# ENUMERATION OF NON-ISOMORPHIC SOLUTIONS OF BALANCED INCOMPLETE BLOCK DESIGNS.

By HARI KINKAR NANDI
*Statistical Laboratory, Calcutta.*

### INTRODUCTION

Two solutions for a balanced incomplete block design with parameters $v, b, r, k, \lambda$ are said to be isomorphic if they are identical except for the renaming of blocks and varieties. The problem of the enumeration of all possible non-isomorphic solutions has been discussed by Fisher (1940), Cox (1940) and recently by Q. M. Husain (1945) who has tried his method on certain symmetrical designs only (i.e. $v = b$, $r = k$, $\lambda = 2$). A variant of that method having general application has been applied in this paper to the unsymmetrical designs : (A) $v = 6$, $b = 10$, $r = 5$, $k = 3$, $\lambda = 2$, and (B) $v = 10$, $b = 15$, $r = 6$, $k = 4$, $\lambda = 2$, which are in fact the designs obtained by block section from those considered by Husain.

In the symmetrical design, any two blocks have just $\lambda$ treatments in common which helps in the building up of a design step by step. In the unsymmetrical cases, we require the frequency-distribution of the number of treatments common to a particular block (called the initial block) and the other $(b-1)$ blocks. A general solution has been given in section 2 of the paper from which it is seen that the frequency distribution is unique for designs with $\lambda=1$, and also for those obtained by block-section from symmetrical designs with $\lambda=2$.

The actual enumeration is given in section 3 ; and it has been found that (A) has only one solution with which the solutions given by Fisher and Yates (Tables, 1938), and Bose (1939) are isomorphic, while (B) has three non-isomorphic solutions given in this paper as $[\alpha_{11}]$, $[\alpha_3]$, $[\beta_{21}]$ ; the solution given by Fisher and Yates (Tables, 1938) is isomorphic with $[\alpha_3]$ while that given by Bose (1939) is isomorphic with $[\beta_{21}]$.

Bose (1939) has shown how from a symmetrical design with $v = b$, $r = k$, $\lambda$ we can always obtain by block-section the designs with parameters $v' = v - k$, $b' = b - 1$, $r' = r$, $k' = k - \lambda$, $\lambda' = \lambda$. It remains to be seen how far the reverse process of block-section i.e. adjunction of the blocks of the designs $v^* = k$, $b^* = b-1$, $r^* = r-1$, $k^* = \lambda$, $\lambda^* = \lambda - 1$ to the latter design, is in general possible, giving the symmetrical design. In section 4 it has been shown that adjunction is possible in both the cases considered above, and a general method is given for this purpose. (A) after adjunction leads to only one solution of $v = b = 11$, $r=k = 5$, $\lambda = 2$, while (B) leads to three non-isomorphic solutions thus verifying the conclusion of Hussain (1945) regarding the two symmetrical designs : $v=b= 11$, $r=k=5$, $\lambda=2$ and $v = b = 16$, $r = k = 6$, $\lambda = 2$.

## 2. FREQUENCY DISTRIBUTION OF THE NUMBER OF COMMON TREATMENTS

Let us consider a variable $x$ giving the number of treatments common to the initial block and any other block of the design. Then there are $(b-1)$ positive integral values of this variable. Let $f_i$ stand for the frequency of the value $x = i$ ($i = 0, 1, 2, \ldots, n$, $n \leqslant k$).

Then we must have,

$$\left.\begin{aligned} f_0 + f_1 + f_2 + \ldots + f_n &= b-1 \\ f_1 + 2f_2 + \ldots + nf_n &= k(r-1) \\ f_1 + 2^2 f_2 + \ldots + n^2 f_n &= k(r-1) + k(k-1)(\lambda-1) \end{aligned}\right\} \quad \ldots \quad (1)$$

All possible solutions satisfying (1) can be obtained by adding to any particular solution, the solution,

$$\left.\begin{aligned} f_0 &= -f_3 - 3f_4 - \ldots - (n^2-3n+2)/2 . f_n \\ f_1 &= \phantom{-}3f_3 + 8f_4 + \ldots + (n^2-2n) f_n \\ f_2 &= -3f_3 - 6f_4 - \ldots - (n^2-n)/2 . f_n \text{ etc.} \\ & f_3, f_4, \ldots, f_n \end{aligned}\right\} \quad \ldots \quad (2)$$

