# Markov Dilations of Nonconservative Quantum Dynamical Semigroups and a Quantum Boundary Theory

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To

*Amma* and *Appa*

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# Contents

## Introduction

In classical probability theory, based on Kolmogorov consistency theorem, one can associate a Markov process to any one parameter semigroup of stochastic matrices or transition probability operators. It is indeed the foundation for the theory of Markov processes. Here a quantum version of this theorem has been established. This effectively answers some of the questions raised by P. A. Meyer in his book (see page 220 of [Me]).

It is widely agreed upon that irreversible dynamics in the quantum setting is described by contractive semigroups of completely positive maps on $C^*$ algebras ([Kr], [AL]). In other words these semigroups, known as quantum dynamical semigroups, are non-commutative Markov kernels. We take the view point that a quantum Markov process consists of a 'filtration' and a time indexed family of representations of a $C^*$ algebra reflecting the Markov property with respect to a suitable 'conditional expectation'. See [AFL], [Ku2], [Sc], and [Sa1] for other approaches.

In the modest approach taken here conditioning means truncating operators to subspaces and so naturally enough filtrations are increasing families of subspaces (or projections). This leads us to the notion of weak Markov flows. This notion is quite powerful and encompasses atleast three kinds of dilations in its fold. Classical Markov processes, Sz. Nagy dilation of contraction semigroups on Hilbert spaces([SzF], [Da3], [EL]), and Evans-Hudson flows of Fock space stochastic calculus([P1], [Me]) are included here in a natural way. The main theorem in Section 4 shows that every quantum dynamical semigroup can be realized as expectation semigroup of a weak Markov flow. Moreover there is uniqueness upto unitary equivalence under a natural irreducibility condition. From the operator algebraic point of view this theorem generalizes Stinespring's famous theorem([St], [P1]) showing that every completely positive map on a $C^*$ algebra can be dilated to a representation of the algebra. Similar generalization under varying stringent conditions on the semigroup and the algebra may be found in ([Em], [Vi-S]).

The Markov process corresponding to a semigroup of substochastic matrices or nonconservative transition probability operators has an exit time which may be interpreted as a stop time at which the trajectory of the process goes out of the state space or hits a boundary. There are many ways of continuing the process after the exit time in such a manner that the Markov property and stationarity of transition probabilities are retained. Feller's study of this problem ([Fe1,2,3]) based on resolvents of semigroups and Chung's pathwise approach ([C1,2], [Dy]) are well-known. Now it is natural to ask as to what happens when we have quantum Markov processes relating to quantum dynamical semigroups. The study in this direction was initiated by Davies [Da2] in the semigroup level. We are able to follow the footsteps of Feller and Chung to successfully return from the boundary and continue along a duplicate of the original flow as and when we reach the boundary to have a new quantum Markov flow. Of course, here the exits are governed by quantum stop times. Our investigations indicate the possibility of developing an extensive quantum boundary theory.

Most of the results are based on [BP2] and some subsequent work with K. R. Parthasarathy. Examples of generators of nonconservative dynamical semigroups in Chapter IV are from [BS2]. The thesis is almost self-contained except for the examples which may need some knowledge of Hudson-Parthasarathy theory of quantum stochastic calculus in Fock spaces. The basic references are [Fe1-3], [C1-2], [BP2], [BS2] and the two books on the subject *An Introduction to Quantum Stochastic Calculus* of K. R. Parthasarathy [P1] and *Quantum Probability for Probabilists* of P. A. Meyer [Me].

Briefly the lay out is as follows. In sections 1 and 2 basic definitions relating to quantum dynamical semigroups and weak Markov flows are given and explained through fundamental examples. It is noted here that Kolmogorov construction leads to two different weak Markov flows. First one is a commutative Markov flow but the second one is not and is got through a simple conditioning procedure. The minimality of weak Markov flows associated with Sz. Nagy dilation as presented here appears to be new. Some general moment computations of weak Markov flows are carried out in Section 3. The most important observation in this context is the existence of a reduction algorithm facilitating computations. Existence and uniqueness of a 'subordinate' dilation, under a minimality assumption, are shown for every quantum dynamical semigroup in the next section. The proof is simple and straight-forward for conservative semigroups and the only tool needed is the G.N.S. construction. The nonconservative case is tackled by a trick of extending the semigroup to a conservative semigroup in a larger algebra. In Section 5 it is seen that flows constructed in Section 4 satisfy a cocycle property automatically. A generalization of a result of Arveson leading to an abstract Feynman-Kac cocycle is established. And in Section 6 an interesting family of representations of the centre of the algebra is constructed. This family, which is not really a weak Markov flow, is called the central flow. If the original algebra is abelian then the central flow provides us a unique minimal abelian dilation of the quantum dynamical semigroup. This once again tallies with the Kolmogorov construction.

A special type of perturbation of positive semigroups on von Neumann algebras is carried out in Chapter II. It is to be noted that here onwards we restrict ourselves to von Neumann algebras and work with continuous time semigroups only. Chapter III deals with general quantum stochastic processes and stop times. A quantum stochastic process would simply be a time indexed family of operators and stop times are spectral measures. The meaning of starting with one process and continuing along a new process after a stop time is made precise. It is done through 'gluing' independent processes and filtrations suitably. In Chapter IV first we glue infinitely many copies of a single Markov process exactly analogous to Chung's pathwise approach to realize Feller perturbed semigroups as expectation semigroups of Markov flows. Then in the last section several examples are discussed. Feller's construction of obtaining minimal semigroups out of formal generators has been genralized to the non-commutative set up (See [Da2], [Ch], [CF], [Fa1], and [Mo]). The formal generators appearing here lead to nonconservative minimal semigroups in a natural way. On employing Feller perturbations we can ensure conservativity.

CHAPTER I

# Markov Dilations

## 1 Quantum Dynamical Semigroups

One parameter semigroups of transformations and operators play an important role in describing any dynamics whether it is deterministic or non-deterministic, classical or quantum. The study of classical Markov processes involves positive semigroups acting on an algebra of continuous or bounded measurable functions on a topological or measurable space. In the quantum setting this abelian algebra of functions gets replaced by a general $C^*$ or von Neumann algebra. The most appropriate notion of positivity here seems to be complete positivity ([Kr], [AL]) which, essentially, means that the positivity does not get destroyed by adjoining an independent system with no motion. Positive linear maps on abelian algebras are automatically completely positive([St], [Ta]) and hence there is no need for any extra condition in the context of classical processes. This leads to a formal definition of quantum dynamical semigroups as completely positive one parameter semigroups on $C^*$ algebras.

The dynamics that we consider here may be in discrete time or in continuous time. Accordingly the set of time points, denoted throughout by $\mathbb{T}_+$, will be either the additive semigroup $\mathbb{Z}_+$ of nonnegative integers or $\mathbb{R}_+$ of nonnegative real numbers.

**Definition 1.1** : Let $\mathcal{A}$ be a unital $C^*$ algebra. A family of linear maps $\{T_t : t \in \mathbb{T}_+\}$, of $\mathcal{A}$ into itself is called a *quantum dynamical semigroup* if the following are satisfied:

(i) $T_t$ is *completely positive* for every t, i.e., $((T_t(X_{ij}))) \geq 0$ for $((X_{ij})) \geq 0$ where $((X_{ij}))$ is in $\mathcal{M}_n(\mathcal{A})$, the $*$-algebra of $n \times n$ matrices with entries from $\mathcal{A}$ for $n = 1, 2, \cdots$;

(ii) $T_s(T_t(X)) = T_{s+t}(X)$ for all $X \in \mathcal{A}, s, t \in \mathbb{T}_+$ ;

(iii) $T_0(X) = X$ for all $X \in \mathcal{A}$;

(iv) $T_t(I) \leq I$ for all $t \in \mathbb{T}_+$.

The semigroup is called *conservative* if $T_t(I) = I$ for every $t$.

It follows from (i) and (iv) that every quantum dynamical semigroup is automatically contractive. A priori we do not put any continuity restriction on the semigroup in the variable $t$. In discrete time the dynamical semigroup consists of $\{I, T, T^2, \cdots\}$ for a single contractive completely positive map $T : \mathcal{A} \to \mathcal{A}$.

To begin with one may keep in mind the following examples of quantum dynamical semigroups.

**Example 1.2** : Let $\mathcal{S}$ be a countable set. Suppose that $P(t) = ((p_{ij}(t)))_{i,j \in \mathcal{S}}$, is a family of matrices such that

(i) $p_{ij}(t) \geq 0$ for all $i, j \in \mathcal{S}$ and $t \in \mathbb{T}_+$;

(ii) $p_{ij}(s+t) = \sum_{k \in \mathcal{S}} p_{ik}(s) p_{kj}(t)$ for all $i, j \in \mathcal{S}$ and $t \in \mathbb{T}_+$;

(iii) $p_{ij}(0) = \delta_{ij}$ for all $i, j \in \mathcal{S}$.

(iv) $\sum_j p_{ij}(t) \leq 1$ for all $i \in \mathcal{S}$ and $t \in \mathbb{T}_+$;

Taking $\mathcal{A}$ as the algebra $l_\infty(\mathcal{S})$ of bounded functions on $\mathcal{S}$ define $T_t : \mathcal{A} \to \mathcal{A}$ by

$$(T_t(f))(i) = \sum_j p_{ij}(t) f(j) = (P(t)f)(i).$$

It is easy to show that $\{T_t\}$ is a quantum dynamical semigroup. The semigroup $\{T_t\}$ is conservative if and only if $P(t)$ is stochastic, i.e., $\sum_j p_{ij}(t) \equiv 1$ for every $i$ and $t$. These are the semigroups involved in classical Markov chains or countable state Markov processes.

**Example 1.3** : Let $V(t), t \in \mathbb{T}_+$, be a semigroup of isometric operators on a complex separable Hilbert space $\mathcal{H}$. Define $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$T_t(X) = V(t) X V(t)^*, \quad X \in \mathcal{B}(\mathcal{H}).$$

Clearly $T_t$ is a quantum dynamical semigroup on the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. This semigroup is conservative if $V(t)$ is unitary for every $t$. In such a case note that, $\{V(t) : t \in \mathbb{T}\}$ becomes a group on defining $V(-t) = V(t)^*$. The definition of $T_t$ can then be extended in a natural way to incorporate negative time points. Such semigroups are used to describe reversible dynamics in the Heisenberg picture of quantum theory.

**Example 1.4 :** Let $L$ be a selfadjoint operator on a complex separable Hilbert space $\mathcal{H}$. Define $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$T_t(X) = \mathbb{E} e^{i\omega(t)L} X e^{-i\omega(t)L}, \; X \in \mathcal{B}(\mathcal{H}),$$

where $\omega(t)$ is the standard Brownian motion or, more generally, a real valued stochastic process with stationary independent increments and $\mathbb{E}$ stands for expectation in the strong Bochner sense. It is not very difficult to see that $T_t$ is a conservative quantum dynamical semigroup on $\mathcal{B}(\mathcal{H})$.

A linear map $T$ on the full algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a complex separable Hilbert space $\mathcal{H}$ is called *ultra-weakly continuous* (see [Da3]) if $\operatorname{tr} \rho T(X_n) \to \operatorname{tr} \rho T(X)$ whenever $\operatorname{tr} \rho X_n \to \operatorname{tr} \rho X$ for all trace class operators $\rho$ in $\mathcal{B}(\mathcal{H})$. If $T$ is an ultra-weakly continuous completely positive map on $\mathcal{B}(\mathcal{H})$ then as a consequence of Stinespring's Theorem (Theorem 4.3) one has $T(X) = \sum_{k=1}^{\infty} L_k^* X L_k$, for some $\{L_k \in \mathcal{B}(\mathcal{H}) : \sum L_k^* L_k \text{ converging strongly}\}$. The structure of continuous time quantum dynamical semigroups is not very clear except under some stringent continuity assumptions. We will be dealing with two notions of continuity.

**Definition 1.5 :** Let $\{T_t : t \in \mathbb{R}_+\}$ be a quantum dynamical semigroup on a $C^*$ algebra $\mathcal{A}$. Then $\{T_t\}$ is said to be *strongly continuous* if $\lim_{t\to s} ||T_t(X) - T_s(X)|| = 0$ for $X \in \mathcal{A}$ and $s \geq 0$, and is said to be *uniformly continuous* if $\lim_{t\to s} \sup_{||X||\leq 1} ||T_t(X) - T_s(X)|| = 0$ for all $s \geq 0$.

The structure of uniformly continuous quantum dynamical semigroups is well known and is as follows.

**Theorem 1.6 ([GKS] and [Li]) :** Let $\{T_t : t \in \mathbb{R}_+\}$ be an uniformly continuous quantum dynamical semigroup on the $C^*$ algebra $\mathcal{B}(\mathcal{H})$ for some complex separable Hilbert space $\mathcal{H}$. Suppose $T_t$ is ultra-weakly continuous for every $t$. Then the generator $\mathcal{L} = \lim_{t\to 0} \frac{1}{t}(T_t - I)$ is a bounded map on $\mathcal{B}(\mathcal{H})$ and is given by

$$\mathcal{L}(X) = i[H,X] - \frac{1}{2}\sum_k (L_k^* L_k X + X L_k^* L_k - 2L_k^* X L_k) - \frac{1}{2}(BX + XB)$$

where $\{L_k \in \mathcal{B}(\mathcal{H}) : \sum_k L_k^* L_k$ converges strongly$\}$, $H$ and $B$ are bounded self-adjoint operators on $\mathcal{H}$ with $B \geq 0$. The semigroup $\{T_t\}$ is conservative if and only if $B$ is zero.

**Proof** : We refer to [P1], [Li], and [Da1] for the conservative case. The nonconservative case can be treated in a similar fashion. ■

## 2 Weak Markov Flows

There have been several attempts to build quantum analogues of classical Markov processes ( [AFL], [Ku1,2], [Sa1], [EL], [Sc]). Our approach is very close to that of [Em] and [ViS]. The main difference being that we work with general quantum dynamical semigroups on $C^*$ algebras whereas [Em] and [Vi-S] consider conservative semigroups on von Neumann algebras and assume existence of a faithful normal invariant state. Vincent-Smith [Vi-S] has already observed the necessity to consider non-commutative processes which are not unital and have a 'weak Markov property'. We face a similar situation. Classical Markov processes, Sz. Nagy dilation of contraction semigroups to isometric semigroups, and Evans-Hudson flows of Hudson-Parthasarathy quantum stochastic calculus in Fock spaces form three fundamental examples of weak Markov flows. In this section weak Markov flows are defined and explained through these examples.

In the theory of Markov processes a crucial role is played by the notion of conditional expectation. There are various non-commutative generalizations of conditional expectation of classical probability ([AC], [Da1], [Um], [Pe]). We find Definition 5.2 of [EL] most suitable for our purposes.

Let $\mathcal{B}$ be a unital $C^*$ sub-algebra of a unital $C^*$ algebra $\mathcal{A}$. A *conditional expectation* of $\mathcal{A}$ given $\mathcal{B}$ is a projection $\mathbb{E}$ of norm one from $\mathcal{A}$ onto $\mathcal{B}$ such that $\mathbb{E}(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. It has been established in [EL] that any such map $\mathbb{E}$ is automatically completely positive and satisfies the crucial module property: $\mathbb{E}(XY) = \mathbb{E}(X)Y$ for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

It should be noted that given two unital $C^*$ algebras $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{B} \subseteq \mathcal{A}$, there may not exist any map $\mathbb{E}$ as above (p. 84 of [EL], [AC], [Su2]). However no such difficulty is faced in the sequel due to some inter-dependence of pairs of algebras under consideration. In fact a very simple type of conditional expectation maps, which may be described as truncating through projections is all that we need. A crucial difference from the classical and the Fock space set up is that these maps are not unital in the strict sense, i.e., the identity of the Hilbert space is not preserved. For this reason we prefer to call them as weak conditional expectation maps.

Let $\mathcal{H}$ be any complex Hilbert space with scalar product $< \cdot, \cdot >$ linear in the second variable and conjugate linear in the first. By a *weak filtration* $F$ on $\mathcal{H}$ we mean a family $F = \{F(t), t \in \mathbb{T}_+\}$ of orthogonal projection operators nondecreasing in the variable $t$. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ and write

$$\mathcal{B}^F_{t]} = \{F(t)XF(t), X \in \mathcal{B}(\mathcal{H})\}$$

for every $t$. Then $\{\mathcal{B}^F_{t]}, t \in \mathbb{T}_+\}$ is a nondecreasing family of $*$ subalgebras of $\mathcal{B}(\mathcal{H})$. The map $\mathbb{E}^F_{t]} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}^F_{t]}$ defined by

$$\mathbb{E}^F_{t]}(X) = F(t)XF(t)$$

is called the *weak conditional expectation* with respect to $F$ at time $t$.

**Proposition 2.1** : The weak conditional expectation maps $\{\mathbb{E}^F_{t]}, t \in \mathbb{T}_+\}$ satisfy the following :

(i) $\mathbb{E}^F_{t]}$ is a completely positive and contractive linear map;

(ii) $\mathbb{E}^F_{t]}(I) = F(t)$;

(iii) $\mathbb{E}^F_{t]}(X) = X$ for all $X \in \mathcal{B}^F_{t]}$;

(iv) $\mathbb{E}^F_{t]}(XY) = X\mathbb{E}^F_{t]}(Y), \mathbb{E}^F_{t]}(YX) = \mathbb{E}^F_{t]}(Y)X$ for all $X \in \mathcal{B}^F_{t]}, Y \in \mathcal{B}(\mathcal{H})$;

(v) $\mathbb{E}^F_{s]}\mathbb{E}^F_{t]} = \mathbb{E}^F_{s\wedge t]}$ where $s \wedge t = \min(s,t)$.

**Proof** : Immediate. ∎

**Definition 2.2** : Let $\mathcal{A}$ be a unital $C^*$ algebra of operators on a Hilbert space $\mathcal{H}_0$ and let $\{T_t, t \in \mathbb{T}_+\}$ be a quantum dynamical semigroup on $\mathcal{A}$. A triple $(\mathcal{H}, F, j_t)$ is called a *weak Markov flow* with *expectation semigroup* $\{T_t\}$ if $\mathcal{H}$ is a Hilbert space containing $\mathcal{H}_0$ as a subspace, $F$ is a weak filtration on $\mathcal{H}$ with $F(0)$ having range $\mathcal{H}_0$ and $\{j_t, t \in \mathbb{T}_+\}$ is a family of $*$-homomorphisms from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ satisfying the following

(i) $\mathbb{E}^F_{0]} j_0(X) = XF(0)$ and $j_t(X)F(t) = F(t)j_t(X)F(t)$ for all $t \geq 0, X \in \mathcal{A}$;

(ii) $\mathbb{E}^F_{s]} j_t(X) = j_s(T_{t-s}(X))F(s)$ for all $0 \leq s \leq t < \infty, X \in \mathcal{A}$.

The flow is called *subordinate to the filtration* $F$ if $j_t(I) \leq F(t)$ for all $t$. If $j_t(I) = F(t)$ for all $t$ it is said to be *conservative*. If for every $t$, $j_t(I)$ is the identity operator in $\mathcal{H}$ then the flow is called *unital*. Suppose the set $\{j_{t_1}(X_1) \cdots j_{t_n}(X_n)u : t_1, \ldots, t_n \in \mathbb{T}_+, X_1, \ldots, X_n \in \mathcal{A}, u \in \mathcal{H}_0$ and $n = 1, 2, \ldots\}$ is total in $\mathcal{H}$ then the Markov flow $(\mathcal{H}, F, j_t)$ is said to be *minimal*.

The Hilbert space $\mathcal{H}_0$ may be called the initial space. The unit of the initial algebra $\mathcal{A}$ is assumed to be the identity operator on $\mathcal{H}_0$ unless mentioned otherwise. We will be dealing with only weak Markov flows and hence may refer to them simply as Markov flows and we may say $j_t$ is a subordinate flow to mean that it is subordinate to its associated filtration $F$.

In Definition 2.2 condition (i) describes the faithfulness of $j_0$ and adaptedness of the flow to the filtration $F$, whereas condition (ii) describes the Markov property of the flow. In the case of a subordinate Markov flow it follows from (i) that $j_0(I) = F(0)$ and the factor $F(s)$ on the right hand side of (ii) may be dropped to have $\mathbb{E}^F_{s]} j_t(X) = j_s(T_{t-s}(X))$. It may be noted that if $(\mathcal{H}, F, j_t)$ is a weak Markov flow then $(\mathcal{H}, F, j_t(\cdot)F(t))$ is a subordinate weak Markov flow. Classical Markov processes appear in this setting in the following form.

**Example 2.3** : Let $\{P(t, x; E) : t \in \mathbb{T}_+, x \in \mathcal{S},$ and $E \subset \mathcal{S}\}$ be a semigroup of transition probability functions [BG] on a state space $\mathcal{S}$. Suppose $\mu$ is an initial distribution on $\mathcal{S}$ such that

$$\mu P_t \ll \mu \text{ for all } t. \tag{2.1}$$

Now set $\mathcal{A} = L^\infty(\mu)$ considered as the algebra of multiplication operators on $\mathcal{H}_0 := L^2(\mu)$. Then the relevant semigroup $\{T_t : \mathcal{A} \to \mathcal{A}\}$ is given by

$$(T_t(f))(x) = \int_S f(y)P(t,x;dy) \quad \text{for} \quad f \in L^\infty(\mu).$$

Condition (2.1) is put so that $T_t$ maps $\mathcal{A}$ into itself. Let $(\Omega, \mathcal{F}, I\!P_\mu)$ be the Kolmogorov probability space so that the 'path space' $\Omega$ is $\{\omega : I\!T_+ \to \mathcal{S}\}$, and the co-ordinate process $\xi_t$ defined by $\xi_t(\omega) = \omega(t)$, is a Markov process with initial distribution $\mu$ and transition probability function $P(t,x;E)$. Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by $\{\xi_s : 0 \leq s \leq t\}$. Now take $\mathcal{H}$ to be $L^2(I\!P_\mu)$ and $F(t)$ as the projection on to the space of $\mathcal{F}_t$-measurable functions on $\mathcal{H}$. In other words, for $\eta \in \mathcal{H}$

$$F(t)\eta = I\!E_{I\!P_\mu}(\eta|\mathcal{F}_t).$$

We make a clarification here. Note that $F(t)1\!\!1 = 1\!\!1$ where $1\!\!1$ is the constant function with value 1. However our conditional expectation is not $F(t)$ but it is $I\!E^F_{t]}$, mapping $X$ to $F(t)XF(t)$ in $\mathcal{B}(\mathcal{H})$. Here the identity operator gets mapped to $F(t)$ by $I\!E^F_{t]}$.

Now define $*$-homomorphisms $k_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $j_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$k_t(f)\eta = f(\xi_t)\eta,$$

and

$$j_t(f)\eta = F(t)f(\xi_t)\eta$$

for $\eta \in \mathcal{H}$ and $t \in I\!T_+$, i.e., $k_t(f)$ is simply multiplication by $f(\xi_t)$ and $j_t(f)$ is multiplication by $f(\xi_t)$ followed by conditional expectation (classical) given the $\sigma$-algebra $\mathcal{F}_t$. Clearly $k_t$ is a $*$-homomorphism. As $\xi_t$ is $\mathcal{F}_t$-measurable

$$j_t(f)\eta = F(t)f(\xi_t)\eta = f(\xi_t)F(t)\eta,$$

and hence $j_t$ is also a $*$-homomorphism. Now it is not very difficult to show that $(\mathcal{H}, F, k_t)$ and $(\mathcal{H}, F, j_t)$ are weak Markov flows with expectation semigroup $\{T_t\}$. Note that a function $h$ in $L^2(\mu)$ is identified with the function $\omega \to h(\omega(0))$ in $L^2(I\!P_\mu)$. Now for $t \geq 0$,

$$j_t(f)h = I\!E(f(\xi_t)h|\mathcal{F}_t) = f(\xi_t)h = k_t(f)h,$$

and by induction it is clear that for $t_1 \geq t_2 \geq \cdots t_n \geq 0$,

$$j_{t_1}(f_1)\cdots j_{t_n}(f_n)h = f_1(\xi_{t_1})\cdots f_n(\xi_{t_n})h = k_{t_1}(f_1)\cdots k_{t_n}(f_n)h. \tag{2.2}$$

Clearly the set $\{f_1(\xi_{t_1})\cdots f_n(\xi_{t_n})h : t_1 \geq t_2 \geq \cdots t_n \geq 0, f_1, \cdots f_n \in L^\infty(\mu), h \in L^2(\mu)\}$ is total in $L^2(\mathbb{P}_\mu)$. Hence both the flows $j_t$ and $k_t$ are minimal.

The weak Markov flows described above retain almost all the information one would like to have about the process. The significance of $j_t$ is that it is a subordinate Markov flow and has a natural generalization to non-commutative algebras as well, as shown in Section 4. On the other hand $k_t$ is important as it is an *abelian flow*, i.e., $k_s$ and $k_t$ commute for all $s$ and $t$. A generalization of $k_t$ to centres of $C^*$ algebras is exhibited in Section 6. It is obvious that $j_t$ is not abelian and $k_t$ is not subordinate to $F$ in general.

**Remark 2.4** : In Example 2.3 if $\mathcal{S}$ is a toplogical space and for each fixed $t$, $P(t, x; \cdot)$ is continuous in the variable $x$ in the topology of weak convergence of totally finite measures then (2.1) may be dropped and $T_t$ can be considered as a semigroup on the algebra $C_b(\mathcal{S})$ of bounded continuous functions on $\mathcal{S}$ acting on $L^2(\mu)$ by multiplication.

**Example 2.5** : Let $\{R(t) : t \in \mathbb{T}_+\}$ be a contraction semigroup on a complex Hilbert space $\mathcal{H}_0$. Then $\{T_t\}$ defined by

$$T_t(X) = R(t)XR(t)^* \text{ for } X \in \mathcal{B}(\mathcal{H}_0)$$

is a quantum dynamical semigroup on $\mathcal{B}(\mathcal{H}_0)$. Now by a well-known dilation theorem of Sz. Nagy [SzF] there exists a Hilbert space $\mathcal{H}$ containing $\mathcal{H}_0$, with a semigroup $V(t)$ of isometric or unitary operators such that

$$R(t) = PV(t)P|_{\mathcal{H}_0}$$

where $P$ is the projection of $\mathcal{H}$ onto $\mathcal{H}_0$. Define the filtration $F$ on $\mathcal{H}$ by putting $F(t)$ as the projection onto the closed linear span of $\{V(s)u : 0 \leq s \leq t, u \in \mathcal{H}_0\}$

and define $j_t : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H})$ by

$$j_t(X) = V(t)PXPV(t)^* \text{ for } t \in \mathbb{T}_+ \text{ and } X \in \mathcal{B}(\mathcal{H}_0).$$

We claim that $(\mathcal{H}, F, j_t)$ is a subordinate Markov flow on $\mathcal{B}(\mathcal{H}_0)$ with expectation semigroup $\{T_t\}$. Homomorphism and adaptedness properties of $j_t$ are trivial to show. Now fixing $0 \leq s \leq t$, consider arbitrary $a, b \leq s$ and $u, v$ in $\mathcal{H}_0$. We have

$$\begin{aligned} \langle V(a)u, j_t(X)V(b)v\rangle &= \langle u, V(t-a)PXPV(t-b)^*v\rangle \\ &= \langle u, PV(t-a)PXPV(t-b)^*Pv\rangle \\ &= \langle u, R(t-a)XR(t-b)^*v\rangle \\ &= \langle u, R(s-a)T_{t-s}(X)R(s-b)^*v\rangle \\ &= \langle V(a)u, j_s(T_{t-s}(X))V(b)v\rangle. \end{aligned}$$

As the collection of vectors $\{V(a)u : 0 \leq a \leq s \text{ and } u \in \mathcal{H}_0\}$ is total in the range of $F(s)$, we have the Markov property, $F(s)j_t(X)F(s) = j_s(T_{t-s}(X))$.

Recall that in Sz. Nagy theory ([SzF], [Da3]) an isometric dilation $V(t)$ is called *minimal* if $\{V(t)u : t \in \mathbb{T}_+, u \in \mathcal{H}_0\}$ is total in $\mathcal{H}$.

**Theorem 2.6** : Let $\{R(t) : t \in \mathbb{T}_+\}$ be a semigroup of contractions on a Hilbert space $\mathcal{H}_0$ with $R(t_0) \neq 0$ for some $t_0 > 0$. Suppose $V(t)$, $T_t$, and $(\mathcal{H}, F, j_t)$ are as above then $(\mathcal{H}, F, j_t)$ is a minimal Markov flow if and only if $V(t)$ is a minimal isometric dilation of $R(t)$. It is conservative if and only if $R(t)$ is co-isometric.

**Proof** : Assume that $V(t)$ is a minimal dilation of $R(t)$. For $s \in \mathbb{T}_+$ choose $n$ large enough so that $t_0 > \frac{s}{n} = s_0$ (say). As $R(s_0) \neq 0$, there exists $u_0 \in \mathcal{H}_0$ with $||R(s_0)u_0||^2 = 1$. For any two vectors $x, y$, let $|x\rangle\langle y|$ denote the operator $A$ defined by $Az = \langle y, z\rangle x$. Now for any $u \in \mathcal{H}_0$, taking $X = |u\rangle\langle R(s_0)^*u_0|$,

$$j_{s_0}(X)u_0 = V(s_0)XV(s_0)^*u_0 = V(s_0)XR(s_0)^*u_0 = V(s_0)u.$$

Put $Y = |R(s_0)^*u_0\rangle\langle R(s_0)^*u_0|$, and then observe that

$$j_{2s_0}(X)j_{s_0}(Y)u_0 = j_{2s_0}(X)V(s_0)u_0 = V(2s_0)XV(s_0)u = V(2s_0)u,$$

and by induction on $n$,

$$j_{ns_0}(X)j_{(n-1)s_0}(Y)\dots j_{s_0}(Y)u_0 = V(ns_0)u = V(s)u.$$

This proves the minimality of $(\mathcal{H}, F, j_t)$. The converse follows easily from the definition of $F$. If $j_t$ is conservative then so is $T_t$ and hence $R(t)$ is co-isometric.

Now assume that $R(t)$ is co-isometric for every $t$. For $a, b, t \geq 0$ and $u, v \in \mathcal{H}_0$ we have

$$\langle V(t)V(a)u, V(b)v\rangle = \begin{cases} \langle R(a+t-b)u, v\rangle & \text{if } a+t \geq b \\ \langle u, R(b-a-t)v\rangle & \text{if } a+t \leq b \end{cases}$$

These relations along with minimality of $V$ imply

$$V^*(t)V(b)v = \begin{cases} R^*(t-b)v & \text{if } t \geq b \\ V(b-t)v & \text{if } t \leq b \end{cases}$$

and a similar computation using co-isometric property of $R(t)$ shows unitarity of $V(t)$ for every $t$. Now for $0 \leq s \leq t$ and $u \in \mathcal{H}_0$, observe

$$j_t(I)V(s)u = V(t)PV(t)^*V(s)u = V(t)PR(t-s)^*u$$

$$= V(t)R(t-s)^*u = V(t)V(t)^*V(s)u = V(s)u.$$

This proves $j_t(I) = F(t)$ for every $t$. ∎

Theorem 2.6 deliberately avoids the discontinuous contraction semigroup $R$ given by

$$R(t) = \begin{cases} I & \text{for } t = 0 \\ 0 & \text{for } t > 0. \end{cases}$$

The corresponding quantum dynamical semigroup $T_t$, which is

$$T_t(X) = \begin{cases} X & \text{for } t = 0 \\ 0 & \text{for } t > 0, \end{cases}$$

has a unique subordinate minimal Markov dilation $(\mathcal{H}_0, F^0, j_t^0)$, where $F^0(t) \equiv I$ and $j_t^0(X) = T_t(X)$. But for isometric dilations of $R(t)$ one needs Hilbert spaces much larger than $\mathcal{H}_0$ and hence the associated weak Markov flows are not minimal.

**Example 2.7** : Let $\{J_t, t \geq 0\}$ be an Evans-Hudson flow ([P1], [Me], [EH]) determined by structure maps $\{\theta^i_j, i,j \geq 0\}$ on a unital von Neumann algebra of operators on a Hilbert space $\mathcal{H}_0$ so that the quantum stochastic differential equations

$$dJ_t(X) = \sum_{i,j} J_t(\theta^i_j(X))d\Lambda^j_i(t), J_0(X) = X \otimes 1,\ X \in \mathcal{A}$$

are fulfilled in the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes \ell^2)$, $\Gamma$ indicating the boson Fock space over its argument and $\Lambda^j_i$ are fundamental processes of time, creation, conservatin and annhilation (Notation as in [P1]). Let $F(t)$ denote the projection onto the subspace $\mathcal{H}_t = \mathcal{H}_0 \otimes \Gamma(L^2[0,t] \otimes \ell^2) \otimes \Phi_{[t} \subset \mathcal{H}$ where $\Phi_{[t}$ is the Fock vacuum in $\Gamma(L^2[t,\infty) \otimes \ell^2)$. Define $j_t(X) = J_t(X)F(t)$, $t \geq 0$, $X \in \mathcal{A}$. Then $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow with expectation semigroup $T_t = e^{t\theta^0_0}$, $t \geq 0$. However, this need not be a minimal Markov flow.

More examples of weak Markov flows are given in Section 4. We end this section with a simple theorem which tells us how to describe weak Markov flows without referring to quantum dynamical semigroups.

