SEMI-MARTINGALES ASSOCIATED WITH CROSSINGS

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To
Prof J. K. Ghosh,
with regards;
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To my parents

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C O N T E N T S

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INTRODUCTION

In this thesis we study the local behaviour of semi-martingales. Given a continuous semi-martingale and an interval (a,b), we define a new process which mirrors the behaviour of the original process in the interval (a,b). This new process turns out to be a semi-martingale whose jumps during $[0,t]$ are closely related to the number of crossings of (a,b) during $[0,t]$.

Our starting point is indeed P. Levy's martingale characterization of Brownian motion : If (X_t) and $(X_t^2 - t)$ are continuous local martingales then (X_t) must be a Brownian motion. Let now $H_n(x,t)$ denote the n-th Hermite polynomial (defined in Section 1.2). Suppose that (X_t) and $(H_n(X_t,t))$ are continuous local martingales. Levy's characterization says that for $n = 2$, this implies that (X_t) must be a Brownian motion. For $n > 2$, this is not in general true. However by imposing an (analytical) condition (Theorem 1.2.1, A.2), we show that (X_t) must be a Brownian motion. Roughly speaking the condition says that the sojourn time of the candidate in a certain set should have zero Lebesgue measure. In the next section (Section 1.3) we study the sojourn time of a continuous martingale in an interval (a,b). We describe the expected sojourn time of the process in (a,b) during $[0,t]$, in terms of the expected number of crossings of (a,b) during $[0,t]$ and terms which are small - of the order of $(b-a)^2$ (Theorem 1.3.1). Two Corollaries are immediate. Firstly, we get an expected version of Levy's down crossing theorem. Secondly, the ratio of the expected sojourn time in (a,b) during

$[0,t]$ to the expected number of crossings of (a,b) during $[0,t]$ converges to $(b-a)^2$ as $t \longrightarrow \infty$. The idea of 'semi-martingales associated with crossings' follows naturally from Theorem 1.3.1 (see the prefacing remarks in Section 1.4, Chapter I). In Section 1.4 we prove the existence of such a semi-martingale by 'bare-hand techniques' developed in the course of the proof of Theorem 1.3.1.

In Chapter II, we study the crossings of an interval (a,b) by a continuous semi-martingale $X_t = X_0 + M_t + V_t$. For simplicity let us assume that $X_0 < a$ almost surely. Let $\sigma_t = \max\{s \leq t : X_s \in (a,b)^c\}$. We note that σ_t is not a stop time. However X_{σ_t} and $X_t - X_{\sigma_t}$ turn out to be (discontinuous) semi-martingales. This is a simple consequence of Tanaka's formula and a pathwise identity (Lemma 2.3.1). We observe that the jumps of $X_t - X_{\sigma_t}$ occur at the crossing times of (a,b) by X_t and that the size of the jumps is $b-a$. We then determine the local times of $X_t - X_{\sigma_t}$ and X_{σ_t} in terms of the local times of (X_t). For example, the local time of $X_t - X_{\sigma_t}$ at the point $x \in (0, b-a)$ is the local time of X_t at $a+x$ during the upcrossings. It can also be seen (Remark 2.4.2) that the local time of $X_t - X_{\sigma_t}$ is in general discontinuous at zero. In Section 2.5 we determine the martingale and bounded variation parts of the semi-martingale $|X_t - X_{\sigma_t}|$ (Lemma 2.5.1). The martingale part is simply $\int_0^t I_{(a,b)}(X_s)dM_s$ and the continuous bounded variation part is $\int_0^t I_{(a,b)}(X_s)dV_s + \frac{1}{2}(L(t,a) + L(t,b-))$ where $L(t,a)$ is the local time at a of (X_s) at time t. The sum of the jumps of

$|X_t - X_{\sigma_t}|$ is easily seen to be $(b-a) \times$ (number of crossings of (a,b) by (X_t) during $[0,t]$). This generalizes Theorem 1.4.1 of Chapter I. By an application of Ito's formula, we recover Theorem 1.3.1 of Chapter I.

In Chapter III, we give some applications of the theory developed in the previous chapters. In Sections 3.3 and 3.4 we deal exclusively with the Brownian motion. In Section 3.3 we show that when (X_t) is a Brownian motion,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{\int_0^t I_{(a,b)}(X_s)\,ds}{C(t)} = (b-a)^2$$

where $C(t)$ is the number of crossings of (a,b) by X during $[0,t]$ (Theorem 3.3.1). This is the almost sure version of Corollary 1.3.2. We then show that for any smooth function f,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{\int_0^t f''(|X_s - X_{\sigma_s}|)\, I_{(a,b)}(X_s)\,ds}{2\,C(t)} = f(b-a)-f(0)-f'(0)(b-a).$$

In Section 3.4, we derive an asymptotic relationship between local times and crossings. We show that for the Brownian motion $\underset{t \to \infty}{\mathrm{Lt}} \frac{C(t)}{L(t,a)} = \frac{1}{b-a}$ almost surely. This can be thought of as Levy's crossing theorem in the variable t. Although we have stated the preceding two results separately (Theorems 3.3.2 and 3.4.1) they are indeed closely related. We close Section 3.4 with a remark on the link between these two results (Remark 3.4.2). Finally in Section 3.5 we prove Levy's crossing theorem for a continuous semi-martingale. This has been proved earlier for the Brownian motion by a number of authors (see Introduction to Section 3.5 for a list of references and also J.B. Walsh [16] where the crossing theorem is used to study the Markov properties of the local time). In particular N. el Karoui [5] has shown that this theorem is true for semi-martingales (X_t) with $\sum_{s \leq t} |\Delta X_s| < \infty$ for all t, almost surely. However, our approach is different. We use the semi-martingale decomposition for $|X_t - X_{\sigma_t}|$ and show that the martingale part tends to zero as $a \longrightarrow b$.

We conclude this introduction with a few general remarks. We have mainly concentrated on the case when (X_t) is a continuous semi-martingale. However this restriction can be relaxed. For example, when $\sum_{s \leq t} |\Delta X_s| < \infty$ for all t almost surely, it is easily seen that $X_t - X_{\sigma_t}$ is a semi-martingale (see Remark 2.3.4). It should in principle be possible to extend our results to this case also. The random times σ_t play a crucial role in our approach. If σ_t are the random times arising from (X_t) and (Y_t) were another semi-martingale, then the structure of Y_{σ_t} - in general - is not known. The reason for expecting a nice structure for Y_{σ_t} is that it is obtained from Y_t by some kind of time change. Specific cases can however be worked out. For instance in case $Y_t = f(X_t)$ where f is a smooth function, then by Ito's formula applied to $f(X_{\sigma_t})$ we can get an explicit expression for N_{σ_t}, where (N_t) is the martingale part of (Y_t). In this connection it is interesting to note that the relation $(X_t - a)^{-} = X_t - X_{\sigma_t}$, where $\sigma_t = \max\{s \leq t : X_s \in (-\infty, a]\}$, holds pathwise. The random times σ_t are also well known in excursion theory. Our study indicates that a better understanding of the σ_t's would also throw more light on the local behaviour of stochastic processes.

CHAPTER I

Sojourn Times of Semi Martingales

1.1 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t \geq 0}$ be a right continuous filtration on it with $\mathcal{F}_0$ containing all null sets. This notation will remain unchanged for the rest of the thesis. Unless otherwise stated all processes we consider will be adapted to this filtration. Also, by a stop time, we shall mean a stop time with respect to this filtration. Let $(X_t)_{t \geq 0}$ be a continuous semi-martingale i.e. $X_t = X_0 + M_t + V_t$ where (M_t) is a continuous local martingale null at zero and (V_t) a continuous process of bounded variation on compact t-intervals and null at zero. We will denote by $\langle X \rangle_t$ the quadratic variation process of the continuous martingale part of X, and by μ_X the random measure induced by the quadratic variation process on $[0,\infty)$. We use the notation m for lebesgue measure on $\mathbb{R}$, and if D is measurable we write $m \perp D$ to mean $m(D) = 0$. Let $L(t,a)$ denote the local time of the semi-martingale X at time t, at the point $a \in \mathbb{R}$. For the proof of the existence of the local time and some of the properties stated below we refer to Meyer [8, p. 361 - 371] . It is defined by Tanaka's formula as follows. For $a \in \mathbb{R}$,

$$\frac{1}{2} L(t,a) = (X_t - a)^+ - (X_0 - a)^+ - \int_0^t I_{(a,\infty)}(X_s)dX_s \quad \text{..... (1)}$$

for every $t \geq 0$, almost surely.

It is well known that for each $a \in \mathbb{R}$, $t \longrightarrow L(t,a)$ is an increasing process and the set of its points of increase is given by $\{s : X_s = 0\}$. Further, it satisfies the occupation density formula viz. for any bounded borel function f on $\mathbb{R}$, we have almost surely,

$$\int_0^t f(X_s)\, d\langle X\rangle_s = \int_{\mathbb{R}} f(a)\, L(t,a)\, da \qquad \ldots\ldots(2)$$

for all $t \geq 0$.

It is also well known that when (X_t) is a martingale, there is a version of $L(t,a)$ which is jointly continuous in t and a (see Yor [18]).

1.2 Levy's Characterization of Brownian motion

We recall that the time dependent Hermite polynomials are defined for $n \geq 0$ by

$$H_n(x,t) = \frac{(-t)^n}{n!}\, e^{\frac{x^2}{2t}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2t}} \quad -\infty < x < \infty,\ t \geq 0 .$$

Note that $H_1(x,t) \equiv x$, $H_2(x,t) \equiv \frac{1}{2}(x^2 - t)$. It is easily verified that these are parabolic functions - that is, they satisfy $U_t + \frac{1}{2} U_{xx} = 0$. Indeed,

$$\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(x,t) = - \frac{\partial}{\partial t} H_n(x,t) = \frac{1}{2} H_{n-2}(x,t) \qquad \ldots\ldots(3)$$

$$\frac{\partial}{\partial x} H_n(x,t) = H_{n-1}(x,t) \qquad \ldots\ldots(4)$$

The well known characterization of Levy states that if (X_t) is a continuous local martingale, such that $H_2(X_t,t) \equiv \frac{1}{2}(X_t^2 - t)$ is also a local martingale, then (X_t) is indeed a Brownian motion. Few years ago, Prof. K.R. Parthasarathy raised the question whether H_2 can be replaced by H_n for some $n \geq 2$. In this section, we will show that the answer is affirmative, provided we impose an additional restriction, of a local nature on the process (X_t). More precisely we have the following theorem.

Theorem 1.2.1 Let (X_t) be a continuous local martingale with $X_0 \equiv 0$. Assume that

A.1 $H_n(X_t,t)$ is a local martingale

A.2 $m \perp \{ t : H_{n-2}(X_t,t) = 0 \}$ almost surely.

Then, (X_t) is a Brownian motion.

Proof By Lévy's characterization it suffices to show $\langle X_t \rangle = t$. Ito's formula and eqn. (4) applied to $H_n(X_t,t)$ gives

$$H_n(X_t,t) = \int_0^t H_{n-1}(X_s,s)dX_s + \frac{1}{2}\int_0^t H_{n-2}(X_s,s)d\langle X\rangle_s$$

$$- \frac{1}{2}\int_0^t H_{n-2}(X_s,s)ds.$$

The condition A.1 now implies that almost surely,

$$\int_0^t H_{n-2}(X_s,s)d\langle X\rangle_s = \int_0^t H_{n-2}(X_s,s)ds \quad \text{for all } t \geq 0.$$

In particular, the above equation implies that almost surely,

$$\mu_X\{ s \leq t : H_{n-2}(X_s,s) \neq 0 \} = m \{ s \leq t : H_{n-2}(X_s,s) \neq 0 \}$$

for all $t \geq 0$. By A.2, the proof would be complete if we can show that, almost surely, $\mu_X\{ s : H_{n-2}(X_s,s) = 0 \} = 0$.

For $m \geq 1$, let $Y_m(t) = H_m(X_t,t)$. Denoting by μ_m the random measure on $[0,\infty)$ induced by $\langle Y_m \rangle_t$, we get from eqn. (2) applied to Y_{n-2} that almost surely,

$$\mu_{n-2} \{ s : Y_{n-2}(s) = H_{n-2}(X_s,s) = 0 \} = 0 .$$

Since by Ito's formula and eqn. (4)

$$\langle Y_{n-2} \rangle_t = \int_0^t Y^2_{n-3} \, d \langle X \rangle_s$$

this means that almost surely,

$$\{ s : H_{n-2}(X_s,s) = 0 \} \subset \{ s : H_{n-3}(X_s,s) = 0 \}$$

modulo a μ_X $(= \mu_1)$ null set. Proceeding similarly we get finally that

$$\{ s : H_{n-2}(X_s,s) = 0 \} \subset \{ s : H_1(X_s,s) = X_s = 0 \}$$

modulo a μ_X null set. However eqn. (2) implies that almost surely, $\mu_X \{ s : X_s = 0 \} = 0$. This completes the proof. ///

<u>Remark 1.2.1</u> It is obvious from the proof that the only property of Hermite polynomials that we have used is that they are parabolic functions. It is natural to ask whether Theorem 1.2.1 is true if we replace the Hermite polynomials by a general parabolic function. This general formulation was dealt with in Mcgill, Rajeev and Rao [10]. The theorem proved there can be precisely stated as follows.

<u>Theorem 1.2.2</u> Let (X_t) be a continuous local martingale with $X_0 = 0$. Let $f(x,t)$ be a continuous function on $\mathbb{R} \times [0,\infty)$, satisfying the parabolic equation $f_t + \frac{1}{2} f_{xx} = 0$ on $\mathbb{R} \times (0,\infty)$, and the

growth condition : for all $T > 0$, there exists $C = C(T)$, $\alpha = \alpha(T)$ such that $|f(x,t)| \leq C e^{-\alpha|x|^2}$ on $\mathbb{R} \times [0,T]$. Suppose that the following hold :

A.1 $f(X_t,t)$ is a local martingale

A.2 $m \perp \{t : f_t(X_t,t) = 0\}$ almost surely.

Then (X_t) is a Brownian motion.

Remark 1.2.2 We note that if (X_t) is a Brownian motion and f as above then A.1 and A.2 hold.

Remark 1.2.3 Some condition like A.2 is in general necessary. For example, let τ be a stop time for the Brownian motion (B_t) such that $B_\tau \equiv 0$. Let $X_t = B_{t\wedge\tau}$ and $f(x,t) = H_n(x,t)$ for any $n \geq 3$ and odd. Then A.1 holds and A.2 fails. Indeed, this example can be generalized. Let $\tau_0 \equiv 0 < \tau_1 < \tau_2 < \tau_3$ be zero times of the Brownian motion (B_t). Let $f(s,\omega) = I_{[\tau_0,\tau_1) \cup [\tau_2,\tau_3)}(s,\omega)$ and $X_t = \int_0^t f_s dB_s$.