$f_3, f_4, \ldots, f_n$ being arbitrary. The admissible solutions are those giving positive integral values for $f_0, f_1, \ldots, f_n$. For the particular solution we may try,

$$\left.\begin{aligned} f_0 + f_1 + f_2 &= b-1 \\ f_1 + 2f_2 &= k(r-1) \\ f_1 + 4f_2 &= k(r-1) + k(k-1)(\lambda-1) \end{aligned}\right\} \quad \ldots \quad (3)$$

Now we shall consider certain special cases which will be of use later.

*Case I.* $\lambda = 1$. For the particular solution we have from (3), $f_0 = (b-1) - k(r-1)$, $f_1 = k(r-1)$, $f_2 = 0$ the values being positive integral; and from (2) it is seen that this is the unique solution as otherwise $f_0$ will be negative.

*Case II.* $\lambda = 2$ and the class of designs: $v' = v-k$, $b' = b-1$, $r' = r$, $k' = k-\lambda$, $\lambda' = \lambda$ obtained by block-section from $v = b$, $r = k$, $\lambda$. For the particular solution we have

$$f_0 = 0, \quad f_1 = 2(k-2) = 2(r'-2), \quad f_2 = (k-2)(k-3)/2 = (r'-2)(r'-3)/2$$

and here also the solution is unique otherwise $f_0$ will be negative.

It should be noted that this uniqueness of solution is not a general property as is easily seen from the designs: (a) $v = 10$, $b = 30$, $r = 9$, $k = 3$, $\lambda = 2$; and (b) $v = 9$, $b = 18$, $r = 8$, $k = 4$, $\lambda = 3$ each of which has two solutions for frequency distribution namely (a) $f_0 = 8$, $f_1 = 18$, $f_2 = 3$, $f_3 = 0$ and $f_0 = 7$, $f_1 = 21$, $f_2 = 0$, $f_3 = 1$, and (b) $f_0 = 1$, $f_1 = 4$, $f_2 = 12$, $f_3 = 0$ and $f_0 = 0$, $f_1 = 7$, $f_2 = 9$, $f_3 = 1$. The solution given in Fisher and Yates' Table (1938) for the latter design shows both the distributions of '$x$'.

## 3. ENUMERATION

We shall now consider the enumeration of the solutions of the design: $v = 6$, $b = 10$, $r = 5$, $k = 3$, $\lambda = 2$. Suppose any block to be filled up by the treatments 1, 2, 3. In the other nine blocks these treatments are to be placed in such a way that each is replicated four times and each of the three pairs occurs once only. From the results in section 2 it is seen that three of the other nine blocks have just two treatments in common with the initial block and the remaining six have just one treatment in common. Hence the nine blocks can be uniquely filled up as shown below:

(1, 2, 3); (1, 2); (1, 3), (2, 3); (1); (1); (2); (2); (3); (3).

Let the second block be filled up by the new treatment 4. In order to place 4 in the other blocks suppose $y$ blocks be chosen from those containing two treatments (called two-treatment blocks) and $x$ from those containing one treatment (called one-treatment blocks), then $x+y=4$, $x+2y=4$, which give $x=4$, $y=0$, thus yielding the partially filled up design :

(1, 2, 3) ; (1, 2, 4) ; (1, 3) ; (2, 3) ; (1, 4) ; (1) ; (2, 4) ; (2) ; (3, 4) ; (3, 4).

Fill up the third block with the new treatment 5 and then again for 5, $x+y=4$, $x+2y=6$, giving $x=2$, $y=2$. Since there are only two one-treatment blocks they are chosen. To choose the other two two-treatment blocks we will proceed in the following way which will be useful in the enumeration of the second design.

