GENERALIZED INVERSES OF MATRICES OVER RINGS

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Dedicated to two 'g's of mine,

guru and g-inverse

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CONTENTS

CHAPTER 0
INTRODUCTION

For a complex or a real matrix A, a matrix G is called a **generalized inverse** (or **g-inverse**) of A if

(1) $AGA = A$

The theory of generalized inverses over the field of complex numbers is well-studied in the literature (see [2], [11], [33], and [62] for an extensive bibliography). Even for matrices over a general field the above equation carries over. In fact, even for matrices over a general ring, equation (1) makes sense. Hence one can talk of g-inverses of matrices over general rings. Some work on g-inverses of matrices over fields also can be found in the literature.

Our purpose in this thesis is to study g-inverses of matrices over rings. Over a commutative ring (even over an integral domain) because of the nonexistence of inverses for nonzero elements, the usual results on g-inverses of matrices over real or complex fields may not be extendable as discussed below.

Over the real or complex field, more generally over any field, every matrix has a g-inverse. But even on the ring of integers, not every matrix has a g-inverse.

For example, the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

has no generalized inverse over $\mathbf{Z}$.

As early as 1939, von Neumann showed that every matrix over a ring $\mathbf{A}$ has a g-inverse if and only if $\mathbf{A}$ is regular.

Another result which is true over any field is that every matrix over a field has a rank factorization. However, this is not true for a general integral domain. Similar observations lead us to a plethora of problems on g-inverses of matrices over general rings, in particular, over integral domains.

Batigne in [4] gave necessary and sufficient conditions for integer matrices to have integer g-inverses. Bose-Mitra [9] presented the first study of generalized inverses of polynomial matrices. These characterizations depend on the Smith normal form of matrices.

However a matrix over a general integral domain need not have Smith normal form. For example,

let $\mathbf{D}$ be the subring of the ring $\mathbf{R}[X, Y]$ of polynomials in X and Y with coefficients from the field of reals, generated by 1, X^2, XY, and Y^2. The matrix

$$A = \begin{bmatrix} X^2 & XY \\ XY & Y^2 \end{bmatrix}$$

has no rank factorization over $\mathbf{D}$. Thus A does not have Smith normal form.

Thus, if an integral domain is such that matrices over this integral domain do not admit Smith normal form, the results of [4] and [9] are not applicable and new techniques are required for study of g-inverses of matrices over such rings.

Bose and Mitra in [9] and Sontag in [59] have observed that an important area of application of generalized inverses of matrices over general integral domains in System Science is investigation of under-determined and over-determined linear algebraic and differential systems. For example:

a type of underdetermined systems is

$$y = Cx$$

where y is $m \times 1$, C is $m \times n$, and x is $n \times 1$ and the elements of matrices are scalar valued functions of t defined over an interval I. The above equation may be considered as the output equation of a control problem.

A type of overdetermined systems is

$$\frac{d}{dt}x = Ax + Bu$$

where x is $n \times 1$, A is $n \times n$, B is $n \times p$ and u is $p \times 1$ and the elements of matrices are scalar valued functions of a real variable t, defined over an interval I. The above equation may be considered as the "state equation" of the control problem, and one may wish to find an input $u(t)$ which will force $x(t)$ to be the prescribed function of t on I.

Generalized inverses of matrices over general rings, or integral domains could be used in solving equations of the above types. Matrices with coefficients from rings like rational functions with no real poles, polynomials, analytic functions, integers, continuous complex valued functions on compact Hausdorff spaces also appear in multidimensional system theory.

Before we explain some more of the problems considered in this thesis let us give some more definitions.

Definitions :

Let $\mathbf{A}$ be a ring with identity and with involution $a \mapsto \bar{a}$. Let A be an $m \times n$ matrix over $\mathbf{A}$ and consider the Moore - Penrose equations :

(1) $AGA = A$

(2) $GAG = G$

(3) $(AG)^* = AG$

(4) $(GA)^* = GA$

where A^* denotes $(\bar{A})^T$.

If G is an $n \times m$ matrix satisfying (1), then G is called a **generalized inverse (g-inverse, 1 -inverse)** of A. We denote an arbitrary g-inverse of A by A^-

A matrix A is called **regular** if it has a g-inverse.

If G satisfies (1) and (2), it is called a **reflexive g-inverse** of A.

G is called a **Moore-Penrose inverse** of A if it satisfies (1)-(4). We denote Moore-Penrose inverse of A by A^+.

Consider the following equations applicable to square matrices

(5) $AG = GA$

(1^k) $A^k = A^{k+1}G$

where k is a positive integer.

Borrowing the definition from real matrices (see [48] ch.4), for a square matrix A over a ring $\mathbf{A}$, a matrix G over $\mathbf{A}$ is called a **group inverse** of A if (1), (2) and (5) are satisfied. We denote a group inverse of A by $A^\#$.

A matrix G over $\mathbf{A}$ is called a **Drazin inverse** of A if (2), (5) and (1^k) (for some positive integer k) are satisfied.

A matrix G over $\mathbf{A}$ satisfying the conditions (1) and (5) is called a **commuting g-inverse** of A.

A matrix G over $\mathbf{A}$ satisfying conditions (1) and (3) ((1) and (4)) is called (1, 3) inverse ((1, 4) inverse) of A. A reflexive (1, 3) inverse ((1, 4) inverse) of A is called (1,2,3) inverse ((1,2,4) inverse) of A.

Now we shall introduce some notation and give some definitions. Let A be an $m \times n$ matrix, and let $\alpha = (i_1,\ldots,i_r)$, $\beta = (j_1,\ldots,j_r)$ be subsets of $(1,\ldots,m)$ and $(1,\ldots,n)$, respectively.

We denote by A^{α}_{β} the submatrix of A, determined by rows indexed by α, columns indexed by β.

For the next few definitions we consider matrices over commutative rings.

The determinant of a square matrix A is denoted by $|A|$, and $\frac{\partial}{\partial a_{ij}}|A|$ denotes the cofactor of a_{ij} in the expansion of $|A|$.

Cauchy-Binet Formula : Let A, B be matrices of sizes $m \times n$ and $n \times k$, respectively, and r be an integer such that $r \leq \min(m, n, k)$. If α is an r-element subset of $(1,2,\ldots m)$ and β is an r-element subset of $(1,2,\ldots,k)$, then

$$|(AB)^{\alpha}_{\beta}| = \sum_{\gamma} |A^{\alpha}_{\gamma}||B^{\gamma}_{\beta}|$$

where γ runs over all r-element subsets of $(1,2,\ldots,n)$

The determinantal rank (the size of largest nonvanishing minor) is denoted by $\rho(A)$.

For an $m \times n$ matrix A of rank r, we say that A has rank factorization if $A = BC$ where B is $m \times r$ and C is $r \times n$. Of course, $\rho(B)$ and $\rho(C)$ both must equal r.

$C_r(A)$ is the r-th compound matrix of A with rows indexed by r-element subsets of $\{1,......m\}$ and columns indexed by r-element subsets of $\{1,.....n\}$. At several places, α, β, γ are assumed to be r-element subsets of $\{1,2,.......n\}$ without explicit mention.

For an $m \times n$ matrix A, $\mathcal{C}(A)$ stands for the module generated by columns of A and $\mathcal{R}(A)$ stands for the module generated by rows of A.

$A_{\overline{\chi}}$ stands for a g-inverse of A with $\mathcal{C}(A_{\overline{\chi}}) = \mathcal{C}(A)$ (equivalently $\mathcal{C}(A_{\overline{\chi}}) \subset \mathcal{C}(A)$).

$A_{\overline{\rho}}$ stands for a g-inverse of A with $\mathcal{R}(A_{\overline{\rho}}) = \mathcal{R}(A)$ (equivalently $\mathcal{R}(A_{\overline{\rho}}) \subset \mathcal{R}(A)$).

$A_{\overline{\rho}\chi}$ stands for a g-inverse of A with $\mathcal{C}(A_{\overline{\rho}\chi}) = \mathcal{C}(A)$ and $\mathcal{R}(A_{\overline{\rho}\chi}) = \mathcal{R}(A)$. equivalently $\mathcal{C}(A_{\overline{\rho}\chi}) \subset \mathcal{C}(A)$ and $\mathcal{R}(A_{\overline{\rho}\chi}) \subset \mathcal{R}(A)$).

In many cases, for the notation related to modules, we follow Jacobson [23] and [24].

Now we shall give our motivation to the various problems considered in this thesis.

Over an arbitrary field, it is known that a matrix A of rank r has Moore-Penrose inverse if and only if $\rho(A^{*}A) = \rho(AA^{*}) = \rho(A)$ (see [33]). But this can not be extended for matrices over a general ring. For example, over $\mathbf{Z}$, for

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\rho(A^* A) = \rho(AA^*) = \rho(A)$, but A does not have Moore-Penrose inverse.

Bhaskara Rao in 1983 [53] gave necessary and sufficient conditons for a matrix over an integral domain to have a g-inverse, using the $r \times r$ minors of the matrix, where r is the determinantal rank of the matrix.

The above two results lead us to consider the following problem.

Problem 1.

Characterization of matrices which have

(i) Moore-Penrose inverse

(ii) Group inverse

(iii) Drazin inverse

(iv) {1,3} inverse ({1,4} inverse)

over an integral domain.

Continuing in the same vein as in [53], we wish to find necessary and sufficient conditions for a matrix to have Moore-Penrose inverse, group inverse, Drazin inverse, {1, 3} inverse and {1,4} inverse.

Problem 2.

In [53], Rao developed a procedure for constructing a g-inverse using a linear combination of $r \times r$ minors which equals one, where r is the determinental rank of the matrix. A natural question that arises is the following.

Is it possible to construct every g-inverse by this procedure ?

Problem 3.

Kentaro Nomakuchi [28] considered a method of bordering to characterize the class of all g-inverses of a given matrix over the complex field. Adi Ben-Israel [1] obtained a Cramer rule using the bordering technique to find a least-norm solution of the consistent linear system

$$Ax = b$$

over the complex field showing that

$$x_j = \frac{\det \begin{bmatrix} A(j \to b) & U \\ V^* & 0 \end{bmatrix}}{\det \begin{bmatrix} A & U \\ V^* & 0 \end{bmatrix}}$$

where x_j is the j-th component of x, $A(j \to b)$ denotes the matrix obtained by replacing j-th column of A by b, and U and V are matrices whose columns form bases for the kernels of A^* and A respectively. Verghese [17] proved a similar results for finding least-square solution for inconsistent linear systems by making a slight change in Ben-Israel's proof. This leads us to the corresponding problem for the matrices over integral domains.

Characterization of integral domains over which every regular matrix has a bordering of the required type.

Problem 4.

Rank factorization and Smith normal form for regular matrices play an important role in the construction and study of g-inverses. If a regular matrix A over an integral domain has a rank factorization, $A = BC$, it could be seen easily that B has a left inverse F and C has a right inverse E. Also, a g-inverse of A could be obtained by the product EF. Further, many problems become amenable when a regular matrix has a rank factorization.

Also, problem 2, mentioned earlier can be answered positively when the given regular matrix has Smith normal form.

The above two observations lead us to the following problem.

Characterization of integral domains over which every regular matrix has a rank factorization.

This also leads us to a characterization of integral domains over which every regular matrix has Smith normal form.

We show that problems 3 & 4 are related. Our approach leads us to a discussion of some problems related to Serre's conjecture also.

Now we shall give a brief summary of various results obtained in various chapters of this thesis.

In Chapter 1 we record Bhaskara Rao's result proved in [53]. In this, Rao developed a procedure for constructing a g-inverse using a linear combination of $r \times r$ minors which equals one, where r is the determinental rank of the matrix. We investigate problem 2 in Section 1.3 of Ch. 1. In fact, we show that every reflexive g-inverse arises in this fashion. Also, over principal ideal domains every g-inverse arises in this fashion. Also, we derive many other interesting known results in literature quickly, through Rao's characterization.

In Chapter 2, we give necessary and sufficient conditions for a matrix to have Moore-Penrose inverse over an arbitrary integral domain. We give a formula to find the Moore-Penrose inverse whenever it exists. Also, we characterize all matrices which have (1,2,3) and (1,2,4) inverses. Similar to the Cramer rule (which is used to find the solution of a linear system $Ax = y$ in case A is invertible), we obtain a generalized Cramer rule to find Moore-Penrose solution, (1,2,3) solution and (1,2,4) solution, even though the given matrix is not invertibe but satisfies certain sufficient conditions. For the former, we use a formula developed in section 2.2. In Chapter 2 we define "Generalized Moore-Penrose Inverse" and find necessary and sufficient conditions for matrices to have the generalized Moore-Penrose inverse. This generalized Moore-Penrose inverse reduces to weighted Moore-Penrose inverse in a special case. Also, in this chapter we give necessary and sufficient conditions for the existence of Khatri-inverse, pointing out an error in a condition given by Khatri. We treat this problem in the general case of integral domains.

Chapter 3 contains necessary and sufficient conditions for the existence of a group inverse, a new formula for a group inverse when it exists, and necessary and sufficient conditions for the existence of a Drazin inverse . We show that a square matrix A of rank r over an integral domain $\mathbf{D}$ has a group inverse if and only if the sum of all $r \times r$ principal minors of A is an invertible element of $\mathbf{D}$. We also show that the group inverse of A, when it exists, is a polynomial in A with coefficients from $\mathbf{D}$.

In Chapter 4, we observe that the bordering technique can not be used over an arbitrary integral domain. Here we characterize all integral domains over which every regular matrix can be bordered. We also characterize all integral domains over which every regular matrix admits a rank factorization. In fact the two characterizations coincide. Also we extend Quillen's theorem to the integral domain $\mathbf{P}[X_1,X_2,....]$, the polynomial ring generated by countably many variables over any principal ideal domain $\mathbf{P}$.

A natural question that arises from von Neumann's result [63] is "over which types of rings does every matrix admit Moore-Penrose inverse ?". In section 2 of chapter 5 we shall characterize all rings over which every matrix admits a Moore-Penrose inverse. In Section 3 we extend many results which we proved earlier over an integral domain to an arbitrary commutative ring with trivial idempotents and in section 4 we discuss the property of an associative ring satisfying the Rao condition in relation to the characterization of (1,3) and (1,4) inverses. Also, we characterize all regular matrices over a Banach algebra.

CHAPTER 1
GENERALIZED INVERSES OVER INTEGRAL DOMAINS

1.1. Introduction

There are a number of results available in the literature on characterization of regular matrices over special integral domains like the ring of integers ([4], [5]), polynomial rings ([9], [59]), and principal ideal domains ([50], [51]) Bhaskara Rao in [53] gave a characterization of regular matrices over an integral domain using minors of matrices. This characterization is strongerthan the previously known characterizations and many other interesting results can be derived from this Further this characterization is independent of the Smith normal form of matrices.

In fact in [53] Bhaskara Rao showed that a matrix A of rank r over an integral domain has a g-inverse if and only if a linear combination of all the $r \times r$ minors of A equals one In this chapter we shall study various problems that arise out of this result.

First of all in Section 1 2 we recapitulate the results given by Bhaskara Rao [53] and derive some results, known in literature, quickly, through Rao's characterization In [53] a method of computing a g-inverse was described starting from a linear combination of minors which equals one. In section 1.3 we investigate as to which of the g-inverses can be constructed using this method. For example, we show that every reflexive g-inverse arises in this fashion. Also, over principal ideal domains every g-inverse arises in this fashion.

1.2. Characterization

Bhaskara Rao in [53] proved that a matrix A of rank 'r' over $\mathbf{D}$ has a g-inverse if and only if there exists a linear combination of the $r \times r$ minors of A which equals one. Since many of our results throughout this thesis depend on this, we shall see this result in Theorem 1.2.2. As a preliminary to Theorem 1.2.2 let us first consider a special case, namely, $\rho(A) = 1$.

Throughout this section we shall consider matrices over an integral domain $\mathbf{D}$ unless otherwise indicated.

Theorem 1.2.1. Let $A = (a_{ij})$ be an $m \times n$ matrix of rank one over $\mathbf{D}$. Then A is regular if and only if a linear combination of all elements is equal to one. If $\sum_{i,j} a_{ij}\, g_{ji} = 1$, then the matrix G whose $(i, j)^{th}$ element is g_{ij} is a g-inverse of A.

Proof. Suppose G is an $n \times m$ matrix such that $AGA = A$. Since $\rho(A) = 1$, there are indices k and l such that $a_{kl} \neq 0$ and

$$a_{kl} = \sum_{ij} a_{kj}\, g_{ji}\, a_{il} \tag{1.2.1}$$

Again, since $\rho(A) = 1$, every 2×2 minor of A vanishes. So for any k, l, j, and i

$$a_{kj}\, a_{il} = a_{kl}\, a_{ij} \tag{1.2.2}$$

Hence

$$a_{kl} = a_{kl} \sum_{ij} a_{ij}\, g_{ji}$$

i e,

$$\sum_{ij} a_{ij}\, g_{ji} = 1 \qquad (1.2.3)$$

Retracing the steps, we get the proof of the 'if' part □

Theorem 1.2.2. Let A be an $m \times n$ matrix with $\rho(A) = r$. Then the following are equivalent.

(i) A is regular

(ii) $C_r(A)$ is regular

(iii) A linear combination of all $r \times r$ minors of A is equal to one.

We need a result on compound matrices for the proof of this theorem. This result is known in [32, p 187] and [53], but we shall supply a different and simpler proof here

Lemma 1.2.3. Let A be an $m \times n$ matrix with $\rho(A) = r$. Then $\rho(C_r(A)) = 1$

Proof. Consider a rank factorization of $A = BC$ over the field of quotients of $\mathbb{D}$, where B is an $m \times r$ matrix with $\rho(B) = r$ and C is an $r \times n$ matrix with $\rho(C) = r$ So from the multiplicative property of compound matrices we get

$$C_r(A) = C_r(B)\, C_r(C)$$

Note that $C_r(B)$ is an $\binom{m}{r} \times 1$ matrix and $C_r(C)$ is an $1 \times \binom{n}{r}$ matrix. Therefore

$\rho(C_r(A))$ is one over the field of quotients of $\mathbf{D}$. Since determinantal ranks over an integral domain and its quotient field coincide, we get $\rho(C_r(A)) = 1$. □

Proof of Theorem 1.2.2. (i) ⇒ (ii) is clear.

(ii) ⇒ (iii) follows from Theorem 1.2.1, because $\rho(C_r(A)) = 1$ by lemma 1.2.3

(iii) ⇒ (i); Suppose that there exists a linear combination

$$\sum_{\alpha,\ \beta} c_\alpha^\beta\ |A_\beta^\alpha| = 1 \qquad (1.2.4)$$

for some c_α^β from $\mathbf{D}$, where the summation is over all subsets α of $\{1, 2, \ldots, m\}$ and β of $\{1, 2, \ldots, n\}$ consisting of r indices. For any $1 \le k \le m$ and $1 \le l \le n$, we have

$$\sum_{\alpha,\ \beta} a_{kl}\ c_\alpha^\beta\ |A_\beta^\alpha| = a_{kl}. \qquad (1.2.5)$$

For any fixed $\alpha = \{i_1, i_2, \ldots, i_r\}$ and $\beta = \{j_1, j_2, \ldots, j_r\}$, consider the matrix

$$B = \begin{bmatrix} & & a_{i_1 l} \\ & A_\beta^\alpha & \cdot \\ & & a_{i_r l} \\ a_{kj_1} & \cdot\ a_{kj_r} & a_{kl} \end{bmatrix} \qquad (1.2.6)$$

If $k \in \alpha$ or $l \in \beta$ trivially $|B| = 0$. If $k \notin \alpha$ and $l \notin \beta$, then also $|B| = 0$, because $\rho(A) = r$. Then in any case $|B| = 0$. Hence

$$a_{kl}\ |A_\beta^\alpha| = \sum_{i \in \alpha} \sum_{j \in \beta} a_{kj}\ a_{il}\ \frac{\partial |A_\beta^\alpha|}{\partial a_{ij}}. \qquad (1.2.7)$$

Then equation (1.2.5) becomes

$$a_{kl} = \sum_{\alpha,\ \beta} \left\{ \sum_{i\ \in\alpha,\ j\in\beta} a_{kj}\, a_{il}\ \frac{\partial |A^{\alpha}_{\beta}|}{\partial a_{ij}} \right\} c^{\beta}_{\alpha}$$

i e,

$$a_{kl} = \sum_{i,\ j} a_{kj}\, a_{il} \left\{ \sum_{\alpha,\ \beta} c^{\beta}_{\alpha}\ \frac{\partial |A^{\alpha}_{\beta}|}{\partial a_{ij}} \right\} \tag{1.2.8}$$

By taking

$$g_{ji} = \sum_{\alpha,\ \beta} c^{\beta}_{\alpha}\ \frac{\partial |A^{\alpha}_{\beta}|}{\partial a_{ij}}, \tag{1.2.9}$$

the matrix G whose $(i,\ j)$ th element is g_{ij} is a g-inverse of A. □

We shall use the above method to compute a g-inverse of a matrix.

Example. Let $\mathbf{D} = \mathbf{Z}[X,\ Y]$ be the polynomial ring over the integers, and let

$$A = \begin{bmatrix} X^2 & 1 & -X^2 \\ 1 & XY & 0 \\ 2X^2 & 2 & -2X^2 \end{bmatrix}.$$

This matrix is of rank 2. Then we get 2-th compound matrix of A

$$C_2(A) = \begin{bmatrix} |A^{(1,2)}_{(1,2)}| & |A^{(1,2)}_{(1,3)}| & |A^{(1,2)}_{(2,3)}| \\ |A^{(1,3)}_{(1,2)}| & |A^{(1,3)}_{(1,3)}| & |A^{(1,3)}_{(2,3)}| \\ |A^{(2,3)}_{(1,2)}| & |A^{(2,3)}_{(1,3)}| & |A^{(2,3)}_{(2,3)}| \end{bmatrix}$$

$$= \begin{bmatrix} x^3y-1 & x^2 & x^3y \\ 0 & 0 & 0 \\ 2-2x^3y & 2x^2 & -2x^3y \end{bmatrix}$$

Observe that $|A^{(1,2)}_{(2,3)}| - |A^{(1,2)}_{(1,2)}| = 1$ So A is regular. Using the formula (1.2.9) we obtain

$$G = \begin{bmatrix} -xy & 1 & 0 \\ 1 & 0 & 0 \\ -xy & -1 & 0 \end{bmatrix}$$

is a g-iverse of A. By direct computation we can verify easily that $AGA = A$.

Smith normal form theorem. Let $\mathbf{D}$ be a principal ideal domain. Every $m \times n$ matrix A of rank r over $\mathbf{D}$ can be written as

$$U \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} V \qquad (1.2.10)$$

where U and V are invertible matrices, $S = \text{diag}(s_1, s_2, \ldots, s_r)$ s_i's form a complete set of non-associates, $s_i \mid s_{i+1}$ for $1 \leq i \leq r-1$ and the product $s_1 s_2 \ldots s_r$ is the greatest common divisor of all $r \times r$ minors of A. Further S is unique

Now we shall derive a result given by Bhaskara Rao [50] and Bose & Mitra [9] over a principal ideal domain from Theorem 1.2.2.

Corollary 1.2.4. (Bhaskara Rao, [50], Theorem 1). Let $\mathbf{D}$ be a principal ideal domain An $m \times n$ matrix A of rank r is regular if and only if

$$A = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V \qquad (1.2.11)$$

where U and V are invertible and I is the $r \times r$ identity matrix

Proof. First of all, since $\mathbf{D}$ is a principal ideal domain, the matrix A has a Smith normal form, say, $A = U \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} V$. From Theorem 1.2.2 it is clear that A is regular if and only if the greatest common divisor of all $r \times r$ minors of A is one So we get the product $s_1 s_2 \ldots s_r$ to be one, where diag $(s_1, s_2, \ldots s_r)$ is the matrix S in the Smith normal form, which in turn, by modifying U and V if necessary , implies that $s_1 = s_2 = \ldots = s_r = 1$ So we get A in the required form. □

Sontag in [59] proved that over $\mathbf{D} = \mathbf{R}[X_1, X_2, \ldots X_k]^*$, the ring of rational functions $a(X_1, X_2, \ldots X_k)\, b(X_1, X_2, \ldots X_k)^{-1}$ with real coefficients and with $b(X_1, X_2, \ldots X_k) \neq 0$ for all $(X_1, X_2, \ldots X_k)$ in $\mathbf{R}^k$, a matrix is regular if and only if it has constant rank. We shall derive this result from Theorem 1.2.2 in the following corollary

Corollary 1.2.5 (Sontag [59] Theorem 3.). A matrix A over $\mathbf{R}[X_1, X_2, \ldots X_k]^*$ is regular if and only if A has constant rank for all $(X_1, X_2, \ldots X_k)$ in $\mathbf{R}^k$

Proof. Let the determinantal rank of A be r. From Theorem 1.2.2 we get that A is regular if and only if there exists $c_\alpha^\beta(X_1, X_2, \ldots, X_k)$ in $\mathbb{R}[X_1, X_2, \ldots, X_k]^*$, such that $\sum_{\alpha,\beta} c_\alpha^\beta(X_1, X_2, \ldots, X_k)\ |A_\beta^\alpha|(X_1, X_2, \ldots, X_k) = 1$, which implies that $(|A_\beta^\alpha|)_{\alpha,\beta}$ have no common zeroes. So for every $(X_1, X_2, \ldots, X_k)$ there exists α and β such that $|A_\beta^\alpha|(X_1, X_2, \ldots, X_k) \neq 0$. Thus $\rho(A(X_1, X_2, \ldots, X_k)) \geq r$. Since $\rho(A)$ is r, $\rho(A(X_1, X_2, \ldots, X_k)) \leq r$. Hence $\rho(A(X_1, X_2, \ldots, X_k)) = r$ for all $(X_1, X_2, \ldots, X_k)$.