Then A.1 is still true and A.2 of course is false. However the question remains whether when $n \geq 3$ and even, the condition A.2 can be eliminated. In this case if A.1 holds, then $H_n(|X|_t,t) \equiv H_n(X_t,t)$ is also a local martingale. It is well known that $|X_t| = M_t + \varphi_t$ where φ_t is the local time at the origin of X and (M_t) is a martingale. Thus A.1 now contains additional information and perhaps, A.2 follows. We do not know the answer to this question.

Remark 1.2.4 The condition A.2 is essentially analytic in character. It is pertinent to ask whether it can be replaced by a more probabilistic condition. We do not have an answer to this question but we make the following observation, which will also set the tone for subsequent discussions. The condition A.2 says that the sojourn time of the process (X_t, t) in the set $E = \{(x,t) : f_t(x,t) = 0\}$, measured in the usual way i.e. with lebesgue measure is zero. Crucial to the proof of Theorem 1.2.1 was also the fact that this sojourn time be zero on the quadratic variation scale. Perhaps the probabilistic meaning of A.2 is related to such sojourn times, which would also depend on the structure of the set E.

1.3 Sojourn times of Martingales

In keeping with the spirit of the concluding remarks of the previous section, we now take up the analysis of the 'Sojourn time' in a simple situation viz. when (X_t) is a continuous square integrable martingale and $E = [a,b]$. We will show that when the sojourn time is measured on the quadratic variation scale, it is closely related to the number of crossings of the interval $[a,b]$ by the process (X_t). To make things more precise we fix $t > 0$ and $a < b$ for the rest of this section.

Let $U_t([a,b])$ be the number of upcrossings of $[a,b]$ upto time t, i.e. the largest integer k such that there are pairs $(t_i, s_i)_{i=1}^{k}$ with $X_{t_i} < a$ and $X_{s_i} > b$ and $0 \leq t_1 < s_1 < t_2 < \ldots < t_k < s_k < t$. Similarly let $D_t([a,b])$ be the number of down crossings of $[a,b]$ upto time t. Let $C_t([a,b]) = U_t([a,b]) + D_t([a,b])$ be the total number of crossings of $[a,b]$ upto time t.

Let $\tau_t = \inf \{ s > 0,\ X_s \notin [a,b] \} \wedge t$

= first hitting time of $[a,b]^c$ before t

and

$$\sigma_t = \begin{cases} \max \{ u \leq t : X_u \in [a,b]^c \} \\ t \quad \text{if there is no such } u. \end{cases}$$

= the first time beyond which the path remains in $[a,b]$ upto t, if such a time exists, otherwise it is t.

The process is in $[a,b]$ during $[0,\tau_t)$ as well as in $(\sigma_t,t]$. τ_t of course is a stop time, but σ_t need not in general be a stop time. However it is easy to see that σ_t is $\mathcal{F}_t$-measurable. With the above notations we can state the main theorem in this section (Rajeev, [11]).

<u>Theorem 1.3.1</u> Let (X_t) be a continuous square integrable martingale. Then we have, for all $a < b$ and $t > 0$,

$$E\,\mu_X \{ s \leq t : X_s \in (a,b] \} = E(X_{\tau_t} - X_0)^2 + (b-a)^2\, EC_t([a,b]) + E(X_t - X_{\sigma_t})^2.$$

Before proving the theorem, we shall define for each $n \geq 1$ and $k \geq 0$, the stop times σ_k^n and τ_k^n as follows :

$$\sigma_0^n = 0$$
$$\tau_0^n = \inf \{ \sigma_0^n < s \leq t : X_s < a - \tfrac{1}{n} \text{ or } X_s > b + \tfrac{1}{n} \} \wedge t$$
$$\sigma_1^n = \inf \{ \tau_0^n < s \leq t : |X_s - b| < \tfrac{1}{n+1} \text{ or } |X_s - a| < \tfrac{1}{n+1} \} \wedge t$$
$$\vdots$$
$$\sigma_k^n = \inf \{ \tau_{k-1}^n < s \leq t : |X_s - b| < \tfrac{1}{n+1} \text{ or } |X_s - a| < \tfrac{1}{n+1} \} \wedge t$$
$$\tau_k^n = \inf \{ \sigma_k^n < s \leq t : X_s > b + \tfrac{1}{n} \text{ or } X_s < a - \tfrac{1}{n} \} \wedge t$$
$$\vdots$$

Let $k_n(\omega) = \min\{k \geq 1 : \tau_k^n(\omega) = t\}$. Then $k_n(\omega) < \infty$ almost surely because (X_t) is continuous. σ_k^n and τ_k^n are all $\mathcal{F}_t$ stop times bounded by t and $\tau_k^n = t$ for all $k \geq k_n$, $\sigma_k^n = t$ for all $k > k_n$. Let $E_n = \bigcup_{k=0}^{\infty} (\sigma_k^n, \tau_k^n)$. We start with a series of observations.

__F1__ $E_{n+1} \subset E_n$: Indeed if $s \in E_{n+1}$ then for some k, $s \in (\sigma_k^{n+1}, \tau_k^{n+1})$. Hence $a - \frac{1}{n+1} < X_s < b + \frac{1}{n+1}$. But if $s \notin E_n$, it is easy to see that either $X_s \geq b + \frac{1}{n+1}$ or $X_s \leq a - \frac{1}{n+1}$, a contradiction.

__F2__ $\{0 < s < t : X_s \in [a,b]\} = \bigcap_{n=1}^{\infty} E_n$. As noted above if $s \notin E_n$ then either $X_s \geq b + \frac{1}{n+1}$ or $X_s \leq a - \frac{1}{n+1}$ and hence $X_s \notin [a,b]$. If $s \in E_n \; \forall \; n$ then $a - \frac{1}{n} < X_s < b + \frac{1}{n}$ for all n and hence $X_s \in [a,b]$.

__F3__ $\tau_0^n \downarrow \tau_t$: By F1, τ_0^n's are decreasing. If the path remains in $[a,b]$ upto t, then $\tau_n^0 = t = \tau_t$ for large n. If $\tau_t < t$, then $X_s \in [a,b]$ for all $s < \tau_t$. Hence $\tau_0^n \geq \tau_t$ for all n. Hence $\tau_t \leq \underset{n \to \infty}{Lt} \tau_0^n$. Also for all $u < \underset{n \to \infty}{Lt} \tau_0^n$, $X_u \in [a,b]$.

$\therefore \tau_t = \underset{n \to \infty}{Lt} \tau_0^n$.

__F4__ $\sigma_{k_n}^n \uparrow \sigma_t$: By F1, $\sigma_{k_n}^n$ are increasing. If $\tau_t = t$, then $\sigma_t = \sigma_{k_n}^n = t$, for all n. Let now $\tau_t < t$. By definition of σ_t, $\sigma_{k_n}^n \leq \sigma_t$ for all n. Hence $\underset{n \to \infty}{Lt} \sigma_{k_n}^n \leq \sigma_t$. On the other hand if $\underset{n \to \infty}{Lt} \sigma_{k_n}^n < u < t$, then $X_u \in [a,b]$, hence by definition of σ_t, $\sigma_t \leq u$. This implies that $\underset{n \to \infty}{Lt} \sigma_{k_n}^n \geq \sigma_t$ and hence the result.

F5 From the definition of τ_t it follows that

$$X_{\tau_t} = X_0 \quad \text{if} \quad X_0 \notin [a,b]$$
$$= X_t \quad \text{if} \quad X_s \in [a,b] \ \forall s \le t$$
$$= a \text{ or } b \text{ otherwise.}$$

In particular it follows that $|X_{\tau_t} - X_0| \le b - a$.

F6 X_{σ_t} is $\mathcal{F}_t$ measurable : This follows from the fact that σ_t is $\mathcal{F}_t$ measurable and $X(t,w)$ is jointly $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$ measurable. Further

$$X_{\sigma_t} = a \text{ or } b \quad \text{if} \quad X_t \in [a,b],\ \tau_t < t$$
$$= X_t \qquad \text{if} \quad X_t \notin [a,b] \text{ or } \tau_t = t$$

In particular $|X_t - X_{\sigma_t}| \le b - a$.

Proof of Theorem 1.3.1 By F1 and F2 ,

$$\mu_X\{s \le t : X_s \in [a,b]\} = \operatorname*{Lt}_{n \to \infty} \mu_X(E_n).$$

Also since $\mu_X(E_n) \le \langle X\rangle_t$ for all w, the dominated convergence theorem implies that

$$E\,\mu_X\{s \le t : X_s \in [a,b]\} = \operatorname*{Lt}_{n \to \infty} E\,\mu_X(E_n) \qquad \text{.....(5)}$$

Since $\mu_X(E_n) = \sum_{k=0}^{\infty} (\langle X\rangle_{\tau_k^n} - \langle X\rangle_{\sigma_k^n})$ we have

$$E\,\mu_X(E_n) = E(\sum_{k=0}^{\infty} (X_{\tau_k^n} - X_{\sigma_k^n})^2) \qquad \text{.....(6)}$$

Now we analyse the sum inside the expectation in (6). Firstly since σ_k^n and τ_k^n are eventually equal to t, the sum is in fact a finite sum almost surely. Secondly in case $\tau_k^n < t$, $X_{\sigma_k^n} = a - \frac{1}{n+1}$ or $b + \frac{1}{n+1}$ and $X_{\tau_k^n} = a - \frac{1}{n}$ or $b + \frac{1}{n}$. Hence during the interval $[\sigma_k^n, \tau_k^n]$, either there is an upcrossing of one of the intervals $[a - \frac{1}{n+1}, b + \frac{1}{n}]$, $[b + \frac{1}{n+1}, b + \frac{1}{n}]$ or a downcrossing of one of the intervals $[a - \frac{1}{n}, a - \frac{1}{n+1}]$, $[a - \frac{1}{n}, b + \frac{1}{n+1}]$. Hence the sum inside the expectation can be broken up into six parts as follows: An initial term and a final term, a term corresponding to upcrossings of $[a - \frac{1}{n+1}, b + \frac{1}{n}]$, a term corresponding to downcrossings of $[a - \frac{1}{n}, b + \frac{1}{n+1}]$, a term corresponding to downcrossings of $[a - \frac{1}{n}, a - \frac{1}{n+1}]$ and a term corresponding to upcrossings of $[b + \frac{1}{n+1}, b + \frac{1}{n}]$. This gives us,

$$\begin{aligned}\sum_{k=0}^{\infty} (X_{\tau_k^n} - X_{\sigma_k^n})^2 = {} & (X_o - X_{\tau_o^n})^2 + (b - a + \tfrac{1}{n} + \tfrac{1}{n+1})^2\, U_t([a - \tfrac{1}{n+1}, b + \tfrac{1}{n}]) \\ & + (b - a + \tfrac{1}{n} + \tfrac{1}{n+1})^2\, D_t([a - \tfrac{1}{n}, b + \tfrac{1}{n+1}]) \\ & + I_1^n + I_2^n + (X_{\tau_{k_n}^n} - X_{\sigma_{k_n}^n})^2\end{aligned}$$

where $I_1^n \leq (\frac{1}{n} - \frac{1}{n+1})^2\, D_t([a - \frac{1}{n}, a - \frac{1}{n+1}])$

$$I_2^n \leq (\tfrac{1}{n} - \tfrac{1}{n+1})^2\, U_t([b + \tfrac{1}{n+1}, b + \tfrac{1}{n}]).$$

By Doob's lemma on crossings, $E(I_1^n) \to 0$ and $E(I_2^n) \to 0$ as $n \to \infty$. Regarding the last term note that $\tau_{k_n}^n = t$ and if $\sigma_{k_n}^n < t$, then $X_t \in [a - \frac{1}{n}, b + \frac{1}{n}]$. Also by F4, $(X_{\tau_{k_n}^n} - X_{\sigma_{k_n}^n})^2 \to (X_t - X_{\sigma_t})^2$. Hence by the bounded convergence theorem we have, $E(X_{\tau_{k_n}^n} - X_{\sigma_{k_n}^n})^2 \to E(X_t - X_{\sigma_t})^2$. Similarly $E(X_o - X_{\tau_o^n})^2 \to E(X_o - X_{\tau_t})^2$.

Also we note that as $n \to \infty$, $U_t([a - \frac{1}{n+1}, b + \frac{1}{n}]) \uparrow U_t([a,b])$.
Hence $E\, U_t([a - \frac{1}{n+1}, b + \frac{1}{n}]) \to E\, U_t([a,b])$. Similarly
$E\, D_t([a - \frac{1}{n}, b + \frac{1}{n+1}]) \longrightarrow E\, D_t([a,b])$ Hence, we have

$$\underset{n \to \infty}{Lt} E\left(\sum_{k=0}^{\infty} (X_{\tau_k^n} - X_{\sigma_k^n})^2\right) = E(X_{\sigma} - X_{\tau_t})^2 + (b-a)^2 E(U_t([a,b]) + D_t([a,b])) + E(X_t - X_{\sigma_t})^2 \qquad \ldots\ldots(7)$$

The proof is now completed using (5), (6) and (7). ///

Corollary 1.3.1 For a continuous square integrable martingale (X_t) we have

$$\underset{\varepsilon \to 0}{Lt}\ \varepsilon\ E\, C_t([x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}]) = EL(t,x)$$

for all $x \in \mathbb{R}$ and for all $t \geq 0$, where $L(t,x)$ is the local time of (X_t) at time t at the point x.

Proof The proof is immediate from Theorem 1.3.1 and eqn. (2), taking $a = x - \frac{\varepsilon}{2}$, $b = x + \frac{\varepsilon}{2}$ in Theorem 1.3.1 and $f = I_{[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]}$ in eqn. (2). ///

Before stating the next Corollary we observe the following :

Lemma 1.3.1 For a continuous local martingale (X_t) the following statements are equivalent :

(a) $\langle X\rangle_t \uparrow \infty$ as $t \to \infty$ almost surely

(b) $\overline{\underset{t \to \infty}{Lt}}\, X_t = +\infty$ and $\underline{\underset{t \to \infty}{Lt}}\, X_t = -\infty$ almost surely.

(c) $\underline{\underset{t \to \infty}{Lt}}\, X_t < \overline{\underset{t \to \infty}{Lt}}\, X_t$ almost surely.

Proof (a) implies (b) : This is because (b) is true for Brownian motion and hence for $(\cdot_t)$ by time change.

(b) implies (c) is trivial.

(c) implies (a) : This is because $X_t = B(\langle X\rangle_t)$ for some Brownian motion $B(t)$ on an extension space (see N. Ikeda and S. Watanabe, [3, p. 85-86]).

Corollary 1.3.2 Let (X_t) be a continuous square integrable martingale with $\langle X\rangle_t \uparrow \infty$ almost surely as $t \to \infty$. Then, for any $a < b$,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{E\,\mu_X \{ s \le t : X_s \in [a,b] \}}{E\, C_t([a,b])} = (b-a)^2 .$$

Proof By hypothesis on $\langle X\rangle_t$ and lemma 1.3.1 (b) it follows that almost surely, $\underset{t \to \infty}{\mathrm{Lt}} \; C_t([a,b]) = \infty$. The result now follows from Theorem 1.3.1 together with F5 and F6. ///

Remark 1.3.1 Corollary 1.3.1 is an expected version of the well known Levy crossing theorem. We shall prove this in greater generality in Chapter III. One way of looking at the crossing theorem is via the Ito-Maisonneuve excursion theory. We remark that Theorem 1.3.1 can also be derived from excursion theory. We refer to Meyer, [9] for an excursion theoretic proof of Theorem 1.3.1. Corollary 1.3.2 says that in the long run the average time spent in [a,b] per crossing is $(b-a)^2$. This result will be strengthened and generalised in Chapter III when we deal with applications.