**Theorem 2.8** : Let $F$ be a weak filtration of a Hilbert space $\mathcal{H}$. Suppose that $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H}_0)$, where $\mathcal{H}_0$ is the range of $F(0)$, and $j_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a family of $*$-homomorphisms satisfying

(i) $\mathbb{E}^F_{0]} j_t(X)|_{\mathcal{H}_0} \in \mathcal{A}$, for $t \geq 0$, and $X \in \mathcal{A}$;

(ii) $\mathbb{E}^F_{s]} j_t(X) = j_s(\mathbb{E}^F_{0]} j_{t-s}(X)|_{\mathcal{H}_0})F(s)$, for $0 \leq s \leq t$, and $X \in \mathcal{A}$.

Then $T_t : \mathcal{A} \to \mathcal{A}$, defined by

$$T_t(X) = \mathbb{E}^F_{0]} j_t(X)|_{\mathcal{H}_0}$$

is a quantum dynamical semigroup on $\mathcal{A}$. Moreover $(\mathcal{H}, F, j_t)$ is a weak Markov flow with expectation semigroup $T_t$.

**Proof** : For $s,t \geq 0$ and $X \in \mathcal{A}$,

$$T_{s+t}(X) = \mathbb{E}^F_{0]} j_{s+t}(X) = \mathbb{E}^F_{0]} \mathbb{E}^F_{s]} j_{s+t}(X)$$

$$= \mathbb{E}^F_{0]} j_s(T_t(X))F(s) = \mathbb{E}^F_{0]} j_s(T_t(X)) = T_s(T_t(X))F(0).$$

Clearly $T_0(X) = X$ and $T_t$ is contractive and completely positive for every $t$. The second part is straight-forward. ∎

## 3 Moment Computations

It is not unusual in the study of non-commutative processes to have situations where only time ordered moments or correlation functions are tractable. It is the case with general weak Markov flows. However, subordinate weak Markov flows are better-behaved and moment computations are possible through a reduction algorithm. These computations pave our way to show that minimal subordinate Markov flows are completely determined by their expectation semigroups. They also help in understanding continuity properties of weak Markov flows.

Let $\mathcal{A}$ be a unital $C^*$ algebra. For any $X \in \mathcal{A}$ denote by $L_X$ and $R_X$ respectively the linear maps from $\mathcal{A}$ into itself defined by $L_X Y = XY$ and $R_X Y = YX$ for all $Y \in \mathcal{A}$. $L_X$ and $R_Y$ commute with each other for any $X, Y$. For any finite sequence $\underline{t} = (t_1, \dots, t_n)$ in $\mathbb{T}_+$ and $\underline{X} = (X_1, \dots, X_n)$ in $\mathcal{A}$ (of *length* $n$) write $j(\underline{t}, \underline{X}) = j(t_1, t_2, \dots, t_n, X_1, \dots, X_n) = j_{t_1}(X_1) j_{t_2}(X_2) \dots j_{t_n}(X_n)$. In particular, $j(t, X) = j_t(X)$. For $\underline{s} = (s_1, \dots, s_m)$, $\underline{X} = (X_1, \dots, X_m)$, $\underline{t} = (t_1, \dots, t_n)$, $\underline{Y} = (Y_1, \dots, Y_n)$ we have $j(\underline{s}, \underline{X}) j(\underline{t}, \underline{Y}) = j((\underline{s}, \underline{t}), (\underline{X}, \underline{Y}))$ where $(\underline{s}, \underline{t}) = (s_1, \dots, s_m, t_1, \dots, t_n)$, $(\underline{X}, \underline{Y}) = (X_1, \dots, X_m, Y_1, \dots, Y_n)$. Since for every $t$, $j_t$ is an algebra homomorphism we have $j(\underline{s}, t, \underline{X}, Y) j(t, Z) = j(\underline{s}, t, \underline{X}, YZ)$, and $j(s, X) j(s, \underline{t}, Y, \underline{Z}) = j(s, \underline{t}, XY, \underline{Z})$. With these conventions we shall establish a few elementary propositions concerning the operators $j(\underline{t}, \underline{X})$ and their expectation values.

**Proposition 3.1** : Let $(\mathcal{H}, F, j_t)$ be a weak Markov flow with expectation semigroup $\{T_t\}$ on a $C^*$ algebra acting on a Hilbert space $\mathcal{H}_0$. Then the following hold:

(i) $j_t(X) F(t) = F(t) j_t(X) = F(t) j_t(X) F(t)$ for all $t \geq 0, X \in \mathcal{A}$;

(ii) If $0 \leq s \leq t_1 \leq \dots \leq t_n, X_1, X_2, \dots, X_n \in \mathcal{A}$ then

$$\mathbb{E}^F_{s]} j(\underline{t}, \underline{X}) = j(s, T_{t_1 - s} L_{X_1} T_{t_2 - t_1} \cdots L_{X_{n-1}} T_{t_n - t_{n-1}}(X_n)) F(s);$$

(iii) If $t_1 \geq t_2 \geq \cdots \geq t_n \geq s \geq 0$ then

$$\mathbb{E}^F_{s]} j(\underline{t}, \underline{X}) = j(s, T_{t_n - s} R_{X_n} T_{t_{n-1}-t_n} \cdots R_{X_2} T_{t_1 - t_2}(X_1)) F(s).$$

**Proof :** From property (i) in Definition 2.2 we have

$$\begin{aligned} F(t)j(t,X) &= \{j(t,X^*)F(t)\}^* \\ &= \{F(t)j(t,X^*)F(t)\}^* \\ &= F(t)j(t,X)F(t) \\ &= j(t,X)F(t). \end{aligned}$$

This proves (i). To prove (ii) we use property (i) of this proposition and the increasing nature of $F(t)$ repeatedly. Thus

$$\begin{aligned} \mathbb{E}^F_{s]} j(\underline{t}, \underline{X}) &= F(s)F(t_1)j(t_1,X_1)\ldots j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,X_1)F(t_1)F(t_2)j(t_2,X_2)\ldots j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,X_1)j(t_2,X_2)F(t_2)j(t_3,X_3)\ldots j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,\ldots,t_{n-1},X_1,\ldots,X_{n-1})F(t_{n-1})j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,\ldots,t_{n-1},X_1,\ldots,X_{n-1})j(t_{n-1},T_{t_n - t_{n-1}}(X_n))F(s) \\ &= \mathbb{E}^F_{s]} j(t_1,\ldots,t_{n-1},X_1,\ldots,X_{n-2},X_{n-1}T_{t_n - t_{n-1}}(X_n)). \end{aligned}$$

Now (ii) follows by induction on $n$. A similar argument yields (iii). ■

Explicit moment computations are possible for subordinate weak Markov flows due to the following important observation.

**Proposition 3.2 :** Let $(\mathcal{H}, F, j_t)$ be a subordinate Markov flow with expectation semigroup $\{T_t\}$ on a unital $C^*$ algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}_0$. Then for any three time points $a, b, c$ in $\mathbb{T}_+$ with $b \geq a, c$ and $X, Y, Z$ in $\mathcal{A}$,

$$j_a(X)j_b(Y)j_c(Z) = \begin{cases} j_a(XT_{b-a}(Y))j_c(Z) & \text{if } c \leq a \leq b, \\ j_a(X)j_c(T_{b-c}(Y)Z) & \text{if } a \leq c \leq b, \\ j_a(XT_{b-a}(Y)Z) & \text{if } a = c \text{ and } a \leq b. \end{cases}$$

**Proof :** If $c \leq a \leq b$, as $j_t$ is a subordinate flow, $j_c(I) \leq F(c) \leq F(a)$, and

$j_a(I) \leq F(a)$. Now using the Markov property of $j_t$,

$$\begin{aligned} j_a(X)j_b(Y)j_c(Z) &= j_a(X)j_a(I)j_b(Y)j_c(I)j_c(Z) \\ &= j_a(X)F(a)j_b(Y)F(a)j_c(Z) \\ &= j_a(X)(\mathbb{E}^F_{a]}j_b(Y))j_c(Z) \\ &= j_a(XT_{b-a}(Y))j_c(Z). \end{aligned}$$

Other parts can be proved in a similar way. ∎

Consider a finite sequence $\underline{t} = (t_1, \ldots, t_n)$ in $\mathbb{T}_+$. If for some index $k$ in $\{2, \ldots, n-1\}$, $t_k \geq t_{k-1}, t_{k+1}$, then $t_k$ is said to be a *peak*. Whenever $t_k$ is a peak the proposition above can be applied at the triple $(t_{k-1}, t_k, t_{k+1})$ to reduce the length of $\underline{t}, \underline{X}$ in $j(\underline{t}, \underline{X})$. Repeated application of this procedure is a convenient reduction algorithm as should be clear from the computations below.

**Theorem 3.3** : Let $(\mathcal{H}, F, j_t)$ be a subordinate Markov flow as in Proposition 3.2. Then for any sequence $t_1, t_2, \ldots, t_n$ in $\mathbb{T}_+$ and $X_1, \ldots, X_n$ in $\mathcal{A}$ there exists a sequence $s_1, s_2, \ldots, s_m$ in $\mathbb{T}_+$ and $Y_1, \ldots, Y_m$ in $\mathcal{A}$ such that $m \leq n, s_1 = t_1, s_m = t_n$, and either $s_1 < s_2 < \cdots < s_m$ or $s_1 > s_2 > \cdots > s_m$ or $s_1 > s_2 > \cdots > s_k < s_{k+1} < \cdots < s_m$ for some $k$ and $j(\underline{t}, \underline{X}) = j(\underline{s}, \underline{Y})$.

**Proof** : Without loss of generality we may assume that $t_i \neq t_{i+1}$ for $1 \leq i < n$. If $\{t_i\}$ itself is either monotonic increasing or decreasing there is nothing to prove. If $t_1 < \cdots < t_i > t_{i+1}$ then $t_i$ is a peak and on applying Proposition 3.2 at $(t_{i-1}, t_i, t_{i+1})$ we have

$$j(\underline{t}, \underline{X}) = \begin{cases} j((t_1, \ldots t_{i-1}, t_{i+1}, \ldots, t_n), \underline{X}') & \text{for some } \underline{X}' \text{ if } t_{i-1} \neq t_{i+1} \\ j((t_1, \ldots t_{i-1}, t_{i+2}, \ldots, t_n), \underline{X}'') & \text{for some } \underline{X}'' \text{ if } t_{i-1} = t_{i+1} \end{cases}$$

This way the length of the $\underline{t}$ sequence gets reduced and consecutive indices still remain distinct. If $t_1 > \cdots > t_k < t_{k+1} < \ldots < t_{k+l} > t_{k+l+1}$, then $t_{k+l}$ is a peak and the length of the $\underline{t}$ sequence can once again can be reduced. Rest follows by induction on the length. ∎

**Remark 3.4** : The set $\{s_1, \ldots, s_m\}$ of Theorem 3.3 can be shown to be equal to $\{t_r : 1 \leq r \leq n$, and there does not exist a pair $(i,j)$ with $1 \leq i \leq r \leq j \leq n$, such that $t_i, t_j < t_r\}$.

**Corollary 3.5** : Let $(\mathcal{H}, F, j_t)$ satisfy the conditions of Proposition 3.2. Then for any sequence $t_1, t_2, \ldots, t_n$ in $\mathbb{T}_+$ and $X_1, X_2, \ldots, X_n$ in $\mathcal{A}$ there exist $t_1 = s_1 > s_2 > \cdots > s_m \geq 0$, $m \leq n$ and $Y_1, Y_2, \ldots, Y_m$ in $\mathcal{A}$ such that $j(\underline{t}, \underline{X})F(0) = j(\underline{s}, \underline{Y})F(0)$.

**Proof** : From the assumption that $j_t$ is a subordinate flow it follows that $F(0) = j_0(I)$. Hence $j(\underline{t}, \underline{X})F(0)$ can be written as $j((\underline{t}, 0), (\underline{X}, I))F(0)$. Now apply Theorem 3.3 to have

$$j((\underline{t}, 0), (\underline{X}, I))F(0) = j(\underline{s}, \underline{Y})F(0).$$

As $s_1 = t_1$ and $s_m = 0$ and $s_i$'s are nonnegative the possibilities $s_1 < s_2 < \cdots < s_m$ and $s_1 > s_2 > \cdots > s_k < s_{k+1} < \cdots < s_m$ are ruled out. Hence $\underline{s}$ is bound to be monotonically decreasing. ∎

**Theorem 3.6** : Let $(\mathcal{H}, F, j_t)$ be a subordinate Markov flow as in Proposition 3.2. Then for any sequence $t_1, \ldots, t_n$ in $\mathbb{T}_+$ and $X_1, \ldots, X_n$ in $\mathcal{A}$,

$$F(0)j(\underline{t}, \underline{X})F(0) = YF(0)$$

for some $Y$ in $\mathcal{A}$, where $Y$ depends only on $(\underline{t}, \underline{X})$ and the semigroup $\{T_t\}$ and not on the flow $j_t$.

**Proof** : On rewriting $F(0)j(\underline{t}, \underline{X})F(0)$ as $F(0)j((0, \underline{t}), (I, \underline{X}))F(0)$ from Corollary 3.5, we have

$$F(0)j(\underline{t}, \underline{X})F(0) = j_0(Y)F(0) = YF(0)$$

for some $Y$ in $\mathcal{A}$. The operator $Y$ is obtained by a repeated application of Proposition 3.2 and at each stage the changes in $\underline{X}$ are determined by time orderings and independently of the underlying flow $j_t$ as such. ∎

We have not bothered to compute the resulting $\underline{Y}$ explicitly in Theorems 3.4 and 3.6 as it is rather cumbersome. But now we do it for several special cases for ready reference in the future.

**Proposition 3.7** : Let $(\mathcal{H}, F, j_t)$ be a subordinate Markov flow as in Proposition 3.2. Consider $r_1 > r_2 > \cdots > r_n$ and $s_1 > s_2 > \cdots > s_p$ in $\mathbb{T}_+$ with $r_1 = s_1$. Then for $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_p$ in $\mathcal{A}$

$$F(0)j((r_n, r_{n-1}, \ldots, r_1, s_1, \ldots, s_p), (Y_n, \ldots, Y_1, Z_1, \ldots, Z_p))F(0)$$
$$= T_{t_m} N_m T_{t_{m-1}-t_m} N_{m-1} \ldots N_2 T_{t_1-t_2}(Y_1 Z_1),$$

where $\{t_1, \ldots, t_m\}$ is $\{r_1, \ldots, r_n\} \cup \{s_1, \ldots, s_p\}$ rearranged in decreasing order and

$$N_i = \begin{cases} L_{Y_k} & \text{if } t_i = r_k \text{ and } t_i \neq s_l \text{ for all } l; \\ R_{Z_l} & \text{if } t_i = s_l \text{ and } t_i \neq r_k \text{ for all } k; \\ L_{Y_k} R_{Z_l} & \text{if } t_i = r_k = s_l \text{ for some } k, l. \end{cases}$$

**Proof** : This is a straight-forward application of the reduction algorithm applied at the peaks $t_1, t_2, \ldots,$ successively. ∎

**Propositon 3.8** : Let $(\mathcal{H}, F, j_t)$ be a subordinate weak Markov flow as in Proposition 3.2. Then the following hold:

(i) If $0 \leq t \leq r_1 < r_2 < \cdots < r_n$, $Y_1, \ldots, Y_n, X \in \mathcal{A}$ then $j(\underline{r}, t, \underline{Y}, X) = j(r_1, t, Y', X)$ where

$$Y' = L_{Y_1} T_{r_2-r_1} L_{Y_2} T_{r_3-r_2} \cdots L_{Y_{n-1}} T_{r_n-r_{n-1}}(Y_n);$$

(ii) If $0 \leq r_1 < r_2 < \cdots < r_{i-1} \leq t \leq r_i < r_{i+1} < \cdots < r_n$ then $j(\underline{r}, t, \underline{Y}, X) = j(r_1, \ldots, r_{i-1}, t, Y_1, \ldots, Y_{i-1}, Y')$ where

$$Y' = R_X T_{r_i - t} L_{Y_i} T_{r_{i+1}-r_i} \cdots L_{Y_{n-1}} T_{r_n - r_{n-1}}(Y_n).$$

**Proof** : First we prove (i). Note that $r_n$ is a peak in $(\underline{r}, t)$. Hence by the reduction algorithm

$$j(\underline{r}, t, \underline{Y}, X) = j(r_1, \ldots, r_{n-2}, r_{n-1}, t, Y_1, \ldots, Y_{n-2}, Y_{n-1} T_{r_n - r_{n-1}}(Y_n), X).$$

Now (i) follows by induction on $n$. To prove (ii) we apply (i) to the sequence $t \leq r_i < r_{i+1} < \cdots < r_n$ and obtain

$$j(r_i, r_{i+1}, \ldots, r_n, t, Y_i, Y_{i+1}, \ldots, Y_n, X) = j(r_i, t, Y'', X)$$

where

$$Y'' = L_{Y_i} T_{r_{i+1}-r_i} \cdots L_{Y_{n-1}} T_{r_n - r_{n-1}}(Y_n).$$

Now as $r_i$ is a peak in $(r_{i-1}, r_i, t)$ we have

$$j(r_{i-1}, r_i, t, Y_{i-1}, Y'', X) = j(r_{i-1}, t, Y_{i-1}, T_{r_i - t}(Y'')X),$$

which implies (ii). ∎

**Proposition 3.9** : Let $(\mathcal{H}, F, j_t)$ be a subordinate Markov flow as in Proposition 3.2. Suppose that $r_1 > r_2 > \cdots > r_n \geq 0$ and $X, Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n$ are in $\mathcal{A}$. Then

$$F(0)j((r_n, r_{n-1}, \ldots, r_1, t, r_1, \ldots, r_n), (Y_n, Y_{n-1}, \ldots, Y_1, X, Z_1, \ldots, Z_n))F(0)$$

$$= \begin{cases} F(0)\{T_{r_n} L_{Y_n} R_{Z_n} T_{r_{n-1}-r_n} \cdots L_{Y_1} R_{Z_1} T_{t-r_1}(X)\}F(0) & \text{if } t \geq r_1 \text{ ;} \\ F(0)\{T_{r_n} L_{Y_n} R_{Z_n} T_{r_{n-1}-r_n} \cdots & \\ \quad L_{Y_i} R_{Z_i} T_{t-r_i}(T_{r_{i-1}-t}(Y'_{i-1}) X T_{r_{i-1}-t}(Z'_{i-1}))\}F(0) & \text{if } r_{i-1} \geq t \geq r_i \text{ ;} \\ F(0)\{T_t(T_{r_n-t}(Y'_n) X T_{r_n-t}(Z'_n))\}F(0) & \text{if } r_n \geq t \geq 0, \end{cases}$$

where $Y'_i$ and $Z'_i$ depend only on $r_1, \ldots, r_i, Y_1, \ldots, Y_i$, and $Z_1, \ldots, Z_i$ for any $i$.

**Proof** : The first part follows from Proposition 3.7. If $r_{i-1} \geq t \geq r_i$, by (i) in Proposition 3.8 we have

$$j(r_{i-1}, r_{i-2}, \ldots, r_1, t, Y_{i-1}, Y_{i-2}, \ldots, Y_1, X) = j(r_{i-1}, t, Y'_{i-1}, X) \tag{3.1}$$

where $Y'_{i-1}$ depends only on $r_1, \ldots, r_{i-1}, Y_1, \ldots, Y_{i-1}$. Since $t \leq r_2 < r_1$

$$j(t, r_1, r_2, X, Z_1, Z_2) = j(t, r_2, X, T_{r_1 - r_2}(Z_1) Z_2).$$

Repeating this argument upto the pair $r_{i-2}, r_{i-1}$ we get

$$j(t, r_1, r_2, \ldots, r_{i-1}, X, Z_1, \ldots, Z_{i-1}) = j(t, r_{i-1}, X, Z'_{i-1}), \tag{3.2}$$

for some $Z'_{i-1}$. Since $r_i \leq t \leq r_{i-1}$ we have

$$j(r_i, r_{i-1}, t, r_{i-1}, r_i, Y_i, Y'_{i-1}, X, Z'_{i-1}, Z_i)$$

$$= j(r_i, t, r_i, Y_i, T_{r_{i-1}-t}(Y'_{i-1}) X T_{r_{i-1}-t}(Z'_{i-1}), Z_i). \tag{3.3}$$

Combining (3.1)-(3.3) and using the proved first part above for the sequence $r_n, r_{n-1}, \dots, r_i, t, r_i, r_{i+1}, \dots, r_n$ we obtain the required result.

Finally if $t \leq r_n$, as $t \leq r_2 \leq r_1$ once again by the reduction algorithm

$$j(t, r_1, r_2, X, Z_1, Z_2) = j(t, X) j(r_2, T_{r_1 - r_2}(Z_1) Z_2).$$

Repeating this argument we get

$$j(t, r_1, \dots, r_n, X, Z_1, \dots, Z_n) = j(t, r_n, X, Z_n')$$

where $Z_n'$ depends only on $r_1, \dots, r_n, Z_1, \dots, Z_n$. Since $r_n < r_{n-1} < \dots < r_1 \geq t$ and $t \leq r_n$ we have from (i) in Proposition 3.8

$$j(r_n, r_{n-1}, \dots, r_1, t, Y_n, Y_{n-1}, \dots, Y_1, X) = j(r_n, Y_n') j(t, X)$$

where $Y_n'$ depends only on $Y_1, \dots, Y_n, r_1, \dots, r_n$. Combining the two we obtain

$$j(r_n, \dots, r_1, t, r_1, \dots, r_n, Y_n, \dots, Y_1, X, Z_1, \dots, Z_n) = j(r_n, t, r_n, Y_n', X, Z_n').$$

Since $0 \leq r_n \geq t$ we have

$$F(0) j(r_n, t, r_n, Y_n', X, Z_n') F(0) = F(0) T_t (T_{r_n - t}(Y_n') X T_{r_n - t}(Z_n')) F(0).$$

■

**Proposition 3.10** : Let $(\mathcal{H}, F, j_t)$ be a subordinate Markov flow as in Proposition 3.2. Suppose also that it is conservative. If $s_1 > s_2 > \dots > s_p \geq 0$, $r_1 > r_2 > \dots > r_n \geq 0$ and $\{s_1, s_2, \dots, s_p\} \supseteq \{r_1, r_2, \dots, r_n\}$ then for any $Y_1, Y_2, \dots, Y_n$ in $\mathcal{A}$

$$j(\underline{r}, \underline{Y}) F(0) = j(\underline{s}, \underline{\tilde{Y}}) F(0);$$

where $\underline{\tilde{Y}} = (\tilde{Y}_1, \dots, \tilde{Y}_p)$ with

$$\tilde{Y}_i = \begin{cases} Y_k & \text{if } s_i = r_k \text{ for some } k, \\ I & \text{otherwise.} \end{cases}$$

**Proof :** Let $s_{i_1} = r_1, \ldots, s_{i_n} = r_n$. Then

$$\begin{aligned} j(r_k, Y_k) &= j(r_k, I) j(s_{i_k}, Y_k) \\ &= F(s_{i_k}) j(s_{i_k}, Y_k) \\ &= F(s_{i_{k-1}+1}) F(s_{i_{k-1}+2}) \cdots F(s_{i_k - 1}) j(s_{i_k}, Y_k) \\ &= j(s_{i_{k-1}+1}, s_{i_{k-1}+2}, \ldots, s_{i_k - 1}, s_{i_k}, I, I, \ldots, I, Y_k) \end{aligned}$$

from which the required result follows. ■

**Proposition 3.11 :** Suppose $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow with a strongly continuous expectation semigroup $\{T_t, t \in I\!R_+\}$ on a unital $C^*$ algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}_0$. Then for any $u, v \in \mathcal{H}_0$, finite sequences $\underline{r} = (r_1, \ldots, r_n)$, $\underline{s} = (s_1, \ldots, s_p)$ in $I\!R_+$ and $Y_1, \ldots, Y_n, X, Z_1, \ldots, Z_p, \in \mathcal{A}$ the function

$$\phi(t) = \langle j(\underline{r}, \underline{Y}) u, j_t(X) j(\underline{s}, \underline{Z}) v \rangle$$

is continuous in $t \in I\!R_+$.

**Proof:** Since $F(0)u = u$, $F(0)v = v$ we can apply Corollary 3.5 and assume without loss of generality that $r_1 > r_2 > \cdots > r_n$ and $s_1 > s_2 > \cdots > s_p$. Since the flow is conservative we can apply Proposition 3.10 and assume without loss of generality that the sequences $\underline{r}$ and $\underline{s}$ are same and strictly decreasing. Then $\phi(t)$ assumes the form

$$\phi(t) = \langle u, F(0) j(r_n, r_{n-1}, \ldots, r_1, t, r_1, \ldots, r_n, Y_n^*, \ldots, Y_1^*, X, Z_1, \ldots, Z_n) F(0) v \rangle.$$

Now the strong continuity and contractivity properties of $\{T_t\}$ together with Proposition 3.9 imply the continuity of $\phi(t)$ in the intervals $[r_1, \infty)$, $[0, r_n]$ and $[r_i, r_{i-1}]$, $i = n, n-1, \ldots, 2$. ■

# 4 Existence and Uniqueness

Given a quantum dynamical semigroup one would like to realize it as the expectation semigroup of a weak Markov flow. In this section it is established that this can always be achieved. Moreover the flow which does the trick is unique upto unitary equivalence under a natural cyclicity assumption. In other words every quantum dynamical semigroup has a unique dilation to a minimal subordinate Markov flow. Classically, Markov processes are constructed using the Kolmogorov consistency theorem. The corresponding theorem here is the G.N.S. (Gelfand, Naimark, and Segal) construction of Hilbert spaces from positive definite kernels and it is not surprising that this is the main tool we need. The proof given in [BP2] made use of [P2] which, in turn, uses Stinespring's theorem. Here a more direct proof has been presented with constructions similar to that of [Em] and [Vi-S].

Note that given a contraction on a Hilbert space, it is a trivial matter to dilate it to a unitary operator on a larger Hilbert space. In fact, if $A$ is a contraction on a Hilbert space $\mathcal{H}$ then

$$C = \begin{bmatrix} A & (I - AA^*)^{\frac{1}{2}} \\ -(I - A^*A)^{\frac{1}{2}} & A^* \end{bmatrix}$$

is a unitary operator on $\mathcal{H} \oplus \mathcal{H}$. Sz. Nagy and Foias [SzF] achieved a simultaneous dilation of all the contractions in a contraction semigroup to get an isometric semigroup (or unitary group). The famous structure theorem of Stinespring shows that a completely positive map can be dilated to a representation of the algebra. The Markov flows are nothing but dilations of semigroups of completely positive maps to a well-knit family of representations of the algebra.

We begin with the statements of the well-known structure theorems of G.N.S. and Stinespring mentioned above.

**Definition 4.1** : Let $\mathcal{X}$ be a set. Then a map $K : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is said to be a *positive definite kernel* on $\mathcal{X}$ if

$$\sum_{i,j} \bar{c}_i c_j K(x_i, x_j) \geq 0$$

for all choices of $c_i \in \mathbb{C}$, $x_i \in \mathcal{X}, i = 1, 2, \ldots, n$ for $n = 1, 2, \ldots$.

**Theorem 4.2 (G.N.S. construction)** : Let $K$ be a positive definite kernel on a set $\mathcal{X}$. Then there exists a Hilbert space $\mathcal{H}$ and a map $\lambda : \mathcal{X} \to \mathcal{H}$ satisfying the following: (i) $K(x,y) = \langle\lambda(x),\lambda(y)\rangle$ for all $x,y \in \mathcal{X}$; (ii) $\{\lambda(x), x \in \mathcal{X}\}$ is total in $\mathcal{H}$. (iii) If $\mathcal{H}'$ is another Hilbert space with a map $\lambda' : \mathcal{X} \to \mathcal{H}'$, satisfying (i), (ii) with $\mathcal{H}, \lambda$ replaced by $\mathcal{H}', \lambda'$ respectively then there exists a unitary isomorphism $U : \mathcal{H} \to \mathcal{H}'$ such that $U\lambda(x) = \lambda'(x)$ for all $x$ in $\mathcal{X}$.

**Proof :** See [P1] or [EL]. ∎

We refer to the pair $(\mathcal{H}, \lambda)$, of the theorem above, which is unique upto unitary equivalence as the *Gelfand pair* associated with the positive definite kernel $K$.

**Theorem 4.3 (Stinespring)** : Let $\mathcal{A}$ and $\mathcal{A}'$ be two $C^*$ algebras with $\mathcal{A}'$ acting on a Hilbert space $\mathcal{H}_0$. Suppose $T : \mathcal{A} \to \mathcal{A}'$ is a completely positive map, then there exists a triple $(\mathcal{H}, P, j)$, where $\mathcal{H}$ is a Hilbert space containing $\mathcal{H}_0$, P is the projection of $\mathcal{H}$ on to $\mathcal{H}_0$, and $j$ is a representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ such that: (i) $T(X) = Pj(X)P|_{\mathcal{H}_0}$ for all $X$ in $\mathcal{A}$ ; (ii) $\{j(X)Vu : u \in \mathcal{H}_0, X \in \mathcal{A}\}$ is total in $\mathcal{H}$. Moreover if $(\mathcal{H}', P', j')$ is another such triple satisfying (i) and (ii) with $\mathcal{H}, P, j$ replaced by $\mathcal{H}', P', j'$ respectively, then there exists a unitary isomorphism $U : \mathcal{H} \to \mathcal{H}'$ such that $Uu = u$ for $u \in \mathcal{H}_0$ and $j'(\cdot) = Uj(\cdot)U^*$.

**Proof :** We give a brief sketch and refer to [P1] and [St] for more detailed proofs. Define the positive definite kernel $K$ on the Cartesian product $\mathcal{H}_0 \times \mathcal{A}$ by putting

$$K((u,X),(v,Y)) = \langle u, T(X^*Y)v\rangle \text{ for } u,v \in \mathcal{H}_0 \text{ and } X,Y \in \mathcal{A}.$$

The complete positivity of $T$ ensures that $K$ is a positive definite kernel. Let $(\mathcal{H}, \lambda)$ be the associated Gelfand pair constructed using Theorem 4.2. Now identify $u \in \mathcal{H}_0$ with $\lambda(u, I)$ and define $j(X)$ as a linear map satisfying

$$j(X)\lambda(u,Y) = \lambda(u, XY)$$

for all $u$ in $\mathcal{H}_0$ and $X, Y$ in $\mathcal{A}$. ∎

The triplet $(\mathcal{H}, P, j)$ will be referred to as the Stinespring triple associated with the completely positive map $T$.

Let $\mathcal{A}$ be a unital $C^*$ algebra acting on a Hilbert space $\mathcal{H}_0$. Let $\{T_t : t \in \mathbb{T}_+\}$ be a conservative quantum dynamical semigroup on $\mathcal{A}$ . We want to define a specific positive definite kernel on the set $\mathcal{M}$, where

$$\mathcal{M} = \{(\underline{r}, \underline{Y}, u) : u \in \mathcal{H}_0, \underline{r} = (r_1, \ldots, r_n), r_1 > r_2 > \cdots > r_n \geq 0 \text{ in } \mathbb{T}_+,$$
$$Y_1, \ldots Y_n \in \mathcal{A}, \text{ for some } n \geq 1\} \tag{4.1}$$

and then construct a Hilbert space using the G.N.S. construction. We need some notation. Let $\mathcal{M}_{t]}$ and $\mathcal{M}_t$ be the sets defined by

$$\mathcal{M}_{t]} = \{(\underline{r}, \underline{Y}, u) \in \mathcal{M} : t \geq r_1 > r_2 > \cdots > r_n \geq 0\} \tag{4.2}$$

and

$$\mathcal{M}_t = \{(\underline{r}, \underline{Y}, u) \in \mathcal{M} : t = r_1 > r_2 > \cdots > r_n \geq 0\}. \tag{4.3}$$

Consider $(\underline{r}, \underline{Y}, u) \in \mathcal{M}$. Then for any $\underline{s} = (s_1, \ldots, s_p)$ with $s_1 > s_2 > \cdots, s_p \geq 0$ in $\mathbb{T}_+$, we write $\underline{s} \geq \underline{r}$ if $\{s_1, \ldots, s_p\} \supseteq \{r_1, \ldots, r_n\}$ and in such a case define $(\underline{s}, \underline{\tilde{Y}}, u)$ by putting

$$\tilde{Y}_j = \begin{cases} Y_i & \text{if } s_j = r_i \text{ for some } i; \\ I & \text{otherwise,} \end{cases}$$

i.e., we extend $\underline{Y}$ by inserting identity at the extra time points. Now define the map $L : \mathcal{M} \times \mathcal{M} \to \mathbb{C}$ by

$$L((\underline{t}, \underline{Y}, u), (\underline{t}, \underline{Z}, v)) = \langle u, T_{t_n}(Y_n^* T_{t_{n-1}-t_n}(Y_{n-1}^* \cdots Y_2^* T_{t_1-t_2}(Y_1^* Z_1) Z_2 \cdots Z_{n-1}) Z_n) v \rangle$$

and

$$L((\underline{r}, \underline{Y}, u), (\underline{s}, \underline{Z}, v)) = L((\underline{r} \vee \underline{s}, \underline{\tilde{Y}}, u), (\underline{r} \vee \underline{s}, \underline{\tilde{Z}}, v)) \tag{4.4}$$

where $\underline{r} \vee \underline{s}$ is obtained by arranging $\{r_1, r_2, \ldots\} \cup \{s_1, s_2, \ldots\}$ in decreasing order.