Conversely, if A has constant rank over all $(X_1, X_2, \ldots, X_k)$ in $\mathbb{R}^k$, we get that $(|A_\beta^\alpha|)$ have no common zeroes and since $|A_\beta^\alpha|^2(X_1, X_2, \ldots, X_k)$ is strictly non negative we get that $u = \sum_{\alpha,\beta} |A_\beta^\alpha|^2$ has no zero over all $(X_1, X_2, \ldots, X_k)$ in $\mathbb{R}^k$ and is invertible in $\mathbb{R}[X_1, X_2, \ldots, X_k]^*$. So,

$$\sum_{\alpha,\beta} c_\alpha^\beta(X_1, X_2, \ldots, X_k)\ |A_\beta^\alpha|(X_1, X_2, \ldots, X_k) = 1,$$

for $c_\alpha^\beta(X_1, X_2, \ldots, X_k) = u(X_1, X_2, \ldots, X_k)^{-1}\ |A_\beta^\alpha|(X_1, X_2, \ldots, X_k)$ and by Theorem 1.2.2, A is regular. □

1.3 g-inverses and linear combinations of minors which are equal to one

A natural question that arises in view of Theorem 1.2.2 is the following: Suppose that the $m \times n$ matrix A over $\mathbb{D}$ is regular and that G is a g-inverse of A. Then do there exist $(c_\alpha^\beta)_{\alpha,\beta}$ in $\mathbb{D}$ such that (1.2.4) and (1.2.9) hold? We show in this section that if G is a reflexive g-inverse of A, then the answer is in the affirmative and in fact the choice $c_\alpha^\beta = |G_\alpha^\beta|$, for all α,β, satisfies (1.2.4) and (1.2.9). We also prove that over an integral domain over which every regular matrix admits a rank factorization (for example, any principal ideal domain has this property), for every g-inverse G of A, there exist $(c_\alpha^\beta)_{\alpha,\beta}$ satisfying (1.2.4) and (1.2.9).

We shall first show that every right inverse of a matrix, when it exists, arises in this fashion.

Lemma 1.3.1. Let C be an $r \times n$ matrix of rank r and let E be a right inverse of C so that $CE = I$. Then for all j,k

$$e_{jk} = \sum_{\beta:\, j\in\beta} |E^\beta| \frac{\partial}{\partial c_{kj}} |C_\beta|$$

where the summation is over all r-element subsets of $\{1,2,\ldots,n\}$.

Proof. We have

$$\sum_{\beta:\, j\in\beta} |E^\beta| \frac{\partial}{\partial c_{kj}} |C_\beta|$$

$$= \sum_{\beta} \sum_{l=1}^{n} e_{jl} \frac{\partial}{\partial e_{jl}} |E^\beta| \frac{\partial}{\partial c_{kj}} |C_\beta|$$

$$= \sum_{\gamma:\, j\notin\gamma} \sum_{l=1}^{n} e_{jl} (-1)^{k+l} |E^{\gamma}_{(1,2,\ldots,r)-l}| \; |C_\gamma^{(1,\ldots,r)-k}| \qquad (1.3.1)$$

where γ runs over all $(r-1)$ - element subsets of $\{1,2,\ldots,n\}$.

Since, for $j \in \gamma$,

$$\sum_{l=1}^{n} (-1)^l \, |E^{\gamma}_{(1,\ldots,r)-(l)}| \, e_{jl} = 0,$$

the restriction, $j \notin \gamma$, in the summation in (1.3.1) can be removed.

Now observe that $C_{(r-1)}(C)\, C_{(r-1)}(E) = I$ and hence

$$\sum_{\gamma} |C_{\gamma}^{(1,\ldots,r)-k}| \, |E^{\gamma}_{(1,\ldots,r)-l}| = \begin{cases} 0, & \text{if } l \neq k. \\ 1, & \text{if } l = k. \end{cases}$$

This observation, together with (1.3.1), gives

$$e_{jk} = \sum_{\beta: j \in \beta} |E^{\beta}| \frac{\partial}{\partial c_{kj}} |C_{\beta}|$$

and the proof is complete. □

A result similar to Lemma 1.3.1 can clearly be proved if B is an $m \times r$ matrix of rank r, and F is a left-inverse of B. We now prove one of the main results of this section.

Theorem 1.3.2. Let A be an $m \times n$ matrix of rank r and let G be a reflexive g-inverse of A. Then for all i,k,

$$g_{ji} = \sum_{\alpha:\, i \in \alpha} \sum_{\beta:\, j \in \beta} |G^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}|, \qquad (*)$$

where α, β run over all r-element subsets of $(1,\ldots,m)$, $(1,\ldots,n)$ respectively

Proof. Let $A = BC$ be a rank factorization of A over the quotient field of D Using the Cauchy-Binet formula we can show for $i \in \alpha$, $j \in \beta$,

$$\frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}| = \sum_{k=1}^{r} \frac{\partial}{\partial b_{ik}} |B^{\alpha}| \frac{\partial}{\partial c_{kj}} |C_{\beta}|.$$

Let G be a **reflexive** g-inverse of A and let $E = GB$, $F = CG$. Using $AGA = A$, $GAG = G$ and the fact that B, C are of full rank, it follows that $G = EF$ is a rank factorization of G, $CE = I$ and $FB = I$.

We have

$$\sum_{\alpha: i\in\alpha} \sum_{\beta: j\in\beta} |G^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}|$$

$$= \sum_{\alpha: i\in\alpha} \sum_{\beta: j\in\beta} |E^{\beta}||F_{\alpha}| \Big(\sum_{k=1}^{r} \frac{\partial}{\partial b_{ij}} |B^{\alpha}| \frac{\partial}{\partial c_{kj}} |C_{\beta}|\Big)$$

$$= \sum_{k=1}^{r} \sum_{\beta: j\in\beta} |E^{\beta}| \frac{\partial}{\partial c_{kj}} |C_{\beta}| \Big(\sum_{\alpha: i\in\alpha} |F_{\alpha}| \frac{\partial}{\partial b_{ik}} |B^{\alpha}|\Big)$$

$$= \sum_{k=1}^{r} e_{jk} f_{ki}$$

by Lemma 1.3.1 and the subsequent remark. Since $G = EF$ the proof is complete. □

Let A be an $m \times n$ matrix of rank $r \geq 1$ and let G be a g-inverse of A. Let $c^{\beta}_{\alpha} = |G^{\beta}_{\alpha}|$ for all α, β. Since A is of rank r, $C_r(A)$ is of rank 1 and hence

$$|A^{\alpha}_{\beta}||A^{\gamma}_{\delta}| = |A^{\alpha}_{\delta}||A^{\gamma}_{\beta}| \tag{1.3.2}$$

for any r-element subsets α,γ of $\{1,\ldots,m\}$ and β,δ of $\{1,\ldots,n\}$. Also since $AGA = A$,

$$C_r(A)\ C_r(G)\ C_r(A) = C_r(A) \tag{1.3.3}$$

It follows easily from (1.3.2) and (1.3.3) that (1.2.4) is satisfied.

Furthermore, if G is reflexive, then as shown in Theorem 1.3.2, (✱) also holds. We now give an example to show that if G is not reflexive, then (✱) may fail.

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then G is a (non-reflexive) g-inverse of A, and

$$|G_{(1,2)}^{(1,2)}| = |G_{(1,3)}^{(1,3)}| = |G_{(2,3)}^{(2,3)}| = 1,$$

the remaining $|\ G_\alpha^\beta\ |$ being zero. Thus it can be verified that if $H = ((h_{ij}))$ is defined as

$$h_{ji} = \sum_{\alpha : i \in \alpha} \ \sum_{\beta : j \in \beta} |G_\alpha^\beta| \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha| \ ,$$

then $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \neq G.$

In the following theorem we find a correspondence between the collection of reflexive g-inverses and a certain class of sets $(c_\alpha^\beta)_{(\beta\ ,\ \alpha)}$ from $\mathbb{D}$.

Theorem 1.3.3. If $(c_\alpha^\beta)_{(\beta\ ,\ \alpha)}$ are such that $\sum_{\alpha,\beta} c_\alpha^\beta |A_\beta^\alpha| = 1$ and $\rho(\ (c_\alpha^\beta)\ ) = 1$ then $G = (g_{ij})$, where

$$g_{ji} = \sum_{\alpha,\beta} c_\alpha^\beta \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha| \qquad (1.3.4)$$

is a reflexive g-inverse of A. Conversely, every reflexive g-inverse of A can be obtained by the above process.

Proof : First, we shall consider coefficients $(c_\alpha^\beta)_{(\beta,\alpha)}$ such that

$$\sum_{\alpha,\beta} c_\alpha^\beta |A_\beta^\alpha| = 1 \text{ and } \rho(\ (c_\alpha^\beta)\ ) = 1$$

and prove that G obtained by (1.3.4) is of rank r. This we shall accomplish by showing that $G = EF$ where E is an $n \times r$ matrix and F is an $r \times m$ matrix, both, matrices over $\mathbf{F}$, the quotient field of $\mathbf{D}$. This would imply that $\rho(G) \leq r$.

Let $A = B_{m \times r} C_{r \times n}$ be a rank factorization over $\mathbf{F}$ then

$$C_r(A) = (B^\alpha)_{\binom{m}{r} \times 1} (C_\beta)_{1 \times \binom{n}{r}}$$

is a rank factorization of $C_r(A)$ over $\mathbf{F}$. Let

$$(c_\alpha^\beta)_{\binom{m}{r} \times \binom{n}{r}} = (\hat{E}^\gamma)_{\binom{n}{r} \times 1} (\hat{F}_\delta)_{1 \times \binom{m}{r}}$$

be a a rank factorization of (c_α^β) over $\mathbf{F}$. By Cauchy-Binet formula we get

$$\frac{\partial}{\partial a_{ji}} |A_\beta^\alpha| = \sum_{k=1}^{r} \frac{\partial}{\partial b_{jk}} |B^\alpha| \frac{\partial}{\partial c_{ki}} |C_\beta|. \qquad (1.3.5)$$

Define

$$E = (e_{ik}) \quad \text{by} \quad e_{ik} = \sum_{\beta : i \in \beta} |\hat{E}^{\beta}| \frac{\partial}{\partial c_{ki}} |C_{\beta}| \quad 1 \le i \le m\ ,\ 1 \le k \le r \tag{1.3.6}$$

and

$$F = (f_{kj}) \quad \text{by} \quad f_{kj} = \sum_{\alpha : j \in \alpha} (\hat{F}_{\alpha}) \frac{\partial}{\partial b_{jk}} |B^{\alpha}| \quad 1 \le j \le n\ ,\ 1 \le k \le r \tag{1.3.7}$$

Now,

$$g_{ji} = \sum_{\alpha,\beta} c_{\alpha}^{\beta} \frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}|$$

$$= \sum_{\alpha,\beta : j \in \beta, i \in \alpha} |\hat{E}^{\beta}||\hat{F}_{\alpha}| \left(\sum_{k=1}^{r} \frac{\partial}{\partial b_{ik}} |B^{\alpha}| \frac{\partial}{\partial c_{kj}} |C_{\beta}| \right) \text{(from (1.3.5))}$$

$$= \sum_{k=1}^{r} \left\{ \left(\sum_{\beta : j \in \beta} |\hat{E}^{\beta}| \frac{\partial}{\partial c_{kj}} |C_{\beta}| \right) \left(\sum_{\alpha : i \in \alpha} |\hat{F}_{\alpha}| \frac{\partial}{\partial b_{ik}} |B^{\alpha}| \right) \right\}$$

$$= \sum_{k=1}^{r} e_{jk} f_{ki}$$

Thus we have shown that $G = EF$ and $\rho(G) \le r$ (1.3.8)

Anyway, $\rho(G) \ge r$ because G is g-inverse of A. From (1.3.8) we get $\rho(G) = r$. So G is a reflexive g-inverse of A.

In theorem 1.3.2 it has been proved that for a matrix A of rank r, if $G = (g_{ji})$ is a reflexive g-inverse of A, then,

$$g_{ji} = \sum_{\alpha,\beta} |G_{\alpha}^{\beta}| \frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}| \quad \text{for all } i,\ j\ , \tag{1.3.9}$$

$\sum_{\alpha,\beta} |G_{\alpha}^{\beta}||A_{\beta}^{\alpha}| = 1$ and since $\rho(G) = r$, being a reflexive g-inverse of A $\rho(|G_{\alpha}^{\beta}|) = 1$.

□

Let A be an $m \times n$ matrix over $\mathbf{D}$. Let us say that A has <u>generalized inverse</u> <u>construction</u> <u>property</u> **(g.i.c.p. in short)** if it is regular and if for any g-inverse G of A there exists $c_\alpha^\beta \in \mathbf{D}$ such that (1.2.4) and (1.2.9) are satisfied. Here we prove that if A has g.i.c.p. and if M, N are invertible matrices (units) over $\mathbf{D}$, then MA and AN have g.i.c.p. As a consequence it will be shown that if $\mathbf{D}$ is a principal ideal domain, then every regular matrix over $\mathbf{D}$ has g.i.c.p.

Lemma 1.3.4. If A is the $m \times n$ matrix given by $A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, where I is the $r \times r$ identity matrix, then A has g.i.c.p.

Proof. It is not difficult to see that if G is a g-inverse of A, then

$$G = \begin{bmatrix} I & B \\ C & D \end{bmatrix}$$

where B is $r \times (m - r)$, C is $(n - r) \times r$ and D is $(n - r) \times (m - r)$.

Let α,β be r-element subsets of $\{1,\dots,m\}$, $\{1,\dots,n\}$ respectively and let $i \in \alpha$, $j \in \beta$. Note that $\frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|$ is nonzero, in fact ± 1, if and only if $\alpha\backslash\{i\} = \beta\backslash\{j\} \subset \{1,\dots,r\}$. Let

$$c_\alpha^\beta = 1 \quad \text{if} \quad \alpha = \beta = \{1,\dots,r\}$$

If $1 \le i \le r$, $r + 1 \le s \le m$, $\alpha = \{1,2,\dots,i-1,i+1,\dots,r,s\}$, $\beta = \{1,2,\dots,r\}$,

let $$c_\alpha^\beta = g_{is} \frac{\partial}{\partial a_{si}} |A_\beta^\alpha|$$

If $1 \le s \le r$, $r + 1 \le i \le n$, $\alpha = \{1,2,\dots,r\}$ $\beta = \{1,2,\dots,s-1,s+1,\dots,r,i\}$,

let $$c_\alpha^\beta = g_{is} \frac{\partial}{\partial a_{si}} |A_\beta^\alpha|$$

If $r + 1 \le i \le n$, $r + 1 \le s \le m$, $\alpha = \{2,\dots,r,i\}$, $\beta = \{2,3,\dots,r,s\}$, let $\quad c_\alpha^\beta = g_{is} \frac{\partial}{\partial a_{si}} |A_\beta^\alpha|$

Finally for all the remaining pairs (α,β), let $c_\alpha^\beta = 0$. We now show that these c_α^β satisfy (1.2.4) and (1.2.9). Clearly, since $|A_\beta^\alpha|$ is 1 only if $\alpha = \beta = (1,\ldots,r)$ and zero otherwise, and $c_{(1,\ldots,r)}^{(1,\ldots,r)} = 1$, (1.1) is satisfied.

To show that (1.2.9) holds, consider the following cases.

Case (i) : $1 \le i \le r, \qquad 1 \le j \le r.$

Since $\frac{\partial}{\partial a_{ij}} |A_\alpha^\beta| \neq 0$ if and only if $\alpha = \beta = (1,\ldots,r)$ and $i = j$, it follows that

$$\sum_\alpha \sum_\beta c_\alpha^\beta \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha| = g_{ji}$$

Case (ii) : $r + 1 \le i \le m, \; 1 \le j \le r$

We have

$$\sum_{\alpha : i \in \alpha} \; \sum_{\beta : j \in \beta} c_\alpha^\beta \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha| = c_{\alpha'}^{\beta'} \frac{\partial}{\partial a_{ij}} |A_{\beta'}^{\alpha'}| \qquad \text{where } \alpha' = (1,2,\ldots,r)$$

$$\beta' = (1,2,\ldots,j-1,\; j+1,\ldots,r,i).$$

$$= g_{ji}$$

Case (iii) : $1 \le i \le r, \qquad r + 1 \le j \le n$

This is similar to case (ii).

Case (iv) : $r + 1 \le i \le m, \qquad r + 1 \le j \le n$

We have

$$\sum_{\alpha:i\in\alpha} \sum_{\beta:j\in\beta} c_\alpha^\beta \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha| = c_{\alpha'}^{\beta'} \frac{\partial}{\partial a_{ij}} |A_{\beta'}^{\alpha'}| = g_{ji}$$

where $\alpha' = (2,\dots,r,i)$

$\beta' = (2,\dots,r,j)$

Therefore we conclude that A has g.i.c.p. □

Lemma 1.3.5. Let A be an $m \times n$ matrix having g.i.c.p and let M be an $m \times m$ invertible matrix. Then $B = MA$ has g.i.c.p.

Proof. Suppose the rank of A is r. Let $N = M^{-1}$ and let H be a g-inverse of B over $\mathbb{D}$. Then $G = HN^{-1}$ is clearly a g-inverse of A. Since A has g.i.c.p. there exist c_α^β satisfying (1.2.4), (1.2.9). Since $H = GN$, we have, for any i,j,

$$h_{ij} = \sum_{k=1}^{m} g_{ik}\, n_{kj}$$

$$= \sum_{k=1}^{m} n_{kj} \sum_{\alpha} \sum_{\beta} c_\alpha^\beta \frac{\partial}{\partial a_{ki}} |A_\beta^\alpha| \qquad (1.3.10)$$

We denote by N^*, the matrix obtained from N by adding one auxiliary column, indexed by $*$, just after the j-th column of N. The entries of this column are not relevant for the proof. We similarly denote by B^*, the matrix obtained by adding a row, indexed by $*$, immediately after the j-th row of B. Observe that

$$\frac{\partial}{\partial a_{ki}} |A^{\alpha}_{\beta}| = \sum_{\theta} \frac{\partial}{\partial n_{k*}} |N^{*\alpha}_{\theta}| \frac{\partial}{\partial b_{*i}} |B^{*\theta}_{\beta}| \tag{1.3.11}$$

where θ runs over all r-element subsets of $(1,2,\ldots,m,*)$. Substituting (1.3.11) in (1.3.10) we have

$$h_{ij} = \sum_{k} \sum_{\alpha,\beta} c^{\beta}_{\alpha} n_{kj} \sum_{\theta} \frac{\partial}{\partial n_{k*}} |N^{*\alpha}_{\theta}| \frac{\partial}{\partial b_{*i}} |B^{*\theta}_{\beta}|$$

$$= \sum_{\alpha,\beta} c^{\beta}_{\alpha} \sum_{\theta} \frac{\partial}{\partial b_{*i}} |B^{*\theta}_{\beta}| \sum_{k} n_{kj} \frac{\partial}{\partial n_{k*}} |N^{*\alpha}_{\theta}|$$

$$= \sum_{\alpha,\beta} c^{\beta}_{\alpha} \sum_{\theta} \frac{\partial}{\partial b_{*i}} |B^{*\theta}_{\beta}| \, |N^{\alpha}_{\theta \cup (j) \setminus (*)}|$$

$$= \sum_{\alpha} \left\{ \sum_{\beta} c^{\beta}_{\alpha} \sum_{\gamma : j \in \gamma} |N^{\alpha}_{\gamma}| \right\} \frac{\partial}{\partial b_{ji}} |B^{\gamma}_{\beta}|$$

$$= \sum_{\alpha} \sum_{\gamma} \left\{ \sum_{\beta} c^{\beta}_{\alpha} |N^{\alpha}_{\gamma}| \right\} \frac{\partial}{\partial b_{ji}} |B^{\gamma}_{\beta}| , \tag{1.3.12}$$

since, if $j \notin \gamma$, then $\frac{\partial}{\partial b_{ji}} |B^{\gamma}_{\beta}| = 0$.

Let

$$d^{\beta}_{\gamma} = \sum_{\alpha} c^{\beta}_{\alpha} |N^{\alpha}_{\gamma}|$$

Since

$$\sum_{\alpha,\beta} c^{\beta}_{\alpha} |A^{\alpha}_{\beta}| = 1$$

and since $A = NB$, we have

$$\sum_{\alpha,\beta} c^{\beta}_{\alpha} \sum_{\gamma} |N^{\alpha}_{\gamma}||B^{\gamma}_{\beta}| = 1, \tag{1.3.13}$$

where γ runs over r-element subsets of $(1,2,\ldots,m)$. It is clear from (1.3.12), (1.3.13) that d^{β}_{γ} satisfy (1.2.4), (1.2.9) and hence we have shown that B has g.i.c.p. □

We can similarly show that if A has g.i.c.p. and N is invertible then AN has g.i.c.p. Lemma 1.3.5 and this observation immediately leads to the following.

Theorem 1.3.6. Let $A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ be of order $m \times n$ and let M,N be invertible matrices of order $m \times m$, $n \times n$ respectively over $\mathbb{D}$. Then MAN has g.i.c.p.

Theorem 1.3.7. Let $\mathbb{D}$ be a principal ideal domain. Then every regular matrix over $\mathbb{D}$ has g.i.c.p.

Proof. From Corollary 1.2.4, every regular matrix A admits a decomposition of the form

$$A = M \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N \tag{1.3.14}$$

where M,N are invertible. Now the result follows by Theorem 1.3.6. □

Remark. We shall see in Ch.4, that every regular matrix over the integral domain $\mathbb{D}$ admits a decomposition of the form (1.3.14) if and only if over $\mathbb{D}$ every finitely generated projective module is free. It follows by Theorem 1.3.6 that over such integral domains every regular matrix has g.i.c.p.. For example every regular matrix over $\mathbb{D}[X_1,\ldots,X_n]$, the polynomial ring over a principal ideal domain D, has g.i.c.p. We have not been able to decide whether every regular matrix over an integral domain $\mathbb{D}$ has g.i.c.p. and this question seems to merit further investigation.

If at all there is an integral domain $\mathbb{D}$ over which there is a regular matrix A without g.i.c.p. then

a) over $\mathbb{D}$, not every finitely generated projective module is free (For example, Bourbaki [10] p.150 could be a good candidate),

b) m and n must be greater than 2, because if for example $m \leq 2$ then A is either of rank 1 or $A = 0$ or A is of rank 2 and so right invertible. In these cases Theorem 1.2.1 and Theorem 1.3.1 would take care

and

c) $1 < \rho(A) < \min(m, n)$. This follows from Theorem 1.2.1 and 1.3.1.

On the other hand, it can be shown that if every idempotent matrix over an integral domain has g.i.c.p. then every regular matrix has g.i.c.p.

CHAPTER 2
MOORE-PENROSE INVERSE
AND
GENERALIZED CRAMER RULE

2.1. Introducion

Moore-Penrose inverses of matrices have wide-spread applications in subjects like statistics, multidimensional system theory, control theory etc (see [8], [9], [59], and [13]). In [59], matrices over the ring $\mathbb{R}[X_1, X_2, \ldots X_k]^*$, the ring of rational functions of polynomials with real coefficients which admit Moore-Penrose inverse were characterized and in [53] matrices which admit Moore-Penrose inverse were characterized in case the integral domain satisfies the condition "$a_1 = a_1^2 + \ldots + a_n^2$ implies $a_2 = \ldots = a_n = 0$".

