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BAYESIAN SINGLE SAMPLING PLAN BY ATTRIBUTES WITH THREE DECISION CRITERIA FOR DISCRETE PRIOR DISTRIBUTION

R. J. PANDEY
Indian Statistical Institute

SUMMARY. Single sampling plan by attributes with three decision criteria minimising the average amount of inspection at process average quality was developed by Pandey *et al* (1972a). The idea of the above three decision criteria was extended to sampling inspection by variables by Pandey (1972b). In the present paper Bayesian Single Sampling Plans by attributes with three decision criteria for discrete prior distribution have been considered. The plans provide minimum (restricted and unrestricted) value of the cost function which is more general than the cost function considered by Hald (1960a). A double binomial with parameters p' and p'' ($p' < p''$) with weight of w_1 and w_2 respectively where $w_1+w_2 = 1$ and in the limiting case a point binomial with parameter $\bar{p}$ have been assumed as the prior distribution for the lot quality. Some examples and set of plans illustrating the computational procedures are given. The existence of unique solution is shown and it has been observed that the expected total loss is a non-decreasing function of the lot size with decreasing non-negative slopes.

1. Introduction

In practice the costs are more tangible to the industrialist and the corresponding choice is easier to make than the choice of risk points and risks. This is mainly because the various decision costs, difficult though some of them may be to determine, are closer to the kind of data that industrialist can supply on a rational basis than are the various risks and risk points. In practice the plans based on even the rough estimates of costs may be found quite satisfactory as Mr. Tippett (1958, pp. 146) has pointed out. During fifties, based on various decision costs some valuable works to develop economic plans were done. Some of them are by Anscombe (1950), Hamaker (1951), Sitting (1951), Weibull (1951), Champernowne (1953) and Horsnell (1957).

To evaluate the minimum expected value of the desirable loss or cost function, it is appropriate to consider some prior distribution for the item characteristic or the lot quality. Assuming prior distribution for the lot quality several papers on Bayesian sampling inspection such as Barnard (1954), Guthrie and Johns (1959), Wetherill (1960), Pfanzagl (1963), Hald (1960, 1965, 1967a, 1967b, 1968), Johansen (1970), Hald and Thyregod (1971), Thyregod (1972) have appeared. These plans are based on two decision criteria i.e. they are either acceptance-rectification or acceptance-rejection plans. The concept of three decision in sampling inspection was first introduced by Pandey et. al (1972a).

The purpose of this paper is to develop single sampling plans with three decision criteria for attribute inspection which minimises consumer's cost under the assumptions (a) the cost is a linear function of the incoming quality and has the model as given in section 4 (b) the incoming lot quality has a prior distribution given by double binomial (or in limiting case point binomial). It has been observed that the expected total loss is a non-decreasing function of the lot size with decreasing non-negative slopes. The existence of unique solution is shown. Some illustrative optimal plans are provided in tabular form. A comparison between plans with three and two decision criteria has shown a definite advantage in favour of three decision and further, this advantage becomes more and more pronounced as the proportion of incoming lots with deteriorated quality level increases.

2. The three decision criteria

The lot-by-lot acceptance sampling plans by the method of attributes, in which each unit in a sample is inspected on a go-not-go gauge basis for one or more characteristics and the lot-by-lot acceptance sampling plans by the method of variables, in which each unit in a sample is measured for a single characteristic, such as weight or strength, are either acceptance rectification or acceptance rejection plans. We shall refer to these plans as plans with two decision criteria i.e. (i) a lot either being accepted as a good one or being classified as bad one and screened for the purpose of acceptance or (ii) a lot either being accepted as a good one or being classified as bad one and rejected. In practice situation arises where a 100% inspection of too bad lots and replacing or rectifying large number of defectives or out right rejection of moderately good lot may not be an economically valid proposal. In such cases it may be economical to operate the plan on the basis of three-decision criteria i.e. either (i) classify the lots in three quality grades A, B and C and accept grade A as good lot, screen grade B lots and reject a grade C lot out right or (ii) classify the lots in three grades A, B and C as before and accept grade A as good lot, accept grade B as moderately good or salvagable lots and reject grade C lot out right. We shall refer to the three-decision criteria (i) as D and (ii) as D^* respectively.

Acceptance sampling plans by attributes with three decision criteria D are discussed in this paper. Acceptance sampling plans by variables with three decision criteria D^* will be discussed in a separate paper.

The consumer is supposed in this paper, to operate the plan to safeguard his interest against accepting a lot which is too bad. The consumer bears the cost of inspection in the interest of the producer. He is not a narrow-minded consumer in the sense of Tippett (1958, pp. 137). The consumer allows a chance of 100% inspection for lots which are not accepted ordinarily but are not too bad. Every such lot should be accepted after screening. The lots which are too bad (grade C) may involve high cost of screening and neither the consumer, who bears the cost of inspection would allow screening of such lots nor the producer would find the cost of replacing or rectifying too many defectives as attractive.

The outright rejection of any lot may be deemed by the producer as a drastic action on the part of the consumer if screening be the general practice under two-decision criteria. However, the producer may agree to such a proposal provided :

(i) the consumer agrees to accept the moderately good lot (Grade B) either after imposing a penality to the producer or after screening.