**Lemma 4.4** : Let $\underline{Y}^{(i)} = (Y_{1i}, \ldots, Y_{ni})$, $1 \leq i \leq p$, be $n$-tuples with entries from $\mathcal{A}$. Then $A = ((A_{ik}))_{1 \leq i,k \leq p}$ with

$$A_{ik} = T_{r_n}(Y_{ni}^* T_{r_{n-1}-r_n}(Y_{(n-1)i}^* \cdots Y_{2i}^* T_{r_1-r_2}(Y_{1i}^* Y_{1k}) Y_{2k} \cdots Y_{(n-1)k}) Y_{nk})$$

is positive as an operator on $\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_0$ ($p$ copies), for arbitrary $\underline{r} = (r_1, \cdots, r_n)$, $r_1 \geq r_2 \geq \cdots r_n \geq 0$.

**Proof :** The proof is through induction on $n$. For $n = 1$, $A = ((T_{r_1}(Y_i^* Y_k))) \geq 0$ as the operator

$$((Y_i^* Y_k)) = \begin{pmatrix} Y_1^* \\ \vdots \\ Y_p^* \end{pmatrix} (Y_1 \ \cdots \ Y_p)$$

is positive and $T_{r_1}$ is completely positive. Now assuming the result for $n-1$, we have $B = ((B_{ik})) \geq 0$, where

$$B_{ik} = T_{r_{n-1}-r_n}(Y^*_{(n-1)i} \cdots Y^*_{2i} T_{r_1-r_2}(Y^*_{1i} Y_{1k}) Y_{2k} \cdots Y_{(n-1)k}).$$

On taking $C_{ik} = Y^*_{ni} B_{ik} Y_{nk}$ , observe that

$$C = \begin{pmatrix} Y_{n1} & & \\ & \ddots & \\ & & Y_{np} \end{pmatrix}^* B \begin{pmatrix} Y_{n1} & & \\ & \ddots & \\ & & Y_{np} \end{pmatrix} \geq 0.$$

Now the proof is complete by the complete positivity of $T_{r_n}$. ∎

**Proposition 4.5 :** The map $L$ defined by (4.4) is a positive definite kernel on $\mathcal{M}$.

**Proof :** First notice that conservativity of $\{T_t\}$ allows us to conclude

$$L((\underline{s}, \tilde{\underline{Y}}, u), (\underline{s}, \tilde{\underline{Z}}, v)) = L((\underline{r}, \underline{Y}, u), (\underline{r}, \underline{Z}, v))$$

for $\underline{s} \geq \underline{r}$ and $(\underline{r}, \underline{Y}, u), (\underline{r}, \underline{Z}, v)$ in $\mathcal{M}$. Now for $1 \leq i \leq p$ consider arbitrary $c_i \in \mathbb{C}$ and $(\underline{r}^{(i)}, \underline{Y}^{(i)}, u^{(i)})$. From the definition in (4.4) and the observation above, we have

$$\begin{aligned} & \sum \bar{c}_i c_k L((\underline{r}^{(i)}, \underline{Y}^{(i)}, u^{(i)}), (\underline{r}^{(k)}, \underline{Y}^{(k)}, u^{(k)})) \\ = & \sum \bar{c}_i c_k L((\underline{r}^{(i)} \vee \underline{r}^{(k)}, \tilde{\underline{Y}}^{(i)}, u^{(i)}), (\underline{r}^{(i)} \vee \underline{r}^{(k)}, \tilde{\underline{Y}}^{(k)}, u^{(k)})) \\ = & \sum \bar{c}_i c_k L((\underline{r}, \tilde{\underline{Y}}^{(i)}, u^{(i)}), (\underline{r}, \tilde{\underline{Y}}^{(k)}, u^{(k)})) \qquad (4.5) \end{aligned}$$

where $\underline{r} = \underline{r}^{(1)} \vee \underline{r}^{(2)} \cdots \vee \underline{r}^{(p)}$. Denoting the co-ordinates of $\underline{r}$ by $(r_1, \ldots, r_n)$, the entry $\tilde{Y}_l^{(i)}$ by $Y_{li}$ and the vector $u^{(i)}$ by $u_i$, the term on the right hand side of (4.5) becomes

$$\sum_{i,k} \bar{c}_i c_k \langle u_i, A_{ik} u_k \rangle = \langle (\oplus c_i u_i), A(\oplus c_k u_k) \rangle \geq 0,$$

where $A = ((A_{ik}))$ is as in Lemma 4.4. ■

**Definition 4.6** : Two weak Markov flows $(\mathcal{H}, F, j_t)$ and $(\mathcal{H}', F', j_t')$ on a $C^*$ algebra $\mathcal{A}$ are called *unitarily equivalent* if there exists a unitary isomorphism $U : \mathcal{H} \to \mathcal{H}'$, such that $F(t) = U^* F'(t) U$ and $j_t(X) = U^* j_t'(X) U$ for all $t$ and $X$.

Now we are ready to state and prove the basic dilation theorem for conservative quantum dynamical semigroups.

**Theorem 4.7** : Let $\mathcal{H}$ be a unital $C^*$ algebra of operators on a Hilbert space $\mathcal{H}_0$. Suppose that $\{T_t : t \in \mathbb{T}_+\}$ is a conservative quantum dynamical semigroup on $\mathcal{A}$. Then there exists a minimal conservative weak Markov flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ with expectation semigroup $\{T_t\}$. Moreover if $(\mathcal{H}', F', j_t')$ is another such Markov flow then it is unitarily equivalent to $(\mathcal{H}, F, j_t)$.

**Proof** : Define the map $L : \mathcal{M} \times \mathcal{M} \to \mathbb{C}$ as in (4.4). Now from Proposition 4.5 $L$ is a positive definite kernel on $\mathcal{M}$. Hence by G.N.S. construction (Theorem 4.2) there exists a Hilbert space $\mathcal{H}$ and a map $\lambda : \mathcal{M} \to \mathcal{H}$, with

$$\langle \lambda(\underline{r}, \underline{Y}, u), \lambda(\underline{s}, \underline{Z}, v) \rangle = L((\underline{r}, \underline{Y}, u), (\underline{s}, \underline{Z}, v))$$

and $\{\lambda(\underline{r}, \underline{Y}, u) : (\underline{r}, \underline{Y}, u) \in \mathcal{M}\}$ is total $\mathcal{H}$. We shall construct $j_t$ so that $j(\underline{r}, \underline{Y})u = \lambda(\underline{r}, \underline{Y}, u)$. For $\underline{s} \geq \underline{r}$ (notation as in (4.1)-(4.4)), observe that

$$\begin{aligned}
& \|\lambda(\underline{s}, \underline{\tilde{Y}}, u) - \lambda(\underline{r}, \underline{Y}, u)\|^2 \\
= \;& L((\underline{s}, \underline{\tilde{Y}}, u), (\underline{s}, \underline{\tilde{Y}}, u)) - L((\underline{r} \vee \underline{s}, \underline{\tilde{Y}}, u), (\underline{r} \vee \underline{s}, \underline{\tilde{Y}}, u)) \\
& - L((\underline{r} \vee \underline{s}, \underline{\tilde{Y}}, u), (\underline{r} \vee \underline{s}, \underline{\tilde{Y}}, u)) + L((\underline{r}, \underline{Y}, u), (\underline{r}, \underline{Y}, u)) \\
= \;& 0,
\end{aligned}$$

by (4.4). And hence

$$\lambda(\underline{s}, \underline{\tilde{Y}}, u) = \lambda(\underline{r}, \underline{Y}, u). \tag{4.6}$$

Take $F(t)$ as the projection onto $\mathcal{H}_{t]}$, where $\mathcal{H}_{t]}$ is the closed linear span of $\{\lambda(\underline{r}, \underline{Y}, u) : (\underline{r}, \underline{Y}, u) \in \mathcal{M}_{t]}\}$. In view of (4.6), if $(\underline{r}, \underline{Y}, u) \in \mathcal{M}_{t]}$ with $t$ strictly larger than $r_1$ then $\lambda(\underline{r}, \underline{Y}, u) = \lambda((t, \underline{r}), (I, \underline{Y}), u)$ and hence $\mathcal{H}_{t]}$ is also the closed linear span of $\{\lambda(\underline{r},\underline{Y},u) : \lambda(\underline{r},\underline{Y},u) \in \mathcal{M}_t\}$. Also note that the range $\mathcal{H}_{0]}$ of $F(0)$ is isomorphic to $\mathcal{H}_0$ through the isomorphism $V_0 : \mathcal{H}_{0]} \to \mathcal{H}_0$ defined by $V_0(\lambda(0,X,u)) = Xu$. We identify $\mathcal{H}_0$ with $\mathcal{H}_{0]}$ through $V_0$.

Having defined the filtration $F$ we now go on to define the flow $j_t$. For $t \in \mathbb{T}_+$ and unitary $U$ in $\mathcal{A}$ define $j_t^0(U) : \mathcal{H}_{t]} \to \mathcal{H}_{t]}$ by

$$j_t^0(U)\lambda(\underline{r},\underline{Y},u) = \lambda(\underline{r},(UY_1,Y_2,\ldots,Y_n),u)$$

for $(\underline{r},\underline{Y},u) \in \mathcal{M}_t$. It is clear from direct computations using the definition of $\lambda$ and $L$ that $j_t^0(U)$ is an isometry on $\lambda(\mathcal{M}_t)$. Moreover if $U$ and $V$ are two unitary operators in $\mathcal{A}$ then $j_t^0(U)j_t^0(V) = j_t^0(UV)$. Hence $j_t^0(U)$ extends to a unitary operator on $\mathcal{H}_{t]}$. Now as $\mathcal{A}$ is a $C^*$ algebra every element in $\mathcal{A}$ is a linear combination of atmost four unitaries. On extending the definition of $j_t^0$ linearly we have a unital $*$-homomorphism $j_t^0 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{t]})$ satisfying

$$j_t^0(X)\lambda(\underline{r},\underline{Y},u) = \lambda(\underline{r},(XY_1,Y_2,\ldots,Y_n),u) \tag{4.7}$$

for all $(\underline{r},\underline{Y},u) \in \mathcal{M}_t$. Now define $j_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by putting

$$j_t(X) = j_t^0(X)F(t).$$

Clearly $j_t(X)$ leaves the range of $F(t)$ invariant and $j_0(X) = XF(0)$. Now to show Markov property fix $s < t$ and consider arbitrary $(\underline{r},\underline{Y},u)$ and $(\underline{r}',\underline{Z},v)$ in $\mathcal{M}_s$. If $\underline{r} = \underline{r}' = (r_1,\ldots,r_n)$ for some $s = r_1 > r_2 > \cdots > r_n \geq 0$, then we have
$\langle\lambda(\underline{r},\underline{Y},u), j_t(X)\lambda(\underline{r}',\underline{Z},v)\rangle$
$= \langle\lambda(t,\underline{r},I,\underline{Y},u), j_t(X)\lambda(t,\underline{r},I,\underline{Z},v)\rangle$
$= \langle u, T_{r_n}(Y_n^* T_{r_{n-1}-r_n}(Y_{n-1}^* \cdots Y_2^* T_{s-r_2}(Y_1^* T_{t-s}(X)Z_1)Z_2 \cdots Z_{n-1})Z_n)v\rangle$
$= \langle\lambda(\underline{r},\underline{Y},u), \lambda(\underline{r},T_{t-s}(X)Z_1,Z_2,\ldots,Z_n,v)\rangle$
$= \langle\lambda(\underline{r},\underline{Y},u), j_s^0(T_{t-s}(X))\lambda(\underline{r},\underline{Z},v)\rangle$
$= \langle\lambda(\underline{r},\underline{Y},u), j_s(T_{t-s}(X))\lambda(\underline{r}',\underline{Z},v)\rangle.$
Without loss of generality we can assume $\underline{r} = \underline{r}'$ as otherwise we can consider $(\underline{r} \vee \underline{r}',\tilde{\underline{Y}},u),(\underline{r} \vee \underline{r}',\tilde{\underline{Z}},v)$ and make use of (4.6). Now as $\lambda(\mathcal{M}_s)$ is total in the range of $F(s)$ we have $F(s)j_t(X)F(s) = j_s(T_{t-s}(X))$. This proves that $(\mathcal{H},F,j_t)$ is a weak

Markov flow with expectation semigroup $\{T_t\}$. Conservativity of $j_t$ follows from (4.7).

A vector $u$ in $\mathcal{H}_0$ has been identified with $\lambda(0, I, u)$ in $\mathcal{H}$. Consider $(\underline{r}, \underline{Y}, u)$ in $\mathcal{M}_t$. From (4.6), $\lambda(0, I, u) = \lambda(r_n, 0, I, I, u) = \lambda(r_n, I, u)$ and hence

$$
\begin{aligned}
j(\underline{r}, \underline{Y})u &= j(\underline{r}, \underline{Y})\lambda(0, I, u) \\
&= j_{r_1}(Y_1) \cdots j_{r_n}(Y_n)\lambda(r_n, I, u) \\
&= j_{r_1}(Y_1) \cdots j_{r_{n-1}}(Y_{n-1})\lambda(r_n, Y_n, u) \\
&= j_{r_1}(Y_1) \cdots j_{r_{n-1}}(Y_{n-1})\lambda(r_{n-1}, r_n, I, Y_n, u) \\
&\vdots \\
&= \lambda(\underline{r}, \underline{Y}, u).
\end{aligned} \tag{4.8}
$$

From the G.N.S. construction $\{\lambda(\underline{r}, \underline{Y}, u) : (\underline{r}, \underline{Y}, u) \in \mathcal{M}\}$ is total in $\mathcal{H}$. So $(\mathcal{H}, F, j_t)$ is minimal.

Finally we observe that the moment computations in Theorem 3.6 imply that

$$
\begin{aligned}
\langle j(\underline{s}, \underline{X})u, j(\underline{t}, \underline{Y})v\rangle &= \langle u, j(\underline{s}, \underline{X})^* j(\underline{t}, \underline{Y})v\rangle \\
&= \langle j'(\underline{s}, \underline{X})u, j'(\underline{t}, \underline{Y})v\rangle
\end{aligned}
$$

for all $(\underline{s}, \underline{X}, u), (\underline{t}, \underline{Y}, v) \in \mathcal{M}$. This shows that the correspondence $j(\underline{s}, \underline{X})u \to j'(\underline{s}, \underline{X})u$ is isometric and hence extends uniquely to a unitary isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$, proving the unitary equivalence of $(\mathcal{H}, F, j_t)$ and $(\mathcal{H}', F', j'_t)$. ∎

Now we extend Theorem 4.7 to include nonconservative quantum dynamical semigroups.

**Theorem 4.8** : Let $\mathcal{H}$ be a unital $C^*$ algebra of operators on a Hilbert space $\mathcal{H}_0$. Suppose that $\{T_t : t \in \mathbb{T}_+\}$ is a quantum dynamical semigroup on $\mathcal{A}$. Then there exists a minimal subordinate Markov flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ with expectation semigroup $\{T_t\}$. Moreover if $(\mathcal{H}', F', j'_t)$ is another such Markov flow then it is unitarily equivalent to $(\mathcal{H}, F, j_t)$.

**Proof** : Consider the extended $C^*$ algebra $\hat{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ acting on the Hilbert space $\hat{\mathcal{H}}_0 = \mathcal{H}_0 \oplus \mathbb{C}$. For convenience we denote the element $X \oplus c$ of $\hat{\mathcal{A}}$, for $X \in \mathcal{A}$ and

$c \in \mathbb{C}$, by the column vector $\binom{X}{c}$. Define the maps $\hat{T}_t : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ by

$$\hat{T}_t \begin{pmatrix} X \\ c \end{pmatrix} = \begin{pmatrix} T_t(X) + c(I - T_t(I)) \\ c \end{pmatrix}, X \in \mathcal{A}, c \in \mathbb{C}. \tag{4.9}$$

Then $\{\hat{T}_t\}$ is a conservative one parameter semigroup of completely positive linear maps. If $\{T_t\}$ is strongly continuous so is $\{\hat{T}_t\}$. Thus Theorem 4.7 becomes applicable for $\{\hat{T}_t\}$ and we have a conservative weak Markov flow $(\hat{\mathcal{H}}, \hat{F}, \hat{j}_t)$ on $\mathcal{A}$ with expectation semigroup $\{\hat{T}_t\}$. Define the operators $F(t)$ and $j_t(X)$ on $\hat{\mathcal{H}}$ by

$$\begin{aligned} F(t) &= \hat{j}_t \begin{pmatrix} I \\ 1 \end{pmatrix} - \hat{j}_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ j_t(X) &= \hat{j}_t \begin{pmatrix} X \\ 0 \end{pmatrix} \text{ for } t \geq 0 \text{ and } X \in \mathcal{A}. \end{aligned}$$

Before obtaining the required Markov flow we prove the following statements. For $0 \leq s \leq t, X \in \mathcal{A}$ and $c \in \mathbb{C}$

(a) $\{\hat{j}_t\binom{0}{1}\}$ is a family of projections nondecreasing in $t$;

(b) $j_t(X)\hat{j}_0\binom{0}{1} = \hat{j}_0\binom{0}{1}j_t(X) = 0$;

(c) $\{F(t)\}$ is a family of projections nondecreasing in $t$;

(d) Range of $F(0)$ is $\mathcal{H}_0$ and range of $F(t)$ increases to the orthogonal complement of range of $\hat{j}_0\binom{0}{1}$ as $t$ increases to $\infty$;

(e) $F(t)\hat{j}_s\binom{X}{c} = j_s(X - cI)F(s) + cF(s)$.

From the Markov property of $\hat{j}_t$,

$$\begin{aligned} \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\geq \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{j}_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbb{E}^{\hat{F}}_{s]} \hat{j}_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{j}_s \begin{pmatrix} I - T_{t-s}(I) \\ 1 \end{pmatrix} \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \hat{j}_s \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Hence

$$\hat{j}_s\begin{pmatrix}0\\1\end{pmatrix} = \hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_t\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_s\begin{pmatrix}0\\1\end{pmatrix},$$

proving (a). Now make use of (a) to obtain

$$j_t(X)\hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} = \hat{j}_t\begin{pmatrix}X\\0\end{pmatrix}\hat{j}_t\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} = 0 = \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix}j_t(X)$$

and

$$\begin{aligned} F(t)F(s) &= (\hat{j}_t\begin{pmatrix}I\\1\end{pmatrix} - \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix})(\hat{j}_s\begin{pmatrix}I\\1\end{pmatrix} - \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix}) \\ &= \hat{j}_s\begin{pmatrix}I\\1\end{pmatrix} - \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} - \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} + \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} = F(s) = F(s)F(t). \end{aligned}$$

Clearly $F(t)^* = F(t)$. This proves (b) and (c). The range of $\hat{F}(0)$ is $\mathcal{H}_0 \oplus \mathbb{C}$ and hence the range of $F(0)$ is $\mathcal{H}_0$. The second part of (d) follows as $\hat{j}_t\binom{I}{1}$ increases to the identity operator in $\tilde{\mathcal{H}}$ as $t$ increases to $\infty$. Now from (a) and (b)

$$\begin{aligned} F(t)\hat{j}_s\begin{pmatrix}X\\c\end{pmatrix} &= (\hat{j}_t\begin{pmatrix}I\\1\end{pmatrix} - \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix})(\hat{j}_s\begin{pmatrix}X\\0\end{pmatrix} + \hat{j}_s\begin{pmatrix}0\\c\end{pmatrix}) \\ &= \hat{j}_s\begin{pmatrix}X\\0\end{pmatrix} + \hat{j}_s\begin{pmatrix}0\\c\end{pmatrix} - 0 - \hat{j}_0\begin{pmatrix}0\\c\end{pmatrix} \\ &= \hat{j}_s\begin{pmatrix}X - cI\\0\end{pmatrix} + cF(s) \\ &= \hat{j}_s\begin{pmatrix}X - cI\\0\end{pmatrix}F(s) + cF(s). \end{aligned}$$

Let $\mathcal{H}$ be the orthogonal complement of the range of $\hat{j}_0\binom{0}{1}$ in $\hat{\mathcal{H}}$. Making use of (a)-(d) we can restrict $F(t)$ and $j_t(X)$ to $\mathcal{H}$ and verify that $(\mathcal{H}, F, j_t)$ is a subordinate weak Markov flow with expectation semigroup $\{T_t\}$. Denote by $\mathcal{H}_t \subset \hat{\mathcal{H}}$ the closed subspace spanned by all vectors of the form $j(\underline{r}, \underline{Y})u$, with $(\underline{r}, \underline{Y}, u)$ in $\mathcal{M}_{t]}$. We now claim that the range of $F(t)$ is $\mathcal{H}_t$. Indeed, consider $\xi = j(\underline{r}, \underline{Y})u$ with $(\underline{r}, \underline{Y}, u) \in \mathcal{M}_{t]}$. Then

$$F(t)\xi = (\hat{j}_t\begin{pmatrix}I\\1\end{pmatrix} - \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix})\hat{j}_{r_1}\begin{pmatrix}Y_1\\0\end{pmatrix}\hat{j}_{r_2}\begin{pmatrix}Y_2\\0\end{pmatrix}\cdots\hat{j}_{r_n}\begin{pmatrix}Y_n\\0\end{pmatrix}u = \xi,$$

and hence the range of $F(t)$ contains $\mathcal{H}_t$. Now for $(\underline{s}, \underline{X}, u) \in \mathcal{M}_{t]}$, $c_1, \ldots, c_n, a \in \mathbb{C}$ consider $\eta = \hat{j}(\underline{s}, (\frac{X}{\underline{c}}))(\frac{u}{a})$. From the statement (e) proved above,

$$F(t)\eta = (j_{s_1}(X_1 - c_1 I)F(s_1) + c_1 F(s_1))\hat{j}_{s_2}\begin{pmatrix} X_2 \\ c_2 \end{pmatrix}\hat{j}_{s_3}\begin{pmatrix} X_3 \\ c_3 \end{pmatrix}\cdots\hat{j}_{s_n}\begin{pmatrix} X_n \\ c_n \end{pmatrix}\begin{pmatrix} u \\ a \end{pmatrix}.$$

By induction on $n$ we conclude that $F(t)\eta$ is a linear combination of elements of the form $j(\underline{r}, \underline{Y})u$ . The closed linear span of all vectors $\eta$ of the form above is the range of $\hat{F}(t)$ and as the range of $F(t)$ is clearly contained in the range of $\hat{F}(t)$ we conclude that $\mathcal{H}_t$ contains the whole of the range of $F(t)$. This proves the minimality of $j_t$. Uniqueness upto unitary equivalence is proved as before. ∎

Given a quantum dynamical semigroup $T_t$, the unique (upto unitary equivalence) weak Markov flow $(\mathcal{H}, F, j_t)$ constructed in Theorems 4.7 and 4.8 will be called its *associated minimal subordinate Markov flow*. Note that if $(\mathcal{H}', F', j'_t)$ is a weak Markov flow with expectation semigroup $\{T_t\}$ then on restricting the flow $(\mathcal{H}', F', F'(t)j'(\cdot)F'(t))$ to the invariant subspace generated by $\mathcal{H}_0$ we obtain the minimal subordinate Markov flow of $\{T_t\}$. That is, every Markov dilation of $T_t$ contains the flow $(\mathcal{H}, F, j_t)$. For this reason $(\mathcal{H}, F, j_t)$ may be called ***the minimal Markov dilation*** of $\{T_t\}$.

Certain differences between conservative Markov flows and subordinate Markov flows may be noted at this stage. In view of (4.6) and (4.8), if $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow then

$$\langle j(\underline{r}, \underline{Y})u, , j(\underline{s}, \underline{Z})v\rangle = \langle j(\underline{r} \vee \underline{s}, \tilde{\underline{Y}})u, j(\underline{r} \vee \underline{s}, \tilde{\underline{Z}})v\rangle = L((\underline{r}, \underline{Y}, u), (\underline{s}, \underline{Z}, v)) \quad (4.10)$$

These relations fail for general subordinate Markov flows. Nevertheless we have the following useful proposition.

**Proposition 4.9 :** Let $(\mathcal{H}, F, j_t)$ be a subordinate Markov flow on $\mathcal{A}$ with expectation semigroup $T_t$. Suppose $(\underline{r}, \underline{Y}, u)$ and $(\underline{s}, \underline{Z}, v)$ are in $\mathcal{M}_t$ for some fixed $t \geq 0$. Then (4.10) holds.

**Proof :** Note that here we have $r_1 = s_1 = t$ and

$$\langle j(\underline{r}, \underline{Y})u, j(\underline{s}, \underline{Z})v\rangle = \langle u, F(0)j(\underline{r}, \underline{Y})^* j(\underline{s}, \underline{Z})v\rangle$$

Now use Proposition 3.7 and observe that extra left or right multiplications by identity can be inserted wherever needed to obtain (4.10). ∎

Recalling the definition of strong continuity (Definition 1.5) we have a simple but usefull result.

**Theorem 4.10 :** *Let $\{T_t\}$ be a continuous time quantum dynamical semigroup on $\mathcal{A}$ with associated minimal subordinate Markov flow $(\mathcal{H}, F, j_t)$. Then if $T_t$ is strongly continuous the maps $t \to F(t)\psi$ and $t \to j_t(X)\psi$ are continuous for every $X \in \mathcal{A}$ and $\psi \in \mathcal{H}$.*

**Proof :** For conservative quantum dynamical semigroups this is immediate from Proposition 3.11 and the fact that $j_t$ is a homomorphism for every $t$. The result can be extended easily to nonconservative quantum dynamical semigroups using the construction involved in the proof of Theorem 4.8. ∎

Note that the construction in (4.9) is the quantum probabilistic analogue of associating with a substochastic semigroup $P_t = ((p_{ij}(t))), 1 \leq i, j < \infty$ of matrices the stochastic semigroup $\hat{P}_t = ((\hat{p}_{ij}(t)), 0 \leq 1, j < \infty$ where

$$\hat{p}_{ij}(t) = \begin{cases} p_{ij}(t) & \text{if } i \geq 1, \ j \geq 1, \\ 0 & \text{if } i = 0, \ j \geq 1, \\ 1 & \text{if } i = 0, \ j = 0, \\ 1 - \sum_{j=1}^{\infty} p_{ij}(t) & \text{if } i \geq 1, \ j = 0. \end{cases}$$

In other words we have incorporated an absorbing boundary. This is reflected in the increasing nature of the family of projections $\{\hat{j}_t(\begin{smallmatrix}0\\1\end{smallmatrix})\}$. It may also be noted that in general $\{\hat{j}_t(\begin{smallmatrix}I\\0\end{smallmatrix})\}$ is not a commuting family of projections.

We conclude this section with two examples of the construction involved in Theorems 4.7 and 4.8.

**Example 4.11** : Let $\mathcal{A}$ be the commutative von Neumann algebra of $2 \times 2$ diagonal matrices and let $T_t : \mathcal{A} \to \mathcal{A}$ be the semigroup defined by

$$T_t\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} e^{-ct}a + (1 - e^{-ct})b & 0 \\ 0 & b \end{bmatrix}$$

for $a, b \in \mathbb{C}, c > 0$ being a fixed constant. $\mathcal{A}$ acts on $\mathbb{C}^2$ in a natural way. Put $\mathcal{H} = \mathbb{C}^2 \oplus L^2(\mathbb{R}_+)$ with filtration $F$ given by $F(t) = I \oplus \chi_t$ where $I$ is the identity operator in $\mathbb{C}^2$ and $\chi_t$ denotes multiplication by the indicator function $\chi_{[0,t]}$ in $L^2(\mathbb{R}_+)$. Define $j_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$j_t\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = aQ(t) + b(F(t) - Q(t))$$

where $Q(t)$ is the rank one projection onto the subspace generated by the unit vector $e^{-\frac{c}{2}t}e_1 \oplus f_t$ with

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f_t(x) = \sqrt{c}\, e^{-\frac{c}{2}(t-x)}\chi_{[0,t]}(x).$$

A routine computation shows that $F(s)j_t(X)F(s) = j_s(T_{t-s}(X))$ for all $X \in \mathcal{A}$ and $0 \leq s \leq t$. Thus $(\mathcal{H}, F, j_t)$ provides a weak Markov flow with expectation semigroup $\{T_t\}$ satisfying all the properties mentioned in Theorem 4.8. It is instructive to compare this with the Markov flow of classical probability theory associated with the one parameter semigroup of $2 \times 2$ stochastic matrices $\begin{bmatrix} e^{-ct} & 1 - e^{-ct} \\ 0 & 1 \end{bmatrix}$, as in Example 2.3.

**Example 4.12** : Let $H$ be a positive selfadjoint operator on $\mathcal{H}_0$. Consider the nonconservative one parameter semigroup $\{T_t\}$ of completely positive maps on $\mathcal{B}(\mathcal{H}_0)$ defined by

$$T_t(X) = e^{-tH}Xe^{-tH}, t \geq 0, X \in \mathcal{B}(\mathcal{H}_0).$$

Following [HIP] introduce the unitary operators $\{U(s,t), 0 \leq s \leq t < \infty\}$ in the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus L^2(\mathbb{R}_+, \mathcal{H}_0)$, given by

$$U(s,t)\begin{pmatrix} u_0 \\ u \end{pmatrix} = \begin{pmatrix} A(s,t) & B(s,t) \\ C(s,t) & I + D(s,t) \end{pmatrix}\begin{pmatrix} u_0 \\ u \end{pmatrix}$$

where $u_0$ and $u = u(\cdot)$ are the components of an arbitrary element in $\mathcal{H}$ with respect to the direct sum decomposition in the definition of $\mathcal{H}$ and

$$\begin{aligned} A(s,t) &= e^{-(t-s)H} \\ B(s,t)u &= -(2H)^{1/2}\int_0^\infty \chi_{[s,t]}(x)e^{-(t-x)H}u(x)dx, \\ (C(s,t)u_0)(x) &= \chi_{[s,t]}(x)(2H)^{1/2}e^{-(x-s)H}u_0, \\ (D(s,t)u)(x) &= -2\int_0^\infty \chi_{[s,t]}(y)\chi_{[y,t]}(x)He^{-(x-y)H}u(y)dy. \end{aligned}$$

It is known from [HIP] that

$$U(s,t)U(r,s) = U(r,t) \text{ for all } 0 \le r \le s \le t < \infty$$

and $U(s,t)$ is an operator of the form $V(s,t) \oplus I$ in the direct sum decomposition $\mathcal{H} = \mathcal{H}(s,t) \oplus \mathcal{H}(s,t)^\perp$ where $\mathcal{H}(s,t) = \mathcal{H}_0 \oplus L^2([s,t],\mathcal{H}_0)$. Define $F(t)$ to be the projection on the subspace $\mathcal{H}(0,t)$ and put

$$j_t(X) = U(0,t)^* \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U(0,t).$$

Then $j_t(I) = j_t(I)F(t) \le F(t)$. From the fact that $\{U(s,t)\}$ is a time orthogonal dilation of the positive contraction semigroup $\{e^{-tH}\}$ it follows that $F(s)U(0,t)F(s) = U(0,s)\{e^{(t-s)H} \oplus I_{[0,s]} \oplus 0\}$ where the term in $\{\quad\}$ on the right hand side is with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus L^2([0,s],\mathcal{H}_0) \oplus L^2([s,\infty),\mathcal{H}_0)$. Thus

$$\begin{aligned} F(s)j_t(X)F(s) &= U(0,s)^*F(s)U(s,t)^*F(s)\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}F(s)U(s,t)F(s)U(0,s) \\ &= U(0,s)^*\begin{pmatrix} T_{t-s}(X) & 0 \\ 0 & 0 \end{pmatrix}U(0,s)F(s) \\ &= j_s(T_{t-s}(X))F(s). \end{aligned}$$

In other words $(\mathcal{H}, F, j_t)$ is a subordinate Markov flow with expectation semigroup $\{T_t\}$.

## 5 Algebraic Properties of the Minimal Flow

Quantum dynamical semigroups can be better understood by studying their Markov dilations. Of course, minimal subordinate dilations have a special role to play here as they are uniquely determined by their expectation semigroups. Moreover as noted in Section 4 every Markov dilation contains the minimal dilation in it. In this section kernels of homomorphisms $j_t$ of minimal dilation are shown to be intimately connected with certain ideals left invariant by the semigroup $\{T_t\}$. These results help us to construct a 'cocycle' of Markov flows. Then all quantum dynamical semigroups dominated by a given quantum dynamical semigroup are characterized through its minimal Markov flow and a family of positive contractions. This generalizes a result of Arveson ([Ar],[Su1]) on single completely positive maps to quantum dynamical semigroups.

Throughout this section $\mathcal{A}$ is a unital $C^*$ algebra acting on a Hilbert space $\mathcal{H}_0$ and $\{T_t\}$ is a quantum dynamical semigroup on $\mathcal{A}$ with associated minimal subordinate Markov flow $(\mathcal{H}, F, j_t)$.

**Theorem 5.1** : For $s \geq 0$ and $X \in \mathcal{A}$, the following are equivalent :

(i) $j_s(X) = 0$.

(ii) $T_{r_n}(Y_n T_{r_{n-1}}(Y_{n-1} \cdots Y_2 T_{r_1}(Y_1 X Z_1) Z_2 \cdots Z_{n-1}) Z_n) = 0$.

for all $Y_1, \ldots, Y_n, Z_1, \ldots, Z_n$ in $\mathcal{A}$ and $r_1, \ldots, r_n \geq 0$ with $\sum_{i=1}^n r_i \geq s$.

