ON SEQUENTIAL TEST PROCEDURES WITH APPLICATIONS IN IDENTIFICATION AND SELECTION

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ABBREVIATIONS AND NOTATIONS

Abbreviations

rv	random variable
iid	independent and identically distributed
cdf	cumulative distribution function
w.p.1	with probability one
a.s.	almost surely
f.b.p.	free boundary problem
SPRT	sequential probability ratio test
ASN	average sample number

Notations

I_S	indicator function of set S
$I(S,T)$	$I_{\{ST \geq 0\}}$
$N(a,b)$	normal with mean a and variance b
$\varphi(\cdot)$	density function of $N(0,1)$
$\Phi(\cdot)$	cdf of $N(0,1)$
$A \Delta B$	symmetric difference of the sets A and B
E_p	a pxp matrix whose all elements are one
$\wedge$	minimum
$\vee$	maximum
$\sim$	distributed as
$\simeq$	approximately equal to
$\Rightarrow$	implies that; converges in distribution

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CHAPTER I

INTRODUCTION AND SUMMARY

1.1 Introduction

The area of sequential testing of statistical hypotheses is an important part of sequential analysis. The idea of a sequential test goes back to Dodge and Romig (1929) who constructed a double sampling procedure for sampling inspection. They were motivated by the observation that the double sampling plan requires a smaller number of observations on the average when compared with the corresponding single sampling plan. Later schemes like multiple sampling vide Walter Bartky (1943) and interesting practical application of large scale experiments in successive stages vide Mahalanobis (1940) started coming up.

The formal theory in sequential analysis began in about 1943 with the work of A. Wald in America (vide Wald (1945)) and G.A. Barnard (vide Barnard (1946)) in Britain in war time industrial advisory groups. The discovery of Wald's sequential probability ratio test (SPRT) was considered to be most important. An elegant theory of SPRT is given in Wald (1947) and a review with a list of references can be found in Johnson (1961). Barnard (1947) also gives a review of Wald (1947).

The subject of sequential analysis has undergone a rapid development since the formal theory came up. For some more references in this area one may look into Wetherill (1966) and Ghosh (1970).

This thesis deals with the problem of testing of hypotheses, sequentially, arising from identification and selection problems. It also gives a numerical solution to a free boundary problem (f. b. p.) arising from the problem of testing sequentially the sign of the drift parameter of a Wiener process.

The problem of identification or classification of an individual into one of the two categories is well known in statistical literature. If the two categories are completely specified then one can adopt a sequential test with an aim to control the errors of misclassification. This has been done by Rao (1948), Armitage (1950) and Mallows (1953). Sequential techniques are adopted even when the categories are partially specified vide Srivastava (1973) and Ghosh and Mukhopadhyay (1980). A more detailed discussion of these works can be found in Section 2.1 of Chapter 2.

Selection and ranking of populations is another important area of Statistics. A vast literature is available in this area. The sequential methods for selection and ranking are summarised beautifully by Bechhofer, Kiefer and Sobel (1968). Both sequential and non-sequential methods useful for selection and ranking problems can be found in Gupta and Panchapakeshan (1979) as well as in Gibbons, Olkin and Sobel (1977).

The problem of selecting one population ('best' or 'worst' in some well defined sense) out of k-many populations ($k \geq 2$) is most common in selection problems. If the populations are reasonably specified then once again sequential procedures can be adopted with a target of reaching the prespecified probability of correct selection namely P^* The idea of sequential procedures of choosing one out of k-many hypotheses using likelihoods goes back to Wald (1947, Chapter 10 and subsequently by Sobel and Wald (1949), Armitage (1950), Meilijson (1969), Hoel (1971), Robbins (1970), Khan (1973) and recently by Mukhopadhyay (1983) . Some more details on these investions are given in Section 5.1 of Chapter 5 .

A relatively modern tool in sequential analysis is optimal stopping theory which has been in a state of rapid development since about 1960 , however some particular optimal stopping problems have a long history in probability theory. For a modern treatment of this topic one may look into the book by Chow, Robbins and Siegmund (1971) as well as Chapter 2 of Neveu (1972) . Chernoff in a series of papers considered a continuous time optimal stopping problem in connection with a problem of testing the sign of a normal mean (without an indifference zone) in presence of a normal prior of the mean. Some more references regarding this problem are available in Section 6.1 of Chapter 6 .

1.2 Summary of the Results

Chapter 2 deals with an identification problem where the populati are univariate normal differing in their unknown means and the common variance may be known or unknown. A sample of fixed size k is given from π_0 the population to be identified and from the other two populations π_1 and π_2 one can sample sequentially or non-sequentially. This formulation (specially the multivariate version which is considered later) fits quite well in many problems in anthropological surveys.

A parameter δ_0 is introduced to specify the indifference zone $|\mu_1 - \mu_2| \geq \delta_0$ where μ_i denotes the mean of μ_i for $i = 1,2$ and invoking invariance the problem is reduced to a testing problem. The one sided $(\mu_1 \geq \mu_2 + \delta_0)$ and the known variance case has been taken up by Ghosh and Mukhopadhyay (1980).

A truncated invariant SPRT is proposed as a solution as the untruncated SPRT does not terminate with probability one. Numerical results show substantial saving achieved by the truncated invariant SPRT with respect to the most powerful invariant fixed sample procedure. Unlike the one sided case we do not have MLR here and so Theorem 2 of Ghosh (1960) does not any longer lead to the monotonicity of error probabilities of the SPRT in $|\mu_1 - \mu_2|$. However we could bound the error probability by a simple technique. For the most powerful invariant fixed sample test HPKE inequality yields partial monotonicity of the error probabilities. Further investigation may be made to establish monotonicity of the error probabilities both in the sequential and in the non-sequential case.

Chapter 3 deals with a similar problem in multivariate set up. The same technique (as in the univariate case) yields bounds on the error probabilities of the proposed invariant SPRTs The monotonicity of the error probabilities of the corresponding most powerful fixed sample test has been obtained only for the case where the two kinds of error probabilities are kept at the same prescribed level by using the results of Dasgupta (1974) as the HPKE condition does not hold here. The study of monotonicity of the error probabilities both in the sequential and non-sequential case requires further investigation. This chapter devotes a large part to the study of termination properties of the proposed SPRT's.

Chapter 4 deals with the asymptotic distributions of the stopping times of the SPRTs proposed in Chapter 2 and Chapter 3. A general theorem regarding the asymptotic study of stopping times is given first from which the limiting distributions of the SPRT's follow both in the truncated and the untruncated case. The truncated case is partially solved.

Chapter 5 deals with a selection problem of choosing the population with the largest mean among k-populations ($k \geq 2$) with the target of reaching a prespecified probability of correct selection namely P^* The problem is formulated with an indifference zone and following the lines of Mukhopadhyay (1983) an extension of invariant SPRT for choosing one out of k-many hypotheses is suggested. The asymptotic distribution of the stopping time of the proposed procedure and an asymptotic expression for ASN are obtained as $P^* \to 1$. The sequential procedure shows substantial saving in sample size when compared numerically with the corresponding fixed sample procedure . A comparison with the two stage procedure of Bechhofer, Dunnet and Sobel (1954) is also made. It will be interesting to develop a purely sequential and truncated (Paulson type) procedure for this problem.

Chapter 6 solves a free boundary problem numerically (by the method of lines vide Sackett (1971)) arising from the problem of testing the sign of a drift parameter μ of a Wiener process $\{X(t),\ t \in [0,\infty)\}$ in presence of a known normal prior of μ. The problem of testing the sign of μ with cost of incorrect decision $|\mu|$ and cost of sampling c per unit time, has been considered by Chernoff in a series of papers. He reduced the problem to a free boundary problem (f. b. p.) and gives asymptotic expression of the optimal boundary as $t \to \infty$ (with Breakwell (1964)) and as $t \to 0$. The purpose of solving the f.b.p. numerically is to have a complete view of the optimal boundary. These results agree with those of Chernoff and Petkau (1986) who used a different method to solve the same testing problem numerically.

CHAPTER 2

AN INVARIANT SPRT FOR IDENTIFYING A UNIVARIATE NORMAL POPULATION

2.1 Introduction

The problem of identifying or classifying an individual into one of the two categories is well known in statistical literature. There is a comprehensive review on this subject by Dasgupta (1973). However the use of sequential technique in classification is much less common. If the two categories are completely specified then one can adopt a sequential test (may be an SPRT) with an aim to control the errors of misclassification. Such attempt has been made by Rao (1948) and Armitage (1950) where in the later there are k (≥ 2) completely specified categories.

Mallows (1953) studied a similar problem from a slightly different view point. Here he takes observations on a single individual sequentially (assuming that there is a sequence of characters which may be measured progressively) and carried out an SPRT with independent but not identical observations. Here also the categories to which the individual is to be classified is assumed to be completely known.

Srivastava (1973) considered a classification problem where the populations are multivariate normal with common unknown variance-covariance matrix and the difference of the mean vectors is assumed to be known. A sequential procedure with an aim to keep both kinds of error probabilities at the same prescribed level is suggested. Here he samples sequentially from two populations instead of three at a time.

Recently Ghosh and Mukhopadhyay (1980) (henceforth will be referred as GM) have developed two sequential procedures for identifying population π_0, as having the same distribution as one of the two other populations π_1 and π_2 on the basis of samples from the three populations. They assume a sample of fixed size k is given from π_0 while unlimited sampling is permitted from π_1 and π_2. Assuming further, normality of all the three populations, with common known variance σ^2 and the one sided situation $\mu_2 > \mu_1$ (μ_i denotes the mean of π_i for $i = 1,2$) they reduce the problem to a testing problem and then use a truncated invariant SPRT.

We carry out here a similar investigation of both sequential and non-sequential procedures with the object of removing the assumption $\mu_2 > \mu_1$ and known σ^2. This requires substantial modifications in the treatment. Following GM , (1980) we have invoked invariance and used a truncated invariant SPRT as a solution and permitted two kinds of errors to be at two different levels unlike Srivastava (1973). The sampling scheme used here (same as in GM . (1980)) is also different from that of Srivastava (1973). The setup used here fits quite well in anthropological studies vide GM (1980) and Schaafsma and Vanvark (1977,1979).

For some more references on sequential discrimination one may look into Lachenbruch (1975).

In this chapter, the case where $\mu_1 \neq \mu_2$ but σ is known is considered first. A parameter δ_0 is introduced to specify the indifference zone and we proceed to test the following hypotheses (with μ denoting the mean of π_0).

$$H_1 : \mu_1 = \mu, \mu_2 \neq \mu, \quad |\mu_1 - \mu_2| = \delta_0 ,$$

$$H_2 : \mu_1 \neq \mu, \mu_2 = \mu, \quad |\mu_1 - \mu_2| = \delta_0$$

with
$$\left.\begin{aligned} P_{H_1} \text{ (Rejection of } H_1) &= \alpha \\ P_{H_2} \text{ (Rejection of } H_2) &= \beta \end{aligned}\right\} \qquad \ldots(2.1.1)$$

where α and β are preassigned numbers.

Of course the idea is that a reasonable solution of the above problem (2.1.1) will satisfy the following stronger requirements :

$$\left.\begin{aligned} \alpha(\delta) &\underset{\text{def}}{=} P_{(\mu=\mu_1,\mu_1,\mu_2)} \text{ (Reject } H_1) \leq \alpha \text{ if } |\mu_1-\mu_2| = \delta \geq \delta_0 \\ \beta(\delta) &\underset{\text{def}}{=} P_{(\mu=\mu_2,\mu_1,\mu_2)} \text{ (Reject } H_2) \leq \beta \text{ if } |\mu_1-\mu_2| = \delta \geq \delta_0 \end{aligned}\right\} \qquad \ldots(2.1.2)$$

As a solution of (2.1.1), an invariant truncated SPRT of H_1 vs H_2 is proposed as the untruncated SPRT does not terminate w.p. 1. Unlike the one sided case we do not have MLR here and so Theorem 2 of Ghosh (1960) does not any longer lead to the monotonicity of error probabilities in $|\mu_1 - \mu_2|$. However we shall bound the error probabilities sufficiently well to make it plausible that our solution does not have error probabilities greater that α, β for $|\mu_2 - \mu_1| > \delta_0$. Numerical studies reported in Section 2.5, confirm this.

Using the HPKE inequality we are also able to prove monotonicity of $\alpha(\delta)$ $(\beta(\delta))$ (as defined in (2.1.2) for the corresponding most powerful invariant fixed sample test if the cut off constant is negative (positive). These results of sequential as well as non-sequential procedures are given in Section 2.2.

Similar results are proved in Section 2.3 when $\mu_1 \neq \mu_2$ and σ^2 is unknown.

Some alternative simpler procedures are developed in Section 2.4 and in Section 2.5, numerical studies relating to the performance of the proposed procedures are made. Numerical comparisons show substantial saving in sample size for the truncated SPRT when compared with that of the corresponding most powerful invariant fixed sample test. The bounds on error probabilities are found to be conservative. Lastly Section 2.6 gives the proofs of the theorems regarding the termination properties of the untruncated SPRT for the known as well as unknown σ case.

This chapter is a revised version of Ghosh and Ray Chaudhuri (1984).

2.2 Procedures For Common Known Variance Case

Let X,Y,Z with suffixes be the random variables associated with π_o, π_1 and π_2 respectively. We have a sample $X_1, X_2, \ldots X_k$ of size k from π_o, $Y_1, Y_2, \ldots, Y_n$ from π_1 and $Z_1, Z_2, \ldots Z_n, \ldots$ from π_2.

Let I(S,T) denote the indicator function of the set $\{ST \geq 0\}$. Now the hypotheses defined in Section 2.1, can be rewritten as

$$\left.\begin{aligned} H_1 &: \underset{\sim}{\theta} = \theta_1(\delta_o) = (\delta_o, \delta_o, 1), \\ H_2 &: \underset{\sim}{\theta} = \theta_2(\delta_o) = (\delta_o, \delta_o, 0) \end{aligned}\right\} \qquad \ldots(2.2.1)$$

where $\underset{\sim}{\theta} = (|2\mu-\mu_1-\mu_2|, |\mu_1-\mu_2|, I(2\mu-\mu_1-\mu_2, \mu_1-\mu_2))$ ...(2.2.2)

Note that $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n)$ is sufficient for (μ, μ_1, μ_2). We consider the group of transformation $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n) \to (a\bar{X}_k + b, a\bar{Y}_n + b, a\bar{Z}_n + b)$ where $a = \pm 1$ and $-\infty < b < \infty$. Then $u_n = (|R|, |Q|, I(R,Q))$ ...(2.2.3)

is maximal invariant under this group of transformation where

$$R = 2\bar{X}_k - \bar{Y}_n - \bar{Z}_n, \quad Q = \bar{Y}_n - \bar{Z}_n \qquad \ldots(2.2.4)$$

The invariant sufficiency of the above statistic u_n follows from the basic theorem of Hall et al (1965). The distribution of u_n depends on the maximal invariant parameter $\underset{\sim}{\theta}$ defined above, which reduces to $(\delta_o, \delta_o, 1)$ and $(\delta_o, \delta_o, 0)$ under H_1 and H_2 respectively. Now it is required to find a test satisfying condition (2.1.1) of Section 2.1.

Fixed Sample Size Procedure

$$\text{Let } V_{n,k}(\delta_o) = \frac{f_{H_2}(u_n)}{f_{H_1}(u_n)} = \frac{\cosh\left(\frac{\delta_o}{2}\left(\frac{kn}{2n+k}R - nQ\right)\right)}{\cosh\left(\frac{\delta_o}{2}\left(\frac{kn}{2n+k}R + nQ\right)\right)} \qquad \ldots(2.2.5)$$

The fixed sample most powerful invariant test (P_o) of H_1 vs H_2 is as follows

$$\text{Reject } H_1 \text{ if } \ln V_{n,k}(\delta_o) > c \qquad \ldots(2.2.6)$$

The constant c and the sample size n_o are chosen to satisfy (2.1.1). Due to complexity of the distribution of the test statistic $V_{n,k}(\delta_o)$, c and n_o are approximated by computer simulation, for given α, β, k and δ_o. Figure I and Figure II on page 139 give a pictorial view of the rejection region of H_1 (vide (2.2.6)) for the case $\alpha < \beta$ and $\alpha > \beta$ respectively.

Upper and lower bounds of c and n_o can be obtained by using upper and lower bounds of error probabilities which are as follows :

$$\text{Error }(c') \leq P_{H_1}(\ln V_{n,k}(\delta_o) > c) \leq \text{Error }(c) \qquad \ldots(2.2.7)$$

$$\text{Power }(c') \leq P_{H_2}(\ln V_{n,k}(\delta_o) > c) \leq \text{Power }(c) \qquad \ldots(2.2.8)$$

where

$$\text{Error }(c) = \varphi(-c/2\delta_o\sigma_A + \sigma_A\delta_o)\,\varphi(-c/2\delta_o\sigma_B - \sigma_B\delta_o) + \varphi(-c/2\delta_o\sigma_A - \sigma_A\delta_o)\,\varphi(-c/2\delta_o\sigma_B + \sigma_B\delta_o) \qquad \ldots(2.2.9)$$

$$\text{Power }(c) = \varphi(-c/2\delta_o\sigma_A + \sigma_A\delta_o)\,\varphi(-c/2\delta_o\sigma_B + \sigma_B\delta_o) + \varphi(-c/2\delta_o\sigma_A - \sigma_A\delta_o)\,\varphi(-c/2\delta_o\sigma_B - \sigma_B\delta_o) \qquad \ldots(2.2.10)$$

for $c' = c + \ln 2$, $\sigma_A = (4k^{-1} + 2n^{-1})^{-1/2}$, $\sigma_B = (2n^{-1})^{-1/2}$ and $\Phi(x)$ denotes the normal c.d.f. The above inequalities are obtained with the assumption $c \gtrsim 0$. For $c < 0$, similar results can be obtained. These bounds given in (2.2.7) and (2.2.8) are helpful for having a raugh idea of c and n_0 before going for simulation.

The proofs of (2.2.7) and (2.2.8) make use of the following simple observations:

$$(1)\ \frac{1}{2}\exp(|S_1+S_2| - |S_1-S_2|) \le \frac{\cosh(S_1+S_2)}{\cosh(S_1-S_2)} \le \exp(|S_1+S_2| - |S_1-S_2|)$$

$$\text{for } S_1 S_2 \gtrless 0$$

$$(2)\ \{|S_1+S_2| - |S_1-S_2| > c'\} \Rightarrow \{\ln\cosh|S_1+S_2| - \ln\cosh|S_1-S_2| > c\}$$

$$\Rightarrow \{|S_1+S_2| - |S_1-S_2| > c\} \text{ for } c \ge 0.$$

(3) Independence of R and Q.

The observations (1) and (2) are also useful for drawing Figure I and Figure II given on page 139 .

Now one needs a minimum number of observations k_0 from π_0 to have a most powerful invariant fixed sample test subject to (2.1.1) vide Section 2.1 of GM (1980). The same problem can be restated in a slightly different way i,e for fixed k, α and β is δ_0 large enough to ensure the existance of a fixed sample most powerful invariant test ? For the one sided case we already know the solution vide (1) of GM (1980). Now for the two sided case we proceed as follows :

$$\ln V_{n,k}(\delta_o) = \frac{\delta_o}{2}\left|\frac{kn}{2n+k}R-nQ\right| - \frac{\delta_o}{2}\left|\frac{kn}{2n+k}R+nQ\right| + \ln\left(\frac{1+e^{\delta_o\left|\frac{kn}{2n+k}R-nQ\right|}}{1+e^{-\delta_o\left|\frac{kn}{2n+k}R+nQ\right|}}\right)$$

$$=\left(\delta_o\left(\left|\frac{kn}{2n+k}R\right|\wedge|nQ|\right)+r_n\right)1_{(RQ<0)} + \left(-\delta_o\left(\left|\frac{kn}{2n+k}R\right|\wedge|nQ|\right)+r_n\right)1_{(RQ\geq 0)}$$

where $r_n = \ln\left(\dfrac{1+e^{-\delta_o\left|\frac{kn}{2n+k}R-nQ\right|}}{1+e^{-\delta_o\left|\frac{kn}{2n+k}R+nQ\right|}}\right)$

$$\to 0 \text{ a.s. as } n \to \infty \text{ whenever } \mu_1 \neq \mu_2 .$$

Thus as $n \to \infty$, $\ln V_{n,k}(\delta_o) \to \frac{\delta_o k}{2}|2\bar{X}_k-\mu_1-\mu_2|1_{((2\bar{X}_k-\mu_1-\mu_2)(\mu_1-\mu_2)\leq 0)}$

$$- \frac{\delta_o k}{2}|2\bar{X}_k-\mu_1-\mu_2|1_{((2\bar{X}_k-\mu_1-\mu_2)(\mu_1-\mu_2)\geq 0)} \text{ a.s.}$$

whenever $\mu_1 \neq \mu_2$. ...(2.2.11)

By (2.2.11), $P_{H_i}(\ln V_{n,k}(\delta_o)\leq x) \to P_{H_i}(X\leq x)$ for $i = 1,2$ as $n \to \infty$

for all $x \in \mathbb{R}$, ...(2.2.12)

where $X \sim N\left((-1)^i\frac{\delta_o^2 k}{2}, \delta_o^2 k\right)$ under H_i

for $i = 1,2$.

Now for the one sided case vide GM (1980),

$$P_{(\mu=\mu_i,\ \mu_1-\mu_2=\delta)}\left(\frac{-\delta_o kn}{2n+k}(2\bar{X}_k - \bar{Y}_n - \bar{Z}_n) \leq x\right)$$

$$\left.\begin{aligned}&\rightarrow P_{(\mu=\mu_i,\mu_1-\mu_2=\delta)}\left(\frac{-\delta_o k}{2}(2\bar{X}_k-\mu_1-\mu_2)\leq x\right)\\&=\Phi\left(\frac{x-(-1)^i\delta_o^2k/2}{\sqrt{\delta_o^2 k}}\right)\end{aligned}\right\} \quad \ldots(2.2.13)$$

Let c_{1n} and c_{2n} be the cut off constants for the one sided and the two sided case, respectively to keep the error probabilities of first kind at level α. Let β_{1n} and β_{2n} be the corresponding errors of second kind for the one sided and two sided case respectively.

It now follows easily from (2.2.12) and (2.2.13) that

$\lim_{n\to\infty} c_{1n} = \lim_{n\to\infty} c_{2n} = c$ *where c is such that*

$$\Phi\left(\frac{-c-\delta_o^2 k}{\sqrt{\delta_o^2 k}}\right) = \alpha \qquad \text{and}$$

$$\lim_{n\to\infty} \beta_{2n} = \lim_{n\to\infty} \beta_{1n} = \Phi(\tau_\alpha - \delta_o k^{1/2}).$$

Thus it follows that for given α, β, k $\exists$ *a fixed sample test with errors at* level α and β if

$$\beta > \Phi(\tau_\alpha - \delta_o k^{1/2})$$

$$\Leftrightarrow \quad \delta_o > (\tau_\alpha + \tau_\beta)\, k^{-1/2} \qquad \ldots(2.2.14)$$

Since the MLR property does not hold here, it is natural to turn to the HPKE inequality (vide Proposition 2.4 of Perlman and Olkin (1980)) to prove the monotonicity of error probabilities. The version of HPKE inequality that is relevant for us is as follows :

"Let v_i be a σ-finite measure on B_i for $B_i \in \mathcal{B}(\mathbb{R})$ $\forall\, i = 1,2,\ldots, p$. For two points $x = (x_1,\ldots, x_p)$ and $y = (y_1, y_2,\ldots y_p)$ of $\mathbb{R}^p$ define,

$$x \wedge y = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_p \wedge y_p),$$
$$x \vee y = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_p \vee y_p),$$
$$x \geq y \text{ if } x_i \geq y_i \ \forall\ i = 1,2,\ldots, p.$$

Suppose φ_1 and φ_2 are two probability density functions on the rectangle $\prod_{i=1}^{p} B_i$ with respect to the product measure $\prod_{i=1}^{p} v_i$ satisfying the HPKE condition i.e.,

$$\varphi_1(x)\, \varphi_2(y) \leq \varphi_1(x \wedge y)\, \varphi_2(x \vee y) \qquad \ldots(2.2.15)$$

Then for a measurable weakly increasing function h i.e. $h(x) \geq h(y)$ for $x \geq y$,

$$\int h\varphi_1 \leq \int h\varphi_2 ." \qquad \ldots(2.2.16)$$

Let $\mathcal{G} = \left\{ f_{\theta_i(\delta)}(u) \text{ for } \delta \geq \delta_0,\ i = 1,2 \right\}$ where $f_{\theta_i(\delta)}(u)$ denotes the density function of u (for fixed n call $u_n = u$) when $\theta_i(\delta)$ is the true parameter. If a pair of density functions (φ_1, φ_2) from the family $\mathcal{C}_\varphi$ had satisfied (2.2.15) with

(i) $\varphi_1(u) = f_{\theta_2(\delta_1)}(u)$, $\varphi_2(u) = f_{\theta_2(\delta_2)}(u)$ for $\delta_2 > \delta_1 \geq \delta_0$

(ii) $\varphi_1(u) = f_{\theta_1(\delta_2)}(u)$, $\varphi_2(u) = f_{\theta_1(\delta_1)}(u)$ for $\delta_2 > \delta_1 \geq \delta_0$

(iii) $f_{\theta_2(\delta)}(u)/f_{\theta_1(\delta)}(u)$ is a weakly increasing function of u,

then the error probabilities $\alpha(\delta)$, $\beta(\delta)$ of the most powerful invariant test would have been monotone in δ by an easy application of HPKE inequality. Unfortunately the situation is much more complex here. We first collect in Lemma 2.1, the HPKE conditions partially satisfied in our problem. The last assertion in Lemma 2.1 plays a crucial role for obtaining the bounds in sequential case.

<u>Lemma 2.1</u> : Let $\delta_2 > \delta_1 \geq \delta_0$, $S = \{u : RQ \geq 0\}$ and $V(\delta) = f_{\theta_2(\delta)}(u)/f_{\theta_1(\delta)}(u)$. Then

(i) $f_{\theta_1(\delta_1)}(u_1) \cdot f_{\theta_1(\delta_2)}(u_2) \leq f_{\theta_1(\delta_1)}(u_1 \wedge u_2) \cdot f_{\theta_1(\delta_2)}(u_1 \vee u_2)$

for $u_1, u_2 \in S$

(ii) $f_{\theta_2(\delta_1)}(u_1) \cdot f_{\theta_2(\delta_2)}(u_2) \leq f_{\theta_2(\delta_1)}(u_1 \wedge u_2) \cdot f_{\theta_2(\delta_2)}(u_1 \vee u_2)$

for $u_1, u_2 \in S^c$

(iii) $V(\delta)$ is weakly decreasing on S and weakly increasing on S^c when considered as a function of u.

(iv) $V(\delta)$ is decreasing function of δ on S and an increasing function of δ on S^c.

Proof : The proofs of (i) and (iii) follow from direct computation and (ii) is just a reformulation of (i).

The proof of part (iv) follows from the fact that the function $g(t) = \cosh(tS_1)/\cosh(tS_2)$ on $[0, \infty)$ is increasing in t if $|S_1| \geq |S_2|$. This is essentially the MLR property of a non-central chi-variable. □

Theorem 2.1 : Consider procedure P_0 and let $V(\delta) = V_{n,k}(\delta)$, with $V_{n,k}(.)$ as in (2.2.5).

(i) If $c \leq 0$, then $P_{\theta_1(\delta_2)}(\ln V(\delta_0) \leq c) \geq P_{\theta_1(\delta_1)}(\ln V(\delta_0) \leq c)$ where

$\theta_1(\delta_i) = (\delta_i, \delta_i, 1)$ for $i = 1,2$, with $\delta_2 > \delta_1 \geq \delta_0$.

(ii) If $c \geq 0$, then $P_{\theta_2(\delta_2)}(\ln V(\delta_0) > c) \geq P_{\theta_2(\delta_1)}(\ln V(\delta_0) > c)$ where

$\theta_2(\delta_i) = (\delta_i, \delta_i, 0)$ for $i = 1,2$, with $\delta_2 > \delta_1 \geq \delta_0$.

Proof : Let $p = P_{\theta_1(\delta_1)}(\ln V(\delta_0) \leq c)$ with $c \leq 0$.

Then $p = \displaystyle\int_{S \cup S^c} 1_{\{\ln V(\delta_0) \leq c\}} \, f_{\theta_1(\delta_1)}(u) \, du$ (where du stands for $d|R|d|Q|$ on $S \cup S^c$.)

$= \displaystyle\int_{S} 1_{\{\ln V(\delta_0) \leq c\}} \, f_{\theta_1(\delta_1)}(u) \, du$ (as $1_{\{\ln V(\delta_0) \leq c\}}(u) = 0$ on S^c for $c \leq 0$.)

$= p \displaystyle\int_{S \cup S^c} f_{\theta_1(\delta_1)}(u) \, du$

$$= p \int_S \left(1 + \frac{\cosh \frac{\delta_1}{2} \left(\frac{nk}{2m+k} |R| - n|Q|\right)}{\cosh \frac{\delta_1}{2} \left(\frac{nk}{2m+k} |R| + n|Q|\right)}\right)$$

$$\pi^{-1} \cosh \frac{\delta_1}{2} \left(\frac{nk}{2m+k} |R| + n|Q|\right)$$

$$e^{\frac{1}{2}\left(\frac{nk}{2m+k} + n\right) \frac{\delta_1^2}{2} - \frac{1}{2}\left(\frac{nk}{4m+2k} R^2 + \frac{n}{2} Q^2\right)} du$$

$$= p \int_S (1 + V(\delta_1)) f_{\theta_1(\delta_1)}(u) du.$$

Now $\int_S (1_{\{\ln V(\delta_o) \leq c\}} - p(1+V(\delta_1)))\, f_{\theta_1(\delta_1)}(u) du = 0$

$\Rightarrow \int_S (1_{\{\ln V(\delta_o) \leq c\}} - p(1+V(\delta_2)))\, f_{\theta_1(\delta_1)}(u) du \leq 0$ (by part (iv) of Lemma 2.1)

$\Rightarrow \int_S (1_{\{\ln V(\delta_o) \leq c\}} - p(1+V(\delta_2)))\, f_{\theta_1(\delta_2)}(u) du \geq 0$

by HPKE inequality (2.2.16) with $h(u) = 1_{\{\ln V(\delta_o) \leq c\}} - p(1+V(\delta_2))$ (weakly increasing in u by part (iii) of Lemma 2.1) and $\varphi_1 = c_1 f_{\theta_1(\delta_1)}(u)$, $\varphi_2(u) = c_2 f_{\theta_1(\delta_2)}(u)$ for $u \in S$ (where c_1 and c_2 are normalising constants and by part (i) of Lemma 2.1, φ_1, φ_2 satisfy HPKE condition (2.2.15)).

So $P_{\theta_1(\delta_2)}(\ln V(\delta_o) \leq c) \geq p = P_{\theta_1(\delta_1)}(\ln V(\delta_o) \leq c)$. The proof of part (ii) follows by similar reasoning. □

Sequential Procedure

Now the SPRT based on the invariant sufficient sequence u_n (as in (2.2.3)) is to be investigated. For given α and β, we choose $a = \ln(\beta/(1-\alpha))$ and $b = \ln((1-\beta)/\alpha)$. The stopping time N_1 and the decision rule in this case is as follows :

$$\text{At } n^{th} \text{ stage, decide } \pi_o = \pi_2 \text{ if } \ln V_{n,k}(\delta_o) \geq b$$
$$\pi_o = \pi_1 \text{ if } \ln V_{n,k}(\delta_o) \leq a$$

and continue the experiment by taking one more observation from each of the two populations π_1 and π_2 if $a < \ln V_{n,k}(\delta_o) < b$. This SPRT does not terminate with probability one (see Theorem 2.3 in Section 2.6), which emphasises the need of a truncation point. We choose the truncation point at $m_o = 2n_o$, where n_o is the sample size of the best invariant fixed sample procedure P_o, as in GM (1980). The modified procedure R_1 is as follows :

Continue the experiment as in usual SPRT (as defined earlier) until $n < m_o$ and at $n = m_o$, decide

$$\pi_o = \pi_2 \quad \text{if} \quad \ln V_{m_o,k}(\delta_o) > 0$$

$$\pi_o = \pi_1 \quad \text{if} \quad \ln V_{m_o,k}(\delta_o) \leq 0.$$

Now the performance of this truncated SPRT R_1 can be examined. The sufficient condition for monotonicity due to Ghosh (1960) does not hold and that due to Hoel (1970) also seems inapplicable. Moreover as noted earlier vide Lemma 2.1, even for the marginal distribution the HPKE condition holds only partially, whereas to use HPKE inequality in the sequential case one would need the HPKE condition for the joint distribution of the u_n 's. So an altogether different approach is made in the following proposition which yields bounds rather than monotonicity but assumes much less than the HPKE condition.

<u>Theorem 2.2</u> : Suppose g_0, g_1, g_0^* and g_1^* are the joint probability density functions of $X_1, X_2, \ldots, X_n$ under the hypotheses H_0, H_1, H_0^* and H_1^* respectively such that for all $n \geq 1$,

$$\left.\begin{aligned} G_n > B &\Rightarrow G_n^* > B \quad \forall \; B > 1 \\ \text{and } G_n < A &\Rightarrow G_n^* < A \quad \forall \; A < 1 \end{aligned}\right\} \qquad \ldots(2.2.17)$$

$$\text{where } G_n = g_1(X_1, X_2, \ldots, X_n)/g_0(X_1, X_2, \ldots, X_n)$$

$$\text{and } G_n^* = g_1^*(X_1, X_2, \ldots, X_n)/g_0^*(X_1, X_2, \ldots, X_n).$$

For given α and β consider the usual SPRT for H_0 versus H_1, with the usual boundary limits $\beta/(1-\alpha)$ and $(1-\beta)/\alpha$.