(ii) only too bad lots (Grade C) are rejected outright and

(iii) the consumer agrees to bear the cost of inspection (or at least 100% inspection).

Single sampling acceptance plan by attributes with three-decision criteria D is defined by the parameters n, c_1 and c_2 and is to be operated as follows :

Take a random sample of n items from N and let x be the number of defectives in the sample then

$$\begin{aligned} &\text{Accept if} \quad 0 \leqslant x \leqslant c_1 \\ &\text{Screen if} \quad c_1 < x \leqslant c_2 \\ &\text{Reject if} \quad c_2 < x \leqslant n \end{aligned} \qquad \dots (1)$$

Double sampling acceptance plan by attributes with three-decision criteria has been also developed and is discussed in a separate paper by Pandey (1972c).

3. The conditions for applicability

Though the inspection with three decision criteria is not in the current use it is not difficult to find practical situations where this can be gainfully employed. For the applicability of the criteria D all of the following five conditions should be satisfied.

(1) Items are inspected on lot-by-lot basis.

(2) Defective items in accepted lots cause some damage which is measurable in economic terms.

(3) Inspection does not involve any destructive or very costly testing i.e. 100% inspection is permissible.

(4) Any defective item if detected can be either replaced with a good item or rectified.

(5) The condition ($p_u < p_v$) for three-decision criteria as derived in Section 4 in terms of the break-even qualities holds.

It is felt that the conditions (1)-(4) are generally satisfied in those industrial situations where two-decision criteria i.e. acceptance-rectification or acceptance-rejection is in use. Thus, the condition (5) in fact, determines whether a three-decision criteria D can be used.

The economic advantages of the plans with three-decision criteria greatly depend on the weights w_1 and w_2. For higher values of w_2 the plan with threee-decision criteria will be more and more advantageous as compared with the plans with two-decision criteria (fig. 7 and fig. 8).

The acceptance sampling plan with three-decision criteria D^* is applicable in those situations where the following five conditions are satisfied.

(1) Items are inspected on lot-by-lot basis.

(2) Acceptance of any defective item is undesirable but if accepted it does not hamper much the functional requirements.

(3) Inspection involves either a destructive or costly testing i.e. 100% inspection is not permissible.

(4) Any defective item if detected can be either replaced with good item or rectified.

(5) The condition ($p_u < p_v$) for three decision criteria as derived in Section 4 in terms of the break-even qualities holds.

To point out a practical situation where D^* can be applied the author wishes to mention the following experience during his consultancy work.

An organisation (around calcutta) purchases various store items from different local suppliers. One of the store items is a sort of bandages used for medical dressing. This organisation is using sampling inspection for the purpose of acceptance of the lot of bandages. The sample is subjected to the following two types of test :

(i) scouring test (which is a destructive test) and

(ii) test for determining weight per metre.

Any failure on the part of a bandage to meet the specifications in respect of the two characteristics is undesirable but it does not affect much the end use of the bandages. Depending on the test results a lot is either (a) accepted as satisfactory or (b) accepted with some penality to the supplier or (c) declared unsatisfactory and is rejected outright. The organisation has some amount of arbitrariness while deciding the percentage of penality to the supplier. This is mainly because the inspection is actually done on the basis of two-decision criteria and a good deal of arbitrariness arises while classifying the unsatisfactory lots into acceptable lots with penality and lots which are outright rejectable ones. The author feels (as was also checked by using some rough values of various costs) that an acceptance sampling plan by variables with three-decision criteria D^* would be an appropriate plan in this situation. As mentioned earlier, such a plan is the subject matter of a forth coming paper.

4. The model

Let N and n denote the lot size and sample size and let X and x denote number of defectives in the lot and the sample respectively. The acceptance numbers are denoted by c_1 and $c_2 (c_1 < c_2)$.

In this paper we shall consider the following linear cost function.

$$h(X, x, N, n, c_1, c_2) = \begin{cases} nS_1+xS_2+(N-n)A_1+(X-x)A_2 & \text{for } x \leqslant c_1 \\ nS_1+xS_2+(N-n)T_1+(X-x)T_2 & \text{for } c_1 < x \leqslant c_2 \\ nS_1+xS_2+(N-n)R_1+(X-x)R_2 & \text{for } c_2 < x \leqslant n \end{cases} \quad \ldots (2)$$

where nS_1 denotes the cost of inspection and xS_2 denotes the cost proportional to the number of defective items in the sample. In fact S_1 includes sampling and testing costs per item and S_2 denotes additional costs for an inspected defective item including the repair costs per item in case the defective items found in the sample are repaired. Thus the costs nS_1+xS_2 associated with the sample give the costs of sampling inspection.

Costs of acceptance are given by $(N-n)A_1+(X-x)A_2$. The part $(N-n)A_1$ is proportional to the number of items in the remainder of the lot and A_1 usually will be zero or negligible. The part $(X-x)A_2$ is proportional to the number of defective items accepted and hence A_2 will often be considerable. If the accepted items are used as a basic raw material or components to manufacture some product A_2 may include the manufacturing cost or the price of an item, the costs of handling a defective item in assembling and disassembling and damage done to other parts used in the assembly. In case of inspection of finished goods A_2 may include the cost of repair, service and guarantees plus loss of good will.