In this chapter we obtain necessary and sufficient conditions for a matrix to have Moore-penrose inverse in the most general case of an integral domain. We also deal with other types of g-inverses.

In Section 2.2 we give necessary and sufficient conditions for matrices to have Moore-Penrose inverse over an arbitrary integral domain. We also give a formula to find Moore-Penrose inverse whenever it exists. In Section 2.3 we characterize all matrices which have {1,2,3} and {1,2,4} inverses. In Section 2.4 we obtain a Generalized Cramer rule to find Moore-Penrose solution, {1,2,3} solution and {1,2,4} solution. For the former, we use a formula developed in section 2.2. In Section 2.5 we define "Generalized Moore-Penrose inverse" and find necessary and sufficient conditions for matrices to have the generalized Moore-Penrose inverse. This generalized Moore-Penrose inverse reduces to weighted Moore-Penrose inverse in a special case. In Section 2.6 we give necessary and sufficient conditions for the existence of Khatri-inverse, pointing out an error in a condition given by Khatri.

2.2. Moore-Penrose inverse

In this section we examine the question of existence of Moore-Penrose inverse of a matrix A over $\mathbf{D}$. In general a matrix need not have a Moore-Penrose inverse even though it is regular. For example, the matrix

$$A = \begin{bmatrix} X^2 & 1 & -X^2 \\ 1 & XY & 0 \\ 2X^2 & 2 & -2X^2 \end{bmatrix}$$

over $\mathbb{Z}[X, Y]$ is regular as shown in an example given in Ch. 1, but it will follow from our result that it has no Moore-Penrose inverse. It is shown, among other results, that A has Moore-Penrose inverse if and only if $\sum_{\alpha,\beta} |A_\beta^\alpha||\bar{A}_\beta^\alpha|$ is an invertible element of $\mathbf{D}$. In the process we also obtain an explicit formula for the Moore-Penrose inverse when it exists. We first prove certain preliminary results

Lemma 2.2.1. Let A be a nonzero $n \times 1$ matrix over $\mathbf{D}$. Then A has a Moore-Penrose inverse over $\mathbf{D}$ if and only if A^*A is invertible in $\mathbf{D}$.

Proof. First suppose that A admits a Moore-Penrose inverse G. Then $AGA = A$ and since A is a nonzero $n \times 1$ matrix, $GA = I$. Now since $(AG)^* = AG$, we have $G^*A^*A = A$ and hence $(G\,G^*)(A^*A) = I$. Therefore A^*A is invertible in $\mathbb{D}$

Conversely, if $u = A^*A$ is invertible in $\mathbb{D}$, then it is easy to verify that $u^{-1}A^*$ is the Moore-Penrose inverse of A. □

A similar result can be proved if A is a $1 \times m$ matrix.

In the next result we characterize matrices of full rank over $\mathbb{D}$ which admit Moore-Penrose inverse.

Lemma 2.2.2. Let A be an $m \times n$ matrix of rank n over $\mathbb{D}$. Then the following conditions are equivalent.

(i) A has Moore-Penrose inverse

(ii) A^*A is invertible over $\mathbb{D}$.

(iii) $\sum_{\alpha} |\bar{A}^{\alpha}||A^{\alpha}|$ is invertible over $\mathbb{D}$, where α runs over all n-element subsets of $\{1,\ldots,m\}$

Furthermore, the Moore-Penrose inverse, when it exists, is given by $A^+ = (A^*A)^{-1}\ A$

Proof. $(i) \Rightarrow (ii)$. Let $A^+ = G$. Then

$$AGG^*A^*A = AGAGA = A$$

Since A is of full column rank, it admits a left inverse over the field of quotients of $\mathbb{D}$ and hence $(GG^*)(A^*A) = I$. Thus A^*A, which is a square matrix, has a left inverse over $\mathbb{D}$ and hence is invertible over $\mathbb{D}$.

(ii) $\Rightarrow$ *(i)* : It is easy to check that $A^+ = (A^*A)^{-1}A^*$ is the Moore-Penrose inverse of A.

(ii) $\Leftrightarrow$ *(iii)* : Note that a square matrix over $\mathbb{D}$ is invertible if and only if its determinant is invertible in $\mathbb{D}$. But, by Cauchy-Binet formula,

$$|A^*A| = \sum_{\alpha} |A^*_{\alpha}||A^{\alpha}|$$

$$= \sum_{\alpha} |\bar{A}^{\alpha}||A^{\alpha}|$$

where α runs over all n-element subsets of $\{1,\ldots,m\}$ and the result follows. □

A result analogous to Lemma 2.2.2 can be proved if A is of full row rank. The next result gives a necessary and sufficient condition for a matrix to have Moore-Penrose inverse under the assumption that the matrix has a rank factorization.

Theorem 2.2.3. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$ and let $A = BC$ be a rank factorization of A over $\mathbb{D}$. Then the following conditions are equivalent.

(i) A has Moore-Penrose inverse.

(ii) B^*B and CC^* are invertible over $\mathbb{D}$.

(iii) $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbb{D}$ where α, β run over r-element

subsets of $\{1,\dots,m\}$, $\{1,\dots,n\}$ respectively.

Furthermore, the Moore-Penrose inverse, if it exists, is given by

$$A^{+} = C^{*}(CC^{*})^{-1}(B^{*}B)^{-1}B^{*}.$$

Proof. $(i) \Rightarrow (ii)$: Let $A^{+} = G$. Then $BCGG^{*}A^{*}BC = BC$ and hence $CGG^{*}A^{*}B = I$. Therefore $CGG^{*}C^{*}B^{*}B = I$ and hence $B^{*}B$ is invertible over $\mathbf{D}$. Similarly by considering the equation $BCA^{*}G^{*}GBC = BC$ we conclude that CC^{*} is invertible over $\mathbf{D}$.

$(ii) \Rightarrow (i)$: If $B^{*}B$ and CC^{*} are invertible then it is easily verified that $C^{*}(CC^{*})^{-1}(B^{*}B)^{-1}B^{*}$ is the Moore-Penrose inverse of A.

$(ii) \Leftrightarrow (iii)$: For any α, β ; $|A^{\alpha}_{\beta}| = |B^{\alpha}||C_{\beta}|$ and hence

$$\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}| = \sum_{\alpha,\beta} |\bar{B}^{\alpha}||\bar{C}_{\beta}||B^{\alpha}||C_{\beta}|$$

$$= \Big(\sum_{\alpha} |\bar{B}^{\alpha}||B^{\alpha}|\Big)\Big(\sum_{\beta} |\bar{C}_{\beta}||C_{\beta}|\Big) \qquad (2.2.1)$$

Therefore $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}|\,|A^{\alpha}_{\beta}|$ is invertible if and only if both $\sum_{\alpha} |\bar{B}^{\alpha}||B^{\alpha}|$ and $\sum_{\beta} |\bar{C}_{\beta}||C_{\beta}|$ are invertible.

Now the result follows by the implication $(ii) \Leftrightarrow (iii)$ of Lemma 2.2.2. □

Corollary 2.2.4. Let A be an $m \times n$ matrix of rank r over $\mathbf{D}$ and suppose there exists a rank factorization $A = BC$ of A over $\mathbf{D}$. Then A has

Moore-Penrose inverse if and only if $C_r(A)$ has Moore-Penrose inverse.

Proof. Clearly $C_r(A)$ has Moore-Penrose inverse if A has Moore-Penrose inverse. To prove the converse, first observe that since A has a rank factorization, $C_r(A)$ has a rank factorization. By (i) $\Rightarrow$ (iii) of Theorem 2.2.3 applied to $C_r(A)$, if $C_r(A)$ has a Moore-Penrose inverse, the sum of square of all the elements of $C_r(A)$, which is same as $\sum_{\alpha,\beta} |A_\beta^\alpha||\bar{A}_\beta^\alpha|$ is invertible. By (iii) $\Rightarrow$ (i) of Theorem 2.2.3, it follows that A has Moore-Penrose inverse. □

Remark. It is not necessary that every matrix over an integral domain admits a rank factorization. For example; (from [53]) consider the integral domain $\mathbb{D}$, the polynomial ring generated by 1, X^2, XY, and Y^2 over $\mathbb{R}$ (real field) which is a subring of $\mathbb{R}[X, Y]$. The matrix

$$A = \begin{bmatrix} X^2 & XY \\ XY & Y^2 \end{bmatrix}$$

has no rank facorization over $\mathbb{D}$. In fact, as we shall notice in chapter 4. even regular matrices, it need not have a rank factorization. Thus Theorem 2.2.3 does not characterize all matrices which have Moore-Penrose inverse. A characterization of all matrices which have Moore-Penrose inevrse will be given in Theorem 2.2.6 which we now proceed to develop.

Lemma 2.2.5 Let A be an $m \times n$ matrix of rank 1 over $\mathbb{D}$. Then A has Moore-Penrose inverse if and only if $\sum_{i,j} a_{ij}\, \bar{a}_{ij}$ is invertible in $\mathbb{D}$.

Proof. First suppose that $u = \sum_{i,j} a_{ij}\bar{a}_{ij}$ is invertible in $\mathbb{D}$ Then we claim that $G = u^{-1}A^{*}$ is the Moore-Penrose inverse of A . This is seen thus : Clearly G satisfies (3), (4) in Chapter 0. Since A is of rank 1, every 2 x 2 minor of A vanishes, and hence for any i, j, k, l,

$$a_{kj}\, a_{il} = a_{ij}\, a_{kl} \,. \tag{2.2.2}$$

Hence for any i, l,

$$\sum_{j,k} a_{ij}\, g_{jk}\, a_{kl} = u^{-1} \sum_{j,k} a_{ij}\, \bar{a}_{kj}\, a_{kl}$$

$$= u^{-1} \sum_{j,k} a_{il}\, a_{kj}\, \bar{a}_{kj} = a_{il}$$

Therefore $AGA = A$ Similarly it can be shown that $GAG = G$ and the claim is proved.

Conversely, suppose that G is the Moore-Penrose inverse of A. Let r_i denote the i-th row of A and C_j the j-th column of G, $i = 1,2,\ldots,m$ $j = 1,2,\ldots,n$. Define matrices B of order $1 \times mn$ and H of order $mn \times 1$ as

$$B = (r_1,\ldots,r_m) \,, \qquad H = (c_1^{*},\ldots, c_m^{*})^{*}$$

We claim that H is the Moore-Penrose inverse of B . This is proved as follows. Using $AGA = A$ and (2.2.2) it can be seen that

$$\sum_{i,j} a_{ij}\, g_{ji} = 1$$

i.e $BH = 1$. Therefore $BHB = B$, $HBH = H$ and $(BH)^{*} = BH$. The matrix HB , in partitioned form, is

$$HB = \begin{bmatrix} c_1r_1 & \cdots\cdots & c_1r_m \\ c_2r_1 & \cdots\cdots & c_2r_m \\ \cdots & \cdots\cdots & \cdots \\ c_mr_1 & \cdots\cdots & c_mr_m \end{bmatrix}$$

Thus in order to show $(HB)^* = HB$, it is sufficient to show that $(c_i r_j)^* = c_j r_i$ for all i,j.

Note that A admits a rank factorization over the quotient field of $\mathbb{D}$ and since A has rank 1 it follows from Theorem 2.2.2 that $u = \sum_{i,j} \bar{a}_{ij} a_{ij}$ is non zero. Furthermore, as observed in the first part of this proof, $u^{-1}A^*$ is the Moore-Penrose inverse of A, where u^{-1} is the inverse of u over the field of quotients. By the uniqueness of Moore-Penrose inverse we have

$$c_k = u^{-1}r_k^* \quad \text{for } k = 1,2,\ldots m.$$

Therefore for any i, j,

$$\begin{aligned} (c_ir_j)^* &= u^{-1}(r_k^*r_j)^* \\ &= u^{-1}(r_j^*r_i) \\ &= c_jr_i. \end{aligned}$$

Thus $(c_ir_j)^* = c_jr_i$ and hence H is the Moore-Penrose inverse of B. It follows from Lemma 2.2.1 that $\sum_{i,j} \bar{a}_{ij} a_{ij}$ is invertible in $\mathbb{D}$ and the proof is complete □

The following is the main result of this section.

Theorem 2.2.6 Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$. Then the following conditions are equivalent.

(i) A has Moore-Penrose inverse

(ii) $C_r(A)$ has Moore-Penrose inverse.

(iii) $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbf{D}$, where α,β run over all r-element subsets of $\{1,\ldots,m\},\{1,\ldots,n\}$ respectively.

Furthermore, the Moore-Penrose inverse, when it exists is given by $G = ((g_{ij}))$ where

$$g_{ji} = \sum_{\alpha: i\in\alpha} \sum_{\beta: j\in\beta} u^{-1} |\bar{A}^{\alpha}_{\beta}| \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}| \quad \text{and} \quad u = \sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$$

Proof (i) ⇒ (ii) : It is easily verified that if $A^{+} = G$, then $C_r(G)$ is the Moore-Penrose inverse of $C_r(A)$.

(ii) ⇒ (iii) : Suppose $C_r(A)$ has Moore-Penrose inverse. Since the rank of $C_r(A)$ is one, it follows from Lemma 2.2.5 that $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbf{D}$

(iii) ⇒ (i) : Let $u = \sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ so that u^{-1} is an element of $\mathbf{D}$. Let G be the Moore-Penrose inverse of A over the field of quotients of $\mathbf{D}$ (G exists by Theorem 2.2.3). We will show that G is in fact a matrix over $\mathbf{D}$. As noted in the proof of (i) ⇒ (ii), $C_r(G)$ is the Moore-Penrose inverse of $C_r(A)$. Also, since $C_r(A)$ is of rank one, it follows from the proof of Lemma 2.2.5 that $u^{-1}\, C_r(A^{*})$ is the Moore-Penrose inverse of $C_r(A)$. So by the uniqueness of Moore-Penrose inverse,

$$C_r(G) = u^{-1}\, C_r(A^{*})$$

i.e., for all α, β,

$$|G^{\beta}_{\alpha}| = u^{-1}\, |A^{\alpha}_{\beta}| \tag{2.2.3}$$

Since G is, in particular, a reflexive g-inverse of A, by Theorem 1.3.2.

$$g_{ji} = \sum_{\alpha:\ i\in\alpha} \sum_{\beta:\ j\in\beta} |G^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}| \tag{2.2.4}$$

$$= \sum_{\alpha:\ i\in\alpha} \sum_{\beta:\ j\in\beta} u^{-1} |\bar{A}^{\alpha}_{\beta}| \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}| \tag{2.2.5}$$

by (4.3). Therefore G is a matrix over $\mathbb{D}$ and the proof is complete. □

Remark. If A^+ exists and equals G, then (2.2.4), (2.2.5) provide formulae for G. Formula (2.2.4) has been proved by Bruening [12] in the simpler case when A has full row (or column) rank.

Now in the following corollary we shall see an interesting result over the "ring of polynomials over the complex (real) field", that a matrix A of rank r has Moore-Penrose inverse if and only if all its $r \times r$ minors are in the complex (real) field.

Corollary 2.2.7. Let $\mathbb{D} = \mathbb{C}[X_1, \ldots, X_n]$, the polynomial ring over the complex field. An $m \times n$ matrix of rank r over $\mathbb{D}$ has Moore-Penrose inverse if and only if $(|A^{\alpha}_{\beta}|)_{\alpha,\beta}$ are all in $\mathbb{C}$, where α, β run over all r-element subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ respectively.

Proof. Suppose that A has Moore-Penrose inverse and there exists a $|A^{\alpha}_{\beta}|$ such that the degree of the polynomial $|A^{\alpha}_{\beta}|$ is at least one. Let $k = \max_{(\alpha,\beta)} \deg|A^{\alpha}_{\beta}|$, since the coefficient of the highest degree term in $|\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is strictly positive, degree of $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is k^2, which implies $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$

is not invertible in $\mathbb{D}$ and contradicts (*iii*) of Theorem 2.2.6.

Conversely, let $|A^{\alpha}_{\beta}|$ be in $\mathbb{C}$ for every α, and β. Since $\rho(A) = r$, there is at least one pair of (α, β) such that $|A^{\alpha}_{\beta}| \neq 0$ and we get that $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is nonzero and so invertible. So we get that A has Moore-Penrose inverse over $\mathbb{D}$.□

In general, we can conclude that **a matrix A of rank r over $\mathbb{D}[X_1 .....X_n]$ where $\mathbb{D}$ is an integral domain with a nonzero definite involution** (i.e.,$\sum \bar{a}_i a_i = 0 \Rightarrow a_i = 0$) **has Moore-Penrose inverse if and only if all $|A^{\alpha}_{\beta}|$ are in $\mathbb{D}$ and $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible.**

In the following corollary we shall derive a result of Sontag from [59], over $\mathbb{R}[X_1 \ldots X_n]^*$ (see p.19 for the definition), using our result.

Corollary 2.2.8. Let $\mathbb{D} = \mathbb{R}[X_1 .....X_n]^*$. An $m \times n$ matrix A of rank r has Moore-Penrose inverse if and only if A has constant rank for all $(X_1,......X_n)$ in $\mathbb{R}^n$

Proof. The g-inverse constructed in Corollary 1.2.5 in fact, can be seen to be the Moore-Penrose inverse, using Theorem 2.2.6.. □

Batigne in [4] proved that an integral matrix A of rank r has Moore-Penrose inverse if and only if

$$A = P \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} Q$$

where P and Q are permutaion matrices and M is an $r \times r$ invertible matrix.

The above result holds (See [54] & [56]) even for matrices over any integral domain $\mathbf{D}$ that satisfies Rao's condition (introduced by Robinson [56]) :

$$\text{"If } \sum_{i=1}^{n} a_i\bar{a}_i = a_1 \text{ then } a_i = 0 \text{ for } i \neq 1\text{"}$$

We shall derive this result from our Theorem 2.2.6. $\mathbf{Z}$, the ring of integers, $\mathbf{Z}[X_1, \ldots X_n]$ polynomial ring over integers are some examples of integral domains satisfying Rao's condition.

Theorem 2.2.9. Let $\mathbf{D}$ be an integral domain satisfying Rao's condition and A be an $m \times n$ matrix of rank r. Then the following are equivalent.

(i) A has Moore-Penrose inverse

(ii) $\sum_{\alpha,\beta} |\bar{A}_{\beta}^{\alpha}||A_{\beta}^{\alpha}|$ is invertible in $\mathbf{D}$, where α,β run over all r-element subsets of $\{1,\ldots,m\},\{1,\ldots,n\}$ respectively.

(iii) A is in the form $P\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}Q$ where P and Q are permutaion matrices and M is an $r \times r$ invertible matrix.

Proof. (i)$\Leftrightarrow$ (ii) follows from the Theorem 2.2.6.

(iii)$\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). From Rao's condition it folllows that whenever $\sum a_i\bar{a}_i = u$ is invertible there is exactly one i such that $a_i \neq 0$. This is because whenever $\sum a_i\bar{a}_i = u$ is invertible,

$$\bar{a}_i u^{-1} a_i = \sum_{j=1}^{n} (\bar{a}_i u^{-1} a_j)\,(\bar{a}_j u^{-1} a_i)$$

for all i. This tells us that $\bar{a}_i u^{-1} a_j = 0$ for all $j \neq i$ from Rao's conditon. But since $\mathbb{D}$ is an integral domain, for all but atmost one i, $a_i = 0$. Trivially there is at least one i such that $a_i \neq 0$ since $\sum a_i \bar{a}_i = u$ is invertible. Thus there is exactly one i with $a_i \neq 0$. Now, since $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbb{D}$, there exists a unique pair α and β (say, α_1 and β_1) such that $|A^{\alpha_1}_{\beta_1}| \neq 0$ and this is invertible in $\mathbb{D}$ and $|A^{\alpha}_{\beta}| = 0$ whenever $\alpha \neq \alpha_1$ or $\beta \neq \beta_1$. So we get A in the form

$$P \begin{bmatrix} A^{\alpha_1}_{\beta_1} & B \\ C & D \end{bmatrix} Q$$

for some permutaion matrices P, Q, $r \times (n-r)$ matrix B, $(n-r) \times r$ matrix C and $(n-r) \times (n-r)$ matrix D. Since $A^{\alpha_1}_{\beta_1}$ is the only submatrix of A of size $r \times r$ with nonzero determinant, we get that B, C and D are zero matrices. □

It is known that [33] over an arbitrary field a matrix has Moore-Penrose inverse if and only if $\rho(A^*A) = \rho(AA^*) = \rho(A)$. In the following theorem we generalize this result for matrices over an arbitrary integral domain .

Theorem 2.2.10. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$. Then the following are equivalent.

(i) A has Moore-Penrose inverse

(ii) $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbb{D}$, where α,β run over all r-element subsets of $\{1,\ldots,m\},\{1,\ldots,n\}$ respectively.

(iii) $\rho(A^*A) = \rho(AA^*) = \rho(A)$ and A^*A, AA^* are regular.

Proof. (i)⇔ (ii) follows from Theorem 2.2.6.

(ii)⇒(iii) Since $Tr(C_r(AA^*)) = Tr(C_r(A^*A)) = \sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}| \neq 0$, we get that $C_r(AA^*)$ and $C_r(A^*A)$ are nonzero matrices. This implies that $\rho(A^*A) = \rho(AA^*) = r = \rho(A)$ Since $Tr(C_r(AA^*))$ invertible, we get that a linear combination of all the $r \times r$ minors of AA^* is equal to one, and so, AA^* is regular. Similarly we get that A^*A is regular.

(iii)⇒ (i) It is easily verified that $A^*(AA^*)^- A(A^*A)^-A^*$ is the Moore-Penrose inverse of A if condition (iii) is satisfied, where $(AA^*)^-$, $(A^*A)^-$ are some g-inverses of AA^* and A^*A respectively. □

2.3 (1,3) and (1,4) inverses

Now we shall characterize all matrices which have (1,3) inverses in the following Theorem 2.3.1. In Theorem 2.3.2 we shall also give a different characterization which is computationally easier.

Theorem 2.3.1. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$. Then the following are equivalent.

(i) A has a (1,3) inverse .

(ii) $C_r(A)$ has a (1,3) inverse . i.e , there exists a matrix $S = (s^{\beta}_{\alpha})_{\binom{n}{r}\times\binom{m}{r}}$ such that $\sum_{\alpha,\beta} s^{\beta}_{\alpha}|A^{\alpha}_{\beta}| = 1$ and $(C_r(A)S)$ is symmetric.

Proof. (i)⇒(ii) is obvious from the properties of compound matrices.

(ii)⇒(i) Suppose there exists an $\binom{m}{r}\times\binom{n}{r}$ matrix S satisfying (ii). Then we claim that the matrix $G = (g_{ji})$ given by the equation

$$g_{ji} = \sum_{\alpha,\beta} s_{\alpha}^{\beta} \frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}|$$

is a (1,3) inverse. From Theorem 1.2.2. we get that G is a g-inverse of A. To prove that (AG) is symmetric, we shall prove that $(AG)_{ij} = (\bar{A}\bar{G})_{ji}$

Fixing α and β we have,

$$\sum_k a_{ik} \frac{\partial}{\partial a_{jk}} |A_{\beta}^{\alpha}| = \begin{cases} |A_{\beta}^{\alpha}| & \text{if } i=j \text{ and } i\in\alpha \\ 0 & \text{if } i\notin\alpha \\ 0 & \text{if } i,\, j\in\alpha \\ |A_{\beta}^{[\alpha/(j)]\cup(i)}| & \text{if } i\in\alpha,\, j\notin\alpha \end{cases} \qquad (2.3.1)$$

From this we get that

$$(AG)_{ij} = \sum_{\alpha,\beta : i\in\beta} s_{\alpha}^{\beta} |A_{\beta}^{\alpha}| \qquad \text{for } i = j \qquad (2.3.2)$$

and

$$(AG)_{ij} = \sum_{\alpha\beta} s_{\alpha}^{\beta} |A_{\beta}^{[\alpha/(j)]\cup(i)}| \qquad \text{for } i \neq j. \qquad (2.3.3)$$

So symmetry of $C_r(A)S$ gives symmetry of (AG). □

Analogous result for the existence of (1,4) inverse follows :

Theorem (2.3.1)'. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$. Then the following are equivalent.

(i) A has a (1,4) inverse .

(ii) $C_r(A)$ has a (1,4) inverse . i.e., there exists a matrix $S = (s_{\alpha}^{\beta})_{\binom{n}{r}\times\binom{m}{r}}$ such that $\sum_{\alpha\beta} s_{\alpha}^{\beta} |A_{\beta}^{\alpha}| = 1$ and $(SC_r(A))$ is symmetric.