Let us write all the treatments except the new treatment introduced at this stage. Thus put one or two dashes : 1″, 2′, 3′, 4 on them according as up to this stage of placing 5 it occurs once or twice with the corresponding treatment. This is shown above where it will be found that 4 remains undashed. Hence in choosing the two remaining blocks, we must choose two each containing 4, and 2 or 3 (i.e. the blocks (3,4) (2,4). Hence placing the remaining treatment '6' to fill up the block, the unique solution is :

(1, 2, 3) ; (1, 2, 4) ; (1, 3, 5) ; (2, 3, 6) ; (1, 4, 6) ; (1, 5, 6) ; (2, 4, 5) ; (2, 5, 6) ; (3, 4, 5) ; (3, 4, 6).

We next consider the enumeration of the solution of the design : $v=10$, $b=15$, $r=6$, $k=4$, $\lambda=2$. Fill up the initial blocks with the treatments 1, 2, 3, 4. Of the remaining fourteen blocks, eight blocks have one treatment in common with the initial block and the other six have two treatments in common. So the blocks filled with these four treatments will be as follows :

(1, 2, 3, 4) ; (1, 2) ; (1, 3) ; (1, 4) ; (2, 3) ; (2, 4) ; (3, 4) ; (1) ; (1) ; (2) ; (2) ; (3) ; (3) ; (4) ; (4).

Fill up the second block with the new treatments 5, 6. The number of one-treatment blocks ($x$) and two-treatment blocks ($y$) for placing of 5 is given by $x+y=5$, $x+2y=6$, which gives $x=4$, $y=1$. Any one of the two-treatment blocks (1,3), (1,4), (2,3), (2,4), (3,4) can be chosen. But choices of one of (1,3), (1,4), (2,3), (2,4) are isomorphic as can be seen by interchanging the names 3, 4, or 1, 2. Consequently we have two non-isomorphic ways of choosing one two-treatment block and the four one-treatment block are then chosen uniquely giving

[$\alpha$] : (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 5) ; (1, 4) ; (2, 3); (2, 4) ; (3, 4); (1) ; (1); (2, 5) ; (2); (3, 5); (3) ; (4, 5) ; (4, 5).

[$\beta$] : (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3) ; (1, 4) ; (2, 3) ; (2, 4) ; (3, 4, 5) ; (1, 5) ; (1) ; (2, 5) ; (2) ; (3, 5) ; (3) ; (4, 5) ; (4).

At this stage the number of one-treatment blocks ($x$), two-treatment blocks ($y$), and three-treatment blocks ($z$) to which 6 is to be alloted is given by $x+y+z=5$, $x+2y+3z=7$ The admissible solutions are (*i*) $z=0$, $y=2$, $x=3$ ; (*ii*) $z=1$, $y=0$, $x=4$.

*Case* [ $\alpha$ ]. Here solution (*ii*) is obviously impossible and as such the only way of choosing according to (*i*) will be the blocks (1), (2), (3) (3,4), (4,5) giving us the design filled upto '6' as :

(1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 5) ; (1, 4) ; (2, 3) ; (2, 4) ; (3, 4, 6) ; (1, 6) ; (1) ; (2, 5) ; (2, 6) ; (3, 5) ; (3, 6) ; (4, 5, 6) ; (4, 5).

Completing the third block by 7, we see that it can be placed only in one way in the other blocks corresponding to one admissible solution of $x+y+z=5$, $x+2y+3z=9$ giving us

(1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 5, 7) ; (1, 4) ; (2, 3) ; (2, 4, 7) ; (3, 4, 6) ; (1, 6) ; (1, 7) ; (2, 5) ; (2, 6, 7) ; (3, 5) ; (3, 6, 7) ; (4, 5, 6) ; (4, 5, 7).

The fourth block is now completed with 8, 9 and 8 is to be placed in two three-treatment blocks and three two-treatment blocks. The possible ways of choosing these are :

[$\alpha_1$] (1, 6) (2, 3) (2, 5) ; [$\alpha_2$] (1, 5) (2, 3) (3, 5) ; [$\alpha_3$] (1, 6) (2, 5) (3, 5) ; [$\alpha_4$] (1, 7) (2, 3) (2, 5) ; [$\alpha_5$] (1, 7) (2, 3) (3, 5) ; [$\alpha_6$] (1, 7) (2, 5) (3, 5) ; [$\alpha_7$] (2, 3) (2, 5) (3, 5)

Now [$\alpha_7$] can be shown to be impossible while [$\alpha_1$] is isomorphic with [$\alpha_2$], [$\alpha_3$] with [$\alpha_1$] and [$\alpha_6$] with [$\alpha_3$] as the transformation $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,3\,2\,4\,5\,7\,6 \end{pmatrix}$ shows. Also [$\alpha_3$] is isomorphic with [$\alpha_2$] as the transformation $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,5\,6\,3\,4\,7 \end{pmatrix}$ reveals. Hence the two non-isomorphic ways of placing '8' give.