(iii) $j_s(W) = 0$ for all $W \in \mathcal{C}_X$, where $\mathcal{C}_X$ is the smallest closed two sided ideal of $\mathcal{A}$, containing $X$ and invariant under $\{T_t\}_{t\geq 0}$ .

(iv) $j_t(X) = 0$ for all $t \geq s$.

**Proof** : (i) $\Leftrightarrow$ (ii).

Assume $j_s(X) = 0$ and consider the case $\sum_{i=1}^n r_i = s$. Now for vectors $u, v$ in $\mathcal{H}_0$, from Proposition 3.7 or 3.9

$$\langle u, T_{r_n}(Y_n T_{r_{n-1}}(Y_{n-1} \cdots Y_2 T_{r_1}(Y_1 X Z_1) Z_2 \cdots Z_{n-1}) Z_n) v\rangle$$
$$= \langle u, F(0) j_{r_n}(Y_n) \cdots j_{s-r_1}(Y_2) j_s(Y_1) j_s(X) j_s(Z_1) j_{s-r_1}(Z_2) \cdots j_{r_n}(Z_n) F(0) v\rangle$$
$$= 0.$$

If $\sum_{i=1}^{n} r_i > s$, let $k$ be the index such that $\sum_{i=1}^{k-1} r_i < s \leq \sum_{i=1}^{k} r_i$. Taking $a = \sum_{i=1}^{k} r_i - s$ and using the semigroup property of $T_t$ we have

$$T_{r_n}(Y_n T_{r_{n-1}}(Y_{n-1} \cdots Y_2 T_{r_1}(Y_1 X Z_1) Z_2 \cdots Z_{n-1}) Z_n)$$
$$= T_{r_n}(Y_n T_{r_{n-1}}(Y_{n-1} \cdots T_{r_{k+1}}(Y_{k+1} T_a(W) Z_{k+1}) Z_2 \cdots Z_{n-1}) Z_n),$$

where

$$W = T_{r_k - a}(Y_k T_{r_{k-1}}(Y_{k-1} \cdots T_{r_1}(Y_1 X Z_1) \cdots Z_{k-1}) Z_k).$$

But then $W = 0$ from what has been shown earlier as $\sum_{i=1}^{k} r_i - a = s$. Conversely (ii) implies $\langle j(\underline{r}, \underline{Y})u, j_s(X) j(\underline{r}, \underline{Z})v \rangle = 0$ for all $(\underline{r}, \underline{Y}, u)$, $(\underline{r}, \underline{Z}, v)$ in $\mathcal{M}_{s]}$. Here we have used notation from (4.1)-(4.4) of Section 4. This of course means $j_s(X) = 0$.

(i) $\Leftrightarrow$ (iii).

Once again assume (i). Clearly the set of all $W$ such that $j_s(W) = 0$ is a closed ideal of $\mathcal{A}$ containing $X$. The equivalence of (i) and (ii) can now be used to show that if $j_s(W) = 0$, then $\langle j(\underline{r}, \underline{Y})u, j_s(T_t(W)) j(\underline{r}, \underline{Z})v \rangle = 0$ for every $t \geq 0$, and $(\underline{r}, \underline{Y}, u)$, $(\underline{r}, \underline{Z}, v)$ in $\mathcal{M}_{s]}$. This, in turn, means $j_s(T_t(W)) = 0$, i.e., the ideal is invariant under $\{T_t\}$. Trivially (iii) implies (i).

(i) $\Leftrightarrow$ (iv).

On assuming (i) or (ii), $j_t(X) = 0$ for $t \geq s$ is seen by computing

$$\langle j(\underline{r}, \underline{Y})u, j_t(X) j(\underline{r}', \underline{Z})v \rangle$$

for $(\underline{r}, \underline{Y}, u)$ and $(\underline{r}', \underline{Z}, v)$ in $\mathcal{M}_{t]}$. Nothing to prove in the converse statement. ∎

Part (ii) of the above theorem describes the ideal $\mathcal{I}_s = \{X : j_s(X) = 0\}$ completely in terms of the semigroup. Note that $\mathcal{I}_0 = 0$ and $\mathcal{I}_s \subset \mathcal{I}_t \subset T_{t-s}^{-1}(\mathcal{I}_s)$ for $s < t$. In discrete time these ideals can be obtained inductively as follows.

**Theorem 5.2** : Suppose $\{T_t\} = \{I, T, T^2, \ldots\}$ is a discrete quantum dynamical semigroup. Then

$\mathcal{I}_0 = 0$,

$\mathcal{I}_1 =$ The largest two sided ideal of $\mathcal{A}$ contained in the kernel of $T$.

$\mathcal{I}_{s+1} =$ The largest two sided ideal of $\mathcal{A}$ contained in $T^{-1}(\mathcal{I}_s)$, for $s \geq 1$.

**Proof** : Fix $s \geq 0$. Suppose $X$ belongs to an ideal in $T^{-1}(\mathcal{I}_s)$. Then $T(YXZ) \in \mathcal{I}_s$ for all $Y, Z$ in $\mathcal{A}$. From (ii) of Theorem 5.1 we have

$$T(Y_nT(Y_{n-1}\cdots T(Y_1(T(YXZ))Z_1)\cdots Z_{n-1})Z_n) = 0 \text{ for } n \geq s$$

i.e.,

$$T(Y_nT(Y_{n-1}\cdots T(Y_1XZ_1)\cdots Z_{n-1})Z_n) = 0 \text{ for } n \geq s+1$$

and hence $X$ is in $\mathcal{I}_{s+1}$. ■

Theorem 5.1 enables us to construct a cocycle of homomorphisms using the minimal flow. Note that $j_t(\mathcal{A})$, the range of $j_t$, is a $C^*$ algebra for every $t$. Denote this algebra by $\mathcal{A}_t$. Then $\mathcal{A}_0$ can be identified with $\mathcal{A}$. For $s \leq t$ define $j_{s,t} : \mathcal{A}_s \to \mathcal{A}_t$ by

$$j_{s,t}(j_s(X)) = j_t(X) \text{ for } X \in \mathcal{A}. \tag{5.1}$$

**Theorem 5.3** : Under the above set up

(i) $j_{s,t} : \mathcal{A}_s \to \mathcal{A}_t$ are $*$-homomorphisms for $s \leq t$,

(ii) $j_{s_2,s_3} o j_{s_1,s_2} = j_{s_1,s_3}$ for $0 \leq s_1 \leq s_2 \leq s_3$ and $j_{s,s}$ is the identity map on $\mathcal{A}_s$,

(iii) $\mathbb{E}^F_{s_2]} j_{s_1,s_3}(Z) = j_{s_1,s_2}(\mathbb{E}^F_{s_1]} j_{s_1,s_1+(s_3-s_2)}(Z))$ for $0 \leq s_1 \leq s_2 \leq s_3$ and $Z \in \mathcal{A}_{s_1}$.

**Proof** : From Theorem 5.1 if $j_s(X) = 0$ then $j_t(X) = 0$ for $t \geq s$. Hence $j_{s,t}$ as in (5.1) are well defined linear maps. Now (i) and (ii) are obvious. To show part (iii), as the algebra $\mathcal{A}_{s_1}$ is precisely the range of $j_{s_1}$ we can take $Z = j_{s_1}(X)$ for some $X \in \mathcal{A}$. Now

$$\begin{aligned}
\mathbb{E}^F_{s_2]} j_{s_1,s_3}(j_{s_1}(X)) &= \mathbb{E}^F_{s_2]} j_{s_3}(X) \\
&= j_{s_2}(T_{s_3-s_2}(X)) \\
&= j_{s_1,s_2}(j_{s_1}(T_{s_3-s_2}(X))) \\
&= j_{s_1,s_2}(\mathbb{E}^F_{s_1]} j_{s_1+(s_3-s_2)}(X)) \\
&= j_{s_1,s_2}(\mathbb{E}^F_{s_1]} j_{s_1,s_1+(s_3-s_2)}(Z)).
\end{aligned}$$

■

Fix $s \geq 0$. Define the maps $T_t^{(s)} : \mathcal{A}_s \to \mathcal{A}_s$ for $t \geq 0$ by

$$T_t^{(s)}(j_s(X)) = j_s(T_t(X)) \quad \text{for } X \in \mathcal{A}. \tag{5.2}$$

Now on defining the shifted filtration $F^{(s)}$ and $*$-homomorphisms $j_t^{(s)}$ by

$$F^{(s)}(t) = F(s+t) \quad \text{and} \quad j_t^{(s)} = j_{s,s+t},$$

we have the following result.

**Theorem 5.4** : For fixed $s \geq 0$, $\{T_t^{(s)}\}$ is a quantum dynamical semigroup on $\mathcal{A}_s$ and $(\mathcal{H}, F^{(s)}, j_t^{(s)})$ is a weak Markov flow with expectation semigroup $\{T_t^{(s)}\}$.

**Proof** : The well definedness of $T_t^{(s)}$ should be clear from Theorem 5.1 or from the observation

$$T_t^{(s)}(Z) = I\!E^F_{s]} j_{s,s+t}(Z) \text{ for } Z \in \mathcal{A}_s, t \geq 0.$$

Now use Theorem 5.3 to complete the proof. ∎

**Definition 5.6** : A completely positive map $T'$ is said to be *dominated* by another completely positive map $T$ if $T - T'$ is completely positive. And a quantum dynamical semigroup $\{T_t'\}$ on $\mathcal{A}$ is said to be *dominated* by another semigroup $\{T_t\}$ if $T_t - T_t'$ is completely positive for every $t$.

Very often we have two quantum dynamical semigroups one dominating another. In fact Chapter II gives a whole collection of such examples through Feller perturbations. How the flows of such pairs are related is not clear and their study leads us to quantum boundary theory. A theorem of Arveson ([Ar], [Su1], [Mi]) states that given a completely positive map $T$ with associated Stinespring triple $(\mathcal{H}, P, j)$ as in Theorem 4.3, all completely positive maps dominated by $T$ can be obtained as $X \to PCj(X)$ for some positive contraction $C$ in the commutator of $j(\mathcal{A})$. Here we have a semigroup version of this result.

**Theorem 5.7** : Let $T_t$ be a conservative quantum dynamical semigroup on $\mathcal{A}$ with associated minimal Markov flow $(\mathcal{H}, F, j_t)$. Suppose that $\{T_t'\}$ is a quantum dynamical semigroup dominated by $\{T_t\}$. Then there exists a unique family of positive contractions $\{C(t)\}$ on $\mathcal{H}$ such that :

(i) $C(0) = F(0)$ and $C(t) \leq F(t)$ for $t \geq 0$ ;

(ii) $C(t)$ commutes with $j_t(X)$ for $X \in \mathcal{A}$ and $t \geq 0$ ;

(iii) $I\!E^F_{s]}C(t)j_t(X) = C(s)j_s(\{I\!E^F_{0]}C(t-s)j_{t-s}(X)\}|_{\mathcal{H}_0})$ for $0 \leq s \leq t$ and $X \in \mathcal{A}$.

(iv) $T'_t(X) = I\!E^F_{0]}C(t)j_t(X)|_{\mathcal{H}_0}$ for $X \in \mathcal{A}$ and $t \geq 0$.

Conversely, if $\{C(t)\}$ is a family of positive contractions on $\mathcal{H}$ satisfying (i)-(iii) then $\{T'_t\}$ defined by (iv) is a quantum dynamical semigroup and is dominated by $T_t$.

**Proof** : Let $(\mathcal{H}', F', j'_t)$ be the minimal subordinate Markov flow associated with $\{T'_t\}$. Recall the notation introduced in (4.1)-(4.4) and consider the map $D(t) : \mathcal{H}_{t]} \to \mathcal{H}'_{t]}$ defined by

$$D(t)j(\underline{r},\underline{Y})u = j'(\underline{r},\underline{Y})u$$

for $(\underline{r},\underline{Y},u)$ in $\mathcal{M}_t$. For arbitrary $c_i \in \mathbb{C}$ and $(\underline{r}^{(i)},\underline{Y}^{(i)},u^{(i)}) \in \mathcal{M}_t$, as $j'_t$ is a subordinate flow from Proposition 4.9 we have

$$\sum \bar{c}_i c_k \langle j'(\underline{r}^{(i)},\underline{Y}^{(i)})u^{(i)}, j'(\underline{r}^{(k)},\underline{Y}^{(k)})u^{(k)}\rangle$$

$$= \sum \bar{c}_i c_k \langle j'(\underline{r},\tilde{\underline{Y}}^{(i)})u^{(i)}, j'(\underline{r},\tilde{\underline{Y}}^{(k)})u^{(k)}\rangle$$

where $\underline{r} = \underline{r}^{(1)} \vee \underline{r}^{(2)} \vee \cdots \vee \underline{r}^{(n)}$. Define operators $A = ((A_{ik}))$ and $A' = ((A'_{ik}))$ using $T_t$ and $T'_t$ respectively as in Lemma 4.4. and Proposition 4.5. Note that $A' \leq A$ as $T'_t$ is dominated by $T_t$. Now

$$\begin{aligned} ||D(t)\sum c_i j(\underline{r}^{(i)},\underline{Y}^{(i)})u^{(i)}||^2 &= \sum \bar{c}_i c_k \langle j'(\underline{r},\tilde{\underline{Y}}^{(i)})u^{(i)}, j'(\underline{r},\tilde{\underline{Y}}^{(k)})u^{(k)}\rangle \\ &= \langle (\oplus c_i u_i), A'(\oplus c_k u_k)\rangle \\ &\leq \langle (\oplus c_i u_i), A(\oplus c_k u_k)\rangle \\ &= ||\sum c_i j(\underline{r}^{(i)},\underline{Y}^{(i)})u^{(i)}||^2. \end{aligned}$$

Hence $D(t)$ extends to a linear contraction of $\mathcal{H}_{t]}$ onto $\mathcal{H}'_{t]}$. Taking $C(t) = D(t)^* D(t)$ we have a positive contraction on $\mathcal{H}_{t]}$ satisfying

$$\langle C(t)j(\underline{r},\underline{Y})u, j(\underline{s},\underline{Z})v\rangle = \langle j'(\underline{r},\underline{Y})u, j'(\underline{s},\underline{Z})v\rangle \tag{5.3}$$

for $(\underline{r}, \underline{Y}, u)$ and $(\underline{s}, \underline{Z}, v)$ in $\mathcal{M}_t$. (Caution: (5.3) is not true in general for elements in $\mathcal{M}_{t]}$). Extend $C(t)$ to the whole of $\mathcal{H}$ by putting $C(t)\psi = 0$ for $\psi$ in $\mathcal{H}_{t]}^{\perp}$. Now (i) is obvious. To prove (ii) consider $(\underline{r}, \underline{Y}, u)$ and $(\underline{r}, \underline{Z}, v)$ in $\mathcal{M}_t$. We have

$$\begin{aligned}
&\langle C(t)j_t(X)j(\underline{r}, \underline{Y})u, j(\underline{r}, \underline{Z})v\rangle \\
&= \langle j'_t(X)j'(\underline{r}, \underline{Y})u, j'(\underline{r}, \underline{Z})v\rangle \\
&= \langle j'(\underline{r}, \underline{Y})u, j'_t(X^*)j'(\underline{r}, \underline{Z})v\rangle \\
&= \langle C(t)j(\underline{r}, \underline{Y})u, j_t(X^*)j(\underline{r}, \underline{Z})v\rangle \\
&= \langle j_t(X)C(t)j(\underline{r}, \underline{Y})u, j(\underline{r}, \underline{Z})v\rangle.
\end{aligned}$$

Also note that for $u, v \in \mathcal{H}_0$,

$$\langle C(t)j_t(X)u, v\rangle = \langle j'_t(X)u, v\rangle = \langle T'_t(X)u, v\rangle.$$

Hence (iv) is also proved. The proof of (iii) is done in a similar way by considering vectors in $\mathcal{H}_{s]}$ and using the Markov property of $j'$. Observe that (iii) and (iv) along with mathematical induction gives us (5.3). Now minimality of $j_t$ yields uniqueness of the family $\{C(t)\}$.

As for the converse part, the semigroup property follows from (iii). To show that $T'_t$ is completely positive and is dominated by $T_t$ observe

$$T'_t(X) = \mathbb{E}^F_{0]} C(t)^{\frac{1}{2}} j_t(X) C(t)^{\frac{1}{2}}|_{\mathcal{H}_0}$$

and

$$T_t(X) = \mathbb{E}^F_{0]} j_t(X) = T'_t(X) + \mathbb{E}^F_{0]}(1 - C(t))j_t(X)|_{\mathcal{H}_0}.$$

**Remark 5.8** : Let $j_t$ be the Markov flow $j_t(f) = f(x + \omega(t))$ on $L^\infty(\mathbb{R})$ where $\omega$ is the standard Brownian motion. Then for any nonnegative function $V$, the family of multiplication operators $C(t)$ defined by

$$C(t) = e^{-\int_0^t V(x+\omega(s))ds}$$

satisfies conditions of Theorem 5.7 and leads to the celebrated Feynman-Kac perturbation ([CW], [RS] Vol.II). Thus the family $\{C(t)\}$ of Theorem 5.7 may be considered as an abstract Feynman-Kac cocycle.

# 6 Central Flows and Minimal Abelian Dilations

Consider the special case where the initial algebra $\mathcal{A}$ is abelian. Here we would like to obtain an *abelian Markov flow*, that is, a flow $k_t$ such that $k_s(X)$ and $k_t(Y)$ commute for all $s,t$ and $X,Y$. The minimal Markov flow obtained in Section 4 is not abelian in general. It will be shown that the minimal Markov flow $(\mathcal{H}, F, j_t)$ can be modified suitably to get an abelian Markov flow $(\mathcal{H}, F, k_t)$. There is no need to change the Hilbert space or the filtration. The main clue for the construction of abelian Markov flows is given by classical Markov processes. The observation (2.2) is important in this context. This leads us to the following theorem which is a bit of a surprise.

**Theorem 6.1 :** Let $\{T_t\}$ be a quantum dynamical semigroup on a $C^*$ algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}_0$ with associated minimal subordinate Markov flow $(\mathcal{H}, F, j_t)$. Let $\mathcal{Z}(\mathcal{A})$ be the centre of the algebra $\mathcal{A}$. Then there exists a unique family of $*$-homomorphisms $\{k_t : \mathcal{Z}(\mathcal{A}) \to \mathcal{B}(\mathcal{H})\}$, such that

(i) $\mathbb{E}^F_{t]} k_t(Z) = j_t(Z)$ for all $Z \in \mathcal{Z}(\mathcal{A})$ and $t \geq 0$;

(ii) $k_s(Z)$ commutes with $k_t(Z')$ for all $Z, Z' \in \mathcal{Z}(\mathcal{A})$ and $s,t \geq 0$.

Moreover if $\{T_t\}$ is conservative then $k_t$ is unital.

**Proof :** We first consider the case where $\{T_t\}$ is conservative. Take a unitary element $U$ in $\mathcal{Z}(\mathcal{A})$. It is enough to define $k_t(U)$ on vectors of the form $j(\underline{r}, \underline{Y})u$ for $(\underline{r}, \underline{Y}, u) \in \mathcal{M}$. (Notation as in Section 4). Also in view of Proposition 4.9 we can assume $t \in \{r_1, \ldots, r_n\}$, i.e., $t = r_p$ for some $p$. Now define $k_t(U)j(\underline{r}, \underline{Y})u$ by

$$k_t(U)j(\underline{r}, \underline{Y})u = j(\underline{r}, (Y_1, \ldots, Y_{(p-1)}, UY_p, Y_{(p+1)}, \ldots, Y_n))u. \tag{6.1}$$

Consider two vectors $j(\underline{r}, \underline{X})u, j(\underline{s}, \underline{Y})v$ in $\mathcal{H}$ . Without loss of generality we assume $\underline{r} = \underline{s}$. Now

$$\begin{aligned}
&\langle k_t(U)j(\underline{r}, \underline{X})u, k_t(U)j(\underline{r}, \underline{Y})v\rangle \\
&\quad = \langle j(\underline{r}, X_1, \ldots, X_{p-1}, UX_p, \ldots, X_n)u, j(\underline{r}, Y_1, \ldots, Y_{p-1}, UY_p, \ldots, Y_n)v\rangle \\
&\quad = \langle u, T_{r_n}(X_n^* T_{r_{n-1}-r_n}(X_{n-1}^* \cdots \\
&\qquad T_{r_p - r_{p+1}}(X_p^* U^* T_{r_{p-1}-r_p}(X_{p-1}^* \cdots T_{r_1 - r_2}(X_1^* Y_1) \cdots Y_{p-1}) U Y_p) \cdots, Y_{n-1}) Y_n) v\rangle
\end{aligned}$$

$= \langle j(\underline{r}, \underline{X})u, j(\underline{r}, \underline{Y})v \rangle,$

as $U \in \mathcal{Z}(\mathcal{A})$ can be taken across to combine with $U^*$. So $k_t(U)$ defined by (6.1) is an isometry. Moreover for any two unitary operators $U$, $V$ in $\mathcal{Z}(\mathcal{A})$ we have $k_t(UV) = k_t(U)k_t(V)$. Hence $k_t(U)$ extends to a unitary operator on $\mathcal{H}$. As every element in $\mathcal{Z}(\mathcal{A})$ is a linear combination of atmost four unitaries we have linear maps $k_t(Z)$ in $\mathcal{B}(\mathcal{H})$ satisfying

$$k_t(Z)j(\underline{r}, \underline{Y})u = j(\underline{r}, Y_1, \ldots, Y_{(p-1)}, ZY_p, \ldots, Y_n)u \tag{6.2}$$

for $(\underline{r}, \underline{Y}, u) \in \mathcal{M}$ with $r_1 > r_2 > \cdots > r_p = t > r_{p+1} > \cdots > r_n$. Clearly $k_t$ is a $*$-homomorphism for every $t$. To prove (i), observe that for $(\underline{r}, \underline{Y}, u) \in \mathcal{M}_t$, $k_t(Z)j(\underline{r}, \underline{Y})u = j_t(Z)j(\underline{r}, \underline{Y})u$. As $t$ is arbitrary this also implies

$$k(\underline{r}, \underline{Y})u = j(\underline{r}, \underline{Y})u \tag{6.3}$$

for $(\underline{r}, \underline{Y}, u) \in \mathcal{M}$. Now fix $s < t$ and consider $(\underline{r}, \underline{Y}, u) \in \mathcal{M}$. With out loss of generality we assume $s, t$ are in $\{r_1, \ldots, r_n\}$. If $p, q$ are the indices such that $s = r_p$, and $t = r_q$ then we have

$$\begin{aligned} k_s(Z)k_t(Z')j(\underline{r}, \underline{Y})u &= j(\underline{r}, Y_1, \ldots, Y_{(p-1)}, ZY_p, \ldots, Z'Y_q, \ldots, Y_n)u \\ &= k_t(Z')k_s(Z)j(\underline{r}, \underline{Y})u \end{aligned}$$

for $Z, Z' \in \mathcal{Z}(\mathcal{A})$. Hence $k_s(Z)$ and $k_t(Z')$ commute.

Suppose $k_t' : \mathcal{Z}(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$ is another family of $*$-homomorphisms satisfying (i) and (ii). From (i) with $k_t'$ in place of $k_t$ we have

$$F(t)k_t'(Z)F(t) = j_t(Z) \text{ for } Z \in \mathcal{Z}(\mathcal{A}).$$

As $k_t'$ and $j_t$ are representations of $\mathcal{Z}(\mathcal{A})$, this means

$$k_t'(Z)F(t) = j_t(Z) = F(t)k_t'(Z) \tag{6.4}$$

for the following reason. Considering the positive operator $j_t(Z^*Z)$ we have

$$\begin{aligned} j_t(Z^*Z) &= F(t)k_t'(Z^*Z)F(t) \\ &= F(t)k_t'(Z^*)\{F(t) + (1 - F(t))\}k_t'(Z)F(t) \\ &= j_t(Z^*)j_t(Z) + \{((1 - F(t))k_t'(Z)F(t)\}^*\{((1 - F(t))k_t'(Z)F(t)\}. \end{aligned}$$

Hence $(1 - F(t))k_t'(Z)F(t) = 0$, implying (6.4). Now for $(\underline{r}, \underline{Y}, u) \in \mathcal{M}$ with $t = r_p$ for some $p$, remembering that $r_1 > \cdots > r_n$ we obtain

$$\begin{aligned} j(\underline{r}, \underline{Y})u &= j_{r_1}(Y_1)j_{r_2}(Y_2)\ldots j_{r_n}(Y_n)u \\ &= k'_{r_1}(Y_1)F(r_1)j_{r_2}(Y_2)\ldots j_{r_n}(Y_n)u \\ &= k'_{r_1}(Y_1)j_{r_2}(Y_2)\ldots j_{r_n}(Y_n)u \end{aligned}$$

and by induction, $j(\underline{r}, \underline{Y})u = k'(\underline{r}, \underline{Y})u$. Now from commutativity and (6.3)

$$k'(\underline{r}, \underline{Y})u = k(\underline{r}, \underline{Y})u$$

for arbitrary (not necessarily decreasing) $\underline{r} = \{r_1, \ldots, r_n\}$. Hence $k_t'$ and $k_t$ are equal. For all $(\underline{r}, \underline{Y}, u) \in \mathcal{M}$, $k_t(1)j(\underline{r}, \underline{Y})u = j(\underline{r}, \underline{Y})u$. So $k_t(1) = I$.

If $\{T_t\}$ is not conservative consider the extended algebra $\hat{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ and construct a conservative semigroup $\{\hat{T}_t\}$ with associated Markov flow $(\hat{\mathcal{H}}, \hat{F}, \hat{j}_t)$ as in the proof of Theorem 4.8. Now homomorphisms $\hat{k}_t : \mathcal{Z}(\mathcal{A}) \to \mathcal{B}(\hat{\mathcal{H}})$ can be constructed as above. Once again define $k_t(Z)$ as $\hat{k}_t\binom{Z}{0}|_{\mathcal{H}}$ , where $\mathcal{H}$ is the Hilbert space of the minimal Markov flow $j_t$ constructed in Theorem 4.8.

**Remark 6.2 :** In general $\{k_s(1)\}$ is a decreasing family of projections with $k_0(1) = I$.

In Theorem 6.1 it is remarkable that though the semigroup $\{T_t\}$ does not necessarily leave the centre invariant we still have a family of $*$-homomorphisms of the centre which behaves almost like a weak Markov flow. The triple $(\mathcal{H}, F, k_t)$ will be called the *central flow* associated with the semigroup $\{T_t\}$.

**Corollary 6.3 :** If the centre $\mathcal{Z}(\mathcal{A})$ is invariant under $\{T_t\}$ then $(\mathcal{H}, F, k_t)$ is an abelian Markov flow on $\mathcal{Z}(\mathcal{A})$ with expectation semigroup $T_t|_{\mathcal{A}}$.

**Proof :** For $s < t$ and $Z \in \mathcal{Z}(\mathcal{A})$ ,

$$\mathbb{E}^F_{s]}k_t(Z) = \mathbb{E}^F_{s]}\mathbb{E}^F_{t]}k_t(Z) = \mathbb{E}^F_{s]}j_t(Z) = j_s(T_{t-s}(Z)) = k_s(T_{t-s}(Z))F(s)$$

as $T_{t-s}(Z) \in \mathcal{Z}(\mathcal{A})$.

Finally we have the promised theorem showing the existence and uniqueness of minimal abelian dilations for quantum dynamical semigroups on abelian algebras. This theorem may be compared with Theorems 4.7 and 4.8 where minimal subordinate dilations were considered which were not necessarily abelian. In contrast now we have abelian weak Markov flows which are not necessarily subordinate flows.

**Theorem 6.4 :** Let $\{T_t\}$ be a quantum dynamical semigroup on an abelian algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}_0$. Then there exists a minimal abelian weak Markov flow $(\mathcal{H}, F, k_t)$ on $\mathcal{A}$ with expectation semigroup $\{T_t\}$. If $(\mathcal{H}', F', k_t')$ is another minimal abelian weak Markov flow with expectation semigroup $\{T_t\}$ then it is unitarily equivalent to $(\mathcal{H}, F, k_t)$.

**Proof :** The Markov flow $(\mathcal{H}, F, k_t)$ has already been constructed in Theorem 6.1 and Corollary 6.3. Minimality of $k_t$ follows from (6.3) and minimality of $j_t$. To prove uniqueness consider the subordinate weak Markov flow $(\mathcal{H}', F', j_t')$, where $j_t'(X) = F'(t)XF'(t)$. For $(\underline{r}, \underline{Y}, u)$ in $\mathcal{M}$ using adaptedness of $k_t'$ we have

$$\begin{aligned} k'(\underline{r},\underline{Y})u &= k'_{r_1}(Y_1)\cdots k'_{r_n}(Y_n)u \\ &= k'_{r_1}(Y_1)\cdots k'_{r_n}(Y_n)F'(r_n)u \\ &= k'_{r_1}(Y_1)\cdots k'_{r_{n-1}}(Y_{n-1})j'_{r_n}(Y_n)u \\ &\ \vdots \\ &= j'(\underline{r},\underline{Y})u. \end{aligned}$$

By minimality $\{k'(\underline{r},\underline{Y})u : (\underline{r},\underline{Y},u) \in \mathcal{M}\}$ is total in $\mathcal{H}'$. Note that as $k_t'$ is assumed to be non-commutative we are allowed to take $r_i$'s decreasing without loss of generality. Hence $(\mathcal{H}', F', j_t')$ is a minimal subordinate Markov dilation of $T_t$ and from Theorem 4.8 it is unitarily equivalent to $(\mathcal{H}, F, j_t)$. Now the proof is complete using uniqueness of $k_t$ proved in Theorem 6.1. ∎

It may be noted that the abelian Markov flow constructed in Example 2.3 through Kolmogorov consistency theorm has to be unitarily equivalent to the one constructed above if the algebra under consideration is $L^\infty(\mu)$.

In general it may not be possible to extend the central flow $k_t$ of Theorem 6.1 to a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. This should be clear from the following discrete time example.

**Example 6.5** : Consider the algebra $\mathcal{A} = \mathcal{M}_2(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C})$ acting on $\mathbb{C}^2 \oplus \mathbb{C}^3$ and the semigroup $\{I, T, T^2, \ldots,\}$ where $T : \mathcal{A} \to \mathcal{A}$ is the completely positive map given by

$$T\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{matrix} & 0 \\ 0 & B \end{bmatrix} \quad \text{where } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

and $A$ is in $\mathcal{M}_2(\mathbb{C})$. Note that $T(I) = I$ and $T^2 = T$. Then it can easily be shown that $\mathcal{H} = \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}$, $F(0) = I \oplus I \oplus 0$, $F(t) = I \oplus I \oplus I$ for $t \geq 1$. The flow $j_t$ is given by

$$j_0\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad j_t\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} b_{11} & b_{12} & 0 & b_{13} \\ b_{21} & b_{22} & & b_{23} \\ 0 & & B & 0 \\ b_{31} & b_{32} & 0 & b_{33} \end{bmatrix}$$

for $t \geq 1$. The centre $\mathcal{Z}(\mathcal{A})$ consists of matrices $aI \oplus bI$ in $\mathcal{M}_2(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C})$ and $k_t$ is given by $k_0(aI \oplus bI) = aI \oplus bI \oplus aI$ and $k_t(I) = bI \oplus bI \oplus bI$. Clearly $k_0$ cannot be extended to a representation of $\mathcal{A}$ in $\mathcal{M}_2(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C}) \oplus \mathcal{M}_1(\mathbb{C})$.

**Example 6.6** : Consider the semigroup $T_t$ of Example 4.11. The associated minimal abelian flow may be realized as $(\mathcal{H}, F, k_t)$, where

$$k_t\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = j_t\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) + a\chi_{(t,\infty)},$$

and $(\mathcal{H}, F, j_t)$ is as before. We can compare this flow with the related classical Markov process.

CHAPTER II

# Feller Perturbations of Positive Semigroups

## 7 Exit and Entrance Cocycles

Following the spirit of Feller [Fel,2,3] and Chung [C1,2] two types of perturbations of strongly continuous positive semigroups on von Neumann algebras are outlined in the next section. The first type arises from what we call an exit cocycle for the semigroup. The second arises from a dualisation of the first and is based on an entrance cocycle for the same semigroup. The terminology is motivated from considerations of classical Markov processes. Feller perturbations work for general positive semigroups, i.e., complete positivity is not needed. Only continuous time semigroups are considered here after unless mentioned otherwise. In this section exit and entrance cocycles are defined and several examples are given. We restrict ourselves to semigroups on von Neumann algebras rather than general $C^*$ algebras as preduals come into picture.

Let $\mathcal{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}_0$ and let $T_t : \mathcal{A} \to \mathcal{A}, t \geq 0$ be a strongly continuous semigroup of positive linear maps so that the following conditions are fulfilled : (i) $T_0(X) = X$ for all $X \in \mathcal{A}$; (ii) $T_s(T_t(X)) = T_{s+t}(X)$ for all $X \in \mathcal{A}, s, t \geq 0$; (iii) $\lim_{t \to s} T_t(X) = T_s(X)$ for all $X \in \mathcal{A}, s \geq 0$; (iv) $T_t(X) \geq 0$ for all $X \geq 0, X \in \mathcal{A}, t \geq 0$.

