Uncertainty Principles
on
Some Lie Groups

Swagato Kumar Ray

Thesis submitted to the Indian Statistical Institute
in partial fulfillment of the requirements
for the award of the degree of
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ERRATA

1. Page 20, line 13,

$$r_{\bar{l}} = r_l = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s} : 1 \leq i \leq r\}.$$

2. Page 20, line 15, $\bar{l}(X_{j_i}) = 0,\ 1 \leq i \leq 2k.$

3. Page 20, line 19, $l'(X_{j_i}) = 0,\ \ l \leq i \leq 2k.$

4. Page 22, line 5, $l_J(X_{j_i}) = l_{j_i},\ \ 1 \leq i \leq 2k.$

5. Page 32, line 16,

$$\begin{pmatrix} A' & C' \\ 0 & D' \end{pmatrix}.$$

6. Page 32, line 17, $\ldots C'$ is a $r \times 2k$ matrix $\ldots$

7. Page 51, line 20, $J(Y) \circ J(Y) = -\|Y\|^2 Id \ldots$

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Chapter 1

Introduction

The uncertainty principles of Harmonic Analysis say that: **a nonzero function and its Fourier transform cannot both be sharply concentrated**. After the initial work on this phenomenon in 1920's, the last two decades witnessed a spurt of activity in this direction (we refer the reader to a very readable survey [FS]). One may notice two broad phases in this activity, the first concentrating on $\mathbb{R}^n$ where the notion of concentration is given different formulations to see whether the phenomenon still holds. In the later phase $\mathbb{R}^n$ is replaced by other commutative or noncommutative groups, or more generally by homogeneous spaces to see which uncertainty principles remain valid.

In this thesis our main objective is to get analogues of the following theorem due to Cowling and Price on some classes of Lie groups.

Theorem 1.0.1 *Suppose $f : \mathbb{R} \to \mathbb{C}$ be a measurable function and satisfies*

(*i*) $\int_{\mathbb{R}} e^{pa\pi x^2} |f(x)|^p dx < \infty$,

(*ii*) $\int_{\mathbb{R}} e^{qb\pi y^2} |\hat{f}(y)|^q dy < \infty$,

where $min(p, q) < \infty$, $a, b > 0$ and $\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y} dx$ is the Fourier transform of f. If $ab \geq 1$ then $f = 0$ almost everywhere and if $ab < 1$

then there exist nonzero functions satisfying the above conditions.

For the proof of the theorem see [CP].

Their motivation for the result is a classical result due to Hardy which uses L^∞ norm instead of L^p and L^q norms; namely, if $f : \mathbb{R} \to \mathbb{C}$ is a measurable function such that $|f(x)| \leq Ce^{-a\pi x^2}$, $|\hat{f}(y)| \leq Ce^{-b\pi y^2}$ with $a, b, C > 0$, then $f = 0$ almost everywhere if $ab > 1$, $f(x) = Ce^{-a\pi x^2}$ if $ab = 1$ and the case $ab < 1$ is like in the previous theorem (see [HJ]).

Whereas the result of Hardy's states that f and $\hat{f}$ cannot be very rapidly decreasing pointwise, that of Cowling and Price's asserts more; it says that both f and $\hat{f}$ cannot decay very rapidly on an average. Barring the case $ab = 1$ (seems to be something special to Euclidean spaces), we see that Hardy's theorem follows from that of Cowling and Price, as expected. However one of our results (Theorem 1.1.4) shows that for $\mathbb{R}^n$, the case $ab > 1$ of the Cowling-Price theorem can be obtained from Hardy's theorem in an elementary way, without the subtle modification of the Phragmen-Lindelöff theorem originally employed. We should mention that because of the case $ab = 1$, these two theorems still stand as independent theorems at least on $\mathbb{R}^n$. Though these results on $\mathbb{R}$ uses complex variable methods (mainly Phragmen-Lindelöff theorem) via entire extension of the Fourier transform, their extensions to $\mathbb{R}^n$ need only analysis of one complex variable.

The complex analytic techniques used in Cowling-Price theorem motivates us to look at semisimple Lie groups. The decay of the matrix coefficients of the Principal series representations prepares the ground for an approach similar to what is done on the Euclidean spaces. There are, however, two serious obstacles in the way. The first one is the existence of the discrete series and the second one is caused by the zeros of the real analytic function appearing in the Plancherel theorem. While we get around the first obstacle, using the main idea of [CSS], the second obliges us to restrict ourselves to the rank one case. The method of analytic continuation is also found to work for some groups outside of the semisimple class. These 'freaks' include the Euclidean motion group of the plane and the oscillator group.

The situation is different on nilpotent Lie groups. Looking at the example of Heisenberg groups we know that the complex analytic techniques no longer work here. But we could prove an analogue of the Cowling-Price theorem on Heisenberg groups (see [BR]) making use of a technique from [SST] ([SST] obtains an analogue of Hardy's theorem on Heisenberg groups). The logical next step is then the two step, nilpotent Lie groups which are known to be the closest relatives of the Heisenberg groups. Here the problem demands a good parametrization of the essential dual (a measurable subset of the unitary dual $\hat{G}$ with full Plancherel measure) and a clear understanding of the Plancherel theorem. The first one is needed to make precise the meaning of the rapid decay of $\hat{f}$ and the second one for reducing the problem down to the center of the group. It is the first problem for which we use the Kirillov correspondence, although to get hold of the unitary dual of a two step, nilpotent Lie group Kirillov theory is not required as has been pointed out in [ACDS] also. We do a detailed discussion of the parametrization problem to point out that because of the simple structures of the coadjoint orbits of two step, nilpotent Lie groups, the parametrization of the essential dual is by far easier than for general nilpotent Lie groups. Coming to the second problem, we know that the proof of the Plancherel theorem for Heisenberg groups as given in [F1], for instance, does not generalise to nilpotent Lie groups but we could make it work for two step, nilpotent Lie groups and that does the trick for us. It is basically the second problem which prohibits us from going towards general nilpotent Lie groups.

Whatever we have discussed so far about two step, nilpotent Lie groups relies heavily on explicit description of the irreducible, unitary representations via a choice of polarization. Using these descriptions we could describe, more or less explicitly, those eigen functions of the sub-Laplacian which are matrix coefficients of irreducible representations, motivated by the example of $\mathcal{F}_{2n,2}$ as given in [St]. We turn to uncertainty principles proved on Heisenberg groups, which use instead of Fourier expansion, eigen function expansions (see [SST], [T1]). Prominent among those are the analogues of the Heisenberg's inequality (this is the result with which the whole story of uncertainty principles began). As far as we know there are three possible analogues of Heisenberg's inequality available on Heisenberg groups in [SST], [T1] and [GL] (actually [SST] generalises the result of [T1]). Unlike the result of [T1] the results given in [SST] and [GL] use the existence of rotation on Heisenberg groups, but it is now known that for existence of rotation on nilpotent Lie groups, two step is a necessary condition but is not sufficient (see [BJR]). So we could only hope for an analogue of the result proved in [T1] which is free from extra structures of the Heisenberg groups. And that is our final result on two step, nilpotent Lie groups.

This thesis is organised as follows. In section 1.1 we obtain extension on $\mathbb{R}^n$ of Cowling-Price theorem as well as some related uncertainty principles. It is here that we also show the 'equivalence' of Hardy's theorem and the Cowling-Price theorem. Chapter 2 deals with the essential background material for two step nilpotent Lie groups, namely, parametrization of the essential dual, Plancherel theorem and the description of the eigen functions of the sub-Laplacian. In chapter 3 we prove analogues of Cowling-Price theorem and Heisenberg's inequality on all connected, simply connected, two step nilpotent Lie groups. Chapter 4 deals with the extension of the Cowling-Price theorem on two semidirect products, namely, the Euclidean motion group of the plane and the oscillator group. In chapter 5 we prove an analogue of the Cowling-Price theorem on rank one semisimple Lie groups.

1.1 Euclidean Spaces

The Cowling-Price theorem depends on the following result for entire functions.

Lemma 1.1.1 *If $g : \mathbb{C} \to \mathbb{C}$ is an entire function and for $1 \le p < \infty$*

(i) $|g(x+iy)| \le Ae^{\pi x^2}$,

(ii) $(\int_{\mathbb{R}} |g(x)|^p dx)^{1/p} < \infty$,

then $g = 0$.

This was proved in [CP] and the proof uses an L^p analogue of Phragmen-Lindelöff Theorem. Using this lemma we can extend the Cowling-Price theorem on $\mathbb{R}^n$ for $n > 1$.

Theorem 1.1.1 *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that*

(i) $\int_{\mathbb{R}^n} e^{pa\pi\|x\|^2} |f(x)|^p dx < \infty$,

(ii) $\int_{\mathbb{R}^n} e^{qb\pi\|y\|^2} |\hat{f}(y)|^q dy < \infty$,

where $a, b > 0$, and $min(p, q) < \infty$. If $ab \ge 1$ then $f = 0$ almost everywhere. If $ab < 1$ then there exist nonzero functions satisfying the above conditions.

Proof. As in $n = 1$, it is enough to prove the case $a = 1 = b$, otherwise we use dilation. By (i) and (ii) it follows from Hölder's inequality that $f, \hat{f} \in L^1(\mathbb{R}^n)$. Then for $\omega = u + iv \in \mathbb{C}$ and fixed $(y_1, \ldots, y_{n-1})$ we have

$$
\begin{aligned}
&|\hat{f}(y_1, \ldots, y_{n-1}, \omega)| \\
&= |\int_{\mathbb{R}^n} f(x_1, \ldots, x_n) e^{-2\pi i(\sum_1^{n-1} x_i y_i + \omega x_n)} dx_1, \ldots, dx_n| \\
&\le \int_{\mathbb{R}^n} |f(x_1, \ldots, x_n)| e^{2\pi v x_n} dx_1, \ldots, dx_n \\
&= \int_{\mathbb{R}} e^{\pi\|x\|^2} |f(x_1 \ldots x_n)| e^{-\pi\|x\|^2 + 2\pi v x_n} dx_1, \ldots, dx_n \\
&= e^{\pi v^2} \int_{\mathbb{R}^n} e^{\pi\|x\|^2} |f(x_1, \ldots, x_n)| e^{-\pi(\sum_{i=1}^{n-1} x_i^2 + (x_n - v)^2)} dx_1, \ldots, dx_n \\
&\le Const. e^{\pi v^2} \qquad (1.1)
\end{aligned}
$$

by Hölder's inequality and (i). By a standard argument using Lebesgue's dominated convergence theorem, Fubini's theorem and Morera's theorem it follows that $\omega \to \hat{f}(y_1, \ldots, y_{n-1}, \omega)$ is an entire function. We define as in [CP]

$$g(\omega) = e^{\pi\omega^2} \hat{f}(y_1, \ldots, y_{n-1}, \omega).$$

By the relation (1.1)

$$|g(\omega)| \leq Const.|e^{\pi\omega^2}|e^{\pi v^2} = Const.e^{\pi u^2}, \tag{1.2}$$

and by *ii*), $g|\mathbb{R} \in L^q(\mathbb{R}^n)$. Hence by lemma 1.1.1, $g = 0$, and thus $f = 0$ almost everywhere by the uniqueness of the Fourier transform.

The case $ab < 1$ easily follows from the one dimensional case: we take $f(\neq 0) \in L^1(\mathbb{R})$ satisfying the conditions *i*), *ii*) of the theorem for $ab < 1$ and define $F(x_1, \ldots, x_n) = f(x_1) \ldots f(x_n)$.

Note: We notice that the above proof actually gives us the following

Theorem 1.1.2 *Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. Suppose for some k, $1 \leq k \leq n$,*

(*i*) $\int_{\mathbb{R}^n} e^{pa\pi x_k^2} |g(x_1, \ldots, \hat{x}_k, \ldots, x_n)|^p |f(x_1, \ldots, x_n)|^p dx_1 \ldots dx_n < \infty$

(*ii*) $\int_{\mathbb{R}^n} e^{qb\pi y_k^2} |h(y_1, \ldots, \hat{y}_k, \ldots, y_n)|^q |\hat{f}(y_1, \ldots, y_n)|^q dy_1 \ldots dy_n < \infty$,

where $a, b > 0$, $g, h : \mathbb{R}^n \to \mathbb{C}$ are measurable and bounded away from zero such that $1/g \in L^{p'}(\mathbb{R}^{n-1})$, $1/p + 1/p' = 1$, $1/h \in L^{q'}(\mathbb{R}^{n-1})$, $1/q + 1/q' = 1$, $(x_1, \ldots, \hat{x}_k, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. If $ab \geq 1$ then $f = 0$ almost everywhere.

The case $ab > 1$ of the Cowling-Price theorem can actually be deduced from a very powerful result due to Beurling, which says that: if $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}^2} |f(x)||\hat{f}(y)|e^{2\pi|xy|} dx dy < \infty,$$

then $f = 0$ almost everywhere (see [Ho] for proof).

It is the n-dimensional version of Beurling's theorem on which now we are interested in, but that involves a suitable interpretation of the quantity '$|xy|$'. We suggest one such interpretation in the following theorem.

Theorem 1.1.3 *Let $f \in L^1(\mathbb{R}^n)$ and for some k, $1 \le k \le n$,*

$$\int_{\mathbb{R}^{2n}} |f(x_1, \dots, x_n)||\hat{f}(y_1, \dots, y_n)|e^{2\pi|x_k y_k|}dx_1 \dots dx_n dy_1 \dots dy_n < \infty.$$

Then $f = 0$ almost everywhere.

PROOF. We fix $\mathbf{y} = (y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)$. We define

$$g_{\mathbf{y}}(x) = \tilde{\mathcal{F}}_k f(y_1, \dots, y_{k-1}, x, y_{k+1}, \dots, y_n), \quad x \in \mathbb{R},$$

where

$$\begin{aligned}&\tilde{\mathcal{F}}_k f(y_1, \dots, y_{k-1}, x, y_{k+1}, \dots, y_n)\\ &= \int_{\mathbb{R}^{n-1}} f(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n)\\ &\quad\times e^{-2i\pi\langle(x_1,\dots,\hat{x_k},\dots,x_n),(y_1,\dots,\hat{y_k},\dots,y_n)\rangle}dx_1 \dots dx_{k-1}dx_{k+1}\dots dx_n.\end{aligned}$$

Then $\hat{g}_{\mathbf{y}}(y) = \hat{f}(y_1, \dots, y_{k-1}, y, y_{k+1}, \dots, y_n)$. Now

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} |g_{\mathbf{y}}(x)||\hat{g}_{\mathbf{y}}(y)|e^{2\pi|xy|}dxdy\\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}} |f(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n)||\hat{f}(y_1, \dots, y_{k-1}, y, y_{k+1}, \dots, y_n)|\\ &\quad\times e^{2\pi|xy|}dx_1 \dots dx_{k-1}dx_{k+1}\dots dx_n dx dy\\ &< \infty\end{aligned}$$

for almost every $(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)$ by hypothesis. So by Beurling's theorem on $\mathbb{R}$, for almost every $(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)$,

$$\tilde{\mathcal{F}}_k f(y_1, \dots, y_{k-1}, x, y_{k+1}, \dots, y_n) = 0$$

for almost every x. Hence by Fubini's theorem and the uniqueness of Fourier transform, $f = 0$ almost everywhere.

Corollary 1.1.3.1 *Let $f \in L^1(\mathbb{R}^n)$.*

(a) If $\int_{\mathbb{R}^{2n}} |f(x)||\hat{f}(y)|e^{2\pi\|x\|\|y\|}dxdy < \infty$ then $f = 0$ almost everywhere.

(*b*) *If*

$$\int_{\mathbb{R}^{2n}} |f(x_1,\dots,x_n)||\hat{f}(y_1,\dots,y_n)|e^{2\pi\Sigma_j|x_jy_j|}dx_1\dots dx_n dy_1\dots dy_n < \infty$$

then $f = 0$ *almost everywhere.*

(*c*) *Suppose for some* k, $1 \le k \le n$, f *and* $\hat{f}$ *satisfy*

(*i*) $|f(x_1,\dots,x_n)| \le Cg(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)e^{-a\pi|x_k|^p}$,

(*ii*) $|\hat{f}(y_1,\dots,y_n)| \le Ch(y_1,\dots,y_{k-1},y_{k+1},\dots,y_n)e^{-b\pi|y_k|^{p'}}$,

where $p^{-1}+p'^{-1} = 1$, g, $h(\ge 0) \in L^1(\mathbb{R}^{n-1})$. *If* $(ap)^{1/p}(bp')^{1/p'} > 2$ *then* $f = 0$ *almost everywhere.*

Remark 1.1.1 The case $ab = 1$ of the Cowling-Price theorem motivates us to ask the following question. An affirmative answer to this question yields a stronger form of Beurling's result.

Question: Suppose $f \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}^2} |f(x)|^p|\hat{f}(y)|^q e^{2\pi\sqrt{pq}|xy|}dxdy < \infty$, where $1 \le p,q < \infty$. Is $f = 0$ almost everywhere?

We have already noticed that for $ab > 1$ theorem 1.1.1 implies Hardy's theorem. The surprising fact is that the reverse implication is also true. In some sense this observation allows us to get an analogue of theorem 1.1.1 on two step nilpotent Lie groups. We conclude this chapter with this observation.

Let us introduce some notation. Let for $k > 0$ and $x \in \mathbb{R}^n$, $e_k(x) = e^{k\pi\|x\|^2}$ and for $1 \le p,q \le \infty$,

$$E_{p,q}(a,b) = \{f : \mathbb{R}^n \to \mathbb{C} \quad \text{measurable and } \|e_a f\|_p < \infty,\ \|e_b \hat{f}\|_q < \infty\}.$$

Theorem 1.1.4 *The following are equivalent :*

(*i*) *If* $ab > 1$ *then* $E_{\infty,\infty}(a,b) = 0$

(*ii*) *If* $ab > 1$ *then* $E_{p,q}(a,b) = 0$

PROOF. (*ii*) $\Rightarrow$ (*i*) Follows on noting that if $f \in E_{\infty,\infty}(a,b)$ then $f \in E_{p,q}(a',b')$ for $a' < a$ and $b' < b$.

$(i) \Rightarrow (ii)$ Let $f \in E_{p,q}(a,b)$ and $ab > 1$. Without loss of generality we assume that $p < \infty$. Let $g \in C_c(\mathbb{R}^n)$ be such that supp $g \subset \{y \in \mathbb{R}^n : \|y\| < \delta\}$. We choose an $\epsilon > 0$ which is to be specified later. We choose an $x \in \mathbb{R}^n$ such that $\|x\| > \delta/\epsilon$. Then for all y in the support of g we have

$$\|x - y\| \geq \|x\| - \|y\| > \|x\| - \delta > \|x\|(1-\epsilon) \tag{1.3}$$

since $e_a.f \in L^p(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^n)$ we have

$$(|e_a.f| * |g|)(y) \leq C \qquad \text{for all } y \in \mathbb{R}^n,$$

where C is a constant. By (1.3) we have

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{a\pi\|x-y\|^2}|f(x-y)||g(y)|dy \\ \geq \quad & e^{a\pi(1-\epsilon)^2\|x\|^2}\int_{\mathbb{R}^n}|f(x-y)||g(y)|dy, \quad \text{for } \|x\| > \delta/\epsilon. \end{aligned}$$

Hence

$$|(f*g)(x)| \leq (|f| * |g|)(x) \leq Ce^{-a\pi(1-\epsilon)^2\|x\|^2}$$

for all x such that $\|x\| > \delta/\epsilon$. Since $f * g$ is a continuous function, C can be so chosen that

$$|(f*g)(x)| \leq Ce^{-a\pi(1-\epsilon)^2\|x\|^2} \qquad \text{for all } x \in \mathbb{R}^n.$$

Also

$$\|e_b.\widehat{(f*g)}\|_q \leq \|\hat{g}\|_\infty \|e_b.\hat{f}\|_q < \infty$$

as $f \in E_{p,q}(a,b)$. It follows that $f * g \in E_{\infty,q}(a(1-\epsilon)^2, b)$. Suppose $q = \infty$, then we choose an ϵ such that $ab(1-\epsilon)^2 > 1$ and then by (i) $f * g = 0$ almost everywhere. By running g over an approximate identity we get $f = 0$ almost everywhere. Suppose now $q < \infty$. We define $f_1 = f * g$. Let $h \in C_c(\mathbb{R}^n)$ be such that supp $h \subset \{x : \|x\| < \delta_1\}$. We choose an $\epsilon > 0$ and do the same thing as above with $\hat{f}_1$ to get

$$|(\hat{f}_1 * h)(y)| \leq Ce^{-b(1-\epsilon)^2\pi\|y\|^2} \qquad \text{for all } y \in \mathbb{R}^n$$

Let $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then

$$|\mathcal{F}^{-1}(f_1 * h)(x)| \leq C_1 e^{-a(1-\epsilon)^2\pi\|x\|^2}$$

as $\hat{f}_1 \in L^1(\mathbb{R}^n)$. Thus $\hat{f}_1 * h \in E_{\infty,\infty}\left(a(1-\epsilon)^2, b(1-\epsilon_1)^2\right)$. We choose ϵ and ϵ_1 such that $ab(1-\epsilon)^2(1-\epsilon^1)^2 > 1$. By (i), $\hat{f}_1 * h = 0$ almost everywhere. By running h over an approximate identity we get $\hat{f}_1 = 0$. So $f * g = 0$ almost everywhere. By running g over an approximate identity we get $f = 0$ almost everywhere.

Chapter 2

Analysis on two step nilpotent Lie groups

The main objective of this chapter is to prove the Plancherel theorem for connected simply connected two step nilpotent Lie groups. To describe the unitary dual $\hat{G}$ of a two step nilpotent Lie group G, we will follow the orbit method of Kirillov (see [CG] for details). For the Plancherel theorem the following are the important steps:

i) To parametrize the coadjoint orbits of $\mathfrak{g}^*$ or at least to parametrize a set of coadjoint orbits which is of full Plancherel measure.

ii) Given $l \in \mathfrak{g}^*$, to construct a maximal subalgebra $\mathfrak{h}$ subordinate to l, that is $l([\mathfrak{h}, \mathfrak{h}]) = 0$.

For general nilpotent Lie groups i) and ii) have explicit answers by Chevalley-Rosenlicht theorem and Vergne polarizations (see [CG]), but as is only to be expected, on two step nilpotent Lie groups both i) and ii) turn out to be much simpler. After this we will go to explicit construction of irreducible unitary representations of G. In Kirillov theory the representations arise as induced representation, but as we will see, for the two step case they come directly from the Stone-von Neumann theorem. In section 2.1 we obtain the parametrization of the coadjoint orbits and section 2.2 is devoted to the unitary representations. In section 2.3 we prove the Plancherel theorem and the last section deals with the eigen functions of the sub-Laplacian.

2.1 Parametrization of coadjoint orbits

For a Lie algebra $\mathfrak{g}$ (we will always work with Lie algebras over $\mathbb{R}$), we define $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$.

Definition 2.1.1 A Lie algebra $\mathfrak{g}$ is called *two step nilpotent* if $\mathfrak{g}^2 = 0$ and $\mathfrak{g}^1 \neq 0$. The connected simply connected Lie group G corresponding to such a $\mathfrak{g}$ is called a *two step nilpotent lie group.*

We find it more convenient to look at a two step nilpotent Lie algebra in another way. Let

$$B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

be a nondegenerate, alternating, bilinear map. Let $\mathfrak{g} = \mathbb{R}^m \oplus \mathbb{R}^n$, we define

$$[(z, v), (z', v')] = (0, B(v, v')), \tag{2.1}$$

where $z, z' \in \mathbb{R}^m$ and $v, v' \in \mathbb{R}^n$. Then $[.,.]$ is a Lie bracket and $\mathfrak{g}$ is a two step nilpotent Lie algebra with $\mathbb{R}^m$ as the center of $\mathfrak{g}$. If on $G = \mathbb{R}^m \oplus \mathbb{R}^n$ we define the product

$$(z, v).(z', v') = (z + z' + \frac{1}{2}B(v, v'), v + v'), \tag{2.2}$$

then G is a connected, simply connected, two step nilpotent Lie group with $\mathfrak{g}$ as its Lie algebra (as $\gamma(t) = (tz, tv)$ $t \in \mathbb{R}$ is a one parameter subgroup) and $\exp : \mathfrak{g} \to G$ is the identity diffeomorphism. The computation of the Lie bracket is easy; in particular,

$$[v, v'] \equiv [(0, v), (0, v')] = B(v, v').$$

Let $\mathfrak{g}^*$ be the real dual of $\mathfrak{g}$. Then G acts on $\mathfrak{g}^*$ by the coadjoint action, that is $G \times \mathfrak{g}^* \to \mathfrak{g}^*$, $(g, l) \to g.l$ is given by

$$\begin{aligned} (g.l)(X) &= l(Adg^{-1}(X)), \qquad g \in G, l \in \mathfrak{g}^*, X \in \mathfrak{g}, \\ &= l(Ad(\exp Y)(X)), \qquad Y \in \mathfrak{g}, \\ &= l(e^{adY}(X)) \\ &= l(X) + l([Y, X]). \end{aligned}$$

We need to parametrize the orbits under this action. For this it is important to consider the structure of these orbits. let us fix some notation first. Let $l \in \mathfrak{g}^*$, then

O_l = The coadjoint orbit of l.