Costs of screening is given by $(N-n)T_1+(X-x)T_2$ and are composed of a part $(N-n)T_1$, proportional to the number of items in the remainder of the lot, and another part, $(X-x)T_2$, proportional to the number of defective items in the remainder of the lot. T_1 may include the costs of handling and costs of inspection per item whereas T_2 may include the costs of rework or replacement and the costs of delay per defective item caused due to screening.

Costs of outright rejection are $(N-n)R_1+(X-x)R_2$ and are composed of $(N-n)R_1$, proportional to the number of items in the remainder of the lot, and another part $(X-x)R_2$ which is proportional to the number of defective items in the remainder of the lot. R_1 may include the costs of storage per rejected item before disposal, the costs of handling per rejected item during disposal and the costs of delay in availability of the raw material or the components for manufacturing or assembling per rejected item. R_2 is generally zero or negligible.

From Hald (1967a) the cost function (2) becomes

$$h \sim \begin{cases} n(S_1+S_2p)+(N-n)(A_1+A_2p) & \text{for} \quad x \leqslant c_1 \\ n(S_1+S_2p)+(N-n)(T_1+T_2p) & \text{for} \quad c_1 < x \leqslant c_2 \\ n(S_1+S_2p)+(N-n)(R_1+R_2p) & \text{for} \quad c_2 < x \leqslant n \end{cases} \quad \ldots \quad (3)$$

disregarding the terms of order $\sqrt{n}$. The average cost can be written as

$$K(N, n, c_1, c_2) = \int K(N, n, c_1, c_2, p)dW(p) \quad \ldots \quad (4)$$

where $W(p)$ denotes the cumulative distribution of p and

$$K(N, n, c_1, c_2, p) = n(S_1+S_2p)+(N-n)[(A_1+A_2p)P_a(p) +(T_1+T_2p)P_s(p)+(R_1+R_2p)P_r(p)] \quad \ldots \quad (5)$$

The probabilities $P_a(p)$, $P_s(p)$ and $P_r(p)$ are defined as follows :

$$P_a(p) = B(c_1, n, p) \quad \ldots \quad (6)$$

$$P_s(p) = B(c_2, n, p)-B(c_1, n, p) \quad \ldots \quad (7)$$

$$P_r(p) = 1-B(c_2, n, p) \quad \ldots \quad (8)$$

where $B(c, n, p) = \sum_{x=0}^{c} \binom{n}{x} p^x(1-p)^{n-x}$. Since it is assumed that the process average varies at random according to the cumulative distribution function $W(p)$, (5) represents the average costs over all the lots with a given process average i.e. a conditional average and (4) gives the overall average.

To simplify the notations let us introduce four cost functions :

$$k_s(p) = S_1+S_2p \qquad \ldots \quad (9)$$

$$k_a(p) = A_1+A_2p \qquad \ldots \quad (10)$$

$$k_t(p) = T_1+T_2p \qquad \ldots \quad (11)$$

$$k_r(p) = R_1+R_2p \qquad \ldots \quad (12)$$

defined for $0 \leqslant p \leqslant 1$. The corresponding averages k_s, k_a, k_t and k_r are defined by

$$k = \int_0^1 k(p)dW(p) \qquad \ldots \quad (13)$$

We shall make the following assumptions regarding the functions (9), (10), (11) and (12).

(1) All these functions are non-negative and none of them is identical to zero.

(2) $k_a(0) < k_t(0) < k_r(0)$ i.e. for a 100% good lot cost of acceptance per item is the least and the cost of screening per item will be quite low whereas the cost of rejection per item would be considerable.

(3) $k_a(1) > k_t(1) > k_r(1)$ i.e. for a 100% defective lot cost of acceptance per item is the highest and the cost of screening would be considerable whereas the cost of rejection per item is the least.

Since $A_1 < T_1 < R_1$ and $A_1+A_2 > T_1+T_2 > R_1+R_2$ it follows that $A_2-T_2 > T_1-A_1 > 0$ and $T_2-R_2 > R_1-T_1 > 0$ and the equations $k_a(p) = k_t(p)$ and $k_t(p) = k_r(p)$ have the solution $p_u = (T_1-A_1)/(A_2-T_2)$, $0 \leqslant p_u \leqslant 1$ and $p_v = (R_1-T_1)/(T_2-R_2)$, $0 \leqslant p_v \leqslant 1$ respectively. (We shall assume that $p_u < p_v$ Fig. 1).

