are the ones defined below.

(1) The <u>Banach norm</u> or the <u>spectral norm</u> of A, which we shall denote as $||A||_B$, is defined as

$$||A||_B = \sup \{||Ax|| : x \in V, ||x|| = 1\} .$$

Let A^* denote the adjoint of A. Then A^*A is a positive operator. The norm $||A||_B$ can also be characterised as the positive square root of the maximum eigenvalue of A^*A.

(2) The Frobenius norm or the Hilbert-Schmidt norm of A, which we shall denote by $||A||_F$, is defined as

$$||A||_F = \operatorname{tr}(A^*A)^{1/2} ,$$

where $\operatorname{tr} A$ denotes the trace of A.

If A is the matrix $(a_{ij})_{1\le i,j\le n}$ then we have

$$||A||_F = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$$

The relation between these two norms is given by the inequalities

$$||A||_B \le ||A||_F \le n^{1/2} ||A||_B \qquad (1.1)$$

We shall use these norms in our analysis, switching from one to the other when convenient.

2. The maps Σ and Eig

We denote by $\Sigma(A)$ the spectrum of an element A of $\mathcal{L}(V)$, i.e. the subset of the complex plane $\mathbb{C}$ whose elements are eigenvalues of A. The cardinality of $\Sigma(A)$ is at most the dimension n of V. In particular, $\Sigma(A)$ is a closed subset of $\mathbb{C}$.

Let $\lambda_1, \ldots, \lambda_n$ be the n eigenvalues of A, each counted as many times as its multiplicity. (By multiplicity we shall always mean the algebraic multiplicity. We denote by Eig(A) the unordered n-tuple $\{\lambda_1, \ldots, \lambda_n\}$.

Let Π_n denote the group of permutations on n symbols. The group Π_n acts naturally on the space $\mathbb{C}^n$, giving rise to a quotient space $\mathbb{C}^n|\Pi_n$ which we denote as $\mathbb{C}^n_{sym}$. An unordered n-tuple $\{x_1,\ldots,x_n\}$ of complex numbers is a point in $\mathbb{C}^n_{sym}$. It is the equivalence class of the n-tuple $(x_1,\ldots,x_n)$ in $\mathbb{C}^n$, where two points are regarded as equivalent if they can be obtained from each other by a permutation of their coordinates.

Thus Σ and Eig are maps from $\mathcal{E}(V)$ into the space of closed subsets of $\mathbb{C}$ and the space $\mathbb{C}^n_{sym}$, respectively. Both these spaces are metric spaces with the metrics defined below.

Let X, Y be closed subsets of $\mathbb{C}$. Let

$$v(X,Y) \qquad \sup_{x \in X} \operatorname{dist}(x,Y)$$

$$= \sup_{x \in X} \inf_{y \in Y} |x-y|$$

Let

$$h(X,Y) = \sup(v(X,Y),\ v(Y,X)) .$$

This is called the <u>Hausdorff distance</u> between X and Y and it defines a metric on the class of all closed subsets of $\mathbb{C}$.

On $\mathbb{C}^n_{sym}$ we can define two natural metrics d and δ as follows. For any two elements $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$ of $\mathbb{C}^n_{sym}$ let

$$d(\{x_1,\ldots,x_n\},\ \{y_1,\ldots,y_n\})$$

$$\inf_{\sigma \in \Pi_n} \{ \sup_{1 \le i \le n} |x_i - y_{\sigma(i)}| \} \quad ,$$

$$\delta(\{x_1,\ldots,x_n\}\ ,\ \{y_1,\ldots,y_n\})$$

$$= \inf_{\sigma \in \Pi_n} \Big(\sum_{1 \le i \le n} |x_i - y_{\sigma(i)}|^2 \Big)^{1/2} .$$

The relation between these two metrics is given by the inequalities

$$d \le \delta \le n^{1/2}\ d$$

The relation between the Hausdorff distance and these metrics is given by the following proposition.

<u>Proposition 2.1</u> Let $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$ be two points in $\mathbb{C}^n_{sym}$. Let X,Y be subsets of $\mathbb{C}$ consisting respectively of the distinct elements from these n-tuples. Then

$$h(X,Y) \le d(\{x_1,\ldots,x_n\}\ ,\ \{y_1,\ldots,y_n\}) \qquad (2.2)$$

<u>Proof</u> With the notations used above in defining h, we have

$$v(X,Y) = \sup_{1\le i\le n}\ \inf_{1\le j\le n}\ |x_i - y_j|$$

$$= \sup_{1\le i\le n}\ \inf_{\sigma \in \Pi_n}\ |x_i - y_{\sigma(i)}|$$

Note that for all $i = 1,2,\ldots,n$ and $\alpha \in \Pi_n$ we have

$$\inf_{\sigma \in \Pi_n} |x_i - y_{\sigma(i)}| \leq |x_i - y_{\alpha(i)}|$$

$$\leq \sup_{1 \leq j \leq n} |x_j - y_{\alpha(j)}|$$

So,

$$\sup_{1 \leq i \leq n} \inf_{\sigma \in \Pi_n} |x_i - y_{\sigma(i)}|$$

$$\leq \inf_{\alpha \in \Pi_n} \sup_{1 \leq j \leq n} |x_j - y_{\alpha(j)}| .$$

In other words,

$$v(X,Y) \leq d(\{x_1,\ldots,x_n\} , \{y_1,\ldots,y_n\})$$

The right hand side of this inequality is symmetric in X and Y. So, it dominates $v(Y,X)$ as well. Hence it dominates $h(X,Y)$. □

The following example shows that strict inequality may hold in (2.2). (I should thank Professor Chandler Davis for clarifying this point to me).

<u>Example 2.2.</u> Let

$$\{x_1, x_2, x_3\} = \{2, -1, i\sqrt{3}\} ,$$

$$\{y_1, y_2, y_3\} = \{1, -2, -i\sqrt{3}\} .$$

Then

$$h(X,Y) = 2$$

$$d(\{x_1,x_2,x_3\} , \{y_1,y_2,y_3\}) = \sqrt{7}$$

<u>Remark 2.3.</u> Proposition 2.1 shows, in particular, that for $A, B \in \mathcal{E}(V)$ we have

$$h(\Sigma(A), \Sigma(B)) \leq d(\mathrm{Eig}(A), \mathrm{Eig}(B)). \qquad (2.3)$$

3. The symmetriser map

In the preceding section we defined the map Eig : $\mathcal{E}(V) \to \mathbb{C}^n_{sym}$. We wish to study the qualitative and quantitative behaviour of this map. The space $\mathbb{C}^n_{sym}$ is not easy to work with. However, it is homeomorphic to $\mathbb{C}^n$ as the following well known proposition shows. In the proof of the proposition we construct this homeomorphism explicitly. The key to our approach lies in exploiting this homeomorphism.

<u>Proposition 3.1</u> The spaces $\mathbb{C}^n_{sym}$ and $\mathbb{C}^n$ are homeomorphic.

<u>Proof</u> For $i = 1,2,\ldots,n$ let $s_i(x_1,\ldots,x_n)$ be the ith elementary symmetric function of the n variables $x_1,\ldots,x_n$, defined as

$$s_i(x_1,\ldots,x_n) = \sum_{1\leq r_1<r_2<\ldots<r_i\leq n} x_{r_1}x_{2}\cdots x_{r_i}$$

Let $: \mathbb{C}^n \to \mathbb{C}^n$ be the map defined as

$$S(x_1,\ldots,x_n) = (s_1(x_1,\ldots,x_n),\ldots,s_n(x_1,\ldots,x_n)).$$

The map is continuous and thus induces a continuous map $\bar{S}: \mathbb{C}^n_{sym} \to \mathbb{C}^n$ such that the diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "S"] \arrow[d, "\Pi_n"'] & \mathbb{C}^n \\ \mathbb{C}^n_{sym} \arrow[ur, "\bar{S}"'] & \end{tikzcd}$$

commutes.. Note that $s_1(x_1,\ldots,x_n), \ldots , s_n(x_1,\ldots,x_n)$ are the coefficients (upto a sign) of the monic polynomial of degree n of which $x_1,\dots,x_n$ are the roots. So the map $\bar{S}$ is bijective. The map Π_n, being a quotient map for the action of a finite group, is open. The map S, being a nonconstant holomorphic map, is open too. Hence $\bar{S}$ is an open map.

Remark 3.2 This identification between $\mathbb{C}^n_{sym}$ and $\mathbb{C}^n$, via the symmetriser map $\bar{S}$, allows us to look upon $\mathbb{C}^n_{sym}$ as a complex manifold and a vector space.

Remark 3.3 If the characteristic polynomial of an element A of $\mathcal{E}_0(V)$ is written as

$$t^n - \phi_1(A)t^{n-1} + \dots + (-1)^n \phi_n(A)$$

and, if

$$\mathrm{Eig}(A) = \{\lambda_1,\dots,\lambda_n\}$$

then,

$$\bar{S}(\{\lambda_1,\dots,\lambda_n\}) = (\phi_1(A),\dots,\phi_n(A)) .$$

Remark 3.4 Proposition 3.1 can be generalised. If W is any finite group generated by reflections in $\mathbb{C}^n$ then the spaces $\mathbb{C}^n|W$ and $\mathbb{C}^n$ are homeomorphic. In Appendix 1 we display these homeomorphisms explicitly for some groups of this type.

4. The map Φ.

We can now outline our approach to the problem. Let X be any interval on the real line (or any connected subset of a Banach space). Let $f : X \to \mathcal{E}(V)$ be a C^1 - map. We wish to study the behaviour of the compositie map Eig o f from X into $\mathbb{C}^n_{sym}$. Since $\bar{S}$ is a homeomorphism from $\mathbb{C}^n_{sym}$ into $\mathbb{C}^n$, we can, equivalently, study the behaviour of the map $\bar{S} \circ \mathrm{Eig} \circ f$. Denote the composite map $\bar{S} \circ \mathrm{Eig}$ by Φ. Then the study of the variation of the spectrum of an operator-valued differentiable map splits up naturally into two parts. First we study the map $\Phi : \mathcal{E}(V) \to \mathbb{C}^n$ and then the map $f: X \to \mathcal{E}(V)$. This separation of the problem into its "universal" and "particular" parts allows great flexibility which will become apparent in later sections. The following diagram explains the situation.

$$\begin{tikzcd}
X \arrow[r, "f"] & \mathcal{E}(V) \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{Eig}"'] & & \mathbb{C}^n & \\
& & \mathbb{C}^n_{sym} \arrow[ur, "\bar{S}"'] & & \mathbb{C}^n \arrow[ul, "S"'] \arrow[ll, "\Pi_n"]
\end{tikzcd}$$

Following this approach we obtain estimates for the map Φ. As remarked earlier, $\Phi(A)$ has as its components the coefficients of the characteristic polynomial of A. Our estimates can then be converted into estimates for the map Eig using the results of Ostrowski on roots of polynomials.

5. Estimates for the derivative of ψ

We shall use some elementary properties of exterior products (Grassman products) of vector spaces. (See, e.g., Lang [13]). Some of these properties are briefly recalled below.