B_l = The skew symmetric matrix corresponding to l, that is, given a basis $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+n}\}$ of $\mathfrak{g}$ through the center, we consider the matrix $B_l = (B_l(i, j)) = (l([X_i, X_j])$.

r_l = The radical of the bilinear form B_l, that is,

$$r_l = \{X \in \mathfrak{g} : l([X, Y]) = 0 \quad \text{for all } Y \in \mathfrak{g}\}.$$

Clearly r_l is an ideal of $\mathfrak{g}$ and $\mathfrak{z}(= \mathbb{R}^m) \subset r_l$.

$\tilde{r}_l = \text{span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_{m+n}\} \cap r_l$.

$\tilde{B}_l = B_l \mid \mathbb{R}^n \times \mathbb{R}^n$ that is restriction of B_l on the complement of the center of $\mathfrak{g}$.

Theorem 2.1.1 *Let $l \in \mathfrak{g}^*$. Then $O_l = l + r_l^\perp$ where $r_l^\perp = \{h \in \mathfrak{g}^* : h \mid r_l = 0\}$ (that is, the coadjoint orbits are hyperplanes).*

Proof. Let $l' \in O_l$. Then $l' = l \circ Ad(\exp X)$ for some $X \in \mathfrak{g}$. Then for $Y \in r_l$

$$(l - l')(Y) = l(Y) - l'(Y) = l(Y) - l(Y) - l([X, Y]) = 0.$$

Thus $l' = l + (l' - l) \in l + r_l^\perp$. Hence $O_l \subseteq l + r_l^\perp$.

Let $\{X_1, \ldots, X_k, X_{k+1}, \ldots, X_{m+n}\}$ be a basis of $\mathfrak{g}$ passing through r_l in the sense that $span_{\mathbb{R}}\{X_1, \ldots, X_K\} = r_l$. Let $l' \in l + r_l^\perp$ and $l'(X_i) = l'_i$, $l(X_i) = l_i, 1 \leq i \leq m+n$. We want to get hold of an $X \in \mathfrak{g}$ such that

$$l(X_i) + l([X, X_i]) = l'(X_i), \quad k+1 \leq i \leq m+n$$

that is

$$l([X, X_i]) = l'_i - l_i, \quad k+1 \leq i \leq m+n.$$

Expressing $X = \sum_{j=1}^{k} a_j X_j + \sum_{j=k+1}^{m+n} \alpha_j X_j$, we are looking for the solutions of

$$\sum_{j=k+1}^{m+n} \alpha_j l([X_j, X_i]) = l'_i - l_i, \quad k+1 \leq i \leq m+n,$$

which is a system of $m+n-k$ linear equations in $m+n-k$ unknowns. Since the matrix $L = (L_{ij}) = (l([X_{k+i}, X_{k+j}]))$ is just the matrix of the bilinear form corresponding to the linear functional $\bar{l}$ on $\mathfrak{g}/r_l$, L is invertible. So the above system has a unique solution $(\alpha_1, \ldots, \alpha_{m+n-k})$ say. Then for any $Y \in r_l$, we have

$$\exp(Y + \sum_{j=1}^{m+n-k} \alpha_j X_{k+j})^{-1}.l = l'.$$

So $l' \in O_l$ and hence $O_l = l + r_l^\perp$. This completes the proof.

Note 2.1.1 : By theorem 2.1.1, $l' \in O_l$ if and only if $r_l = r_{l'}$ and $l \mid r_l = l' \mid r_{l'}$.

From now on $\mathfrak{g}$ stands for a two step nilpotent Lie group with $\dim \mathfrak{g} = n$. Let $\mathcal{B} = \{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ such that

$$\text{span}_{\mathbb{R}}\{X_1, \ldots, X_m\} = \text{center of } \mathfrak{g} = \mathfrak{z}.$$

So B_l is the $n \times n$ matrix whose $(i,j)-$th entry is $l([X_i, X_j])$, $1 \leq i, j \leq n$. Let $\mathcal{B}^* = \{X_1^*, \ldots, X_n^*\}$ be the dual basis of $\mathfrak{g}^*$. This is a Jordan-Hölder basis, that is $\mathfrak{g}^* = \text{span}_{\mathbb{R}}\{X_1^*, \ldots, X_j^*\}$ is $Ad^*(G)$ stable for $1 \leq j \leq n$.

Let $l \in \mathfrak{g}^*$ and $X_i \in \mathcal{B}$.

Definition 2.1.2 i is called a *jump index* for l if the rank of the $i \times n$ submatrix of B_l, consisting of first i rows, is strictly greater than the rank of the $(i-1) \times n$ submatrix of B_l, consisting of first $(i-1)$ rows.

Since an alternating bilinear form has even rank the number of jump indices must be even. The set of jump indices are denoted by $J = \{j_1, \ldots, j_{2k}\}$. Notice that $j_1 \geq m+1$. The subset of $\mathcal{B}$ corresponding to J is then $\{X_{j_1}, \ldots, X_{j_{2k}}\}$. Notice that if i is a jump index then rankB_l^i =rank$B_l^{i-1}+1$, where B_l^i is the submatrix of B_l consisting of first i rows.

Note 2.1.2 : These jump indices depend on l and on the order of the basis as well. But ultimately we will restrict ourselves to 'generic linear functionals' and they will have the same jump indices.

Now we are going spell out what we mean by *generic linear functionals*. This is also a basis dependent definition. We work with the basis $\mathcal{B}$ chosen above. Let us fix some notations. Let $R_i(l)$ =rankB_l^i and $R_i = \text{Max}\{R_i(l)/l \in \mathfrak{g}^*\}$.

Definition 2.1.3 A linear functional $l \in \mathfrak{g}^*$ is called *generic* if $R_i(l) = R_i$ for all $i, 1 \leq i \leq n$.

Let $\mathcal{U} = \{l \in \mathfrak{g}^* : l \text{ is generic}\}$

Example 2.1.1 : Let $\mathfrak{g} = \text{span}_{\mathbb{R}}\{X_1, X_2, X_3, X_4, X_5\}$. The nonzero brackets are given by

$$[X_5, X_3] = X_1, \quad [X_5, X_4] = X_2.$$

Clearly $\mathfrak{z} = \text{span}_{\mathbb{R}}\{X_1, X_2\}$. This Lie algebra arises from the bilinear form

$$B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^2$$

$$B\left((a_1, a_2, a_3), (a_1', a_2', a_3')\right) = (a_3 a_1' - a_1 a_3', a_3 a_2' - a_3' a_2).$$

Let $l = \sum_{i=1}^{5} l_i X_i^*$. Then

$$B_l = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -l_1 \\ 0 & 0 & 0 & 0 & -l_2 \\ 0 & 0 & l_1 & l_2 & 0 \end{pmatrix}$$

and $R_1(l) = R_2(l) = 0, R_3(l) = 1$ if $l_1 \neq 0, R_4(l) = 1$ if $l_1 \neq 0, R_5(l) = 2$ if one of l_1 or l_2 is nonzero. Clearly $R_1 = R_2 = 0, R_3 = R_4 = 1, R_5 = 2$. Thus

$$\mathcal{U} = \{l \in \mathfrak{g}^* : l_1 = l(X_1) \neq 0\}$$

and 3, 5 are jump indices. We call this Lie algebra $q\mathfrak{f}_{3,2}$ and the corresponding group $QF_{3,2}$.

Example 2.1.2 : Let $\mathfrak{f}_{3,2} = \text{span}_{\mathbb{R}}\{X_1, \ldots, X_6\}$ with nontrivial brackets

$$[X_4, X_5] = X_1, \quad [X_4, X_6] = X_2, \quad [X_5, X_6] = X_3.$$

Thus $\mathfrak{z} = \text{span}_{\mathbb{R}}\{X_1, X_2, X_3\}$. Let $l = \sum_{i=1}^{6} l_i X_i^* \in \mathfrak{g}^*$. Then

$$B_l = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_1 & l_2 \\ 0 & 0 & 0 & -l_1 & 0 & l_3 \\ 0 & 0 & 0 & -l_2 & -l_3 & 0 \end{pmatrix}$$

Thus $R_1(l) = R_2(l) = R_3(l) = 0$. $R_4(l) = 1$ if $l_1 \neq 0$ or $l_2 \neq 0$.

$$R_5(l) = \begin{cases} 2 & \text{if } l_1 \neq 0 \\ 0 \text{ or} 1 & \text{if } l_1 = 0 \end{cases}$$

and

$$R_6(l) = \begin{cases} 2 & \text{if } l_1 \neq 0 \\ 0 \text{ or} 2 & \text{if } l_1 = 0 \end{cases}$$

Thus $R_1 = R_2 = R_3 = 0, R_4 = 1, R_5 = R_6 = 2$. Hence $\mathcal{U} = \{l \in \mathfrak{g}^* : l_1 = l(X_1) \neq 0\}$ and $4, 5$ are jump indices.

Example 2.1.3 : let n be an even number. Let $\Lambda^2(\mathbb{R}^n)$ = The set of $n \times n$ skew symmetric matrices. $\Lambda^2(\mathbb{R}^n)$ is an Euclidean space with the inner product $\langle y, y' \rangle = \frac{1}{2}\sum_{j,k} y_{jk}y'_{jk} = \frac{1}{2}Tr(y(y')^t)$ for $y, y' \in \Lambda^2(\mathbb{R}^n)$. Consider the map

$$\Lambda : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \Lambda^2(\mathbb{R}^n)$$

$$\Lambda(x, x')_{jk} = x_j x'_k - x_k x'_j.$$

Then Λ is alternating, bilinear and nondegenerate. Let $\mathfrak{f}_{n,2} = \Lambda^2(\mathbb{R}^n) \oplus \mathbb{R}^n$ with the induced Lie bracket. If $\{e_i : 1 \leq i \leq n\}$ is the canonical orthonormal basis for $\mathbb{R}^n$, then $\{E_{ij} = \Lambda(e_i, e_j) : j > i\}$ is an orthonormal basis for $\Lambda^2(\mathbb{R}^n)$. So we can identify $\Lambda^2(\mathbb{R}^n)$ with $\mathbb{R}^{\frac{n}{2}(n-1)}$ with respect to the above basis. Let $l \in \Lambda^2(\mathbb{R}^n)^*$. Thus $l(A) = \langle L, A \rangle = \frac{1}{2}Tr(LA^t)$ for all $A \in \Lambda^2(\mathbb{R}^n)$, and for some $L \in \Lambda^2(\mathbb{R}^n)$, representing l. Hence for $x, y \in \mathbb{R}^n$,

$$\begin{aligned} B_l(x, y) &= \langle L, \Lambda(x, y) \rangle_{\Lambda(\mathbb{R}^n)} \\ &= \frac{1}{2}Tr(L.\Lambda(x, y)^t) \\ &= -\frac{1}{2}Tr(L.\Lambda(x, y)) \\ &= -\frac{1}{2}\langle L(x), y \rangle_{\mathbb{R}^n}. \end{aligned}$$

Thus

$$x \in \tilde{r}_l \;\Rightarrow\; B_l(x, y) = 0 \qquad \text{for all } y \in \mathbb{R}^n$$

$$\Rightarrow \quad \langle L(x), y\rangle = 0 \qquad \text{for all } y \in \mathbb{R}^n$$

$$\Rightarrow \quad L(x) = 0$$

$$\Rightarrow \quad x \in \ker L.$$

Thus if $L \in \Lambda^2(\mathbb{R}^n)$ is such that $\det L \neq 0$, then $r_l = \mathfrak{z}$. Thus basis indices outside the center are all jump indices and $\mathcal{U} = \{L \in \Lambda^2(\mathbb{R}^n) : \det L \neq 0\}$.

Note 2.1.3 : If $l \in \mathfrak{g}^*$ is such that $\tilde{B}_l$ is an invertible matrix, then $r_l = \mathfrak{z}$ and then $m+1, \ldots, n$ are jump indices and then

$$\mathcal{U} = \{l \in \mathfrak{g}^* : \tilde{B}_l \text{ is an invertible matrix}\}.$$

Clearly, if codimension of $\mathfrak{z}$ in $\mathfrak{g}$ is odd then this cannot happen. Following [MW], we call the two step nilpotent Lie algebras, *MW algebras* if there exist $l \in \mathfrak{g}^*$ such that $\tilde{B}_l$ is nondegenerate (or the corresponding matrix is invertible). So Heisenberg algebras and $\mathfrak{f}_{n,2}$ (n even) are MW algebras.

Remark 2.1.1 : Since for any $l \in \mathfrak{g}^*$, we have $g.l \mid \mathfrak{z} = l \mid \mathfrak{z}$ where $g.l = l \circ Adg^{-1}$, we get $R_i(l) = R_i(g.l)$, $1 \leq i \leq n$ and hence,

(i) $\mathcal{U}$ is a $G-$invariant Zariski open subset of $\mathfrak{g}^*$. So $\mathcal{U}$ is union of orbits.

(ii) If j is a jump index for some $l \in \mathcal{U}$, then j is a jump index for all $l \in \mathcal{U}$.

(iii) Let $l \in \mathcal{U}$, then the number of jump indices for l is the same as the dimension of O_l (as a manifold). For, the rank of the matrix B_l is equal to the number of jump indices ($= 2k$, say) and the dimension of the radical r_l is the nullity of the matrix of B_l, which is $n - 2k$. Since $\mathfrak{g}/r_l$ is diffeomorphic to O_l (see [CG]), we have $\dim O_l = 2k$.

(iv) Every orbit in $\mathcal{U}$ is of maximum dimension though every maximum dimensional orbit may not be in $\mathcal{U}$.

Example 2.1.4 : We consider $QF_{3,2}$. Here maximum dimensional orbits are two dimensional. Let $l \in \mathfrak{g}^*$ be such that $l_1 = l(X_1) = 0$ and $l_2 = l(X_2) \neq 0$. then $\dim O_l = 2$ as $4, 5$ are jump indices but $l \notin \mathcal{U}$.

Our aim is to parametrize the orbits in $\mathcal{U}$. We will see that they constitute a set of full Plancherel measure. We again describe some notation

$N = \{1, \ldots, m, n_1, \ldots . n_r\} \subset \{1, \ldots, n\}$ is the complement of J in $\{1, \ldots, n\}$,

$V_J = \text{span}_{\mathbb{R}}\{X_{j_i} : 1 \leq i \leq 2k, j_i \in J\}$,

$V_N = \text{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{n_i} : 1 \leq i \leq r, n_i \in N\}$,

$V_J^* = \text{span}_{\mathbb{R}}\{X_{j_1}^*, \ldots, X_{j_{2k}}^*\}$,

$V_N^* = \text{span}_{\mathbb{R}}\{X_1^*, \ldots, X_m^*, X_{n_i}^* : n_i \in N\}$,

$\tilde{V}_N^* = \text{span}_{\mathbb{R}}\{X_{n_i}^* : n_i \in N\}$.

Now we come to the main theorem of this section. A basic tool here is the theorem 2.1.1.

Theorem 2.1.2 *(i) V_N^* intersects every orbit in $\mathcal{U}$ at a unique point.*
(ii) There exist a birational homeomorphism $\Psi : (V_N^* \cap \mathcal{U}) \times V_J^* \to \mathcal{U}$.

PROOF. (i) Let $l \in \mathcal{U}$. We first try to describe $\tilde{r}_l$. Denoting by $\tilde{\rho}_i(l)$ the $i-$th row of the matrix $\tilde{B}_l$, every vector $\tilde{\rho}_{n_i}(l)$, $n_i \in N$, is a unique linear combination of j_s-th rows of $\tilde{B}_l$, $1 \leq s \leq 2k$ that is

$$\tilde{\rho}_{n_i}(l) = \sum_{s=1}^{2k} c_s^i(l) \tilde{\rho}_{j_s}(l)$$

where the scalars $c_s^i(l)$ depend rationally on l, in fact they depend only on $l \mid \mathfrak{z}$. Also if $j_s > n_i$, $c_s^i(l) = 0$. Thus

$$\begin{aligned}
\tilde{\rho}_{n_i}(l) &= (l([X_{n_i}, X_{m+1}]), \ldots, l([X_{n_i}, X_n])) \\
&= \sum_{s=1}^{2k} c_s^i(l)\,(l([X_{j_s}, X_{m+1}]), \ldots, l([X_{j_s}, X_{m+1}])) \\
&= \left(l([\sum_{s=1}^{2k} c_s^i(l) X_{j_s}, X_{m+1}]), \ldots, l([\sum_{s=1}^{2k} c_s^i(l) X_{j_s}, X_n]) \right).
\end{aligned}$$

So

$$\left(l([X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s}, X_{m+1}]), \ldots, l([X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s}, X_n]) \right) = 0.$$

Hence $\tilde{X}_{n_i} = X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s} \in \tilde{r}_l$. Since $\{\tilde{X}_{n_i} : 1 \le i \le r\}$ are linearly independent vectors in $\tilde{r}_l$ we have $\tilde{r}_l = \mathrm{span}_{\mathbb{R}}\{\tilde{X}_{n_i} : i \le i \le r\}$. We need to exhibit a unique $\bar{l} \in V_N^*$ (that is, $\bar{l}(X_{j_i}) = 0$, $1 \le j \le 2k$) such that $\bar{l} \in O_l$; so $\bar{l}$ has to satisfy $r_l = r_{\bar{l}}$ and $l \mid r_l = \bar{l} \mid r_{\bar{l}}$ by note 2.1.1.

We define $\bar{l} \mid \mathfrak{z} = l \mid \mathfrak{z}$. For any such $\bar{l}$,

$$r_{\bar{l}} = r_l = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s} : 1 \le j \le r\}.$$

We also define

$$\bar{l}(X_{j_i}) = 0, \quad 1 \le i \le k$$

and

$$\bar{l}(X_{n_i}) = l(X_{n_i}) - \sum_{s=1}^{2k} c_s^i(l) l(X_{j_s}), \quad 1 \le i \le r.$$

Thus $\bar{l} \mid r_{\bar{l}} = l \mid r_l$. So $\bar{l} \in O_l$.

Suppose there exist $l' \in \mathfrak{g}^*$ such that $l'(X_{j_i}) = 0, 1 \le i \le k$ and $r_{l'} = r_l$ with $l' \mid r_{l'} = l \mid r_l$. Then $l' \mid \mathfrak{z} = l \mid \mathfrak{z} = \bar{l} \mid \mathfrak{z}$ in particular. Now for all i, $1 \le i \le r$,

$$l'(X_{n_i}) = l'\left(X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s} \right) \quad \text{as } l'(X_{j_i}) = 0$$

$$\begin{aligned} &= l\left(X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s}\right) \quad \text{as } l' \mid r_{l'} = l \mid r_l \\ &= \bar{l}(X_{n_i}) \quad \text{by definition of } \bar{l}. \end{aligned}$$

This completes the proof of (i).

(ii) Let $(l_N, l_J) \in (V_N^* \cap \mathcal{U}) \times V_J^*$ where

$$l_N = \sum_{i=1}^{m} l_i X_i^* + \sum_{i=1}^{r} l_{n_i} X_{n_i}^* \quad \text{and} \quad l_J = \sum_{i=1}^{2k} l_{j_i} X_{j_i}^*$$

Since $l_N \in \mathcal{U}$, there exist constants $c_s^i(l_N) = c_s^i(l_1, \ldots, l_m)$ such that $\tilde{r}_{l_N} = \text{span}_{\mathbb{R}}\{X_{n_i} - \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) X_{j_s} : 1 \leq i \leq r\}$. Now we define Ψ by putting

$$\begin{aligned} \Psi(l_N, l_J)(X_i) &= l_i, \qquad 1 \leq i \leq m, \\ \Psi(l_N, l_J)(X_{j_i}) &= l_{j_i}, \qquad 1 \leq i \leq 2k, \\ \Psi(l_N, l_J)(X_{n_i}) &= l_{n_i} + \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) l_{j_s}. \end{aligned}$$

As

$$\tilde{r}_{\Psi(l_N, l_J)} = \text{span}_{\mathbb{R}}\{X_{n_i} - \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) X_{j_s} : 1 \leq i \leq r\} = \tilde{r}_{l_N},$$

and

$$\begin{aligned} &\Psi(l_N, l_J)\left(X_{n_i} - \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) X_{j_s}\right) \\ &= l_{n_i} + \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) l_{j_s} - \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) l_{j_s} \\ &= l_N(X_{n_i} - \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) X_{j_s}), \end{aligned}$$

it follows that $\Psi(l_N, l_J) \in O_{l_N} \subseteq \mathcal{U}$. Thus Ψ is well defined. It is easy to describe $\Psi^{-1} : \mathcal{U} \to (V_N^* \cap \mathcal{U}) \times V_J^*$. Let $l \in \mathcal{U}$ with $l(X_i) = l_i, 1 \leq i \leq m$,

$l(X_{n_i}) = l_{n_i}, 1 \leq i \leq r$ and $l(X_{j_i}) = l_j, i \leq i \leq 2k$. Then $\Psi^{-1}(l) = (l_N, l_J)$ is defined by the conditions

$$\begin{aligned} l_N(X_i) &= l_i, \qquad 1 \leq i \leq m, \\ l_N(X_{n_i}) &= l_{n_i} - \sum_{s=1}^{2k} c_s^i(l_1, \ldots, l_m) l_{j_s}, \qquad 1 \leq i \leq r, \\ l_J(X_{j_i}) &= l_{j_i}, \qquad 1 \leq i \leq r. \end{aligned}$$

Clearly Ψ is birational. This completes the proof.

Note 2.1.4 : If we keep l_N fixed, then $\Psi(l_N, l_J)$ is a polynomial in $(l_{j_1}, \ldots, l_{j_{2k}})$ and the graph of this polynomial is O_{l_N}.

Example 2.1.5 : Let $G = QF_{3,2}$. Then $N = \{1, 2, 4\}$ and $J = \{3, 5\}$, $\mathcal{U} = \{l \in q\mathfrak{f}_{3,2}^* : l_1 = l(X_1) \neq 0\}$, $n_1 =$ First nonjump index outside center $= 4$, $V_N^* \cap \mathcal{U} = \{l \in q\mathfrak{f}_{3,2}^* : l_1 = l(X_1) \neq 0, l_3 = l(X_3) = l_5 = l(X_5) = 0\}$, $\tilde{\rho}_4(l) = \frac{l_2}{l_1}\tilde{\rho}_3(l)$. Thus $c_3^4(l_1, l_2) = \frac{l_2}{l_1}$ and $c_5^4(l_1, l_2) = 0$. Thus

$$\Psi((l_1, l_2, l_4), (l_3, l_5)) = (l_1, l_2, l_3, l_4 + \frac{l_2}{l_1} l_3, l_5)$$

where $(l_1, l_2, l_3) \in V_N^* \cap \mathcal{U}$ and $(l_3, l_5) \in V_J^*$ with the obvious interpretation.

Example 2.1.6 : Let $G = F_{3,2}$. Then $N = \{1, 2, 3, 6\}$, $J = \{4, 5\}$ $\mathcal{U} = \{l \in \mathfrak{f}_{3,2}^* : l_1 = l(X_1) \neq 0\}$, $V_N^* \cap \mathcal{U} = \{l \in \mathfrak{f}_{3,2}^* : l_1 = l(X_1) \neq 0, l_4 = l(X_4) = l_5 = l(X_5) = 0\}$, $n_1 =$ the first jump index outside center $= 6$.

$$\tilde{\rho}_6(l) = -\frac{l_3}{l_1}\tilde{\rho}_4(l) + \frac{l_2}{l_1}\tilde{\rho}_5(l).$$

Thus $c_4^6(l) = -l_3/l_1, c_5^6(l) = l_2/l_1$. So

$$\Psi((l_1, l_2, l_3, l_6), (l_4, l_5)) = (l_1, l_2, l_3, l_4, l_5, l_6 - \frac{l_3}{l_1} l_4 + \frac{l_2}{l_1} l_5),$$

where, as before $(l_1, l_2, l_3, l_6) \in V_N^* \cap \mathcal{U}$ and $(l_4, l_5) \in V_J^*$.