(4) $k_s(p) \geqslant k_m(p)$ for $0 \leqslant p \leqslant 1$ where $k_m(p)$ is defined as follows :

If $p_u < p_v$, $k_m(p)$ is given by (14)

$$k_m(p) = \begin{cases} k_a(p) & \text{for} \quad p < p_u \\ k_t(p) & \text{for} \quad p_u \leqslant p < p_v \\ k_r(p) & \text{for} \quad p \geqslant p_v \end{cases} \qquad \ldots \quad (14)$$

and if $p_u \geqslant p_v$, $k_m(p)$ is given by (15)

$$k_m(p) = \begin{cases} k_a(p) & \text{for} \quad p < p_w \\ k_r(p) & \text{for} \quad p \geqslant p_w \end{cases} \qquad \ldots \quad (15)$$

where $p_w = (R_1-A_1)/(A_2-R_2)$, $0 \leqslant p_w \leqslant 1$

when $p_u \geqslant p_v$ the three-decision criteria D reduces to two-decision criteria of acceptance-rejection plan (see fig. 1 and fig. 2)

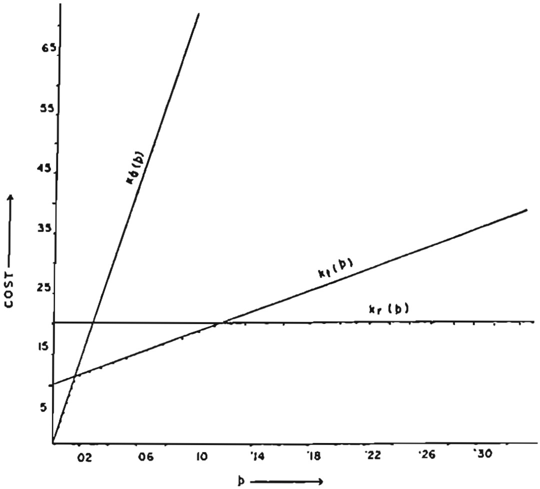


Fig. 1. Cost functions when $p_u < p_v$, $ka(p) = 720p$, $kr(p) = 10-85p$, $kr(p) = 20$, $p_u = 0.0158$, $p_v = 0.1176$.

Further, it can be noted that under assumptions 1-4 the three-decision criteria is always better than the two-decision criteria if $p_u < p_v$. In cases where $p_u \geqslant p_v$ it is economic to use two-decision criteria (Fig. 2).

In this paper we shall assume that $p_u < p_v$. For $p_u < p_v$ the average minimum cost k_m is defined as

$$k_m = \int_0^{p_u} k_a(p)dW(p) + \int_{p_u}^{p_v} k_t(p)dW(p) + \int_{p_v}^{1} k_r(p)dW(p) \qquad \dots \quad (16)$$

k_m given by (16) represents the average cost per item when the following decision rule is used.

Accept all lots from processes with $p < p_u$

Screen all lots from processes with p, $p_u \leqslant p < p_v$

Reject all lots from processes with $p \geqslant p_v$

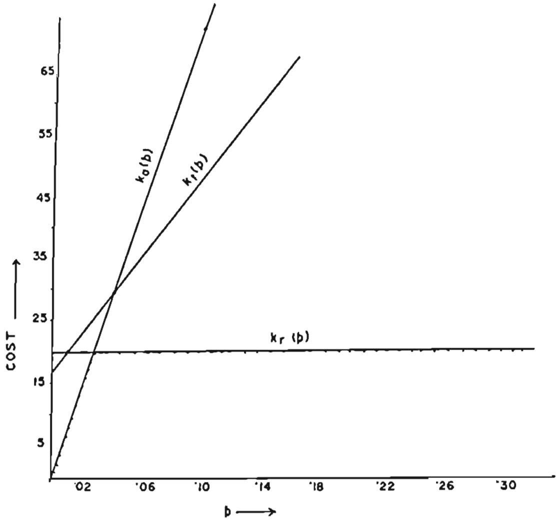


Fig. 2. Cost functions when $p_u > p_v$, $ka(p) = 720p$, $kr(p) = 17+300p$, $kr(p) = 20$, $p_u = 0.0405$, $p_v = 0.0100$.

p_u and p_v are called break-even qualities. The break-even qualities p_u and p_v are used to classify the lots in three-quality grades as follows :

$$\begin{aligned} &\text{Grade A}: \quad \text{lot of quality } p < p_u \\ &\text{Grade B}: \quad \text{lot of quality } p,\ p_u \leqslant p < p_v \qquad \dots \quad (17) \\ &\text{Grade C}: \quad \text{lot of quality } p \geqslant p_v \end{aligned}$$

Following Hald (1967a) we define the standardised form of (4) as

$$R(N, n, c_1, c_2) = (K(N, n, c_1, c_2) - Nk_m)/(k_s - k_m) \qquad \dots \quad (18)$$

which gives

$$R = n + \frac{(N-n)}{(k_s-k_m)}\Big[(A_2-T_2)\Big\{ \int_0^{p_u} (p_u-p)(1-P_a(p))dW(p) +$$

$$\int_{p_u}^{1}(p-p_u)P_a(p)dW(p)\Big\}+(T_2-R_2)\Big\{\int_{0}^{p_v}(p_v-p)P_r(p)dW(p)+$$

$$\int_{p_v}^{1}(p-p_v)(1-P_r(p))dW(p)\Big\}\Big] \quad \ldots \quad (19)$$

In this paper we shall confine to discrete prior distribution. The case of continuous prior distribution has been dealt in another paper by Pandey (1973). Assume that the prior distribution is a double binomial or as a limiting case a point binomial. The double binomial distribution is a weighted average of two binomials with parameters p' and p'', $p' < p''$ and weights w_1 and w_2, $w_1+w_2 = 1$ i.e. the process average has a two point distribution. To justify, at least with practical point of view, the assumption of double binomial or point binomial as the prior distribution of the lot quality p, the following may be added :