[$\alpha_1$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 5, 7) ; (1, 4, 8, 9) ; (2, 3, 8) ; (2, 4, 7) ; (3, 4, 6) ; (1, 6, 8) ; (1, 7) ; (2, 5, 8) ; (2, 6, 7) ; (3, 5) ; (3, 6, 7, 8) ; (4, 5, 6) ; (4, 5, 7, 8).

[$\alpha_3$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 5, 7) ; (1, 4, 8, 9) ; (2, 3) ; (2, 4, 7, 8) ; (3, 4, 6) ; (1, 6, 8) ; (1, 7) ; (2, 5, 8) ; (2, 6, 7) ; (3, 5, 8) ; (3, 6, 7, 8) ; (4, 5, 6) ; (4, 5, 7)

In placing 9 in [$\alpha$] we have to choose two two-treatment blocks and three three-treatment blocks, and as there are only two two-treatment blocks, they will be filled up. Then the three three-treatment blocks can be chosen in two non-isomorphic ways namely [$\alpha_{11}$] (2,3,8),(2,6,7), (4,5,6) and [$\alpha_{12}$] (2,5,8) (3,4,6) (2,6,7) Of these, the placing of 9 in [$\alpha_{12}$] is isomorphic with that of 8 in [$\alpha_2$] or [$\alpha_3$] . Again [$\alpha_3$] admits of only one placing of 9. Hence from [$\alpha$] two non-isomorphic solutions are obtained :

[$\alpha_{11}$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 5, 7) ; (1, 4, 8, 9) ; (2, 3, 8, 9) ; (2, 4, 7, 10) ; (3, 4, 6, 10 ) ; (1, 6, 8, 10) ; (1, 7, 9, 10) ; (2, 5, 8, 10) ; (2, 6, 7, 9) ; (3, 5, 9, 10) (3, 6, 7, 8) ; (4, 5, 6, 9) ; (4, 5, 7, 8).

[$\alpha_3$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 5, 7) ; (1, 4, 8, 9) ; (2, 3, 9, 10) ; (2, 4, 7, 8) ; (3, 4, 6, 10) ; (1, 6, 8, 10) ; (1, 7, 9, 10) ; (2, 5, 8, 10) ; (2, 6, 7, 9) ; (3, 5, 8, 9) (3, 6, 7, 8) ; (4, 5, 6, 9) ; (4, 5, 7, 10).

*Case* [$\beta$]. Here for placing of 6 solution (*i*) will give its placing isomorphic with that of 5 in [$\alpha$] and according to [*ii*] placing of 6 will yield

(1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3) ; (1, 4) ; (2, 3) ; (2, 4) ; (3, 4, 5, 6) ; (1, 5) ; (1, 6) ; (2, 5) ; (2, 6) ; (3, 5) ; (3, 6) ; (4, 5) ; (4, 6) .

Filling up the third block with 7 and 8 other five blocks which all contain two treatments to accomodate 7 can be chosen in the following way. We have to choose one two-treatment block containing 1 i.e. we have to choose either (14) or (15) or (16). If we choose (14) placing of 7 will be isomorphic with that of 5 in [$\alpha$] while choice of (15) or (16) are isomorphic as is easily seen by interchanging 5, 6. So there are only three non-isomorphic ways of choosing the five blocks :

[$\beta_1$] (1, 5) (2, 4) (2, 5) (3, 6) (4, 6) [$\beta_2$] (1, 5) (2, 4) (2, 6) (3, 6) (4, 5) [$\beta_3$] (1, 5) (2, 4) (2, 6) (3, 5) (4, 6).

leading to the partially filled-up designs :

[$\beta_1$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 7, 8) ; (1, 4) ; (2, 3) ; (2, 4, 7) ; (3, 4, 5, 6) ; (1, 5, 7) ; (1, 6) ; (2, 5, 7) ; (2, 6) ; (3, 5) ; (3, 6, 7) ; (4, 5) ; (4, 6, 7).

