# Symmetric Traveling Salesman Problem : Some New Insights

M. Usha

Indian Statistical Institute
1998

# Symmetric Traveling Salesman Problem : Some New Insights

**M. Usha**



A thesis submitted to the Indian Statistical Institute in partial fulfillment
of the requirements for the degree of Doctor of Philosophy
Chennai
1998

## ACKNOWLEDGMENTS

I wish to convey my thanks to those who helped me in writing this thesis. I am greatly indebted to Professor T. S. Arthanari, my supervisor, for his constant guidance, motivation and encouragement. He has been very supportive of my efforts and was always ready to discuss new ideas.

Professor. T. Parthasarathy helped me out during the most critical times with a lot of encouragement. He remained supportive throughout this process and showed keen interest in my work even though our research interests are different. I record here my sincere gratitude to him. I thank Professor S. R. Mohan for his useful suggestions in improving presentation of this thesis.

I thank the Indian Statistical Institute,(ISI), for the fellowship to conduct my research. I thank all my teachers at ISI Calcutta, in particular, Professor. B. V. Rao, who were very helpful during the early stages of my research. I thank all staff and faculty at the Stat-Math Unit and SQC Unit, ISI Bangalore for their support and encouragement. My special thanks to everyone at ISI, Chennai for extending all facilities to conduct my research. I also thank the library staff and administrative staff at ISI Calcutta, Bangalore and Chennai for their help during my stay.

I got a lot of support from outside ISI. Professor M. R. Rao of I.I.M, Bangalore suggested a lot of new ideas and discussed with me some of the problems I was working on. He has been extremely kind in sparing his valuable time to discuss with me. Dr. Chandru, Dr. Ashok and Anil of I.I.Sc were always helpful and were available to listen to any of my ideas. My sincere thanks to all of them. I thank the Institute of Mathematical Sciences, Chennai for giving me permission to use their computer facilities, without which I would have been severely handicapped.

The support I received from each and every member in the family and my friends has been immeasurable. All of them have been a source of great strength to me. I would not have completed my thesis without their encouragement.

## Abstract

This thesis is a study on a 'compact' formulation of the symmetric traveling salesman problem $STSP$. Arthanari(1982) posed the $STSP$ as a multistage decision problem. We call this formulation as the Multistage-insertion($MI$) formulation. We study properties of this formulation in detail. We also obtain the linear description of the projection of the $MI$ polytope and prove its equivalence to the classic subtour elimination polytope, $SEP$. We discuss the equivalence of the $MI$ formulation to the Cycle-shrink,($CS$), formulation proposed by Carr(1996). Both the $MI$ and $CS$ formulations are 'compact formulations' which use fewer number of constraints. We also study structure of small $SEP$ and $MI$ polytopes. The gist of the thesis is presented below in a chapter-wise summary.

Chapter-1 is an introductory chapter in which we present a brief introduction to concepts from combinatorial optimisation problems, graph theory, polyhedral combinatorics and linear programming which are used in the thesis.

Chapter-2 is an introduction to the Traveling Salesman Problem( $TSP$). We discuss various formulations of the TSP such as the classic Dantzig, Fulkerson and Johnson ($DFJ$), Bellmann's dynamic programming formulation, Miller, Tucker , Zellin($MTZ$) and Gavish, Graves formulation. The $STSP$ polytope is defined and we introduce different facets of this polytope. This chapter has a brief introduction to the graphical traveling salesman problem and discusses the separation problem for the $STSP$ polytope.

In chapter-3 we present the Multistage-insertion formulation of the symmetric traveling salesman problem given by Arthanari(1982). We have a $(n-3)$ stage decision problem,in which in stage $(k-3), 4 \leq k \leq n$,we decide on where to insert $k$. We give the formulation and state properties. There is a 1-1 correspondence between $n$-tours and the integer feasible solutions to the $MI$ problem. The vector of slack variables in the $MI$ problem is the edge-tour incidence vector. We define two polytopes $\zeta(n)$ and $\mathcal{U}(n)$ and show $\mathcal{U}(n)$ to be at least as tight as the subtour elimination polytope $SEP(n)$. $\mathcal{U}(n)$

is the orthogonal projection of the $MI$ polytope, $\zeta(n)$. We briefly state the $MI$ formulation for the asymmetric traveling salesman problem, $ATSP$, and give some properties.

In chapter-4, we obtain the linear description of $\mathcal{U}(n)$ and show $\mathcal{U}(n)$ is equivalent to $SEP(n)$. The linear description of $\mathcal{U}(n)$ is obtained by applying the results of Padberg and Sung(1991). We explicitly work out generators for $\mathcal{U}(n), n = 6$. The results are given in Appendix -I.

In chapter-5 we discuss another polynomial sized formulation proposed by Carr(1996) called the Cycle-shrink ,($CS$). The $CS$ is equivalent to $SEP(n)$. We show it is equivalent to $MI$.

In chapter-6 we study small subtour polytopes for $n \leq 7$. We present some criteria to characterise hamiltonian cycles. We use the results to give a necessary and sufficient condition for a feasible solution to $MI$ to lie within $STSP(n)$. We apply these results to small polytopes and give computational results. Appendix-II gives these results.

Chapter-7 summarises all the results presented in the thesis and addresses problems for further research.

# Contents

# Chapter 1

# Introduction

This thesis focuses on the Symmetric Traveling Salesman Problem($STSP$) and the Multistage-insertion,($MI$), formulation. The traveling salesman problem ,$TSP$, is to find the shortest route of a traveling salesman starting from a home city, visiting a given list of cities and returning to the home city. If the distance from going from city $i$ to city $j$ is the same as returning from city $j$ to city $i$, we call the problem the Symmetric Traveling Salesman Problem, $STSP$. In terms of graph theory, the problem can be posed as the shortest Hamiltonian cycle problem in a complete graph on $n$ vertices.

In this chapter we give a brief introduction to combinatorial optimisation problems. We also state some concepts from graph theory, polyhedral combinatorics and linear algebra used in this work.

## 1.1 Combinatorial Optimisation Problems

Integer and combinatorial optimisation deal with problems of maximising or minimising a function of several variables subject to inequality or equality constraints and integer restriction on some or all variables. In a combinatorial optimisation problem, generally we have a finite ground set $E$, a weight $c_e$ associated with each element of $E$ and a family $\mathcal{F}$ of feasible subsets. We wish to find $S \in \mathcal{F}$ for which $c(S) = \sum_{e \in S} c_e$ is maximised. The members of $\mathcal{F}$ can be represented by vectors, usually the 0-1 incidence vectors(will be defined in section 1.2) of the sets $S \in \mathcal{F}$. We define a polyhedron $P$ to be the convex hull of the incidence vectors of the sets $S$, obtain a linear system sufficient to define $P$ and apply linear programming techniques to solve the optimisation problem.

An instance of a problem is a single occurrence of such a problem which is specified by providing a certain input. The size of the instance is the number of characters required to represent the instance. Each problem has associated with it a fixed encoding scheme and the input length for an instance of a problem is defined to be the number of symbols in the description of the instance obtained from the encoding scheme. Problems which can be answered with a 'yes' or 'no' are called decision problems. Algorithms are general step by step procedures for solving problems. The class **P** is the set of all decision problems which can be solved 'polynomially'. That is, for each problem $P \in \mathbf{P}$, there must exist an algorithm and a polynomial $p(l)$ such that an instance of $P$ whose encoding is of length $l$ can be solved by the algorithm in at most $p(l)$ elementary steps. We refer the reader to Garey and Johnson(1979) for a more exhaustive discussion of the topic. The most important class of problems is in the class $NP$. These are problems for which a 'yes' answer can be verified in a polynomial amount of time, provided some extra information is given. This extra information is called a certificate and for each instance, its length must be polynomially bounded in the length of the corresponding input. Trivially $P \subseteq NP$. A problem is $NP$- *complete* if it is in $NP$, and showing that it is in $P$ would imply $P = NP$. More specifically, a problem is $NP$- complete if a polynomially bounded algorithm for solving it could be used once as a subroutine to obtain a polynomially bounded algorithm for every problem in $NP$. Karp(1972) showed many classical problems to belong to the $NP$-complete class. A problem is $NP-hard$ if there is a $NP$-complete problem that can be polynomially reduced to it. The traveling salesman problem, $TSP$, was one of the first problems to be proven $NP-hard$ by Karp in 1972. Garey and Johnson(1979), Papdimitriou and Steiglitz(1982) and Shmoys and Tardos(1995) are excellent references on Computational Complexity.

## 1.2 Graph Theory

In this section we give basic results from graph theory which we use in the thesis.

A graph $G$ is an ordered pair of $(V, E)$, where $V$ is a set of a finite, nonempty elements called nodes or vertices and $E$ is a family of two-element subsets of $V$ called edges. An edge is denoted by $e = [u, v]$ where $u, v$ are elements of $V$. If $[u, v] = [v, u]$ for all $u, v \in V$, then $G$ is called an undirected graph else, it is called a directed graph or a diagraph. A complete graph $K_n = (V_n, E_n)$ on $n$ nodes is a graph in which every pair of nodes $u, v$ has an edge $[u, v]$ between them. An undirected graph $K_n$ has $\frac{n(n-1)}{2}$ edges. A weighted graph is a graph with weights associated with edges.

An edge $e \in E$ meets or is incident to a node $v \in V$ if $e$ has $v$ as one of its endpoint. The number of edges incident to a node $v$ is called the degree of node $v$. Two edges are said to be adjacent if they have a node in common.

A path is a sequence of edges traversing several nodes such that each edge has one node in common with its predecessor and successor in the sequence. A cycle is a path where the starting node of the path is the same as the ending node of the path. A Hamiltonian cycle is a cycle traversing each node in a graph exactly once. An Eulerian cycle is a cycle in which each edge in a graph is traversed exactly once. The $TSP$ is to find the shortest Hamiltonian cycle in a graph.

Let $U \subseteq V$, define $E(U) = \{[i, j] | [i, j] \in E, i, j \in U\}$, $E(U)$ is the set of edges with both end points in $U$. Define $\delta(U) = \{[u, v] \in E | u \in U, v \notin U\}$. For singleton sets denote $\delta(\{w\}) = \delta(w)$. If $V' \subseteq V$ and $E' \subseteq E(V')$, then $G' = (V', E')$ is said to be a subgraph of $G = (V, E)$. If $V = V'$ then $G'$ is a spanning subgraph . If $E' = E(V')$, then $G'$ is the subgraph induced by $V'$.

A connected graph is a graph such that every pair of nodes can be reached by a path. A component of a graph is a maximal connected subgraph of the graph. A clique is a complete subgraph of the graph. A connected graph has only one component. An acyclic graph is called a forest. A tree is a connected forest. A spanning tree is a collection of edges without cycles and connecting all nodes.

A cut is a subset of edges $e = [u, v]$ where $u \in S, v \in V \backslash S, S \subset V, S \neq \phi, S \neq V$. A ($s$-$t$)- cut for $s, t \in V$ is a cut with $s \in S$ and $t \in V \setminus S$. Given two non-empty sets $S \subseteq V$ and $T \subseteq V \setminus S$ we define a cut as

$$\{S : T\} = \{[u, v] \in E | u \in S, v \in T\}.$$

A graph can be represented by its $m \times n$ node-edge incidence matrix $A = ((a_{ij}))$ where $a_{ij} = 1$, if edge $e_j$ is incident with node $i$ and $a_{ij} = 0$ otherwise.

For every subset $F \subseteq E$ we associate a vector $x^F \in R^E$ called the incidence vector of $F$ defined as

$$x_e^F = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$$

Also we define $x(F) = \sum_{e \in F} x_e$ for any vector $x \in R^E$.

A matching in a graph is a collection $M$ of edges such that no two edges in $M$ are adjacent. A $b$ matching is a collection of edges $M \subseteq E$ such that at most $b_v$ edges are incident with node $v$.

A network $G = (N, \mathcal{A})$ is a diagraph consisting of distinct vertices $s$ and $t$, where $s$ is called the source and $t$ is called the sink, $N = V \bigcup \{s\} \bigcup \{t\}$, and $\mathcal{A} = \{e = [i, j] \mid i, j \in N \ \& \ i \neq t, j \neq s\}$. With each arc $[i, j] \in \mathcal{A}$, we associate a nonnegative integer $c_{ij}$ called the capacity of the arc $[i, j]$. A flow in a network $G$ is an integer valued function $f$ defined on $\mathcal{A}$ such that capacity constraints and flow conservation equations are satisfied, i.e.,

$$\begin{aligned} 0 \leq f_e & \leq c_e \ \forall \ e \in \mathcal{A} \\ \sum_{e \in \delta^+(i)} f_e - \sum_{e \in \delta^-(i)} f_e & = 0 \ \forall \ i \in V \end{aligned}$$

where $\delta^+(i) = \{j \mid [i, j] \in \mathcal{A}\}$ and $\delta^-(i) = \{j \mid [j, i] \in \mathcal{A}\}$

The maximum-flow problem is to maximise the flow out of the source or, equivalently, the flow into the sink, subject to the constraints of flow out is equal to flow in for all the other nodes.

For more details about graphs and networks we refer to standard texts as Berge(1973),Bondy and Murthy(1985), Nemhauser and Wolsey(1988) and Bazaara et. al (1990), Ahuja et.al(1993).

## 1.3 Polyhedral Theory

Polyhedral combinatorics studies combinatorial problems with the help of polyhedra. We now discuss the main ideas of Polyhedral combinatorics: Let $\mathcal{F}$ be a collection of subsets of a finite set $E$, let $c : E \longrightarrow R$, and suppose we need to find

$$max\{\sum_{e \in S} c(e) | S \in \mathcal{F}\} \tag{1.3.1}$$

Complete enumeration cannot be resolved to in general for solving (1.3.1) due to the size of $\mathcal{F}$. Denote by $\chi^S$ the incidence vector of $S$ in $R^{|E|}$, i.e, $\chi^{(e)} = 1$ if $e \in S$ and 0 otherwise. Since (1.3.1) means maximising a linear function over a set of vectors, we can equally well maximise over the convex hull of these vectors:

$$max\{c^T x | x \in conv\{\chi^S | S \in \mathcal{F}\}\} \tag{1.3.2}$$

This convex hull is a polytope and there exists a matrix $A$ and vector $b$ such that

$$conv\{\chi^S | S \in \mathcal{F}\} = \{x \in R^{|E|} | Ax \leq b\} \tag{1.3.3}$$

Hence (1.3.2) is equal to

$$max\{c^T x | Ax \leq b\} \tag{1.3.4}$$

The original combinatorial problem is thus formulated as a linear programming problem, and we can appeal to linear programming methods to study this problem. However, in order to apply LP techniques (discussed in the section 1.4), we should be able to find matrix $A$ and vector $b$ satisfying (1.3.3). This is one of the main theoretical problems in polyhedral combinatorics.

The fact that Khachiyan's(1979) ellipsoidal method for solving LP problems in polynomial time does not require explicitly $A$ revived interest in polyhedral combinatorics. We refer to Stoer and Witzgall(1970), Pulleyblank(1980) Bachem and Grötschel(1982), Grötschel and Padberg(1979), Padberg and Grötschel(1985) Grötschel and Pulleyblank(1986), Nemhauser and Wolsey(1988) and Schrijver (1995) for a good exposition and motivation on polyhedral theory. In this section we summarize the necessary polyhedral concepts and terminology.

### 1.3.1 Elementary Linear Algebra

In linear programming we are given a description of the feasible set of points $P = \{x \in R^n | Ax = b, x \geq 0\}$ and we solve a linear program using either the simplex method or one of the interior point methods.

Integer programming is different. In this we are given an implicit description of the set $S \subseteq Z^n_+$of feasible points. The main objective here is to find a linear equality description of the set.

We work with the $n$-dimensional Euclidean space denoted by $R^n$. We call $x \in R^n$ a linear combination of vectors $x_1, x_2, \ldots, x_k$ if $x = \sum_{i=1}^{k} \lambda_i x_i$, where $\lambda_1, \ldots, \lambda_k \in R$. If in addition, $\lambda_i's$ satisfy $\sum_{i=1}^{k} \lambda_i = 1$ then $x$ is an affine combination of $x_i's$. If $x$ is an affine combination such that $\lambda_i \geq 0$, for $i = 1, \ldots, k$, then $x$ is called a convex combination of $x_1, \ldots, x_k$.

Let $0 \neq S \subseteq R^n$. Then the set of all linear(resp. affine, convex) combination of finitely many vectors in $S$ is called the linear(resp. affine, convex) hull of $S$ and is denoted by $lin(S), (resp. aff(S), conv(S))$. A set with $S = lin(S)(resp. aff(S), conv(S))$ is called a linear subspace(resp. affine subspace, convex subspace).

A set of points $x_1 \ldots, x_k \in R^n$ is linearly(resp. affinely) independent if the unique solution of $\sum_{i=1}^{k} \lambda_i x_i = 0 (resp. \sum_{i=1}^{k} \lambda_i x_i = 0, \sum_{i=1}^{k} \lambda_i = 0)$ is $\lambda_i = 0, i = 1, \ldots, k$. Otherwise $S$ is called linearly(resp. affinely) dependent. Every linearly(resp. affinely) independent set $S$ contains at most $n$(resp.

$(n+1))$ elements.

The cardinality of the largest linearly(resp. affinely) independent subset of $S$ is the rank(resp. affine rank) of $S \subseteq R^n$. The dimension of $S$ is the affine rank of $S$ minus one. A set $S \subseteq R^n$ is full dimensional if $dim(S) = n$.

If $\{x \in R^n | Ax = b\} \neq 0$, the maximum number of affinely independent solutions of $Ax = b$ is $n + 1 - rank(A)$.

$H \subseteq R^n$ is a subspace if $x \in h$ implies $\lambda x \in H$ for all $\lambda \in R^1$ and if $x, y \in H$ implies $x + y \in H$. If $H \subseteq R^n$ is a subspace, then $\{x \in R^n | xy = 0 \ for \ y \in H\}$ is a subspace. This subspace is called the orthogonal subspace of $H$ and is denoted by $H^{\perp}$. If $p \in R^n$ and $H$ is a subspace, the projection of $p$ on $H$ is the vector $q \in H$ such that $p - q \in H^{-}$. The projection of $S$ on $H$ is denoted by

$$proj_H(S) = \{q \mid q \ is \ the \ projection \ of \ p \ \ on \ H \ for \ some \ p \in S\}$$

A set $H \subseteq R^n$ is called a half space if there is a vector $a \in R^n$ and a scalar $a_o \in R$ such that $H = \{x \in R^n | ax \leq a_o\}$. An inequality is called valid with respect to $S \subseteq R^n$ if $S \subseteq \{x \in R^n | ax \leq a_o\}$. A valid inequality $ax \leq a_o$ is called supporting if $S \bigcap \{x \in R^n | ax = a_o\} \neq \phi$. A valid inequality $ax \leq a_O$ is said to be proper valid inequality if $S$ is not contained in the hyperplane $\{x | a^T x = b\}$.

A polyhedron $P \subseteq R^n$ is the set of points that satisfy a finite number of linear inequalities, i.e. $P$ can be represented in the form
$P = \{x \in R^n | Ax \leq b\}$. A bounded polyhedron is a polytope.

A subset $F$ of a polyhedron $P$ is called a face of $P$ if
$F = \{x \in P | ax = a_O\}$, where $ax \leq a_o$ is a supporting inequality with respect to $P$. A face $F$ is proper if $F \neq P$.

Two valid inequalities $ax \leq a_o$ and $bx \leq b_o$ are equivalent if

$$\{x \in P \ | ax = a_o\} = \{x \in P | bx = b_o\}$$

A facet $F$ is a maximal proper face , i.e; $dim(F) = dim(P) - 1$.

A polyhedron $P$ can be expressed as follows by Weyl's theorem

**Theorem 1.3.1** *(Weyl's theorem) Every polyhedron can be written as*

$$P = conv(V) + cone(D)$$

*where $V = (x_1, \ldots, x_k)$ is a nonempty finite set of vectors in $R^n$ called vertices and $D = \{d_1, \ldots, d_k)$, a finite set of linearly independent vectors called extreme rays or directions.*

### 1.3.2 Facet identification

In combinatorial optimisation, polyhedra are usually given as the convex hulls of finite sets of integral points. Finding an inequality system which defines such a polyhedron is a major challenge. Moreover, finding such inequality systems with as few inequalities as possible is a problem. Hence facet defining inequalities gain importance . The first step to find a facet is to find a valid inequality which is proper for the polyhedron $P$. Then one determines whether or not the valid inequality $ax \leq a_o$ is a facet of the polyhedron. We have the following theorems(Grötschel (1985))

**Theorem 1.3.2** *Let $P \subseteq R^n$ be a polyhedron and assume that $A$ is an $(m, n)$ matrix, $b \in R^m$ such that $aff(P) = \{x \in R^n \mid Ax = b\}$. Let $F$ be a nonempty face of P, then the following statements are equivalent:*
*(a) $F$ is a facet of $P$*
*(b) $F$ is a maximal proper face of $P$*
*(c) $dim(F) = dim(P) - 1$*
*(d) There exists an inequality $\{c^T x \leq c_o\}$ valid with respect to $P$ with the following three properties*
*(d1) $F \subseteq \{x \in P \mid c^T x = c_o\}$*
*(d2) There exists $\bar{x} \in P$ with $c^T \bar{x} < c_o$, i.e. the inequality is proper.*
*(d3) If any other inequality $d^T x \leq d_o$ valid with respect to $P$ satisfies $F \subseteq \{x \in P \mid d^T x = d_o\}$, then there exists a scalar $\alpha \geq 0$ and a vector $\lambda \in R^m$ such that*

$$d^T = \alpha c^T + \lambda^T A$$
$$d_o = \alpha c_o + \lambda^T b$$

Condition (c) provides a direct method to exhibit a set of $k = dim(P)$ vectors and prove they are affinely independent. (d) is an indirect method of establishing a facet.

## 1.4 Linear Programming

Linear programming(LP) is concerned with problems in which a linear objective function in terms of decision variables is to be optimised(minimised or maximised) with a set of linear equations, inequalities and sign restrictions imposed on the decision variables. A standard form linear programming problem is described as follows:

$$\begin{aligned} (P1) \quad min \quad & z = cx \\ s.t \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where $x$ is $n \times 1$ column vector, $c$ is a $1 \times n$ row vector and $A$ is a $m \times n$ matrix. We can assume without loss of generality rank$(A) = m$. the objective function to be minimised is given by $z = cx$. Any LP problem can be posed in the standard form. A basis $B$ is an $m \times m$ nonsingular submatrix of $A$. Define $P = \{x \in R^n | Ax = b, x \geq 0\}$ to be the feasible region of the linear problem. When $P$ is not void, the LP is said to be consistent. We call $x \in P$ to be a feasible solution. Given any basis $B$, a feasible solution $x$ can be decomposed as $x^T = (x_B^T, x_N^T)$ with $x_B = B^{-1}b$ and $x_N = 0$. If $x_B \geq 0$ then $x$ is called a basic feasible solution. Each basic feasible solution corresponds to a vertex( or extreme point) of the polyhedron $P$. A feasible solution $x^*$ is said to be an optimal solution if $cx^* \leq cx$ for all $x \in P$. We denote $P^* = \{x^* \in P | x^*$ is an optimal solution$\}$ as the optimal solution set. We have the following fundamental theorem in linear programming.

**Theorem 1.4.1** *A point $x \in P$ is an extreme point of $P$ if and only if the columns of $A$ corresponding to the positive components of $x$ are linearly independent.*

**Corollary 1.4.1** *A point $x \in P$ is an extreme point of $P$ iff $x$ is a basic feasible solution corresponding to some basis $B$.*

**Duality**
Given a linear programming problem $(P1)$ we can always write a dual program $D$ corresponding to $P1$ as follows

$$\begin{array}{lll} (D) & max & wb \\ & s.t & wA \leq c \\ & & w \; unrestricted \; in \; sign \end{array}$$

$P1$ is called the primal and $D$ the dual. We have the following:

**Lemma 1.4.1** *The dual of dual is the primal*

**PRIMAL-DUAL RELATIONSHIPS**

**The relationship between objective values**

Consider $P1$ and $D$ . Let $x_0$ and $w_0$ be any feasible solutions to the primal and dual problems respectively. Then we have

$$cx \geq wb$$

**Theorem 1.4.2** *(Fundamental Theorem of Duality) Let $P1$ and $D$ be a pair of primal-dual problems. Then exactly one of the following statements is true.*
*1. $P1$ and $D$ possess optimal solutions $x^*$ and $w^*$ with $cx^* = w^*b$*
*2. If one of $P1$ and $D$ have unbounded objective value then the other is infeasible.*
*3. Both $P1$ and $D$ are infeasible.*

Another useful theorem is the theorem on Complementary Slackness which is given below.

**Theorem 1.4.3** *(Complementary Slackness) If $x^*$ and $w^*$ are optimal solutions to $P1$ and $D$ then*

$$(c - w^*A)x^* = 0$$

## 1.5 Contents of the thesis

In the second chapter we have given an introduction to the Traveling Salesman Problem, various formulations, facet defining inequalities, the graphical traveling salesman problem and the separation problem for $TSP$.

Chapter three introduces the multistage-insertion formulation ,($MI$), of the symmetric traveling salesman problem. Theoretically interesting properties of this formulation are presented. We show the slack variables that arise out of the $MI$ formulation are precisely the edge tour incidence vectors. We study the relationship of this formulation and the subtour relaxation and show the projection of the $MI$ formulation, $\mathcal{U}(n)$, is at least as tight as the subtour polytope. We introduce the $MI$ formulation for the asymmetric traveling salesman problem also.

In chapter four we derive the linear description of $\mathcal{U}(n)$. We also show $\mathcal{U}(n)$ is equivalent to the subtour polytope.

Chapter 5 discusses another polynomially sized formulation of the $STSP$ which was introduced by Carr(1995), called cycle-shrink($CS$). We show how this formulation is equivalent to the $MI$ formulation.

Chapter 6 is a study on small polytopes. We work out the extreme points of small subtour and $MI$ polytopes. We also study the structure of fractional extreme points of these polytopes. We give a method to generate cutting planes to eliminate some of the fractional points.

Chapter 7 is a brief conclusion which shows some future directions for research of topics discussed in this thesis.

**APPENDICES**
**APPENDIX -I** : We work out the linear description of $\mathcal{U}(6)$.
**APPENDIX -II** : In this appendix we give the extreme points of small $SEP$ and $MI$ polytopes. We obtain cutting planes for $SEP(6)$ and $SEP(7)$. We also give the extreme points of small $DFJ$ and Asymmetric $MI$ polytopes.

# Chapter 2

# The Traveling Salesman Problem

The traveling salesman problem is typical of problems of its genre and is one of the most challenging problems in combinatorial optimisation. Bellmore and Nemhauser(1966) in their survey of the $TSP$ classify solution techniques and give description of some of the proven methods to solve $TSP$. Most successful approaches to solve tough combinatorial optimisation problems were first formulated for the $TSP$ and $TSP$ was one of the first problems to be proven $NP-Hard$ by Karp(1972). New algorithmic techniques have been first developed for $TSP$ to check for their effectiveness. Examples of these are the branch and bound, Lagrangean relaxation, simulated annealing, Lin-Kernighan type methods. The $TSP$ has a variety of applications in vehicle routing, X-ray crystallography etc.

Lawler et.al(1985) motivated considerable research in this area. The more recent developments are given in a review by Jünger et.al(1995). A good bibliographical survey is given in Jünger et. al (1997). Burkard et. al(1998) give a survey of well-solvable cases of the $TSP$.

In this thesis we concentrate on formulations of the $STSP$. We also give structure of small polytopes associated with our study. In this chapter we give preliminaries of topics discussed in the thesis. Section-2.1 describes various formulations of the $TSP$. In section 2.2, we describe the most well known facets of the $STSP$. We give a brief introduction to the graphical traveling salesman problem in section 2.3 and in section 2.4 we discuss the separation problem for the $STSP$.

## 2.1 Various Formulations of the TSP

In this section we describe some of the known formulations of the traveling salesman problem . Dantzig, Fulkerson and Johnson $(DFJ)(1954)$ gave a linear programming approach to solve the TSP. Their classical formulation, an assignment problem on a graph $G = (V, E)$, with additional restrictions is as follows:

$$min \quad \sum_i \sum_j c_{ij} x_{ij} \tag{2.1.1}$$

$$st \quad \sum_i x_{ij} = 1 \qquad j = 1, \ldots, n \tag{2.1.2}$$

$$\sum_j x_{ji} = 1 \qquad i = 1, \ldots, n \tag{2.1.3}$$

$$\sum_{i,j \in S} x_{ij} \leq | S | -1 \quad \forall S \subseteq V; 2 \leq | S | \leq n-1 \tag{2.1.4}$$

$$x_{ij} \geq 0 \qquad \forall\, i, j \tag{2.1.5}$$

$$x_{ij} \ integer \qquad \forall\, i, j \tag{2.1.6}$$

where $V = \{1, 2, \ldots, n\}$. We thus have a formulation of the $TSP$ involving $n(n-1)$ variables and $\mathcal{O}(2^n)$ constraints. The constraints (2.1.4) are referred to as subtour elimination constraints. They can be equivalently described as

$$x(\delta(S)) \geq 2 \quad \forall\, S \subseteq V, \;\; S \neq \phi; \; S \neq V \tag{2.1.7}$$

The symmetric version of the above formulation gives rise to the subtour elimination polytope which is given by

SUBTOUR ELIMINATION POLYTOPE $SEP$ :
$SEP$ is the polytope defined by the set of all $x \in R^E$ such that the following

hold:

$$x_e \geq 0 \quad \forall\, e \in E \tag{2.1.8}$$
$$x(\delta(v)) = 2 \quad \forall\, v \in V \tag{2.1.9}$$
$$x(E(S)) \leq |\, S \,| - 1 \quad \forall\, S \subseteq V\ , S \neq \phi\ \ and\ \ S \neq V \tag{2.1.10}$$

**Dynamic Programming Formulation**

A dynamic programming formulation of the $TSP$ problem, due to Bellman(1961) is given below:

Consider the $TSP$ as a multistage decision problem. Without loss in generality, fix origin of the tour as some city , say 0. Suppose at a certain stage of an optimal tour starting at 0, one has reached a city $i$ and there are cities $j_1, j_2, \ldots, j_k$ to be visited before returning to 0. Since the tour is optimal, the path from $i$ through $j_1, j_2, \ldots, j_k$ in some order and then to 0 must be of minimum length. Define

$$f(i; j_1, j_2, \ldots, j_k) \equiv \textit{length of a path of minimum length from } i \textit{ to } 0 \textit{ which passes once and only once through each of the remaining } k \textit{ unvisited cities } j_1, j_2, \ldots, j_k \tag{2.1.11}$$

If we obtain $f(0; j_1, j_2, \ldots, j_n)$, and a path which has this length then the $TSP$ has been solved. Let $c_{ij}$ be the distance between the $i$th and $j$th cities. As a consequence, we have

$$f(i; j_1, j_2, \ldots, j_k) = min_{1 \leq m \leq k}\{c_{ij_m} + f(j_m; j_1, j_2, \ldots, j_{m-1}, j_{m+1}, \ldots, j_k\} \tag{2.1.12}$$

The above equation is an application of Bellman's principle of optimality of the theory of dynamic programming. The iterative procedure given above is initiated through the use of

$$f(i; j) = c_{ij} + c_{j0}$$

from which we obtain $f(i; j_1)$, $f(i; j_2)$. This in turn yields $f(i; j_1, j_2)$. We go on applying (2.1.12) recursively and terminate when an optimal tour with length $f(0; j_1, \ldots, j_n)$ is obtained. The sequence of values of $m$ which minimises $R.H.S$ of (2.1.12) gives the desired minimal path.

