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A TROTTER TYPE FORMULA FOR SEMIMARTINGALES*

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SUMMARY. Let S_1, S_2 be $d \times d$ matrix valued semimartingales and let $Q_i(s, t)$, $s < t$ be given by $Q_i(s, t) = I + \int_s^t Q_i(s, u-)dS_i(u)$. This paper investigates the limits of

$$\prod_{i : t_i \leqslant 1} (Q_i(t_i, t_{i+1} \wedge t) Q_2(t_i, t_{i+1} \wedge t))$$

and

$$\prod_{i : t_i \leqslant t} \exp(S_1(t_{i+1} \wedge t) - S_1(t_i)) \exp(S_2(t_{i+1} \wedge t) - S_2(t_i))$$

as the norm $\sup_i (t_{i+1} - t_i)$ of the partition $t_0 = 0 < l_1 < \ldots < l_i < \ldots t_i \uparrow \infty$, goes to zero. Here $[t]$ are allowed to be stop times. The expressions are shown to converge in the Emery topology on the space of semimartingales. Conditions on $[t]$ are also given which ensure a.s. convergence.

1. Introduction

In Berger [1], the following analogue of Trotter's formula was considered. Let S_1, S_2 be semimartingales. Let $Q_i(s, t)$ for $s < t$ be given by

$$Q_i(s, t) = I + \int_s^t Q_i(s, u-) dS_i(u).$$

For a partition $\Delta = \{t_0 = 0 < t_1 < \ldots\}$ of $[0, \infty)$, consider

$$Z_\Delta(t) = Q_1(t_0, t_1)\, Q_2(t_0, t_1)\, Q_1(t_1, t_2)\, Q_2(t_1, t_2) \ldots Q_1(t_k, t) Q_2(t_k, t)$$

for $t_k < t \leqslant t_{k+1}$. What is the limit of Z_Δ as norm Δ goes to zero ? Berger [1] showed that when S_1, S_2 are given by

$$dS_i = A_i dW + B_i dt$$

for (constant) matrices A_i, B_i, and a Brownian motion W, then Z_Δ converges in law to Z, where law of Z is indentified via S_1, S_2,

In Karandikar [5] the author had considered another analogue on the lines of Masani [9]. Let

$$Y_\Delta(t) = \prod_{i=0}^{k} \exp(S_1(t_{i+1} \wedge t) - S_1(t_i)) \exp(S_2(t_{i+1} \wedge t) - S_2(t_i))$$

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for $t_k < t < t_{k+1}$. When S_1, S_2 are continuous semimartingales and $\{t_i\}$ are stop times (chosen in a particular way), then it was shown that Y_Δ converges a.s.

In this paper, we consider both these questions for (r.c.l.l.) semimartingales. We prove that Z_Δ, Y_Δ converge in the Emery topology on the space of semimartingales, as $\|\Delta\| = \sup\{t_{i+2}-t_i : i \geqslant 1\} \to 0$. The partition $\{t_i\}$ is allowed to be deterministic, or consisting of stop times. Conditions on a sequence Δ^n of partitions are given when one has a.s. convergence of Z_{Δ^n} Y_{Δ^n}.

The paper is organized as follows. The necessary background on Emery topology is given in Section 2. In Section 3, we prove our basic result Theorem 3.2 which is an extension of some results in Emery [2], Karandikar [7]. In Section 4, the two versions of Trotter formula are deduced from Theorem 3.2.

The material presented in Sections 2 and 3 is of independent interest for other problems in stochastic calculus.

2. Emery topology and stability of solution to SDE

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(\mathcal{F}_t)$ be a filteration satisfying usual hypothesis. In what follows, adapted, stoptime, martingale, local martingale will be understood to be with reference to $(\mathcal{F}_t)$.

Let $\mathcal{V}^+$ be the class of increasing adapted processes V with $V(0) \geqslant 0$ and let

$$\mathcal{V} = \{B_1 - B_2 : B_1, B_2 \in \mathcal{V}^+\}.$$

Let $\mathcal{J}$ be the class of locally bounded predictable processes. Let $\mathcal{SM}$ be the class of semimartingales and $\mathcal{M}^2_{loc}$ be the class of locally square integrable martingales. Recall that every $X \in \mathcal{SM}$ admits a decomposition

$$X = M + A,\ M \in \mathcal{M}^2_{loc},\ A \in \mathcal{V} \quad \dots \quad (2.1)$$

Definition. A process $V \in \mathcal{V}^+$ is said to be a *dominating process* of a semimartingle X, written as $X <<< V$, if for some decomposition $X = M + A$ as in (2.1), one has

$$V - \{2(< M, M > + [M, M])^{\frac{1}{2}} + |A|\} \in \mathcal{V}^+. \quad \dots \quad (2.2)$$