If V is a vector space of dimension n then its kth exterior power is a vector space of dimension $\binom{n}{k}$. This vector space is denoted as $\Lambda^k V$, $1 \leq k \leq n$. The exterior product of k vectors $v_1,\ldots,v_k$ in V is an element of $\Lambda^k V$ denoted as $v_1 \wedge v_2 \wedge \ldots \wedge v_k$. This product is linear in each of its variables and is alternating i.e. it vanishes in case $v_i = v_j$ for some $i \neq j$. If $e_1,\ldots,e_n$ is a basis for V then the set

$$\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\}$$

is a basis for $\Lambda^k V$. If V has an inner product $\langle \cdot,\cdot \rangle$ then an inner product on $\Lambda^k V$ can be defined as follows. For two k-vectors $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ and $w_1 \wedge w_2 \wedge \ldots \wedge w_k$ in $\Lambda^k V$ define the inner product between them as

$$\langle (v_1 \wedge \ldots \wedge v_k), (w_1 \wedge \ldots \wedge w_k) \rangle = \det (\langle v_i, w_j \rangle)_{1 \leq i,j \leq k}$$

where det denotes the determinant. This defines an inner product on all basis vectors in $\Lambda^k V$ and can be extended bilinearly to all of $\Lambda^k V$. If $e_1,\ldots,e_n$ is an orthonormal basis for V then $\{e_{i_1} \wedge \ldots \wedge e_{i_k},\ 1 \leq i_1 < \ldots < i_k \leq n\}$ is an orthonormal basis for $\Lambda^k V$ with this inner product. We denote the space $\mathcal{E}(\Lambda^k V)$ as $(\mathcal{E}^k(V))$. For $A \in \mathcal{E}(V)$ and $v_1,\ldots,v_k \in V$ define

$$\Lambda^k A(v_1 \wedge \ldots \wedge v_k) = Av_1 \wedge \ldots \wedge Av_k \quad ,$$

and extend $\Lambda^k A$ to all of $\Lambda^k V$ linearly. This defines $\Lambda^k A$ as an element of $(\mathcal{E}^k(V))$. The map $\Lambda^k : \mathcal{E}(V) \to (\mathcal{E}^k(V))$ has the following two properties which we shall use.

$$\Lambda^k(AB) = \Lambda^k(A)\Lambda^k(B) \qquad \text{for all} \quad A,B \in \mathcal{E}(V),$$

$$(\Lambda^k(A))^* = \Lambda^k(A^*) \qquad \text{for all} \quad A \in \mathcal{E}(V).$$

Now, let $A \in \mathcal{E}(V)$ and let

$$\chi_A(t) = t^n - \phi_1(A)t^{n-1} + \cdots + (-1)^n \phi_n(A)$$

be the characteristic polynomial of A. It is well known that $\phi_k(A)$ can be characterised as the trace of the operator $\Lambda^k A$ and also as the sum of all $k \times k$ principal minors in a matrix representation of A. In other words,

$$\phi_k(A) = \operatorname{tr} \Lambda^k A$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \det \begin{vmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_k} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{i_k i_1} & \cdots & a_{i_k i_k} \end{vmatrix}$$

for $k = 1, 2, \ldots, n$.

Remark 5.1. From this expression it is clear that the map Φ is holomorphic. Hence the map Eig is also holomorphic. The inequality (2.3) then implies that the map Σ is continuous.

Let $D\,\Phi(A)$ denote the derivative of Φ at A. This is a linear map from the Banach space $\mathcal{E}_o(V)$ into the Banach space $\mathbb{C}^n$. In the next few paragraphs we estimate the norm of this linear map.

Lemma 5.2. Let $\Delta_n : M(n) \to \mathbb{C}$ be the map which takes an nxn matrix to its determinant. The derivative of this map at a point A of $M(n)$ is given by the linear map $\tilde{A} : M(n) \to \mathbb{C}$ defined as

$$\tilde{A}(X) = \sum_{i,j=1}^{n} A_{ij}\, x_{ij}$$

where A_{ij} is the cofactor of the element a_{ij} of A and x_{ij} are the elements of the matrix X.

Proof: Identifying $M(n)$ with $\mathbb{C}^{n^2}$ and computing the Jacobian leads immediately to the statement.

Remark 5.3. If the operator $A \in \mathcal{E}_o(V)$ has the matrix $(a_{ij})_{1\le i,j\le n}$ in the orthonormal basis $e_1, e_2, \ldots, e_n$ of V then the operator $\Lambda^{n-1}(A)$ has the matrix $(A_{ij})_{1\le i,j\le n}$ in the orthonormal basis $\{e_{i_1} \wedge \ldots \wedge e_{i_{n-1}} \cdot 1 \le i_1 < \ldots < i_{n-1} \le n\}$ for $\Lambda^{n-1}(V)$. Thus the Frobenius norm of $D\,\Delta_n(A)$ - the derivative of Δ_n at A - is the same as the Frobenius norm of $\Lambda^{n-1}(A)$. This is not true for the Banach norm. In Appendix 2 we show how the Banach norm of $D\,\Lambda^k$ may be computed. The inequality we get is rather weak but is of independent interest and could perhaps be strengthened . This is why we use the Frobenius norm for our estimate of $D\Phi$.

<u>Lemma 5.4.</u> Let $y_1,\ldots,y_n$ be n nonnegative real variables subject to the constraint

$$y_1 + \ldots + y_n = M.$$

Then the function f defined as

$$f(y_1,\ldots,y_n) = \sum_{1\le i_1<i_2<\ldots<i_k\le n} y_{i_1}\cdots y_{i_k}$$

attains a maximum value given by

$$f_{max} = \binom{n}{k}\frac{M^k}{n^k}.$$

<u>Proof.</u> Let

$$g(y_1,\ldots,y_n) = \sum y_{i_1}\cdot\ldots\cdot y_{i_k}$$

where the summation runs over all permutations of k distinct indices $i_1, i_2,\ldots,i_k$ chosen from $1,2,\ldots,n$. Then

$$f(y_1,\ldots,y_n) = \frac{1}{k!}\; g(y_1,\ldots,y_n).$$

A simple calculation using Lagrange's method of undetermined multipliers shows that g attains a maximum value given by

$$g_{max} = \frac{n!}{(n-k)!}\;\frac{M^k}{n^k}$$

This proves the lemma.

<u>Proposition 5.5.</u> If V is a vector space of dimension n, then for every $A \in \mathcal{L}(V)$ we have

$$||\Lambda^k A||_F^2 \le \frac{1}{n^k}\binom{n}{k}\;||A||_F^{2k}.$$

<u>Proof.</u> By the properties of the function Λ^k mentioned in the beginning of this section we have

$$||\Lambda^k A||_F^2 = tr(\Lambda^k A)^* (\Lambda^k A)$$
$$tr\ \Lambda^k(A^*A) .$$

Let A^*A have eigenvalues $a_1, a_2, \ldots, a_n$. Notice that these are nonnegative real numbers. The eigenvalues of $\Lambda^k (A^*A)$ are $a_{i_1} a_{i_2} \cdots a_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$. So, we have

$$||A||_F^2 = \sum_{i=1}^{n} a_i ,$$

$$||\Lambda^k A||_F^2 = \sum_{\leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k}$$

An application of Lemma 5.4 completes the proof.

In view of Remark 5.3 we have

Corollary 5.6 For every $A \in \mathcal{E}(V)$, where V has dimension n, we have

$$||D\Delta_n(A)||_F \leq n^{1-n/2} ||A||_F^{n-1}$$

Our crucial estimate follows from this.

Theorem 5.7 : Let $A \in \mathcal{E}(V)$, where V has dimension n, and let $\phi_k(A) = tr(\Lambda^k A)$. Then

$$||D\phi_k(A)|| \leq k^{2-k/2} \binom{n}{k} ||A||_F^{k-1} ,$$

for $k = 1, 2, \ldots, n$.

Proof Denote the span of a set of vectors $v_1,\ldots,v_r$ by $[v_1,\ldots,v_r]$. Let $e_1,\ldots,e_n$ be an orthonormal basis for V. Let

$$Q_{i_1\cdots i_k} : [e_{i_1},\ldots,e_{i_k}] \to V$$

and

$$P_{i_1\cdots i_k} : V \to [e_{i_1},\ldots,e_{i_k}]$$

denote the inclusion map and the orthogonal projection map respectively. Let

$$\phi_{i_1\cdots i_k}(A) = P_{i_1\cdots i_k} A\, Q_{i_1\cdots i_k}$$

and

$$\psi_{i_1\cdots i_k}(A) = \Delta_k \begin{vmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_k} \\ \cdots & \cdots & \cdots \\ a_{i_k i_1} & \cdots & a_{i_k i_k} \end{vmatrix}$$

for $1 \le i_1 < i_2 < \cdots < i_k \le n$. Then

$$\phi_k(A) = \sum_{1\le i_1 < \cdots < i_k \le n} \psi_{i_1\cdots i_k}(A)$$

$$= \sum_{1\le i_1 < \cdots < i_k \le n} \Delta_k\ \phi_{i_1\cdots i_k}(A) .$$

By the chain rule of differentiation we have

$$D\psi_{i_1\cdots i_k}(A) = D\Delta_k(\phi_{i_1\cdots i_k}(A))\, D\phi_{i_1\cdots i_k}(A) .$$

$$D\psi_{i_1 \cdots i_k}(A) \;=\; D\Delta_k(\phi_{i_1 \cdots i_k}(A))\, D\phi_{i_1 \cdots i_k}(A) \,.$$

Since $\phi_{i_1 \cdots i_k}$ is a linear map we have

$$D\phi_{i_1 \cdots i_k}(A) \;=\; \phi_{i_1 \cdots i_k} \qquad \text{for all} \quad A \in \mathcal{E}(V) \tag{5.1}$$

Note that $\phi_{i_1 \cdots i_k}(A)$ is a $k \times k$ submatrix of the $n \times n$ matrix A.

Hence

$$||\phi_{i_1 \cdots i_k}(A)||_F \;\leq\; ||A||_F \tag{5.2}$$

For $i,j = 1,2,\ldots,n$ let E_{ij} *be the matrix whose (ij)th entry is* 1 and the rest of whose entries are all zero. Such matrices form an orthonormal basis for $M(n)$. In this basis $\phi_{i_1 \cdots i_k}$ has k^2 entries 1 and the remaining entries 0 So,

$$||\phi_{i_1 \cdots i_k}||_F \qquad k \tag{5.3}$$

Using (5.1) - (5.3) and Corollary 5.6 we obtain

$$||D\psi_{i_1 \cdots i_k}(A)||_F \;\leq\; k^{2-k/2}\, ||A||_F^{k-1}$$

The theorem follows. $\square$

Remark : Since $(\psi_1,\ldots,\psi_n)$ we have

$$||D\Phi(A)||_F \leq \sum_{k=1}^{n} k^{2-k/2} \binom{n}{k} ||A||_F^{k-1} .$$

For later use it is more convenient to have these estimates in a slightly different form.

Proposition 5.8 With notations as in Theorem 5.7, we have

$$||D\phi_k(A)||_B \leq k^{1-k/2} \binom{n}{k} ||A||_F^{k-1}$$

Proof We have

$$||P_{i_1\ldots i_k}||_B = ||Q_{i_1\ldots i_k}||_B \quad 1$$

Hence,

$$||\psi_{i_1\ldots i_k}||_B \leq 1.$$

Using (1.1) and Corollary (5.6) we have, therefore,

$$\begin{aligned} ||D\psi_{i_1\ldots i_k}(A)||_B &\leq ||D\Delta_k(\phi_{i_1\ldots i_k}(A))||_B \; ||\psi_{i_1\ldots i_k}||_B \\ &\leq ||D\Delta_k(\phi_{i_1\ldots i_k}(A))||_B \\ &\leq ||D\Delta_k(\phi_{i_1\ldots i_k}(A))||_F \\ &\leq k^{1-k/2} \; ||\phi_{i_1\ldots i_k}(A)||_F^{k-1} \\ &\leq k^{1-k/2} ||A||_F^{k-1} . \end{aligned}$$

The statement follows from this inequality.

Using this we can now obtain the distance between $\Phi(A)$ and $\Phi(B)$ or any two elements A and B of $\mathcal{E}(V)$. For this we use the mean value theorem. (See, e.g., Dieudonne[7]).

The Mean Value Theorem Let $X = [a,b]$ be any interval on the real line and let E be a Banach space. Let $f:X \to E$ be any differentiable map. Then we have

$$||f(x_2)-f(x_1)|| \leq |x_2-x_1| \sup_{x \in X} ||Df(x)||_B$$

for all $a < x_1 < x_2 < b$.