Let $\alpha^* = P_{H_0^*}$ (Rejection of H_0), $\beta^* = P_{H_1^*}$ (Rejection of H_1) and N the stopping time of the SPRT. Then Wald's inequalities hold for α^* and β^*, namely

$$\left.\begin{array}{c} \alpha^* \leq \frac{\alpha}{1-\beta}(1-\beta^*), \quad \beta^* < \frac{\beta}{1-\alpha}(1-\alpha^*) \text{ and} \\ \alpha^* + \beta^* \leq \alpha + \beta \end{array}\right\} \qquad \ldots(2.2.18)$$

if the untruncated SPRT terminates with probability one.

Moreover if the SPRT is truncated at m_0 and the decision at m_0 is taken in favour of H_1 and H_0 according as $G_{m_0} > 1$ or $G_{m_0} \leq 1$ respectively then

$$\left.\begin{array}{l} \alpha^* \leq \frac{\alpha}{1-\beta}(1-\beta^*) - \frac{\alpha}{1-\beta} P_{H_1^*}(N \geq m_0, G_{m_0} > 1) + P_{H_0^*}(N \geq m_0, G_{m_0} > 1) \\ \beta^* \leq \frac{\beta}{1-\alpha}(1-\alpha^*) - \frac{\beta}{1-\alpha} P_{H_0^*}(N \geq m_0, G_{m_0} \leq 1) + P_{H_1^*}(N \geq m_0, G_{m_0} \leq 1). \end{array}\right\} \ldots(2.2.19)$$

<u>Remark 2.1</u>. The first set of inequalities (2.2.18) is well known to be conservative. The second set of inequalities (2.2.19) suggests that α^* and β^* are unlikely to exceed α, β when $P_{H_0^*}(N \geq m_0)$ and $P_{H_1^*}(N \geq m_0)$ are small compared with α and β.

Though the above Theorem is self evident, it has some useful applications.

We shall now see how it provides bounds for error probabilities for the rule R. Let H_o, H_1, H_o^* and H_1^* be the hypotheses corresponding to the parameter points $\theta_1(\delta_o)$, $\theta_2(\delta_o)$, $\theta_1(\delta^*)$ and $\theta_2(\delta^*)$ respectively for $\delta^* > \delta_o$. Let f_o, f_1, f_o^* and f_1^* be the density functions of u_n under the hypotheses H_o, H_1, H_o^* and H_1^* respectively. Then from part (iv) of Lemma 2.1, it is evident that condition (2.2.17) is satisfied for the truncated SPRT R_1. Hence the bounds given in (2.2.19) are valid for R_1.

Thus if the probabilities $P_{\theta_i(\delta^*)}(N_1 \geq m_o)$ for $i = 1,2$ are small compared with α,β, the error probabilities at δ^* are unlikely to exceed α,β, stipulated for δ_o. Monte Carlo studies confirm that the truncation probabilities are small (provided n_o is not too small) and α^*, β^* are less than α,β respectively. Of course $P_{\theta_i(\delta^*)}(N_1 \geq m_o)$ can be bounded as in Wald (1947, section 3.8) and the bounds tend to zero as δ^* tends to infinity.

2.3 Procedures For Unknown Common Variance Case

In this case $\sigma^{-1}|\mu_1 - \mu_2| \geq \delta_o$ is considered with δ_o a positive real number as in the known variance case. The following hypotheses are to be tested,

$$H_1 : \sigma^{-1}(\mu - \mu_1) = 0,\ \sigma^{-1}(\mu - \mu_2) \neq 0,\ \sigma^{-1}|\mu_1 - \mu_2| = \delta_o,$$

$$H_2 : \sigma^{-1}(\mu - \mu_1) \neq 0,\ \sigma^{-1}(\mu - \mu_2) = 0,\ \sigma^{-1}|\mu_1 - \mu_2| = \delta_o$$

with the prescribed error levels α and β as given in (2.1.1). The problem now can be reduced by invariance as in Section 2.2. Here $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n, T_n)$ is sufficient for $(\mu, \mu_1, \mu_2, \sigma^2)$ where

$$T_n = \sum_1^k (X_i - \bar{X}_k)^2 + \sum_1^n (Y_i - \bar{Y}_n)^2 + \sum_1^n (Z_i - \bar{Z}_n)^2 \qquad \ldots(2.3.1)$$

Consider the group of transformation

$$(\bar{X}_k, \bar{Y}_n, \bar{Z}_n, T_n) \to (a\bar{X}_k + b,\ a\bar{Y}_n + b,\ a\bar{Z}_n + b,\ a^2 T_n)$$

where $-\infty < b < \infty$, $-\infty < a < \infty$ and $a \neq 0$.

$$\left.\begin{aligned} \text{Define}\quad t_{1n} &= (2\bar{X}_k - \bar{Y}_n - \bar{Z}_n)/\sqrt{T_n} \\ t_{2n} &= (\bar{Y}_n - \bar{Z}_n)/\sqrt{T_n}\,, \end{aligned}\right\} \qquad \ldots(2.3.2)$$

$$\text{then}\quad W_n = (t_{1n}, t_{2n}, I(t_{1n}, t_{2n})) \qquad \ldots(2.3.3)$$

is maximal invariant under this group of transformation and the invariant sufficiency follows from the basic theorem of Hall et al (1965). Here also the maximal invariant parameter $\underset{\sim}{\theta}$ reduces to $(\delta_o, \delta_o, 1)$ and $(\delta_o, \delta_o, 0)$ under H_1 and H_2 respectively where

$$\underset{\sim}{\theta} = (\sigma^{-1}|2\mu-\mu_1-\mu_2|,\ \sigma^{-1}|\mu_1-\mu_2|,\ I(\sigma^{-1}(2\mu-\mu_1-\mu_2),\ \sigma^{-1}(\mu_1-\mu_2))) \ldots(2.3.4)$$

$$\text{Let}\quad Q_{n,k}(\delta_o) = f_{H_2}(W_n)/f_{H_1}(W_n) \qquad \ldots(2.3.5)$$

$$= \frac{\int_0^\infty \cosh\left(\frac{\delta_o}{2}\sqrt{T}\left(\frac{kn}{2n+k}\, t_{1n} - n\, t_{2n}\right)\right) g(t_{1n}, t_{2n}, T)\,dT}{\int_0^\infty \cosh\left(\frac{\delta_o}{2}\sqrt{T}\left(\frac{kn}{2n+k}\, t_{1n} + n\, t_{2n}\right)\right) g(t_{1n}, t_{2n}, T)\,dT} \qquad \ldots(2.3.6)$$

where

$$g(t_{1n}, t_{2n}, T) = \exp(-\frac{1}{2} T(\frac{kn}{4n+2k} t_{1n}^2 + \frac{n}{2} t_{2n}^2 + 1))T^{\frac{2n+k-3}{2}} \quad ...(2.3.7)$$

The most powerful invariant fixed sample procedure as in Section 2.2 can be defined with the test statistic $Q_{n,k}(\delta_o)$ in place of $V_{n,k}(\delta_o)$. Here also one needs to have a minimum number of observations k_o from π_o for having a fixed sample most powerful invariant procedure subject to (2.1.1). Obtaining an exact value of k_o involves tedious numerical computation. However an upper bound of k_o may be obtained by a much simpler method as follows :

For given α and β, consider the harder problem with $\alpha'=\beta'=\alpha\wedge\beta$. For this harder problem the probability of correct identification is

$$P_{H_1}(t_{1n}\ t_{2n} \gtrless 0) \geq (\Phi(\delta_o(4k^{-1} + 2n^{-1})^{-1/2}))^2 \quad \text{for } t_{1n}, t_{2n} \text{ as in (2.3.2).}$$

Now the minimum value of k (say k_1) necessary for having a solution in n for the following equation

$$(\Phi(\delta_o(4k^{-1} + 2n^{-1})^{-1/2}))^2 = 1 - \alpha' \quad ...(2.3.8)$$

yields an upper bound of k_o. The solution in n (say n_o^*) for equation in (2.3.8) (for $k \geq k_1$) is an upper bound of n_o, where n_o denotes the sample size of the most powerful invariant fixed sample procedure.

The monotonicity of the error probabilities for this fixed sample procedure can be shown by direct computation for the case $\alpha = \beta$. This result is comparable to that of Schaafsma and Vanvark (1977) where they show that the likelihood ratio test (for the same problem as in this section with $k = 1$) has monotone error probabilities.

A truncated invariant SPRT as in the previous section can also be defined with the test statistic $Q_{n,k}(\delta_o)$ in place of $V_{n,k}(\delta_o)$ as the untruncated SPRT does not terminate w.p. 1 (see Theorem 2.4 in Section 2.6). One may choose the truncation point as $m_o = 2n_o^*$. Here also the error probabilities of the truncated SPRT can be bounded as in the known σ case vide the following lemma.

Lemma 2.2. For $B > 1$ and $A < 1$ and $\delta^* > \delta_o$, we have

(i) $\quad Q_{n,k}(\delta_o) \leq A \implies Q_{n,k}(\delta^*) < A$

(ii) $\quad Q_{n,k}(\delta_o) \geq B \implies Q_{n,k}(\delta^*) \geq B$.

Proof. For part (i), $Q_{n,k}(\delta_o) \leq A$ implies $t_{1n}\, t_{2n} \geq 0$ and

$$\int_0^\infty \left(1 - A^{-1} \frac{C_2(\delta_o,T)}{C_1(\delta_o,T)}\right) C_1(\delta_o,T)\, g\,(t_{1n}, t_{2n}, T)dT \geq 0$$

where $\quad C_1(\delta_o,T) = \cosh\left(\frac{\delta_o}{2} \sqrt{T}\left(\frac{kn}{2n+k} t_{1n} + n\, t_{2n}\right)\right)$

$$C_2(\delta_o,T) = \cosh\left(\frac{\delta_o}{2} \sqrt{T}\left(\frac{kn}{2n+k} t_{1n} - n\, t_{2n}\right)\right).$$

From the proof of part (iv) of Lemma 2.1, $\frac{C_1(\delta,T)}{C_2(\delta,T)}$ is an increasing function of δ, (for fixed T) when $t_{1n}\ t_{2n} \geq 0$. Hence

$$\int_0^\infty \left(1 - A^{-1}\,\frac{C_2(\delta^*,T)}{C_1(\delta^*,T)}\right) C_1(\delta_0,T)\ g\,(t_{1n},\ t_{2n},\ T)dT \geq 0.$$

To show $$\int_0^\infty \left(1 - A^{-1}\,\frac{C_2(\delta^*,T)}{C_1(\delta^*,T)}\right) C_1(\delta^*,T)\ g\,(t_{1n},\ t_{2n},\ T)dT \geq 0.$$

Let $$\varphi_1(T) = \begin{cases} a_1\,C_1\,(\delta_0,T)\ g\,(t_{1n},\ t_{2n},\ T) & T \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_2(T) = \begin{cases} a_2\,C_1\,(\delta^*,T)\ g\,(t_{1n},\ t_{2n},\ T) & T \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where a_1, a_2 are constants, so that φ_1 and φ_2 become probability density functions. Since φ_1 and φ_2 satisfy MLR condition and $h(T) = 1 - A^{-1}\,\frac{C_2(\delta^*,T)}{C_1(\delta^*,T)}$ is an increasing function of T, (for fixed δ^*) the proof of part (i) follows immediately (see Lehmann (1959), page 74). Proof of part (ii) follows by similar reasoning. □

We havn't carried out numerical comparison of the truncated SPRT to the fixed sample procedure. But we feel that the behaviour is similar to that of the known σ case.

2.4 Some Other Procedures

Since μ here has only two choices namely μ_1 and μ_2, one may look at the following formulation based on sampling from π_1 and π_0 only.

One may test H_1 : $|\mu - \mu_1| = 0$ versus H_2 : $|\mu - \mu_1| = \delta_o$. Here the rejection of H is considered as identifying π_o with π_2.

Let P_1 denote the best fixed sample invariant procedure in this case and n_1 the sample size required for keeping the two kinds of error at pre-assigned level α and β. If α and β are small enough to have

$$\Phi(-\tau_\beta) - \Phi(-2\tau_{\alpha/2} - \tau_\beta) \simeq \Phi(-\tau_\beta) = \beta \quad \text{then}$$

$$n_1 = k\left(\frac{\delta_o^2 k}{(\tau_{\alpha/2} + \tau_\beta)^2} - 1\right)^{-1}.$$

One can also look at a similar procedure P_2 to test the hypotheses H_1 : $|\mu - \mu_2| = \delta_o$ versus H_2 : $|\mu - \mu_2| = 0$ with the prescribed error probabilities as given in (2.1.1). Here the acceptance of H_1 leads to identification of π_o with π_1. As in procedure P_1, the sample size n_2 of this procedure P_2 is given by $k(\delta_o k/(\tau_{\beta/2} + \tau_\alpha)^2 - 1)^{-1}$ if α and β are small enough to have,

$$\Phi(-\tau_\alpha) - \Phi(-2\tau_{\beta/2} - \tau_\alpha) \simeq \Phi(-\tau_\alpha).$$

Though P_1 and P_2 seem to be an unsatisfactory way of discriminati (since one of the populations is not sampled at all), they are simple to use. A simulation study is made to judge the performance of the procedures P_0, P_1 and P_2. The results reported in Table 2.3 of Section 2.5, seem to favour P_0, decisively.

2.5 Numerical Studies

In this section numerical studies are made for the procedures described in Section 2.2 and in Section 2.4. The sample size n_o of the procedure P_0 is first estimated from the Monte-Carlo experiments and then the truncated SPRT R_1 is studied.

Table 2.1 shows the estimated type I and type II errors $\hat{\alpha}$ *and* $\hat{\beta}$ respectively, which are brought near to the given values of $\alpha = .05$ and $\beta = .025$ by adjusting c and n. Fractional values of n have not been allowed here and the nearest integer exceeding this is considered (for this reason our $\hat{\alpha}$ and $\hat{\beta}$ are little lower than α and β respectively, in some cases). The procedure P_0 is used 1000 times for each $\hat{\alpha}$ and $\hat{\beta}$ given in Table 2.1.

Table 2.2 shows that performance of the truncated SPRT R_1 as defined in Section 2.2. For $\alpha = .05$ and $\beta = .025$, $2n_o$ has been used as the truncation point where n_o is the simulated sample size of procedure P_0 as given in Table 2.1. The results in Table 2.2 are based on 200 repetitions of the rule R_1. The ASN and the type I and type II errors of R_1 (involving δ_o in the test statistic) are studied both for $|\mu_1 - \mu_2| = \delta_o$ and $|\mu_1 - \mu_2| > \delta_o$. Only the results for $|\mu_1 - \mu_2| = \delta_o$ are reported in Table 2.2 for different values of δ_o. Numerical results indicate a saving in sample size (as reflected by Columns 3 and 4 of Table 2.2) when compared with n_o. The saving is more prominent for smaller $\delta_o^2 k$. As $\delta_o^2 k$ increases the results indicate a tendency for ASN of R_1 to approach towards a constant multiple of n_o, namely $\frac{2}{3} n_o$. The simulated type I and type II errors are found to be much lower than α and β, and the number of cases leading truncation is also very few. The bounds (2.2.18) given in Theorem 2.2 seem to be quite conservative.

The simulation studies (not presented here) show a decline in misclassification probabilities as well as in ASN when the rule R_1 (involving δ_o in the test statistic) is used for the samples having $|\mu_1 - \mu_2| > \delta_o$.

In Chapter 4 asymptotic distribution of the stopping time N_1 (truncated as well as untruncated) is obtained as $k \to \infty$. The simulated performance of R_1 (given in Table 2.2) is discussed in Chapter 4, keeping in view these asymptotic results.

Table 2.3 shows the sample sizes of three different test procedures P_0, P_1 and P_2. Here n_i denotes the sample size of procedure P_i, for $i = 0, 1, 2$. For P_0 the observations are taken from each of the two populations π_1 and π_2 at each draw, while for $P_1(P_2)$ only one observation is taken at a time from $\pi_1(\pi_2)$. For this reason $2n_o$ is compared with n_1 and n_2. Results in Table 2.3, show savings in sample size for P_0 when compared with P_1 and P_2.

TABLE 2.1

Values of n_o and c for Procedure P_0

$\hat{\alpha}$ and $\hat{\beta}$ are the Simulated Type I and Type II Errors

k	δ_o	n_o	c	$\hat{\alpha}$	Estimated s.e. of $\hat{\alpha}$	$\hat{\beta}$	Estimated s.e. of $\hat{\beta}$
1	4.0000	3	- 0.50	.0485	.00679	.0205	.00448
	4.5000	1	- 0.50	.0495	.00686	.0200	.00443
5	1.7889	11	- 0.50	.0485	.00679	.0285	.00526
	2.2889	3	- 0.40	.0445	.00652	.0240	.00484
10	1.2649	22	- 0.50	.0485	.00679	.0225	.00469
	1.7649	4	- 0.50	.0495	.00686	.0245	.00488
20	0.8944	44	- 0.57	.0515	.00699	.0255	.00495
	1.3944	6	- 0.45	.0465	.00666	.0245	.00488
50	0.5657	108	- 0.50	.0480	.00676	.0255	.00495
	1.0657	9	- 0.45	.0515	.00699	.0260	.00503
100	0.4000	216	- 0.55	.0515	.00699	.0220	.00464
	0.9000	12	- 0.60	.0470	.00669	.0265	.00508
200	0.7828	16	- 0.55	.0490	.00683	.0245	.00488

Here k = Size of the fixed sample given from π_o.

$\delta_o = |\mu_1 - \mu_2|$

n_o = The estimated sample size of procedure P_0 from Monte Carlo experiments, needed to keep $\alpha = .05$ and $\beta = .025$.

c = The cut-off constant for P_0 needed to control α and β as above.

TABLE 2.2

Performance of the Rule R_1

k	δ_o	n_o	ASN of R_1	Estimated s.e. of ASN	P_{H_1} (Accept H_1)	Number of truncation
1	4.0000	3	1.02	0.0099	1	0
	4.5000	1	1.009	0.0064	1	0
5	1.7889	11	2.46	0.0813	.995	0
	2.2889	3	1.429	0.0427	.995	0
10	1.2649	22	4.1299	0.1315	.99	0
	1.7649	4	3.465	0.1230	.99	5
20	0.8944	44	9.425	0.3847	.985	0
	1.3944	6	4.155	0.1510	.97	1
50	0.5657	108	15.344	0.5382	.99	0
	1.0657	9	5.094	0.1650	1	0
100	0.4000	216	30.5	1.2282	.985	0
	0.9000	12	8.035	0.2922	.96	1
200	0.7828	16	10.6299	0.3854	.975	0

k, δ_o, n_o are as defined in Table 2.1.

TABLE 2.3

Sample Size Behaviour of P_0, P_1 and P_2

k	δ_0	n_0	n_1	n_2
1	4.0000	3	25	17
	4.5000	1	4	3
5	1.7889	11	122	84
	2.2889	3	8	7
10	1.2649	22	243	168
	1.7649	4	10	10
20	0.8944	44	486	336
	1.3944	6	14	13
50	0.5657	108	1213	840
	1.0657	9	19	19
100	0.4000	216	2426	1680
	0.9000	12	24	23

k, δ_0 are as defined in Table 2.1. n_i denotes the sample size of procedure P_i, needed to keep $\alpha = .05$ and $\beta = .025$, for i = 0,1,2.

Here $2n_0$ is compared with n_1 and n_2.

2.6 Termination Properties Of The SPRTs

In this section we prove that the untruncated invariant SPRTs defined in Section 2.2 and Section 2.3 do not terminate w.p. 1. Firstly Theorem 2.3 deals with the known σ case and finally Theorem 2.4 deals with the case of unknown σ.

Theorem 2.3 $P_{(\mu,\ \mu_1,\ \mu_2)}\ (N_1 = \infty) \geq 0$, for fixed (μ, μ_1, μ_2).

Proof. Note $N_1 = n \Longrightarrow \ln V_{n,k}(\delta_o) \geq b$ or $\ln V_{n,k}(\delta_o) \leq a$

$$\Longrightarrow \frac{\delta_o}{2}\left(\left| \frac{kn}{2n+k} R - n Q \right| - \left| \frac{kn}{2n+k} R + n Q \right| \right) \geq b$$

$$\text{or} \quad \frac{\delta_o}{2}\left(\left| \frac{kn}{2n+k} R + n Q \right| - \left| \frac{kn}{2n+k} R - n Q \right| \right) \geq -a$$

$$\Longrightarrow \delta_o \left(\left|\frac{kn}{2n+k} R\right| \wedge | n Q | \right) \geq b$$

$$\text{or} \quad \delta_o \left(\left|\frac{kn}{2n+k} R\right| \wedge | n Q | \right) \geq -a$$

$$\Longrightarrow \delta_o \frac{kn}{2n+k} |R| \geq b \wedge (-a)$$

Let $N_1^* = \inf \left\{ n : \frac{\delta_o kn}{2n+k} |R| \geq b \wedge (-a) \right\}$.

Then $N_1 \geq N_1^*$ and

$$P_{(\mu,\ \mu_1,\ \mu_2)}\ (N_1^* = \infty) > 0 \quad \text{(by Theorem 2 of GM (1980))}$$

imply the required result. □

For the unknown σ case let us define,

$$\left.\begin{aligned} N_2 &= \inf\left\{n : Q_{n,k}(\delta_o) \geq B \text{ or } Q_{n,k}(\delta_o) \leq A\right\} \\ &= \infty \qquad\qquad\qquad \text{otherwise} \end{aligned}\right\} \qquad ...(2.6.1)$$

with $Q_{n,k}(\delta_o)$ as given in (2.3.6) and A and B are real numbers s.t. $A < 1$ and $B > 1$.

Theorem 2.4 $P_\theta(N_2 = \infty) > 0$ for $\theta = (\mu, \mu_1, \mu_2, \sigma)$ fixed and $\sigma^{-1} \mid \mu_1 - \mu_2 \mid > 0$.

Proof. We shall first bound N_2 by a smaller stopping time and then prove the required result for the later.

$$\text{Observe } Q_{n,k}(\delta_o) = \frac{\int_0^\infty \cosh(s_{1n} T^{1/2}) e^{-T/2} T^{\frac{n^*-3}{2}} dT}{\int_0^\infty \cosh(s_{2n} T^{1/2}) e^{-T/2} T^{\frac{n^*-3}{2}} dT} \qquad ...(2.6.2)$$

$$\left.\begin{aligned} &\text{where } s_{in} = 2^{-1} t_n^{-1} \delta_o \left| \frac{kn}{2n+k} t_{1n} + (-1)^i n\, t_{2n} \right| \\ &\quad \text{with } t_n = \left(\frac{kn}{4n+2k} t_{1n}^2 + \frac{n}{2} t_{2n}^2 + 1 \right)^{1/2} \\ &\quad \text{and } t_{in} \text{ as in (2.3.2) for } i = 1, 2 \text{ and} \\ &\quad\quad n^* = 2n + k. \end{aligned}\right\} \qquad ...(2.6.3)$$

$$\text{Define } h(x) = \int_0^\infty e^{tx - t^2/2}\, t^{n^*-2}\, dt \quad \text{for } x \in \mathbb{R} \qquad ...(2.6.4)$$

$$Q_{n,k}(\delta_o) = \frac{h(s_{1n})}{h(s_{2n})} \qquad \ldots (2.6.5$$

Now $$Q_{n,k}(\delta_o) = \frac{h(s_{1n}) + h(-s_{1n})}{h(s_{2n}) + h(-s_{2n})}$$

(by substituting $\sqrt{T} = t$ in (2.6.2))

$$= Q'_{n,k}(\delta_o) \left(\frac{1 + \frac{h(-s_{1n})}{h(s_{1n})}}{1 + \frac{h(-s_{2n})}{h(s_{2n})}} \right)$$

If $Q_{n,k}(\delta_o) \geq B$

$\Rightarrow s_{1n} > s_{2n}$

$$\Rightarrow \frac{h(-s_{1n})}{h(s_{1n})} \leq \frac{h(-s_{2n})}{h(s_{2n})}$$

(as $h(.)$ is an increasing function)

$\Rightarrow Q_{n,k}(\delta_o) \leq Q'_{n,k}(\delta_o)$

Similarly $Q_{n,k}(\delta_o) \leq A \Rightarrow s_{2n} > s_{1n} \Rightarrow Q^{-1}_{n,k}(\delta_o) \leq Q'^{-1}_{n,k}(\delta_o)$.

This fact will be used to obtain a lower bound for N_2.

Now define

$$\left.\begin{array}{ll} N_2' = \inf\left\{ n \;\middle|\; \left|\ln Q'_{n,k}(\delta_o)\right| > \ln(B \wedge A^{-1}) \right\} & \\ = \infty & \text{otherwise} \end{array}\right\} \quad \ldots(2.6.6)$$

It is easy to see $N_2' \leq N_2$ $\quad \ldots(2.6.7)$

$$\text{Now } \ln Q'_{n,k} = \ln h(s_{1n}) - \ln h(s_{2n})$$

$$= (s_{1n} - s_{2n}) \frac{h'(s)}{h(s)} \text{ (by Mean Value Theorem)} \quad \ldots(2.6.8)$$

where $s \in (s_{1n} \wedge s_{2n}, s_{1n} \vee s_{2n})$

$$\text{Now } \frac{h'(s)}{h(s)} = \frac{\int_0^\infty e^{ts - t^2/2}\, t^{n^*-1}\, dt}{\int_0^\infty e^{ts - t^2/2}\, t^{n^*-2}\, dt} = (n^*)^{1/2} \frac{J_{n^*,1}(s')}{J_{n^*,2}(s')}$$

where $J_{n,\lambda}(x) = \int_0^\infty e^{nxt - nt^2/2}\, t^{n-\lambda}\, dt$ for $n \geq \lambda$ and $s' = s/\sqrt{n^*}$.

As s' is bounded, by (3.3.14) of Wijsman (1979) (page 256)

$$e^{-2c} < \frac{J_{n^*,1}(s')}{J_{n^*,2}(s')} < e^{2c} \quad \text{for } n^* \geq 2 \text{ and } \forall\, s' \text{ (bounded)} \ldots(2.6.9)$$

where c is a positive constant.

From (2.6.8) and (2.6.9)

$$| \ln Q'_{n,k}(\delta_o) | < (n^*)^{1/2} \, | s_{1n} - s_{2n} | \, e^{2c} \qquad \ldots(2$$

$$\left.\begin{aligned} \text{Let } N_2'' &= \inf \left\{ n : (n^*)^{1/2} \, |s_{1n} - s_{2n}| > c' \right\} \\ &= \infty \qquad\qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(2$$

for $c' = e^{-2c} \ln(\theta \wedge \bar{A}^{-1})$.

Then it follows from (2.6.6), (2.6.7), (2.6.10) and (2.6.11)

that $\quad N_2'' \leq N_2 \qquad \ldots(2.6$

$$\begin{aligned} \text{Now } (n^*)^{1/2} |s_{1n} - s_{2n}| &= (2n+k)^{1/2} \, t_n^{-1} \, \delta_o \left| \frac{kn}{2n+k} t_{1n} \right| \wedge | n \, t_{2n} | \\ &\leq \frac{\delta_o kn}{(2n+k)^{1/2}} \, |t_{1n}| t_n^{-1} \\ &\leq \frac{\delta_o kn}{(2n+k)^{1/2}} \, |t_{1n}| \left(\frac{n}{2} t_{2n}^2 \right)^{-1/2} \\ &= \delta_o k \left(\frac{2n}{2n+k} \right)^{1/2} \frac{|t_{1n}|}{|t_{2n}|} \\ &\leq \delta_o k \, \frac{|2\bar{X}_k - \bar{Y}_n - \bar{Z}_n|}{|\bar{Y}_n - \bar{Z}_n|} \qquad \ldots(2.6.13 \end{aligned}$$

$$\left.\begin{aligned} \text{Let } N_2^* &= \inf \left\{ n : \delta_o k \, \frac{|2\bar{X}_k - \bar{Y}_n - \bar{Z}_n|}{|\bar{Y}_n - \bar{Z}_n|} > c' \right\} \\ &= \infty \qquad\qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(2.6.14$$

Now (2.6.11), (2.6.12), (2.6.13) and (2.6.14) imply $N_2^* \leq N_2$ ...(2.6.15)

By Theorem 2 of GM (1980), for any positive number a_o,

$$P_\theta \left\{ \delta_o k \bar{\sigma}^1 |2\bar{X}_k - \bar{Y}_n - \bar{Z}_n| < a_o \quad \forall\, n \geq 1 \right\} > 0 \qquad ..(2.6.16)$$

and for fixed $\varepsilon > 0$ (to be chosen suitably later)

$$P_\theta \left\{ \bar{\sigma}^1 |\bar{Y}_n - \bar{Z}_n - (\mu_1 - \mu_2)| < \varepsilon \quad \forall\, n \geq 1 \right\} > 0 \qquad ...(2.6.17)$$

By hypothesis $\theta_1 = \bar{\sigma}^1 |\mu_1 - \mu_2| > 0$.

Choose $\varepsilon < \theta_1$, then $\bar{\sigma}^1 |\bar{Y}_n - \bar{Z}_n - (\mu_1 - \mu_2)| < \varepsilon$

$$\Rightarrow \bar{\sigma}^1 |\bar{Y}_n - \bar{Z}_n| > \theta_1 - \varepsilon > 0 \qquad ...(2.6.18)$$

Thus (2.6.17) and (2.6.18) imply

$$P_\theta \left\{ \bar{\sigma}^1 |\bar{Y}_n - \bar{Z}_n| > \theta_1 - \varepsilon \quad \forall\, n \geq 1 \right\} > 0 \qquad ...(2.6.19)$$

From (2.6.16),

$$P_\theta \left\{ \delta_o k \bar{\sigma}^1 |2\bar{X}_k - \bar{Y}_n - \bar{Z}_n| < c' (\theta_1 - \varepsilon) \quad \forall\, n \geq 1 \right\} > 0 \qquad ...(2.6.20)$$

Independence of the events described in (2.6.19) and (2.6.20) together with (2.6.19) and (2.6.20) imply $P_\theta (N_2^* = \infty) > 0$ ..(2.6.21)

The theorem now follows from (2.6.15) and (2.6.21). □

Remark 2.2 : The proof of Theorem 2.4 uses the normality of $\bar{X}_k$, $\bar{Y}_n$ and $\bar{Z}_n$ only to establish the independence of $\bar{Y}_n + \bar{Z}_n$ and $\bar{Y}_n - \bar{Z}_n$. It is interesting to note that the theorem follows even without using the normality assumption, as given below. Let X_k', Y_n', Z_n', μ_1', μ_2' stand for $\sigma^{-1}\bar{X}_k$, $\sigma^{-1}\bar{Y}_n$, $\sigma^{-1}\bar{Z}_n$, $\sigma^{-1}\mu_1$ and $\sigma^{-1}\mu_2$ respectively.

Define

$$f(X_k', Y_n', Z_n') = \frac{\delta_0 K\,|2X_k' - Y_n' - Z_n'|}{|Y_n' - Z_n'|} \quad \text{for } Y_n' \neq Z_n'$$

Then f is a continuous function in each of its argument on the set $Y_n' \neq Z_n'$. Thus for given any ε $\exists\, \delta'$ s.t.

$$\left\{|Y_n' - \mu_1'| < \delta',\ |Z_n' - \mu_2| < \delta'\right\} \Rightarrow \left\{|f(X_k', Y_n', Z_n') - f(X_k', \mu_1', \mu_2')| < \varepsilon\right. \qquad \ldots (1^*)$$

(Here $|\mu_1' - \mu_2'| > 0$ which implies that one can choose δ' small enough to have Y_n' and Z_n' sufficiently apart so that $f(X_k', Y_n', Z_n')$ is defined.)

Call $\quad A_{\delta'}(\varepsilon) = \left\{|Y_n' - \mu_1'| < \delta',\ |Z_n' - \mu_2'| < \delta' \quad \forall\, n \geq 1\right\}$

Note $\quad P_{\theta}\,(A_{\delta'}(\varepsilon)) > 0 \quad$ (by Theorem 2 of GM (1980)) $\qquad \ldots (2^*)$

Now for any $\varepsilon < a_0$ $\qquad$ (a_0 as in (2.6.16))

$$P_{\theta}\left\{f(X_k', \mu_1', \mu_2') < a_0 - \varepsilon\right\} > 0 \qquad \ldots (3^*)$$

Also $\quad A_{\delta'}(\varepsilon) \cap \left\{f(X_k', \mu_1', \mu_2') < a_0 - \varepsilon\right\}$

$$\Rightarrow \left\{f(X_k', Y_n', Z_n') < a_0 \quad \forall n \geq 1\right\} \qquad \ldots (4^*)$$

(by using (1*)).

Independence of $A_{\delta'}(\varepsilon)$ and $\left\{f(X_k', \mu_1', \mu_2') < a_0 - \varepsilon\right\}$ together with (2*), (3*) and (4*) imply (2.6.21).

CHAPTER 3

SOME INVARIANT SEQUENTIAL AND NON-SEQUENTIAL RULES FOR IDENTIFYING A MULTIVARIATE NORMAL POPULATION

3.1 Introduction

This chapter deals with the multivariate version of the problem taken up in the previous chapter. As mentioned earlier the set up fits quite well in anthropological studies (vide GM (1980) and Schaafsma and Vanvark (1977, 1979)), a multivariate extension has a wider scope.