Note that every semimartingale X admits a dominating process, one choice being

$$V = 2(< M, M > + [M, M])^{\frac{1}{2}} + |A|,$$

where $X = M + A$ is as in (2.1). Here as usual, for $X \in \mathcal{SM}$, $[X, X]$ denotes its quadratic variation process and for $M \in \mathcal{M}^2_{loc}$, $< M, M >$ denotes its predictable quadratic variation process. The following is an easy consequence of the Metivier-Pellaumail inequality (see Karandikar [7]). Let $X <<< V$. Then for any stop time τ,

$$E\|X\|_{\tau-}^{*2} \leqslant 2\, EV_{\tau-}^2 \qquad \ldots \quad (2.3)$$

where

$$\|f\|_t^* := \sup_{s<t} \|f(s)\|,$$

The importance of the notion $X <<< V$ is contained in the following proposition. See Karandikar [7], [8] for its proof, which is elementary.

Proposition 2.1.(*a*) *Let* $X <<< U$, $Y <<< V$ *and* $Z = X+Y$. *Then* $\exists W : Z <<< W$ *and* $W \leqslant U+V$.

(*b*) *Let* $X <<< U$ *and* $f \in \mathcal{J}$. *Let* $Z = \int f dX$. *Then* $\exists W : Z <<< W$ *and* $W_t \leqslant \theta_t(f, U)$, *where*

$$\theta_t(f, U) = \sqrt{2}\left\{\left(\int_0^t \|f\|^2 dU^2\right)^{\frac{1}{2}} + \int_0^t \|f\| dU\right\}.$$

Combining (2.3) with (*b*) above, one has the following. Let $X <<< V$, $f \in \mathcal{J}$. Then for all stop times τ,

$$E\|\int f\, d\, X\|_{\tau-}^* \leqslant 2E\theta_{\sigma-}^2(f, V). \qquad \ldots \quad (2.4)$$

The Emery topology: For measurable process f, g, let

$$d_{cp}(f, g) := \sum_{m=1}^{\infty} 2^{-m} E\{|f-g|_m^* \wedge 1\}.$$

Note that $d_{cp}(f_n, f) \to 0$ iff $|f_n - f|_m^* \to 0$ in probability for all t. For $X, Y \in \mathcal{SM}$ define

$$\rho(X, Y) = \inf\{d_{cp}(V, 0) : (X-Y) <<< V\}.$$

Using part (a), Proposition 2.1, it follows that ρ is a metric on $\mathcal{SM}$. It is shown in Karandikar [7], that ρ is a complete metric on $\mathcal{SM}$ and that the induced topology is the same as the one introduced by Emery [3].

Fast convergence in d_{cp} *and* ρ: For processes f_n, f, we say that $f_n \xrightarrow{\circ} f$ if

$$\sum_n |f_n - f|_t^{*2} < \infty \text{ a.s. for all } t. \qquad \ldots \quad (2.5)$$

It is easy to see that $d_{cp}(g_m, g) \to 0$ implies the existence of a subsequence $g_{m_k} \xrightarrow{\circ} g$: just choose m_k such that $\sum_k d_{cp}(g_{m_k}, g) < \infty$. Of course, $f_n \xrightarrow{\circ} f$ in turn implies $|f_n - f|_t^* \to 0$ a.s. and as a consequence $d_{cp}(f_n, f) \to 0$. Thus $\xrightarrow{\circ}$ can be termed as *fast convergence* in d_{cp}.

Similarly, we define for $X_n, X \in \mathcal{SM}, X_n \xrightarrow{*} X$ if $\exists V_n \in \mathcal{V}^+$, $(X_n - X) <<< V_n$ such that

$$\sum_n V_n^2(t) < \infty \text{ a.s. for all } t.$$

The following results on $\xrightarrow{o}, \xrightarrow{*}, d_{cp}, \rho$ are proved in Karandikar [7]. See also Emery [3], [4].