Many other formulations have been proposed ever since. There have been consistent efforts in trying to reduce the number of variables used in a formulation. Miller, Tucker and Zemlin(1960) proposed a mixed-integer model for a more general $TSP$ on $V = \{1, \ldots, n\}$ nodes. This was also known as the "clover-leaf" model for the $TSP$. The formulation goes as follows: Denote city 1 as the home city, the salesman has to visit the other $n-1$ cities exactly once, returning to his home city exactly $t$ times, including his final return. He must visit no more than $p$ cities different from his home city in one tour. The problem has the following linear programming formulation :

$$\min \quad \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$s.t \quad \sum_{i=2}^{n} x_{i1} = t \tag{2.1.13}$$

$$\sum_{i=2}^{n} x_{1i} = t \tag{2.1.14}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 2, \ldots, n \tag{2.1.15}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 2, \ldots, n \tag{2.1.16}$$

$$u_i - u_j + px_{ij} \leq p - 1, \quad 2 \leq i \neq j \leq n \tag{2.1.17}$$

$$x_{ij} = 0 \text{ or } 1 \qquad \forall\, i, j \;\; u_i \geq 0, \;\forall\, i \tag{2.1.18}$$

$$x_{ij} \text{ integer} \qquad \forall\, i, j \tag{2.1.19}$$

This formulation uses $O(n^2)$ constraints and $(n^2 - 1)$ variables. For $t = 1$ and $p \geq n-1$, $MTZ$ models the standard formulation.

Gavish and Graves(1978) propose a formulation using $O(n^2)$ binary variables $x_{ij}$, $O(n^2)$ continuous variables $z_{ij}$ and $n^2 + 3n$ constraints. The variables describe the flow of a single commodity to node one from every other node. The subtour constraint in this case are given by

$$\sum_{j} z_{ij} - \sum_{j \neq i} z_{ji} = 1 \quad i = 2, \ldots n \tag{2.1.20}$$

$$z_{ij} \leq (n-1)x_{ij} \quad i = 2, \ldots n, \;\; j = 1, \ldots, n \tag{2.1.21}$$

$$z_{ij} \geq 0 \quad for\ all\ i,j \tag{2.1.22}$$

**Multi-commodity flow formulation**

We present a formulation(Wong(1980)) for the $TSP$ using the flow of $2(n-1)$ commodities, $Y^k = (y_{ij}^k), k = 2, \ldots, n$ and $Z^k = (z_{ij}^k), k = 2, \ldots, n$ as follows

$$minimise \qquad \sum_i \sum_j c_{ij} x_{ij}$$

$$s.t \quad (2.1.2), (2.1.3), (2.1.4), (2.1.5)$$

$$\sum_j (y_{ij}^k - y_{ji}^k) = \left\{ \begin{array}{ll} 1 & \text{if } i = 1 \\ -1 & \text{if } i = k \\ 0 & \text{if } i \neq 1\ and\ k \end{array} \right\} \quad k = 2, \ldots, n \tag{2.1.23}$$

$$\sum_j (z_{ij}^k - z_{ji}^k) = \left\{ \begin{array}{ll} 1 & \text{if } i = 1 \\ -1 & \text{if } i = k \\ 0 & \text{if } i \neq 1\ and\ k \end{array} \right\} \quad k = 2, \ldots, n \tag{2.1.24}$$

$$y_{ij}^k \leq x_{ij}, z_{ij}^k \leq x_{ij} \quad for\ all\ i,j,k \tag{2.1.25}$$

$$y_{ij}^k \geq 0, z_{ij}^k \geq 0 \quad for\ all\ i,j,k \tag{2.1.26}$$

Constraints (2.1.23 ) and (2.1.24 ) ensure that a unit of commodity $Y^k$ travels from node 1( source of $Y^k$) to node $k$ (sink of $Y^k$) while one unit of commodity $Z^k$ travels from node $k$ to node 1.

Langevin et. al(1990) present several single commodity , two-commodity and multi commodity flow formulations of the $TSP$. They compare the optimal value of the linear programming relaxations of different formulations. They also present relations between the formulations with that of the classical *DFJ* formulation.

Arthanari(1982) proposed the $STSP$ as a multistage decision problem. His formulation uses only polynomial number of constraints but additional variables $x_{ijk}$. We call this formulation the Multistage-insertion formulation (*MI*). In this thesis we study this formulation in detail and show its equivalence to the classic subtour elimination formulation.

Padberg and Sung(1991) make an analytical comparison of the $MTZ$, $FGG$ and *Claus* formulations with that of the classical $DFJ$ formulation. They obtain linear descriptions of the projection of the respective polytopes in the space of $x_{ij}$ variables and compare the formulations. They show all these formulations to be weaker than the $DFJ$ formulation. We describe their methodology and apply it to obtain the linear description of $\mathcal{U}(n)$ of the $MI$ formulation in chapter-4. They also suggested extension of their work to the $TDTSP$ problem.

The $TDTSP$ is a generalisation of the $TSP$ where the cost of any given arc is dependent on its position in the tour. The $TDTSP$ was introduced by Fox in 1973. Picard and Queyranne(1978) presented a formulation based on a quadratic assignment model. Fox, Gavish and Graves(1980) give a formulation of the TDTSP which used only $\mathcal{O}(n)$ constraints. Gouveia and Voß(1995) classify various $TDTSP$ formulations . They also compare well known formulations of the $TSP$.

Carr(1995) proposed the Cycle-shrink($CS$) formulation of the $TSP$. $CS$ is a polynomial sized linear programming relaxation of the $TSP$. Carr shows the equivalence of $CS$ to the $SEP$. We discuss this formulation in Chapter-5 and show its equivalence to the $MI$ formulation.

## 2.2 Facets of $STSP(n)$

Let $\mathcal{T}_n$ be the set of all tours in $K_n$. Then the polytope

$$STSP(n) = conv\{x^T \in R^{E_n} | T \in \mathcal{T}_n\}$$

is called the Symmetric Traveling Salesman Polytope. The dimension of $STSP(n) = \frac{n(n-3)}{2}$.

Due to the complexity of the $TSP$ polytope, considerable amount of research has been devoted to characterising new classes of facet defining inequalities for the $TSP$ polytope. In this section we summarise properties and known families of facet defining inequalities. For further details we refer to Padberg and Grötschel(1985) and Jünger et. al(1995). Jünger et. al(1997) in their annotated bibliography give good references of the work done so far. The known families of facets of $STSP(n)$ are described below

**Trivial facets**

$$x_e \leq 1 \quad \forall e \in E \quad , n \geq 4$$
$$x_e \geq 0 \quad \forall e \in E \quad , n \geq 5$$

**Subtour Elimination Constraints**

Dantzig, Fulkerson and Johnson $(DFJ)(1954)$ introduce a class of inequalities known as the subtour elimination constraints. They are given by

$$x(E(S)) \leq \mid S \mid -1, 2 \leq \mid S \mid \leq n-1, S \subseteq V_n$$

For $\mid S \mid = 2$, these inequalities are the trivial facets. However $DFJ$ do not address the question of whether these inequalities are facet defining or not. Grötschel and Padberg(1979b) show the subtour elimination constraints to be facet defining for $STSP(n)$, for $2 \leq \mid S \mid \leq n-2$.

**2-matching inequalities**

Edmonds(1965) defines a class of inequalities called the 2-matching inequalities to give a complete description of the polytope associated with the 2-matching problem. They are given by

$$x(H) + x(E') \leq \mid H \mid + \frac{\mid E' - 1}{2}$$

for all $H \subseteq V$ and all $E' \subseteq E$ satisfying

$$\begin{aligned} \mid e \bigcap H \mid &= 1, \quad \forall e \in E' \\ e_i \bigcap e_j &= \phi, \quad e_i \neq e_j \in e' \\ \mid E' \mid &\geq 3 \end{aligned}$$

The set $H$ is called the handle and $E'$ teeth. Chvatal(1973) generalised this class of inequalities and defined a class of inequalities and called them comb inequalities. Now they are referred to as Chvatal combs.

**Comb inequalities**

Grötschel and Padberg(1979a,b) generalised the Chvatal combs to a larger class of inequalities they called comb inequalities and showed they were facet defining for $STSP(n)$. The comb inequalities are defined for $S = \{H, T_1, T_2, \ldots, T_s\}$. The set $H$ is called the handle and sets $T_i$, teeth. The comb inequality is

$$x(E(H)) + \sum_{j=1}^{s} x(E(T_j)) \leq \mid H \mid + \sum_{j=1}^{s}(\mid T_j \mid -1) - \frac{s+1}{2}$$

for all $H, T_1, \ldots, T_s \subseteq V_n$ satisfying

$$\begin{aligned} \mid T_j \bigcap H \mid &\geq 1, \quad j = 1, \ldots, s \\ \mid T_j \setminus H \mid &\geq 1, \quad j = 1, \ldots, s \\ T_i \bigcap T_j &= \phi, \quad 1 \leq i \leq j \leq s \\ s &\geq 3 \quad \textit{and odd} \end{aligned}$$

**Chain inequalities**

Padberg and Hong(1980) describe a further generalisation of the comb inequalities called chain inequality. The chain inequalities are defined for $S = \{H, T_1, T_2, \ldots, T_s\}$ with the following conditions

$$\begin{aligned} \mid T_j \bigcap H \mid &= \phi \quad \textit{for } j = 1, \ldots, p \\ \mid T_j \bigcap H \mid &\geq 1, \quad j = p+1, \ldots, s \\ T_i \bigcap T_j &= \phi, \quad 1 \leq i \leq j \leq s \\ s &\geq 3 \quad \textit{and odd} \end{aligned}$$

Let $R \subseteq H$ satisfying $\mid R \mid = p$ and $R \bigcap T_i = \phi, i = 1, \ldots s$, then the chain inequality is given by

$$x(E(H)) + \sum_{i=1}^{s} x(E(T_i)) + \sum_{i=1}^{p} x(E(R : T_i)) \leq \mid H \mid + \mid R \mid + \sum_{i=1}^{s}(\mid T_i \mid -1) - \frac{s-p+1}{2}$$

Only a proof of validity of the chain inequalities is given .

**Clique tree inequalities**
A new class of inequalities which generalize the comb inequalities was introduced by Grötschel and Pulleyblank(1986). The new class of inequalities are called clique tree inequalities and were shown to be facet defining for $STSP(n), n \geq 11$. A clique tree is defined on a connected graph $S = \{H_1, \ldots, H_r, T_1, \ldots, T_s\}$ composed of cliques (the sets $H_i$ are called handles and sets $T_j$ are called teeth)which satisfy the following properties
(1) The cliques are partitioned into two sets handles and teeth
(2) no two teeth intersect
(3) no two handles intersect
(4) each tooth contains at least one node not belonging to any handle
(5) each handle intersects an odd number of teeth
(6) if a tooth $T$ and a handle $H$ have nonempty intersection,then $H \bigcap T$ is an articulation set of the clique-tree

The inequality is given by

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq \sum_{i=1}^{r} \mid H_i \mid + \sum_{j=1}^{s} (\mid T_j \mid - t_j) - \frac{s+1}{2}$$

where $t_j$ is the number of handles intersected by tooth $T_j$.

**Bipartition and Ladder inequalities**
Boyd and Cunningham(1991) introduced a new class of valid inequalities for $n \geq 7$ called bipartition inequalities. Let $\{H_1, \ldots, H_r\}$ be mutually disjoint sets of handles and $\{T_1, \ldots, T_{t+m}\}$ be mutually disjoint nonempty sets of teeth. Assume, $t + m \geq 1$, satisfying the following conditions
$H_i$ intersects $2k_i + 1$ teeth, where $k_i$ is a positive integer., $1 \leq i \leq r$.

$$T_j \setminus (\bigcup H_i) \neq \phi \qquad for\ 1 \leq j \leq t$$
$$T_j \setminus (\bigcup H_i) = \phi \quad for\ t+1 \leq j \leq t+m$$

The bipartition inequality associated with $H_1, \ldots, H_r, T_1, \ldots, T_{t+m}$ is

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{t} x(E(T_j)) + \sum_{j=t+1}^{t+m} \frac{d_j}{d_j - 1} x(E(T_j)) \leq$$

$$\sum_{i=1}^{r} | H_i | + \sum_{i=1}^{r} k_i + \sum_{j=1}^{t}(| T_j | -d_j - 1) + \sum_{j=t+1}^{t+m} \frac{d_j}{d_j - 1}(| T_j | -d_j)$$

If $r = 2, t = 1, m = 2$, we get an example of a bipartition inequality called the *envelope inequality*.

A ladder inequality is defined on a family of $S$ sets

$$S = \{H_1, H_2, P_1, P_2, T_1, \ldots, T_t, D_1, \ldots, D_m\}$$

with $t, m \geq 0$. The sets $H_i, P_i, T_i$ and $D_i$ are called handles, pendent teeth, regular teeth and degenerate teeth respectively. A ladder inequality associated with the family is

$$\begin{aligned} \sum_{i=1}^{2} x(E(H_i)) + \sum_{i=1}^{2} x(E(P_i)) + \sum_{i=1}^{t} x(E(T_i)) + \sum_{i=1}^{m} 2x(E(D_i)) \\ + x(E(P_1 \bigcap H_1 : P_2 \bigcap H_2) \\ \leq \sum_{i=1}^{2} | H_i | + \sum_{i=1}^{2} | P_i | + \sum_{i=1}^{t} | T_i | + \sum_{i=1}^{m} 2 | D_i | -2t - 3m - 4 \end{aligned}$$

The proof that ladder inequalities are facet-defining for $STSP(n), n \geq 8$ is given in Boyd, Cunningham, Queyranne and Wang(1995).

## 2.3 The Graphical Traveling Salesman Relaxation

Cornuéjols et al.(1985) introduce a new generalization of the $TSP$ called the Graphical Traveling Salesman, Problem($GTSP$). In the standard $TSP$, we have a graph $G = (V, E)$ where $V$ is the set of vertices(cities) and $E$ is the set of edges between two cities, each edge $[i, j] \in E$ has a cost $c_{ij}$ associated with it. In the $TSP$ we find a Hamiltonian cycle with minimum weight. The $GTSP$ is to find a shortest tour, starting from home city, visit a given list of cities at least once and then return to the home city so that no edge is used more than once.

In $GTSP$ tour $t_G$ of $G$ is a multiset of edges such that
(i) Degree of $t_G$ for every $v \in V$ is positive and even
(ii) The subgraph of $G$ induced by edges of $t_G$ is connected.

Denote the set of all tours on $n$ vertices by $\mathcal{T}_n^*$. Denote the weighted incidence vector of $t_G \in \mathcal{T}_n^*$ by $\chi^t$. The convex hull of the set of incidence vectors of the elements of $\mathcal{T}_n^*$ is the graphical traveling salesman polytope, $GTSP(n)$

$$GTSP(n) = conv\{\chi^{t_G} | t_G \in \mathcal{T}_n^*\}$$

**Definition 2.3.1** *An inequality $fx \geq f_o$ defined on $R^n$ is tight triangular(TT) if the following are satisfied:*
*a) the coefficients $f_e$ satisfy the triangular inequality.*
*b)For all $w \in V_n$, there exists $u, v \in V_n$ such that $f_{uv} = f_{uw} + f_{vw}$*

Almost every inequality facet-defining for $GTSP(n)$ is tight-triangular. We refer to Naddef and Rinaldi(1988) for a detailed discussion on $GTSP$. We now give a few inequalities defined in the $TT$ form. We give an example of an inequality in $TT$ form below.

**Crown Inequalities**
Naddef and Rinaldi(1992) discovered a new class of valid inequalities called Crown inequalities and showed that they are facet-defining for $STSP(n)$(with $n \geq 8$).

For any integer $k \geq 2$, let $C(k) = (V_c, E_c)$ be the graph with the following vertex and edge sets, $V_c = \{v_i | i \in \{1, \ldots, 4k\}\}$
$E_c = \{[v_i, v_{i+1}] | i \in \{1, \ldots, 4k\}\}$, where $[i]$ stand for $((i-1)\text{mod}4k) + 1)$. $C(k)$ is called a crown configuration. A simple crown inequality associated with $C(k)$ is the inequality $fx \geq f_0$ such that $f_0 = 12k(k-1) - 2$, and for every $i \in \{1, \ldots, 4k\}$

$$\begin{aligned} f(v_i, v_{[i+j]}) &= 4k - 6 + |j| \quad for\ 1 \leq |j| \leq 2k-1 \\ &= 2(k-1) \quad for\ j = 2k \end{aligned}$$

Generalisations of clique-tree inequalities lead to Bipartition inequalities[Boyd and Cunningham(1991)], Hyperstar inequalities [Fleischmann(1987)] and the binested inequalities [Naddef,(1992)]. We refer to Naddef(1990) for a complete survey on these classes of inequalities.

### 2.3.1 Extension of inequalities

The very complex structure of $STSP(n)$ makes it very difficult to describe all inequalities known to define facets of this polytope. Naddef and Rinaldi(1992) define some operations on inequalities which allow derivation of new inequalities from those which have already been characterised. Two such operations are the zero node-lifting and the edge cloning. Both these operations preserve the facet defining nature of inequalities.

The operation of zero node-lifting is as follows. Let $fx \geq f_o$ be a facet defining inequality in $TT$ form on $K_n = (V_n, E_n)$. Add $k$ more vertices to $V_n$ obtaining the set $V_{n+k}$. We zero node-lift node $u$ to obtain the inequality $f^*x \geq f_o$ where :

$$\begin{aligned} i) f^*_e &= f_e \quad \forall e \in E_n \\ ii) f^*_{ij} &= f_{uj} \quad \forall i \in V_{n+k} \setminus V_n \ \forall j \in V_n \\ iii) f^*_{ij} &= 0 \quad \forall i, j \in V_{n+k} \setminus V_n \end{aligned}$$

Let $fx \geq f_o$ be a $TT$ inequality defined on $R^n$ and $e \in E_n$. We say the inequality $f^*x^* \geq f^*_o$ defined on $R^{E_n+2h}$, with $h \geq 1$ is obtained from $fx \geq f_o$ by cloning the edge $e(h\ times)$ if

$$\begin{aligned} f^*_o &= f_o + 2hc_e \\ f^*(u_i, u_{n+j}) &= f(u_i, u_{n-1}) \quad 1 \leq i \leq n-2 \\ & \qquad\qquad\qquad 1 \leq j \leq 2h-1, j\ odd \\ &= f(u_i, u_n) \quad 1 \leq i \leq n-2 \\ & \qquad\qquad\qquad 2 \leq j \leq 2h, j\ even \\ f^*(u_{n+i}, u_{n+j}) &= 2c_e \quad -1 \leq i < j \leq 2h, j-1\ even \\ &= c_e \quad -1 \leq i < j \leq 2h, j-1\ odd \end{aligned}$$

By using the above operations Naddef and Rinaldi(1988) show the $PWB$ inequalities are facet defining for $STSP(n)$. Boyd et al.(1995) show the ladder inequalities and their extensions obtained by zero node-lifting to define facets of $STSP(n)$.

## 2.4 Separation Problem for $STSP(n)$

Given a point $\bar{x}$ in the space of a polyhedron $P$, the separation problem for a class $\mathcal{I}$ of valid inequalities for $P$ is to produce an inequality of $\mathcal{I}$ violated by $\bar{x}$ or to prove that all the inequalities of $\mathcal{I}$ are satisfied by $\bar{x}$.

An exact separation algorithm for a family $\mathcal{I}$ of inequalities is one that solves the separation problem, a heuristic is one which may find a violated inequality, but that in case it cannot find any, is unable to guarantee that no violated inequalities exist in $\mathcal{I}$.

Separation problem is known to be solvable only for two classes of facet defining inequalities of $STSP(n)$. They are the subtour elimination and the 2-matching inequalities. The separation for these classes can be solved using the Gomory-Hu algorithm and the Padberg -Rao algorithm respectively.

An excellent introduction to the separation problem and cutting plane algorithms associated with them is given in Padberg and Grötschel(1985). In this paper they state the facet-identification problem and bring out the relationship between optimisation problems and facet identification. They also give heuristics to separate subtour elimination constraints, 2-matching constraints and a simple heuristic to separate comb constraints.More recent references include Applegate et. al (1997), Naddef and Thienel(1999).

Let $x^*$ be a point we want to separate from $STSP(n)$. The support graph of $x^*$ is given by $G_{x^*} = (V_n, E, x^*)$, where $E$ contains all edges of $E_n$ corresponding to a positive component of $x^*$.

**Separation for subtour elimination constraints**

Lets start with a relaxation for the $STSP$ involving only the degree constraints and non-negativity constraints. Let

$$P_1 = \{x \in R^{|E_n|} \mid x \; satisfies \; degree \; and \; nonnegativity \; constraints\}$$

We have the following

**Proposition 2.4.1** *If* $x^* \in P_1$ *then* $\sum_{e \in E(W)} x_e^* = \mid W \mid -1 + \epsilon$ *if and only if*

$$\sum_{e\in\delta(W)} x^*_e = 2 - 2\epsilon.$$

To check whether a subtour elimination inequality is violated by $x^*$ or not it suffices to solve

$$\xi = min\left\{ \sum_{e\in\delta(U)} x^*_e \mid U \subset V, 3 \leq\mid U \mid\leq \frac{\mid V \mid}{2} \right\}$$

and check whether $\xi < 2$ or not. Let

$$\xi_j = min\{ \sum_{e\in\delta(U)} x^*_e \mid \{1, 2, \ldots, j-1\} \subset U, j \in \bar{U}, 3 \leq\mid U \mid\leq\mid V \mid -3\}$$

for $j = 2, \ldots, \mid V \mid -2$ Then $\xi = min_{j=2,\ldots \mid V \mid -2}\xi_j$. Imposing the condition $\{2, \ldots, j-1\} \subset U$ in the $1-j$ cut problem is done by replacing the capacities $x^*_{ik}$ by $\infty$ for $k = 2, \ldots, j-1$ The separation algorithm is to solve the maximum $1-j$ problem for $j = 2, \ldots, \mid V \mid -2$. The Gomory-Hu algorithm is based on the computation of $n-1$ max-flow problems on weighted graphs. We refer to J"unger et al(1995) for a complete list of efficient algorithms to find a min-cut in graphs.

*MI* relaxation is shown in Chapter-4 to optimize over the *SEP*, and hence the separation problem for the class of subtour elimination inequalities can be achieved in polynomial time using *MI* relaxation. Also Carr(1995) achieves the same.

### Separation for 2-matching inequalities

Padberg and Rao(1982) propose an algorithm which computes the minimum weight odd cut of a labeled weighted graph in polynomial time. Grötschel and Holland(1987) give a heuristic separation algorithm for the separation of the 2-matching inequalities. They also give an effective implementation of the exact Padberg Rao algorithm.

### Separation of comb inequalities and Clique tree inequalities

There exist no exact algorithms to separate the comb and clique tree inequalities. Efficient facet identification procedures have been described in

Padberg and Rinaldi(1990). They define shrinkable sets and obtain sufficient conditions for a set to be shrinkable. They discuss certain graph reduction techniques and give a heuristic which runs faster than the Padberg Rao algorithm to separate 2- matching inequalities. They also give heuristics for identification of comb and clique tree inequalities.

Carr(1995) describes an exact separation algorithm for the class of bipartition inequalities when the number of handles and teeth are fixed. He defines a backbone to be a set of $k$ nodes of the graph that is shared by a subset of inequalities in $\mathcal{I}$. For a fixed backbone, the most violated inequality in the corresponding set is found in polynomial time. Carr(1996) describes a polynomial time separation algorithm for a class of valid inequalities for $STSP(n)$ obtained by zero node-lifting of any fixed inequality defined on subsets of $V$ having $k$ nodes.

## 2.5 Conclusions

This chapter discussed some of the important concepts of the Traveling Salesman Problem. In chapter 3 through to chapter 5 we present our work on the Multistage-insertion formulation and show its equivalence to the subtour elimination relaxation and the cycle-shrink formulation.

# Chapter 3

# The Multistage-insertion formulation

After Dantzig, Fulkerson and Johnson(1954) proposed their integer programming formulation for the $TSP$, there has been a race to provide with alternate formulations to solve the $TSP$. Researchers have been providing formulations using fewer constraints and implying the $DFJ$ formulation. One such is the Multistage-insertion formulation ($MI$) proposed by Arthanari(1982). We present this formulation in this chapter and give its properties.

Arthanari(1982) posed the Symmetric Traveling Salesman Problem($STSP$) as a multistage decision problem and gave the $MI$ formulation. He showed that the slack variables that arise in this formulation are precisely the edge-tour incidence vectors. This formulation uses $O(n^3)$ variables but only a quadratic number of constraints. Bellman(1962), Held and Karp(1962) were the first to consider a multi stage decision (dynamic programming ) approach to TSP. However their formulations are different from that of Arthanari(1982).

In this chapter we show that the formulation we have, defines the $SEP(n)$,(where $SEP(n)$ denotes the subtour elimination polytope on $n$ vertices), for all $n$ using a polynomial number of constraints In section-3.2 we give the $MI$ formulation. We state some properties and give results on this formulation. We how $\mathcal{U}(n) \subseteq SEP(n)\ \forall\ n$. In section 3.3 we characterise integer optimal solutions of the $MI$ problem and in section 3.4 we give the $MI$ formulation or the asymmetric traveling salesman problem.

## 3.1 Notations and Definitions

Let $n$ denote the number of cities.

**Definition 3.1.1** *$t = (1, i_1, \ldots, i_{k-1}, 1)$ is a k-tour in case $(i_1, \ldots, i_{k-1})$ is a permutation of $(2, \ldots, k), k \leq n$.*

Let $C_{ijk} = c_{ik} + c_{jk} - c_{ij}$ *for* $4 \leq k \leq n;\ \ 1 \leq i < j \leq k-1$.

**Definition 3.1.2** *The length of a k-tour is defined as $C(t)$ given by*

$$C(t) = \sum_{r=1}^{k-2} c_{i_r i_{r+1}} + c_{1 i_1} + c_{i_{k-1} 1}$$

Let $\mathcal{T}_k$ denote the set of all $k$-tours and $\mathcal{T}_{ijk}$ denote the set of all $k$-tours in which edge$[i,j]$ appears that is $i$ and $j$ are adjacent to each other in every $k$-tour in $\mathcal{T}_{ijk}$. Then we have $\mathcal{T}_k = \bigcup\limits_{1 \leq i < j \leq k}^{\cdot} \mathcal{T}_{ijk}$.

Let $F_{ij}^k$ be a mapping from $\mathcal{T}_{ijk-1}$ to $\mathcal{T}_k$ such that for $t \in \mathcal{T}_{ijk}$, $t = (1, i_1, \ldots, i, j, \ldots, i_{k-1}, 1); F_{ij}^k(t) = (1, i_1, \ldots, i, k, j, \ldots, i_{k-1}, 1) \in \mathcal{T}_k$, i.e, $F_{ij}^k(t)$ is the $k$-tour obtained from the $(k-1)$-tour $t$ by inserting $k$ between $i$ and $j$.

We start with the 3-tour $t = (1, 2, 3, 1)$.

**Example 3.1.1** *Take n = 5. Consider $t = (1, 4, 3, 2, 1) \in \mathcal{T}_4$. Then t belongs to each one of $\mathcal{T}_{144}, \mathcal{T}_{344}, \mathcal{T}_{234}, \mathcal{T}_{124}. F_{14}^5(t) = (1, 5, 4, 3, 2, 1) \in \mathcal{T}_5$.*

We now state some results.

**Proposition 3.1.1** *Let* $t_1, t_2 \in \mathcal{T}_{ijk-1}$*. If* $C(t_1) \leq C(t_2)$ *then*

$$C(F_{ij}^k(t_1)) \leq C(F_{ij}^k(t_2))$$

**Proposition 3.1.2** $\mathcal{T}_{k+1} \; = \; \bigcup_{1 \leq i < j \leq k} \{F_{ij}^{k+1}(t) | t \in \mathcal{T}_{ijk}\}$

**Proposition 3.1.3** $min_{t \in \mathcal{T}_{k-1}} C(t) \; = \; min_{1 \leq i < j \leq k} \; \{min_{t \in \mathcal{T}_{ijk}} C(t) + C_{ijk+1}\}$

**Remark 3.1.1** *The symmetric traveling salesman problem is to find an optimal n-tour, given* $c_{ij}, 1 \leq i < j \leq n$*, with* $c_{ij} = c_{ji}$*. Proposition (3.1.3) assures an optimal n-tour, if we have a subset of* $(n-1)$ *tours which includes for each* $1 \leq i < j \leq n-1$*, a* $(n-1)$ *tour in which i and j are adjacent and it minimises the length of the tour among all such* $(n-1)$ *tours in which i and j are adjacent. However, finding such* $(n-1)$ *tours may not be an easy task.*

Thus we really have a $(n-3)$ stage decision problem, in which in stage $(k-3), 4 \leq k \leq n$, we decide on where to insert $k$. In the beginning we have a 3-tour $(1,2,3,1)$. In the first stage we decide on where to insert 4 among the available pairs $[1,2]$ , $[2,3]$, and $[1,3]$. Depending on this decision we have certain available pairs for the second stage insertion.