In practice, generally a process is set to operate at a good or acceptable quality level (AQL) and as soon as it deteriorates to operate at a bad or rejectable quality level (RQL) corrective actions are taken to restore the good level. A process is desired and hence generally designed, to operate at the good level as long as possible (i.e. low value of p' with high value of w_1) and only occasional deterioration of the process to the bad level (i.e. high value of p'' with low value of $w_2 = 1-w_1$) may be considered as a normal phenomena. However, situations may arise where the process may continue to turn out large number of bad lots (i.e. $w_1 < w_2$) over a considerable period. The consumer may not have any other sources of supply and he may adopt a plan with three decision criteria to safeguard his economic interests.

In view of the above a double binomial (as also assumed by Hald (1960)) may be used as a prior distribution for the quality of the lots. Alternatively as a limiting case a well controlled process may be stable around the process average (say $\bar{p}$) and for the lots produced by this process we may assume point binomial (as also assumed by Dodge and Romig (1929)) as a prior distribution of the lot quality.

A necessary condition for a sampling plan with three decision to exist under the double binomial as a prior distribution is that $p' < p_u < p_v < p''$ (i.e. $\nu_{ij} > 0$ for $i, j = 1, 2$). Assuming that $p' < p_u < p_v < p''$ the standardised cost function can be written as

$$R(N, n, c_1, c_2) = n+(N-n)\{\nu_{11}(1-P_a(p'))+\nu_{12}P_a(p'')+$$
$$\nu_{21}P_r(p')+\nu_{22}(1-P_r(p''))\} \quad \ldots \quad (20)$$

where
$$\nu_{11} = w_1\{k_l(p')-k_a(p')\}/(k_s-k_m)$$
$$\nu_{12} = w_2\{k_a(p'')-k_l(p'')\}/(k_s-k_m)$$
$$\nu_{21} = w_1\{k_r(p')-k_l(p')\}/(k_s-k_m)$$
$$\nu_{22} = w_2\{k_l(p'')-k_r(p'')\}/(k_s-k_m)$$

Thus the average decision loss per item is a linear combination of the probabilities $1-P_a(p')$, $P_a(p'')$, $P_r(p')$ and $1-P_r(p'')$.

5. Choice of risks

The consumer's and producer's risks are most widely used for characterising systems of sampling plans. The consumer's risk is defined as the probability of acceptance of a lot or process with deteriorated or lot tolerance percent defective (LTPD) quality. The producer's risk, a concept opposite to the consumer's risk is defined as the probability of rejecting a lot of good quality or acceptable quality level (AQL) by the consumer. The well known systems of plans which provide lot quality protection in terms of the consumer's risk are the Dodge and Romig's (1929) LTPD systems of plans. When a plan with three decision criteria is operated for a receiving inspection any misclassification of an inferior quality lot or process as a superior quality lot or process entails a risk to the consumer. To provide a lot quality protection in case of the plans with three decision criteria on the line of the Dodge and Romig's LTPD systems of plans, it may be therefore, logical to specify the consumer's risk in terms of the probabilities of misclassification of the above type. When a plan with the three decision criteria is operated the misclassification entailing a risk to the consumer would arise in two ways : (a) when a fairly good grade B lot is classified as grade A or (b) when a fairly bad grade C lot is classified as grade B or grade A lot. The consumer's risk, therefore, need be specified at two quality levels—one of grade B and the other of grade C. For this purpose the two quality levels p_1 and p_2 ($p_1 < p_2$) are chosen depending on the consumer's specification as follows :

The quality level p_1 is specified in such a way that the probability of misclassifying a lot of this quality under the plan as grade A is quite low β_1 (say .01 to 0.10) and the probability of classifying it correctly as grade B is fairly high. Similarly the quality level p_2 is specified by the consumer in such a way that the probability of misclassifying a lot of this quality as grade B or grade A is quite low β_2 (say .01 to .10) and the probability of classifying it correctly as grade C is fairly high (Fig. 3).

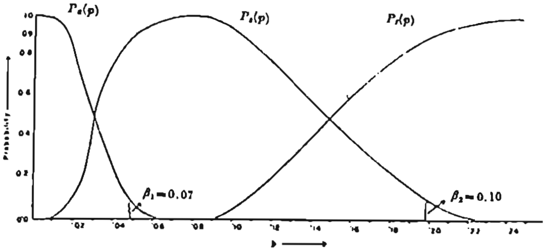


Fig. 3. Operating characteristic curve for the plan (n, c_1, c_2) with $n = 220$, $c_1 = 6$, $c_2 = 36$.

The risk β_1 is the probability of misclassifying a lot of quality p_1 as grade A. The risk β_2 is the probability of misclassifying a lot of quality p_2 either as grade B or A. These risks β_1 and β_2 are mathematically written as

$$\beta_1 = B(c_1, n, p_1) \qquad \dots \quad (21)$$

and

$$\beta_2 = B(c_2, n, p_2) \qquad \dots \quad (22)$$

using the sample criteria (1).

