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A SIMPLE APPROACH TO ANALYSIS OF CERTAIN NON-ORTHOGONAL DATA

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SUMMARY. An alternative and simple approach to analysis of certain types of non-orthogonal data is put forward. The emphasis is on identifying first, certain "non-orthogonal" treatment contrasts, out of a set of $(v-1)$ mutually orthogonal treatment contrasts. These contrasts are then estimated and sum of squares due to them are obtained. Sum of squares due to the treatment contrasts which are free from block effects (i.e., orthogonal to all the block contrasts) are obtained in the usual way. The two sum of squares, then add to the adjusted treatment sum of squares. The method of analysis suggested in this note is, in many situations, particularly preferable to the conventional missing plot technique for randomised block designs. An illustration for p missing observations affecting p treatments in p different bocks in a randomised block design (RBD) is given at the end.

1. Introduction

The method of fitting constants for the analysis of non-orthogonal data, does not take into account the amount or degree of non-orthogonality. All non-orthogonal data are put on the same footing and are analysed in the same general way. But, as we observe in this paper, if some data are non-orthogonal because of one or two observations missing leading to a few treatment contrasts being 'non-orthogonal' to block contrasts, then these can be analysed with more ease and simplicity, in a different way. Thus if the degree or amount of non-orthogonality is not much, the approach suggested herein will lead to a simpler analysis of non-orthogonal data. The method may be preferable in many situations, particularly to the conventional missing plot technique for the randomised block designs. Throughout this note we consider only connected block designs.

2. The basic result

Consider the analysis of a block design under the usual fixed effect additive model. The reduced normal equations for the treatment effects are given by

$$C\hat{\tau} = Q.$$

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Let
$$\begin{bmatrix} \frac{1}{\sqrt{v}}\mathbf{1}' \\ \cdots\cdots\cdots \\ P_{\overline{v-1}\times v} \end{bmatrix}$$
be an orthogonal matrix, where $\mathbf{1}'$ is a row vector of unities of appropriate order.

Clearly, $\boldsymbol{PP'} = I_{v-1}$, $\boldsymbol{P'P} = \boldsymbol{I}_v - v^{-1}\boldsymbol{J}_v$. Then one can easily work out (see for example Kiefer, 1958).

$$\boldsymbol{P}\cdot\hat{\boldsymbol{\tau}} = (\boldsymbol{PCP'})^{-1}\boldsymbol{PQ}$$

$$\Rightarrow \hat{\boldsymbol{\tau}} = \boldsymbol{P'}(\boldsymbol{PCP'})^{-1}\boldsymbol{PQ}[\because \mathbf{1}'\hat{\boldsymbol{\tau}} = 0]$$

$$\Rightarrow \boldsymbol{P'}(\boldsymbol{PCP'})^{-1}\boldsymbol{P} \text{ is a } g\text{-inverse of } \boldsymbol{C} \text{ (denoted by } \overline{\boldsymbol{C}}).$$

$$\text{S.S. due to treatment (adj.)} = \boldsymbol{Q'\overline{C}Q}$$
$$= \boldsymbol{Q'P'}(\boldsymbol{PCP'})^{-1}\boldsymbol{PQ}.$$

(The above S.S. is invariant w.r.t. any choice of P).

Let us now choose $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{L}_{t\times v} \\ \cdots\cdots\cdots \\ \boldsymbol{M}_{\overline{v-1-t}\times v} \end{bmatrix}$ in such a manner that

(i) $$\boldsymbol{LN} = 0$$

(ii) $$\boldsymbol{Lr^{\delta}M'} = 0$$

where $r^{\delta} = \text{Diag}(r_1, \ldots, r_v)$.

Then one can easily verify

$$(\boldsymbol{PCP'})^{-1} = \begin{bmatrix} (\boldsymbol{Lr^{\delta}L'})^{-1} & \vdots & \mathbf{0} \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ \mathbf{0} & \vdots & (\boldsymbol{MCM'})^{-1} \end{bmatrix}$$

where
$$\boldsymbol{C} = \boldsymbol{r^{\delta}} - \boldsymbol{Nk^{-\delta}N'}$$
$$\boldsymbol{k^{-\delta}} = \text{Diag}(k_1^{-1}, \ldots, k_b^{-1}).$$

Therefore S.S. due to the treatments (adj.)

$$= \boldsymbol{Q'P'}(\boldsymbol{PCP'})^{-1}\boldsymbol{PQ}$$

$$= \boldsymbol{T'L'}(\boldsymbol{Lr^{\delta}L'})^{-1}\boldsymbol{LT} + \boldsymbol{Q'M'}(\boldsymbol{MCM'})^{-1}\boldsymbol{MQ}$$

$$= \text{S.S. due to contrasts } \{\boldsymbol{L\tau}\} + \text{S.S. due to contrasts } \{\boldsymbol{M\tau}\}.$$

We can call contrasts $\{\boldsymbol{L\tau}\}$ as "orthogonal" and those of $\{\boldsymbol{M\tau}\}$ as "non-orthogonal".