In the second stage we decide on where to insert 5 among the available pairs. For instance, our decision in the first stage is to introduce 4 between $i_4$ and $j_4$. Then the available pairs are

$$A_5 \; = \; \{[1,2],[1,3],[2,3]\} \bigcup \{[i_4,4],[j_4,4]\} \; - \; \{[i_4,j_4]\}$$

In general $A_k$ depends on the decisions made in preceding stages, $4 \leq k \leq n$. We have

$$A_k \; = \; A_{k-1} \bigcup \{[i_{k-1},k-1],[k-1,j_{k-1}]\} - \{[i_{k-1},j_{k-1}]\}$$

for some $[i_{k-1},j_{k-1}] \in A_{k-1}$. $A_k$ gives the set of all $[i,j]$ such that they are adjacent in the $(k-1)$ tour, which results out of the decisions made in the preceding stages.

The associated total cost of these decision made at different stages is

$$C_{i_4j_44} + C_{i_5j_55} + \ldots + C_{i_nj_nn}.$$

We are interested in finding optimal $[i_4, j_4], \ldots, [i_n, j_n]$ such that the total cost is minimum. This finally produces an $n-$tour. The length of this tour given by $(c_{12} + c_{13} + c_{23}) + \sum_{k=4}^{n} C_{i_kj_kk}$. Here $(c_{12} + c_{13} + c_{23})$ is the length of the initial 3-tour which is independent of the decisions subsequently made.

## 3.2 Mathematical Programming formulation of the *STSP*

In this section we describe the 0-1 integer programming formulation of the *STSP* given by Arthanari(1982).

For $1 \leq i < j \leq k-1$, we define $x_{ijk}$ by

$$x_{ijk} = \begin{cases} 1 & \text{if in stage } (k-3) \text{ the decision is to insert } k \text{ between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

Therefore there are $\tau_n = \sum_{4}^{n} \frac{(k-1)(k-2)}{2}$ variables.

**Definition 3.2.1** *Given*
$X = (x_{124}, x_{134}, x_{234}; x_{125}, \ldots, x_{345}; \ldots; x_{12n}, x_{13n}, \ldots, x_{n-2,n-1,n}) \in B^{\tau_n}$, *where* $B^{\tau_n} = \{0, 1\}^{\tau_n}$. *We say* $X$ *is a feasible decision vector in case,*

*(i) For every* $k = 4, \ldots, n$

$$\sum_{1 \leq i < j \leq k-1} x_{ijk} = 1 \qquad (3.2.1)$$

*that is* $k$ *is inserted between* $i$ *and* $j$ *for exactly one pair* $[i, j]$ *and*
*(ii)* $x_{ijk} = 1 \Rightarrow T_{k-1}(X) \in \mathcal{T}_{ijk-1}$ , *where* $T_{k-1}(X)$ *is the* $(k-1)-$*tour resulting from the preceding decisions, that is, depending on* $(x_{124}, \ldots, x_{k-3,k-2,k-1})$,

*denoted by $X/k-1$. In other words, $X$ is a feasible decision vector if* $x_{i_kj_kk} = 1 \Longrightarrow [i_k, j_k] \in A_k, 4 \leq k \leq n$.

**Example 3.2.1** *For $n = 6$, let $x_{124} = 1, x_{145} = 1, x_{236} = 1, x_{ijk} = 0$ for all other $1 \leq i < j \leq k-1$, $4 \leq k \leq 6$ then* $X = (x_{124}, x_{134}, x_{234}; x_{125}, \ldots, x_{345}; x_{126}, \ldots, x_{456}) =$ $(1,0,0;0,0,0,1,0,0;0,0,1,0,0,0,0,0,0,0)$ *is a feasible decision vector as* $\sum_{1 \leq i < j \leq k-1} x_{ijk} = 1$; *for $k = 4,5$ and $6$ and $x_{124} = 1$, requires $T_3(X) \in \mathcal{T}_{123}$. This is true as $T_3(X) = (1,2,3,1)$ Similarly $x_{145} = 1 \Longrightarrow T_4(X) = (1,4,2,3,1) \in \mathcal{T}_{144}$ and $x_{236} = 1 \Longrightarrow T_5(X) = (1,5,4,2,3,1) \in \mathcal{T}_{235}$. However, $X =$ $(100, 100000, 0010000000)$ is not a feasible decision vector as $T_4(X) = (1,4,2,3,1)$, resulting from $x_{124} = 1 \notin \mathcal{T}_{124}$ as required for $x_{125} = 1$.*

Let $\Im$ be the set of all feasible decision vectors. We can state the multistage decision process as:

**PROBLEM 0**:Find $X^* \in \Im$ such that $\mathcal{C}(X^*) = min_{X \in \Im} \mathcal{C}(X)$ where

$$\mathcal{C}(X) = \sum_{k=4}^{n} \sum_{1 \leq i < j \leq k-1} C_{ijk} x_{ijk}$$

We shall now show , how $X \in \Im$ can be expressed as a set of linear equalities and inequalities along with $X \in B^{r_n}$.

Notice that we already have

$$\sum_{1 \leq i < j \leq k-1} x_{ijk} = 1 \ for \ X \in \Im$$

The above equation ensures that at stage $k$ city $k$ is inserted between exactly one pair of cities $i$ and $j$. In addition , $x_{ijk}$ can not be 1 if $[i,j] \notin T_{k-1}(X)$.

Condition (ii) of Definition 3.2.1 states that $x_{i_kj_kk} = 1 \Longrightarrow [i_k, j_k] \in A_k; 4 \leq k \leq n$. We express this as linear inequality as follows:

For all $X$, we have $[1,2], [1,3]$ and $[2,3] \in T_3(X)$ as the initial tour is always $(1,2,3,1)$. And the edges are available in all sets $A_k, 4 \leq k \leq n$

unless $x_{ijk} = 1$. Since we begin with the 3-tour and at most one of the $x_{ijk} = 1, 4 \leq k \leq n$ for each $[i,j]; 1 \leq i < j \leq 3$ we have the following constraint

$$\sum_{4 \leq k \leq n} x_{ijk} \leq 1$$

Intuitively, the above inequality ensures that between a pair of successive cities $i$ and $j$ where $1 \leq i < j \leq 3$, at most one city $k$ may be inserted.

Now consider other $[i,j]'s$, $for\ 4 \leq j \leq n-1$ and $1 \leq i < j$, $x_{ijk}$ can not be 1 unless $[i,j]$ is an edge in the $(k-1)$- tour resulting from earlier decisions given by $X/k-1$. However $[i,j]$ is created only in one of the two ways, given below:

Either (i) $x_{rij} = 1$ for some $1 \leq r < i$ or

(ii) $x_{isj} = 1$ for some $i+1 \leq s < j$. Therefore , if

$$\sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = 1$$

then edge $[i,j]$ is present at the $k^{th}$ stage and hence $x_{ijk}$ can either be 0 or 1 for any $k \geq j+1$. If

$$\sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = 0$$

then the edge $[i,j]$ is not available for insertion from the $k^{th}$ stage; $k \geq j+1$ and $\sum_{j+1 \leq k \leq n} x_{ijk} = 0$. Hence we have

$$\sum_{j+1 \leq k \leq n} x_{ijk} \leq \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj}$$

$$\Longrightarrow - \sum_{1 \leq r \leq i-1} x_{rij} - \sum_{i+1 \leq s \leq j-1} x_{isj} + \sum_{j+1 \leq k \leq n} x_{ijk} \leq 0$$

The above inequality ensures that if city $k$ has been inserted between $i$ and $j$ with $i < j$, then $j$ must have been inserted between some $r$ and $i$ with $r < i$, or $j$ must have been inserted between some $i$ and $s$ with $i < s < j$.

Now Problem- 0 can be given a 0-1 programming formulation as given below:

PROBLEM 1:

$$minimise \sum_{k=4}^{n} \sum_{1\leq i<j<k-1} C_{ijk}x_{ijk}$$

subject to

$$\sum_{1\leq i<j\leq k-1} x_{ijk} = 1 \qquad 4 \leq k \leq n \qquad (3.2.2)$$

$$\sum_{k=4}^{n} x_{ijk} \leq 1 \qquad 1 \leq i < j \leq 3 \qquad (3.2.3)$$

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^{n} x_{ijk} \leq 0 \quad 4 \leq j \leq n-1; 1 \leq i < j \qquad (3.2.4)$$

$$x_{ijk} = 0 \; or \; 1 \quad 1 \leq i < j \leq k;\; 4 \leq k \leq n \qquad (3.2.5)$$

**Remark 3.2.1** *The objective function is the same as in* PROBLEM 0.

Let $E$ denote the matrix corresponding to (3.2.2), $E$ is a $(n-3) \times \tau_n$ matrix of the following form.

$$E = \begin{bmatrix} e_{\frac{3\times 2}{2}} & 0 & \cdot & \cdot & 0 & \\ 0 & \cdot & \cdot & \cdot & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & e_{\frac{(n-1)(n-2)}{2}} & \end{bmatrix}$$

where $e_k$ is a vector each of whose coordinates is 1.

Let $A$ be the matrix of coefficients corresponding to (3.2.3)-(3.2.4). Relaxing the integer constraints with $0 \leq x_{ijk} \leq 1$ and adding the following constraints

$$-\sum_{r=1}^{i-1} x_{rin} - \sum_{s=i+1}^{n-1} x_{isn} \leq 0 \qquad i = 1, \ldots, n-1 \qquad (3.2.6)$$

we get the following problem.

PROBLEM 2:

$$\min C'X$$

subject to

$$\begin{bmatrix} E & \mathbf{0} \\ A & \mathbf{I} \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} e_{n-3} \\ e_3 \\ 0 \end{bmatrix}$$

$$X, U \geq 0. \tag{3.2.7}$$

Note that the upper bound 1 on $x_{ijk}$ is not explicitly stated. Also (3.2.6) are always satisfied as $x_{ijk}$ are non-negative. However adding these constraints help us bring out the connection between the slack variables of Problem-2 and the edge-tour incident vectors of $n$-tours given by integer $X$ feasible to Problem -2. Here $A$ is the matrix corresponding to inequality constraints without non-negativity constraints.

**Theorem 3.2.1** *Any integer feasible solution to* PROBLEM -2 *is a basic solution and has the following property.*
*Let the submatrix of $A$ corresponding to the columns of $x_{i_k j_k k} = 1$, $k = 4, \ldots, n$ be denoted by $\mathbf{Q}$. Then any row of $\mathbf{Q}$ is such that either*
*(i)All entries in a row are zeroes.*
*or (ii)Exactly one of the elements is +1 and the rest are zeroes in the row.*
*or(iii) There is a -1 and a +1 in the row and the rest are zeroes.*
*or (iv) There is a -1 in the row and the rest are zeroes.*
*Moreover any such solution corresponds to a $n$-tour.*

**Proof:** Consider the square matrix $\mathbf{B}$ obtained by taking the columns corresponding to $x_{i_k j_k k} = 1; 4 \leq k \leq n$ and the columns corresponding to the slack variables $u_{ij}$. We can write $B$ as a partitioned matrix as given below

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}$$

e Q is the submatrix of A corresponding to the columns $x_{i_k j_k k} = 1; 4 \leq k \leq n$.
inverse of $B$ has the following form.

$$B^{-1} = \begin{bmatrix} I & 0 \\ -Q & I \end{bmatrix}$$

$Q_{ij}$ denote the row corresponding to the pair $[i, j]$.
(i) : $1 \leq i < j \leq 3$
this case either no $x_{ijk}$ is positive for pair $[i, j]$ or at most one of them is
al to 1 in any integer feasible solution. This implies either
$Q_{ij}$ is a zero vector where we have an instance of (i)
) $Q_{ij}$ has a single 1 and rest zeroes, where we have an instance of (ii).
act in these rows there can be no -1's.
e(ii) : $1 \leq i < j; 4 \leq j \leq n-1$
ng the fact that for any $[i, j]$ at most one of the $x_{ijk}$ can be equal to 1 in
integer feasible solution to the problem, there can be at most one +1 in
of these rows.

wever this +1 cannot occur without a -1 in the same row since

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^{n} x_{ijk} \leq 0$$

all $x_{rij}$ or $x_{isj} = 0$ then $\sum_{k=j+1}^{n} x_{ijk} = 1$ and cannot satisfy this constraint.
at least one of the $x_{rij}$ or $x_{isj} = 1$. But for any $k$ at most one $x_{i_k j_k k} = 1$.So
ere is exactly one -1 in row $Q_{ij}$. This leads to an instance of (*iii*).On the
er hand if $x_{rij}$ as well as $x_{isj}$ are zeroes then $x_{ijk}$ must all be zeroes . We
ve an instance of (i).

nally if one of the $x_{rij}$ or $x_{isj} = 1$ and all $x_{ijs} = 0$ we have an instance of
v).

w we prove that any such solution corresponds to a tour. Consider $x_{i_k j_k k} =$
$4 \leq k \leq n$. Insert in the 3-tour $(1, 2, 3, 1)$, city 4 between $[i_4, j_4]$ and obtain
-tour. Assume introducing $5, \ldots, k$ in this manner in the $4, \ldots, k-1$ tour

respectively we obtain a $k$ tour. We shall show that introducing $(k+1)$ in the unique $k$-tour obtained will result in a $(k+1)$-tour.

We need to show that $[i_{k+1}, j_{k+1}]$ is a pair available in

$$A_k(X) \stackrel{\text{def}}{=} \{[1,2],[1,3],[2,3]\} \bigcup_{r=4}^{k} \{[i_r, r],[j_r, r]\} - \bigcup_{r=4}^{k} \{[i_r, j_r]\};$$
$$1 \le i_{k+1} < j_{k+1} \le k$$

If $[i_{k+1}, j_{k+1}] \notin A_k(X)$, Then it must be either
(a) be $[i_r, j_r]$ for some $4 \le r \le k$ or
(b) $[i_{k+1}, j_{k+1}]$ is $[i, r]$ with $i_r \ne i \ne j_r$, $4 \le r \le k$

However (a) cannot happen as for any pair $[i,j], x_{ijk} = 1$ for at most one $r$ and already $x_{i_r j_r r} = 1$, $4 \le r \le k$.

If (b) happens then the constraint corresponding to $[i, r]$ will be violated and $X$ cannot be feasible for the problem. This leads to a contradiction. Hence $[i_{k+1}, j_{k+1}] \in A_k(X)$. Hence any such solution corresponds to a tour.
Hence the result □

We have the following theorem

**Theorem 3.2.2** *Thus there is a 1-1 correspondence between n-tours and the integer feasible solutions to* PROBLEM-2.

We have proved in the previous theorem that every integer solution corresponds to a $n$ tour. It remains to show every tour corresponds to an integer solution to PROBLEM-2.

Take $n = 4$, we have three 4-tours and three integer feasible solutions. The correspondence between them is shown in the table below:

| *Tour* | | *Integer solution* | |
|---|---|---|---|
| | $x_{124}$ | $x_{134}$ | $x_{234}$ |
| $t_1 = (14231)$ | 1 | 0 | 0 |
| $t_2 = (12341)$ | 0 | 1 | 0 |
| $t_3 = (12431)$ | 0 | 0 | 1 |

Suppose the result is true for $n = k-1$, that is, every $(k-1)$ tour corresponds to an integer solution to the PROBLEM-2. In other words, for every $t_{k-1} \in \mathcal{T}_{k-1}$, we have an integer solution $X/k-1 = (x_{124}, \ldots, x_{k-3,k-2,k-1})$ to PROBLEM-2. We now show the result is true for $n = k$. Let $t_k = (1, i_1, \ldots, i_r, k, i_s, \ldots, i_{k-2}, 1)$, $\quad 1 \leq i_l \leq k-2,\ 1 \leq l \leq k-2$ be a $k$ tour with $k$ being inserted between some $i_r$ and $i_s$. We need to show this tour corresponds to an integer solution.

Define $x_{i_r i_s k} = 1$ and $x_{ijk} = 0$ for $1 \leq i < j \leq k-1,\ i, j \neq i_r, i_s$

We now show $X = (X/k-1, x_{12k}, x_{13k}, \ldots, x_{i_r i_s k}, \ldots, x_{k-2,k-1,k})$ is a feasible integer solution to PROBLEM-2. First of all we note that $x_{i_r i_s l} = 0$ for all $4 \leq l \leq k-1$. We need to show a) $\sum\limits_{1 \leq i < j \leq l-1} x_{ijk} = 1$. This is true from definition of $X$

We now consider the following cases:

case (i) $1 \leq i_r < i_s \leq 3$

Since $k$ is inserted between $i_r$ and $i_s$ and $x_{i_r i_s l} = 0$ for all $4 \leq l \leq k-1$, we have $\sum\limits_{l=4}^{k-1} x_{i_r i_s l} = 0$ for $1 \leq i_r < i_s \leq 3$, which in turn implies that $\sum\limits_{l=4}^{k-1} x_{i_r i_s l} + x_{i_r i_s k} = 1$ . Hence $X$ satisfies (3.2.2)

case (ii) $1 \leq i_r < i_s;\ \ 4 \leq i_s \leq k-2$

Since $k$ is inserted between $i_r$ and $i_s$ , this implies that either $x_{r i_r i_s} = 1$ for some $r$, $1 \leq i_r \leq r-1$ or $x_{i_r s i_s} = 1$ for $i_r + 1 \leq s \leq i_s - 1$ and $\sum\limits_{l=i_s+1}^{k-1} x_{i_r i_s l} = 0$.

Hence we have

$$-\sum_{r=1}^{i_r-1} x_{r i_r i_s} - \sum_{s=i_r+1}^{i_s-1} x_{i_r s i_s} + \sum_{l=i_s+1}^{k-1} x_{i_r i_s l} + x_{i_r i_s k} = 0$$

Hence $X$ satisfies (3.2.3)

Hence the theorem. □

**Lemma 3.2.1** *Let $U$ denote the vector of slack variables in* PROBLEM - 2.*Let $(X, U)$ be any integer feasible solution to* PROBLEM-2.*Then $U$ is the* *edge-tour incidence vector of the n-tour given by* $(X, U)$.

$$u_{ij} = \begin{cases} 1 & \text{if edge [i,j] is present in the n-tour} \\ 0 & \text{otherwise} \end{cases}$$

**Proof:** Consider $1 \leq i < j \leq 3$ then from equation (3.2.3) we have

$$u_{ij} = 1 - \sum_{k=4}^{n} x_{ijk}$$

$u_{ij} = 0 \Longrightarrow \sum_{k=4}^{n} x_{ijk} = 1$, which implies that for some $4 \leq k \leq n, x_{ijk} = 1$, i.e, $[i, j]$ is not in the solution.

Conversely, suppose $[i, j]$ is not in the solution, then we have $\sum_{k=4}^{n} x_{ijk} = 1$ for some $4 \leq k \leq n$ which implies that $u_{ij} = 0$.

Now consider $1 \leq i < j; 4 \leq j \leq n - 1$

$$u_{ij} = \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} - \sum_{j+1 \leq k \leq n} x_{ijk}$$

$[i, j]$ is not present in the solution if

$$\sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = 0 \Longrightarrow \sum_{j+1 \leq k \leq n} x_{ijk} = 0$$

$$or \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = 1 \; and \sum_{j+1 \leq k \leq n} x_{ijk} = 1$$

Hence $u_{ij} = 0$ if $[i, j]$ is not present in the solution. Conversely, if $u_{ij} = 0$ we show that $[i, j]$ is not present in the solution.

$$u_{ij} = 0 \Longrightarrow \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} - \sum_{j+1 \leq k \leq n} x_{ijk} = 0$$

$$\Longrightarrow \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = \sum_{j+1 \leq k \leq n} x_{ijk}$$

There are two cases

a)

$$\sum_{1\le r\le i-1} x_{rij} + \sum_{i+1\le s\le j-1} x_{isj} = 0$$

This implies that edge $[i,j]$ is not created upto the $j$ stage and hence is not available for insertion of $k; j+1 \le k \le n$.Hence $[i,j]$ is not in the solution.

b) $\sum_{1\le r\le i-1} x_{rij} + \sum_{i+1\le s\le j-1} x_{isj} = 1 = \sum_{j+1\le k\le n} x_{ijk}$ which implies that edge $[i,j]$ is created before stage $j$, but then some $k,\ j+1 \ \le k \le n$ is inserted between $[i,j]$. Hence, $[i,j]$ is not in the solution.

Hence, $u_{ij} = 0\ iff\ [i,j]$ is not in the solution. □

**Lemma 3.2.2** *Corresponding to any feasible solution to* PROBLEM-2,*we have*

*(i)* $\sum_{1\le i<j\le n} u_{ij} = n$

*(ii)* $0 \le u_{ij} \le 1, \quad \forall 1 \le i < j \le n$

**Proof:**(i)We shall show that this is true for any feasible solution $(X,U)$ to PROBLEM -2.As $(X,U)$ is feasible we have

$$EX = e_{n-3} \tag{3.2.8}$$

$$AX + \mathrm{I}U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} \tag{3.2.9}$$

Now sum the last $\frac{n(n-1)}{2}$ terms of (3.2.9) .We get

$$-\sum_{k=4}^{n} \sum_{1\le i<j\le k-1} x_{ijk} + \sum_{1\le i<j\le n} u_{ij} = 3. \tag{3.2.10}$$

But $\sum_{k=4}^{n} \sum_{1\le i<j\le k-1} x_{ijk} = n-3$ as obtained from the sum of the first $(n-3)$ rows of (3.2.8).

(ii) case (a): $1 \leq i < j \leq 3$

We have $0 \leq x_{ijk} \Longrightarrow 0 \leq \sum_{k=4}^{n} x_{ijk}$. Also $\sum_{k=4}^{n} x_{ijk} \leq 1$

Hence we have

$$1 \geq 1 - \sum_{k=4}^{n} x_{ijk} \geq 0$$
$$\Longrightarrow 0 \leq u_{ij} \leq 1$$

case(b): $1 \leq i < j; 4 \leq j \leq n-1$

$$u_{ij} = \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} - \sum_{j+1 \leq k \leq n} x_{ijk}$$

Hence

$$(i) \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = 1, \sum_{j+1 \leq k \leq n} x_{ijk} = 1 \Longrightarrow u_{ij} = 0$$
$$(ii) \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = 1, \sum_{j+1 \leq k \leq n} x_{ijk} = 0 \Longrightarrow u_{ij} = 1$$
$$(iii) \sum_{1 \leq r \leq i-1} x_{rij} + \sum_{i+1 \leq s \leq j-1} x_{isj} = 0, \sum_{j+1 \leq k \leq n} x_{ijk} = 0 \Longrightarrow u_{ij} = 0$$

Hence $0 \leq u_{ij} \leq 1$. □

Observe that $\mathcal{C}' = (\mathcal{C}_{124}, \ldots, \mathcal{C}_{12n}, \ldots, \mathcal{C}_{(n-2)(n-1)n})$ is such that $\mathcal{C}' = -c'A$. Consider any solution $(X, U)$ to PROBLEM-2.Then

$$\mathbf{A}X + \mathbf{I}U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix}$$

Premultiply both sides by $c'$. Now

$$\begin{aligned} c'U &= c' \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - c'\mathbf{A}X \\ &= c' \begin{bmatrix} e_3 \\ 0 \end{bmatrix} + \mathcal{C}'X. \end{aligned} \tag{3.2.11}$$

But $c' \begin{bmatrix} e_3 \\ 0 \end{bmatrix} = c_{12} + c_{13} + c_{23}$ is a constant given $X$. Therefore it is sufficient to minimise $c'U$ in order to minimise $C'X$.
Now we have PROBLEM-3 which is equivalent to PROBLEM-2.

PROBLEM-3:

$$\text{minimise } c'U$$

$$\begin{aligned} \begin{bmatrix} E & \mathbf{0} \\ A & \mathbf{I} \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} &= \begin{bmatrix} e_{n-3} \\ e_3 \\ 0 \end{bmatrix} \\ X, U &\geq 0 \end{aligned} \tag{3.2.12}$$

**Remark 3.2.2** *Any $n-$tour corresponds to an integer basic feasible solution. But there are basic feasible solutions which are non-integer as illustrated in the following example.*

**Example 3.2.2** *Let $n = 5$, for the basis matrix $B$ is given as below :*

$$\begin{bmatrix} x_{124} & x_{134} & x_{135} & x_{245} & u_{12} & u_{23} & u_{14} & u_{34} & u_{15} & u_{25} & u_{35} & u_{45} \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

*the inverse of B is*

$$
\begin{bmatrix}
0.5 & 0.5 & 0 & -0.5 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 \\
0.5 & -0.5 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & 0.5 & 0 & 0.5 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & -0.5 & 1 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & -0.5 & 0 & 0.5 & 0 & 0 & 0.5 & 1 & 0 & 0 & 0 & 0 \\
-0.5 & 0.5 & 0 & 0.5 & 0 & 0 & -0.5 & 0 & 1 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 \\
-0.5 & 0.5 & 0 & 0.5 & 0 & 0 & -0.5 & 0 & 0 & 0 & 1 & 0 \\
0.5 & 0.5 & 0 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

*There is a basic feasible solution to* PROBLEM-3 *with corresponding* $x_{124} = x_{134} = x_{135} = x_{245} = 1/2$,
$u_{12} = 1/2, u_{13} = 0, u_{23} = 1, u_{14} = 1, u_{24} = 0, u_{34} = 1/2$,
$u_{15} = u_{25} = u_{35} = u_{45} = 1/2$.

Let

$$\zeta(n) = \{X | EX = e_{n-3}, AX \leq \begin{bmatrix} e_3 \\ 0 \end{bmatrix}, \ X \geq 0\} \tag{3.2.13}$$

$$\mathcal{U}(n) = \{U | U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - AX \geq 0, X \in \zeta(n)\} \tag{3.2.14}$$

**Remark 3.2.3** *Let*

$$U^s = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - AX^s \geq 0$$

*for any integer* $X^s \in \zeta(n)$. *Then* $U^s$ *is an extreme point of* $\mathcal{U}(n)$.

**Proof:** Let $U, V \in \mathcal{U}(n) - \{U^s\}$.We shall show that $\lambda U + (1-\lambda)V$ *for* $\lambda \in (0,1)$ belongs to $\mathcal{U}(n) - \{U^s\}$. We have $U \neq U^s \neq V$. As $U, V \in \mathcal{U}(n) - \{U^s\}$ there exists $X, Y$ such that

$$U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - AX \geq 0$$

and

$$V = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - AY \geq 0$$

Note that $X \neq X^s \neq Y$.
Now $U + (1-\lambda)V \in \mathcal{U}(n)$ since $\mathcal{U}(n)$ is a convex set. We want to prove that $\lambda U + (1-\lambda)V \in \mathcal{U}(n) - \{U^s\}$.
Suppose this is not true. Then $\lambda U + (1-\lambda)V = U^s$.Since $X^s$ is integer $U^s$ is also integer. We know that

$$\sum_{1 \leq p < q \leq n} U^s_{pq} = \sum_{1 \leq p < q \leq n} U_{pq} = \sum_{1 \leq p < q \leq n} V_{pq} = n$$

as these correspond to feasible solutions to PROBLEM-2. Also notice that for any feasible solution $(X, U)$ to PROBLEM-2,$0 \leq u_{ij} \leq 1$.Therefore if any coordinate of $U^s$ is zero the corresponding coordinates of $U$ as well as $V$ have to be zero,as $\lambda, (1-\lambda) > 0,$ *and* $U, V \geq 0$.
Thus $U = V = U^s$,which leads to a contradiction as $U, V \in \mathcal{U}(n) - \{U^s\}$. Hence the result.

$\mathcal{U}(n)$ is the orthogonal projection of the polytope $\zeta(n)$. It is expected that some of the projected extreme points are no longer extremal in the projection as shown in the following two examples.

**Example 3.2.3** *Consider the fractional basic feasible solution given earlier for the 5-city problem in Example(3.2.2). The fractional components of the vector U for this solution can be written as a convex combination of extreme point solutions given by*
*(a)*$x_{134} = 1, x_{125} = 1, u_{23} = u_{14} = u_{34} = u_{15} = u_{25} = 1$ *and*
*(b)*$x_{134} = x_{345} = 1, u_{12} = u_{23} = u_{14} = u_{35} = u_{45} = 1$ *with equal weightage.*

Here we have an example of a slack variable vector $U$ corresponding to a fractional basic feasible solution to PROBLEM-3 which need not be an extreme point of $\mathcal{U}(n)$. However a question that still remains is whether the set of all extreme points of $\mathcal{U}(n)$ is the set of all $U's$ corresponding to integer feasible solutions? The answer is NO as shown by the following example.

**Example 3.2.4** *Consider the Petersen's graph $G = (V, E)$ where*
$V = \{1, 2, \ldots, 10\}$
$E = \{[1,2], [1,5], [1,9], [2,3], [2,7], [3,4], [3,10], [4,5], [4,8], [5,6], [6,7], [7,8], [8,9], [9,10], [6,10]\}$

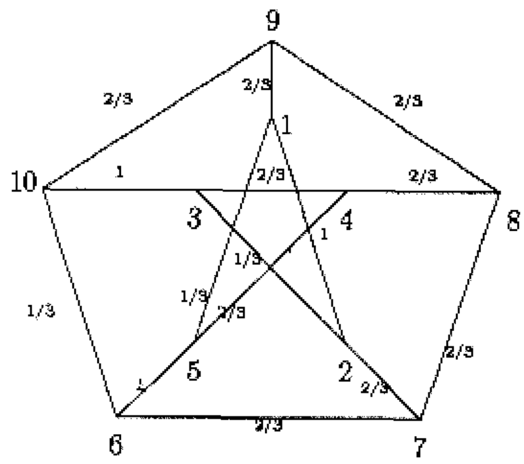



Fig: 3.2.1 : Petersen graph with values given as in Example 3.2.4

Let

$$c_{ij} = \begin{cases} -1 & \text{if } [i,j] \in E \\ 0 & \text{otherwise} \end{cases}$$

Consider the 10-city $STSP$ on the above graph. It is well known that Petersen's graph is non-Hamiltonian i.e. there is no 10-tour available only using the edges of the graph $G$. Any tour uses 10 edges of the complete graph $K_{10}$. So, an optimal tour for this problem will have an objective value of at least -9 since it has to use an edge not in $E$.