Now consider any differentiable map $f:X \to \mathcal{E}(V)$. Let $\phi_k:X \to \mathbb{C}$ be the composite map

$$F_k(x) = \phi_k(f(x)) \qquad k = 1,2,\ldots,n$$

Then by the chain rule of differentiation and the mean value theorem have

$$\begin{aligned} |F_k(x_2)-F_k(x_1)| &\leq |x_2-x_1| \sup_{x \in X} ||DF_k(x)||_B \\ &\leq |x_2-x_1| \sup_{x \in X} ||D\phi_k(f(x)||_B||Df(x)||_B \end{aligned} \tag{5.4}$$

or all $a < x_1 < x_2 < b$, $1 \leq k \leq n$.

This, together with the results obtained above, can be used to obtain results for any special cases one may be interested in. For example, we have:

__Theorem 5.9__ Let A, B be any two elements of $\mathcal{E}(V)$ where V is an n-dimensional unitary space. Then

$$|\phi_k(A)-\phi_k(B)| \leq k^{1-k/2} \binom{n}{k} (m + \|B-A\|_F)^{k-1} \|B-A\|_F \tag{5.5}$$

for all $k = 1,2,\ldots,n$, where

$$m = \min(\|A\|_F, \|B\|_F) .$$

__Proof__ Consider the map $f: \mathbb{R} \to \mathcal{E}(V)$ defined as

$$f(x) = A + x(B-A) .$$

Then,

$$f(0) = A , f(1) = B$$

and

$$(Df(x))(y) = y(B-A) \quad \text{for all} \quad x,y \in \mathbb{R} .$$

So, we have, using (5.4) and Proposition 5.8,

$$|\phi_k(B)-\phi_k(A)| \leq k^{1-k/2} \binom{n}{k} \sup_{0\leq x\leq 1} \|A+x(B-A)\|_F^{k-1} \|B-A\|_B$$

$$\leq k^{1-k/2} \binom{n}{k} (\|A\|_F + \|B-A\|_F)^{k-1} \|B-A\|_F$$

Interchanging A and B this leads to (5.5). $\square$

<u>Remark 5.10</u> Notice that the last factor, $||B-A||_F$, on the right hand side of (5.5) can be replaced by the potentially smaller quantity $||B-A||_B$, as steps in the proof show.

Since Φ is made up of components ϕ_k , we have,

$$||\Phi(A)-\Phi(B)||$$

$$\leq \sum_{k=1}^{n} k^{1-k/2} \binom{n}{k}(m+||B-A||_F)^{k-1} \; ||B-A||_F$$

where,

$$m = \min(||A||_F , \; ||B||_F) .$$

6. Ostrowski's Theorems

Results in this section are quoted from Ostrowski [15]. They give estimates for the distance between the roots of two complex polynomials in terms of the coefficients of the polynomials.

<u>Theorem 6.1</u> Let

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n ,$$

$$g(z) = z^n + b_1 z^{n-1} + \cdots + b_n$$

be two monic polynomials with complex coefficients and with roots $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_n$ respectively. Let

$$\mu_1 = \max_{1\le k\le n} |\alpha_k| ,$$

$$\mu_2 = \max_{1\le k\le n} |\beta_k| ,$$

$$\mu = \max (\mu_1,\mu_2) ,$$

$$\theta = \{ \sum_{k=1}^{n} |b_k-a_k| \mu^{n-k}\}^{1/n}$$

Then the roots α_k and β_k can be arranged in such a way that

$$|\alpha_k-\beta_k| < (2n-1)\theta$$

for all $k = 1,2,\ldots,n$. $\square$

<u>Remarks</u> The expression θ involves the roots α_k, β_k. We can replace μ by an expression involving the coefficients as follows. Let

$$\Gamma = \max_{1\le k\le n} (|a_k|^{1/k}, |b_k|^{1/k}) .$$

Then

$$\mu \le 2\Gamma$$

Thus we have, for a suitable arrangement of the roots,

$$|\alpha_k-\beta_k| < 2^{1/n} (2n-1) \{ \sum_{k=1}^{n} |b_k-a_k| \Gamma^{n-k}\}^{1/n} \tag{6.1}$$

for all $k = 1,2,\ldots,n$.

We have seen earlier that there is a homeomorphism $\bar{S}:\mathbb{C}^n_{sym} \to \mathbb{C}^n$. Inequality (6.1) gives an estimate for the distance between two points $\{\alpha_1,\dots,\alpha_n\}$ and $\{\beta_1,\dots,\beta_n\}$ in $\mathbb{C}^n_{sym}$ in terms of the distance between their images in $\mathbb{C}^n$ under this map.

Another theorem which gives "relative error bounds" is quoted below.

<u>Theorem 6.2</u> Let

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n$$

$$g(z) = z^n + b_1 z^{n-1} + \dots + b_n$$

be two monic polynomials with complex coefficients. Suppose $a_n \neq 0$, *i.e.*, the n roots $\alpha_1,\dots,\alpha_n$ of $f(z)$ are all nonzero. Suppose there exists a constant R such that

$$0 < 4\, n\, R^{1/n} \leq 1$$

and

$$|b_k - a_k| \leq R|a_k| \quad \text{for all} \quad 1 \leq k \leq n\,.$$

Then the n roots $\beta_1,\dots,\beta_n$ of $g(z)$ can be ordered in such a way that we have for all $k = 1,2,\dots,n$

$$\left| \frac{\beta_k}{\alpha_k} - 1 \right| \leq 8\, n\, R^{1/n}$$

$\square$

7. Distance between the eigenvalues of two operators

Our results in Section 5, together with Ostrowski's theorems, yield some estimates of the distance between the eigenvalues of two operators.

Note that if A has eigenvalues $\alpha_1, \ldots, \alpha_n$ then we have

$$\max_k |\alpha_k| \leq ||A||_B \leq ||A||_F \qquad (7.1)$$

Using this fact, Theorem 5.9 and Theorem 6.1, we obtain

Theorem 7.1 For any two operators A and B on an n-dimensional unitary space, we have

$$d(\mathrm{Eig}(A), \ \mathrm{Eig}(B))$$

$$\leq (2n-1) \{ \sum_{k=1}^{n} k^{1-k/2} \binom{n}{k} (m+||B-A||_F)^{k-1} M^{n-k} \}^{1/n} ||B-A||_F^{1/n} , \qquad (7.2)$$

where

$$m = \min(||A||_F \ , ||B||_F) \ ,$$

$$M = \max(||A||_F \ , ||B||_F)$$

Remarks In (7.2) we have used the Frobenius norm for all operators. This inequality can be strengthened slightly. First note that because of (7.1) and the definition of μ_1 and μ_2 in Theorem 6.1 we can replace M in (7.2) by $M_1 = \max(||A||_B \ , ||B||_B)$. Because of Remark 5.10 we can also replace the last factor $||B-A||_F^{1/n}$ on the right hand side of (7.2) by $||B-A||_B^{1/n}$. Thus we have

$$d(\mathrm{Eig}(A),\ \mathrm{Eig}(B))$$

$$\leq (2n-1)\Big\{ \sum_{k=1}^{n} k^{1-k/2} \binom{n}{k} (m + \|B-A\|_F)^{k-1} M_1^{n-k} \Big\}^{1/n} \|B-A\|_B^{1/n} \,, \tag{7.3}$$

where

$$M_1 \quad \max(\|A\|_B \,,\ \|B\|_B).$$

Sacrificing some strength in favour of elegance we can write everything in terms of the Banach norm:

$$d(\mathrm{Eig}(A),\ \mathrm{Eig}(B))$$

$$\leq (2n-1) \Big\{ \sum_{k=1}^{n} k^{1-k/2} \binom{n}{k} n^{(k-1)/2} (m_1 + \|B-A\|_B)^{k-1} M_1^{n-k} \Big\}^{1/n} \times$$

$$\times \ \|B-A\|_B^{1/n} \tag{7.4}$$

where,

$$m_1 = \min(\|A\|_B \,,\ \|B\|_B),$$

$$M_1 = \quad \|A\|_B \,,\ \|B\|_B) \,.$$

The above inequalities give basis-free estimates for d(Eig(A), Eig(B)) for any two linear operators A and B. These estimates compare favourably with the results of Ostrowski ([15] pp. 282-283) and of Henrici [9]. We state their results below.

Ostrowski's Bound

For an $n \times n$ matrix $A = (a_{ij})$ define

$$|||A||| = \frac{1}{n} \sum_{1 \leq i,j \leq n} |a_{ij}|$$

This gives a norm on $M(n)$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two elements of $M(n)$. Let

$$K = \max_{1 \leq i,j \leq n} (|a_{ij}| , |b_{ij}|).$$

Then

$$d(\mathrm{Eig}(A),\mathrm{Eig}(B)) \leq 2n(n+2)K^{1-1/n}|||A-B|||^{1/n} \tag{7.5}$$

Henrici's Bound

Henrici's results depend on a certain measure of nonnormality which he defines as follows. Let $A \in \mathcal{E}_0(V)$ and let T be a Schur triangular form of A, i.e. T is an upper triangular matrix such that for some unitary matrix U we have

$$T = U A U^*$$

In general, such a T is not unique. Write

$$T = D + N$$

where D and N are, respectively, the diagonal and the nilpotent parts of T. If ν is any matrix norm, define the ν-departure from normality of A by where the infimum is taken with respect to all N that appear in the various Schur triangular reductions for A. Note that $\Delta_\nu(A) = 0$ if and only if A is normal.

For any real number $y \geq 0$ let $g = g(y)$ be the unique nonnegative solution of the equation

$$g + g^2 + \ldots + g^n = y$$

Henrici's result can be stated as follows. Let A be a nonnormal matrix and let $B-A \neq 0$. Let ν be any norm that majorises the Banach norm. Let

$$y = \frac{\Delta_\nu(A)}{\nu(B-A)}$$

Then

$$d(\mathrm{Eig}(A),\mathrm{Eig}(B)) \leq (2n-1)\, \frac{y}{g(y)}\, \nu(B-A). \tag{7.6}$$

When ν is the Frobenius norm, Henrici gives an upper bound for Δ_F, viz.

$$\Delta_F(A) \leq \left(\frac{n^3-n}{12}\right)^{1/4} \,||A^*A-AA^*||_F^{1/2} . \tag{7.7}$$

An upper bound for y and a lower bound for $g(y)$ may then be substituted in (7.6). For that we have to use the relations

$$\frac{y}{g(y)} \leq \frac{y}{n^{-1}y} = n, \quad \text{for} \quad 0 \leq y \leq n ,$$

$$\frac{y}{g(y)} \leq \frac{y}{(n^{-1}y)^{1/n}} \leq y, \quad \text{for} \quad y \geq n . \tag{7.8}$$

(See Henrici [9], p.32).

The bounds (7.2), (7.5) and (7.6) are of the same order. Note that in (7.6) the order $\frac{1}{n}$ enters through $g(y)$. As the following example shows this order cannot, in general, be improved.

Example 7.2 Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & 0 \end{pmatrix}$$

Then

$$||A-B||_F = |\varepsilon| ,$$

$$d(\mathrm{Eig}(A), \mathrm{Eig}(B)) = |\varepsilon^{1/3}|$$

The same phenomenon can be displayed for matrices of any order by perturbing an upper Jordan matrix by ε in the southwest corner.

Ostrowski's bound uses a norm which is dependent on the choice of a particular basis - the norm he uses is not a unitary invariant one. Henrici's bound has the drawback that the departure from normality is not easily amenable to computation. The use of (7.7) and (7.8) considerably weakens (7.6). Further, (7.6) can be used to study rates of convergence only when A is fixed and $B-A$ is bounded away from zero (because of the definition of y). Our bound does not have these drawbacks. It gives a basis - free estimate valid for all linear operators A and B. A comparison of the numerical performance of these bounds is given by the following two examples.