**Definition 7.1** : Let $\mathcal{F}_b(\mathbb{R}_+)$ be the family of all bounded Borel subsets of $\mathbb{R}_+$. A map $S : \mathcal{F}_b(\mathbb{R}_+) \to \mathcal{A}_+$, the cone of nonnegative elements in $\mathcal{A}$, is called an *$\mathcal{A}_+$-valued Radon measure* on $\mathbb{R}_+$, if, for any sequence $\{E_i\}$ of disjoint elements in $\mathcal{F}_b(\mathbb{R}_+)$ such that $\bigcup_i E_i \in \mathcal{F}_b(\mathbb{R}_+), S(\bigcup_i E_i) = \sum_i S(E_i)$ where the right hand side converges in the strong sense.

**Definition 7.2** : An $\mathcal{A}_+$-valued Radon measure $S$ on $\mathbb{R}_+$ is called an *exit cocycle* for the semigroup $\{T_t\}$ if

$$T_t(S(E)) = S(E+t) \text{ for all } E \in \mathcal{F}_b(\mathbb{R}_+), t \geq 0. \tag{7.1}$$

**Remark 7.3** : The strong continuity of the semigroup $\{T_t\}$ and the fact that $T_0$ is identity imply that every exit cocycle is nonatomic, i.e., $S(\{t\}) = 0$ for all $t \geq 0$.

**Example 7.4** : Choose and fix an element $B$ in $\mathcal{A}_+$. Define

$$S_B(E) = \int_E T_t(B)dt \text{ for } E \in \mathcal{F}_b(\mathbb{R}_+). \tag{7.2}$$

Then the semigroup property and positivity of $\{T_t\}$ imply that $S_B$ is an exit cocycle.

Another class of exit cocycles is obtained by the following definition.

**Definition 7.5** : Let $A \in \mathcal{A}$. Then $A$ is called *excessive* for the semigroup $\{T_t\}$ if $T_t(A) \leq A$ for all $t \geq 0$. If $T_t(A) = A$ for all $t \geq 0$, $A$ is said to be *harmonic*.

**Example 7.6** : Let $A \in \mathcal{A}$ be excessive for $\{T_t\}$. Define a Radon measure $S$ by putting

$$S([a,b]) = T_a(A) - T_b(A) \text{ for } 0 \leq a \leq b < \infty. \tag{7.3}$$

Since $A$ is excessive and $T_t$ is positive we have $S([a,b]) = T_a(A - T_{b-a}(A)) \geq 0$. And as $T_t(S([a,b])) = S([a+t, b+t])$, it follows that $S$ is an exit cocycle.

It should be noted that in this example if $\mathcal{L}$ is the generator of $\{T_t\}$ and $A$ is in the domain of $\mathcal{L}$ then $S([a,b]) = \int_a^b T_s(-\mathcal{L}(A))ds$ reduces to Example 7.4. If $B \in \mathcal{A}_+$ is harmonic and $\mu$ denotes the Lebesgue measure in $\mathbb{R}_+$ then $S_B(E) = \mu(E)B$ which is a special case of Example 7.4. Note that the identity operator is always excessive for a quantum dynamical semigroup.

**Example 7.7** : If we replace the von Neumann algebra $\mathcal{A}$ by a $C^*$ algebra the definitions given in the preceding discussions are meaningful. For example let $\mathcal{A}$ denote the $C^*$ algebra of bounded continuous functions on $\mathbb{R}_+$ and let $\{T_t\}$ be the semigroup of translation operators defined by

$$(T_t f)(x) = f(x+t), \quad t \geq 0, f \in \mathcal{A}.$$

Define the Radon measure $S$ by

$$S([a,b])(x) = (b+x)^\delta - (a+x)^\delta, \quad 0 \leq x < \infty$$

for some fixed $\delta, 0 < \delta < 1$. Then

$$\frac{d}{dx} S([a,b]) = \delta((b+x)^{\delta-1} - (a+x)^{\delta-1}) \leq 0$$

and hence $\sup_x S([a,b])(x) = b^\delta - a^\delta < \infty$. Clearly $S([a,b])(x) \geq 0$. The cocycle property is obvious. This cocycle if expressed as $\int_a^b \phi(x+s)ds$ then $\phi(x) = \delta x^{\delta-1}$ is unbounded and $\phi \notin \mathcal{A}$. On the other hand if $S([a,b]) = T_a\psi - T_b\psi$ then $\psi(x) = c - x^\delta$ for some constant c, is unbounded and does not belong to $\mathcal{A}$. This shows that there are exit cocycles not covered by Example 7.4 and 7.6.

**Example 7.8** : Let $\mathcal{A}$ be the $C^*$ algebra of all bounded continuous functions on the real line $\mathbb{R}$ and $\{T_t\}$ be the semigroup defined by

$$(T_t f)(x) = \mathbb{E} f(x + B(t)), \quad t \geq 0, f \in \mathcal{A},$$

where $B(t)$ denotes the standard Brownian motion process on $\mathbb{R}$. Define $S$ by

$$S([0,t])(x) = \mathbb{E}|x + B(t)| - |x|, \quad x \in \mathbb{R}, t \geq 0.$$

From Tanaka's formula (page 137 in [CW]) we know that

$$d|x + B(t)| = \operatorname{sgn}(x + B(t))dB(t) + dL(t,x)$$

where $L(t,x)$ is the local time at $-x$. $L(t,x)$ is jointly continuous in the variables $t$ and $x$ and $L(t,x)$ is nondecreasing in $t$ for fixed $x$. Thus $S([0,t])(x)$ is increasing in $t$ and continuous in $(t,x)$. Since $B(t)$ has a symmetric distribution it follows that

$S([0,t])(x) = S([0,t])(-x)$. When $x \geq 0$ we have

$$\begin{aligned} S([0,t])(x) &= \int_{-\infty}^{\infty} (|x + y\sqrt{t}| - |x|)(2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} dy \\ &= (2\pi)^{-\frac{1}{2}} \left\{ \int_{-xt^{-\frac{1}{2}}}^{\infty} y\sqrt{t} e^{-\frac{y^2}{2}} dy - \int_{-\infty}^{-xt^{-\frac{1}{2}}} (2x + y\sqrt{t}) e^{-\frac{y^2}{2}} dy \right\} \\ &\leq (2\pi)^{-\frac{1}{2}} \sqrt{t} \left\{ \int_{-xt^{-\frac{1}{2}}}^{\infty} y e^{-\frac{y^2}{2}} dy - \int_{-\infty}^{-xt^{-\frac{1}{2}}} y e^{-\frac{y^2}{2}} dy \right\} \\ &= \pi^{-\frac{1}{2}} \sqrt{2t} e^{-\frac{x^2}{2t}} \end{aligned}$$

which shows that $\sup_x S([0,t])(x) < \infty$. The cocycle property is now immediate from the standard properties of Brownian motion.

We now proceed to a brief discussion on entrance cocycles. Let $\mathcal{A}_*$ denote the predual of the von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_0)$. Then $\mathcal{A}_*$ is a subalgebra of the algebra of all trace class operators on $\mathcal{H}_0$.

**Definition 7.9 :** An $\mathcal{A}_*$-valued nonnegative Radon measure $\psi$ on $\mathbb{R}_+$ is called an *entrance cocycle* for the semigroup $\{T_t\}$ if

$$\text{tr}\, \psi(E) T_s(X) = \text{tr}\ \ \psi(E+s)X \text{ for all } s \geq 0, E \in \mathcal{F}_b(\mathbb{R}_+), X \in \mathcal{A}. \tag{7.4}$$

Imitating Example 7.4 and 7.6 we can obtain examples of entrance cocycles provided there exists a semigroup $\{\pi_t\}$ in $\mathcal{A}_*$ satisfying

$$\text{tr}\, \pi_t(\rho) X = \text{tr}\, \rho T_t(X) \ \text{ for all } \ X \in \mathcal{A}, \rho \in \mathcal{A}_*, t \geq 0. \tag{7.5}$$

In such a case we have the following examples.

**Example 7.10 :** Let $\rho \in \mathcal{A}_*$ be positive. Define the Radon measure $\psi_\rho$ by

$$\psi_\rho(E) = \int_E \pi_s(\rho) ds \quad \text{for} \quad E \in \mathcal{F}_b(\mathbb{R}_+). \tag{7.6}$$

Then $\psi_\rho$ is an entrance cocycle for the semigroup $\{T_t\}$.

**Example 7.11** : Suppose $\rho_0 \in \mathcal{A}_*$ is excessive for $\{\pi_t\}$. Define $\psi$ by

$$\psi([a,b]) = \pi_a(\rho_0) - \pi_b(\rho_0). \tag{7.7}$$

Then $\psi$ is an entrance cocycle.

# 8 Perturbed Semigroups

In his analysis of Kolmogorov equations Feller [Fe 1,2,3] constructed a class of substochastic semigroups called minimal semigroups and outlined a method of constructing new semigroups including stochastic ones by perturbing their resolvents (or Laplace transforms) appropriately. The same goal was achieved more directly by a pathwise approach in the works of Chung [C1,2] and Dynkin [Dy]. Non-commutative minimal semigroups and their perturbations were introduced by Davies while studying neutron diffusion equations [Da]. Much progress has been achieved in understanding these semigroups by Chebotarev ([Ch],[CF]), Mohari[Mo], Fagnola [Fa1] et al. Section 13 outlines the basic ideas involved and provides some truely non-commutative examples. Here we have a more general method of perturbations of positive semigroups on von Neumann algebras based on exit and entrance cocycles of Section 7. Boundary theory is essentially the study of Markov dilations of these perturbed semigroups.

Let $\{T_t : t \in R_+\}$ be a positve strongly continuous semigroup on a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$. Suppose that $S$ is an exit cocycle for $T_t$ and $\omega$ is a state (always assumed to be normal) on $\mathcal{A}$. The Feller perturbation of $T_t$ will depend upon this pair $(S, \omega)$. To this end we introduce the Radon measure $\mu$ defined by

$$\mu(E) = \omega(S(E)), \quad E \in \mathcal{F}_b(\mathbb{R}_+) \tag{8.1}$$

and some notation. For any $t \geq 0, n = 0,1,2,...$ define the linear maps $T_t^{(n)}$ on $\mathcal{A}$ by

$$T_t^{(n)}(X) = \begin{cases} T_t(X) & \text{if } n = 0, \\ \int_0^t S(dt_1)\omega(T_{t-t_1}(X)) & \text{if } n = 1, \\ \int_{\Delta_n(t)} S(dt_1)\mu(dt_2)...\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X)) & \text{if } n \geq 2 \end{cases} \tag{8.2}$$

for all $X \in \mathcal{A}$, where

$$\Delta_n(t) = \{(t_1, t_2, ..., t_n) : t_i \geq 0 \text{ for each } i, t_1 + \cdots + t_n \leq t\}.$$

For $0 \leq s \leq t < \infty$ and $0 \leq m \leq n$ define

$$T_{s,t}^{m,n}(X) = \begin{cases} T_t(X) & \text{if } m = n = 0 \\ \int_{\Delta_{m,n}(s,t)} S(dt_1)\mu(dt_2)...\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X)) & \text{otherwise} \end{cases} \quad (8.3)$$

for all $X \in \mathcal{A}$, where

$$\Delta_{m,n}(s,t) = \begin{cases} \{(t_1, t_2, ..., t_n) : & t_1 + \cdots + t_m \leq s < t_1 + \cdots + t_{m+1}, \\ & t_1 + \cdots + t_n \leq t, \ t_i \geq 0 \text{ for every } i.\} \quad \text{if } m < n, \\ \Delta_m(s) & \text{if } m = n. \end{cases}$$

**Proposition 8.1** : For each $X \in \mathcal{A}$ the infinite series

$$\hat{T}_t(X) = \sum_{n=0}^{\infty} T_t^{(n)}(X) \quad (8.4)$$

converges in operator norm. The convergence is uniform in $t$ over bounded intervals.

**Proof** : It follows from Remark 7.3 and the definition of $\mu$ in (8.1) that $\mu$ is nonatomic. Hence $\lim_{s \downarrow 0} \mu([0,s]) = \mu(\{0\}) = 0$. Choose and fix $t_0 > 0$ such that $\mu([0,t_0]) < 1$. We shall now estimate $\mu^{*^n}([0,t]) = \mu^{\otimes^n}(\{(t_1,...,t_n) : t_1 + \cdots + t_n \leq t\})$. Let $t_1 + \cdots + t_n \leq t$ and $r = \#\{i : 1 \leq i \leq n, t_i \geq t_0\}$. Then $t \geq t_1 + \cdots + t_n \geq rt_0$ and, in particular, $r \leq [\frac{t}{t_0}] + 1 = j$, say. Hence

$$\begin{aligned} \mu^{*^n}([0,t]) &\leq \sum_{r=0}^{j} \binom{n}{r} \mu([0,t_0])^{n-r} \mu([t_0,t])^r \\ &\leq n^j \mu([0,t_0])^{n-j} \sum_{r=0}^{j} \binom{j}{r} \mu([0,t_0])^{j-r} \mu([t_0,t])^r \\ &= \mu([0,t])^j n^j \mu([0,t_0])^{n-j}. \end{aligned}$$

From (8.2) we have for $n \geq 1$

$$||T_t^{(n)}(X)|| \leq ||\omega|| \, ||X|| \, ||S([0,t])|| \sup_{0 \leq s \leq t} ||T_s|| \mu([0,t])^j (n-1)^j \mu([0,t_0])^{n-1-j}$$

which implies the required result. ■

In order to show that $\{\hat{T}_t\}$ is a semigroup we need the following lemma.

**Lemma 8.2** : For any $s, t \in \mathbb{R}_+$ and $X \in \mathcal{A}$ the following hold:

(i) $T_s^{(m)}(T_t^{(n)}(X)) = T_{s,s+t}^{m,m+n}(X)$ for $m, n \geq 0$;

(ii) $\sum_{m+n=k} T_s^{(m)}(T_t^{(n)}(X)) = T_{s+t}^{(k)}(X)$ for $k \geq 0$.

**Proof** : First we prove (i). Clearly (i) holds when $m = n = 0$. When $m, n \geq 1$ we have

$$T_s^{(m)}(T_t^{(n)}(X))$$
$$= \int_{\Delta_m(s)} S(ds_1)\mu(ds_2)...\mu(ds_m)\omega(T_{s-(s_1+\cdots+s_m)}(T_t^{(n)}(X)))$$
$$= \int_{\Delta_m(s)} S(ds_1)\mu(ds_2)...\mu(ds_m)$$
$$\times \int_{\Delta_n(t)} \omega(T_{s-(s_1+\cdots+s_m)}(S(dt_1)))\mu(dt_2)...\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X)).$$

Consider the change of variables

$$s_{m+1} = s - (s_1 + \cdots + s_m) + t_1, s_{m+2} = t_2, ..., s_{m+n} = t_n.$$

Then the cocycle property of $S$ and the definition of $\mu$ imply

$$\omega(T_{s-(s_1+\cdots+s_m)}(S(dt_1))) = \mu(ds_{m+1})$$

and under the change of variables, the conditions $t_1 \geq 0$ and $t_1 + \cdots + t_n \leq t$ become $s \leq s_1 + \cdots + s_{m+1}$ and $s_1 + \cdots + s_{m+n} \leq s + t$ respectively. By the nonatomicity of $S$ we may as well write $s < s_1 + \cdots + s_{m+1}$ so that

$$T_s^{(m)}(T_t^{(n)}(X)) = \int_{\Delta_{m,m+n}(s,s+t)} S(ds_1)\mu(ds_2)\cdots\mu(ds_{m+n})\omega(T_{t+s-(s_1+\cdots+s_{m+n})}(X))$$

and (8.3) shows that the right hand side is the same as $T_{s,s+t}^{m,m+n}(X)$. When $m = 0$, $n \geq 1$ we have

$$T_s^{(m)}(T_t^{(n)}(X))$$
$$= T_s(\int_{\Delta_n(t)} S(dt_1)\mu(dt_2)\cdots\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X))$$
$$= \int_{\Delta_n(t)} T_s(S(dt_1))\mu(dt_2)\cdots\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X)).$$

Changing the variables to $s_1 = s + t_1, s_2 = t_2, ..., s_n = t_n$ yields the required result as before. When $m \geq 1, n = 0$ the semigroup property of $\{T_t\}$ implies

$$T_s^{(m)}(T_t^{(n)}(X)) = \int_{\Delta_m(s)} S(ds_1)\mu(ds_2)\cdots\mu(ds_m)\omega(T_{s+t-(s_1+\cdots+s_m)}(X))$$

and completes the proof of (i).

Property (ii) is obvious for $k = 0$. When $k \geq 1$ property (i) together with the observation that $\Delta_k(s+t)$ is the disjoint union of $\{\Delta_{m,k}(s, s+t), 0 \leq m \leq k\}$ for all $s$ and $t$ implies (ii). ■

**Theorem 8.3** : Let $T_t : \mathcal{A} \to \mathcal{A}$ be a positive strongly continuous semigroup of linear maps. Suppose $\omega$ is a state on $\mathcal{A}$ and $S$ is an exit cocycle for $\{T_t\}$. Then the family $\{\hat{T}_t\}$ defined by (8.4) is also a positive strongly continuous semigroup of linear maps on $\mathcal{A}$. If $\{T_t\}$ is completely positive so is $\{\hat{T}_t\}$.

**Proof** : Clearly $\hat{T}_0(X) = T_0(X) = X$ for all $X \in \mathcal{A}$. For $0 \leq s, t < \infty$ and $X \in \mathcal{A}$ we have from Lemma 8.2.

$$\begin{aligned} \hat{T}_s(\hat{T}_t(X)) &= \sum_{m,n\geq 0} T_s^{(m)}(T_t^{(n)}(X)) \\ &= \sum_{k\geq 0}\sum_{m+n=k} T_s^{(m)}(T_t^{(n)}(X)) \\ &= \sum_{k\geq 0} T_{s+t}^{(k)}(X) = T_{s+t}(X). \end{aligned}$$

Thus $\{\hat{T}_t\}$ is a semigroup. By (8.2), $\{T_t^{(n)}\}$ is strongly continuous in $t$ and linear on $\mathcal{A}$ for each $n$ and Proposition 8.1 implies the same property for $\{\hat{T}_t\}$. If $\{T_t\}$ is positive or completely positive so is each $\{T_t^{(n)}\}$ and hence $\{\hat{T}_t\}$ also shares the same property. ■

The semigroup $\{\hat{T}_t\}$ occurring in Theorem 8.3 is called the *Feller perturbation* of $\{T_t\}$ determined by the exit cocycle $S$ and the state $\omega$. It is to be noted that $\hat{T}_t$ may not be contractive even if $T_t$ is. The discussion after Remark 8.8 indicates an important special case where contractivity of perturbed semigroup is easily ensured.

**Remark 8.4** : Theorem 8.3 holds good when $\mathcal{A}$ is a $C^*$ algebra and the proof remains the same.

**Remark 8.5** : From the proof of Proposition 8.1 it follows that $\nu = \delta_0 + \mu + \mu^{*^2} + \ldots$ is a Radon measure on $I\!R_+$ where $\delta_0$ is the Dirac measure at 0 and $\mu$ is defined by (8.1). This shows that the perturbed semigroup $\hat{T}_t$ can be expressed as

$$\hat{T}_t(X) = T_t(X) + \int_0^t (S * \nu)(ds)\omega(T_{t-s}(X))$$

when $S * \nu$ is the positive operator-valued Radon measure defined by

$$S * \nu([0,t]) = \int_0^t S(ds)\nu([0,t-s]).$$

If $X$ is in the domain of the generator $\mathcal{L}$ of $\{T_t\}, u, v$ are vectors in the Hilbert space $\mathcal{H}_0$ (with $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_0)$) and $\langle u, S * \nu([0,t])v\rangle$ is differentiable at the origin then

$$\frac{d}{dt}\langle u, \hat{T}_t(X)v\rangle|_{t=0} = \langle u, \mathcal{L}(X)v\rangle + \omega(X)\frac{d}{dt}\langle u, S * \nu([0,t])v\rangle|_{t=0}.$$

In particular, if $S(E) = S_B(E) = \int_E T_s(B)ds, B \in \mathcal{A}_+$ then

$$\frac{d}{dt}\langle u, \hat{T}_t(X)v\rangle|_{t=0} = \langle u, \{\mathcal{L}(X) + \omega(X)B\}v\rangle.$$

In order to compare the perturbed semigroup $\{\hat{T}_t\}$ with Feller's construction we shall compute its resolvent. At this stage it is useful to recollect the well known Hille-Yosida theorem which makes precise the one to one correspondence between a semigroup and its resolvent.

**Theorem 8.6 (Hille-Yosida [Y], [Dy])** : Let $\mathcal{X}$ be a Banach space. Let $\{R_\lambda\}_{\lambda>\beta}$ be a family of operators in $\mathcal{X}$, with $\beta \geq 0$ a fixed scalar, satisfying the following:

(i) $R_\lambda R_\mu = (\mu - \lambda)^{-1}(R_\lambda - R_\mu)$ for $\lambda, \mu > \beta, \lambda \neq \mu$;

(ii) $||R_\lambda|| \leq M(\lambda - \beta)^{-1}$ for all $\lambda > \beta$ and some positive constant $M$ ;

(iii) $\text{s-}\lim_{\lambda \to \mu} R_\lambda(X) = R_\mu(X)$ for all $\mu > \beta, X \in \mathcal{X}$;

(iv) Range of $R_\lambda$ is dense in $\mathcal{X}$ for some $\lambda > \beta$.

Then there exists a unique strongly continuous semigroup $\{T_t\}$ of operators in $\mathcal{X}$ such that $||T_t|| \leq Me^{\beta t}$ and $R_\lambda(X) = \int_0^\infty e^{-\lambda t} T_t(X)dt$ for all $\lambda > \beta, X \in \mathcal{X}$.

Conversely, if $T_t : \mathcal{X} \to \mathcal{X}, t \geq 0$ is a strongly continuous semigroup of operators then there exist constants $M, \beta \geq 0$ such that $||T_t|| \leq Me^{\beta t}$ for all $t \geq 0$ and $R_\lambda(X) = \int_0^\infty e^{-\lambda t} T_t(X)dt$ defines a family of operators satisfying (i) - (iv). The semigroup $\{T_t\}$ is contractive if and only if $M$ and $\beta$ can be chosen to be 1 and 0 respectively.

**Proof** : We omit the proof (See page 30, Vol. I, [Dy]). ∎

**Theorem 8.7** : Let the semigroup $\{T_t\}$ in Theorem 8.3 satisfy the inequalities $||T_t|| \leq Me^{\beta t}$ for $M > 0, \beta \geq 0$ and all $t \geq 0$. Let $\{R_\lambda, \lambda > \beta\}$ be its resolvent. Then there exists a $\hat{\beta} \geq 0$ such that the resolvent $\{\hat{R}_\lambda, \lambda > \hat{\beta}\}$ is given by

$$\hat{R}_\lambda(X) = R_\lambda(X) + \frac{\omega(R_\lambda(X))}{1 - \omega(A_\lambda)}(A_\lambda) \tag{8.5}$$

where

$$A_\lambda = \int_0^\infty e^{-\lambda t} S(dt), \;\; \lambda > \hat{\beta}. \tag{8.6}$$

**Proof** : Choose and fix a $t_0$ such that $\mu([0, t_0]) = a < 1$. Let $b = ||S([0, t_0])||$. By the cocycle property

$$||S([nt_0, (n+1)t_0])|| = ||T_{nt_0}(S([0, t_0]))|| \leq bMe^{\beta n t_0}, n \geq 0.$$

Hence

$$\begin{aligned} ||\int_0^\infty e^{-\lambda t} S(dt)|| &\leq \sum_{n \geq 0} ||\int_{nt_0}^{(n+1)t_0} e^{-\lambda t} S(dt)|| \\ &\leq b + \sum_{n=1}^\infty bMe^{-(\lambda-\beta)nt_0} \\ &\leq b + \frac{bMe^{-(\lambda-\beta)t_0}}{1 - e^{-(\lambda-\beta)t_0}} \\ &< \infty \text{ for all } \lambda > \beta. \end{aligned}$$

Since $\mu(E) = \omega(S(E))$ we have

$$\int_0^\infty e^{-\lambda t}\mu(dt) \leq a + \frac{bMe^{-(\lambda-\beta)t_0}}{1 - e^{-(\lambda-\beta)t_0}}.$$

Since $a < 1$ we conclude the existence of a constant $\hat{\beta} \geq 0$ such that

$$\omega(A_\lambda) = \int_0^\infty e^{-\lambda t}\mu(dt) < 1 \quad \text{for all} \quad \lambda > \hat{\beta}.$$

Thus, for $X \in \mathcal{A}, \lambda > \hat{\beta}$, we have

$$\begin{aligned}
& \hat{R}_\lambda(X) \\
= & \int_0^\infty e^{-\lambda t}\hat{T}_t(X)dt \\
= & R_\lambda(X) + \sum_{n\geq 1}\int_0^\infty e^{-\lambda t}\{\int_{\Delta_n(t)} S(dt_1)\mu(dt_2)\cdots\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X))\}dt \\
= & R_\lambda(X) + \sum_{n\geq 1}\omega(R_\lambda(X))\omega(A_\lambda)^{n-1}A_\lambda \\
= & R_\lambda(X) + \frac{\omega(R_\lambda(X))}{1-\omega(A_\lambda)}A_\lambda.
\end{aligned}$$

**Remark 8.8** : As a direct consequence of the cocycle property of $S$ it follows that

$$R_\lambda(A_\mu) = \frac{A_\lambda - A_\mu}{\mu - \lambda} \quad \text{for} \quad \lambda, \mu > \beta, \lambda \neq \mu.$$

Using this relation we can verify that $\hat{R}_\lambda$ satisfies the resolvent identity. So we could as well have defined $\hat{R}_\lambda$ directly by (8.5) and (8.6), used the Hille-Yosida theorem and recovered the semigroup $\hat{T}_t$.

Feller in [Fe2] had taken this kind of resolvent approach. We shall compare the formula for $\hat{R}_\lambda$ with that of Feller. Let $A \in \mathcal{A}_+$ be excessive for $\{T_t\}$ and let $S$ be defined as in Example 7.6 so that $S(t) = S([0,t]) = A - T_t(A)$. Then

$$A_\lambda = \int_0^\infty e^{-\lambda t}S(dt) = \lambda\int_0^\infty e^{-\lambda t}(A - T_t(A))dt,$$

i.e., $A_\lambda = A - \lambda R_\lambda(A)$. Hence (8.5) becomes

$$\hat{R}_\lambda(X) = R_\lambda(X) + \frac{\omega(R_\lambda(X))}{1 - \omega(A) + \lambda\omega(R_\lambda(A))}(A - \lambda R_\lambda(A)). \tag{8.7}$$

Now consider the special case $A = qI$ for some $q > 0$ and $\omega(X) = \mathrm{tr}\,\rho X$ for some density matrix $\rho$ describing the state. Then

$$\begin{aligned} \hat{R}_\lambda(X) &= R_\lambda(X) + \frac{\mathrm{tr}\,\rho\, R_\lambda(X)}{1 - q + q\lambda\ \mathrm{tr}\ \rho R_\lambda(I)} q(I - \lambda R_\lambda(I)) \\ &= R_\lambda(X) + \frac{\mathrm{tr}\,\rho R_\lambda(X)}{m + \lambda\ \mathrm{tr}\ \rho R_\lambda(I)}(I - \lambda R_\lambda(I)) \end{aligned} \tag{8.8}$$

where $m = \frac{1-q}{q}$. If $\{T_t\}$ is contractive and $m \geq 0$ then $\{\hat{T}_t\}$ is also contractive. When $m = 0$ and $\{T_t\}$ is not conservative it follows that $\lambda \hat{R}_\lambda(I) = I$ and hence $\{\hat{T}_t\}$ is conservative. When $\mathcal{A} = \ell_\infty$ and $\{T_t\}$ is the minimal semigroup of substochastic matrices associated with a Kolmogorov equation, formula (8.8) coincides with the expression (8.1) in [Fe2]. This suggests that the density matrix $\rho$ in (8.8) mediates the transition from a "boundary point" back into the "state space" of the Markov flow and $\frac{m}{m+1}$ is the probability that it is stuck in the boundary. Of course, it is desirable to have a clearer picture of the manner in which $\rho$ mediates the transition.

Finally we come to Feller Perturbations based on entrance cocycles. As the analysis is similar to that of exit coclycles proofs have been ommitted.

Let $\psi$ be an entrance cocycle for the semigroup $\{T_t\}$ and let $Z$ be a fixed positive element in $\mathcal{A}$. In analogy with (8.4) define

$$\check{T}_t(X) = T_t(X) + \sum_{n\geq 1} \int_{\Delta_n(t)} \mathrm{tr}(\psi(dt_1)X)\mathrm{tr}(\psi(dt_2)Z) \cdot \mathrm{tr}(\psi(dt_n)Z) T_{t-(t_1+\cdots+t_n)}(Z) \tag{8.9}$$

for $t \geq 0, X \in \mathcal{A}$.

**Theorem 8.9** : The series on the right hand side of (8.9) converges in norm and $\{\check{T}_t\}$ is a strongly continuous positive semigroup. If $\{T_t\}$ is completely positive so is $\{\check{T}_t\}$.

**Proof** : This is exactly along the same lines of the proof of Theorem 8.3. ■

**Theorem 8.10** : Let $R_\lambda$ and $\check{R}_\lambda$ be the resolvents of $\{T_t\}$ and $\{\check{T}_t\}$ respectively for $\lambda > \gamma$ for some $\gamma \geq 0$. Then

$$\check{R}_\lambda(X) = R_\lambda(X) + \frac{\alpha_\lambda(X)}{1 - \alpha_\lambda(Z)} R_\lambda(Z) \text{ for } X \in \mathcal{A}$$

where $\alpha_\lambda$ is the positive linear functional on $\mathcal{A}$ given by

$$\alpha_\lambda(X) = \int_0^\infty e^{-\lambda t} \operatorname{tr}(\psi(dt)X)$$

**Proof** : This is obtained by a direct computation. ■

We conclude this section with some remarks on perturbations of direct sums and tensor products of semigroups. Suppose $\mathcal{A}_i$ is a von Neumann algebra of operators in a Hilbert space $\mathcal{H}_i$ and $\{T_t^{(i)}\}$ is a positive strongly continuous semigroup of linear maps on $\mathcal{A}_i$ for each $i = 1, 2$. Let $\omega_i$ be a state in $\mathcal{A}_i$ and let $S_i$ be an exit cocycle for each $i$. For the semigroup $T_t = T_t^{(1)} \oplus T_t^{(2)}, S = S_1 \oplus S_2$ is an exit cocycle and for any $0 \leq p \leq 1, \omega = p\omega_1 \oplus (1-p)\omega_2$ is a state on $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Expressing any element of $\mathcal{A}$ as a column vector $\binom{X}{Y}, X \in \mathcal{A}_1, Y \in \mathcal{A}_2$ we see that the perturbed semigroup $\{\hat{T}_t\}$ associated with the pair $(S, \omega)$ has its resolvent $\hat{R}_\lambda$ given by

$$\hat{R}_\lambda \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} R_\lambda^{(1)}(X) \\ R_\lambda^{(2)}(Y) \end{pmatrix} + \frac{p\omega_1(R_\lambda^{(1)}(X)) + (1-p)\omega_2(R_\lambda^{(2)}(Y))}{1 - \{p\omega_1(A_\lambda^{(1)}) + (1-p)\omega_2(A_\lambda^{(2)})\}} \begin{pmatrix} A_\lambda^{(1)} \\ A_\lambda^{(2)} \end{pmatrix} \qquad (8.10)$$

where $R_\lambda^{(i)}$ is the resolvent of $\{T_t^{(i)}\}$ and $A_\lambda^{(i)} = \int_0^\infty e^{-\lambda t} S_i(dt)$. When $\mathcal{A}_2 = \mathbb{C}, p = 0, \omega_2(c) = c$ and $S_1([0,t]) = I - T_t^{(1)}(I), \{T_t^{(1)}\}$ being contractive (8.10) reduces to

$$\hat{R}_\lambda \begin{pmatrix} X \\ c \end{pmatrix} = \begin{pmatrix} R_\lambda^{(1)}(X) \\ \lambda^{-1}c \end{pmatrix} + \lambda^{-1}c \begin{pmatrix} I - \lambda R_\lambda^{(1)}(I) \\ 0 \end{pmatrix}.$$

This is the resolvent of a semigroup which is the quantum probabilistic analogue of a Markov chain with an absorbing boundary point as described after Theorem 4.10.

Just like direct sums we can also perturb tensor products of semigroups. Indeed, let $T_t = T_t^{(1)} \otimes T_t^{(2)}$ in $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Then there exists an exit cocycle $S$ for $\{T_t\}$ such that $S([0,t]) = S_1([0,t]) \otimes S_2([0,t])$ for all $t$. It should be noted that

$S([a,b]) \neq S_1([a,b]) \otimes S_2([a,b])$. It is also interesting to note that $A_1 \otimes A_2$ is excessive for $\{T_t\}$ if $A_i$ is excessive for $\{T_t^{(i)}\}, i = 1,2$. Indeed,

$$\begin{aligned}(T_t^{(1)} \otimes T_t^{(2)})(A_1 \otimes A_2) &= T_t^{(1)}(A_1) \otimes T_t^{(2)}(A_2) \\ &\leq A_1 \otimes T_t^{(2)}(A_2) \\ &\leq A_1 \otimes A_2.\end{aligned}$$

If $H$ is harmonic for $\{T_t^{(2)}\}$ then $S([0,t]) = S_1([0,t]) \otimes H$ defines an exit cocycle for $\{T_t\}$. If $T_t^{(2)} \equiv$ identity we can express the resolvent of the perturbed semigroup $\{\hat{T}_t\}$ associated with the exit cocycle $S$ and any state $\omega$ on $\mathcal{A}$ as

$$\hat{R}_\lambda(X \otimes Y) = R_\lambda^{(1)}(X) \otimes Y + \frac{\omega(R_\lambda^{(1)}(X) \otimes Y)}{1 - \omega(A_\lambda^{(1)} \otimes H)} A_\lambda^{(1)} \otimes H.$$

A similar analysis can be done with entrance cocycles.