Note 2.1.5 : For each coadjoint orbit in $\mathcal{U}$, we choose their representatives from $V_N^* \cap \mathcal{U}$. Notice that $V_N^* \cap \mathcal{U}$ can be identified with the Cartesian product of $\tilde{V}_N^*$ and a Zariski open subset $\mathcal{U}'$ of $\mathfrak{z}^*$, where $\mathcal{U}' = \{l \in \mathfrak{z}^* / R_i(l) = R_i, 1 \leq i \leq m\}$. In the next section our aim will be to construct irreducible unitary representations corresponding to elements in $V_N^* \cap \mathcal{U}$ by the orbit method of Kirillov.

2.2 Polarization and unitary representation

We begin with a brief discussion of Kirillov theory, for details see [CG].

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. G acts on $\mathfrak{g}^*$ by the coadjoint action. Given any $l' \in O_l$, the coadjoint orbit of l, there exist a subalgebra $\mathfrak{h}_{l'}$ of $\mathfrak{g}$ which is maximal with respect to the property

$$l'([\mathfrak{h}_{l'}, \mathfrak{h}_{l'}]) = 0. \tag{2.3}$$

Thus we have a character $\chi_{l'} : \exp(\mathfrak{h}_{l'}) \longrightarrow \mathbb{T}$ given by

$$\chi_{l'}(\exp X) = e^{2\pi i l'(X)}, X \in \mathfrak{h}_{l'}.$$

Let $\pi_{l'} = ind_{\exp(\mathfrak{h}_{l'})}^{G} \chi_{l'}$. Then

(1) $\pi_{l'}$ is an irreducible unitary representation of G.

(2) If $\mathfrak{h}'$ is another subalgebra maximal with respect to the property $l'([\mathfrak{h}', \mathfrak{h}']) = 0$, then $ind_{\exp(\mathfrak{h}_l')}^{G} \chi_l \cong ind_{\exp(\mathfrak{h}')}^{G} \chi_{l'}$.

(3) $\pi_{l_1} \cong \pi_{l_2}$ if and only if l_1 and l_2 belong to the same coadjoint orbit.

(4) Any irreducible unitary representation π of G is equivalent to π_l for some $l \in \mathfrak{g}^*$.

So we have a map $\kappa : \mathfrak{g}^*/Ad^*(G) \rightarrow \hat{G}$, which is a bijection. A subalgebra corresponding to $l \in \mathfrak{g}^*$, maximal with respect to (2.3) is called a *polarization*. It is known that the maximality of $\mathfrak{h}$ with respect to (2.3) is equivalent to the following *dimension condition*

$$\dim \mathfrak{h} = \frac{1}{2}(\dim \mathfrak{g} + \dim r_l).$$

Now suppose $\mathfrak{g}$ is a two step nilpotent Lie algebra and $l \in \mathfrak{g}^*$. The following technique for construction of a polarization corresponding to l, seems to be standard: we consider the bilinear form $\tilde{B}_l$ on the complement of the center, we restrict $\tilde{B}_l$ on its nondegenerate subspace, then on that subspace one can choose a basis with respect to which $\tilde{B}_l$ is the canonical symplectic form. With a little modification the basis can be chosen to be orthonormal as well. This is essentially what was done in [MR1], [BJR], [St], [Pa]. We will set down the basis change explicitly; our main ingredient for that is the following lemma.

Lemma 2.2.1 *Let $B : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a nondegenerate, alternating, bilinear form. Then there exist an orthonormal basis $\{W_i, Y_i / 1 \leq i \leq k\}$ of $\mathbb{R}^n$ such that $B(X_i, Y_j) = \delta_{i,j}\lambda_j(B), B(X_i, X_j) = B(Y_i, Y_j) = 0, 1 \leq i, j \leq k, n = 2k$ where $\pm i\lambda_j(B)$ are eigenvalues of the matrix of B.*

As a consequence we have the following.

Corollary 2.2.0.1 *Let $l \in \mathfrak{g}^*$. Then there exist an orthonormal basis*

$$\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l), Y_1(l), \ldots, Y_k(l)\} \qquad (2.4)$$

of $\mathfrak{g}$ such that

a) $r_l = span_{\mathbb{R}}\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l)\}$.

b) $l([W_i(l), Y_j(l)]) = \delta_{ij}\lambda_j(l), 1 \leq i, j \leq k$ *and*

$$l([W_i(l), W_j(l)]) = l([Y_i(l), Y_j(l)]) = 0, 1 \leq i, j \leq k.$$

c) $span_{\mathbb{R}}\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l)\} = \mathfrak{h}$ *is a polarization for* l.

PROOF. We choose a basis $\mathcal{B} = \{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}$ of $\mathfrak{g}$ such that $\text{span}_{\mathbb{R}}\{X_1, \ldots, X_m\} = \mathfrak{z}$. We define the Euclidean inner product on $\mathfrak{g}$ such that $\mathcal{B}$ is an orthonormal basis. Let $l \in \mathfrak{g}^*$ and suppose $\dim r_l = m + r$ and $\dim O_l = 2k = n - m - r$. We get hold of $r_l = \text{span}_{\mathbb{R}}\{X_1, \ldots, X_m, \tilde{X}_{n_i} = X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s}, 1 \leq i \leq r\}$. we use Gram- Schmidt orthogonalization on r_l to get an orthonormal basis $\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l)\}$. On $r_l^{\perp}$, the orthogonal complement of r_l, $\tilde{B}_l$ is nondegenerate. By lemma (2.2.1) we get an orthonormal basis $\{W_1(l), \ldots, W_k(l), Y_1(l), \ldots, Y_k(l)\}$ of $r_l^{\perp}$ such that $l([W_i(l), Y_j(l)]) = \delta_{ij}\lambda_j(B_l)$ and $l([W_i(l), W_j(l)]) = l([Y_i(l), Y_j(l)]) = 0$. If we define $\lambda_j(B_l) = \lambda_j(l), 1 \leq j \leq k$ then *a*), and *b*) follow. c) follows by observing that $\mathfrak{h}$ satisfies (2.3) and the dimension condition.

Note 2.2.1 : we call the above basis an *almost symplectic basis.* Given $X \in \mathfrak{g}$ and a basis (2.4) we write

$$X = \sum_{j=1}^{m} x_j X_j(l) + \sum_{j=1}^{r} z_j Z_j(l) + \sum_{j=1}^{k} w_j W_j(l) + \sum_{j=1}^{k} y_j Y_j(l) \equiv (x, z, w, y).$$

Since we are going to use induced representations we need to describe nice sections of G/H and a $G-$invariant measure on G/H. In our situation we will always have that H is a normal subgroup of G. We identify G and $\mathfrak{g}$ via the exponential map. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ containing $\mathfrak{z}$ and $H = \exp \mathfrak{h}$.

We take $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}, \ldots, X_n\}$ a basis of $\mathfrak{g}$ such that

$$\mathfrak{z} = \text{span}_{\mathbb{R}}\{X_1, \ldots, X_m\}, \quad \mathfrak{h} = \text{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}\}.$$

If $L_g(x) = g^{-1}x$ and $R_g(x) = xg$, $x, g \in G$, then it is clear from the group multiplication that the Jacobian matrix for either of the transformations is upper triangular with diagonal entries 1. Thus we have

Lemma 2.2.2 *Let $\mathfrak{g}$, $\mathfrak{h}$, $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}, \ldots, X_n\}$ be as before. Then*

i) $dx_1 \ldots dx_n$ is a left and right invariant measure on G.

ii) $\sigma : G/H \to G$ given by

$$\sigma\left(\exp(\sum_{i=1}^{n} t_i X_i)H\right) = \exp\left(\sum_{i=1}^{n-m-k} t_i X_{m+k+i}\right),$$

is a section for G/H.

iii) $dx_{m+k+1} \ldots dx_n$ is a left $G-$invariant measure on G/H.

Now we come to the construction of representations corresponding to $l \in V_N^\bullet \cap \mathcal{U}$. Let $\dim r_l = m+r$ and $\dim O_l = 2k$ so $m+r+2k = n$. We choose an almost symplectic basis (2.4) of $\mathfrak{g}$ corresponding to l and get hold of $\mathfrak{h}_l$ as in corollary 2, c). On $H_l = exp(\mathfrak{h}_l)$ we have the character $\chi_l : H_l \longrightarrow \mathbb{T}$. Let $\pi_l = ind_{H_l}^G \chi_l$. We do not use the standard model for the induced representation as given in chapter 2 of [CG], rather using the continuous section σ given in lemma 2.2.2 and computing the unique splitting of a typical group element

$$(x, z, w, y) = (0, 0, 0, y)(x - \frac{1}{2}[(0, 0, 0, y), (0, z, w, 0)], z, w, 0),$$

corresponding to σ, the representation π_l is realised on $L^2(\mathbb{R}^k)$ and is given by

$$\begin{aligned} & (\pi_l(x, z, w, y)f)(\bar{y}) \qquad f \in L^2(\mathbb{R}^k), \\ = \quad & e^{2\pi i(l(x)+l(z)+l(w)-(1/2)\sum_{i=1}^{k} y_i w_i \lambda_i(l) + \sum_{i=1}^{k} \bar{y}_i w_i \lambda_i(l))} f(\bar{y} - y), \qquad (2.5) \end{aligned}$$

for almost every $\bar{y} \in \mathbb{R}^k$. At this point we indulge ourselves a little to stop to show that, for two step nilpotent lie groups, the Kirillov theory can be totally bypassed. The conclusions 3) and 4) listed at the beginning of the section can be reached through a straight forward application of the Stone-von Neumann theorem. This fact is most likely known to experts, our justification for including it here is that we know of no source pointing it out clearly.

Suppose π' is an irreducible unitary representation of G acting on the Hilbert space $\mathcal{H}_{\pi'}$, with the condition that $\pi'(\exp X) = e^{2\pi i l(X)} I$ where $X \in \mathfrak{z}$ and $l \in \mathfrak{z}^*$. As before we get hold of an almost symplectic basis (2.4) (note that r_l is actually determined by $l \mid \mathfrak{z}$). We again write elements of the Lie algebra and the group as well by (x, z, w, y). Then using (2.2) it is easy to show that π' has to satisfy the following properties:

a) $\pi'(0, z, 0, 0)\pi'(0, z_1, 0, 0) = \pi'(0, z + z_1, 0, 0)$,

b) $\pi'(0, z, 0, 0)\pi'(0, 0, w, y) = \pi'(0, 0, w, y)\pi'(0, z, 0, 0)$,

c) $\pi'(0, 0, w, 0)\pi'(0, 0, w_1, 0) = \pi'(0, 0, w + w_1, 0)$,

d) $\pi'(0, 0, 0, y)\pi'(0, 0, 0, y_1) = \pi'(0, 0, 0, y + y_1)$,

e) $\pi'(0, 0, w, 0)\pi'(0, 0, 0, y) = e^{2\pi i \sum_{i=1}^{k} w_i y_i \lambda_i(l)} \pi'(0, 0, 0, y)\pi'(0, 0, w, 0)$.

From a) and b), it follows by Schur's lemma that,

$$\pi'(0, z, 0, 0) = e^{2\pi i \tilde{l}(z)} \quad \tilde{l} \in \text{span}_{\mathbb{R}}\{Z_1(l), \ldots, Z_r(l)\}^*.$$

By c)-e) and Stone-von Neumann theorem $\mathcal{H}_{\pi'}$ is unitarily equivalent to $L^2(\mathbb{R}^k)$ and

$$\begin{aligned} (\pi'(0, 0, 0, y)f)(\bar{y}) &= f(\bar{y} - y), \qquad f \in L^2(\mathbb{R}^k), \\ (\pi'(0, 0, w, 0)f)(\bar{y}) &= e^{2\pi i \sum_{i=1}^{k} w_i \bar{y}_i \lambda_i(l)} f(\bar{y}), \quad \bar{y} \in \mathbb{R}^k, \end{aligned}$$

for almost every $\bar{y} \in \mathbb{R}^k$.Then by using the fact that

$$\begin{aligned} & (x, z, w, y) \\ = \; & (x - (1/2)[(0, z, 0, 0), (0, 0, w, 0)] - (1/2)[(0, z, w, 0), (0, 0, 0, y)], 0, 0, 0) \\ & (0, z, 0, 0)(0, 0, w, 0)(0, 0, 0, y), \end{aligned}$$

we get that for almost every $\bar{y} \in \mathbb{R}^k$

$$\begin{aligned} & (\pi'(x, z, w, y)f)(\bar{y}) \\ & = e^{2\pi i [l(x) + \tilde{l}(z) + \sum_{i=1}^{k} w_i \bar{y} \lambda_i(l) - (1/2) \sum_{i=1}^{k} w_i y_i \lambda_i(l)]} f(\bar{y} - y). \end{aligned} \qquad (2.6)$$

If $\tilde{l} = l' \mid \text{span}_{\mathbb{R}}\{Z_1(l), \ldots, Z_r(l)\}$ then it follows from (2.5) that $\pi' = ind_{H_l}^{G} \chi_{l'}$ where $l' \in \mathfrak{g}^*$ is such that

$$l' \mid \text{span}_{\mathbb{R}}\{X_1, \ldots, X_m\} = l,$$
$$l' \mid \text{span}_{\mathbb{R}}\{W_1(l), \ldots, Y_k(l)\} = 0.$$

We have noted above that every unitary irreducible representation of G is of the form (2.5). The assertion about equivalences among the representations now is an immediate consequence of the uniqueness of the Stone-von Neumann theorem. For, if π_{l_1} and π_{l_2} are given by (2.5), then the analysis a)-e) on π_{l_1} and π_{l_2} would show that $\pi_{l_1} \cong \pi_{l_2}$ if and only if

i) $l_1 \mid \mathfrak{z} = l_2 \mid \mathfrak{z}$ and hence $r_{l_1} = r_{l_2}, \lambda_i(l_1) = \lambda_i(l_2)$ for all i,

ii) $l_1 \mid r_{l_1} = l_2 \mid r_{l_2}$,

which are equivalent to the condition $O_{l_1} = O_{l_2}$.

Now we give an example where we will carry out the construction of an almost symplectic basis.

Example 2.2.1 : Consider $\mathfrak{gf}_{3,2}$. Let $l \in V_N^* \cap \mathcal{U}$ that is $l_1 = l(X_1) \neq 0$ and $l_3 = l(X_3) = l(X_5) = l_5 = 0$ (see example 2.1.1). Then the eigen values of the matrix

$$\tilde{B}_l = \begin{pmatrix} 0 & 0 & -l_1 \\ 0 & 0 & -l_2 \\ l_1 & l_2 & 0 \end{pmatrix}$$

are $\{0, \pm i\sqrt{l_1^2+l_2^2}\}$. Then the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{l_2}{\sqrt{l_1^2+l_2^2}} & \frac{l_1}{\sqrt{l_1^2+l_2^2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{l_1}{\sqrt{l_1^2+l_2^2}} & \frac{l_1}{\sqrt{l_1^2+l_2^2}} & 0 \end{pmatrix}$$

changes the basis $\{X_1, \ldots, X_5\}$ to $\{X_1, X_2, Z_1(l), W_1(l), Y_1(l)\}$ where

$$Z_1(l) = \frac{-l_2X_3 + l_1X_4}{\sqrt{l_1^2+l_2^2}}, W_1(l) = X_5, Y_1(l) = \frac{l_1X_3 + l_2X_4}{\sqrt{l_1^2+l_2^2}}.$$

Then $l([W_1(l), Y_1(l)]) = \sqrt{l_1^2+l_2^2}$ and $Z_1(l) \in r_l$.

2.3 The Plancherel theorem

The elementary proof of the Plancherel theorem on a two step nilpotent Lie group G, that we want to give, proceeds through asking the following question: *suppose* $f \in L^1(G) \cap L^2(G)$ *and suppose* $l \in V_N^* \cap \mathcal{U}$. *What is the relation between* $\hat{f}(\pi_l)$ *and* $\mathcal{F}_1 f(l \mid \mathfrak{z}, v)$? Where $\hat{f}$ is the operator valued group Fourier transform, (z, v) are elements of the group with $z \in \mathfrak{z}$ and v belongs to $\text{span}_{\mathbf{R}}\{X_{m+1}, \ldots, X_n\}$ and $\mathcal{F}_1 f(l \mid \mathfrak{z}, v)$ means the partial (Euclidean) Fourier transform of f in the central variables at the point $l \mid \mathfrak{z}$?

The motivation comes from the Heisenberg groups, which we denote by H_n. Let $\mathfrak{h}_n = \text{span}_{\mathbf{R}}\{Z, W_1, \ldots, W_n, Y_1, \ldots, Y_n\}$, with the only nonzero Lie brackets $[W_i, Y_i] = Z$, $i = 1, \ldots, n$, be the Lie algebra of H_n. Then $V_N = \text{span}_{\mathbf{R}}\{Z\}$ and $V_J = \text{span}_{\mathbf{R}}\{W_1, \ldots, Y_n\}$ and $V_N^* \cap \mathcal{U} = \{l \in \mathfrak{h}^* : l(Z) = \lambda \neq 0\}$. Then it can be proved easily (see [F1]) that for $f \in L^1(H_n) \cap L^2(H_n)$ and $l \in V_N^* \cap \mathcal{U}$

$$\|\hat{f}(\pi_l)\|_{HS}^2 = |\lambda|^{-n} \int_{\mathbf{R}^{2n}} |\mathcal{F}_1 f(\lambda, w, y)|^2 dwdy. \qquad (2.7)$$

We are looking for an analogue of this. As with the Heisenberg groups, the Plancherel theorem for G will just fall out of that analogue.

Definition 2.3.1 For $l \in \mathfrak{g}^*$ we define

$$Pf(l) = \sqrt{\det((B_l')_{js})}$$

called the *Pfaffian* of l, where $(B_l')_{is} = l([X_{j_i}, X_{j_s}]), X_{j_i}, X_{j_s} \in V_J$.

Note 2.3.1 : If J is the set of jump indices for l, then B_l' is nondegenerate on V_J and then $Pf(l)$ is the Pfaffian of B_l' (see [J]). It is easy to show that

a) $\det((B_l')_{is})$ is always a square of a polynomial and hence $Pf(l)$ is a homogeneous polynomial in $l \mid \mathfrak{z}$.

b) $Pf(l) \neq 0$ if $l \in \mathcal{U}$ and is Ad^*G invariant.

Example 2.3.1 : Let $\mathfrak{g} = q\mathfrak{f}_{3,2}$. Then $J = \{3,5\}$, $V_N^* \cap \mathcal{U} = \{l \in \mathfrak{g}^* : l(X_1) = l_1 \neq 0, l_3 = l(X_3) = l_5 = l(X_5) = 0\}$ and

$$B_l' = \begin{pmatrix} 0 & -l_1 \\ l_1 & 0 \end{pmatrix}$$

and hence $Pf(l) = l_1$.

Example 2.3.2 : Let $\mathfrak{g} = \mathcal{F}_{3,2}$. Then $J = \{4,5\}$, $V_N \cap \mathcal{U} = \{l \in \mathfrak{g}^* : l_1 = l(X_1) \neq 0, l_4 = l(X_4) = l(X_5) = l_5 = 0\}$, and

$$B_l' = \begin{pmatrix} 0 & l_1 \\ -l_1 & 0 \end{pmatrix}$$

and hence $Pf(l) = l_1$.

To find an analogue of (2.7), it is necessary to find the Jacobian of a transformation which we are now going to describe.

Let $l \in \tilde{V}_N^* = span_{\mathbb{R}}\{X_{n_1}^*, \ldots, X_{n_r}^*\}$. Notice that for H_n and $F_{n,2}$, where n is even, $\tilde{V}_N^* = \{0\}$, so the transformation we are going to describe, appears only for those two step nilpotent Lie groups where there does not exist any $l \in \mathfrak{g}^*$ such that $r_l = \mathfrak{z}$. Suppose $l_{n_i} = l(X_{n_i}), 1 \leq i \leq r$ and we also have $l(X_{j_i}) = 0, 1 \leq i \leq 2k$. From $\tilde{B}_l$ we have constructed an orthonormal basis $\{Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l), Y_1(l), \ldots, Y_k(l)\}$ with respect to which the matrix of $\tilde{B}_l$ is of the following form

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{S} \end{pmatrix} \tag{2.8}$$

where the $2k \times 2k$ matrix S is given by

$$\begin{pmatrix} & \mathbf{0} & & \lambda_1(l) & & \\ & & & & \ddots & \\ & & & & & \lambda_k(l) \\ -\lambda_1(l) & & & & & \\ & \ddots & & & \mathbf{0} & \\ & & -\lambda_k(l) & & & \end{pmatrix} \tag{2.9}$$

Let $l(Z_i(l)) = \bar{l}_i, 1 \leq i \leq r$. We consider the map

$$\phi : \tilde{V}_N^*(\cong \mathbb{R}^r) \to \mathbb{R}^r$$
$$\phi(l_{n_1}, \ldots, l_{n_r}) = (\bar{l}_1, \ldots, \bar{l}_r). \tag{2.10}$$

We need to find the modulus of the Jacobian determinant of ϕ. To see what it should be, we compute that in the following example.

Example 2.3.3 : Consider $\mathfrak{g} = q\mathfrak{f}_{3,2}$. Let $l \in V_N^* \cap \mathcal{U}$. Then eigenvalues of $\tilde{B}_l$ are $\{0, \pm i\sqrt{l_1^2 + l_2^2}\}$, and $Z_1(l) = \frac{-l_2 X_3 + l_1 X_4}{\sqrt{l_1^2+l_2^2}}$ (see exemple 2.2.1). Thus

$$\phi : \tilde{V}_N^*(\cong \mathbb{R}) \to \mathbb{R}$$
$$\phi(l_4) = \frac{l_1 l_4}{\sqrt{l_1^2 + l_2^2}}.$$

ϕ is actually linear. So the modulus of the Jacobian determinant of $\phi = \frac{|l_1|}{\sqrt{l_1^2+l_2^2}} = \frac{|Pf(l)|}{\sqrt{l_1^2+l_2^2}}$

Lemma 2.3.1 *The modulus of the Jacobian determinant of ϕ is given by*

$$|\det J_\phi| = \frac{|Pf(l)|}{\lambda_1(l)\lambda_2(l)\dots\lambda_k(l)},$$

where J_ϕ is the Jacobian matrix of ϕ.

PROOF. : First we systematically describe the transformations which gave the almost symplectic basis. We restrict ourselves only to the complement of the center, because it is there that the change of basis takes place.

$$A_1 : \{X_{m+1}, X_{m+2}, \dots, X_n\} \to \{X_{n_1}, \dots, X_{n_r}, X_{j_1}, \dots, X_{j_{2k}}\}$$

$$A_2 : \{X_{n_1}, \dots, X_{n_r}, X_{j_1}, \dots, X_{2k}\} \to \{\tilde{X}_{n_1}, \dots, \tilde{X}_{n_r}, X_{j_1}, \dots, X_{j_{2k}}\}$$

$$A_3 : \{\tilde{X}_{n_1}, \dots, \tilde{X}_{n_r}, X_{j_k}, \dots, X_{j_{2k}}\} \to \{Z_1(l), \dots, Z_r(l), W_1(l), \dots, W_k(l),$$
$$Y_1(l), \dots, Y_k(l)\}$$

where$\tilde{X}_{n_i} = X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s}, 1 \le i \le r$. A_1 is just a rearrangement of basis and hence is given by an orthogonal matrix. A_2 is clearly given by a lower triangular matrix with diagonal entries equal to one. The matrix of A_3 looks like

$$\begin{pmatrix} A' & 0 \\ C' & D' \end{pmatrix}$$

where A' is a $r \times r$ matrix, C' is a $2k \times r$ matrix and D' is a $2k \times 2k$ matrix, because A_3 is obtained from the following operations:

i) Gram-Schmidt orthogonalisation of $\{\tilde{X}_{n_i} : 1 \le i \le r\}$.

ii) Finding the orthogonal complement of the span of $\{\tilde{X}_{n_i} : 1 \le i \le r\}$.

iii) Choosing an almost symplectic basis on the nondegenerate subspace of $\tilde{B}_l$.