Example 7.3 Let

$$A = \begin{pmatrix} 1 & 10^{-3} \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 + \frac{10^{-1}}{\sqrt{3}} & 10^{-3} \\ \frac{10^{-1}}{\sqrt{3}} & 1 - \frac{10^{-1}}{\sqrt{3}} \end{pmatrix}$$

In this case,

$$\Delta_F(A) = 10^{-3}, \qquad \|B-A\|_F = 10^{-1}$$

$$y = 10^{-2}, \qquad g(y) = .009902,$$

$$K = 1 + \frac{10^{-1}}{\sqrt{3}}, \qquad |||A-B||| = \frac{\sqrt{3}}{2}\, 10^{-1}.$$

$$\|A\|_F = \sqrt{7.000001}, \quad \|B\|_F = \sqrt{2.010001}.$$

With this data, an upper bound for d(Eig (A), Eig (B)) given by Ostrowski is 4.842546, that given by Henrici is 0.302969 and the one given by our bound (7.2) is 1.978569.

Example 7.4 Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 + \frac{10^{-2}}{\sqrt{2}} & 2 \\ 1 & 1 - \frac{10^{-2}}{\sqrt{2}} \end{pmatrix}$$

In this case,

$$\|A\|_F = \sqrt{7}, \qquad \|B\|_F = \sqrt{7.0001},$$

$$\|A-B\|_F = 10^{-2}, \qquad |||A-B||| = \frac{10^{-2}}{\sqrt{2}},$$

$$K = 2.$$

The quantity $\Delta_F(A)$ is not known as easily as it was in the preceding example. Using (7.7) we have

$$\Delta_F(A) \leq \sqrt{3} \quad ,$$

$$y \leq \sqrt{3} \; 10^2$$

So, by (7.8)

$$\frac{y}{g(y)} \leq \sqrt{3} \; 10^2$$

Using this data, Ostrowski's bound for d(Eig (A), Eig (B)) is 1.9027, that of Henrici is 5.1962, whereas our bound (7.2) gives 0.8457 as an upper bound for this quantity.

Thus, in both of the examples considered above we get a smaller upper bound for d(Eig(A), Eig(B)) than that obtained from Ostrowski's estimate. In the first example where $\Delta_F(A)$ is explicitly known Henrici's bound is lower than ours. In the second example where $\Delta_F(A)$ is not known Henrici's bound is higher than ours.

In more special cases, our method can be adapted to give special results. We give one such example below. Note that both the Banach norm and the Frobenius norm are not only unitary invariant but also have the following property which we shall call <u>biunitary invariance</u>

$$||UAV|| = ||A|| \quad \text{for all unitary operators } U, V \; .$$

<u>Theorem 7.5</u> Let U,V <u>be two unitary operators on an n-dimensional space and let</u> K <u>be a skew - Hermitian operator such that</u> $VU^{-1} = \exp K$. <u>Then</u>

$$d(\text{Eig } U, \text{Eig } V) \leq 2(2n-1) \, n^{1/2n} \, ||K||_B^{1/n}$$

Proof Let

$$f(x) = (\exp xK)U \quad , \qquad x \in \mathbb{R} .$$

Then

$$f(0) = U, \quad f(1) \quad V \; ,$$

$$(Df(x))(y) = yK(\exp xK)U \; , \quad \text{for all} \quad x,y \in \mathbb{R}$$

So, the biunitary invariance of the Banach norm implies

$$||Df(x)||_B = ||K||_B \qquad \text{for all} \quad x \in \mathbb{R}.$$

Now that the Frobenius norm of the unitary operator $f(x)$ is $\sqrt{n}$.

Thus we have, from Proposition 5.8,

$$||D\phi_k(f(x))||_B \leq k^{1-k/2} \binom{n}{k} n^{(k-1)/2}$$

$$\leq n^{1/2} \binom{n}{k}$$

for $k = 1,2,\ldots,n$ and for all $x \in \mathbb{R}$.

So, by (5.4) we get

$$|\phi_k(V) - \phi_k(U)| \leq n^{1/2} \binom{n}{k} ||K||_B$$

Since the eigenvalues of a unitary operator are bounded by 1 we have, using Theorem 6.1,

$$d(\text{Eig } U, \text{Eig } V) \leq (2n-1) \left(\sum_{k=1}^{n} n^{1/2} \binom{n}{k} \right)^{1/n} ||K||_B^{1/n}$$

$$\leq 2(2n-1)\, n^{1/2n} \, ||K||_B^{1/n} . \qquad \square$$

Remark Of course, this estimate is weaker than that of Parthasarathy [16], or the one that would follow from the Hoffman-Wielandt inequality [10]. However, that is but to be expected as their estimates were derived for special cases using more special properties.

We now use Ostrowski's second theorem (Theorem 6.2) in conjunction with Theorem 5.9 to conclude

Theorem 7.6 Let A be an invertible operator on an n-dimensional space V. Let B be any other operator on V. Suppose there exists a constant R such that

$$0 \leq 4n\, R^{1/n} \leq 1$$

for which we have

$$(\|A\|_F + \|B-A\|_F)^{k-1}\, \|B-A\|_F \leq \frac{R}{k^{1-k/2} \binom{n}{k}} \tag{7.9}$$

for $k = 1, 2, \ldots, n$.

Then the eigenvalues $\{\alpha_1, \ldots, \alpha_n\}$ of A and $\{\beta_1, \ldots, \beta_n\}$ of B can be arranged in such a way that

$$\left| \frac{\beta_k}{\alpha_k} - 1 \right| < 8\, nR^{1/n}$$

for all $k = 1, 2, \ldots, n$.

Rem 4   Since

$$\max_{1 \le k \le n} k^{1-k/2} = 1$$

and

$$\max_{1 \le k \le n} \binom{n}{k} = \binom{n}{I(n/2)}$$

where I denotes the integral part, we can replace (7.9) by the stronger but more elegant condition

$$(||A||_F + ||B-A||_F)^{n-1} \, ||B-A||_F \le \frac{R}{\binom{n}{I(n/2)}} .$$

CHAPTER II-A

THE QUEST FOR A BETTER ORDER

We have derived in Chapter I an estimate for d(Eig (A), Eig (B)) for any two linear operators A and B on an n-dimensional vector space. This estimate shows that when $||A-B||$ is small the distance between the eigenvalues of A and those of B is comparatively large, being of the order $||A-B||^{1/n}$. Example 7.2 in Chapter I shows that this is the best possible order, in general. It might, however turn out that when A and B are restricted to special classes of operators this order is better. Some examples of such inequalities are the ones given by the estimates of Weyl, Hoffman-Wielandt, Parthasarathy, Kahan. We know of no other results of this type covering large and interesting special classes of operators.

In this chapter we identify some more classes of operators for which the order can be improved. We show that when A and B lie in the Lie algebra $\underline{so}(n,\mathbb{C})$ of complex skew symmetric matrices of order $n = 2r$ or $2r+1$ or when they lie in the symplectic Lie algebra $\underline{sp}(r,\mathbb{C})$ then the distance between the squares of their eigenvalues is of order $||A-B||^{1/r}$. It will become clear during the course of our discussion why it is more natural to consider squares of eigenvalues in these cases. When they all lie outside a circle around the origin, the distance between the eigenvalues themselves is of this order.

These results are deduced from some very simple observations combined with our earlier results. This shows the advantages of formulating the problem the way we have done.

1. Carrollian n-tuples.

Definition 1.1 We shall call an n-tuple $(x_1,\dots,x_n)$ of complex numbers a Carrollian n-tuple if it satisfies the following condition : a complex number x is a member of this n-tuple if and only if $-x$ is also its member with the same multiplicity as that of x.

Remark 1.2 Note that when n is odd a Carrollian n-tuple must contain 0 with an odd multiplicity. A Carrollian n-tuple can be arranged as

$$(x_1,\dots,x_r, -x_1,\dots, -x_r) \quad \text{when} \quad n = 2r$$

and as

$$(0, x_1,\dots,x_r, -x_1,\dots, -x_r) \quad \text{when} \quad n = 2r+1 .$$

(Here, some of the x_i may be zero too). We will think of a Carrollian n-tuple as having been thus arranged.

Let $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a monic polynomial of degree n with complex coefficients. Let $n = 2r$ or $2r+1$. If $\alpha_1,\dots,\alpha_n$ are the n roots of $f(z)$ then we know that

$$a_k = (-1)^k s_k(\alpha_1,\dots,\alpha_n) , \qquad 1 \le k \le n$$

where,

$$s_k = \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}$$

Now suppose the roots $\alpha_1,\dots, \alpha_n$ form a Carrollian n-tuple. Then we must have

$$a_{2j} = (-1)^j s_j(\alpha_1^2, \ldots, \alpha_r^2) \qquad 1 \le j \le r ,$$

$$a_{2j+1} = 0 \qquad 0 \le j \le r$$

(See Remark 1.2).

Thus, $\alpha_1^2, \ldots, \alpha_r^2$ are the r roots of the polynomial

$$F(z) = z^r + a_2 z^{r-1} + \ldots + a_{2j} z^{r-j} + \ldots + a_{2r}$$

of degree r.

This observation leads immediately to the following corollary of Ostrowski's Theorem (Theorem 6.1 in Chapter I).

Theorem 1.3 Let $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, $g(z) = z^n + b_1 z^{n-1} + \ldots + b_n$ be two polynomials of degree $n = 2r$ or $2r+1$. Suppose the respective roots $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ of $f(z)$ and $g(z)$ are both Carrollian n-tuples. Let

$$\gamma = \max_{1 \le k \le n} (|\alpha_k|^2 , |\beta_k|^2) ,$$

$$\theta' = \{ \sum_{k=1}^{r} |b_{2k} - a_{2k}| \gamma^{r-k} \}^{1/r}$$

Then the roots can be arranged in such a way that

$$|\alpha_k^2 - \beta_k^2| < (2r-1)\theta' \tag{1.1}$$

for all $k = 1,2,\ldots,n$.

An inequality for the roots themselves can be obtained using the following lemma.

Lemma 1.4 Let x_1, x_2 be two complex numbers each of modulus not less than c. Then either $|x_1-x_2| \geq c$ or $|x_1+x_2| \geq c$.

Proof Let $x_1 = c_1 e^{i\phi_1}$, $x_2 = c_2 e^{i\phi_2}$, where $c_1, c_2 \geq c$. Suppose

$$c > |x_1-x_2| = |c_1 - c_2 e^{i(\phi_2-\phi_1)}|$$

Then,

$$\begin{aligned} x_1 + x_2 &= |c_1 + c_2 e^{i(\phi_2-\phi_1)}| \\ & \quad |2c_1 - (c_1 - c_2 e^{i(\phi_2-\phi_1)})| \\ &\geq |2c_1 - |c_1 - c_2 e^{i(\phi_2-\phi_1)}|| \\ & \quad c \end{aligned}$$

$\square$

Corollary 1.5 Let the notations be as in Theorem 1.3. Suppose the roots $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_n$ are either all located outside the circle of radius c around the origin or they contain zero with the same multiplicity and the nonzero ones are located outside this circle. Then they can be arranged so that :

$$|\alpha_k - \beta_k| < \frac{(2r-1)}{c} \theta' \tag{1.2}$$

for all $k = 1,2,\ldots,n$.

Proof It is enough to consider the case when n is even and none of the roots is zero. Suppose the roots have been arranged so as to satisfy (1.1). By Lemma 1.4 we have for every k, either $|\alpha_k-\beta_k| \geq c$ or $|\alpha_k+\beta_k| \geq c$. Assume the latter. Then

$$|\alpha_k-\beta_k| \leq \frac{(2r-1)\theta'}{|\alpha_k+\beta_k|} \leq \frac{(2r-1)\theta'}{c}$$

Notice that the left hand side is the distance between the roots α_k and β_k and also that between the roots $-\alpha_k$ and $-\beta_k$. Assume the former. Then

$$|\alpha_k+\beta_k| \leq \frac{(2r-1)\theta'}{c}$$

The left hand side is now the distance between the roots α_k and $-\beta_k$ and also that between the roots $-\alpha_k$ and β_k. This gives an arrangement of all the $2r$ roots. $\square$

2. Operators with Carrollian Spectra

An operator A will be said to have a Carrollian spectrum if Eig(A) is a Carrollian n-tuple.