[$\beta_2$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 7, 8) ; (1, 4) ; (1, 3) ; (2, 4, 7) ; (3, 4, 5, 6) ;
(1, 5, 7) ; (1, 6) ; (2, 5) ; (2, 6, 7) ; (3, 5) ; (3, 6, 7) ; (4, 5, 7) ; (4, 6).

[$\beta_3$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 7, 8) ; (1, 4) ; (2, 3) ; (2, 4, 7) ; (3, 4, 5, 6) ;
(1, 5, 7) ; (1, 6) ; (2, 5) ; (2, 6, 7) ; (3, 5, 7) ; (3, 6) ; (4, 5) ; (4, 6, 7).

As regards placing of 8, we have to choose one three-treatment block ; and it may be noted that except the choice of the block (2, 4, 7,) all other choices make the placing of 8 isomorphic with that of 5 in [$\alpha$]. Hence 8 is placed uniquely in the above three solutions. For placing of 9 each of the three admits of two non-isomorphic ways, but again in case of [$\beta_2$] the second way of placing 9 is isomorphic with that of 7 in [$\beta_1$] ; and in the case of [$\beta_3$] the second way is isomorphic with that of 7 in [$\beta_2$]. Thus we obtain four non-isomorphic solutions ; two from [$\beta_1$] and one each from [$\beta_2$] and [$\beta_3$]

[$\beta_{11}$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 7, 8) ; (1, 4, 9, 10) ; (2, 3, 9, 10) ; (2, 4, 7, 8) ;
(3, 4, 5, 6) ; (1, 5, 7, 9) ; (1, 6, 8, 10) ; (2, 5, 7, 10) ; (2, 6, 8, 9) ; (3, 5, 8, 10) ;
(3, 6, 7, 9) ; (4, 5, 8, 9) ; (4, 6, 7, 10) ;

[$\beta_{12}$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 7, 8) ; (1, 4, 9, 10) ; (2, 3, 9, 10) ; (2, 4, 7, 8) ;
(3, 4, 5, 6) ; (1, 5, 7, 9) ; (1, 6, 8, 10) ; (2, 5, 7, 10) ; (2, 6, 8, 9) ; (3, 5, 8, 9) ;
(3, 6, 7, 10) ; (4, 5, 8, 10) ; (4, 6, 7, 9).

[$\beta_{21}$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 7, 8) ; (1, 4, 9, 10) ; (2, 3, 9, 10) ; (2, 4, 7, 8) ;
(3, 4, 5, 6) ; (1, 5, 7, 9) ; (1, 6, 8, 10) ; (2, 5, 8, 10) ; (2, 6, 7, 9) ; (3, 5, 8, 9) ;
(3, 6, 7, 10) ; (4, 5, 7, 10) ; (4, 6, 8, 9).

[$\beta_{31}$] (1, 2, 3, 4) ; (1, 2, 5, 6) ; (1, 3, 7, 8) ; (1, 4, 9, 10) ; (2, 3, 9, 10) ; (2, 4, 7, 8) ;
(3, 4, 5, 6) ; (1, 5, 7, 9) ; (1, 6, 8, 10) ; (2, 5, 8, 10) ; (2, 6, 7, 9) ; (3, 5, 7, 10) ;
(3, 6, 8, 9) ; (4, 5, 8, 9) ; (4, 6, 7, 10).

Now it remains to be seen whether choice of any other block of the above solutions as the initial makes them isomorphic with the [$\alpha$] set. Taking the last block of each solution as the initial block, it is revealed that [$\beta_{11}$] is isomorphic with [$\alpha_3$] while [$\beta_{12}$], [$\beta_{31}$] are isomorphic with [$\alpha_{11}$]. The following transformations establish this isomorphism.