Here π_0, π_1 and π_2 denote p-variate normal populations with unknown means μ, μ_1 and μ_2 respectively and the common variance - covariance matrix Σ may be known or unknown. A sample of fixed size is given from π_0 which is to be identified with one of two other populations π_1 and π_2 from which sampling can be done sequentially or non-sequentially. The case where sequential sampling is permitted from all the three populations is also considered.

Let X, Y, Z with suffixes denote random variables associated with π_0, π_1 and π_2 respectively.

The problem can be formulated in the following way. Test

$$\left.\begin{array}{ll} H_1 & \mu = \mu_1 \quad \text{versus} \\ H_2 & \mu = \mu_2 \end{array}\right\} \qquad \ldots(3.1.1)$$

with the restriction

$$\left.\begin{array}{l} P_{H_1}(\text{Rejection of } H_1) = \alpha \\ P_{H_2}(\text{Rejection of } H_2) = \beta \end{array}\right\} \qquad \ldots(3.1.$$

A parameter Δ_0 is introduced (as in the univariate case) to specify the indifference zone and the following hypotheses are tested

$$\left.\begin{array}{l} H_1 : \mu = \mu_1,\ \mu \neq \mu_2, \quad ||\mu_1 - \mu_2||_{\Sigma} = \Delta_0 \\ H_2 : \mu \neq \mu_1,\ \mu = \mu_2, \quad ||\mu_1 - \mu_2||_{\Sigma} = \Delta_0 \end{array}\right\} \qquad \ldots(3.1.3$$

where $\quad ||\mu_1 - \mu_2||_{\Sigma} = ((\mu_1 - \mu_2)' \Sigma^{-1} (\mu_1 - \mu_2))^{1/2} \qquad \ldots(3.1.4$

It is natural to expect that any reasonable procedure for testing the hypotheses described in (3.1.3) will also work (in fact in a better way perhaps) when the true $||\mu_1 - \mu_2||_{\Sigma} > \Delta_0$.

The following three schemes are considered here :

(S1) Three fixed samples of size k (k predetermined, $k \geq k_0$), n_0 and n_0 are taken from π_0, π_1 and π_2 respectively. Here k_0 denotes the minimum sample size from π_0, needed for the identification problem subject to condition (3.1.2) (vide Section 2.1 of GM (1980)). Clearly k_0 depends on α, β and Δ_0

(S2) A sample of fixed size k ($k \geq k_0$ as given in S1) is taken from π_0 where π_1 and π_2 are sampled sequentially.

(S3) All the three populations are sampled sequentially.

Under sampling scheme S1, the best invariant fixed sample procedure is considered. This procedure has error probabilities monotonically decreasing as $||\mu_1 - \mu_2||_\Sigma$ increases when $\alpha = \beta$ (vide Dasgupta 1974).

Under sampling schemes S2 and S3, the invariant SPRTs based on the maximal invariant are considered once with $(\overline{X}_k, \overline{Y}_n, \overline{Z}_n)$ and once with $(\overline{X}_n, \overline{Y}_n, \overline{Z}_n)$ as sufficient statistic for (μ, μ_1, μ_2) for the known Σ case. These procedures are discussed in Section 3.2. Procedures for the case of unknown Σ are given in Section 3.3. The error probabilities of all these sequential procedures can be bounded as in the univariate case vide Theorem 2.2 of Chapter 2. The termination properties of all these sequential procedures are studied in Section 3.4.

This chapter is a revised version of a part of Ray Chaudhuri (1985).

3.2 Procedures For Known Σ Case

If Σ is known, it may be assumed to be I_p without loss of generality.

Now the hypotheses described in (3.1.3) can be restated as

$$\left.\begin{aligned} H_1 &: \underset{\sim}{\theta} = (\Delta_o, \Delta_o, 1) \\ H_2 &: \underset{\sim}{\theta} = (\Delta_o, \Delta_o, -1) \end{aligned}\right\} \qquad \ldots(3.2.1)$$

where $\underset{\sim}{\theta} = (||2\mu - \mu_1 - \mu_2||, \; ||\mu_1 - \mu_2||, \; \frac{(2\mu-\mu_1-\mu_2)'(\mu_1-\mu_2)}{||2\mu-\mu_1-\mu_2||\cdot||\mu_1-\mu_2||})$ ...(3.2

and $||X|| = ||X||_{I_p}$

Let $\bar{X}_k$, $\bar{Y}_n$, $\bar{Z}_n$ denote the usual sample mean vectors of π_0, π_1 and π_2 respectively.

Here $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n)$ is sufficient for (μ, μ_1, μ_2) with k and n defined as follows for different schemes. For scheme S1, $k \geq k_0$; $n = n_0$ where both k and n_0 are fixed, k is predetermined and n_0 is determined subject to (3.1.2). Here such a choice of n_0 is possible as $k \geq k_0$, k_0 as described in scheme S1. For scheme S2, k is predetermined, $k \geq k_0$ as in scheme S2 and $n = 1,2,3,\ldots$ and for scheme S3, $k = n = 1,2,\ldots$.

The group of transformation applied here is $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n) \longrightarrow (B\bar{X}_k + C, B\bar{Y}_n + C, B\bar{Z}_n + C)$ where B is $p \times p$ orthogonal matrix and C is a $p \times 1$ scalar vector. The maximal invariant under this transformation is $A_n = (||2\bar{X}_k - \bar{Y}_n - \bar{Z}_n||^2, \; ||\bar{Y}_n - \bar{Z}_n||^2, (2\bar{X}_k - \bar{Y}_n - \bar{Z}_n)'(\bar{Y}_n - \bar{Z}_n))$. By the basic theorem of Hall et al (1965) A_n is invariantly sufficient for $\underset{\sim}{\theta}$. Let Ψ_{pxp} be orthogonal such that $\Psi(\mu_1 - \mu_2) = (\Delta_0, 0, 0 \ldots 0)'$.

Then $S = \Psi(2\bar{X}_k - \bar{Y}_n - \bar{Z}_n)$ is normally distributed with mean $(\Delta_0, 0, 0, \ldots, 0)'$ under H_1 and $(-\Delta_0, 0, 0, \ldots, 0)'$ under H_2 and variance-covariance matrix $(4k^{-1} + 2n^{-1})I_p$. And $T = \Psi(\bar{Y}_n - \bar{Z}_n)$ is independent of S and is normally distributed with mean $(\Delta_0, 0, 0, \ldots, 0)$ and variance-covariance matrix $2n^{-1}I_p$.

The distribution of $A_n = (||S||^2, ||T||^2, S'T)$ is noncentral Wishart (vide Section 3 of Anderson and Girshick (1944)) with the density function

$$f_{H_m}(A_n) = \frac{e^{-\frac{1}{2}k_n^2 - \frac{1}{2}\sum_{i=1}^{2} b_{ii}^{(m)}}}{2^{p-\frac{1}{2}(p-2)}\,\pi^{1/2}\,\Gamma(\frac{p-1}{2})} \,|b_{ij}^{(m)}|^{\frac{p-3}{2}} \,(k_n^2 b_{11}^{(m)})^{-\frac{p-2}{4}}\, I_{\frac{1}{2}(p-2)}(k_n\sqrt{b_{11}^{(m)}})$$

for $m = 1,2,$ ...(3.2.3)

where $k_n^2 = \Delta_o^2\,(\sigma_S^{-1}+\sigma_T^{-1})$, $\sigma_S = 4k^{-1} + 2n^{-1}$, $\sigma_T = 2n^{-1}$ ...(3.2.4)

$$\left.\begin{aligned}
b_{11}^{(m)} &= \left(\frac{\sigma_T}{\sigma_S} S'S + \frac{\sigma_S}{\sigma_T} T'T + (-1)^{m+1}\, 2S'T\right)(\sigma_S + \sigma_T)^{-1} \\
b_{12}^{(m)} &= ((-1)^{m+2}\sigma_T\, S'S + (\sigma_T-\sigma_S)S'T + (-1)^{m+1}\sigma_S\, T'T)(\sigma_S+\sigma_T)^{-1}(\sigma_S\sigma_T)^{-1/2} \\
b_{22}^{(m)} &= (S'S + (-1)^{m+2}2S'T + T'T)\,(\sigma_S + \sigma_T)^{-1}
\end{aligned}\right\} \quad ...(3.2.5)$$

$$|b_{ij}^{(m)}| = b_{11}^{(m)}\, b_{22}^{(m)} - (b_{12}^{(m)})^2 \qquad \text{for } m = 1,2.$$

$I_{\frac{1}{2}(p-2)}(\,\cdot\,)$ is a Bessel function of imaginary argument. The trace $\sum_{i=1}^{2} b_{ii}^{(m)}$ and the determinant $|b_{ij}^{(m)}|$ both remain unchanged under the two hypotheses.

Define $Z_m = k_n(b_{11}^{(m)})^{1/2}$

$$= \Delta_o\left(\frac{S'S}{\sigma_S^2} + \frac{T'T}{\sigma_T^2} + (-1)^{m+1}\frac{2S'T}{\sigma_S\sigma_T}\right)^{1/2} \quad (3.2.6)$$

for $m = 1,2.$

Then the test statistic reduces to

$$W_{n,k}(\Delta_o) = \frac{f_{H_2}(A_n)}{f_{H_1}(A_n)} \quad ..(3.2.7)$$

$$= \frac{(Z_2)^{\frac{p-2}{2}} I_{\frac{1}{2}(p-2)}(Z_2)}{(Z_1)^{\frac{p-2}{2}} I_{\frac{1}{2}(p-2)}(Z_1)} \quad ..(3.2.8)$$

$$= \frac{w_p(Z_2)}{w_p(Z_1)} \quad ...(3.2.9)$$

where $w_p(x) = \int_0^1 \cosh\,(xt)(1-t^2)^{\frac{p-3}{2}}\,dt$ ..(3.2.10)

The equality of (3.2.8) and (3.2.9) is an easy consequence of the series representation of cosh(x) and $I_{\frac{1}{2}(p-2)}(x)$ (vide Whittaker and Watson (1958) page 373).

Remark 3.1 : One may obtain this form (as in 3.2.9) of density ratio of maximal invariant A_n by integrating over the group of transformation (vide Wijsman (1967, 1979)). One may avoid the complicated series expansion of density by this method.

For scheme S1 the procedure is as follows :

reject H_1 if $\ln W_{n,k}(\Delta_0) > c$ ...(3.2.11)

where c and n_0 are chosen to satisfy (3.1.2). If $\alpha = \beta$ then $c = 0$ and $\ln W_{n,k}(\Delta_0) > 0$

$$<=> \frac{S'T}{\sigma_S \sigma_T} < 0.$$

By Theorem 2.1 of Dasgupta (1974), both types of error probabilities are monotonically decreasing function of $||\mu_1 - \mu_2||$ when $\alpha = \beta$.

But the density of the maximal invariant in the multivariate case does not satisfy HPKE condition on the critical region for the $\alpha \neq \beta$ case. Thus the monotonicity of error probabilities for this case does not follow by reasoning as in the univariate case.

Now to implement the fixed sample rule of S1 (vide 3.2.11), one needs the value of k_0 or at least an upper bound of k_0. Derivation of exact value of k_0 involves tedious numerical calculation as the distribution of $W_{n,k}(\Delta_0)$ is extremely complicated, whereas an upper bound of k_0 can be obtained by a much simpler method as given below.

If $\alpha \neq \beta$, consider the harder problem with $\alpha' = \beta' = \alpha \wedge \beta$ (if $\alpha = \beta$, then $\alpha' = \beta' = \beta$). The probability of correct identification for this harder problem is

$$P_{H_1}\left(\frac{S'T}{\sigma_S\sigma_T} < 0\right) \geq \left(\Phi\left(\frac{\Delta_o}{p(4k^{-1}+2n^{-1})^{1/2}}\right)\right)^{2p}$$ (by using independence of S and T)

Now for having a solution in n for

$$\left(\Phi(\Delta_o p^{-1}(4k^{-1}+2n^{-1})^{-1/2})\right)^{2p} = 1-\alpha' \qquad ..(3.2.12)$$

one needs to have $k \geq \left\lceil 4\tau_{\alpha_p}^2 \, p \, \Delta_o^{-2} \right\rceil = k_1$ (say) ..(3.2.13)

where τ_{α_p} is s.t. $\Phi(\tau_{\alpha_p})^p = (1-\alpha')^{1/2p}$ and $\lceil x \rceil$ is the smallest integer $\geq x$. One may take $k \geq k_1$ to implement the fixed sample rule for scheme S1.

$$\text{Call } n_1 = \left\lceil \left(\frac{\Delta_o^2}{2\tau_{\alpha_p}^2 p} - 2k^{-1}\right)^{-1} \right\rceil \text{ for } k \geq k_1 \qquad ..(3.2.14)$$

Then n_1 is an upper bound of n_o

Both these bounds k_1 and n_1 are conservative.

For scheme S2, the truncated invariant SPRT with test statistic $W_{n,k}(\Delta_o)$ is considered with the usual boundaries $\frac{\beta}{1-\alpha}$ and $\frac{1-\beta}{\alpha}$ Here the untruncated SPRT does not terminate with probability one (by Theorem 3.1 in Section 3.4), which emphasises the need for a truncation point. One may choose the truncation point $m_o = 2n_1$, with n_1 as in (3.2.14).

For scheme S3, the invariant SPRT with usual boundaries is studied. The test statistic in this case is $W_{n,n}(\Delta_o)$. This SPRT terminates with probability one which is ensured by Theorem 3.2 in Section 3.4.

Both kinds of error probabilities of the invariant SPRTs for schemes S2 and S3, can be bounded as given in Theorem 2.2 of Chapter 2. For applying Theorem 2.2 the following lemma 3.1 is needed.

<u>Lemma 3.1</u>. For $A < 1$, $B > 1$ and $\Delta^* > \Delta_o > 0$,

(i) $W_{n,k}(\Delta_o) \leq A \Rightarrow W_{n,k}(\Delta^*) \leq A$ and

(ii) $W_{n,k}(\Delta_o) \geq B \Rightarrow W_{n,k}(\Delta^*) \geq B$.

The proof of Lemma 3.1 follows in exactly similar lines as the proof of Lemma 2.2 of Chapter 2. Thus Lemma 3.1 ensures the fulfilment of condition (2.2.17) of Theorem 2.2 of the previous chapter and the following bounds can be obtained.

For scheme S2, we have

$$\left.\begin{aligned} \alpha^* &\leq \frac{\alpha}{1-\beta}(1-\beta^*) - \frac{\alpha}{1-\beta} P_{H_2}^*(N_1 \geq m_o,\ W_{m_o,k} > 1) \\ &\qquad + P_{H_1}^*(N_1 \geq m_o,\ W_{m_o,k} > 1) \\ \beta^* &\leq \frac{\beta}{(1-\alpha)}(1-\alpha^*) - \frac{\beta}{1-\alpha} P_{H_1}^*(N_1 \geq m_o,\ W_{m_o,k} \leq 1) \\ &\qquad + P_{H_2}^*(N_1 \geq m_o,\ W_{m_o,k} \leq 1) \end{aligned}\right\} \quad \ldots(3.2.15)$$

where N_1 is the stopping time of the untruncated SPRT.

H_1^* : $\underset{\sim}{\theta} = (\Delta^*, \Delta, 1)$, H_2^* : $\underset{\sim}{\theta} = (\Delta^*, \Delta^*, -1)$ with $\Delta^* > \Delta_o$,

(with $\underset{\sim}{\theta}$ as in (3.2.2) and m_o the truncation p

and $\alpha^* = p_{H_1^*}$ (Rejection of H_1)

$\beta^* = p_{H_2^*}$ (Rejection of H_2)

For scheme S3, the bounds are much simpler (as in page 46 of Wald (1947))

For then

$$\left.\begin{array}{l} \alpha^* \leq \frac{\alpha}{1-\beta}(1-\beta^*),\ \beta^* \leq \frac{\beta}{1-\alpha}(1-\alpha^*) \text{ and thus} \\ \alpha^* + \beta^* \leq \alpha + \beta. \end{array}\right\} \quad \ldots(3.2.16$$

3.3 Procedures for Unknown Σ Case

Σ being not known, the situation here is more complicated. The hypotheses tested here are as follows :

$$\left.\begin{array}{l} H_1 : \underset{\sim}{\theta} = (\Delta_o, \Delta_o, 1) \\ H_2 : \underset{\sim}{\theta} = (\Delta_o, \Delta_o, -1) \end{array}\right\} \quad \ldots(3.3.1)$$

where $$\underset{\sim}{\theta} = \left(||2\mu-\mu_1-\mu_2||_\Sigma,\ ||\mu_1-\mu_2||_\Sigma,\ \frac{(2\mu-\mu_1-\mu_2)'\Sigma^{-1}(\mu_1-\mu_2)}{||2\mu-\mu_1-\mu_2||_\Sigma ||\mu_1-\mu_2||_\Sigma}\right), \ldots(3.3.2)$$

Here $(\bar{X}_k, \bar{Y}_n, \bar{Z}_n, S_n)$ is sufficient for $(\mu, \mu_1, \mu_2, \Sigma)$

where $$S_n = \sum_1^k (X_i-\bar{X}_k)(X_i-\bar{X}_k)' + \sum_1^n (Y_i-\bar{Y}_n)(Y_i-\bar{Y}_n)' + \sum_1^n (Z_i-\bar{Z}_n)(Z_i-\bar{Z}_n)'.$$

Here n and k for different schemes are as defined in Section 3.2.

The group of transformation considered is

$(\bar{X}_k, \bar{Y}_n, \bar{Z}_n, S_n) \rightarrow (B\bar{X}_k + C, B\bar{Y}_n + C, B\bar{Z}_n + C, BS_nB')$ where B is pxp nonsingular matrix and C is px1 scalar vector

Now $B_n = (Y_{1n}'S_n^{-1}Y_{1n}, Y_{2n}'S_n^{-1}Y_{2n}, Y_{1n}'S_n^{-1}Y_{2n})$ is maximal invariant and by the basic theorem of Hall et al (1965), B_n is invariantly sufficient for $\underset{\sim}{\theta}$ where

$$Y_{1n} = \frac{(2\bar{X}_k - \bar{Y}_n - \bar{Z}_n)}{\sqrt{4k^{-1}+2n^{-1}}}, \qquad Y_{2n} = \frac{\bar{Y}_n - \bar{Z}_n}{\sqrt{2n^{-1}}}$$

The density of B_n under both hypotheses are given as follows (vide Sitgreaves (1952))

$$f_{H_m}(B_n) = \frac{\Gamma(\frac{n^*+1}{2})\, e^{-\frac{1}{2}\Delta_o^2(k_1^2+k_2^2)}\, |B|^{\frac{p-3}{2}}}{\Gamma(\frac{n^*-p+2}{2})\,\Gamma(\frac{n^*-p+1}{2})\,\Gamma(\frac{p-1}{2})\,\Gamma(\frac{1}{2})\,|I+B|^{(\frac{n^*+2}{2})}}$$

$$\sum_{j=0}^{\infty} \frac{\Gamma(\frac{n^*+2}{2}+j)}{j!\,\Gamma(\frac{p}{2}+j)} (\tfrac{1}{2})^j (U_m)^{2j} \qquad \ldots(3.3.4)$$

$$U_m \quad \Delta_o(k_1^2 b_{11}^* + 2k_1k_2 b_{12}^*(-1)^{m+1} + k_2^2 b_{22}^*)^{1/2} \quad \text{for } m = 1,2. \qquad \ldots(3.3.5)$$

$$\left.\begin{aligned} b_{11}^* &= b^{-1}(b_{11}+b_{11}b_{22}-b_{12}^2) \\ b_{22}^* &= b^{-1}(b_{22}+b_{11}b_{22}-b_{12}^2) \\ b_{12}^* &= b^{-1}b_{12} \\ \text{where } b &= 1+b_{11}+b_{12}+b_{11}b_{12}-b_{12}^2 \end{aligned}\right\} \qquad \ldots(3.3.6)$$

$$\left.\begin{aligned} b_{11} &= Y_{1n}' S_n^{-1} Y_{1n},\ b_{22} = Y_{2n}' S_n^{-1} Y_{2n},\ b_{12} = Y_{1n}' S_n^{-1} Y_{2n}, \\ B &= \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix},\ n^* = 2n + k-3,\ \Delta_o = \|\mu_1 - \mu_2\|_\Sigma \\ k_1 &= (4k^{-1}+2n^{-1})^{-1/2}\ ;\quad k_2 = (2n^{-1})^{-1/2} \end{aligned}\right\} \quad ..(3.3.7)$$

Following Sitgreaves (1952) we have

$$B^* = \begin{pmatrix} b_{11}^* & b_{12}^* \\ b_{12}^* & b_{22}^* \end{pmatrix} = Y_n'(S_n + Y_n Y_n')^{-1} Y_n \quad \text{where} \quad Y_n = (Y_{1n}, Y_{2n}) \quad ...(3.3.8)$$

The test statistic reduces to,

$$V_{n,k}(\Delta_o) = \frac{f_{H_2}(B_n)}{f_{H_1}(B_n)} = \frac{\sum_{j=0}^{\infty} \Gamma(\frac{n^*+2}{2}+j)(j!\,\Gamma(\frac{p}{2}+j))^{-1}(\frac{1}{2})^j\, U_2^{2j}}{\sum_{j=0}^{\infty} \Gamma(\frac{n^*+2}{2}+j)(j!\,\Gamma(\frac{p}{2}+j))^{-1}(\frac{1}{2})^j\, U_1^{2j}} \quad ..(3.3.9)$$

$$= \frac{\int_0^\infty \int_0^1 \cosh(\sqrt{2}U_2 tv)(1-v^2)^{\frac{p-3}{2}} t^{n^*+1} e^{-t^2}\, dv\, dt}{\int_0^\infty \int_0^1 \cosh(\sqrt{2}U_1 tv)(1-v^2)^{\frac{p-3}{2}} t^{n^*+1} e^{-t^2}\, dv\, dt} \quad ..(3.3.10)$$

The equality of (3.3.9) and (3.3.10) is once again an easy consequence of the series representation of cosh(x) and the fact that $\frac{\Gamma(j+\frac{1}{2})2^{2j}}{(2j)!} = \frac{\Gamma(\frac{1}{2})}{j!\,2^j}$ for all nonnegative integer j.

Here also one may obtain the density ratio (of the form given in 3.3.10) by integrating over the group of transformation.

The procedures for schemes S1, S2 and S3 are similar to the procedures for the known Σ case with $V_{n,k}(\Delta_o)$ in place of $W_{n,k}(\Delta_o)$.

For scheme S1, $V_{n_o,k}(\Delta_o) > 1 \iff b_{12}^* < 0$

$$\iff Y_{1n}' S_n^{-1} Y_{2n} < 0.$$

Now $Y_{1n}' S_n^{-1} Y_{2n} = Y_{1n}' P'(P'^{-1} S_n^{-1} P^{-1}) P Y_{2n}$ where P is $p \times p$ nonsingular s.t. $P \Sigma P' = I_p$ and $P(\mu_1 - \mu_2) = (\Delta_o, 0, \ldots, 0)'$. Invoking part (ii) of Theorem 2.2 of Dasgupta (1974), the monotonicity of both types of error for $\alpha = \beta$ case, can be obtained.

For scheme S2, the usual truncated invariant SPRT with the test statistic $V_{n,k}(\Delta_o)$ can be used. The necessity of truncation is ensured by Theorem 3.3 of Section 3.4.

For scheme S3, the usual invariant SPRT with test statistic $V_{n,n}(\Delta_o)$ terminates with probability one. Theorem 3.4 of Section 3.4 ensures this.

Error probabilities of both kinds of the truncated SPRT (of scheme S2) as well as the untruncated SPRT (of scheme S3) can be bounded as in the known Σ case. For that the fulfilment of condition 2.2.17 of Theorem 2.2 of Chapter 2 is necessary, which is assured by the following Lemma 3.2.

Lemma 3.2 : For $A < 1$, $B > 1$ and $\Delta^* > \Delta_o > 0$,

(i) $V_{n,k}(\Delta_o) \leq A \Rightarrow V_{n,k}(\Delta^*) \leq A$

(ii) $V_{n,k}(\Delta_o) \geq B \Rightarrow V_{n,k}(\Delta^*) \geq B$.

Proof: The test statistic can be written as

$$V_{n,k}(\Delta_o) = \frac{\int_0^\infty \cosh(\sqrt{2}U_2 u) f(u) du}{\int_0^\infty \cosh(\sqrt{2}U_1 u) f(u) du}$$

where $f(u) \geq 0$ for $0 < u < \infty$. The proof now follows in the exactly similar lines as the proof of Lemma 2.2 of Chapter 2.

3.4 Termination Properties of the SPRTs for Various Schemes

This section supplies the proofs of four Theorems as mentioned in preceeding sections. Let us first prove Theorem 3.1.

Theorem 3.1 : Let $N_1 = \inf \{ n : W_{n,k}(\Delta_o) \geq B$ or $W_{n,k}(\Delta_o) \leq A \}$

$= \infty$ otherwise

Then $P_\theta(N_1 = \infty) > 0 \quad \forall \theta = (\mu, \mu_1, \mu_2)$ fixed.

Proof : Let $W_{n,k}(\Delta_o) < A$

$$\Rightarrow W^{-1}_{n,k}(\Delta_o) > A^{-1} > 1$$

$$\Rightarrow \frac{\int_0^1 \cosh(Z_1 t)(1-t^2)^{\frac{p-3}{2}} dt}{\int_0^1 \cosh(Z_2 t)(1-t^2)^{\frac{p-3}{2}} dt} > A^{-1}$$

$\Longrightarrow$ $Z_1 > Z_2$ where Z_1 and Z_2 are as given in (3.2.6).

Let $u_1 = \dfrac{S'S}{\sigma_S^2}$, $u_2 = \dfrac{T'T}{\sigma_T^2}$, $u = \dfrac{S'T}{\sqrt{S'S,T'T}}$ and in this case $0 \leq u \leq 1$.

Now $\displaystyle\int_0^1 \frac{\cosh(Z_1 t)}{\cosh(Z_2 t)} f(t)dt > A^{-1}$ where $f(t) = \dfrac{\cosh(Z_2 t)(1-t^2)^{\frac{p-3}{2}}}{\int_0^1 \cosh(Z_2 t)(1-t^2)^{\frac{p-3}{2}} dt}$

$\Rightarrow$ $\dfrac{\cosh Z_1}{\cosh Z_2} > A^{-1}$ as $\dfrac{\cosh Z_1 t}{\cosh Z_2 t}$ is an increasing function of t for $Z_1 > Z_2$.

$$\Rightarrow \frac{\cosh\left(\Delta_o(u_1 + u_2 + 2\sqrt{u_1 u_2}u)\right)^{1/2}}{\cosh\left(\Delta_o(u_1 + u_2 - 2\sqrt{u_1 u_2}u)\right)^{1/2}} > A^{-1}$$

$$\Rightarrow \frac{\cosh\left(\Delta_o(\sqrt{u_1} + \sqrt{u_2})\right)}{\cosh\left(\Delta_o(\sqrt{u_1} - \sqrt{u_2})\right)} > A^{-1}$$

Thus $N_1 = n \Rightarrow \dfrac{\cosh\left(\Delta_o(\sqrt{u_1} + \sqrt{u_2})\right)}{\cosh\left(\Delta_o(\sqrt{u_1} - \sqrt{u_2})\right)} > B \wedge A^{-1}$

$$\Rightarrow \quad 2\,\Delta_o \sqrt{u_1} > \log\left(B \wedge A^{-1}\right) \qquad \text{.. (3.4.1)}$$

(following similar lines as in proof of Theorem 2.3 of Chapter 2).

$$\left.\begin{aligned} \text{Let } M &= \inf \left\{ n : \frac{\Delta_o kn}{2n+k} \|S\| > \ln (B \wedge A^{-1}) \right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \qquad (3.4.2)$$

Noting that $\sqrt{u_1} = J_S^{-1} \|S\|$ where $J_S = 4k^{-1}+2n^{-1}$ and from (3.4.1) we have $M \leq N_*$. ..(3.4.3)

Now by Theorem 2 of GM (1980) we have for any positive number a,

$$P_\theta \left\{ \frac{kn}{2n+k} |S_j| < ap^{-1/2} \quad \forall\, n \geq 1 \right\} > 0 \qquad \forall_j = 1,2,\ldots p \,. \qquad ..(3.4.4)$$

where $S' = (S_1, S_2, \ldots S_p)$.

Now noting that the events $\left\{ \frac{kn}{2n+k} |S_j| < a\, p^{-1/2} \;\forall\, n \geq 1 \right\}$, $j = 1,2,\ldots p$ are independent (as Σ is I_p here) we have

$$P_\theta \left\{ \frac{kn}{2n+k} |S_j| < ap^{-1/2} \quad \forall\, j = 1,2,\ldots p \text{ and } \forall\, n \geq 1 \right\} > 0 \qquad (3.4.5)$$

$$\Rightarrow \quad P_\theta \left\{ \frac{kn}{2n+k} \|S\| < a \quad \forall\, n \geq 1 \right\} > 0 \qquad ..(3.4.6)$$

$$\Rightarrow \quad P_\theta \left\{ M = \infty \right\} > 0 \qquad ...(3.4.7)$$

The proof now follows from (3.4.3) and (3.4.7). $\square$

From now onwards we shall write $X_n \rightarrow$ C.D. as $n \rightarrow \infty$ to mean that X_n converges in distribution to a continuous random variable as $n \rightarrow \infty$

<u>Theorem 3.2</u> : Let $N_2 = \inf \left\{ n : W_{n,n}(\Delta_o) \geq B \text{ or } W_{n,n}(\Delta_o) \leq A \right\}$

$$= \infty \qquad \text{otherwise}$$

Then $P_\theta (N_2 < \infty) = 1 \;\forall\, \theta$ fixed, where $\theta = (\mu, \mu_1, \mu_2)$.

<u>Proof</u> : It is enough to show $P_\theta (A < W_{n,n}(\Delta_o) < B) \longrightarrow 0$, as $n \to \infty$

Theorem 3.7 of Ghosh (1970) says it is enough to have convergence of $n^{-1/2} \ln W_{n,n}(\Delta_o)$ to a continuous r.v. (in distribution) or to $+\infty$ or to $-\infty$ in probability. For then $n^{-1/2} \ln A$ and $n^{-1/2} \ln B$ both go to zero and the convergence of $P_\theta (A < W_{n,n}(\Delta_o) < B)$ to zero is immediate.

Now $W_{n,n}(\Delta_o) = \dfrac{w_p(nZ_{2n})}{w_p(nZ_{1}n)}$ where $w_p(\cdot)$ as in (3.2.10)

and $Z_{mn} = n^{-1} Z_m$ (with $k = n$ in Z_m given in (3.2.6))

$$= \Delta_o (6^{-2} S'S + 2^{-2} T'T + (-1)^{m+1} 6^{-1} S'T)^{1/2} \text{ for } m = 1,2. \quad (3.4.8)$$

$$\left.\begin{aligned} \text{with} \quad S &= (2\bar{X}_n - \bar{Y}_n - \bar{Z}_n) \sim N_p((2\mu - \mu_1 - \mu_2), 6n^{-1} I_p) \\ T &= (\bar{Y}_n - \bar{Z}_n) \sim N_p (\mu_1 - \mu_2), 2n^{-1} I_p) \end{aligned}\right\} \quad \ldots(3.4.9)$$

The approximation formula (3.3.4) of page 255 of Wijsman (1979) simplifies the situation as follows ,

$$\log A < \log W_{n,n}(\Delta_o) < \log B$$

$$\Rightarrow \log A - c < nZ_{2n} - \frac{1}{2}(p-1) \log (1+nZ_{2n})$$

$$- (nZ_{1n} - \frac{1}{2}(p-1) \log (1+nZ_{1n})) < \log B + c \quad \ldots(3.4.10)$$

with c a positive real number.

Let $Z_{mn} \longrightarrow a_m$ a.s. as $n \to \infty$ for $m = 1,2$. Then the possible cases are

(1) $a_1 \neq a_2$ (2) $a_1 = a_2$. Since $Z_{mn} \geq 0$ $\forall n$, $a_m \geq 0$ for $m = 1,2$.

If $a_m = 0$ then $n^{1/2}Z_{mn} \longrightarrow$ C.D as $n \to \infty$ and thus

$$n^{-1/2}\ln(1+nZ_{mn}) = n^{-1/2}\ln(n^{1/2}) + n^{-1/2}\ln(n^{-1/2} + n^{1/2}Z_{mn}) = o_p(1)$$

If $a_n > 0$ then $n^{-1/2}\ln(1+nZ_{mn}) = n^{-1/2}\ln(n) + n^{-1/2}\ln(n^{-1}+Z_{mn})$

$$\longrightarrow 0 \text{ a.s. as } n \to \infty.$$

Thus the large sample behaviour of $n^{1/2}(Z_{2n} - Z_{1n})$ is of main interest.

Let us now take up two different cases :

<u>Case 1</u> : $a_1 \neq a_2 \Rightarrow n^{1/2}|Z_{2n} - Z_{1n}| \longrightarrow \infty$ a.s. as $n \to \infty$ implying the required result.

<u>Case 2a</u>: $a_1 = a_2 = 0 \Rightarrow n^{1/2}(Z_{2n} - Z_{1n}) \longrightarrow$ C.D. as $n \to \infty$ as in th case the distribution of $n^{1/2}(Z_{2n} - Z_{1n})$ is free of n for each fixed

<u>Case 2b</u>: $a_1 = a_2 > 0$

$$a_1 = a_2 \Rightarrow (ES)'(ET) = 0$$

Now $Z_{2n} - Z_{1n} = (Z_{2n} + Z_{1n})^{-1} \Delta_o^2 \, 3^{-1}(-S'T)$

$$n^{1/2}S'T = n^{1/2}(S-ES)'(T-ET) + n^{1/2}S'ET + n^{1/2}T'ES - n^{1/2}(ES)'(ET)$$

$\longrightarrow$ C.D. as $n \to \infty$ and $Z_{2n}+Z_{1n} \to 2a_1$ a.s. as $n \to \infty$.