Using Theorem 5.9 of Chapter I, Theorem 1.3 and Corollary 1.5 we have

Theorem 2.1 Let $A, B \in \mathcal{E}(V)$, where $\dim V = n = 2r$ or $2r+1$. Suppose both A and B have Carrollian spectra. Then we have

$$d(\mathrm{Eig}(A^2), \mathrm{Eig}(B^2))$$

$$\leq (2r-1) \left\{ \sum_{k=1}^{r} (2k)^{1-k} \binom{n}{2k} (m+||B-A||_F)^{2k-1} M^{2(r-k)} \right\}^{1/r} ||B-A||_F^{1/r} \qquad (2.1)$$

where

$$m = \min(||A||_F, ||B||_F),$$

$$M = \max(||A||_F, ||B||_F).$$

If A, B have 0 as one of their eigenvalues with the same multiplicity and if the rest of their eigenvalues lie outside a circle of radius c around the origin then we have

$$d(\mathrm{Eig}(A), \mathrm{Eig}(B)) \leq \frac{R}{c},$$

where R denotes the righthand side of (2.1). □

Remarks 1. Inequality (2.1) may be strengthened slightly in the same way as we obtained (7.3) in Chapter I.

2. The left hand side of (2.1) is the distance in $\mathbb{C}^n_{\mathrm{sym}}$ between the squares of the eigenvalues of A and the squares of the eigenvalues of B. This may be larger than the distance between their eigenvalues in many cases.

3. Entries of a Carrollian n-tuple are determined essentially upto their squares. So, it is natural to consider the distance between their squares.

Examples Important examples of classes of operators whose spectra are Carrollian are provided by three of the four classical Lie algebras. (See e.g. Helgason [8]). We define them below.

Let A^t denote the transpose of the matrix A. We call symmetric if $A^t = A$ and skew-symmetric if $A^t = -A$. Let I_n denote the $n \times n$ identity matrix and let J denote a $(2r) \times (2r)$ matrix with a block decomposition

$$J = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$$

Let

$$\underline{so}(n, \mathbb{C}) = n \times n \text{ complex skew symmetric matrices}$$

$$\underline{sp}(r, \mathbb{C}) = \{A : A^t = -JAJ^{-1}\} .$$

It is easy to see that $A \in \underline{sp}(r, \mathbb{C})$ if A has a block decomposition

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^t \end{pmatrix}$$

where A_1, A_2, A_3 are $r \times r$ matrices and A_2, A_3 are symmetric. Since

$$\mathrm{Eig}(A^t) = \mathrm{Eig}(A) = \mathrm{Eig}(JAJ^{-1})$$

and

$$\mathrm{Eig}(-A) = -\mathrm{Eig}(A)$$

matrices in $\underline{so}(n, \mathbb{C})$ and $\underline{sp}(r, \mathbb{C})$ have Carrollian spectra.

Remarks The four complex classical Lie algebras are enumerated as

$$\underline{a}_r = \underline{s\ell}(r+1, \mathbb{C}),$$

$$\underline{b}_r = \underline{so}(2r+1, \mathbb{C}),$$

$$\underline{c}_r = \underline{sp}(r, \mathbb{C}),$$

$$\underline{d}_r = \underline{so}(2r, \mathbb{C}).$$

The Lie algebra $s\ell(n, \mathbb{C})$ consists of $n \times n$ complex matrices of trace 0. Since every matrix A can be reduced to a matrix with trace 0 by subtracting from it the scalar matrix $(\frac{\text{tr } A}{n})I$, the case of $\underline{s\ell}(n, \mathbb{C})$ is equivalent to considering the case of all matrices. This was done in Chapter I. To the other three Lie algebras the more special result derived above applies. The subscript r in this classification is the rank of the Lie algebra. This is also the exponent which occurs in our estimate (2.1). If we look at the spaces $\mathbb{C}^n|W$ where W are the Weyl groups of $\underline{b}_r$, $\underline{c}_r$ and $\underline{d}_r$ we find that natural metrics on them are defined using squares of the coordinates. (See Appendix 1). This connection could be more than fortuitous.

CHAPTER II-B

THE QUEST FOR A BETTER PATH

We now turn our attention in a different direction. Our main result in Chapter I was obtained by linking two matrices A and B by the linear path $A+t(B-A)$, $0 \leq t \leq 1$. This path may be wasteful from the point of view of estimating spectral variation. We introduce a geometric approach to the problem and using this obtain some old results. It is likely that using these ideas significant improvements for the case of arbitrary matrices might be obtained. An approach along these lines and the attendant difficulties are indicated.

3. Orbits and their tangent spaces

Let $M(n)$ be the space of all $n \times n$ complex matrices. Let $GL(n)$ denote the multiplicative group of all invertible matrices and $U(n)$ that of all unitary matrices. (A matrix shall, henceforth, mean an $n \times n$ complex matrix).

We know that $GL(n)$ is a Lie group with Lie algebra $M(n)$. The natural adjoint action of $GL(n)$ on $M(n)$ is defined as the map

$$A \to gAg^{-1}, \qquad A \in M(n), \quad g \in GL(n).$$

The subset O_A of $M(n)$ defined as

$$O_A = \{gAg^{-1} : g \in GL(n)\}$$

is called the <u>orbit</u> of A under this action. In other words O_A consists of all matrices similar to A.

It is well known that O_A is a smooth submanifold of the manifold $M(n)$.)(See, e.g., Helgason [8]). The tangent space to O_A at the point A will be denoted by $T_A O_A$. This is a linear subspace of the tangent space to $M(n)$ at A - which is $M(n)$ itself.

The space $M(n)$ has also got a Hilbert space structure defined by the inner product

$$< A, B > \; = \; \operatorname{tr} AB^*$$

The norm arising from this inner product is the Frobenius norm. We denote by $S^{\perp}$ the orthogonal complement of a subspace S of $M(n)$. Our next proposition identifies the subspaces $T_A O_A$ and $(T_A O_A)^{\perp}$ of $M(n)$.

As usual we denote by $[A,B]$ the Lie bracket $AB-BA$. The <u>centraliser</u> of A in $M(n)$ is defined as the set

$$\underline{Z}(A) = \{X \in M(n) \quad [A,X] = 0\}$$

<u>Proposition 3.1</u> For every $A \in M(n)$ we have

$$T_A O_A = \operatorname{Span} \{ [A,X] : X \in M(n)\} ,$$

$$(T_A O_A)^{\perp} = \underline{Z}(A^*)$$

<u>Proof</u> Every differentiable curve in O_A passing through A can be written locally, as:

of all matrices similar to A.

It is well known that O_A is a smooth submanifold of the manifold M(n).)(See, e.g., Helgason [8]). The tangent space to O_A at the point A will be denoted by $T_A O_A$. This is a linear subspace of the tangent space to M(n) at A - which is M(n) itself.

The space M(n) has also got a Hilbert space structure defined by the inner product

$$< A, B > \; = \; \mathrm{tr} \; AB^*$$

The norm arising from this inner product is the Frobenius norm. We denote by S the orthogonal complement of a subspace S of M(n). Our next proposition identifies the subspaces $T_A O_A$ and $(T_A O_A)^{\perp}$ of M(n).

As usual we denote by $[A,B]$ the Lie bracket $AB-BA$. The <u>centraliser</u> of A in M(n) is defined as the set

$$\underline{Z}(A) = \{X \in M(n) \quad [A,X] = 0\}$$

<u>Proposition 3.1</u> For every $A \in M(n)$ we have

$$T_A \, O_A = \mathrm{Span} \, \{ [A,X] : X \in M(n)\} \quad ,$$

$$(T_A O_A)^{\perp} = \underline{Z}(A^*)$$

<u>Proof</u> Every differentiable curve in O_A passing through A can be written locally, as:

$$\underline{Z}(A^*) = \underline{Z}(A) \quad \text{if and only if } A \text{ is normal.}$$

So, we have

$$M(n) = T_A O_A \oplus \underline{Z}(A), \quad A \text{ normal} . \tag{3.2}$$

When A is not normal we can write a three-component decomposition

$$M(n) = T_A O_A \oplus (\underline{Z}(A) \cap \underline{Z}(A^*)) \oplus \underline{Y}(A), \tag{3.3}$$

where

$$\underline{Y}(A) = \{T_A O_A \oplus (\underline{Z}(A) \cap \underline{Z}(A^*))\}^{\perp} \tag{3.3}$$

The reason for writing these decompositions shall become clear in a moment.

4. Application to spectral variation

In this section we use the metric δ on $\mathbb{C}^n_{sym}$. (See Section 2, in Chapter I). The norm in this section is always the Frobenius norm. So we drop the subscript F from $||\cdot||_F$.

Since similar matrices have identical eigenvalues, we have

$$\delta(\mathrm{Eig}(A), \mathrm{Eig}(B)) = 0 \quad \text{when} \quad B \in O_A . \tag{4.1}$$

If $[A,B] = 0$, there exists a unitary matrix U such that UAU^{-1} and UBU^{-1} are upper triangular. Since the Frobenius norm is unitary invariant, this implies

$$\delta(\mathrm{Eig}(A), \mathrm{Eig}(B)) \leq ||A-B|| , \quad \text{when} \quad B \in \underline{Z}(A). \tag{4.2}$$

The relations (3.2), (4.1) and (4.2) suggest that the variation of the spectrum of a normal matrix can be estimated componentwise in two orthogonal directions. To make this precise, we use the following lemma.

<u>Lemma 4.1</u> Let H be a Hilbert space (or a Riemannian manifold). Let $\phi : H \to \mathbb{R}$ be a C^1 function and $\gamma : [0,1] \to H$ a piecewise C^1 curve. Suppose the following conditions are satisfied

(i) $\gamma(0) = x_o$, $\gamma(1) = x_1$

$\phi(x_o) = 0$

(ii) For all $0 \leq t \leq 1$ the tangent space to H at $\gamma(t)$ splits as a direct sum

$$T_{\gamma(t)} H = T_t^{(1)} \oplus T_t^{(2)}$$

in such a way that

$$v^{(1)}\phi = 0 \quad \text{for all} \quad v^{(1)} \in T_t^{(1)}$$

$$v^{(2)}\phi \leq c||v^{(2)}|| \quad \text{for all} \quad v^{(2)} \in T_t^{(2)}$$

(Here $v^{(1)}\phi$ and $v^{(2)}\phi$ denote the directional derivatives of ϕ in these two directions).

Let $P_t^{(1)}$, $P_t^{(2)}$ denote, respectively, the orthogonal projections onto these two subspaces of $T_{\gamma(t)} H$

Then

$$\phi(x_1) \le c \int_0^1 ||P_t^{(2)} \quad \gamma'(t)||dt$$

where $\gamma'(t)$ denotes the derivative of γ at t.

Proof We have

$$\phi(x_1) = \int_0^1 \gamma'(t) \, (\phi)dt$$

$$= \int_0^1 (P_t^{(1)} \quad \gamma'(t)) \, (\phi)dt + \int_0^1 (P_t^{(2)} \quad \gamma'(t))(\phi)dt$$

$$\le 0 + c \int_0^1 ||P_t^{(2)} \quad \gamma'(t)||dt$$

by condition (ii). $\square$

Remark The statement of the lemma remains valid if the function ϕ is C^1 on a dense open subset G of H and γ is a curve which intersects the complement of G at only a finite number of points. In such a case we will say that is generically C^1 and γ is a curve adapted to ϕ.