Notice that for $l \in \tilde{V}_N^*$, $l(X_{j_i}) = 0, 1 \le i \le 2k$; thus $l(\tilde{X}_{n_i}) = l(X_{n_i}), 1 \le i \le r$. Hence

$$|\det J_\phi| = |\det A'|.$$

Since $|\det A_1 . \det A_2 . \det A_3| = 1$, we have $|\det A_3| = 1$. But

$$|\det A_3| = |\det A'||\det D'|.$$

So

$$|\det J_\phi| = |\det D'|^{-1}.$$

If we write $\tilde{B}_l$ in terms of the basis $\{\tilde{X}_{n_1}, \ldots, \tilde{X}_{n_r}, X_{j_1}, \ldots, X_{j_{2k}}\}$, then the matrix of $\tilde{B}_l$ looks like

$$\begin{pmatrix} 0 & 0 \\ 0 & B'_l \end{pmatrix}$$

where $(B'_l)_{is} = l([X_{j_i}, X_{j_s}])$. Thus clearly

$$|\det B'_l| = |Pf(l)|^2.$$

Because of A_3 the above matrix changes to

$$\begin{pmatrix} 0 & 0 \\ 0 & D'B'_l(D')^t \end{pmatrix}$$

which is nothing but the matrix in (2.9). So

$$|\det D'|^2 = \frac{|\lambda_1(l) \ldots \lambda_k(l)|^2}{|Pf(l)|^2}$$
$$\Rightarrow |\det D'| = \frac{|\lambda_1(l) \ldots \lambda_k(l)|}{|Pf(l)|}.$$

Thus

$$|\det J_\phi| = \frac{|Pf(l)|}{|\lambda_1(l) \ldots \lambda_k(l)|}$$

as claimed.

Now we come to the analogue of (2.7). Given $f \in L^1(G) \cap L^2(G)$ and $\pi \in \hat{G}$ we define the so called group Fourier transform by

$$\hat{f}(\pi) = \int_G f(g)\pi(g^{-1})d\mu(g)$$

where μ is a Haar measure of the group. The above integral is interpreted as a bounded linear operator on $\mathcal{H}_\pi$ (where $\mathcal{H}_\pi$ is the Hilbert space associated to π) given by

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \int_G f(g)\langle \pi(g^{-1})\xi, \eta\rangle d\mu(g), \qquad \xi, \eta \in \mathcal{H}_\pi.$$

Given $l \in V_N^* \cap \mathcal{U}$, we get hold of an almost symplectic basis (2.4) and because of the orthonormal basis change, $dxdzdwdy$ is the normalized Haar measure we started with, where

$$(x, z, w, y) = \sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{r} z_i Z_i(l) + \sum_{i=1}^{k} w_i W_i(l) + \sum_{i=1}^{k} y_i Y_i(l).$$

The representation π_l corresponding to l, we are going to work with are given by (2.6). Let $dl_{n_1} \dots dl_{n_r}$ denotes the usual Lebesgue measure on $\tilde{V}_N^*$ (after we identify $\tilde{V}_N^*$ with $\mathbb{R}^r$ through the basis $\{X_{n_1}^*, \dots, X_{n_r}^*\}$).

Theorem 2.3.1 *Let $f \in L^1(G) \cap L^2(G)$. Then*

$$|Pf(l)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 dl_{n_1} \dots dl_{n_r}$$
$$= \int_{\mathbb{R}^{r+2k}} |\mathcal{F}_1 f(l_1, \dots, l_m, x_{n_1}, \dots, x_{n_r}, u, v)|^2 dx_{n_1} \dots dv \qquad (2.11)$$

for almost every $l \in V_N^* \cap \mathcal{U}$, where

$$\mathcal{F}_1 f(l_1, \dots, l_m, x_{n_1}, \dots, x_{n_r}, u, v)$$
$$= \int_{\mathbb{R}^m} f(x_1, \dots, x_m, x_{n_1}, \dots, x_{n_r}, u, v) e^{-2\pi i \sum_{j=1}^m l_j x_j} dx_1 \dots dx_m$$

and $l(X_i) = l_i, 1 \le i \le m$.

Proof. : Let $\phi \in L^2(\mathbb{R}^k)$. Then from (2.6),

$$(\hat{f}(\pi_l)\phi)(\bar{y})$$
$$= \int_{\mathbb{R}^{m+r+2k}} f(x, z, w, y)(\pi_l(-x, -z, -w, -y)\phi)(\bar{y}) dxdzdwdy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{m+r+2k}} f(x,z,w,y) e^{2\pi i[-l(x)-l(z)-\sum_{j=1}^{k} w_j \bar{y}_j \lambda_j(l) - (1/2)\sum_{j=1}^{k} w_j y_j \lambda_j(l)]} \\
&\qquad \times \phi(\bar{y}+y)\,dx dz dw dy \\
&= \int_{\mathbb{R}^{r+m+2k}} f(x,z,w,y-\bar{y}) e^{2\pi i[-l(x)-l(z)-\sum_{j=1}^{k} w_j \bar{y}_j \lambda_j(l) - (1/2)\sum_{j=1}^{k} w_j (y_j-\bar{y}_j)\lambda_j(l)]} \\
&\qquad \times \phi(y)\,dx dz dw dy
\end{aligned}
$$

(by the change of variable $y' = y + \bar{y}$)

$$
\begin{aligned}
&= \int_{\mathbb{R}^{r+m+2k}} f(x,z,w,y-\bar{y}) e^{2\pi i[-l(x)-l(z)-(1/2)\sum_{j=1}^{k} w_j y_j \lambda_j(l) - (1/2)\sum_{j=1}^{k} w_j \bar{y}_j \lambda_j(l)]} \\
&\qquad \times \phi(y)\,dx dz dw dy \\
&= \int_{\mathbb{R}^{r+m+2k}} f(x,z,w,y-\bar{y}) e^{-2\pi i l(x)} e^{-2\pi i l(z)} e^{-\pi i \sum_{j=1}^{k}(y_j+\bar{y}_j) w_j \lambda_j(l)} \\
&\qquad \times \phi(y)\,dx dz dw dy.
\end{aligned}
$$

Let

$$
K_l^f(y,\bar{y}) = \int_{\mathbb{R}^{m+r+k}} f(x,z,w,y-\bar{y}) e^{-2\pi i l(x)} e^{-2\pi i l(z)} e^{-\pi i \sum_{j=1}^{k}(y+\bar{y})\lambda_j(l) w_j}\, dx dz dw.
$$

Since $f \in L^1(G) \cap L^2(G)$, it follows that $K_l^f \in L^2(\mathbb{R}^k \times \mathbb{R}^k)$ for almost every $l \in \tilde{V}_N^* \cap \mathcal{U}$. Let $l \mid \mathfrak{z} = (l_1, \ldots, l_m)$ and $l \mid \mathrm{span}_{\mathbb{R}}\{Z_1(l), \ldots, Z_r(l)\} = (\bar{l}_1, \ldots, \bar{l}_r)$. Then

$$
\begin{aligned}
&K_l^f(y,\bar{y}) \\
&= \mathcal{F}_{123} f(l_1, \ldots, l_m, \bar{l}_1, \ldots, \bar{l}_r, \frac{y_1+\bar{y}_1}{2}\lambda_1(l), \ldots, \frac{y_k+\bar{y}_k}{2}\lambda_k(l), y-\bar{y})
\end{aligned}
$$

where $\mathcal{F}_{123}$ stands for the partial Fourier (Euclidean) transform in the variables x, z, w. Thus $\hat{f}(\pi_l)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^k)$ with the kernel K_l^f. Hence

$$
\begin{aligned}
&\|\hat{f}(\pi_l)\|_{HS}^2 \\
&= \int_{\mathbb{R}^{2k}} |k_l^f(y,\bar{y})|^2 dy d\bar{y} \\
&= \int_{\mathbb{R}^{2k}} |\mathcal{F}_{123} f(l_1, \ldots, \bar{l}_r, \frac{y_1+\bar{y}_1}{2}\lambda_1(l), \ldots, \frac{y_k+\bar{y}_k}{2}\lambda_k(l), y-\bar{y})|^2 dy d\bar{y}.
\end{aligned}
$$

If we do the change of variables

$$\begin{aligned} u_i &= \frac{y_i + \bar{y}_i}{2}\lambda_i(l), \quad 1 \le i \le k, \\ v_i &= y_i - \bar{y}_i, \quad 1 \le i \le k, \end{aligned}$$

then the modulus of the Jacobian determinant is $|\lambda_1(l)\ldots\lambda_k(l)|$ and the above integral reduces to

$$|\lambda_1(l)\ldots\lambda_k(l)|^{-1}\left(\int_{\mathbb{R}^{2k}} |\mathcal{F}_{123}f(l_1,\ldots,l_m,\bar{l}_1,\ldots\bar{l}_r,u,v)|^2 du dv\right),$$

where $u = (u_1,\ldots,u_k)$ and $v = (v_1,\ldots,v_k)$. By applying the Euclidean Plancherel theorem in the variable u we get

$$\|\hat{f}(\pi_l)\|^2 = |\lambda_1(l)\ldots\lambda_k(l)|^{-1}\int_{\mathbb{R}^{2k}} |\mathcal{F}_{12}f(l_1,\ldots,l_m,\bar{l}_1,\ldots,\bar{l}_r,u,v)|^2 du dv.$$

If we integrate both sides of the above equation on $\tilde{V}_N^*$ with respect to the usual Lebesgue measure and use change of variables by the map ϕ defined in (2.10), we get

$$\begin{aligned} &\int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 dl_{n_1}\ldots dl_{n_r} \\ =\ & |\lambda_1(l)\ldots\lambda_k(l)|^{-1}\frac{|\lambda_1(l)\ldots\lambda_k(l)}{|Pf(l)|} \\ &\times \int_{\mathbb{R}^{r+2k}} |\mathcal{F}_{12}f(l_1,\ldots,l_m,l_{n_1},\ldots,l_{n_r},u,v)|^2 dl_{n_1}\ldots dl_{n_r} du dv. \end{aligned}$$

Then by applying the Euclidean Plancherel theorem on the variables $(l_{n_1},\ldots,l_{n_r}) \in \mathbb{R}^r$ we get

$$\begin{aligned} &|Pf(l)|\int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 dl_{n_1}\ldots dl_{n_r} \\ =\ & \int_{\mathbb{R}^{r+2k}} |\mathcal{F}_1 f(l_1,\ldots,l_m,x_{n_1},\ldots,x_{n_r},u,v)|^2 dx_{n_1}\ldots dx_{n_r} du dv. \end{aligned}$$

This completes the proof.

Theorem 2.3.2 *(Plancherel theorem) For* $f \in L^1(G)\cap L^2(G)$

$$\int_{V_N^*\cap\mathcal{U}} \|\hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| dl = \|f\|_{L^2(G)}^2,$$

where dl is the standard Lebesgue measure on $V_N^*(\cong \mathbb{R}^{m+r})$ with respect to the basis $\{X_1^*, \ldots, X_m^*, X_{n_1}^*, \ldots, X_{n_r}^*\}$.

Proof. : Regarding $V_N^* \cap \mathcal{U}$ as the Cartesian product of $\mathcal{U}'$ and $\mathbb{R}^r$ as in note 2.1.5, we integrate both sides of (2.11) with respect to the standard Lebesgue measure on $\mathfrak{z}^*$ (upon identification with $\mathbb{R}^m$ via the basis $\{X_1^*, \ldots, X_m^*\}$) to get

$$\begin{aligned}
&\int_{V_N^* \cap \mathcal{U}} \|\hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| dl \\
= &\int_{\mathcal{U}'} \left(|Pf(l)| \int_{\dot{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 dl_{n_1} \ldots dl_{n_r} \right) dl_1 \ldots dl_m \\
= &\int_{\mathcal{U}'} \left(\int_{\mathbb{R}^{r+2k}} |\mathcal{F}_c f(l_1, \ldots, l_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 dx_{n_1} \ldots dv \right) dl_1 \ldots dl_m \\
&\qquad\qquad \text{(by (2.11))} \\
= &\int_{\mathbb{R}^m} \int_{\mathbb{R}^{r+2k}} |f(x_1, \ldots, x_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 dx_1 \ldots dx_m dx_{n_1} \ldots dx_{n_r} du dv,
\end{aligned}$$

by using the Euclidean Plancherel theorem in the outer integral, ($\mathcal{U}'$ is a set of full Lebesgue measure in $\mathfrak{z}^*$). The last integral is, of course, $\|f\|_{L^2(G)}^2$ and the proof is complete.

Note 2.3.2 : The situation is simpler if we consider the case of MW groups. In this case $V_N^* \cap \mathcal{U} \subseteq \mathfrak{z}^*$ is Zariski open and for $l \in \mathcal{U} \subseteq \mathfrak{z}^*$, the representation π_l is given by

$$\begin{aligned}
&(\pi_l(x, z, y) f)(\bar{y}) \\
= \; &e^{2\pi i [l(x) + \sum_{j=1}^k \bar{y}_j \bar{w}_j \lambda_j(l) - (1/2) \sum_{j=1}^k y_j w_j \lambda_j(l)]} f(\bar{y} - y)
\end{aligned}$$

where $\bar{y} \in \mathbb{R}^k$, $f \in L^2(\mathbb{R}^k)$ and $\dim \mathfrak{g}/\mathfrak{z} = 2k$. Then it follows from the calculations done in theorem (2.3.1) that

$$\|\hat{f}(\pi_l)\|_{HS}^2 = \frac{1}{|\lambda_1(l) \ldots \lambda_k(l)|} \int_{\mathbb{R}^{2k}} |\mathcal{F}_1 f(l_1, \ldots, l_m, u, v)|^2 du dv.$$

Clearly $|\lambda_1(l)\dots\lambda_k(l)| = |Pf(l)|$, since $\tilde{B}_l$ is nondegenerate. The Plancherel theorem again follows from integrating both sides on $U \subseteq \mathfrak{z}^*$. *So the change of variables through the map ϕ is not needed for MW groups.*

2.4 Infinitesimal representations and the sub-Laplacian

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra with a basis $\mathcal{B}$ as in section 2.1.

In this section we consider elements of $\mathfrak{g}$ as left invariant differential operators acting on $C^\infty(G)$, that is given $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, the differential operator X acts on f by the rule

$$(Xf)(g) = \frac{d}{dt}|_{t=0} f(g.\exp tX). \tag{2.12}$$

We define

$$\mathcal{L} = -\sum_{i=1}^{n-m} X_{m+i}^2 \tag{2.13}$$

and as on the Heisenberg groups, call it the sub-Laplacian of G.

Given an irreducible, unitary representation π of G, we define a function

$$\phi^\pi_{u,v} : G \to \mathbb{C}, \qquad u, v \in \mathcal{H}_\pi$$
$$\phi^\pi_{u,v}(g) = \langle \pi(g)u, v\rangle \tag{2.14}$$

called the matrix functions of π, where $\mathcal{H}_\pi$ is the Hilbert space associated with π.

In this section our aim is to find: *which matrix functions of representations are joint eigen functions of $\mathcal{L}$ and $\{X_i : 1 \le i \le m\}$?*

Given $\pi \in \hat{G}$ and $X \in \mathfrak{g}$, we define

$$d\pi(X)(u) = \frac{d}{dt}\Big|_{t=0} \pi(\exp tX)u \tag{2.15}$$

only for those vectors $u \in \mathcal{H}_\pi$ such that the above derivative exits for all $X \in \mathfrak{g}$. In that case u is called a C^∞ *vector for* π and $d\pi$ is called the *infinitesimal representation* corresponding to π. It can be shown that $d\pi(X)$ is a skew adjoint operator (usually unbounded) with domain $C^\infty(\pi)$, the space of C^∞ vectors for π, which is a dense subspace for $\mathcal{H}_\pi$ and $d\pi$ defines a representation of $\mathfrak{g}$ on $C^\infty(\pi)$. By the universal property $d\pi$ extends to a representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which can be viewed as the algebra of all left invariant differential operators acting on $C^\infty(G)$. If $A \in \mathcal{U}(\mathfrak{g})$ then it follows that

$$A\langle\pi(g)u, v\rangle = \langle\pi(g)d\pi(A)u, v\rangle.$$

Thus if u is an eigen vector for $d\pi(A)$ then $\phi^\pi_{u,v}$ is an eigen function for A. Since for $1 \le i \le m$, $X_i \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$, the center of the universal enveloping algebra, then $d\pi(X_i)$ acts as a scalar (see [CG]) and hence $\phi^\pi_{u,v}$ is an eigen function for X_i for any u, v. Thus our job reduces to finding the eigen functions of $d\pi(\mathcal{L})$ which are also matrix functions of π. Looking at the case of the Heisenberg groups and the group $F_{2n,2}$ (see [St]) it is reasonable to expect that $d\pi(\mathcal{L})$ is closely related to the Hermite operator and, indeed, that is the case.

It is possible to be little bit more explicit about (2.12). Using exponential coordinates we coordinatise G by the above chosen basis. Given $x = \sum_{i=1}^n x_i X_i$ and $x' = \sum_{i=1}^n x'_i X_i$ we define

$$[x, x']_p = \langle [x, x'], X_p\rangle, \quad 1 \le p \le m,$$

where $\langle.,.\rangle$ is the Euclidean inner product on $\mathfrak{g}$ such that $\{X_i : 1 \le i \le n\}$ is an orthonormal basis. Then it follows that, for $1 \le i \le m$

$$\begin{aligned} & (X_i f)(x) \\ = & \left.\frac{d}{dt}\right|_{t=0} f(x.\exp tX_i) \\ = & \frac{\partial f}{\partial x_i}(x), \qquad (2.16) \\ & \qquad (2.17) \end{aligned}$$

and for $m+1 \leq i \leq n$

$$
\begin{aligned}
& (X_i f)(x) \\
= \ & \frac{d}{dt}\Big|_{t=0} f(x.\exp tX_i) \\
= \ & \frac{d}{dt}\Big|_{t=0} f\left(\sum_{j=1}^{m}(x_j + \frac{t}{2}[x, X_i]_j)X_j + \sum_{j=m+1}^{n-m} \tilde{x}_j X_j\right) \\
& \quad \text{where } \tilde{x}_j = \begin{cases} x_j & \text{if } j \neq i \\ x_i + t & \text{if } j = i \end{cases} \\
= \ & \left(\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^{m}[x, X_i]_j \frac{\partial}{\partial x_j}\right) f(x).
\end{aligned} \tag{2.18}
$$

Now we start with a representation $\pi_l \in \hat{G}$ such that $l \mid \mathfrak{z} \neq 0$. We get hold of an almost symplectic basis (2.4) with $\dim r_l = m + r$ and $\dim O_l = 2k$, so $n = 2k + m + r$. The representations π_l are realized on $L^2(\mathbb{R}^k)$ and are given by (2.6). It is known from theorem 4.1.1 of [CG] that $C^\infty(\pi_l) = \mathcal{S}(\mathbb{R}^k)$, the Schwartz class functions on $\mathbb{R}^k$. Now we want to compute the effect of $d\pi_l$ on elements of the almost symplectic basis (2.4).

Lemma 2.4.1 *For $\phi \in \mathcal{S}(\mathbb{R}^k)$ and $\xi \in \mathbb{R}^k$*

i) $d\pi_l(Z_j(l))\phi(\xi) = 2\pi i \bar{l}_j \phi(\xi), \qquad 1 \leq j \leq r.$

ii) $d\pi_l(W_j(l))\phi(\xi) = 2\pi i \xi_j \lambda_j(l)\phi(\xi), \qquad 1 \leq j \leq k.$

iii) $d\pi_l(Y_j(l))\phi(\xi) = -\frac{\partial \phi}{\partial \xi_j}(\xi), \qquad 1 \leq j \leq k.$

iv) $d\pi_l(\mathcal{L})\phi(\xi) = \{4\pi^2 \sum_{j=1}^{r} \bar{l}_j^2 + L_l\}\phi(\xi)$ *where*

$$L_l = \sum_{j=1}^{k}(-\frac{\partial^2}{\partial \xi_j^2} + 4\pi^2 \lambda_j(l)^2 \xi_j^2).$$

PROOF. We calculate directly

$$
\begin{aligned}
d\pi_l(Z_j(l))\phi(\xi) &= \frac{d}{dt}\Big|_{t=0} \pi_l(\exp tZ_j(l))\phi(\xi) \\
&= \frac{d}{dt}\Big|_{t=0} e^{2\pi i t l(Z_j(l))}\phi(\xi)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi i l(Z_j(l))\phi(\xi) \\
&= 2\pi i \bar{l}_j \phi(\xi), \\
d\pi_l(W_j(l))\phi(\xi) &= \frac{d}{dt}\Big|_{t=0} \pi_l(\exp tW_j(l))\phi(\xi) \\
&= \frac{d}{dt}\Big|_{t=0} e^{2\pi i t\xi_j\lambda_j(l)}\phi(\xi) \\
&= 2\pi i \xi_j \lambda_j(l)\phi(\xi), \\
d\pi_l(Y_j(l))\phi(\xi) &= \frac{d}{dt}\Big|_{t=0} \pi_l(\exp tY_j(l))\phi(\xi) \\
&= \frac{d}{dt}\Big|_{t=0} \phi(\xi - te_j) \\
&= -\frac{\partial\phi}{\partial\xi_j}(\xi).
\end{aligned}
$$

We notice that in terms of the almost symplectic basis (2.4), $\mathcal{L}$ is given by

$$
\mathcal{L} = -\sum_{j=1}^{r} Z_j(l)^2 - \sum_{j=1}^{k}(W_j(l)^2 + Y_j(l)^2). \tag{2.19}
$$

(2.19) follows from the facts that $\mathcal{L}$ and the right hand side of (2.19) are both left invariant, and at the identity the two differential operators agree by virtue of the invariance of the Euclidean Laplacian under an orthonormal basis change. Now by $i), ii)$ and $iii)$ we have

$$
\begin{aligned}
d\pi_l(\mathcal{L})\phi(\xi) &= \left(-\sum_{j=1}^{r} d\pi_l(Z_j(l))^2 - \sum_{j=1}^{k}(d\pi_l(W_j(l))^2 + d\pi_l(Y_j(l))^2)\right)\phi(\xi) \\
&= \left(\sum_{j=1}^{r} 4\pi^2 l(Z_j(l)^2 + \sum_{j=1}^{k}(4\pi^2\xi_j^2\lambda_j(l)^2 - \frac{\partial^2}{\partial\xi_j^2}\right)\phi(\xi) \\
&= \left(4\pi^2\sum_{j=1}^{r}\bar{l}_j^2 + L_l\right)\phi(\xi).
\end{aligned}
$$

This completes the proof.

Because of $iv)$ now it is easy to describe the eigen functions of $d\pi_l(\mathcal{L})$. Let $\mu(l) = 4\pi^2\sum_{j=1}^{r}\bar{l}_j^2$. Then $d\pi_l(\mathcal{L}) = \mu(l) + L_l$, and $\mu(l) \geq 0$. If ϕ is an eigen function of L_l with eigen value $c(l)$, then ϕ is an eigen function of $d\pi_l(\mathcal{L})$

with eigen value $c(l)+\mu(l)$. If ϕ_j^l is an eigen function of $-\frac{\partial^2}{\partial x^2}+4\pi^2\lambda_j(l)^2x^2$ on $\mathbb{R}$, then clearly

$$\phi^l(\xi_1,\ldots,\xi_k)=\phi_i^l(\xi_1)\ldots\phi_k^l(\xi_k)$$

is an eigen function of L_l. Since for $s\in\mathbb{N}$, the $s-$th normalized hermite function h_s is an eigen function of $-\frac{d^2}{dx^2}+x^2$ with eigen value $2s+1$, it is clear that

$$h_s^l(x)=(2\pi\lambda_j(l))^{\frac{1}{4}}h_s(\sqrt{2\pi}\lambda_j(l)^{\frac{1}{2}}x)$$

is an eigen function of $-\frac{d^2}{dx^2}+4\pi^2\lambda_j(l)^2x^2$ with eigen value $2\pi\lambda_j(l)(2s+1)$ and also $\|h_s^l\|_2=1$. So for $(\alpha_1,\ldots,\alpha_k)\in\mathbb{N}^k$ we define

$$h_\alpha^l(\xi_1,\ldots,\xi_k)=\Pi_{j=1}^k h_{\alpha_j}^l(\xi_j), \tag{2.20}$$

where

$$h_{\alpha_j}^l(\xi_j)=(2\pi\lambda_j(l))^{\frac{1}{4}}h_{\alpha_j}(\sqrt{2\pi}\lambda_j(l)^{\frac{1}{2}}\xi_j).$$

Then

$$L_l(h_\alpha^l)=\left(\sum_{j=1}^k 2\pi\lambda_j(l)(2\alpha_j+1)\right)h_\alpha^l. \tag{2.21}$$

Thus

$$d\pi_l(\mathcal{L})(h_\alpha^l)=\left(\mu(l)+\sum_{j=1}^k 2\pi\lambda_j(l)(2\alpha_j+1)\right)h_\alpha^l. \tag{2.22}$$

In the next chapter we will use (2.22) to get an analogue of Heisenberg's inequality on two step nilpotent Lie groups.