1.4 A Semi-Martingale associated with crossings

In the light of the comments made in Remark 1.3.1 above, we now make a closer scrutiny of Theorem 1.3.1. We proceed as follows.

$$\text{Let} \quad Y_t = (b-a)^2 \, C_t([a,b]) + (X_t - X_{\sigma_t})^2$$

and
$$Y'_t = Y_t - \int_0^t I_{(a,b)}(X_s) \, d\langle X\rangle_s .$$

It is easily seen that (Y_t) and (Y'_t) are continuous $\mathcal{F}_t$ adapted processes. Assuring that $X_0 \notin [a,b]$ almost surely, Theorem 1.3.1 says that $E\,Y'_t = 0$ for all t. If it were true that $E\,Y'_\tau = 0$ for all bounded stop times τ, then it would follow that (Y'_t) is a martingale or equivalently that (Y_t) is a semi-martingale. We will show that this is indeed true in a more general set up, in the next chapter. Thus (Y_t) is a semi-martingale associated with the crossings of $[a,b]$ by X. The term $(X_t - X_{\sigma_t})^2$ corresponds to the unfinished crossing at time t. In this section we will show that the process

$$Z_t = (b-a)\, C_t([a,b]) + |X_t - X_{\sigma_t}|$$

is a non-negative submartingale and determine its martingale and bounded variation parts. This also follows from the more general result proved in Chapter II. The reason for treating the special case here is twofold. Firstly the proof is elementary in the sense that it does not use deep semi-martingale results. Secondly the proof uses the techniques developed in the course of the proof of

Theorem 1.3.1. As in Section 1.3 we assume that (X_t) is a continuous square integrable martingale. From the observations made in Section 1.3 (see F7) it is easy to see that the process Z_t defined above is a continuous $\mathcal{F}_t$ adapted process. We fix $t > 0$ and define two sequences of stop times, (η_k^d) and (η_k^u) as follows :

$$\eta_0^d = \inf \{ s > 0 : X_s > b \} \wedge t$$
$$\eta_1^d = \inf \{ s > \eta_0^d : X_s < a \} \wedge t$$
$$\vdots$$
$$\eta_{2k}^d = \inf \{ s > \eta_{2k-1}^d : X_s > b \} \wedge t$$
$$\eta_{2k+1}^d = \inf \{ s > \eta_{2k}^d : X_s < a \} \wedge t$$
$$\vdots$$

Let $k^d(\omega) = \min \{ k : \eta_{2k+1}^d = t \}$. Then, for all $n < k^d$, $\eta_{2n+1}^d < t$. Similarly we define (η_k^u) as

$$\eta_0^u = \inf \{ s > 0 : X_s < a \} \wedge t$$
$$\eta_1^u = \inf \{ s > \eta_0^u : X_s > b \} \wedge t$$
$$\vdots$$
$$\eta_{2k}^u = \inf \{ s > \eta_{2k-1}^u : X_s < a \} \wedge t$$
$$\eta_{2k+1}^u = \inf \{ s > \eta_{2k}^u : X_s > b \} \wedge t$$
$$\vdots$$

Let $k^u(\omega) = \min \{ k : \eta_{2k+1}^u = t \}$. Then for all $n < k^u$, $\eta_{2n+1}^u < t$.

Let $$\theta^d(s,\omega) = \sum_{k=0}^{\infty} I_{(\eta_{2k}^d,\, \eta_{2k+1}^d]}(s)$$

$$\theta^u(s,\omega) = \sum_{k=0}^{\infty} I_{(\tau^u_{2k},\, \tau^u_{2k+1}]}(s)$$

and $\qquad \theta(s,\omega) = \theta^u(s,\omega) - \theta^d(s,\omega)$.

We note that θ is a bounded predictable function and $\theta^u(s)\theta^d(s) \equiv 0$. We note that $\theta(s) = 0$ for $s \in [0,\tau]$ where τ is the first exit time of (X_t) from $[a,b]$. We recall that $L(t,a)$ denotes the local time process of (X_t).

Let $\quad \varphi(s,\omega) = I_{[a,b]}(X_s)$

We can now state the main result in this section (Rajeev [12]).

<u>Theorem 1.4.1</u> Let (X_t) be a continuous square integrable martingale and $a < b$. Then almost surely,

$$(b-a)\, C_t([a,b]) + |X_t - X_{\sigma_t}| = \int_0^t \varphi(s)\,\theta(s)\,dX_s + \tfrac{1}{2}(L(t,a)+L(t,b)) \quad \ldots(8)$$

for all $t \geq 0$ and hence the process in the left side is an $\mathcal{F}_t$ semi-martingale.

To prove the theorem we need the following lemma which can be considered an L_2-version of Levy's crossing theorem.

<u>Lemma 1.4.1</u> For any $b \in \mathbb{R}$ and $t > 0$,

i) $\quad \underset{\epsilon \to 0}{Lt}\ \epsilon\, D_t([b,b+\epsilon]) \overset{L_2}{=} \frac{1}{2} L(t,b)$

ii) $\quad \underset{\epsilon \to 0}{Lt}\ \epsilon\, U_t([b-\epsilon,b]) \overset{L_2}{=} \frac{1}{2} L(t,b)$.

<u>Proof</u> For $\epsilon > 0$, we define the following sequence of stop times :

$$\sigma_1^\varepsilon = \inf\{ s > 0 : X_s > b + \varepsilon \}$$

$$\tau_1^\varepsilon = \inf\{ s > \sigma_1^\varepsilon : X_s < b \}$$

$$\vdots$$

$$\sigma_k^\varepsilon = \inf\{ s > \tau_{k-1}^\varepsilon : X_s > b + \varepsilon \}$$

$$\tau_k^\varepsilon = \inf\{ s > \sigma_k^\varepsilon : X_s < b \}$$

$$\vdots$$

Let $f^\varepsilon(t,\omega) = \sum_{1}^{\infty} I_{(\sigma_k^\varepsilon, \tau_k^\varepsilon]}(t,\omega)$. Then f^ε is a bounded $\mathcal{F}_t$ - predictable function and we have for any $t > 0$

$$\int_0^t f^\varepsilon(s)dX_s = \sum_{k=1}^{\infty} (X_{\tau_k^\varepsilon \wedge t} - X_{\sigma_k^\varepsilon \wedge t})$$

$$= -\varepsilon D_t([b, b+\varepsilon]) + (X_t - b - \varepsilon)f^\varepsilon(t,\omega)$$

$$- (X_0 - (b + \varepsilon))^+ \qquad \ldots\ldots \quad (9)$$

As $\varepsilon \to 0$, almost surely, $f^\varepsilon(t,\omega) \to I_{\{x > b\}}(X_t)$ for almost every t, $d\langle X\rangle$. This can be seen as follows : Fix any (t,ω). If $X_t < b$, then $f^\varepsilon(t,\omega) = 0$ for all ε. If $X_t > b$, then for sufficiently small ε, there exists $k = k(\varepsilon)$ such that $t \in [\sigma_k^\varepsilon, \tau_k^\varepsilon]$. In other words $f^\varepsilon(t,\omega) = 1$ for sufficiently small ε. Since $\mu_X\{t : X_t = b\} = 0$, the proof of the claim is complete. Hence $\int_0^t f^\varepsilon(s)dX_s \to \int_0^t I_{\{x > b\}}(X_s)dX_s$ in L_2. Also it is easy to see that $(X_t - b - \varepsilon) f^\varepsilon(t,\omega) \to (X_t - b)^+$ almost surely and hence in L_2 by dominated convergence. Similarly $(X_0 - (b+\varepsilon))^+ \to (X_0 - b)^+$ in L_2. Now taking $\varepsilon \to 0$ in eqn. (9) and using the Tanaka formula (1), the proof of i) is complete. The proof of ii) is similar. ///

Proof of Theorem 1.4.1 We first note that if $a_n \uparrow a$, then $L(t,a_n) \longrightarrow L(t,a)$ in L_2. This is an easy consequence of the Tanaka formula. Fix a sequence (a_n) strictly increasing to a. By Lemma 1.4.1 we can choose a_n' such that $a_n < a_n' < a_{n+1}$ and such that

$$\| (a_n' - a_n) D_t([a_n,a_n']) - L(t,a_n) \|_{L_2} < \frac{1}{2^n} .$$

$$\therefore \operatorname*{Lt}_{n \to \infty} (a_n' - a_n) D_t([a_n,a_n']) \overset{L_2}{=} \frac{1}{2} L(t,a)$$

Similarly fix a sequence (b_n) strictly increasing to b, and for each n, fix b_n' such that $b_{n+1} < b_n' < b_n$ and

$$\operatorname*{Lt}_{n \to \infty} (b_n - b_n') U_t([b_n',b_n]) \overset{L_2}{=} \frac{1}{2} L(t,b)$$

Now as in Section 1.3, we define a sequence of stop times (σ_k^n), (τ_k^n) as follows :

$$\sigma_0^n \equiv 0$$

$$\tau_0^n = \inf \{ 0 < s \le t : X_s < a_n \text{ or } X_s > b_n \} \wedge t$$

$$\vdots$$

$$\sigma_k^n = \inf \{ \tau_{k-1}^n < s \le t : |X_s - a| < |a - a_n'| \text{ or } |X_s - b| < |b - b_n'| \} \wedge t$$

$$\tau_k^n = \inf \{ \sigma_k^n < s \le t : X_s < a_n \text{ or } X_s > b_n \} \wedge t$$

$$\vdots$$

Let $k_n(\omega) = \min \{ k : \tau_k^n = t \}$. Then $k_n(\omega) < \infty$ almost surely, $\tau_k^n = t$ for all $k \ge k_n$, and $\sigma_k^n = t$ for all $k > k_n$. Let $E_n = \bigcup_{k=0}^{\infty} (\sigma_k^n, \tau_k^n)$. Then it is easy to see that the properties F1-F6 of Section 1.3 hold for the E_n's defined above.

Let $\varphi_n(s,w) = \sum_{k=0}^{\infty} I_{(\sigma_k^n,\, \tau_k^n]}(s)$. Then φ_n's are a sequence of bounded predictable functions and almost surely $\varphi_n(s) \to \varphi(s)$ for almost every s, $d\langle X\rangle$, because of F-2, Section 1.3. Hence

$$\int_0^t \varphi_n(s)\theta(s)dX_s \xrightarrow{L_2} \int_0^t \varphi(s)\theta(s)dX_s$$

But $$\int_0^t \varphi_n(s)\theta(s)dX_s = \int_0^t \varphi_n(s)\theta^u(s)dX_s - \int_0^t \varphi_n(s)\theta^d(s)dX_s$$

$$= I_1 - I_2 .$$

We now look at the integral I_1 more closely. The integrand in I_1 is

$$\varphi_n(s)\,\theta^u(s) = \left(\sum_{k=0}^{\infty} I_{(\sigma_k^n,\tau_k^n]}(s)\right)\left(\sum_{j=0}^{\infty} I_{(\eta_{2j}^u,\eta_{2j+1}^u]}(s)\right)$$

$$= \sum_{j,k} I_{(\sigma_k^n \vee \eta_{2j}^u,\; \tau_k^n \wedge \eta_{2j+1}^u]}(s) .$$

We note that all the sums involved are finite almost surely. Hence

$$\int_0^t \varphi_n(s)\,\theta^u(s)dX_s = \sum_{j,k}\left(X_{(\tau_k^n \wedge \eta_{2j+1}^u)\wedge t} - X_{(\sigma_k^n \vee \eta_{2j}^u)\wedge t}\right)$$

Each summand on the right side takes one of the values $(b-a)$, $(a_n - a_n')$, $(b-a_n')$ or $(a_n - a)$ except when $(j,k) = (0,0)$ or $(k_n, 2k^u)$. Accordingly we can write

$$\int_0^t \varphi_n(s)\,\theta^u(s)dX_s = \sum_{i=1}^{6} S_i .$$

To explain S_i, let us denote by $U([a,b])$ etc. the number of upcrossings of $[a,b]$ etc. by (X_t) in the time interval

$(\tau_o^n \wedge \eta_1^u) \wedge t, (\sigma_{k_n}^n \vee \eta_{2k^u}^u) \wedge t]$. We denote by $U([c,d],[a,b])$ a number of upcrossings of $[c,d]$ during the time interval $[(\tau_o^n \wedge \eta_1^u) \wedge t, (\sigma_{k_n}^n \vee \eta_{2k^u}^u) \wedge t]$ which contain at least one crossing (up or down) of $[a,b]$. A similar notation is used for the downcrossings. We also note that for all w and for sufficiently large n, there are no upcrossings of $[a,b]$ or downcrossings of $[a_n,a_n']$ in either of the time intervals $[(\sigma_o^n \vee \eta_o^u) \wedge t, (\tau_o^n \wedge \eta_1^u) \wedge t]$ or $[(\sigma_{k_n}^n \vee \eta_{2k^u}^u, t]$. With the above notation we have,

$$S_1 = X_{(\tau_o^n \wedge \eta_1^u) \wedge t} - X_{(\sigma_o^n \wedge \eta_o^u) \wedge t} .$$

$$S_2 = (b-a) \{ U([a,b]) - U([a_n',b]) - D([a_n,a_n'],[a,b]) \}$$

$$S_3 = (a_n - a_n') \{ D([a_n,a_n']) - D([a_n,b_n']) - D([a_n,a_n'],[a,b]) \}$$

$$S_4 = (b-a_n') \{ U([a_n',b_n]) + D([a_n,a_n'],[a,b]) \}$$

$$S_5 = (a_n - a) \{ D([a_n,b_n']) + D([a_n,a_n'],[a,b]) \}$$

$$S_6 = X_{(\tau_{k_n}^n \wedge \eta_{2k^u+1}^u) \wedge t} - X_{(\sigma_{k_n}^n \vee \eta_{2k^u}^u) \wedge t}$$

since $\tau_o^n \downarrow \tau_t$, the first exit time from $[a,b]$ before t, S_1 tends to zero almost surely. Since by F4, Section 1.3, $\sigma_{k_n}^n \uparrow \sigma_t$ and $\tau_{k_n}^n = t$, $\eta_{2k^u+1}^u = t$, S_6 converges to $(X_t - X_{\sigma_t \vee \eta_{2k^u}^u})$ almost surely and in L_2. It is easy to see that $S_2 + S_4$ increases to $(b-a)\, U_t([a,b])$. Regarding the first term in S_3, we note that

$$(a_n - a_n')\, D([a_n,a_n']) = (a_n - a_n')\, D_t([a_n,a_n'])$$

$$\xrightarrow{L_2} -\tfrac{1}{2}\, L(t,a)$$

as noted in the beginning of the proof. The other two terms in S_3 are bounded by $(a_n - a_n') D_t([a,b])$ and since $a_n - a_n' \to 0$, these two terms tend to zero almost surely and in L_2. Similarly S_4 converges to zero almost surely and in L_2.

$$\therefore \operatorname*{Lt}_{n\to\infty} \int_0^t \varphi_n(s)\, \theta^u(s)\, dX_s \overset{L_2}{=} (b-a)\, U_t([a,b])$$

$$+ \; (X_t - X_{\sigma_t \vee \eta^u_{2k^u}}) - \tfrac{1}{2} L(t,a).$$

In an analogous fashion we get

$$\operatorname*{Lt}_{n\to\infty} \int_0^t \varphi_n(s)\, \theta^d(s)\, dX_s \overset{L_2}{=} (a-b)\, D_t([a,b])$$

$$+ \; (X_t - X_{\sigma_t \vee \eta^d_{2k^d}}) + \tfrac{1}{2} L(t,a)$$

subtracting the above two equations, the proof is complete. ///

<u>Remark 1.4.1</u> The above proof proves equation (8) for each t, almost surely. The null set in question may depend on t. However since both sides of (8) are continuous process, it follows by the usual arguments that (8) holds almost surely for all $t \geq 0$.