Proposition 2.2. *Let $f_n, g_n, f, g \in \mathcal{J}$ and $X_n, Y_n, X, Y \in \mathcal{SM}$*

(i) $f_n \xrightarrow{*} f, X_n \xrightarrow{*} X \Rightarrow \int f_n dX_n \xrightarrow{*} \int f dX.$

(ii) $d_{cp}(f_n, f) \to 0, \rho(X_n, X) \to 0 \Rightarrow \rho(\int f_n dX_n, \int f dX) \to 0.$

(iii) $X_n \xrightarrow{*} X \Rightarrow X_n \xrightarrow{o} X.$

(iv) $X_n \xrightarrow{*} X \Rightarrow [X_n - X, X_n - X]^{\frac{1}{2}} \xrightarrow{o} 0.$

(v) $X_n \xrightarrow{*} X, Y_n \xrightarrow{*} Y \Rightarrow [X_n, Y_n] \xrightarrow{*} [X, Y]$

Let $\mathcal{D}$ denote the class of r.c.l.l. adapted processes. Let $L(a, b)$ denote the space of $a \times b$ matrices, (a, b positive integers). We will say $\boldsymbol{X} = (X_{ij}) \in \mathcal{SM}(L(a, b))$ if $X_{ij} \in \mathcal{SM}$ for all i, j. $\mathcal{J}(L(a, b)), \mathcal{D}(L(a, b))$ are defined analogously. For $\boldsymbol{X} = (X_{ij}), \boldsymbol{Y} = (Y_{ij})$, we set $\rho(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{ij} \rho(X_{ij}, Y_{ij})$ if $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{SM}$ and $d_{cp}(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{ij} d_{cp}(X_{ij}, Y_{ij})$.

Let $G : \mathcal{D}(L(d, d)) - \mathcal{D}(L(d, d))$ be a mapping such that for some $A \in \mathcal{V}^+$

$$|G(Z_1) - G(Z_2)|_t^* \leqslant A_t |Z_1 - Z_2|_t^*, \forall Z_1, Z_2 \in \mathcal{D}(L(d, d)). \quad \dots \quad (2.6)$$

Let $\boldsymbol{X} \in \mathcal{SM}(L(d, d))$, $\boldsymbol{Y} \in \mathcal{D}(L(d, d))$. Consider the SDE

$$\boldsymbol{Z} = \boldsymbol{Y} + \int G(\boldsymbol{Z})_- d\boldsymbol{X}. \quad \dots \quad (2.7)$$

Here, $(\int G(\boldsymbol{Z})_- d\boldsymbol{X})_{ij} := \sum_{k=1}^{d} \int (G(\boldsymbol{Z})_-)_{ik} \, d\boldsymbol{X}_{kj}$. It is well known that (2.7) admits a unique solution. We will state a result on stability of (2.7). For its proof, see Emery [2], or Karandikar [7].

Theorem 2.3. *Let G satisfy* (2.6). *Let* $\boldsymbol{X}_n, \boldsymbol{X} \in \mathcal{SM}(L(d, d)), \boldsymbol{Y}_n, \boldsymbol{Y} \in \mathcal{D}(L(d, d))$. *Let $\boldsymbol{Z}$ be a solution to* (2.7) *and let $\boldsymbol{Z}_n$ satisfy*

$$\boldsymbol{Z}_n = \boldsymbol{Y}_n + \int G_n(\boldsymbol{Z}_n)_- d\boldsymbol{X}_n$$

where $G_n : \mathcal{D}(L(d, d)) \to \mathcal{D}(L(d, d))$. Suppose

$$d_{cp}(\boldsymbol{Y}_n, \boldsymbol{Y}) \to 0, \rho(\boldsymbol{X}_n, \boldsymbol{X}) \to 0, d_{cp}(G_n(\boldsymbol{Z}_n), G(\boldsymbol{Z}_n)) \to 0.$$

Then $d_{cp}(\boldsymbol{Z}_n, \boldsymbol{Z}) \to 0$. If $\rho(\boldsymbol{Y}_n, \boldsymbol{Y}) \to 0$, then $\rho(\boldsymbol{Z}_n, \boldsymbol{Z}) \to 0$. Further, if $\boldsymbol{X}_n \xrightarrow{*} \boldsymbol{X}, \boldsymbol{Y}_n \xrightarrow{*} \boldsymbol{Y}, G_n(\boldsymbol{Z}_n) - G(\boldsymbol{Z}_n) \xrightarrow{*} 0$, then $\boldsymbol{Z}_n \xrightarrow{*} \boldsymbol{Z}$.

3. Convergence of Riemann sums and products

In order to prove the Trotter type formulae, we need to recast and extend some results in Emery [2], Karandrikar [7] on convergence of random Riemann sums and products.

A partition Δ (of $[0,\infty)$) is a sequence of stoptimes $\{t_i\}$ such that $0 = t_0 < t_1 < t_2, \ldots, t_i \to \infty$. For a partition $\Delta = \{t_i\}$, let

$$|\Delta| := \sup_i |t_{i+1} - t_i|.$$

Note that $|\Delta|$ is in general a random variable. For a partition Δ, we define $J^\Delta : \mathfrak{D} - \mathfrak{D}$ by

$$J^\Delta X = \sum_{i=1}^{\infty} X(t_i) 1_{[t_i, t_{i+1})}.$$

Note that given $\epsilon > 0$ and $X_1. \ldots, X_m \in \mathfrak{D}$, we can get a partition Δ such that $|J^\Delta X_j - X_j| \leqslant \epsilon$, $1 \leqslant j \leqslant m$.