The concept of producer's risks say, α_1 and α_2 at quality level p_1' and p_2' (say) respectively (Fig. 3) can be similarly defined. Mathematically

$$\alpha_1 = \sum_{x=c_1+1}^{n} b(x, n, p_1') \qquad \dots \quad (23)$$

and

$$\alpha_2 = \sum_{x=c_2+1}^{n} b(x, n, p_2') \qquad \dots \quad (24)$$

where p_1' denotes the quality level such that when a plan with three decision criteria is operated the probability of misclassification of a lot of quality p_1' (grade A) as either grade B or C is quite low say α_1 whereas the probability of classifying it correctly as grade A is fairly high. Similarly, p_2' denotes the quality level such that the probability of misclassifying a lot of quality p_2' (grade B) as grade C is quite low say α_2 whereas the probability of classifying it correctly as grade B is fairly high. However, since the plans considered in this paper are on the line of Dodge and Romig's LTPD systems of plans (consumer's risks plan), the notion of the producer's risks is not relevant to the discussions in the subsequent sections.

From amongst the large number of plans satisfying specified values of the lot tolerance percent defective (LTPD), consumer's risk ($\beta = .10$), lot size (N) and process average ($\bar{p}$) the Dodge and Romig's LTPD systems of plans select the unique plan which minimizes average amount of inspection at process average quality $\bar{p}$. Similar to the above plans, Pandey *et al* (1972a) developed plans with three decision criteria for inspection by attributes. From amongst the large number of plans satisfying the specified values of the parameters their plans select the unique plan which minimises average amount of inspection at process average quality. In the present paper the criteria of minimum total average cost per lot has been used to select the unique plan.

As explained earlier in most practical situations the quality of items being produced by a process may be assumed either to be fluctuating on two levels in the manner as described in section 4 or to be stable around an average level. Further, it may be desired in some cases to have sampling inspection plans which either have some specified probability of misclassification (consumer's risk) at some specified level of quality or are free from any such restriction. Motivated by the above practical considerations, the subsequent sections we shall devote to the following three areas of the single sampling plan with three-decision criteria

(1) *Double binomial restricted Bayes solution—Bayesian single sampling plans with double binomial as a prior distribution and having some restriction on the consumer's risks i.e. the plans are required to satisfy closely the specified value of the consumer's risks* (β_1 and β_2). These plans are called restricted Bayesian plans with double binomial as the prior distribution and are discussed in the Section 6.

(2) *Double binomial unrestricted Bayes solution—Bayesian single sampling plan with double binomial as a prior distribution.* In this case the plans (n, c_1, c_2) minimising R given by (20) and having no restriction regarding the consumer's risk, would be developed. In context of the plans of type (1), these plans will be referred to as unrestricted Bayesian single sampling plans by attributes (Section 7).

(3) *Point binomial Bayes solution.* The limiting case of double binomial i.e. point binomial as a prior distribution is discussed in Section 8.

6. Double binomial restricted Bayes solution

In this section we shall assume that the prior distribution is a double binomial with parameters p' and p'', $p' < p''$ and weights w_1 and w_2, $w_1 + w_2 = 1$ i.e. the process average has a two point distribution. The standardised average cost is given by (20). The problem is to find (n, c_1, c_2) minimising (20) and satisfying (21) and (22). Let S be the set of plans (n, c_1, c_2) satisfying (21) and (22). For any plan (n, c_1, c_2) in S, if any one of n, c_1 and c_2 is fixed the other two are uniquely determined. For example for an arbitrary $c_1 = 0$ say, $B(c_1, n, p_1) \leqslant \beta_1$ gives unique $n = 52$ and $B(c_2, n, p_2) \leqslant \beta_2$ gives $c_2 = 2$ where $p_1 = .05$, $p_2 = .10$ $\beta_1 = .07$ and $\beta_2 = .10$. Thus a plan in S can be uniquely defined according to any one of n, c_1 and c_2. We have chosen c_1 for this purpose. Let $S(c_1)$ denote a plan (n, c_1, c_2) and $R_1(c_1)$ be the value of (20). Using the notations of (6), (7) and (8) we write $R_1(c_1)$ from (20) as

$$R_1(c_1) = n_{c_1}+(N-n_{c_1})(\nu_{11}(1-B_{c_1}(p'))+\nu_{12}B_{c_1}(p'')+$$
$$\nu_{21}(1-B_{c_1}(c_2, p'))+\nu_{22}B_{c_1}(c_2, p'')) \quad \ldots \quad (25)$$

where n_{c_1} denotes the sample size in a plan (n, c_1, c_2) and

$$B_{c_i}(p) = \sum_{x=0}^{c_i} \binom{n}{x} p^x(1-p)^{n-x} \quad \text{and} \quad B_{c_i}(c_2, p) = \sum_{x=0}^{c_2^{(c_i)}} \binom{n}{x} p^x(1-p)^{n-x}$$

with $c_2^{(c_i)}$ denoting the value of the second acceptance number c_2 when the first acceptance number c_1 equals c_i. The value of c_1 minimising R_1 is determined from the inequality

$$\Delta R_1(c_1-1) \leqslant 0 < \Delta R_1(c_1) \quad \ldots \quad (26)$$

We get

$$\Delta R_1(c_1) = (1-\nu_{11}-\nu_{21}+\nu_{11}B_{c_1+1}(p')-\nu_{12}B_{c_1+1}(p'')$$
$$+\nu_{21}B_{c_1+1}(c_2, p')-\nu_{22}B_{c_1+1}(c_2, p''))\Delta n_{c_1}$$
$$-(N-n_{c_1})(\nu_{11}\Delta B_{c_1}(p')-\nu_{12}\Delta B_{c_1}(p'')$$
$$+\nu_{21}\Delta B_{c_1}(c_2, p')-\nu_{22}\Delta B_{c_1}(c_2, p'')) \quad \ldots \quad (27)$$