[$\beta_{11}$] $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & & 7 & 8 & 9 & 10 \\ 8 & 10 & 5 & 2 & 6 & 1 & 3 & 9 & 7 & 4 \end{pmatrix}$

[$\beta_{12}$] $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 5 & 3 & 7 & 1 & 2 & 8 & 4 & \end{pmatrix}$

[$\beta_{31}$] $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 5 & 9 & 3 & 8 & 2 & 1 & 7 & 6 & 4 \end{pmatrix}$

Hence, we get three solutions in all, for the design ; $v=10$, $=15$, $r=6$, $k=4$, $\lambda=2$ given in the paper as [$\alpha_{11}$], [$\alpha_3$], [$\beta_{21}$].

## 4. PROCESS OF BLOCK ADJUNCTION

Husain (1945) has shown that the symmetrical design, $v=b=11$, $r=k=5$, $\lambda=2$ has only one solution while $v=b=16$, $r=k=6$, $\lambda=2$ has three non-isomorphic solutions. This, along with the conclusions arrived at above regarding the designs obtained by block-section, raises the problem whether it is always possible to pass from the unsymmetrical to the corresponding symmetrical design by suitable adjunction of blocks ; and if possible how do the non-isomorphic solutions of the two stand in relation to one another.

It is obvious that the designs whose blocks need to be adjoined is the design obtained from the symmetrical design by block-intersection. So the necessary conditions for the reverse process of block-section are : (1) the design of block-intersection must exist, and (2) the distribution of $x$, the number of treatments common to the initial block and any other block in the two designs to be adjoined must be complementary in the sense that if in the former $f_i = l$, then in the later $f_{\lambda-i}$ must also be equal to $l$. If adjunction is possible, there may arise two situations ; (*i*) more than one non-isomorphic ways of adjunction may be possible ; (*ii*) after adjunction two non-isomorphic solutions may become isomorphic, so that the number of solutions in the unsymmetrical case may be less than, equal to or greater than the number of solutions in the symmetrical case.

It is easy to see that the necessary conditions enumerated above are satisfied in the case of the designs (A) and (B) considered in this paper. The former requires for adjunction to blocks of all possible pairs of five new treatments and one block containing these five treatments to get the design ; $v=b=11$, $r=k=5$, $\lambda=2$, while the later, all possible pairs of six new treatments to obtain ; $v=b=16$, $r=k=6$, $\lambda=2$. For actual adjunction the following method may be attempted with slight variation in complex cases and if it succeeds, adjunction is possible ; otherwise not.

(A) The solution is [$\alpha$] :

| | | | |
|---|---|---|---|
| (1, 2, 3) [7, 8] | (2, 3, 6) [10, 11] | (2, 4, 5) [8, 11] | (3, 4, 6) [8, 9] |
| (1, 2, 4) [9, 10] | (1, 4, 6) [7, 11] | (2, 5, 6) [7, 9] | [7, 8, 9, 10, 11] |
| (1, 3, 5) [9, 11] | (1, 5, 6) [8, 10] | (3, 4, 5) [7, 10] | |

To the first block we adjoin the treatment pair (7, 8). The second having two treatments in common with the first, two other new varieties (9, 10) are adjoined to it. The third block has two treatments in common with the first but only one with the second. Hence the treatment pair to be adjoined to the third will contain the new treatment 11 and one of 9, 10. In this way treatment-pairs are allotted to the remaining blocks keeping in mind that any two new blocks have just two treatments in common. In the above, the adjoined pairs are shown in [ ] and it will be found that the conditions of balanced incomplete block designs are satisfied.

(B) The solutions are [$\alpha_{11}$], [$\alpha_3$], [$\beta_{11}$]. Here, in the first place, a new treatment pair is adjoined to the first block. Then out of the six other blocks which have two treatments in common with the first, two blocks are chosen which have also two treatments in common with each other, to which new treatment-pairs are assigned. In all the three cases this is possible ; and then following the above method as in the case of (A) we have only one way of adjoining to each of the three :