However the following fractional solution to the problem has objective function value -10.
$x_{134} = x_{135} = x_{356} = x_{147} = x_{178} = x_{348} = x_{478} = x_{139} = x_{189} = x_{389} = x_{3,6,10} = 1/3$; $x_{234} = x_{245} = x_{256} = x_{267} = x_{3910} = 2/3$;
$u_{12} = u_{56} = u_{310} = 1$;
$u_{34} = u_{45} = u_{27} = u_{67} = u_{48} = u_{78} = u_{19} = u_{89} = u_{910} = 2/3$;
$u_{23} = u_{15} = u_{610} = 1/3$; and other $u_{ij}$'s are zeroes. The values of $u_{ij}$ are shown along the edges in Figure 3.2.1.

As $u_{ij}$ add up to 10 and the distance associated with the edges in the Petersen's graph is -1, we have -10 as the objective function value corresponding to this solution. It is not possible to write this solution as a convex combination of $U$ vectors corresponding to tour solutions, which have objective function value at least -9.

**Theorem 3.2.3** *$\mathcal{U}(n) \subseteq SEP(n)$ where $SEP(n)$ is defined as in section - 2.*

**Proof :** The proof is by induction on $n$. Consider the constraints that define Problem-2, other than the non-negativity restrictions. We introduce the following notation to facilitate the induction proof.

Let $u_{ij}^n$ be the slack variables associated with the constraint corresponding to the pair$[i, j]$, when we have $n$ cities in all. Recall that $\mathcal{U}(n)$ is the set of all $U$, such that there exists $X$, such that $(X, U)$ is feasible for Problem-2. We have introduced a superscript for $U$ now. Let $U^n$ be the vector of slack variables $(u_{ij}^n)$.

We have,

$$\sum_{1 \leq i < j \leq k-1} x_{ijk} = 1 \qquad 4 \leq k \leq n \qquad (3.2.15)$$

$$\sum_{k=4}^{n} x_{ijk} + u_{ij}^{n} = 1 \qquad 1 \leq i < j \leq 3 \quad (3.2.16)$$

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^{n} x_{ijk} + u_{ij}^{n} = 0 \quad 4 \leq j \leq n-1; 1 \leq i < j \quad (3.2.17)$$

$$-\sum_{r=1}^{i-1} x_{rin} - \sum_{s=i-1}^{n-1} x_{isn} + u_{i,n}^{n} = 0 \qquad i = 1, \ldots, n-1 \quad (3.2.18)$$

Now consider the problem with the number of cities equal to $n - 1$, with the first $n - 1$ cities. We have the corresponding equality constraints, after introducing $u_{ij}^{n-1}$, the slack variables,

$$\sum_{1 \leq i < j \leq k-1} x_{ijk} = 1 \qquad 4 \leq k \leq n-1 \quad (3.2.19)$$

$$\sum_{k=4}^{n-1} x_{ijk} + u_{ij}^{n-1} = 1 \qquad 1 \leq i < j \leq 3 \quad (3.2.20)$$

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i-1}^{j-1} x_{isj} + \sum_{k=j+1}^{n-1} x_{ijk} + u_{ij}^{n-1} = 0 \quad 4 \leq j \leq n-2; 1 \leq i < j \quad (3.2.21)$$

$$-\sum_{r=1}^{i-1} x_{rin} - \sum_{s=i+1}^{n-2} x_{isn} + u_{i,n-1}^{n-1} = 0 \qquad i = 1, \ldots, n-2 \quad (3.2.22)$$

Compare these two sets of constraints, we notice that, given a non-negative solution $(X, U^n)$ for the $n - city$ problem, we have, $(X/\ (n-1),\ U^{n-1})$ given below is a non-negative solution to the problem with first $(n-1)$ cities:

$$X/(n-1) = \qquad (x_{123}, \ldots, x_{n-3,n-2,n-1}) \quad (3.2.23)$$

$$u_{ij}^{n-1} = \ u_{ij}^{n} + x_{ijn} \ \forall\ 1 \leq i < j \leq n-2 \quad (3.2.24)$$

**Basis for Induction** We first prove that the result is true for $n = 4$. i.e., $\mathcal{U}(4) \subseteq SEP(4)$. We have $u_{ij}^{4}$ as the slack variables. From equations (3.2.15)

(3.2.18) we have the following

$$u_{ij}^4 = \quad 1 - x_{ij4}; 1 \leq i < j \leq 3 \tag{3.2.25}$$

$$u_{i4}^4 = \sum_{r=1}^{i-1} x_{ri4} + \sum_{s=i+1}^{3} x_{is4}; 1 \leq i \leq 3 \tag{3.2.26}$$

Notice that all $u_{ij}^4$ are non-negative. Now we show that the degree constraints are satisfied for all $i$.

$i = 1$:

$$u_{12}^4 + u_{13}^4 + u_{14}^4 = 1 - x_{124} + 1 - x_{134} + x_{124} + x_{134} = 2 \tag{3.2.27}$$

Similarly checked for $i = 2$ and 3.

$i = 4$ :

$$u_{14}^4 + u_{24}^4 + u_{34}^4 = x_{124} + x_{134} + x_{124} + x_{234} + x_{134} + x_{234} = 2 \tag{3.2.28}$$

We have the following remark:

**Remark 3.2.4** *If degree constraints are satisfied for all $v \in V_n$, then the subtour elimination constraint (2.1.10) is superfluous for all $S$ with $|S| \geq \lfloor V_n/2 \rfloor + 1$. When each node is of degree 2 and $x(E(S)) \geq |S|$, then every $v \in S$ is in a subtour and there can be no edges between $S$ and $V_n \setminus S$. Hence $x(E(S)) = |S|$ and $x(E(V_n \setminus S)) = |V_n \setminus S|$. Thus it suffices to check (2.1.10) for either $S$ or $V_n \setminus S$. subtour elimination constraints for $S$*

From Remark (3.2.4) we see, it suffices to verify the subtour elimination constraints, in cut form, for $|S| = 2, S \subseteq V_4$.

Let $i_1, i_2, i_3$ be a permutation of $(1, 2, 3)$.

$$u^4(\delta(S)) = \quad 2 + 2x_{i_1 i_2 4} \text{ for } S = \{i_1, i_2\} \text{ or } S = \{i_3, 4\}. \tag{3.2.29}$$

Thus we have, $u^4(\delta(S)) \geq 2$ as $x_{i_1 i_2 4} \geq 0$. Hence for $n = 4$ we have the result.

Let us assume $\mathcal{U}(n-1) \subseteq SEP(n-1)$ . We shall show that $\mathcal{U}(n) \subseteq SEP(n)$. Since we are going to deal with value of cuts corresponding to subsets of $V_n$, hereon we assume symmetry of the notation of subscripts denoting the edges, i.e, $u_{ij} = u_{ji}$. We show that constraints

$$x(\delta(S)) \geq 2 \ \forall \ S \subseteq V_n$$

hold for the required nonempty proper subsets $S$ of $V_n$. i.e. Note $\mid S \mid$ is between 2 and $\lfloor |V_n|/2 \rfloor + 1$. Let

$$\delta^n = \sum_{r \in S, s \in \bar{S}} u^n_{rs} = u^n(\delta(S)) \tag{3.2.30}$$

be the value of the cut corresponding to a subset $S$, given $(X, U^n)$ feasible for the $n-city$ problem, with $U^n \in \mathcal{U}(n)$. We need to show that

$$\delta^n \geq 2 \tag{3.2.31}$$

Without loss of generality let $n \in \bar{S}$, where $\bar{S}$ is the complement of set $S$. Define

$$P = \{[i,j] | x_{ijn} > 0\} \tag{3.2.32}$$

Let $S = \{i_1, i_2, \ldots, i_m\}$ *and* $\bar{S} = \{j_1, j_2, \ldots, j_l, n\}$

Now consider $U^{n-1}$ derived from $U^n$ and $X$ using equation (3.2.23) and (3.2.24). We have by the feasibility of $U^{n-1}, U^{n-1} \in \mathcal{U}(n-1)$. And by induction hypothesis $U^{n-1} \in SEP(n-1)$. Therefore we have,

$$\delta^{n-1} = \sum_{r=1}^{m} \sum_{s=1}^{l} u^{n-1}_{i_r j_s} \geq 2 \tag{3.2.33}$$

We need to show that

$$\delta^n = \sum_{r=1}^{m} \sum_{s=1}^{l} u^n_{i_r j_s} + \sum_{r=1}^{m} u^n_{i_r n} \geq 2 \tag{3.2.34}$$

Take any point $i_r \in S$. We have

$$\sum_{s=1}^{l} u^n_{i_r j_s} = \sum_{[i_r, j_s] \in P} u^n_{i_r j_s} + \sum_{[i_r, j_s] \notin P} u^n_{i_r j_s} \tag{3.2.35}$$

If $[i_r, j_s] \in P$ then $x_{i_r j_s n} > 0$ and

$$u^n_{i_r j_s} = u^{n-1}_{i_r j_s} - x_{i_r j_s n} \qquad (3.2.36)$$

$$otherwise \quad u^n_{i_r j_s} = u^{n-1}_{i_r j_s} \qquad (3.2.37)$$

Hence

$$\sum_{s=1}^{l} u^n_{i_r j_s} = \sum_{[i_r, j_s] \in P} u^{n-1}_{i_r j_s} - \sum_{[i_r, j_s] \in P} x_{i_r j_s n} + \sum_{[i_r, j_s] \notin P} u^{n-1}_{i_r j_s} \qquad (3.2.38)$$

$$u^n_{i_r n} = \sum_{[i_r, i_q] \in P} x_{i_r i_q n} + \sum_{[i_r, j_s] \in P} x_{i_r j_s n} \qquad (3.2.39)$$

$$\sum_{s=1}^{l} u^n_{i_r j_s} + u_{i_r n} = \sum_{[i_r, j_s] \in P} u^{n-1}_{i_r j_s} - \sum_{[i_r, j_s] \in P} x_{i_r j_s n} + \sum_{[i_r, j_s] \notin P} u^{n-1}_{i_r j_s} + \sum_{[i_r, i_q] \in P} x_{i_r i_q n} + \sum_{[i_r, j_s] \in P} x_{i_r j_s n} \qquad (3.2.40)$$

Therefore,

$$\delta^n = \sum_{r=1}^{m} \left[ \sum_{s=1}^{l} u^{n-1}_{i_r j_s} + + \sum_{[i_r, i_q] \in P} x_{i_r i_q n} \right] \qquad (3.2.41)$$

$$= \delta^{n-1} + \sum_{r=1}^{m} \sum_{[i_r, i_q] \in P} x_{i_r i_q n} \geq 2 \qquad (3.2.42)$$

Hence $\delta^n \geq 2 \;\; \forall \; n$.

We can check that the degree constraints are satisfied, as follows:
If $S$ is a singleton set, say $S = \{i\}, i \neq n$, then $u^n(\delta(S)) = u^n(\delta(i))$is still greater than equal to 2 , as the preceding arguments go through for $m = 1$, the cardinality of $S$. However, notice that for no $i$ the strict inequality can hold, as it will contradict the fact $\sum_{1 \leq i < j \leq n} u^n_{ij} = n$.

Hence the theorem. □

## 3.3 Characterisation of integer optimal solutions

Let $m = n_{C_2}$ *and* $l = (n-1)!/2$. Let $\lambda$ be a $l \times 1$ vector. $e_l$ be a $l \times 1$ vector of ones. Let $(X^*, U^*)$ be an optimal solution to PROBLEM -3. It has been shown that if PROBLEM-3 has an integer optimal solution then we have found an optimal tour of the $STSP$ with distances $c_{ij}$. The following result characterises integer optimal solutions to PROBLEM-3.

**Theorem 3.3.1** PROBLEM - 3 *has an integer optimal solution iff the problem* $P(U^*)$ *has a feasible solution where*

$$\begin{aligned} &P(U^*): \\ &T\lambda = \; U^* \\ &e_l'\lambda = \; 1 \\ &\lambda \geq 0. \end{aligned} \tag{3.3.1}$$

*where* $(X^*, U^*)$ *is an optimal solution to* PROBLEM -3,*and* $T$ *is the edge-tour incidence matrix.*

**Proof** :(If part) Suppose $\lambda$ is a feasible solution to $P(U^*)$, then since $U^*$ is optimal for PROBLEM -3 $c'U^*$ is a minimum.
Case (i): $\lambda$ is an integer vector, $T\lambda = U^* \Longrightarrow U^*$ is the edge-tour incidence vector corresponding to some tour $t^*$ or $U^*$ is an integer vector.So $U^*$ is optimal for PROBLEM-3.Also no other feasible solution to $P(U^*)$ exists.
Case(ii): $\lambda$ is a non-integer vector. Let $k$ be the number of nonzero $\lambda_j$. We have $k \geq 2$. Let $J = \{j|\lambda_j > 0\}$ and $T_J$ be the submatrix corresponding to the column indices in $J$ and $\lambda_J$ is similarly defined. We have, $T_J\lambda_J = U^*$. Therefore,

$$c'U^* = c'T_J\lambda_J = \sum_{j \in J}(c't_j) \tag{3.3.2}$$

where $t_j$ is the $j^{th}$ column of $T$.
CLAIM : $c't_j = c'U^*$ for all $j \in J$.
Proof of the claim: $c'U^* \leq c't_j$ for all $j$ as $U^*$ is optimal for PROBLEM-3.If

for some $j \in J, c'U^* < c't_j$ then from (3.3.2) there should exist a $j \in J$ such that $c'U^* > c't_j$ which contradicts the optimality of $U^*$. Hence the claim.

Therefore for every $j \in J$, the corresponding tour $t_j$ is optimal for PROBLEM - 3,with objective function value $c'U^*$.

[only if part] Suppose PROBLEM-3 has no integer optimal solution then we shall show that $P(U^*)$ has no feasible solution.

For every tour $t$ with integer solution$(X, U)$ for PROBLEM-3, $c'U > c'U^*$. Now consider the system with

$$F = \{(w, w_o) | T'w + ew_o \geq 0, U^{*'}w + w_o < 0\} \tag{3.3.3}$$

where $T$ is the edge-tour incidence matrix as defined earlier.

CLAIM: $F \neq \phi$ In fact $w = c$ *and* $w_o = -min_{u \in \mathcal{U}_I} c'U = -c'\hat{U}$, where $\mathcal{U}_I = \{u | (X, U)$ is an integer solution to PROBLEM-3$\}$.

We require $T'w + ew_o \geq 0$ and $U^*w + w_o < 0$. Notice that columns of $T$ are $U$ vectors from $\mathcal{U}_I$, as they are edge-tour incidence vectors. Therefore we require

$$\begin{aligned} (a) \quad c'U + w_o &\geq 0 \;\forall\; U \in \mathcal{U}_I \\ c'U &\geq w_o = c'\hat{U} \end{aligned} \tag{3.3.4}$$

By definition of $w_o$ the inequalities are met. We also require

$$\begin{aligned} (b) \quad U^{*'}c + w_o &< 0 \\ i.e\; U^{*'}c < -w_o &= min_{u \in \mathcal{U}_I} c'U = c'\hat{U} \end{aligned} \tag{3.3.5}$$

This is from our supposition of non existence of integer optimal solutions to PROBLEM-3.Therefore $F \neq \phi$. Hence the claim.

$F \neq \phi$ implies the alternate system

$$\begin{aligned} T\lambda &= U^* \\ e'\lambda &= 1 \\ \lambda &\geq 0 \end{aligned}$$

has no solution from Farkas' lemma. The result follows. □

**Remark 3.3.1** *The above theorem simply states : Let $(X^*, U^*)$ be an optimal solution for the PROBLEM-3. $P(U^*)$ is feasible implies that $U^*$ is a tour or can be written as a convex combination of tours, which in turn implies $U^* \in STSP(n)$. Hence an integer optimal solution exists. Conversely, if $(X^*, U^*)$ is integer than $P(U^*)$ has a solution which is equal to $\lambda_j = 1$ for $T_j = U^*$.*

As defined in section-3.1 let $\mathcal{T}_n$ denote the set of all $n$-tours. Let $\mathcal{T}_{ijn}$ denote the set of all $n$-tours in which edge $[i, j]$ occurs. Let $\mathcal{F}(U^*)$ denote the set of $n$-tours that are covered by $U^*$,i.e.

$$\mathcal{F}(U^*) = \{t | \forall \; edge \; [i,j] \; in \; t, \; U^*_{ij} > 0\} \tag{3.3.6}$$

$$= \mathcal{T}_n - \cup_{[i,j] \ni U^*_{ij}=0} \mathcal{T}_{ijn} \tag{3.3.7}$$

We have the following theorem

**Theorem 3.3.2** *Let $E(U^*)$ be the set of edges with $U^*_{ij} > 0$. Then*
*(a)$G(U^*) = (N, E(U^*))$ is non-Hamiltonian.*
*(b)$\mathcal{F}(U^*)$ is empty.*
*(c)The problem $P(U^*)$-ext:*

$$\begin{aligned} min \quad & e'X_a + X_{a_o} \\ sub \quad & T\lambda + IX_a = U^* \\ & e'\lambda + X_{a_o} = 1 \\ & \lambda \geq 0, X_a \geq 0, X_{a_o} \geq 0 \end{aligned} \tag{3.3.8}$$

*has a unique solution $(\lambda^*, X^*_a, X^*_{a_o}), \lambda^* = 0, X^*_a = U^*$ and $X^*_{a_o} = 1$ with the objective function value $(n+1)$.*

Proof: (a) $\Longrightarrow$ (b)
$G(U^*)$ is non Hamiltonian $\Longrightarrow$ every tour in $\mathcal{T}_n$ has at least one edge $[i, j]$ with $U^*_{ij} = 0$. Therefore

$$\mathcal{F}(U^*) = \mathcal{T}_n - \bigcup_{[i,j] \ni U^*_{ij}=0} \mathcal{T}_{ijn} = \phi. \tag{3.3.9}$$

as required.

(b) $\Longrightarrow$ (c)

$\mathcal{F}(U^*) = \phi$. We shall show that $P(U^*)$-ext has the property that every feasible solution $(\lambda, X_a, X_{a_o})$, has $\lambda = 0$.

Suppose not. Let there exist a $(\lambda, X_a, X_{a_o})$ feasible for $P(U^*)$-ext with $\lambda \neq 0$. Say $\lambda_l > 0$. Let $t_l$ be the corresponding tour with $T_l$ as edge-tour incidence vector. Notice that at least for one edge $[i,j]$ in $T_l, U^*_{ij} = 0$. Otherwise this would contradict nullity of $\mathcal{F}(U^*)$. Therefore there exists an edge $[i,j]$ in $T_l \ni U^*_{ij} = 0$.

Now consider row $r$ corresponding to $[i,j]$.

$$(T\lambda)_r + X_{a_r} = U^*_{ij} = 0 \tag{3.3.10}$$

But $(T\lambda)_r > 0$ as $T$ is a 0 or 1 matrix, $\lambda \geq 0$ *and* $\lambda_l > 0$ with $T_{rl} = 1$.

Therefore $LHS > 0$ but the $RHS = 0$. Thus we get a contradiction. Hence our supposition is false. (c) $\Longrightarrow$ (a)

We have $X_a = U^*, \lambda = 0$ *and* $X_{a_o} = 1$. Suppose there is a Hamiltonian circuit in $G(U^*)$ then let that tour be $t_l$ with $T_l$ as the edge-tour incidence vector. Let $\lambda_l$ be the corresponding variable in $P(U^*)$ -ext.

Now consider $\lambda$ given by

$$\lambda_j = \begin{cases} 0 & \text{if } j \neq 0 \\ \theta & \text{otherwise} \end{cases}$$

where $\theta = min_{[i,j] \ in \ t_l}\{U^*_{ij}\} > 0$ and $X_a = U^* - \theta T_l$ & $X_{a_o} = 1 - \theta$. Now $(\lambda, X_a, X_{a_o})$ as given above is feasible for $P(U^*)$-ext.

Now the objective function corresponding to feasible solution is

$$\begin{aligned} e'X_a + X_{a_o} &= e'U^* - \theta e'T_l + 1 - \theta \\ &= n - n\theta + 1 - \theta \\ &= (n+1)(1-\theta) \\ &< n + 1 \ \ as \ \ \theta > 0 \end{aligned} \tag{3.3.11}$$

contradicting the fact that the optimal objective function value is $(n+1)$ for $P(U^*)$-ext. Hence (c) $\Longrightarrow$ (a). □

**Remark 3.3.2** *Theorem 3.3.2 gives equivalent conditions for the graph $G(U^*)$ to be non-Hamiltonian. We use this theorem to find a necessary and sufficient condition for any feasible solution to PROBLEM-3 to lie in $STSP(n)$. This condition is given in chapter-6.*

## 3.4 $MI$ formulation for $ATSP$

In this section we give the $MI$ formulation for the asymmetric traveling salesman problem, $ATSP$, and give some properties. The study on symmetric traveling salesman polytopes can be extended to the asymmetric case.

The asymmetric traveling salesman problem is to find an optimal $n$-tour, given $c_{ij}, 1 \leq i \neq j \leq n$, with $c_{ij} \neq c_{ji}$.Here we have a $(n-2)$ stage decision problem, in which starting with a 2-tour $(1, 2, 1)$, we decide inserting city $k$ at stage $(k-2), 3 \leq k \leq n$. Analogous to the symmetric traveling salesman problem we have $A_k$ which depends on the decisions made in preceding stage $(k-2)$, $3 \leq k \leq n$.

$$x_{ijk} = \begin{cases} 1 & \text{if in stage } (k-2) \text{ the decision is to insert } k \text{ between } i \text{ and } j \\ & \quad 1 \leq i \neq j \leq k-1 \\ 0 & \text{otherwise} \end{cases}$$

We can define $X$ similarly as before:

**Definition 3.4.1** *Given $X = (x_{123}, \ldots, x_{n-2,n-1,n}) \in R^t$ a 0 or 1 vector, where $t = \sum_{k=3}^{n}(k-1)P_2$. We say $X$ is a* feasible decision vector *in case $x_{i_k j_k k} = 1$ then $[i_k, j_k] \in A_k$, $3 \leq k \leq n$.*

The 0-1 integer programming formulation is given below.

PROBLEM $ASMI$:

$$minimise \sum_{k=3}^{n} \sum_{1 \leq i \neq j \leq k-1} C_{ijk} x_{ijk}$$

subject to

$$\sum_{1\le i\ne j\le k-1} x_{ijk} = 1 \qquad 3 \le k \le n \tag{3.4.1}$$

$$\sum_{k=3}^{n} x_{ijk} \le 1 \qquad 1 \le i \ne j \le 2 \tag{3.4.2}$$

$$-\sum_{r=1,r\ne i}^{j-1} x_{irj} + \sum_{k=j-1}^{n} x_{ijk} \ \le\ 0 \tag{3.4.3}$$

$$-\sum_{s=1,s\ne i}^{j-1} x_{sij} + \sum_{k=j+1}^{n} x_{jik} \ \le\ 0 \tag{3.4.4}$$

$$3 \le j \le n-1; 1 \le i < j$$

$$-\sum_{r=1}^{n-1} x_{irn} \ \le\ 0 \tag{3.4.5}$$

$$-\sum_{s=1}^{n-1} x_{sin} \ \le\ 0 \tag{3.4.6}$$

$$i = 1, \ldots, n-1$$

$$x_{ijk} = 0 \text{ or } 1 \qquad 1 \le i < j \le k; 3 \le k \le n \tag{3.4.7}$$

In matrix form we have

$$\min C'X$$

subject to

$$\begin{bmatrix} E^{AS} & 0 \\ A^{AS} & I \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} e_{n-2} \\ e_2 \\ 0 \end{bmatrix}$$

$$X, U \ge 0. \tag{3.4.8}$$

where $E^{AS}$ is the matrix corresponding to equality constraints and $A^{AS}$ corresponds to inequality constraints. We have the following theorem:

**Theorem 3.4.1** *There is a 1-1 correspondence between n-tours and the integer feasible solutions to ASM1.*

Let

$$\zeta^{AS}(n) = \{X | E^{AS}X = e_{n-2},\ X \geq 0\} \tag{3.4.9}$$

$$\mathcal{U}^{AS}(n) = \{U | U = \begin{bmatrix} e_2 \\ 0 \end{bmatrix} - A^{AS}X \geq 0, X \in \zeta^{AS}(n)\} \tag{3.4.10}$$

**Remark 3.4.1** *Most of the results true for the $STSP$ can be verified for the $ATSP$.*

As a computational exercise we computed extreme points of small $ASMI$ and $DFJ$ polytopes. The results are presented in Appendix-II.

## 3.5 Conclusions

In this chapter we have studied properties of $MI$ and have shown that $\mathcal{U}(n) \subsetneq SEP(n)$ without explicitly giving a linear description of $\mathcal{U}(n)$. We have briefly described the $MI$ formulation for the $ATSP$. We believe the asymmetric formulation would also give interesting results.

A natural consequence of this work would be to derive the linear description of $\mathcal{U}(n)$ and study how it compares with the $SEP(n)$. In the next chapter we work out the linear description of $\mathcal{U}(n)$ and prove it to be equivalent to $SEP(n)$.

# Chapter 4

# Equivalence of $SEP$ and $MI$

Padberg and Sung(1991) give an analytical comparison of different formulations of the Asymmetric Traveling Salesman Problem. They obtain linear descriptions of projected sets and compare with that of the $DFJ$ formulation. In this chapter we apply the techniques described by them to obtain a linear description of $\mathcal{U}(n)$ and show the linear description obtained is equivalent to the $SEP$.

In section-4.1 we state the results we apply from Padberg and Sung(1991) to work out the linear description of $\mathcal{U}(n)$. In section 4.2 we obtain the linear description of $\mathcal{U}(n)$.

## 4.1 Affine Transformations of Polyhedra

In this section we give results from Padberg and Sung(1991). There are different ways of formulating a combinatorial optimisation problem. Let $A$ and $B$ be two formulations of a given problem that are stated in the same space of variables and let $P_A$ and $P_B$ be the respective polyhedra. we say formulation $A$ is '*better*' than formulation $B$ if $P_A \subset P_B$. Different formulations of a given problem can be stated in terms of different sets of variables. Suppose, $G$ is a formulation that models the same problem as $A$

, but, in a higher dimensional space. Let $Q_G$ be the polyhedra defined by $G$. Suppose we have an affine transformation $T$ which maps integer(mixed-integer) points of $Q_G$ onto integer(mixed-integer) points of $P_A$. Formulation $G$ is *better* than formulation $A$ if $T(Q_G) \subset P_A$, where $T(Q_G)$ is the image of $Z_G$ under the transformation $T$.