Let $(a_1,\ldots,a_n)$ be a fixed point in $\mathbb{C}^n$ with distinct coordinates. Then the function

$$\phi(x_1,\ldots,x_n) = \min_{\sigma \in \Pi} \sum_{i=1}^{n} |a_i - x_i|^2)^{1/2}$$

is generically C^1. It is not differentiable on the hyperplanes defined by the following conditions:

(1) points with not all coordinates distinct ,

(2) points $(x_1,\ldots,x_n)$ for which the minimum in the definition of ϕ is attained for two different permutations σ .

Outside these hyperplanes ϕ behaves as the ordinary Euclidean distance in $\mathbb{C}^n$.

Now let A_o be a fixed matrix with distinct eigenvalues and let $\phi(A) = \delta(\mathrm{Eig}\ A_o, \mathrm{Eig}\ A)$. Then ϕ is generically a C^1 function on $M(n,\mathbb{C})$. It is not differentiable on the set of matrices whose eigenvalues constitute an n tuple of type (1) or (2) above. These conditions being algebraic the set of matrices satisfying them is nowhere dense and closed. Thus outside a finite number of algebraic surfaces, ϕ is a C^1 function on $M(n)$.

Now we can prove the following theorem.

<u>Theorem 4.2</u> Let A_o be a normal matrix with distinct eigenvalues. Let $A : [0,1] \to M(n)$ be a piecewise C^1 curve with the following properties

(i) $A(t)$ is normal for all $0 \leq t \leq 1$,

(ii) $A(0) = A_o$, $A(1) = A_1$,

(iii) $A(t)$ is adapted to the generically C^1 function $\phi(A) = \delta(\mathrm{Eig}\ A_o, \mathrm{Eig}\ A)$.

Let $P_t^{(1)}$ and $P_t^{(2)}$ denote, respectively, the orthogonal projection operators onto the subspaces $T_{A(t)}\, O_{A(t)}$ and $\underline{Z}(A(t))$ of $M(n)$, for $0 \leq t \leq 1$. Then

$$\delta(\text{Eig } A_o, \text{ Eig } A_1) \leq \int_o^1 ||P_t^{(2)} A'(t)||dt \leq \int_o^1 ||A'(t)||dt \tag{4.3}$$

where $A'(t)$ denotes the derivative of $A(t)$.

The last inequality in (4.3) is strict whenever

$$L \{t : P_t^{(1)} A'(t) \neq 0\} > 0$$

where L is the Lebesgue measure on $[0,1]$.

<u>Proof</u> We apply Lemma 4.1, and the remark following it, to the Hilbert space $M(n)$, the function $\phi(A)$ and the curve $A(t)$. By (3.2) the space splits into two orthogonal components $T_t^{(1)} = T_{A(t)} \ O_{A(t)}$ and $T_t^{(2)} = \underline{Z}(A(t))$ for all $0 \leq t < 1$. Choose and fix a t in $[0,1]$. Since $\phi(A(t)) = \phi(B)$ for all $B \in O_{A(t)}$, the derivative of ϕ in the direction of $O_{A(t)}$ is zero, i.e.

$$v^{(1)}\phi = 0 \quad \text{for all } v^{(1)} \in T_t^{(1)} .$$

Now consider the orthogonal direction $T_t^{(2)}$ Let

$$\psi(A) = \delta(\text{Eig } A(t), \text{ Eig } A) ,$$

$$h(A) = \phi(A(t)) + \psi(A) = \delta(\text{Eig } A_o, \text{ Eig } A(t)) + \delta(\text{Eig } A(t), \text{ Eig } A).$$

Then note that

$$\phi(A(t)) = h(A(t))$$

and

$$\phi(A) \leq h(A) \quad \text{for all} \quad A.$$

Hence,

$$v^{(2)}\phi \leq v^{(2)} h \quad \text{for all} \quad v^{(2)} \in T_t^{(2)}$$

(In fact this inequality will hold for the derivative in any direction, so in particular for the direction $T_t^{(2)}$). But,

$$v^{(2)} h = v^{(2)}\psi \quad \text{for all} \quad v^{(2)} \in T_t^{(2)}$$

since for a fixed t, $\phi(A(t))$ is a constant.

By (4.2) we have

$$v^{(2)}\psi \leq ||v^{(2)}|| \quad \text{for all} \quad v^{(2)} \in T_t^{(2)}$$

So,

$$v^{(2)}\phi \leq ||v^{(2)}|| \quad \text{for all} \quad v^{(2)} \in T_t^{(2)}$$

Hence, by Lemma 4.1 we have

$$\phi(A_1) \leq \int_0^1 ||P_t^{(2)} A'(t)||dt$$

Note that

$$||P_t^{(2)} A'(t)|| \leq ||A'(t)||$$

where strict inequality holds whenever $P_t^{(1)}A'(t) \neq 0$.

This proves the theorem completely. $\square$

Remarks (1) A path satisfying the conditions of the theorem can be constructed as follows. For the normal matrices A_o and A_1 let

$$A_i = U_i D_i U_i^{-1} \qquad i = 1,2, .$$

where D_i are diagonal and U_i are unitary. Let $D(t) = D_1 + t(D_1 - D_o)$. Then if D_o has distinct entries then so does $D(t)$ except at a finite number of points. This linear path cuts the hyperplanes where $\delta(\text{Eig } D_o, \text{Eig } D)$ is not C^1 only at a finite number of points. Now let $U(t)$ be any C^1 curve joining U_1 and U_2. Then the curve $A(t) = U(t) D(t) U(t)^{-1}$ satisfies the conditions of the theorem.

(2) Since $\delta(\text{Eig } A_o, \text{Eig } A)$ is a continuous function of A an inequality of the type

$$\delta(\text{Eig } A_o, \text{Eig } A) \leq f(||A_o - A||)$$

where f is a continuous function, holds for all A if it holds for a dense set. By perturbing the matrix A, if necessary, we can therefore assume that A lies in the dense open set where $\phi(A)$ is C^1. In the next few paragraphs we will make this assumption without mentioning it. In the same way, to avoid repetition, for a fixed matrix A_o, "a curve passing through A_o" will mean a curve adapted to the function $\delta(\text{Eig } A_o, \text{Eig } A)$.

We deduce two corollaries.

Corollary 4.3 Let A_o, A_1 be Hermitian matrices. Then

$$\delta(\text{Eig }(A_o), \text{Eig}(A_1)) \leq ||A_o - A_1|| \tag{4.4}$$

The inequality is strict whenever $[A_o, A_1] \neq 0$.

<u>Proof</u> The curve $A(t) = A_o + t(A_1-A_o)$ satisfies the conditions of the theorem. We have

$$||P_t^{(2)} A'(t)|| \leq ||A'(t)|| = ||A_1-A_o||$$

So, (4.4) follows from (4.3).

Further, $P_t^{(1)} A'(t) \neq 0$ if and only if

$$[A_1-A_o, A_o + t(A_1-A_o)] \neq 0$$

which, in turn, holds if and only if $[A_o,A_1] \neq 0$. $\square$

<u>Corollary 4.4</u> Let U_o, U_1 be unitary matrices and let K be a skew-Hermitian matrix such that $U_1U_o^{-1}$ $\exp K$. Then

$$\delta(\text{Eig } U_o, \text{Eig } U_1) \leq ||K|| \qquad (4.5)$$

The inequality is strict whenever $[U_o, U_1] \neq 0$

<u>Proof</u> The curve $U(t) = (\exp t K)U_c$ joints U_o and U_1 and satisfies the conditions of the theorem. We have

$$U'(t) = K(\exp t K)U_o .$$

So,

$$||P_t^{(2)} U'(t)|| \leq ||U'(t)|| = ||K||$$

So, (4.5) follows from (4.3).

Further, the condition

$$[K \exp(tK)U_o , \exp(tK)U_o] \neq 0$$

is readily seen to be equivalent to

$$[K, U_o] \neq 0$$

which, in turn, is equivalent to

$$[U_o, U_1] \neq 0 \qquad \square$$

<u>Remarks</u> Inequalities (4.4) and (4.5) follow also from the Hoffman-Wielandt Theorem which asserts the validity of (4.4) when A_o and A_1 are normal. They prove this using the spectral theorem and Birkhoff's characterisation of the extreme points of the convex set of doubly stochastic matrices. At the moment, we are unable to deduce their inequality using this method. While normal matrices have several nice analytic properties, their set is not endowed with a rich geometric structure and so does not lend itself easily to our geometric method. Nevertheless, the following calculation is instructive. Though it does not lead to the Hoffman-Wielandt inequality it brings in commutators in an interesting manner.

Let A_o, A_1 be normal matrices. Then we can write $A_i = U_i D_i U_i^{-1}$, $i = 1,2$, where U_i are unitary matrices and D_i are diagonal matrices. Again, let K be a skew-Hermitian matrix such that $U_1 U_o^{-1} = \exp K$. Let

$$U(t) = (\exp t\,K)U_o \ ,$$

$$D(t) = D_o + t(D_1 - D_o) \ ,$$

and

$$A(t) = U(t)\,D(t)\,U(t)^{-1}$$

Then $A(t)$ is a path joining A_o and A_1 and satisfying the conditions of Theorem 4.1. We have

$$U'(t) = KU(t) \ ,$$

$$(U^{-1})'(t) = -U(t)^{-1}\,U'(t)\,U(t)^{-1} = -U(t)^{-1}\,K \ .$$

So,

$$\begin{aligned} A'(t) &= KU(t)D(t)U(t)^{-1} + U(t)(D_1-D_o)U(t)^{-1} \\ &\quad - U(t)D(t)U(t)^{-1}\,K \\ &= [K,A(t)] + U(t)(D_1-D_o)U(t)^{-1} \end{aligned}$$

The first term on the right hand side belongs to $T_{A(t)}\,O_{A(t)}$. So, we have

$$||P_t^{(2)}\,A'(t)|| \leq ||U(t)(D_1-D_o)U(t)^{-1}|| = ||D_1-D_o|| \ .$$

Hence, by Theorem 4.1

$$\delta(\mathrm{Eig}(A_o), \mathrm{Eig}(A_1)) \leq ||D_o-D_1|| \ .$$

Of course, this inequality follows from the very definition of the metric δ. So we do not get anything new. However, it is interesting to note how the expression for $A'(t)$ involves two terms, one of which lies entirely in the component $T_{A(t)}\,O_{A(t)}$.

For nonnormal matrices the situation is more involved. We now have the third component in decomposition (3.3) to take care of. Using relations (2.1) and (7.2) of Chapter I, we can write an inequality of the type

$$\delta(\text{Eig } A_o, \text{ Eig } A_1) \leq f(M, ||A_o - A_1||) \qquad (4.6)$$

valid for all matrices A_o and A_1, where $M = \max((||A_o||, ||A_1||)$ and f is a continuous monotonically increasing function of each of its variables determined explicitly by our earlier analysis. As we pointed out, the inequality (4.6) is rather weak. Decomposition (3.3) now suggests how this "bad part" can be isolated. Indeed, let $A:[0,1] \to M(n)$ be any differentiable curve such that $A(0) = A_o$ and $A(1) = A_1$. Let $P_t^{(1)}$, $P_t^{(2)}$ and $P_t^{(3)}$ be the respective projection operators onto the three subspaces occuring in the decomposition (3.3) of $M(n)$ corresponding to the matrix $A(t)$. Let

$$M_1 = \sup_{0 \leq t \leq 1} (||A(t)||, ||A'(t)||).$$

Then, we have with these notations

<u>Theorem 4.4</u>

$$\delta(\text{Eig } A_o, \text{ Eig } A_1) \leq \int_o^1 ||P_t^{(2)} A'(t)|| dt + \int_o^1 f(M_1, P_t^{(3)} A'(t)|| dt .$$

A suitable choice of the path $A(t)$ would thus give a better inequality than (4.6). As the examples given above indicate this choice would depend on the nature of A_o and A_1.