Thus $n^{1/2}(Z_{2n} - Z_{1n}) \longrightarrow$ C.D. (precisely a normal distribution) as $n \to \infty$, implying the required result. □

<u>Theorem 3.3</u> : Let

$$N_3 = \inf\left\{n : V_{n,k}(\Delta_o) \geq B \text{ or } V_{n,k}(\Delta_o) \leq A\right\}.$$
$$= \infty \qquad \text{otherwise}$$

Then $P_\theta(N_3 = \infty) > 0 \quad \forall\, \theta = (\mu, \mu_1, \mu_2, \Sigma)$ fixed and $\|\mu_1 - \mu_2\|_\Sigma > 0$.

<u>Proof of Theorem 3.3</u> : The proof is similar to that of Theorem 2.4. Firstly we bound N_3 (from below) by N_3' (arguing as in 2.6.7) where

$$\left.\begin{aligned} N_3' &= \inf\left\{n : |\ln V'_{n,k}(\Delta_o)| \geq \ln(B \wedge A^{-1})\right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(3.4.11)$$

when
$$V'_{n,k}(\Delta_o) = \frac{\int_0^\infty\int_0^1 \exp(\sqrt{2}\,U_2\, tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\, dv\,dt}{\int_0^\infty\int_0^1 \exp(\sqrt{2}\,U_1\, tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\, dv\,dt}$$

$$= \frac{\int_0^1\int_0^\infty \exp(\sqrt{2}\,U_2\, tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\, dt\,dv}{\int_0^1\int_0^\infty \exp(\sqrt{2}\,U_1\, tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\, dt\,dv} \qquad \ldots(3.4.12)$$

Let
$$h(U_m) = \ln \int_0^1\int_0^\infty \exp(\sqrt{2}\,U_m\, tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\, dt\,dv$$

for $m = 1,2$.

Then
$$\ln V'_{n,k}(\Delta_o) = h(U_2) - h(U_1)$$
$$= (U_2 - U_1)\, h'(U) \quad \text{for } U \in (U_1 \wedge U_2,\ U_1 \vee U_2) \qquad \ldots(3.4.13)$$

$$\text{Now} \quad h'(U) = \frac{\int_0^1 \int_0^\infty \exp(\sqrt{2}\,U\,tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\,(tv\sqrt{2})\,dtdv}{\int_0^1 \int_0^\infty \exp(\sqrt{2}\,U\,tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\,dtdv}$$

$$\leq \sqrt{2}\; \frac{\int_0^1 \int_0^\infty \exp(\sqrt{2}\,U\,tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+2}\, e^{-t^2}\,dtdv}{\int_0^1 \int_0^\infty \exp(\sqrt{2}\,U\,tv)(1-v^2)^{\frac{p-3}{2}}\, t^{n^*+1}\, e^{-t^2}\,dtdv}$$

$$= \sqrt{2n_1^*}\; \frac{\int_0^1 \int_0^\infty \exp(\sqrt{2}n_1^* U'\,sv)(1-v^2)^{\frac{p-3}{2}}\, s^{n_1^*-1}\, e^{-n_1^* s^2}\,dvds}{\int_0^1 \int_0^\infty \exp(\sqrt{2}n_1^* U'\,sv)(1-v^2)^{\frac{p-3}{2}}\, s^{n_1^*-2}\, e^{-n_1^* s^2}\,dvds}$$

$$\left\{ \begin{array}{ll} \text{by substituting} & s = t/\sqrt{n_1^*} \\ & U' = U/\sqrt{n_1^*} \\ \text{where} & n_1^* = n^*+3 = 2n + k \,. \end{array} \right.$$

$$= (2n+k)^{1/2}\; \frac{\int_0^1 J_{n_1^*,1}\,(U'v)\,(1-v^2)^{\frac{p-3}{2}}\,dv}{\int_0^1 J_{n_1^*,2}\,(U'v)\,(1-v^2)^{\frac{p-3}{2}}\,dv} \qquad \ldots(3.4.14$$

$$\text{where} \quad J_{n,\lambda}(x) = \int_0^\infty e^{nxt - 1/2\,nt^2}\, t^{n-\lambda}\,dt \qquad \ldots(3.4.15$$

for $n \geq \lambda$

as in (3.3.5) of Wijsman (1979)

Now note that U' is bounded and thus by (3.3.14) of Wijsman (1979) and arguing as in (2.6.9) (of Chapter 2) we have

$$e^{-2c} \leq \frac{J_{n^*_{1,1}}(U'v)}{J_{n^*_{1,2}}(U'v)} \leq e^{2c} \quad \forall v \in (0,1) \text{ and } \forall n^*_1 \geq 2. \qquad \ldots(3.4.16)$$

(here we have $n^*_1 \geq 3$)

where c is a positive constant.

(3.4.13), (3.4.14) and (3.4.16) together imply

$$|\ln V'_{n,k}(\Delta_o)| \leq (2m+k)^{1/2} \; |U_2-U_1| \; e^{2c} \qquad \ldots(3.4.17)$$

Now $|U_2-U_1| \leq 2\,\Delta_o \; (k_1\sqrt{b^*_{11}}) \wedge (k_2\sqrt{b^*_{22}})$

$$\leq 2\,\Delta_o\,k_1\sqrt{b^*_{11}}$$

$$= 2\,\Delta_o k_1^2 \; \|2\bar{X}_k - \bar{Y}_n - \bar{Z}_n\|_{(S_n+Y_nY_n')} \quad \text{(Vide (3.3.8)}$$

$$= 2\Delta_o k_1^2 \left((2\bar{X}_k-\bar{Y}_n-\bar{Z}_n)'B'(BS_nB'+BY_nY_n'B')^{-1}B(2\bar{X}_k-\bar{Y}_n-\bar{Z}_n)\right)^{1/2}$$

(where B is $p \times p$ nonsingular matrix such that

$B\,\Sigma\,B' = I_p$ and $B(\mu_1-\mu_2) = p^{-1/2}(\Delta, \Delta, \ldots \Delta)$ ...(3.4.18)

with $\Delta = \|\mu_1-\mu_2\|_\Sigma > 0$ (by hypothesis))

$$\leq 2\,\Delta_o k_1^2 \;\; \|B\,(2\bar{X}_k-\bar{Y}_n-\bar{Z}_n)\| \;.\left(\text{largest eigen value of } (BS_nB' + BY_nY_n'B')^{-1}\right)^{1/2} \qquad \ldots(3.4.19)$$

Now the largest eigen value of $(BS_nB' + BY_nY_n'B')^{-1}$

$$\leq \text{trace } (BS_nB' + BY_nY_n'B')^{-1}$$

$$= \sum_{j=1}^{p} A_{jj} \Big/ \sum_{j=1}^{p} a_{jj}A_{jj} \quad (\text{where } A_{jj} = (j,j)^{th} \text{ cofactor of } B(S_n+Y_nY_n')B' \text{ and}$$

$$a_{jj} = (jj)^{th} \text{ element of } B(S_n+Y_nY_n')B').$$

$$\leq \left(\bigwedge_{j=1}^{p} a_{jj}\right)^{-1}$$

$$\leq \left(\bigwedge_{j=1}^{p} k_2^2(B_j'(\bar{Y}_n - \bar{Z}_n))^2\right)^{-1} \text{ where } B_j' \text{ is the } j^{th} \text{ row of } B.$$

...(3.4.

Thus $$|\ln V'_{n,k}(\Delta_o)| \leq (2m+k)^{1/2} 2\Delta_o k_1^2 k_2^{-1} e^{2c} \frac{||B(2\bar{X}_k-\bar{Y}_n-\bar{Y}_n)||}{\left(\bigwedge_{j=1}^{p}(B_j'(\bar{Y}_n-\bar{Z}_n))^2\right)^{1/2}}$$

$$\leq \Delta_o' \, k \, e^{2c} \frac{||B(2\bar{X}_k-\bar{Y}_n-\bar{Z}_n)||}{\bigwedge_{j=1}^{p}|B_j'(\bar{Y}_n-\bar{Z}_n)|}$$ ...(3.4.

using (3.3.7)

Let $$N_3'' = \inf\left\{n \cdot \quad \Delta_o' k \frac{||B(2\bar{X}_k-\bar{Y}_n-\bar{Z}_n)||}{\bigwedge_{j=1}^{p}|B_j'(\bar{Y}_n-\bar{Z}_n)|} \geq \ln(B\Lambda \bar{A}^{-1})e^{-2c}\right\}$$

$= \infty$ otherwise

...(3.4.22

Clearly $N_3 \leq N_3 \leq N_3$ ...(3.4.23)

Now from (3.4.6) for any positive number a_o

$$P_\theta \left\{ \Delta_o k \quad \|2\bar{X}_k - \bar{Y}_n - \bar{Z}_n\|_\Sigma < a_o \quad \forall\, n \geq 1 \right\} > 0 \qquad \text{...(3.4.24)}$$

Proceeding as in the proof of Theorem 2 of GM (1980) we have for fixed positive ε (to be chosen suitably later) and $\forall\, j = 1,\ldots p$,

$$P_\theta \left\{ |B_j'(\bar{Y}_n - \bar{Z}_n) - \Delta p^{-1/2}| < \varepsilon \quad \forall\, n \geq 1 \right\} > 0 \qquad \text{...(3.4.25)}$$

Now by hypotheses $\Delta > 0$, and choose $\varepsilon < \Delta p^{-1/2}$, then

$$|B_j'(\bar{Y}_n - \bar{Z}_n) - \Delta p^{-1/2}| < \varepsilon \Rightarrow |B_j'(\bar{Y}_n - \bar{Z}_n)| > \Delta p^{-1/2} - \varepsilon > 0 \qquad \text{...(3.4.26)}$$

$$\forall\, j = 1,2,\ldots p$$

Thus (3.4.25) and (3.4.26) together imply $\forall\, j = 1,2,\ldots p$,

$$P_\theta \left\{ |B_j'(\bar{Y}_n - \bar{Z}_n)| > \Delta p^{-1/2} - \varepsilon \quad \forall\, n \geq 1 \right\} > 0 \qquad \text{...(3.4.27)}$$

Note that the events $\left\{ |B_j'(\bar{Y}_n - \bar{Z}_n)| > \Delta p^{-1/2} - \varepsilon \ \forall\, n \geq 1 \right\}$ $j = 1,2,\ldots p$ are independent, which implies

$$P_\theta \left\{ \bigwedge_{j=1}^{p} |B_j'(\bar{Y}_n - \bar{Z}_n)| > \Delta p^{-1/2} - \varepsilon \quad \forall\, n \geq 1 \right\} > 0 \qquad \text{...(3.4.28)}$$

Now the independence of the two events described in (3.4.24) and (3.4.28) and the fact $\|B(2\bar{X}_k - \bar{Y}_n - \bar{Z}_n)\| = \|2\bar{X}_k - \bar{Y}_n - \bar{Z}_n\|_\Sigma$ imply

$$P_\theta (N_3'' = \infty) > 0 \qquad \text{...(3.4.29)}$$

The proof now follows from (3.4.23) and (3.4.29). □

Theorem 3.4 : Let $N_4 = \inf\{n : V_{n,n}(\Delta_0) \geq B \text{ or } V_{n,n}(\Delta_0) \leq A\}$

$= \infty$ otherwise

then $P_\theta(N_3 < \infty) = 1$ for all fixed $\theta = (\mu, \mu_1, \mu_2, \Sigma)$.

For proving Theorem 3.4, we need the following two lemmas.

Lemma 3.3 : If x lies in a bounded subset of $\mathbb{R}$ then

$$L_1(x) \leq \ln \int_0^\infty \exp(-\tfrac{1}{2} nt^2) w_p(nxt)\, t^{3n-3} dt \leq L_2(x) \quad \text{where}$$

$$L_1(x) = \frac{3n}{2}\ln 3 - \frac{3n}{2} - \frac{1}{2}\ln(n) + \ln w_p(3^{1/2}\, nx) + \text{constant}$$

$$L_2(x) = \frac{3n}{2}\ln 3 - \frac{3n}{2} - \frac{1}{2}\ln(n) + \frac{nx^2}{4}\left(1+\frac{|x|}{(x^2+12)^{1/2}}\right)+\ln w_p(3^{1/2} nx)+\text{co}$$

where $w_p(x)$ is given as in (3.2.10).

Remark 3.1 : The purpose of this lemma is to provide bounds for $\ln \int_0^\infty \exp(-\frac{1}{2} nt^2) w_p(nxt)\, t^{3n-3} dt$ which are easier to tackle especially for the case when the bounded subset of x is not away from zero. For the case when x is known to be bounded and away from zero, one may look into (3.3.18) of Wijsman (1979).

Lemma 3.4 : If $X_{mn} = \left(\frac{1}{6}b_{11}^* + \frac{1}{2}b_{22}^* + (-1)^m \frac{2b_{12}^*}{\sqrt{2}\cdot 6}\right)^{1/2} \rightarrow 0$ a.s. as $n \rightarrow \infty$ for $m = 1,2$ then $n^{1/2} X_{mr} \rightarrow$ C.D. as $n \rightarrow \infty$ for $m = 1,2$. Here b_{ij}^* is as in (3.3.6 or 3.3.8) with $k = n$.

Proof of Lemma 3.3 :

Consider $\int_0^\infty \exp(-\frac{1}{2} nt^2)\, w_p\,(nxt)t^{3n-3}dt$

$$=\frac{1}{2}\int_0^1 (J_{n,3}(|xv|) + J_{n,3}(-|xv|))\,(1-v^2)^{\frac{p-3}{2}}\,dv$$

(by changing the order of integration)

where $J_{n,3}(y) = \int_0^\infty \exp(-\frac{1}{2} nt^2 + nyt)t^{3n-3}\,dt$ for $n \geq 1$ ...(3.4.30)

By the approximation formula (3.3.13) of page 256 of Wijsman (1979) one gets

$$|\ln J_{n,3}(y) - n(\beta(y) - \frac{3}{2}) + \frac{1}{2}\ln(n)| < c \qquad ...(3.4.31)$$

for y belonging to a bounded subset of IR, $n \geq 1$ and c is a real constant.

Here $\beta(y) = \frac{1}{2} y\alpha(y) + 3\ln\alpha(y)$ ...(3.4.32)

$\alpha(y) = \frac{1}{2}(y + (y^2 + 12)^{1/2})$ ...(3.4.33)

Thus $\frac{c_1}{2}\left\{\exp(n\beta(|xv|) - \frac{3n}{2} - \frac{1}{2}\ln(n)) + \exp(n\beta(-|xv|) - \frac{3n}{2} - \frac{1}{2}\ln(n))\right\}$ ...(3.4.34)

$\leq \frac{1}{2}\left\{J_{n,3}(|xv|) + J_{n,3}(-|xv|)\right\}$ ...(3.4.35)

$\leq \frac{c_2}{2}\left\{\exp(n\beta(|xv|) - \frac{3n}{2} - \frac{1}{2}\ln(n)) + \exp(n\beta(-|xv|) - \frac{3n}{2} - \frac{1}{2}\ln(n))\right\}$

(with c_1, c_2 both constant) ...(3.4.36)

Using Taylor's expansion of $\beta(|xv|)$ and $\beta(-|xv|)$ around the point zero upto the second order term, one gets the following lower bound of (3.4.34) (and hence a lower bound of (3.4.35)) as

$$\frac{c_1}{2}\exp\left\{-\frac{3n}{2}-\frac{1}{2}\ln(n)+n\beta(0)\right\}\left\{\exp\left(n\left(|xv|\beta'(0)+\frac{|xv|^2}{2!}\beta''(\theta_1|xv|)\right)\right)\right.$$

$$\left.+\exp\left(n\left(-|xv|\beta'(0)+\frac{|xv|^2}{2!}\beta''(-\theta_2|xv|)\right)\right)\right\} \quad \ldots(3.4.37)$$

where $\beta(0)=\frac{3\ln 3}{2}$, $\beta'(0)=3^{1/2}$, $\beta''(x)=\frac{\alpha(x)}{(x^2+12)^{1/2}}=\frac{1}{2}\left(1+\frac{x}{(x^2+12)^{1/2}}\right)$

θ_1 and θ_2 both lie between 0 and 1. ...(3.4.38)

Thus a lower bound of (3.4.37) and hence a further lower bound of (3.4.35) can be obtained as

$$c_1\exp\left(n\left(\beta(0)-\frac{3}{2}\right)-\frac{1}{2}\ln(n)+n\frac{|xv|^2}{2!}\beta''(-\theta_2|xv|)\right)\cosh(\sqrt{3}nxv)$$

$$\geq c_1\exp\left(n\frac{3\ln 3}{2}-\frac{3n}{2}-\frac{\ln(n)}{2}\right)\cosh(\sqrt{3}nxv) \quad \ldots(3.4.39)$$

Similarly an upper bound of (3.4.36) (and hence an upper bound of (3.4.35)

$$c_2\exp\left(n\frac{3\ln 3}{2}-\frac{3n}{2}-\frac{\ln(n)}{2}+\frac{nx^2}{2}\cdot\frac{\alpha(|x|)}{(x^2+12)^{1/2}}\right)\cosh(\sqrt{3}nxv) \quad \ldots(3.4.40)$$

where $\frac{\alpha(|x|)}{(x^2+12)^{1/2}}=\frac{1}{2}\left(1+\frac{|x|}{(x^2+12)^{1/2}}\right)$.

Multiplying (3.4.39) and (3.4.40) by $(1-v^2)^{\frac{p-3}{2}}$ and integrating out v over $(0,1)$, one gets the required result.

Proof of Lemma 3.4 :

As noted in (3.3.8), $B^* = \begin{pmatrix} b_{11}^* & b_{12}^* \\ b_{12}^* & b_{22}^* \end{pmatrix}$

$$= Y_n' \bar{P}_n^{-1} Y_n \quad \text{with} \quad k = n. \qquad ..(3.4.41)$$

where,

$$Y_n' = \begin{pmatrix} Y_{1n}' \\ , \\ Y_{2n} \end{pmatrix} = n^{1/2} \begin{pmatrix} 6^{-1/2}(2\bar{X}_n - \bar{Y}_n - \bar{Z}_n)' \\ 2^{-1/2}(\bar{Y}_n - \bar{Z}_n)' \end{pmatrix} \qquad ..(3.4.42)$$

$$P_n = S_n + Y_n Y_n' = \sum_{i=1}^{n}((X_i - \bar{X}_n)(X_i - \bar{X}_n)' + (Y_i - \bar{Y}_n)(Y_i - \bar{Y}_n)' + (Z_i - \bar{Z}_n)(Z_i - \bar{Z}_n)')$$
$$+ Y_n Y_n' \qquad ...(3.4.43)$$

Thus $n X_{mn}^2 = \left(\frac{Y_{1n}}{\sqrt{6}} + (-1)^m \frac{Y_{2n}}{\sqrt{2}}\right)' (n^{-1}P_n)^{-1} \left(\frac{Y_{1n}}{\sqrt{6}} + (-1)^m \frac{Y_{2n}}{\sqrt{2}}\right)$ and the hypotheses of Lemma 3.4 says $n^{-\frac{1}{2}} E\left(\sqrt{6}^{-1} Y_{1n} + (-1)^m \sqrt{2}^{-1} Y_{2n}\right) = 0$.
Thus it can be shown that $nX_{mn}^2 \to$ C.D. as $n \to \infty$ by elementary argument and hence Lemma 3.4 follows.

Proof of Theorem 3.4 :

As mentioned in the proof of Theorem 3.2 it is enough to show that $n^{-1/2}\ln(V_{n,n}(\Delta_o)) \to$ C.D. or $|n^{-1/2}\ln(V_{n,n}(\Delta_o))| \to \infty$ in probability as $n \to \infty$.

Now $$V_{n,n}(\Delta_o) = \frac{\int_0^\infty \exp(-\frac{1}{2}nt^2)\, w_p(nU_{2n}t)\, t^{3n-3}\, dt}{\int_0^\infty \exp(-\frac{1}{2}nt^2)\, w_p(nU_{1n}t)\, t^{3n-3}\, dt}$$

where $U_{mn} = n^{-1/2} U_m$ (with U_m as in (3.3.5) with $k = n$)

$$= \Delta_o \left(\frac{b_{11}^*}{6} + \frac{b_{22}^*}{2} + (-1)^m \frac{2b_{12}^*}{\sqrt{2.6}}\right)^{1/2}$$

$$= \Delta_o \left\|\frac{Y_{1n}}{\sqrt{6}} + (-1)^m \frac{Y_{2n}}{\sqrt{2}}\right\|_{P_n} \quad \text{for } m = 1,2 \qquad \ldots(3.4.$$

with Y_{1n}, Y_{2n}, P_n as in (3.4.42) and (3.4.43) respectively and $w_p(x)$ is as in (3.2.10).

Let $U_{mn} \to b_m$ a.s. as $n \to \infty$ for $m = 1,2$. Then there are two cases namely,

<u>Case 1</u> : $b_1 \neq b_2$

<u>Case 2</u> : $b_1 = b_2$.

<u>Subcase 1a</u> : $b_m > 0$ for $m = 1,2$ and $b_1 \neq b_2$.

By formula (3.3.18) of Wijsman (1979) of page 257,

$$n^{1/2}(\beta(U_{2n}) - \beta(U_{1n})) - cn^{-1/2} \leq n^{-1/2}\ln V_{n,n}(\Delta_o) \leq n^{1/2}(\beta(U_{2n}) - \beta(U_{1n}))+$$

where U_{mn} belongs to a bounded subset of IR and $\beta(\cdot)$ is as given in (3.4.32)

Now $\beta(U_{mn}) \to \beta(b_m)$ a.s. as $n \to \infty$ $\forall\, m = 1,2$ and $b_1 \neq b_2 \Rightarrow$ $n^{1/2}|\beta(U_{2n}) - \beta(U_{1n})| \to \infty$ a.s. as $n \to \infty$ (as the $\beta(\cdot)$ function is continuous and strictly increasing).

Thus $|n^{-1/2} \ln V_{nn}(\Delta_o)| \to \infty$ a.s. as $n \to \infty$.

<u>Subcase 1b</u> : $b_1 = 0, \quad b_2 > 0$

From Lemma 3.3 and formula (3.3.18) of Wijsman (1979) one gets,

$$n^{-1/2} \ln V_{n,n}(\Delta_o) \geq n^{-1/2}(-L_2(U_{1n}) + (n(\beta(U_{2n}) - \tfrac{3}{2}) - \tfrac{p}{2}\ln(n)) - c)$$

$$= n^{-1/2}\left[\left\{-\frac{3n\ln 3}{2} + \frac{3n}{2} + \frac{1}{2}\ln(n) - \frac{nU_{1n}^2}{4}\left(1+\frac{U_{1n}}{(U_{1n}^2+12)^{1/2}}\right) - \ln w_p(\sqrt{3}nU_{1n})\right\}\right.$$

$$\left. + \; n\,\beta(U_{2n}) - \frac{3n}{2} - \frac{p}{2}\ln(n)\right] + o_p(1)$$

$$= n^{-1/2}\left\{\frac{1-p}{2}\ln(n) - \frac{nU_{1n}^2}{4}\left(1+\frac{U_{1n}}{(U_{1n}^2+12)^{1/2}}\right)\right\} - n^{-1/2}\ln w_p(\sqrt{3}nU_{1n})$$

$$+ n^{1/2}\left\{\beta(U_{2n}) - \frac{3}{2}\ln 3\right\} + o_p(1).$$

By Lemma 3.4, $n^{1/2}U_{1n} \to$ C.D. as $n \to \infty$.

Thus the first term goes to zero in probability, the second term converges in distribution to a continuous random variable (vide formula 3.3.4 of Wijsman (1979)) and the third term goes to ∞ a.s. as

$$\beta(U_{2n}) \to \beta(b_2) > \beta(0) = \frac{3}{2}\ln 3.$$

Thus $n^{-1/2}\ln V_{n,n}(\Delta_o) \to \infty$ in P as $n \to \infty$.

<u>Subcase 1c</u> : $b_2 = 0, \quad b_1 > 0.$

By Lemma 3.3 and formula (3.3.18) of Wijsman (1979),

$$n^{-1/2}\ln V_{n,n}(\Delta_o) \leq n^{-1/2}\left\{L_2(U_{2n}) - (n(\beta(U_{1n}) - \tfrac{3}{2}) - \tfrac{p}{2}\ln(n)) + c\right\}$$

Now reasoning as in Subcase 1b, it follows that

$$n^{-1/2}\ln V_{n,n}(\Delta_o) \to -\infty \text{ in P as } n \to \infty.$$

<u>Subcase 2a</u> : $b_1 = b_2 = 0$

By Lemma 3.3.

$$L_1(U_{2n}) - L_2(U_{1n}) \leq \ln V_{n,n}(\Delta_0) \leq L_2(U_{2n}) - L_1(U_{1n})$$

Now $n^{-1/2}(L_1(U_{2n}) - L_2(U_{1n})) = n^{-1/2}(\ln w_p(\sqrt{3}\, nU_{2n}) - \ln w_p(\sqrt{3}\, nU_{1n}) + o_p($

By Lemma 3.4 and formula (3.3.14) of Wijsman (1979), one can ensure $n^{-1/2}(L_1(U_{2n}) - L_2(U_{1n})) \to$ C.D. as $n \to \infty$ and $n^{-1/2}(L_2(U_{2n} - L_1(U_{1n}$ converges to the same C.D. as $n \to \infty$, which implies the required result.

<u>Subcase 2b</u> : $b_1 = b_2 > 0$.

$$n^{-1/2}\ln V_{n,n}(\Delta_0) = n^{1/2}(\beta(U_{2n}) - \beta(U_{1n})) + o_p(1)$$

(by (3.3.18) of page 257 of Wijsman (1979)).

$$= n^{1/2}(U_{2n} - U_{1n})\, \beta'(U_n) + o_p(1)$$

where $U_n \in (U_{1n} \wedge U_{2n}, U_{1n} \vee U_{2n})$

Now $U_n \to b_1$ a.s. (as U_{1n} and U_{2n} both converge to b_1 a.s.)

$\Rightarrow \beta'(U_n) \to \beta'(b_1)$ a.s. (as $\beta'(.) = \alpha(.)$ is a continuous function)

and $\beta'(b_1) = \alpha(b_1) > 0$.

$$n^{1/2}(U_{2n} - U_{1n}) = \frac{-4\Delta_0^2(12)^{-1/2}\, n^{1/2}\, b_{12}^*}{(U_{2n} + U_{1n})}$$

$$= -\frac{4\Delta_0^2}{(12)^{1/2}} \frac{n^{1/2}\, b_{12}}{(U_{1n}+U_{2n})(1+b_{11}+b_{22}+b_{11}b_{22}-b_{12}^2)}$$

As the denominator converges to a positive constant a.s. as $n \to \infty$ and $n^{1/2}\, b_{12} \to$ C.D. as $n \to \infty$ (by standard argument) we have $n^{-1/2}\ln V_{n,n}(\Delta_0) \to$ C.D. as $n \to \infty$.

Thus the proof of Theorem 3.4 follows. □

Remark 3.3 : One can make a similar comment on the proof of Theorem 3.3 as in Remark 2.2 of Chapter 2 with

$$f(B\overline{X}_k,\ B\overline{Y}_n,\ B\overline{Z}_n)$$

$$= \ .\ \Delta_o\ \ k \cdot \frac{||B(2\overline{X}_k - \overline{Y}_n - \overline{Z}_n)||}{\bigwedge_{j=1}^{p} |B_j'(\overline{Y}_n - \overline{Z}_n)|} \quad \text{for} \quad B_j'\,\overline{Y}_n \neq B_j'\,\overline{Z}_n \quad \forall\ j = 1,2,\ldots,p$$

(where B, B_j' are as in (3.4.18) and (3.4.20) respectively.)

in place of $f(X_k', Y_n', Z_n')$.

CHAPTER 4

ASYMPTOTIC DISTRIBUTIONS OF STOPPING TIMES

4.1 Introduction

In Sequential Analysis obtaining the exact distribution of a stopping time is in general a tedious task. Especially in case of an SPRT it is practically impossible to obtain the distribution of a stopping time analytically (except in a few cases like SPRT with Bernoulli r.v.). Thus it is natural to turn to asymptotic study or to the Monte Carlo study of stopping times.

Asymptotic distributions of stopping times arising in the area of Sequential Analysis, have been obtained by Siegmund (1968), Bhattachar and Mallik (1973) (henceforth will be referred as BM) and Ghosh and Mukhopadhyay (1975). Siegmund (1968) extends some results of Heyde (1967 1967b) on limit theorems of random walk. The result in BM is based on the asymptotic normality of sample sum with random index. They also give an alternate proof of Siegmund's (1968) result. The idea of Ghosh and Mukhopadhyay (1975) is similar to that of BM but the stopping rules there need not be expressed in terms of sample sum. They have made use of asymptotic normality of U-statistics with random indices (vide Sproule (1969)) to obtain asymptotic normality of stopping times arising in sequential estimation problems.

More recently the method of nonlinear renewal theory adopted in Sequential Analysis gives a revealing way of studying the second order asymptotic behaviour of stopping times. The work of Lai and Siegmund

(1977, 1979) and that of Woodroofe (1982) make a major step in this area. One may look into Chapter 8 and Chapter 9 of Siegmund (1985) for a complete discussion in this area.

In this chapter a general theorem studying the asymptotic distribution of a class of stopping times is given first. This is then used to obtain the asymptotic distribution (as $k \to \infty$ with k the size of the fixed sample from π_0) of stopping times of the SPRTs discussed in the preceding chapters. The general theorem here can be thought of as a version of Theorem 2 (Theorem 1) of GM (Ghosh and Mukhopadhyay (1975)) based on the ideas of Anscombe (1952) with little modification suitable for the present context.

The main theorem is given in Section 4.2. Section 4.3 and Section 4.4 deal with its applications to the stopping times (both truncated and untruncated) of the invariant SPRT in the multivariate known Σ case and in the univariate known σ case respectively. For an elaborate discussion on truncated SPRT (with Brownian motion approximation) one may look into Chapter 3 and Chapter 10 of Siegmund (1985).

This chapter is a revised version of a part of Ray Chaudhuri (1985).

4.2 The Main Result

This section gives the main theorem regarding the asymptotic distribution of a class of stopping times.

<u>Theorem 4.1</u>. Let $\{W_n^r\}_{n \geq 1}$ denote a sequence of random variables for $r \in [0, \infty)$.

Let $\{b_r\}$ be a real sequence s.t. $b_r \longrightarrow \infty$ as $r \longrightarrow \infty$

$$\left.\begin{aligned} \text{Let } \tau_r &= \inf\{n : W_n^r \geq b_r\} \\ &= \infty \quad \text{otherwise} \end{aligned}\right\} \quad \ldots(4.2.$$

Suppose the following conditions hold :

(A1) $\mu > 0$ s.t. $b_r^{-1} \tau_r \longrightarrow \mu^{-1}$ in P as $r \longrightarrow \infty$.

For any sequence of positive integer $\{m_r\}$ for which $b_r^{-1} m_r \longrightarrow \mu^{-1}$ as $r \longrightarrow \infty$,

(A2) $\exists$ a distribution function $F(\cdot)$ and a real sequence $\{\mu_r\}$ converging to $\mu(\mu$ as given in (A1)) as $r \longrightarrow \infty$, such that the following holds for all continuity points t of F,

$$P\left\{b_r^{-1/2}(W_{m_r} - m_r \mu_r) \leq t\right\} \longrightarrow F(t) \quad \text{as } r \longrightarrow \infty . \quad \ldots(4.2.$$

(A3) For given any ε and η $\exists$ r_0 (large) and c_0 (small) such that $\forall\, r \geq r_0$

$$P\left\{\left|\frac{W^r_{m_r}}{m_r} - \frac{W^r_{m'}}{m'}\right| < \varepsilon m_r^{-1/2} \;\forall\, m' : |m' - m_r| < c_0\, m_r\right\} > 1-\eta \qquad \ldots(4.2.3$$

Then (a) $P(\{\tau_r > n_{r,x}\} \,\Delta\, \{W^r_{n_{r,x}} < b\}) \longrightarrow 0$ as $r \longrightarrow \infty$

where $n_{r,x} = \lceil b_r \mu_r^{-1} - b_r^{1/2} \mu^{-1} x \rceil$, with x a continuity of F. $\lceil y \rceil$ denotes the smallest integer greater than or equal to y and $A \Delta B$ denotes the symmetric difference of the two sets A and B.

(b) Moreover for all sequences $\{n_r\}$ s.t. $b_r^{-1} n_r \to \mu^{-1}$ as $r \to \infty$

$$\mu\, b_r^{-1/2}(\tau_r - b_r \mu_r^{-1}) = -b_r^{-1/2}(W_{n_r}^r - n_r \mu_r) + o_p(1)$$

and hence the limiting distribution of $-\mu b_r^{1/2}(\tau_r - n_r \mu_r)$ is F.

<u>Remark 4.1</u> : In applications of Theorem 4.1, μ_r cannot be replaced by μ in general.

<u>Remark 4.2</u> : Observe that if (A2) and (A3) are satisfied for one sequence $\{m_r\}$ s.t. $b_r^{-1} m_r \to \mu^{-1}$ as $r \to \infty$, then (A2) and (A3) are satisfied for all sequences $\{n_r\}$ s.t. $b_r^{-1} n_r \to \mu^{-1}$ as $r \to \infty$, with the same μ and F.