<u>Remark 1.4.2</u> We have used Tanaka's formula in deriving Theorem 1.4.1. Conversely by letting $b \to \infty$ in Theorem 1.4.1, we can get back Tanaka's formula. Note that for fixed t, w, $D_t([a,b]) = 0$ for large b. $L(t,b)$ is also zero for large b, since it is supported on the set $\{s : X_s = b\}$. Strictly speaking, letting $b \to \infty$ gives Tanaka's formula only in the interval $[T,\infty)$

where τ is the first hit of $(-\infty, a)$ by (X_t) But on $[0, \tau)$, $X_{\tau \wedge t} - X_0$ itself does the j b.

Remark 1.4.3 Theorem 1.4.1 says that the process $(b-a)\, C_t([a,b]) + |X_t - X_{\sigma_t}|$ is a non negative submartingale whose increasing part in the Doob-Meyer decomposition is $\frac{1}{2}(L(t,a) + L(t,b))$. The asymptotic behaviour of the process (X_t) gets reflected in the convergence of this submartingale. For example if (X_t) is Brownian motion then $\overline{\lim_{t \to \infty}}\, X_t = \pm \infty$, so that this submartingale converges to ∞ almost surely as $t \to \infty$ for all $a < b$.

Remark 1.4.4 In addition to the previous remark we note that $(b-a)\, C_t([a,b])$ is an increasing left continuous process which equals the sum of the jumps of $|X_t - X_{\sigma_t}|$. Thus even though $|X_t - X_{\sigma_t}|$ would not be a semi-martingale in the usual sense i.e. with rcll paths, we can consider it as a semi-martingale with left continuous paths. To avoid this technicality we restrict ourselves to crossings of open intervals when we make a more detailed study of crossings in the next chapter.

CHAPTER - II

Semi-martingales Associated with Crossings

2.1 Introduction

In this chapter we will study the crossings of an interval by a continuous semi-martingale. The results of the previous chapter give a broad indication of the nature of the results we wish to prove. In the notation of the previous chapter, we have proved that when (X_t) is a continuous square integrable martingale, then $|X_t - X_{\sigma_t}|$ is a " semi-martingale " . By applying Ito's formula to $|X_t - X_{\sigma_t}|$, it is easy to see that $(X_t - X_{\sigma_t})^2$ is a semi-martingale whose sum of the jumps upto time t is $(b-a)^2 C_t([a,b])$ (see Rajeev [12] and also the introductory comments of Section 1.4). The natural question then is whether $X_t - X_{\sigma_t}$ itself is a semi-martingale or equivalently whether X_{σ_t} is a semi-martingale. The answer to this question is indeed yes not only when (X_t) is a martingale but also in the case when it is a semi-martingale. This is not difficult to see and is in fact an immediate consequence of Tanaka's formula. Our main aim in this chapter is to describe the semi-martingale $|X_t - X_{\sigma_t}|$ or equivalently to describe the local times of $X_t - X_{\sigma_t}$ in terms of the local times of (X_t), when (X_t) is a continuous semi-martingale (see Rajeev [13]). An essential tool in this analysis is the theory of local times of semi-martingales (see Meyer [8, p. 361 - 371], and Yor [19]). Unlike in the previous chapter we will consider crossings of the open interval (a,b) rather than the closed interval $[a,b]$. This is mainly for technical reasons, as noted at the end of the first chapter (Remark 1.4.4). Of course for the Brownian motion it makes no difference.

2.2 Preliminaries

Let (X_t) be a continuous process adapted to the filtration $\mathcal{F}_t$. For $a < b$, we define the following sequence of stop times :

$$\sigma_0 = \inf \{ s > 0 : X_s \leq a \}$$

$$\sigma_1 = \inf \{ s > \sigma_0 : X_s \geq b \}$$

$$\vdots$$

$$\sigma_{2k} = \inf \{ s > \sigma_{2k-1} : X_s \leq a \}$$

$$\sigma_{2k+1} = \inf \{ s > \sigma_{2k} : X_s \geq b \}$$

$$\vdots$$

Similarly let

$$\tau_0 = \inf \{ s > 0 : X_s \geq b \}$$

$$\tau_1 = \inf \{ s > \tau_0 : X_s \leq a \}$$

$$\vdots$$

$$\tau_{2k} = \inf \{ s > \tau_{2k-1} : X_s \geq b \}$$

$$\tau_{2k+1} = \inf \{ s > \tau_{2k} : X_s \leq a \}$$

$$\vdots$$

Observe that either $\sigma_{2k+1} = \tau_{2k}$ or $\sigma_{2k} = \tau_{2k+1}$. But to avoid cumbersome notation we have defined the sequences (σ_i), (τ_i) separately.

Let $\quad e^U(s) = \sum_{k=0}^{\infty} I_{(\sigma_{2k}, \sigma_{2k+1}]}(s)$

$$\theta^d(s) = \sum_{k=0}^{\infty} I_{(\tau_{2k}, \tau_{2k+1}]}(s)$$

$$\theta(s) = \theta^u(s) - \theta^d(s) .$$

We note that $\theta(s) = 1$ during an upcrossing of (a,b) and $\theta(s) = -1$ during a downcrossing. For $t \geq 0$, let

$$U(t) = \max\{k : \sigma_{2k+1} \leq t\}$$

$$D(t) = \max\{k : \tau_{2k+1} \leq t\}$$

$$C(t) = U(t) + D(t)$$

In the notation of Chapter I, $U(t) = U_t((a,b))$, $D(t) = D_t((a,b))$ and $C(t) = C_t((a,b))$, i.e. $U(t)$ and $D(t)$ are respectively the number of upcrossings and downcrossings of (a,b) in time t and $C(t)$ is the total number of crossings of (a,b) in time t.

Let $\qquad \tau = \inf\{s > 0 : X_s \notin (a,b)\}$

$$\tau_t = \tau \wedge t$$

$$\sigma_t = \begin{cases} \max\{s \leq t : X_s \in (a,b)^c\} \\ t \qquad \text{if } X_s \in (a,b) \text{ for all } s \leq t . \end{cases}$$

τ_t is of course a stop time. As noted earlier, σ_t and X_{σ_t} are $\mathcal{F}_t$ measurable.

We now recall some facts regarding local times of semi - martingales. Let (X_t) be an $\mathcal{F}_t$- semi-martingale with rcll paths, i.e. $X_t = X_0 + M_t + V_t$ where (M_t) is a local martingale and (V_t)

$\mathcal{F}_t$- adapted process of bounded variation on $[0,t]$, for $t \geq 0$. We denote by (X_t^c) the continuous martingale part of . ΔX_s will denote the jump of X at time s. $\mathcal{P}$ will denote predictable σ-field of the filtration $\mathcal{F}_t$. Let $L(t,x)$ denote local time of the process (X_t) at the point $x \in \mathbb{R}$ at time t. for each $x \in \mathbb{R}$, $L(t,x)$ is an increasing continuous process t, satisfying the Tanaka formula viz. almost surely,

$$(X_t - x)^+ = (X_0 - x)^+ + \int_0^t I_{(x,\infty)}(X_{s-})\, dX_s + \sum_{0<s\leq t} I_{(x,\infty)}(X_{s-})(X_s - x)^-$$

$$+ \sum_{0<s\leq t} I_{(-\infty,x]}(X_{s-})(X_s - x)^+ + \frac{1}{2} L(t,x) \quad \ldots (1)$$

note that the third and fourth terms on the RHS in the above uation are zero when X is continuous.

Further for each $x \in \mathbb{R}$, almost surely the points of increase the function $t \to L(t,x)$ is given by the set $\{s : X_s(\omega) = X_{s-}(\omega)\}$, nd for all bounded borel functions f on $\mathbb{R}$, the following occupation ensity formula holds : almost surely,

$$\int_0^t f(X_s)\, d\langle X^c\rangle_s = \int_{-\infty}^{\infty} f(a)L(t,a)\, da \quad \ldots (2)$$

for all $t \geq 0$.

It is easy to check from (1) above that for each $x \in \mathbb{R}$, almost surely, $L(t,x)$ satisfies

$$(X_t - x)^- = (X_0 - x)^- - \int_0^t I_{(-\infty, x]}(X_{s-})dX_s + \sum_{0<s\leq t} I_{(x,\infty)}(X_{s-})(X_s - x)^-$$

$$+ \sum_{0<s\leq t} I_{(-\infty, x]}(X_{s-})(X_s - x)^+ + \frac{1}{2} L(t,x) \quad \ldots\ldots (3)$$

We refer to Meyer [8, p. 361-371] for the proofs of the above properties of $L(t,x)$. The behaviour of the local time as a joint function of the triple (t,x,ω) has been studied by M. Yor. In particular he has obtained results on the joint continuity properties of $L(t,x)$ in the pair (t,x). For future reference we state the following result (see Yor [19]).

<u>Theorem 2.2.1</u> Let (X_t) be a semi-martingale such that for all t, $\sum_{s\leq t} |\Delta X_s| < \infty$ almost surely. Then almost surely,

$X_t - \sum_{0<s\leq t} \Delta X_s = X_0 + M_t + V_t$ where (M_t) is a continuous local martingale and (V_t) is a continuous adapted process of bounded variation. Let $L(t,x)$ denote the local time process of (X_t). Then there exists a version of $L(t,x)$ which is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ measurable and such that for all ω, it is jointly continuous from the right in x and continuous in t. If we denote this version again by $L(t,x)$, then almost surely, the limits $L(t-,x-)$ and $L(t,x-)$ exist and are equal. Further, almost surely

$$L(t,x) - L(t,x-) = 2 \int_0^t I_{\{X_s = x\}} \, dV_s \qquad \ldots\ldots (4)$$

for $x \in \mathbb{R}$ and $t \geq 0$.

2.3 The Semi-martingales $X_t - X_{\sigma_t}$ and X_{σ_t}

From now on we fix a continuous semi-martingale $X_t = X_0 + M_t + V_t$. We also fix $a < b$. Let $Y^1_t = X_t - X_{\sigma_t}$ and $Y^2_t = X_{\sigma_t}$. We note that Y^1_t and Y^2_t are $\mathcal{F}_t$ adapted and depend on the interval (a,b). However they are no longer continuous. We shall show in Theorem 2.3.1 below that they are indeed semi-martingales. To prove the theorem we need the following easy lemma.

Lemma 2.3.1 For all ω and for all $t \geq 0$, the following pathwise identity holds :

$$(X_{\tau_t} - X_0) + (b-a)\,[U(t) - D(t)] + (X_t - X_{\sigma_t})$$

$$= (X_t - a)^+ - (X_0 - a)^+ - (X_t - b)^+ + (X_0 - b)^+ .$$

Proof. The proof follows by direct verification for each of the possible positions of X_0 and X_t with respect to the interval (a,b). ///

Theorem 2.3.1 The processes $Y^1_t = X_t - X_{\sigma_t}$ and $Y^2_t = X_{\sigma_t}$ are both semi-martingales. More precisely we have, almost surely

$$Y^1_t = \int_{\tau_t}^{t} I_{(a,b]}(X_s)\,dX_s - \frac{1}{2}\,(L(t,b) - L(t,a)) - (b-a)(U(t)-D(t)) \quad \ldots\ldots (5)$$

and
$$Y^2_t = X_0 + \int_0^{\tau_t} I_{(a,b]}(X_s)\,dX_s + \int_0^{t} I_{(a,b]^c}(X_s)\,dX_s$$

$$+ \frac{1}{2}\,(L(t,b) - L(t,a)) + (b-a)(U(t)-D(t)) \quad \ldots\ldots (6)$$

for all $t \geq 0$.

Proof. By Lemma 2.3.1 and Tanaka's formula (1),

$$- X_a) + (b - a)\,(U(t) - D(t)) + (X_t - X_{\sigma_t})$$

$$\int_0^t I_{(a,b]}(X_s)\,dX_s - \frac{1}{2}\,(L(t,a) - L(t,b)) .$$

.e $(b - a)(U(t) - D(t))$ is an adapted cadlag process of bounded iation, the above equation shows that Y^1_t and hence Y^2_t are i-martingales. The required decompositions for Y^1_t and Y^2_t are n immediate.

Remark 2.3.1 We note that $|Y^1_t| \leq b - a$ for all t. rther $Y^1_t \geq 0$ whenever $\theta^u(t) = 1$, i.e. during an upcrossing and ≤ 0 whenever $\theta^d(t) = 1$, i.e. during a downcrossing. Also, $= 0$ if $X_t \notin (a,b)$. In particular, $Y^1_t = 0$ if $X_t = a$ or b.

Remark 2.3.2 The sum of the jumps of Y^1 upto time t — i. $\sum_{s\leq t} \Delta Y^1_s$ — is precisely $-(b - a)(U(t) - D(t))$. Similarly sum of the jumps of the process Y^2 in time t is $(b-a)(U(t)-D(t))$. ice $|U(t) - D(t)| \leq 1$, this implies that Y^1 and Y^2 are indeed ecial semi-martingales (see Meyer [8, p. 310 - 316]). The jump es for these processes are precisely the times of crossings of (a,b) X, and in both cases the size of the jumps have absolute values - a. The decomposition of Y^1 and Y^2, given by equations (5) and) above, into a continuous martingale part and a bounded variation rt is unique. However the canonical decomposition of Y^1 and Y^2 n which the bounded variation part is predictable, may be different n that given by equations (5) and (6).