Chapter 3

Uncertainty principles on two step nilpotent Lie groups

The principal results in this chapter are the analogues of the Cowling-Price theorem and the Heisenberg's inequality for two step nilpotent Lie groups. Along the way we also give a proof of a (p, q) version of Hardy's theorem for two step nilpotent Lie groups.

Hardy's theorem for Heisenberg groups was proved in [SST] and its L^p−analogue (Cowling-Price theorem) and the (p, q) version was proved in [BR]. An analogue of Hardy's theorem on two step nilpotent Lie groups was proved in [ACDS]. An analogue of Heisenberg's inequality for Heisenberg groups was proved in [T1] (see also [SST] and [GL] for two other variants).

Remark 3.0.1 Our treatment in this chapter tacitly assumes that G is not MW. So for the case of MW groups the treatment needs only obvious modifications using the description of $\|\hat{f}(\pi_l)\|_{HS}$ given in note 2.3.2.

3.1 Extensions of Hardy's theorem

In the case of Heisenberg groups, Hardy's theorem and Cowling-Price theorem (the (p, q) version as well) actually reduces to the corresponding problems on the center of the group by an application of (2.7). For two step nilpotent Lie groups we have a reasonable analogue of (2.7), namely theorem 2.3.1. So it is expected that the same technique may work here also; and it does, as we shall show presently. Since we are going to talk about exponential decay of the group Fourier transform, we need a growth parameter on the dual, where exponential maps make sense, but that has been addressed in section 2.1. In our parametrisation the dual is essentially a vector subspace (actually a Zariski open subset of that subspace) of $\mathfrak{g}^*$, which is good enough for us.

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra with basis $\mathcal{B}$ as before. G is the corresponding connected, simply connected, Lie group. We write elements of $\mathfrak{g}$ (as well as that of G) by $(x, v) \equiv \sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{n-m} v_i X_{m+i}$. The set $V_N^* \cap \mathcal{U}$ serves as the effective dual (that is, it is a set of full Plancherel measure in $\hat{G}$) of G and we put Euclidean norm there such that $\{X_1^*, \dots, X_m^*, X_{n_i}^* : 1 \leq i \leq r\}$ is an orthonormal basis. We write elements of V_N^* as

$$(\lambda, \gamma) \equiv \sum_{i=1}^{m} \lambda_i X_i^* + \sum_{i=1}^{r} \gamma_i X_{n_i}^*.$$

First we prove the (p, q) version of Hardy's theorem.

Theorem 3.1.1 *Let $f : G \to \mathbb{C}$ be a measurable function. Suppose*

(i) $|f(x, v)| \leq C g(v) e^{-a\pi \|x\|^p}$,

(ii) $\|\hat{f}(\pi_{\lambda,\gamma})\|_{HS} \leq C h(\gamma) e^{-b\pi \|\lambda\|^q}$,

where $C > 0$, $p \geq 2$, $1/p + 1/q = 1$ and g, h are nonnegative functions with $g \in L^1(\mathbb{R}^{n-m}) \cap L^2(\mathbb{R}^{n-m})$ and $h \in L^1(\mathbb{R}^r) \cap L^2(\mathbb{R}^r)$. If $(ap)^{1/p}(bq)^{1/q} > 2$, then $f = 0$ almost everywhere.

PROOF. We notice that because of (i), $f \in L^p(G)$, for all p and as a result

$\hat{f}$ in (ii) make sense. We define

$$\tilde{f}(x,v) = \overline{f(-x,v)}, \quad \text{and}$$
$$h(x) = \int_{\mathbb{R}^{n-m}} (f_v * \tilde{f}_v)(x)dv, \tag{3.1}$$

where $f_v(x) = f(x,v)$ and $*$ is the convolution on $\mathbb{R}^m$. If $f \in L^1(G)$ then $h \in L^1(\mathbb{R}^m)$ and the Euclidean Fourier transform of h is given by

$$\begin{aligned}\hat{h}(\lambda) &= \int_{\mathbb{R}^m} h(x)e^{-2\pi i\langle\lambda,x\rangle}dx \\ &= \int_{\mathbb{R}^{n-m}} |\mathcal{F}_1 f(\lambda,v)|^2 dv \\ &= |Pf(\lambda)| \int_{\dot{V}_N} \|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^2 d\gamma \qquad \text{(by (2.11))}. \end{aligned} \tag{3.2}$$

By (ii) we get

$$\begin{aligned}\hat{h}(\lambda) = |\hat{h}(\lambda)| &\le C'|Pf(\lambda)|e^{-2b\pi\|\lambda\|^q} \\ &\le Be^{-2b'\pi\|\lambda\|^q} \qquad \text{(as } Pf(\lambda) \text{ is a polynomial)}\end{aligned} \tag{3.3}$$

where $b' < b$ is such that $(ap)^{1/p}(b'q)^{1/q} > 2$. Now (writing $\exp x = e^x$)

$$\begin{aligned}|h(x)| &\le \int_{\mathbb{R}^n} |f(x-y,v)||f(-y,v)|dydv \\ &\le C^2 \int_{\mathbb{R}^n} |g(v)|^2 \exp(-a\pi(\|x-y\|^p + \|y\|^p))dydv \\ &= C' \int_{\mathbb{R}^m} \exp(-a\pi(\|x-y\|^p + \|y\|^p))dy \\ &\le C' \int_{\mathbb{R}^m} \exp(-a\pi 2^{1-(p/2)}\{\|x-y\|^2 + \|y\|^2\}^{p/2})dy \\ &\qquad \text{(by Jensen's inequality)} \\ &\le C' \int_{\mathbb{R}^m} \exp(-a\pi 2^{1-(p/2)}\{(\|x\| - \|y\|)^2 + \|y\|^2\}^{p/2})dy \\ &= C' \int_{\mathbb{R}^m} \exp(-a\pi 2^{1-(p/2)}\{2(\|y\| - \|x\|/2)^2 + \|x\|^2/2\}^{p/2})dy \\ &\le C' \int_{\mathbb{R}^m} \exp(-a\pi 2^{1-(p/2)}\{2^{(p/2)} \mid \|y\| - \|x\|/2 \mid^p + 2^{-p/2}\|x\|^p\})dy\end{aligned}$$

(by the elementary inequality,

$$
\begin{aligned}
& \left.(1+x^2)^{p/2} \geq 1+x^2 \text{ for } p \geq 2, x \geq 0\right) \\
= \;& C' \exp(-a\pi 2^{1-p}\|x\|^p) \int_{\mathbf{R}^m} \exp(-2a\pi \mid \|y\| - \|x\|/2 \mid^p) dy \\
\leq \;& A e^{-a'\pi 2^{1-p}\|x\|^p} \qquad (3.4)
\end{aligned}
$$

where $a' < a$ is such that $(a'p)^{1/p}(b'q)^{1/q} > 2$. Since $(a'2^{1-p}p)^{1/p}(2b'q)^{1/q} = (a'p)^{1/p}(b'q)^{1/q}$, we get that h satisfies the conditions of corollary 1.1.3.1, c), by (3.3) and (3.4). So $h = 0$ almost everywhere. By (3.2), $\|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS} = 0$ for almost every (λ, γ). By Plancherel theorem $f = 0$ almost everywhere. This completes the proof.

Our next result, theorem 3.1.2 is the analogue of the theorem 1.1.1 on two step nilpotent Lie groups. We use a trick here we employed in theorem 1.1.4. First we need a lemma.

Lemma 3.1.1 *Let G be a two step nilpotent Lie group. Then there exist a constant C such that*

$$\|(x,v).(x_1,v_1)^{-1}\| \geq \|(x,v)\| - \|(x_1,v_1)\| - C\|(x,v)\|\|(x_1,v_1)\|, \qquad (3.5)$$

for all $(x,v),(x_1,v_1) \in G$.

PROOF. Since $[.,.]$ is a bilinear map on a finite dimensional vector space there exist a constant C such that

$$\|[v,v_1]\| \leq C\|v\|\|v_1\|.$$

The lemma follows from the above and the formula of the group multiplication (2.2).

Theorem 3.1.2 *Let $f \in L^1(G) \cap L^2(G)$ and satisfies*

(i) $\int_G e^{pa\pi\|(x,v)\|^2}|f(x,v)|^p dx dv < \infty$,

(ii) $\int_{V_N^*} e^{qb\pi\|(\lambda,\gamma)\|^2}\|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^q |Pf(\lambda)| d\lambda d\gamma < \infty$,

where $1 \leq p \leq \infty$ and $2 \leq q < \infty$. If $ab > 1$ then $f = 0$ almost everywhere.

PROOF. : We first prove the case $p = \infty$ and later, use this result for the case $1 \le p < \infty$.

Case 1 $\quad p = \infty$

In this case we interpret (i) as

$$|f(x,v)| \le Ae^{-a\pi\|(x,v)\|^2}. \tag{3.6}$$

We define $\tilde{f}$ and h as in theorem 3.1.1. Then writing $e^x = \exp x$

$$\begin{aligned}
|h(x)| &\le \int_{\mathbb{R}^{n-m}} |(f_v * \tilde{f}_v)(x,v)|dv \\
&\le \int_{\mathbb{R}^n} |f(x-y,v)||f(-y,v)|dydv \\
&\le A^2 \int_{\mathbb{R}^n} \exp(-a\pi[2\|v\|^2 + \|x-y\|^2 + \|y\|^2])dydv \\
&\le A^2 \int_{\mathbb{R}^n} \exp(-a\pi[2\|v\|^2 + (\|x\| - \|y\|)^2 + \|y\|^2])dydv \\
&= A^2 \exp\left(-a\pi\frac{\|x\|^2}{2}\right) \int_{\mathbb{R}^n} \exp(-a\pi[2\|v\|^2 + 2(\|y\| - \frac{\|x\|}{2})^2]\Big) dydv \\
&\le A_1 e^{-(a'\pi/2)\|x\|^2}
\end{aligned} \tag{3.7}$$

where $a' < a$ with $a'b > 1$ (the integral in the last line but one being a polynomial in $\|x\|$). Choosing $b' < b$ such that $a'b' > 1$ we have, on the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^m} \exp(\frac{q}{2}\pi 2b'\|\lambda\|^2)|\hat{h}(\lambda)|^{q/2}d\lambda \\
= &\int_{\mathbb{R}^m} \exp(q\pi b'\|\lambda\|^2)\left(|Pf(\lambda)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^2 d\gamma\right)^{q/2} d\lambda \quad \text{by (3.2)} \\
= &\int_{\mathbb{R}^m} \left(\int_{\tilde{V}_N^*} \exp(2b'\pi\|\gamma\|^2)\|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^2 \exp(-2b'\pi\|\gamma\|^2)d\gamma\right)^{q/2} \\
&\quad \times \exp(q\pi b'\|\lambda\|^2)|Pf(\lambda)|^{q/2}d\lambda \\
\le &\int_{\mathbb{R}^m} \exp(qb'\pi\|\lambda\|^2)\left\{\left(\int_{\tilde{V}_N^*} \exp(\frac{q}{2}2b'\pi\|\gamma\|^2)\|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^q d\gamma\right)^{2/q}\right.
\end{aligned}$$

$$\times \left(\int_{\hat{V}_N^*} \exp(-2b'\pi\alpha\|\gamma\|^2) d\gamma \right)^{1/\alpha} \Bigg\}^{q/2} |Pf(\lambda)|^{q/2} d\lambda$$

(by Hölder's inequality, where $2/q + 1/\alpha = 1$)

$$\begin{aligned}
&= B \int_{\mathbb{R}^m} \int_{\hat{V}_N^*} \exp(qb'\pi\|(\lambda,\gamma)\|^2) \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^q |Pf(\lambda)|^{q/2} d\lambda d\gamma \\
&= B \int_{V_N^*} \exp(qb\pi\|(\lambda,\gamma)\|^2) \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^q \\
&\qquad \times \{\exp((b'-b)\pi\|(\lambda,\gamma)\|^2) |Pf(\lambda)|^{q/2}\} d\lambda d\gamma \\
&< \infty \qquad \text{(by } (ii)). \qquad (3.8)
\end{aligned}$$

Since $(a'/2)2b' = a'b' > 1$, by theorem 1.1.1 for the case $p = \infty$ and $q/2$ (which is ≥ 1 as $q \geq 2$) we get that $h = 0$ almost everywhere and thus $\|\hat{f}(\lambda,\gamma)\|_{HS} = 0$ for almost every (λ,γ) and thus $f = 0$ almost everywhere by the Plancherel theorem.

Case 2 $\quad p < \infty$

Let $e_k(x,v) = e^{k\|(x,v)\|^2}$ for $k \in \mathbb{R}^+$. Suppose $g \in C_c(G)$ is such that $\operatorname{supp} g \subset \{(x_1,v_1) : \|(x_1,v_1)\| \leq \frac{1}{m}\}$, where $m \in \mathbb{N}$. We choose $(x,v) \in G$ with $\|(x,v)\| > 1$. Thus, if $(x_1,v_1) \in \operatorname{supp} g$ we have $\|(x_1,v_1)\| \leq \|(x,v)\|/m$ and hence by lemma 3.1.1

$$\begin{aligned}
\|(x,v)(x_1,v_1)^{-1}\| &\geq \|(x,v)\| - \|(x_1,v_1)\| - C\|(x,v)\|\|(x_1,v_1)\| \\
&\geq \|(x,v)\| - \frac{\|(x,v)\|}{m} - \frac{C}{m}\|(x,v)\| \\
&= \|(x,v)\|(1 - \frac{d}{m}), \qquad (3.9)
\end{aligned}$$

where $d = 1 + C$. Thus for $(x,v) \in G$ with $\|(x,v)\| > 1$ we have

$$\begin{aligned}
&(e_{a\pi}|f| * |g|)(x,v) \\
&= \int_{\operatorname{supp} g} e^{a\pi\|(x,v)(x_1,v_1)^{-1}\|} |f((x,v)(x_1,v_1)^{-1})||g(x_1,v_1)| dx_1 dv_1 \\
&\geq e^{a\pi(1-(d/m))^2\|(x,v)\|^2} (|f| * |g|)(x,v) \qquad \text{(by (3.9))}. \qquad (3.10)
\end{aligned}$$

By (i) we have that $e_{a\pi}|f|$ is a L^p function ($p < \infty$) on G and $g \in C_c(G)$, thus $e_{a\pi}|f| * |g|$ is a bounded continuous fonction. Thus from (3.10) we have that

$$|(f * g)(x, v)| \leq \beta e^{-a\pi(1-(d/m))^2} \|(x, v)\|^2,$$

for all $(x, v) \in G$ with Euclidean norm greater than 1. By continuity of $f * g$ we have

$$|(f * g)(x, v)| \leq \beta e^{-a\pi(1-(d/m))^2 \|(x,v)\|^2}, \tag{3.11}$$

for all $(x, v) \in G$ (possibly with a different constant). Since

$$\begin{aligned} \|\widehat{(f * g)}(\pi_{(\lambda,\gamma)})\|_{HS} &\leq \|\hat{g}(\pi_{(\lambda,\gamma)})\|_{Op} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS} \\ &\leq \|g\|_{L^1(G)} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}, \end{aligned}$$

from (ii) we get that

$$\int_{V_N^*} e^{qb\pi\|(\lambda,\gamma)\|^2} \|\widehat{(f * g)}(\pi_{(\lambda,\gamma)})\|_{HS}^q |Pf(\lambda)| d\lambda d\gamma < \infty. \tag{3.12}$$

We choose m so large that $ab(1 - (d/m))^2 > 1$. Then by (3.11) and (3.12) we are reduced to case 1. Hence $f * g = 0$ almost every where. Now by choosing g from an approximate identity we get $f = 0$ almost every where. This completes the proof.

Note 3.1.1 : The conditions on p and q can be relaxed a bit in theorem 3.1.2 if G is one of the Heisenberg groups, or more generally, H-type groups. For Heisenberg groups the following is true.

Theorem 3.1.3 *Let* $f \in L^1(H_n) \cap L^2(H_n)$. *Suppose for* $a, b > 0$ *and* $min(p, q) < \infty$

(i) $\int_{H_n} e^{pa\pi\|(z,t)\|^2} |f(z, t)|^p dz dt < \infty$,

(ii) $\int_{\mathbb{R}} e^{qb\pi\lambda^2} \|\hat{f}(\pi_\lambda)\|_{HS}^q |\lambda|^n d\lambda < \infty$.

a) If $q \geq 2$, *then* $f = 0$ *for* $ab > 1$.

b) If $1 \leq q < 2$, *then for* $p = \infty$, $f = 0$ *if* $ab \geq 2$ *and for* $p < \infty$, $f = 0$ *if* $ab > 2$.

PROOF. : We notice that a) is just theorem 3.1.2 for H_n.

b) We define $\tilde{f}$ and h as in the previous theorems, then for $p = \infty$,

$$|h(t)| \leq Ce^{-(a/2)\pi t^2} \tag{3.13}$$

as usual. Since in this case we have

$$\hat{h}(\lambda) = |\lambda|^n \|\hat{f}(\pi_\lambda)\|_{HS}^2,$$

from (ii) we get

$$\begin{aligned}
\infty \quad &> \int_{\mathbb{R}} e^{qb\pi\lambda^2} \|\hat{f}(\pi_\lambda)\|_{HS}^q |\lambda|^n d\lambda \\
&= \int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} |\lambda|^{-nq/2} |\lambda|^n d\lambda \\
&= \int_{|\lambda|\leq 1} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} |\lambda|^{n(1-(q/2))} d\lambda \\
&\quad + \int_{|\lambda|>1} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} |\lambda|^{n(1-(q/2))} d\lambda \\
&\geq \int_{|\lambda|>1} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} d\lambda.
\end{aligned}$$

In the above we have used the facts that the first integral is over a compact set and in the second, $q < 2$. As the integrand in the last inequality is a continuous function of λ, we have

$$\int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} d\lambda < \infty.$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^q d\lambda &= \int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} \hat{h}(\lambda)^{q/2} d\lambda \\
&\leq \|h\|_{L^1}^{q/2} \int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} d\lambda < \infty. \qquad (3.14)
\end{aligned}$$

By (3.13) and (3.14) we get $h = 0$ almost every where if $(a/2)b \geq 1$ that is $ab \geq 2$. This proves the theorem for $p = \infty$. The case $p < \infty$ follows as the case 2 of theorem 3.1.2.

The proof actually uses the nature of the Pfaffian arising in the Plancherel theorem, in that $\{\lambda \in \mathbb{R} : |Pf(\lambda)| = |\lambda|^n \leq 1\}$ is a compact subset of $\mathfrak{z}^*$. This property of the Pfaffian we are using is never available for two step nilpotent Lie groups which are not *MW*. Even for *MW* groups, we may not have this property of the Pfaffian. To see that, let $G = \mathcal{F}_{4,2}$, which is a *MW* group (see example 2.1.3), and let $l \in \mathfrak{z}^* = \Lambda^2(\mathbb{R}^4)^*$ be given by the matrix

$$\begin{pmatrix} 0 & 0 & n & 0 \\ 0 & 0 & 0 & \frac{1}{n} \\ -n & 0 & 0 & 0 \\ 0 & -\frac{1}{n} & 0 & 0 \end{pmatrix}$$

so that $|Pf(l)| = |\det l|^{1/2} = 1$. But $\|l\|_{\Lambda^2(\mathbb{R}^4)} = \sqrt{2(n^2 + \frac{1}{n^2})} > n$. Thus $\{l \in \mathfrak{z}^* / |Pf(l)| \leq 1\}$ is an unbounded set in $\mathfrak{z}^*$. In fact Example 2 of [MW] shows that any arbitrary homogeneous polynomial can arise as the Pfaffian of the generic linear functionals of a *MW* group, so it is not possible to enclose the zero set of that polynomial inside a compact set. Thus our proof does not say anything for $q < 2$ on a general two step nilpotent Lie group; nevertheless it goes through for H-type groups. We briefly outline the basic features of H-type groups, to bring out this point.

Let $\mathcal{B}$ and $\mathcal{Z}$ be Euclidean spaces and consider the vector space sum $\mathfrak{g} = \mathcal{Z} \oplus \mathcal{B}$. Suppose there exist a linear map $J : \mathcal{Z} \longrightarrow End(\mathcal{B})$ satisfying the conditions

$$a) \|J(Y)(X)\| = \|Y\|\|X\|, \qquad \text{for all } Y \in \mathcal{Z}, X \in \mathcal{B},$$
$$b) J(Y) \circ J(Y) = -Id, \qquad \text{for all } Y \in \mathcal{Z}, Y \neq 0.$$

We define a Lie bracket on $\mathfrak{g}$ as follows: $\mathcal{Z}$ is defined to be central in $\mathfrak{g}$ and for $X, X' \in \mathcal{B}$,

$$\langle [X, X'], Y \rangle = \langle J(Y)(X), X' \rangle, \qquad \text{for all } Y \in \mathcal{Z},$$
$$\langle [X, X'], X'' \rangle = 0, \qquad \text{for all } X'' \in \mathcal{B}.$$

Then $\mathfrak{g}$ is a two step nilpotent Lie algebra called an H-type Lie algebra and the connected, simply connected Lie group corresponding to $\mathfrak{g}$ is called an H-type group (for details on structures of H-type groups, we refer to [KP], [CDKR]).

Let $\lambda(\neq 0) \in \mathcal{Z}^* \subset \mathfrak{g}^*$ where the inclusion stands for the trivial extension to $\mathcal{B}$. Let $\dim \mathcal{Z} = m$ and $\dim \mathcal{B} = 2n$. We notice from $a)$ that for fixed X, $Y \to J(Y)(X)$ is $\|X\|$ times an isometry, so that

$$\langle J(Y)(X), J(Y)(X')\rangle = \|X\|^2\langle Y, Y'\rangle. \tag{3.15}$$

Using the inner product on $\mathcal{Z}$ we can identify $\mathcal{Z}^*$ with $\mathcal{Z}$ and write $\lambda = \sum_{j=1}^{m} \lambda_j e_j$ where $\{e_j\}$ is an orthonormal basis for $\mathcal{Z}$. We look at the skew symmetric bilinear form B_λ on $\mathcal{B} \times \mathcal{B}$; here

$$B_\lambda(X, X') = \lambda([X, X']) = \langle J(\lambda)(X), X'\rangle.$$

Then B_λ is nondegenerate, for, if $B_\lambda(X, X') = 0$ for all $X' \in \mathcal{B}$, then $J(\lambda)(X) = 0$ and hence $X = 0$. Therefore $V_N^* = \mathfrak{z}^*$ and $V_N^* \cap \mathcal{U} = \{\lambda \in \mathfrak{z}^* : \lambda \neq 0\}$ by note 2.1.3. Let $\lambda_0 = (1/\|\lambda\|).\lambda$. Then $B_\lambda(X, X') = \|\lambda\| B_{\lambda_0}(X, X')$. Since $|\det B_{\lambda_0}(X_i, X_j)| = 1$ (as $J(\lambda_0)$ is an isometry) we have $|Pf(\lambda)| = \|\lambda\|^n$. Thus $\{\lambda \in \mathfrak{z}^* : |Pf(\lambda)| \leq 1\} \subset V_N^*$ is again compact, and we get the case $q < 2$ exactly as on H_n.

3.2 Heisenberg's inequality

The classical inequality of Heisenberg for L^2 functions on $\mathbb{R}$ says that

$$\left(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} |y|^2 |\hat{f}(y)|^2 dy\right)^{1/2} \geq C\|f\|_2^2 \tag{3.16}$$

where $\hat{f}$ is defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x} dx$$

and C is a constant independent of f.

In this section our aim is to extend the result proved in [T1] for all connected, simply connected, step two nilpotent Lie groups.