### 4.1.1 Properties of polyhedral cones

A set $C \subseteq R^n$ is called a *cone* if $v^1, v^2 \in C$ implies $\lambda_1 v^1 + \lambda_2 v^2 \in C$ for all scalars $\lambda_1, \lambda_2 > 0$. A halfline(or ray) $(v)$ is the set of points $\{\lambda v \in R^n | \forall \lambda \geq 0\}$. A halfline $(v)$ is called an extreme ray of $C$ if for any $v^1, v^2 \in C, v = \lambda v^1 + (1-\lambda)v^2$ with $0 < \lambda < 1$ implies that $v^1, v^2$ are positive multiples of $v$. A set of generators of $C$ is a set of halflines that spans $C$. We can write $C$ as

$$C = \{x \in R^n | Ax \geq 0\}$$

Define the lineality space $L$ of $C$ to be the set of all vectors $x$ such that $x \in C$ and $-x \in C$ and is given by

$$L = \{x \in R^n | Ax = 0\}$$

Define

$$\begin{aligned} L^{\perp} &= \{y \in R^n | xy = 0, \ \forall \, x \in L\} \\ C^0 &= C \bigcap L^{\perp} \end{aligned}$$

The following Lemma and theorem state a criterion for $x \in C$ to define an extreme ray of $C$

**Lemma 4.1.1** *Let $C = \{x \in R^n | Ax \geq 0\}$ be such that rank$(A) = n$ and let $M$ denote the index set of all rows of $A$. $(x)$ is an extreme ray of $C$ if and only if $x \in C, x \neq 0$ and there exists $I \subseteq M$ such that (i)$| I | = n-1$, (ii) $a^i x = 0$ for all $i \in I$ and (iii) the rows $a^i$ with $i \in I$ are linearly independent.*

**Theorem 4.1.1** *Let $C = \{x \in R^n | Ax \geq 0\}, F = \{x \in R^n | Ax = 0\}$ and $\dim F = d$. A halfline $(x)$ is an extreme ray of $C$ if and only if (i) $x \in C^0$*

*and (ii) there exist exactly $n-d-1$ linearly independent rows $a^i$ of $A$ such that $a^i x = 0$.*

A solution of *minimal support* is a nonzero solution with least number of nonzero components to the solution satisfying the requirement (ii) of theorem 4.1.1. We have the following remark:

**Remark 4.1.1** *A full generator system for the conical part of a cone $C$ with $\dim F = d$ can be obtained by determining for each system of $n-d-1$ linearly independent rows of $A$ a nonzero solution of minimal support to $Ax \geq 0$ that is not in $F$ provided it exists.*

To find a full generator system for $C$ we can work with smaller dimensional cones. We have the following proposition which gives the 'intersection property' of cones

**Proposition 4.1.1** *Let*

$$C_1 = \{x^1 \in R^p | A_1 x^1 \geq 0\}, \ C_2 = \{x^2 \in R^q | A_2 x^2 \geq 0\}$$

*and $C_3$ be the intersection cone of cones $C_1$ and $C_2$ embedded naturally in $R^{p+q}$, i.e.,*

$$C_3 = \{(x^1, x^2) \in R^{p+q} | A_1 x^1 \geq 0, \ A_2 x^2 \geq 0\}$$

*Let $R_i$ denote the generators of $C_i; i = 1, 2, 3$. Then*

$$R_3 = \{(u, 0) \in R^{p+q} | u \in R_1\} \bigcup \{(0, v) \in R^{p+q} | v \in R_2\}$$

### 4.1.2 Affine transformations of polyhedra

We consider affine transformations of full rank that map $R^n$ into $R^m$ where $m \leq n$. Given the linear description of a polyhedron $Z \subseteq R^n$ we are interested in finding a linear description of the image of $Z$ under a given affine transformation. Restrict the 'feasible' points in the image of $Z$ to be in some set $Q \subseteq R^m$. Let

$$x = f + Lz \tag{4.1.1}$$

be the affine transformation from $R^n$ into $R^m$, i.e., $f \in R^m$ and $L$ is an $m \times n$ matrix having full row rank. Partition $L$ into $L_1$ and $L_2$ such that $L_1$ is an $m \times m$ matrix of rank $m$ that corresponds to the first $m$ columns of $L$. Let

$$Z = \{z \in R^n | Az = b, Dz \leq d\} \tag{4.1.2}$$

$$W = \{x \in Q | \exists z \in Z \text{ such that } x = f + Lz\} \tag{4.1.3}$$

where $A$ is a $p \times n$ matrix, $D$ is a $q \times n$ matrix, $f \in R^m, Q \subseteq R^m$ is an arbitrary set, $L = (L_1, L_2)$ is an $m \times n$ matrix having full row rank and $A$ and $D$ are partitioned as $A = (A_1, A_2)$ and $D = (D_1, D_2)$ according to the partition of $L$. Define

$$C = \{(u, v) \in R^{p+q} | u(A_2 - A_1 L_1^{-1} L_2) + v(D_2 - D_1 L_1^{-1} L_2) = 0, v \geq 0\} \tag{4.1.4}$$

$$W_c = \{x \in Q | (uA_1 + vD_1) L_1^{-1} x \leq ub + vd + (uA_1 + vD_1) L_1^{-1} f \quad \forall\ (u, v) \in C\} \tag{4.1.5}$$

**Theorem 4.1.2** $W = W_c$

**Corollary 4.1.1** *Let*
$Z = \{z \in R^n | Az = b, Dz \leq d, z \geq 0\}$
where $A$ is a $p \times n$ matrix and $D$ is a $q \times n$ matrix and let $W$ be defined as above, then $W = W_c$ where $W_c$ and $C$ are as given below:

$$W_c = \{x \in Q | (uA_1 + vD_1 - w) L_1^{-1} x \leq ub + vd + (uA_1 + vD_1 - w) L_1^{-1} f \quad \forall\ (u, v, w) \in C\} \tag{4.1.6}$$

$$C = \{(u, v, w) \in R^{p+q+m} | u(A_2 - A_1 L_1^{-1} L_2) + v(D_2 - D_1 L_1^{-1} L_2) + w L_1^{-1} L_2 \geq 0, v \geq 0 w \geq 0\} \tag{4.1.7}$$

We can replace the requirement "for all $(u, v) \in C$" by the requirement "for all $(u, v)$ in a generator system of $C$". This way we get a finite set of inequalities for $W$.

Padberg and Sung compare three compact formulations, the $MTZ$, $FGG$ and $Claus$ formulations of the asymmetric traveling salesman problem with the $DFJ$ formulation. Let $P_M$, $P_F$ and $P_C$ denote the projections of the $MTZ$, $FGG$ and $Claus$ formulations respectively. We summarise results below:

**Result 4.1.1** *(i) $P_{S,t}$ is a proper subset of $P_M$, where $P_{S,t}$ is the subtour polytope redefined to satisfy a modified subtour constraint.*
*(ii) The subtour polytope is a proper subset of $P_F$*
*(iii) $P_C$ is equivalent to the subtour polytope.*

## 4.2 Linear Description of $\mathcal{U}(n)$

In this section we obtain the linear description of $\mathcal{U}(n)$ by applying the results discussed in the previous sections. Since the system of equations which defines $\mathcal{U}(n)$ is not a transformation of full rank, we define a polytope $\mathcal{U}'(n)$ in $R^{m'}, m' = \frac{(n-1)(n-2)}{2}$, work out its linear description and then obtain the linear description of $\mathcal{U}(n)$ with the help of some more inequalities. Define

$$\zeta'(n) = \{X \in R^{\tau_n} | EX = [e_{n-3}], \bar{A}X \leq \begin{bmatrix} e_3 \\ 0 \\ 0 \end{bmatrix}, X \geq 0\} \tag{4.2.8}$$

$$\mathcal{U}'(n) = \{U | U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - \bar{A}X \geq 0, X \in \zeta'(n)\} \tag{4.2.9}$$

where $E$ and $\bar{A}$ are the matrices of coefficients corresponding to equality constraints (3.2.2) and inequality constraints (3.2.3)-(3.2.4) respectively. Also every integer point of $\zeta'(n)$ corresponds to a tour of the $STSP(n)$. We say $\mathcal{U}'(n)$ is the linear transformation of $\zeta'(n)$.

The affine transformation which maps points in $R^{\tau_n}$ to $R^{m'}$ is given by

$$U = f + LX \tag{4.2.10}$$

where

$$f = \begin{bmatrix} e_3 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

and $L = -\bar{A}$.

Partition $L$ into $\begin{bmatrix} L_1 & L_2 \end{bmatrix}$ such that $L_1$ is a $m' \times m'$ matrix of full rank $m'$ corresponding to the first $m'$ columns of $L$. We can write $\mathcal{U}'(n)$ as

$$\mathcal{U}'(n) = \{U \in R^{m'} | \exists\ X \in \zeta'(n),\ such\ that\ U = f + LX\} \quad (4.2.11)$$

and $\bar{A}$ and $E$ as $\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \end{bmatrix}$ and $E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$ according to the partitioning of $L$.

Let $p = (n-3)$. The cone in consideration is given by

$$\begin{aligned} C = \{(v, w, z) \in R^{p+m'+m'} \quad | \quad & v(E_2 - E_1 L_1^{-1} L_2) \\ & +w(\bar{A}_2 - \bar{A}_1 L_1^{-1} L_2) + z L_1^{-1} L_2 \geq 0; w, z \geq 0\} \end{aligned} \quad (4.2.12)$$

and the projected polytope is

$$\begin{aligned} W_c = \{U \in \mathcal{U}'(n) \quad | \quad & (vE_1 + w\bar{A}_1 - z)L_1^{-1} U \leq v \begin{bmatrix} e_p \\ \cdot \end{bmatrix} + w \begin{bmatrix} e_3 \\ 0 \end{bmatrix} \\ & +(vE_1 + wA_1 - z)L_1^{-1} f\ \forall\ (v, w, z)\ \ generators\ \ of\ \ C\} \end{aligned} \quad (4.2.13)$$

We have the following theorem which is a direct application of theorem 4.1.2.

**Theorem 4.2.1**

$$\mathcal{U}'(n) = W_c$$

Hence to find the linear description of $\mathcal{U}'(n)$ it suffices to find the description of $W_c$. The matrix $\mathcal{E} = (E_2 - E_1 L_1^{-1} L_2)$ is a $p \times (\tau_n - m')$ matrix of rank

$(p-1)$ with first row having all entries as $+1$. Also $\bar{A}_2 - \bar{A}_1 L_1^{-1} L_2 = 0$. Now we have

$$C = \{(v, w, z) \in R^{(p+m'+m')} | v\mathcal{E} + z\bar{A}_2 \geq 0, w, z \geq 0\} \tag{4.2.14}$$

The vector $(0, \xi^i, 0)$ where $\xi^i$ is a unit vector with 1 in the $i^{th}$ place generates the space $w \geq 0$. By intersection property of the cones we can work with the smaller cone

$$C' = \{(v, z) \in R^{(p+m')} | B \begin{bmatrix} v^t \\ z^t \end{bmatrix} \geq 0\} \tag{4.2.15}$$

where

$$v = (v_1, \ldots, v_p) \; \& \; z = (z_{12}, z_{13}, \ldots, z_{n-2,n-1}) \tag{4.2.16}$$

$$B = \begin{bmatrix} \mathcal{E}^t & \bar{A}_2^t \\ 0 & I \end{bmatrix} \tag{4.2.17}$$

The matrix $B$ is of rank $(p + m' - 1)$.

$W_c$ can be simplified as follows:

$$\begin{array}{ll} L.H.S: & (vE_1 + w\bar{A}_1 - z)L_1^{-1}U \\ RHS: & \sum_{l=2}^{p} v_l + z_{12} + z_{13} + z_{23} - 2v_1 \end{array} \tag{4.2.18}$$

for all generators of $C$.

To get a full generator system of $C$ and the linear description of $W_c$ it suffices to find a generator system for $C'$. We find extreme rays of $C'$ by finding solutions to all homogeneous equation systems corresponding to $(p + m')$ variables and $(p + m' - 2)$ linearly independent rows of $B$. We have given the solutions below: Define

$$v_1 = 0 \tag{4.2.19}$$

$$v_{s+1} = -min_{r_s+1 \leq r \leq r_{s+1}} \mathbf{b}_r z \tag{4.2.20}$$

where

$$\begin{aligned} \mathbf{b}_r &= (b_{r,p+1}, \ldots, b_{r,p+m'}) \\ r_s &= \tau_{n-1} - \tau_{n-s}; 1 \le s \le n-4 \\ r_{n-3} &= \tau_{n-1} \end{aligned} \tag{4.2.21}$$

Substituting the generators thus obtained, we get the linear description of $W_c$ which is not minimal.

Let $S \subseteq V_{n-1}$. Define

$$z_{ij} = 1;\ [i,j] \in E(S) \tag{4.2.22}$$

We can prove

$$\sum_{l=2}^{p} v_l + z_{12} + z_{13} + z_{23} = \mid S \mid -1 \tag{4.2.23}$$

Hence we get

$$\sum_{[i,j]\in S} u_{ij} \le \mid S \mid -1 \tag{4.2.24}$$

which are precisely the subtour elimination constraints.

So far we have only considered the case when $2 \le \mid S \mid \le (n-1), S \subseteq V_{n-1}$. We have not considered sets which include $n$. The subsets which include $n$ also satisfy the subtour elimination constraints. Once we have $u_{ij}$'s determined for $1 \le i < j \le n-1$ we can see that $u_{in} = 2 - \sum_{j=i;1\le j\le n-1} u_{ij}$ from the fact that degree constraints are satisfied.

Let $S = \{i_1, i_2, \ldots, i_k\}$, & $S^c = \{j_1, j_2, \ldots, j_l\}; l \ge 1$ because we need to consider sets of cardinality less than or equal to $(n-1)$, Now let $S' = S \bigcup \{n\}$. We need to prove $u(\delta(S')) \ge 2;\ \forall\ S', 2 \le \mid S' \mid \le n-1$
Now by definition

$$u(\delta(S')) = \sum_{r=1}^{k}\sum_{s=1}^{l} u_{i_r j_s} + \sum_{s=1}^{l} u_{j_s n}$$

$$
\begin{aligned}
&= \sum_{r=1}^{k}\sum_{s=1}^{l} u_{i_r j_s} + \sum_{s=1}^{l}[2 - \sum_{r=1}^{k} u_{i_r j_s} - \sum_{p=1,p\neq s}^{l} u_{j_p j_s}] \\
&= 2l - \sum_{s=1}^{l}\sum_{p=1,p\neq s}^{l} u_{j_p j_s}
\end{aligned}
\tag{4.2.25}
$$

But $\sum_{s=1}^{l}\sum_{p=1,p\neq s}^{l} u_{j_p j_s} = 2.u(E(S^c))$
and $u(E(S^c)) \leq l - 1$, by previous result. Hence

$$
\begin{aligned}
u(\delta(S')) &= 2l - u(E(S^c)) \\
&\geq 2l - 2(l-1) \\
&\geq 2
\end{aligned}
$$

Hence $u(\delta(S')) \geq 2 \ \forall \ S'$. Thus we have shown that subtour elimination constraints are all satisfied for all strict subsets of $V_n$. The generators $(0, \xi^i, 0)$ give rise to non negativity constraints.

Hence $\mathcal{U}(n) \ \equiv \ SEP(n) \ \ \forall \, n.$ □

We have explicitly worked out the generators for $n = 6$. We present these results in Appendix-I.

## 4.3 Conclusions

In the previous chapter we showed $\mathcal{U}(n)$ is at least as tight as $SEP(n)$. In this chapter, we obtained the linear description of $\mathcal{U}(n)$ and showed its equivalence to $SEP(n)$.

We now have a quadratic constraint formulation of the $STSP$ which is equivalent to the exponential constraint formulation $SEP$. The advantage of this formulation is that we can start with this formulation as well and add all the facet defining inequalities to this constraint set. This would result in fewer inequalities to obtain a complete description of $STSP(n)$ for large $n$. We have computed extreme points of small $SEP$ and $MI$ polytopes. We present these results in Chapter-6.

In the next chapter, we compare another compact formulation of the $STSP$, called Cycle-shrink, which uses only polynomial number of constraints. We compare this formulation with the $MI$.

# Chapter 5

# Equivalence of $MI$ and $CS$ Formulations

In this chapter we compare two 'compact' formulations, the Multistage insertion($MI$)and the Cycle-shrink ($CS$), of the $STSP$. Formulations are 'compact' in the sense that the number of constraints and variables is a polynomial function of the number of cities, $n$, in the problem. The Multistage insertion formulation and its properties were discussed in Chapters 3 and 4. Carr(1995,96) proposed a new formulation called the Cycle-shrink($CS$) of the $STSP$ which was polynomially sized. He showed the equivalence of $SEP$ to the Cycle-shrink formulation. In chapter 4 we showed the equivalence of the $MI$ formulation to $SEP$. The equivalence of $MI$ and $CS$ follow from the above statements. In this chapter we directly show the equivalence of $CS$ to $MI$ without using the result that $CS$ is equivalent to $SEP$. In section-5.1 we state the $CS$ formulation, in section-5.2 we show the equivalence of the $CS$ formulation to $MI$.

## 5.1 The Cycle-Shrink Relaxation ($CS$)

Cycle-shrink is a polynomial sized linear programming relaxation developed by Carr(1995,96) that implies the validity of all the subtour elimination constraints. The importance of Cycle-shrink lies in the way it is used to separate naturally defined classes of $TSP$ inequalities.

To formulate cycle-shrink, additional variables are used as follows. We have $K_n = (V_n, E_n)$ as the complete graph on $n$ vertices. First, rename the edge variables in the usual formulation of the $TSP$ as $x_e^0$ for each $e \in E_n$. Arbitrarily label the vertices in $V_n$ with the integers from 1 through $n$. Any such labeling imposes a total ordering on the vertices, which is used in the cycle-shrink formulation. Construct the family of graphs $G_i = (V_i, E_i)$ for $i \in V$ such that $G_i$ is the subgraph of the complete graph $K_n$ which is induced by the set $V_i$ of all those vertices whose labels are greater than $i$. Then on each graph $G_i$, create additional variables $x_e^i$ for each $e \in E_i$.

Consider an incidence vector $x^0$ of a Hamiltonian cycle $H^0(x^0)$ in $K_n$. Let $H^1(x^0)$ be the Hamiltonian cycle on $G_1$ formed by removing vertex 1 from $H^0(x^0)$ and linking the neighbors of 1 in $H^0(x^0)$ with an edge. Let $H^i(x^0)$ be the Hamiltonian cycle on $G_i$ formed by removing vertex $i$ from $H^{i-1}(x^0)$ and linking the neighbors of $i$ in $H^{i-1}(x^0)$ with an edge. The values that we want the additional variables of cycle-shrink to have, given the values of $x^0$, can be determined by considering the family of Hamiltonian cycles $\mathcal{H}^i$.

Then the values that we want the additional variables of cycle-shrink to have in order to represent the Hamiltonian cycle $H^0$ are as follows.

$(x_e^0 \mid e \in E_n)$ is the incidence vector of $H^0$.
$(x_e^i \mid e \in E_i)$ is the incidence vector of $H^i$ for all $i \in \{1, \ldots, n-3\}$.

A complete feasible solution $x$ for cycle-shrink can thus be represented by

$$x := (x^0, x^1, \ldots, x^{n-3}) \tag{5.1.1}$$

where for each $i \in \{0, 1, \ldots, n-3\}$, $x^i$ is a vector having a component for each edge in $G_i$.

Carr(1995,96) defines cycle-shrink relaxation to be the following linear program:

$$\textit{PROBLEM CS}: \quad \textit{minimise} \sum_{e \in E_n} c_e x_e^o \tag{5.1.2}$$

$$\textit{subject to} \quad x_e^0 \geq 0 \qquad e \in E_n \tag{5.1.3}$$

$$\sum_{e\in\delta(\{j\})\bigcap E_i} x_e^i = 2 \quad \forall i \in \{0,\ldots,n-3\}; \forall j \in V_i \tag{5.1.4}$$

$$x_e^i \geq x_e^{i-1} \qquad \forall i \in \{0,\ldots,n-3\}; \forall e \in E_i \tag{5.1.5}$$

Cycle-shrink is a valid relaxation of the $TSP$. Call the part of a feasible cycle-shrink solution given by $x^k$ as the k-th *level* of this feasible solution.

### 5.1.1 Theorems About Cycle-Shrink

**Theorem 5.1.1** *If $x^o$ is feasible for cycle-shrink, then $x^o$ satisfies all the subtour elimination constraints*

**Theorem 5.1.2** *The projection of the cycle-shrink polytope onto the space of original variables is exactly the subtour polytope.*

The Cycle-shrink formulation has $\tau_{n+1} + (n+3)(n-2)/2$ constraints, not counting non negativity restrictions while the number of variables is $\tau_{n+1}$(all non negative).

Carr(1995,96) shows that the Cycle-shrink is a compact description of the $SEP$.

**Remark 5.1.1** *We now have two formulations, the $MI$ and $CS$ of the $STSP$ which have polynomial number of constraints and both the formulations are compact descriptions of the $SEP$. Therefore it would be very interesting to compare these two formulations and check for their equivalence. We do this in the next section.*

## 5.2 Equivalence of Multistage-insertion and Cycle-shrink Formulations

In this section we show the equivalence of the Multistage-insertion and the Cycle-shrink polytopes. We redefine the variables used in the Multistage-insertion formulation to compare with the Cycle-shrink formulation.

**Remark 5.2.1** *In the MI formulation we begin with a 3-tour on* $(1, 2, 3, 1)$ *and build up the n tour by insertion of a node at each stage, in the CS we start with a n tour and then shrink to a 3-tour* $(n-2, n-1, n)$.

Let $u_{ij}^{k-3}$ denote the value of the edge $[i, j]$ at the $k^{th}$ stage $1 \le i < j \le k$; $4 \le k \le n$. We start with the 3-city tour with $u_{12}^0 = u_{13}^0 = u_{23}^0 = 1$. The decision vector is

$$U = (u_{12}^0, u_{13}^0, u_{23}^0, \ldots, u_{12}^{n-3}, \ldots, u_{n-1,n}^{n-3}) \quad (5.2.1)$$

Note that $U$ and $x$ defined by Cycle-shrink have the same dimension. In chapter-2 we have defined $u_{ij}^{n-1} = u_{ij}^n + x_{ijn}$. To be consistent with the notation of Cycle-shrink we redefine the relation as

$$x_{ijk} = u_{ij}^{k-4} - u_{ij}^{k-3}; 4 \le k \le n \quad (5.2.2)$$

From equation (3.2.11) we have, $\sum_{1 \le i < j \le n} c_{ij} u_{ij}^{n-3}$ is equivalent to minimising $\sum_{k=4}^{n} \sum_{1 \le i < j \le k-1} C_{ijk} x_{ijk}$ We can therefore reformulate PROBLEM - $CS$ in terms of $u_{ij}^l; 1 \le l \le n-3$ as:

PROBLEM CSMI:

$$\text{minimise} \sum_{1\leq i<j\leq n} c_{ij} u_{ij}^{n-3}$$

subject to

$$\sum_{1\leq i<j\leq k-1} u_{ij}^{k-3} = k-2; 4 \leq k \leq n \qquad (5.2.3)$$

$$u_{ij}^{n-3} \geq 0; 1 \leq i < j \leq 3 \qquad (5.2.4)$$

$$\sum_{r=1}^{i-1} u_{ri}^{j-3} + \sum_{s=i+1}^{j-1} u_{is}^{j-3} + u_{ij}^{j-3} = 2; 1 \leq i < j; 4 \leq j \leq n \qquad (5.2.5)$$

Now

$$\sum_{r=1}^{i-1} u_{ri}^{j-3} + \sum_{s=i+1}^{j-1} u_{is}^{j-3} + u_{ij}^{j-3} = \sum_{[i,k]\in\delta(i)\bigcap E_j'} u_{ik}^{j-3}$$

where

$$E_j' = \{[i,k] \mid 1 \leq i < k \leq j\}$$

Let $G_i' = (V_i', E_i')$ be defined for another ordering on vertices, as the graph induced by set $V_i'$ of all vertices whose labels are lesser than $i$. On each graph $G_i'$ additional variables are created. $H^0(x^0)$ is the Hamiltonian cycle on $K_n$. Let $H^n(x^0)$ be the Hamiltonian cycle formed by removing vertex $n$ from $H^0(x^0)$ and linking neighbours of $n$ in $H^0(x^0$ with an edge. Iteratively, $H^i(x^0)$ is the Hamiltonian cycle on $G_i'$ formed by removing edge $i$ from $H^{i+1}(x^0)$ and linking neighbours of $i$ in $H^{i+1}(x^0)$ with an edge.

With this ordering we see that (5.2.5) is equivalent to constraint (5.1.3).

The non negativity constraints $x_{ijk} \geq 0$ give rise to

$$u_{ij}^{k-4} \geq u_{ij}^{k-3};\ 1 \leq i < j \leq k; 4 \leq k \leq n \qquad (5.2.6)$$

We can see that constraint (5.2.6) is equivalent to (5.1.4). Hence we have the Cycle-shrink formulation equivalent to the Multistage-insertion formulation. □

## 5.3 Conclusions

We have presented two compact descriptions of the $STSP$ and proved their equivalence. Both of them imply validity of the subtour elimination constraints. Based on $CS$ a polynomial time separation algorithm is described for a class of valid inequalities for $STSP(n)$. Alternatively one may use $MI$ to also separate over classes of inequalities in polynomial time. These algorithms can give useful hints to develop efficient heuristic algorithms to solve the $TSP$. We address these problems for future research.

# Chapter 6

# On Small Polytopes

The study of polytopes in Combinatorial Optimisation is a very challenging problem. Over the years researchers have been trying to find complete descriptions for CO polytopes.

The symmetric traveling salesman polytope, $STSP(n)$, is one of the most interesting and complex combinatorial optimisation polytope. Finding complete description of $STSP(n)$ for moderately sized $n$ itself has been a major challenge. Norman(1955), in an abstract announced that, for $n \leq 7$, $STSP(n)$ is completely determined by degree, non negativity, subtour elimination and comb constraints. However, Boyd and Cunningham(1991) show that this is not true for $n = 7$. They give the example of a point which satisfies comb inequalities but does not lie in $STSP(7)$. They introduce the bipartition inequalities for $STSP(n)$ and give a complete description of $STSP(7)$ using non negativity, degree, subtour elimination, comb and envelope inequalities which are 3437 in number. Christof, Jünger and Reinelt(1991) give a complete description of $STSP(8)$. They characterise $STSP(8)$ in terms of 194187 inequalities belonging to 24 classes, three of which are unknown. Christof and Reinelt(1996) completely describe $STSP(9)$ with 42104442(192 classes) and give a possibly incomplete set of 51043900866 inequalities in 15379 classes for $STSP(10)$. Statistics of facet structure of $STSP(n)$ for $n \leq 10$ are given in Table 6.1.1

| | No. of tours | No. of different facets | No. of facet classes |
|---|---|---|---|
| 3 | 1 | 0 | 0 |
| 4 | 3 | 3 | 1 |
| 5 | 12 | 20 | 2 |
| 6 | 60 | 10 | 4 |
| 7 | 360 | 3437 | 6 |
| 8 | 2520 | 194187 | 24 |
| 9 | 20160 | 42,104,442 | 192 |
| 10 | 181,440 | $\geq$ 51,043,900,866 | 15,379 |

Table 6.1.1 : Facet structure of $STSP(n)$

Euler and Verge(1995) obtain complete descriptions of small assymetric traveling salesman polytopes. They use a refined version of Chernikova's algorithm to determine a complete and irredundant description of small assymmetric polytopes. The motivation for their work is the belief that study of small polytopes would lead to better description of the general asymmetric traveling salesman problem and better cutting plane methods. Boyd and Cunningham(1991) suggest considering explicitly computing the facets of $STSP(n)$. The above mentioned papers motivated us to study small polytopes of $SEP$ and $MI$, by explicitly computing their extreme points and studying the structure of the extreme points.

In this chapter we present results on the structure of extreme points of $SEP(n), n \leq 7$ and $MI(n), n \leq 6$ . We also give a necessary and sufficient condition for an extreme point of the $MI$ polytope to lie within $STSP(n)$ by characterising Hamiltonian cycles. This is done in Section 6.1. In Section 6.2 we give a method to obtain cutting planes to eliminate fractional extreme points. The computational results are presented in Section 6.3.

## 6.1 Characterising Hamiltonian Cycles

In this section we give certain characterisations of Hamiltonian cycles. We use these to give a necessary and sufficient condition for a feasible solution to the $MI$ problem to lie in $STSP(n)$.

Given a Hamiltonian graph, we give a necessary and sufficient condition for a sub-graph of the graph, which satisfies certain conditions to be Hamiltonian in the following theorem.

**Theorem 6.1.1** *Let $G' = (V', E'), V' = \{v_1, \ldots, v_{n'}\}$. Consider the graph $G = (V, E)$ where*

$$V = V' \bigcup \{v_{n'+1}, \ldots, v_n\}$$

$$E = E' \bigcup \{[v_{n'}, v_{n'+1}], \ldots, [v_{n-1}, v_n]\} \bigcup_{[v, v_{n'}] \in E'} \{[v_n, v]\}$$

*Then $G$ is Hamiltonian iff $G'$ is Hamiltonian.*

Proof: Suppose $G$ is Hamiltonian. Then we show that $G'$ is also Hamiltonian. Let $H$ in $G$ be a Hamiltonian cycle. $H$ should cover all nodes in $V'$ and $\{v_{n'+1}, \ldots, v_n\}$. The only way $H$ can cover all the nodes in $V - V'$ is by the path $(v_{n'+1}, \ldots, v_n)$. Also $[v_{n'}, v_{n'+1}]$ is the only edge connecting $v_{n'+1}$ to $V'$.

Construct a path $(w_1, \ldots, w_{n'})$ of nodes in $V'$ as part of $H$ such that $w_1 = v_{n'}$ & $w_{n'} = v$ for such $v$ such that $[v_n, v] \in E$. The path $(w_1, \ldots, w_{n'})$ with $[v, v_{n'}]$ is a Hamiltonian cycle in $G'$. Hence $G'$ is Hamiltonian.

Now suppose $G'$ is Hamiltonian $\Longrightarrow$ there exists a Hamiltonian cycle $H' = (w_1, \ldots, w_{n'}, w_1)$ of nodes in $V'$. Let $w_i = v_{n'}$. $[w_{i-1}, w_i]$ & $[w_i, w_{i+1}]$ are edges in $H'$. Include the path $(v_{n'+1}, \ldots, v_n)$ along with the edges $[v_{n'}, v_{n'+1}]$ & $[v_n, w_{i+1}]$ to $H'$. This gives a Hamiltonian cycle in $G$. Hence $G$ is Hamiltonian. □

Let $\mathcal{H}_n$ be the set of all Hamiltonian cycles of $K_n$, the complete graph with $n$ vertices. Let $x^o$ be the average of all $x^H$ for $H \in \mathcal{H}_n$. As there are $(n-2)!$ Hamiltonian cycles with an edge common to all of them and the sum over all $H$ of any coordinate of $x^H = (n-2)!$, we have

$$x^o = \frac{2}{(n-1)!} \sum_{H \in \mathcal{H}_n} x^H = \frac{2}{(n-1)} \rho$$

where $\rho$ is a vector of ones. We have the following Lemma:

**Lemma 6.1.1** $\| x^o - x^H \| = \sqrt{\frac{n(n-3)}{n-1}} \quad \forall \; H \in \mathcal{H}_n$

We know the slack variables $U$ of the $MI$ problem are the edge-tour incidence vectors. Now consider a $U$ optimal to the $MI$ problem but not integer. If $U \in STSP(n)$ then

$$
\begin{aligned}
U &= \sum_{j\in J} \lambda_j x^{H(j)}; \quad 0 < \lambda_j < 1, \sum \lambda_j = 1 \qquad (6.1.1)\\
\|x^o - U\| &= \| \sum \lambda_j x^{H(j)} - x^o \|^2 \qquad (6.1.2)\\
&= \| \frac{2}{n-1}\rho - \sum_{j\in J} \lambda_j x^{H(j)} \| \leq \sqrt{\frac{n(n-3)}{n-1}} \qquad (6.1.3)
\end{aligned}
$$

However if $\|x^o - U\| \leq \sqrt{\frac{n(n-3)}{n-1}}$ we cannot conclude that $U \in STSP(n)$ as illustrated in the following example.

**Example 6.1.1** *Consider the graph* $G = (V, E)$, *where* $V = \{1, 2, \ldots, 12\}$. Consider the solution $U$ given by

$$
u_{ij} = \begin{cases} 2/3 & [i,j] = [1,2],[2,3],[3,4],[5,6],[6,7],[1,5],[2,7],[4,8],[7,8],[8,9],[1,9],[4,5] \\ 1/3 & [i,j] = [3,10],[3,12],[6,10],[6,12],[9,10],[9,12] \\ 1 & [i,j] = [10,11] \ \& \ [11,12] \end{cases}
$$

as given in Figure 6.1.1. From Theorem 6.1.1 we have $G$ is not Hamiltonian as it is got by adding vertices 11 and 12 and edges $[3, 12]$, $[6, 12]$, $[9, 12]$, $[10, 11]$, & $[11, 12]$ to the Petersen's graph which is known to be non Hamiltonian. $n = 12$ & $\sum u_{ij} = 12$; $U$ is optimal for $MI$. We have,

$$x^o = \frac{2}{11}\rho \ \ \& \ \ \sqrt{\frac{n(n-3)}{n-1}} = 3.13$$

$$\|x^o - U\| = 2.0730 < 3.13$$

Hence we have the example of a point for which $\|x^o - U\| < \sqrt{\frac{n(n-3)}{n-1}}$ but $U \in STSP(n)$.