Remark 2.3.4 In the case of a semi-martingale (X_t) with $\sup_t |\Delta X_s| < \infty$ almost surely, for all t, and σ_t defined as in section 2.2, analogs of Lemma 2.3.1 and Theorem 2.3.1 are still true. In this case we can write (see Yor [ ]) $X_t - \sum_{s\leq t} \Delta X_s = X_0 + M_t + V_t$, where (M_t) is a continuous local martingale and (V_t) is a continuous process of bounded variation. Denoting $X_t - X_{\sigma_t}$ by Y_t and assuming $X_0 \notin (a,b)$ almost surely we have,

$$- \sum_{s\leq t} \Delta Y_s + Y_t = (X_t - a)^+ - \sum_{s\leq t} \Delta (X_s - a)^+ - (X_t - b)^+$$

$$+ \sum_{s\leq t} \Delta (X_s - b)^+$$

$$= \int_0^t I_{(a,b)}(X_s)\,dM_s + \int_0^t I_{(a,b)}(X_s)\,dV_s$$

$$+ \frac{1}{2}\left(L(t,a) - L(t,b)\right)$$

where we have used Tanaka's formula in the last step.

2.4 Local times of $X_t - X_{\sigma_t}$ and X_{σ_t}

In this section we will determine the structure of the local times of the semi-martingales Y^1 and Y^2 in terms of the local times of X. The Y^1 process takes values in the interval $(-(b-a), (b-a))$. It lives in $[0, b-a)$ during an upcrossing of (a,b) by X and in $(-(b-a), 0]$ during a downcrossing. Also $Y_t = 0$ whenever $X_t = a$ or b. We will see below that these aracteristics of the process Y^1 get reflected in the structure f its local time. In particular the discontinuity of Y^1 at the ossing times is reflected in a discontinuity at the origin of its ocal time. The case of the Y^2 process is analogous and somewhat mplementary to that of the Y^1 process.

Let $L_1(t,x)$ denote the local time of the Y_1 process. Th n we have the following lemma.

<u>Lemma 2.4.1</u>

(i) For $x \in [0, b-a)$,

$$(Y^1_t - x)^+ = \int_{T_t}^{t} I_{(a,b]}(X_s) I_{(x,\infty)}(Y^1_{s-}) dX_s$$

$$- (b - a - x)\, U(t) + \frac{1}{2} L_1(t,x) \qquad \ldots\ldots (7)$$

(ii) For $x \in (-(b-a), 0]$,

$$(Y^1_t - x)^- = - \int_{T_t}^{t} I_{(a,b]}(X_s)\, I_{(-\infty,x]}(Y^1_{s-}) dX_s$$

$$+ \frac{1}{2}\left(\int_0^t I_{(-\infty,x]}(Y^1_{s-}) L(ds,b) - \int_0^t I_{(-\infty,x]}(Y^1_{s-}) L(ds,a)\right)$$

$$- (b - a + x)\, D(t) + \frac{1}{2} L_1(t,x) \qquad \ldots\ldots (8)$$

<u>Remark 2.4.1</u> Observe that in the case $x < 0$, the second erm on the right side of (8) reduces to zero, whereas when $x = 0$ t reduces to $\frac{1}{2}(L(t,b) - L(t,a))$.

<u>Proof of Lemma 2.4.1</u> Tanaka's formula (1) applied to Y^1 the point $x \in [0, b-a)$ gives

$$(Y^1_t - x)^+ = (Y^1_0 - x)^+ + \int_0^t I_{(x,\infty)}(Y^1_{s-})dY^1_s$$

$$+ \sum_{0<s\leq t} I_{(x,\infty)}(Y^1_{s-})\,(Y^1_s - x)^-$$

$$+ \sum_{0<s\leq t} I_{(-\infty,x]}(Y^1_{s-})(Y^1_s - x)^+ + \frac{1}{2} L_1(t,x)$$

$$= I_0 + I_1 + I_2 + I_3 + \frac{1}{2} L_1(t,x) .$$

as $Y_0 = 0$, $I_0 = 0$ for $x \in [0, b-a)$. From eqn. (5) we get,

$$I_1(t) = \int_0^t I_{(x,\infty)}(Y^1_{s-})dY^1_s$$

$$= \int_{\tau_t} I_{(a,b]}(X_s)\, I_{(x,\infty)}(Y^1_{s-})dX_s$$

$$+ \frac{1}{2}\left(\int_0^t I_{(x,\infty)}(Y^1_{s-})L(ds,a) - \int_0^t I_{(x,\infty)}(Y^1_{s-})L(ds,b)\right)$$

$$- (b-a)\left(\int_0^t I_{(x,\infty)}(Y^1_{s-})U(ds) - \int_0^t I_{(x,\infty)}(Y^1_{s-})D(ds)\right) .$$

As $x \in [0, b-a)$, $Y^1_{s-} > x$ implies that $X_s \neq a$ or b. Moreover as the measures $L(ds,a)$, $L(ds,b)$ are supported on the sets $\{s : X_s = a\}$, $\{s : X_s = b\}$ respectively, the second term in the above is zero. Also $Y^1_{s-} > x$ implies that $D(s)$ has no jumps s, so that $\int_0^t I_{(x,\infty)}(Y^1_{s-})\, U(ds) = 0$. Hence,

$$I_1(t) = \int_{\tau_t}^{t} I_{(a,b]}(X_s)\, I_{(x,\infty)}(Y^1_{s-})\,dX_s - (b-a)U(t) .$$

Since the jumps of Y^1 occur at the crossing times

$_{2k+1}$ or τ_{2k+1}), it is easy to see that almost surely, for

$\in [0, b-a)$, $I_2(t) = x\,U(t)$, $I_3(t) \equiv 0$. This proves the first

rt of the lemma. The proof of (8) is similar using eqn. (3)

$(Y^1_t - x)^-$. ///

We now come to the main theorem in this section.

<u>Theorem 2.4.1</u>

(i) For $x \in (0, b-a)$, almost surely,

$$L_1(t,x) = \int_0^t e^u(s)\, L(ds, a+x) \qquad \text{..... (9)}$$

(ii) For $x \in (-(b-a),0)$, almost surely,

$$L_1(t,x) = \int_0^t e^d(s)\, L(ds, b+x) \qquad \text{..... (10)}$$

(iii) For $x = 0$, almost surely,

$$L_1(t,0) = L(t,a). \qquad \text{..... (11)}$$

<u>Proof</u>. We first prove eqn. (9). Let $x \in (0, b-a)$. Since

most surely the measure $L_1(ds,x)$ is supported on the set

$\{ Y^1_{s-} = Y^1_s = x \}$ and since $x > 0$, we conclude that $L_1(ds,x)$

supported on the upcrossing intervals viz. $\bigcup_{k \geq 0} (\sigma_{2k}, \sigma_{2k+1})$.

n other words,

$$L_1(t,x) = \sum_k \int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, L_1(ds,x)$$

$$= \sum_k (L_1(t\wedge\sigma_{2k+1}, x) - L_1(t\wedge\sigma_{2k}, x)) \qquad \ldots\ldots (12)$$

Now we fix k and analyse the semi-martingale Y^1 in the interval $(\sigma_{2k}, \sigma_{2k+1})$. Let $Z_1(t) = (Y_t^1 - x)^+$ and $Z_2(t) = (X_t - (a+x))^+$. Then it is easy to see that $Z_1(t) = Z_2(t)$ for all $t \in (\sigma_{2k}, \sigma_{2k+1})$. Thus we have almost surely,

$$\int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, dZ_1(s) = \int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, dZ_2(s) \qquad \ldots\ldots (13)$$

for all $t \in (\sigma_{2k}, \sigma_{2k+1})$. But by the Tanaka formula,

$$\int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, dZ_2(s) = \int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, I_{(a+x,\infty)}(X_s)dX_s$$

$$+ \frac{1}{2}(L(t\wedge\sigma_{2k+1}, a+x) - L(t\wedge\sigma_{2k}, a+x))$$

$$= \int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, I_{(a,b)}(X_s)\, I_{(x,\infty)}(Y_{s-}^1)dX_s$$

$$+ \frac{1}{2}(L(t\wedge\sigma_{2k+1}, a+x) - L(t\wedge\sigma_{2k}, a+x)).$$

On the other hand it follows from eqn. (7) that,

$$\int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, dZ_1(s) = \int_0^t I_{(\sigma_{2k},\sigma_{2k+1})}(s)\, I_{(a,b)}(X_s)\, I_{(x,\infty)}(Y_{s-}^1)dX_s$$

$$+ \frac{1}{2}(L_1(t\wedge\sigma_{2k+1}, x) - L_1(t\wedge\sigma_{2k}, x))$$

all $t \in [0, \sigma_{2k+1})$. The above observations together with ... (13) now implies that almost surely,

$$L_1(t \wedge \sigma_{2k+1}, x) - L_1(t \wedge \sigma_{2k}, x) = L(t \wedge \sigma_{2k+1}, a+x) - L(t \wedge \sigma_{2k}, a+x)$$

all $t \geq 0$. Hence eqn. (12) becomes

$$L_1(t,x) = \sum_k (L(t \wedge \sigma_{2k+1}, a+x) - L(t \wedge \sigma_{2k}, a+x))$$

is proves (9).

Proof of (10) : Let $x \in (-(b-a), 0)$. Now $L_1(t,x)$ is upported on the downcrossing intervals, i.e.,

$$L_1(t,x) = \sum_k (L_1(t \wedge \tau_{2k+1}, x) - L_1(t \wedge \tau_{2k}, x))$$

aking now $Z_1(t) = (Y_t^1 - x)^-$ and $Z_2(t) = (X_t - (b+x))^-$ it is y to see that $Z_1(t) = Z_2(t)$ for all $t \in (\tau_{2k}, \tau_{2k+1})$. Applying qn. (3) to Z_2 and eqn. (8) to Z_1, we can show as above that

$$L_1(t \wedge \tau_{2k+1}, x) - L_1(t \wedge \tau_{2k}, x) = L(t \wedge \tau_{2k+1}, b+x) - L(t \wedge \tau_{2k}, b+x).$$

is complete the proof of (10).

Proof of (11) : Let $x = 0$. We note that ,

$$L_1(t,0) = \int_0^t \theta^4(s) L_1(ds,0) + \int_0^t \theta^3(s) L_1(ds,0) .$$

oceeding as in proof of eqn. (9) (or alternatively using right ntinuity of $L_1(t,\cdot)$) we can show that ,

$$\int_0^t e^U(s)L_1(ds,0) = \sum_k L_1(t\wedge\sigma_{2k+1},0) - L_1(t\wedge\sigma_{2k},0)$$

$$= \sum_k L(t\wedge\sigma_{2k+1},a) - L(t\wedge\sigma_{2k},a)$$

$$= \int_0^t e^U(s)L(ds,a)$$

$$= L(t,a).$$

he last equality is due to the fact that $L(ds,a)$ has no support n the downcrossing intervals. Hence to prove (11) it is sufficient to show that almost surely, $L_1(t\wedge\tau_{2k+1},0) - L_1(t\wedge\tau_{2k},0) = 0$ or all $t \geq 0$ and for all k. To this end we fix k and as in roof of eqn. (10) we compare the expressions for $(X_t - b)^-$ and $(Y_t^1)^-$ given by eqns. (3) and (8) respectively. We note that $(X_t - b)^- = (Y_t^1)^-$ for $t \in (\tau_{2k}, \tau_{2k+1})$. Following Remark 2.4.1, e deduce that

$$L(t\wedge\tau_{2k+1},b) - L(t\wedge\tau_{2k},b) + L_1(t\wedge\tau_{2k+1},0) - L_1(t\wedge\tau_{2k},0)$$

$$= L(t\wedge\tau_{2k+1},b) - L(t\wedge\tau_{2k},b)$$

i.e., $$L_1(t\wedge\tau_{2k+1},0) - L_1(t\wedge\tau_{2k},0) = 0 .$$

is proves eqn. (11) and completes the proof of Theorem 2.4.1. ///

Remark 2.4.2 Applying eqn. (4) of Theorem 2.2.1 to the mi-martingale Y^1 and noting that $\sum_{0<s\leq t} \Delta Y_s^1 = (b-a)(U(t) - D(t))$ d using eqn. (5), we have for $x \in (0, b-a)$,

$$L_1(t,x) - L_1(t,x-) = 2\int_{\tau_t}^{t} I_{(a,b]}(X_s)\, I_{\{x\}}(Y_s^1)\,dW_s$$

$$= 2\int_{\tau_t}^{t} e^{U}(s)\, I_{\{a+x\}}(X_s)\,dW_s$$

Hence for $x \in (0, b-a)$, $L_1(t, .)$ is continuous at x if $L(t, .)$ is continuous at $a+x$. A similar comment is true when $x \in (-(b-a),0)$. When $x = 0$, again using eqn. (5) we have,

$$L_1(t,0) - L_1(t,0-) = 2\int_{\tau_t}^{t} I_{(a,b]}(X_s)\, I_{\{0\}}(Y_s^1)\,dW_s$$

$$+ \int_0^t I_{\{0\}}(Y_s^1)L(ds,a) - \int_0^t I_{\{0\}}(Y_s^1)L(ds,b)$$

$$= 2\int_0^t I_{\{b\}}(X_s)\,dW_s + L(t,a) - L(t,b)$$

$$= L(t,b) - L(t,b-) + (L(t,a) - L(t,b))$$

$$= L(t,a) - L(t,b-) .$$

Hence, $\qquad L_1(t,0-) = L(t,b-).$

This shows that in general $L_1(t, .)$ has discontinuity at the origin.

Let $L_2(t,x)$ denote the local time of the process $Y^2(t) = X_{\sigma_t}$. Then we have the following theorem.

Theorem 2.4.2 For $x \geq b$, we have almost surely,

$$L_2(t,x) = L(t,x)$$

for all $t \geq 0$. For $x = a$,

$$L_2(t,a) \equiv 0$$

for all $t \geq 0$. When $x < a$, almost surely,

$$L_2(t,x) = L(t,x)$$

for all $t \geq 0$. Finally, when $x \in (a,b)$, almost surely,

$$L_2(t,x) = L(\tau_t,x)$$

for all $t \geq 0$.

Remark 2.4.3 The theorem is intuitively obvious because whenever the Y^2 process is outside (a,b) it is exactly the same as the original process. The proof is a straightforward application of this fact.

Proof of Theorem 2.4.2 We shall prove only the case $x \geq b$. The proof for the case $x \leq a$ is similar using the Tanaka formula for $(Y^2_t - x)^-$. In the case $x \in (a,b)$, the result follows from the fact that the Y^2 process equals the original process in the time interval $[0, \tau_t)$ and does not spend any time in (a,b) after τ_t.