CHAPTER III

# Gluing Adapted Processes and Filtrations

## 9 Processes and Stop Times

Just as a classical stochastic process is a family of random variables indexed by time, a quantum stochastic process may be viewed as a family of operators on some Hilbert space. Some regularity conditions are put at times for technical convenience. In the classical theory of Markov processes the notion of stop times plays a predominant role. However it seems there is not much literature on quantum stop times ([Hu], [BW], [AS], [PS]). Hudson[Hu] looks upon stop times as a spectral measure on $\mathbb{R}_+$. The same idea was used fruitfuly by Parthasarathy and Sinha [PS] to prove a strong factorisability property of Fock spaces. In this short section adapted processes and stop times are formally defined. Some simple examples of stop times are provided. We also introduce a simple notion of integration of operator valued functions with respect to a spectral measure. This integral is used as a tool to construct new processes and filtrations in subsequent sections.

**Definition 9.1** : By a *bounded process* $\boldsymbol{X} = \{X(t), t \geq 0\}$ in a Hilbert space $\mathcal{H}$ we mean a family of bounded operators in $\mathcal{H}$ satisfying the following: (i) the map $t \rightarrow X(t)$ is weakly measurable; (ii) $\sup\limits_{0 \leq s \leq t} ||X(s)|| < \infty$ for every $t$. Such a process is called *contractive, isometric* or *co-isometric* according as all the operators $X(t), t \geq 0$ possess the same property. If $F$ is a weak filtration in $\mathcal{H}$ then $\boldsymbol{X}$ is said to be *adapted* to $F$ if

$$X(t)F(t) = F(t)X(t)F(t) \text{ for every } t.$$

A *stop time* in $\mathcal{H}$ is a spectral measure on the closed interval $[0, \infty] = \mathbb{R}_+ \cup \{\infty\}$ with values in the lattice of projections in $\mathcal{H}$. A stop time $P$ is called a *stop time for the bounded process* $\boldsymbol{X}$ in $\mathcal{H}$ if $X(t)P([0,t]) = P([0,t])X(t)$ for every $t$. $P$ is called an *F-adapted stop time* for the bounded process $\boldsymbol{X}$ if, in addition,

$$P([0,t])F(t) = F(t)P([0,t]) \text{ for every } t.$$

Note that the definition says $X$ is $F$ adapted if $X(t)$ leaves the range of $F(t)$ invariant for every $t$. Naturally enough classical stop times are our first set of examples.

**Example 9.2** : Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mu)$ be a filtered probability space and let $\tau$ be a stop time in $\Omega$ so that $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for every $t$. Let $P(t)$ be multiplication by the indicator of $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ in $L^2(\mu)$. Conditions of commutativity put between $P$ and $X$ are redundant here. Observe that the family of projections $\{F(t)\}$, where $F(t)$ is projection onto the subspace of square integrable functions measurable with respect to $\mathcal{F}_t$, does not constitute a process in the classical sense.

**Example 9.3** : Let $(\mathcal{H}, F, j_t)$ be a subordinate weak Markov flow on $\mathcal{A}$. Suppose $Q$ is a projection in $\mathcal{A}$. Then observe that for $t \geq 0$

$$\inf\{j_r(Q) : 0 \leq r \leq t\} \leq j_t(Q) \leq j_t(1) \leq F(t).$$

So we have a family of projections $\{P(t)\}$ defined by

$$P(t) = F(t) - \inf\{j_r(Q) : 0 \leq r \leq t\}.$$

Also $P(s) \leq P(t)$ for $s \leq t$, and hence $P$ is an $F$ adapted stop time. The stop time $P$ can be interpreted as a quantum analogue of the notion of hitting time of classical Markov processes. We may say that the process is observed until it goes outside the 'domain' $Q$. But one should be a bit cautious here as $j_r(Q)$'s may not commute.

**Example 9.4** : Let $(\mathcal{H}, F, k_t)$ be a central flow as in the discussion before Corollary 6.3. Then from Remark 6.2 $k_t(1)$ is a decreasing family of projections. On setting

$$P(t) = I - k_t(1) \text{ for } t \geq 0,$$

we have an $F$ adapted stop time $P$. Moreover for every $X$ in the initial algebra $\mathcal{A}$, $k_t(X)P(t) = P(t)k_t(X) = 0$, and hence $P$ is a stop time for the process $\{k_t(X), t \geq 0\}$. These stop times play an important role in boundary theory and will be called as exit times.

Now recall that the Hahn-Hellinger theorem [P1] allows us to decompose a spectral measure into a direct sum of canonical spectral measures. We define an integral of operator valued functions with respect to canonical spectral measures and then generalize it by simply taking direct sums. This helps in visualizing the integral.

Let $(\Omega, \mathcal{F}, \mu)$ be a totally finite standard measure space and let $P^\mu$ denote the canonical spectral measure on $\Omega$ so that $P^\mu(E)$ is the operator of multiplication by the indicator function $\chi_E$ of $E \in \mathcal{F}$ in the Hilbert space $L^2(\mu)$. Suppose $k$ is a Hilbert space and $X : \Omega \to \mathcal{B}(k)$ is a map satisfying the following:

(i) the map $\omega \to \langle u, X(\omega)v \rangle$ on $\Omega$ is measurable for every $u, v \in k$;

(ii) $\sup_\mu \|X(\cdot)\|_k < \infty$.

Note that the Hilbert space $L^2(\mu) \otimes k$ is isomorphic to the Hilbert space $L^2(\mu, k)$ where

$$L^2(\mu,k) = \{f \mid f : \Omega \to k, \int_\Omega \|f(\omega)\|_k^2 \mu(d\omega) < \infty\}$$

with

$$\langle f, g \rangle = \int_\Omega \langle f(\omega), g(\omega) \rangle_k \mu(d\omega),$$

$f, g, \ldots$ denoting equivalence classes modulo $\mu$-null sets. Making use of this identification between $L^2(\mu) \otimes k$ and $L^2(\mu, k)$ we define the operator $\int_\Omega P^\mu(d\omega)X(\omega)$ on $L^2(\mu) \otimes k$ by

$$\{\int_\Omega P^\mu(d\omega)X(\omega)f\}(\omega') = X(\omega')f(\omega'),\ \omega' \in \Omega, \tag{9.1}$$

Then $\int_\Omega P^\mu(d\omega)X(\omega)$ is a bounded operator on $L^2(\mu) \otimes k$ with

$$\| \int_\Omega P^\mu(d\omega)X(\omega)\| \le \sup_\mu \|X(\cdot)\|_k. \tag{9.2}$$

It is natural to denote the operator given by (9.1) as $\int_\Omega P^\mu(d\omega) \otimes X(\omega)$ but we drop the symbol $\otimes$ for notational convenience.

Now suppose that $P$ is any spectral measure on the standard Borel space $(\Omega, \mathcal{F})$ with values in the lattice of orthogonal projections in a Hilbert space $\mathcal{H}$. By a part of the Hahn-Hellinger theorem [P1] there exist totally finite measures $\{\mu_\alpha, \alpha \in S\}$ on $(\Omega, \mathcal{F})$ and a unitary operator $U : \mathcal{H} \to \oplus_\alpha L^2(\mu_\alpha)$ such that $UPU^{-1} = \oplus_\alpha P^{\mu_\alpha}$.

Let now $X : \Omega \to \mathcal{B}(k)$ be a weakly measurable map satisfying $\sup_P \|X(\cdot)\|_k < \infty$. Then we define the integral of $X(\cdot)$ with respect to $P$ by

$$\int_\Omega P(d\omega)X(\omega) = U^{-1}\{\oplus_\alpha \int_\Omega P^{\mu_\alpha}(d\omega)X(\omega)\}U. \tag{9.3}$$

Then the left hand side yields an operator on $\mathcal{H} \otimes k$ with

$$\| \int_\Omega P(d\omega)X(\omega)\| \leq \sup_\mu \|X(\cdot)\|_k. \tag{9.4}$$

**Proposition 9.5** : Let $(\Omega, \mathcal{F})$ be a standard Borel space and let $\mathcal{H}, k$ be Hilbert spaces. Suppose $P$ is a spectral measure on $\mathcal{F}$ with values in the lattice of orthogonal projections in $\mathcal{H}$. Let $\mathcal{N}$ be the $*$ unital algebra of all weakly measurable maps of the form $X : \Omega \to \mathcal{B}(k)$ satisfying the condition $\sup_P \|X(\cdot)\|_k < \infty$. Then the following hold:

(i) the map $X \to \int_\Omega P(d\omega)X(\omega)$ is a $*$ unital homomorphism from $\mathcal{N}$ into $\mathcal{B}(\mathcal{H} \otimes k)$ such that (9.4) holds;

(ii) for any $u, u' \in \mathcal{H}, v, v' \in k$

$$\langle u \otimes v, \int_\Omega P(d\omega)X(\omega)u' \otimes v'\rangle = \int_\Omega \langle u, P(d\omega)u'\rangle\langle v, X(\omega)v'\rangle.$$

**Proof** : This is immediate when $\mathcal{H} = L^2(\mu)$ and $P = P^\mu$. Rest follows from (9.2) and (9.3). ∎

## 10 Gluing Adapted Processes

Given two classical stochastic processes $\{\xi(t)\}$ and $\{\eta(t)\}$ with $t \geq 0$ and a stop time $\tau$ for $\{\xi(t)\}$ we can glue them at time $\tau$ and obtain a new process $\{\zeta(t)\}$ by defining

$$\zeta(t) = \begin{cases} \xi(t) & \text{if } t < \tau, \\ \eta(t-\tau) & \text{if } t \geq \tau. \end{cases} \tag{10.1}$$

Here initially we are observing the process $\zeta$. It is observed upto the stop time $\tau$ and then the second process $\eta$ is initiated. Now a quantum analogue of this construction is introduced using the integral defined in Section 9. This method of obtaining new processes is quite general and works with any two bounded processes. However, we will be gluing only Markov processes to obtain Markov dilations of perturbed semigroups.

Let $\boldsymbol{X}_i$ be a bounded process in the Hilbert space $\mathcal{H}_i, i = 1,2$ and let $P_1$ be a stop time for $\boldsymbol{X}_1$. Then the *glued process* $\boldsymbol{X}_1 o_{P_1} \boldsymbol{X}_2$ is defined by

$$\boldsymbol{X}_1 o_{P_1} \boldsymbol{X}_2(t) = X_1(t)(1 - P_1(t)) + \int_{[0,t]} P_1(dt_1) X_2(t - t_1) \tag{10.2}$$

where the first term is actually the ampliated operator $X_1(t)(1 - P_1(t)) \otimes I_2, P_1(t) = P_1([0,t])$ and $I_2$ is the identity operator in $\mathcal{H}_2$. By Proposition 9.5 it follows that $\boldsymbol{X}_1 o_{P_1} \boldsymbol{X}_2$ is a bounded process in $\mathcal{H}_1 \otimes \mathcal{H}_2$. When the stop time $P_1$ is clear in a context we shall write $\boldsymbol{X}_1 o \boldsymbol{X}_2$ for $\boldsymbol{X}_1 o_{P_1} \boldsymbol{X}_2$. The event that the process $\boldsymbol{X}_1$ is stopped at a time not exceeding $t$ is described by the projection $P_1(t)$. Since $X_1(t)$ and $P_1(t)$ commute with each other we may express the first term on the right hand side of (10.2) also as $P_1((t,\infty])X_1(t)P_1((t,\infty])$.

Normally $P_1(\{0\}) = 0$ so that $\boldsymbol{X}_1 o \boldsymbol{X}_2(0) = X_1(0)P_1((0,\infty]) = X_1(0)$, i.e., the glued process starts at $X_1(0)$. Otherwise $\boldsymbol{X}_1 o \boldsymbol{X}_2(0) = X_1(0)P_1((0,\infty]) + P_1(\{0\})X_2(0)$. This may be interpreted as an instantaneous change from $X_1(0)$ to $X_2(0)$ (with some probability in a given state).

**Proposition 10.1** : Let $P_i$ be a stop time in $\mathcal{H}_i, i = 1,2$. Then $P_1 o_{P_1} P_2$ is a stop time in $\mathcal{H}_1 \otimes \mathcal{H}_2$. If, in addition, $P_i$ is a stop time for the bounded process $\boldsymbol{X}_i, i = 1,2$ then $P_1 o_{P_1} P_2$ is a stop time for the glued process $\boldsymbol{X}_1 o_{P_1} \boldsymbol{X}_2$.

**Proof** : We have from (10.2)

$$\begin{aligned} P_1 o_{P_1} P_2(t) &= P_1(t)P_1((t,\infty]) + \int_{[0,t]} P_1(dt_1)P_2(t - t_1) \\ &= \int_{0 \le t_1 + t_2 \le t} P_1(dt_1)P_2(dt_2) \\ &= P_1 \otimes P_2(\{(t_1,t_2) : 0 \le t_1 + t_2 \le t; t_1, t_2 \ge 0\}). \end{aligned} \tag{10.3}$$

This proves the first part. The second part is immediate from Proposition 9.5 and the definition of a stop time for a bounded process. ∎

We denote the stop time $P_1 o_{P_1} P_2$ by $P_1 o P_2$ and call it the *cumulative stop time* of $P_1$ followed by $P_2$. In other words we wait till the stop time $P_1$ first and subsequently wait till $P_2$ so that the total waiting time is $P_1 o P_2$. Such a view is useful in gluing more than two processes.

**Proposition 10.2** : Let $X_i$ be a bounded process in $\mathcal{H}_i, i = 1, 2, 3$ and let $P_i$ be a stop time for $X_i, i = 1, 2$. Then

$$\begin{aligned}
&\{(X_1 o_{P_1} X_2) o_{P_1 o P_2} X_3\}(t) = \{X_1 o_{P_1}(X_2 o_{P_2} X_3)\}(t) \\
&= X_1(t)P_1((t,\infty]) + \int_{0\leq t_1 \leq t} P_1(dt_1)P_2((t-t_1,\infty])X_2(t-t_1) \\
&\quad + \int_{0\leq t_1+t_2\leq t} P_1(dt_1)P_2(dt_2)X_3(t-t_1-t_2) \qquad (10.4)
\end{aligned}$$

for all $t \geq 0$ in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

**Proof:** By repeated application of (10.2) we have

$$\begin{aligned}
&\{X_1 o_{P_1}(X_2 o_{P_2} X_3)\}(t) = X_1(t)P_1((t,\infty]) + \int_{[0,t]} P_1(dt_1)(X_2 o_{P_2} X_3)(t-t_1) \\
= \;& X_1(t)P_1((t,\infty]) + \int_{[0,t]} P_1(dt_1)\{X_2(t-t_1)P_2((t-t_1,\infty]) \\
& + \int_{[0,t-t_1]} P_2(dt_2)X_3(t-t_1-t_2)\}
\end{aligned}$$

which agrees with the right hand side of (10.4) owing to the fact that $P_2$ is a stop time for $X_2$. Similarly by Proposition 9.5 we have

$$\begin{aligned}
& \{(X_1 o_{P_1} X_2) o_{P_1 o P_2} X_3\}(t) \\
= \;& (X_1 o_{P_1} X_2)(t)(I - P_1 o P_2(t)) + \int_{[0,t]} (P_1 o P_2)(dt_2)X_3(t-t_2) \\
= \;& \{X_1(t)P_1((t,\infty]) + \int_{[0,t]} P_1(dt_1)X_2(t-t_1)\}\{I - \int_{[0,t]} P_1(dt_1)P_2(t-t_1)\} \\
& + \int_{0\leq t_1+t_2\leq t} P_1(dt_1)P_2(dt_2)X_3(t-t_1-t_2) \\
= \;& X_1(t)P_1((t,\infty]) + \int_{[0,t]} P_1(dt_1)(I - P_2(t-t_1))X_2(t-t_1) \\
& + \int_{0\leq t_1+t_2\leq t} P_1(dt_1)P_2(dt_2)X_3(t-t_1-t_2),
\end{aligned}$$

which once again agrees with the right hand side of (10.4). ■

In view of Proposition 10.2 we can now take the liberty of denoting the left hand side of (10.4) as $\boldsymbol{X}_1 o \boldsymbol{X}_2 o \boldsymbol{X}_3$ whenever the concerned stop times $P_1$ and $P_2$ are unambiguously fixed.

Consider a sequence of triples $\mathcal{H}_n, \boldsymbol{X}_n, P_n, n = 1, 2, \ldots$, where $\mathcal{H}_n$ is a Hilbert space, $\boldsymbol{X}_n$ is a bounded process and $P_n$ is a stop time for $\boldsymbol{X}_n$ for each $n$. Let $\mathcal{H}_{n]} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. Define the operators $\hat{X}_{n+1}(t), X^0_{n+1}(t)$ and $X_{n+1]}(t)$ in $\mathcal{H}_{n+1]}$ by

$$\begin{aligned}\hat{X}_{n+1}(t) &= \int_{t_1+\cdots+t_n \leq t < t_1+\cdots+t_{n+1}} P_1(dt_1)\cdots P_{n+1}(dt_{n+1}) X_{n+1}(t - \overline{t_1 + \cdots t_n}) \\ &= \int_{t_1+\cdots+t_n \leq t} P_1(dt_1)\ldots P_n(dt_n) \\ &\quad \times X_{n+1}(t - \overline{t_1 + \cdots t_n}) P_{n+1}(t - \overline{t_1 + \cdots t_n}, \infty])\end{aligned} \tag{10.5}$$

for $n \geq 1$,

$$\hat{X}_1(t) = \int_{t<t_1} P_1(dt_1) X_1(t) = X_1(t) P_1((t, \infty]), \tag{10.6}$$

$$X^0_{n+1}(t) = \int_{t_1+\cdots+t_n \leq t} P_1(dt_1)\cdots P_n(dt_n) X_{n+1}(t - \overline{t_1 + \cdots t_n}) \tag{10.7}$$

$$X_{n+1]}(t) = \sum_{m=0}^{n-1} \hat{X}_{m+1}(t) + X^0_{n+1}(t) \tag{10.8}$$

for $n \geq 1$, where the m-th term which looks like an operator in $\mathcal{H}_{m+1]}$ is, indeed, ampliated to $\mathcal{H}_{n+1]}$. It is to be emphasized that $\boldsymbol{X}_{n+1]} = \boldsymbol{X}_1 o \boldsymbol{X}_2 o \ldots \boldsymbol{X}_{n+1}$, the glued process obtained from the sequence $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_{n+1}$ through the stop times $P_1, P_2, \ldots, P_n$. Define the spectral measures $P_{n]}$ in $\mathcal{H}_{n]}$ by

$$P_{n]}(E) = \int_{\substack{t_1+\ \ +t_n \in E \\ 0 \leq t_i \leq \infty}} P_1(dt_1)\ldots P_n(dt_n), \tag{10.9}$$

for any Borel set $E \subset [0, \infty]$, and denote their ampliations by the same symbols. Then, for any fixed $t$, $P_{n]}(t) = P_{n]}([0, t])$ is a decreasing sequence in $n$ and

$$\hat{X}_{n+1}(t) = (P_{n]}(t) - P_{n+1]}(t)) \hat{X}_{n+1}(t) (P_{n]}(t) - P_{n+1]}(t)), \tag{10.10}$$

$$X^0_{n+1}(t) = P_{n]}(t) X^0_{n+1}(t) P_{n]}(t) \tag{10.11}$$

with the understanding that $P_{0]}(t) \equiv I$. We have the estimates

$$\|\hat{X}_{n+1}(t)\| \leq \sup_{0\leq s\leq t} \|X_{n+1}(s)\|, \tag{10.12}$$

$$\|X_{n+1]}(t)\| \leq \sup_{1\leq j\leq n+1} \sup_{0\leq s\leq t} \|X_j(s)\|. \tag{10.13}$$

Let now $\phi_n$ be a unit vector in $\mathcal{H}_n, n = 1, 2, \ldots$ Consider the countable tensor product $\mathcal{H} = \bigotimes_{n=1}^{\infty} \mathcal{H}_n$ defined with respect to the stabilizing sequence $\{\phi_n\}$. Assume that

$$\sup_n \sup_{0\leq s\leq t} \|X_n(s)\| < \infty \text{ for all } t \geq 0. \tag{10.14}$$

On ampliating $\hat{X}_{n+1}(t)$ to $\mathcal{H}$ we see from (10.10) and (10.12) that the infinite series

$$X_{\infty]}(t) = \sum_{m=0}^{\infty} \hat{X}_{m+1}(t) \tag{10.15}$$

converges strongly and

$$\|X_{\infty]}(t)\| \leq \sup_n \sup_{0\leq s\leq t} \|X_n(s)\|. \tag{10.16}$$

Roughly speaking, $X_{\infty]}$ is the glued process $X_1 o X_2 o \ldots$ Note that the infinitely glued process $X_{\infty]}$ depends on the stabilizing sequence $\{\phi_n\}$. The next two propositions describe the basic properties of the operation of gluing a finite or countable number of bounded processes.

**Proposition 10.3** : Let $X_n, Y_n$ be bounded processes in the Hilbert space $\mathcal{H}_n$ for each $n = 1, 2, \ldots$ satisfying (10.14) and let $P_n$ be a stop time for both $X_n$ and $Y_n$ for each $n$. Then the following holds for all $2 \leq n \leq \infty$:

(i) $X_{n]} + Y_{n]} = (X + Y)_{n]}$;

(ii) $X_{n]} Y_{n]} = (XY)_{n]}$;

(iii) $(X_{n]})^* = (X^*)_{n]}$;

(iv) $X_{n]}$ is positive or contractive according as each $X_i$ is positive or contractive.

**Proof** : Immediate from Proposition 9.5 and the definition of glued processes. ■

**Proposition 10.4** : In Proposition 10.3 suppose that $X_n$ is the process $I_n$ where $I_n(t) \equiv I$ in $\mathcal{H}_n$ for each $n$. Then $I_{n]}(t) \equiv I$ for $2 \leq n < \infty$. Define the probability measures $\{\nu_n\}$ on $[0, \infty]$ associated with the stabilizing sequence $\{\phi_n\}$ by

$$\nu_n(E) = \langle \phi_n, P_n(E)\phi_n \rangle \text{ for every Borel set } E \subset [0, \infty].$$

Then $I_{\infty]}(t) \equiv I$ if and only if

$$\lim_{n \to \infty} (\nu_1 * \nu_2 * \cdots * \nu_n)([0, t]) = 0 \tag{10.17}$$

for all $0 \leq t < \infty$.

**Proof** : The first part is immediate from the relations $\hat{I}_{n+1}(t) = P_{n]}(t) - P_{n+1]}(t)$, $P_{0]}(t) = I$ and the fact that $X^0_{n+1}(t)$ in (10.7) becomes $P_{n]}(t)$. To prove the sufficiency in the second part consider an element $u = u_1 \otimes u_2 \otimes \cdots \otimes u_k \otimes \phi_{k+1} \otimes \phi_{k+2} \otimes \cdots$ in $\mathcal{H}$ and observe that (10.7) yields

$$\|X^0_{n+1}(t)u\|^2 = (\lambda_1 * \cdots * \lambda_k * \nu_{k+1} * \nu_{k+2} * \cdots * \nu_n)([0, t]) \tag{10.18}$$

for $n > k$, when $X_i(t) \equiv I$ for all $i$, $\lambda_i$'s being the measures defined by $\lambda_i(E) = \langle u_i, P_i(E)u_i \rangle$, for any Borel subset $E$ of $[0, \infty]$. Now (10.17) implies that the left hand side of (10.18) converges to 0 as $n \to \infty$. Since vectors of the form $u$ are total in $\mathcal{H}$ it follows that $X^0_{n+1}(t) \to 0$ strongly as $n \to \infty$ for every $t$. Now (10.8) and (10.15) together with the first part imply that $\hat{I}_{\infty]}(t) = I$ for all $t \geq 0$.

To prove the necessity of (10.17) observe that $\nu_1 * \nu_2 * \cdots * \nu_n([0, t])$ decreases monotonically in $n$ for every fixed $t \geq 0$. Suppose that $\lim_{n \to \infty} \nu_1 * \cdots * \nu_n([0, t_0]) = \delta > 0$ for some $t_0 > 0$. Then (10.8) implies that for the unit vector $u = \phi_1 \otimes \phi_2 \otimes \cdots$ in $\mathcal{H}$

$$\begin{aligned} \|I_{\infty]}(t_0)u\|^2 &= \lim_{n \to \infty} \|\sum_{m=0}^{n-1} \hat{I}_{m+1}(t_0)u\|^2 \\ &= 1 - \delta < 1. \end{aligned}$$

In other words $I_{\infty]}(t_0)$ is a proper projection . ■

**Remark 10.5** : From Proposition 10.3 and 10.4 it is clear that for $2 \leq n < \infty$ the bounded process $X_{n]}$ is isometric, co-isometric or unitary according as each

$X_i, i = 1, 2, \ldots$ has the same property. If the measures $\{\nu_n\}$ defined in Proposition 10.4 satisfy the condition (10.17) then $X_{\infty]}$ is isometric, co-isometric or unitary according as each $X_i, i = 1, 2, \ldots$ has the same property.

**Proposition 10.6** : Let $\mathcal{H}_n, X_n, P_n, n = 1, 2, \ldots$ be as in Proposition 10.3. Suppose that the maps $t \to X_n(t)$ are strongly right continuous for each $n$. Then $X_{n]}(t)$ is strongly right continuous in $t$ for every $2 \leq n \leq \infty$. If $X_n(t)$ is strongly continuous in $t$ and $P_n$ has no atoms in $\mathbb{R}_+$ for every $n$ then $X_{n]}(t)$ is strongly continuous in $t$ for every $2 \leq n \leq \infty$.

**Proof** : Consider an element $u = u_1 \otimes u_2 \otimes \ldots$ in $\mathcal{H}$ where each $u_n$ is a unit vector and $u_n = \phi_n$ for all $n$ exceeding some $n_0$. From (10.10) and (10.15) we have

$$\|X_{\infty]}(t)u\|^2 = \sum_{n=0}^{\infty} \|\hat{X}_{n+1}(t)u\|^2. \tag{10.19}$$

Consider a fixed bounded interval $[0, T]$ and observe that (10.10), (10.12) and (10.14) imply the existence of a positive constant $C$ depending on $T$ such that

$$\begin{aligned} \|\hat{X}_{n+1}(t)u\|^2 &\leq C\|(P_{n]}(t) - P_{n+1]}(t))u\|^2 \\ &= C\{(\mu_1 * \cdots * \mu_n)([0,t]) - (\mu_1 * \cdots * \mu_{n+1})([0,t])\} \end{aligned} \tag{10.20}$$

for all $0 \leq t \leq T$ where $\mu_i(\cdot) = \langle u_i, P_i(\cdot)u_i \rangle$. Note that when $n = 0$ the right hand side of the inequality above is to be interpreted as $C(1 - \mu_1([0,t]))$. It follows from (10.19) and (10.20) that the right hand side of (10.19) converges uniformly in $t \in [0, T]$. Thus, in order to prove the first part of the proposition, it suffices to show that the map $t \to \hat{X}_{n+1}(t)u$ is strongly right continuous. We have

$$\begin{aligned} & \{\hat{X}_{n+1}(t+h) - \hat{X}_{n+1}(t)\}u \\ = & \{\int_{[0,t+h]} P_{n]}(ds) X_{n+1}(t+h-s) P_{n+1}((t+h-s, \infty])\}u \\ & -\{\int_{[0,t+h]} P_{n]}(ds) X_{n+1}(t-s) P_{n+1}((t-s, \infty])\}u \\ = & \int_{[0,t]} P_{n]}(ds) X_{n+1}(t+h-s) P_{n+1}((t+h-s, \infty])u \end{aligned}$$

$$- \int_{[0,t]} P_{n]}(ds) X_{n+1}(t-s) P_{n+1}((t-s,\infty])u$$
$$+ \int_{(t,t+h]} P_{n]}(ds) X_{n+1}(t+h-s) P_{n+1}((t+h-s,\infty])u.$$

Thus

$$\begin{aligned} &\|\{\hat{X}_{n+1}(t+h) - \hat{X}_{n+1}(t)\}u\|^2 \\ \leq &\int_{[0,t]} < u_{n]}, P_{n]}(ds)u_{n]} > \|\{X_{n+1}(t+h-s)P_{n+1}((t+h-s,\infty]) \\ &-X_{n+1}(t-s)P_{n+1}((t-s,\infty])\}u_{n+1}\|^2 + C\langle u_{n]}, P_{n]}((t,t+h])u_{n]}\rangle, \end{aligned}$$

where $C$ is the positive constant mentioned earlier, $u_{n]} = u_1 \otimes \cdots \otimes u_n, h > 0, t+h \leq T$. Since $X_{n+1}(t)$ and $P_{n+1}((t,\infty])$ are both right continuous for every $n$ the right continuity of $\hat{X}_n(t)$ in $t$ follows from the inequality above. It is to be noted that we have used the fact that vectors of the form $u$ described at the beginning are total in $\mathcal{H}$. The second part of the proposition is proved in the same manner. ■

## 11 Gluing Filtrations

As filtrations are adapted processes in their own right they can be glued the way general adapted proceesses are glued in Section 10. However filtrations glued this way do not give new filtrations as we no longer have an increasing family of projections. This anomaly is taken care of by a minor modification while gluing filtrations. In the end we have a filtration which may be considered as the natural filtration for glued processes.

Consider a sequence $(\mathcal{H}_n, F_n), n = 1,2,\ldots$, where $\mathcal{H}_n$ is a Hilbert space and $F_n$ is a weak filtration in $\mathcal{H}_n$ such that the map $t \to F_n(t)$ is strongly right continuous. Let $\boldsymbol{X}_n$ be a bounded process adapted to $F_n$ and let $P_n$ be a stop time for $\boldsymbol{X}_n$ so that $P_n$ is adapted to $F_n$ in the sense of Definition 9.1. We choose the stabilizing sequence of unit vectors $\{\phi_n\}$ such that $\phi_n$ is in the range of $F_n(0)$ for each $n = 1,2,\ldots$. Denote by $\Phi([m,n]) = |\phi_m\rangle\langle\phi_m| \otimes |\phi_{m+1}\rangle\langle\phi_{m+1}| \otimes \cdots \otimes |\phi_n\rangle\langle\phi_n|$ the one dimensional projection (The operators of the form $|x\rangle\langle y|$ have been defined in the proof of Theorem 2.6) ampliated to $\mathcal{H}_{n]}$ or $\mathcal{H}$ whenever necessary and using the same symbol for operators and their ampliations introduce the operators $\tilde{F}_{n+1]}(t)$ by

$$\begin{aligned}\tilde{F}_{n+1]}(t) &= \sum_{m=0}^{n-1} \hat{F}_{m+1}(t)\Phi([m+2,n+1]) \\ &\quad + \int_{t_1+\cdots+t_n\leq t} P_1(dt_1)\ldots P_n(dt_n)F_{n+1}(t-\overline{t_1+\cdots+t_n}) \quad (11.1)\end{aligned}$$

for $1 \leq n < \infty$ and the operators $\tilde{F}_{\infty]}(t)$ in $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes \ldots$ by

$$\tilde{F}_{\infty]}(t) = \sum_{m=0}^{\infty} \hat{F}_{m+1}(t)\Phi([m+2,\infty)). \quad (11.2)$$

Here we have used the notations (10.5) and (10.6). It is important to compare (11.1) and (11.2) with (10.8) and (10.15).

**Proposition 11.1 :** $\tilde{F}_{n+1]}$ defined by (11.1) is a right continuous weak filtration in $\mathcal{H}_{n+1]}$ for every $2 \leq n < \infty$. If $X_n$ is an $F_n$-adapted bounded process in $\mathcal{H}_n$ and $P_n$ is an $F_n$-adapted stoptime for every $n$ then $X_{n+1]}$ is $\tilde{F}_{n+1]}$-adapted for every $1 \leq n < \infty$.