In some situations, as considered in this note, it is possible to choose $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{L} \\ \boldsymbol{M} \end{bmatrix}$ in such a manner that

$$\text{(c)} \quad \begin{cases} \text{(i)} & \boldsymbol{LN} = 0, \\ \text{(ii)} & \boldsymbol{Lr^{\delta\prime}L} \text{ is diagonal}, \\ \text{(iii)} & \boldsymbol{Lr^{\delta}M'} = 0. \end{cases}$$

Then, S.S. due to treatments (adj.) = S.S. due to the "orthogonal" treatment contrasts calculated in the usual manner (as in orthogonal-design)

$$+\boldsymbol{Q'M'}(\boldsymbol{MCM'})^{-1}\boldsymbol{MQ}.$$

Thus the main computational problem reduces to obtaining $(\boldsymbol{MCM'})^{-1}$ only. In some special cases (one given here), $\boldsymbol{MCM'}$ also turns out to be diagonal and hence its inverse is obtained easily. A formula for $V(\boldsymbol{l'}\hat{\boldsymbol{\tau}})$ of an estimable function $\boldsymbol{l'\tau}$ in a block design satisfying the conditions (c) is clearly given by

$$V(\boldsymbol{l'}\hat{\boldsymbol{\tau}}) = \boldsymbol{\delta}' \left[\begin{array}{c:c} (\boldsymbol{Lr^{\delta}L'})^{-1} & \boldsymbol{0} \\ \hdashline \boldsymbol{0} & (\boldsymbol{MCM'})^{-1} \end{array} \right] \boldsymbol{\delta}\sigma^2,$$

where $\boldsymbol{\delta} = \boldsymbol{Pl}$.

3. Missing observations in RBD and choice of L, M

3.1. Consider a randomised block design with v treatments in r blocks. Suppose m observations affecting p treatments are missing. Then with the $(v-p)$ unaffected treatments, we can write down $(v-p-1)$ mutually orthonormal contrasts which will belong to our L. We also add to L some more contrasts, if possible, involving the 'p' affected treatments, inclusion of which will not affect the conditions (c). The remaining mutually orthonormal contrasts involving the p affected treatments and $(v-p)$ unaffected treatments form our M. Thus our objective is to add as many contrasts as possible to L which will satisfy the conditions (c) and the remaining contrasts will form the matrix M.

Consider for example the following design.

block	treatments					
1	1	2	3	4	5	6
2	3	2	1	4	6	5*
3	6	4	1*	3*	2	5
4	5	2	1	4	3	6

*indicates the missing plots.

Here we can choose $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{L} \\ \cdots \\ \boldsymbol{M} \end{bmatrix}$ as follows :

$$\boldsymbol{L} = \begin{array}{c} \text{treatments} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\boldsymbol{M} = \begin{array}{c} \text{treatments} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \end{array}$$

It is easily verified that this $\boldsymbol{P}$ satisfies the conditions (c), and that

$$\boldsymbol{MCM'} = \frac{1}{10} \begin{bmatrix} 27 & -1 \\ -1 & 33 \end{bmatrix}$$

Thus S.S. due to treatment (adj.)

$$= \left[\frac{T_2^2+T_4^2+T_6^2}{4} - \frac{(T_2+T_4+T_6)^2}{12} \right]$$

$$+ \left[\frac{T_1^2+T_3^2}{3} - \frac{(T_1+T_3)^2}{6} \right] + \boldsymbol{Q'M'(MCM')^{-1}MQ}.$$

3.2. We shall now show that sometimes $\boldsymbol{MCM'}$ turns out to be diagonal and hence its inverse is obtained easily. Consider for example a RBD for v treatments in r blocks with p missing observations affecting p treatments in p different blocks. Let, without loss of generality, the treatment i be missing in block i, $i = 1, 2, \ldots, p$.

Here, we take

$$
\begin{array}{c@{}c}
 & \text{treatments} \\
 & \begin{array}{cc} 1\,2\ldots p & \qquad\qquad p+1\ldots v \end{array} \\
M_{p\times v} = & \left[\begin{array}{c:c}
\begin{array}{c}(p-1)\text{ mutually}\\ \text{orthonormal contrasts}\end{array} & 0 \\
\hdashline
\sqrt{\dfrac{v-p}{vp}}, \ldots, \sqrt{\dfrac{v-p}{vp}} & -\sqrt{\dfrac{p}{v(v-p)}}, \ldots, -\sqrt{\dfrac{p}{v(v-p)}}
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c@{}c}
 & \text{treatments} \\
 & \begin{array}{cc} 1\,2\ldots p & \qquad p+1\ldots v \end{array} \\
L_{\overline{v-p-1}\times v} = & \left[\begin{array}{c:l}
0 & \begin{array}{l}(v-p-1)\\ \text{mutually orthonormal}\\ \text{contrasts.}\end{array}
\end{array}\right]
\end{array}
$$

This $P = \begin{bmatrix} L \\ \ldots \\ M \end{bmatrix}$ clearly satisfies the conditions (c).

One can now easily see that

$$
MCM' = (v-1)^{-1}\left[\begin{array}{c:c}
\{(v-1)(r-1)-1\}I_{p-1} & \mathbf{0} \\
\hdashline
\mathbf{0}' & (v-1)(r-1)+p-1
\end{array}\right]
$$

Reference

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