We first consider the case $x = b$. The Tanaka formula (eqn. (1)) gives

$$(Y^2_t - b)^+ = (Y^2_0 - b)^+ + \int_0^t I_{(b,\infty)}(Y^2_{s-})dY^2_s + \sum_{0<s\leq t} I_{(b,\infty)}(Y^2_{s-})(Y^2_s - b)^-$$

$$+ \sum_{0<s\leq t} I_{(-\infty,b]}(Y^2_{s-})(Y^2_s - b)^+ + \frac{1}{2} L_2(t,b)$$

$$= I_0 + I_1 + I_2 + I_3 + \frac{1}{2} L_2(t,b) \quad \text{......} (14)$$

is easy to see that $I_0 = (X_0 - b)^+$ and $I_2 = I_3 = 0$. Since

$> b$ iff $X_{s-} > b$, it follows using eqn. (5), Theorem 2.3.1, that

$$I_1 = \int_0^t I_{(b,\infty)}(Y^2_{s-})dY^2_s$$

$$= \int_0^t I_{(a,b]}(X_s)\, I_{(b,\infty)}(Y^2_{s-})dX_s$$

$$+ \int_0^t I_{(a,b]^c}(X_s)\, I_{(b,\infty)}(Y^2_{s-})dX_s$$

$$- \frac{1}{2}\left(\int_0^t I_{(b,\infty)}(Y^2_{s-})L(ds,a) - \int_0^t I_{(b,\infty)}(Y^2_{s-})L(ds,b)\right)$$

$$+ (b-a)\left(\int_0^t I_{(b,\infty)}(Y^2_{s-})U(ds) - \int_0^t I_{(b,\infty)}(Y^2_{s-})D(ds)\right)$$

$$= \int_0^t I_{(b,\infty)}(X_{s-})dX_s$$

ce (14) reduces to

$$(X_t - b)^+ = (X_0 - b)^+ + \int_0^t I_{\{X_{s-} > b\}} dX_s + \frac{1}{2} L_2(t,b) .$$

follows that $L_2(t,b) = L(t,b)$. The same arguments also prove the

$x > b$. ///

…emi-martingales $f(|X_t - X_{\sigma_t}|)$

In this section we determine the martingale and bounded …ion parts of the semi-martingale $f(|X_t - X_{\sigma_t}|)$ when f is …h function. These generalize our results on Brownian motion in …st chapter. We first consider the semi-martingale $|X_t - X_{\sigma_t}|$. We have the following lemma.

Lemma 2.5.1 For all $a < b$, almost surely,

$$\ldots)D(t) + |X_t - X_{\sigma_t}| = \int_0^t \theta(s)\, I_{(a,b)}(X_s)\, dX_s + \tfrac{1}{2}(L(t,a)+L(t,b-)) \quad \ldots\ldots (15)$$

…ll $t \geq 0$.

Proof. Combining eqn. (7) of Lemma 2.4.1 with eqn. (11) of …ma 2.4.1 gives,

$$\ldots^+ = \int_{\tau_t}^t I_{(a,b]}(X_s)\, I_{(0,\infty)}(Y^1_{s-})\, dX_s - (b-a)U(t) + \tfrac{1}{2} L(t,a)$$

…rly eqn. (8) together with eqn. (11) gives,

$$\ldots - \int_{\tau_t}^t I_{(a,b]}(X_s)\, I_{(-\infty,0]}(Y^1_{s-})\, dX_s - (b-a)D(t) + \tfrac{1}{2} L(t,b) .$$

$$\ldots - X_{\sigma_t}| = |Y^1_t| = (Y^1_t)^+ + (Y^1_t)^-$$

$$= \int_{\tau_t}^t \left(I_{(0,\infty)}(Y^1_{s-}) - I_{(-\infty,0]}(Y^1_{s-})\right) I_{(a,b]}(X_s)\, dX_s$$

$$- (b-a)\, D(t) + \tfrac{1}{2}\left(L(t,a) + L(t,b)\right)$$

$$= \int_{\tau_t}^{t} (I_{(0,\infty)}(Y^1_{s-}) - I_{(-\infty,0)}(Y^1_{s-}))\, I_{(a,b)}(X_s)\,dX_s$$

$$- (b-a)\,C(t) + \frac{1}{2}(L(t,a) + L(t,b-))$$

(Using eqn. (4), Theorem 2.2.1)

$$= \int_0^t \theta(s,\omega)\, I_{(a,b)}(X_s)\,dX_s - (b-a)C(t)$$

$$+ \frac{1}{2}(L(t,a) + L(t,b-))$$

This completes the proof. ///

To analyse the general case $f(|X_t - X_{\sigma_t}|)$ we use the Ito formula. We recall that if (X_t) is a semi-martingale and f is a C^2 - function, then

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d\langle X^c\rangle_s$$

$$+ \sum_{0<s\leq t} [(f(X_s) - f(X_{s-})) - f'(X_{s-})\,\Delta X_s] \qquad \ldots\ldots (16)$$

We now apply this formula to the semi-martingale $|X_t - X_{\sigma_t}|$ to get the following theorem.

<u>Theorem 2.5.1</u> Let f be a C^2 - function. Then,

$$(f(b-a) - f(0))\,C(t) + f(|X_t - X_{\sigma_t}|) - f(0)$$

$$= \int_0^t f'(|X_s - X_{\sigma_s}|_-)\,\theta(s)\, I_{(a,b)}(X_s)\,dX_s$$

$$+ \frac{1}{2}\int_0^t f''(|X_s - X_{\sigma_s}|_-)\, I_{(a,b)}(X_s)\, d\langle X^c\rangle_s$$

$$+ \frac{1}{2} f'(0)(L(t,a) + L(t,b-)) \qquad \ldots\ldots (17)$$

<u>Proof</u>. Ito's formula for the semi-martingale $Z(t) = |X_t - X_{\sigma_t}|$

$$f(Z_t) = f(0) + \int_0^t f'(Z_{s-})dZ_s + \frac{1}{2}\int_0^t f''(Z_{s-})d\langle Z^c\rangle_s$$

$$+ \sum_{0<s\leq t}\left[f(Z_s) - f(Z_{s-}) - f'(Z_{s-})\Delta Z_s\right]$$

$$= f(0) + I_1 + I_2 + I_3$$

Lemma 2.5.1 it follows that

$$I_2(t) = \frac{1}{2}\int_0^t f''(Z_{s-})\, I_{(a,b)}(X_s)d\langle X^c\rangle_s$$

$\sum_{0<s\leq t}\Delta Z_s = -(b-a)C(t)$ we get

$$I_3(t) = -(f(b-a) - f(0))\,C(t) + (b-a)\,f'(b-a)\,C(t).$$

n using Lemma 2.5.1 we have,

$$I_1 = \int_0^t f'(Z_{s-})\,\theta(s)\,I_{(a,b)}(X_s)dX_s + \frac{1}{2}\int_0^t f'(Z_{s-})L(ds,a)$$

$$+ \frac{1}{2}\int_0^t f'(Z_{s-})L(ds,b-) - (b-a)\,f'(b-a)\,C(t)$$

note that by eqn. (4) Theorem 2.2.1, $L(ds,b-) = L(ds,b) - I_{\{b\}}(X_s)dV_s$.

ther $Z_{s-} = |X_s - X_{\sigma_s}|_- = 0$ whenever $X_s = a$ or b, except at the

s of Z_s, which are only finite in number in $[0,t]$. Since

,a), $L(\cdot,b)$ and $V(\cdot)$ are all continuous in t, we get

$$I_1 = \int_0^t f'(Z_{s-})\,\theta(s)\, I_{(a,b)}(X_s)\,dX_s$$

$$+ \frac{1}{2} f'(0)(L(t,a) + L(t,b-)) - (b-a)\, f'(b-a)\, C(t).$$

together I_1, I_2 and I_3 the proof is complete. ///

As a special case of this theorem we have

<u>Corollary 2.5.1</u> Let (X_t) be a continuous square integrable

gale. Then,

$$E\left(\int_0^t I_{(a,b)}(X_s)\,d\langle X\rangle_s\right) = E(X_{T_t} - X_0)^2 + (b-a)^2 EC(t) + E(X_t - X_{\sigma_t})^2$$

<u>Proof</u>. The proof is immediate if we take $f(x) = x^2$ in

m.2.5.1. ///

<u>Remark 2.5.1</u> Since for Brownian motion we have almost surely

,b] } = $C_t((a,b))$, the above corollary compares with Theorem 1.3.1.

CHAPTER III

Some Applications of Semi-Martingales Associated with Crossings

3.1 Introduction

This chapter is devoted to applications of the theory developed in the previous two chapters. In Section 3.2, we document some folklore and use these to compute the expected value of some Brownian functionals. Our first application concerns the Brownian motion and is a generalization of Corollary 1.3.1. We first prove an almost sure version of this corollary, i.e. we show that almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{\int_0^t I_{(a,b)}(X_s)\,ds}{C_t((a,b))} = (b-a)^2$$

Using this result, we obtain a further generalization, namely that when f is a smooth function, almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{\int_0^t I_{(a,b)}(X_s)\, f''(|X_s - X_{\sigma_s}|)\,ds}{2\, C_t((a,b))} = f(b-a)-f(0)-f'(0)(b-a).$$

The next two applications deals with Levy's crossing theorem. We had already obtained an 'expected' version of this theorem for martingales in Corollary 1.3.2. In Section 3.4 we obtain a result closely related to the crossing theorem. The crossing theorem says

$t\,(b-a)\,C_t((a,b)) \sim L(t,a)$ as $b \to a$. The parameter t of course fixed. We will show in Section 3.4 that the above tionship holds even if we let $t \to \infty$, i.e. $(b-a)C_t((a,b)) \sim L(t,a)$ $t \to \infty$. In our words $\lim_{t \to \infty} \frac{L(t,a)}{C_t((a,b))} = (b-a)$ almost surely.

Lastly in Section 3.5 we prove Levy's crossing theorem in se of a continuous semi-martingale. Several proofs of this rem are now known (see introductory comments of Section 3.5). particular N. el. Karoui has proved the theorem even when the i-martingale is discontinuous. However our approach is different. prove the theorem by letting $a \uparrow b$ in Lemma 2.5.1.

Expectations of Brownian local time : Some folklore

Let $(X_t)_{t \geq 0}$ be a continuous square integrable martingale th $\langle X \rangle_t \uparrow \infty$ almost surely. Let $\tau_c = \inf\{s \geq 0 : X_s = c\}$. e note that $\tau_c < \infty$ almost surely (Lemma 1.3.1). As usual we note by $L(t,x)$ the local time process of (X_t). We wish to lculate $EL(\tau_c,x)$ for $x \in \mathbb{R}$. We do this using the Tanaka rmula (see also F. Knight [6, p. 110]) An explicit formula this is the content of Theorem 3.2.1. As a corollary we calculate plicitly the second moment of a certain class of stochastic integrals. a further corollary we derive the well known result that there exists smooth function f such that $f(X_t)$ is a martingale. We start th a lemma whose proof uses the earlier techniques (Corollary 2.5.1).

Lemma 3.2.1 Let (X_t) be a continuous square integrable martingale, such that almost surely, $\langle X\rangle_t \uparrow \infty$ and $X_0 = y$. Then for any real number a we have,

(i) $E\,\mu_X\{s \leq \tau_c : X_s \in (a, \infty)\} < \infty$ in case $a \leq c$ and $y \leq c$.

(ii) $E\,\mu_X\{s \leq \tau_c : X_s \in (-\infty, a)\} < \infty$ in case $c < a$ and $c < y$ almost surely.

Proof. We will prove only case (i), the proof of case (ii) being similar. We note that when $a = c$ or $y = c$, (i) is trivially true. So let $a < c$ and $y < c$. Let $a < b < c$. Let (σ_k), (τ_k) be the crossing times for the interval (a,b), i. e. $\sigma_k = \inf\{t > \tau_{k-1}, X_t \geq b\}$, $\tau_k = \inf\{t > \sigma_k : X_t \leq a\}$ (we take $\tau_{-1} \equiv 0$). Let $f(t,w) = \sum_{k=0}^{\infty} I_{(\sigma_k,\tau_k]}(t)$ ($= \theta^d$ in our earlier notation). Let $Y(t) = X(t \wedge \tau_c)$ and $Y'(t) = Y_t - Y_0$. Then since f is a predictable process we have,

$$\int_0^t f(s,w)\,dY'(s) = \sum_{k=0}^{\infty} Y_{t\wedge\tau_k} - Y_{t\wedge\sigma_k}$$

$$= -(y-b)^+ - (b-a)D(t\wedge\tau_c) + (Y_t - b)f(t\wedge\tau_c)$$

where $D(t\wedge\tau_c)$ is the number of downcrossings in time $t\wedge\tau_c$, of (a,b) by (X_t). We note that $D(t\wedge\tau_c) \uparrow D(\tau_c)$ as $t \to \infty$. Further $(a-b) \leq (Y_t - b)\,f(t\wedge\tau_c) \leq c-b$ for all t and $(Y_t-b)f(t\wedge\tau_c) \to c-b$ as $t \to \infty$. Hence taking expectations and letting $t \to \infty$ in the equation we get

$$(b-a)\, E\, D(\tau_c) = (c-b) - (y-b)^+ \qquad \ldots\ldots (1)$$

er we note that $U(\tau_c) = D(\tau_c) + I_{(-\infty, a]}(X_\tau)$ where τ is first exit time from (a,b). Hence we get from (1) above that

$$(b-a)^2\, E\, D(\tau_c) = 2(b-a)\left[(c-b) - (y-b)^+\right] + (b-a)^2 P(X_\tau \le a)\,. \qquad \ldots\ldots (2)$$

ying Corollary 2.5.1 to $(X_{t \wedge \tau_c})$ and letting $t \to \infty$ et

$$E\, \mu_X \{ s \le \tau_c : X_s \in (a,b) \} = E(X_\tau - y)^2 + 2(b-a)\left[(c-b)-(y-b)^+\right] + (b-a)^2 P(X_\tau \le a) \qquad \ldots\ldots (3)$$

easy to see that the RHS in (3) is bounded $\forall\, a \le b \le c$. e letting $b \to c$ in LHS of (3) we get the required result. ///

Remark 3.2.1 We note that (i) of Lemma 3.2.1 deals with the first of the following four cases, viz. (1) $a \le c$, $y \le c$; $a \le c$, $y > c$; (3) $a > c$, $y > c$; (4) $a > c$, $y \le c$. However it easy to see that $E\, \mu_X \{ s \le \tau_c : X_s \in (a, \infty) \}$ is infinity case (2) and (3) and zero in case (4). The situation is similar ase (ii) of Lemma 3.2.1.

(contd.)

Theorem 3.2.1 Let (X_t) be a continuous square integrable martingale with $X_0 = y$ almost surely. Then,

(i) If $y \le c$, $E\,L(\tau_c, x) = 2(c-x) \wedge (c-y) \qquad x \le c$

$\qquad\qquad\qquad\qquad = 0 \qquad x > c$

(ii) If $y > c$, $E\,L(\tau_c, x) = 2(x-c) \wedge (y-c) \qquad x \ge c$

$\qquad\qquad\qquad\qquad = 0 \qquad x < c.$

Proof. Lemma 3.2.1 (i) allows us to take expectation in eqn. (1), Section 2.2 at the random time $t = \tau_c$. The first part of Theorem 3.2.1 follows immediately if we note that stochastic integrals with respect to martingales have mean zero. The 2nd part is proved similarly using Lemma 3.2.1 (ii) and eqn. (3) Section 2.2.

Remark 3.2.2 We note that to prove Theorem 3.2.1 we do not really need the lemma. We can take expectations in Tanaka's formula at time $t \wedge \tau_c$ and then let $\to \infty$, observing that this is indeed justified.