This analysis gives rise to an interesting question in approximation theory and an equivalent question in matrix equations. What are the projections onto the subspaces occuring in (3.3)? The answer depends on the solution to either of the following problems.

<u>Problem 1</u> Given two matrices A and B, find a matrix X such that $||B-[A,X]||$ is minimal.

<u>Problem 2</u> Given A,B find an X such that

$$[A^*, B - [A,X]] = 0 \tag{4.7}$$

Such an X is not unique. However $[A,X]$ is unique.

The equivalence of these problems is readily seen using the characterisation of $T_A O_A$ and $(T_A O_A)^{\perp}$ given by Proposition 3.1.

When A is in the Jordan form, $\underline{Z}(A)$ is characterised as "triangularly striped" matrices, (see [18]), where the position and the length of stripes depend on the Jordan structure of A. In this case (4.7) can be solved explicitly. However, for our problem that is of little use since the Jordan form is obtained by a similarity operation and the norm is invariant only under unitary operations.

Appendix 1

In Chapter I we made use of a homeomorphism between the spaces $\mathbb{C}^n$ and $\mathbb{C}^n/\Pi_n$ where Π_n is the group of permutations on n symbols. It is a consequence of a theorem of Chevalley that if G is a finite group generated by reflections then $\mathbb{C}^n$ and $\mathbb{C}^n/G$ are homeomorphic. Here, we construct these homeomorphisms explicitly for some groups of this type.

Let V be a finite-dimensional vector space. An element of g of GL(V) is called a reflection if $g^2 = 1$ and g leaves a hyperplane in V pointwise fixed. A group G of linear transformations on V is called a finite reflection group if it is a finite group generated by reflections. Let S be the symmetric algebra of V. The operations of G extend to automorphisms of S as follows. For $g \in G$, $P \in S$ and $x \in V$ define

$$(gP)(x) = P(g^{-1}x) .$$

An element P of S such that $gP = P$ is called an invariant of G.

The theorem of Chevalley [3] says that if G is a finite reflection group in an n-dimensional vector space V (over a field of characteristic 0) then the algebra of invariants of G is generated by n algebraically independent homogeneous elements.

The algebra of invariants of G can be regarded as the polynomial algebra over V/G. Since the polynomial algebra over V also has n algebraically independent homogeneous generators viz., the n elementary symmetric functions, this means that the polynomial algebras over V and V|G are isomorphic. Hence the varieties V and V|G are isomorphic.

Finite reflection groups have been enumerated completely by Coxeter. We consider the actions on $\mathbb{C}^n$ of finite reflection groups which are the Weyl groups of the classical Lie algebras. (See, e.g. Bourbaki [2]).

Let G be one of these groups. We will construct a map $S: \mathbb{C}^n \to \mathbb{C}^n$ such that the induced map $\bar{S}: \mathbb{C}^n/G \to \mathbb{C}^n$ is a homeomorphism. We will also give a natural metric on the space $\mathbb{C}^n|G$.

1. Let $G = \Pi_n$ be the permutation group. (Note that this group is generated by transpositions. The transposition of coordinates x_i and x_j is a reflection keeping the hyperplane $x_i - x_j = 0$ in $\mathbb{C}^n$ fixed). This case was considered in Section 3 of Chapter I.

2. Let G be the semi-direct product of Π_n and $(\mathbb{Z}/2\mathbb{Z})^n$. The group G acts on $\mathbb{C}^n$ by permutations and sign changes of the coordinates. Let $\{x_1,\ldots,x_n\}$ be the image of the point $(x_1,\ldots,$ $(x_1,\ldots,x_n) \in \mathbb{C}^n$ in the quotient space $\mathbb{C}^n/G$. Then $\{x_1,\ldots,x_n\} = \{y_1,\ldots,y_n\}$ if and only if there is an element σ of Π_n such that $x_i = \pm y_{\sigma(i)}$ for $i = 1,2,\ldots,n$. Let

$$S((x_1,\ldots,x_n)) = (s_1(x_1^2,\ldots,x_n^2),\ldots,s_n(x_1^2,\ldots,x_n^2))$$

where s_i, $i = 1,2,\ldots,n$ are the elementary symmetric functions. This defines a map $S: \mathbb{C}^n \to \mathbb{C}^n$. By an obvious modification of the argument given in Proposition 3.1 the induced map $\bar{S}$ from $\mathbb{C}^n/G$ to $\mathbb{C}^n$ is a homeomorphism. Let

$$d(\{x_1,\ldots,x_n\}, \{y_1,\ldots,y_n\}) = \min_{\sigma \in \Pi_n} \max_{1 \le i \le n} |x_i^2 - y_{\sigma(i)}^2|$$

Then d gives a metric on $\mathbb{C}^n/G$.

3. Let G be the semi-direct product of Π_n and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ The group G acts on $\mathbb{C}^n$ by permutations and an even number of sign

changes of coordinates. Thus two points $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$ are identified with each other in $\mathbb{C}^n|G$ if and only if there is an element σ of Π_n such that $x_i = \pm y_{\sigma(i)}$ for $i = 1,2,\ldots,n$ and the product $x_1x_2\ldots x_n$ is equal to the product $y_1y_2\ldots y_n$. In this case, let

$$S((x_1,x_2,\ldots,x_n))$$

$$= (s_1(x_1^2,\ldots,x_n^2), \ldots , s_{n-1}(x_1^2,\ldots,x_n^2), s_n(x_1,\ldots,x_n))$$

where s_i the elementary symmetric functions. This defines a map $S: \mathbb{C}^n \to \mathbb{C}^n$ and the induced map $\bar{S}: \mathbb{C}^n/G \to \mathbb{C}^n$ is a homeomorphism. A metric on the space $\mathbb{C}^n/G$ is given by

$$d(\{x_1,\ldots,x_n\} , \{y_1,\ldots,y_n\})$$
$$= \max (\min_{\sigma \ \Pi_n} \max_{1\leq i\leq n} |x_i^2-y_{\sigma(i)}^2| , |x_1\ldots x_n-y_1\ldots y_n|)$$

Remarks The group G in case 1 is the Weyl group of the Lie algebra $\underline{a}_n$, in case 2 that of the Lie algebras $\underline{b}_n$ and $\underline{c}_n$ and in case 3 it is the Weyl group of the Lie algebra $\underline{d}_n$.

In each of these cases an Ostrowski type theorem comparing the metric in $\mathbb{C}^n/G$ defined above with the metric in $\mathbb{C}^n$ may be obtained by a small modification of the original Ostrowski's theorem.

Besides the four classical Lie algebras named above, there are five exceptional ones. For the simplest one $\underline{g}_2$ we can do the analysis easily. In this case the group G is the dihedral group of order 12. This group is defined abstractly as follows. It has two generators R_1 and R_2 which satisfy the relations

$$R_1^2 = R_2^2 = (R_2R_1)^6 = 1.$$

If we let G act on the space $\mathbb{C}^2$, then the action of these generators has to be

$$R_1 \ : \ (x_1,x_2) \to (x_2,x_1)$$

$$R_2 \ : \ (x_1,x_2) \to (w^{-1}x_2, wx_1)$$

where w is the primitive sixth root of unity. So $\mathbb{C}^2/G$ consists of equivalence classes $\{x_1,x_2\}$ of points (x_1,x_2) in $\mathbb{C}^2$ where two points (x_1,x_2) and (y_1,y_2) are identified if (x_1,x_2) is a permutation of $(w^i y_1, w^{-i}x_2)$ for some $i = 1,2,\ldots,6$. In this case let

$$S((x_1,x_2)) = (x_1^6 + x_2^6 , x_1 x_2) .$$

Then the map $S: \mathbb{C}^2 \to \mathbb{C}^2$ induces a homeomorphism $\bar{S}: \mathbb{C}^2 \to \mathbb{C}^2/G$. A natural metric on $\mathbb{C}^n/G$ is defined as

$$d(\{x_1,x_2\} , \{y_1,y_2\})$$

$$= \max\Big(\min_{\sigma \in \Pi_2} \max_{i=1,2} |x_i^6 - y^6_{(i)}| , |x_1x_2-y_1y_2|\Big).$$

For the other exceptional Lie algebras a corresponding analysis does not seem to be as easy, primarily because the description of their Weyl groups is very complicated and indirect. (see, e.g. [2]).

Appendix 2

We find an estimate for the Banach norm of the derivative of the k-th exterior power map. (In this appendix the norm, $||\cdot||$, of a linear operator will mean the Banach norm).

To represent the action of the multilinear operators under consideration we shall use the following form of the polar decomposition theorem. We give a proof which is different from the standard proof that uses extraction of square roots of positive operators.

Proposition 1 Let A be a linear operator on an n-dimensional unitary space V. Then there exist orthonormal bases $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ such that

$$A\, e_i = c_i f_i \qquad i = 1, 2, \dots, n \; ,$$

where

$$||A|| \qquad c_1 \geq c_2 \geq \cdots \quad c_n \geq 0.$$

Proof We have $||A|| = \sup\{||Ax|| : ||x|| = 1\}$. Since the unit ball in V is compact, there exists a unit vector e_1 for which

$$||Ae_1|| = ||A|| = c_1 \; .$$

Let V_1 be the orthogonal complement of e_1 and let A_1 be the restriction of A to V_1. By the same argument, there exists a unit vector e_2 for which

$$||A_1\, e_2|| = ||A_1|| = c_2 \; , \text{ say .}$$

Continue this process to get an orthonormal basis $e_1, \dots, e_n$ for V.

If $C_n > 0$, define the vectors f_i as

$$f_i = \frac{Ae_i}{C_i} \qquad i = 1,2,\ldots,n .$$

If $C_k > 0$ but $C_{k+1} = 0$ for some $1 < k < n$, then A has rank k. In this case choose the first k vectors $f_1,\ldots,f_k$ as above and choose $f_{k+1},\ldots,f_n$ to be orthonormal vectors in the orthogonal complement of the range of A.

We claim that $\{f_i,\ i = 1,2,\ldots,n\}$ is an orthonormal basis for V. We only have to show that they are mutually orthogonal. By our construction, the function

$$\psi(x,y) = ||(A(xe_1 + ye_2)||^2$$

of two complex variables x and y subject to the constraint $|x|^2 + |y|^2 = 1$ attains a maximum when $|x| = 1$. Let

$$x = x_1 + ix_2 , \quad y = y_1 + iy_2 ,$$

$$< f_1, f_2 > = u + iv$$

be the respective decompositions into real and imaginary parts. Then the above statement means that the function

$$\psi(x_1,x_2,y_1,y_2) = C_1^2(x_1^2+x_2^2) + C_2^2(y_1^2+y_2^2)$$

$$+ 2u(x_1y_1+x_2y_2) + 2v(x_1y_2-x_2y_1)$$

of the four real variables x_1,x_2,y_1,y_2 subject to the constraint

$$x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1$$

attains a maximum when $y_1 = y_2 = 0$.