Lemma 3.1. *Let $\Delta^n = \{t_i^n\}$ be a sequence of partitions such that $|\Delta^n| \to 0$ in probability. Let us write J^n for J^{Δ^n}. Suppose $d_{cp}(Y_n, Y) \to 0$. Then $d_{cp}(Y_n - J^n Y_n, 0) \to 0$.*

Proof. By usual subsequence arguments, it suffices to show that if $|\Delta^n| \to 0$ a.s. and $|Y_n - Y|_t^* \to 0$ a.s. for all t, then $d_{cp}(Y_n - J^n Y_n, 0) \to 0$. Now

$$\begin{aligned} |Y_n - J^n Y_n|_t^* &\leqslant |Y_n - Y|_t^* + |J^n Y_n - J^n Y|_t^* + |Y - J^n Y|_t^* \\ &\leqslant 2|Y_n - Y|_t^* + |Y - J^n Y|_t^* \\ &\to 0 \text{ a.s.} \end{aligned}$$

as $|\Delta^n| \to 0$ *a.s.* This completes the proof. □

Remark 1. The subsequence argument runs as follows. Let $a_n = d_{cp}(Y_n - J^n Y_n, 0)$. Let $\{n'\}$ be any subsequence. Get a subsequence $\{n''\}$ of $\{n'\}$ such that $|\Delta^{n''}| \to 0$ and $|Y_{n''} - Y|_t^* \to 0$ *a.s.* for all t. From what is proved above it follows that $a_{n''} \to 0$. Thus every subsequence of $\{a_n\}$ has a further subsequence converging to zero. Thus $a_n \to 0$.

Recall that $L(a, b)$ is the space of $a \times b$ matrices. An element of $L(a, b)$ will be denoted by $x = (x^{jk})$. We fix a function $f : L(a, b) \to L(d, d)$ such that $f(0) = 0$, the partial deviations

$$\frac{\partial}{\partial x^{jk}} f = f_{jk}, \frac{\partial}{\partial x^{lm}} f_{jk} = f_{jk,lm}$$

exist and are continuous, $1 \leqslant j, l \leqslant a$, $1 \leqslant k, m \leqslant b$.

Define $g : L(a, b) \times L(a, b) \to L(d, d)$ by

$$g(x, y) = f(x+y) - f(x) - \sum_{jk} f_{jk}(x) y^{jk} - \frac{1}{2} \sum_{jk} \sum_{lm} f_{jk,lm}(x) y^{jk} y^{lm}.$$

The following is an easy consequence of Taylor's theorem (see Karandikar [7]). For $\alpha > 0$, there exists a constant C_α such that for $\|x\| \leqslant \alpha$, $\|y\| \leqslant \alpha$, one has

$$\|g(x, y)\| \leqslant C_\alpha \|y\|^2. \qquad \ldots \quad (3.1)$$

Let $F : L(a, b) \to L(d, d)$ be defined by $F(A) = I + f(A)$.

Let $\Delta^n = \{t_i^n\}$, Y_n be as in the statement of Lemma 3.1. Define X_n, Z_n by

$$X_n(t) := \sum_{i=0}^{\infty} f(Y^n(t_{i+1}^n \wedge t) - Y^n(t_i^B \wedge t)) \qquad \ldots \quad (3.2)$$

$$Z_n(t) := \prod_{i=0}^{\infty} F(Y^n(t_{i+1}^n \wedge t) - Y^n(t_i^u \wedge t)) \qquad \ldots \quad (3.3)$$

Note that for each t, ω, these are finite sums and products respectively. Also, $\prod_{i=0}^{\infty} B_i$ denotes the ordered product $B_0 B_1 B_2, \ldots B_i, B_{i+1}, \ldots$.