To obtain the bounds for the lot size for which the plan (n, c_1, c_2) satisfying (26) is the optimal plan we shall define the following auxiliary function from (27)

$$N_{c_1} = n_{c_1}+\Big(\frac{1}{\nu_{22}}-\frac{\nu_{11}}{\nu_{22}}-\frac{\nu_{21}}{\nu_{22}}+\frac{\nu_{11}}{\nu_{22}}B_{c_1+1}(p')-\frac{\nu_{12}}{\nu_{22}}B_{c_1+1}(p'')$$
$$+\frac{\nu_{21}}{\nu_{22}}B_{c_1+1}(c_2, p')-B_{c_1+1}(c_2, p'')\Big)\frac{\Delta n_{c_1}}{U} \quad \ldots \quad (28)$$

where

$$U = \left(\frac{\nu_{11}}{\nu_{22}}\Delta B_{c_1}(p') - \frac{\nu_{12}}{\nu_{22}}\Delta B_{c_1}(p'') + \frac{\nu_{21}}{\nu_{22}}\Delta B_{c_1}(c_2, p') - \Delta B_{c_1}(c_2, p'')\right)$$

The plan (n, c_1, c_2) is the locally optimal plan for the lot size N if

$$N_{c_1-1} \leqslant N < N_{c_1},\ \Delta B_{c_1-1} > 0 \quad \text{and} \quad \Delta\ B_{c_1} > 0 \qquad \ldots \quad (29)$$

For two plans (n_1, c_1', c_2') and (n_2, c_1'', c_2'') satisfying (26) and having overlapping N—intervals according to (30) solving the equation $R_1(N, n_1, c_1', c_2') = R_1(N, n_2, c_1'', c_2'')$ for N we get

$N_{12} =$

$$\frac{(n_2 - n_1)(1-\nu_{11}) + n_2(\delta_1(n_2, c_1'') - \nu_{21} + \delta_2(n_2, c_2'')) - n_1(\delta_1(n_1, c_1') - \nu_{21} + \delta_2(n_1, c_2'))}{\delta_1(n_2, c_1'') + \delta_2(n_2, c_2'') - \delta_1(n_1, c_1') - \delta_2(n_1, c_2')} \qquad \ldots \quad (30)$$

where $$\delta_i(n, c_j) = \nu_{i1}B(c_j, n, p') - \nu_{i2}B(c_j, n, p'') \quad i = 1, 2 \qquad \ldots \quad (31)$$

It is clear from (20) that R for a given (n, c_1, c_2) is an increasing linear function of N and we have that

$$R(N, n_1, c_1', c_2') \lesseqgtr R(N, n_2, c_1'', c_2'') \text{ according as } N \lesseqgtr N_{12} \qquad \ldots \quad (32)$$

To discuss the uniqueness of the solution write the regret function $R_1(c_1)$ as follows :

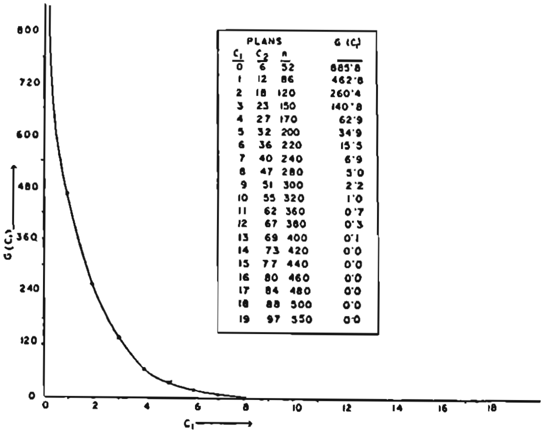
$$R_1(c_1) = \begin{cases} N & 0 \leqslant N \leqslant n \\ n_{c_1} + (N - n_{c_1})G(c_1) & n < N \end{cases} \qquad \ldots \quad (33)$$

where $G(c_1)$ denotes the expected decision loss (standardised) as is defined in (34)

$$G(c_1) = \nu_{11}(1 - B_{c_1}(p')) + \nu_{12}B_{c_1}(p'') + \nu_{21}(1 - B_{c_1}(c_2, p')) + \nu_{22}B_{c_1}(c_2, p'') \quad \ldots \quad (34)$$