In the above Theorems, the choice of (s_{α}^{β}) which plays an important role is a little difficult to find during the computation of G. In the following Theorems we shall give a different set of necessary and sufficient conditions, in which case, the choice of (s_{α}^{β}) is a little easier.

Theorem 2.3.2. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$. Then the following are equivalent.

(i) A has a (1,2,3) inverse.

(ii) There exists a $\binom{m}{r}\times\binom{n}{r}$ symmetric matrix S such that

$Tr[C_r(A)SC_r(A)^*] = 1.$

Proof. (i) $\Rightarrow$ (ii) Suppose A has a (1,2,3) inverse, say G. Then

$$\sum_{\alpha,\beta}|G_{\alpha}^{\beta}||A_{\beta}^{\alpha}| = 1.$$

But $G = GAG = GG^*A^*$, so that

$$|G_{\alpha}^{\beta}| = \sum_{\gamma,\delta}|G_{\delta}^{\beta}||G^{*\delta}_{\gamma}||A^{*\gamma}_{\alpha}|$$

and we get

$$\sum_{\alpha,\beta,\gamma}|(GG^*)_{\gamma}^{\beta}||A^{*\gamma}_{\alpha}||A_{\beta}^{\alpha}| = 1$$

i.e.,

$$\sum_{\alpha,\beta,\gamma}|A_{\beta}^{\alpha}||(GG^*)_{\gamma}^{\beta}||A^{*\gamma}_{\alpha}| = 1 \qquad (2.3.4)$$

which implies that for $S = C_r(GG^*)$, we get $Tr[C_r(A)SC_r(A)^*]=1$. (2.3.5)

(ii)$\Rightarrow$(i) Let S be a symmetric matrix satisfying the given condition .

Then we claim that G obtained by the equation

$$g_{ij} = \sum_{\alpha,\beta,\gamma} s_\gamma^\beta |A^*{}_\alpha^\gamma| \frac{\partial}{\partial a_{ji}} |A_\beta^\alpha| \qquad \begin{matrix} 1 \le i \le n \\ 1 \le j \le m \end{matrix} \tag{2.3.6}$$

is (1,2,3) inverse. By Theorem 1.2.2 we get that G is a g-inverse. From (2.3.6) we get $\rho(SC_r(A)^*) = \rho(C_r(A)) = 1$, so from Theorem 1.3.3, we get that G is a reflexive g-inverse of A. Now,

$$(AG)_{ij} = \sum_{k=1}^{m} a_{ik} g_{kj}$$

$$= \sum_{k=1}^{m} a_{ik} \sum_{\alpha,\beta,\gamma} s_\gamma^\beta |A^*{}_\alpha^\gamma| \frac{\partial}{\partial a_{jk}} |A_\beta^\alpha|$$

$$= \sum_{\alpha,\beta,\gamma : j \in \alpha} s_\gamma^\beta |A^*{}_\alpha^\gamma| \sum_{k \in \beta} a_{ik} \frac{\partial}{\partial a_{jk}} |A_\beta^\alpha|$$

and (2.3.1) gives us

$$(AG)_{ii} = \sum_{\alpha,\beta,\gamma : i \in \alpha,} |A_\beta^\alpha| \, s_\gamma^\beta |A^*{}_\alpha^\gamma| \qquad \text{for } i = j \tag{2.3.7}$$

$$(AG)_{ij} = \sum_{\alpha,\beta,\gamma : j \in \alpha, i \notin \alpha} |A_\beta^{[\alpha/\{j\}]\cup\{i\}}| \, s_\gamma^\beta |A^*{}_\alpha^\gamma| \qquad \text{for } i \ne j \tag{2.3.8}$$

From (2.3.7) , (2.3.8) and symmetry of $C_r(A)SC_r(A)^*$ we get $(AG)_{ij} = (AG)_{ji}$. □

Analogous result for the existence of (1,2,4) inverse follows :

Theorem (2.3.2)' Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$. Then the following are equivalent.

(i) A has a (1,2,4) inverse.

(ii) There exists a $\binom{m}{r}\times\binom{n}{r}$ symmetric matrix S such that

$$Tr[C_r(A)^* S\ C_r(A)] = 1.$$

The above theorems lead us to a result similar to the result proved in Theorem 2.2.10.

Theorem 2.3.3. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$. Then the following are equivalent.

(i) A has {1,2,3} inverse.({1,2,4} inverse).

(ii) There exists a $\binom{m}{r}\times\binom{n}{r}$ symmetric matrix S such that

$$Tr[C_r(A)S\ C_r(A)^* = 1.\ (\ Tr[C_r(A)^* S\ C_r(A)] = 1.)$$

(iii) $\rho(A^*A) = \rho(A)$ and A^*A is regular. ($\rho(AA^*)$)$=\rho(A)$ and AA^* is regular).

Proof is similar to Theorem 2.2.10. □

If $\mathbb{D}$ is the complex field, for example, S in the above theorem can be chosen to be a diagonal matrix with all diagonal elements being $[C_r(AA^*)]^{-1}$. Now we shall consider some integral domains over which the conditions in the above theorem become simpler. $\mathbf{Z}$, the ring of integers, and any principal ideal domain with trivial involution satisfy the hypothesis of the theorem below.

Theorem 2.3.4. Let $\mathbb{D}$ be a principal ideal domain such that for any symmetric elements $p_1,\ldots,p_n$ for which the ideal $(p_1,\ldots,p_n\ ) = \mathbb{D}$, there exist symmetric elements $q_1,\ \ldots,q_n$ with $\sum_{i=1}^{n} p_i q_i = 1$. Then the following are equivalent.

(i) A has a (1,2,3) inverse.

(ii) $\left(\sum_{\alpha} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|\right)_{\beta}$ are relatively prime, where α runs over all r-element subsets of $\{1,2,\ldots,m\}$ and β runs over all r-element subsets of $\{1,2,\ldots,n\}$.

Proof. Over any principal domain every regular matrix has a rank factorization. Let $A = BC$ be a rank factorization of A. Since $AGG^{*}A^{*}A = A$, we get that BB^{*} has inverse and that C has left inverse. So we get

$$u = \sum_{\alpha} |B^{*}{}_{\alpha}|\ |B^{\alpha}| \text{ is invertible} \tag{2.3.9}$$

and that there exists $k_{\beta} \in \mathbf{D}$ such that $\sum_{\beta} k_{\beta}\ |C_{\beta}| = 1$. (2.3.10)

From the properties of $\mathbf{D}$ we get symmetric elements p_{β}'s such that

$$\sum_{\beta} p_{\beta} |\bar{C}_{\beta}|\ |C_{\beta}| = 1 \tag{2.3.11}$$

by multiplying (2.3.9), (2.3.11) and u^{-1} we get that

$$\sum_{\beta} s_{\beta}\left(\sum_{\alpha} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|\right) = 1$$

where $s_{\beta} = u^{-1}p_{\beta}$ is symmetric in $\mathbf{D}$

(ii)⇒(i) If $\left(\sum_{\alpha} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|\right)_{\beta}$ are symmetric and relatively prime, then by the condition on $\mathbf{D}$, we can get s_{β} symmetric elements in $\mathbf{D}$ such that

$$\sum_{\beta} s_{\beta}\left(\sum_{\alpha} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|\right) = 1.$$

So proof of (ii)⇒ (i) becomes easier as in case of (iii)⇒(ii) of earlier theorem by taking $S = diag\ (s_{\beta})$. □

Analogous result for the existence of (1,2,4) inverse follows

Theorem (2.3.4)' Let $\mathbb{D}$ be a principal ideal domain such that for any symmetric elements $p_1, \ldots, p_n$ for which the ideal $(p_1, \ldots, p_n) = \mathbb{D}$, there exist symmetric elements $q_1, \ldots, q_n$ with $\sum_{i=1}^{n} p_i q_i = 1$. Then the following are equivalent.

(i) A has a (1,2,4) inverse.

(ii) $\left(\sum_{\beta} |\bar{A}_{\beta}^{\alpha}||A_{\beta}^{\alpha}|\right)_{\alpha}$ are relatively prime, where α runs over all r-element subsets of $\{1,2,\ldots,m\}$ and β runs over all r-element subsets of $\{1,2,\ldots,n\}$.

2.4 Generalized Cramer rule for finding various solutions.

In this section we shall give a method similar to the Cramer rule for finding various types of solutions, which we shall call "Generalized Cramer Rule". We start with finding Gy, where G is a g-inverse of A.

Theorem 2.4.1. Let A be an $m \times n$ matrix over an integral domain $\mathbb{D}$ and $Ax = y$ be a consistent linear system. Then Gy is a solution for the system $Ax = y$ where G is the matrix obtained by the equation (1.2.9), given in the above Theorem, and

$$(Gy)_j = \sum_{\alpha,\beta} c_{\alpha}^{\beta} \, |A(j \rightarrow y)_{\beta}^{\alpha}| \qquad (2.4.1)$$

where $A(j \rightarrow y)$ is the matrix A with j-th column replaced by y.

Proof. Since $g_{ji} = \sum_{\alpha,\beta} c_\alpha^\beta \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|$,

$$(Gy)_j = \sum_{i=1}^{m} g_{ji} y_i$$

$$= \sum_{i=1}^{m} \Big(\sum_{\alpha,\beta} c_\alpha^\beta \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha| \Big) y_i$$

$$= \sum_{\alpha,\beta : j\in\beta} c_\alpha^\beta \Big(\sum_{i\in\alpha} \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|\, y_i \Big)$$

$$= \sum_{\alpha,\beta : j\in\beta} c_\alpha^\beta\, |A(j\to y)_\beta^\alpha|$$

□

Corollary 2.4.2. Let $Ax=y$ be a linear system over an integral domain $\mathbf{D}$ such that G is the Moore-Penrose inverse of A, then Gy, the Moore-Penrose solution (called least square-minimum norm solution if $\mathbf{D}$ is the field of complex numbers) is given by

$$(Gy)_j = \sum_{\alpha,\beta\ :\ j\in\beta} u^{-1} |\bar{A}_\beta^\alpha| |A(j\to y)_\beta^\alpha| \tag{2.4.2}$$

Proof. Easily proved as in the case of Theorem 2.4.1. □

Remark. More generally, we can rewrite (2.4.2) as

$$u(Gy)_j = \sum_{\alpha,\beta\ :\ j\in\beta} |\bar{A}_\beta^\alpha| |A(j\to y\ )_\beta^\alpha|$$

and we can say that a linear system $Ax = y$ has Moore-Penrose solution if and only if u divides $\sum_{\alpha,\beta\ :\ j\in\beta} |\bar{A}^{\alpha}_{\beta}||A(j\to y)^{\alpha}_{\beta}|$ for all $1\le j\le m$. Equivalently,

$$\left\{ \sum_{\alpha,\beta\ :\ j\in\beta} |\bar{A}^{\alpha}_{\beta}||A(j\to y)^{\alpha}_{\beta}| \right\}_{1\le j\le m}$$

are all in (u), the ideal generated by u.

We shall call the method given in Theorem 2.4.1 and Corollary 2.4.2 and the remark above as "Generalized Cramer Rule", which reduces to the known Cramer rule in case A is invertible.

Remark. If $\mathbb{D}$ is an integral domain satisfying the Rao condition, equivalently $\sum_{i=1}^{n} a_i\bar{a}_i = 1$ implies that a_i's are zero except for one a_i (see (*iii*) ⇒ (*ii*) of Theorem 2.2.9) matrix A in Corollary 2.4.2 has exactly one non zero minor, say $A^{\alpha_0}_{\beta_0}$ such that $|A^{\alpha_0}_{\beta_0}|$ is invertible. In this case Generalized Cramer Rule takes a very simple form and

$$(Gy)_j = |\bar{A}^{\alpha_0}_{\beta_0}|^{-1}|A(j\to y)^{\alpha_0}_{\beta_0}|$$

Remark. A (1,2,3) solution (also called minimum norm solution over the complex field) and (1,2,4) solution (called least square solution over the complex field) can be obtained by the Generalized Cramer Rule as in the earlier case.

Remark. This Generalized Cramer Rule depends only on minors of the given matrix and this is easier and more general than the Cramer rule for matrices using the method of Bordering obtained by Ben-Isreal in [1] .

2.5 Generalized Moore-Penrose Inverse

We consider matrices over an integral domain $\mathbb{D}$ with involution $a \to \bar{a}$, unless indicated otherwise. Let A, M and N be matrices of order $m \times n$, $m \times m$ and $n \times n$ respectively, where M,N are invertible. An $n \times m$ matrix G is called the Generalized Moore-Penrose Inverse of A with respect to M,N if the conditions

(1) $AGA = A$ (2) $GAG = G$

(3') $(MAG)^* = MAG$ (4') $(NGA)^* = NGA$

are satisfied, where $*$ denotes induced involution over matrices (i.e., $A^* = (\bar{A})^T$). We denote a Generalized Moore-Penrose inverse of A with respect to M and N by A^+_{MN}.

The main results of this section consist of, for any matrix A over $\mathbb{D}$ and invertible matrices M, N of corresponding sizes,

(1) necessary and sufficient conditions for the existence of an $A^+_{M,N}$;

(2) the uniqueness of and a formula for $A^+_{M,N}$, whenever it exists; and

(3) a generalized Cramer rule to find a generalized Moore-Penrose solution with respect to M and N.

We shall start with a necessary and sufficient condition for the existence of $A^+_{M,N}$ for a column ($m \times 1$) matrix A over $\mathbb{D}$.

Lemma 2.5.1. Let A be a nonzero column matrix over $\mathbb{D}$. If M is an invertible matrix of **size** $m \times m$ and N is the 1×1 identity matrix then $A^+_{M,N}$ exists if and only if $A^* MA$ is symmetric and an invertible element in $\mathbb{D}$.

Proof. First, suppose that $A^{+}_{M,N}$ exist and let $G = A^{+}_{M,N}$. Then $AGA = A$, and since A is a nonzero $m \times 1$ matrix over an integral domain, $GA = 1$. Now, since MAG is symmetric, we have $M^{-1}G^{*}A^{*}M^{*}A = A$ and hence $GM^{-1}G^{*}A^{*}M^{*}A = 1$. Therefore $A^{*}MA$ is invertible in $\mathbb{D}$. By multiplying with A from the right and A^{*} from the left, the equation

$$G^{*}A^{*}M^{*} = MAG$$

gives $A^{*}MA$ is symmetric.

Conversely, if $A^{*}MA$ is invertible and symmetric in $\mathbb{D}$, then it is easy to verify that $(A^{*}MA)^{-1}A^{*}M^{*}$ is an $A^{+}_{M,N}$. □

For a row matrix we give a similar result in the following lemma, without proof.

Lemma 2.5.2. Let A be a nonzero row matrix over $\mathbb{D}$. If N is an invertible matrix of size $n \times n$ and M is the 1×1 identity matrix then $A^{+}_{M,N}$ exists if and only if $AN^{-1}A^{*}$ is symmetric and an invertible element in $\mathbb{D}$. In this case $N^{-1}A^{*}(AN^{-1}A^{*})$ is an $A^{+}_{M,N}$.

Now we shall give necessary and sufficient conditions for a matrix A to have a generalized Moore-Penrose inverse with respect to M and N, when A has a rank factorization over $\mathbb{D}$.

Theorem 2.5.3. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$ and let $A = BC$ be a rank factorization for A. If M and N are invertible matrices of size $m \times m$

and $n \times n$ respectively, then the following are equivalent:

(i) A has a generalized Moore-Penrose inverse with respect to M, N.

(ii) B^*MB and $CN^{-1}C^*$ are symmetric and invertible over $\mathbb{D}$.

(iii) $\sum_{\alpha\beta} |A^{\alpha}_{\beta}|\ |(N^{-1}A^*M)^{\beta}_{\alpha}|$ is invertible in $\mathbb{D}$, where α, β run over r-element subsets of $\{1,2,\ldots\ldots,m\}$ and $\{1,2,\ldots\ldots,n\}$ respectively, and A^*MA and $AN^{-1}A^*$ are symmetric matrices.

Proof. (i) $\Rightarrow$ (ii) Suppose $A^{+}_{M,N}$ exists and let $G = A^{+}_{M,N}$. Then

$$AGM^{-1}G^*A^*M^*A = A. \tag{2.5.1}$$

(Since MAG is symmetric). Since $A = BC$ is a rank factorization over $\mathbb{D}$, B is a full column rank matrix and C is a full row rank matrix over the field of quotients of $\mathbb{D}$, B is left cancellable and C is right cancellable over $\mathbb{D}$. So from (2.5.1) we get

$$CGM^{-1}G^*C^*B^*M^*B = I \tag{2.5.2}$$

which implies B^*M^*B is invertible. Similarly by considering the equation

$$AN^{-1}A^*G^*N^*GA = A \tag{2.5.3}$$

we get $CN^{-1}C^*$ is invertible.

Since G is a reflexive g-inverse of A, G has a rank factorization $G = UV$ over $\mathbb{D}$ (for example, take $U = GB$ and $V = CG$) such that $CU = I$ and $VB = I$. From the condition (3') we get

$$MBV = U^*B^*M^* \tag{2.5.4}$$

By multiplying B^* on the left and B on the right, we get B^*M^*B is symmetric. Similarly by considering the condition (4') we get

$$NUC = C^*U^*N^* \tag{2.5.5}$$

from which we can get $CN^{-1}C^*$ is symmetric. Hence (i) $\Rightarrow$ (ii).

and $n \times n$ respectively, then the following are equivalent:

(i) A has a generalized Moore-Penrose inverse with respect to M, N.

(ii) B^*MB and $CN^{-1}C^*$ are symmetric and invertible over $\mathbb{D}$.

(iii) $\sum_{\alpha\beta} |A^{\alpha}_{\beta}|\ |(N^{-1}A^*M)^{\beta}_{\alpha}|$ is invertible in $\mathbb{D}$, where α, β run over r-element subsets of $\{1,2,\ldots\ldots,m\}$ and $\{1,2,\ldots\ldots,n\}$ respectively, and A^*MA and $AN^{-1}A^*$ are symmetric matrices.

Proof. (i) $\Rightarrow$ (ii) Suppose $A^{+}_{M,N}$ exists and let $G = A^{+}_{M,N}$. Then

$$AGM^{-1}G^*A^*M^*A = A. \tag{2.5.1}$$

(Since MAG is symmetric). Since $A = BC$ is a rank factorization over $\mathbb{D}$, B is a full column rank matrix and C is a full row rank matrix over the field of quotients of $\mathbb{D}$, B is left cancellable and C is right cancellable over $\mathbb{D}$. So from (2.5.1) we get

$$CGM^{-1}G^*C^*B^*M^*B = I \tag{2.5.2}$$

which implies B^*M^*B is invertible. Similarly by considering the equation

$$AN^{-1}A^*G^*N^*GA = A \tag{2.5.3}$$

we get $CN^{-1}C^*$ is invertible.

Since G is a reflexive g-inverse of A, G has a rank factorization $G = UV$ over $\mathbb{D}$ (for example, take $U = GB$ and $V = CG$) such that $CU = I$ and $VB = I$. From the condition (3') we get

$$MBV = U^*B^*M^* \tag{2.5.4}$$

By multiplying B^* on the left and B on the right, we get B^*M^*B is symmetric. Similarly by considering the condition (4') we get

$$NUC = C^*U^*N^* \tag{2.5.5}$$

from which we can get $CN^{-1}C^*$ is symmetric. Hence (i) $\Rightarrow$ (ii).

and

$$N^{-1}A^*(AN^{-1}A^*)^-A(A^*MA)^-A^*M^*$$

is an $A^+_{M,N}$

Corollary 2.5.4 Let A be a matrix of rank 1 and $A = BC$ be a rank factorization of A over $\mathbb{D}$. Then $A^+_{M,N}$ is unique whenever it exists.

Proof. Assume $A^+_{M,N}$ (say G) exists. Since it is a reflexive g-inverse, it can be written in the form $G = UV$ where $CU = I$ and $VB = I$. From (2.5.4) and (2.5.5) we get

$$V = (B^*MB)^{-1}B^*M^* \qquad (2.5.8)$$

$$U = N^{-1}C^*(CN^{-1}C^*)^{-1} \qquad (2.5.9)$$

and

$$G = N^{-1}C^*(CN^{-1}C^*)^{-1}(B^*MB)^{-1}B^*M^*$$

$$= N^{-1}C^*B^*M^*(CN^{-1}C^*)^{-1}(B^*MB)^{-1} \quad \text{(since } (CN^{-1}C^*)^{-1},$$

$(B^*MB)^{-1}$ are in $\mathbb{D}$)

$$= N^{-1}A^*M^*[Tr(CN^{-1}C^*B^*MB)]^{-1}$$

$$= N^{-1}A^*M^*[Tr(N^{-1}C^*B^*MBC)]^{-1}$$

$$= N^{-1}A^*M^*[Tr(N^{-1}A^*MA)]^{-1} \qquad (2.5.10)$$

So $A^+_{M,N}$ is unique, whenever it exists □

Corollary 2.5.5 Let A be an $m \times n$ matrix of rank r and M, N be invertible matrices over $\mathbb{D}$ such that A^*M^*A and $AN^{-1}A^*$ are symmetric. Also, let $A = BC$ be a rank factorization of A over $\mathbb{D}$ Then A has a generalized Moore-

Penrose inverse with respect to M and N if and only if $C_r(A)$ has a generalized Moore-Penrose inverse with respect to $C_r(M)$, and $C_r(N)$.

Proof. From the properties of compound matrices, it is clear that $C_r(A)$ has a generalized Moore-Penrose inverse with respect to $C_r(M)$, and $C_r(N)$ whenever A has a generalized Moore-Penrose inverse with respect to M and N.

To prove the converse, first observe that, $C_r(A)$ has a rank factorizaton as A has a rank factorization. By $(i) \Rightarrow (iii)$ of Theorem 2.5.3, applied to $C_r(A)$, we get that the summation $\sum_{\alpha,\beta} C_r(A)_{\alpha,\beta}(C_r(N)^{-1}C_r(A)^*C_r(M))_{\beta,\alpha}$, which is the same as $\sum_{\alpha,\beta} |A^{\alpha}_{\beta}||(N^{-1}A^*M)^{\beta}_{\alpha}|$, is invertible Since A^*M^*A and $AN^{-1}A^*$ are symmetric and $\sum_{\alpha,\beta} |A^{\alpha}_{\beta}||(N^{-1}A^*M)^{\beta}_{\alpha}|$ is invertible, by $(iii) \Rightarrow (i)$ of Theorem 2.5.3., A has a generalized Moore-Penrose inverse with respect to M and N □

Remark. It is easy to observe that the existence of a generalized Moore-Penrose inverse for $C_r(A)$ with respect to $C_r(M)$, and $C_r(N)$ is not a sufficient condition for the existence of a generalized Moore-Penrose inverse for A with respect to M and N. For example, let $\mathbb{D} = \mathbb{C}$, the field of complex numbers with respect to conjugation as the involution, and let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad \text{and} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $C_2(A) = 1$, $C_2(M) = -1$, $C_2(N) = 1$ and $C_2(A)$ has a generalized Moore-Penrose inverse with respect to $C_2(M)$, and $C_2(N)$ But A has no generalized Moore-

Penrose inverse with respect to M and N.

Now we shall consider matrices which may not have any rank factorization over an integral domain $\mathbb{D}$ and we shall give necessary and sufficient conditions for them to have generalized Moore-Penroses inverse with respect to M and N. First we shall consider a matrix A of rank 1 and in the following Lemma obtain a necessary conditon for the existence of $A^{+}_{M,N}$.

Lemma 2.5.6. Let A be an $m \times n$ matrix of rank 1, and M and N be invertible matrices over $\mathbb{D}$. Then $[Tr(N^{-1}A^{*}MA)]$ is invertible whenever $A^{+}_{M,N}$ exists.

Proof. Let $G = A^{+}_{M,N} = (g_{ij})$ exist. Let r_i denote the i-th row of A and c_j the j-th column of G, where $i = 1,2, \ldots, m$, $j = 1,2, \ldots, n$. Define matrices $\hat{A}$ of order $1 \times mn$ and $\hat{G}$ of order $mn \times 1$ such that

$$\hat{A} = (r_1, r_2, \ldots, r_m)$$

$$\hat{G} = (c_1^T, c_2^T, \ldots, c_m^T)^T.$$

We claim that $\hat{G}$ is a Moore-Penrose inverse of $\hat{A}$ with respect to I and $M^{T^{-1}} \otimes N$ (where $\otimes$ stands for the tensor product of matrices).