| [$\alpha'_{11}$] | [$\alpha'_3$] | [$\beta'_{11}$] |
|---|---|---|
| (1, 2, 3, 4) [11, 12] | (1, 2, 3, 4) [11, 12] | (1, 2, 3, 4) [11, 12] |
| (1, 2, 5, 6) [13, 15] | (1, 2, 5, 6) [13, 15] | (1, 2, 5, 6) [13, 14] |
| (1, 3, 5, 7) [14, 16] | (1, 3, 5, 7) [14, 16] | (1, 3, 7, 8) [13, 15] |
| (1, 4, 8, 9) [13, 14] | (1, 4, 8, 9) [13, 14] | (1, 4, 9, 10) [14, 15] |
| (2, 3, 8, 9) [15, 16] | (2, 3, 9, 10) [13, 16] | (2, 3, 9, 10) [13, 16] |
| (2, 4, 7, 10) [13, 16] | (2, 4, 7, 8) [15, 16] | (2, 4, 7, 8) [14, 16] |
| (3, 4, 6, 10) [14, 15] | (3, 4, 6, 10) [14, 15] | (3, 4, 5, 6) [15, 16] |
| (1, 6, 8, 10) [11, 16] | (1, 6, 8, 10) [11, 16] | (1, 5, 7, 9) [11, 16] |
| (1, 7, 9, 10) [12, 15] | (1, 7, 9, 10) [12, 15] | (1, 6, 8, 10) [12, 16] |
| (2, 5, 8, 10) [12, 14] | (1, 5, 8, 10) [12, 14] | (2, 5, 8, 10) [11, 15] |

| $[\alpha'_{11}]$ | $[\alpha_{3}]$ | $[\beta_{31}]$ |
|---|---|---|
| (2, 6, 7, 9) [11, 14] | (2, 6, 7, 9) [11, 14] | (2, 6, 7, 9) [12, 15] |
| (3, 5, 9, 10) [11, 13] | (3, 5, 8, 9) [11, 15] | (3, 5, 8, 9) [12, 14] |
| (3, 6, 7, 8) [12, 13] | (3, 6, 7, 8) [12, 13] | (3, 6, 7, 10) [11, 14] |
| (4, 5, 6, 9) [12, 16] | (4, 5, 6, 9) [12, 16] | (4, 5, 7, 10) [12, 13] |
| (4, 5, 7, 8) [11, 15] | (4, 5, 7, 10) [11, 13] | (4, 6 8, 9) [11, 13] |
| [11, 12, 13, 14, 15, 16] | [11, 12, 13, 14, 15, 16] | [11, 12, 13, 14, 15, 16] |

Let us now consider the following transpositions in the six cases :—

$[\alpha'_{11}]$ : Interchange (11, 5) and (12, 6) and make the substitution

$$\begin{pmatrix} 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16 \\ 13, & 14, & 15, & 16, & 8, & 10, & 9, & 12, & 7, & 11 \end{pmatrix}$$

$[\alpha'_{3}]$ : Interchange (11, 6), (12, 5) and (2, 3) then make the substitution

$$\begin{pmatrix} 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16 \\ 10, & 14, & 15, & 16, & 8, & 13, & 11, & 9, & 12, & 7 \end{pmatrix}$$

$[\beta'_{31}]$ : Interchange (11, 5) and (12, 6) and then substitute

$$\begin{pmatrix} 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16 \\ 12, & 13, & 14, & 15, & 9, & 10, & 7, & 8, & 11, & 16 \end{pmatrix}$$

After these transformations, it will be found that the symmetrical designs $[\alpha'_{11}]$, $[\alpha'_{3}]$, $[\beta'_{31}]$ are isomorphic respectively with the solutions (II), (III), (I) given by Husain (1945).

## 5. TEST OF ISOMORPHISM

It will be seen that when considering the placings and adjunctions, we reject the isomorphic ones, but that does not dispose of all the isomorphic solutions. So when the final solutions have been obtained we have to test their isomorphisms.

In the first place, test the isomorphisms of the broad types. Here we have two broad types $[\alpha]$ and $[\beta]$. We take the blocks of each solution of $(\beta)$, one by one and correspond it to the initial block of type $[\alpha]$ and see whether $(\beta)$ is isomorphic with $[\alpha]$. This method has revealed, as shown above, that out of the four solutions from $(\beta)$, three are isomorphic with $[\alpha]$. Secondly, to point out the isomorphism of the two solutions from the same set we can adopt the following method.

Let the first solution remain as it is, and identify the blocks of the second, one by one, with the initial block of the first solution. Then consider the placings of other treatments in the second solution with reference to the treatments in the initial blocks. If this set of placings can be made to correspond to the set of placings in the first solution, the two are isomorphic ; otherwise not.