Theorem 3.2. *Suppose* $\mathbf{Y}_n, \mathbf{Y} \in \mathcal{SM}(L(a, b))$ *be such that* $\rho(\mathbf{Y}_n, Y) \to 0$ *and let* $|\Delta^n| \to 0$. *Then*

$$\rho(X_n, X) \to 0 \text{ and } \rho(Z_n, Z) \to 0$$

where

$$X(t) = \sum_{jk} f_{jk}(0) Y^{jk}(t) + \frac{1}{2} \sum_{jk, lm} f_{jklm}(0) Y^{jk}(t) Y^{lm}(t) + \sum_{s \leqslant t} g(0, \Delta \mathbf{Y}). \qquad \ldots \quad (3.4)$$

and $Z = \epsilon(X)$, *i.e.* Z *is the unique solution to*

$$Z(t) = I + X(0) + \int_0^t Z(s-) dX(s).$$

Proof. Note that for any $\mathbf{S} \in \mathcal{SM}(L(a, b))$ and bounded stop times $\sigma \leqslant \tau$, Itô's formula yields

$$f(\mathbf{S}_\tau - \mathbf{S}_\sigma) = \sum_{jk} \int 1_{(\sigma, \tau]} f_{jk} (\mathbf{S}_{u-} - \mathbf{S}_\sigma)\, dS_u^{jk}$$

$$+ \frac{1}{2} \sum_{jk, lm} \int 1_{(\sigma, \tau]} f_{jk,lm} (\mathbf{S}_{u-} - \mathbf{S}_\sigma)\, d[S^{jk}, S^{lm}]_u$$

$$+ \sum_{\sigma < u \leqslant \tau} g(\mathbf{S}_{u-} - \mathbf{S}_\sigma, \Delta \mathbf{S}_u)$$

Using (3.5) for $S = Y^n$, $\tau = t_{i+1}^n \wedge t$, $\sigma = t_i^n \wedge t$ and summing over i, we get

$$X_n = W^{1,n} + W^{2,n} + W^{3,n}$$

where

$$W^{1,n} = \sum_{jk} \int f_{jk} (Y_n - J^n Y_n)_- \, dY_n^{jk}$$

$$W^{1,n} = \sum_{jk,lm} \int f_{jk,lm} (Y_n - J^n Y_n)_- \, d[Y_n^{jk}, Y_n^{lm}]$$

$$W^{3,n} = \sum_{s \leq t} g((\Delta Y_n - J^n Y_n)(s_-), \Delta Y^n(s)).$$

Continuity of f_{jk}, $f_{jk,lm}$ and Lemma 3.1 implies that

$$f_{jk} (Y_n - J^n Y_n) \to f_{jk}(0)$$

and

$$f_{jk,lm} (Y_n - J^n Y_n) \to f_{jk,lm}(0)$$

in d_{cp}. Further, $[Y_n^{jk}, Y_n^{lm}] \to [Y^{jk}, Y^{lm}]$ in ρ-metric. Hence

$$W^{1,n} \xrightarrow{\rho} \sum_{jk} f_{jk}(0) Y^{jk} \qquad \ldots \quad (3.6)$$

$$W^{2,n} \xrightarrow{\rho} \sum_{jk,lm} f_{jk,lm}(0) [Y^{jk}, Y^{lm}] \qquad \ldots \quad (3.7)$$

To complete the proof, we need to show that

$$W^{3,n} \xrightarrow{\rho} \sum_{s \leq t} g(0, \Delta Y s)). \qquad \ldots \quad (3.8)$$

Let

$$U^n(s) = \|g((Y_n - J^n Y_n)(s_-), \Delta Y^n(s)) - g(0, \Delta Y(s))\|.$$

We will show that

$$\sum_{s \leq t} U^n(s) \to 0 \text{ in probability} \qquad \ldots \quad (3.9)$$

which implies (3.8).

Using subsequence arguments as in Remark 1, it suffices to prove (3.9) under the assumptions $Y_n \to Y$, $|\Delta^n| \to 0$ a.s. Then $|Y_n - Y|_t^* \to 0$ as well and in view of Lemma 3.1 and continuity of g, one has that outside a fixed ω null set, say $\omega \notin N$,

$$U^n(s, \omega) \to 0 \text{ for all } s. \qquad \ldots \quad (3.10)$$

Let $K(t, \omega) := \sup_n \|Y_n\|^* (t, \omega) + \|Y\|^* (t, \omega)$. Then with $C(t, \omega) = C_\alpha$ for $\alpha = K(t, \omega)$ (C_α as in (3.1)) one has for $s \leq t$,

$$U^n(s) \leq C(t)\{\|\Delta Y_n(s)\|^2 + \|\Delta Y(s)\|\|^2\}$$
$$\leq C(t) U(s)$$

where

$$U(s) = \sup_n \|\Delta Y_n(s)\|^2 + \|\Delta Y(s)\|^2.$$

Now (3.9) would follow from dominated convergence theorem and (3.10) if we prove that

$$\sum_{s \leq t} U(s) < \infty \text{ a.s. for all } t.$$

Note that

$$U(s) \leqslant 3\|\Delta Y(s)\|^2 + 2 \sup_n \|\Delta Y_n(s) - \Delta Y(s)\|^2$$

$$\leqslant 3\|\Delta Y(s)\|^2 + \sum_n \|\Delta(Y_n - Y)(s)\|^2.$$

Hence

$$\sum_{s<t} U(s) \leqslant 3 \sum_{jk} \{[Y^{jk}, Y^{jk}](t) + \sum_n [Y_n^{jk} - Y^{jk}, Y_n^{jk} - Y^{jk}](t)$$

which is finite *a.s.* as $Y^n \xrightarrow{*} Y$ (see part (*iv*) of Proposition 2.2). We have also used the fact tht for any semimartingale S, $\sum_{s \leq t} (\Delta S(s))^2 \leqslant [S, S](t)$.