<u>Remark 4.3</u> : Let $\tau_r' = \inf \left\{ n : W_n^r + c \geq b_r \right\}$ where W_n^r, b_r are as in Theorem 4.1 and c is a real constant. Suppose (A1) (with τ_r' in place of τ_r), (A2) and (A3) are satisfied. Then $P(\{\tau_r' > n_{r,x}\} \Delta \{W_{n_{r,x}}^r < b_r\}) \to 0$ as $r \to \infty$. The proof is along similar lines as the proof of Theorem 4.1.

We now proceed to the proof of Theorem 4.1. Let us first state a lemma.

<u>Lemma 4.1</u> : Let $\{U_r, r \in [0, \infty)\}$ and $\{V_r, r \in [0,\infty)\}$ be two stochastic processes satisfying the following conditions,

(1) $P\{U_r \leq t\} \longrightarrow G(t)$ as $r \longrightarrow \infty$,

for all continuity point t of G, where G is a distribution functi

(2) For all continuity point t of G and for all $\varepsilon \geq 0$,

$$\lim_{r \to \infty} P\{V_r < t - \varepsilon, U_r > t\} = 0$$

$$\lim_{r \to \infty} P\{V_r > t, U_r < t - \varepsilon\} = 0$$

Then $V_r - U_r = o_p(1)$.

The proof of Lemma 4.1 follows from the proof of Lemma 1 of Ghosh (1971).

<u>Proof of Theorem 4.1</u> :

<u>Proof of Part (a)</u> : For simplicity in notation let us denote $n_{r,x}$ by n_r in the proof of Part (a).

$$P(\{\tau_r > n_r\} \triangle \{W^r_{n_r} < b_r\})$$

$$= P\{\tau_r < n_r, W^r_{n_r} < b_r\} \text{ (By the definition of } \tau_r \text{ as given in (4.2}$$

$$\leq P\{\tau_r < n_r, W^r_{n_r} < b_r, |\tau_r b_r^{-1} \mu_r - 1| < \varepsilon_1\}$$

$$+ P\{|\tau_r b_r^{-1} \mu_r - 1| \geq \varepsilon_1\} \qquad \ldots(4.2.4)$$

where $0 < \varepsilon_1 < 1$, is to be chosen suitably later.

For any fixed $\varepsilon_1 > 0$, the second term of (4.2.4) goes to zero as $r \to \infty$ (by (A1) and the fact that $\mu_r \to \mu$ as $r \to \infty$).

Fix $\varepsilon_2 > 0$. Let n_1 be the smallest integer less than or equal to $(1-\varepsilon_1)b_r\,\mu_r^{-1}$. Thus n_1 is less than n_r for large r

Now the first term on (4 2.4)

$$\leq P\left\{n_1 < \tau_r < n_r,\ W^r_{n_r} < b_r\right\}$$

$$\leq P\left\{\max_{n_1<j<n_r} W^r_j \geq b_r,\ W^r_{n_r} < b_r - \varepsilon_2\, b_r^{1/2}\right\}$$

$$+ P\left\{b_r - \varepsilon_2\, b_r^{1/2} \leq W^r_{n_r} < b_r\right\} \qquad \ldots(4.2.5)$$

The second term of (4.2.5), can be made as small as we please if $\varepsilon_2 > 0$, is chosen sufficiently small and then $r \longrightarrow \infty$ (by using (A2) and the fact that x is a continuity point of F).

The first term on (4.2.5) $\leq P\left\{\max_{n_1<j<n_r} W^r_j - W^r_{n_r} \geq \varepsilon_2\, b_r^{1/2}\right\}$

$$\leq P\left\{\max_{n_1<j<n_r} j\left(\frac{W^r_j - j\mu_r}{j} - \frac{W^r_{n_r} - n_r\,\mu_r}{j}\right) > \varepsilon_2\, b_r^{1/2}\right\}$$ (As $n_r > j$ and for large $r, \mu_r > 0$)

$$= P\left\{\max_{n_1<j<n_r} j\left(\frac{W^r_j}{j} - \frac{W^r_{n_r}}{n_r}\right) + \left(W^r_{n_r} - n_r\,\mu_r\right)\left(\frac{j}{n_r} - 1\right) > \varepsilon_2\, b_r^{1/2}\right\}$$

$$\leq P\left\{b_r^{-1/2} \max_{n_1<j<n_r} n_r \left|\frac{W^r_j}{j} - \frac{W^r_{n_r}}{n_r}\right| \geq \varepsilon_2/2\right\}$$

$$+ P\left\{b_r^{-1/2}\left|W^r_{n_r} - n_r\,\mu_r\right| \cdot \left|\max_{n_1<j<n_r}\left(\frac{j}{n_r} - 1\right)\right| \geq \varepsilon_2/2\right\}$$

The first term on (4.2.6) goes to zero by (A3) and the fact $b_r^{-1} n_r \longrightarrow \mu^{-1}$ as $r \longrightarrow \infty$. The second term on (4.2.6) goes to zero by (A2) and the fact that $\max_{n_1 < j < n_r} (\frac{j}{n_r} - 1)$ can be made arbi - trarily small by first making ε_1 sufficiently small and then making $r \longrightarrow \infty$. Thus Part (a) is proved.

Proof of Part (b) :

Observe $P\left\{\tau_r > n_{r,x}, W^r_{n_{r,x}} \geq b\right\} = 0$ (by definition of τ_r)

$$\Rightarrow P\left\{-\mu b_r^{-1/2}(\tau_r - b_r \mu_r^{-1}) < x - \mu b_r^{-1/2},\ b_r^{-1/2}(W^r_{n_{r,x}} - n_{r,x}\mu_r) > x \mu_r \mu^{-1}\right\} \quad \ldots(4.2.7)$$

Using Part (a),

$$P\left\{-\mu b_r^{-1/2}(\tau_r - b_r \mu_r^{-1}) > x,\ b_r^{-1/2}(W^r_{n_{r,x}} - n_{r,x}\mu_r) < x \mu_r \mu^{-1} + \mu b_r^{-1/2}\right\} \longrightarrow 0 \text{ as } r \longrightarrow \infty \quad \ldots(4.2.8)$$

Now condition (2) of Lemma 4.1 with $U_r = b_r^{-1/2}(W^r_{n_r} - n_r \mu_r)$ and $V_r = -\mu b_r^{-1/2}(\tau_r - b_r \mu_r^{-1})$ can be seen to be satisfied using (4.2.7), (4.2.8) (A2) and the fact $b_r^{-1/2}(W^r_{n_r} - n_r \mu_r) - b_r^{-1/2}(W^r_{n_{r,x}} - n_{r,x}\mu_r) = o_p(1)$ (which follows from (A2), (A3) and Remark 4.2). Condition (1) of Lemma 4.1 follows from (A2) and thus the proof of Part(b) follows from Lemma 4.1.

4.3 Application to SPRT for the Multivariate known Σ Case :

This section gives the asymptotic distribution of the stopping time of the invariant SPRT proposed (in Chapter 3) for identifying a multivariate normal distribution (with known Σ) for the cases $\mu = \mu_1$ and $\mu = \mu_2$. Since the original problem is an identification problem these two cases are most important.

The asymptotic study of the SPRT (for the known Σ case) for the sampling scheme S2 (as in Chapter 3) is made here as $k \to \infty$ where k is the size of the fixed sample available from π_0. The case for scheme S3 (for $\alpha \wedge \beta \to 0$ instead of $k \to \infty$ as in S2) is much simple and follows from the existing results in the literature without any further modification (as well as from Theorem 4.1 as a particular case with $W_n^r = W_n$, $\mu_r = \mu$, $b_r = r$).

The version of N_1 (of Chapter 3) considered here is

$$\left.\begin{aligned} N_k &= \inf\left\{n : \left| \ln W_{n,k}(\Delta_0) \right| \geq b_k\right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(4.3.1)$$

where $\ln W_{n,k}(\Delta_0) = \ln w_p(Z_2) - \ln w_p(Z_1)$ ...(4.3.2)

as in (3.2.9) with Z_1, Z_2 as in (3.2.6), $w_p(\cdot)$ as in (3.2.10), and

$$b_k \to \infty \quad \text{as} \quad k \to \infty \quad \text{s.t.} \quad k^{-1} b_k \to a_1 > 0 . \qquad \ldots(4.3.3)$$

Let $\nu = \mu_1 - \mu_2$

$$\theta_k = \Delta_o \, || \nu || - 2b_k \, k^{-1}$$

$$\theta = \Delta_o \, || \nu || - 2a_1 \qquad \ldots(4.3.4)$$

$$\sigma_p = (2a_1 \, \Delta_o^3 \, || \nu ||)^{1/2} \, (\Delta_o \, || \nu || - 2a_1)^{-2}$$

$$\text{for} \quad \Delta_o \, || \nu || - 2a_1 > 0$$

<u>Theorem 4.2</u> : For $\mu = \mu_1$ or $\mu = \mu_2$, $k^{-1/2} (N_k - b_k \, \theta_k^{-1})$ is asymptotically (as $k \to \infty$) normal with mean zero and variance σ_p^2 if $\Delta_o \, || \nu || > 2a_1$.

<u>Proof of Theorem 4.2</u> : For $\mu = \mu_1$, it is enough to consider

$$\left.\begin{aligned} N_k' &= \inf \left\{ n : \ln W_{n,k} (\Delta_o) \leq - b_k \right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(4.3.5)$$

as $P_{\mu = \mu_1} (N_k = N_k') \to 1$ as $k \to \infty$.

We now approximate N_k' by two other stopping times M_k and L_k which are simpler to handle

$$\left.\begin{aligned} M_k &= \inf \left\{ n : Z_2 - Z_1 + 2^{-1} (p-1) \ln \frac{1+Z_1}{1+Z_2} - c \leq - b_k \right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \ldots (4.3.6$$

$$\left.\begin{aligned} L_k &= \inf \left\{ n : Z_2 - Z_1 + 2^{-1} (p-1) \ln \frac{1+Z_1}{1+Z_2} + c \leq - b_k \right\} \\ &= \infty \end{aligned}\right\} \ldots(4.3.7$$

where Z_1, Z_2 are as in (3.2.6).

$M_k \leq N_k' \leq L_k$ by (4.3.5), (4.3.6), (4.3.7) and the approximation formula 3.3.4 of Wijsman (1979).

Let us first study M_k.

$$M_k = \inf\left\{n : (2k^{-1}n+1)(Z_1-Z_2+2^{-1}(p-1)\ln\frac{1+Z_2}{1+Z_1} - 2k^{-1}n\,(b_k-c) \geq b_k - c\right\}$$
$$= \infty \qquad \text{otherwise}$$

...(4.3.8)

First we shall show (a) and (b) of Theorem 4.1 are satisfied with

$r = k$, $b_r = b_k' = b_k - c$,

$$W_n^r = W_n^k = (2k^{-1}n+1)\,(Z_1-Z_2+2^{-1}(p-1)\ln\frac{1+Z_2}{1+Z_1}) - 2k^{-1}\,n\,b_k'$$

$$\tau_r = \tau_k = M_k$$

$$\mu_r = \mu_k = \theta_k' = \Delta_o\,||\,\lambda\,|| - 2k^{-1}\,b_k' \qquad \text{...(4.3.9)}$$

$$\mu = \theta = \Delta_o\,||\,\lambda\,|| - 2a_1 > 0 \text{ by hypotheses.}$$

$F(x) = \Phi(x/\ \Delta_o(\theta^{-1}(4a_1\,\theta^{-1}+2))^{1/2})$, $\Phi(\cdot)$ denotes the normal c.d.f.

Now (A1) (with $\tau_r = \tau_k = M_k$), (A2) and (A3) with terms defined in (4.3.9) are satisfied vide Lemma 4.2, Lemma 4.3 and Lemma 4.4 given below.

Thus (a) and (b) of Theorem 4.1 hold with terms as described in (4.3.9)

Now from part (b) one gets, as $k \to \infty$,

$$\theta\, b_k'^{-1/2}\,(M_k - b_k'\,\theta'^{-1}) \Rightarrow N\,(0,\ \Delta_o^2\,\theta^{-1}(4a_1\,\theta^{-1}+2)$$

$$\Rightarrow\ \theta k^{-1/2}\,(M_k - b_k'\,\theta_k'^{-1}) \Rightarrow N\,(0,\ \Delta_o^2\,a_1\,\theta^{-1}\,(4a_1\theta^{-1}+2))$$

$$\Rightarrow\ k^{-1/2}\,(M_k - b_k\,\theta_k^{-1}) \Rightarrow N\,(0,\ \Delta_o^2\,a_1\,\theta^{-3}\,(4a_1\theta^{-1}+2)) \qquad \text{...(4.3.10)}$$

as $k^{-1/2}\,(b_k'\,\theta_k'^{-1} - b_k\,\theta_k^{-1}) \to 0$ as $k \to \infty$

Similarly one can show

$$k^{-1/2}(L_k - b_k \theta_k^{-1}) \Rightarrow N(0, \Delta_o^2 a_1 \theta^{-3}(4a_1\theta^{-1}+2)) \quad \ldots(4.3.11)$$

Observe

$$\Delta_o^2 a_1 \theta^{-3}(4a_1 \theta^{-1} + 2) = \Delta_o^2 a_1(\Delta_o \|\nu\| - 2a_1)^{-4}(2\Delta_o \|\nu\|)$$
$$= \sigma_p^2$$

Thus the proof for the case $\mu = \mu_1$ follows from (4.3.10), (4.3.11) and the fact that $M_k \leq N_k' \leq L_k$.

The proof for the case $\mu = \mu_2$ follows along similar lines. □

Let us now provide a motivation for the lemmas mentioned in the proof of Theorem 4.2.

$$\text{Let } S = 2\bar{X}_k - \bar{Y}_n - \bar{Z}_n \; ; \; T = \bar{Y}_n - \bar{Z}_n \quad \ldots(4.3.12)$$

$$\text{Then } Z_1 = \Delta_o \left\| \left(\frac{S-\nu}{\sigma_S} + \frac{T-\nu}{\sigma}\right) + \nu\left(\frac{1}{\sigma_S} + \frac{1}{\sigma_T}\right) \right\| \quad \ldots(4.3.13)$$

for σ_S, σ_T as in (3.2.4), Z_1 as in (3.2.6) and ν as in (4.3.4).

For $\mu = \mu_1$, the first term on the RHS of (4.3.13) is expected to be smaller compared to the second for large n and k. Thus making first order expansion about $\nu(\sigma_S^{-1} + \sigma_T^{-1})$ and doing the same with Z_2, we get

$$Z_1 - Z_2 = 2\Delta_o \sigma_S^{-1} \|\nu\|^{-1} S'\nu + R_{n,k} \quad \ldots(4.3.14)$$

where $2\Delta_o^{-1} R_{n,k} = U_{n,k}' A_{n,k}(\nu'\nu + a_{n,k})^{-1/2} + 2U_{nk}'\nu((\nu'\nu + a_{n,k})^{-1/2}$
$- \|\nu\|^{-1}) - V_{n,k}' B_{n,k}(\nu'\nu + b_{n,k})^{-1/2} + 2V_{n,k}'\nu((\nu'\nu + b_{n,k})^{-1/2} - \|\nu\|^{-1})$

...(4.3.15)

with $U_{n,k} = \frac{S-\nu}{\sigma_S} + \frac{T-\nu}{\sigma_T}$; $V_{n,k} = \frac{S-\nu}{\sigma_S} - \frac{T-\nu}{\sigma_T}$

$$A_{n,k} = (\frac{1}{\sigma_S} + \frac{1}{\sigma_T})^{-1} U_{n,k}; \quad a_{n,k} = \theta^{(1)}_{n,k} A'_{n,k}(A_{n,k}+2\nu)$$

$$B_{n,k} = (\frac{1}{\sigma_T} - \frac{1}{\sigma_S})^{-1} V_{n,k}; \quad b_{n,k} = \theta^{(2)}_{n,k} B'_{n,k}(B_{n,k}-2\nu) \qquad \ldots(4.3.16)$$

$0 < \theta^{(j)}_{n,k} < 1$ for $j = 1,2$ (appears from the first order expansion)

If $\{n_k\}$ is a sequence of positive integer s.t. $k^{-1}n_k \to a$ $(a > 0)$ as $k \to \infty$ then it is easy to see from (4.3.15) and (4.3.16) that

$$R_{n_k,k} = o_p\,(n_k^{1/2}) \quad \text{for} \quad \mu = \mu_1 \qquad \ldots(4.3.17)$$

Also for $\mu = \mu_1$ and $||\nu|| > 0$, $\ln(\frac{1+Z_2}{1+Z_1}) \to \ln(\frac{a}{1+a})$ a.s. as $k \to \infty$. $\qquad \ldots(4.3.18)$

where Z_1, Z_2 as given in (3.2.6) having n_k in place of $\bar{n}$.

Similar results as in (4.3.14) - (4.3.18) can also be obtained for $\mu = \mu_2$.

These facts will be used in the proofs of Lemma 4.3 and Lemma 4.4. They also motivate Lemma 4.2 but the proof of Lemma 4.2 runs along a different line.

<u>Lemma 4.2</u> : $b_k'^{-1} M_k \rightarrow \theta^{-1}$ a.s. as $k \rightarrow \infty$

<u>Proof of Lemma 4.2</u> . Proceeding along similar lines as in Theorem 3.1 (of Chapter 3), one can show that M_k does not terminate with probabil one for fixed k. However, $b_k'^{-1} M_k$ does admit a limit as $k \rightarrow \infty$. To show that choose ε_1 and ε_2 both positive s.t.

$$2\varepsilon_2 + \Delta_o \| \gg \|^{-1} \varepsilon_1 < \Delta_o \| \gg \| - 2a_1 \text{ and } \varepsilon_2 < a_1 \qquad \ldots(4$$

Define $B_k' = \{ | S'T - (2\mu - \bar{Y}_n - \bar{Z}_n)'(\bar{Y}_n - \bar{Z}_n) | < \varepsilon_1 \ \forall\ k \geq k' \}$ ...(4

Then for given any η and ε_1 (as in (4.3.19), we can choose k_1 large s.t.

$$P(B_{k_1}) \geq 1 - \eta \qquad \ldots(4$$

Let k_2 be chosen using (4.3.3) s.t.

$$| k^{-1} b_k' - a_1 | < \varepsilon_2 \ \forall\ k \geq k_2. \qquad \ldots(4.$$

Let $k_o = k_1 \vee k_2$

Let $M_k' = \inf \{ n : 2\ \Delta_o \| 2\bar{X}_k - \bar{Y}_n - \bar{Z}_n \| \sigma_S^{-1} k^{-1} > a_1 - \varepsilon_2 \}$

$= \infty$ otherwise.

Then $\| 2\bar{X}_k - \bar{Y}_n - \bar{Z}_n \| \sigma_S^{-1} k^{-1} \rightarrow 0$ a.s. as $k \rightarrow \infty$ (with n fixed) which implies $M_k' \rightarrow \infty$ a.s. as $k \rightarrow \infty$

By (4.3.22) and proceeding as in the proof of Theorem 3.1 we have $M_k' \leq M_k \ \forall\ k \geq k_o$. Thus $M_k \rightarrow \infty$ a.s. as $k \rightarrow \infty$.

$$\left.\begin{aligned} \text{Let } M_k'' &= \inf\Big\{n : n(Z_1+Z_2)^{-1}\,((2\mu-\bar{Y}_n-\bar{Z}_n)'(\bar{Y}_n-\bar{Z}_n)-\varepsilon_1)\,\Delta_o^2\,(2+n^{-1}k)^{-1} \\ &\qquad +(2k)^{-1}(p-1)\ln((1+Z_2)(1+Z_1)^{-1}) > a_1+\varepsilon_2\Big\} \\ &= \infty \qquad\qquad \text{otherwise} \end{aligned}\right\}$$

For fixed k, and $n \to \infty$, $n^{-1} Z_i \to 2^{-1} \Delta_o \,||\nu||$ a.s. for $i = 1,2$, and $(2\mu - \bar{Y}_n - \bar{Z}_n)' (\bar{Y}_n - \bar{Z}_n) \to \nu'\nu$ a.s. Thus the choice of $\varepsilon_1, \varepsilon_2$ assures $p(M''(k) < \infty) = 1$.

Now on B_{k_o}, $M_k \leq M_k''$ $\forall\, k \geq k_o$

Thus, $p(B_{k_o}, M_k < \infty \;\forall\, k \geq k_o) > 1 - \eta$ ...(4.3.23)

Thus, we now concentrate on B_{k_o}

$$\text{On } B_{k_o},\; W^k_{M_k} > b_k' > W^k_{M_k} - 1. \qquad \text{...(4.3.24)}$$

$$\text{Now } M_k^{-1} W^k_{M_k} = (2k^{-1} + M_k^{-1})\left(Z_1 - Z_2 + 2^{-1}(p-1)\ln\frac{1+Z_2}{1+Z_1}\right) - 2k^{-1} b_k'$$

where Z_1, Z_2 (defined in (3.2.6)) both have M_k in place of n.

$$(2k^{-1} + M_k^{-1})(Z_1 - Z_2) = 2^{-1}\sigma_S (Z_1 - Z_2)$$

$$= 2^{-1} \Delta_o \left(||S + \frac{\sigma_S}{\sigma_T}T|| - ||S - \frac{\sigma_S}{\sigma_T}T||\right) \qquad \text{...(4.3.25)}$$

(Here σ_S, σ_T (as in 3.2.4)) both have M_k in place of n)

$$= 2\Delta_o S'T\left(||\frac{\sigma_T}{\sigma_S} + T|| + ||\frac{\sigma_T}{\sigma_S} - T||\right)^{-1} \qquad \text{...(4.3.26)}$$

Expression in (4.3.26) is more convenient to handle as $\frac{\sigma_T}{\sigma_S}$ is bounded above by 1.

Now, $\left\| \frac{\sigma_T}{\sigma_S} S + T \right\| + \left\| \frac{\sigma_T}{\sigma_S} S - T \right\|$

$$= \left\| \frac{\sigma_T}{\sigma_S} (S - \nu) + (T - \nu) + \nu\left(1 + \frac{\sigma_T}{\sigma_S}\right) \right\|$$

$$+ \left\| \frac{\sigma_T}{\sigma_S} (S - \nu) - (T - \nu) + \nu\left(\frac{\sigma_T}{\sigma_S} - 1\right) \right\| \qquad \ldots(4.3.27)$$

(For $\mu = \mu_1$, $ES = ET = \nu$)

$$\rightarrow 2 \, \|\nu\| \quad \text{a.s.} \quad \text{as} \quad k \rightarrow \infty \qquad \ldots(4.3.28)$$

For (4.3.28) add and subtract $2 \|\nu\|$ to (4.3.27) and then break up $-2 \|\nu\| = -\left(\left(1 + \frac{\sigma_T}{\sigma_S}\right) \|\nu\| + \left(\frac{\sigma_T}{\sigma_S} - 1\right) \|\nu\|\right)$. This

expression of $-2 \|\nu\|$ together with (4.3.27) can be shown to converge to zero a.s. as $k \rightarrow \infty$.

Thus $(2k^{-1} + M_k^{-1}) (Z_1 - Z_2) \rightarrow 2 \; \Delta_0 \; \nu'\nu \; (2\|\nu\|)^{-1} = \Delta_0 \|\nu\|$ a.s. as $k \rightarrow \infty$

from (4.3.26) and (4.3.28). $\ldots(4.3.29)$

Now $(2k^{-1} + M_k^{-1}) \left| \ln\left(\frac{1+Z_2}{1+Z_1}\right)\right| \leq (2k^{-1} + M_k^{-1}) \ln(1+|Z_2 - Z_1|)$

$$= (2k^{-1} + M_k^{-1}) \ln\left((2k^{-1} + M_k^{-1}) (1+|Z_2 - Z_1|)\right) - (2k^{-1}+M_k^{-1}) \ln(2k^{-1}+M_k^{-1})$$

$$\rightarrow 0 \quad \text{a.s.} \quad \text{as} \quad k \rightarrow \infty \ \text{(using (4.3.29))} \qquad \ldots(4.3.30)$$

Thus from (4.3.29), (4.3.30) and (4.3.3), we have

$$M_k^{-1} \, W_{M_k}^k \rightarrow \Delta_0 \, \|\nu\| - 2a_1 \quad \text{a.s.} \quad \text{as} \quad k \rightarrow \infty \qquad \ldots(4.3.31)$$

Similarly $M_k^{-1} \, W_{M_k - 1}^k \rightarrow \Delta_0 \, \|\nu\| - 2a_1$ a.s. as $k \rightarrow \infty$ $\ldots(4.3.32)$

Thus from (4·3·24), (4·3·31) and (4·3·32), $\exists$ a P-null set N_o s.t.

$$\text{on} \quad N_o^c \cap B_{k_o}, \; b_k'^{-1} M_k \longrightarrow (\Delta_o \, \| \; \| - 2a_1)^{-1} = \theta^{-1} \text{ as } k \to \infty \quad \cdots (4\cdot3\cdot33)$$

Thus $P(\lim_{k\to\infty} b_k'^{-1} M_k = \theta^{-1}) \geq P(\lim_{k\to\infty} b_k'^{-1} M_k = \theta^{-1}, B_{k_o}) \geq 1-\eta$ and the fact that η is arbitrary implies Lemma 4·2. □

Lemma 4·3 : Let $\{m_k\}$ be any sequence of integers s.t. $b_k'^{-1} m_k \to \theta^{-1}$ as $k \to \infty$. Then $b_k'^{-1/2}(W^k_{m_k} - m_k \theta_k')$ is asymptotically normal with mean 0 and variance $\Delta_o^2 \theta^{-1}(4a_1 \theta^{-1} + 2)$.

Proof of Lemma 4·3

$$b_k'^{-1/2}(W^k_{m_k} - m_k \theta_k')$$

$$= b_k'^{-1/2}\Big(m_k \frac{\Delta_o (S-\nu)'\nu}{\|\nu\|} + (2k^{-1}m_k + 1) R_{m_k,k}$$

$$+ (2k^{-1}m_k + 1)\left(\frac{p-1}{2}\right) \ln\left(\frac{1+Z_2}{1+Z_1}\right)\Big)$$

(by 4·3·9 and 4·3·14)

$$= b_k'^{-1/2} m_k \frac{\Delta_o (S-\nu)'\nu}{\|\nu\|} + o_p(1) \quad \text{(by (4·3·17), (4·3·18), (4·3·3) and the choice of } m_k)$$

$$\Rightarrow N(0, \Delta_o^2 \theta^{-1}(4a_1 \theta^{-1} + 2))$$

<u>Lemma 4·4</u> : For given ε and η $\exists$ k_o (large) and c_o (small) s.t.

$$\forall\, k \geq k_o\,,\quad P\left\{\left|\frac{W^k_{m_k}}{m_k} - \frac{W^k_{m'}}{m'}\right| < \varepsilon\, m_k^{-1/2}\ \forall m' : |m' - m_k| < c_o m_k\right\} > 1-\eta \qquad \ldots(4{\cdot}3{\cdot}34)$$

for m_k s.t. $b_k^{-1} m_k \rightarrow \theta^{-1}$ as $k \rightarrow \infty$

<u>Proof of Lemma 4·4</u> :

$$\text{Note}\quad \frac{W^k_{m_k}}{m_k} = \Delta_o \,\|\, \nu \,\|^{-1} (2\bar{X}_k - \bar{Y}_{m_k} - \bar{Z}_{m_k})'\nu + (2k^{-1} + m_k^{-1})\, R_{m_k,k}$$

$$+ (k^{-1} + 2^{-1}\, m_k^{-1})\,(p-1)\,\ln\left(\frac{1+Z_{2,m_k}}{1+Z_{1,m_k}}\right) \qquad \ldots(4{\cdot}3{\cdot}35)$$

where $Z_{i,m} = \left\| \dfrac{2\bar{X}_k - \bar{Y}_m - \bar{Z}_m}{4k^{-1} + 2m^{-1}} + (-1)^{i+1} \dfrac{\bar{Y}_m - \bar{Z}_m}{2m^{-1}} \right\|$ for $i = 1,2$.

For proving Lemma 4·4 it is enough to check (4·3·34) with $t_{j,m}$ in place of $m^{-1} W^k_m$ for each $j = 1,2,3$ where $t_{j,m}$ denotes the j^{th} term on the RHS of (4·3·35). Now (4·3·34) with $t_{1,m}$ (in place of $m^{-1} W^k_m$) follows immediately from Theorem 3 of Anscombe (1952). (4·3·34) with $t_{2,m}$ (in place of $m^{-1} W^k_m$) follows from Lemma 4·5 (given below), (4·3·16) and Theorem 3 of Anscombe (1952). For $t_{3,m}$ once again Lemma 4·5 implies the required condition

Thus the proof of Lemma 4·4 follows. $\square$

<u>Lemma 4.5</u> : Let $\{m_k\}$ be a sequence of integer s.t. $m_k \to \infty$ as $k \to \infty$.

Let $\{X_{m_k}\}_{k \geq 1}$, $\{Y_{m_k}\}_{k \geq 1}$ be *two sequences of random variable such* that the following conditions hold :

(1) For all $\delta > 0$, $\exists\, \lambda$ (depending on δ) s.t. $P(|m_k^{1/2} X_{m_k}| > \lambda) < \delta \;\forall k$.

(2) For given any ε and η (both positive real numbers) $\exists\, k_o$ (large) and c_o (small) s.t. $\forall k \geq k_o$.

$$P\left\{|X_{m_k} - X_{m'}| < \varepsilon\, m_k^{-1/2} \;\forall m' : |m' - m_k| < c_o\, m_k\right\} > 1 - \eta$$

(3) $Y_{m_k} \to$ constant a.s. as $k \to \infty$

Then for given any ε and η (both positive real numbers) $\exists\, k_o$ (large) and c_o (small) s.t. $\forall k \geq k_o$

$$P\left\{|X_{m_k} Y_{m_k} - X_{m'} Y_{m'}| < \varepsilon\, m_k^{-1/2} \;\forall m' : |m' - m_k| < c_o\, m_k\right\} > 1 - \eta .$$

<u>Proof of Lemma 4.5</u> :

$$|X_{m_k} Y_{m_k} - X_{m'} Y_{m'}| = |X_{m_k} Y_{m_k} - X_{m_k} Y_{m'} + X_{m_k} Y_{m'} - X_{m'} Y_{m'}|$$

$$\leq |X_{m_k}|\, |Y_{m_k} - Y_{m'}| + |Y_{m'}|\, |X_{m_k} - X_{m'}| \qquad \ldots(4.3.36)$$

Now for given ε and η, $\exists\, k_o$ and c_o s.t. $\forall k \geq k_o$,

$$P\left\{|X_{m_k}|\, |Y_{m_k} - Y_{m'}| < \frac{\varepsilon}{2}\, m_k^{-1/2} \;\forall m' : |m' - m_k| < c_o m_k\right\} > 1 - \eta/2 \qquad \ldots(4.3.37)$$

by (1) and (2)).

and $P\left\{|Y_{m'}|\,|X_{m_k} - X_{m'}| < 2^{-1}\,\varepsilon m_k^{-1/2}\ \forall\ m' : |m' - m_k| < c_0 m_k\right\} > 1-\eta/2$ ...(

(by (2) and (3)).

The proof now follows from (4.3.36), (4.3.37) and (4.3.38).

Theorem 4.3 : Let N_k be a stopping time such that $k^{-1/2}(N_k - \nu_k)$ converges in distribution to F (F a distribution function). Let m_{ok} denote a sequence s.t. $k^{-1} m_{ok} \to a$ $(a > 0)$ and $k^{-1} N_k \to b$ $(b > 0)$ a.s. as $k \to \infty$. Then for $b < a$, $N_k \wedge m_{ok}$ has the same limiting distribu F while for $b > a$, $N_k \wedge m_{ok}$ is asymptotically degenerate at m_{ok}.

Proof of Theorem 4.3. Case 1 : $a > b$

In this case, we shall show $N_k \wedge m_{ok} - N_k = o_p(1)$.

For that, $P(N_k - N_k \wedge m_{ok} > 0)$

$$= P(N_k \wedge m_{ok} < N_k)$$

$$= P(m_{ok} < N_k)$$

$$= P(k^{-1/2}(N_k - \nu_k) > k^{-1/2}(m_{ok} - \nu_k)).$$

$$\longrightarrow 0 \text{ as } k^{-1/2}(m_{ok} - \nu_k) \longrightarrow \infty \text{ (for } a > b)$$

and $k^{-1/2}(N_k - \nu_k) \Rightarrow F$ as $k \to \infty$

Thus $k^{-1/2}(N_k \wedge m_{ok} - \nu_k) \Rightarrow F$ as $k \to \infty$.

Case 2 : $a < b$

In this case we shall show $N_k \wedge m_{ok} - m_{ok} = o_p(1)$

For that, $P(m_{ok} - N_k \wedge m_{ok} > 0)$

$$= P(N_k \wedge m_{ok} < m_{ok})$$

$$= P(N_k < m_{ok})$$

$$= P(k^{-1/2}(N_k - \nu_k) < k^{-1/2}(m_{ok} - \nu_k))$$

$$\longrightarrow 0 \text{ as } k^{-1/2}(m_{ok} - \nu_k) \to -\infty \quad (\text{for } a < b)$$

and $k^{-1/2}(N_k - \nu_k) \Rightarrow F$ as $k \to \infty$.

Thus $N_k \wedge m_{ok}$ is asymptotically degenerate at m_{ok}.

Remark 4.5 : Theorem 4.3 gives us the asymptotic behaviour of N_1 when truncated. The case a = b (a,b are as in Theorem 4.3) remains open. Theorem 4.2 gives the asymptotic distribution of N_1 (untruncated) for the case $\alpha = \beta$. For $\alpha \neq \beta$ one can obtain a similar result.

4.4 Application to SPRT for the univariate known σ case

This section gives similar results as in the previous section for the SPRT proposed (in Section 2.2 of Chapter 2 as N_1) for identifying a univariate normal population (with known σ). Here also we consider two cases namely $\mu = \mu_1$ and $\mu = \mu_2$.