Since A is of rank 1, every 2×2 minor of A vanishes and hence for all i,j,k,l

$$a_{ij}\, a_{kl} = a_{kj}\, a_{il} \tag{2.5.11}$$

Using $AGA = A$ and (2.5.11) it can be seen that $\sum_{i,j} a_{ij} g_{ji} = 1$ i.e., $\hat{A}\hat{G} = 1$. Therefore $\hat{A}\hat{G}\hat{A} = \hat{A}$, $\hat{G}\hat{A}\hat{G} = \hat{G}$ and $(\hat{A}\hat{G})^{*} = \hat{A}\hat{G}$.

Now we shall prove $(M^{T^{-1}} \otimes N)(\hat{G}\hat{A})$ is symmetric. First, consider a rank factorization $A = BC$, where B is a column matrix and C is a row matrix over the field of quotients of $\mathbb{D}$. Let $G = UV$, where U and V are matrices over the field of quotients of $\mathbb{D}$ such that $CU = 1$ and $VB = 1$. By (2.5.8) and (2.5.9) we get

$$V = (B^*MB)^{-1}B^*M^*$$

$$U = N^{-1}C^*(CN^{-1}C^*)^{-1}$$

But, $$A = B^T \otimes C \text{ and } G = V^T \otimes U$$

and

$$\begin{aligned}(M^{T^{-1}} \otimes N)(\hat{A}\hat{G}) &= M^{T^{-1}} \otimes N[(V^T \otimes U)(B^T \otimes C)] \\ &= M^{T^{-1}}V^TB^T \otimes NUC \\ &= (M^{*-1}G^*A^*) \otimes NGA \qquad (2.5.12)\end{aligned}$$

is symmetric. So $\hat{G}$ is a Moore-Penrose inverse of $\hat{A}$ with respect to 1 and $M^{T^{-1}} \otimes N$. Therefore, by Lemma 2.5.2 we get $\hat{A}(M^T \otimes N)^{-1}\hat{A}^*$, which is equal to $Tr(N^{-1}A^*MA)$, is invertible in $\mathbb{D}$. □

Lemma 2.5.7. Let A be an $m \times n$ matrix of rank 1 over $\mathbb{D}$ and M and N be invertible matrices of appropriate sizes. Then $A^+_{M,N}$ exists if and only if $Tr(N^{-1}A^*MA)$ is invertible in $\mathbb{D}$ and A^*M^*A and $AN^{-1}A^*$ are symmetric. Also, $A^+_{M,N}$ is unique whenever it exists.

Proof. (if part) Let $A^+_{M,N}$ exist over $\mathbb{D}$. The symmetry of A^*M^*A, $AN^{-1}A^*$ and the invertiblity of $Tr(N^{-1}A^*MA)$ over the field of quotients of $\mathbb{D}$ follow from Theorem 2.5.3. From Lemma 2.5.5 we get $Tr(N^{-1}A^*MA)$ is invertible in $\mathbb{D}$. Hence the proof of the "if" part of the Lemma.

(Only if part) Suppose $Tr(N^{-1}A^*MA)$ is invertible and A^*M^*A, $AN^{-1}A^*$ are symmetric then we claim that

$$G = [Tr(N^{-1}A^*MA)]^{-1}N^{-1}A^*M^*$$

is an $A^+_{M,N}$

$$AGA = A[Tr(N^{-1}A^*MA)]^{-1}N^{-1}A^*M^*A$$

$$= [Tr(N^{-1}A^*MA)]^{-1}AN^{-1}A^*M^*A$$

$$= (B^*MB)^{-1}(CN^{-1}C^*)^{-1}B(CN^{-1}C^*)(B^*M^*B)C$$

(by taking $A = BC$ a rank factorization over the field of quotients) Since A^*M^*A is symmetric , B^*M^*B is symmetric over the field of quotients of $\mathbf{D}$. So we get $AGA = BC = A$. Similarly it can be verified easily that $GAG = G$, and that MAG and NGA are symmetric. So G is an $A^+_{M,N}$.

Uniqueness follows from corollary 2.5.4. □

In the following theorem we shall obtain necessary and sufficient conditions for the existence of $A^+_{M,N}$ even if A does not have a rank factorization over $\mathbf{D}$.

Theorem 2.5.8. Let A be an $m \times n$ matrix of rank r over $\mathbf{D}$ and M and N be are invertible matrices of appropriate sizes. Then the following are equivalent

(i) A has generalized Moore-Penrose inverse with respect to M, N

(ii) $C_r(A)$ has a generalized Moore-Penrose inverse with respect to $C_r(M)$ and $C_r(N)$, also, A^*MA, $AN^{-1}A^*$ are symmetric.

(iii) $Tr[C_r(N^{-1}A^*MA)]$ is invertible in $\mathbf{D}$ and A^*MA and $AN^{-1}A^*$ are symmetric

In this case $A^+_{M,N}$ is unique and

$$g_{ij} = \sum_{\alpha,\beta} (Tr[C_r(N^{-1}A^*MA)])^{-1} |(N^{-1}A^*M^*)^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ji}} |A^{\alpha}_{\beta}| \qquad (2.5.13)$$

gives $A^{+}_{M,N}$

Proof : $(i) \Rightarrow (ii)$ can be verified easily as in the case of $(i) \Rightarrow (ii)$ of Theorem 2.5.3 and $(ii) \Rightarrow (iii)$ follows from Lemma 2.5.7.

$(iii) \Rightarrow (i)$ Suppose $Tr[C_r(N^{-1}A^*MA)]$ is invertible in $\mathbf{D}$ and A^*MA and $AN^{-1}A^*$ are symmetric. By $(iii) \Rightarrow (i)$ of Theorem 2.5.3 we get $G = A^{+}_{M,N}$ over the field of quotients of $\mathbf{D}$. But $C_r(G)$ is a unique generalized Moore-Penrose inverse with respect to $C_r(M)$ and $C_r(N)$ (by Lemma 2.5.7), so

$$C_r(G) = (Tr[C_r(N^{-1}A^*MA)])^{-1} C_r(N^{-1}) C_r(A^*) C_r(M^*).$$

From Theorem 1.3.2 we get

$$g_{ij} = \sum_{\alpha,\beta} (Tr[C_r(N^{-1}A^*MA)])^{-1} |(N^{-1}A^*M^*)^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ji}} |A^{\alpha}_{\beta}|$$

for all $i = 1,2, \ldots n$, $j = 1,2,\ldots,m$, which implies that G is a matrix over $\mathbf{D}$. Also, from Theorem 2.3.2 and the uniqueness of $C_r(A)^{+}_{C_r(M),C_r(N)}$, we get the uniqueness of $A^{+}_{M,N}$.

□

Remark. If $Ax = y$ is a given linear system over $\mathbf{D}$ and G is a generalized Moore-Penrose inverse with respect to M, N, then Gy is a generalized Moore-Penrose solution. In fact it is easily verified using 2.5.13 that

$$(Gy)_i = \sum_{\alpha,\beta,\ i \in \beta} (Tr[C_r(N^{-1}A^*MA)])^{-1} |(N^{-1}A^*M^*)^{\beta}_{\alpha}| |A(i \rightarrow y)^{\alpha}_{\beta}|$$

where $A(i \rightarrow y)$ is the matrix obtained by replacing the i-th column of A by y. If

M, N are positive definite matrices in $\mathbb{C}$ (complex numbers with usual conjugation as involution), then the generalized Moore-Penrose inverse is referred to as the minimum N-norm M-least square g-inverse of A (see [49]) which alwayas exists and in which case Gy is called the minimum N-norm M-least square solution.

2.6 Khatri inverse

We consider matrices over an integral domain with an involution $a \to \bar{a}$, unless indicated otherwise. Let A, M and N be matrices of order $m \times n$, $m \times m$ and $n \times n$ respectively, where M and N are invertible (not necessarily symmetric). An $n \times m$ matrix G is called the Khatri-inverse of A (see Rao and Mitra [49]) with respect to M, N if the conditions

(1) $AGA = A$ **(2)** $GAG = G$

(3″) $(AG)^{*}M = M(AG)$ **(4″)** $(GA)^{*}N = N(GA)$

are satisfied, where A^{*} denotes $(\bar{A})^{T}$ If M and N are positive definite over the complex field, then G is called minimum N-norm M-least square g-inverse of A, denoted by A^{+}_{MN} ([49], p 52).

The Khatri-inverse is unique whenever it exists; this can be seen by suitably manipulating equations (1)-(4″). In [29] and [49] it has been claimed that Khatri-inverse exists if and only if $\rho(A^{*}MA) = \rho(AN^{-1}A^{*}) = \rho(A)$, where ρ denotes the rank The condition can be seen to be necessary from equations (1)-(4″). However the condition is not sufficient as can be seen from the example given below The error persists in [49], p 69 where the result is given as an exercise.

Example. Let $\mathbb{D} = \mathbb{C}$, the field of complex numbers,

$$A = [\,1 \quad 2\,]^*, \qquad M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This A has no Khatri-inverse with respect to M, N although $\rho(A^*MA) = \rho(AN^{-1}A^*) = \rho(A)$. In [29], $G = N^{-1}A^*(AN^{-1}A^*)^- A(A^*MA)^- A^*M$ is given as the formula for the Khatri-inverse. In this example the matrix G obtained by using the above formula is $\frac{1}{9}[\,1 \quad 4\,]$, and it does not satisfy (3″) and (4″).

Now we shall give necessary and sufficient conditions for the existence of the Khatri-inverse. It easily follows from the definition that if the Khatri-inverse of A with respect to M and N exists, then $\rho(A^*MA) = \rho(AN^{-1}A^*) = \rho(A)$.

Theorem 2.6.1. Let A be an $m \times n$ matrix of rank r and let $A = BC$ be a rank factorization of A over $\mathbb{D}$. Then the following statements are equivalent :

(i) A has a Khatri-inverse with respect to M and N

(ii) B^*MB, $CN^{-1}C^*$ are invertible and

$$(B^*MB)^{-1}B^*M = (B^*M^*B)^{-1}B^*M^* \qquad (2.6.1)$$

and

$$N^{-1}C^*(CN^{-1}C^*)^{-1} = N^{*-1}C^*(CN^{*-1}C^*)^{-1} \qquad (2.6.2)$$

Proof : Let G be the Khatri inverse of A. Since G is a reflexive g-inverse it can be written in the form $G = UV$ such that

$$VB = I \text{ and } CU = I \qquad (2.6.3)$$

From the condition (1) and (3″) we get

$$AGM^{-1}(AG)^*MA = A$$

i.e.,

$$BCGM^{-1}G^*C^*B^*MBC = BC$$

which implies

$$(CGM^{-1}G^*C^*)(B^*MB) = I$$

So that (B^*MB) is invertible. By (3'') we get $MBV = V^*B^*M$ and hence

$$(B^*MB)V = B^*M \text{ (since } B^*V^* = I). \tag{2.6.4}$$

Also,

$$MB = V^*B^*MB \text{ (since } VB = I \text{)} \tag{2.6.5}$$

From (2.6.4) and (2.6.5) we get

$$V = (B^*MB)^{-1}B^*M = (B^*M^*B)^{-1}B^*M^*$$

since B^*MB is invertible. Similarly we can prove $CN^{-1}C^*$ is invertible and

$$U = N^{-1}C^*(CN^{-1}C^*)^{-1} = N^{*-1}C^*(CN^{*-1}C^*)^{-1}$$

Conversly, if *(ii)* of the theorem holds then it can be verified that

$$G = N^{-1}C^*(CN^{-1}C^*)^{-1}(B^*MB)^{-1}B^*M$$

is the Khatri- inverse of A with respect to M and N. □

Corollary 2.6.2. Let A be an $m \times n$ matrix of rank 1 with a rank factorization over $\mathbb{D}$. Then A has a Khatri-inverse with respect to M and N if and only if $Tr(N^{-1}A^*MA)$ is nonzero and the matrices B,C,M,N satisfy (2.6.1),(2.6.2). In which case

$$G = [Tr(N^{-1}A^*MA)]^{-1}N^{-1}A^*M$$

is the Khatri-inverse.

Proof. Let $A = BC$ be a rank factorization of A. Note that $Tr(N^{-1}A^*MA) =$

$|CN^{-1}C^*||B^*MB|$ whereas $N^{-1}C^*(CN^{-1}C^*)^{-1}(B^*MB)^{-1}B^*M = Tr(N^{-1}A^*MA)^{-1}N^{-1}A^*M$ and the result follows from Theorem 2.6.1. □

Corollary 2.6.3. Let A be an $m \times n$ matrix of rank r with a rank factorization $A = BC$ over $\mathbf{D}$. Then A has Khatri-inverse with respect to M,N if and only if $\sum_{\alpha,\beta} |(N^{-1}A^*M)^{\beta}_{\alpha}||A^{\alpha}_{\beta}|$ is invertible and (2.6.1), (2.6.2) are satisfied.

Proof. By Cauchy-Binet formula we get

$$|CN^{-1}C^*||B^*MB| = \sum_{\alpha,\beta,\gamma,\eta} |C_{\beta}||N^{-1\beta}_{\gamma}||C^{*\gamma}||B^*_{\eta}||M^{\eta}_{\alpha}||B^{\alpha}|$$

$$= \sum_{\alpha,\beta} |(N^{-1}A^*M)^{\beta}_{\alpha}||A^{\alpha}_{\beta}|$$

and hence the result follows from Theorem 2.6.1. □

We now prove the main result of this section.

Theorem 2.6.4. Let A be an $m \times n$ matrix. Then the following are equivalent

(i) A has a Khatri-inverse with respect to M and N

(ii) $\rho(A^*MA) = \rho(AN^{-1}A^*) = \rho(A)$, and A^*MA, $AN^{-1}A^*$ are regular and also the equations

$$A(A^*MA)^-A^*M = A(A^*M^*A)^-A^*M^* \tag{2.6.6}$$

$$N^{-1}A^*(AN^{-1}A^*)^-A = N^{*-1}A^*(AN^{*-1}A^*)^-A \tag{2.6.7}$$

are satisfied for any choice of g-inverses.

Furthermore, if (ii) is satisfied,

$$G = N^{-1}A^{*}(AN^{-1}A^{*})^{-}A(A^{*}MA)^{-}A^{*}M \qquad (2.6.8)$$

is the Khatri-inverse for any choice of g-inverses.

Proof. If (i) satisfied, then as remarked earlier, $\rho(A^{*}MA) = \rho(AN^{-1}A^{*}) = \rho(A)$ holds and also it can be verified easily that GMG^{*}, $G^{*}NG$ are g-inverses of $A^{*}MA$ and ANA^{*} respectively. Therefore the matrices $A(A^{*}MA)^{-}A^{*}$, $A(A^{*}M^{*}A)^{-}A^{*}$, $A^{*}(AN^{-1}A^{*})^{-}A$ and $A^{*}(AN^{*-1}A^{*})^{-}A$ are invariant under the choice of g-inverse. It follows that (2.6.6) and (2.6.7) over $\mathbf{D}$ are equivalent to (2.6.1) and (2.6.2) over the quotient field of $\mathbf{D}$ and (ii) satisfied by Theorem 2.6.1. Conversly, if (ii) is satisfied then it can be verified that G given in (2.6.8) is the Khatri-inverse. □

Analogous to the result proved in Theorem 2.5.8, in Theorem 2.6.4 also we can get that $u = Tr[C_r(N^{-1}A^{*}MA)]$ is invertible in $\mathbf{D}$.

Theorem 2.6.5. Let A be an $m \times n$ matrix of rank r over $\mathbf{D}$ and let $u = Tr[C_r(N^{-1}A^{*}MA)]$. If $G = (g_{ij})$, the Khatri-inverse of A with respect M,N exists, then

$$g_{ij} = \sum_{\alpha,\beta} u^{-1}|(N^{-1}A^{*}M)^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ji}}|A^{\alpha}_{\beta}| \qquad 1 \le i \le n,\ 1 \le j \le m \quad (2.6.9)$$

Proof. Let G be the Khatri-inverse of A with respect to M and N. Then it easily follows from multiplication property of the compound matrix that $C_r(G)$ is the Khatri-inverse of $C_r(A)$ with respect to $C_r(M)$ and $C_r(N)$. Since $C_r(A)$ is of rank one and since the Khatri-inverse is unique whenever it exists, we have from Corollary 2.6.2, which is valid over the quotient field of $\mathbb{D}$, that,

$$|G_{\alpha}^{\beta}| = u^{-1}|(N^{-1}A^{*}M)_{\alpha}^{\beta}|$$

for all subsets α, β of $\{1,...m\}$, $\{1,.....n\}$ respectively, it follows from Theorem 1.3.2 that G must be given by (2.6.9). Since u is invertible in $\mathbb{D}$, terms given in the expression (2.6.9) are well defined over $\mathbb{D}$. □

CHAPTER 3
GROUP AND DRAZIN INVERSES

3.1 Introduction.

It is well known that over the field of real numbers a square matrix A has a group inverse if and only if $\text{Rank}(A) = \text{Rank}(A^2)$ and that every matrix has a Drazin inverse (See [2] and [49]) In this chapter we shall investigate the problems of existence of group inverse and Drazin inverse for matrices over general integral domains.

The main results of this chapter consist of, for a square matrix over an integral domain,

(1) necessary and sufficient conditions for the existence of a group inverse,

(2) a new formula for finding a group inverse when it exists, and

(3) necessary and sufficient conditions for the existence of a Drazin inverse

For the existence of a group inverse we find necessary and sufficient conditions in terms of its $r \times r$ minors akin to the results of Chapters 1 and 2

We also generalize some results from Rao and Mitra [49] ch. 4 for matrices over integral domains Incidentally we give an interesing necessary and sufficient condition for $\text{Rank}\,(A) = \text{Rank}\,(A^2)$.

Now we shall recall some notation. For an $m \times n$ matrix A, A^- stands for a generalized inverse of A, $\mathcal{C}(A)$ stands for the module generated by columns of A and $\mathcal{R}(A)$ stands for the module generated by rows of A.

A^-_χ stands for a g-inverse of A with $\mathcal{C}(A^-_\chi) = \mathcal{C}(A)$ (equivalently $\mathcal{C}(A^-_\chi) \subset \mathcal{C}(A)$)

A^-_ρ stands for a g-inverse of A with $\mathcal{R}(A^-_\rho) = \mathcal{R}(A)$ (equivalently $\mathcal{R}(A^-_\rho) \subset \mathcal{R}(A)$)

$A^-_{\rho\chi}$ stands for a g-inverse of A with $\mathcal{C}(A^-_{\rho\chi}) = \mathcal{C}(A)$ and $\mathcal{R}(A^-_{\rho\chi}) = \mathcal{R}(A)$. (equivalently $\mathcal{C}(A^-_{\rho\chi}) \subset \mathcal{C}(A)$ and $\mathcal{R}(A^-_{\rho\chi}) \subset \mathcal{R}(A)$).

3.2. Existence of $A^-_{\rho\chi}$

We shall start this section by generalizing Lemma 4.1.1 given in [49]. Let $\mathbf{D}$ be an integral domain. We consider matrices over $\mathbf{D}$.

Lemma 3.2.1. Let A, P and Q be matrices over the integral domain $\mathbf{D}$. Then A has a g-inverse of the form PCQ for some C if and only if

(i) $\rho(QAP) = \rho(A)$

and (ii) QAP is regular

In which case C is a g-inverse of QAP. A g-inverse with the above properties is unique whenever $\rho(A) = \rho(P) = \rho(Q)$

Proof (only if part) : First note that the Cauchy-Binet formula gives us that $\rho(DE) \leq \min(\rho(D), \rho(E))$. Let PCQ for some C be a g-inverse of A. Then $A = A(PCQ)A = A(PCQ)A(PCQ)A$. So $\rho(A) \leq \rho(QAP)$. Again since $A = A(PCQ)A$, we have that $QAP = QAPCQAP$. So $\rho(QAP) \leq \rho(A)$. Thus we have (i) and (ii).

(if part) : Let C be a g-inverse of QAP. So $(QAP)C(QAP) = QAP$. Since $\rho(QAP) = \rho(A)$ we have that $\rho(A) = \rho(QA) = \rho(AP)$. If A and QA are considered as matrices over the field of quotients $\mathbf{F}$ of $\mathbf{D}$, $\rho(A) = \rho(QA)$ gives us a matrix D over $\mathbf{F}$ such that $A = DQA$. Similarly there exists a matrix E over $\mathbf{F}$ such that $A = APE$. Now $(QAP)C(QAP) = QAP$ gives us $APCQA = DQAPCQAPE = DQA = A$. So we are done.

A similar argument gives the uniqueness also (see the last part of the proof of Lemma 4.1.1. of [49]) □

Theorem 3.2.2. The following statements are equivalent for a square matrix A

(i) A^-_{χ} exists .

(ii) A^-_{ρ} exists

(iii) $A^-_{\rho\chi}$ exists

(iv) $\rho(A) = \rho(A^2)$ and A^2 is regular.

(v) $\rho(A) = \rho(A^2)$ and A^3 is regular

(vi) $\rho(A) = \rho(A^3)$ and A^3 is regular

Proof. $(i) \Rightarrow (iv)$ follows from Lemma 3.2.1 by taking $P = A$ and $Q = I$.

$(iv) \Rightarrow (v)$ Let us verify that $(A^2)^- A (A^2)^-$ is a g-inverse of A^3. Since $\rho(A) = \rho(A^2)$, there exists a matrix E over the quotient field of $\mathbf{D}$ such that $A = A^2E$. So

$$A^3(A^2)^- A(A^2)^- A^3 = AA^2(A^2)^- A^2E(A^2)^- A^3$$

$$= AA^2E(A^2)^-A^3$$

$$= A^2(A^2)^-A^3$$

$$= A^3.$$

So $(A^2)^-A(A^2)^-$ is a g-inverse of A^3.

$(v)\Rightarrow(vi)$ is clear.

$(vi)\Rightarrow(iii)$ follows from Lemma 3.2.1. In fact $A^-_{\rho\chi} = A(A^3)^-A$.

$(iii)\Rightarrow(i)$ is trivial.

$(ii)\Rightarrow(iv)\Rightarrow(v)\Rightarrow(vi)\Rightarrow(iii)\Rightarrow(ii)$ hold by similar arguments □

Remark. The two concepts $A^\#$ and $A^-_{\rho\chi}$ are identical and $A^\#$(so, $A^-_{\rho\chi}$) is unique. For, firstly, that $A^\#$ is an $A^-_{\rho\chi}$ follows because

$$A^\# = A^\#AA^\#$$

$$=A^\#AA^\#AA^\#$$

$$=AA^{\#3}A$$

So $A^\#$ is an $A^-_{\rho\chi}$

Secondly $A^-_{\rho\chi}$ is an $A^\#$ follows because, as observed in the proof of $(vi)\Rightarrow(iii)$ it is enough to verify the equations (1),(2) and (5) for $G = A(A^3)^-A$ Equation (1) is clear by the definition of a g- inverse. For equation (2),if E is a matrix over the field of quotients such that $A = A^3E$, then

$$GAG = A(A^3)^-A^3(A^3)^-A^3E$$

$$= A(A^3)^-A^3E$$

$$= A(A^3)^-A$$

$$= G$$

Equation (5), i.e., $AG = GA$ also follows similarly. The uniqueness of $A^{\#}$ is easily proven from its definition.

Remark. From the previous remark, the existence of $A^{\#}$ is equivalent to all the six statements of Theorem 3.2.2. Also, since trivially existence of a commuting g-inverse of A is equivalent to the existence of $A^{\#}$, the six statements of Theorem 3.2.2 are equivalent to the existence of a commuting g-inverse of A.

Remark. $A^{-}_{\rho\chi} = A^{-}_{\chi} A A^{-}_{\rho}$ when $\rho(A) = \rho(A^2)$, and A^2 is regular.

Remark. More genarally, for a regular matrix A, there is a g-inverse of the form PCQ for some C if and only if there are g-inverses G_1 and G_2 of the form PD and EQ respectively. In fact $G_1 A G_2$ serves our purpose.

3.3. Existence of the group inverse of A in terms of its minors.

In Theorem 1.2.2 we saw that a matrix A of rank r over $\mathbf{D}$ is regular if and only if a linear combination of all the $r \times r$ minors is one. In chapter 2 we showed that a matrix of rank r over $\mathbf{D}$ has a Moore-Penrose inverse if and only if a particular linear combination of all the $r \times r$ minors is one). The aim of this section is to give a similar condition for the existence of $A^{\#}$. We shall show that $A^{\#}$ exists if and only if $\sum_{\alpha} u|A^{\alpha}_{\alpha}| = 1$ for some u.