**Proof :** We prove the result first when $n = 1$. From Proposition 9.5 it follows easily that the operator

$$\tilde{F}_{2]}(t) = F_1(t)(1-P_1(t))|\phi_2\rangle\langle\phi_2| + \int_{[0,t]} P_1(dt_1)F_2(t-t_1)$$

with $\phi_2$ in the range of $F_2(0)$ is a projection. For $0 \leq s \leq t < \infty$ we have

$$\begin{aligned}&\tilde{F}_{2]}(t)\tilde{F}_{2]}(s) \\ &= F_1(t)(1-P_1(t))(1-P_1(s))F_1(s)|\phi_2\rangle\langle\phi_2| \\ &\quad + F_1(t)\textstyle\int_{(t,\infty]} P_1(dt_1)|\phi_2\rangle\langle\phi_2|\int_{[0,s]} P_1(ds_1)F_2(s-s_1) \\ &\quad + \textstyle\int_{[0,t]} P_1(dt_1)F_2(t-t_1)\int_{(s,\infty]} P_1(ds_1)F_1(s)|\phi_2\rangle\langle\phi_2| \\ &\quad + \textstyle\int_{[0,t]} P_1(dt_1)F_2(t-t_1)\int_{[0,s]} P_1(ds_1)F_2(s-s_1) \\ &= (1-P_1(t))F_1(s)|\phi_2\rangle\langle\phi_2| + 0 + \textstyle\int_{(s,t]} P_1(ds_1)F_1(s)|\phi_2\rangle\langle\phi_2| \\ &\quad + \textstyle\int_{[0,s]} P_1(ds_1)F_2(s-s_1) \\ &= \tilde{F}_{2]}(s)\end{aligned}$$

which shows that $\tilde{F}_{2]}(t)$ is increasing in $t$. In other words $\tilde{F}_{2]}$ is a filtration. To prove the adaptedness of $\boldsymbol{X}_{2]}$ with respect to $\tilde{F}_{2]}$ observe that

$$\begin{aligned}
&X_{2]}(t)\tilde{F}_{2]}(t)\\
&= \{X_1(t)(1-P_1(t)) + \textstyle\int_{[0,t]} P_1(dt_1)X_2(t-t_1)\}\tilde{F}_{2]}(t)\\
&= X_1(t)(1-P_1(t))F_1(t)|\phi_2\rangle\langle\phi_2| + \textstyle\int_{[0,t]} P_1(dt_1)X_2(t-t_1)F_2(t-t_1)\\
&= |\phi_2\rangle\langle\phi_2|F_1(t)X_1(t)(1-P_1(t))F_1(t)|\phi_2\rangle\langle\phi_2|\\
&\quad + \textstyle\int_{[0,t]} P_1(dt_1)F_2(t-t_1)X_2(t-t_1)F_2(t-t_1)\\
&= \tilde{F}_{2]}(t)X_{2]}(t)\tilde{F}_{2]}(t),
\end{aligned}$$

which proves the claim for $n = 1$. Now assume that the Proposition is true for $n \leq k$. Then on gluing $\tilde{F}_{k+1]}$ with $F_{k+2}$ using the cumulative stop time $P_{k+1]} = P_1 o P_2 o \ldots o P_{k+1}$ given by

$$P_{k+1]}(t) = \int_{t_1+\cdots+t_{k+1}\leq t} P_1(dt_1)\cdots P_{k+1}(dt_{k+1})$$

we have a new filtration $G$ given by

$$G(t) = \tilde{F}_{k+1]}(t)(I - P_{k+1]}(t))|\phi_{k+2}\rangle\langle\phi_{k+2}| + \int_{[0,t]} P_{k+1]}(dt_{k+2})F_{k+2}(t-t_{k+2})$$

which is easily verified to be the same as $\tilde{F}_{k+2]}(t)$. Hence $\tilde{F}_{k+2]}$ is also a filtration. Since $\boldsymbol{X}_{k+2]} = \boldsymbol{X}_{k+1]} o \boldsymbol{X}_{k+2}$ a repetition of the earlier argument shows that $\boldsymbol{X}_{k+2]}$ is $\tilde{F}_{k+2]}$-adapted. The strong right continuity of $\tilde{F}_{n]}(t)$ in $t$ is proved exactly as in Proposition 10.6. ■

**Proposition 11.2** : In Proposition 11.1 suppose that the sequence of probability measures $\{\nu_n\}$ in the closed interval $[0,\infty]$ defined by $\nu_n(\cdot) = \langle\phi_n, P_n(\cdot)\phi_n\rangle, n = 1,2,\ldots$ satisfies (10.17). Then $\tilde{F}_{\infty]}$ defined by (11.2) is a strongly right continuous weak filtration in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$. If $\boldsymbol{X}_n$ is an $F_n$-adapted process and $P_n$ is an $F_n$-adapted stop time for $\boldsymbol{X}_n$ for every $n$ then $\boldsymbol{X}_\infty]$ is $\tilde{F}_{\infty]}$ - adapted.

**Proof**: Proceeding along the lines of the proof of Proposition 10.4 we conclude that $\tilde{F}_{\infty]}(t) = \text{s-}\lim_{n\to\infty} \tilde{F}_{n+1]}(t), X_{\infty]}(t) = \text{s-}\lim_{n\to\infty} X_{n+1]}(t)$. The strong right continuity of $\tilde{F}_{\infty]}(t)$ in $t$ is proved exactly as in Proposition 10.6. Rest is immediate from Proposition 11.1. ■

CHAPTER IV

# Glued Markov Flows

## 12 Boundary Theory

In classical probability theory a Markov process governed by a nonconservative or substochastic semigroup of transition probability operators on a state space $\mathcal{S}$ is interpreted as a Markov process whose trajectories may get out of the space $\mathcal{S}$ (or hit the boundary) at an exit time depending on the individual trajectory. Such an exit time provides a natural stop time at which the trajectory may be stopped at the boundary with probability $p$ or continued with probability $q = 1 - p$ along a new independent trajectory of the original flow starting from a point $x \in \mathcal{S}$ chosen according to a suitable entrance probability law. Such a procedure can be repeated ad infinitum. The aim of the present section is to quantize this idea or, equivalently, express it in the language of operators in a Hilbert space by adopting the gluing mechanism described in Chapter III with respect to suitable exit times for a nonconservative Markov flow mediated by a quantum dynamical semigroup $\{T_t\}$ on a unital von Neumann algebra and thereby obtain a new Markov flow whose expectation semigroup $\{\hat{T}_t\}$ is a Feller perturbation of $\{T_t\}$.

Before embarking on this we explain the construction of Chung[C1] in greater detail to facilitate the intuition. Let $\mathcal{S}$ be a countable set. Consider a matrix $Q = ((q_{ij}))_{i,j \in \mathcal{S}}$ satisfying

$$\text{(i)} \quad q_{ij} \geq 0 \quad \text{if } i \neq j,$$
$$\text{(ii)} \quad -\infty < \textstyle\sum_j q_{ij} \leq 0 \quad \text{for all } i.$$

The first problem is to find all semigroups of substochastic matrices $((p_{ij}(t)))$ such that

$$\lim_{t \downarrow 0} \frac{1}{t}(p_{ij}(t) - \delta_{ij}) = q_{ij} \quad \text{for all } i, j. \tag{12.1}$$

And the second problem is to realize these semigroups through explicit probabilistic constructions. To avoid some trivialities we assume $q_{ii} < 0$ (note the strict inequality) for all $i$. Now Feller's minimal semigroup $((\bar{p}_{ij}(t)))$ is a special solution of (12.1). It is given by

$$\bar{p}_{ij}(t) = \sum_{n \geq 0} p_{ij}^{(n)}(t),$$

where,

$$p_{ij}^{(0)}(t) = \delta_{ij} e^{q_{ii}(t)} \text{ and } p_{ij}^{(n+1)}(t) = \sum_{k \neq i} \int_0^t e^{q_{ii}(t-s)} q_{ik} p_{kj}^{(n)} ds \text{ for } n \geq 0.$$

This semigroup is minimal in the sense that if $((p_{ij}(t)))$ is another semigroup of substochastic matrices satifying (12.1) then

$$\bar{p}_{ij}(t) \leq p_{ij}(t) \text{ for every } i, j, \text{ and } t.$$

Now we can have a Markov process $\xi$ on the state space $\mathcal{S}$ with $((\bar{p}_{ij}(t)))$ as its semigroup of transition probability matrices. Note that here substochasticity implies that $\xi$ would be defined only upto a stop time $\tau$ called the first infinity of $\xi$. Consider a sequence $\xi^{(1)}, \xi^{(2)}, \ldots$, of mutually independent Markov processes having a fixed initial distribution $\{p_i\}$, transition probabilities $((\bar{p}_{ij}(t)))$ and associated stop times $\tau^{(1)}, \tau^{(2)}, \ldots$. Define a new process $\hat{\xi}$ by

$$\hat{\xi}(t) = \begin{cases} \xi^{(1)}(t) & 0 \leq t < \tau^{(1)} \\ \xi^{(n+1)}(t - (\tau^{(1)} + \cdots \tau^{(n)})) & \tau^{(1)} + \cdots \tau^{(n)} \leq t < \tau^{(1)} + \cdots \tau^{(n+1)} \\ & \text{for } n \geq 1 \end{cases}$$

Then $\hat{\xi}$ is a Markov process on $\mathcal{S}$. In this construction stopping at the boundary is not allowed and the process returns instantly to the original state space according to the law $\{p_i\}$ after exit. This makes the process conservative. $\hat{\xi}$ has the same initial distribution $\{p_i\}$ and a new semigroup of transition probability matrices $((\hat{p}_{ij}(t)))$ given by

$$\hat{p}_{ij}(t) = \bar{p}_{ij}(t) + \sum_{n \geq 0} \int_0^t (\sum_k p_k \bar{p}_{kj}(t-s)) dL_i * L^{*^n}(s) \tag{12.2}$$

where

$$L_i(s) = 1 - \sum_k \bar{p}_{ik}(s), \quad L(s) = \sum_i p_i L_i(s)$$

and $*$ stands for convolution in $\mathbb{R}_+$. Comparing (12.2) with Remark 8.5 it is clear that $\hat{p}_{ij}(t)$ is the Feller perturbation of $\bar{p}_{ij}(t)$ determined by the exit cocycle obtained from the excessive operator $I$ and the state $\{p_i\}$. It is self-evident that $\hat{p}_{ij}(t)$ is another solution of (12.1). Quantum analogue of this construction is now a simple matter of putting together results of previous chapters.

**Definition 12.1:** Let $(\mathcal{H}, F, j_t)$ be a weak Markov flow on a unital von Neumann algebra $\mathcal{A}$ of operators in a Hilbert space $\mathcal{H}_0$ with expectation semigroup $\{T_t\}$. A spectral measure $P$ on the closed interval $[0, \infty]$ with values in the lattice of orthogonal projections in $\mathcal{H}$ is called an *exit time* for the flow $(\mathcal{H}, F, j_t)$ if the following conditions hold:

(i) $j_t(X)P([0,t]) = 0$ for all $t \geq 0, X \in \mathcal{A}$; (12.3)

(ii) $P([0,t])F(t) = F(t)P([0,t])$ for all $t \geq 0$;

(iii) If $S_P$ denotes the positive operator-valued Radon measure defined on $\mathbb{R}_+$ by

$$S_P(\{0\}) = 0, S_P((a,b]) = \mathbb{E}^F_{0]}P((a,b])|_{\mathcal{H}_0} \tag{12.4}$$

then $S_P(E) \in \mathcal{A}$ and

$$j_s(S_P(E)) = \mathbb{E}^F_{s]}P(E+s) \text{ for all } s \geq 0, E \in \mathcal{F}_b(\mathbb{R}_+). \tag{12.5}$$

Condition (ii) expresses the adaptedness of the stop time $P$ and for any initial state $\lambda$ on $\mathcal{A}$, $\lambda(S_P([0,t]))$ is the probability that "hitting the boundary" occurs at or before time $t$. Condition (i) can be interpreted as the fact that if the system or flow goes out of $\mathcal{A}$ before time $t$ the event $j_t(X)$ for any projection $X$ in $\mathcal{A}$ cannot occur at time $t$. Condition (iii) emphasizes the covariant nature of the exit time under the flow.

**Proposition 12.2** : If $\mathcal{H}, F, j_t$ and $P$ are as in Definition 12.1 then the Radon measure $S_P$ satisfying (12.4) and (12.5) is an exit cocycle for the expectation semigroup $\{T_t\}$ of the flow $(\mathcal{H}, F, j_t)$.

**Proof** : Taking conditional expectation $\mathbb{E}_{0]}^F$ in (12.5) we have from the Markov property of the flow

$$T_s(S_P(E))F(0) = \mathbb{E}_{0]}^F j_s(S_P(E)) = \mathbb{E}_{0]}^F \mathbb{E}_{s]}^F P(E+s)$$
$$= \mathbb{E}_{0]}^F P(E+s) = S_P(E+s)F(0).$$

∎

Let $\mathcal{H}_n, F_n, j_t^{(n)}, P_n, n = 1,2,\ldots$ be copies of $\mathcal{H}, F, j_t, P$ in Definition 12.1. Note that equation (12.3) together with its adjoint and condition (ii) of Definition 12.1 imply that the exit time $P$ is also an $F$-adapted stop time for the bounded process $\{j_t(X), t \geq 0\}$ for every $X \in \mathcal{A}$. Choose and fix a unit vector $\phi$ in the range of $F(0)$ in $\mathcal{H}$. Let $\hat{\mathcal{H}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$ where the countably infinite tensor product is taken with respect to the stabilizing sequence $\{\phi_n\}$ with $\phi_n = \phi$ in the n-th copy for each $n$. Using $\{P_n\}$ we make an infinite gluing of the processes $\{j_t^{(n)}(X)\}$ for each $X \in \mathcal{A}$ as in Section 10 to obtain the processes

$$\begin{aligned}\hat{j}_t(X) &= j_t^{(1)}(X)P_1((t,\infty]) \\ &+ \sum_{n\geq 1}\int_{t_1+\cdots+t_n \leq t < t_1+\cdots+t_{n+1}} P_1(dt_1)\cdots P_{n+1}(dt_{n+1}) j^{(n+1)}_{t-(t_1+\cdots+t_n)}(X).\end{aligned} \tag{12.6}$$

By (12.3) we have $j_t(X)P((t,\infty]) = j_t(X)$ and (12.6) can be expressed as

$$\hat{j}_t(X) = \sum_{n\geq 0} \hat{j}_t^{(n)}(X) = \sum_{n\geq 0}\sum_{m=0}^{n} \hat{j}_{s,t}^{m,n}(X) \quad \text{for } 0 \leq s \leq t, X \in \mathcal{A} \tag{12.7}$$

where

$$\hat{j}_t^{(n)}(X) = \begin{cases} j_t^{(1)}(X) & \text{if } n = 0, \\ \int_{\Delta_n(t)} P_1(dt_1)\ldots P_n(dt_n) j^{(n+1)}_{t-(t_1+\cdots+t_n)}(X) & \text{if } n \geq 1 \end{cases}$$

and for $0 \leq m \leq n$

$$\hat{j}_{s,t}^{(m,n)}(X) = \begin{cases} j_t^{(1)}(X) & \text{if } m = n = 0, \\ \int_{\Delta_{m,n}(s,t)} P_1(ds_1)\ldots P_n(ds_n) j^{(n+1)}_{t-(s_1+\cdots+s_n)}(X) & \text{otherwise,} \end{cases}$$

where $\Delta_n$ and $\Delta_{m,n}(s,t)$ are as in (8.2) - (8.3). It is useful to compare the two expressions above with (8.2) and (8.3) and interpret $\hat{j}_t^{(n)}$ as a description of the glued process at time $t$ under the knowledge that exactly $n$ exits have occurred upto time $t$. Similarly $\hat{j}_{s,t}^{m,n}$ describes the glued process under the knowledge that exactly $n$ exits upto time $t$ and $m$ exits upto time $s$ have been made.

**Theorem 12.3** : Let $\mathcal{H}, F, j_t, T_t, P, S_P$ be as in Definition 12.1 and let $\phi$ be a unit vector in the range of $F(0)$. Define the maps $\hat{j}_t : \mathcal{A} \to \mathcal{B}(\hat{\mathcal{H}})$ by (12.6). Let $\hat{F} = \hat{F}_{\infty]}$ be the glued filtration in $\hat{\mathcal{H}}$ defined by (11.2). Then $(\hat{\mathcal{H}}, \hat{F}, \hat{j}_t)$ is a weak Markov flow with expectation semigroup $\{\hat{T}_t\}$ which is the Feller perturbation of $\{T_t\}$ determined by the exit cocycle $S_P$ and the vector state $\omega$ with density matrix $|\phi\rangle\langle\phi|$.

**Proof** : It follows from Proposition 10.3 that for each $t$, $\hat{j}_t$ is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}(\hat{\mathcal{H}})$. From (12.3) and (12.6) we have $\hat{j}_0(X) = j_0^{(1)}(X)(1 - P_1(0)) = j_0^{(1)}(X)$ which is $j_0(X)$ in the first copy of $\mathcal{H}$ ampliated to $\hat{\mathcal{H}}$. Thus

$$\begin{aligned} \hat{F}(0)\hat{j}_0(X)\hat{F}(0) &= F(0)j_0(X)F(0) \otimes \Phi([2,\infty)) \\ &= X\hat{F}(0). \end{aligned}$$

Since the measure $\mu$ defined by $\mu(\cdot) = \langle\phi, P(\cdot)\phi\rangle$ is not degenerate at 0 it is clear that $\lim_{n\to\infty} \mu^{*^n}([0,t]) = 0$ for every $t \geq 0$. Hence by Proposition 11.2 the process $\{\hat{j}_t(X)\}$ is adapted to the filtration $\hat{F}$ for every $X \in \mathcal{A}$.

Fixing $0 \leq s \leq t$ and using (12.6), (12.7) and (11.2) we obtain

$$\begin{aligned} &\hat{F}(s)\hat{j}_t(X)\hat{F}(s) \\ &= \{\sum_{k\geq 0} \hat{F}_{k+1}(s)\Phi([k+2,\infty))\}\{\sum_{0\leq m\leq n<\infty} \hat{j}_{s,t}^{m,n}(X)\}\{\sum_{k\geq 0}\hat{F}_{k+1}(s)\Phi([k+2,\infty))\} \\ &= \sum_{0\leq m\leq n<\infty} Z_{m,n} \end{aligned} \tag{12.8}$$

where

$$Z_{n,n} = \int_{t_1+\cdots+t_n\leq s} P_1(dt_1)\cdots P_n(dt_n)\mathbb{E}^{(n+1)}_{s-\overline{t_1+\cdots+t_n}]}(j^{(n+1)}_{t-\overline{t_1+\cdots+t_n}}(X))\Phi([n+2,\infty)) \tag{12.9}$$

and for $m < n$

$$Z_{m,n} = \int_{\substack{t_1+\cdots+t_m \leq s < t_1+\cdots+t_{m+1} \\ t_1+\cdots+t_n \leq t}} P_1(dt_1)\cdots P_m(dt_m)$$

$$\times F_{m+1}(s - \overline{t_1 + \cdots + t_m}) P_{m+1}(dt_{m+1}) F_{m+1}(s - \overline{t_1 + \cdots + t_m})$$

$$\times\ \mu(dt_{m+2})\cdots\mu(dt_n)\langle\phi, T_{t-\overline{t_1+\cdots+t_n}}(X)\phi\rangle\Phi([m+2,\infty)). \tag{12.10}$$

From the Markov property of $j_t$ it follows that (12.9) can be expressed as

$$Z_{n,n} = \int_{t_1+\cdots+t_n \leq s} P_1(dt_1)\cdots P_n(dt_n)$$
$$\times j^{(n+1)}_{s-\overline{t_1+\cdots+t_n}}(T_{t-s}(X)) F_{n+1}(s - \overline{t_1 + \cdots + t_n})\Phi([n+2,\infty)). \tag{12.11}$$

In (12.10) make the change of variables:

$$s_1 = t_1 + \cdots + t_{m+1} - s, s_2 = t_{m+2}, \ldots, s_{n-m} = t_n$$

and use (12.5) in the form

$$\mathbb{E}^{(m+1)}_{s-\overline{t_1+\cdots+t_m]}}\{P_{m+1}(dt_{m+1})\} = j^{(m+1)}_{s-\overline{t_1+\cdots+t_m}}(S_P(ds_1)).$$

Then we obtain

$$Z_{m,n} = \int_{t_1+\cdots+t_m \leq s} P_1(dt_1)\cdots P_m(dt_m)$$

$$\times j^{(m+1)}_{s-\overline{t_1+\cdots+t_m}}\{\int_{s_1+\cdots+s_{n-m} \leq t-s} S_P(ds_1)\mu(ds_2)\cdots\mu(ds_{n-m})\langle\phi, T_{t-s-\overline{s_1+\ldots+s_{n-m}}}(X)\phi\rangle\}$$

$$\times\ \ F_{m+1}(s - \overline{t_1 + \cdots + t_m})\Phi([m+2,\infty)). \tag{12.12}$$

Plugging the expressions (12.11) and (12.12) in (12.8), first summing over the variable $n - m$ from 0 to $\infty$ and then over the variable $m$ from 0 to $\infty$ we obtain

$$\hat{F}(s)\hat{j}_t(X)\hat{F}(s) = \hat{j}_s(\hat{T}_{t-s}(X))\hat{F}(s),$$

where $\{\hat{T}_t\}$ is the Feller perturbation of $\{T_t\}$ based on the exit cocycle $S_P$ and the vector state $\phi$. ∎

**Remark 12.4:** Theorem 12.3 can be easily adapted to the case of Feller perturbations based on $S_P$ and a state determined by a density matrix of the form $\rho = \sum_n p_n|\phi_n\rangle\langle\phi_n|$ where $\{\phi_n\}$ is an orthonormal sequence in the range of $F(0)$ and $\{p_n\}$ is a probability sequence, i.e., $p_n \geq 0$ for each $n$ and $\sum_n p_n = 1$. We do this as follows. Put $\mathcal{H}_0' = \mathcal{H}_0 \otimes \mathcal{H}_0, \mathcal{A}' = \{I_o \otimes X, X \in \mathcal{A}\}, T_t'(I_0 \otimes X) = I_0 \otimes T_t(X)$ for all $t\epsilon 0, X \in \mathcal{A}$ where $I_0$ is the identity operator in $\mathcal{H}_0$. Let $\mathcal{H}' = \mathcal{H}_0 \otimes \mathcal{H}, F'(t) = I_0 \otimes F(t), j_t'(I_0 \otimes X) = I_0 \otimes j_t(X), P' = I_0 \otimes P, S'_{P'} = I_0 \otimes S_P$. Then $(\mathcal{H}', F', j_t')$ is a weak Markov flow on $\mathcal{A}'$ with exit time $P'$ and expectation semigroup $\{T_t'\}$. $S'_{P'}$ is an exit cocycle for $\{T_t'\}$. Now consider the vector state on $\mathcal{A}'$ determined by the unit vector $\phi' = \sum_n \sqrt{p_n}\phi_n \otimes \phi_n$ in $\mathcal{H}_0'$. We may view $\phi'$ as an element in the range of $F'(0)$ and construct the infinitely glued flow $(\hat{\mathcal{H}}', \hat{F}', \hat{j}_t')$ according to Theorem 12.3. This glued flow is a weak Markov flow on $\mathcal{A}'$ with expectation semigroup $\{\hat{T}_t'\}$ where $\hat{T}_t'(I_0 \otimes X) = I_0 \otimes \hat{T}_t(X), \{\hat{T}_t\}$ being the Feller perturbation of $\{T_t\}$ based on $(S_P, \rho)$.

It is interesting to note that for any initial state $\omega$ on $\mathcal{A}$ and projection $Q \in \mathcal{A}$ the probability that according to the glued flow $(\hat{\mathcal{H}}', \hat{F}', \hat{j}_t')$ exactly $m$ exits occur upto time $s, n$ exits occur upto time $t$ and the event $Q$ occurs at time $t$ is equal to

$$\int_{\substack{s_1+\dots+s_m \leq s < s_1+\dots+s_{m+1} \\ s_1+\dots+s_n \leq t}} tr\rho(T_{t-(s_1+\dots+s_n)}(Q))(\omega o S_P)(ds_1)\mu(ds_2)\dots\mu(ds_n)$$

where $0 \leq s \leq t < \infty$ and $\mu = tr\rho S_P$.

# 13 Examples

In this section few examples of weak Markov flows with exit times are exhibited. Mostly they are based on non-unital Evans-Hudson flows. Non-commutative generaliztions of Feller's work ([Da2], [CF], [Ch], [Fa1,2], [Mo]) on minimal semigroups become relevant in this context. Here one is interested in all quantum dynamical semigroups having the same formal generator in a fixed domain. The natural constructions starting with the formal generator lead to a semigroup called the minimal semigroup which may or may not be conservative. Various criteria are known to determine this. We discuss a few examples of formal generators from [BS2]. In case the minimal semigroup is non-conservative Feller perturbations lead us to many other solutions of the original problem. Dilating these semigroups could be done through gluing Markov flows of the minimal semigroup.

**Example 13.1** : Using isometric cocycles arising naturally from the theory of quantum stochastic differential equations in the Fock spaces one can construct many examples of non-unital flows with an exit time. Indeed, let $\mathcal{H} = \mathcal{H}_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes k)$, and as in [Mo] consider an isometric cocycle $U = \{U(s,t), 0 \leq s \leq t < \infty\}$ obeying the q.s.d.e

$$U(s,s) = 1, dU(s,t) = U(s,t)\{\sum_{i,j} L^i_j d\Lambda^j_i(t)\}$$

where $\{L^i_j\}$ is a family of operators in $\mathcal{H}_0$. (See [P1] for notation). By the cocycle property $U(0,s)U(s,t) = U(0,t)$ for all $0 \leq s \leq t < \infty$. Define

$$J_t(X) = U(0,t)XU(0,t)^*, X \in \mathcal{B}(\mathcal{H}_0)$$

where we denote an operator and its ampliation by the same symbol. Then $J_t(I) = U(0,t)U(0,t)^*$ is a projection. For any $0 \leq s \leq t < \infty, \psi \in \mathcal{H}$ we have

$$\begin{aligned} \langle \psi, J_t(I)\psi \rangle &= \|U(0,t)^*\psi\|^2 \\ &= \|U(s,t)^*U(0,s)^*\psi\|^2 \\ &\leq \|U(0,s)^*\psi\|^2 \\ &= \langle \psi, J_s(I)\psi \rangle. \end{aligned}$$

This shows that $\{J_t(I)\}$ is a family of projections decreasing in $t$. Using the strong continuity of $J_t(X)$ in $t$ we conclude the existence of a spectral measure $P$ on $[0,\infty]$ such that $P([0,t]) = 1 - J_t(I)$ for all $t$ where 1 and $I$ denote the identity operators in $\mathcal{H}$ and $\mathcal{H}_0$ respectively. Let $\{T_t\}$ be the semigroup of completely positive linear maps on $\mathcal{B}(\mathcal{H}_0)$ satisfying $T_t(X) = \mathbb{E}_{0]}J_t(X)$ for all $t \geq 0, X \in \mathcal{A}$ where $\mathbb{E}_{0]}$ is the Fock vacuum conditional expectation. Let $S$ be the positive operator-valued

Radon measure determined by $S([0,t]) = I - T_t(I)$ for all $t \geq 0$. Denoting by $\mathbb{E}_{s]}$ the usual conditional expectation with respect to the Fock vacuum vector $\Phi_{[s}$ in $\Gamma(L^2([s,\infty)\otimes k)$ we have

$$\begin{aligned} \mathbb{E}_s P((s,t+s]) &= \mathbb{E}_{s]}(P([0,t+s]) - P([0,s])) \\ &= \mathbb{E}_{s]}(J_s(I) - J_{s+t}(I)) \\ &= J_s(I - T_t(I)) \\ &= J_s(S([0,t])). \end{aligned} \tag{13.1}$$

Let $F(t)$ denote the projection on to the subspace $\mathcal{H}_0 \otimes \Gamma(L^2[0,t]\otimes k)\otimes \Phi_{[t} \subset \mathcal{H}$. Define

$$j_t(X) = J_t(X)F(t).$$

Then $(\mathcal{H}, F, j_t)$ is a subordinate weak Markov flow on $\mathcal{B}(\mathcal{H}_0)$ with expectation semigroup $\{T_t\}$. Furthermore

$$\begin{aligned} j_t(X)P([0,t]) &= J_t(X)F(t)(1 - J_t(I)) \\ &= F(t)J_t(X)(1 - J_t(I)) \\ &= 0 \end{aligned}$$

and (13.1) implies

$$\begin{aligned} j_s(S([0,t])) &= J_s(S([0,t]))F(s) \\ &= \{\mathbb{E}_{s]}P((s,s+t])\}F(s) \\ &= \mathbb{E}^F_{s]}P((s,s+t]). \end{aligned}$$

In other words $P$ is an exit time for $(\mathcal{H}, F, j_t)$.

More generally, consider a family of non-unital Evans-Hudson flows $J_{s,t}$ , $s \leq t$, on a unital von Neumann algebra $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H}_0)$, taking values in $\mathcal{A}_{[s} = \mathcal{A}_0 \otimes \mathcal{B}(\Gamma(L_2([s,\infty), k]))$ with structure maps $\{\theta^i_j\}$ so that

$$d_t J_{s,t}(X) = J_{s,t}(\theta^i_j(X))d\Lambda^j_i(t), \quad J_{s,s}(X) = X$$

for $s \leq t$. Extend the domain of definition of $J_{0,s}$ from $\mathcal{A}_0$ to $\mathcal{A}_{[s}$ by putting

$$J_s(X \otimes Z) = J_{0,s}(X)\hat{Z}$$

for $X \in \mathcal{A}_0$ and $Z \in \mathcal{B}(\Gamma(L_2([s,\infty),k))$, where $\hat{Z}$ is the ampliation of $Z$ to an element of $\mathcal{A}_{[0} = \mathcal{A}_0 \otimes \mathcal{B}(\Gamma(L_2([0,s),k))) \otimes \mathcal{B}(\Gamma(L_2([s,\infty),k)))$. Then

$$\begin{aligned} J_t(X) &= J_{0,t}(X), \quad X \in \mathcal{A}_0 \\ J_t(X) &= J_s(J_{s,t}(X)), \quad X \in \mathcal{A}_0, 0 \leq s \leq t. \end{aligned}$$

This shows, in particular, that

$$J_t(I) = J_s(J_{s,t}(I)), \quad 0 \leq s \leq t.$$

Since $J_{s,t}$ is a contractive $*$-homomorphism it follows that $\{J_t(I)\}$ is a family of projections which is decreasing and strongly continuous in $t$. Thus there exists a spectral measure $P$ on $[0,\infty]$ such that $P([0,t]) = 1 - J_t(I), 1$ being the identity operator in $\mathcal{H}$. As before define $j_t(X) = J_t(X)F(t)$. Then $(\mathcal{H}, F, j_t)$ yields a weak Markov flow with exit time $P$.

**Example 13.2** : The simplest example of a nonconservative flow with exit time is constructed from a given conservative flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ with expectation semigroup $\{T_t\}$ as follows. Consider a classical Poisson process with intensity $\lambda_0$ whose probability measure $\mu$ in the path space yields the Hilbert space $\mathcal{H}_1 = L^2(\mu)$.

Let $P_1([0,t])$ be the projection in $\mathcal{H}_1$ which is multiplication by the indicator function of the event that the Poisson path undergoes a jump in the interval $[0,t]$. Let $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_1$ and let $\tilde{P}$ be the spectral measure in $[0,\infty]$ determined by

$$\tilde{P}([0,t]) = 1 \otimes P_1([0,t]) \quad \text{for all } t \geq 0.$$

Define

$$\tilde{j}_t(X) = j_t(X) \otimes P_1((t,\infty]), \quad t \geq 0, \quad X \in \mathcal{A}.$$

Note that $\tilde{P}(\{\infty\}) = 0$. If $F_1(t)$ is the projection on to the subspace of functions of the Poisson path upto time $t$ and $\tilde{F}(t) = F(t) \otimes F_1(t)$ it follows from the fact that the Poisson process has independent increments, that

$$\tilde{F}(s)\tilde{j}_t(X)\tilde{F}(s) = \tilde{j}_s(T_{t-s}(X))e^{-\lambda_0(t-s)}$$

for all $0 \leq s \leq t < \infty$. In other words we have a weak Markov flow $(\tilde{\mathcal{H}}, \tilde{F}, \tilde{j}_t)$ with expectation semigroup $\{e^{-\lambda_0 t}T_t\}$. It is easily verified that $\tilde{P}$ is an exit time for this flow.