This proves $X_n \to X$. For convergence of Z_n, note that,

$$Z_n = I + \int (J^n Z_n)_- dX_n$$

and $Z = \epsilon(X)$ satisfies by definition (since $X(0) = 0$)

$$Z(t) = I + \int Z_- dX.$$

In view of Theorem 2.3, the conclusion $\rho(Z_n, Z) \to 0$ would follow once we show

$$d_{ep}(J^n Z_n - Z_n, 0) \to 0 \quad \ldots \quad (3.12)$$

since we have already shown $\rho(X_n, X) \to 0$. Once again using subsequence arguments (see Remark 1), suffices to show (3.12) under the additional assumptions

$$|\Delta| \to 0 \text{ a.s.}, X_n \xrightarrow{*} X. \quad \ldots \quad (3.13)$$

Now

$$\|J^n Z_n - Z_n\|_t^* = \|J^n Z_n(X_n - J^n X_n)\|_t^* \quad \ldots \quad (3.14)$$

$$\leqslant \|J^n Z_n\|_t^* \|X_n - J^n X_n\|_t^* \quad \ldots \quad (3.15)$$

It is shown in [7] that for a sequence of stop times σ_i, $\sigma_i \uparrow \infty$, one has

$$\sup_n E\|Z_n\|_{\sigma_i -}^{*2} < \infty \text{ for all } i. \quad \ldots \quad (3.16)$$

And from $X_n \xrightarrow{*} X$, we get $X_n \xrightarrow{\circ} X$ and Lemma 3.1 yields $\|J^n X_n - X_n\|^* \to 0$ in probability for all t. This implies

$$\|J^n X_n - X_n\|_{\sigma_i -}^{*2} \to 0 \text{ in probability for all } t \quad \ldots \quad (3.17)$$

and now (3.15), (3.16) imply

$$\|J^n Z_n - Z_n\|^*_{\sigma_i -} \to 0 \text{ in probability.}$$

This completes the proof.

Remark 2. If $f_{jk,lm}$ are locally Lipschitz and $Y_n \xrightarrow{*} Y$ and Δ satisfy

$$\|Y_n - J^n Y_n\| \leqslant \epsilon_n$$

for a sequence ϵ_n such that $\sum_n \epsilon_n^2 < \infty$, then one has $X_n \xrightarrow{*} X$, $Z_n \xrightarrow{*} Z$. This is proved in Karandikar [7]. The proof is very similar to the proof given above. This would extend some of the results on a.s. approximation proved in Karandikar [6] for the case of continuous semimartingales.

It may be noted that a special case of Theorem 3.2 with $Y_n = Y$ and ρ-convergence of X_n to X, Z_n to Z replaced by d_{cp} convergence is proved in Emery [2].

4. The trotter type formulae

Let $S_1, S_2, \ldots, S_p \in \mathcal{SM}$ $(L(d, d))$ with $S_i(0) = 0$. Suppose that

$$P\{\boldsymbol{I} + \Delta \boldsymbol{S}_i(t) = 0 \text{ for some } t \geqslant 0\} = 0,\ 1 \leqslant i \leqslant p.$$

Let $\boldsymbol{R}_i = \epsilon(\boldsymbol{S}_i)$, $1 \leqslant i \leqslant p$. i.e. R_i is the unique solution to

$$\boldsymbol{R}_i(t) = \boldsymbol{I} + \int_0^t \boldsymbol{R}_i(s\_) d\boldsymbol{S}_i(s). \qquad \ldots \quad (4.1)$$

It is easy to check, and well known that $\boldsymbol{R}_i(t)$ is invertible. For $s < t$, define

$$\boldsymbol{Q}_i(s, t) = [\boldsymbol{R}_i(s)]^{-1} \boldsymbol{R}_i(t).$$

Then $\{\boldsymbol{Q}_i(s, t)\}$ satisffies (1.1). Let $\Delta^n = \{t_i^n\}$ be a sequence of partitions. Let

$$\boldsymbol{Z}_n(t) = \left\{ \prod_{r=0}^{\infty} \prod_{j=1}^{p} \boldsymbol{Q}_j (t_i^n \wedge t, t_i^n \wedge t) \right\}. \qquad \ldots \quad (4.2)$$