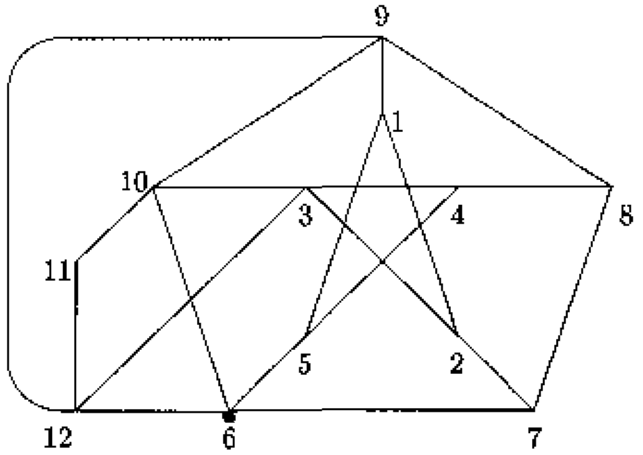



*Figure : 6.1.1 : Support graph for the example described in Example 6.1.1*

Arthanari(1981) proposed the $d^\infty$ norm to check whether a point lies in $STSP(n)$ or not. Given a feasible $U$ for the $MI$ problem and a $H \in \mathcal{H}_n$ define

$$d^\infty(U,H) \quad = \quad max_{1 \leq e \leq \frac{n(n-1)}{2}} \mid (u_e - x_e^H) \mid \tag{6.1.4}$$

We have the following

**Theorem 6.1.2** $U \in STSP(n)$ iff $min_{H \in \mathcal{H}_n} d^\infty(U,H) < 1$ .

Proof: If $min_{H \in \mathcal{H}_n} d^\infty(U,H) < 1$ then for some $H^*$ we have

$$d^\infty(U,H^*) = \epsilon < 1 \tag{6.1.5}$$

$\Longrightarrow \forall$ e such that $x_e^{H^*} = 1, u_e > 0$ otherwise

$$d^\infty(u, H^*) = 1$$

Therefore edge set of $H^*$ is a subset of the edges with $u_e > 0$ .Hence,by Theorem 3.3.2 $G$ is Hamiltonian. Also $\forall e$ such that $u_e = 1, x_e^{H^*} = 1$ otherwise $d^\infty(U, H^*) = 1$.

Let $r$ be $\#\{e \mid u_e = 1\}$. The set of remaining $(n - r)$ edges have $0 < u_e < 1$.

If $G$ is Hamiltonian we have $U \in conv\{x^H \mid H \in \mathcal{H}_n\}$. Therefore there exists $H^{(1)}, \ldots, H^{(s)} \in \mathcal{H}_n$ such that

$$U = \sum_{l=1}^{s} \lambda_l x^{H^{(l)}};\ 0 < \lambda_l < 1, \sum \lambda_l = 1$$

We shall show that there exists $H$ with $d^\infty(U, H) < 1$. Consider

$$\begin{aligned} d^\infty(U, H) &= max_e \mid u_e - x_e^H \mid \\ &= max_e \mid \sum_l \lambda_l x_e^{H^{(l)}} - x_e^H \mid \\ &= max_e \mid \sum_l \lambda_l (x_e H^{(l)} - x_e^H) \mid \end{aligned}$$

Let $H = H^{(l_o)}$ for some $l_o, 1 \leq l_o \leq s$.
Define

$$d_e^l = x_e^{H^{(l)}} - x_e^{H^{(l_o)}} \tag{6.1.6}$$

We have

$$d_e^{l_o} = 0 \ \forall \ e \tag{6.1.7}$$

$$d^\infty(U, H) = max_e \mid \sum_{l \ni d_e^l \neq 0} \lambda_l d_e^l \mid \tag{6.1.8}$$

Also

$$(x_e^{H^{(l)}} - x_e^{H^{(l_o)}}) \neq 0 \Longrightarrow d_e^l \pm 1 \tag{6.1.9}$$

Therefore

$$d^\infty(U, H) = max_e \mid \sum_{l \ni d_e^l = 1} \lambda_l - \sum_{l \ni d_e^l = -1} \lambda_l \mid < 1$$

Therefore $min\ d^\infty(U, H) < 1$ Hence $G$ is Hamiltonian iff $min_{H \in \mathcal{H}_n} d^\infty(U, H) < 1$. □

**Remark 6.1.1** *If $U$ is integer, then we have $U = H^*$ for some Hamiltonian cycle $H^*$. Hence, $d^\infty(U, H^*) = 0 \Longrightarrow min_{H\in\mathcal{H}_n} d^\infty(U, H) = 0 < 1$.*

We apply the above norm to find out whether a given set of extreme points to the $MI$ problem lie in $STSP(n)$ or not. The computational results for small polytopes are given in Section-6.3.

## 6.2 Neighbourhoods of extreme points

Let $\mathcal{P}$ be a polytope defined by the set of inequalities $Dz \leq d$, where $D$ is a $p \times q$, $p \leq q$ matrix and $d \in R^q$. $\mathcal{P}$ is either the $SEP(n), n \leq 7$ or $MI(n), n \leq 6$ polytope in our study. $\mathcal{P}$ has both integer and fractional extreme points. Let $x = (z, s)'$, where $s$ corresponds to the slack variables. We have $Cx = d$ as the defining system of equations now. Let $x^* = (z^*, s^*)'$ be a fractional extreme point of $\mathcal{P}$. Let $B$ be the basis, which consists of linearly independent columns of $C$, which corresponds to $x^*$, i.e,

$$x^*_N = B^{-1}d$$

where $N$ is the set of basic variables. Let $R$ be the set of non-basic variables. We define a neighbourhood, $N(x^*, R)$, of $x^*$ with respect to $R$, to consist of the extreme points obtained from $x^*$ by the insertion of a non-basic variable belonging to $R$, and followed by the removal of a basic variable in $N$ by the minimum ratio method, as in case of the simplex algorithm. In case of the $SEP$ polytope the set $R$ has cardinality $\frac{n(n-3)}{2}$, which is polynomial. Let $N_B(x^*) = \{y^* | y^* \text{ is a neighbour of } x^*\}$ be the set of neighbours of $x^*$. Due to the high degeneracy of the $TSP$ polytope, $N_B(x^*, R)$ may not be the same as $N_B(x^*)$. Let $I_{N_B}(x^*) \subseteq N_B(x^*)$ be the set of integer elements of $N_B(x^*)$. Since integer extreme points of $SEP$ and $MI$ are tours, $I_{N_B}(x^*)$ consists of tours if $\mathcal{P}$ is $SEP$ or $MI$. Let $T_{N_B}$ be the tour-defining variable incidence matrix. The defining variable being $x_{ij}$ in case of $SEP$ and $x_{ijk}$ in case of $MI$. Consider the system of inequalities defined below

$$\begin{aligned} T_{N_B} f &= \lambda \\ z' f &> \lambda \\ f &\geq 0, f \in R^q \end{aligned} \tag{6.2.1}$$

If a feasible solution $f$ exists to the above system of inequalities, then the inequality $fz \leq \lambda$ is a cutting plane which cuts off the fractional point $x^*$ and has the integer neighbours of $x^*$ lying on it( ensure feasible $f$ is such that all integer extreme points satisfy $fz \leq \lambda$). Add $fz \leq \lambda$ to the defining set of inequalities of $\mathcal{P}$ and compute optimal solutions. We illustrate this method for $SEP(6), SEP(7)$ and $MI(6)$ in the next section. We have not worked out neighbours of extreme points for $SEP(n)$; $n \geq 7$. We are currently working on whether we can give an iterative method to give sharper cutting planes to cut off fractional points. Since the size of the neighbourhoods of $SEP$ would be polynomial we hope some efficient local search method can be developed.

### 6.2.1 Neighbourhood graphs

The neighbourhood graph $\Gamma(n)$ is defined to have vertex set formed by all tours of $K_n$. A tour $T_1$ is adjacent to a tour $T_2$ if they belong to the same neighbourhood $N(x^*, R)$ for some $x^*$ and $R$. Clearly, $T_1$ is adjacent to $T_2$ if and only if $T_2$ is adjacent to $T_1$. Let $dist(T_1, T_2)$ be the distance from $T_1$ to $T_2$, which is defined to be the length of the shortest path from $T_1$ to $T_2$. The diameter of the $\Gamma(n)$ is the least positive integer $d$ such that the distance from a vertex to another vertex in $\Gamma(n)$ does not exceed $d$. The above quantities can be computed given the extreme points and neighbourhoods. We are currently working on small locally searchable neighbourhoods of $TSP$. We refer the reader to Gutin(1997) for work on small neighbourhoods of $TSP$.

## 6.3 Computational Results

We give here computational results on small polytopes. The complete structure of small $SEP$ and $MI$ has been given. This gives a better understanding of the polytopes.

### 6.3.1 New Inequalities for $MI$

We introduce some new inequalities for the $MI$ formulation. Consider $1 \leq i < j \leq k-2$; $5 \leq k \leq n$. We have the following

$$x_{ijk-1} + x_{ijk} \leq 1; \quad 1 \leq i < j \leq k-2 \tag{6.3.1}$$

If $x_{ijk-1} = 1$, then we have pairs $[i, k-1]$ and $[j, k-1]$ formed, so that, the other pairs $[l, k-1], 1 \leq l \leq k-2, l \neq i, j$ are not available for insertion of $k$. Hence we have the inequality

$$x_{ijk-1} + x_{ijk} + \sum_{l=1, l\neq i,j}^{k-2} x_{l,k-1,k} \leq 1; \; 5 \leq k \leq n, 1 \leq i < j \leq k-2 \tag{6.3.2}$$

We refer to these inequalities as $\mathcal{U}_f$-inequalities, as they eliminate fractional extreme points of $MI$. We illustrate how $\mathcal{U}_f$-inequalities eliminates fractional extreme points of $MI(5)$ and $MI(6)$ in this section. Arthanari(1997) gives more inequalities for the $MI$ problem. We refer to his paper for details.

### 6.3.2 5 city polytopes

In this section we compare the $SEP(5)$ and $MI(5)$ polytopes. It is well known that $SEP(5) \equiv STSP(5)$. From the equivalence of $SEP$ and $MI$, it naturally follows that $MI \equiv STSP$. We verified that $MI(5)$ completely describes $STSP(5)$. The data matrices and extreme points are given in Appendix-II. We summarise results in Table : 6.3.1.

We computed $min_{H \in \mathcal{H}_n} d^{\infty}(U, H)$ for the fractional extreme points of $MI(5)$, we see $min_{H \in \mathcal{H}_n} d^{\infty}(U, H) < 1$ for all $U$, hence all the extreme points lie within $STSP(n)$. Hence $STSP(5)$ is completely described using either $SEP(5)$ or $MI(5)$.

| Formulation | # constraints | # Extreme points | # Fractional Extreme points |
|---|---|---|---|
| $SEP$ | 15 | 12 | 0 |
| $MI$ | 8 | 18 | 6 |

Table 6.3.1 : Extreme points of the 5-city polytopes

Notice that even though $MI(5)$ completely describes $STSP(5)$, it has some fractional extreme points. Even though these fractional points lie within $STSP(n)$, eliminating them to get only integer optimal solutions for $MI(5)$ helps to understand the structure of $MI$. To achieve this, we worked out neighbours of a fractional extreme point. The results are presented in the Appendix. Applying the Neighbourhood method discussed in Section-6.2, we obtain the following inequalities.

$$x_{124} + x_{125} + x_{345} \leq 1 \qquad (6.3.3)$$

$$x_{134} + x_{135} + x_{245} \leq 1 \qquad (6.3.4)$$

$$x_{234} + x_{235} + x_{145} \leq 1 \qquad (6.3.5)$$

We see that these are equivalent to $\mathcal{U}_f$-inequalities. These inequalities are sufficient to eliminate the fractional extreme points of $MI(5)$ and get only integer extreme points. These integer optimal solutions correspond to the tours.

### 6.3.3 6 city polytopes

In this section we study six-city polytopes.

**$SEP(6)$ and its structure**

The data matrix and the number of extreme points are given in Appendix.

*Number of extreme points*: There are 120 extreme points of the polytope. Of this 60 points are integer which correspond to tours of $STSP(6)$ while other 60 are fractional. We label the tours as $T_1, \ldots, T_{60}$ and the fractional points as $NIP_1, \ldots, NIP_{60}$.

*Structure of any noninteger extreme point of $SEP(6)$*

Let $P_m = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ be a permutation of $\{1, \ldots, 6\}$. Total number of circular permutations possible is equal to $\frac{(6-1)!}{2}$.The non-integer extreme point corresponding to $P_m$ has the following support matrix.

$$x_{ij} = \begin{cases} 1/2 & [\text{i,j}] = [i_1, i_2],[i_3, i_4],[i_4, i_5],[i_6, i_1], [i_2, i_6],[i_3, i_5] \\ 1 & [\text{i,j}] = [i_1, i_4],[i_2, i_3],[i_5, i_6] \\ 0 & otherwise \end{cases}$$

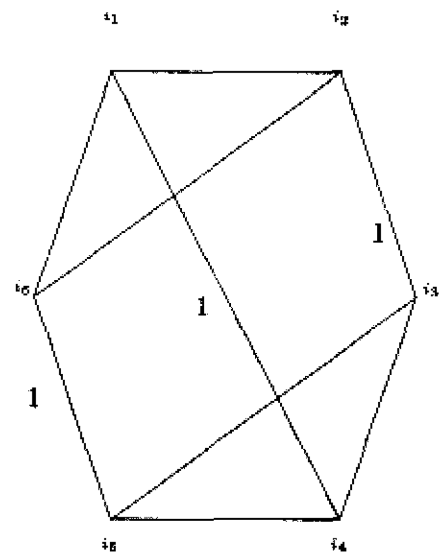
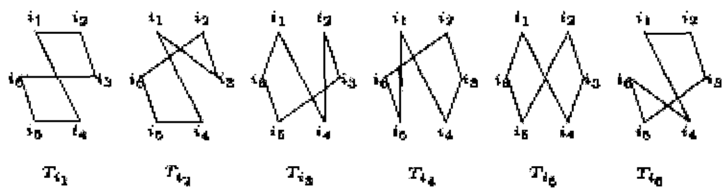



*Fig :6.3.1: Structure of a non-integer extreme point of* $SEP(6)$

Substituting $i_k = k$, for all $k = 1, \ldots, 6$ in the permutation $P_m$ we see that we get the non-integer point $NIP_{60}$. Similarly the structure of all the non-integer extreme points can be obtained for all permutations $P_j, j = 1, 60$.

*Elimination of non-integer points*

Using the Neighbourhood method, we computed neighbours of all fractional extreme points of $SEP(6)$. Each fractional extreme point has 9-integer neighbours. The cutting planes are found out. Interestingly, the cutting planes obtained thus are equivalent to the comb-inequalities. The neighbours of the fractional point in Fig:6.3.1 is worked out and they are described in Fig:6.3.2.



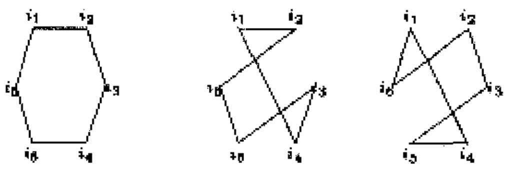



*Fig :6.3.2: Neighbours of the non-integer point given in Fig:6.3.1*

The tours $T_{i_1}, \ldots, T_{i_6}$ are obtained by a non-basic edge entering the basis whereas tours $T_{i_7}, T_{i_8}$ & $T_{i_9}$ are obtained by allowing the non-basic slack variables to enter the basis.

The computational results are given in the Appendix.

### 6.3.4 $MI(6)$ and its structure

The data matrix and the extreme points are given in the appendix. We have 666 extreme points of $MI(6)$. We computed $min_{H \in \mathcal{H}_n} d^{\infty}(U, H)$ for all these extreme points. On adding $\mathcal{U}_f$-inequalities we get 60-fractional points which lie outside $STSP(n)$. To these fractional points we apply the Neighbourhood-method and find out cutting planes. The results are given in the Appendix. The results are summarised below.

| No. of constraints | No. of Extreme Points | Points outside $STSP(6)$ |
|---|---|---|
| 13 | 666 | 72 |
| $13 + 9\,\mathcal{U}_f$ inequalities | 300 | 60 |
| 13 + 9 + 45 Inequalities by Neighbourhood-method | 702 | None |

**Remark 6.3.1** *The number of extreme points increase on addition of the 45 inequalities. However, the new extreme points that are created are such that their corresponding $U \in STSP(n)$. Hence, an increase in number of extreme points of $MI$ does not imply that new points outside $STSP(n)$ are created.*

## 6.3.5 $SEP(7)$

$SEP(7)$ is defined by a set of 63 inequalities. There are 4140 extreme points of which 360 are integer which correspond to tours of $STSP(7)$ and 3780 fractional. The fractional extreme points can be classified into two classes $F71$ and $F72$. The structure of extreme points belonging to the above classes given below. Any extreme point can be classified into one of these classes.

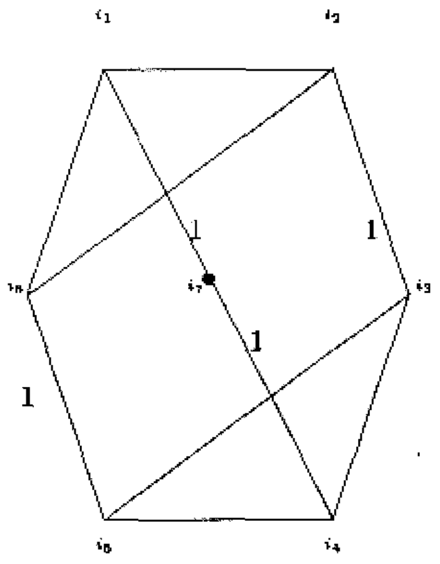



*Fig : 6.3.3 : Support graph of an extreme point in* $F71$, *The edges with weight 1 are depicted. The other edges have weight 0.5*

*Structure of fractional extreme points in* $F71$
Let $P_m = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$ be a permutation of $\{1, \ldots, 7\}$. The fractional extreme point corresponding to $P_m$ has the following support matrix. ($x_{[ij]}$ denotes weight on edge$[i, j]$).

$$x_{ij} = \begin{cases} 1/2 & [i,j] = [i_1,i_2],[i_3,i_4],[i_4,i_5],[i_6,i_1],\ [i_2,i_6],[i_3,i_5] \\ 1 & [i,j] = [i_2,i_3],[i_5,i_6],[i_1,i_7],[i_4,i_7] \\ 0 & otherwise \end{cases} \quad (6.3.6)$$

Note this extreme point has the vertex $i_7$ introduced between vertices $i_1$ and $i_4$ of the fractional point of $SEP(6)$ described in Fig: 6.3.1. Every such fractional point of $SEP(6)$ gives rise to 3 fractional points of $SEP(7)$. The six vertices can be chosen in $7_{C_2}$ ways , hence the total number of fractional points belonging to $F71 = 7_{C_2} \times 60 \times 3 = 1260$.

*Structure of fractional extreme points in $F72$*

The fractional points in $F72$ have the following support matrix.

$$x_{ij} = \begin{cases} 1/2 & [i,j] = [i_1,i_2],[i_2,i_6],[i_1,i_6],[i_1,i_4],\ [i_1,i_7],[i_3,i_4],[i_5,i_7],[i_3,i_5] \\ 1 & [i,j] = [i_2,i_3],[i_5,i_6],[i_4,i_7] \\ 0 & otherwise \end{cases}$$

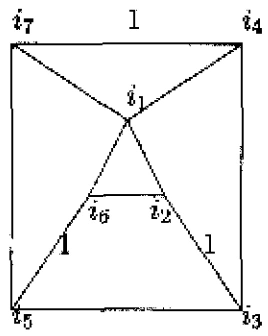



*Fig 6.3.4 : Support graph of extreme point in the class F72, the edges $[i_4, i_7]$, $[i_2, i_3]$ and $[i_5, i_6]$ have weights 1, whereas the remaining edges in the figure have weight 0.5*

The points $i_1, i_2, i_6$ form a triangle. These can be chosen in $7_{P_3}$ ways, the other vertices can be arranged in $\frac{4!}{2}$ ways. Hence $| F72 |= 2520$.

The neighbours of the extreme point described above are calculated by the neighbourhood method and the cutting plane is found. They are given in the Appendix. We depict this is Figure 6.3.5.

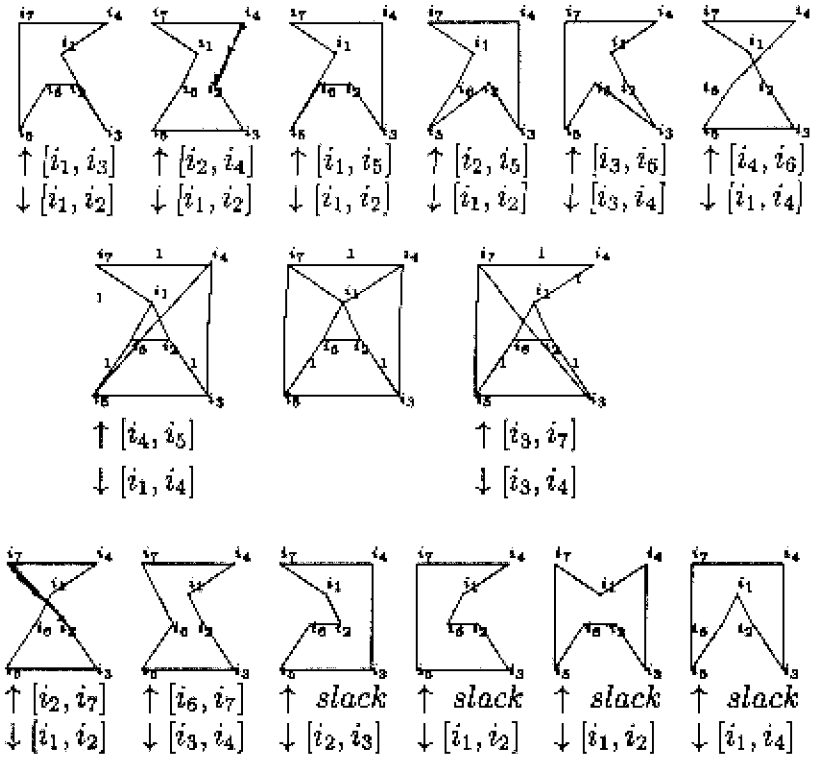



*Fig : 6.3.5 :The Neighbours of an extreme point in F72. The point in the centre is the extreme point. The integer neighbours are given above and below the point , whereas, the fractional neighbours are adjacent to it. Here ↑ denotes the edge entering the basis and ↓ denotes the edge leaving the basis.*

The cutting plane generated by the neighbourhood-method for the point is

$$x_{i_1 i_2} + x_{i_2 i_6} + x_{i_1 i_6} + x_{i_1 i_7} + x_{i_1 i_4} + x_{i_4 i_7} + x_{i_5 i_6} + x_{i_2 i_3} \leq 5$$

This inequality has all the integer-neighbours lying on it and cuts off the fractional neighbours.

**Remark 6.3.2** *The above inequality corresponds to the comb inequality given* by $H = \{i_1, i_2, i_6\}, T_1 = \{i_1, i_4, i_7\}, T_2 = \{i_2, i_3\}$ & $T_3 = \{i_5, i_6\}$

Equation 6.3.7 cuts off the point defined by the following support matrix.

$$x_{ij} = \begin{cases} 1/2 & [i,j] = [i_1, i_2],[i_2, i_6],[i_1, i_6],[i_1, i_4],\ [i_1, i_7],[i_4, i_5],[i_3, i_7],[i_3, i_5] \\ 1 & [i,j] = [i_2, i_3],[i_5, i_6],[i_4, i_7] \\ 0 & otherwise \end{cases}$$

**Remark 6.3.3** *Inequality 6.3.7 cuts off the fractional point described in 6.3.6 also. Each cutting plane obtained thus cuts off 3 fractional points. Hence the total number of cutting planes introduced by the above method* = 1260.

**Remark 6.3.4** *The cutting planes introduced above are effective in cutting off the fractional points of $MI$, but some of the existing fractional points have become extreme points. Perhaps this is because we have fractional points as neighbours. The comb inequalities are not sufficient for a complete description of $SEP(7)$. The example Boyd and Cunningham(1991) give to introduce envelope inequalities is a convex combination of an extreme point from F72 and a tour. We are working on whether an effective local search method can be developed to give sharper cuts.*

## 6.4 Conclusions

We have presented results on small polytopes. As mentioned earlier, we believe this would help to develop better heuristics to solve $TSP$. We are working on whether the neighbourhood method could be used to develop a branch and bound algorithm to solve $TSP$.

# Chapter 7

# Conclusions

This thesis concentrates on the $MI$ formulation for the symmetric traveling salesman problem though it briefly introduces the $MI$ formulation for the asymmetric traveling salesman problem. As an extension of this work we would like to work out similar results for the asymmetric case and study how it compares with the $DFJ$ formulation.

The Multistage-insertion formulation ($MI$) was shown to be a polynomial sized formulation which implies validity of the subtour elimination constraints. We also showed the equivalence of this formulation to the Cycle-shrink ($CS$) formulation.

Carr(1996) used the Cycle-shrink to separate over classes of inequalities and developed a polynomial separation technique. We are working on how the Multistage-insertion could be used for separation. We are working on how the $MI$ can be used to separate over more general classes of inequalities. The question whether all comb inequalities can be separated in polynomial time remains open. We address these questions in our future research.

Kipp Martin(1991) uses separation algorithms to generate mixed-integer reformulations. We are working on whether we can have a mixed-integer reformulation for the subtour elimination polytope and how it compares with $\mathcal{U}(n)$. This paper suggests several open questions: Does there exist a polynomial size LP formulation for any problem that can be solved in polynomial time? For example, a polynomial size description of the matching polytope is unknown. Can the $MI$ be helpful in expressing the matching polytope with a polynomial size linear program?

We also plan to extend our work on small polytopes and see whether we can obtain complete descriptions of $STSP$ using $MI$ formulation.

# APPENDIX-I

In this Appendix, we illustrate the method of finding a linear desciption for $\mathcal{U}(n)$ for $n = 6$.

$X = (x_{126}, x_{136}, x_{236}, \ldots, x_{456}, \ldots, x_{124}, x_{134}, x_{234})$
$\tau_6 = 15;\ m' = 10\ \&\ p = 3$

The matrix

$$\begin{bmatrix} E \\ \bar{A} \end{bmatrix}$$

is given as below :

$$\left[\begin{array}{cccccccccc|ccccccccc}
 & & & & E_1 & & & & & & & & & E_2 & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
 & & & & \bar{A}_1 & & & & & & & & & \bar{A}_2 & & & & &
\end{array}\right]$$

$-L_1 = \bar{A}_1 = I_{10\times 10}$

The matrix

$$\mathcal{E} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and

$$B = \left[\begin{array}{ccccccccccccc}
v_1 & v_2 & v_3 & z_{12} & z_{13} & z_{23} & z_{14} & z_{24} & z_{34} & z_{15} & z_{25} & z_{35} & z_{45} \\
\hline
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\
1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0
\end{array}\right]$$

**Generators of $C'$**

(i) $(0, \xi^i, 0)$ give rise to non negativity constraint.
(ii) Define $v_1 = 0$.