The version of N_1 (of Section 2.2) considered here is

$$\left.\begin{aligned} N_{1k} &= \inf\left\{n : |\ln V_{n,k}(\delta_o)| \geq b_k\right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(4.4.1)$$

with $\ln V_{n,k}(\delta_o)$ as in (2.2.5) and b_k as in (4.3.3).

Let $\nu_1 = \mu_1 - \mu_2$

$$\theta_{1k} = \delta_o |\nu_1| - 2b_k k^{-1}$$

$$\theta_1 = \delta_o |\nu_1| - 2a_1 \qquad (a_1 = \lim_{k \to \infty} k^{-1} b_k \text{ as in } (4\cdot3\cdot3)) \qquad \ldots(4\cdot4\cdot2)$$

$$\sigma_1 = (2a_1 \delta_o^3 \, |\nu_1|)^{1/2} (\delta_o |\nu_1| - 2a_1)^{-2}$$

$$\text{for } \delta_o |\nu_1| > 2a_1$$

<u>Theorem 4·4</u> : For $\mu = \mu_1$ or $\mu = \mu_2$, $k^{-1/2}(N_{1k} - b_k \theta_{1k}^{-1})$ is asymptotically (as $k \to \infty$) normal with mean zero and variance σ_1^2 if $\delta_o |\nu_1| - 2a_1 > 0$.

<u>Proof of Theorem 4·4</u> : For $\mu = \mu_1$, it is enough to consider the one sided stopping rule namely

$$\left.\begin{aligned} N'_{1k} &= \inf \left\{ n : \ln V_{nk}(\delta_o) \leq - b_k \right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(4\cdot4\cdot3)$$

as $P_{\mu = \mu_1}(N_{1k} = N'_{1k}) \longrightarrow 1$ as $k \to \infty$

Observe $\ln V_{n,k}(\delta_o) = \ln\left(\frac{\cosh P_2}{\cosh P_1}\right)$

$$= P_2 - P_1 + \ln\left(\frac{1+e^{-2P_2}}{1+e^{-2P_1}}\right) \text{ where}$$

$$P_i = \delta_o \left| \frac{kn}{4n+2k} R + (-1)^{i+1} \frac{n}{2} Q \right| \quad \text{for } i = 1,2$$

with R and Q as in (2·2·4).

Thus $P_2 - P_1 - \ln 2 < \ln V_{n,k}(\delta_0) < P_2 - P_1 + \ln 2$ ...(4.4.4)

$$\left.\begin{aligned} \text{Let } M_{1k} &= \inf \{n : P_2 - P_1 - \ln 2 \leq - b_k\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \quad ...(4.4.5)$$

and

$$\left.\begin{aligned} L_{1k} &= \inf \{n : P_2 - P_1 + \ln 2 \leq - b_k\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \quad ...(4.4.6)$$

From (4.4.3) - (4.4.6), $M_{1k} \leq N_{1k} \leq L_{1k}$

Note $M_{1k} = \inf \left\{n : (2k^{-1}n + 1)(P_1 - P_2) - 2k^{-1}n(b_k - \ln 2) \geq b_k - \ln 2\right\}$

$= \infty$ otherwise

...(4.4.7)

Thus M_{1k} is now comparable with M_k given in (4.3.8). Now proceeding as in the proof of Theorem 4.2 (in fact in an easier way) one can show (a) and (b) of Theorem 4.1 hold wit. terms similar to these given in (4.3.9) .

The proof of Theorem 4.4 now follows along exactly similar lines as the proof of Theorem 4.2 .

Remark 4.6 : Here also Theorem 4.3 gives the asymptotic distribution of N_1 (of Section 2.2) when truncated for the case $a \neq b$ (a, b as in the statement of Theorem 4.3). One may have similar comments as in Remark 4.5 for the univariate case too.

Remark 4.7 : One may make use of the identity

$$|u+v| - |u-v| = 2(|u| \wedge |v|)\, 1_{(uv \geq 0)} - 2(|u| \wedge |v|)\, 1_{(uv < 0)}$$

for proving Theorem 4.4 . This will simplify some of the complicated calculations.

Remark 4.8 : Simulation results given in Table 2.2 of Chapter 2 may be compared with the mean and variance of the limiting distribution of N_1 (vide Theorem 4.4) . For smaller values of $(\delta_0|\nu_1| - 2a_1)^{-1}$ the simulated values are found to be closed to the theoretical values given by Theorem 4.4. This is probably because of the fact that the variance of the limiting distribution of N_1 (namely σ_1^2 as in (4.4.2)) is proportional to $(\delta_0|\nu_1| - 2a_1)^{-4}$ and smaller variance has led to the smaller sampling fluctuations .

Remark 4.9 : The stopping time (N_{ok} say) of the invariant SPRT for the univariate one sided case ($\mu_1 \geq \mu_2 + \delta$ vide GM(1980)) also admits a similar limiting distribution as $k \to \infty$. In fact the limiting distribution can be obtained $\forall\ \mu : |2\mu - \mu_1 - \mu_2|\,\delta > 2a_1$ (i.e., not only for $\mu = \mu_i$, $i = 1,2$ as given in the two sided case) .

For the one sided case the truncation point can be found explicitly vide GM (1980) as

$$m_{ok} = k\left[(\delta^2 k/(\tau_\alpha + \tau_\beta)^2) - 1\right]^{-1} .$$

Assume (as in (4.3.3)) $\lim\limits_{k \to \infty} k^{-1} \ln((1-\beta_k)/\alpha_k) = \lim\limits_{k \to \infty} k^{-1}(1-\alpha_k)/\beta_k)$

$$= a_1 > 0 .$$

Thus $-k^{-1} \ln \alpha_k$ and $k^{-1} \ln \beta_k$ both tend to a_1 as $k \to \infty$

Now $\tau_{\alpha_k}^{-2} \ln \alpha_k \to -\frac{1}{2}$ as $k \to \infty \Rightarrow k^{-1} \tau_{\alpha_k}^2 \to 2a_1$ as $k \to \infty$ and

similarly $k^{-1} \tau_{\beta_k}^2 \longrightarrow 2a_1$ as $k \to \infty$.

Thus for $\delta^2 > 8a_1$, $k^{-1} m_{ok} \longrightarrow ((\delta^2/8a_1) - 1)^{-1}$.

Now by Theorem 4.1, N_{ok} is asymptotically normal if $\delta|\nu_o| > 2a_1$ where $\nu_o = 2\mu - \mu_1 - \mu_2$ and $N_{ok} \wedge m_{ok}$ admits the same asymptotic distribution if $a_1 (\delta|\nu_o| - 2a_1)^{-1} < (\delta^2/8a_1 - 1)^{-1}$

$\Rightarrow \delta|\nu_o| - \delta^2/8 \quad > a_1$

For $|\nu_o| \geq \delta$, $\delta^2 > 2a_1 \Rightarrow \delta|\nu_o| > 2a_1$. Thus N_{ok} is asymptotically normal if $\delta^2 > 2a_1$ for $|\nu_o| \geq \delta$. Now $N_{ok} \wedge m_{ok}$ is asymptotically normal if $\delta^2 > \frac{8}{7} a_1$ which holds when $\delta^2 > 8a_1$. Thus the extra condition needed for $N_{ok} \wedge m_{ok}$ to have asymptotically normal distribution when $|\nu_o| \geq \delta$, is the condition under which $k^{-1} m_{ok}$ admits a finite positive limit as $k \to \infty$, i.e., when $\delta^2 > 8a_1$.

<u>Remark 4.10</u> : The next chapter gives an asymptotic study of ASN using the technique of Lai (1975). This kind of study is not carried out here as $Ek^{-1}N_k$ as well as $E\, k^{-1}N_{1k}$ is infinite $\forall\, k$.

Thus $-k^{-1}\ln\alpha_k$ and $k^{-1}\ln\beta_k$ both tend to a_1 as $k\to\infty$

Now $\tau_{\alpha_k}^{-2}\ln\alpha_k \to -\frac{1}{2}$ as $k\to\infty \Rightarrow k^{-1}\tau_{\alpha_k}^{2}\to 2a_1$ as $k\to\infty$ and

similarly $k^{-1}\tau_{\beta_k}^{2} \longrightarrow 2a_1$ as $k\to\infty$

Thus for $\delta^2 > 8a_1$, $k^{-1} m_{ok} \longrightarrow ((\delta^2/8a_1) - 1)^{-1}$.

Now by Theorem 4.1, N_{ok} is asymptotically normal if $\delta|\nu_o| > 2a_1$ where $\nu_o = 2\mu - \mu_1 - \mu_2$ and $N_{ok}\wedge m_{ok}$ admits the same asymptotic distribution if $a_1(\delta|\nu_o| - 2a_1)^{-1} < (\delta^2/8a_1 - 1)^{-1}$

$\Rightarrow \delta|\nu_o| - \delta^2/8 \quad > a_1$

For $|\nu_o| \geq \delta$, $\delta^2 > 2a_1 \Rightarrow \delta|\nu_o| > 2a_1$. Thus N_{ok} is asymptotically normal if $\delta^2 > 2a_1$ for $|\nu_o| \geq \delta$. Now $N_{ok}\wedge m_{ok}$ is asymptotically normal if $\delta^2 > \frac{8}{7}a_1$ which holds when $\delta^2 > 8a_1$. Thus the extra condition needed for $N_{ok}\wedge m_{ok}$ to the asymptotically normal distribution when $|\nu_o| \geq \delta$, is the condition under which $k^{-1} m_{ok}$ admits a finite positive limit as $k\to\infty$, i.e., when $\delta^2 > 8a_1$.

<u>Remark 4.10</u> : The next chapter gives an asymptotic study of ASN using the technique of Lai (1975). This kind of study is not carried out here as $Ek^{-1}N_k$ as well as $E\,k^{-1}N_{1k}$ is infinite $\forall\, k$.

Following Mukhopadhyay (1983), the present problem may be treated as a k-hypotheses testing problem with a target of attaining the given probability of correct selection P^*. The sequential procedure suggested here is an extension of an invariant SPRT to more than two hypotheses.

The basic idea of choosing one out of $k(k \geq 2)$ many hypotheses using likelihoods, goes back to Wald (1947, Chapter 10). Sobel and Wald (1949) used a combination of two SPRTs to decide one out of three hypotheses concerning the unknown mean of a normal distribution. Meilijson (1969) followed the reasoning of Sobel and Wald (1949) for choosing one out of $k(k \geq 2)$ decisions (regarding the unknown mean of a normal population) but applied it to Anderson's (1960) modification of SPRT. The form of the procedure is similar to that of Paulson (1963) but requires less number of observations (than that of Paulson (1963)). Armitage (1950) extended the idea of Wald's SPRT to $k(k \geq 2)$ many hypotheses and gave interesting applications. Robbins (1970) made use of similar technique to define a general stopping time for estimating an integer mean of a normal distribution. Later Khan (1973) developed this idea, emphasising on its application to sequential distinguishability problem.

Recently Mukhopadhyay (1983) suggested a similar sequential procedure for selecting the normal population having the largest mean among k normal populations, when the common variance is known. This sequential procedure showed substantial saving in sample size when compared (asymptotically as $P^* \to 1$) with the corresponding fixed sample procedure. This fact encouraged the author to investigate the performance of a similar sequential procedure when the common variance σ^2 is unknown.

Here the problem is first reduced to a k-hypotheses testing problem by invariance technique and then an extension of invariant SPRT to k hypotheses is used. Asymptotic distribution of the stopping time of the proposed procedure is obtained as $P^* \to 1$. The limiting distribution is precisely the maximum of (k-1) normal variates whose joint distribution is (k-1) variate normal. Asymptotic expression of the ASN is also obtained following the technique of Lai (1975). This asymptotic expression of the ASN shows substantial saving in sample size when compared with the corresponding fixed sample procedure. We also compare our procedure with the two stage procedure of Bechhofer, Dunnet and Sobel (1954) (henceforth will be denoted by BDS). As the formulation of the indifference zone in BDS differs from the present one, the comparison has been made as follows :

$$\text{Let } \Omega^{B}_{\delta^*} = \left\{ (\underset{\sim}{\mu}, \sigma) \varepsilon\, R^k \times R^+ : \mu_{[k]} - \mu_{[k-1]} \geq \delta^* \right\} \qquad ..(I2)$$

denote the parameter space considered by BDS. Suppose σ is bounded above by a known constant 'a' Then our indifference zone is contained in that of BDS if '$a\delta = \delta^*$' Thus in this case our procedure provides more protection i.e. it guarantees $P_{(\underset{\sim}{\mu},\sigma)}$ (Correct Selection) $\geq P^*$, (approximately for large P^* vide Theorem 5.2) for a larger set of parameters. If in addition $a^{-2}\sigma^2 \geq e(\delta,k,p^*)$ (as given in (5.4.1)) then our ASN (asymptotically as $P^* \to 1$) is also smaller than that of BDS.

This comparison is discussed in detail in Section 5.4 while in Section 5.3 asymptotic behaviour of the proposed procedure is studied. In the beginning, Section 5.2 deals with the formulation of the problem and the statement of the procedure with some of its properties.

This chapter is a revised version of Ray Chaudhuri (1986).

5.2 Formulation of the Problem and Statement of the Procedure

Let $X_{i1}, X_{i2}, \ldots$ denote a sequence of iid random variables from $\pi_i, i = 1,2,\ldots k$. The samples from different populations are assumed to be independent. For the parameter space Ω_δ (as in I1) the configuration $\frac{\mu_{[1]}}{\sigma} = \frac{\mu_{[2]}}{\sigma} = \ldots = \frac{\mu_{[k-1]}}{\sigma} = \frac{\mu_{[k]}}{\sigma} - \delta$ is considered as a least favourable configuration (LFC).

Here a sequential procedure is proposed which assures the probability of correct selection P^* under the LFC. Since for any reasonable procedure it is natural to expect to perform in a still better way (than the LFC) for other configurations, the problem is formulated with an aim to attain P^* under LFC. The problem is looked into as a k-hypotheses testing problem. The hypotheses are as follows,

$$H_i : \mu_{[k]} = \mu_i, \ \sigma^{-1}\mu_{[1]} = \sigma^{-1}\mu_{[2]} = \ldots = \sigma^{-1}\mu_{[k-1]} = \sigma^{-1}\mu_{[k]} - \delta$$

for $i = 1,2,\ldots k$.

This can be re-written as

$H_1 : (\theta_j = -\delta \ \forall \ j = 2,3,\ldots k)$

$H_i : (\theta_i = \delta, \ \theta_l = 0 \ \forall \ l = 2,\ldots,k$ and $l \neq i)$ for $i = 2,3,\ldots k$

where $\underset{\sim}{\theta} = (\theta_2, \theta_3, \ldots \theta_k) = (\sigma^{-1}(\mu_2-\mu_1), \sigma^{-1}(\mu_3-\mu_1), \ldots \sigma^{-1}(\mu_k-\mu_1))$ ..(5.2.1)

Let us call $\Omega_{H_i}(\delta) = \left\{(\underset{\sim}{\mu},\sigma) \in \Omega_\delta : \theta = \theta_{H_i}\right\}$ where θ_{H_i} is the value of θ under H_i, for $i = 1,2,\ldots k$. ..(5.2.2)

$$\text{Let } S_n = \sigma^{-1}\left\{\sum_{i=1}^{k}\sum_{m=1}^{n}(X_{im}-\bar{X}_{in})^2\right\}^{1/2}$$

$$\bar{X}_{in} = n^{-1}\sum_{m=1}^{n} X_{im} \qquad ..(5.2.3)$$

$$U_n' = \left(\frac{\bar{X}_{jn}-\bar{X}_{in}}{S_n}\sigma^{-1}\left(\frac{k(n-1)n}{2}\right)^{1/2}, \ j = 2,3,\ldots k\right).$$

Under the transformation $X \rightarrow aX+b$, $0 < a < \infty$, $-\infty < b < \infty$. U_n' is maximal invariant and the invariant sufficiency follows from the basic theorem of Hall et al (1965). The distribution of U_n is noncentral multivariate t as given below (vide Kshirsagar (1961)).

$$f(U_n) = e^{-\frac{1}{2}\theta_n' R^{-1}\theta_n}(\nu\pi)^{-\frac{m_0}{2}}|R|^{-1/2}\left(\Gamma\left(\frac{\nu}{2}\right)\right)^{-1}(1+\nu^{-1}U_n' R^{-1} U_n)^{\frac{1}{2}(\nu+m_0)}$$

$$\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{\nu+m_0+m}{2}\right)}{m!}\left(\left(\frac{2}{\nu(1+\nu^{-1}U_n' R^{-1} U_n)}\right)^{1/2}\theta_n' R^{-1} U_n\right)^m$$

where $\theta_n' = \left(\frac{n}{2}\right)^{1/2}\underset{\sim}{\theta}$

$R^{-1} = 2(I_{k-1} - k^{-1}E_{k-1})$ where E_p is a

pxp matrix whose all elements are one.

$\nu = k(n-1)$

$m = k-1$ ...(5.2.4)

The stopping rule considered here is

$$\left.\begin{array}{l} N = \inf\left\{ n \geq 1 : \sup\limits_{j: j\neq i} \dfrac{f_{j,n}}{f_{i,n}} \leq \left(\dfrac{1-P^*}{k-1}\right) \text{ for some } i, \text{ where } i,j = 1,2,\ldots k \right\} \\ \quad = \infty \qquad\qquad \text{otherwise} \end{array}\right\} \quad \ldots(5.2.5)$$

Here, $f_{i,n}$ denotes the density of U_n under H_i for $i = 1,2,\ldots k$.

The termination of N w.p.1 is guaranteed by the following theorem.

<u>Theorem 5.1</u> : $P_{(\underset{\sim}{\mu},\sigma)}(N < \infty) = 1$ where $(\underset{\sim}{\mu},\sigma) \in \Omega_{\hat{\delta}}$.

<u>Proof of Theorem 5.1</u> : For any $(\underset{\sim}{\mu},\sigma) \in \Omega_{\delta}$, $\mu_{[k]} = \mu_i$ for some $i = 1,2,\ldots k$. Let us first suppose that $\mu_{[k]} = \mu_1$.

Now, $N = \inf\left\{ n : \sup\limits_{j: j\neq 1} \dfrac{f_{j,n}}{f_{i,n}} \leq \left(\dfrac{1-P^*}{k-1}\right) \text{ for some } i, \text{ where } i,j = 1,2\ldots k \right\}$

$$\sup_{j: j\neq i} \frac{f_{j,n}}{f_{i,n}} = \sup_{j: j\neq i} \frac{\sum\limits_{m=0}^{\infty} (m!)^{-1}\, \Gamma\left(\frac{\nu+m_0+m}{2}\right)(\sqrt{2n}\, x_{j,n})^m}{\sum\limits_{m=0}^{\infty} (m!)^{-1}\, \Gamma\left(\frac{\nu+m_0+m}{2}\right)(\sqrt{2n}\, x_{i,n})^m}$$

$$\text{where } x_{i,n} = \frac{\theta'_{n,i}\, \bar{R}^{-1} U_n}{(\nu + U_n' \bar{R}^{-1} U_n)^{1/2}}\, n^{-1/2} \qquad \ldots(5.2.6)$$

with $\theta_{n,i}$ denoting θ_n under H_i, $i = 1,2,\ldots k$.

$$= \sup_{j: j\neq i} \frac{\int_0^{\infty} e^{-nt^2/2 + x_{j,n} nt}\, t^{(kn-2)}\, dt}{\int_0^{\infty} e^{-nt^2/2 + x_{i,n} nt}\, t^{(kn-2)}\, dt}$$

$$= \frac{\int_0^\infty e^{-nt^2/2 + nt(\sup_{j: j\neq i} x_{j,n})} t^{(kn-2)} dt}{\int_0^\infty e^{-nt^2/2 + nt(x_{i,n})} t^{(kn-2)} dt}$$

$$= \frac{J_{n,2}(\sup_{j: j\neq i} x_{j,n})}{J_{n,2}(x_{i,n})} \text{ where } J_{n,2}(x) = \int_0^\infty e^{-nt^2/2 + ntx} t^{kn-2} dt$$

for $n \geq 2$, $k \geq 2$.

Observe $|x_{i,n}| \leq (\delta^2(\frac{k-1}{k}))^{1/2}$ $\forall$ $i = 1,2,\ldots k$.

Using the approximation formula 3.3.14 of Wijsman (1979) one gets $\forall$ $n \geq 2$,

$$|\ln J_{n,2}(x_{i,n}) - n\Psi(x_{i,n})| < c, \quad c \text{ is a constant and } k \text{ fixed} \quad \ldots(5.2.7)$$

$$\text{Here, } \Psi(x) = \beta(x) - \frac{k}{2} - \frac{1}{2n}\ln(n) \quad \ldots(5.2.8)$$

$$\beta(x) = \frac{1}{2} x\alpha(x) - k\ln\alpha(x) \quad \ldots(5.2.9)$$

$$\alpha(x) = \frac{1}{2}(x + (x^2+4k)^{1/2}) \quad \ldots(5.2.10)$$

$$\left.\begin{aligned} \text{Let } M &= \inf\left\{n : n(\beta(x_{i,n}) - \beta(\sup_{j: j\neq i} x_{j,n})) \geq \ln\left(\frac{1-P^*}{k-1}\right) - 2c \text{ for some } i,\ ij = 1,2,\ldots k.\right\} \\ &= \infty \quad \text{otherwise} \end{aligned}\right\} \ldots(5.2.11)$$

$$\left.\begin{aligned} M' &= \inf\left\{n : n(\beta(x_{i,n}) - \beta(\sup_{j: j\neq i} x_{j,n})) \geq \ln\left(\frac{1-P^*}{k-1}\right) + 2c \text{ for some } k,\ ij = 1,2,\ldots k\right\} \\ &= \infty \quad \text{otherwise} \end{aligned}\right\} \ldots(5.2.12)$$

Then $M \leq N \leq M'$ ..(5.2.13)

Thus for Theorem 5.1, it is enough to show

$$P_{(\underset{\sim}{\mu},\sigma)}(M' < \infty) = 1 \qquad ..(5.2.14)$$

Now for, $\mu_{[k]} = \mu_1$, let $\underset{\sim}{\theta} = (-\delta_2, -\delta_3, \ldots, -\delta_k)$ where $\delta_j \geq \delta$

$\forall\ j = 2,3,\ldots k$ as $-\delta_j = \dfrac{\mu_j - \mu_1}{\sigma}$ and $(\underset{\sim}{\mu},\sigma) \in \Omega_\delta$.

Thus $n^{-1/2} U_n' \longrightarrow \Sigma^{-1/2}(-\delta_2, -\delta_3, \ldots, -\delta_k)$ a.s. as $n \longrightarrow \infty$.

Also $\Sigma^{-1}(-\delta, -\delta, \ldots, -\delta)\, R^{-1}(-\delta_2, -\delta_3, \ldots, -\delta_k)' = k^{-1}\delta \sum_{j=2}^{k} \delta_j$

$$(\delta \Sigma^{-1}) e_j' R^{-1}(-\delta_2, -\delta_3, \ldots, -\delta_k) = (k^{-1}\delta \sum_{j=2}^{k} \delta_j) - \delta_j \delta \quad \forall j = 2,\ldots k$$

where e_j is a $(k-1)$ dimensional vector whose $(j-1)^{th}$ element is one and all other elements are zero.

Thus $$x_{1,n} \to \frac{k^{-1}\delta \sum_{j=2}^{k} \delta_j}{(k + \sum_{j=2}^{k} \delta_j^2 - k^{-1}(\Sigma\delta_j)^2)^{1/2}} \text{ a.s.}$$

$$x_{j,n} \to \frac{(k^{-1}\delta \sum_{j=2}^{k} \delta_j) - \delta_j \delta}{(k + \sum_{j=2}^{k} \delta_j^2 - k^{-1}(\Sigma\delta_j)^2)^{1/2}} \text{ a.s. } \forall\ j = 2,3,\ldots k.$$

Thus for $\mu_{[k]} = \mu_1$, $\beta(x_{1,n}) - \beta(\sup_{j:j\neq 1} x_{j,n}) \longrightarrow c_\theta$ a.s. as $n \to \infty$, ..(5.2.15)

where c_θ is a positive constant depending on θ.

$$\Rightarrow \max_{1\leq i\leq k} n \left\{\beta(x_{i,n}) - \beta(\sup_{j:j\neq i} x_{j,n})\right\} \longrightarrow \infty \text{ a.s. which proves (5.2.14)}$$

If $\mu_{[k]} = \mu_j$, $j = 2,3,\ldots,k$, the proof of Theorem 5.1 follows by exactly similar reasoning. □

Now we define procedure R as follows : Take N (as defined in 5.2.5) number of observations from each population and select π_i as having the largest mean if H_i is accepted, for $i = 1,2,\ldots k$.

About the probability of correct selection (CS) of the procedure R we have the following theorem.

Theorem 5.2 : $\forall(\underset{\sim}{\mu},\sigma) \varepsilon \Omega_\delta$, $P_{(\underset{\sim}{\mu},\sigma)}$ (correct selection) $\geq 2P^* - 1$ ...(5.2.16)

Moreover for the slippage configuration i.e for $(\underset{\sim}{\mu},\sigma)$ s.t.

$\sigma^{-1}(\mu_{[1]},\mu_{[2]},\ldots,\mu_{[k]}) = (t,t,\ldots t,t+\delta^*)$ where $\delta^* \geq \delta$ and $t \varepsilon R$,

$$P_{(\underset{\sim}{\mu},\sigma)} \text{ (correct selection) } \geq P^* \qquad \ldots(5.2.17)$$

Proof of Theorem 5.2 : Let $(\underset{\sim}{\mu},\sigma) \varepsilon \Omega_\delta$ be such that μ_1 is the largest among all other μ_i's, (μ_i denotes the ith coordinate of $\underset{\sim}{\mu}$).

$$\text{Call } \theta_{(\underset{\sim}{\mu},\sigma)} = 2^{-1/2}\,\sigma^{-1}(\mu_2-\mu_1,\ \mu_3-\mu_1,\ldots,\ \mu_k-\mu_1)' \qquad \ldots(5.2.18)$$

$$= 2^{-1/2}(-\delta_2,\ -\delta_3,\ldots,-\delta_k)' \qquad \ldots(5.2.19)$$

where $\delta_j \geq \delta$ $\forall$ $j = 2,3,\ldots k$.

Now $\theta'_{(\underset{\sim}{\mu},\sigma)} R^{-1} \theta_{(\underset{\sim}{\mu},\sigma)} \geq \theta'_{(\underset{\sim}{\mu},\sigma)}\ \theta_{(\underset{\sim}{\mu},\sigma)}$. smallest eigen value of R^{-1}.

(R^{-1} as given in (5.2.4)).

$$= \theta'_{(\underset{\sim}{\mu},\sigma)}\ \theta_{(\underset{\sim}{\mu},\sigma)} 2k^{-1}$$

$$= k^{-1} \sum_{=2}^{k} \delta_j^2 \geq k^{-1}(k-1)\delta^2 \qquad .(5.2.20)$$

Thus for given $\theta_{(\underset{\sim}{\mu},\sigma)}$ one can find

$$\theta_{(\underset{\sim}{\mu},\sigma)^{s}_{jo}} = (0,0,\ldots,0,\ 2^{-1/2}\Delta\ ,0,\ldots,0)' \ \forall\ jo = 2,\ldots k. \qquad ..(5.2.21)$$

with $2^{-1/2}\Delta$ at the $(jo-1)^{th}$ position where

$$\Delta = (k(k-1)^{-1}\ \theta'_{(\underset{\sim}{\mu},\sigma)}\ \bar{R}^{1}\ \theta_{(\underset{\sim}{\mu},\sigma)})^{1/2} \qquad ..(5\ 2.22)$$

Then $\theta_{(\underset{\sim}{\mu},\sigma)}'\ \bar{R}^{1}\ \theta_{(\underset{\sim}{\mu},\sigma)} = \theta'_{(\underset{\sim}{\mu},\sigma)^{s}_{jo}}\ \bar{R}^{1}\ \theta_{(\underset{\sim}{\mu},\sigma)^{s}_{jo}}\ \forall jo = 2,3,\ldots k$ ...(5.2.23)

Also for each $i_o \neq 1$, $\theta_{(\underset{\sim}{\mu},\sigma)_{jo}} = 2^{-1/2}(\delta_2^{*},\ \delta_3^{*},\ldots,\delta_k^{*})'$

where $\delta_{jo}^{*} = \delta_{jo}$ and $\delta_j^{*} = \delta_{jo} - \delta_j\ \ \forall j = 2,\ldots k$ and $j \neq jo$ ..(5.2.24)

Here also $\theta'_{(\underset{\sim}{\mu},\sigma)_{jo}}\ \bar{R}^{1}\ \theta_{(\underset{\sim}{\mu},\sigma)_{jo}} = \theta'_{(\underset{\sim}{\mu},\sigma)}\ \bar{R}^{1}\ \theta_{(\underset{\sim}{\mu},\sigma)}$ ...(5.2.25)

Let $S_{jo} = \left\{\text{Selection of } \pi_{jo}\right\}$, $jo = 2,3,\ldots k$ ...(5.2.26)

For proving (5.2.16) it is enough to prove

$$P_{(\underset{\sim}{\mu},\sigma)}(S_{jo}) \leq \frac{2(1-P^{*})}{k-1} \qquad \forall jo = 2,3,\ldots,k. \qquad ..(5.2.27)$$

$$P_{(\underset{\sim}{\mu},\sigma)}(S_{jo}) = \sum_{n=1}^{\infty} \int_{\{S_{jo},\ N=n\}} f_{(\underset{\sim}{\mu},\sigma)}(U_1,U_2,\ldots U_n)\,d(U_1,U_2,\ldots,U_n)$$

where $f_{(\underset{\sim}{\mu},\sigma)}(U_1,\ldots U_n)$ denotes the joint density of $(U_1,U_2,\ldots U_n)$ when $(\underset{\sim}{\mu},\sigma)$ is the true parameter.

$$= \sum_{n=1}^{\infty} \int_{\{S_{jo},N=n\}} \frac{f_{(\underset{\sim}{\mu},\sigma)}(U_1,\ldots U_n)}{f_{(\underset{\sim}{\mu},\sigma)_{jo}}(U_1,\ldots U_n) \vee f_{(\underset{\sim}{\mu},\sigma)^{s}_{jo}}(U_1,U_2,\ldots U_n)}\ f_{(\underset{\sim}{\mu},\sigma)_{jo}}(U_1,\ldots U_n) \vee f_{(\underset{\sim}{\mu},\sigma)^{s}_{jo}}(U_1,U_2,\ldots U_n)\ d(U_1,U_2,\ldots U_n) \qquad \ldots(5.2.28)$$

Now, $$\frac{f_{(\underset{\sim}{\mu},\sigma)}(U_1,U_2,\ldots U_n)}{f_{(\underset{\sim}{\mu},\sigma)_{jo}}(U_1,U_2,\ldots U_n) \vee f_{(\underset{\sim}{\mu},\sigma)^s_{jo}}(U_1,U_2,\ldots U_n)}$$

$$= \frac{f_{(\underset{\sim}{\mu},\sigma)}(U_n)}{f_{(\underset{\sim}{\mu},\sigma)_{jo}}(U_n) \vee f_{(\underset{\sim}{\mu},\sigma)^s_{jo}}(U_n)} \quad \text{(as } U_n \text{ is invariantly sufficient)}.$$

$$= \frac{\int_0^\infty e^{-nt^2/2+ntx_{(\underset{\sim}{\mu},\sigma),n}}\, t^{kn-2}\, dt}{\int_0^\infty e^{-nt^2/2+nt(x_{(\underset{\sim}{\mu},\sigma)_{jo,n}} \vee x_{(\underset{\sim}{\mu},\sigma)^s_{jo,n}})}\, t^{kn-2}\, dt} \qquad \ldots(5.2.29)$$

where $$x_{(\underset{\sim}{\mu},\sigma),n} = \frac{\theta'_{(\underset{\sim}{\mu},\sigma)}\bar{R}^{1}U_n}{(\nu+U_n'\bar{R}^{1}U_n)^{1/2}} \qquad .(5.2.30)$$

$$x_{(\underset{\sim}{\mu},\sigma)_{jo,n}} = \frac{\theta'_{(\underset{\sim}{\mu},\sigma)_{jo}}\bar{R}^{1}U_n}{(\nu+U_n'\bar{R}^{1}U_n)^{1/2}} \qquad ..(5.2.31)$$

$$x_{(\underset{\sim}{\mu},\sigma)^s_{jo,n}} = \frac{\theta'_{(\underset{\sim}{\mu},\sigma)^s_{jo}}\bar{R}^{1}U_n}{(\nu+U_n'\bar{R}^{1}U_n)^{1/2}} \qquad \ldots(5.2.32)$$

Now $$U_n'\bar{R}^{1}(\theta_{(\underset{\sim}{\mu},\sigma)}-\theta_{(\underset{\sim}{\mu},\sigma)_{jo}}) = U_n'\bar{R}^{1}\,\bar{2}^{1/2}(-\delta_{jo},-\delta_{jo},\ldots -2\delta_{jo}, -\delta_{jo}\ldots-\delta_{jo})$$

$$= -2^{1/2}U_{jo,n}\delta_{jo} \qquad ..(5.2.33)$$

where $U_{jo,n}$ is the j_o^{th} coordinate of U_n.