In other words, if

$$\bar{\boldsymbol{Q}}(s, t) = \boldsymbol{Q}_1(s, t)\, \boldsymbol{Q}_2(s, t), \ldots, \boldsymbol{Q}_p(s, t)$$

then for $t_k^n \leqslant t \leqslant t_{k \times 1}^n$,

$$\boldsymbol{Z}_n(t) = \bar{\boldsymbol{Q}}(0, t_1^n)\, \bar{\boldsymbol{Q}}(t_1^n, t_2^n) \ldots \bar{\boldsymbol{Q}}(t_{k-1}^n, t_k^n) \bar{\boldsymbol{Q}}(t_k^n, t). \qquad \ldots \quad (4.3)$$

For $L(d, d)$ valued semimartingales $\boldsymbol{X} = (X^{ij})$ and $\boldsymbol{Y} = (Y^{ij})$, $[\boldsymbol{X}, \boldsymbol{V}]\, \boldsymbol{Y}$ is defined by

$$[\boldsymbol{X}, Y]^{ij} = \sum_k [X^{ik}, Y^{kj}].$$

Theorem 4.1, *Let* $|\Delta^n| \to 0$ *in probability. Then*

$$\boldsymbol{Z}_n \xrightarrow{p} \boldsymbol{Z}$$

where $\boldsymbol{Z} = \epsilon(\boldsymbol{X})$ *and*

$$\boldsymbol{X}(t) = \boldsymbol{S}_1(t)+\ldots+\boldsymbol{S}_p(t)+\sum_{i<j}[\boldsymbol{S}_i, \boldsymbol{S}_j](t)+\sum_{s \leq t} G(\Delta \boldsymbol{S}_1 \ldots, \Delta \boldsymbol{S}_p)(s) \quad \ldots \quad (4.4)$$

where for $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_p \in L(d, d)$

$$G(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_p) = \prod_{i=1}^{p} (\boldsymbol{I}+\boldsymbol{A}_i)-\boldsymbol{I}-\prod_{i=1}^{p} \boldsymbol{A}_i-\prod_{i<j} \boldsymbol{A}_i\boldsymbol{A}_j.$$

Proof. Note that for $s < t$

$$\begin{aligned}\boldsymbol{Q}_i(s, t) &= [\boldsymbol{R}_i(s)]^{-1} \boldsymbol{R}_i(t) \\ &= \boldsymbol{I}+[\boldsymbol{R}_i(s)]^{-1} \{\boldsymbol{R}_i(t)-\boldsymbol{R}_i(s)\},\end{aligned}$$

Hence

$$\begin{aligned}\boldsymbol{Y}_{n,i}(t) &= \sum_{i=0}^{\infty} \{\boldsymbol{Q}_i(t_i^n \wedge t, t_{i+1}^n \wedge t)-\boldsymbol{I}\} \\ &= \sum_{i=0}^{\infty} (\boldsymbol{R}_i(t_i^n \wedge t))^{-1}(\boldsymbol{R}_i(t_{i+1}^n \wedge t)-\boldsymbol{R}(t_i^n \wedge t)) \\ &= \int_0^t (J^n\boldsymbol{R}_i^{-1})_- d\boldsymbol{R}_i\end{aligned}$$

Since $|\Delta^n| \to 0$ in probability, $J^n \boldsymbol{R}_i^{-1} \to R_i^{-1}$ in d_{cp} by Lemma 3.1 and hence

$$\int (J^n\boldsymbol{R}^{-1})_- d\boldsymbol{R}_i \xrightarrow{a} \int (\boldsymbol{R}_i^{-1})_- d\boldsymbol{R}_i.$$

But $\int (\boldsymbol{R}^{-1} - d\boldsymbol{R}_i = \boldsymbol{S}_i$. Hence $\boldsymbol{Y}_{n,i} \to \boldsymbol{S}_i$ in ρ. Identify $L(pd, d) \simeq L(d, d) \otimes \ldots \otimes L(d, d)$ and let $\boldsymbol{Y}_n(t) = \boldsymbol{Y}_{n,1}(t), \ldots, \boldsymbol{Y}_{n,p}(t))$. Let $F : L(pd, d) \to L(d,d)$ be defined by

$$F(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_p) = \prod_{i=1}^{p} (\boldsymbol{I}+\boldsymbol{A}_i),$$

for $(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_p) \in L(pd, d) = L(d, d)^{\otimes p}$. Let $f = F-\boldsymbol{I}$. Note that Z_n defined by (4.2) is consistent with (3.3) for the choice of Y_n and F given above. Since f is twice continuously differentiable, we conclude from Theorem 3.2 that Z_n converges in ρ. The limit $\boldsymbol{Z}$ of $\boldsymbol{Z}_n$'s is identified by computing the various quantities in equation (3.4.) Note that here, $\boldsymbol{Y}(t) = (\boldsymbol{S}_1(t), \ldots, \boldsymbol{S}_p(t))$ and $g(0, (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_p)) = G(A_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_p)$.