We state (3.16) in a slightly different way. Let $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplacian on $\mathbb{R}^n$. Then $\widehat{(\Delta f)}(y) = 4\pi^2\|y\|^2 \hat{f}(y)$ for any Schwartz class function on $\mathbb{R}^n$. We may relate Δ to the character $\gamma_y(x) = e^{2\pi i y.x}$ of $\mathbb{R}^n$ by

$$d\gamma_y\left(\frac{\partial}{\partial x_j}\right) = \frac{d}{ds}\Big|_{s=0} \gamma_y(se_j) = \frac{d}{ds}\Big|_{s=0} e^{2\pi i s y_j} = 2\pi i y_j,$$

and hence $d\gamma_y(\Delta) = 4\pi^2\|y\|^2$. Thus we have

$$\widehat{(\Delta f)}(y) = d\gamma_y(\Delta)\hat{f}(y).$$

Since $d\gamma_y(\Delta)$ is a positive, self adjoint operator, it has a (visible) square root, which is *multiplication by* $2\pi\|y\|$. Thus we define

$$\widehat{(\Delta^{\frac{1}{2}} f)}(y) = 2\pi\|y\|\hat{f}(y) = (d\gamma_y(\Delta))^{\frac{1}{2}}\hat{f}(y),$$

for all Schwartz class functions on $\mathbb{R}^n$. Since the Fourier transform is an isomorphism on Schwartz class functions, the operator $(\Delta)^{\frac{1}{2}}$ is defined completely. Then we can restate (3.16) as

$$\left(\int_{\mathbb{R}^n} \|x\|^2 |f(x)|^2 dx\right)^{1/2} \left(|\widehat{(\Delta^{\frac{1}{2}}(f))}(y)|^2 dy\right)^{1/2} \geq C\|f\|_2^2, \tag{3.17}$$

for all f of Schwartz class on $\mathbb{R}^n$, where C is a constant independent of f. It is (3.17), whose analogue on connected, simply connected, two step nilpotent Lie groups we are looking for. As in the case of Heisenberg groups, here also the proof, in principle, is close to the proof on $\mathbb{R}^n$ (see [F2]) having the same basic ingredients, namely, integration by parts, Cauchy-Schwartz inequality and the Plancherel theorem.

We call *a function f on G a Schwartz class function if $f \circ \exp$ is a Schwartz class function on* $\mathfrak{g}$. We denote the Schwartz class functions by $\mathcal{S}(G)$.

The main result of this section is the following analogue of Heisenberg's inequality for two step nilpotent Lie groups.

Theorem 3.2.1 *Let G be a connected, simply connected, step two nilpotent Lie group and $f \in \mathcal{S}(G)$. Then*

$$\left(\int_G \|v\|^2 |f(x,v)|^2 dx dv\right)^{1/2} \left(\int_{V_N^* \cap \mathcal{U}} \|\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l)\|_{HS}^2 |Pf(l)| dl\right)^{1/2}$$
$$\geq C \|f\|_{L^2(G)}^2 \tag{3.18}$$

where C is a constant independent of f and $\mathcal{L}$ is the sub-Laplacian.

Let us explain the meaning of $\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l)$. Given $X \in \mathfrak{g}$, we view this as a left invariant ($(Xf)_g = X(f_g)$, where $f_g(x) = f(gx)$) differential operator on $C^\infty(G)$ with the action given by (2.12). Then in view of our definition of the group Fourier transform, we have for $f \in \mathcal{S}(G)$

$$\widehat{(Xf)}(\pi_l) = d\pi_l(X) \circ \hat{f}(\pi_l), \tag{3.19}$$

where $d\pi_l(X)$ is given by (2.15). We view elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, as the algebra of all left invariant differential operators on $C^\infty(G)$. Since $d\pi_l$ is a representation of $\mathfrak{g}$, it extends to a representation of $\mathcal{U}(\mathfrak{g})$ realized on $C^\infty(\pi_l)$. By (3.19) we have

$$\widehat{(\mathcal{L}f)}(\pi_l) = d\pi_l(\mathcal{L}) \circ \hat{f}(\pi_l),$$

as $\mathcal{L} \in \mathcal{U}(\mathfrak{g})$. It is known that if π_l is realized on $L^2(\mathbb{R}^k)$ then $C^\infty(\pi_l) = \mathcal{S}(\mathbb{R}^k)$ (see [CG], theorem 4.1.1). In section 4, chapter 2 we have seen that the eigen functions of $d\pi_l(\mathcal{L})$ are parametrized by $\mathbb{N}^k$ and are given by (2.20). Let $\{t_i(l) > 0 : i = 0, \ldots\}$ be the real numbers such that there exist $\alpha \in \mathbb{N}^k$ with

$$t_i(l) = \mu(l) + \sum_{j=1}^{k} 2\pi\lambda_j(l)(2\alpha_j + 1). \tag{3.20}$$

Let $E_i(l) = \operatorname{span}_{\mathbb{R}}\{h_\alpha^l : d\pi_l(\mathcal{L})(h_\alpha^l) = t_i(l) h_\alpha^l\}$, that is, $E_i(l)$ is the eigen space corresponding to the eigen value $t_i(l)$, which is clearly finite dimensional. If $P_i(l) : L^2(\mathbb{R}^k) \to E_i(l)$ is the projection, we have

$$d\pi_l(\mathcal{L}) = \sum_{j=0}^{\infty} t_j(l) P_j(l). \tag{3.21}$$

Thus we define

$$d\pi_l(\mathcal{L})^{\frac{1}{2}} = \sum_{j=0}^{\infty} t_j(l)^{\frac{1}{2}} P_j(l), \tag{3.22}$$

and

$$d\pi_l(\mathcal{L})^{-\frac{1}{2}} = \sum_{j=0}^{\infty} t_j(l)^{-\frac{1}{2}} P_j(l). \tag{3.23}$$

Analogous to the Euclidean spaces, we define

$$\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l) = d\pi_l(\mathcal{L})^{\frac{1}{2}} \circ \hat{f}(\pi_l), \tag{3.24}$$

for all $f \in \mathcal{S}(G)$ and $l \in V_N^* \cap \mathcal{U}$. Thus the statement in theorem 3.2.1 makes sense.

It follows from (3.20) that the eigen values of $d\pi_l(\mathcal{L})^{-\frac{1}{2}}$ are bounded by $\lambda_0(l)^{-\frac{1}{2}}$ where $\lambda_0(l) = \min\{\lambda_j(l) : 1 \leq j \leq k\}$. As a consequence

Lemma 3.2.1 *The operator $d\pi_l(\mathcal{L})^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^k)$.*

Let us consider the following elements of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$,

$$D_j(l) = Y_j(l) - iW_j(l), \qquad 1 \leq j \leq k, \tag{3.25}$$

$$\bar{D}_j(l) = Y_j(l) + iW_j(l), \qquad 1 \leq j \leq k. \tag{3.26}$$

Because of lemma 2.4.1 we have

$$\begin{aligned} d\pi_l(D_j(l))\phi(\xi) &= (d\pi_l(Y_j(l) - id\pi_l(W_j(l)))\,\phi(\xi) \\ &= \left(-\frac{\partial}{\partial \xi_j} + 2\pi\lambda_j(l)\xi_j\right)\phi(\xi), & (3.27) \\ d\pi_l(\bar{D}_j(l))\phi(\xi) &= (d\pi_l(Y_j(l)) + id\pi_l(W_J(l)))\,\phi(\xi) \\ &= \left(-\frac{\partial}{\partial \xi_j} - 2\pi\lambda_j(l)\xi_j\right)\phi(\xi). & (3.28) \end{aligned}$$

If h_s is the s-th normalized hermite function on $\mathbb{R}$, then we have

$$\left(-\frac{d}{dx} + x\right) h_s = (2s+2)^{\frac{1}{2}} h_{s+1},$$

$$\left(\frac{d}{dx}+x\right)h_s=(2s)^{\frac{1}{2}}h_{s-1}, \quad s\geq 1,$$

(see [T2]). Thus if $h_s^c(x)=c^{1/4}h_s(c^{1/2}x)$, then

$$\left(-\frac{d}{dx}+cx\right)h_s^c=c^{1/2}(2s+2)^{1/2}h_{s+1}^c,$$

$$\left(\frac{d}{dx}+cx\right)h_s^c=c^{1/2}(2s)^{1/2}h_{s-1}^c.$$

Using this with (3.27) and (3.28) we get for $\alpha\in\mathbb{N}^k$

$$d\pi_l(D_j(l))(h_\alpha^l) = (2\pi\lambda_j(l))^{1/2}(2\alpha_j+2)^{1/2}h_{\alpha+e_j}, \tag{3.29}$$

$$d\pi_l(\bar{D}_j(l))(h_\alpha^l = -(2\pi\lambda_j(l))^{1/2}(2\alpha_j)^{1/2}h_{\alpha-e_j}^l, \tag{3.30}$$

where

$$\alpha+e_j = (\alpha_1,\ldots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\ldots,\alpha_k)\in\mathbb{N}^k,$$

$$\alpha-e_j = (\alpha_1,\ldots,\alpha_{j-1},\alpha_j-1,\alpha_{j+1},\ldots,\alpha_k)\in\mathbb{N}^k.$$

Lemma 3.2.2 *The operators $d\pi_l(D_j(l))\circ d\pi_l(\mathcal{L})^{-\frac{1}{2}}$ and $d\pi_l(\bar{D}_j(l))\circ d\pi_l(\mathcal{L})^{-\frac{1}{2}}$ are bounded operators on $L^2(\mathbb{R}^k), 1\leq j\leq k$.*

PROOF. We consider the orthonormal basis $\{h_\alpha^l:\alpha\in\mathbb{N}^k\}$ of $L^2(\mathbb{R}^k)$. By (3.23), (3.29) and (3.30) we have

$$d\pi_l(D_j(l))\circ d\pi_l(\mathcal{L})^{-\frac{1}{2}}(h_\alpha^l)=\left(\frac{2\pi\lambda_j(l)(2\alpha_j+2)}{\mu(l)+\sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}}h_{\alpha+e_j}^l,$$

and

$$d\pi_l(\bar{D}_j(l))\circ d\pi_l(\mathcal{L})^{-\frac{1}{2}}(h_\alpha^l)=-\left(\frac{2\pi\lambda_j(l)2\alpha_j}{\mu(l)+\sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}}h_{\alpha-e_j}^l.$$

Since

$$\left(\frac{2\pi\lambda_j(l)(2\alpha_j+2)}{\mu(l)+\sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}}\leq\sqrt{2},$$

and

$$\left(\frac{2\pi\lambda_j(l)2\alpha_j}{\mu(l)+\sum_{p=1}^{k}2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}} \le 1,$$

the operators $d\pi_l(D_j(l)) \circ d\pi_l(\mathcal{L})^{-\frac{1}{2}}$ and $d\pi_l(\bar{D}_j(l)) \circ d\pi_l(\mathcal{L})^{-\frac{1}{2}}$ are bounded operators on $L^2(\mathbb{R}^k)$. This completes the proof.

Suppose $f \in \mathcal{S}(G)$ and let $l \in V_N^* \cap \mathcal{U}$ be arbitrary but fixed. So we have an almost symplectic basis (2.4) of $\mathfrak{g}$. Let $l \mid \mathfrak{z} = \lambda$. We define

$$\mathcal{F}_c f(\lambda, v) = \int_{\mathbb{R}^m} f(x, v) e^{-2\pi i \lambda(x)} dx, \tag{3.31}$$

that is, the partial Fourier transform in the central component. So $v \to \mathcal{F}_c f(\lambda, v)$ is a Schwartz class function on $\mathbb{R}^{n-m}$. On Euclidean spaces, differentiation and multiplication are intertwined by the Fourier transform. On two step groups, as analogues of differentiation we consider the operators $D_l(l)$ and $\bar{D}_j(l)$ and as analogue of Fourier transform we consider the partial Fourier transform defined in (3.31). Want to find, who plays the role of multiplication?

Let $f \in \mathcal{S}(G)$ and $X_j \in \mathcal{B} \subset \mathfrak{g}$, $m+1 \le j \le n$. By (2.18) it is clear that $X_j f \in \mathcal{S}(G)$ and an easy calculation shows that

$$\mathcal{F}_c(X_j f)(\lambda, v) = \left(\frac{\partial}{\partial x_j} + \pi i B_\lambda(v, X_j)\right)(\mathcal{F}_c f)(\lambda, v).$$

Thus using the basis in (2.4) we have

$$\begin{aligned} &\mathcal{F}_c(W_j(l)f)(\lambda, z, w, y) \\ &= \left(\frac{\partial}{\partial w_j} - \pi i \lambda_j(l) y_j\right)(\mathcal{F}_c f)(\lambda, z, w, y), \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} &\mathcal{F}_c(Y_j(l)f)(\lambda, z, w, y) \\ &= \left(\frac{\partial}{\partial y_j} + \pi i \lambda_j(l) w_j\right)(\mathcal{F}_c f)(\lambda, z, w, y), \end{aligned} \tag{3.33}$$

for $1 \leq j \leq k$. Thus writing

$$V_j(l) = \left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right) - \pi\lambda_j(l)(y_j - iw_j), \tag{3.34}$$

$$\bar{V}_j(l) = \left(\frac{\partial}{\partial y_j} + i\frac{\partial}{\partial w_j}\right) + \pi\lambda_j(l)(y_j + iw_j), \tag{3.35}$$

we have from (3.32) and (3.33)

$$\mathcal{F}_c(D_j(l)f)(\lambda, z, w, y) = V_j(l)(\mathcal{F}_c f)(\lambda, z, w, y), \tag{3.36}$$

$$\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, z, w, y) = \bar{V}_j(l)(\mathcal{F}_c f)(\lambda, z, w, y). \tag{3.37}$$

Thus $V_j(l)$ and $\bar{V}_j(l)$ play the role of multiplication.

Now we come to the proof of theorem 3.2.1.

Proof of theorem 3.2.1.

Let $f \in \mathcal{S}(G)$ and $l \mid \mathfrak{z} = \lambda$. Now

$$\begin{aligned}&\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, z, w, y)|^2 dz dw dy \\ = &\int_{\mathbb{R}^{n-m}} \mathcal{F}_c f(\lambda, z, w, y)\overline{\mathcal{F}_c f(\lambda, z, w, y)} dz dw dy.\end{aligned}$$

Since

$$\begin{aligned}&\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(x_j + iy_j)g(x, y) \\ = \; &2g(x, y) + (x_j + iy_j)\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)g(x, y),\end{aligned}$$

we have from the above equality

$$\begin{aligned}&\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, z, w, y)|^2 dz dw dy \\ = &\int_{\mathbb{R}^{n-m}} \frac{1}{2}\left\{\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)(y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y)\right. \\ &\left. -(y_j + iw_j)\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)\mathcal{F}_c f(\lambda, z, w, y)\right\}\end{aligned}$$

$$
\begin{aligned}
&\quad\times\overline{\mathcal{F}_cf(\lambda,z,w,y)}dzdwdy\\
&= \frac{1}{2}\int_{\mathbb{R}^{n-m}}\left(\frac{\partial}{\partial y_j}-i\frac{\partial}{\partial w_j}\right)(y_j+iw_j)\mathcal{F}_cf(\lambda,z,w,y)\\
&\quad\times\overline{\mathcal{F}_cf(\lambda,z,w,y)}dzdwdy\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\left(\frac{\partial}{\partial y_j}-i\frac{\partial}{\partial w_j}\right)\mathcal{F}_cf(\lambda,z,w,y)\\
&\quad\times\overline{\mathcal{F}_c(\lambda,z,w,y)}dzdwdy\\
&= -\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\mathcal{F}_cf(\lambda,z,w,y)\\
&\quad\times\overline{\left(\frac{\partial}{\partial y_j}+i\frac{\partial}{\partial w_j}\right)\mathcal{F}_cf(\lambda,z,w,y)dzdwdy}\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\left(\frac{\partial}{\partial y_j}-i\frac{\partial}{\partial w_j}\right)\mathcal{F}_cf(\lambda,z,w,y)\\
&\quad\times\overline{\mathcal{F}_cf(\lambda,z,w,y)}dzdwdy\\
&\quad\Big(\text{by integration by parts in the first integral}\Big)\\
&= -\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\mathcal{F}_cf(\lambda,z,w,y)\\
&\quad\times\overline{(\bar{V}_j(l)-\pi\lambda_j(l)(y_j+iw_j))\mathcal{F}_cf(\lambda,z,w,y)}dzdwdy\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)(V_j(l)+\pi\lambda_j(l)(y_j-iw_j))\mathcal{F}_cf(\lambda,z,w,y)\\
&\quad\times\overline{\mathcal{F}_cf(\lambda,z,w,y)}dzdwdy\\
&\quad\Big(\text{by (3.34) and (3.35)}\Big)\\
&= -\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\mathcal{F}_cf(\lambda,z,w,y)\overline{(\bar{V}_j(l)\mathcal{F}_cf)(\lambda,z,w,y)}dzdwdy\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)(V_j(l)\mathcal{F}_cf)(\lambda,z,w,y)\overline{\mathcal{F}_cf(\lambda,z,w,y)}dzdwdy\\
&= -\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\mathcal{F}_cf(\lambda,z,w,y)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,z,w,y)}dzdwdy\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\mathcal{F}_c(D_j(l)f)(\lambda,z,w,y)\overline{\mathcal{F}_cf(\lambda,z,w,y)}dzdwdy\\
&\quad\Big(\text{by (3.36) and (3.37)}\Big). \qquad (3.38)
\end{aligned}
$$

Let us recall, if l varies over $V_N^\bullet \cap \mathcal{U}$ then $l \mid \mathfrak{z} = \lambda$ varies over the Zariski open subset $\mathcal{U}'$ of $\mathfrak{z}^*$ (see note 2.1.5). Hence

$$\int_{\mathfrak{z}}\int_{\mathbb{R}^{n-m}} |f(x,v)|^2 dx dv$$
$$= \int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, v)|^2 d\lambda dv$$
$$\Big(\text{by Euclidean Plancherel theorem on } \mathfrak{z}\Big)$$
$$= \int_{\mathcal{U}'}\left(\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, z, w, y)|^2 dz dw dy\right) d\lambda$$

$\Big($by Fubini's theorem and the orthogonal basis change on $\mathbb{R}^{n-m}$ by $T_l : \mathrm{span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_n\} \to \mathrm{span}_{\mathbb{R}}\{Z_1(l), \ldots, Y_k(l)\}\Big)$

$$= \int_{\mathcal{U}'}\left(-\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, z, w, y)} dz dw dy\right.$$
$$\left. -\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c(D_j(l)f)(\lambda, z, w, y)\overline{\mathcal{F}_c f(\lambda, z, w, y)} dz dw dy\right) d\lambda$$
$$\Big(\text{by } (3.38)\Big)$$
$$= \int_{\mathcal{U}'}\left(-\frac{1}{2}\int_{\mathbb{R}^{n-m}} T_l^{-1}(y_j + iw_j)\mathcal{F}_c f(\lambda, v)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, v)} dv\right.$$
$$\left. -\frac{1}{2}\int_{\mathbb{R}^{n-m}} T_l^{-1}(y_j + iw_j)\mathcal{F}_c(D_j(l)f)(\lambda, v)\overline{\mathcal{F}_c f(\lambda, v)} dv\right) d\lambda$$
$$\Big(\text{by change of variables}\Big)$$
$$\leq \frac{1}{2}\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |T_l^{-1}(y_j + iw_j)|^2 |\mathcal{F}_c f(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}$$
$$\times\left\{\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}\right.$$
$$\left. + \left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}\right\}$$

$\Big($by Cauchy-Schwartz inequality and nonnegativity of the integral$\Big)$

$$\leq \frac{1}{2}\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} \|v\|^2 |\mathcal{F}_c f(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}$$
$$\times\left\{\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}\right.$$

$$+ \left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}\Bigg\}$$

$$= \frac{1}{2}\left(\int_{\mathfrak{z}}\int_{\mathbb{R}^{n-m}} \|v\|^2 |f(x,v)|^2 dx dv\right)^{\frac{1}{2}}$$

$$\times \Bigg\{\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}$$

$$+ \left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}\Bigg\}, \tag{3.39}$$

by the Euclidean Plancherel theorem on $\mathfrak{z}$.

By theorem (2.3.1) we have

$$\int_{\bar{V}_N^\bullet} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^2 dl_{n_1}\dots dl_{n_r} = |Pf(l)|^{-1}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, v)|^2 dv$$

where $l \mid \mathfrak{z} = \lambda$ and $l \mid \tilde{V}_N = \gamma = (l_{n_1}, \dots, l_{n_r})$. Thus

$$\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(\bar{D}(l)f)(\lambda, v)|^2 dv d\lambda\right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathcal{U}'}\int_{\bar{V}_N^\bullet} \|(\widehat{\bar{D}_j(l)f})(\pi_l)\|_{HS}^2 |Pf(l)| dl_{n_1}\dots dl_{n_r} d\lambda\right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathcal{U}'}\int_{\bar{V}_N^\bullet} \|d\pi_l(\bar{D}_j(l)) \circ \hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| dl_{n_1}\dots dl_{n_r} d\lambda\right)^{\frac{1}{2}}$$

$$= \Bigg(\int_{\mathcal{U}'}\int_{\bar{V}_N^\bullet} \|d\pi_l(\bar{D}_j(l)) \circ d\pi_l(\mathcal{L})^{-\frac{1}{2}} \circ d\pi_l(\mathcal{L})^{\frac{1}{2}} \circ \hat{f}(\pi_l)\|_{HS}^2$$

$$\times |Pf(l)| dl_{n_1}\dots dl_{n_r} d\lambda\Bigg)^{\frac{1}{2}}$$

$$\leq \Bigg(\int_{\mathcal{U}'}\int_{\bar{V}_N^\bullet} \|d\pi_l(\bar{D}_j(l) \circ d\pi_l(\mathcal{L})^{-\frac{1}{2}}\|_{Op}^2 \|(\widehat{\mathcal{L}^{\frac{1}{2}}f})(\pi_l)\|_{HS}^2$$

$$\times |Pf(l)| dl_{n_1}\dots dl_{n_r} d\lambda\Bigg)^{\frac{1}{2}}$$

$$\leq 2\left(\int_{\mathcal{U}'}\int_{\bar{V}_N^\bullet} \|(\widehat{\mathcal{L}^{\frac{1}{2}}f})(\pi_l)\|_{HS}^2 |Pf(l)| dl_{n_1}\dots dl_{n_r} dl_1\dots dl_m\right)^{\frac{1}{2}}$$

$\big($by lemma 3.2.2$\big)$.

Similarly as above we can show that

$$\left(\int_{\mathcal{U}'}\int_{\mathbf{R}^{n-m}}|\mathcal{F}_c(D_j(l)f)(\lambda,v)|^2 dv d\lambda\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathcal{U}'}\int_{\tilde{V}_N^{\bullet}}\|\widehat{(\mathcal{L}^{\frac{1}{2}}f)}(\pi_l)\|_{HS}^2|Pf(l)|dl_{n_1}\dots dl_{n_r}dl_1\dots dl_m\right)^{\frac{1}{2}}$$

$$= \left(\int_{V_N^{\bullet}\cap\mathcal{U}}\|\widehat{(\mathcal{L}^{\frac{1}{2}}f)}(\pi_l)\|_{HS}^2|Pf(l)|dl\right)^{\frac{1}{2}}.$$

Thus from (3.39) we have

$$\int_G|f(x,v)|^2 dx dv$$

$$\leq C\left(\int_G\|v\|^2|f(x,v)|^2 dx dv\right)^{\frac{1}{2}}\left(\int_{V_N^{\bullet}\cap\mathcal{U}}\|\widehat{(\mathcal{L}^{\frac{1}{2}}f)}(\pi_l)\|_{HS}^2|Pf(l)|dl\right)^{\frac{1}{2}}.$$

where C is a constant independent of f. This completes the proof.

Chapter 4

Uncertainty principles on some semidirect products

In this chapter, we work with two well known solvable Lie groups, the Euclidean motion group of the plane and the oscillator group. Our aim is to get analogues of the Cowling-Price theorem for these two groups and also of a theorem of Morgan (see [HJ] or theorem 4.1.2) which is a slightly weaker version of 1.1.3.1, c). As shown in [S], here it is possible to extend the proofs given for the Euclidean spaces, the reason being that in both the cases, the *important* series of irreducible unitary representations can be analytically continued.

4.1 The Euclidean motion group of the plane

The Euclidean motion group of the plane, denoted by $M(2)$, is a semidirect product $\mathbb{R}^2 \times_\gamma SO(2)$, where

$$\gamma : SO(2) \to Aut(\mathbb{R}^2) \quad \text{is given by}$$

$$\gamma(e^{i\theta})(x,y) = (x,y)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and the product is given by

$$(x, y, e^{i\theta})(x_1, y_1, e^{i\theta_1}) = \left((x, y) + \gamma(e^{i\theta})(x_1, y_1), e^{i\theta+\theta_1}\right).$$

If we identify $\mathbb{R}^2$ with $\mathbb{C}$ that is $(x, y) \to x + iy$, then the above product takes the form

$$(z, \alpha)(z_1, \alpha_1) = (z + \alpha z_1, \alpha\alpha_1), \tag{4.1}$$

where $\alpha = e^{i\theta}$. In terms of 4.1, $dzd\beta$ is a Haar measure of $M(2)$. All irreducible, unitary, infinite dimensional representations of $M(2)$ are realized on $L^2(\mathbb{T})$ and the equivalence classes of them are parametrized by $\{r \in \mathbb{R} : r \in \mathbb{R}^+\}$ and are given by

$$\pi_r : M(2) \to \mathcal{U}(L^2(\mathbb{T}))$$

$$(\pi_r(z, \beta)f)(\alpha) = e^{2\pi i \mathrm{Re}(r\bar{\alpha}z)} f(\bar{\beta}\alpha), \quad f \in L^2(\mathbb{T}), \quad \alpha \in \mathbb{T}. \tag{4.2}$$

π_{-r} can be defined similarly by replacing r by $-r$ in (4.2), but π_r and π_{-r} are unitarily equivalent. It turns out that the family $\{\pi_r : r \in \mathbb{R}^+\}$ is a set of full Plancherel measure in the dual of $M(2)$ and the Plancherel measure is given by $Const.rdr$ (see [Su] for details).