With the same set up as in Theorem 3.2.1, let $\psi(x,y,c) = E\,L(\tau_c, x)$. Here y refers to the initial point of (X_t). Let

$$L_X = \left\{ f : E\left(\int_0^{\tau_c} f(X_s)\,dX_s\right)^2 = E \int_0^{\tau_c} f^2(X_s)\,d\langle X\rangle_s < \infty \right\}$$

$$L_X' = \left\{ f : \int_{\mathbb{R}} f^2(x)\ \psi(x,y,c)\,dx < \infty \right\} .$$

Corollary 3.2.1 $L_x = L'_x$.

Proof. The proof is immediate if we substitute '$t = T_c$' the occupation density formula (eqn. (2), Section 2.2) and n take expectations. ///

Remark 3.2.2 Let f be a smooth function, (X_t) a wnian motion and $X_0 \equiv 0$. Let $c > 0$. Then it is easy to see at

$$E\int_0^{T_c} f''(X_s)\, I_{[0,c]}(X_s)\,ds = \int_0^c f''(x)\, \varphi(x,0,c)\,dx$$

$$= 2\left[f(c) - f(0) - c f'(0)\right]$$

will use this fact in the next section.

In the context of the results of Chapter I, Section 1.2, is natural to ask the following question : For a smooth function and (X_t) a continuous martingale, can $f(X_t)$ be a martingale ? have the following lemma.

Lemma 3.2.2 Let (X_t) be a continuous local martingale th $\langle X\rangle_t \to \infty$ almost surely as $t \to \infty$ and $X_0 \equiv y$ almost ely. Let f be a C^2 - function. Then the following are uivalent.

(a) $f(X_t)$ is a local martingale.

(b) $f(x) = ax + b$ for some a and b.

Proof. (b) $\Rightarrow$ (a) is trivial. To prove (a) $\Rightarrow$ (b) it is fficient - by time change - to consider the case when (X_t) is a ward n motion. Let $a < y < b$ and $\tau = \inf\{ s > 0 : X_s = a \text{ or } b \}$. to's formula and eqn. (2), Section 2.2, (a) $\Rightarrow$ almost surely, that

$$\int_a^b f''(x)\, L(\tau,x)\,dx = 0 . \qquad \ldots\ldots (4)$$

is easy to see that for the Brownian motion the Tanaka formula plies that

$$\begin{aligned} E\, L(\tau,x) &= 0 \qquad \text{if } x \in (-\infty, a) \cup (b, \infty) \\ &= \frac{y-a}{b-a}(b-x) \qquad y \leq x \leq b . \\ &= \frac{b-y}{b-a}(x-a) \qquad a \leq x < y . \end{aligned}$$

nce taking expectations in (4) and using the above expression for $L(\tau,x)$ we get,

$$\frac{(b-y)}{(b-a)} \int_a^y f''(x)(x-a)\,dx + \frac{y-a}{b-a} \int_y^b f''(x)(b-x)\,dx = 0.$$

viding by $y - a$ and letting $a \longrightarrow y$ we get,

$$\int_y^b f''(x)(b-x)\,dx = 0 .$$

s implies that $f(b) = f(y) + (b-y) f'(y)$ and completes the of. ///

Remark 3.2.3 The condition on f in the above lemma can be changed. Let f be a convex function on $\mathbb{R}$. By applying the Ito's formula for convex functions (see Meyer [8, p. 361 - 371]), we get instead of eqn. (4) above, the following : almost surely,

$$\int_{-\infty}^{\infty} E\, L(\tau,x)\, \mu(dx) = 0$$

where $\mu(dx) = f''(x)$ in the sense of distributions. Hence $\mu \equiv 0$. This implies that $f'_g(x) \equiv$ constant where f'_g denotes the left derivative of f at x. Hence Lemma 3.2.2 is true in this case also.

3.3 A Ratio Limit Theorem

We now take a fresh look at Corollary 1.3.1 in the light of the theory developed in Chapter II. It is natural to ask whether an almost sure version of Corollary 1.3.1 holds. The answer is indeed yes and we have the following theorem (J. Rajeev and B.V. Rao [14]).

Theorem 3.3.1 Let (X_t) be a Brownian motion and $a < b$. Then almost surely,

$$\operatorname*{Lt}_{t\to\infty} \frac{\int_0^t I_{(a,b)}(X_s)\,ds}{C_t((a,b))} = (b-a)^2$$

Proof. Let $\sigma_1 = \inf\{ s > 0 : X_s = a \}$ and for $i = 1,2,3,\ldots,$ let

$$\sigma_{2i} = \inf\{ s > \sigma_{2i-1} \quad X_s = b \}$$

and

$$\sigma_{2i+1} = \inf\{ s > \sigma_{2i} \quad X_s = a \}$$

$$Y_0 = \int_0^{\sigma_1} I_{(a,b)}(X_s)ds \quad \text{and} \quad Y_i = \int_{\sigma_i}^{\sigma_{i+1}} I_{(a,b)}(X_s)ds \quad \text{for} \quad i \geq 1.$$

e strong markov property of Brownian motion, it follows that Y_i, 1 are i.i.d. It follows from Remark 3.2.2 that $E\,Y_i = (b-a)^2$. By the strong law of large numbers, almost surely

$$\frac{Y_1 + \ldots + Y_n}{n} \longrightarrow (b-a)^2 \qquad \ldots\ldots (5)$$

bserve that if $\sigma_n \leq t < \sigma_{n+1}$, then

$$Y_1 \leq \int_0^t I_{(a,b)}(X_s)ds \leq \sum_{i=0}^{n} Y_i \quad \text{and} \quad n-1 \leq C_t((a,b)) < n .$$

e inequalities combined with (5) above complete the proof. ///

Remark 3.3.1 Prof. Meyer has drawn our attention to K. Burdzy, Pitman and M. Yor [1] where a more general result in the context urt processes is obtained. We also refer to Ito-Mckean [4, p.228-] for other ratio ergodic theorems involving local times of rrent diffusions.

Let (X_t) be a Brownian motion. Taking expectations in eqn. (17), rem 2.5.1 and dividing throughout by $E\,C(t)$, we get

$$\underset{t \to \infty}{\mathrm{Lt}} \; \frac{1}{2} \; \frac{E \int_0^t f''(|X_s - X_{\sigma_s}|)\, I_{(a,b)}(X_s)ds}{E\,C(t)}$$

$$= f(b-a) - f(0) - \frac{1}{2} f'(0) \underset{t \to \infty}{\mathrm{Lt}} \frac{E(L(t,a)+L(t,b))}{E\,C(t)}$$

...we use the notation $C(t)$ for $C_t((a,b))$. It is easy to see from ...n. (15), Lemma 2.5.1 that the limit in the RHS above is $2(b-a)$.

...ow we have the following lemma.

Lemma 3.3.1 Let (X_t) be a Brownian motion, $a < b$ and f a ...ooth function. Then,

$$f(b-a) - f(0) - f'(0)(b-a) = \operatorname*{Lt}_{t\to\infty} \frac{1}{2} \frac{E\left(\int_0^t f''(|X_s - X_{\sigma_s}|_-)\, I_{(a,b)}(X_s)\,ds\right)}{E\, C_t((a,b))}.$$

We now look at the almost sure version of the above result. More ...recisely we will prove the following theorem (Rajeev [12]).

Theorem 3.3.2 Let (X_t) be a Brownian motion, $a < b$ and f a ...ooth function. Then, almost surely,

$$f(b-a) = f(0) + f'(0)(b-a) + \operatorname*{Lt}_{t\to\infty} \frac{\int_0^t f''(|X_s - X_{\sigma_s}|_-)\, I_{(a,b)}(X_s)\,ds}{2\, C_t((a,b))}. \quad \ldots\ldots(6)$$

Proof. We assume without loss of generality that $X_0 = a$ almost ...urely. Let $\tau_0 \equiv 0$, and

$$\tau_{2k} = \inf\{s > \tau_{2k-1} : X_s \leq a\} \qquad k = 1,2,\ldots$$

$$\tau_{2k+1} = \inf\{s > \tau_{2k} : X_s \geq b\} \qquad k = 0,1,2,\ldots$$

...et

$$Y_k(s) = X_{\tau_{2k+1}\wedge s} - X_{\tau_{2k}\wedge s} \qquad k = 0,1,2,\ldots$$

$$Z_k(s) = X_{\tau_{2k}\wedge s} - X_{\tau_{2k-1}\wedge s} \qquad k = 1,2,\ldots$$

so that

$$f''(|X_s - X_{\sigma_s}|)\, I_{(a,b)}(X_s) = f''(Y_k(s))\, I_{(0,b-a)}(Y_k(s)),\quad \tau_{2k} \le s < \tau_{2k+1}$$

$$= f''(-Z_k(s))\, I_{(-(b-a),0)}(Z_k(s)),\quad \tau_{2k-1} \le s < \tau_{2k}$$

s,

+1

$$f''(|X_s - X_{\sigma_s}|)\, I_{(a,b)}(X_s)\,ds = \sum_k \int_0^\infty f''(Y_k(s)) I_{(0,b-a)}(Y_k(s)) I_{[\tau_{2k},\tau_{2k+1})}(s)\,ds$$

$$+ \sum_k \int_0^\infty f''(-Z_k(s)) I_{(-(b-a),0)}(Z_k(s)) I_{[\tau_{2k-1},\tau_{2k})}(s)\,ds$$

re on the right side the first sum is over $2k \le n$ and the second sum over $2k \le n+1$. Now by the strong Markov property of the Brownian tion, the two sums in the RHS are in fact sums of i.i.d. random riables. Further during an upcrossing (resp. downcrossing) Y_k esp. Z_k) behaves like a Brownian motion started at zero and stopped $b-a$. Hence by Remark 3.2.2

$$E\int_0^\infty f''(Y_k(s))\, I_{(0,b-a)}(Y_k(s))\, I_{[\tau_{2k},\,\tau_{2k+1})}(s)\,ds$$

$$= E\int_0^\infty f''(-Z_k(s)) I_{(-(b-a),0)}(Z_k(s)) I_{[\tau_{2k-1},\tau_{2k})}(s)\,ds$$

$$= 2\left[f(b-a) - f(0) - f'(0)(b-a)\right] .$$

The strong law of large numbers now implies that almost surely,

$$\underset{n \to \infty}{\mathrm{Lt}} \frac{1}{2n} \int_0^{\tau_{n+1}} f''(|X_s - X_{\sigma_s}|_-) I_{(a,b)}(X_s)\,ds = f(b-a) - f(0) - f'(0)(b-a). \quad \ldots\ldots(7)$$

Now we observe that for fixed ω and $\tau_n \le t < \tau_{n+1}$,

$$\frac{1}{C(t)} \int_0^t f''(|X_s - X_{\sigma_s}|_-) I_{(a,b)}(X_s)\,ds = (1 - \tfrac{1}{n}) \cdot \frac{1}{n-1} \int_0^{\tau_n} f''(|X_s - X_{\sigma_s}|_-) I_{(a,b)}(X_s)\,ds$$

$$+ \frac{1}{C(t)} \left(\int_{\tau_n}^t f''(|X_s - X_{\sigma_s}|_-) I_{(a,b)}(X_s)\,ds \right.$$

The second term in the RHS can be dominated by $C_0 \left(\frac{1}{C(t)} \int_{\tau_n}^t I_{(a,b)}(X_s)\,ds\right)$ where C_0 is a constant. By Theorem 3.3.1, this tends to zero almost surely as $t \to \infty$. From eqn. (7) it follows that the first term in the RHS above converges to the required limit as $t \to \infty$. This completes the proof of the theorem. ///

<u>Remark 3.3.2</u> We note that Theorem 3.3.1 follows by taking $f(x) = x^2$ in the above theorem.

<u>Remark 3.3.3</u> It is interesting to compare the above result with the usual Taylor's formula of differential calculus. The remainder term is now probabilistic. Indeed we can go a little further. By the usual Taylor's formula of differential calculus there exists $\xi \in (0, b-a)$ such that $f(b-a) - f(0) - f'(0)(b-a) = f''(\xi) \frac{(b-a)^2}{2}$ Hence we get

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{\int_0^t f''(|X_s - X_{\sigma_s}|_-)\, I_{(a,b)}(X_s)\,ds}{2\, C(t)} = \frac{1}{2} (b-a)^2 f''(\xi) .$$

ng Theorem 3.3.1 we can write this as

$$\operatorname*{Lt}_{t \to \infty} \frac{\int_0^t f''(|X_s - X_{\sigma_s}|_-) \, I_{(a,b)}(X_s)\,ds}{\int_0^t I_{(a,b)}(X_s)\,ds} = f''(\xi) \, .$$

nally if g were any continuous function we can choose f such that $= g$ to get the following : For any continuous function g on $\mathbb{R}$, ere exists $\xi \in (0, b-a)$ such that almost surely,

$$\operatorname*{Lt}_{t \to \infty} \frac{\int_0^t g(|X_s - X_{\sigma_s}|_-) \, I_{(a,b)}(X_s)\,ds}{\int_0^t I_{(a,b)}(X_s)\,ds} = g(\xi) \, .$$

<u>Remark 3.3.4</u> It is worth noting that the last term in the right side of eqn. (6), Theorem 3.3.2 is almost surely constant, because the other terms are so. This can also be seen by observing that $\bigcap_{t>0} \sigma(X_s, s \geq t)$ is trivial. Thus in particular Lemma 3.3.1 and Theorem 3.3.2 together imply the following ratio ergodic theorem :

lmost surely,

$$\operatorname*{Lt}_{t \to \infty} \frac{\int_0^t f''(|X_s - X_{\sigma_s}|_-) \, I_{(a,b)}(X_s)\,ds}{C(t)}$$

$$= \operatorname*{Lt}_{t \to \infty} \frac{E\int_0^t f''(|X_s - X_{\sigma_s}|_-) I_{(a,b)}(X_s)\,ds}{E\,C(t)}$$

Further we note that, as in the above remark, we can replace $C(t)$ by $\int_0^t I_{(a,b)}(X_s)ds$ and f^n by any continuous function g in the above limit. We also note that in this case the limit is nothing but $\int_0^{b-a} g(x)\, M(dx)$ where M is the probability measure on $(0, b-a)$ given by $M(dx) = \frac{1}{(b-a)^2}\psi(x, 0, b-a)dx$, where in the notation of Section 3.2, $\psi(x, y, c) = E\, L(\tau_c, x)$ (see also Remark 3.2.3).