We now look at another type of Trotter formula which corresponds to Masani's version in the deterministic case. Once again let $S_1, S_2, \ldots, S_p \in \mathcal{SM}(L(d, d))$ and $\Delta^n = \{t_i^n\}$ be a sequence of partitions. Consider

$$Z_n(t) = \prod_{i=0}^{\infty} \left\{ \prod_{j=1}^{p} \exp\left(S_j(t_{i+1}^n \wedge t) - S_j(t_i^n \wedge t)\right)\right\}. \quad \ldots \quad (4.5)$$

In this case we have

Theorem 4.2. *Let* $|\Delta^n| \to 0$ *in probabioity. Then* $Z_n \xrightarrow{p} Z$ *where* $Z = \epsilon(X)$ *and*

$$X(t) = \sum_{i=1}^{p} S_i(t) + \frac{1}{2}\sum_{i=1}^{p} [S_i, S_i](t) + \sum_{i<j} [S_i, S_j](t) + \prod_{s<t} G(\Delta S_1(s), \ldots, \Delta S_p(s)) \quad \ldots \quad (4.6)$$

where

$$G(A_1, \ldots, A_p) = \prod_{i=1}^{p} \exp(A_i) - I - \sum_{i=1}^{p}\left(A_i + \frac{1}{2}A_i^2\right) - \sum_{i<j} A_i A_j. \quad \ldots \quad (4.7)$$

Proof. This follows from Theorem 3.2 by taking $Y_n = Y = (S_1, \ldots, S_p)$ and $f : L(pd, d) \to L(d, d)$ as

$$f(A_1, \ldots, A_p) = \left\{ \prod_{i=1}^{p} \exp(A_i)\right\} - I,$$

for $(A_1, \ldots, A_p) \in L(pd, d) \cong (L(d, d))^{\otimes k}$. □

Remark 3. In Theorems 4.1 and 4.2, if Δ^n is such that $J^n \equiv J^{\Delta^n}$ satisfy $|J^n S_t - S_t| \leqslant \epsilon_n$ with $\sum_{n=1}^{\infty} \epsilon_n^2 < \infty$, then the convergence of Z_n to Z is stronger, namely $Z_n \xrightarrow{*} Z$. In particular, $|Z_n - Z|^t \to 0$ a.s. This would follow from Theorem 4.6 in Karandikar [7].

Remark 4. Let $S \in \mathcal{SM}\, L(d, d)$ with $S(0) = 0$. Suppose S is *continuous*. Consider the exponential equation for the stratenovich integral $o\, dS$.

$$Z(t) = I + \int_0^t Z(s-)\, o\, dS(s). \quad \ldots \quad (4.8)$$

It is straightforward to verify that (4.8) is equivalent to

$$Z(t) = I + \int_0^t Z(s-)\, dS(s) + \frac{1}{2}\int_0^t Z(s-)\, d(S, S)(s). \quad \ldots \quad (4.9)$$

If we denote the solution to (4.8) by $\mathcal{E}(S)$, then one gets

$$\mathcal{E}(S) = \epsilon \left(S + \frac{1}{2}[S, S]\right). \quad \ldots \quad (4.10)$$

Thus the result 4.1 can be recast for $\mathcal{E}$. Note that in view of (4.10) Theorem 4.2 implies that if $|\Delta^n| \to 0$ in probability

$$\prod_{i=0}^{\infty} \exp\left(S(t_{i+1}^n \wedge t) - S(t_i^n \wedge t)\right) \to \mathcal{E}(S). \quad \ldots \quad (4.11)$$

Remark 5. For an r.c.l.l. semimartingale S, it follows from Theorem 4.2 that the limit in (4.11) exists and further that if we denote the limit by $\mathcal{E}(S)$, then

$$\mathcal{E}(S) = \epsilon(S') \quad \ldots \quad (4.12)$$

where

$$S'(t) = S(t) + \frac{1}{2}[S, S](t) + \sum_{s \leq t} \left\{\exp(\Delta S(s)) - I - \Delta S(s) - \frac{1}{2}(\Delta S(s))^2\right\}.$$

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