For $f \in L^1(M(2))$, we define the group Fourier transform by

$$\hat{f}(r) \stackrel{\text{def.}}{=} \hat{f}(\pi_r) = \int_{M(2)} f(z, \beta)\Pi_r((z, \beta)^{-1})dzd\beta$$

where the integral is interpreted in the weak sense. If $f \in L^1(M(2)) \cap L^2(M(2))$, then $\hat{f}(\pi_r) \in \mathcal{HS}(L^2(\mathbb{T}))$, the algebra of Hilbert-Schmidt operators on $L^2(\mathbb{T})$.

First, for our use here we state an equivalent version of Lemma 1.1.

Lemma 4.1.1 *Suppose $g : \mathbb{C} \to \mathbb{C}$ is an entire function and for $1 \le p < \infty$,*

(*i*) $|g(x + iy)| \le Ae^{a\pi x^2}$,

(*ii*) $(\int_{\mathbb{R}} |g(x)|^p dx)^{1/p} < \infty$, *where $a > 0$. Then $g = 0$.*

Using this we prove

Theorem 4.1.1 *Let $f \in L^1(M(2)) \cap L^2(M(2))$ and*

(i) $\int_{M(2)} e^{pa\pi|z|^2}|f(z,\alpha)|^p dz d\alpha < \infty$,

(ii) $\int_{\mathbb{R}^+} e^{qb\pi r^2}\|\hat{f}(r)\|_{HS}^q r dr < \infty$,

where $a, b > 0$, $1 \leq q < \infty$, $1 \leq p \leq \infty$. If $ab > 1$ then $f = 0$ almost everywhere.

Proof. Let $\{e_n : n \in \mathbb{Z}\}$ be the canonical orthonormal basis for $L^2(\mathbb{T})$. We define

$$\begin{aligned} \Phi^r_{m,n}(z,\beta) &= \langle \pi_r(z,\beta)e_m, e_n\rangle_{L^2(\mathbb{T})} \qquad ((z,\beta) \in M(2), \quad r > 0) \\ &= \int_{\mathbb{T}} e^{2\pi i \mathrm{Re}(r\bar{\alpha}z)} e_m(\bar{\beta}\alpha)\overline{e_n(\alpha)}d\alpha, \end{aligned}$$

the matrix functions of the representation π_r. Now for $h \in L^2(\mathbb{T})$,

$$(\Pi_\omega(z,\beta)h)(\alpha) = e^{2\pi i\omega \mathrm{Re}(\bar{\alpha}z)}h(\bar{\beta}\alpha) \quad \omega = u + iv \in \mathbb{C}$$

continues to be a *nonunitary* representation of $M(2)$ and we get the complex extension of the function $r \to \Phi^r_{m,n}(z,\beta)$, for fixed (z,β), m, n. Further

$$|\Phi^\omega_{m,n}(z,\beta)| = |\langle \Pi_\omega(z,\beta)e_m, e_n\rangle| \leq \int_{\mathbb{T}} e^{-2\pi v \mathrm{Re}(\bar{\alpha}z)}d\alpha, \tag{4.3}$$

for fixed $m, n, (z,\beta)$. From (4.3), $\omega \to \Phi^\omega_{m,n}(z,\beta)$ is an entire function by a standard argument. Also we have the estimate

$$\begin{aligned} &|f(z,\beta)\Phi^\omega_{m,n}(-\bar{\beta}z,\bar{\beta})| \\ &\leq |f(z,\beta)| \int_{\mathbb{T}} e^{2\pi v \mathrm{Re}(-\bar{\alpha}\bar{\beta}z)}d\alpha \quad \text{where}(\omega = u + iv \in \mathbb{C}) \\ &= |f(z,\beta)|e^{2\pi|v||z|}. \end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbf{C}\times\mathbf{T}} |f(z,\beta)||\Phi^{\omega}_{m,n}(-\bar{\beta}z,\bar{\beta})|dzd\beta \\
&\leq \int_{\mathbf{C}\times\mathbf{T}} |f(z,\beta)|e^{2\pi|v||z|}dzd\beta \\
&= e^{\pi|v|^2/a}\int_{\mathbf{C}\times\mathbf{T}} \left(|f(z,\beta)|e^{a\pi|z|^2}\right)\left(e^{-a\pi(|z|-|v|/a)^2}\right)dzd\beta \\
&\leq C_1 e^{\pi v^2/a}(A+B|v|+K|v|^2) \\
&\quad \text{(by (i) and Holder's inequality, } A,B,K>0) \\
&\leq C_1 e^{k\pi|v|^2}, \qquad (4.4)
\end{aligned}$$

for some k, such that $b > k > 1/a$. A routine argument using Morera's theorem and dominated convergence theorem now shows that the complex extension of the function $r \to \langle \hat{f}(r)e_m, e_n\rangle$, $r \in \mathbb{R}^+$, which we write as

$$\langle \hat{f}(\omega)e_m, e_n\rangle = \int_{\mathbf{C}\times\mathbf{T}} f(z,\beta)\Phi^{\omega}_{m,n}(-\bar{\beta}z,\bar{\beta})dzd\beta,$$

is an entire function of the complex variable ω, for fixed m, n. We note further that $\langle \hat{f}(r)e_m, e_n\rangle = \langle \hat{f}(-r)e_m, e_n\rangle$ for $r \in \mathbb{R}^+$. Since $|\langle \hat{f}(r)e_m, e_n\rangle| \leq \|\hat{f}(r)\|_{HS}$, we have from (ii)

$$\int_{\mathbb{R}} e^{qb\pi r^2}|\langle \hat{f}(r)e_m, e_n\rangle|^q r dr < \infty.$$

Since $|\langle \hat{f}(r)e_m, e_n\rangle|$ is a continuous function of r,

$$\int_{\mathbb{R}} e^{qb\pi r^2}|\langle \hat{f}(r)e_m, e_n\rangle|^q dr < \infty. \qquad (4.5)$$

From (4.4) we have

$$|\langle \hat{f}(\omega)e_m, e_n\rangle| \leq C_1 e^{k\pi v^2} \quad \text{where } \omega = u + iv. \qquad (4.6)$$

We define

$$g(\omega) = e^{k\pi\omega^2}\langle \hat{f}(\omega)e_m, e_n\rangle.$$

Then g is an entire function. The estimates (4.5) and (4.6) give

$$|g(u+iv)| \leq C_1 e^{k\pi(u^2-v^2)} e^{k\pi v^2} = Ce^{k\pi u^2}, \tag{4.7}$$

$$\int_{\mathbb{R}} |g(r)|^q dr = \int_{\mathbb{R}} e^{qk\pi r^2} |\langle \hat{f}(r)e_m, e_n\rangle|^q dr < \infty \quad \text{as } k < b. \tag{4.8}$$

By (4.7) and (4.8) it follows that g satisfies conditions of lemma 4.1.1 and hence $g = 0$. So $\langle \hat{f}(\omega)e_m, e_n\rangle = 0$ for all ω. But m, n are arbitrary and hence $\|\hat{f}(r)\|_{HS} = 0$, for all $r > 0$, which implies $f = 0$ by the Plancherel theorem. This completes the proof.

Remark 4.1.1 A. Sitaram brought to our notice the preprint [EKK] which proves Theorem 4.1.1 on $M(n)$ but the proof is similar to ours.

Next we take up Morgan's theorem. This theorem in some sense is complementary to Hardy's theorem and is a special case of corollary 1.1.3.1, c). We start with the statement of Morgan's theorem for $\mathbb{R}$.

Theorem 4.1.2 *Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function such that*

(*i*) $|f(x)| \leq Ce^{-a\pi|x|^p}$

(*ii*) $|\hat{f}(y)| \leq Ce^{-(A(a)+\epsilon)\pi|y|^q}$

where $p > 2$, $1/p + 1/q = 1$, $a, \epsilon > 0$ *and* $A(a) = 2^q\{q(pa)^{q-1}\sin\alpha\}^{-1}$ *with* $\alpha = \pi(q-1)/2$. *Then* $f = 0$ *almost everywhere.*

Note 4.1.1 - It is easy to see that $(ap)^{1/p}\{(A(a) + \epsilon)q\}^{1/q} > 2$. Hence Morgan's theorem actually follows from corollary 1.1.3.1, c).

As in the case of Hardy's theorem and theorem 1.1.1, a result on entire functions is responsible for Morgan's theorem. We state the result in the following lemma.

Lemma 4.1.2 *Suppose that* $q \in (1,2)$, $\alpha = \pi(q-1)/2$, $\sigma > 0$, $A(q,\sigma) = \sigma/\sin\alpha$. *If the order of an entire function F does not exceed q and* $B > A(q,\sigma)$ *and* (*i*) $F(ix) = O(e^{\sigma|x|^q})$

(*ii*) $F(x) = O(e^{-B|x|^q})$

as $|x| \to \infty$. *Then* $F = 0$.

For a proof of the above see [HJ].

It is easy to get an n dimensional analogue of theorem 4.1.2 which we now describe.

Theorem 4.1.3 *Let* $f : \mathbb{R}^n \to \mathbb{C}$ *be a measurable function such that*

(*i*) $|f(x)| \leq Ce^{-a\pi\|x\|^p}$,

(*ii*) $|\hat{f}(y)| \leq Ce^{-(A(a)+\epsilon)\pi\|y\|^q}$,

where $p > 2$, $1/p + 1/q = 1$, $a, \epsilon > 0$ *and* $A(a) = 2^q\{q(pa)^{q-1}\sin\alpha\}^{-1}$ *with* $\alpha = \pi(q-1)/2$. *Then* $f = 0$ *almost everywhere.*

PROOF. The proof is basically to reduce the problem in one dimension and then apply theorem 4.1.2. Fix $x \in \mathbb{R}^{n-1}$ and define

$$g_x(y) = \int_{\mathbb{R}^{n-1}} f(t, y)e^{-2\pi i\langle t, x\rangle}dt.$$

By *i*) we have that $g_x \in L^1(\mathbb{R})$ and $\widehat{g_x}(\xi) = \hat{f}(x, \xi)$. Now it is easy to see that g_x satisfies conditions of theorem 4.1.2. Thus $g_x = 0$ almost everywhere. By uniqueness of the Fourier transform and Fubini's theorem, $f = 0$ almost everywhere.

Our result now is an analogue of Morgan's theorem on $M(2)$.

Theorem 4.1.4 *Let* $f \in L^1(M(2)) \cap L^2((M(2))$ *and satisfy*

(*i*) $|f(z, \alpha)| \leq Ce^{-a\pi|z|^p}$

(*ii*) $\|\hat{f}(r)\|_{HS} \leq Ce^{-(A(a)+\epsilon)\pi|r|^q}$,

where $p > 2$, $1/p + 1/q = 1$, $\alpha = \pi(q-1)/2$, $a > 0$ *and* $A(a) = 2^q\{q(pa)^{q-1}\sin\alpha\}^{-1}$. *Then* $f = 0$ *almost everywhere.*

PROOF. For $\omega = u + iv \in \mathbb{C}$, if we define $\langle\hat{f}(\omega)e_m, e_n\rangle$ as in the previous theorem, then

$$|\langle\hat{f}(\omega)e_m, e_n\rangle|$$

$$
\begin{aligned}
&\leq \int_{\mathbf{C}\times\mathbf{T}} |f(z,\beta)||\Phi^{\omega}_{m,n}(-\bar{\beta}z,\bar{\beta})|dzd\beta \\
&\leq \int_{\mathbf{T}}\int_{\mathbf{C}\times\mathbf{T}} |f(z,\beta)|e^{2\pi v Re(-\alpha\bar{\beta}z)}dzd\alpha d\beta \\
&\leq \int_{\mathbf{T}}\int_{\mathbf{C}\times\mathbf{T}} |f(z,\beta)|e^{a'\pi|Re(-\alpha\bar{\beta}z)|^p+\sigma(a')2^q\pi|v|^q}dzd\alpha d\beta \\
&\qquad \text{(where } a' < a,\ \sigma(a') = \tfrac{1}{q(pa')^{q-1}} \\
&\leq e^{\sigma(a')2^q\pi|v|^q}\int_{\mathbf{T}}\int_{\mathbf{C}\times\mathbf{T}} |f(z,\beta)|e^{a'\pi|z|^p}dzd\alpha d\beta \\
&\leq Ce^{\sigma(a')2^q\pi|v|^q}\int_{\mathbf{T}}\int_{\mathbf{C}\times\mathbf{T}} e^{-(a-a')\pi|z|^p}dzd\alpha d\beta \qquad \text{(by (i))} \\
&\leq Ce^{\sigma(a')2^q\pi|v|^q}. \qquad (4.9)
\end{aligned}
$$

As in the previous theorem, by Lebesgue's dominated convergence theorem and Morera's theorem it follows that $g(\omega) = \langle \hat{f}(\omega)e_m, e_n\rangle$ is an entire function. By (ii) and (4.9) it follows that

$$
\left.\begin{aligned}
|g(ix)| &\leq Ce^{\sigma(a')2^q\pi|x|^q} \\
|g(x)| &\leq Ce^{-(A(a)+\epsilon)\pi|x|^q}
\end{aligned}\right\} \qquad (4.10)
$$

Choosing $a' < a$ satisfying

$$
\frac{2^q\pi}{q(pa)^{q-1}\sin\alpha} + \epsilon\pi > \frac{2^q\pi}{q(pa')^{q-1}\sin\alpha},
$$

g satisfies conditions of lemma 4.1.2 with $\sigma = \sigma(a')2^q$. Hence $g = 0$. Thus $\|\hat{f}(r)\|_{HS} = 0$ for all r. Hence $f = 0$ almost everywhere, by the Plancherel theorem. This completes the proof.

4.2 The Oscillator group

The oscillator group is the semidirect product of H_1 (the one dimensional Heisenberg group) and $\mathbb{R}$ with respect to the homomorphism

$$
\begin{aligned}
&\gamma : \mathbb{R} \longrightarrow \mathrm{Aut}(H_1) \\
&\gamma(r)(x,\xi,t) = (x\cos r + \xi\sin r, -x\sin r + \xi\cos r, t).
\end{aligned}
$$

Since γ has a cocompact kernel, $G = H_1 \times_\gamma \mathbb{R}$ is a type 1, unimodular group with H_1 as a regularly embedded, closed normal subgroup (see [KL]). If we denote the elements of G by (x, ξ, t, r) where $(x, \xi, t) \in H_1$ and $r \in \mathbb{R}$ then $dx d\xi dt dr$ is a Haar measure of the group G. The Lie algebra of G is given by $\mathfrak{g} = \text{span}_{\mathbb{R}}\{W, X, Y, Z\}$ with the only nonzero Lie brackets

$$[W, X] = Y, [W, Y] = -X, [X, Y] = Z$$

of the basis elements.

To find the dual of the group we proceed by Mackey's *little group method* (see [L]). For details about representation theory of G we refer to [Str], [Q], [MR2]. For $\lambda \in \mathbb{R} \setminus \{0\}$, we consider $\pi_\lambda \in \widehat{H_1}$. Since $\pi_\lambda | Z(H_1) = (\pi_\lambda \circ \gamma(r)) | Z(H_1)$ for all $r \in \mathbb{R}$ where $Z(H_1)$ is the center of H_1, by Stone-von Neumann theorem there is an operator $W(r) \in \mathcal{U}(L^2(\mathbb{R}))$ satisfying

$$\pi_\lambda(\gamma(r)(x, \xi, t) = W(r) \circ \pi_\lambda(x, \xi, t) \circ W(r)^*,$$

for all $(x, \xi, t) \in H_1$, and $W(r)$ is unique upto a scalar. Since $\mathbb{R}$ has no nontrivial multiplier (see [P]), $r \to W(r)$ can be chosen to be a true unitary representation of $\mathbb{R}$. We do not need an explicit description of $W(r)$, but we remark that the description is easy if we start with the Fock space description of π_λ (see [F3]) instead of the Schrödinger model. By the little group method we get a family of irreducible unitary representations of G given by

$$\pi_{\lambda,s} : G \to \mathcal{U}(L^2(\mathbb{R})),$$

$$\pi_{\lambda,s}(x, \xi, t, r) = \chi_s(r)\pi_\lambda(x, \xi, t) \circ W(r),$$

where $\chi_s(r) = e^{2\pi i s r}$. Instead of π_λ, if we start with a nontrivial character of H_1 given by $\Gamma_{b,\beta}(x, \xi, t) = e^{2\pi i (bx + \beta\xi)}$, then the stabilizer is $G_\Gamma = H_1 \times \mathbb{Z}$ and hence we get another family of irreducible unitary representations of G given by $Ind_{H_1 \times \mathbb{Z}}^G(\Gamma \times \alpha)$ where $\alpha \in \hat{\mathbb{Z}}$. Again if we start with the trivial character of H_1 we get characters of G given by $\Gamma_a(x, \xi, t, r) = e^{2\pi i a r}$. Since this is a regular semidirect product the above said families of representations exhaust $\hat{G}$. It follows from proposition 1., section 2 of [MR2] that the representations $\{\pi_{\lambda,s} : \lambda \in \mathbb{R} \setminus \{0\}, s \in \mathbb{R}\}$ constitute a set with full Plancherel measure, and the Plancherel measure is given by $|\lambda| d\lambda ds$ (see also [KL], Theorem 3.1).

Now we prove an analogue of theorem 1.1.1 in this case.

Theorem 4.2.1 *Let $f \in L^1(G) \cap L^2(G)$ and*

(i) $\int_G e^{pa\pi \|(x,\xi,t,r)\|^2} |f(x,\xi,t,r)|^p dx d\xi dt dr < \infty$,

(ii) $\int_{\mathbb{R}} e^{qb\pi s^2} \|\hat{f}(\lambda, s)\|_{HS}^q ds < A_\lambda$,

where A_λ is a constant depending on λ only, and $a, b > 0$, $1 \le q < \infty$, $1 \le p \le \infty$. If $ab \ge 1$ then $f = 0$ almost everywhere.

PROOF. Let $\{e_m : n \in \mathbb{Z}\}$ be any orthonormal basis of $L^1(\mathbb{R})$. We define

$$\begin{aligned} \Phi_{m,n}^{\lambda,s}(x,\xi,t,r) &= \langle \pi_{\lambda,s}(x,\xi,t,r) e_m, e_n \rangle_{L^2(\mathbb{R})} \\ &= e^{2\pi i s r} \langle \pi_\lambda(x,\xi,t) \circ W(r)(e_m), e_n \rangle_{L^2(\mathbb{R})}, \end{aligned}$$

the (m,n)-th matrix coefficient of the operator $\pi_{\lambda,s}(x,\xi,t,r)$. If for $\omega = u + iv \in \mathbb{C}$ we define $\Phi_{m,n}^{\lambda,s} = e^{2\pi i \omega r} \langle \pi_\lambda(x,\xi,t) e_m, e_n \rangle_{L^2(\mathbb{R})}$ then for fixed m, n, λ and (x,ξ,t,r) the function $\omega \to \Phi_{m,n}^{\lambda,s}$ is entire. By definition of the group Fourier transform we have

$$\langle \hat{f}(\lambda,s) e_m, e_n \rangle_{L^2(\mathbb{R})} = \int_G f(x,\xi,t,r) \Phi_{m,n}^{\lambda,s}((x,\xi,t,r)^{-1}) dx d\xi dt dr.$$

Now from the trivial estimate

$$|f(x,\xi,t,r) \Phi_{m,n}^{\lambda,\omega}((x,\xi,t,r)^{-1})| \le e^{2\pi v r} |f(x,\xi,t,r)|,$$

it follows from (i) and Hölder's inequality that

$$|\langle \hat{f}(\lambda,\omega) e_m, e_n \rangle| \le C e^{\frac{\pi}{a} v^2} \le C e^{\pi b v^2} \tag{4.11}$$

and that the function $\omega \to \langle \hat{f}(\lambda,\omega) e_m, e_n \rangle$ for fixed λ, m, n is an entire function. Let

$$g(\omega) = e^{b\pi\omega^2} \langle \hat{f}(\lambda,\omega) e_m, e_n \rangle, \qquad \omega \in \mathbb{C}.$$

Then g is an entire function and

$$|g(u+iv)| \leq Ce^{b\pi u^2} \tag{4.12}$$

by (4.11). Since $|\langle \hat{f}(\lambda, s)e_m, e_n\rangle| \leq \|\hat{f}(\lambda, s)\|_{HS}$, we have from $ii)$ that

$$\int_{\mathbb{R}} e^{qb\pi s^2} |\langle \hat{f}(\lambda s)e_m, e_n\rangle|^q ds < A_\lambda,$$

that is

$$\left(\int_{\mathbb{R}} |g(r)|^q dr\right)^{\frac{1}{q}} < \infty. \tag{4.13}$$

By (4.12) and (4.13) we have that g satisfies the conditions of lemma 4.1.1 and hence $g = 0$. Thus $\|\hat{f}(\lambda, s)\|_{HS} = 0$ for all (λ, s). Thus $f = 0$ almost everywhere by the Plancherel theorem. This completes the proof.

Remark 4.2.1 We have not been able to settle the following question. Should the answer to the question in the affirmative, we would get a more natural candidate for the Cowling-Price theorem on the oscillator group.

Question- With the notation as above, let $f \in L^1(G) \cap L^2(G)$ and

$$i) \int_G e^{pa\pi\|(x,\xi,t,r)\|^2} |f(x,\xi,t,r)|^p dx d\xi dt dr < \infty,$$

$$ii) \int_{\mathbb{R}} e^{qb\pi\lambda^2} \|\hat{f}(\lambda, s)\|_{HS}^q |\lambda| d\lambda < A_s$$

where A_s is a constant depending on s only and $a, b > 0$, $1 \leq q < \infty$. Does $ab > 1$ imply $f = 0$ almost everywhere?

Chapter 5

Semisimple Lie Groups

In this chapter our aim is to prove an analogue of Theorem 1.1.1 on rank one semisimple Lie groups.

In the first section we set up the required notation and state the Plancherel theorem for a rank one semisimple Lie group. In section 5.2 we prove the proposed analogue of the Cowling- Price theorem in the rank one case.

5.1 Notations and preliminaries

We refer to [GV] and [K] for all unexplained notation and facts in the following. Let G be a connected, non compact, real semisimple Lie group with finite center, and $\mathfrak{g}$ denotes the Lie algebra of G. Let K be a fixed maximal compact subgroup of G with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, be the Cartan decomposition of $\mathfrak{g}$. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and $\mathfrak{a}^*$ denotes the real dual of the vector space $\mathfrak{a}$.

The set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ. It consists of all $\alpha \in \mathfrak{a}^*$ such that the vector space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{a}\}$$

has dimension $m_\alpha > 0$. We choose a system of positive roots Σ^+ and with respect to Σ^+, the positive Weyl chamber

$$\mathfrak{a}^+ = \{X \in \mathfrak{a} : \alpha(X) > 0 \quad \text{for all } \alpha \in \Sigma^+\}.$$

We denote by $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ and obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The corresponding decomposition $G = KAN$ is the Iwasawa decomposition of the group giving the unique expression

$$g = \mathbf{k}(g)\mathbf{a}(g)\mathbf{n}(g)$$

when $g \in G$, $\mathbf{k}(g) \in K$, $\mathbf{a}(g) \in A$, $\mathbf{n}(g) \in N$. We denote by $\exp : \mathfrak{g} \to G$ the exponential map of the group. The inverse of the diffeomorphism $\exp : \mathfrak{a} \to A$ is called $\log : A \to \mathfrak{a}$. Let M' and M denote the normalizer and centralizer of $\mathfrak{a}$ in K respectively, that is

$$M' = \{k \in K : Adk(\mathfrak{a}) \subset \mathfrak{a}\},$$

$$M = \{k \in K : Adk(H) = H \quad \text{for all } H \in \mathfrak{a}\}.$$

Then M is a normal subgroup of M' and the quotient group $W = M'/M$ is a finite group, called *Weyl group* of the pair $(\mathfrak{g}, \mathfrak{k})$. W acts on $\mathfrak{a}$ by the following rule

$$w.H = Adw(H), \quad w \in W, H \in \mathfrak{a}.$$

It follows that W acts as a group of orthogonal transformations (preserving the Cartan-Killing form) on $\mathfrak{a}$. Each $w \in W$ permutes the Weyl chambers, and the action of W on the Weyl chambers is simply transitive.