As an illustration, consider the isomorphism of $[\beta_{11}]$ and $[\beta_{31}]$. The two initial blocks of the two solutions are made to correspond. Then the other six treatments occupy the following positions with reference to the four treatments in the initial block.

| Treatments | Blocks of $[\beta_{11}]$ | Blocks of $[\beta_{31}]$ |
|---|---|---|
| 5 | (12), (34), $(1)_a$ $(2)_a$ $(3)_a$ $(4)_a$ | (12), (34), $(1)_a$ $(2)_a$ $(3)_a$ $(4)_a$ |
| 6 | (12), (34), $(1)_b$ $(2)_b$ $(3)_b$ $(4)_b$ | (12), (34), $(1)_b$ $(2)_b$ $(3)_b$ $(4)_b$ |
| 7 | (13), (24), $(1)_a$ $(2)_a$ $(3)_a$ $(4)_a$ | (13), (24), $(1)_a$ $(2)_a$ $(3)_a$ $(4)_a$ |
| 8 | (13), (24), $(1)_b$ $(2)_b$ $(3)_b$ $(4)_b$ | (13), (24), $(1)_b$ $(2)_b$ $(3)_b$ $(4)_b$ |
| 9 | (14), (23), $(1)_a$ $(2)_a$ $(3)_a$ $(4)_a$ | (14), (23), $(1)_a$ $(2)_a$ $(3)_a$ $(4)_a$ |
| 10 | (14), (23), $(1)_b$ $(2)_b$ $(3)_b$ $(4)_b$ | (14), (23), $(1)_b$ $(2)_b$ $(3)_b$ $(4)_b$ |

The two blocks, each containing 1 or 2 or 3 or 4 are distinguished as 'a' or 'b' in the order of their positions. Let us concentrate on the patterns of the one-treatment blocks.

| Treatments | $[\beta_{12}]$ | | | | $[\beta_{31}]$ | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | *a* | *a* | *a* | *a* | *a* | *a* | *a* | *a* |
| 6 | *b* | *b* | *b* | *b* | *b* | *b* | *b* | *b* |
| 7 | *a* | *a* | *b* | *b* | *a* | *b* | *a* | *b* |
| 8 | *b* | *b* | *a* | *a* | *b* | *a* | *b* | *a* |
| 9 | *a* | *b* | *a* | *b* | *a* | *b* | *b* | *a* |
| 10 | *b* | *a* | *b* | *a* | *b* | *a* | *a* | *b* |

If we interchange 1, 4, the sets of two treatments blocks correspond but the pattern in $[\beta_{31}]$ becomes

| | | | | |
|---|---|---|---|---|
| 7 | *b* | *b* | *a* | *a* |
| 8 | *a* | *a* | *b* | *b* |
| 5 | *a* | *a* | *a* | *a* |
| 6 | *b* | *b* | *b* | *b* |
| 9 | *a* | *b* | *a* | *b* |
| 10 | *b* | *a* | *b* | *a* |

Interchange (a, b) in the third and fourth positions, and we see that the transformed pattern is indentical with $[\beta_{12}]$ and we get the following substitution :

$$\begin{pmatrix} 1 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 7 & 8 & 6 & 5 \end{pmatrix}$$

to establish the isomorphism of $[\beta_{31}]$ with $[\beta_{12}]$. The isomorphism of the symmetrical designs are tested by considering the isomorphism of all possible block-cuts.

## References


1. Bose, R. C. (1939) On the construction of balanced incomplete block design. *Annals of Eugenics*, 9, 358-399.
2. Cox, Gertrude M. (1940) Enumeration and construction of balanced incomplete configuration. *Annals of Math. St.*, 11, 72-85.
3. Fisher, R. A. (1940) An examination of the different possible solutions of a problem in incomplete block. *Annals of Eugenics*, 10, 52-75.
4. Fisher & Yates (1938) Statistical Tables for Biological, Agricultural & Medical Research. Olive & Boyd, London.
5. Husain, Q. M. (1945) On the totality of solutions of symmetrical incomplete block designs. *Sankhyā*, 7(2), 204-208.


*Paper received* : 14 *October*, 1943.