First we shall prove the condition for matrices of rank 1.

Lemma 3.3.1. If A is a square matrix of rank 1 over an integral domain $\mathbb{D}$, then A has a group inverse if and only if the trace of A ($Tr(A)$ for short) is invertible in $\mathbb{D}$. In this case the group inverse $A^{\#} = (Tr(A)^{-2}A)$.

Proof. Let A be a matrix of rank 1 over $\mathbb{D}$. Over the field of quotients we can write $A = xy^T$ where x and y are $n \times 1$ matrices over the field. Note that y^Tx is the trace of A.

(if part) : Suppose $Tr(A)$ is invertible in $\mathbb{D}$. Then we shall prove that $G = (Tr(A))^{-2}A$ is the group inverse of A :

$$\begin{aligned} AGA &= A(Tr(A))^{-2}AA \\ &= xy^T(y^Tx)^{-2}xy^Txy^T \\ &= xy^T = A \end{aligned}$$

Similarly we can prove that $GAG = G$ and $AG = GA$. So $G = (Tr(A))^{-2}A$ is the group inverse of A

(only if part) : Suppose that A has a group inverse. Then $\rho(A) = \rho(A^2) = 1$ and A^2 is regular (by Lemma 3.2.2) If $B = A^2$, then the (i,j)-th element of B is

$$b_{ij} = \sum_k a_{ik}\, a_{kj} \qquad (3.3.1)$$

Since B is regular and $\rho(B) = 1$, there exists $g_{ji} \in \mathbb{D}$ such that

$$\sum_{i,j} g_{ji}\, b_{ij} = 1 \qquad (3.3.2)$$

Substituting (3.3.1) in (3.3.2) we get

$$\sum_{i,j,k} g_{ji}\, a_{ik}\, a_{kj} = 1$$

Since $\rho(A) = 1$, we have $a_{ik}a_{kj} = a_{kk}a_{ij}$.

So,
$$\left(\sum_k a_{kk}\right)\left(\sum_{i,j} g_{ji}\, a_{ij}\right) = 1. \qquad (3.3.3)$$

(3.3.3) now implies that $\sum_k a_{kk} = Tr(A)$ is invertible in $\mathbb{D}$. □

Thereom 3.3.2 : Let A be an $n \times n$ matrix of rank r over an integral domain $\mathbb{D}$. Then $\rho(A) = \rho(A^2)$ and A^2 is regular if and only if $\sum_\gamma |A^\gamma_\gamma|$, where γ runs over all r -element subsets of $\{1,2,\ldots\ldots.n\}$ is invertible in $\mathbb{D}$.

Proof. (only if part) : Let $\rho(A) = \rho(A^2) = r$ and A^2 be regular. So $\rho([C_r(A)]^2) = \rho(C_r(A^2)) = 1$ and $C_r(A^2)$ is regular. From the only if part of Lemma 3.3.1 we get that $Tr(C_r(A)) = \sum_\gamma |A^\gamma_\gamma|$ is invertible in $\mathbb{D}$.

(if part) : Let $\sum |A^\gamma_\gamma|$ be invertible in $\mathbb{D}$. First we shall prove that $\rho(A) = \rho(A^2) = r$. Suppose $\rho(A) \neq \rho(A^2)$, Then $\rho(A^2) < r$ (since $\rho(A) = r$) , and

$$|(A^2)^\alpha_\beta| = 0 \qquad (3.3.4)$$

for all r -element subsets α , β of $\{1,2 \ldots\ldots.n\}$.

But,

$$|(A^2)^\alpha_\beta| = \sum_\gamma |A^\alpha_\gamma||A^\gamma_\beta|$$

$$= \sum_\gamma |A^\gamma_\gamma||A^\alpha_\beta| \text{ , since } \rho(C_r(A)) = 1$$

$$= |A_\beta^\alpha| \sum_\gamma | A_\gamma^\gamma|. \qquad (3.3.5)$$

Since $\sum_\gamma | A_\gamma^\gamma|$ is invertible in $\mathbf{D}$, from (3.3.4) and (3.3.5) we get $| A_\beta^\alpha| = 0$ for all r -element subsets α and β. This contradicts the fact that $\rho(A) = r$. So we must have $\rho(A) = \rho(A^2) = r$.

Now it remains to prove that A^2 is regular. Since $\sum_\gamma |A_\gamma^\gamma| = u$ is invertible in $\mathbf{D}$, we have,

$$u^{-2}\Big(\sum_\alpha |A_\alpha^\alpha|\Big)\Big(\sum_\beta | A_\beta^\beta|\Big) = 1;$$

i.e.

$$\sum_{\alpha,\beta} u^{-2}|A_\alpha^\alpha||A_\beta^\beta| = 1,$$

and

$$\sum_{\alpha,\beta} u^{-2}|A_\alpha^\beta||A_\beta^\alpha| = 1 \text{ , because } \rho(C_r(A)) = 1. \qquad (3.3.6)$$

By the Cauchy -Binet formula we have

$$|(A^2)_\alpha^\alpha| = \sum_\beta |A_\beta^\alpha||A_\alpha^\beta| \qquad (3.3.7)$$

By substituting (3.3.7) into (3.3.6) we get

$$\sum_\alpha u^{-2}|(A^2)_\alpha^\alpha| = 1 \qquad (3.3.8)$$

So, from Theorem 1.2.2. by taking $c_\alpha^\beta = 0$ for $\alpha \neq \beta$ and u^{-2} for $\alpha = \beta$, it is clear that A^2 is regular. Hence we have proved the theorem. □

Theorem 3.3.3. Let A be an $n \times n$ matrix over $\mathbf{D}$ such that $\rho(A) = r$. Then the following are equivalent

(i) A has a group inverse.

(ii) $C_r(A)$ has a group inverse.

(iii) $\sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible in $\mathbf{D}$.

(iv) $\rho(A) = \rho(A^2)$ and A^2 is regular.

Proof. (i) ⇒ (ii) is trivial from the properties of compound matrices. (ii)⇒ (iii) follows from Lemma 3.3.1, (iii)⇒(iv) is a part of Theorem 3.3.2, (iv) ⇒(i) follows from Lemma 3.2.2.

□

Corollary 3.3.4. Let A be an $n \times n$ matrix of rank r over $\mathbf{D}$. Then the following are equivalent

(i) A has Moore-Penrose inverse

(ii) (A^*A) has group inverse and $\rho(A^*A) = \rho(A)$.

(iii) (AA^*) has group inverse and $\rho(AA^*) = \rho(A)$

Proof. (i) ⇒ (ii): Suppose that A has Moore-Penrose inverse, then by Theorem 2.2.6 we get $\sum_{\alpha,\beta} |\bar{A}_{\beta}^{\alpha}||A_{\beta}^{\alpha}|$ is invertible in $\mathbf{D}$, which gives $\rho(A^*A) = \rho(A)$. But,

$$\sum_{\alpha,\beta} |\bar{A}_{\beta}^{\alpha}||A_{\beta}^{\alpha}| = \sum_{\alpha,\beta} |A^{*\beta}_{\alpha}||A_{\beta}^{\alpha}|$$

$$= \sum_{\beta} |A^*A_{\beta}^{\beta}|.$$

So from (iii)⇒ (i) of Theorem 3.3.3 we get A^*A has group inverse.

(ii) ⇒ (i) By (i) ⇒ (iii) of the Theorem 3.3.3 we get

$$\sum_{\beta}|(A^{*}A)^{\beta}_{\beta}| = \sum_{\alpha,\beta}|\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$$

is invertible which implies that A has Moore-Penrose inverse. $(i)\Leftrightarrow(iii)$ is similar.

□

We know that over a field the group inverse of a matrix A, whenever it exists can be written as a polynomial in A with coefficients from the field (See [2] and [49]). We shall prove this result in case of integral domains also.

Theorem 3.3.5. Let A be a square matrix of order n over $\mathbb{D}$ for which $A^{\#}$ exists over $\mathbb{D}$. Then $A^{\#}$ is a polynomial in A with coefficients from $\mathbb{D}$.

Proof. Let the characteristic polynomial of A be

$$|\lambda I - A| = p_r\lambda^{n-r}+p_{r-1}\lambda^{n-r+1}+\ldots\ldots+\lambda^{n}$$

where r is the rank of A and $(-1)^{k}p_k$ is the sum of all the principal minors of order k. Observe that $(-1)^{r}p_r$ is the sum of all the $r\times r$ principal minors of A, which is invertible in $\mathbb{D}$ (by our Theorem 3.3.3 above). Now by the Cayley-Hamilton Theorem,

$$p_rA^{n-r}+\ \ldots\ldots+A^{n} = 0$$

So

$$A^{n-r}= a_{r-1}A^{n-r+1}+a_{r-2}A^{n-r+2}+\ldots\ldots\ldots+a_oA^{n} \qquad (3.3.9)$$

where $a_k= \frac{-p_k}{p_r}$ for $0< k \le r-1$ and $a_o= \frac{-1}{p_r}$. (Observe that $a_{r-1}, \ldots, a_o$ are elements of $\mathbb{D}$.)

Multiplying both sides of (3.3.9) by $(A^{\#})^{n-r+1}$ we get

$$A^{\#} = a_{r-1}AA^{\#}+a_{r-2}A+\ldots\ldots\ldots+a_0A^{n-r-1}, \qquad (3.3.10)$$

and multiplying both sides of 3.3.10 by A, we get

$$A^{\#}A = a_{r-1}A+a_{r-2}A^2+\dots\dots\dots+a_0A^{n-r}. \tag{3.3.11}$$

Substituting (3.3.11) in (3.3.10) gives us

$$A^{\#}= (a^2{}_{r-1}+a_{r-2})A+(a_{r-1}a_{r-2}+a_{r-3})A^2+\dots\dots\dots+a_0A^{n-r}$$

and this is a polynomial in A over $\mathbf{D}$ □

Incidentally, from the proof of the equivalence of (*iii*) and (*iv*) of Theorem 3.3.3 we obtain the following remarkable condition for $\rho(A)$ to be equal to $\rho(A^2)$.

Theorem 3.3.6. Let A be a square matrix of rank r over $\mathbf{D}$. Then $\rho(A) = \rho(A^2) = r$ if and only if the sum of all the $r \times r$ principal minors of A is nonzero

Proof. We observe that for any α and β (r-element subsets of $\{1,2,\dots\dots,n\}$)

$$|A^{2\alpha}_{\beta}| = \left(\sum_{\gamma}|A^{\gamma}_{\gamma}|\right) \, | \, A^{\alpha}_{\beta}| \, , \tag{3.3.12}$$

where γ runs over all r-element subsets of $\{1,2,\dots\dots\dots n\}$. Since $\mathbf{D}$ is an integral domain, we get that

$$\rho(A) = \rho(A^2) \text{ if and only if } \sum_{\gamma}|A^{\gamma}_{\gamma}| \neq 0.$$ □

3.4. New formulae

We have seen in the previous section that if $\sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible in $\mathbf{D}$, then $A^{\#}$ exists We shall give in this section a method of finding $A^{\#}$ whenever it exists

First of all, observe that from second remark at the end of Section 2, it follows that A has a commuting g-inverse if and only if $\sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible.

Thereom 3.4.1. Let A be a matrix of rank r over $\mathbf{D}$. Then

(i) If $u = \sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible in $\mathbf{D}$, then $G = (g_{ij})$ defined by

$$g_{ji} = \sum_{\gamma} u^{-1} \frac{\partial}{\partial a_{ij}} |A_{\gamma}^{\gamma}|$$

is a commuting g-inverse of A.

(ii) If $u = \sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible in $\mathbf{D}$, then $G = (g_{ij})$ where

$$g_{ji} = \sum_{\alpha,\beta} u^{-2} |A_{\alpha}^{\beta}| \frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}|$$

is the group inverse of A.

Proof (i) : First we shall prove that $G = (g_{ij})$ obtained from the formula

$$g_{ji} = \sum_{\gamma} u^{-1} \frac{\partial}{\partial a_{ij}} |A_{\gamma}^{\gamma}| \qquad (3.4.1)$$

is a commuting g-inverse of A Note that G is a g-inverse of A over $\mathbf{D}$ (by (1 2 4) and (1 2 9) , taking $c_{\alpha}^{\beta} = 0$ for $\alpha \neq \beta$ and $c_{\alpha}^{\alpha} = u^{-1}$)

Now we shall prove that G commutes with A.

i.e. $$(AG)_{ij} = (GA)_{ij} \text{ for all } i, j . \tag{3.4.2}$$

For $i = j$,

$$(AG)_{ii} = \sum_k a_{ik} g_{ki} \qquad 1 \le k \le n$$

$$= \sum_k a_{ik} \sum_{\gamma : i \in \gamma} u^{-1} \frac{\partial}{\partial a_{ik}} |A_\gamma^\gamma|$$

$$= \Big(\sum_{\gamma : i \in \gamma} |A_\gamma^\gamma| \Big) u^{-1} \tag{3.4.3}$$

Similarly we get $(GA)_{ii} = \Big(\sum_{\gamma : i \in \gamma} |A_\gamma^\gamma| \Big) u^{-1}$. So $(GA)_{ii} = (AG)_{ii}$.

For $i \ne j$

$$(AG)_{ij} = \sum_k a_{ik} g_{kj}$$

$$= \sum_k a_{ik} \sum_{\gamma : j \in \gamma} u^{-1} \frac{\partial}{\partial a_{jk}} |A_\gamma^\gamma|$$

$$= \sum_{\gamma : j \in \gamma} u^{-1} \sum_k a_{ik} \frac{\partial}{\partial a_{jk}} |A_\gamma^\gamma|$$

$$= \sum_{\gamma : j \in \gamma, i \notin \gamma} |A_\gamma^{\gamma \setminus (j) \cup (i)}| \, u^{-1}$$

because for $i \in \gamma$ $\quad \sum_k a_{ik} \frac{\partial}{\partial a_{jk}} |A_\gamma^\gamma| = 0$.

So,

$$(AG)_{ij} = u^{-1} \sum_{\alpha, \beta \; i \in \alpha \; j \notin \alpha \; j \in \beta \; i \notin \beta} |A_\beta^\alpha| \tag{3.4.5}$$

Similarly we get

$$(AG)_{ij} = u^{-1} \sum_{\gamma : i \in \gamma, j \notin \gamma} |A_{\gamma \setminus (i) \cup (j)}^\gamma|$$

$$= u^{-1} \sum_{\alpha,\beta \,:\, i\in\alpha, j\notin\alpha, i\notin\beta, j\in\beta} |A^{\alpha}_{\beta}|$$

$$= (AG)_{ij}$$

hence G commutes with A

Now we shall prove part (ii) of the theorem. Since $u = \sum_{\gamma} |A^{\gamma}_{\gamma}|$ is invertible in $\mathbf{D}$,

$$\Big(\sum_{\gamma} u^{-1}|A^{\gamma}_{\gamma}|\Big)^{2} = u^{-2}\sum_{\alpha,\beta} |A^{\alpha}_{\alpha}|\ |A^{\beta}_{\beta}| = 1$$

Since $\rho(C_r(A)) = 1$,

$$|A^{\alpha}_{\alpha}|\ |A^{\beta}_{\beta}| = |A^{\alpha}_{\beta}|\ |A^{\beta}_{\alpha}|$$

and hence

$$\sum_{\alpha,\beta} u^{-2}|A^{\alpha}_{\beta}|\ |A^{\beta}_{\alpha}| = 1$$

We claim that the matrix $G = (g_{ij})$ obtained from the formula

$$g_{ji} = \sum_{\alpha,\beta} u^{-2}|A^{\beta}_{\alpha}|\frac{\partial}{\partial a_{ij}}\ |A^{\alpha}_{\beta}|$$

is the group inverse of A.

Note that $C_r(A^{\#})$ is the group inverse of $C_r(A)$. But, by lemma 3.3.1, we get

$$C_r(A)^{\#} = [Tr(C_r(A))]^{-2}C_r(A)$$
$$= u^{-2}C_r(A)$$

Therefore $|A^{\#\beta}_{\ \alpha}| = u^{-2}|A^{\beta}_{\alpha}|$.

Since $A^{\#}$ is a refexive g-inverse of A, by (1 3 2) we get

$$(A^{\#})_{ij} = \sum_{\alpha,\beta} u^{-2}|A^{\#\beta}_{\alpha}|\frac{\partial}{\partial a_{ij}}|A^{\alpha}_{\beta}|$$

$$= \sum_{\alpha,\beta} u^{-2}|A^{\beta}_{\alpha}|\frac{\partial}{\partial a_{ij}}|A^{\alpha}_{\beta}|$$

$$= \sum_{\alpha,\beta} u^{-2}|A^{\beta}_{\alpha}|\frac{\partial}{\partial a_{ij}}|A^{\alpha}_{\beta}|$$

So we get $G = A^{\#}$. Hence the proof. □

Remark. Theorem 3.4.1. provides a direct proof of *(iii)* ⇒ *(iv)* of Theorem 3.3.3

5. Drazin inverse

In this section we shall give necessary and sufficient conditions for a square matrix over $\mathbf{D}$ to have a Drazin inverse over $\mathbf{D}$.

Theorem 3.5.1. Let A be a matrix over $\mathbf{D}$. Then A has a Drazin inverse over $\mathbf{D}$ (satisfying (2) (5) and (1^k)) if and only if for that k, $\rho(A^k) = \rho(A^{k+1})$ and A^{2k+1} is regular. Also the Drazin inverse when it exists is unique.

Proof. (only if part) : Let A have a Drazin inverse, say G, over $\mathbf{D}$. Condition (1^k) gives us that $\rho(A^k) = \rho(A^{k+1})$ and also,

$$A^{2k+1} = A^{2k+2}G$$

$$= A^{2k+1}GA \text{ (from condition (5))}$$

$$= A^{2k+1}G^{2k+1}A^{2k+1} \text{ (from condition (2))}$$

So A^{2k+1} is regular.

(if part) : Let k. be a positive integer for which $\rho(A^k) = \rho(A^{k+1})$ and A^{2k+1} is regular. We shall prove that $G = A^k(A^{2k+1})^- A^k$ is a Drazin inverse of A.

Since $\rho(A^k) = \rho(A^{k+j})$ for all positive integers j, there exist matrices D, E and F over the field of quotients of $\mathbf{D}$ such that

$$A^{k+1} = DA^{2k+1}$$

$$A^k = EA^{2k+1}$$

$$A^k = A^{2k+1}F$$

So

$$AG = AA^k(A^{2k+1})^- A^k$$

$$= D(A^{2k+1})F = A^{k+1}F$$

Similarly

$$GA = A^k(A^{2k+1})^- A^k A$$

$$= A^k(A^{2k+1})^- AA^k$$

$$= A^{k+1}F.$$

Hence $AG = GA$., i.e. (5) holds.

$$A^{k+1}G = A^{k+1}A^k(A^{2k+1})^- A^k$$

$$= A^{2k+1}(A^{2k+1})^- A^{2k+1}F$$

$$= A^{2k+1}F$$

$$= A^k.$$

Hence (1^k)

$$G^2A = G(GA) = (A^k(A^{2k+1})^- A^k)A^{k+1}F \text{ , (since } GA = A^{k+1}F)$$

$$= A^k(A^{2k+1})A^k = G.$$

Hence (2). Thus G is a Drazin inverse of A.

Now we shall prove that the Drazin inverse is unique when it exists. First observe that if G satisfies (1^k) then G satisfies (1^m) for all $m \geq k$. If F and G are two Drazin inverses of A we can choose a k such that F and G both satisfy the conditions (2),(5) and (1^k). By repeated applications of (5) and (1^k) we get

$$G^{k+1}A^{2k+1}F^{k+1} = G^{k+1}A^k = G.$$

and

$$G^{k+1}A^{2k+1}F^{k+1} = A^{k+1}F^{k+1} = F.$$

So $F = G$

□

Remark. Let us observe that if A has Drazin inverse over $\mathbb{D}$ and if the index of A is p, then $\rho(A^p) = \rho(A^{p+1})$ and that A^{2p+1} is regular. If A has a Drazin inverse H over $\mathbb{D}$, then considering A as a matrix over the field of quotients of $\mathbb{D}$, A^{2p+1} has a g-inverse over this field. So A has a Drazin inverse G over this field and G satisfies (2),(5) and (1^p). By the uniqueness of Drazin inverse over the field we have then $G = H$. So H satisfies (2) ,(5) and (1^p). Theorem 3.5.1. gives our statement. Also we have the following result: A has Drazin inverse over the integral domain $\mathbb{D}$ if and only if A^{2p+1} is regular, where p is the index of A

Remark. Over an integral domain $\mathbb{D}$, for a given matrix A over $\mathbb{D}$ there need not exist an integer k such that $\rho(A^k) = \rho(A^{k+1})$ and A^{2k+1} is reglar over $\mathbb{D}$

For example , Take $\mathbf{D} = \mathbf{Z}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\rho(A) = \rho(A^k) = 1$ for all positive integers k. But A^k is not regular for $k \geq 2$.

Remark. Note that A has a Drazin inverse with index p if and only if A^p has group inverse (It is easy to verify that G^p is a group inverse of A^p, and in fact p is the smallest positive integer for which A^p has a group inverse). Conversely, if A^p has a group inverse, then A^{3p} is regular with $\rho(A) = \rho(A^{3p})$ which implies that A^{2p+1} is regular

Now we shall prove the following theorem

Theorem 3.5.2 Let A be a matrix over $\mathbf{D}$ with index p and $\rho(A^p) = s$. Then the following are equivalent.

(*i*) A has a Drazin inverse

(*ii*) $C_s(A)$ has a Drazin inverse

(*iii*) $C_s(A^p)$ has a group inverse

(*iv*) $Tr(C_s(A^p))$ is invertible over $\mathbf{D}$ and A^{2p} is regular.

(*v*) A^p has a group inverse.

(*vi*) A^{p+n} is regular for all positive integers n.

(*vii*) A^{2p} is regular

Proof. : (*i*)$\Rightarrow$(*ii*) follows from the properties of $C_s(A)$.

(*ii*) $\Rightarrow$(*iii*) Since $C_s(A)$ has a Drazin inverse with index $\leq$ k, $C_s(A^k)$ has a group inverse (from the last remark following Theorem 3.5.2).

(*iii*) $\Rightarrow$ (*iv*) is trivial by Lemma 3 3 1

$(iv) \Rightarrow (v)$ holds from Theorem 3.3.2.

$(v) \Rightarrow (vi)$ If n is a positive integer, choosing m such that $n \leq (m-1)p$, we have $\rho(A^p) = \rho(A^{mp}) = \rho(A^{p+n})$ and since A^{mp} is regular, $A^{k+n}$is regular.

$(vi) \Rightarrow (vii)$ is obvious

$(vii) \Rightarrow (i)$ holds from first remark following Theorem 3.5.1. □

It is known that over a field every matrix has a decomposition (See [2], ch 4 and [49], ch 4)of the form

$A = A_1+A_2$ with the properties

(i) $\rho(A_1)=\rho(A_1{}^2)$

(ii) A_2 is nilpotent.

and

(iii) $A_1A_2 = A_2A_1 = 0$

We shall now investigate whether over an integral domain also such a decomposition as above (or similar) exists for every matrix.

Observe that over a field condition (i) is equivalent to

(i') A_1 has a group inverse.

In the following Theorem we shall give a necessary and sufficient condition for a square matrix over an integral domain to have a decomposition satisfying the properties (i') , (ii) and (iii).

Theorem 3.5.3 A square matrix A over **D** has a decomposition $A = A_1+A_2$ satisfying (i') (ii) and (iii) if and only if A has a Drazin inverse. Such a decomposition is unique.

Proof. (if part) If A has a Drazin inverse K over $\mathbf{D}$, then, by defining $A_1 = AKA = K^{\#}$ and $A_2 = A - A_1$, one can check as in the proof of Theorem 10, ch 4 of [2] that A_1 and A_2 satisfy *(i')* *(ii)* and *(iii)*.

(only if part) : Suppose that A has a decomposition of the form $A = A_1 + A_2$ with *(i')* *(ii)* and *(iii)*. Then there is a positive integer m such that $A_2^m = 0$ For this m, $A^m = A_1^m$ Since $A_1^m = A^m$, the index of $A = d \leq m$. Since A_1 has a group inverse, $A_1^m = A^m$ has group inverse Since some power of A has a group inverse, A has a Drazin inverse.