Let $P_1 = \{z_{ij} | z_{ij} = 1\}$ Let $F_1 = \{z_{12}, z_{13}, z_{23}\}; F_2 = \{z_{14}, z_{24}, z_{34}\} \& F_3 = \{z_{15}, z_{25}, z_{35}, z_{45}\}$

The LHS for the linear inequalities is given by $\sum_{[i,j] \ni z_{ij} \in P_1} u_{ij}$. We now list the various options available for $v_2$ & $v_3$ given distribution of elements of $P_1$ and the RHS of the inequalities.

| $P_1$ | $F_1$ | $F_2$ | $F_3$ | $v_2$ | $v_3$ | RHS | Comments/Examples |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | $u_{12}$ or $u_{13}$ or $u_{23} \leq 1$ |
| | 0 | 1 | 0 | 0 | 1 | 1 | $u_{14}$ or $u_{24}$ or $u_{34} \leq 1$ |
| | 0 | 0 | 1 | 1 | 0 | 1 | $u_{15}$ or $u_{25}$ or $u_{35}$ or $u_{45} \leq 1$ |
| 2 | 2 | 0 | 0 | 0 | 0 | 2 | $u_{12} + u_{13} \leq 2$ |
| | 1 | 1 | 0 | 0 | 1 | 2 | $u_{12} + u_{14} \leq 2$ |
| | 1 | 0 | 1 | 1 | 0 | 2 | $u_{12} + u_{15} \leq 2$ |
| | 0 | 2 | 0 | 0 | 2 | 2 | $u_{14} + u_{24} \leq 2$ |
| | 0 | 1 | 1 | 1 | 1 | 2 | $u_{14} + u_{15} \leq 2$ |
| | 0 | 0 | 2 | 2 | 0 | 2 | $u_{15} + u_{25} \leq 2$ |
| 3 | 3 | 0 | 0 | 0 | -1 | 2 | $u_{12} + u_{13} + u_{23} \leq 2$ |
| | 2 | 1 | 0 | 0 | 0,1 | 2,3 | $u_{12} + u_{13} + u_{14} \leq 2$ |
| | | | | | | | $u_{12} + u_{13} + u_{24} \leq 3$ |
| | 2 | 0 | 1 | 1 | 0 | 3 | $u_{12} + u_{13} + u_{15} \leq 3$ |
| | 1 | 2 | 0 | 0 | 2,1 | 3,2 | $u_{12} + u_{14} + u_{24} \leq 2$ |
| | 1 | 0 | 2 | 2,1 | 0 | 3,2 | $u_{12} + u_{15} + u_{25} \leq 2$ |
| | 0 | 3 | 0 | 0 | 2 | 2 | $u_{14} + u_{24} + u_{34} \leq 2$ |
| | 0 | 1 | 2 | 2 | 1 | 3 | $u_{14} + u_{15} + u_{25} \leq 3$ |
| | 0 | 2 | 1 | 1 | 2 | 3 | $u_{14} + u_{24} + u_{15} \leq 3$ |
| | 0 | 0 | 3 | 2 | 0 | 2 | $u_{15} + u_{25} + u_{35} \leq 2$ |
| 4 | 3 | 1 | 0 | 0 | 0 | 3 | $u_{12} + u_{13} + u_{23} + u_{i4} \leq 3, i = 1,2,3$ |
| | 3 | 0 | 1 | 1 | -1 | 3 | $u_{12} + u_{13} + u_{23} + u_{i5} \leq 3, i = 1,2,3$ |
| | 2 | 1 | 1 | 0,1 | 1,2 | 2,3,4 | $u_{12} + u_{13} + u_{14} + u_{15} \leq 2$ |
| | 2 | 2 | 0 | 0 | 2,1 | 3,4 | $u_{12} + u_{13} + u_{14} + u_{24} \leq 3$ |
| | 2 | 0 | 2 | 1,2 | 0 | 3,4 | $u_{12} + u_{13} + u_{15} + u_{25} \leq 3$ |
| | 1 | 3 | 0 | 0 | 2 | 3 | $u_{12} + u_{14} + u_{24} + u_{34} \leq 3$ |
| | 1 | 2 | 1 | 1 | 1,2 | 3,4 | $u_{12} + u_{14} + u_{24} + u_{15} \leq 3$ |
| | 1 | 1 | 2 | 1,2 | 1 | 3,4 | $u_{12} + u_{14} + u_{15} + u_{25} \leq 3$ |
| | 1 | 0 | 3 | 2 | 0 | 3 | $u_{12} + u_{15} + u_{25} + u_{35} \leq 3$ |
| | 0 | 3 | 1 | 1 | 2 | 3 | $u_{14} + u_{24} + u_{34} + u_{15} \leq 3$ |
| | 0 | 1 | 3 | 2 | 1 | 3 | $u_{14} + u_{15} + u_{25} + u_{35} \leq 3$ |
| | 0 | 2 | 2 | 2 | 2 | 4 | $u_{14} + u_{24} + u_{15} + u_{25} \leq 4$ |
| 5 | 3 | 2 | 0 | 0 | 1 | 4 | $u_{12} + u_{13} + u_{23} + u_{14} + u_{24} \leq 4$ |
| | 3 | 1 | 1 | 0,1 | 1 | 4,5 | $u_{12} + u_{13} + u_{23} + u_{14} + u_{15} \leq 4$ |
| | 3 | 0 | 2 | 1,2 | 1 | 4,5 | $u_{12} + u_{13} + u_{23} + u_{15} + u_{25} \leq 4$ |
| | 2 | 1 | 2 | 1,2 | 1,0 | 4,5 | $u_{12} + u_{13} + u_{14} + u_{15} + u_{25} \leq 4$ |
| | 2 | 2 | 1 | 1,0 | 1,2 | 4,5 | $u_{12} + u_{13} + u_{14} + u_{24} + u_{15} \leq 4$ |
| | 2 | 3 | 0 | 0 | 2 | 4 | $u_{12} + u_{13} + u_{14} + u_{24} + u_{34} \leq 4$ |
| | 2 | 0 | 3 | 2 | 0 | 4 | $u_{12} + u_{13} + u_{15} + u_{25} + u_{35} \leq 4$ |
| | 1 | 3 | 1 | 1 | 2 | 4 | $u_{12} + u_{14} + u_{24} + u_{34} + u_{15} \leq 4$ |
| | 1 | 1 | 3 | 2 | 1 | 4 | $u_{12} + u_{14} + u_{15} + u_{25} + u_{35} \leq 4$ |
| | 0 | 3 | 2 | 2 | 2 | 4 | $u_{14} + u_{24} + u_{34} + u_{15} + u_{25} \leq 4$ |
| | 0 | 2 | 3 | 2 | 2 | 4 | $u_{14} + u_{24} + u_{15} + u_{25} + u_{35} \leq 4$ |
| | 0 | 1 | 4 | 2 | 1 | 3 | $u_{14} + u_{15} + u_{25} + u_{35} + u_{45} \leq 3$ |
| | 1 | 0 | 4 | 2 | 0 | 3 | $u_{12} + u_{15} + u_{25} + u_{35} + u_{45} \leq 3$ |
| 6 | 0 | 2 | 4 | 2 | 2 | 4 | $u_{14} + u_{24} + u_{15} + u_{25} + u_{35} + u_{45} \leq 4$ |
| | 0 | 3 | 3 | 2 | 2 | 4 | $u_{14} + u_{24} + u_{34} + u_{15} + u_{25} + u_{35} \leq 4$ |
| | 1 | 1 | 4 | 2 | 2 | 4 | $u_{12} + u_{14} + u_{15} + u_{25} + u_{35} + u_{45} \leq 4$ |
| | 1 | 2 | 3 | 2 | 2 | 5 | $u_{12} + u_{14} + u_{24} + u_{15} + u_{25} + u_{35} \leq 5$ |
| | 1 | 3 | 2 | 2 | 2 | 5 | $u_{12} + u_{14} + u_{24} + u_{34} + u_{15} + u_{25} \leq 5$ |
| | 2 | 0 | 4 | 2 | 0 | 4 | $u_{12} + u_{13} + u_{15} + u_{25} + u_{35} + u_{45} \leq 4$ |
| | 2 | 1 | 3 | 2 | 1,0 | 4,5 | $u_{12} + u_{13} + u_{14} + u_{15} + u_{25} + u_{35} \leq 4$ |
| | 2 | 2 | 2 | 1,2 | 1,2 | 4,5 | $u_{12} + u_{13} + u_{14} + u_{24} + u_{15} + u_{25} \leq 4$ |
| | 2 | 3 | 1 | 1 | 2 | 5 | $u_{12} + u_{13} + u_{14} + u_{24} + u_{34} + u_{15} \leq 4$ |
| | 3 | 1 | 2 | 1,2 | 1 | 5,5 | $u_{12} + u_{13} + u_{23} + u_{14} + u_{15} + u_{25} \leq 4$ |
| | 3 | 2 | 1 | 1 | 1 | 5 | $u_{12} + u_{13} + u_{23} + u_{14} + u_{24} + u_{15} \leq 5$ |
| | 3 | 0 | 3 | 1 | -1 | 3 | $u_{12} + u_{13} + u_{23} + u_{15} + u_{25} + u_{35} \leq 3$ |
| | 3 | 3 | 0 | -1 | 1 | 3 | $u_{12} + u_{13} + u_{23} + u_{14} + u_{24} + u_{34} \leq 3$ |

| $F$ | $F_1$ | $F_2$ | $F_3$ | $v_2$ | $v_3$ | RHS | Comments/Examples |
|---|---|---|---|---|---|---|---|
| 7 | 0 | 3 | 4 | 2 | 2 | 4 | $u_{14}+u_{24}+u_{34}+u_{16}+u_{26}+u_{35}+u_{46}\leq 4$ |
| | 1 | 3 | 3 | 2 | 2 | 4 | $u_{13}+u_{14}+u_{24}+u_{34}+u_{16}+u_{25}+u_{36}\leq 4$ |
| | 1 | 2 | 4 | 2 | 2 | 4 | $u_{12}+u_{14}+u_{24}+u_{16}+u_{26}+u_{36}+u_{45}\leq 4$ |
| | 2 | 3 | 2 | 1,2 | 2 | 5,6 | $u_{12}+u_{13}+u_{14}+u_{24}+u_{34}+u_{16}+u_{25}\leq 5$ |
| | 2 | 2 | 3 | 1,2 | 1,2 | 4,5,6 | $u_{12}+u_{13}+u_{14}+u_{24}+u_{16}+u_{26}+u_{36}\leq 5$ |
| | 2 | 1 | 4 | 2 | 1 | 5 | $u_{12}+u_{13}+u_{14}+u_{16}+u_{26}+u_{36}+u_{46}\leq 5$ |
| | 3 | 0 | 4 | 3 | -1 | 4 | $u_{12}+u_{13}+u_{23}+u_{15}+u_{25}+u_{35}+u_{46}\leq 4$ |
| | 3 | 1 | 3 | 1,2 | 0 | 4,5 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{16}+u_{26}+u_{36}\leq 4$ |
| | 3 | 2 | 2 | 1,2 | 1 | 5,6 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{24}+u_{15}+u_{26}\leq 5$ |
| | 3 | 3 | 1 | 0 | 1 | 4 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{24}+u_{34}+u_{15}\leq 4$ |
| 8 | 1 | 3 | 4 | 2 | 2 | 5 | $u_{12}+u_{14}+u_{24}+u_{34}+u_{15}+u_{25}+u_{36}+u_{46}\leq 5$ |
| | 2 | 2 | 4 | 2 | 2 | 6 | $u_{12}+u_{13}+u_{14}+u_{24}+u_{16}+u_{25}+u_{35}+u_{46}\leq 6$ |
| | 3 | 1 | 4 | 1,2 | 1 | 5,6 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{16}+u_{25}+u_{35}+u_{46}\leq 5$ |
| | 2 | 3 | 3 | 1,2 | 2 | 5,6 | $u_{12}+u_{13}+u_{14}+u_{24}+u_{34}+u_{16}+u_{25}+u_{36}\leq 5$ |
| | 3 | 3 | 2 | 1 | 1 | 5 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{24}+u_{34}+u_{15}+u_{46}\leq 5$ |
| | 3 | 2 | 3 | 1,2 | 1 | 5,6 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{24}+u_{16}+u_{26}+u_{35}\leq 5$ |
| 9 | 2 | 3 | 4 | 2 | 1 | 5 | $u_{12}+u_{13}+u_{14}+u_{24}+u_{34}+u_{15}+u_{26}+u_{36}+u_{46}\leq 5$ |
| | 3 | 2 | 4 | 1,2 | 1 | 5,6 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{24}+u_{16}+u_{25}+u_{35}+u_{46}\leq 5$ |
| | 3 | 3 | 3 | 1 | 1 | 5 | $u_{12}+u_{13}+u_{23}+u_{14}+u_{24}+u_{34}+u_{15}+u_{26}+u_{36}\leq 5$ |
| 10 | 3 | 3 | 4 | 1 | 1 | 5 | Since all $x_e^* > 0$ the corresponding linear inequality is strict Hence $u(E(S)) \leq 4$ is the constraint |

We see for the above choice of generators we get all the subtour elimination constraints for sets $S \subseteq V_5$. Since the variables $u_{ij}$ satisfy degree constraints we can conclude $SEP$ constraints are satisfied for $S \subseteq V_6;\ 2 \leq \mid S \mid \leq 5$

# APPENDIX-II

Finding extreme points of a polytope and facet enumeration problems are very challenging problems. The double-description method is a most commonly used algorithm to compute extreme points. Euler and Verge(1995) use a refined version of Chenikova's (1965) algorithm to compute extreme points of the five city asymmetric symmetric traveling salesman problem. In this dissertation we used the algorithm given in Arthanari(1981) to compute the extreme points of the polytope.

We now give the algorithm we have used to compute extreme points of the polytopes. Using this algorithm we determine all the extreme points of $SEP(5), SEP(6), SEP(7), MI(5)$ & $MI(6)$. Programs in Fortran were written to find the extreme points of a system of linear equations using the algorithm and computing neighbours of extreme points and cutting planes using the neighbourhood method discussed in Section 6.4.

**Algorithm to find all extreme points of a system of linear equations**

We consider the problem of finding all the extreme points of

$$\begin{aligned} DW &= 0 \\ W &\geq 0 \end{aligned}$$

Let $C = \{W \setminus D = 0, W \geq 0\}$. Consider the matrix $(D', I')$. The algorithm produces a series of transformations of this matrix that generates the required extreme points of the cone of nonnegative solutions of the given system of equations.Let $Q$ denote the matrix that is transformed at any stage of the computation and $\bar{Q}$ the resulting matrix. $Q$ is partitioned into $(U', L'); Q = (U', L')$. Initially $U' = D'$ and $L' = I$.

**Algorithm -1**

*Step 0:* If all rows of the matrix $U$ are zeroes then the matrix $L$ gives the extreme vectors of the cone of nonnegative solutions of the system of equations. Otherwise go to Step 1.

*Step 1:* If a row of $U$ has neither zeroes nor any sign change then $W = 0$ is the only solution to the system. Otherwise go to Step 2.

*Step 2:* Consider row $r$ for processing. Let $R = \{j \setminus q_{rj} = 0\}$. Let $v$ be the number of elements of $R$. The first $v$ columns of the new matrix $\bar{Q}$ are $\mathbf{q_j}$ for $j \in R$, where $\mathbf{q_j}$ denotes the $j^{th}$ column of $Q$. Go to Step $2^{'}$.

*Step* $2^{'}$ : If $Q$ has only two columns and $q_{r1}q_{r2} < 0$, adjoin the column $| q_{r2} | \mathbf{q_1} + | q_{r1} | \mathbf{q_2}$ to the matrix $\bar{Q}$. Go to Step 4. Othersiwe go to Step 3.

Let $S = \{(s,t) \setminus q_{rs}q_{rt} < 0, s < t\}$

*Step 3:* Let $I(s,t) = \{i \setminus i \in L, q_{is} = q_{it} = 0\}$, for an $(s,t) \in S$, where $L$ also denotes the set of indices of the rows in $L$ namely $\{m+1, \ldots, m+n+1\}$.

1. If $I(s,t) = \phi$ then $\mathbf{q_s}$ and $\mathbf{q_t}$ do not contribute a column to matrix $\bar{Q}$. Go to Step 4.

2. If $I(s,t) \neq \phi$, check for a $u$ not equal to $s$ or $t$ such that $q_{iu} = 0$ for all $i \in I(s,t)$. If such a $u$ exists, then $\mathbf{q_s}, \mathbf{q_t}$ do not contribute a column to the matrix $\bar{Q}$. Go to Step 4. Otherwise, choose $\alpha_1 =| q_{rs} |, \alpha_2 =| q_{rt} |$. Adjoin the column $\alpha_1\mathbf{q_s} + \alpha_2\mathbf{q_t}$ to the matrix $\bar{Q}$.

*Step 4:* When all pairs in $S$ have been examined and the additional columns(if any) have been added, we say Row $r$ has been processed. Now if $\bar{Q} = \phi$, stop; $W = 0$ is the only solution to the system. Otherwise, let $Q = \bar{Q}$ go to *Step* 0.

**5-city polytopes**

$SEP(5)$: The constraint matrix for $SEP(5)$ is given in Table:A1. The extreme points of this polytope are given in Table:A2.

| $x_{12}$ | $x_{13}$ | $x_{23}$ | $x_{14}$ | $x_{24}$ | $x_{34}$ | $x_{15}$ | $x_{25}$ | $x_{35}$ | $x_{45}$ | $b_i$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 2 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

Table:A1: Data Matrix of $SEP(5)$

| Tour | $x_{12}$ | $x_{13}$ | $x_{23}$ | $x_{14}$ | $x_{24}$ | $x_{34}$ | $x_{15}$ | $x_{25}$ | $x_{35}$ | $x_{45}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $T_1$ | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| $T_2$ | 1 | 0 | 1 | 1 | 0 | .0 | 0 | 0 | 1 | 1 |
| $T_3$ | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| $T_4$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| $T_5$ | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $T_6$ | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| $T_7$ | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| $T_8$ | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| $T_9$ | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| $T_{10}$ | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| $T_{11}$ | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| $T_{12}$ | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 |

Table:A2: Extreme points of $SEP(5)$

We note that all the extreme points are integer extreme points which correspond to the tours of the $STSP(5)$.

$MI(5)$: The constraint matrix is given in Table :A3. There are 8 rows in the constraint matrix. The extreme points of $MI(5)$ are given in Table : A4. The neighbours were calculated for the extreme point 13 using neighbourhood method. They are given in Table : A5. The cutting planes obtained for this point by neighbourhood method is given in Table:A6. Similarly we can work out inequalities for the other fractional points also. We see they are same as $U_f$ inequalities.

| $x_{124}$ | $x_{134}$ | $x_{234}$ | $x_{125}$ | $x_{135}$ | $x_{235}$ | $x_{145}$ | $x_{245}$ | $x_{345}$ | $b(i)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| -1 | -1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| -1 | 0 | -1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Table : A3: Constraint matrix for $MI(5)$

| No. $x_{124}$ | $x_{134}$ | $x_{234}$ | $x_{125}$ | $x_{135}$ | $x_{235}$ | $x_{145}$ | $x_{245}$ | $x_{345}$ | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | .0 | 1.0 | .0 | 0 | .0 | .0 | .0 | .0 | 1.0 |
| 2 | 1.0 | .0 | .0 | .0 | .0 | .0 | .0 | 1.0 | 0 |
| 3 | .0 | .0 | 1.0 | 0 | .0 | .0 | .0 | .0 | 1.0 |
| 4 | 1.0 | 0 | .0 | 0 | .0 | .0 | 1.0 | .0 | .0 |
| 5 | 1.0 | .0 | .0 | .0 | .0 | 1.0 | .0 | .0 | .0 |
| 6 | 1.0 | 0 | 0 | .0 | 1.0 | .0 | .0 | .0 | .0 |
| 7 | .0 | 0 | 1.0 | .0 | .0 | .0 | .0 | 1.0 | .0 |
| 8 | .0 | 1.0 | 0 | .0 | .0 | .0 | 1.0 | 0 | .0 |
| 9 | .0 | 1.0 | .0 | .0 | .0 | 1.0 | .0 | .0 | .0 |
| 10 | .0 | 1.0 | .0 | 1.0 | .0 | .0 | .0 | .0 | .0 |
| 11 | 0 | .0 | 1.0 | .0 | 1.0 | .0 | .0 | .0 | .0 |
| 12 | 0 | .0 | 1.0 | 1.0 | .0 | .0 | .0 | .0 | .0 |
| 13 | 0 | .5 | .5 | .0 | .0 | .5 | .5 | .0 | .0 |
| 14 | .5 | .0 | .5 | .0 | .0 | .5 | .5 | .0 | .0 |
| 15 | .0 | .5 | .5 | .0 | .0 | .0 | 0 | .5 | .0 |
| 16 | .5 | .5 | .0 | .0 | .5 | .0 | 0 | .5 | .0 |
| 17 | .5 | .0 | .0 | .5 | .0 | .0 | 0 | .0 | .5 |
| 18 | .5 | .0 | .5 | .5 | 0 | .0 | 0 | .0 | .0 |

Table : A4 : Extreme points of $MI(5)$

| | $x_{124}$ | $x_{134}$ | $x_{234}$ | $x_{125}$ | $x_{135}$ | $x_{235}$ | $x_{145}$ | $x_{245}$ | $x_{345}$ |
|---|---|---|---|---|---|---|---|---|---|
| Point | 0 | 0.5 | 0.5 | 0 | 0 | 0.5 | 0.5 | 0 | 0 |
| Neighbour1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| Neighbour2 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| Neighbour3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| Neighbour4 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| Neighbour5 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Neighbour6 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Neighbour7 | 0.5 | 0 | 0.5 | 0 | 0 | 0.5 | 0.5 | 0 | 0 |

Table : A5 : Neighbours of extreme point 13 of $MI(5)$

| Ineq | $x_{124}$ | $x_{134}$ | $x_{234}$ | $x_{125}$ | $x_{135}$ | $x_{235}$ | $x_{145}$ | $x_{245}$ | $x_{345}$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0. | 0. | 1. | 0. | 0. | 1. | 1. | 0. | 0. | 1. |

Table : A6 : Cutting plane for the extreme point 13 of $MI(5)$

### 6-city Polytopes

$SEP(6)$: The constraint matrix for $SEP(6)$ is given in Table:A7. There are 31 rows. There are 120 extreme points for this polytope. There are 60 integer extreme points which correspond to the 60 tours and 60 fractional extreme points. The integer extreme points are given in Table: A8a and the fractional extreme points are given in Table:A8b.

| $x_{12}$ | $x_{13}$ | $x_{23}$ | $x_{14}$ | $x_{24}$ | $x_{34}$ | $x_{15}$ | $x_{25}$ | $x_{35}$ | $x_{45}$ | $x_{16}$ | $x_{26}$ | $x_{36}$ | $x_{46}$ | $x_{56}$ | $b_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 2 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 2 |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 2 |
| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 2 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 2 |

Table:A7: Constraint Matrix of $SEP(6)$

| Tour | $x_{12}$ | $x_{13}$ | $x_{23}$ | $x_{14}$ | $x_{24}$ | $x_{34}$ | $x_{15}$ | $x_{25}$ | $x_{35}$ | $x_{45}$ | $x_{16}$ | $x_{26}$ | $x_{36}$ | $x_{46}$ | $x_{56}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_1$ | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| $T_2$ | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| $T_3$ | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| $T_4$ | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| $T_5$ | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| $T_6$ | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 |
| $T_7$ | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 |
| $T_8$ | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| $T_9$ | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 |
| $T_{10}$ | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| $T_{11}$ | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 |
| $T_{12}$ | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 |
| $T_{13}$ | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 |
| $T_{14}$ | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| $T_{15}$ | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| $T_{16}$ | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 |
| $T_{17}$ | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 |
| $T_{18}$ | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 |
| $T_{19}$ | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| $T_{20}$ | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 |
| $T_{21}$ | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 |
| $T_{22}$ | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| $T_{23}$ | .0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 |
| $T_{24}$ | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 |
| $T_{25}$ | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 |
| $T_{26}$ | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 |
| $T_{27}$ | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 |
| $T_{28}$ | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| $T_{29}$ | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 |
| $T_{30}$ | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 |
| $T_{31}$ | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 |
| $T_{32}$ | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 |
| $T_{33}$ | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| $T_{34}$ | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 |
| $T_{35}$ | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 |
| $T_{36}$ | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 |
| $T_{37}$ | .0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| $T_{38}$ | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 |
| $T_{39}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 |
| $T_{40}$ | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 |
| $T_{41}$ | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 |
| $T_{42}$ | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 |
| $T_{43}$ | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 |
| $T_{44}$ | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 |
| $T_{45}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 |
| $T_{46}$ | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 |
| $T_{47}$ | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 |
| $T_{48}$ | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 |
| $T_{49}$ | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| $T_{50}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 |
| $T_{51}$ | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 |
| $T_{52}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 |
| $T_{53}$ | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 |
| $T_{54}$ | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 |
| $T_{55}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 |
| $T_{56}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 |
| $T_{57}$ | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 |
| $T_{58}$ | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 |
| $T_{59}$ | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 |
| $T_{60}$ | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 |

Table :A8a: Extreme points of $SEP(6)$ : Tours

|  | $x_{12}$ | $x_{13}$ | $x_{23}$ | $x_{14}$ | $x_{24}$ | $x_{34}$ | $x_{15}$ | $x_{25}$ | $x_{35}$ | $x_{45}$ | $x_{16}$ | $x_{26}$ | $x_{36}$ | $x_{46}$ | $x_{56}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $NIP_1$ | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.0 |
| $NIP_2$ | 0.5 | 0.0 | 1.0 | 0.5 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 1.0 | 1.0 | 0.0 | 0.5 | 0.0 | 0.5 |
| $NIP_3$ | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 1.0 | 0.5 | 0.5 | 0.0 | 0.0 |
| $NIP_4$ | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 | 0.5 |
| $NIP_5$ | 0.5 | 0.0 | 1.0 | 0.0 | 0.0 | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 |
| $NIP_6$ | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 0.0 | 1.0 | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 |
| $NIP_7$ | 0.0 | 0.0 | 0.5 | 0.5 | 0.0 | 1.0 | 0.5 | 1.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.5 | 0.0 | 0.0 |
| $NIP_8$ | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 | 0.5 |
| $NIP_9$ | 0.5 | 0.0 | 0.0 | 0.0 | 1.0 | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 |
| $NIP_{10}$ | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.0 | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 |
| $NIP_{11}$ | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 1.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.0 |
| $NIP_{12}$ | 0.5 | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 | 0.0 | 1.0 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 0.0 | 0.5 |
| $NIP_{13}$ | 0.0 | 1.0 | 0.5 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 |
| $NIP_{14}$ | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 |
| $NIP_{15}$ | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.5 | 1.0 | 0.5 | 0.0 | 0.0 |
| $NIP_{16}$ | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 |
| $NIP_{17}$ | 0.0 | 0.0 | 0.5 | 0.5 | 0.0 | 1.0 | 1.0 | 0.5 | 0.5 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 |
| $NIP_{18}$ | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.5 | 1.0 | 0.5 | 0.0 | 0.0 |
| $NIP_{19}$ | 0.5 | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 1.0 | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 |
| $NIP_{20}$ | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 0.5 |
| $NIP_{21}$ | 0.5 | 1.0 | 0.0 | 0.0 | 0.0 | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 | 1.0 | 0.5 | 0.5 | 0.0 |
| $NIP_{22}$ | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 0.5 |
| $NIP_{23}$ | 0.5 | 0.0 | 0.0 | 1.0 | 0.0 | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 | 1.0 | 0.5 | 0.5 | 0.0 |
| $NIP_{24}$ | 0.5 | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 | 1.0 | 0.0 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 |
| $NIP_{25}$ | 0.5 | 0.0 | 0.0 | 1.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 |
| $NIP_{26}$ | 0.5 | 0.0 | 0.0 | 0.0 | 1.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 |
| $NIP_{27}$ | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 |
| $NIP_{28}$ | 1.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 |
| $NIP_{29}$ | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 |
| $NIP_{30}$ | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 |
| $NIP_{31}$ | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 |
| $NIP_{32}$ | 1.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 |
| $NIP_{33}$ | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 | 0.0 | 1.0 | 0.5 | 0.5 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 |
| $NIP_{34}$ | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 |
| $NIP_{35}$ | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 | 0.5 |
| $NIP_{36}$ | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 | 0.5 |
| $NIP_{37}$ | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.5 | 1.0 | 0.0 |
| $NIP_{38}$ | 0.0 | 1.0 | 0.5 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.5 | 1.0 | 0.0 |
| $NIP_{39}$ | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.5 | 0.0 | 0.5 | 1.0 | 0.0 |
| $NIP_{40}$ | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.0 | 0.5 | 1.0 | 0.0 |
| $NIP_{41}$ | 0.5 | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 |
| $NIP_{42}$ | 0.5 | 0.0 | 1.0 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 |
| $NIP_{43}$ | 0.5 | 1.0 | 0.0 | 0.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 |
| $NIP_{44}$ | 0.5 | 0.0 | 1.0 | 0.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 |
| $NIP_{45}$ | 1.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 |
| $NIP_{46}$ | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 0.0 | 1.0 | 0.5 |
| $NIP_{47}$ | 1.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 |
| $NIP_{48}$ | 0.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 | 0.0 | 0.0 | 1.0 | 0.5 |
| $NIP_{49}$ | 0.5 | 1.0 | 0.0 | 0.0 | 1.0 | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 |
| $NIP_{50}$ | 0.5 | 0.0 | 1.0 | 1.0 | 0.0 | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 0.0 | 0.0 | 0.5 | 0.5 | 1.0 |
| $NIP_{51}$ | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 |
| $NIP_{52}$ | 0.0 | 0.5 | 1.0 | 1.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 |
| $NIP_{53}$ | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 |
| $NIP_{54}$ | 0.0 | 1.0 | 0.5 | 0.5 | 1.0 | 0.0 | 0.0 | 0.5 | 0.5 | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 |
| $NIP_{55}$ | 1.0 | 0.0 | 0.5 | 0.5 | 0.0 | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.0 | 1.0 |
| $NIP_{56}$ | 0.0 | 1.0 | 0.5 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.0 | 1.0 |
| $NIP_{57}$ | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.0 | 0.5 | 0.0 | 1.0 |
| $NIP_{58}$ | 0.0 | 0.5 | 1.0 | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.0 | 0.5 | 0.5 | 0.0 | 0.5 | 0.0 | 1.0 |
| $NIP_{59}$ | 0.5 | 1.0 | 0.0 | 0.0 | 1.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 1.0 |
| $NIP_{60}$ | 0.5 | 0.0 | 1.0 | 1.0 | 0.0 | 0.5 | 0.0 | 0.0 | 0.5 | 0.5 | 0.5 | 0.5 | 0.0 | 0.0 | 1.0 |

Table:A8b:Extreme points of $SEP(6)$ : Non-integer points

For the fractional extreme point described in Section-6.4, we compute neighbours by the neighbourhood method. Table: A9 gives all the neighbours.

| Tour | $i_1i_2$ | $i_1i_3$ | $i_2i_3$ | $i_1i_4$ | $i_2i_4$ | $i_3i_4$ | $i_1i_5$ | $i_2i_5$ | $i_3i_5$ | $i_4i_5$ | $i_1i_6$ | $i_2i_6$ | $i_3i_6$ | $i_4i_6$ | $i_5i_6$ | $b_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{i_1}$ | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 4 |
| $T_{i_2}$ | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 4 |
| $T_{i_3}$ | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 4 |
| $T_{i_4}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 4 |
| $T_{i_5}$ | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 4 |
| $T_{i_6}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 4 |
| $T_{i_7}$ | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 4 |
| $T_{i_8}$ | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 4 |
| $T_{i_9}$ | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 4 |
| $NIP$ | 0.0 | 0.5 | 0.0 | 0.5 | 1.0 | 0.5 | 1.0 | 0.5 | 0.0 | 0.0 | 0.0 | 0.5 | 1.0 | 0.0 | 0.5 | 4.5 |

Table: A9 : Neighbours of non-integer point

We see all the neighbours are integer or tours. We compute cutting planes using the neighbourhood-method. There are two solutions. They are given in Table: A10.

| Ineq | $i_1i_2$ | $i_1i_3$ | $i_2i_3$ | $i_1i_4$ | $i_2i_4$ | $i_3i_4$ | $i_1i_5$ | $i_2i_5$ | $i_3i_5$ | $i_4i_5$ | $i_1i_6$ | $i_2i_6$ | $i_3i_6$ | $i_4i_6$ | $i_5i_6$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ineq:1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| Ineq:2 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |

Table : A10 : The cutting plane for removing Extreme point

Table A11 gives the neighbourhood structure of $SEP(6)$. $N_i$ is the neighbourhood set of fractional point $i$. We see that $\mid N_i \mid = 9$, $\forall i$, and each tour appears as neighbour for eactly 9 non-integer points.