Introducing a Lagrange multiplier λ, we see that at an extremum we must have

$$c_1^2 x_1 + u y_1 + v y_2 + \lambda x_1 = 0 ,$$

$$c_1^2 x_2 + u y_2 - v y_1 + \lambda x_2 = 0 ,$$

$$c_2^2 y_1 + u x_1 - v x_2 + \lambda y_1 = 0 ,$$

$$c_2^2 y_2 + u x_2 + v x_1 + \lambda y_2 = 0 .$$

Eliminating λ we get

$$u(y_1 y_2 - x_1 x_2) + v(y_2^2 - x_1^2) = 0$$

$$u(y_1^2 - x_1^2) + v(y_1 y_2 + x_1 x_2) = 0 ,$$

$$u(y_1 y_2 - x_1 x_2) - v(y_1^2 - x_2^2) = 0 ,$$

$$u(y_2^2 - x_2^2) - v(y_1 y_2 + x_1 x_2) = 0$$

These equations, in turn, lead to

$$u(x_1^2 + x_2^2 - y_1^2 - y_2^2) = 0$$

$$v(x_1^2 + x_2^2 - y_1^2 - y_2^2) = 0 .$$

Using the constraint $x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1$, we see athat at an extremum we must have

$$u(2x_1^2 + 2x_2^2 - 1) = 0 ,$$

$$v(2x_1^2 + 2x_2^2 - 1) = 0 .$$

But, we know that $x_1^2 + x_2^2 = 1$ is an extremum point. So

$$u = v = 0 .$$

Thus f_1, f_2 are orthogonal. The same argument shows that $f_1, \ldots, f_n$ are mutually orthogonal. $\square$

Consider now the map $A \to \Lambda^k(A)$ from $\mathcal{L}(V)$ to $\mathcal{L}(\Lambda^k V)$. Then $D\Lambda^k(A)$ is a linear map from $\mathcal{L}(V)$ to $\mathcal{L}(\Lambda^k V)$. Our next theorem gives a bound for the norm of this linear map.

<u>Theorem 2</u> Let $A \in \mathcal{L}(V)$, where $\dim V = n$. Then

$$||D\Lambda^k(A)|| \leq C_{n,k} \, ||A||^{k-1}$$

where,

$$\begin{aligned} C_{n,k} &= 1 \qquad \text{for } k = 1 \\ &= 2(k-1)^{1/2} \binom{n}{k}^{1/2} \qquad \text{for } k = 2, \ldots, n \end{aligned}$$

<u>Proof</u> We have,

$$||D\Lambda^k(A)|| = \sup_{\substack{\alpha \in \mathcal{L}(V) \\ ||\alpha|| = 1}} ||D\Lambda^k(A))(\alpha)||$$

$$= \sup_{\substack{\alpha \in \mathcal{L}(V) \\ ||\alpha||=1}} \sup_{\substack{z \in \Lambda^k V \\ ||z||=1}} ||(D\Lambda^k(A)(\alpha))(z)|| \tag{1}$$

Choose e_i, f_i, c_i, $i = 1,2,\ldots,n$ as given by Proposition 1. If $z \in \Lambda^k V$ has norm 1, we can write

$$z = \sum_{1 \leq i_1 < \ldots < i_k \leq n} t_{i_1 \ldots i_k} \, e_{i_1} \wedge \ldots \wedge e_{i_k}$$

where

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} |t_{i_1 \ldots i_k}|^2 = 1 \tag{2}$$

Let

$$\alpha e_i = \sum_{h=1}^{n} a_{ih} f_h \qquad i = 1,2,\ldots,n.$$

If $||\alpha|| = 1$, we must have

$$\sum_{h=1}^{n} |a_{ih}|^2 \leq 1 \qquad i = 1,2,\ldots,n \qquad (3)$$

For convenience we adopt the following notation : -

$\sum\limits_{k=1}^{n}{}^{*} \; x_1 \ldots x_{k-1}\, y_k\, x_{k+1} \ldots x_n$ will denote the sum of terms obtained

from $x_1 x_2 \ldots x_n$ by successively replacing x_k by y_k, $k = 1,2,\ldots,n$.

Since $(D\Lambda^k(A))(\alpha) \quad \left.\frac{d}{dt}\right|_{t=o} \Lambda^k(A+t\alpha)$

and Λ^k is multilinear, we have

$(D\Lambda^k(A)(\alpha))(z)$

$$= \sum_{1\leq i_1 <\ldots< i_k \leq n} t_{i_1\ldots i_k} \sum_{p=1}^{k}{}^{*} Ae_{i_1} \wedge \cdots \wedge \alpha e_{i_p} \wedge \cdots \wedge Ae_{i_k}$$

$$= \sum_{1\leq i_1 <\ldots< i_k \leq n} t_{i_1\ldots i_k} \Big\{ \sum_{p=1}^{k}{}^{*} \sum_{h=1}^{n} c_{i_1} \ldots c_{i_{p-1}} a_{i_p h} c_{i_{p+1}} \ldots c_{i_k}$$

$$\cdot f_{i_1} \wedge \cdots \wedge f_{i_{p-1}} \wedge f_h \wedge f_{i_{p+1}} \cdots \wedge f_{i_k} \Big\}$$

(4)

Denote this expression by X for brevity. To find a bound for (1) it suffices to find the supremum of $||X||$ subject to conditions (2) and (3).

We have,

We have,

$$||x||^2 = \sum_{\substack{1\le i_1<\dots<i_k\le n\\ 1\le j_1<\dots<j_k\le n}} t_{i_1\cdots i_k}\bar{t}_{j_1\cdots j_k}$$

$$\{ \sum_{p,q=1,\dots,k}^{*} \sum_{h,m=1,\dots,n} (c_{i_1}\cdots c_{i_p}\cdots c_{i_k})(c_{j_1}\cdots \hat{c}_{j_q}\cdots c_{j_k}) a_{i_p h}\bar{a}_{j_q m}$$

$$< f_{i_1}\wedge\cdots\wedge f_h\wedge\cdots\wedge f_{i_k},\ f_{j_1}\wedge\cdots\wedge f_m\wedge\cdots\wedge f_{j_k} > \} \qquad (5)$$

where the bar denotes complex conjugation, the circumflex denotes that the index under it has been omitted and where in the inner product on the right hand side f_h and f_m have been inserted in place of f_{i_p} and f_{j_q} respectively. Since f_i form an orthonormal basis, most of the terms on the right hand side vanish. In fact, the inner product in (5) is nonvanishing only when the following three conditions are fulfilled:

(i) the indices $i_1,\dots,h,\dots,i_k$ are distinct; (ii) the indices $j_1,\dots,m,\dots,j_k$ are distinct; (iii) the two sets of indices $\{i_1,\dots,h,\dots,i_k\}$ and $\{j_1,\dots,m,\dots,j_k\}$ are identical.

Denote the expression inside the braces on the right hand side of (5) by S. Then S is nonvanishing only in the following situations:

<u>Case a</u> : $\{i_1,\dots,i_k\} = \{j_1,\dots,j_k\}$, $h = m, p = q, p$ takes all values between 1 and k, h takes all values between 1 and n except $i_1,\dots,i_{p-1},i_{p+1},\dots,i_k$. If the value of S in this case is S_a we have

$$|S_a| \le ||A||^{2(k-1)} \sum_{h,p} |a_{i_p h}|^2 \quad ,$$

where h and p vary as indicated above. So, by (3) we have

$$|S_a| \leq k||A||^{2(k-1)} \quad .$$

<u>Case b</u> : the set $\{i_1,\ldots,i_k\}$ does not contain precisely one index j_t from the set $\{j_1,\ldots,j_k\}$ and the latter does not contain the index i_s from the former. Then the terms in the sum S are nonvanishing only in the following subcases:

(i) $p = s$, $q = t$, $h = m$, h varies from 1 to n avoiding the values $i_1,\ldots,i_{s-1}$, $i_{s+1},\ldots,i_k$. If the value of S in this case is denoted as $S_{b(i)}$ we have, using (3) and the Schwarz inequality,

$$|S_{b(i)}| \leq ||A||^{2(k-1)} \quad .$$

(ii) $p = s$, $q \neq t$, $h = j_t$, $m = j_q$. In this case

$$|S_{b(ii)}| \leq ||A||^{2(k-1)} \sum_{\substack{q=1,\ldots,k \\ q\neq t}} |a_{i_s j_t} \bar{a}_{j_q j_q}|$$

$$\leq (k-1) ||A||^{2(k-1)} .$$

(iii) $p \neq s$, $q = t$, $m = i_s$, $h = i_p$. Again, in this case

$$|S_{b(iii)}| \leq (k-1) ||A||^{2(k-1)} .$$

(iv) $p \neq s$, $q \neq t$, $p = q$, $h = j_t$, $m = i_s$. In this case

$$|S_{b(iv)}| \leq (k-2) ||A||^{2(k-1)} .$$

Adding all these, we see that the value S_b of S in case b is bounded as

$$|S_b| \leq 3(k-1)||A||^{2(k-1)}$$

Case c Two indices i_r, i_s do not occur in the set $\{j_1,\ldots,j_k\}$ and two indices j_t, j_u do not occur among $\{i_1,\ldots,i_k\}$. Then the terms in the sum S are nonvanishing only in the following four cases

(i) $p = r, \quad h = j_t, \; q = u, \quad m = i_s$

(ii) $p = r, \quad h = j_u, \; q = t, \; m = i_s$

(iii) $p = s, \quad h = j_t, \; q = u, \quad m = i_r$

(iv) $p = s, \quad h = j_u, \; q = t, \quad m = i_r$.

So we have for the value S_c of S in Case c,

$$|S_c| < 4||A||^{2(k-1)}$$

For all other choices of indices the inner product on the right hand side of (5) vanishes. So,

$$|S| \leq ||A||^{2(k-1)} \quad \max\{k,(3k-1),4\}$$

$$\leq 4(k-1)\,||A||^{2(k-1)}, \qquad k = 2,3,\ldots,n\,.$$

Hence,

$$||X||^2 \leq \binom{n}{k}\;|S|$$

$$\leq 4(k-1)\binom{n}{k}||A||^{2(k-1)}, \qquad k = 2,3,\ldots,n\,.$$

The theorem follows. $\square$

<u>Remark</u> The constants $C_{n,k}$ occuring in the statement of the theorem could perhaps be improved. It is not clear to us how to do that.

In particular, the inequality derived above shows that when B is close to A we have

$$||A^kB-A^kA|| \leq C_{n,k} ||A||^{k-1} ||B-A|| + O(||B-A||^2) .$$

REFERENCES

1. H.Baumgärtel, Endlichdimensionale Analytische Störungstheorie, Akademie - Verlag, Berlin, 1972.

2. N.Bourbaki, Groupes et Algèbres de Lie, Chapitres 4-6, Hermann, Paris, 1968.

3. C.Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math. 77(1955), 778-782.

4. C.Davis, The rotation of eigenvectors by a perturbation, J. Math. Anal. and Appl. 6(1963), 159-173.

5. ——————, The rotation of eigenvectors by a perturbation II, ibid, 11(1965), 20-27.

6. —————— and W.Kahan, The rotation of eigenvectors by a perturbation III, SIAM J. Numer. Anal. 7(1970), 1-46.

7. J.Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.

8. S.Helgason, Differential Geometry and Symmetric Spaces, Academic Press New York, 1962.

9. P. Henrici, Bounds for iterates, inverses, spectral variation and fields of values of nonnormal matrices, Numer. Math. 4(1962), 24-39.

10. A.J. Hoffman and H.W. Wielandt, The variation of the spectrum of a normal matrix, Duke Math. J. 20(1953), 37-39.

11. W.Kahan, Spectra of nearly Hermitian matrices, Proc. Amer. Math. Soc. 48(1975), 11-17.

12. T.Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1976.

13. S. Lang, Algebra, Addison-Wesley, Reading, 1970.

14. V.B.Lidskii, The proper values of the sum and product of symmetric matrices, Dokl. Akad. Nauk. SSR 75(1950) 769-772, (in Russian).

15. A.M. Ostrowski, Solution of Equations and Systems of Equations, 2nd ed., Academic Press, New York, 1966.

16. K.R.Parthasarathy, Eigenvalues of matrix-valued analytic maps J.Austral. Math. Soc. (Series A) 26(1978), 179-197.

17. F.Rellich, Perturbation Theory of Eigenvalue Problems, Gordon and Breach, New York, 1969.

18. D.A. Suprunenko and R.I.Tyshkevich, Commutative Matrices, Academic Press, New York, 1968.

19. H.Weyl, Das asymptotische Verteilungsgestz der Eigenwerte ... , Math. Ann. 71(1911), 441-479.