3.4 An Asymptotic relationship between Local times and Crossings

In this section we look at the asymptotic relationship between crossings and local time of the Brownian motion. As already noted in the introduction this can be looked at as the analogue of Levy's crossing theorem in the variable t. Our main result is the following :

Theorem 3.4.1 Let (X_t) be a Brownian motion and $a < b$. Then almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{C_t((a,b))}{L(t,a)} = \frac{1}{b-a}.$$

Proof. We will show that almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{L(t,a) + L(t,b)}{2\, C(t)} = (b-a) \qquad \text{..... (8)}$$

and

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{L(t,a) - L(t,b)}{2\, C(t)} = 0 \qquad \text{..... (9)}$$

Adding eqns. (8) and (10) the proof is complete. We will prove only eqn. (8) using Lemma 2.5.1. The proof of (9) is similar using

Theorem 2.3.1. We assume without loss that $X_0 = a$. Let (τ_n) be the successive crossing times of (a,b) as defined in the proof of Theorem 3.3.2. Observe that

$$\int_0^{\tau_{n+1}} \theta(s)\varphi(s)dX_s = \sum_{k=1}^{n+1} (-1)^{k-1}\int_{\tau_{k-1}}^{\tau_k} \varphi(s)dX_s$$

$$= \sum_k \int_0^{\infty} I_{(0,b-a)}(Y_k(s))dY_k(s)$$

$$= \sum_k \int_0^{\infty} I_{(-(b-a),0)}(Z_k(s))dZ_k(s)$$

where θ and φ are as in Lemma 2.5.1 and Y_k, Z_k are as in Theorem 3.3.2 :

$$Y_k(s) = X_{\tau_{2k+1}\wedge s} - X_{\tau_{2k}\wedge s} \quad k = 0,1,2,\ldots$$

$$Z_k(s) = X_{\tau_{2k}\wedge} \quad - X_{\tau_{2k-1}\wedge s} \quad k = 1,2,\ldots$$

The strong Markov property of the Brownian motion and the strong law of large numbers imply that almost surely,

$$\frac{1}{n}\int_0^{\tau_{n+1}} \theta(s)\varphi(s)dX_s \longrightarrow 0 \quad \text{as} \quad n \to \infty .$$

Hence by Lemma 2.5.1 we conclude that

$$\operatorname*{Lt}_{n\to\infty} \frac{L(\tau_{n+1},a) + L(\tau_{n+1},b)}{2n} = (b-a).$$

If now $\tau_n < t \leq \tau_{n+1}$, then $n-1 < C(t) \leq n$ and hence

$$\frac{1}{2}\left(\frac{n-1}{n}\right)\frac{L(\tau_n,a)+L(\tau_n,b)}{n-1} < \frac{1}{2}\frac{(L(t,a)+L(t,b))}{C(t)}$$

$$\leq \frac{1}{2}\left(\frac{n}{n-1}\right)\frac{L(\tau_{n+1},a)+L(\tau_{n+1},b)}{n}$$

and eqn. (8) follows on letting $n \to \infty$. ///

The following intuitively obvious result can now be derived (see also Ito-McKean [4 , p. 228]).

Corollary 3.4.1 For any a,b, almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{L(t,a)}{L(t,b)} = 1 .$$

Proof. The proof is immediate from eqn. (9) and Theorem 3.4.1. ///

Remark 3.4.1 Theorem 3.4.1 and its Corollary together imply that for any x,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{C_t((a,b))}{L(t,x)} = \frac{1}{b-a}$$

Corollary 3.4.2 Let $a < b$ and $c < d$. Then almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{C_t((a,b))}{C_t((c,d))} = \frac{d-c}{b-a}$$

Remark 3.4.2 We note that Theorem 3.3.2 and Theorem 3.4.1 can both be derived from the following fact :

For $x \in [0, b-a)$, $\displaystyle \operatorname*{Lt}_{t \to \infty} \frac{L_1(t,x)}{C(t)} = (b - a - x)$

For $x \in (-(b-a), 0)$, $\displaystyle \operatorname*{Lt}_{t \to \infty} \frac{L_1(t,x)}{C(t)} = (b - a + x)$.

Here $L_1(t,x)$ is the local time of $X_t - X_{\sigma_t}$. We note that for $x = 0$, this is just Theorem 3.4.1. This is proved via the explicit identification of $L_1(t,x)$ given by Theorem 2.4.1, and the fact that the increments of the local time of Brownian motion over successive crossing times are independent and identically distributed.

Now let f be any bounded borel function on $\mathbb{R}$. Then since $\langle (X_t - X_{\sigma_t})^c \rangle = \int_0^t I_{(a,b)}(X_s)ds$, the occupation density formula gives

$$\int_0^t f(|X_s - X_{\sigma_s}|) I_{(a,b)}(X_s)ds = \int_{-(b-a)}^{(b-a)} f(|x|)L_1(t,x)dx \,.$$

We note that the increments of the RHS over successive crossing times are i.i.d. Hence using the strong law and arguments as in proof of Theorem 3.3.2 we get

$$\operatorname*{Lt}_{t \to \infty} \frac{\int_0^t f(|X_s - X_{\sigma_s}|) I_{(a,b)}(X_s)ds}{2\, C(t)} = \int_0^{b-a} f(x)(b-a-x)dx.$$

We refer to Ito-Mckean 4, p. 228 for an unconditional version of these results.

3.5 Levy's Crossing theorem

We now use Lemma 2.5.1 to prove Levy's crossing theorem for a continuous semi-martingale (X_t) . Levy first conjectured the result in the following form : If (X_t) is the reflected Brownian motion, then $\lim_{\varepsilon \to 0} \varepsilon D_t((0,\varepsilon)) = L(t,0)$. The first proof of this striking result, using excursions of the Brownian motion seems to be in Ito-McKean [4, p.48-50]. Subsequently several proofs have been given in the literature. Chung and Durret [2] use theta functions, Williams [17] uses elementary arguments involving poisson processes, Ito (see N. Ikeda and S. Watanabe [3, p. 123-131]) uses excursion theory, Maisonneuve [7] uses regenerative systems. Yet another proof is given in Stroock [15]. In this context Meyer [8, p. 311-376] discusses the uniform integrability of the random variables $\varepsilon\, U_t((0,\varepsilon))$ when (X_t) is a martingale. On the other hand N. el. Karoui [5] has shown that this result is true for general semi-martingales both continuous and discontinuous. Here we treat only the case of a continuous semi-martingale using a technique which is suggested quite naturally by Lemma 2.5.1. The idea is to exploit the structure of the semi-martingale $|X_t - X_{\sigma_t}|$ and show that the martingale part tends to zero as $b \to a$. To do this we use some estimates for moments of stochastic integrals viz. $E(\int_0^t I_{(a,b)}(X_s)dX_s)^{2k}$. Such estimates have been obtained by Yor (see Yor [19]) to establish the joint continuity properties of local times for semi-martingales. These also play a crucial role in the proof given in Karoui [5]

We recall that a continuous semi-martingale $X_t = X_0 + M_t + V_t$ belongs to H^p iff $\|X\|_p < \infty$, where

$$\|X\|_p^p = E\{ <M>_\infty^{p/2} + (\int_0^\infty |dV_s|)^p \}$$

The following lemma due to Yor gives us the required estimates.

<u>Lemma 3.5.1</u> Let $p \geq 1$ and (X_t) a continuous semi-martingale in H^p. Then

$$E(\int_0^\infty I_{(a,b)}(X_s)d<X^c>_s)^p \leq K.(b-a)^p \|X\|_p^p$$

<u>Theorem 3.5.1</u> Let (X_t) be a continuous semi-martingale. Then,

(i) almost surely, $\underset{a \uparrow b}{Lt}\ (b-a)C_t((a,b)) = L(t,b-)$ uniformly on compact t-intervals.

(ii) If $(X_t) \in H^p$, $p \geq 1$, then

$$\underset{a \uparrow b}{Lt}\ (b-a)\ C_t((a,b)) \overset{L_p}{=} L(t,b-)$$

uniformly on compact t-intervals.

<u>Proof</u>. We first prove ii). Let $(X_t) \in H_p$, $p \geq 1$. From Lemma 2.5.1 we have,

$$|(b-a)C(t)-L(t,b-)| \leq |\int_0^t e(s)\ I_{(a,b)}(X_s)dM_s|$$

$$+|\int_0^t e(s)\ I_{(a,b)}(X_s)dV_s|$$

$$+|X_t - X_{\sigma_t}| + \frac{1}{2}|L(t,a) - L(t,b-)| \qquad \text{..... (10)}$$

where $\theta(s)$ is as in Chapter II, Section 2.2. By the dominated convergence theorem it is easy to see that, $\sup_{s\leq t} |\int_0^s \theta(s)\, I_{(a,b)}(X_s)\,dV_s|$ and $\sup_{s\leq t} |X_s - X_{\sigma_s}|$ tend to zero in L^p as $a \uparrow b$. Also by the B-D-G inequalities we have

$$E(\sup_{s\leq t} |\int_0^s \theta(s)\, I_{(a,b)}(X_s)\,dM_s)^p \leq E(\int_0^\infty I_{(a,b)}(X_s)\,d\langle M\rangle_s)^{p/2}$$

where the RHS tends to zero as $a \longrightarrow b$ by dominated convergence. Further, using eqns. (1) and (2) of Section 2.2 we have

$$L(t,a) - L(t,b-) = L(t,a) - L(t,b) + 2\int_0^t I_{(X_s = b)}\,dV_s$$

$$= 2\ (X_t - a)^+ - (X_t - b)^+ - (X_0 - a)^+ + (X_0 - b)^+$$

$$- \int_0^t I_{(a,b)}(X_s)\,dV_s - \int_0^t I_{(a,b]}(X_s)\,dM_s \qquad \ldots\ldots (11)$$

It is easy to see that the first three terms in the RHS tend to zero in L^p uniformly in t as $a \longrightarrow b$. Applying the B-D-G inequalities we can show as above that

$$E(\sup_{s\leq t} |\int_0^s I_{(a,b]}(X_s)\,dM_s|)^p \longrightarrow 0 \quad \text{as} \quad a \longrightarrow b\,.$$

Hence from (11), $\sup_{s\leq t} |L(t,a) - L(t,b-)| \longrightarrow 0$ in L^p as $a \longrightarrow b$. Hence each of the terms in the right side of (10) tend to zero in L^p, uniformly on compact t-intervals as $a \longrightarrow b$. This proves ii).

We now prove i). By stopping (X_t) suitably we assume without loss that it is in H^4. We use the following ad hoc notation :
$C_\varepsilon(t) = C_t((b-\varepsilon, b))$, $\sigma^n_t = \sigma_t(b - \frac{1}{n}, b)$, $C_n(t) = C_t((b - \frac{1}{n}, b))$ and
$\varphi_n(s) = I_{(b-\frac{1}{n}, b)}(X_s)$. Then from Lemma 2.5.1 we have ,

$$
\begin{aligned}
\left|\frac{1}{n} C_n(t) - L(t,b-)\right| \leq & \left|\int_0^t \theta(s)\varphi_n(s)dM_s\right| \\
& + \left|\int_0^t \theta(s)\varphi_n(s)dV_s\right| \\
& + |X_t - X_{\sigma^n_t}| + \frac{1}{2}\left|L(t, b - \tfrac{1}{n}) - L(t,b-)\right| \qquad \ldots\ldots(12)
\end{aligned}
$$

It is easy to see that almost surely, $\sup_{s\leq t} \left|\int_0^s \theta(s)\varphi_n(s)dV_s\right|$ and $\sup_{s\leq t} |X_s - X_{\sigma^n_s}|$ tend to zero uniformly as $n \to \infty$.

Let
$$Y_n(t) = \int_0^t \theta(s)\varphi_n(s)dM_s$$

By the B-D-G inequality and Lemma 3.5.1,

$$E(\sup_{s\leq t} |Y_n(s)|^4) \leq K.\frac{1}{n^2} \|X\|_2^2$$

A standard application of the Borel - Cantelli lemma now shows that $\sup_{s\leq t} |Y_n(s)| \to 0$ almost surely as $n \to \infty$. By eqns. (1) and (2) Section 2.2, we have

$$L(t,b-\tfrac{1}{n}) - L(t,b-) = L(t,b-\tfrac{1}{n}) - L(t,b) + 2\int_0^t I_{(X_s=b)}\,dV_s$$

$$= 2\ (X_t - b + \tfrac{1}{n})^+ - (X_t - b)^+$$

$$- (X_0 - b + \tfrac{1}{n})^+ + (X_0 - b)^+$$

$$- \int_0^t \varphi_n(s)dV_s - \int_0^t \varphi_n(s)dM_s \qquad \ldots\ldots (13)$$

It is easy to see that $\sup\limits_{s\le t} |(X_t - b + \frac{1}{n})^+ - (X_t - b)^+|$, $\sup\limits_{s\le t} |(X_0 - b + \frac{1}{n})^+ - (X_0 - b)^+|$ and $\sup\limits_{s\le t} |\int_0^t \varphi_n(s)dV_s|$ tend to zero almost surely as $n \to \infty$. If we let $Y_n'(t) = \int_0^t \varphi_n(s)dM_s$, then we can show as in the case for Y_n's above that $\sup\limits_{s\le t} |Y_n'(s)| \to 0$ almost surely. Hence from (13), $\sup\limits_{s\le t} |L(t,b-\frac{1}{n}) - L(t,b-)| \to 0$ almost surely and hence from (12), $\sup\limits_{s\le t} |\frac{1}{n} C_n(t) - L(t,b-)| \to 0$ almost surely. Now we note that if $\frac{1}{n+1} \le \varepsilon \le \frac{1}{n}$, then,

$$\frac{1}{n+1} C_n(t) \le \varepsilon C_\varepsilon(t) \le \frac{1}{n} C_{n+1}(t)$$

for all t. Hence $\sup\limits_{s\le t} |\varepsilon C_\varepsilon(t) - L(t,b-)| \to 0$ almost surely and the proof is complete. ///

Corollary 3.5.1 Let (X_t) be a continuous semi-martingale. Then,

(i) almost surely,

$$\underset{b \downarrow a}{\mathrm{Lt}} \ (b-a)\, C_t((a,b)) = L(t,a)$$

uniformly on compact t intervals.

(ii) If $(X_t) \in H^p$, $p \geq 1$, then

$$\underset{b \downarrow a}{\mathrm{Lt}} \ (b-a)\, C_t((a,b)) \overset{L^p}{=} L(t,a)$$

uniformly on compact t-intervals.

Proof. The proof follows if we apply the theorem to the semi-martingale $(-X)$ and the interval $(-b, -a)$ and note that (see Yor [19]) for any continuous semi-martingale (X_t) with local time L,

$$L(t,x-) = \underset{\varepsilon \to 0}{\mathrm{Lt}} \ \frac{1}{\varepsilon} \int_0^t I_{(x-\varepsilon,x)}(X_s)\, d\langle X^c\rangle_s$$

$$L(t,x) = \underset{\varepsilon \to 0}{\mathrm{Lt}} \ \frac{1}{\varepsilon} \int_0^t I_{(x,x+\varepsilon)}(X_s)\, d\langle X^c\rangle_s \ . \qquad ///$$

Remark 3.5.1 Let f be a C^1-function. Writing

$$(f(b-a)-f(0))C_t((a,b)) = \frac{f(b-a)-f(0)}{b-a} \cdot (b-a)C_t((a,b))$$

it follows immediately from the above theorem that almost surely,

$$\underset{a \uparrow b}{\mathrm{Lt}} \ (f(b-a)-f(0))C_t((a,b)) = f'(0)L(t,b-).$$

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