Every element of G can also be written as $g = k_1 a k_2$ for some $k_1, k_2 \in K$ and $a \in A$, but this representation is not unique. If $g = k_1 a k_2 = k_1' a' k_2'$ then there is a $w \in W$ such that $a' = w.a$. Let $A^+ = \exp \mathfrak{a}^+$ and $\overline{A^+}$ denotes the closure of A^+ in G. Then every element $g \in G$ can be uniquely written as $g = k_1 a k_2$, $k_1, k_2 \in K$, $a \in \overline{A^+}$ that is $G = K\overline{A^+}K$, called the *polar decomposition* of G.

Everything above depends on the choice of $\mathfrak{a}$ and $\mathfrak{a}^+$, but for any two choices of $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{a}^+$, $\mathfrak{b}^+$ there is an element $k \in K$ such that $Ad(k)\mathfrak{a} = \mathfrak{b}$

and $Ad(k)\mathfrak{a}^+ = \mathfrak{b}^+$. Thus in particular any two maximal abelian subspaces of $\mathfrak{p}$ have same dimensions, called the *real rank* of G.

The Haar measure 'dx' of G with respect to the polar decomposition is given by $dx = J(a)dk_1 da dk_2$, where

$$J(a) = \Pi_{\alpha\in\Sigma^+}(e^{\alpha(\log a)} - e^{-\alpha(\log a)})^{m_\alpha}$$

that is

$$\int_G f(x)dx = \int_K \int_{\overline{A^+}} \int_K f(k_1 a k_2) J(a) dk_1 da dk_2$$

where da, dk are Haar measures on A and K respectively.

Given $g \in G$, we define

$$\|g\|_G = B(\log g_P, \log g_P)^{\frac{1}{2}}$$

using Cartan decomposition where B is the Cartan-Killing form. Since B is nondegenerate and positive definite, on $\mathfrak{a}$ we have an inner product induced by the Cartan-Killing form. We extend this inner product on $\mathfrak{a}^*$(the real dual of $\mathfrak{a}$) by duality, that is we set

$$\langle\lambda, \mu\rangle = \langle H_\lambda, H_\mu\rangle \quad \lambda, \mu \in \mathfrak{a}^*, H_\lambda, H_\mu \in \mathfrak{a}$$

where H_λ is the unique element in $\mathfrak{a}$ such that $\lambda(H) = \langle H_\lambda, H\rangle$ for all $H \in \mathfrak{a}$. Let $\mathfrak{a}^*_\mathbb{C}$ denote the complexification of $\mathfrak{a}^*$ that is all complex valued linear functionals on $\mathfrak{a}$ which are real linear. Let $\lambda : \mathfrak{a} \longrightarrow \mathbb{C}$ be a real linear functional. Then $\lambda_\mathbb{R} : \mathfrak{a} \longrightarrow \mathbb{R}$ and $\lambda_I : \mathfrak{a} \longrightarrow \mathbb{R}$, given by

$$\lambda_\mathbb{R}(H) = \text{Real part of} \lambda(H) \text{ for all} H \in \mathfrak{a},$$

$$\lambda_I(H) = \text{Imaginary part of} \lambda(H) \text{ for all} H \in \mathfrak{a}$$

are real valued linear functionals on $\mathfrak{a}$ and $\lambda = \lambda_\mathbb{R} + i\lambda_I$. Using the inner product of $\mathfrak{a}^*$, for $\lambda, \lambda' \in \mathfrak{a}^*_\mathbb{C}$ we define

$$\langle\lambda, \lambda'\rangle = [\langle\lambda_\mathbb{R}, \lambda'_\mathbb{R}\rangle - \langle\lambda_I, \lambda'_I\rangle] + i[\langle\lambda_I, \lambda'_\mathbb{R}\rangle + \langle\lambda_\mathbb{R}, \lambda'_I\rangle].$$

This inner product we denote by the same symbol as it extends the inner product of $\mathfrak{a}^*$.

Now we describe the series of representations of G, called *spherical principal series representations.* Let $G = KAN$ be the Iwasawa decomposition of G and M the centralizer of $\mathfrak{a}$ in K. Since M normalizes N also, we have a subgroup $P = MAN$ of G, called a *minimal parabolic subgroup.* Let $\xi \in \hat{M}$ with representation space $\mathcal{H}_\xi$, a finite dimensional space as M is closed in K. Let $\lambda \in \mathfrak{a}^*$. We define

$$\mu_{\xi,\lambda} : MAN \longrightarrow \mathcal{U}(\mathcal{H}_\xi) \quad \text{by}$$

$$\mu_{\xi,\lambda}(man)(v) = e^{(i\lambda+\rho)(\log a)}\xi(m)(v) \quad \text{for all } v \in \mathcal{H}_\xi$$

where $m \in M$, $a \in A$, $n \in N$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. From $\mu_{\xi,\lambda}$ we get by induction a representation $\pi_{\xi,\lambda}$ of G realized on the Hilbert space $\mathcal{H}_{\xi,\lambda}$, where

$$\mathcal{H}_{\xi,\lambda} = \{f : K \longrightarrow \mathcal{H}_\xi : \quad i) f \text{ is measurable}$$
$$ii) f(km) = \xi(m^{-1})(f(k))$$
$$iii) \int_K \|f(k)\|^2_{\mathcal{H}_\xi} dk < \infty\}$$

and

$$(\pi_{\xi,\lambda}(g)f)(k) = e^{-(i\lambda+\rho)(H(g^{-1}k))} f(\mathbf{k}(g^{-1}k)).$$

From the description of $\pi_{\xi,\lambda}$ it follows that the action of K is by left translation. It can be shown that $\pi_{\xi,\lambda}$ defined above is a strongly continuous, unitary representation. We also notice that from *ii*) and *iii*) of the definition of $\mathcal{H}_{\xi,\lambda}$, it follows that $\mathcal{H}_{\xi,\lambda}$ is isomorphic to the vector valued L^2-space, $L^2(K/M, d\dot{k}, \mathcal{H}_\xi)$ where $d\dot{k}$ is a Haar measure on K/M. It is known that given $\xi \in \hat{M}$ there exist an open dense subset $O_\xi \subset \mathfrak{a}^*$ such that for $\lambda \in O_\xi$, $\pi_{\xi,\lambda}$ is irreducible. Now we come to the equivalence of the above series of representations. The Weyl group W acts on the equivalence class $\xi \in \hat{M}$ by conjugation, $(gM.\xi)(m) = \xi(g^{-1}mg)$, $g \in M'$. Since W acts on $\mathfrak{a}$ by adjoint action, it acts on $\mathfrak{a}^*$ by coadjoint action. It turns out that

$\pi_{\xi,\lambda} \cong \pi_{\xi',\lambda'}$ if and only if there exist $w \in W$ such that $\xi' = w.\xi$ and $\lambda' = w.\lambda$.

Now, let ξ be the trivial representation of M on $\mathbb{C}$. By π_λ we will denote the representation $\pi_{\xi,\lambda}$ where ξ is the trivial representation. These are called the *class 1 principal series representations* as they contain the trivial representation of K as a subrepresentation with multiplicity one. Let $\{e_0, e_1, \ldots\}$ be an orthonormal basis of $L^2(K/M)$ consisting of K-finite vectors where we always choose as e_0 to be the constant function $\mathbf{1} : K/M \to \mathbb{C}$, $\mathbf{1}(x) = 1$ for all $x \in K/M$. We notice that since the vectors e_i's are K-finite, $e_i \in C^\infty(K)$ as functions on K, in particular they are all bounded functions. Let

$$\begin{aligned}
\Phi_\lambda(x) &= \langle \pi_\lambda(x)e_0, e_0\rangle_{L^2(K)}, \qquad x \in G \\
&= \int_K e^{-(i\lambda+\rho)(H(g^{-1}k))} dk \\
&= \int_K e^{(i\lambda-\rho)(H(gk))} dk \qquad \text{(see p. 104 of [GV])}
\end{aligned}$$

be the so called *elementary spherical functions*. Let $\mathcal{D}_K(G)$ denotes the set of all left invariant differential operators acting on $C^\infty(G)$ which are also right invariant under the action of K. The elementary spherical functions on G are the K-biinvariant eigen functions ϕ of every differential operator $D \in \mathcal{D}_K(G)$, normalized by the condition $\phi(e) = 1$. It can be shown that for $\lambda \in \mathfrak{a}^*$ that is $\lambda = \lambda_\mathbb{R} + i\lambda_I$, the integral defining Φ_λ make sense and it is a result of Harish Chandra that $\Phi_\lambda, \lambda \in \mathfrak{a}^*_\mathbb{C}$ exhaust the set of elementary spherical functions on G. Moreover for $\lambda \in \mathfrak{a}^*$ and $a \in \bar{A}^+$, we have the following estimate

$$|\Phi_{i\lambda}(a)| \le e^{\lambda^+(\log a)}. \tag{5.1}$$

where λ^+ is the image of λ in the fundamental Weyl chamber, under the action of the Weyl group. Now suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$. As we said above

$$\Phi_\lambda(x) = \int_K e^{-(i\lambda+\rho)(H(x^{-1}k))} dk$$

and if $\lambda = \lambda_{\mathbf{R}} + i\lambda_I$, then

$$\begin{aligned} & \int_K |e^{-\{i(\lambda_{\mathbf{R}}+i\lambda_I)+\rho\}(H(x^{-1}k))}| dk \\ = & \int_K e^{(\lambda_I-\rho)(H(x^{-1}k))} dk \\ = & \Phi_{-i\lambda_I}(x). \end{aligned}$$

The integrand being continuous on K, the integral is well defined. For fixed $x \in G$, the function $\lambda \longrightarrow \Phi_\lambda(x)$ is clearly continuous by dominated convergence theorem and hence entire by Morera's theorem. From K-biinvariance of Φ_λ and (5.1) we have

$$|\Phi_\lambda(x)| \leq C e^{\lambda_I^{+}(\log a)}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^{*}.$$

Going back to a principal series representation $\pi_{\xi,\lambda}$, we define, for an orthonormal basis $\{e_i^\xi / i \in \mathbb{N}\}$ of $L^2(K/M, \mathcal{H}_\xi)$ consisting of K-finite vectors

$$\begin{aligned} \Phi_{\xi,\lambda}^{m,n}(g) &= \langle \pi_{\xi,\lambda}(g) e_m^\xi, e_n^\xi \rangle \\ &= \int_K \langle \pi_{\xi,\lambda}(g)(e_m^\xi)(k), e_n^\xi(k) \rangle_{\mathcal{H}_\xi} dk \\ &= \int_K e^{-(i\lambda+\rho)(H(g^{-1}k))} \langle e_m^\xi(\mathbf{k}(g^{-1}k)), e_n^\xi(k) \rangle_{\mathcal{H}_\xi} dk. \end{aligned}$$

Since the vector valued functions e_i^ξ are K-finite, they are in $C^\infty(K, \mathcal{H}_\xi)$ and hence bounded in particular. Then it follows as before, that for $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$, the integral defining $\Phi_{\xi,\lambda}^{m,n}$ make sense, in fact for each fixed $g \in G$ the function $\lambda \longrightarrow \Phi_{\xi,\lambda}^{m,n}(g)$ extends as an entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$. Again writing $\lambda = \lambda_{\mathbf{R}} + i\lambda_I$, we get

$$|\Phi_{\xi,\lambda}^{m,n}(g)| \leq \text{Const.} \int_K e^{(\lambda_I-\rho)(H(g^{-1}k))} dk = \text{Const.}\Phi_{-i\lambda_I}(g). \qquad (5.2)$$

From now onwards we continue with the hypothesis that $\dim \mathfrak{a} = 1$. In this case some further simplifications can be made. The set Σ^+ consists of two elements at the most, α and possibly 2α. Hence $\rho = \frac{1}{2}(m_\alpha + m_{2\alpha})$ and $B(H,H) = 2\alpha(H)^2(m_\alpha + 4m_{2\alpha})$. We choose $H_0 \in \mathfrak{a}^+$ such that $\alpha(H_0) = 1$. Then

$$B(H_0, H_0) = 2(m_\alpha + 4m_{2\alpha}) = \delta \text{ (say)}.$$

We identify $\mathfrak{a}$ with $\mathbb{R}$ by this choice of H_0. Under this identification, the usual inner product of $\mathbb{R}$ becomes $\langle a, b\rangle = \delta ab$ and inner product on $\mathbb{R}^*$ is $\langle a, b\rangle = \frac{1}{\delta}ab$.

Given an integrable function $f \in G$, we define the so called *group Fourier transform* by

$$\pi(f) = \hat{f}(\pi) = \int_G f(g)\pi(g)d\mu(g),$$

a bounded linear operator on $\mathcal{H}_\pi$. We follow [W] for a description of the Plancherel measure for a rank one semisimple Lie group G. $\hat{G}_d$ will stand for the collection of equivalence classes of square integrable, unitary, irreducible representations of G called the discrete series representations. By a theorem of Harish Chandra these representations occur as subquotients (subrepresentation of a quotient representation) of certain nonunitary principal series representations (see [W]). We shall not go into the details of these embeddings.

Theorem 5.1.1 *For $f \in L^1(G) \cap L^2(G)$,*

$$\|f\|_2^2 = \sum_{U \in \hat{G}_d} d_U \|\hat{f}(U)\|_{HS}^2 + \frac{1}{2}\sum_{\xi \in \hat{M}} \int_{-\infty}^{\infty} \|\hat{f}(\pi_{\xi,\lambda})\|_{HS}^2 Q(\xi,\lambda)d\lambda,$$

where d_U's are 'formal degrees' of the representations U and $Q(\xi,\lambda)$ is a nonnegative function with the property $Q(\xi,\lambda) \neq 0$ for $\lambda \neq 0$, having the following form

$$Q(\xi,\lambda) = [W(M, A_K)]q(\chi_\xi, \lambda),$$

where $[W(M, A_K)]$ is a constant and χ_ξ is the trace of the representation $\xi \in \hat{M}$. Finally $q(\chi_\xi,\lambda)$ is of the form

$$P(\chi_\xi,\lambda)\coth\left(\frac{\pi\lambda}{2}\right) \qquad or \qquad R(\chi_\xi,\lambda)\tanh\left(\frac{\pi\lambda}{2}\right),$$

depending on ξ (we shall not need the exact expression for this dependence) and P, R are polynomials in λ for a fixed $\xi \in \hat{M}$.

5.2 The uncertainty principle

Before we can embark on our extension of the theorem of Cowling and Price to a group G as above, we need the following lemma which is essentially in [CSS].

Lemma 5.2.1 *Let $f \in L^1(G) \cap L^2(G)$. Suppose f decays sufficiently rapidly that $\hat{f}(\pi_{\xi,\lambda})$ makes sense for all $\xi \in \hat{M}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ (see remark below), and*

$$\hat{f}(\pi_{\xi,\lambda}) = 0 \quad \text{for all} \xi \in \hat{M}, \lambda \in \mathfrak{a}^*_{\mathbb{C}}$$

then $f = 0$ almost everywhere.

For a *rapidly decreasing* integrable function f, even though $\hat{f}(\pi_{\xi,\lambda})$ may fail to make sense as an operator on $\mathcal{H}_{\xi,\lambda}$ when $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, (note that, in general, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \setminus \mathfrak{a}^*$, $\pi_{\xi,\lambda}$ is not unitary), we however use the notation $\hat{f}(\pi_{\xi,\lambda})$ in the sense of an infinite matrix whose elements are

$$\hat{f}(\pi_{\xi,\lambda})_{m,n} = \int_G f(x)\langle \pi_{\xi,\lambda}(x)e_m, e_n\rangle_{\mathcal{H}_{\xi,\lambda}} d\mu(x) \qquad e_m, e_n \in \mathcal{H}_{\xi,\lambda}.$$

With this preparation we come to the main theorem of this chapter.

Theorem 5.2.1 *Let $f \in L^1(G) \cap L^2(G)$, satisfying the following conditions*

(*i*) $\int_G |f(x)|^p e^{p\bar{a}\|x\|_G^2} d\mu(x) < \infty$

(*ii*) $\int_{\mathfrak{a}^*} e^{q\bar{b}\|\lambda\|^2} \|\hat{f}(\pi_{\xi,\lambda}\|_{HS}^q Q(\xi,\lambda) d\lambda < C_\xi$

where $q < \infty$, $\bar{a}, \bar{b} > 0$ and C_ξ is a constant which depends on ξ only. If $\bar{a}\bar{b} > \frac{1}{4}$ then $f = 0$ almost everywhere.

PROOF. Let $\xi \in \hat{M}$ be fixed and $\{e_i^\xi / i \in \mathbb{N}\}$ be an orthonormal basis of $L^2(K/M, \mathcal{H}_\xi)$ consisting of K-finite vectors, as before. We choose $m, n \in \mathbb{N}$

and keep it fixed. Our aim is to show that for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the function

$$F(\lambda) = \int_G f(x)\Phi_{\xi,\lambda}^{m,n}(x)d\mu(x), \tag{5.3}$$

is zero. Then by lemma 5.2.1 we will conclude that $f = 0$ almost everywhere. But the first thing is to show that (5.3) is well defined. If $x = k_1ak_2$ and $a = \exp(H)$, $H \in \mathfrak{a}$ then $\|x\|_G = \|k_1ak_2\|_G = \|H\|$ and from the explicit expression of $J(a)$ it follows that there is a constant A_1 such that $|J(a)| \leq Const.e^{A_1\|H\|}$. Using this and $i)$ the function F is well defined, for

$$\begin{aligned}
&\int_G |f(x)||\Phi_{\xi,\lambda}^{m,n}(x)|d\mu(x), \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^* \\
\leq\ & \text{Const.} \int_G |f(k_1ak_2)|e^{\lambda_I^+(\log a)}|J(a)|dk_1dadk_2 \\
& \quad \text{(by (5.2))} \\
=\ & \text{Const.} \int_G e^{\bar{a}\|k_1ak_2\|_G^2}|f(k_1ak_2)|e^{-\bar{a}\|k_1ak_2\|_G^2}e^{\lambda_I^+(\log a)}|J(a)|dk_1dadk_2 \\
\leq\ & \text{Const.} \left(\int_G e^{p\bar{a}\|x\|_G^2}|f(x)|^p d\mu(x)\right)^{\frac{1}{p}} \left(\int_{\mathfrak{a}} e^{p'\{-\bar{a}\|H\|^2+\lambda_I^+(\log a)+A_1\|H\|\}}dH\right)^{\frac{1}{p'}} \\
& \text{(By Hölder's inequality, } \tfrac{1}{p}+\tfrac{1}{p'}=1\text{, and } dH \text{ denotes} \\
& \text{the Lebesgue measure on } \mathfrak{a}) \\
\leq\ & \text{Const.} \left(\int_{\mathfrak{a}} e^{p'\{-\bar{a}\|H\|^2+\langle H,H_{\lambda_I^+}\rangle+A_1\|H\|\}}dH\right)^{\frac{1}{p'}} \qquad \text{(By } (i)) \\
\leq\ & \text{Const.} \left(\int_{\mathfrak{a}} e^{p'\{-a'\|H\|^2+\langle H,H_{\lambda_I^+}\rangle\}}dH\right)^{\frac{1}{p'}} \\
& \text{(where } 0 < a' < \bar{a} \text{ is such that } a'\bar{b} > \tfrac{1}{4} \text{ and} \\
& e^{-\bar{a}\|H\|^2+A_1\|H\|} \leq Const.e^{-a'\|H\|^2}).
\end{aligned}$$

Now

$$\begin{aligned}
&\int_{\mathfrak{a}} e^{p'\{-a'\|H\|^2+\langle H,H_{\lambda_I^+}\rangle\}}dH \\
=\ & e^{(p'\|H_{\lambda_I^+}\|^2)/4a'} \int_{\mathfrak{a}} e^{-p'a'\|H-\frac{1}{2a'}H_{\lambda_I^+}\|^2}dH \\
=\ & Const.e^{\frac{p'\|H_{\lambda_I^+}\|^2}{4a'}}
\end{aligned}$$

by translation invariance of Lebesgue measure and the fact $\|H_{\lambda_I^+}\| = \|H_{\lambda_I}\|$. Thus we have that F is well defined and

$$|F(\lambda)| \leq Const.e^{\frac{1}{4a'}\|\lambda_I\|^2} \quad \text{for all } \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{5.4}$$

By the identification of $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathbb{C}$, (5.4) can be written as

$$|F(\lambda)| \leq Const.e^{\frac{1}{4a'\delta}\lambda_I^2}. \tag{5.5}$$

To see that F is a continuous function, for $\lambda_n \longrightarrow \lambda$ in $\mathbb{C}$, we define

$$f_n(x) = f(x)\Phi^{m,n}_{\xi,\lambda_n}(x).$$

Then $f_n \longrightarrow f.\Phi^{m,n}_{\xi,\lambda}$ pointwise by continuity of $\lambda \longrightarrow \Phi^{m,n}_{\xi,\lambda}(x)$ for each fixed x. We also have

$$|f_n(x)| \leq \text{Const.}|f(x)|e^{(\lambda_n)_I^+(\log a)},$$

for $x = k_1 a k_2$, $a \in \overline{A^+}$. As $\{\lambda_n\}$ is convergent, there is a constant $\bar{c}$ such that

$$|f_n(x)| \leq \text{Const.}|f(x)|e^{\bar{c}\|x\|_G} \quad \text{for all } n, \text{ for all } x \in G.$$

By *i*) and Hölder's inequality, it follows that the right hand side of the above inequality is an integrable function. By dominated convergence theorem F is a continuous function. As $\lambda \longrightarrow \Phi^{m,n}_{\xi,\lambda}(x)$ is an entire function, by Fubini's theorem and Morera's theorem F is an entire function. We define

$$g(\lambda) = e^{\frac{b}{\delta}\lambda^2}F(\lambda), \qquad \lambda \in \mathbb{C}.$$

Then clearly g is an entire function and

$$|g(\lambda_{\mathbb{R}} + i\lambda_I)| \leq Const.e^{\frac{b}{\delta}(\lambda_{\mathbb{R}}^2 - \lambda_I^2)}e^{\frac{1}{4a'\delta}\lambda_I^2} \leq Const.e^{\frac{b}{\delta}\lambda_{\mathbb{R}}^2} \tag{5.6}$$

as $1/4a'\delta < \bar{b}/\delta$. Now we show that $g \mid \mathbb{R} \in L^q(\mathbb{R})$. The argument here uses the expression for the Plancherel measure. For $r > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}} |g(\lambda)|^q d\lambda &= \int_{\mathbb{R}} e^{\frac{qb}{\delta}\lambda^2}|F(\lambda)|^q d\lambda \\ &= \int_{|\lambda|\leq r} e^{\frac{qb}{\delta}\lambda^2}||F(\lambda)|^q d\lambda + \int_{|\lambda|>r} e^{\frac{qb}{\delta}\lambda^2}|F(\lambda)|^q d\lambda.\end{aligned}$$

So it is enough to show that the second integral is finite. Now

$$\begin{aligned}
\int_{|\lambda|>r} e^{\frac{q\bar{b}}{\delta}\lambda^2}|F(\lambda)|^q d\lambda &= \int_{|\lambda|>r} e^{\frac{q\bar{b}}{\delta}\lambda^2}|\langle \hat{f}(\pi_{\xi,\lambda})e_m^{\xi}, e_n^{\xi}\rangle|^q d\lambda \\
&\leq \int_{|\lambda|>r} e^{q\bar{b}\|\lambda\|^2}\|\hat{f}(\pi_{\xi,\lambda})\|_{HS}^q d\lambda \\
&= \int_{|\lambda|>r} e^{q\bar{b}\|\lambda\|^2}\|\hat{f}(\pi_{\xi,\lambda})\|_{HS}^q Q(\xi,\lambda)\frac{1}{Q(\xi,\lambda)} d\lambda.
\end{aligned}$$

From the description of the Plancherel measure it follows that the function $\frac{1}{Q(\xi,\lambda)}$ is bounded for $|\lambda| > r$. So by *ii)* $g \in L^2(\mathbb{R})$ and hence by (5.6), g satisfies the conditions of lemma 1.1.1, so $g = 0$. It follows that $F = 0$. This completes the proof.

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