Uniqueness of the decomposition follows as in the real case. □

CHAPTER 4
BORDERING, RANK FACTORIZATION AND SERRE'S CONJECTURE

4.1. Intoduction

Kentaro Nomakuchi in [28] presented a characterization of Generalized inverses of matrices over the field of complex numbers using Bordered matrices. Specifically, Nomakuchi showed that if A is an $m \times n$ matrix of rank r over the field of complex numbers, there exists an invertible matrix

$T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix}$ of size $(m+n-r) \times (m+n-r)$ where P and Q are matrices of size $m \times (m-r)$ and $(n-r) \times n$ respectively. Nomakuchi in fact showed that all g-inverses of A can be obtained by looking at the inverses of matrices T in

$$\mathfrak{B}(A) = \left\{ T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix} \middle/ P, Q \text{ are matrices of size } m \times (m-r) \text{ and } (n-r) \times n \text{ respeoctively , and } T \text{ is invertible} \right\} \quad (4.1.1)$$

The above results hold good even for matrices over any field. But, over an arbitrary ring it may not be possible to find a bordered matrix of the above kind for every matrix, as the next example shows. From our Theorem 4.2.4, it follows that even for a regular matrix over an arbitrary integral domain it may not be possible to find a bordered matrix of the above kind.

Example. Consider the matrix $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$ over the ring of integers $\mathbf{Z}$. This is a 2 x 2 matrix of rank 1. For this A there is no bordered matrix

$$T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix},$$

where T is an invertible 3 x 3 matrix over $\mathbf{Z}$, because $|T|$ is divisible by 2 whatever be P, Q and R. Hence $\mathfrak{B}(A) = \emptyset$ over the ring of integers.

In Theorem 4.2.4 we shall give necessary and sufficient conditions on an integral domain $\mathbf{D}$ so that every regular matrix over $\mathbf{D}$ has a bordered matrix of the above type. Towards the end of this section using our results and Quillen and Suslin theorem we shall show that $\mathbf{F}[X_1, X_2, \ldots\ldots\ldots]$ is projective free (An integral domain is said to be projective free if every finitely generated projective module is free) for any principal ideal domain $\mathbf{P}$, thus extending Quillen-Suslin result to infinitely many variables.

4.2. Bordering and g-inverses

In this section we shall give necessary and sufficient conditions on an integral domain $\mathbf{D}$ so that every regular matrix has nonempty $\mathfrak{B}(A)$. If $T \in \mathfrak{B}(A)$ we can find a g-inverse of A as shown in the following theorem

Theorem 4.2.1 Let A be an m x n matrix of rank "r".

Let $T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix} \in \mathfrak{B}(A)$ and $T^{-1} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ where G_{11}, G_{12}, G_{21}, and G_{22} are matrices corresponding to the partition in T. Then G_{11} is a g-inverse of A

Proof: Proof of this theorem can be borrowed directly from the proof given by Kentaro Nomakuchi in [28] by considering T over the field of quotients of $\mathbb{D}$ □

Now we shall prove a lemma which is useful in proving the main theorem of this chapter

Lemma 4.2.2. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$ and suppose that P is an $m \times m\text{-}r$ matrix such that $T = [\, A, \quad P \,]$ has a right inverse. Then A is regular, and P has a left inverse P_L^{-1} such that $P_L^{-1}A = 0$ and $P_L^{-1}P = I_{m-r}$.

Proof. Suppose T has a right inverse. Then there exists a linear combination $\sum_{\alpha} |T_{\alpha}^{m}|\, c^{\alpha}$ of $m \times m$ minors of T which equls one.

i.e.,

$$\sum_{\alpha} |T_{\alpha}^{m}|\, c^{\alpha} = 1 \qquad (4.2.1)$$

Since $\rho(A) = r$, $\rho([\, A, \quad P \,]) = m$ and A, P are of size $m \times n$, $m \times m\text{-}r$ respectively, we get $\rho(P)$ is $m\text{-}r$, also, $|T_{\alpha}^{m}|$ could be nonzero only if α contains the indices $n+1,\ n+2, \ldots, m+n-r$. Let $\alpha' = \alpha \setminus \{n+1,\ n+2, \ldots, m+n-r\}$ whenever $|T_{\alpha}^{m}|$ is nonzero. Then

$$|T^m_\alpha| = \sum_\gamma sgn(\gamma)|P^\gamma_{m-r}||A^{\gamma^c}_{\alpha'}|$$

where $\gamma = (\gamma_1, \gamma_2, \ldots \gamma_{m-r})$ an $(m-r)$-elements subset of $(1,2,\ldots\ldots,m)$,

$sgn(\gamma) = (-1)^{\sum_{i=1}^{m-r}\gamma_i + (n+i)}$ and $\gamma^c = (1,2,\ldots m)\setminus\gamma$ (by Laplace expansion). Hence by considering only the nonzero $|T^m_\alpha|$, (4.2.1) can be rewritten as

$$\sum_\alpha \Big(\sum_\gamma sgn(\gamma)|P^\gamma_{m-r}||A^{\gamma^c}_{\alpha'}|\Big)\, c^\alpha = 1 \qquad (4.2.2)$$

So,

$$\sum_\gamma \Big(\sum_\alpha sgn(\gamma)|A^{\gamma^c}_{\alpha'}|\, c^\alpha\Big)|P^\gamma_{m-r}| = 1 \qquad (4.2.3)$$

and the matrix obtained P_L^{-1} by

$$(P_L^{-1})_{ij} = \sum_\gamma \Big(\sum_\alpha sgn(\gamma)|A^{\gamma^c}_{\alpha'}|\, c^\alpha\Big)\frac{\partial}{\partial p_{ji}}|P^\gamma_{m-r}|. \qquad (4.2.4)$$

$$= \sum_\alpha c^\alpha \frac{\partial}{\partial t_{j,n+i}}|T^m_\alpha| \qquad (4.2.5)$$

Clearly, P_L^{-1} obtained by (4.2.4) is a left inverse of P. Since the matrix T^* obtained by replacing $(n+i)$–th column by k-th column of A is of rank strictly less than m, we get

$$(P_L^{-1}A)_{ik} = \sum_j \Big(\sum_\alpha c^\alpha \frac{\partial}{\partial t_{j,n+i}}|T^m_\alpha|\Big)a_{jk} = \sum_\alpha |T^{*m}_\alpha|\, c^\alpha = 0$$

i.e., $P_L^{-1}A = 0$. Since the left hand side in (4.2.2) is a linear cobination of $r \times r$ minors of A, we get that A is regular. Hence the proof. □

Lemma 4.2.3. Let A be an $m \times n$ matrix of rank r over $\mathbb{D}$ and let Q be an n-$r \times n$ matrix such that $T = \begin{bmatrix} A \\ Q \end{bmatrix}$ has left inverse. Then A is regular and

Q has a right inverse Q_R^{-1} such that $AQ_R^{-1} = 0$ and $QQ_R^{-1} = I_{n-r}$.

Proof is similar to Lemma 4.2.2.

In the following theorem we shall characterize integral domains over which every regular matrix A has nonempty $\mathfrak{B}(A)$

Theorem 4.2.4. Let $\mathbb{D}$ be an integral domain . Then the following are equivalent

(i) Every finitely generated projective module over $\mathbb{D}$ is free.

(ii) Every regular matrix has a rank factorization. (See p.6 for defn.)

(iii) For every regular matrix A , $\mathfrak{B}(A) \neq \emptyset$

Remark. *(i)* above is the statement of Serre's conjucture for integral domains

Proof. *(i)* $\Rightarrow$ *(ii)* : Let every finitely generated projective module over $\mathbb{D}$ be free. Let A be an $m \times n$ regular matrix of rank k. Consider A as a module homomorphism from $\mathbb{D}^n$ into $\mathbb{D}^m$. Since A is regular there exists a matrix $G : \mathbb{D}^m \rightarrow \mathbb{D}^n$ such that $AGA = A$. We observe that AG is an idempotent linear map on $\mathbb{D}^m$ into $\mathbb{D}^m$ and $Range(A) = Range(AG)$ (= S, say). Now observe that for any idempotent linear map $T : \mathbb{D}^m \rightarrow \mathbb{D}^m$, $Range(T)$ is projective. So we get that S is projective and by the hypothesis it is free. Suppose that S is isomorphic to $\mathbb{D}^l$ for some integer l through an isomorphism $\phi : S \rightarrow \mathbb{D}^l$. Let $C = \phi A$ and $B = i\phi^{-1}$, where $i : S \rightarrow \mathbb{D}^m$ is the inclusion map. Observe that $A = BC$, where B is an $m \times l$

matrix and C is an $l \times n$ matrix. Now we shall prove that $k= l$.

Since S is isomorphic to $\mathbf{D}^l$, $C : \mathbf{D}^n \to \mathbf{D}^l$ is onto. Since S is a direct summand of $\mathbf{D}^n$, there exists a matrix $C' : \mathbf{D}^l \to \mathbf{D}^n$ such that CC' is identity. Since B is an injective map and S is a direct summand of $\mathbf{D}^n$, there is a matrix B' such that $B'B$ is identity. Now, since $CC'=I_l$ and $B'B = I_l$, also $A = BC$, we get $k = l$. Hence $A = BC$ is a rank factorization.

$(ii) \Rightarrow (iii)$ Suppose every regular matrix has a rank factorization. We shall prove that for every regular matrix A, $\mathfrak{B}(A)$ is nonempty. Let A be an $m \times n$ matrix of rank r and G be a reflexive g-inverse of A. Then $I_m - AG$ and $I_n - GA$ are idempotent matrices and so, they are regular of rank $m-r$ and $n-r$ respectively (In fact, over an integral domain if an indempotent matrix has a rank facorization then $\rho(A) = trace(A)$. More generally, over a commutative ring if an idempotent marix A has a rank factorization BC such that B has a left inverse and C has a right inverse then $\rho(A) = trace(A)$). Since every regular matrix over $\mathbf{D}$ has rank factorization, $I_m - AG$ and $I_n - GA$ have rank factorizations. Let

$$I_m - AG = B_{m \times m-r} C_{m-r \times m} \tag{4.2.6}$$

and

$$I_n - GA = P_{n \times n-r} Q_{n-r \times n} \tag{4.2.7}$$

be rank factorizations. Since $I_m - AG$ and $I_n - GA$ areidempotent matrices, we get that $CB = I_{m-r}$ and $QP = I_{n-r}$. Using (4.2.6) and (4.2.7) we also get that $CA = 0$, $GB = 0$, $AP = 0$ and $QG = 0$. Hence we get that

$\begin{bmatrix} A & B \\ Q & 0 \end{bmatrix}$ is an $m+n-r \times m+n-r$ matrix with inverse $\begin{bmatrix} G & P \\ C & 0 \end{bmatrix}$

Hence $\mathfrak{B}(A)$ is nonempty.

$(iii) \Rightarrow (i)$: Let X be a finitly generated projective module with $X \oplus Y \simeq \mathbb{D}^n$ for some module Y and some integer n. Let $A : \mathbb{D}^n \to \mathbb{D}^n$ be the natural projection onto X and $\rho(A) = r$. Then A and $B = I-A$ are idempotent matrices and so B is regular. From (iii) we get that $\mathfrak{B}(B)$ is nonempty and let

$T = \begin{bmatrix} B & P \\ Q & S \end{bmatrix} \in \mathfrak{B}(B)$ with inverse $T^{-1} = \begin{bmatrix} G & E \\ F & H \end{bmatrix}$. Then we get that $\begin{bmatrix} B \\ Q \end{bmatrix}$ has a left inverse By Lemma 4.2.3 we can obtain a right inverse Q_R^{-1} of Q such that $BQ_R^{-1} = 0$ and $QQ_R^{-1} = I_{n-r}$ Since $[\, G \quad E \,]$ is a left inverse of $\begin{bmatrix} B \\ Q \end{bmatrix}$, we get that

$$GB + EQ = I. \tag{4.2.8}$$

By multipying both sides of 4.2.8 on the right by Q_R^{-1} we get that

$$E = Q_R^{-1}. \tag{4.2.9}$$

Since $(I- GB)(I-B) = I- B$ and $(I- B)(I- GB) = I - GB$, we get that $Range(I- GB) = Range(I- B)$ and this inturn $Range(I- GB) = Range(A) = X$. From 4.2.8 we get that $Range(I-GB) = Range(EQ) = Range(E)$ last equality is because Q has a left inverse. But $Range(E)$ is free because E has a left inverse. Thus, X is free. □

Corollary 4.2.5. Over an integral domain $\mathbb{D}$ if every finitely generated projective module is free then every $m \times k$ regular matrix of rank k can be completed to an $m \times m$ invertible matrix

Proof follows from $(i) \Rightarrow (iii)$ of the above theorem □

Remark. From the proof of the above theorem, it is clear that a regular matrix A over $\mathbb{D}$ has nonempty $\mathfrak{B}(A)$ if and only if its kernel and cokernel are free. In other words $\mathfrak{B}(A)$ is nonempty if and only if $I_n - GA$ and $I_m - AG$ have rank factorizations for any g-inverse G of A.

Corollary 4.2.6. Over an integral domain $\mathbb{D}$, the statement that "every regular matrix has the Smith normal form" is equivalent to any of *(i)*, *(ii)* & *(iii)* of Theorem 4.2.4.

Proof. If a regular matrix has Smith normal form, it is easily verified that it has a rank factorization. Conversely, if A is a regular matrix, from condition *(ii)* of Theorem 4.2.4, A has a rank factorization, say, $A = BC$. Then from Corollary 4.2.5, the matrix B can be completed to an invertible matrix P of size $m \times m$ and C can be completed to an invertible matrix Q of size $n \times n$ and we get that $A = P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q$. Hence the corollary. □

Remark. *(iii)* of Theorem 4.2.4, namely "for every regular matrix A, $\mathfrak{B}(A)$ is nonempty" is equivalent to the statement that "for every $m \times n$ regular matrix A of rank r, there is an $m \times m-r$matrix P such that $[\ A \quad P\ ]$ is right invertible.

Remark. Corollary 4.2.6 generalizes a result (Theorem 1. of [59]) of Sontag. In [59] Sontag showed that over the ring of polynomials in several variables with complex coefficients, being regular is same as having Smith

CHAPTER 5
GENERALIZED INVERSES OVER RINGS

1. Introduction

In this chapter we are concerned with generalized inverses of matrices over various types of rings.

In the second section of this chapter we shall characterize those rings over which every matrix has Moore-Penrose inverse.

In the third section we shal see that with respect to g-inverses, rings with trivial idempotents behave almost like integral domains.

In section 4, we shall observe that the Rao condition on a ring is equivalant to the condition " If G is a (1,3) inverse of an $m \times n$ matrix A, then $AG = diag(e_1, e_2, \ldots e_n)$, where $e_1, e_2, \ldots e_n$ are symmetric idempotents" which generalizes the Robinson, Puystjens and Van Geel [56] result.

In sections 5 and 6 we shall discuss about the existence of a g-inverse for a matrix over rings of $\mathbf{A}$ -valued functions on topological spaces.

2. Moore-Penrose inverse over rings

It is known that over a field every matrix has a g-inverse. But, it is not necessary that over an arbitrary ring, every matrix has a g-inverse (see the example given in chapter 0). As early as 1936, von Neumann found necessary and sufficient conditions on a ring so that every matrix over that ring has a g-inverse [63]. See Brown and McCoy [11] for an elegant proof of this result. von Neumann's result is the following.

Theorem 5.2.1. Let $\mathbf{A}$ be a ring, not necessarily commutative. Every matrix over $\mathbf{A}$ has a g-inverse if and only if $\mathbf{A}$ is regular i.e., for every a in $\mathbf{A}$ there is a g in $\mathbf{A}$ such that $aga = a$.

A natural question which arises is **"what are the rings over which all matrices have Moore-Penrose inverses ?"**. The following theorem answers this question.

Theorem 5.2.2. Let $\mathbf{A}$ be a ring with unity and an involution $a \to \bar{a}$. Then every matrix A over $\mathbf{A}$ has Moore-Penrose inverse if and only if $\mathbf{A}$ is regular and is nonzero definite with respect to the involution '-' in the sence that

$$\sum a_i\bar{a}_i = 0 \;\Rightarrow a_i = 0 \quad \text{for all } i. \tag{5.2.1}$$

Proof. Suppose that every matrix has Moore-Penrose inverse over $\mathbf{A}$. Then by Theoerm 5.2.1. we get that $\mathbf{A}$ is regular. Now we shall prove that $\sum a_i\bar{a}_i = 0 \;\Rightarrow a_i = 0$ for all i. Let $A = (a_1, a_2, \ldots\ldots a_n)$ be such that $A^*A = 0$. Multiplying by A^{*+}, we get

$$A^{*+}A^*A = 0.$$

Since $A^{*+}A^* = A^{+*}A^* = (AA^+)^* = AA^+$ we get $A = AA^+A = 0$

Conversely, let $\mathbf{A}$ be regular and satisfy condition (5.2.1). Let A be an $m \times n$ matrix over $\mathbf{A}$. First we shall prove that A satisfies $*$-cancellation property, for any arbitrary choice of A. i.e.,

$$A^*AB = A^*AC \;\Rightarrow B = C.$$

Let x be any $n \times 1$ nonzero matrix and $A^*Ax = 0$, then we get $x^*A^*Ax = 0$. Since $\mathbf{A}$ satisfies 5.2.1., we get $Ax = 0$. Hence $A^*AB = A^*AC \;\Rightarrow B = C$. We shall show that $A^*(AA^*)^-A(A^*A)^-A^*$ is the Moore-Penrose inverse of A. Since $AA^*(AA^*)^-AA^* = AA^*$ and $A^*(AA^*)^-A$, $A(AA^*)^-A^*$ are invariant under any g-inverses of AA^*, by the $*$-cancellation property of A we get $AA^*(AA^*)^-A = A$ and $A(AA^*)^-A^*A = A$. So $A(A^*(AA^*)^-A(A^*A)^-A^*)A = A$. Similarly it could be verified that $A^*(AA^*)^-A(A^*A)^-A^*$ is a reflexive g-inverse and $A(A^*(AA^*)^-A(A^*A)^-A^*)$, $(A^*(AA^*)^-A(A^*A)^-A^*)A$ are symmetric. Therefore $A^*(AA^*)^-A(A^*A)^-A^*$ is the Moore-Penrose inverse of A. □

Remark. The above (5.2.1) is in fact also a necessary and sufficient condition for every matrix A over $\mathbf{A}$ to have a {1,3} inverse (or for every matrix to have a {1,4} inverse). Cao,Chang-Guang proved a similar result in [14].

5.3. Rings with Trivial Idempotents.

We say that a commutative ring $\mathbf{A}$ has **trivial idempotents** if it has only 0 and 1 as the idempotents. In this section we shall give necessary and sufficient conditions for a matrix to have a generalized inverse over $\mathbf{A}$, when $\mathbf{A}$ has only the trivial idempotents. In Ch. 1. we have seen that, over an integral domain, a necessary and sufficient condition for a matrix to have a g-inverse is that $C_r(A)$ has a g-inverse. But this condition is not sufficient over an arbitrary ring. For example, let $\mathbf{A} = \mathbf{Z}_{12}$, the ring of integers modulo 12,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

has no g-inverse even though $C_2(A) = 4$ which is regular.

Now we state a theorem from [52]. **Throughout this section $\mathbf{A}$ is a commutative ring with identity having trivial idempotents unless otherwise indicated.**

Theorem 5.3.1. (Bhaskara Rao [52]). Let r be the determinantal rank of a nonzero matrix A over a commutative ring $\mathbf{A}$ with identity. In the following, $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$. If 0 and 1 are the only idempotents in $\mathbf{A}$ then $(vi) \Rightarrow (i)$

(i) There exist (c_α^β) in $\mathbf{A}$ such that $\sum c_\alpha^\beta \ |A_\beta^\alpha| = 1$.

(ii) There exist (c_α^β) in $\mathbf{A}$ such that $a_{ij}\left(\sum c_\alpha^\beta \ |A_\beta^\alpha|\right) = a_{ij}$ for all i, j.

(iii) A is regular.

(iv) $C_r(A)$ is regular.

(v) There exist (c_α^β) in $\mathbf{A}$ such that $A_\delta^\gamma\left(\sum c_\alpha^\beta \ |A_\beta^\alpha|\right) = A_\delta^\gamma$ for all γ and δ.

(vi) There exist (c_α^β) in $\mathbf{A}$ such that $\sum c_\alpha^\beta \ |A_\beta^\alpha|$ is a nonzero idempotent.

Proof. Since all 2 x 2 minors of $C_r(A)$ can be written as a linear combination of all $r+k$ x $r+k$ minors of A, $k \geq 1$ [31], every 2 x 2 minor of $C_r(A)$ vanishes and we get $\rho(C_r(A))$ is one. Now the proof follows the steps of Theorem 1.2.2 □

Now we shall derive several interesting consequence of this Theoerm

Corollary 5.3.2. Let $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ be a $k \times l$ matrix such that $\rho(A) = \rho(P)$, where P is $m \times n$, Q is $m \times l\text{-}n$, R is $k\text{-}m \times n$ and S is $k\text{-}m \times l\text{-}n$ matrix over **A**. If P is regular , so is A.

Proof. If P is regular, then by Theorem 5.3.1, there exist (c_δ^γ) in **A** such that $\sum c_\delta^\gamma \ |P_\gamma^\delta| = 1$, where δ runs over all r-elements subsets of $(1,2,\dots\dots m)$ and γ runs over all r- elements subsets of $(1,2,\dots\dots n)$. Then choose $d_\alpha^\beta = c_\alpha^\beta$ if α is a subset of $(1,2,\dots\ m)$ and β is a subset of $(1,2,\dots\dots n)$, and $d_\alpha^\beta = 0$ otherwise. Since $|A_\beta^\alpha| = |P_\beta^\alpha|$ for $\alpha \subset (1, 2,\dots\dots m)$ and $\beta \subset (1,2,\dots n)$, we get

$$\begin{bmatrix} G_{nxm} & 0 \\ 0 & 0 \end{bmatrix} \text{ is a g-inverse of } A,$$

where G is a g-inverse of P whose (i, j)th entry is $g_{ij} = \sum_{\alpha,\beta} c_\alpha^\beta \frac{\partial}{\partial a_{ji}} |P_\beta^\alpha|$. □

Corollary 5.3.3. Let A be an $m \times n$ regular matrix of rank r over **A**. If

$$\rho\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \rho(A),$$

then D is uniquely determined for given choice of B and C.

In fact $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ must be of the form $\begin{bmatrix} A & AE \\ FA & FAE \end{bmatrix}$ for some matrices F and E.

Proof. By Theorem 5.3.1. find (c_α^β) such that $\sum c_\alpha^\beta \; |A_\beta^\alpha| = 1$. Let G be defined by

$$g_{ij} = \sum_{\alpha,\beta} c_\alpha^\beta \frac{\partial}{\partial a_{ji}} |A_\beta^\alpha|.$$

By Corollary 5.3.2. we get that $\begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}$ is a g-inverse of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, therefore

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} AGA & AGB \\ CGA & CGB \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

determines D uniquely from B and C. The second part follows easily. □

Now we shall look at group inverses and generalize Theorem 3.3.3 to rings with trivial idempotents.

Theorem 5.3.4. Let A be a square matrix over $\mathbf{A}$. Then the following statements are equivalent.

(i) A has group inverse

(ii) $\rho(A^2) = \rho(A)$ and A^2 is regular

(iii) $\sum_\gamma |A_\gamma^\gamma|$ is invertible in $\mathbf{A}$.

Proof. (i)⇒(ii). Let G be the group inverse of A. Then $AGA = A$ gives us $A^2G = A$ which in turn implies that $\rho(A^2) = \rho(A)$. Since $AG = GA$, $A^2G^2A^2 = A^2$ and A^2 is regular

$(ii) \Rightarrow (iii)$ Suppose $\rho(A^2) = \rho(A)$ and A^2 is regular. From Theorem 5.3.1, there exist c_α^β in $\mathbf{A}$ such that

$$\sum_{\alpha,\beta} c_\alpha^\beta \quad |A^2{}_\beta^\alpha| \quad = 1$$

$$\sum_{\alpha,\beta,\gamma} c_\alpha^\beta \quad |A_\gamma^\alpha||A_\beta^\gamma| \quad = 1$$

and

$$\sum_\gamma |A_\gamma^\gamma| \sum_{\alpha,\beta} c_\alpha^\beta \quad |A_\beta^\alpha| \quad = 1$$

from which we get that $\sum_\gamma |A_\gamma^\gamma|$ is invertible.

$(iii) \Rightarrow (i)$ Proof is similar to the case of part (i) of Theorem 3.4.1. G obtained by

$$g_{ji} = \sum_\gamma u^{-1} \frac{\partial}{\partial a_{ij}} |A_\gamma^\gamma|$$

is a commuting g-inverse of A and GAG is the group inverse of A. □

Remark. It can also be shown that *(i)*, *(ii)* and *(iii)* of the above theorem are equivalent to the statement "$C_r(A)$ has group inverse".