**7-city polytopes**

$SEP(7)$: The constraint matrix for $SEP(7)$ is given in Table : A12. The neighbours for an extreme point in $F72$ described in section 6.4 are computed. They are given in Table A13. The cutting plane computed is given in Table: A14.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | 11 | 47 | 46 | 17 | 60 | 18 | 45 | 58 | 16 |
| $N_2$ | 18 | 12 | 60 | 59 | 47 | 11 | 24 | 32 | 3 |
| $N_3$ | 24 | 48 | 46 | 30 | 58 | 32 | 47 | 60 | 29 |
| $N_4$ | 32 | 28 | 58 | 56 | 46 | 24 | 11 | 18 | 1 |
| $N_5$ | 17 | 59 | 60 | 11 | 48 | 12 | 34 | 42 | 7 |
| $N_6$ | 12 | 18 | 48 | 47 | 59 | 17 | 43 | 54 | 13 |
| $N_7$ | 34 | 43 | 40 | 44 | 42 | 54 | 59 | 48 | 38 |
| $N_8$ | 54 | 52 | 42 | 36 | 43 | 34 | 17 | 12 | 5 |
| $N_9$ | 56 | 58 | 30 | 24 | 46 | 26 | 36 | 40 | 20 |
| $N_{10}$ | 26 | 32 | 46 | 45 | 30 | 56 | 44 | 52 | 27 |
| $N_{11}$ | 36 | 42 | 44 | 43 | 40 | 52 | 56 | 46 | 37 |
| $N_{12}$ | 52 | 54 | 40 | 34 | 44 | 36 | 30 | 26 | 19 |
| $N_{13}$ | 10 | 45 | 43 | 15 | 51 | 13 | 47 | 49 | 17 |
| $N_{14}$ | 9 | 19 | 43 | 45 | 55 | 15 | 48 | 50 | 18 |
| $N_{15}$ | 26 | 47 | 44 | 29 | 49 | 27 | 45 | 51 | 30 |
| $N_{16}$ | 23 | 27 | 44 | 47 | 29 | 57 | 46 | 53 | 32 |
| $N_{17}$ | 33 | 37 | 46 | 48 | 38 | 53 | 57 | 44 | 42 |
| $N_{18}$ | 35 | 48 | 38 | 46 | 37 | 50 | 55 | 49 | 40 |
| $N_{19}$ | 13 | 9 | 56 | 51 | 45 | 10 | 25 | 27 | 2 |
| $N_{20}$ | 27 | 23 | 57 | 49 | 47 | 25 | 10 | 13 | 4 |
| $N_{21}$ | 15 | 51 | 55 | 10 | 43 | 9 | 35 | 37 | 6 |
| $N_{22}$ | 53 | 50 | 37 | 33 | 45 | 35 | 15 | 9 | 8 |
| $N_{23}$ | 49 | 57 | 29 | 25 | 44 | 23 | 33 | 38 | 21 |
| $N_{24}$ | 50 | 53 | 36 | 35 | 48 | 33 | 29 | 23 | 22 |
| $N_{25}$ | 51 | 59 | 30 | 36 | 21 | 19 | 34 | 25 | 44 |
| $N_{26}$ | 55 | 60 | 29 | 40 | 20 | 23 | 36 | 24 | 45 |
| $N_{27}$ | 7 | 25 | 35 | 6 | 24 | 2 | 55 | 20 | 10 |
| $N_{28}$ | 3 | 2 | 35 | 25 | 34 | 6 | 51 | 19 | 9 |
| $N_{29}$ | 21 | 41 | 36 | 51 | 30 | 28 | 30 | 29 | 49 |
| $N_{30}$ | 22 | 31 | 29 | 55 | 40 | 39 | 35 | 26 | 50 |
| $N_{31}$ | 6 | 24 | 34 | 7 | 25 | 3 | 59 | 21 | 11 |
| $N_{32}$ | 2 | 3 | 34 | 24 | 35 | 7 | 60 | 22 | 12 |
| $N_{33}$ | 20 | 39 | 40 | 60 | 29 | 31 | 41 | 30 | 58 |
| $N_{34}$ | 19 | 28 | 30 | 59 | 39 | 41 | 40 | 31 | 54 |
| $N_{35}$ | 28 | 19 | 41 | 21 | 59 | 51 | 6 | 3 | 14 |
| $N_{36}$ | 31 | 22 | 39 | 20 | 60 | 58 | 7 | 2 | 16 |
| $N_{37}$ | 10 | 20 | 33 | 21 | 11 | 4 | 57 | 7 | 26 |
| $N_{38}$ | 8 | 39 | 37 | 49 | 17 | 14 | 41 | 15 | 51 |
| $N_{39}$ | 11 | 21 | 36 | 20 | 10 | 1 | 56 | 6 | 24 |
| $N_{40}$ | 7 | 41 | 42 | 58 | 15 | 16 | 39 | 17 | 60 |
| $N_{41}$ | 14 | 5 | 39 | 6 | 56 | 49 | 21 | 1 | 28 |
| $N_{42}$ | 16 | 8 | 41 | 7 | 58 | 57 | 20 | 4 | 31 |
| $N_{43}$ | 17 | 49 | 56 | 37 | 6 | 5 | 36 | 10 | 43 |
| $N_{44}$ | 15 | 57 | 58 | 42 | 7 | 8 | 33 | 11 | 48 |
| $N_{45}$ | 4 | 1 | 33 | 10 | 36 | 21 | 49 | 5 | 23 |
| $N_{46}$ | 8 | 16 | 57 | 15 | 42 | 41 | 37 | 14 | 53 |
| $N_{47}$ | 1 | 4 | 38 | 11 | 33 | 20 | 58 | 8 | 26 |
| $N_{48}$ | 5 | 14 | 56 | 17 | 37 | 39 | 42 | 16 | 52 |
| $N_{49}$ | 16 | 28 | 1 | 2 | 52 | 50 | 22 | 5 | 39 |
| $N_{50}$ | 14 | 31 | 4 | 3 | 54 | 53 | 19 | 8 | 41 |
| $N_{51}$ | 8 | 23 | 5 | 9 | 26 | 22 | 50 | 1 | 33 |
| $N_{52}$ | 4 | 53 | 14 | 13 | 32 | 31 | 27 | 16 | 57 |
| $N_{53}$ | 5 | 26 | 8 | 12 | 23 | 19 | 54 | 4 | 36 |
| $N_{54}$ | 1 | 52 | 16 | 18 | 27 | 28 | 32 | 14 | 56 |
| $N_{55}$ | 9 | 22 | 19 | 23 | 12 | 8 | 53 | 3 | 35 |
| $N_{56}$ | 2 | 50 | 28 | 27 | 18 | 16 | 31 | 13 | 55 |
| $N_{57}$ | 12 | 19 | 22 | 26 | 9 | 5 | 52 | 2 | 34 |
| $N_{58}$ | 3 | 54 | 31 | 32 | 13 | 14 | 26 | 18 | 59 |
| $N_{59}$ | 18 | 27 | 50 | 52 | 2 | 1 | 26 | 9 | 45 |
| $N_{60}$ | 13 | 32 | 53 | 54 | 3 | 4 | 23 | 12 | 47 |

Table : A11: Neighbourhood sets of non-integer points

| $x_{ij}$ | $b_i$ |
|---|---|
| 1 1 0 1 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 0 | 2 |
| 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 | 2 |
| 0 1 1 0 0 1 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 | 2 |
| 0 0 0 1 1 1 0 0 0 1 0 0 0 1 0 0 0 0 1 0 0 | 2 |
| 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 0 0 0 0 1 0 | 2 |
| 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 1 | 2 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 | 2 |
| 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 | 1 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 | 1 |
| 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 1 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 1 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 | 2 |
| 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 0 1 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 0 1 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 | 2 |
| 0 0 1 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 0 0 1 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 0 0 1 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 | 2 |
| 0 0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 0 0 0 1 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 | 2 |
| 0 0 0 0 1 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 0 0 0 0 1 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 | 2 |
| 0 0 0 0 0 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 | 2 |
| 0 0 0 0 0 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 | 2 |
| 0 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 0 | 2 |
| 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 0 0 | 2 |
| 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 0 0 0 | 2 |
| 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 0 0 0 0 0 0 | 2 |
| 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 | 2 |
| 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 | 2 |
| 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 | 2 |
| 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 | 2 |
| 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 | 2 |
| 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 | 2 |
| 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 1 0 | 2 |
| 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 1 0 | 2 |
| 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 1 0 | 2 |
| 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 1 0 | 2 |
| 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1 | 2 |
| 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 0 1 | 2 |
| 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 1 | 2 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1 | 2 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 | 2 |

Table : A12 : Constraint matrix for $SEP(7)$

| | 12 | 13 | 23 | 14 | 24 | 34 | 15 | 25 | 35 | 45 | 16 | 26 | 36 | 46 | 56 | 17 | 27 | 37 | 47 | 57 | 67 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pt | 1 | 0.5 | 0.5 | 0.5 | 0 | 0 | 0 | 0 | 0.5 | 1 | 0 | 0.5 | 0 | 0.5 | 0 | 0 | 0 | 0.5 | 0 | 0.5 | 1 |
| N1 | 1 | 1 | 0 | 0 | 0.5 | 0 | 0 | 0 | 0.5 | 1 | 0 | 0.5 | 0 | 0.5 | 0 | 0 | 0 | 0.5 | 0 | 0.5 | 1 |
| N2 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| N3 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| N4 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| N5 | 1 | 0 | 1 | 0.5 | 0 | 0 | 0 | 0 | 0.5 | 1 | 0.5 | 0 | 0 | 0.5 | 0 | 0 | 0 | 0.5 | 0 | 0.5 | 1 |
| N6 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| N7 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| N8 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| N9 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| N10 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| N11 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| N12 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| N13 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| N14 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |

Table : A13 : Neighbourhood of extreme point in $F72$

| 12 | 13 | 23 | 14 | 24 | 34 | 15 | 25 | 35 | 45 | 16 | 26 | 36 | 46 | 56 | 17 | 27 | 37 | 47 | 57 | 67 | RHS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 5 |

Table : A14 : Cutting plane for the extreme point in $F72$

**Extreme points of $DFJ$ and $ASMI$ polytopes**

| $x_{12}$ | $x_{21}$ | $x_{13}$ | $x_{31}$ | $x_{23}$ | $x_{32}$ | RHS |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 |

Table: A15a : Constraint matrix for $DFJ(3)$

| $x_{12}$ | $x_{21}$ | $x_{13}$ | $x_{31}$ | $x_{23}$ | $x_{32}$ |
|---|---|---|---|---|---|
| .0 | 1.0 | 1.0 | .0 | .0 | 1.0 |
| 1.0 | .0 | .0 | 1.0 | 1.0 | .0 |

Table : A15b : Extreme points of $DFJ(3)$

| $x_{12}$ | $x_{21}$ | $x_{13}$ | $x_{31}$ | $x_{23}$ | $x_{32}$ | $x_{14}$ | $x_{41}$ | $x_{24}$ | $x_{42}$ | $x_{34}$ | $x_{43}$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 2 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 2 |

Table : A16a : constraint matrix for $DFJ(4)$

| $x_{12}$ | $x_{21}$ | $x_{13}$ | $x_{31}$ | $x_{23}$ | $x_{32}$ | $x_{14}$ | $x_{41}$ | $x_{24}$ | $x_{42}$ | $x_{34}$ | $x_{43}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | .0 | .0 | 1.0 | .0 | .0 | .0 | .0 | 1.0 | .0 | .0 | 1.0 |
| .0 | 1.0 | 1.0 | .0 | .0 | .0 | .0 | .0 | .0 | 1.0 | 1.0 | .0 |
| .0 | 1.0 | .0 | .0 | .0 | 1.0 | 1.0 | .0 | .0 | .0 | .0 | 1.0 |
| 1.0 | .0 | .0 | .0 | 1.0 | .0 | .0 | 1.0 | .0 | .0 | 1.0 | .0 |
| .0 | .0 | .0 | 1.0 | 1.0 | .0 | 1.0 | .0 | .0 | 1.0 | .0 | .0 |
| .0 | .0 | 1.0 | .0 | .0 | 1.0 | .0 | 1.0 | 1.0 | .0 | .0 | .0 |
| .5 | .5 | .0 | .5 | .5 | .0 | .5 | .0 | .0 | .5 | .5 | .5 |
| .5 | .5 | .5 | .0 | .0 | .5 | .0 | .5 | .5 | .0 | .5 | .5 |
| .0 | .5 | .5 | .5 | .0 | .5 | .5 | .0 | .5 | .5 | .0 | .5 |
| .5 | .0 | .5 | .5 | .5 | .0 | .0 | .5 | .5 | .5 | .5 | .0 |
| .5 | .0 | .0 | .5 | .5 | .5 | .5 | .5 | .5 | .0 | .0 | .5 |
| .0 | .5 | .5 | .0 | .5 | .5 | .5 | .5 | .0 | .5 | .5 | .0 |

Table : A16b : Extreme points of $DFJ(4)$

| $x_{123}$ | $x_{213}$ | RHS |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |

Table : A17 : Constraint matix for $ASMI(3)$

| $x_{123}$ | $x_{213}$ | $x_{124}$ | $x_{214}$ | $x_{134}$ | $x_{314}$ | $x_{234}$ | $x_{324}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| -1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | -1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | -1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| -1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Table : A18a: Constraint matrix for $ASMI(4)$

| $x_{123}$ | $x_{213}$ | $x_{124}$ | $x_{214}$ | $x_{134}$ | $x_{314}$ | $x_{234}$ | $x_{324}$ |
|---|---|---|---|---|---|---|---|
| 1.0 | .0 | .0 | .0 | .0 | .0 | .0 | 1.0 |
| .0 | 1.0 | .0 | .0 | .0 | .0 | 1.0 | .0 |
| .0 | 1.0 | .0 | .0 | .0 | 1.0 | .0 | .0 |
| .0 | 1.0 | 1.0 | .0 | .0 | .0 | .0 | .0 |
| 1.0 | .0 | .0 | .0 | 1.0 | .0 | .0 | .0 |
| 1.0 | .0 | .0 | 1.0 | .0 | .0 | .0 | .0 |
| .5 | .5 | .0 | .5 | .5 | .0 | .0 | .0 |
| .5 | .5 | .5 | .0 | .0 | .5 | .0 | .0 |
| .5 | .5 | .0 | .0 | .0 | .5 | .5 | .0 |
| .5 | .5 | .5 | .0 | .0 | .0 | .5 | .0 |
| .7 | .3 | .3 | .0 | .0 | .3 | .3 | .0 |
| .5 | .5 | .0 | .0 | .5 | .0 | .0 | .5 |
| .5 | .5 | .0 | .5 | .0 | .0 | .0 | .5 |
| .3 | .7 | .0 | .3 | .3 | .0 | .0 | .3 |

Table : A18b :Extreme points of $ASMI(4)$

| $x_{ij3}$ | $x_{ij4}$ | $x_{ij5}$ | RHS |
|---|---|---|---|
| 1 1 | 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 | 1 1 1 1 1 1 | 0 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 0 | 0 0 0 0 0 0 | 1 1 1 1 1 1 1 1 1 1 1 1 | 1 |
| 1 0 | 1 0 0 0 0 0 | 1 0 0 0 0 0 0 0 0 0 0 0 | 1 |
| 0 1 | 0 1 0 0 0 0 | 0 1 0 0 0 0 0 0 0 0 0 0 | 0 |
| -1 0 | 0 0 1 0 0 0 | 0 0 1 0 0 0 0 0 0 0 0 0 | 0 |
| 0 -1 | 0 0 0 1 0 0 | 0 0 0 1 0 0 0 0 0 0 0 0 | 0 |
| 0 -1 | 0 0 0 0 1 0 | 0 0 0 0 1 0 0 0 0 0 0 0 | 0 |
| -1 0 | 0 0 0 0 0 1 | 0 0 0 0 0 1 0 0 0 0 0 0 | 0 |
| 0 0 | -1 0 -1 0 0 0 | 0 0 0 0 0 0 1 0 0 0 0 0 | 0 |
| 0 0 | 0 -1 0 -1 0 0 | 0 0 0 0 0 0 0 1 0 0 0 0 | 0 |
| 0 0 | 0 -1 0 0 -1 0 | 0 0 0 0 0 0 0 0 1 0 0 0 | 0 |
| 0 0 | -1 0 0 0 0 -1 | 0 0 0 0 0 0 0 0 0 1 0 0 | 0 |
| 0 0 | 0 0 0 -1 0 -1 | 0 0 0 0 0 0 0 0 0 0 1 0 | 0 |
| 0 0 | 0 0 -1 0 -1 0 | 0 0 0 0 0 0 0 0 0 0 0 1 | 0 |

Table : A19 : Constraint matrix for $ASMI(5)$

| n | Formulation | Inequalities | Extreme points | Remarks |
|---|---|---|---|---|
| 3 | DFJ | 12 | 2 | All tours |
| | ASMI | 3 | 2 | All tours |
| 4 | DFJ | 30 | 12 | 6 tours<br>6 fractional |
| | ASMI | 8 | 14 | 6 tours<br>8 fractional<br>2 written as convex combination of tours |
| 5 | DFJ | 55 | 384 | 24 tours<br>360 fractional |
| | ASMI | 23 | 1150 | 24 tours |

Table A20 : Summary of extreme points for small $ASMI$ and $DFJ$ polytopes

# BIBLIOGRAPHY


1. R. K. Ahuja, T. L. Magnanti and J. B. Orlin, *Network Flows : Theory, Algorithms and Applications*, Prentice Hall, (1993)

2. D. Applegate, R. Bixby, V. Chvátal and W. Cook, Projection Cuts in the TSP, *To appear in Computational Combinatorial Optimisation*, (1999)

3. T.S. Arthanari, On the Traveling Salesman Problem, Presented at the XI Symposium on Mathematical Programming held at Bonn, West Germany, (1982).

4. T.S. Arthanari and Y. Dodge,*Mathematical Programming in Statistics*, John Wiley and Sons, (1981)

5. T.S. Arthanari and M. Usha, An Alternate Formulation of the Symmetric Traveling Salesman Problem and its Properties, Discrete Applied Mathematics 98, 173-190, (2000)

6. T.S. Arthanari and M. Usha, Equivalence of $\mathcal{U}(n)$ and $SEP(n)$ and its implications, *Preprint*, (1997) *communicated for publication.*

7. T.S. Arthanari and M. Usha, On the Equivalence of the Multistage-insertion and Cycle-shrink Formulations of the Symmetric Traveling Salesman Problem,(1998) *To appear in OR Letters*

8. T.S. Arthanari, On the Symmetric Traveling Salesman Problem and Separation,*Preprint*, (1998)

9. A. Bachem and M. Grötschel, New Aspects of Polyhedral Theory in Modern Applied Mathematics,ed. B. Korte, (1982)

10. M. S. Bazaara, J. J. Jarvis and H.D. Sherali, *Linear Programming and Network Flows*, John Wiley and Sons, (1990)

11. R.E.Bellman, Dynamic Programming Treatment of the Traveling Salesman Problem. J.Assoc.Comput. Mach. 9, 61-63 (1962)

12. M. Bellmore and G. L. Nemhauser, The Traveling Salesman Problem; A Survey, Operations Research, 6, 538-556 (1968)

13. C. Berge, *Graphs and Hypergraphs*, North -Holland, Amsterdam, (1973)

14. J. Bondy and U. Murthy, *Graph Theory and Applications*, North-Holland, New York, (1985)

15. S.C. Boyd and W. R. Pulleyblank, Optimizing over the Subtour Polytope of the Traveling Salesman Problem, *Math Prog*, 49, 163-187(1990)

16. S. Boyd and W.H. Cunningham, Small Traveling Salesman Polytopes, Math. Oper. Res. 16, 259-271 (1991)

17. S. Boyd,W. H. Cunningham, M. Queyranne, Y. Wang, Ladders for the Traveling Salesmen, SIAM J. Optim. 5, 408-420 (1995)

18. R. E. Burkard, V. G. Deineko, R. van Dal, J. A. A. van der Veen and G. J. Woeginger, Well-solvable Cases of the Traveling Salesman Problem : A Survey, SIAM Rev, 40, No. 3, 496-546 (1998)

19. R. Carr, Polynomial Separation Procedures and Facet Determination for Inequalities of the Traveling Salesman Polytope, Ph.D Thesis, Carnegie Mellon University,(1995)

20. R. Carr, Separating Over Classes of TSP Inequalities Defined by 0 Node-Lifting in Polynomial Time, Preprint, (1996)

21. T. Christof, M. Jünger and G. Reinelt, A Complete Description of the Traveling Salesman Problem on 8 nodes, Oper. Res. Letters. 10, 497-500 (1991)

22. T. Christof and G. Reinelt, Combinatorial Optimisation and Small Polytopes, TOP, 4, 1-53 (1996)

23. N. Christofides, *Graph Theory: An Algortihmic Approach*, Academic Press(1975a)

24. V. Chavátal, Edmonds Polytopes and Weakly Hamiltonian Graphs. Math. Program. 5,29-40 (1973)

25. G. Cornuéjols, J. Fonlupt, D. Naddef, The Traveling Salesman Problem on a Graph and some Related Polyhedra, Math Program. 33, 1-27 (1985)

26. G.Dantzig, D. Fulkerson, and S. Johnson, Solution of a Large Scale Traveling Salesman Problem, Operations Research 2, 393-410 (1954)

27. J. Edmonds, Maximum Matching and a Polyhedron with 0,1 vertices, J. Res. Nat. Bur. Stand. B 69,125-130 (1965)

28. R. Euler and H. Le Verge, Complete Linear Descriptions of Small Asymmetric Traveling Salesman Polytopes, Disc. Appl. Math. 62, 193-208 (1995)

29. B. Fleischmann, Cutting Planes for the Symmetric Traveling Salesman Problem, Technical Report, Universität Hamburg (1987)

30. K.R.Fox, B.Gavish and S.C.Graves, A $n$-constraint Formulation of the (Time-Dependent) Traveling Salesman Problem, Operations Research 28, 1018-1021 (1980)

31. K. R. Fox, *Production Scheduling on Parallel Lines with Dependencies*, Ph. D. Thesis, John Hopkins University (1973)

32. M. R. Garey and D. S. Johnson,*Computer and Intractibility: A Guide to the Theory of NP-completeness*, Freeman (1979)

33. B. Gavish and S.C. Graves, The Traveling Salesman Problem and Related Problems, Woking Paper GR- 078-78, Operations Research Center, Massachusetts Institute of technology (1978)

34. R.E. Gomory and T. C. Hu, Multiterminal Network Flows, Journal of SIAM, 9, 551-570 (1961)

35. L.Gouveia and S.Voß, A Classification of Formulations for the (time-dependent) Traveling Salesman Problem, European Jour.of Oper. Res. 83, 69-82 (1995)

36. M. Grötschel and O. Holland, A Cutting Plane Algorithn for the Linear Ordering Problem, Oper. Res. 32, 1195-1220 (1987)

37. M. Grötschel and Padberg, On the Symmetric Traveling Salesman Problem I : inequalities, Math Program. 16, 265-280(1979a)

38. M. Grötschel and Padberg, On the Symmetric Traveling Salesman Problem II : Lifting theorems and facets, Math Program. 16, 281-302(1979b)

39. M. Grötschel and Padberg, Polyhedral Theory in E.Lawler et. al eds., *The Traveling Salesman Problem*, John Wiley and Sons(1985)

40. M. Grötschel and Pulleyblank, Clique-tree Inequalities and the Symmetric Traveling Salesman Problem, Math. Oper. Res. 11, 537-569 (1986)

41. G. Gutin, Exponential Neighbourhood Local Search for the Traveling Salesman Problem, *Preprint*, (1997)

42. M.Held and R.M.Karp A Dynamic Programming Approach to sequencing problems. SIAM Appl. Math. 10 196-210,(1962)

43. M. Held and R. Karp, The Traveling Salesman Problem and Minimum Spanning Trees, Operations Research, 18, 1138-1162 (1970)

44. M. Held and R. Karp, The Traveling Salesman Problem and Minimum Spanning Trees, part II, Math. Prog., 6, 6-25 (1970)

45. M. Jünger et.al., The Traveling Salesman Problem, in Handbooks in OR & MS Vol. 7, Elseiver Science B.V.(1995).

46. M. Jünger et al., The Traveling Salesman Problem, in Annotated Bibliographies in Combinatorial Optimisation, eds. M.Dell' Amico, I. Maffioli, Martello, 199-221 (1997)

47. N. Karmarkar, A Polynomial Time Algortihm for Linear Programming, Combinatorica, 4, 375-395 (1984)

48. R. M. Karp, Reducibility among Combinatorial Problems, in Complexity of Computer Computations, R. E. Miller and J. W. Thatcher eds. Plenum Press, 85-103(1972)

49. L. G. Khacian, A Polynomial Algorithm in Linear Programming, Soviet Mathematics Doklady 20, 191-194 (1979)

50. R. Kipp Martin, Using Separation Algorithms to generate Mixed Integer Model Reformulations, Oper. Res. Letters,10,119-128(1991)

51. A. Langevin, F. Soumis and J. Desrosiers,Classification of Traveling Salesman Problem Formulations,OR Letters, 9, 127-132 (1990)

52. E. Lawler, et.al.(eds).,*The Traveling Salesman Problem*, Wiley, New York,(1985).

53. C. Miller, A.Tucker and R. Zemlin, Integer Programming Formulation and Traveling Salesman PRoblems, Jour Assoc. Comput. Mach. , 7, 326-329 (1960)

54. D. Naddef, Handles and Teeth is the Symmetric Traveling Salesman Polytope, in W. Cook and P. D. Seymour (eds.), *Polyhedral Combinatorics*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 1. A. M. S. 61-74 (1990)

55. D. Naddef and G. Rinaldi, The Symmetric Traveling Salesman Polytope: New Facets from the Graphical Relaxation, Tech. Report 248, IASI-CNR, Rome (1988)

56. D. Naddef and G. Rinaldi, The Crown Inequalties for the Symmetric traveling Salesman Polytope ,Math. of Oper Res, 17,308-326(1992)

57. D. Naddef, The Binested Inequalities for the Symmetric Traveling Salesman Polytope. Math. Oper. Res. 17, 882-900 (1992).

58. D. Naddef and S. Thienel, Efficient Separation Routines for the Symmetric Traveling Salesman Problem I : General Tools and Comb Separation, RR ID-IMAG Grenoble, downloadable at:
http://www-apache.imag.fr/naddef/files/separationI.ps (1999).

59. D. Naddef and S. Thienel, Efficient Separation Routines for the Symmetric Traveling Salesman Problem II : Separating Multi handle Inequalities, RR ID-IMAG Grenoble, downloadable at: http://www-apache.imag.fr/naddef/files/separationII.ps (1999).

60. G. L. Nemhauser and L. A. Wolsey,*Integer and Combinatorial Optimisation* ,John Wiley & Sons, Chichester, (1988)

61. R. Z. Norman, On the Convex Polyhedra of the Symmetric Traveling Salesman Problem(abstract), Bull. of Amer. Math. Soc. 61, 559 (1955)

62. M. Padberg and M.Gr ötschel, Polyhedral Computations in E. Lawler, et.al.(eds).,*The Traveling Salesman Problem*, Wiley, New York,(1985).

63. M.W. Padberg and S. Hong, On the Symmetric Traveling Salesman Problem : A Combinatorial Study, *Math Prog. Study* 12, 78-107(1980)

64. M. Padberg and M. R. Rao, Odd Minimum Cut Sets and b-matchings, Math. Oper. Res. 7, 67-80 (1982)

65. M. Padberg and G. Rinaldi, Facet Identification for the Symmetric Traveling Salesman Problem, Math. Program. 47, 219-257(1990)

66. M. W. Padberg and L.A. Wolsey, Trees and Cuts,*Annals of Discrete Mathematics*, 17, 511-517(1983)

67. M. Padberg, and T. Y. Sung, An Analytical Comparison of Different Formulations of the Traveling Salesman Problem, Math Prog 52, 315-357(1991)

68. C. H. Papadimitriou and K. Steiglitz,*Combinatorial Optimisation: Algorithms and Complexity*, Prentice-Hall (1982)

69. J. C. Picard and M. Queyranne, The Time-dependent Traveling Salesman Problem and its Application to the Tardiness Problem in One Machine Scheduling, Oper. Res. 26, 86 -110 (1978)

70. W. R. Pulleyblank, Polyhedral Combinatorics, in Mathematical Programming The State of the Art, eds. A. Bachem, M. Grötschel, B. Korte,Springer Verlag(1982)

71. W. R. Pulleyblank ed.,*Progress in Combinatorial Optimisation*, Academic Press (1984)

72. G. Reinelt, *The Traveling Salesman Problem-Computational Solutions*, Lecture Notes in Computer Science,Springer (1994)

73. A. Schrijver, Polyhedral Combinatorics, R. Graham et. al eds. Handbook of Combinatorics, Elseiver Science, 1649-1704( 1995)

74. D. B. Shmoys and É.Tardos, Computational Complexity in R. Graham et. al eds. Handbook of Combinatorics, Elseiver Science, 1599-1643( 1995)

75. J. Stoer and C. Witzgall, *Convexity and Optimisation in Finite Dimensions*, Springer, Berlin(1970)

76. M. Usha, New Inequalities for the $MI$ polytope,*Private communication*, (1998)

77. H. Weyl, Elementare Theorie der Konvexin Polyeder, commentarii Mathematici Helvetici 7, 509-533 (1935)

78. R. T. Wong, Integer Programming Formulations of the Traveling Salesman Problem, Proc. of the IEEE International conference of Circuits and Computers, 149-152 (1980)