Also on $\left\{S_{j_o},\ N=n\right\}$ $\frac{f_{1,n}}{f_{j_o,n}} \le \frac{1-p^*}{k-1}$ where $f_{i,n}$ is as in (5.2.5).

$$\Rightarrow \; U_n' R^{-1}(\theta_{n,1}-\theta_{n,j_0}) = -2^{1/2} U_{j_0,n}\delta < 0 \qquad ..(5.2.34)$$

(where $\theta_{n,1}$ is as in 5.2.6)

$$\Rightarrow \; U_{j_0,n} > 0 \qquad .(5.2.35)$$

$$\Rightarrow \; -2^{1/2} U_{j_0,n}\delta_{j_0} \leq -2^{1/2} U_{j_0,n}\delta \text{ as } \delta_{j_0} \geq \delta \qquad ..(5.2.36)$$

$$\Rightarrow \; x_{(\underset{\sim}{\mu},\sigma),n} - x_{(\underset{\sim}{\mu},\sigma)_{j_0},n} \leq x_{1,n} - x_{j_0,n} \leq 0 \qquad ...(5.2.37)$$

by (5.2.30), (5.2.31), (5.2.33), (5.2.34), (5.2.36) and (5.2.6).

Now suppose $x_{(\underset{\sim}{\mu},\sigma)_{j_0},n} \geq x_{(\underset{\sim}{\mu},\sigma)^s_{j_0},n}$ then

(5.2.29), (5.2.37) and the fact that $x_{(\underset{\sim}{\mu},\sigma)^s_{j_0},n} \geq x_{j_0,n}$ on $\{S_{j_0}, N=n\}$

together with Lemma 5.1 (given below) imply the

term on (5.2.29) $\leq \frac{1-P^*}{k-1}$ ..(5.2.38)

If $x_{(\underset{\sim}{\mu},\sigma)_{j_0},n} < x_{(\underset{\sim}{\mu},\sigma)^s_{j_0},n}$ then

$$(5.2.37) \Rightarrow \; x_{(\underset{\sim}{\mu},\sigma),n} - x_{(\underset{\sim}{\mu},\sigma)^s_{j_0},n} \leq x_{1,n} - x_{j_0,n} \leq 0 \qquad ...(5.2.39)$$

Thus, using Lemma 5.1 as in the earlier case we have the

term on (5.2.29) $\leq \frac{1-P^*}{k-1}$ ...(5.2.40)

Thus (5.2.38) and (5.2.40) and the fact that

$$\sum_{n=1}^{\infty} \int_{\{S_{j_0},\, N=n\}} \left(f_{(\underset{\sim}{\mu},\sigma)_{j_0}}(U_1,U_2,\ldots U_n) \vee f_{(\underset{\sim}{\mu},\sigma)^s_{j_0}}(U_1,U_2,\ldots U_n) \; d(U_1,U_2,\ldots U_n)\right) \leq 2$$

imply (5.2.27)

Thus (5.2.16) holds for $(\underset{\sim}{\mu},\sigma)$ s.t. μ_1 is the largest coordinate.

For other cases also, the proof follows by similar reasoning.

For the slippage configuration, $\theta(\underset{\sim}{\mu},\sigma)_{j_0} = \theta(\underset{\sim}{\mu},\sigma)^s_{j_0}$ and thus the proof of (5.2.17) follows in a much simpler way.

Remark 5.1. Although the above theorem says $P_{(\underset{\sim}{\mu},\sigma)}(C.S.) \geq 2P^*-1$ $\forall(\underset{\sim}{\mu},\sigma)\varepsilon\Omega_\delta$, the author believes that $P_{(\underset{\sim}{\mu},\sigma)}(C.S.) \geq P^*$. Moreover, if the boundary constant $\frac{1-P^*}{k-1}$ is replaced by the more conservative value $\frac{(1-P)}{2(k-1)}$ the same proof shows $P_{(\underset{\sim}{\mu},\sigma)}(C.S.) \geq P^*$ $\forall(\underset{\sim}{\mu},\sigma)\varepsilon\Omega_\delta$, and for the slippage configuration $P_{(\underset{\sim}{\mu},\sigma)}(C.S.) \geq 2^{-1}(P^*+1) > P^*$. This fact makes the author feel that the use of $\frac{1-P^*}{2(k-1)}$ in place of $\frac{1-P^*}{k-1}$ gives more control to the error than actually needed, requiring a larger sample size. However, for large P^*, this change in the boundary constant becomes negligible as

$$\ln\left(\frac{1-P^*}{2(k-1)}\right)/\ln\left(\frac{1-P^*}{k-1}\right) \to 1 \text{ as } P^* \to 1.$$

Thus the asymptotic distribution of N and the asymptotic expression of ASN remain unchanged, even if we use $\frac{(1-P^*)}{2(k-1)}$ in place of $(\frac{1-P^*}{k-1})$.

Remark 5.2 : For the known σ case (considered by Mukhopadhyay (1983)) one can have a stronger result than Theorem 5.2 i.e, $\forall(\underset{\sim}{\mu},\sigma)\in\Omega_\delta$, $P_{(\underset{\sim}{\mu},\sigma)}(C.S.) \geq P^*$.

Lemma 5.1 : For $x_1-x_0 \geq y_1-y_0 \geq 0$ and $x_1 \geq y_1$

$$\frac{\int_0^\infty e^{ntx_1} g(t)dt}{\int_0^\infty e^{nty_1} g(t)dt} \geq \frac{\int_0^\infty e^{ntx_0} g(t)dt}{\int_0^\infty e^{nty_0} g(t)dt} \qquad (5.2.41)$$

where $g(t)$ is a real valued continuous function s.t.

$$0 < \int_0^\infty e^{ntx} g(t)\, dt < \infty \quad \forall x \in R \text{ and } g(t) \geq 0 \text{ on } R^+$$

Proof of Lemma 5.1 : Let (for $i = 0,1$) $h_i(t) = e^{nty_i} g(t) \left(\int_0^\infty e^{nty_i} g(t)\, dt \right)^{-1}$

$$\text{for } 0 < t < \infty$$

$$= 0 \qquad \text{otherwise}$$

Then $h_1(t)/h_o(t)$ is a non-decreasing function of t on R^+ and the fact $\exp(ntx_1 - nty_1)$ is a non-decreasing function of t on R^+ implies,

$$\text{LHS of (5.2.41)} = \int_0^\infty \exp(ntx_1 - nty_1)\, h_1(t)\, dt$$

$$\geq \int_0^\infty \exp(ntx_1 - nty_1)\, h_o(t)\, dt \quad \text{(As in page 74 of Lehmann(1959))}$$

$$\geq \int_0^\infty \exp(ntx_o - nty_o)\, h_o(t)\, dt$$

$$= \text{RHS of (5.2.41)}$$

5.3 Asymptotic Study of N

This section is devoted to study the asymptotic behaviour of N. Firstly the asymptotic distribution of N is obtained (using Theorem 4.1 of Chapter 4) and then the asymptotic expression (based on the ideas of Lai (1975)) for $E_{(\underset{\sim}{\mu},\sigma)}(N)$ is given (for $(\underset{\sim}{\mu},\sigma) \in \Omega_\delta$) as $P^* \to 1$.

Here the symbol '$\Rightarrow$' will be used to denote 'converges in distribution' as well as ''implies that'' at suitable places.

<u>Theorem 5.3</u> : Under H_{i_o}, $1 \leq i_o \leq k$, i_o fixed, (a) and (b) of Theorem 4.1 are satisfied with $r = -\ln\left(\frac{1-P^*}{k-1}\right)$; $\tau_r = N$;

$$W_n^r = W_n = n\left\{ \beta(x_{i_o}, n) - \beta(\sup_{j:j\neq i_o} x_j, n) \right\} = W_{n,i_o} \text{ (say)}$$

$$b_r = r = -\ln\left(\frac{1-P^*}{k-1}\right)$$

$$\mu_r = \mu = \beta_1(\delta)$$

where $\beta_1(\delta) = \beta(\delta^2(\frac{k-1}{k})(k+\delta^2(\frac{k-1}{k}))^{-1/2}) - \beta(-\frac{\delta^2}{k}(k+\delta^2(\frac{k-1}{k}))^{-1/2})$

with $\beta(x)$ as in (5.2.9).

<u>Proof of Theorem 5.3</u> : As noted in (5.2.13), that $M \leq N \leq M'$ (where M and M' are given in (5.2.11) and (5.2.12) respectively) the theorem follows if the same is true with $\tau_r = M$ and $\tau_r = M'$.

Let us first concentrate on $M = \bigwedge_{i=1}^{k} M_i$ where

$$\left.\begin{aligned} M_i &= \inf\left\{n : n(\beta(x_{i,n}) - \beta(\sup_{j:j\neq i} x_{j,n})) \geq \ln\left(\frac{1-P^*}{k-1}\right) - 2c\right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\} \qquad \ldots(5.3.1)$$

Look at
$$\left.\begin{aligned} M_{i_o} &= \inf\left\{n : W_{n,i_o} + 2c \geq r\right\} \\ &= \infty \qquad \text{otherwise} \end{aligned}\right\}$$

Clearly $M_{i_o} \rightarrow \infty$ as $r \rightarrow \infty$ and $P_{H_{i_o}}(M_{i_o} < \infty) = 1$, by Theorem 5.1.

Now $W_{M_{i_o},i_o} + 2c \geq r > W_{M_{i_o}-1,i_o} + 2c$

$$\Rightarrow M_{i_o}^{-1}(W_{M_{i_o},i_o}+2c) \geq M_{i_o}^{-1} r > M_{i_o}^{-1}(W_{M_{i_o}-1,i_o}+2c)$$

$$\Rightarrow r^{-1}\beta_1(\delta)M_{i_o} \rightarrow 1 \text{ a.s. as } n^{-1}W_{n,i_o} \rightarrow \beta_1(\delta) \text{ a.s.}$$

as $n \rightarrow \infty$ under H_{i_o} and $\beta_1(\delta) > 0$.

Thus (A1) of Theorem 4.1 is satisfied with $\tau_r = M_{i_o}$

Since 'c' in the definition of M_i, is just a constant, it is enough to verify (A2) and (A3) with W_{n,i_o} (vide Remark 4.5 of Chapter 4).

Lemma 5.2 and Lemma 5.3 (given below ensure that W_{n,i_o} satisfies (A2) and (A3). Thus (a) and (b) hold for $\tau_r = M_{i_o}$ and hence the same is true with $\tau_r = M$ as

$$P_{H_{i_o}}(\tau_r = M_i) \rightarrow 0 \ \forall \ i \neq i_o \quad \text{as} \quad r \rightarrow \infty.$$

The result with $\tau_r = M'$ follows in exactly similar lines. Thus the proof of Theorem 5.3 is complete except for Lemma 5.2 and Lemma 5.3 which are given below. □

<u>Lemma 5.2</u> : Under $H_{i_o}, 1 \leq i_o \leq k$,

$$n^{-1/2}(W_{n,i_o} - n\mu) = n^{1/2}(\beta(x_{i_o,n}) - \beta(\sup_{j:j\neq i_o} x_{j,n}) - \beta_1(\delta))$$

is asymptotically (as $n \rightarrow \infty$) distributed as $F(.)$ where $F(.)$ is the distribution function of $Y = \min_{2\le j\le k} Y_j$ with

$$(Y_2, Y_3, \ldots, Y_k) \sim N_{k-1}(0, V) \text{ and } V = \frac{a^2}{2} I_{k-1} + \frac{1}{2}\left\{(1+\delta^2/2k)(b+\overline{k-1}\ a)^2 + a(b+\overline{k-1}\ a) + ab\right\} E_{k-1}$$

where a, b are given in (5.3.7).

Proof of Lemma 5.2 : Let us first consider $i_0 = 1$.

$$\text{Then } W_{n,1} = n(\beta(x_{1,n}) - \beta(\sup_{j:j\neq 1} x_{j,n}))$$

$$= \min_{j:j\neq 1} n(\beta(x_{1,n}) - \beta(x_{j,n})).$$

To obtain the asymptotic joint distribution of $(\beta(x_{1,n}) - \beta(x_{j,n})$:

$j = 2, 3, \ldots, k)$

we proceed as follows :

$$\text{Under } H_1, n^{1/2}\left(\left(\frac{\overline{X}_{2n} - \overline{X}_{1n}}{\sigma}, \frac{\overline{X}_{3n} - \overline{X}_{1n}}{\sigma}, \ldots \frac{\overline{X}_{kn} - \overline{X}_{1n}}{\sigma}, \frac{T_n}{\sigma^2}\right) - (-\delta, -\delta, -\delta, 1)\right)$$

$$\Rightarrow N_k(0, V_1) \text{ where}$$

$$V_1 = \begin{pmatrix} E_{k-1} + I_{k-1} & \underset{\sim}{0}_{k-1} \\ \underset{\sim}{0}'_{k-1} & 2/k \end{pmatrix} \qquad ..(5.3.2)$$

$\underset{\sim}{0}_p$ is a p dimensional vector whose all elements are zero and $T_n = \dfrac{\sum_{i=1}^{k}\sum_{m=1}^{n}(X_{im} - \overline{X}_{in})^2}{k(n-1)}$.

$$\text{Let } Y_{j,n} = \frac{\overline{X}_{jn} - \overline{X}_{1n}}{(2T_n)^{1/2}}, \qquad (5.3.3)$$

then by Theorem 4.2.5 of Anderson (1972).

$$n^{1/2}\sum_{j=2}^{k} t_j(Y_{j,n}+\delta 2^{-1/2}) \Rightarrow N(0,t'V_2t) \quad \forall\, t \in R^{k-1} \text{ where}$$

$$V_2 = \tfrac{1}{2}(I_{k-1}+(1+\delta^2/2k)E_{k-1}) \tag{5.3.4}$$

$$\Rightarrow n^{1/2}((Y_{2,n},Y_{3,n},\ldots Y_{k,n})+2^{-1/2}(\delta,\delta,\ldots\delta))$$

$$\Rightarrow N_{k-1}(0,V_2).$$

Now, as defined earlier in (5.2.6), $x_{i,n} = \dfrac{\theta'_{n,i}\bar{R}^{-1}U_n}{n^{1/2}(\nu+U_n'\bar{R}^{-1}U_n)^{1/2}}$

$$= \frac{\theta_i'\bar{R}^{-1}Y_n}{((\frac{n-1}{n})k+Y_n\bar{R}^{-1}Y_n)^{1/2}} \quad \ldots(5.3.5)$$

as $Y_n = n^{-1/2}\,U_n$ and $\theta_i = n^{-1/2}\theta_{n,i}$

Define $$x^*_{i,n} = \frac{\theta_i\bar{R}^{-1}Y_n}{(k+Y_n\bar{R}^{-1}Y_n)^{1/2}} \quad \ldots(5.3.6)$$

Now, the limiting distribution of $(\beta(x_{1,n})-\beta(x_{j,n}):\ j = 2,\ldots k)$ is same as the limiting distribution of $(\beta(x^*_{1,n})-\beta(x^*_{j,n}):j=2,\ldots k)$ as $x_{i,n}-x^*_{i,n} = o(n^{-1/2})\ \forall\ i = 1,2,\ldots k$. Thus it is enough to get the limiting distribution of $(\beta(x^*_{1,n})-\beta(x^*_{j,n}):\ j = 2,3,\ldots k)$.

Now $x^*_{i,n}$ is a differentiable function of $(Y_{j,n}, j = 2,\ldots k)$ $\forall\ i = 1,2,\ldots k$ and $\beta(x)$ (vide (5.2.9)) is also a nice differentiable function of x. Thus by repeated application of Theorem 4.2.5 of Anderson (1972), one gets

$$n^{1/2}\sum_{j=2}^{k} c_j(\beta(x^*_{1,n})-\beta(x^*_{j,n})-\beta_1(\delta)) \Rightarrow N(0,c'Vc) \text{ where}$$

$$\left.\begin{aligned}
c' &= (c_2,c_3,\ldots c_k) \in R^{k-1}\\
V &= B_2 V_2 B_2 \text{ where}\\
B_2 &= a\,E_{k-1}+bI_{k-1} \text{ with}\\
a &= \delta 2^{1/2}(k+\delta^2(\tfrac{k-1}{k}))^{-3/2}\,((1+\delta^2/2k)\alpha_2(\delta)-\alpha_1(\delta)\\
b &= \delta 2^{1/2}(k+\delta^2(\tfrac{k-1}{k}))^{-1}\,(-\alpha_2(\delta))\\
\alpha_1(\delta) &= \alpha(\delta^2(\tfrac{k-1}{k})(k+\delta^2(\tfrac{k-1}{k}))^{-1/2})\\
\alpha_2(\delta) &= \alpha(-\delta^2/k(k+\delta^2(\tfrac{k-1}{k}))^{-1/2}) \text{ where}
\end{aligned}\right\} \quad \ldots(5.3.7)$$

$\alpha(x)$ is as defined in (5.2.9).

B_2 is clearly nonsingular and V_2 as defined in (5.3.4) is positive definite which implies that V is positive definite. Here

$$V = \frac{a^2}{2} I_{k-1}+ \frac{1}{2}\left\{(1+\delta^2/2k)(b+\overline{k-1}\ a)^2+a(b+\overline{k-1}\ a)+ab\right\} E_{k-1} \quad \ldots(5.3.8)$$

where a and b are as defined in (5.3.7).

Thus $n^{1/2}((\beta(x_{1,n})-\beta(x_{j,n})-\beta_1(\delta)):\ j=2,3,\ldots k) \Rightarrow N_{k-1}(0,V)$.

The proof for other valves of i_o, follows in exactly similar lines and one can find that the limiting distribution of $(n^{1/2}((\beta(x_{i_o,n})-\beta(x_{i,n})-\beta_1(\delta)),\ i = 1,2,\ldots k,\ i\neq i_o)$ under H_{i_o} is same for all $i_o = 1,2,\ldots k$. Thus the limiting distribution of

$$n^{-1/2}(W_{n,i_o}-n\beta_1(\delta)) = n^{1/2} \bigwedge_{\substack{i=1\\ i\neq i_o}}^{k}(\beta(x_{i_o,n})-\beta(x_{i,n}))-\beta_1(\delta))$$

under H_{i_o} is F. Thus the proof of Lemma 5.2 is complete. □

Lemma 5.3 : For given ε and η (both positive real numbers) $\exists\, n_o$ (large) and c_o (small) such that $\forall\, n \geq n_o$

$$P_{H_{i_o}}\left\{\left|\frac{W_{n,i_o}}{n}-\frac{W_{n',i_o}}{n'}\right| < \varepsilon\, n^{-1/2}\ \forall\, n' \quad |n-n'| < c_o n\right\} \geq 1-\eta \qquad \ldots(5.3.9)$$

Proof of Lemma 5.3 : First let us consider $i_o = 1$.

Note that $n^{1/2}\left|(\beta(x_{1,n})-\beta(x_{j,n}))-(\beta(x_{1,n'})-\beta(x_{j,n'}))\right| < \varepsilon$

$$\forall\, j = 2,3,\ldots k \text{ and } \forall\, n' \ |n-n'| < c_o n$$

$$\Rightarrow \left|\frac{W_{n,1}}{n}-\frac{W_{n',1}}{n}\right| < \varepsilon\, n^{-1/2}\ \forall\, n' : |n-n'| < c_o n.$$

Thus to prove (5.3.9) it is enough to show, for given ε and η $\exists\, n_o$ and c_o such that $\forall\, n \geq n_o$,

$$P\left\{n^{1/2}\left|(\beta(x_{1,n})-\beta(x_{j,n}))-(\beta(x_{1,n'})-\beta(x_{j,n'}))\right| < \varepsilon\ \forall_j = 2,\ldots k \quad \forall\, n' \quad |n-n'| < c_o n\right\} \geq 1-\eta \qquad ..(5.3.10)$$

For (5.3.10) it is sufficient to show, for given ε and η $\exists\, n_o$ and c_o such that, $\forall\, i = 1,2,\ldots k$, $\forall\, n \geq n_o$,

$$P\left\{n^{1/2}\left|\beta(x_{i,n})-\beta(x_{i,n'})\right| < \varepsilon\ \forall\, n' : |n-n'| < c_o n\right\} \geq 1-\eta \qquad \ldots(5.3.11)$$

Now $\beta(x_{i,n})-\beta(x_{i,n'}) = \alpha(Z_{in})(x_{i,n}-x_{i,n'})$ $\qquad \ldots(5.3.12)$

$\forall\, i = 1,2,\ldots k$, where $Z_{i,n}\ \varepsilon(x_{i,n} \wedge x_{i,n'}, x_{i,n} \vee x_{i,n'})$

Now for all $j = 2,3,\ldots k$, as $n \to \infty$ (under H_1)

$$\alpha(Z_{j,n}) \to \alpha_2(\delta) \text{ a.s.}$$

$$\text{and} \quad \alpha(Z_{1,n}) \to \alpha_1(\delta) \text{ a.s} \qquad \ldots(5.3.13)$$

where $\alpha_1(\delta)$ and $\alpha_2(\delta)$ are given in (5.3.7) and both are positive.

Also from (5.3.5) and (5.3.6) one can see $\forall\, i = 1,2,\ldots k$ that

$$n^{1/2}(x_{i,n} - x^*_{i,n}) \to 0 \text{ a.s. as } n \to \infty. \qquad \ldots(5.3.14)$$

Now by (5.3.12), (5.3.13), (5.3.14) it is enough to show for (5.3.11) that for given ε and η $\exists\, n_0$ and c_0 s.t.

$\forall\, i = 1,2,\ldots k$, and $\forall\, n \geq n_0$,

$$P\left\{n^{1/2}\left|x^*_{i,n} - x^*_{i,n'}\right| < \varepsilon \;\forall\, n' : \left|n-n'\right| < c_0 n\right\} \geq 1-\eta \qquad \ldots(5.3.15)$$

Now $x^*_{i,n}$ as defined in (5.3.5) can be looked into as

$$x^*_{i,n} = f_i(Y_{2,n}, Y_{3,n}, \ldots Y_{k,n}) \quad \forall\, i = 1,2,\ldots k.$$

where each f_i is a differentiable function of $(Y_{2,n} \ldots Y_{k,n})$. Here each $Y_{j,n}$ $\forall\, j = 2,\ldots k$, is again a differentiable function of $(\overline{X}_{j,n} - \overline{X}_{1,n})$ and T_n (T_n as in (5.3.2)). From Theorem 3 of Anscombe (1952) (which guarantees similar condition as (5.3.15) for T_n and $(\overline{X}_{j,n} - \overline{X}_{1,n})$ $\forall\, j = 2,3,\ldots,k$) and by repeated application of Taylor's Theorem of several variable one can show (5.3.15) holds. Thus (5.3.9) holds for $i_0 = 1$.

The proof for other values of i_0 follows along similar lines.

<u>Asymptotic Study For ASN</u>

As already noted in (5.2.15), we have

$$\forall\, (\underset{\sim}{\mu},\sigma) \in \Omega_\delta,\; \max_{1\leq i\leq k} \left(\beta(x_{i,n}) - \beta(\sup_{j: j\neq i} x_{j,n})\right) \to c_\theta\, (> 0) \text{ a.s.} \qquad \ldots(5.3.16)$$

$$\text{as } n \to \infty.$$

Moreover if $\mu_{[k]} = \mu_{i_o}$ then

$$\max_{1\le i\le k} \left(\beta(x_{i,n})-\beta(\sup_{j:j\ne i} x_{j,n})\right)-\left(\beta(x_{i_o,n})-\beta(\sup_{j:j\ne i_o} x_{j,n})\right) \rightarrow 0 \text{ a.s} \quad \ldots(5.3.17)$$

as $n \rightarrow \infty$

(5.3.16) and (5.3.17) play important role to prove that $r^{-1}M$ and $r^{-1}M'$ both converge to c_θ^{-1} a.s. as $r \rightarrow \infty$ (M, M' as in (5.2.11) and (5.2.12) respectively). This fact together with (5.2.13) imply $r^{-1}N \rightarrow c_\theta^{-1}$ a.s. as $r \rightarrow \infty$ which motivates the following theorem.

<u>Theorem 5.4</u> : $\forall\ (\underset{\sim}{\mu},\sigma) \in \Omega_\delta$, $E_{(\underset{\sim}{\mu},\sigma)}(r^{-1}N) \rightarrow c_\theta^{-1}$ as

$$r = -\ln\left(\frac{1-P^*}{k-1}\right) \rightarrow \infty$$

<u>Proof of Theorem 5.4</u> : Fix $(\underset{\sim}{\mu},\sigma) \in \Omega_\delta$, then $\mu_{[k]} = \mu_{i_o}$ for some $i_o = 1,2,\ldots k$.

$$\text{Consider } M_{i_o} = \inf\left\{n : W_{n,i_o} \ge r'\right\} \quad = \infty, \text{ otherwise}$$

where $W_{n,i_o} = n(\beta(x_{i_o,n})-\beta(\sup_{j:j\ne i_o} x_{j,n}))$ and $r' = r-2c$.

Then $W_{M_{i_o},i_o} \ge r' > W_{M_{i_o}-1,i_o}$

$$\Rightarrow M_{i_o}^{-1}\ W_{M_{i_o},i_o} \ge M_{i_o}^{-1}\ r' > M_{i_o}^{-1}\ W_{M_{i_o}-1,i_o} \quad \ldots(5.3.18)$$

$$\Rightarrow M_{i_o}^{-1} r' \longrightarrow c_\theta \text{ a.s. as } r \longrightarrow \infty \text{ (as } M_{i_o} \longrightarrow \infty \text{ as } r \longrightarrow \infty,$$

and both sides of (5.3.18) converge to c_θ a.s. as $r \longrightarrow \infty$ by (5.3.16)).

$$\Rightarrow r'^{-1} M_{i_o} \longrightarrow c_\theta^{-1} \text{ a.s.}$$

We now proceed as in the proof of Theorem 6 of Lai (1975)

By Fatou's lemma $\liminf_{r \to \infty} E_{(\underset{\sim}{\mu},\sigma)} (r^{-1} M_{i_o}) \geq c_\theta^{-1}$ ...(5.3.19)

For given any small $\varepsilon > 0$, define for $i = 1,2,\ldots k$

$$L_i = \sup\left\{ n \geq 1 : \left|\overline{X}_{in} - \mu_i\right| > \varepsilon \text{ or } \left|\overline{X^2_{in}} - \mu_i^2 - \sigma^2\right| > \varepsilon \right\}$$

where $\overline{X}_{in} = n^{-1} \sum_{j=1}^{n} X_{ij}$, $\overline{X^2_{in}} = n^{-1} \sum_{j=1}^{n} X_{ij}^2$.

Then $E_{(\underset{\sim}{\mu},\sigma)} (L_i) < \infty \ \forall\, i = 1,2,\ldots k$ (by Lemma 6 of Lai (1975) and the fact $E_{(\underset{\sim}{\mu},\sigma)} (X_{ij}{}^4) < \infty \ \forall\, j \geq 1$, and $i = 1,2,\ldots k$.)

Let $L = \max_{1 \leq i \leq k} L_i$. Then $E_{(\underset{\sim}{\mu},\sigma)} (L) < \infty$.

Write $n^{-1} W_{n,i_o} = g(\overline{X}_{1n}, \overline{X}_{2n}, \ldots \overline{X}_{kn}, \overline{X^2_{1n}}, \ldots \overline{X^2_{kn}})$

where g is a continuous function in each of its argument and

clearly $g(\mu_1, \mu_2, \ldots \mu_k, \mu_1^2 + \sigma^2, \ldots \mu_k^2 + \sigma^2) = c_\theta$

Define $\rho(\varepsilon) = \min\left\{ g(u_1, u_2, \ldots u_k, v_1, v_2, \ldots v_k) : |u_i - \mu_i| \leq \varepsilon \text{ and } |v_i - \mu_i^2 - \sigma^2| \leq \varepsilon \ \forall\, i = 1,2,\ldots k \right\}$

Thus $\rho(\varepsilon) \rightarrow c_\theta\ (> 0)$ as $\varepsilon \rightarrow 0$

Now $M_{i_o} \leq (L+1)\, 1_{(L+1 \geq M_{i_o})} + M_{i_o}\, 1_{(M_{i_o} > L+1)}$ (Here 1_S denotes the indicator function of S)

On $\{M_{i_o} > L+1\}$, $(M_{i_o}-1)\,\rho(\varepsilon) \leq W_{M_{i_o}-1,\, i_o} < r'$

$$\Rightarrow M_{i_o} - 1 < r'/\rho(\varepsilon) \quad \text{(for sufficiently small } \varepsilon,\ \rho(\varepsilon) > 0)$$

Thus $M_{i_o} \leq (L+1)\, 1_{(L+1 \geq M_{i_o})} + \left(\frac{r'}{\rho(\varepsilon)} + 1\right) 1_{(M_{i_o} > L+1)}$

$$\leq L + \frac{r'}{\rho(\varepsilon)} + 2$$

$$\Rightarrow \limsup_{r \to \infty} E_{(\underset{\sim}{\mu},\sigma)} \left(\frac{M_{i_o}}{r'}\right) \leq \frac{1}{\rho(\varepsilon)}$$

Now letting $\varepsilon \to 0$ we have $\limsup_{r \to \infty} E_{(\underset{\sim}{\mu},\sigma)} \left(\frac{M_{i_o}}{r'}\right) \leq c_\theta^{-1}$ ...(5.3.20)

Thus from (5.3.19) and (5.3.20) we have $\lim_{r \to \infty} E_{(\underset{\sim}{\mu},\sigma)} \left(\frac{M_{i_o}}{r'}\right) = c_\theta^{-1}$ ...(5.3.21)

Recall from (5.2.11),
$$M = \begin{cases} \inf\left\{n : \max\limits_{1 \leq i \leq k} W_{n,i} \geq r'\right\} \\ = \infty \quad \text{otherwise} \end{cases}$$

$$\Rightarrow r'^{-1} M \to c_\theta^{-1} \text{ a.s.} \quad \text{...(5.3.22)}$$

(From (5.3.16), (5.3.17) and by similar reasoning as in the proof of $r'^{-1} M_{i_o} \to c_\theta^{-1}$ a.s).

Observe $M = \bigwedge_{i=1}^{k} M_i \Rightarrow M \leq M_{i_o}$ ...(5.3.23)

$$\text{Thus} \quad c_\theta^{-1} \le \liminf_{r \to \infty} E_{(\underset{\sim}{\mu},\sigma)} (r'^{-1} M) \text{ (by Fatou's Lemma and (5.3.22))}$$

$$\le \limsup_{r \to \infty} E_{(\underset{\sim}{\mu},\sigma)} (r'^{-1} M)$$

$$\le \limsup_{r \to \infty} E_{(\underset{\sim}{\mu},\sigma)} (r'^{-1} M_{i_0}) \quad \text{(By (5.3.23))}$$

$$= c_\theta^{-1}$$

Thus $E_{(\underset{\sim}{\mu},\sigma)} (r'^{-1} M) \to c_\theta^{-1}$ and as $rr'^{-1} \to 1$ as $r \to \infty$.

We have $E_{(\underset{\sim}{\mu},\sigma)} (r^{-1} M) \to c_\theta^{-1}$. ...(5.3.24)

Similarly for M' defined in (5.2.12), $E_{(\underset{\sim}{\mu},\sigma)} r^{-1} M' \to c_\theta^{-1}$ as $r \to \infty$ ...(5.3.25)

The proof now follows from (5.3.24), (5.3.25) and (5.2.13). □

<u>Corollary 5.1</u> : $\lim_{r \to \infty} E_{H_i} (r^{-1} N) = (\beta_1(\delta))^{-1} \;\forall\; i = 1,2,\ldots k$ where

$$\beta_1(\delta) = \beta(\delta^2(\tfrac{k-1}{k})(k+\delta^2(\tfrac{k-1}{k}))^{-1/2}) - \beta(\tfrac{-\delta^2}{k}(k+\delta^2(\tfrac{k-1}{k}))^{-1/2}) \quad \text{with}$$

$\beta(x)$ as in (5.2.9) and $r = -\ln (\frac{1-p^*}{k-1})$.

<u>Proof of Corollary 5.1</u> : The proof follows at once from Theorem 5.4 noting the fact that $c_\theta = \beta_1(\delta) \;\forall\; (\underset{\sim}{\mu},\sigma) \in \bigcup_{i=1}^{k} \Omega_{H_i}$ where

Ω_{H_i} is as given in (5.2.2) □

Remark 5.3 : It can be shown by standard argument that

$$\delta^2 \geq \beta_1(\delta) \geq \delta^2 - \frac{\delta^2}{2}\frac{\delta^2}{(2k+\delta^2(\frac{k-1}{k}))} + \frac{k}{3}\left(\frac{\delta^2}{(2k+\delta^2(\frac{k-1}{k})}\right)^3$$

$$\Rightarrow \beta_1(\delta)/\delta^2 \to 1 \text{ as } \delta^2 \to 0.$$

Thus for small δ, $\beta_1(\delta)$ can be approximated by δ^2. The numerical tables in Section 5.4 also verify this.

Remark 5.4 : Following as in the proof of Theorem 6 of Lai (1975) one can show for any $\alpha > 0$, $E_{(\mu,\sigma)}(r^{-1} N)^{\alpha} \to c_{\theta}^{-\alpha}$ as $r \to \infty$.

Remark 5.5 : For the known σ case, ASN is asymptotically equal to $r\delta^{-2}$ vide Mukhopadhyay (1983, page 177). From Corollary 5.1, ASN for unknown σ case (under H_1) is $E_{H_1}(N) \simeq r(\beta_1(\delta))^{-1} \geq r\,\delta^{-2}$ (by Remark 5.3) which is expected.

5.4 : Numerical Study for the Procedure R

This section is devoted to the asymptotic comparison (as $P^* \to 1$) of procedure R with the corresponding fixed sample procedure R_0 and the BDS's two stage procedure, numerically.