Corollary 5.3.5. If $A^\#$ is the group inverse of A, then $A^\#$ is a polynomial in A over the ring $\mathbf{A}$.

Proof is similar to that of Theorem 3.3.5. □

Using Theorem 5.3.1 we shall now generalize Theorem 2.2.6 for matrices over rings with trivial idempotents. We need a result due to Puystjens [45] .

First we define

Definition : A matrix A is said to be $*$-cancellable if

$$A^*AB = 0 \text{ implies that } AB = 0$$

$$AA^*B = 0 \text{ implies that } A^*B = 0$$

Lemma 5.3.6. Over a commutative ring if a matrix A has Moore-Penrose inverse, then A is $*$-cancellabe.

Proof. See [45] of Puystjens. □

Theorem 5.3.7. Let A be an $m \times n$ matrix of rank r over $\mathbb{A}$. Then the following are equivalent.

(i) A has Moore-Penrose inverse .

(ii) $\rho((A^*A)^2) = \rho(A)$ and $(A^*A)^{\#}$ exist.

(iii) $\sum_{\alpha,\beta} |\bar{A}_\beta^\alpha||A_\beta^\alpha|$ is invertible.

Proof. (i) ⇒ (ii) : Suppose that A has Moore-Penrose inverse over $\mathbb{A}$. Then we get $\rho(A^*A) = \rho(A)$ and A and $C_r(A)$ are $*$-cancellable by Lemmma 5.3.6. In case $C_r((A^*A)^2) = 0$, we get $C_r((A^*A)^2) = C_r(A^*)\ C_r(A)\ C_r(A^*)\ C_r(A) = 0$ which implies $C_r(A) = 0$. Therefore we get $C_r((A^*A)^2) \neq 0$ and $\rho((A^*A)^2) = \rho(A)$. If A^+ is the Moore-Penrose inverse of A, then $A^+A^{*+}A^+A^{*+}$is a g-inverse of $(A^*A)^2$ and hence $(A^*A)^2$ is regular. So we get (A^*A) has group inverse, by Theorem 5.3.4.

(ii) ⇒ (iii) : Suppose $\rho((A^*A)^2) = \rho(A)$ and $(A^*A)^{\#}$ exist. Since

$$\sum_{\beta} |(A^{*}A)^{\beta}_{\beta}| = \sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}| ,$$

by Theorem 5.3.4 we get $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible.

(iii)⇒ *(i)* : By Theorem 2.3.1, we get that G obtained by the equation

$$g_{ji} = \sum_{\alpha,\beta} u^{-1} |\bar{A}^{\alpha}_{\beta}| \frac{\partial}{\partial a_{ij}} |A^{\alpha}_{\beta}|$$

is a (1,3) inverse and similarly it is a (1,4) inverse which ensures the existence of Moore-Penrose inverse. Note that G itself may not be a reflexive g-inverse but GAG definitely is a Moore-Penrose inverse. □

5.4. Rings with Rao condition

Let $\mathbf{A}$ be an associative ring with identity. Following Robinson [56] we say that $\mathbf{A}$ satisfies

Rao Condition : If $\sum_{i} a_i\bar{a}_i = a_1$ in $\mathbf{A}$ then $a_i = 0$ for $i \neq 1$

In 1983 Bhaskara Rao [53] showed that over an integral domain with trivial involution, satisfying the above condition, only matrices with Moore-Penrose inverses are those which are permutationally equivalent to $\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$ with M invertible. In Theorem 5.4.2 we shall see how a matrix which has (1,3) ((1,4)) inverses over an associative ring $\mathbf{A}$ satisfying Rao condition, looks like. In the following lemma we shall first see that an idempotent symmetric matrix over a ring $\mathbf{A}$ satisfying Rao conditon is permutationally equivalent to a diagonal matrix with symmetric idempotent entries.

Lemma 5.4.1. Let E be an $n \times n$ symmetric idempotent matrix over an associative ring $\mathbf{A}$ satisfying Rao condition. Then $E = diag(e_1, e_2, \ldots, e_n)$, where $e_1, e_2, \ldots, e_n$ are symmetric idempotents.

Proof. Let (i, j)th entry of E be e_{ij}. Since $E^2 = E = E^*$, $EE^* = E$ and we get

$$e_{ii} = \sum e_{ik}\bar{e}_{ik}.$$

By Rao condition we get $e_{ik} = 0$ for $i \neq k$ and e_{ii} is a symmetric idepmotent. □

Theorem 5.4.2. Let $\mathbf{A}$ be an associative ring with identity. then the following are equivalent

(i) $\mathbf{A}$ satsfies Rao condition

(ii) If G is a (1,3) inverse of an $m \times n$ matrix A, then $AG = diag(e_1, e_2, \ldots, e_n)$, where $e_1, e_2, \ldots, e_n$ are symmetric idempotents.

(iii) If $\sum_i a_i\bar{a}_i = 1$ in $\mathbf{A}$, then $\bar{a}_i a_j = 0$ whenever $i \neq j$.

(ii)' If G is a (1,4) inverse of an $m \times n$ matrix A, then $GA = diag(f_1, f_2, \ldots, f_n)$, where $f_1, f_2, \ldots, f_n$ are symmetric idempotents.

Proof. (i) $\Rightarrow$ (ii) is trivial by earliar Lemma.

(ii)$\Rightarrow$ (iii) : If $\sum_i a_i\bar{a}_i = 1$ in $\mathbf{A}$, let $A = [a_1\ a_2 \ldots\ a_n]^*$. Note that $G = [a_1\ a_2\ \ldots\ a_n]^*$ is a (1,3) inverse of A. Also, by (ii) we get $\bar{a}_i a_j = 0$ for $i \neq j$.

(iii) $\Rightarrow$ (i) : Let $\sum_i a_i\bar{a}_i = a_1$ in $\mathbf{A}$. Since a_1 is symmetric and

$$1 = (1-a_1)(1-\bar{a}_1) + \sum_{i \neq 1} a_i\bar{a}_i + \sum_i a_i\bar{a}_i ,$$

we get $a_i\bar{a}_j = 0$ for every i and j except for $i = j = 1$ and $(1- a_1)\bar{a}_j$ for every j which implies that a_1 is symmetric idempotent and $a_j = 0$ for $j \neq 1$ (i) ⇒(ii)' ⇒ (iii) ⇒ (i) follow similarly

□

In the above theorem if the only symmetric idempotents in $\mathbf{A}$ are 0 and 1 it reduces to the theorem given below

Theorem 5.4.3. Let $\mathbf{A}$ be an associative ring with trivial symmetric idempotents then the following are equivalent

(i) $\mathbf{A}$ satisfies Rao condition

(ii) If G is a (1,3) inverse of an $m \times n$ matrix A, then $AG = diag(e_1, e_2, \ldots, e_n)$, where $e_1, e_2, \ldots, e_n$ are either 0 or 1

(iii) If $\sum_i a_i\bar{a}_i = 1$ in $\mathbf{A}$, then then $\bar{a}_i a_j = 0$ whenever $i \neq j$.

(ii)' If G is a (1,4) inverse of an $m \times n$ matrix A, then $GA = diag(f_1, f_2, \ldots, f_n)$, where $f_1, f_2, \ldots, f_n$ are either 0 or 1

Theorem 5.4.4. Let $\mathbf{A}$ be an associative ring with trivial idempotents. Then the following are equivalent

(i) $\mathbf{A}$ satisfies Rao condition

(ii) If an $m \times n$ matrix A has (1,3) inverse, then there exist a permutation matrix P such that $A = P\begin{bmatrix} M \\ 0 \end{bmatrix}$ where M has a right inverse

(ii)' If an $m \times n$ matrix A has (1,4) inverse, then there exist a permutation matrix Q such that $A = [\,N \quad 0\,]Q$ where N has a left inverse.

Proof. $(i) \Rightarrow (ii)$ Suppose G be a (1,3) inverse of A. Then by $(i) \Rightarrow (ii)$ of earlier theorem we get $AG = diag(e_1, e_2, \ \ e_n)$, where e_i are 0 or 1. Therfore

there exist a pemutation matrix such that $AG = P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P'$. Since $AGA = A$, we get

$$A = AGA = P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P'A = P \begin{bmatrix} M \\ 0 \end{bmatrix} \quad \text{where } M = [\ I \quad 0\ ]\ P'A.$$

Since

$$P'AGP = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

we get that M has a right inverse Similarly we can prove $(i) \Rightarrow (ii)'$ $(ii) \Rightarrow (i)$ and $(ii)' \Rightarrow (i)$ follow from the earliar theorem □

D W Robinson Et Al. [56] proved that **A** satisies Rao condition if and only if matrices which have Moore-Penrose inverse are permutationally equivalent to $\begin{bmatrix} M_{r \times s} & 0 \\ 0 & 0 \end{bmatrix}$, where $M_{r \times s}$ is an invertible matrix

Remark. If **A** is a commutative ring in the above theorems it is very easy to verify that r, s which correspond to the sizes of M and N are equal to the determinantal rank of the matrix.

5.5. $\mathbf{A}$-valued Functions over Topological Spaces

In this section we shall consider matrices with coefficients from an additive category of functions from a set X into a commutative ring $\mathbf{A}$ with trivial idempotents. We denote this category by $\mathfrak{F}(X, \mathbf{A})$. Note that $\mathfrak{F}(X, \mathbf{A})$ is also a commutative ring with identity. We consider mainly the rings $\mathfrak{C}(X, \mathbf{A})$ of continuous functions on X, $\mathfrak{D}(X, \mathbf{A})$ of differentiable functions on X whenever X is a topological space and $\mathbf{A}$ is also a toplogical space and differentiation of functions is well defined . We have the real line or complex numbers as a candidate for $\mathbf{A}$ in this case.

In the following theorem we shall consider matrices with constant rank over $\mathbf{A}$, i.e., $\rho(A(x))$ is fixed for all x in X, and characterize those matrices which have Moore-Penrose inverse. We say that a matrix A over $\mathfrak{F}(X, \mathbf{A})$ is of constant rank if $\rho(A(x))$ is constant for all x in X whenever $A(x)$ is considered as a matrix over $\mathbf{A}$.

Theorem 5.5.1. Let X be a topological space and A an $m \times n$ matrix of constant rank r over the ring $\mathfrak{F}(X, \mathbf{A})$. Then the following are equivalent

(i) A is regular

(ii) $C_r(A)$ is regular.

(iii) There exist c_α^β in $\mathfrak{F}(X, \mathbf{A})$ such that $\sum_{\alpha,\beta} c_\alpha^\beta |A_\beta^\alpha| = 1$, where α, β run over all r-elements subsets of $\{1,2,......m\}$ and $\{1,2,......n\}$ respectively.

Proof. Proof of $(i) \Rightarrow (ii)$, $(iii) \Rightarrow (i)$ are similar to the cases $(iii) \Rightarrow (iv)$ and $(i) \Rightarrow (iii)$ of Theorem 5.2.1.

$(ii) \Rightarrow (iii)$: Let $C_r(A)$ be regular and $(c^{\beta}_{\alpha})_{\alpha,\beta}$ be a g-inverse of $C_r(A)$, where α, β run over all r-elements subsets of $\{1,2,\ldots m\}$ and $\{1,2,\ldots n\}$. Since $C_r(A(x))$ is of rank one for every x in X and $(c^{\beta}_{\alpha}(x))_{\alpha,\beta}$ is a g-inverse of $C_r(A(x))$, we get $\sum_{\alpha,\beta} c^{\beta}_{\alpha}|A^{\alpha}_{\beta}|(x) = 1$ for every x in X. Hence $\sum_{\alpha,\beta} c^{\beta}_{\alpha}|A^{\alpha}_{\beta}| = 1$. □

Similar results can be obtained over the rings $\mathfrak{C}(X, \mathbf{A})$, $\mathfrak{D}(X, \mathbf{A})$, where $\mathbf{A}$ is a ring with trivial idempotents. **In the following theorems we shall consider the case $\mathbf{T} = \mathfrak{C}(X, \mathbf{R})$, $\mathfrak{C}(X, \mathbf{C})$, $\mathfrak{D}(X, \mathbf{R})$, $\mathfrak{D}(X, \mathbf{C})$ and derive conditions on matrices which have Moore-Penrose inverse and group inverse.**

Theorem 5.5.2. Let A be an $m \times n$ matrix of rank r over $\mathbf{T}$. Then A is of constant rank (ofcourse, r) over X if and only if $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible.

Proof. Let A be a matrix of constant rank r over X, we get that $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|(x)$ is nonzero for every x in X. Since the mapping $x \to x^{-1}$ is continuous and differentiable on $\mathbf{A}\setminus\{0\}$ for $\mathbf{A} = \mathbf{C}$ or $\mathbf{R}$, we get $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbf{T}$

Conversely, if $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbf{T}$, then

$\left(\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|\right)^{-1}(x) \left(\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|\right)(x) = 1$ for all x in X which implies that

$\left\{x \in X \mid |A^{\alpha}_{\beta}|(x) = 0 \text{ for every } \alpha, \beta \right\}$ is empty. Hence we get that $\rho(A(x)) = r =$

$\rho(A)$ and A is of constant rank □

Remark. In the above theorems the hypothesis that A has constant rank over X is essential. For, if X is not a connected space one can construct an easy example of a matrix A in $\mathbb{C}(X, \mathbb{R})$ which has Moore-Penrose inverse but $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is not invertible and not even a matrix of constant rank.

Now we shall give a necessary and sufficient condition for a matrix A of rank r over $\mathbb{C}(X, \mathbb{R})$ to have a g-inverse

Remark. From the codition (iii) of the above theorem it is necessary that a regular matrix over $\mathbb{C}(X, \mathbb{R})$ must be of constant rank

Theorem 5.5.3. Let A be a square matrix of rank r over $\mathbb{T}$ in which case X be a connected topological space Then the following are equivalent.

(i) A is regular

(ii) $C_r(A)$ is regular.

(iii) There exist (c^{β}_{α}) in $\mathbb{C}(X, \mathbb{R})$ such that $\sum_{\alpha,\beta} c^{\beta}_{\alpha}|A^{\alpha}_{\beta}| = 1$, where α, β run over all r-elements subsets of $\{1,2,\ldots,m\}$ and $\{1,2,\ldots,n\}$ respectively.

(iv) A is of constant rank

(v) A has Moore-Penrose inverse

Proof. Proof of $(i) \Rightarrow (ii)$ is clear

$(ii) \Rightarrow (iii)$ Let $C_r(A)$ be regular and $(c^{\beta}_{\alpha})_{\alpha,\beta}$ be a g-inverse of $C_r(A)$, where α, β run over all r-elements subsets of $\{1,2, \ldots m\}$ and $\{1,2, \ldots n\}$. Since $C_r(A(x))$ is of rank one or zero for every x in X and $(c_\alpha^\beta(x))_{\alpha,\beta}$ is a g-inverse of $C_r(A(x))$, we get

$$\sum_{\alpha,\beta} c_\alpha^\beta |A_\beta^\alpha|(x) = 1 \text{ or } 0 \text{ for all } x \text{ in X.}$$

Since X is connected, we get that

$$\sum_{\alpha,\beta} c_\alpha^\beta |A_\beta^\alpha|(x) = 1 \text{ for all } x \text{ in X or } 0 \text{ for all } x \text{ in X.}$$

But $\{ x \in X : \rho(A(x)) = 1 \}$ is nonempty (since $\rho(A) = r$), so we get that

$$\sum_{\alpha,\beta} c_\alpha^\beta |A_\beta^\alpha|(x) = 1 \quad \text{for all } x \text{ in X.}$$

Hence $\sum_{\alpha,\beta} c_\alpha^\beta |A_\beta^\alpha| = 1$

(iii) $\Rightarrow$ (iv) can be proved as in the proof of Theorem 5.5.2.

(iv) $\Rightarrow$ (v) If A is of constant rank, from Theorem 5.5.2, we get that $\sum_{\alpha,\beta} |\bar{A}_\beta^\alpha||A_\beta^\alpha|$ is invertible. Hence

$$g_{ji}(x) = \Big(\sum_{\alpha,\beta} u^{-1} |\bar{A}_\beta^\alpha| \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|\Big)(x)$$

determines the Moore-Penrose inverse of A at x for every x in X. Therefore

$$g_{ji} = \sum_{\alpha,\beta} u^{-1} |\bar{A}_\beta^\alpha| \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|$$

determines the Moore-Penrose inverse of A over $\mathbf{T}$

(v) $\Rightarrow$ (i) is trivial □

Remark. The above theorem is true even for $\mathbf{T} = \mathbb{R}[X_1, X_2, \ldots, X_n]^*$(see ch 1 , for definition) That (*iii*) ⇒ (*iv*) follows , because $\left(\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|\right)^{-1}$ is in $\mathbb{R}[x_1, x_2, \ldots x_n]^*$ whenever $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ does not have a zero. Thus, our Theorem 5.5.3 generalizes a result of Sontag([59], Theorem 3).

Theorem 5.5.4. Let A be a square matrix of rank r over $\mathbf{T}$ such that A has constant rank over X. Then A has group inverse if and only if $\sum_{\alpha} |A^{\alpha}_{\alpha}|$ is invertible in $\mathbf{T}$

Proof is similar to the case of the Moore-Penrose inverse □

5.6. Generalized inverses over Banach algebras

In the finite dimensional case, all questions of controllability, observability, and stabilizability for linear systems have been reduced to simple forms by many mathematicians (Example. [57], [62]), but due to the complexity of infinite-dimensional systems, all the above mentioned three questions become manifold and difficult. Chen Wanyi [65] gave necessary and sufficient conditions for right and left invertibility of a matrix over a commutative Banach algebra in connection with finding necessary and sufficient conditions for controllability of a given control system. In this section we shall consider matrices with constant rank over the carrier space of a given commutative Banach algebra with unit, and characterize all regular matrices. This caracterization is useful in solving the linear systems like, state equation of a control problem. Now we shall state Gelfand-Naimark theorem which we used without proof

Theorem 5.6.1. If $\mathbf{B}$ is a commutative Banach algebra, then the map $\mathcal{F}: b \in \mathbf{B} \rightarrow \hat{b} \in C(X)$ is a homomorphism of $\mathbf{B}$ into the C^*-algebra $C(X)$ of all continuous functions on X, where X is the set of all nonzero homomorhisms of $\mathbf{B}$ into $\mathbb{C}$. If $\mathbf{B}$ is unital, then X is compact and the spectrum of an element b in $\mathbf{B}$ is $Sp_{\mathbf{B}}(b) = \hat{b}(X)$. (This X is called the carrier space of the Banach algebra).

Corollary 5.6.2. For an element b in a unital Banach algebra $\mathbf{B}$, $\hat{b}$ in $C(X)$ is invertible if and only if b is invertible in $\mathbf{B}$.

Proof. Since $Sp_{\mathbf{B}}(b) = \hat{b}(X)$, in case b is not invertible, then $0 \in Sp_{\mathbf{B}}(b) = \hat{b}(X)$ which implies that $\hat{b}$ is not invertible. Therefore $\hat{b}$ in $C(X)$ is invertible if and only if b is invertible in $\mathbf{B}$. □

Now we shall see that a matrix A has Moore-Penrose inverse whenever $\hat{A}$ the matrix corresponding to A in $C(X)$ has constant rank over X.

Theorem 5.6.3. Let A be an $m \times n$ matrix of rank r over a commutative Banach algebra $\mathbf{B}$ with unit and X be the carrier space of $\mathbf{B}$. Then the following are equivalent

(i) A has Moore-Penrose inverse and $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbf{B}$.

(ii) A is regular and there exist c^{β}_{α} in $\mathbf{B}$ such that $\sum_{\alpha,\beta} c^{\beta}_{\alpha}|A^{\alpha}_{\beta}| = 1$, where α, β run over all r-elements subsets of $\{1,2,\ldots,m\}$ and $\{1,2,\ldots,n\}$

respectively

(*iii*) There does not exist any linear functional x in X such that $x(|A^{\alpha}_{\beta}|) = 0$ for all α, β

Proof. (*i*) $\Rightarrow$ (*ii*) is trivial

(*ii*) $\Rightarrow$ (*iii*) Let $A = (f_{ij})$ be a regular matrix and there exist c^{β}_{α} in $\mathbf{B}$ such that $\sum_{\alpha,\beta} c^{\beta}_{\alpha}|A^{\alpha}_{\beta}| = 1$. Then we get $\sum_{\alpha,\beta} \hat{c}^{\beta}_{\alpha}|\hat{A}^{\alpha}_{\beta}| = 1$ in $\mathcal{C}(X)$ which implies $\hat{A}$ has constant rank over X and

$$\bigcap_{\alpha,\beta}\{x \in X \,/\, |\hat{A}^{\alpha}_{\beta}|(x) = 0\} = \emptyset$$

which gives the result.

(*iii*) $\Rightarrow$ (*i*). Suppose for a matrix $A = (f_{ij})$, there does not exist any linear functional x in X such that $x(|A^{\alpha}_{\beta}|) = 0$ for all α, β , then we get

$$\bigcap_{\alpha,\beta}\{x \in X \,/\, |\hat{A}^{\alpha}_{\beta}|(x) = 0\} = \emptyset \,.$$

Therefore $\hat{u} = \sum_{\alpha,\beta} |\bar{\hat{A}}^{\alpha}_{\beta}||\hat{A}^{\alpha}_{\beta}|$ is invertible in $\mathcal{C}(X)$, where $u = \sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ in $\mathbf{B}$.

Hence we get $u = \sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ is invertible in $\mathbf{B}$ by corollry 5.6.2. From Theorem 5.5.2 we get

$$\hat{g}_{ji} = \sum_{\alpha,\beta} \hat{u}^{-1}\, |\bar{\hat{A}}^{\alpha}_{\beta}|\, \frac{\partial}{\partial f_{ij}} |\hat{A}^{\alpha}_{\beta}| \qquad (5.5.1)$$

determines the Moore-Penrose inverse of $\hat{A}$ over $\mathcal{C}(X)$ and hence

$$g_{ji} = \sum_{\alpha,\beta} u^{-1}\, |\bar{A}^{\alpha}_{\beta}|\, \frac{\partial}{\partial f_{ij}} |A^{\alpha}_{\beta}|$$

determines the Moore-Penrose inverse of A over $\mathbf{B}$ □

Corollary 5.6.4.(Chen Wanyi [65]). Let A be an $m \times n$ matrix over a commutative Banach algebra with unit. Then the following are equivalent.

(i) There exist an $n \times m$ matrix G such that $AG = I_m$

(ii) For each x in X, $\hat{A}(x)$ considered as a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^n$ is onto

(iii) For each x in X, the rank of the scalar matrix $\hat{A}(x)$ is m.

Proof. (i)$\Rightarrow$ (ii) follows from (ii)$\Rightarrow$ (iii) of Theorem 5.6.3., since $|AG| = \sum_{\beta} |A_{\beta}||G^{\beta}| = 1$ (iii)$\Rightarrow$(i) follows from (iii)$\Rightarrow$ (i) of Theorem 5.6.3 as follows :

Since $\sum_{\alpha,\beta} |\bar{A}^{\alpha}_{\beta}||A^{\alpha}_{\beta}| = |AA^*|$ is invertible we get that $G = A^*(AA^*)^{-1}$ satisfies the condition (i) of the corrollary. (ii) $\Leftrightarrow$ (iii) is trivial. □

Theorem 5.6.5. Let A be a square matrix of rank r over a commutative Banach algebra $\mathbb{B}$ with unit and let the carrier space X of $\mathbb{B}$ be connected. Then the following are equivalent.

(i) A has group inverse.

(ii) $\sum_{\gamma} |A^{\gamma}_{\gamma}|$ is invertible, where γ runs over r- elements subsets of {1,2,.....n}

(iii) There does not exist any linear functional x in X such that $x(tr(C_r(A))) = 0$

Proof. (i) $\Rightarrow$ (ii) : Since A has group inverse, A^2 is a regular and since the carrier space is connected we get that $\hat{A}^2$ is of constant rank over X. Therefore $\sum_{\gamma} |\hat{A}^{\gamma}_{\gamma}|(x) \neq 0$ for every x in X, we get $\sum_{\gamma} |\hat{A}^{\gamma}_{\gamma}|$ is invertible in C(X) and hence $\sum_{\gamma} |A^{\gamma}_{\gamma}|$ is invertible in $\mathbf{B}$.

(ii)⇒*(i)* can verified easily.

(ii) ⇔ *(iii)* is trivial. □

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