**Example 13.3** : Let $T_t$ be a uniformly continuous nonconservative quantum dynamical semigroup of ultra-weakly continuous maps on $\mathcal{B}(\mathcal{H}_0)$ for some complex separable Hilbert space $\mathcal{H}_0$. Then by Theorem 1.6 the generator $\mathcal{L}$ of $T_t$ has the form

$$\mathcal{L}(X) = i[H_0, X] - \frac{1}{2}\sum_k (L_k^* L_k X + X L_k^* L_k - 2L_k^* X L_k) - \frac{1}{2}(BX + XB) \quad (13.2)$$

where $H_0, L_k$ and $B$ are bounded operators in $\mathcal{H}_0, H_0$ is selfadjoint, $B$ is positive and $\sum_k L_k^* L_k$ is strongly converent. We shall now construct a concrete Markov flow whose expectation semigroup has generator $\mathcal{L}$. To this end consider $\mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_0)$ and represent any element in it in the form of a matrix $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ where $X_{ik} \in \mathcal{B}(\mathcal{H}_0)$ for each $i, k$. Define the operators

$$H = \begin{pmatrix} H_0 & -\frac{1}{2i}\sqrt{B} \\ \frac{1}{2i}\sqrt{B} & 0 \end{pmatrix}$$

$$L^{(1)} = \begin{pmatrix} 0 & 0 \\ \sqrt{B} & I \end{pmatrix}, L^{(k)} = \begin{pmatrix} L_{k-1} & 0 \\ 0 & 0 \end{pmatrix}, k = 2, 3, \ldots$$

Consider the standard Evans-Hudson flow $\tilde{J}_t$ induced by a unitary cocycle in the Hilbert space

$$\tilde{\mathcal{H}} = (\mathcal{H}_0 \oplus \mathcal{H}_0) \otimes \Gamma(L^2(\mathbb{R}_+) \otimes \ell^2)$$

satisfying $\mathbb{E}_{s]}\tilde{J}_t(\tilde{X}) = \tilde{J}_s(\tilde{T}_{t-s}(\tilde{X}))$ for all $\tilde{X} \in \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_0), 0 \leq s \leq t < \infty$ where $\tilde{T}_t$ has generator $\tilde{\mathcal{L}}$ given by

$$\tilde{\mathcal{L}}(\tilde{X}) = i[H, \tilde{X}] - \frac{1}{2}\sum_k (L^{(k)^*} L^{(k)} \tilde{X} + \tilde{X} L^{(k)^*} L^{(k)} - 2L^{(k)^*} \tilde{X} L^{(k)})$$

and $\mathbb{E}_{s]}$ is Fock vacuum conditional expectation. When $\tilde{X} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ an easy computation shows that

$$\tilde{\mathcal{L}}(\tilde{X}) = \begin{pmatrix} \mathcal{L}(X) & 0 \\ 0 & 0 \end{pmatrix}$$

where $\mathcal{L}(X)$ is given by (13.2). Let $\tilde{F}(t) = 1_{t]} \otimes |\Phi_{[t} >< \Phi_{[t}|$ where $1_{t]}$ is the identity operator in $(\mathcal{H}_0 \oplus \mathcal{H}_0) \otimes \Gamma(L^2[0,t] \otimes \ell^2)$ and $\Phi_{[t}$ is the vacuum vector in $\Gamma(L^2[t,\infty) \otimes \ell^2)$. Put

$$\tilde{j}_t(X) = \tilde{J}_t(\tilde{X})F(t), \quad X \in \mathcal{A}.$$

Now on identifying $X$ with $\tilde{X}$ we get a weak Markov flow $(\tilde{\mathcal{H}}, \tilde{F}, \tilde{j}_t)$ with expectation semigroup $\{T_t\}$ with initial space $\mathcal{H}_0 \oplus \mathcal{H}_0$ (Here the unit of the initial algbra is $I \oplus 0$). However the family of projections $\{P(t), t \geq 0\}$, defined by

$$P(t) = 1 - \tilde{J}_t\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)$$

is not increasing in general and hence does not constitute an exit time for the flow $(\tilde{\mathcal{H}}, \tilde{F}, \tilde{j}_t)$.

Now we come to non-commutative generalizations of Feller's minimal semigroup and related ideas. Let $\mathcal{H}_0$ be a complex separable Hilbert space. Consider a strongly continuous semigroup $C(t)$ on $\mathcal{H}_0$ and a family $\{L_k : k \geq 1\}$ of densely defined closed operators on $\mathcal{H}_0$. We denote the generator of the semigroup $C(t)$ by $Y$ and assume that we have a core $\mathcal{D}$ of $Y$ satisfying,

$$\text{(i)} \quad \mathcal{D} \subset D(L_k) \quad \text{for all} \quad k; \tag{13.3}$$

$$\text{(ii)} \quad \langle f, Yg\rangle + \langle Yf, g\rangle + \sum_{k\geq 1}\langle L_k f, L_k g\rangle = 0 \quad \text{for all } f, g \text{ in } \mathcal{D}. \tag{13.4}$$

Then by [Fa1], $D(Y) \subset D(L_k)$ for all $k$ and (13.4) is satisfied for all $f, g$ in $D(Y)$. Hence without loss of generality we assume $\mathcal{D} = D(Y)$ and (13.3) and (13.4) are satisfied.

For $X$ in $\mathcal{B}(\mathcal{H}_0)$ define a bilinear form $\mathcal{L}(X)$ on $\mathcal{D} \times \mathcal{D}$ by

$$\langle f, \mathcal{L}(X)g\rangle = \langle f, XYg\rangle + \langle Yf, Xg\rangle + \sum_k \langle L_k f, XL_k g\rangle \tag{13.5}$$

for all $f, g$ in $\mathcal{D}$. We are interested in the class $\mathcal{C}$ of all quantum dynamical semigroups $T_t$ on $\mathcal{B}(\mathcal{H}_0)$ satisfying

$$\frac{d}{dt}\langle f, T_t(X)g\rangle|_{t=0} = \langle f, \mathcal{L}(X)g\rangle \tag{13.6}$$

for all $f, g$ in $\mathcal{D}$. This may be compared with (12.1). It is known ([Ch], [Da2]) that $\mathcal{C}$ has a distinguished element $T_t^{\min}$, called *the minimal semigroup*. This semigroup is characterized by the fact that $T_t^{\min} \in \mathcal{C}$ and $T_t^{\min}(X) \leq T_t(X)$ for all $X \in \mathcal{B}(\mathcal{H}_0)_+$ and $T_t$ in $\mathcal{C}$.

**Remark 13.4** : If $T_t^{\min}$ is conservative then $\mathcal{C}$ has no other element and $T_t^{\min}$ is the only semigroup satisfying (13.6). Indeed if $T_t \in \mathcal{C}$ then for $0 \leq X \leq I$,

$$T_t^{\min}(X) \leq T_t(X) \leq I - T_t(I - X) \leq I - T_t^{\min}(I - X) = T_t^{\min}(X),$$

and hence $T_t(X) = T_t^{\min}(X)$ for all $X$.

Whenever $T_t^{\min}$ is nonconservative there are many solutions of (13.6) and at least some of these semigroups can be got through Feller perturbations described in the second chapter. Gluing would be the natural way to obtain their Markov dilations. But normally one does not know the minimal semigroup explicitly and it is a delicate matter to decide its nonconservativity. Various criteria have been developed for this purpose. See [Ch],[Fa1] or [Mo]. Here we state two of them, both quantum versions of results in[Fe1].

Define a form $Q_\lambda(X)$ in $\mathcal{D} \times \mathcal{D}$ for $X$ in $\mathcal{B}(\mathcal{H}_0)$ by

$$\langle f, Q_\lambda(X)g\rangle = \int_0^\infty \exp(-\lambda t)(\sum_k \langle L_k C(t)f, XL_k C(t)g\rangle)dt \tag{13.7}$$

for all $f, g$ in $\mathcal{D}$. It follows from some elementary estimates that $Q_\lambda$ extends to a contractive completely positive map on $\mathcal{B}(\mathcal{H}_0)$. Moreover for $\lambda > 0$ and $X \in \mathcal{B}(\mathcal{H}_0)$,

$Q_\lambda(X) = X$ if and only if

$$\langle f, \mathcal{L}(X)g\rangle = \lambda\langle f, Xg\rangle \ \ \forall \ f, g \in \mathcal{D}.$$

For every scalar $\lambda > 0$, define a subset $\beta_\lambda$ of $\mathcal{B}(\mathcal{H}_0)$ by

$$\beta_\lambda \equiv \{X \in \mathcal{B}(\mathcal{H}_0) : 0 \leq X \leq I \text{ and } \langle f, \mathcal{L}(X)g\rangle = \lambda\langle f, Xg\rangle \ \ \forall \ f, g \in \mathcal{D}\}. \quad (13.8)$$

Now conservativity of $T_t^{\min}$ can be determined using the following theorem.

**Theorem 13.5** : The following are equivalent.

(i) $T_t^{\min}$ is conservative ;

(ii) $\beta_\lambda = \{0\}$ for some $\lambda > 0$;

(iii) $\text{s-}\lim_{n\to\infty} Q_\lambda^n(I) = 0$.

**Proof** : We refer to [BS2], [Ch], [Fa1], and [MS]. ∎

Now we begin with examples of generators leading to conservative minimal semigroups.

**Example 13.6** : Suppose that the generator $Y = iH$ for a symmetric operator $H$ and $L_k = 0$ for all $k$. Clearly (13.3) and (13.4) are satisfied. As $Q_\lambda(X) = 0$ for all $X$ in $\mathcal{B}(\mathcal{H}_0), Q_\lambda(X) = X$ has no non-trivial solution. Hence $T_t^{\min}$ is conservative.

It should be noted that $Y$ can be a generator of a contraction semigroup with out $H$ being self-adjoint. In fact it can easily be shown using a result from XII.9.8, page 1258 of [DS] that $Y$ with above form is a generator of a strongly continuous contraction semigroup if and only if $H$ is a maximally symmetric operator with its positive deficiency index, i.e. the dimension of $\ker(H^* - i)$ is zero. Then $C(t)$ is a semigroup of isometries and $T_t^{\min}$ is given by $T_t^{\min}(X) = C(t)^* X C(t)$. The operator $H$ is self-adjoint if and only if $C(t)$ is unitary for all $t$.

**Example 13.7** : Assume that $Y$ is a generator of a strongly continuous contraction semigroup satisfying

$$|\langle f, Yf\rangle + \langle Yf, f\rangle| \leq K\|f\|^2 \quad (13.9)$$

for some constant $K$ and all $f$ in $\mathcal{D}$. Note that for all $f$ in $\mathcal{D} = D(Y)$,

$$\begin{aligned}\langle f, Q_\lambda(I)f\rangle &= -\int_0^\infty \exp(-\lambda t)(\langle C(t)f, YC(t)f\rangle + \langle YC(t)f, C(t)f\rangle)dt \\ &\leq K\int_0^\infty \exp(-\lambda t)\|C(t)f\|^2 dt \\ &\leq \frac{K}{\lambda}\|f\|^2.\end{aligned}$$

By iteration $\|Q_\lambda^n(I)\| \leq (\frac{K}{\lambda})^n$ for all $n$ and by (iii) of Theorem 13.5 $T_t^{\min}$ is conservative.

**Remark 13.8** : Observe that if $Y = -\frac{1}{2}\sum_k L_k^* L_k + iH$, where $H$ is a symmetric operator and the series $\sum_k L_k^* L_k$ converges strongly to a bounded operator then (13.9) is satisfied on $D(H)$.

In the remaining examples the family $\{L_k : k \geq 1\}$ consists of a single operator $L$ and $Y$ is equal to $-\frac{1}{2}L^* L$. Clearly $Y$ is a generator of a contraction semigroup and conditions (13.3) and (13.4) are automatically satisfied.

**Example 13.9** : Let $L$ be a normal operator, that is, $\mathcal{D}(L^*L) = \mathcal{D}(LL^*)$ and $L^*L = LL^*$. Then a simple computation shows that $Q_\lambda(I) = L^*L(\lambda + L^*L)^{-1}$. Let $L^* = U|L^*|$ be the polar decomposition of $L^*$. By normality of $L^*$ the operators $U$ and $|L^*|$ commute (see [Ka]) on $D(L^*)$. Hence, on $D(L^*)$,

$$\begin{aligned}Q_\lambda(I) &= LL^*(\lambda + L^*L)^{-1} = LU|L^*|(\lambda + |L^*|^2)^{-1} = LU(\lambda + |L^*|^2)^{-1}|L^*| \\ &= L(\lambda + |L^*|^2)^{-1}U|L^*| = L(\lambda + L^*L)^{-1}L^*.\end{aligned}$$

Let $R(t)$ denote the semigroup $e^{-\frac{\lambda}{2}t}C(t)$. Now for $f, g$ in $D(L^*L)$.

$$\begin{aligned}\langle f, Q_\lambda^2(I)g\rangle &= \int_0^\infty \langle LR(t)f, Q_\lambda(I)LR(t)g\rangle dt \\ &= \int_0^\infty \langle LR(t)f, L(\lambda + L^*L)^{-1}L^*LR(t)g\rangle dt \\ &= \int_0^\infty \langle R(t)f, L^*L(\lambda + L^*L)^{-1}L^*LR(t)g\rangle dt.\end{aligned}$$

Hence $Q_\lambda^2(I) = (L^*L)^2(\lambda + L^*L)^{-2}$. By iteration $Q_\lambda^n(I) = (L^*L)^n(\lambda + L^*L)^{-n}$ and therefore $\text{s-}\lim_{n\to\infty} Q_\lambda^n(I) = 0$ and consequently $T_t^{\min}$ is conservative. Observe that in general $Q_\lambda^n(I)$ does not converge to zero in norm.

**Example 13.10** : Let $\mathcal{H}_0$ be the Hilbert space $l_2$ on $\{0,1,2,\dots\}$ with standard orthonormal basis $\{e_0, e_1, \dots\}$. Let $V$ be the isometry on $\mathcal{H}_0$ defined by $Ve_n = e_{n+1}$ for $n \geq 0$. Define $L$ as the Cayley transform of $V$, that is,

$$\begin{aligned} D(L) &= \text{Range}\,(I - V). \\ L &= i(I+V)(I-V)^{-1} \text{ on } D(L). \end{aligned}$$

Then the operator $L^*$ is given by $D(L^*) = D(L) + \mathbb{C}e_0$, $L^*f = Lf$ for $f \in D(L)$ and $L^*e_0 = -ie_0$. Consider the form $\mathcal{L}(X)$ as before with $Y = -1/2L^*L$. It can easily be seen that the form equation $\mathcal{L}(X) = \lambda X$ is satisfied for $\lambda = 2$ by the operator $X$ defined by $\langle e_m, Xe_k\rangle = \binom{m+k}{k}\left(\frac{1}{2}\right)^{m+k+1}$, $m, k \geq 0$. Also the operator $X$ is a positive bounded operator. Hence in this case $T_t^{\min}$ is not conservative.

**Example 13.11** : Let $\mathcal{H}_0 = L^2([0,1])$. Define the operator $L$ on $\mathcal{H}_0$ by

$D(L) = \{f \in \mathcal{H}_0 : f \text{ is absolutely continuous, } f' \in \mathcal{H}_0 \text{ and } f(0) = f(1) = 0\}$

$Lf = if'$ for $f \in D(L)$.

Then $L$ is a symmetric operator with deficiency indices $(1,1)$. The operators $L^*$ and $Y = -\frac{1}{2}L^*L$ are given by

$$\begin{aligned} D(L^*) &= \{f \in \mathcal{H}_0 : f \text{ is absolutely continuous, } f' \in \mathcal{H}_0\}, \\ L^*f &= if' \text{ for } f \in D(L^*); \\ D(Y) &= \{f \in \mathcal{H}_0 : f \text{ is absolutely continuous, } f' \in \mathcal{H}_0, f(0) = f(1) = 0, \\ &\qquad f' \text{ is absolutely continuous, } f'' \in \mathcal{H}_0\}, \\ Yf &= \frac{1}{2}f'' \text{ for } f \in D(Y). \end{aligned}$$

Now for $\lambda > 0$ consider the multiplication operators $A_\lambda$ and $B_\lambda$ defined by

$$\begin{aligned} A_\lambda f(x) &= e^{-\sqrt{2\lambda}x} f(x), \\ B_\lambda f(x) &= e^{+\sqrt{2\lambda}x} f(x). \end{aligned}$$

Clearly $A_\lambda, B_\lambda \geq 0$ and $\|A_\lambda\| = \|B_\lambda\| = 1$. Observe that $A_\lambda, B_\lambda$ leave $D(L), D(L^*)$ and $D(Y)$ invariant. For $f \in D(Y)$

$$\begin{aligned}
& (A_\lambda Y + Y A_\lambda + L^* A_\lambda L) f(x) \\
= \; & e^{-\sqrt{2\lambda}x} \frac{f''(x)}{2} + \frac{1}{2}[e^{-\sqrt{2\lambda}x} f''(x) - 2\sqrt{2\lambda} e^{-\sqrt{2\lambda}x} f'(x) + 2\lambda e^{-\sqrt{2\lambda}x} f(x)] \\
& -[e^{-2\sqrt{2\lambda}x} f''(x) - \sqrt{2\lambda} e^{\sqrt{2\lambda}x} f'(x)] \\
= \; & \lambda e^{-\sqrt{2\lambda}x} f(x) \\
= \; & \lambda A_\lambda f(x).
\end{aligned}$$

Hence it follows that $\mathcal{L}(A_\lambda) = \lambda A_\lambda$ in the form sense. In a similar way $\mathcal{L}(B_\lambda) = \lambda B_\lambda$. Thus $A_\lambda$ and $B_\lambda$ are two linearly independent elements in $\beta_\lambda$.

Example 13.9 shows in particular that $Y = -\frac{1}{2}L^* L$ with $L$ self-adjoint leads to a conservative $T_t^{\min}$. In contrast to this we now show that all non-selfadjoint maximal symmetric operators give rise to nonconservative minimal semigroups. It may also be noted that Example 13.10 is a special case of this result.

**Theorem 13.12** : Let $L$ be a non-selfadjoint maximal symmetric operator on a Hilbert space $\mathcal{H}_0$. Then for $Y = -\frac{1}{2}L^* L$, the set $\beta_\lambda$ has atleast $n^2$ linearly independent elements, where $n$ is the non-zero deficiency index of $L$.

**Proof:** As $L$ is maximal symmetric deficiency indices of $L$ are of the form $(m, 0)$ with $m > 0$ or of the form $(0, n)$ with $n > 0$. Clearly $Y$ remains unchanged by changing $L$ to $-L$ and hence with out loss of generality we can take deficiency indices of $L$ as $(0, n)$ with $n > 0$.

Now by von Neumann's structure theorem for maximal symmetric operators ([DS]), $\mathcal{H}_0, L$ can be taken to be $L^2(\mathbb{R}_+) \otimes k$ and $M \otimes I$ respectively, where $k$ is a $n$ dimensional Hilbert space and $M$ is the canonical elementary symmetric operator on $L^2(\mathbb{R}_+)$ defined by

$$\begin{aligned}
D(M) &= \left\{ \begin{array}{ll} f \in L^2(\mathbb{R}_+): & f \text{ is absolutely continuous,} \\ & f' \in L^2(\mathbb{R}_+), \text{ and } f(0) = 0 \end{array} \right\} \\
Mf &= if' \text{ for } f \in D(M).
\end{aligned}$$

The operator $M^*$ is given by

$$\begin{aligned} D(M^*) &= \{f \in L^2(\mathbb{R}_+) : f \text{ is absolutely continuous, } f' \in L^2(\mathbb{R}_+)\}, \\ M^*f &= if' \text{ for } f \in D(M^*). \end{aligned}$$

Now for $\lambda > 0$ consider the operator $A_\lambda$ on $L^2(\mathbb{R}_+)$, defined by $(A_\lambda f)(x) = e^{-\sqrt{2\lambda}x} f(x)$. Clearly $A_\lambda$ is a positive bounded operator with $\|A_\lambda\| = 1$. By direct computation we can verify that,

$$\langle f, A_\lambda(-\frac{1}{2}M^*M)g\rangle + \langle(-\frac{1}{2}M^*M)f, A_\lambda g\rangle + \langle Mf, A_\lambda Mg\rangle = \lambda\langle f, A_\lambda g\rangle$$

for $f, g \in D(M^*M)$. This ensures that $A_\lambda \otimes B$ is an element of $\beta_\lambda$ for any positive contraction $B$ in $\mathcal{B}(k)$.

## Notes and Comments

Meyer ([Me], App. 2) gives a justification for imposing complete positivity on semigroups from a probabilistic point of view. Lindblad's structure theorem for uniformly continuous semigroups (Theorem 1.6) is valid for more general von Neumann algebras (See [Li], [EL]). Generators of quantum dynamical semigroups under weaker notions of continuity have been studied by Davies[Da4], Evans[Ev1] et al. Sections 1-4 are based on [BP2]. Results in sections 5 and 6 have not been published elsewhere. The methods employed in Section 5 have been borrowed to some extent from [Su1]. A thorough study of minimal dilations of non-commutative dynamical semigroups, leading to probabilistic interpretations, is yet to be done. For example, in the case of classical Markov chains the dimension of the range of $F(t)$ is clearly the number of all possible paths of length $t$ having non-zero probability. We do not have any such interpretation here. May be, a 'classification of states' is possible as in the theory of classical Markov chains. Remark 5.8 pertaining to Feynman-Kac cocycles is due to Parthasarathy. In the context of Theorem 6.4 one may recall that Parthasarathy and Sinha showed that Evans-Hudson flows on abelian algebras are abelian [P1]. However this result does not answer the question as to when can we extend the central flow $k_t$ to a weak Markov flow on the full algebra.

Chapters II and III are adapted from [BP2]. In general, how to ensure contractivity of Feller perturbed semigroups is not clear. A structure theorem determining all possible cocycles of a given positive semigroup would go a long way in understanding these perturbations. In Chapter III a quantum version of gluing of processes through stop times has been described. There are many other commonly used methods of obtaining new processes from the old in classical probability. For example, given two processes $\xi$ and $\eta$ with $\eta$ positive and increasing we can have a new process $\psi$ as $\xi$ evaluated at time $\eta$, i. e., for all $t \geq 0$, $\psi(t) = \xi_{\eta(t)}$. We do not know how to quantize these constructions.

The important constructions of Section 12 and examples of exit times in the final section are taken from [BP2] with minor changes. Examples of generators of non-conservative semigroups are adapted from [BS2]. Significant contributions to Feller boundary theory were made by Reuter, Ledermann, Kato et al. Exact references may be found in [C1,2]. A general boundary theory should allow more than one point in the boundary. The set of boundary points has to be topologised in a suitable way, as in the classical theory. Moreover exact mechanisms by which the process visits the boundary and comes back have to be made clearer.

## References


[Ac] Accardi, L. : Nonrelativistic quantum mechanics as a noncommutative Markov process, Advances in Math. **20**, 329-366(1976).

[AC] Accardi, L., Cecchini, C. : Conditional expectations in von Neumann algebras and a theorem of Takesaki, J. Funct. Anal. **45**, 245-273 (1982).

[AFL] Accardi, L., Frigerio, A., Lewis, J.T. : Quantum Stochastic Processes, Publ. RIMS, Kyoto Univ. **18**, 97-133 (1982).

[AL] Alicki, R., Lendi, K. : *Quantum Dynamical Semigroups and Applications*, Springer LNP **286**, (1987)Berlin.

[Ar] Arveson, W.B. : Notes on extensions of $C^*$ algebras, Duke Math. J., **44**, 329-355 (1977).

[AS] Accardi, L., Sinha, Kalyan B. : Quantum stop times, Quant. Prob. and Appl. **IV**, Springer LNM **1396**, 68-72 (1989).

[Be] Belavkin, V.P. : Quantum stochastic calculus and quantum nonlinear filtering, Preprint Centro Mathematico V. Volterra, Dip di Matematica, Universita di Roma II(1989)Roma.

[BG] Bluementhal, R.M., Getoor, R. K. : *Markov Processes and Potential Theory*, Acad. Press (1968) New York.

[Bh] Bhat, B.V. Rajarama : On a characterization of velocity maps in the space of observables, Pacific J. of Math. **152**, 1-14 (1992).

[Bi] Biane, Ph. : Quantum random walks on the dual of $SU(n)$, Probab. Th. Rel. Fields **89**, 117-129 (1991).

[BL] Barchielli, A., Lupieri, G. : Semigroups of positive-definite maps on $*$ bi-algebras, Quant. Prob. Rel. topics **VII**, 15-29, World Scientific (1992) Singapore.

[BP1] Bhat, B.V. Rajarama, Parthasarathy, K.R. : Generalized harmonic oscillators in Quantum Probability, Seminaire de Probabilities **XXV**, Springer LNM - **1485**, 39-51 (1991).

[BP2] Bhat, B.V. Rajarama, Parthasarathy, K.R. : Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory, Indian Statistical Institute preprint (1993), New Delhi.

[BPS] Bhat, B.V. Rajarama, Pati, V., Sunder, V.S. : On some convex sets and their extreme points (to appear in Mathematische Annalen).

[BS1] Bhat, B.V. Rajarama, Sinha, Kalyan B. : A stochastic differential equation with time dependent and unbounded operator coefficients, J. Func. Anal., Vol. **114**, 12-31 (1993).

[BS2] Bhat, B.V. Rajarama, Sinha, Kalyan B. : Examples of unbounded generators leading to nonconservative minimal semigroups, Indian Statistical Institute preprint (1993), New Delhi.

[BW] Barnett, C., Wilde, I. F. : Quantum stopping times, Quant. Prob. Rel. Topics **VI**, 127-136 World Scientific (1992) Singapore.

[C1] Chung, K.L. : *Markov Chains with Stationary Transition Probabilities,* Springer (1960) Berlin.

[C2] Chung, K.L. : *Lectures on the Boundary Theory for Markov Chains,* Princeton Univ. Press (1970) Princeton.

[CF] Chebotarev, A.M., Fagnola, F. : Sufficient conditions for conservativity of quantum dynamical semigroups, J. Funct. Anal., to appear.

[Ch] Chebotarev, A.M. : The theory of the conservative dynamical semigroups and its applications, preprint MIEM 1 (1990).

[CW] Chung, K.L., Williams, R.J.:*Introduction to Stochastic Integration,* Birkhäuser (1983) Boston.

[Da1] Davies, E.B. : *Quantum Theory of Open Systems,* Acad. Press (1976) New York.

[Da2] Davies, E.B. : Quantum dynamical semigroups and the neutron diffusion equation, Rep. Math. Phys. **11**, 169-189 (1977).

[Da3] Davies, E.B. : *One-Parameter Semigroups,* Acad. Press (1980), New York.

[Da4] Davies, E.B. : Generators of dynamical semigroups, J. Funct. Anal. **34**, 421-432 (1979).

[Di] Dixmier, J. : *Von Neumann Algebras,* North Holland (1981) Amsterdam.

[DS] Dunford, N., Schwartz, J.T. : *Linear Operators,* Part I-III, Inter Science (1963) New York.

[Dy] Dynkin, E.B. : *Markov Processes,* Vol. I and II, Springer-Verlag (1965) Berlin.

[EH] Evans, M.P., Hudson, R.L. : Multidimensional diffusions, Quant. Prob. and Appl.-**III**, Springer LNM **1303**, 69-88 (1988).

[EL] Evans, D.E., Lewis, J.T. : *Dilations of Irreducible Evolutions in Algebraic Quantum Theory,* Commun. Dublin Inst. Adv. Studies, Ser A no.**24** (1977).

[Em] Emch, G. G. : Minimal dilations of CP flows, $C^*$ algebras and applications to physics, Springer LNM **650**, 156-159 (1978).

[Ev1] Evans, D.E. : Irreducible quantum dynamical semigroups, Comm. Math. Phys. **54**, 293-297 (1977).

[Ev2] Evans, D.E. : Qunatum dynamical semigroups, symmetry groups and locality, Acta Appl. Mathematicae (1984).

[Fa1] Fagnola, F. : Chebotarev's sufficient conditions for conservativity of quantum dynamical semigroups, preprint (1993) Trento.

[Fa2] Fagnola, F. : Unitarity of solutions to quantum stochastic differential equations and conservativity of the associated semigroups, Quant. Prob. Rel. Topics **VII**, 139-148, World Scientific(1992)Singapore.

[Fe1] Feller, W. : Boundaries induced by non-negative matrices, Trans. Amer. Math. Soc. **83**, 19-54 (1956).

[Fe2] Feller, W. : On boundaries and lateral conditions for the Kolmogorov differential equations, Annal. Math. **65**, 527-570 (1957).

[Fe3] Feller, W. : Notes to my paper "On boundaries and lateral conditions for the Kolmogorov differential equations", Annal. Math. **68**, 735-736 (1958).

[Fr] Frigerio, A. : Covariant Markov dilations of quantum dynamical semigroups, Publ. RIMS. Kyoto Univ. **21**, 657-675 (1985).

[GKS] Gorini, V., Kossakowski, A., Sudarshan, E.C.G. : Completely positive dynamical semigroups of $n$-level systems, J. Math. Phys. **17**, 821-825 (1976).

[He] Hegerfeldt, G.C. : Noncommutative analogs of probabilistic notions and results, J. Funct. Anal. **64**, 436-456 (1985).

[HIP] Hudson, R.L., Ion, P.D.F., Parthasarathy, K.R. : Time-orthogonal unitary dilations and non-commutative Feynman-Kac formulae, Commun. Math. Phys. **83**, 261-280 (1982).

[HL] Hudson, R.L., Lindsay, J.M. : On Characterizing quantum stochastic evolutions, Math. Proc. Camb. Phil. Soc. **102**, 363-369 (1987).

[HP] Hudson, R.L. Parthasarathy, K.R. : Stochastic dilations of uniformly continuous completely positive semigroups, Acta Appl. Mathematicae **2**, 353-398 (1984).

[Ho] Holevo, A.S. : Stochastic representation of quantum dynamical semigroups, Trudy Mat. Inst. A.N.SSSR, **191**, 130-139 (1989) (Russian).

[Hu] Hudson, R.L. : The strong Markov property for canonical Wiener processes, J. Func. Anal. **34**, 266-281 (1979).

[Jo] Journé, J.L. : Structure des cocyles Markoviens sur léspace de Fock, Probab. Th. Rel. Fields **75**, 291-316 (1987).

[Ka] Kato, T. : *Perturbation Theory for Linear Operators,* 2nd edition, Springer (1976) Berlin.

[KM] Kümmerer, B., Maassen, H. : The essentially commutative dilations of dynamical semigroups on $M_n$, Commun. Math. Phys. **109**, 1-22 (1987).

[Kr] Kraus, K. : General state changes in quantum theory, Ann. Phys. **64**, 311-335 (1971).

[Ku1] Kümmerer, B. : Markov dilations on $W^*$-algebras, J. Funct. Anal. **63**, 139-177 (1985).

[Ku2] Kümmerer, B. : Survey on a theory of noncommutative stationary Markov processes, Quant. Prob. and Appl.-**III**, Springer LNM **1303** , 154-182 (1988) Berlin.

[Li] Lindblad, G. : On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119-130 (1976).

[LP] Lewis, J.T., Pule, J.V. : Dynamical Theories of Brownian Motion, Proc. Int. Symp. on Mathematical Problems in Theoretical Physics, Springer LNP **39**, 516-519 (1975).

[Ma] Maassen, H. : Quantum Markov processes on Fock space described by integral kernels, Springer LNM **1136**, 361-374 (1985) Berlin.

[Me] Meyer, P.A. : *Quantum Probability for Probabilists,* Springer LNM **1538**, (1993) Berlin.

[Mi] Mittal, N. : Completely positive maps and quantum dynamical semigroups, M. Phil. dissertation, Univ. of Delhi (1986).

[Mo] Mohari, A. : Quantum Stochastic Calculus with infinite degrees of freedom and its applications (Ph.D. Thesis), Indian Statistical Institue (1991) New Delhi.

[MS] Mohari, A., Sinha, Kalyan B. : Stochastic dilation of minimal quantum dynamical semigroups, Proc. of Ind. Acad. of Sci. (Math. Sci.), **102**, 159-173 (1992).

[P1] Parthasarathy, K.R. : *An Introduction to Quantum Stochastic Calculus,* Monographs in Mathematics, Birkhauser Verlag (1991) Basil.

[P2] Parthasarathy, K.R. : A continuous time version of Stinespring's theorem on completely positive maps, Quant. Prob. and Appl. **V**, Springer LNM **1442**, 296-300 (1990) Berlin.

[Pe] Petz, D. : Conditional expectation in quantum probability, Quant. Prob. and Appl. **III**, Springer LNM **1303**, 251-260 (1988).

[PS] Parthasarathy, K.R. Sinha, K.B. : Stop times in Fock space stochastic calculus, Probab. Th. Rel. Fields. **75**, 317-349 (1987).

[PSc] Parthasarathy, K.R., Schmidt, K. : *Positive Definite kernels, Continuous Tensor Products and Central Limit Theorems of Probability Theory,* Springer LNM **272** (1972) Berlin.

[RS] Reed, M., Simon, B. : *Methods of Mathematical Physics,* Vol. **I-IV**, Acad. Press (1972) New York.

[Sa1] Sauvegeot, J-L. : Markov quantum semigroups admit covariant Markov $C^*$ dilations,Commun.Math.Phys. **106**,91-103(1986).

[Sa2] Sauvegeot, J-L. : First exit times in quantum processes, Quant. Prob. and Appl.-**III**, springer LNM **1303**, 285-299 (1988).

[Sc] Schürmann, M. : *White Noise on Bialgebras,* (to appear in the series LNM Springer).

[Sp] Speicher, R. : Survey of stochastic integration on the full Fock space, Quantum Probab. Rel. Topics **VI**, 421-437. World Scientific (1992) Singapore.

[St] Stinespring, W.F. : Positive functions on $C^*$ algebras, Proc. Amer. Math Soc. **6**, 211-216 (1955).

[Su1] Sunder, V.S. : Completely positive maps of $C^*$ algebras, Unpublished expository notes, Indian Statistical Institute, (1991) Bangalore.

[Su] Sunder, V.S. : *An Invitation to von Neumann Algebras,* Springer (1987) Berlin.

[SzF] Sz.-Nagy, B., Foias, C. : *Harmonic Analysis of Operators on Hilbert Space,* North-Holland (1970) Amsterdam.

[Ta] Takesaki, M. : *Theory of Operator Algebras* **I**, Springer (1979) New York.

[Um] Umegaki, H. : Conditional expectation in an operator algebra I, Tôhoku Math. J. (2) **6**, 177-181 (1954).

[Va] Varadarajan, V.S. : *Geometry of Quantum Theory,* Second Edition, Springer (1985) Berlin.

[ViS] Vincent-Smith, G. F. : Dilations of a dissipative quantum dynamical system to a quantum Markov process, Proc. London Math. Soc.(3), **49**, 58-72 (1984).

[VDN] Voiculescu, D.V., Dykema, K.J., Nica, A. : *Free Random Variables,* CRM Monograph Series, Amer. Math. Soc. (1992) Providence.

[vW] von Waldenfels, W. : Illustration of the central limit theorem by independent addition of spins, Sém, Prob. XXIV, LNM **1426**, 349-356 (1990).

[Y] Yosida, K. : *Functional analysis,* 4th edition, Springer (1974) Berlin.