Let us first take up the comparison of R with R_0. The procedure R_0 suggests to take sample of size n_0 from each one of the k populations and select the population corresponding to the maximum of $\bar{X}_{in}\sqrt{T_n^{-1}}$ i.e. maximum of $\bar{X}_{in}$ for $i = 1,2,\ldots,k$, (T_n as in (5.3.2)). This procedure is invariant under the transformation $X \to aX+b$, $a > 0$, $-\infty < b < \infty$.

One can easily verify that the required sample size n_o for attaining $P_{H_i}(CS) = P^* \; \forall \; i = 1,2,\ldots k$ is same as that of the corresponding selection procedure for the known σ case (due to Bechhofer (1954)). Thus here also $n_o = \delta^{-2} \tau_t^2$ where τ_t is tabulated in Table 4.1 of Gibbons et al (1977).

$$\text{Thus} \quad \frac{E_{H_i}(N)}{n_o} \simeq \frac{-\ln(\frac{1-P^*}{k-1})\, \delta^2}{\tau_t^2 \, \beta_1(\delta)}$$

$$= e(\delta, k, P^*) \text{ (say)} \qquad \ldots(5.4.1)$$

$$\forall \; i = 1,2,\ldots k.$$

From now onwards we shall write EN for $E_{H_i} N$.

By Remark 5.3 for small values of δ, the ratio n_o^{-1} EN is approximately equal to that for the known σ case (vide (2.7) of Mukhopadhyay (1983)). This implies the efficiency of the sequential procedure R to the corresponding fixed sample procedure R_o (for unknown σ case) is approximately same as that for the known σ case.

Values of $e(\delta,k,P^*)$ are computed for different values of δ, k and P^*. The values show substantial saving in sample size for procedure R with respect to R_o.

Now to compare R with BDS's two stage procedure one has to put special consideration as the formulation in BDS is different from the present one. So here we proceed as follows :

Let $0 < \sigma \leq a$ and suppose a is known. Consider BDS in the region $\left\{ (\underset{\sim}{\mu}, \sigma) : \mu_{[k]} - \mu_{[k-1]} \geq \delta^* \right\}$ where $\delta^* = a\delta$ ...(5.4.2)

Then $\mu_{[k]} - \mu_{[k-1]} \geq \delta^* \Rightarrow \mu_{[k]} - \mu_{[k-1]} \geq a\delta$

$$\Rightarrow \mu_{[k]} - \mu_{[k-1]} \geq \sigma\delta$$

Thus for this region (given in (5.4.2)) BDS puts more protection than our procedure R, and we restrict our comparison to this parametric region only.

Now, call the sample size in BDS, N_B. Then

$E(N_B) \geq \tau_t^2 \frac{\sigma^2}{\delta^{*2}}$ (vide the footnote at page 174 of BDS)

Thus $\frac{EN}{EN_B} \leq \frac{r}{\beta_1(\delta)} \left(\frac{\delta^*}{\sigma}\right)^2 \tau_t^{-2}$ with $r = -\ln\left(\frac{1-P^*}{k-1}\right)$

If $\sigma = a$, then $\frac{EN}{EN_B} \leq \frac{r}{\beta_1(\delta)} \frac{\delta^2}{\tau_t^2} = e(\delta, k, P^*)$ ...(5.4.3)

Tables (given below) show all the tabulated values of $e(\delta,k,P^*)$ are less than one. In fact the highest and the lowest calculated values of $e(\delta,k,P^*)$ are .64639 (= e(1,3,.9)) and .33808 (= e(1,10,.999)) respectively. Thus for sufficiently small δ and large k and P^*, procedure R shows saving in sample size w.r.t. BDS (when $\sigma = a$).

If $\sigma < a$, then $(EN)(EN_B)^{-1} \leq 1$ if

$$a^{-1}\sigma \geq (e(\delta,k,P^*))^{1/2}$$
$$= \nu(\delta,k,P^*) \quad \text{(say)}$$

From the tables, the numerical values of $\nu(\delta,k,P^*)$ are found to lie between .58145 $(=(.33808)^{1/2})$ to .80523 $(=(.64839)^{1/2})$. Thus for sufficiently small δ and large k and P^*, the tables show low values of $\nu(\delta,k,P^*)$ which indicates in favour of R in a reasonably large set of values for $\bar{a}^{1}\sigma$.

Tables below show values of $\frac{-\ln\left(\frac{1-P^*}{k-1}\right)}{\beta_1(\delta)} \frac{\delta^2}{\tau_t^2}$ for

δ = .1, .2, .3, .4, .5, 1.

k = 3, 4, ...10

P^* = .900, .950, .975, .990, .999

TABLE 1 : k = 3

δ^2	$\beta_1(\delta)$	P^*= .900	P^*= .950	P^*= .975	P^* = .990	P^*= .999
.01	.0099917	.60250	.50242	.44837	.40534	.35229
.04	.039867	.60401	.50367	.44949	.40635	.35317
.09	.089334	.60649	.50674	.45134	.40802	.35462
.16	.157922	.60992	.50860	.45389	.41033	.35663
.25	.245000	.61428	.51224	.45714	.41326	.35918
1	.928459	.64839	.54068	.48252	.43621	.37912

TABLE 2 : k = 4

δ^2	$\beta_1(\delta)$	P^*= .900	P^* = .950	P^*= .975	P^*= .990	P^*= .999
.01	.0099937	.56636	.48130	.46129	.39625	.34822
.04	.039900	.56742	.48220	.46215	.39699	.34887
.09	.089500	.56916	.48369	.46357	.39821	.34994
.16	.158434	.57159	.48575	.46556	.39991	.35144
.25	.246219	.57469	.48839	.46808	.40208	.35334
1	.944858	.59903	.50907	.48790	.41911	.36831

TABLE 3 : k = 5

δ^2	$\beta_1(\delta)$	$P^*=.900$	$P^*=.950$	$P^*=.975$	$P^*=.990$	$P^*=.999$
.01	.0099959	.54622	.47019	.42617	.39016	.34514
.04	.039920	.54709	.47094	.42685	.39078	.34569
.09	.089600	.54844	.47210	.42790	.39174	.34654
.16	.158743	.55032	.47372	.42937	.39309	.34773
.25	.246969	.55270	.47577	.43123	.39479	.34923
1	.955032	.57171	.49213	.44606	.40836	.36124

TABLE 4 : k = 6

δ^2	$\beta_1(\delta)$	$P^*=.900$	$P^*=.950$	$P^*=.975$	$P^*=.990$	$P^*=.999$
.01	.0099963	.53320	.46117	.42015	.38614	.34313
.04	.039933	.53389	.46177	.42070	.38665	.34357
.09	.089665	.53499	.46272	.42157	.38744	.34428
.16	.158950	.53652	.46404	.42277	.38855	.34526
.25	.247458	.53847	.46573	.42431	.38996	.34652
1	.961991	.55406	.47921	.43660	.40125	.35655

TABLE 5 k = 7

δ^2	$\beta_1(\delta)$	$P^* = .900$	$P^* = .950$	$P^* = .975$	$P^* = .990$	$P^* = .999$
.01	.0099946	.52328	.45625	.41622	.38321	.34118
.04	.039943	.52375	.45665	.41659	.38355	.34149
.09	.089713	.52467	.45746	.41733	.38422	.34209
.16	.159098	.52596	.45858	.41836	.38517	.34293
.25	.247814	.52761	.46002	.41967	.38638	.34401
1	.967062	.54081	.47153	.43017	.39604	.35261

TABLE 6 : k = 8

δ^2	$\beta_1(\delta)$	$P^* = .900$	$P^* = .950$	$P^* = .975$	$P^* = .990$	$P^* = .999$
.01	.0099969	.51616	.45114	.41313	.38112	.34010
.04	.039950	.51664	.45156	.41352	.38148	.34042
.09	.089748	.51745	.45227	.41416	.38207	.34095
.16	.159209	.51856	.45324	.41505	.38289	.34169
.25	.248083	.51999	.45448	.41619	.38394	.34263
1	.970927	.53145	.46450	.42537	.39241	.35018

TABLE 7 : k = 9

δ^2	$\beta_1(\delta)$	$P^* = .900$	$P^* = .950$	$P^* = .975$	$P^* = .990$	$P^* = .999$
.01	.0099973	.51014	.44712	.41011	.37910	.33909
.04	.039951	.51062	.44755	.41050	.37946	.33941
.09	.089776	.51127	.44811	.41102	.37994	.33984
.16	.159296	.51225	.44898	.41181	.38067	.34050
.25	.248293	.51351	.45007	.41282	.38160	.34133
1	.973975	.52363	.45894	.42095	.38913	.34806

TABLE 8 : k = 10

δ^2	$\beta_1(\delta)$	$P^* = .900$	$P^* = .950$	$P^* = .975$	$P^* = .990$	$P^* = .999$
.01	.0099975	.50613	.44411	.408102	.37709	.33808
.04	.039960	.50651	.44444	.40841	.37738	.33834
.09	.089799	.50713	.44499	.40891	.37784	.33876
.16	.159366	.50801	.44577	.40962	.37850	.33934
.25	.248461	.50913	.44675	.41063	.37933	.34009
1	.976442	.51821	.45471	.41784	.38609	.34615

CHAPTER 6

NUMERICAL SOLUTION FOR BAYES SEQUENTIAL PROBLEM OF TESTING THE SIGN OF THE DRIFT PARAMETER OF A WIENER PROCESS

6.1 Introduction

So far we have been dealing with invariant SPRT and its extension in the context of classification and selection problem.

In this chapter we deal with the problem of testing sequentially the sign of the drift parameter μ of a Wiener process $\{X(t),\ t\in[0,\infty)\}$.

Here the data consists of a Wiener process $X(t)$ with unknown drift μ and known variance σ^2 per unit time. The unknown drift μ has a prior distribution which is normal with fixed mean μ_0 and variance σ_0^2. The problem is to test

$$\left.\begin{array}{ll} H_1 : \mu \geq 0 & \text{versus} \\ H_2 : \mu < 0 & \end{array}\right\} \qquad \ldots (6.1.1)$$

when the cost of incorrect decision is $|\mu|$ and the cost of sampling is c (units of money) per unit time. ... (6.1.2)

Chernoff in a series of paper (1961), (1964 with Breakwell), (1965) and (1972) considered the problem of finding an optimal Bayes procedure for testing the hypotheses described in (6.1.1) with the cost structure given in (6.1.2). A similar problem as in (6.1.1) with indifference zone (which is an interval around $\mu = 0$) has been solved and generalised by Schwarz (1962).

Chernoff (1961) showed that the posterior distribution of μ is given by

$$\mathcal{L}(\mu | X(t'),\ 0 \leq t' \leq t) = N(Y(s),\ s)$$

where

$$Y(s) = (\mu_0 \sigma_0^{-2} + X(t)\sigma^{-2})/(\sigma_0^{-2} + t\,\sigma^{-2}) ,$$

$$s = (\sigma_0^{-2} + t\,\sigma^{-2})^{-1}$$

and $N(a,b)$ is a normal distribution with mean a and variance b. He also showed that $Y(s)$ is a standard Wiener process in $-s$ scale, originating at $(y_0, s_0) = (\mu_0, \sigma_0^2)$.

Let $\quad d(y,s) = c\,\sigma^2\, s^{-1} + s^{1/2}\, \Psi(ys^{-1/2}) - c\,\sigma_0^{-2}\,\sigma^2 \ ,\ \sigma_0^2 \geq s > 0$.

with $\quad \Psi(y) = \varphi(u) - |u|(1 - \Phi(|u|))$

where $\quad \varphi(u) = (2\pi)^{-1/2} \exp(-u^2/2) \qquad$ for $-\infty < u < \infty$.

$$\Phi(u) = \int_{-\infty}^{u} \varphi(x)dx .$$

Chernoff reduced the above testing problem to the following stopping problem : The standard Wiener process $Y(s)$ is observed in $-s$ scale. The statistician may stop at any value of $s (s > 0)$ and pay $d(Y(s), s)$. Problem is to find the stopping rule, that minimizes the expected cost.

By taking,

$$Y^*(s^*) = a\,Y(s)$$
$$s^* = a^2 s$$
$$\text{where } a = c^{-1/3}\,\sigma^{-2/3} ,$$

one can normalize the problem to $\sigma = 1$, $c = 1$ and $Y^*(s^*)$ is a standard Wiener process in $-s^*$ scale, starting at $(a\mu_o, a^2\sigma_o^2)$.

Now $d(y,s) = a^{-1} d^*(y^*,s^*) - c\,\sigma_o^{-2}\sigma^2$ ... (6.1.3)

where $d^*(y^*,s^*) = s^{*1/2}\,\psi(y^* s^{*-1/2}) + (s^*)^{-1}$ ... (6.1.4)

The constant 'a' and $c\,\sigma_o^{-2}\sigma^2$ in the expression of $d(y,s)$ (vide (6.1.3)) has no effect on the optimal procedure. Thus one can work with $d^*(y^*,s^*)$ in stead of $d(y,s)$.

Let

$$\rho(y^*,s^*) = \inf_{\tau} E\left\{d^*(y^* + Y^*(\tau),\ s^* - \tau)\right\} \qquad \text{... (6.1.5)}$$

Where the infimum is taken over the set of all stopping times τ. From now onward we shall work on the above normalized problem and for notational simplicity we shall write s, y and d for s^*, y^* and d.

Chernoff showed that $\rho(y,s)$ satisfies the following free boundary problem,

$$\left.\begin{aligned} \tfrac{1}{2}\rho_{yy}(y,s) &= \rho_s(y,s) \text{ in } C_o \\ \rho(y,s) &= d(y,s) \text{ on } \partial C_o \\ \rho_y(y,s) &= d_y(y,s) \text{ on } \partial C_o \\ \rho_y(o,s) &= 0 \end{aligned}\right\} \qquad \text{... (6.1.6)}$$

where C_0 is the optimal continuation region which is open and ∂C_0 is the boundary of C_0. Some modern techniques useful for obtaining (6.1.6) are available in Friedman (1975,1976).

Chernoff obtained (with Breakwell for $s \to 0$) the following asymptotic expansions for the optimal boundary $\breve{y}(s)$,

$$\tilde{\alpha}(s) = s^{-1/2}\tilde{y}(s) \sim \frac{1}{4} s^{3/2} \left\{1 - \frac{s^3}{12} + \frac{7}{15.16} s^6 \dots\right\} \text{ for } s \to 0 \quad ..(6.1.7)$$

$$\tilde{\alpha}(s) = s^{-1/2}\tilde{y}(s) \sim \left\{\ln s^3 - \ln 8\pi - 6(\ln s^3)^{-1} + \dots\right\}^{1/2} \text{ for } s \to \infty \quad ..(6.1.8)$$

Here we find out the optimal boundary $\tilde{y}(s)$ (which is also the Bayes boundary) numerically by solving the free boundary problem (f.b.p.) given in (6.1.6). The purpose is to throw light on the Bayes boundary. Following Sackett (1971) we use here the method of lines (introduced by Rothe (1930)) for solving the f.b.p. This is given in Section 6.2.

Chernoff and Petkau (1986) used a different numerical method for solving the same testing problem (as given in (6.1.1)). They used the technique of backward induction which is followed by a continuity correction, unlike the present case where we solve the f.b.p. (given in (6.1.6)) numerically. Our results are found to be very close to those of Chernoff and Petkau (1986) as indicated by Table 6.2.

Thus the present chapter deals purely with a numerical investigation unlike the previous ones. The problems in previous chapters are with indifference zone and they were looked into with Neyman-Pearsonian view point while the present problem is a Bayes sequential testing problem without an indifference zone.

This chapter is a revised version of Ghosh, Mallik and Ray Chaudhuri (1986).

6.2 Computation of the Bayes boundary by Method of Lines

$$\text{Let} \quad s' = 2^{-1}s \qquad \ldots (6.2.1)$$

$$U(y,s') = \rho(y,s') - s'^{1/2}\,\psi(ys'^{-1/2}) \qquad \ldots (6.2.2)$$

By using symmetry and (6.2.1), (6.2.2), the f.b.p. (6.1.6) changes to the following free boundary problem :

$$\begin{aligned} &U_{yy}(y,s') = U_{s'}(y,s'), && 0 < y < \tilde{y}(s') \\ &U(\tilde{y}(s'),s') = (2s')^{-1} && s' > 0 \\ &U_y(\tilde{y}(s'), s') = 0 && s' > 0 \\ &U_y(0,s') = \tfrac{1}{2} && s' > 0, \end{aligned} \qquad \ldots (6.2.3)$$

where $\tilde{y}(s')$ is the free boundary.

To remove the singularity at $s' = 0$, Sackett (1971) decomposed $U(y,s')$ as follows,

$$U(y,s') = W(y,s') + V(y,s'), \qquad \ldots (6.2.4)$$

where

$$W(y,s') = \frac{1}{2s'} \sum_{n=0}^{\infty} (-1)^n \frac{n!\,y^{2n}}{(2n)!\,s'^n} \qquad \ldots (6.2.5)$$

Then the free boundary problem given in (6.2.3) reduces to the following free boundary problem,

$$\left.\begin{aligned} &\text{(i)} && V_{yy}(y,s') = V_{s'}(y,s'), && 0 < y < \tilde{y}(s'). \\ &\text{(ii)} && V_y(0,s') = \tfrac{1}{2} && \text{for } s' > 0 \\ &\text{(iii)} && V(\tilde{y}(s'),s') = F(\tilde{y}(s'),s') && \text{for } s' > 0 \\ &\text{(iv)} && V_y(\tilde{y}(s'),s') = G(\tilde{y}(s'),s') && \text{for } s' > 0, \end{aligned}\right\} \qquad \ldots (6.2.6)$$

where

$$F(y,s') = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{(2n)!} y^{2n} s'^{-(n+1)} \qquad \ldots(6.2.7)$$

$$G(y,s') = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{(2n-1)!} y^{(2n-1)} s'^{-(n+1)} \qquad \ldots(6.2.8)$$

<u>The Method of Lines Algorithm</u>

In this algorithm by discretizing the time scale s' we replace the equation (6.2.6) (i) by the system of ordinary differential equations,

$$\frac{\partial^2}{\partial x^2} V_n(x) = \frac{V_n(x) - \tilde{V}_{n-1}(x)}{h}, \quad 0 \leq x \leq s_n, \; n = 1,2,\ldots \qquad \ldots(6.2.9)$$

where s_n is to be chosen so that,

$$\left.\begin{aligned} &\frac{\partial}{\partial x} V_n(0) = \frac{1}{2}, \; V_n(s_n) = F(s_n, nh), \\ &\frac{\partial}{\partial x} V_n(s_n) = G(s_n, nh), \end{aligned}\right\} \qquad \ldots(6.2.10)$$

and

$$\left.\begin{aligned} \tilde{V}_n(x) &= V_n(x), \quad x \leq s_n \\ &= F(s_n, nh) + G(s_n, nh)(x - s_n), \; x > s_n. \end{aligned}\right\} \qquad \ldots(6.2.11)$$

To initiate this procedure, it is necessary to specify $\tilde{V}_0(x)$. The choice of $\tilde{V}_0(x)$ is taken as $x/2$ (with $s_0 = 0$). As described in Sackett (1971) it is reasonable to expect $F(\tilde{y}(s'), s')$ to tend to zero as s' tends to zero, in light of the asymptotic expansion of the optimal boundary given by Breakwell and Chernoff (1964) (vide (6.1.7)). Thus the choice of $\tilde{V}_0(x)$ is justified by (6.2.6(ii)) and (6.2.6(iii)).

Now the equation (6.2.9) has the following explicit solution,

$$V_n(x) = A_n \cosh(xh^{-1/2}) + B_n \sinh(xh^{-1/2}) - \frac{1}{h^{1/2}} \int_0^x \tilde{V}_{n-1} \sinh((x-\xi)h^{-1/2}) d\xi \qquad \ldots(6.2.12)$$

where A_n and B_n are functions of h, n and s_n.

Using the conditions in (6.2.10) and (6.2.12) one can show that s_n is the zero of the following non-linear function,

$$\eta_n(s) = F(s,nh) \sinh(sh^{-1/2}) - h^{1/2} G(s,nh) \cosh(sh^{-1/2}) + \frac{1}{2} h^{1/2} - h^{-1/2} \int_0^s \tilde{V}_{n-1}(\xi) \cosh(\xi h^{-1/2}) d\xi \qquad \ldots(6.2.13)$$

The method of line algorithm is as follows :

1. Find the root of the equation $\eta_n(s) = 0$ and denote it by s_n.
2. Using the conditions in (6.2.10) and (6.2.12) compute A_n and B_n.
3. Using these A_n and B_n, compute $V_n(x)$ from (6.2.12) for $0 \leq x \leq s_n$ and then compute $\tilde{V}_n(x)$.

Sackett (1971) used this method to solve the free boundary problem given in (6.2.6) for $0 < s \leq 2$. We extended his computation for $0 < s' \leq 50$. For, $0 < s' \leq 5$, we used $h = .01$, for $5 < s' \leq 25$, we used $h = .04$ and for, $25 < s' \leq 50$, we used $h = .5$

In Table 6.1 we have tabulated $\hat{\alpha}(s) = \hat{y}(s)s^{-1/2}$ where $\hat{y}(s)$ is the optimal boundary obtained by the method of lines, described as s_n (the zero of the function $\eta_n(s)$ vide (6.2.13)) for various values of n. An estimate of the nominal significance level $\tilde{\beta}(s) = 1 - \Phi(\tilde{\alpha}(s))$ (say $\hat{\beta}(s) = 1 - \Phi(\hat{\alpha}(s))$) and $\hat{\rho}(0,s)$, the estimated Bayes risk obtained by the method of lines are also tabulated in Table 6.1. In Table 6.2 we have tabulated $\hat{\beta}(s)$ given by the method of line boundary as well as by Chernoff-Petkau (1986) boundary. Comparing the second and third columns of Table 6.2 one can conclude that the method of lines and Chernoff-Petkau method are of same accuracy. Chernoff and Petkau (1986) used a continuity correction on the boundary obtained by the method of backward induction, while in the present case no such boundary correction is needed to attain the same level of accuracy.

TABLE 6.1

s	$\hat{\alpha}(s) = \frac{\hat{y}(s)}{s^{1/2}}$	$\hat{\beta}(s) = 1 - \Phi(\hat{\alpha}(s))$	$\hat{\rho}(0,s)$
.4	.06294	.4749	2.74238
.6	.11432	.4545	1.95364
1.0	.23431	.4074	1.34134
2.0	.53502	.2963	.88744
3.0	.77882	.2181	.72498
4.0	.97170	.1656	.63506
5.0	1.12828	.1296	.57539
10.0	1.62829	.0517	.42646
20.0	2.11418	.0173	.32026
30.0	2.39143	.0084	.27365
40.0	2.57008	.0051	.24342
50.0	2.70303	.0034	.22137
60.0	2.81972	.0024	.20575
70.0	2.90822	.0018	.19277
80.0	2.98214	.0014	.18169
90.0	3.04568	.0012	.17201
100.0	3.10135	.0010	.16340

$\hat{y}(s)$ = The optimal boundary obtained by the method of lines.

$\hat{\rho}(0,s)$ = The Bayes risk obtained by the method of lines.

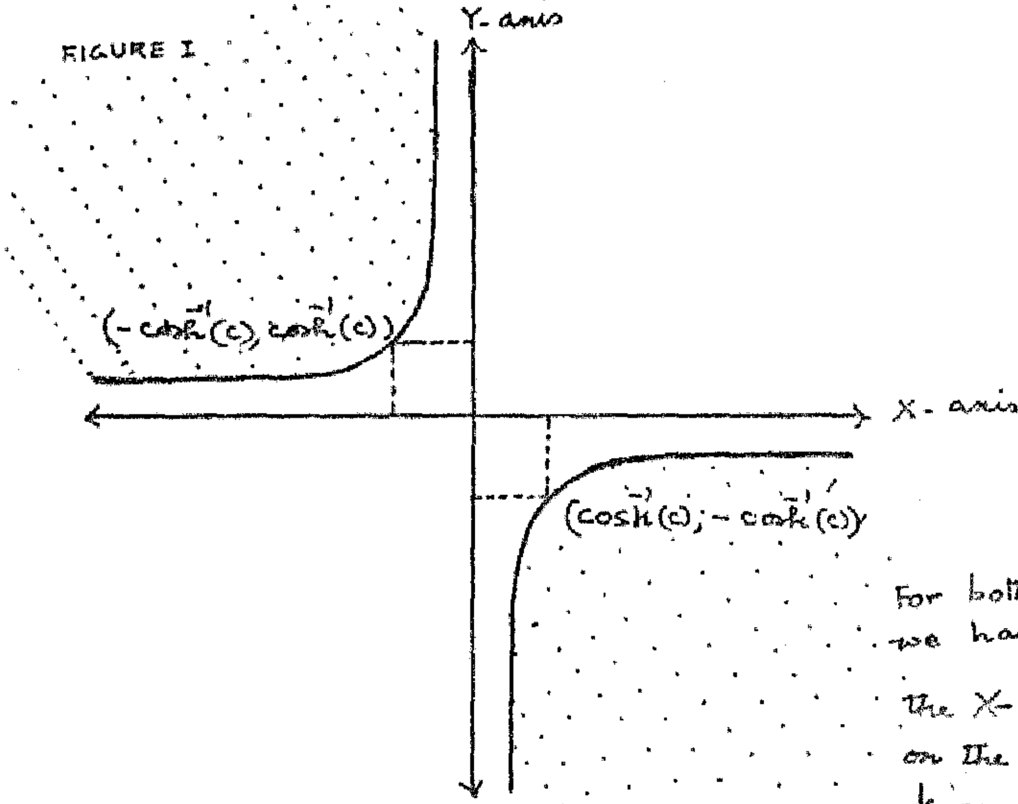
TABLE 6.2

$\hat{\beta}(s) = 1 - \Phi(\hat{\alpha}(s))$

s	Method of Line	Chernoff-Petkau
.4	.4749	.4749
1.0	.4074	.4073
2.0	.2963	.2964
5.0	.1296	.1300
10.0	.0517	.0522
20.0	.0173	.0176
50.0	.0034	.0036
100.0	.0010	.0010

Figure III on the page 140 gives a graphical view of $\hat{\alpha}(s)$ (by method of lines) together with $\tilde{\alpha}(s)$ for $s \to \infty$ (using the first three terms of the RHS of (6.1.8)).

FIGURE I



For both the figures we have $\frac{\delta_0 k n}{2n+k} R$ on the X-axis and $\delta_0 n Q$ on the Y-axis with k, n, δ_0 fixed.

'c' is the cutoff constant as given in (2.2.6).

The shaded region denotes the critical region for P_0 with

(i) $\alpha < \beta$ or $c > 1$ in Fig I

(ii) $\alpha > \beta$ or $c < 1$ in Fig II

FIGURE II

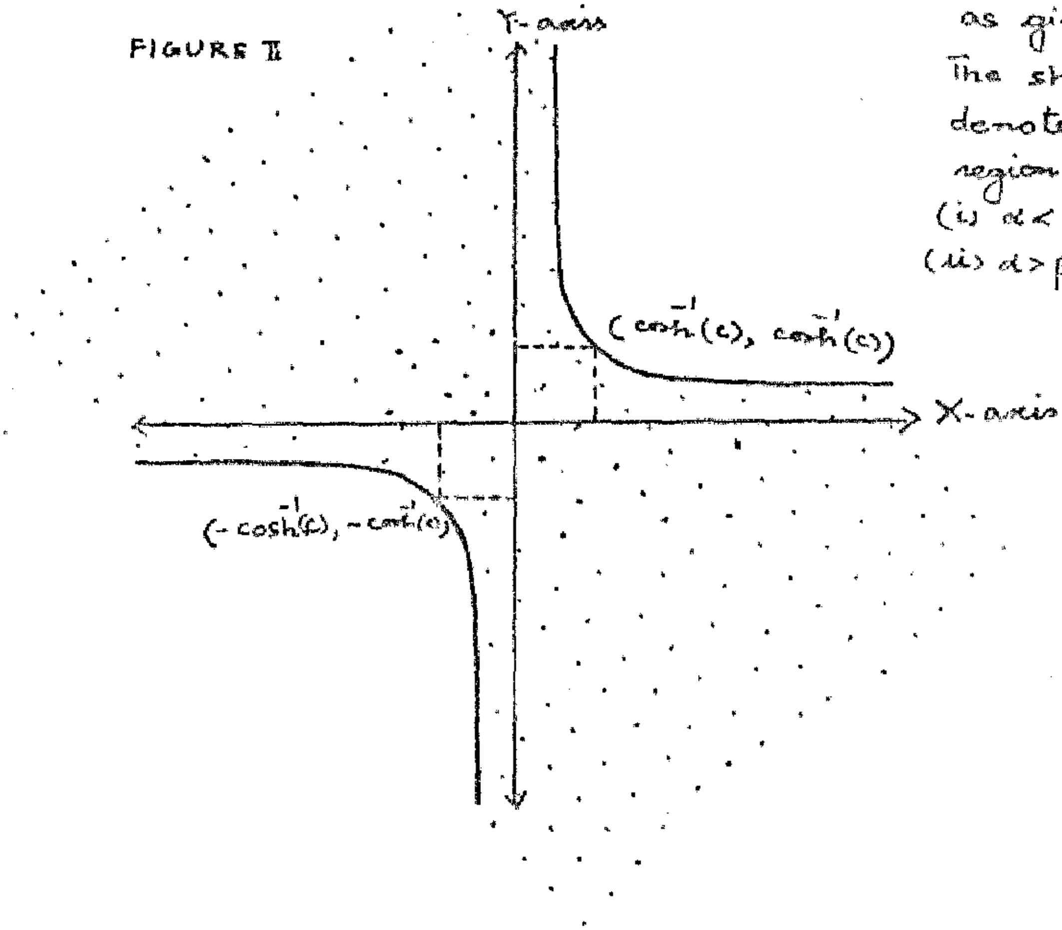
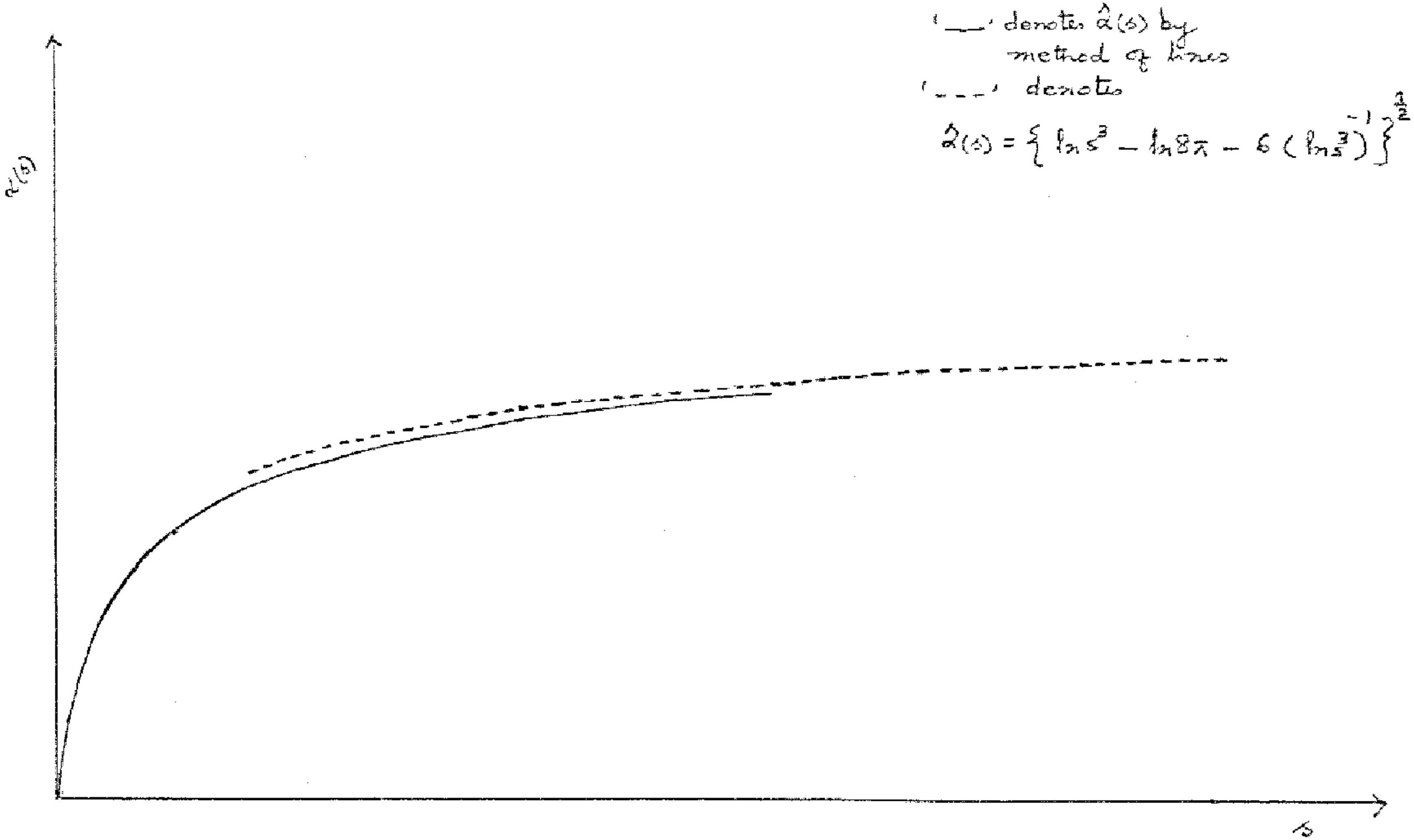


FIGURE III


'___' denotes $\hat{\alpha}(s)$ by method of lines
'- - -' denotes
$\alpha(s) = \{\ln s^3 - \ln 8\pi - 6(\ln s^3)^{-1}\}^{\frac{1}{2}}$
$\alpha(s)$
s

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