Note that the function $R_1(c_1)$ for a fixed (n, c_1, c_2) is a polygonal line consisting of two segments with slopes 1 and $G(c_1)$ when plotted against N. Clearly the minimum of $R_1(c_1)$ would be attained for some plan in the set S for which the slope $G(c_1) < 1$. Thus, to consider the uniqueness of the optimal solution we can assume $G(c_1) < 1$. For a given c_1 the cost function R_1 has one and only one minimum with respect to this value of c_1. Writing $N_{c_1} = n_{c_1} + \frac{1 - G(c_1+1)}{-\Delta G(c_1)}$ we note that $N_{c_1} > n_{c_1}$ for $G(c_1+1) < 1$ and $G(c_1)$ is non-increasing function of

c_1 with increasing slopes as it can be seen from the figure 4. It can be easily shown that N_{c_1} is an increasing function of n. Further, if it is shown that for given plans $S(c_1')$ and $S(c_1'')$ where $c_1'' = c_1'+1$, the regret function $R_1(N, S(c_1'))$ and the regret function $R_1(N, S(c_1''))$ as a function of N have at the most only one point in common and the common point is an increasing function of c_1, the uniqueness of the solution is established.



PLANS c_1	c_2	n	$G(c_1)$
0	6	52	885·8
1	12	86	462·8
2	18	120	260·4
3	23	150	140·8
4	27	170	62·9
5	32	200	34·9
6	36	220	15·5
7	40	240	6·9
8	47	280	3·0
9	51	300	2·2
10	55	320	1·0
11	62	360	0·7
12	67	380	0·3
13	69	400	0·1
14	73	420	0·0
15	77	440	0·0
16	80	460	0·0
17	84	480	0·0
18	88	500	0·0
19	97	550	0·0

Fig. 4. Expected decision loss (standardised) as a function of c_1 for Bayesian Plan with double binomial as a prior distribution $p' = 0.01$, $p'' = 0.15$ and $w_1 = 0.93$, $w_2 = 1-w_1$ ($G(C_1)$ is given in the units of 10^{-3})

Since for the restricted plans of this section the cost functions $R(N, S(c_1'))$ and $R(N, S(c_1''))$ for a fixed c_1' are strictly increasing linear function of N with constant slopes, there can be at the most only one common point say, $N(c_1', c_1'')$ at which $R(N, S(c_1')) = R(N, S(c_1''))$. The R functions for the plans in S (satisfying the two restrictions) for the costs as considered in the example 1 do not intersect (see fig. 5) and hence each of these is the optimal plan for the specified lot sizes. These optimal plans for $c_1 = 0, 1, \ldots 6$ are given in the table 1.

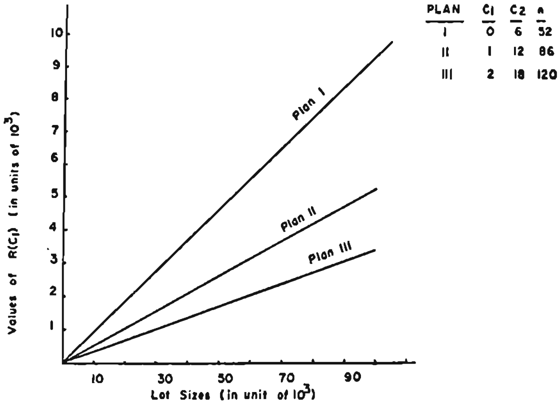


Fig. 5. The values of $R(c_1)$ for varying lot sizes.

TABLE 1. SINGLE SAMPLING PLAN WITH THREE-DECISION CRITERIA FOR DOUBLE BINOMIAL PRIOR DISTRIBUTION WITH $p' = 0.01$ AND $p'' = 0.15$ AND WEIGHT $w_1 = 0.93$ AND $w_2 = 1-w_1$ RESPECTIVELY

lot size	sample size	acceptance number c_1	acceptance number c_2	$\bar{N}$	$K/\bar{N}$	% saving
52– 846	52	0	6	209	15.36	*
847– 1811	86	1	12	1238	10.34	27.5
1812– 2674	120	2	18	2201	9.64	32.4
2675– 3011	150	3	23	2838	9.34	34.5
3012–11910	170	4	27	5989	8.84	38.0
11911–15765	200	5	32	13703	8.53	40.2
15766–62678	220	6	36	31436	8.38	41.2

* Acceptance without inspection is more economic

In case the R functions intersect, the common point $N(c_1', c_1^*)$ is an increasing function of c_1' as is shown below :

From (30) writing $N(c_1', c_1^*)$ as

$$N(c_1', c_1^*) = \frac{n_{c_1^*}(1-G(c_1^*))-n_{c_1'}(1-G(c_1'))}{-\Delta G(c_1')} \quad \dots \quad (35)$$

$$\Delta_{c_1'} N(c_1'c_1^*)\Delta G(c_1'+1)\Delta G(c_1') = (-\Delta G(c_1'))(n_{c_1'+2}(1-G(c_1'+2))$$
$$-n_{c_1'+1}(1-G(c_1'+1)))-(-\Delta G(c_1'+1))(n_{c_1'+1}(1-G(c_1'+1))$$
$$-n_{c_1'}(1-G(c_1'))) \quad \dots \quad (36)$$

Since $G(c_1')$ is non-increasing function of c_1' with *increasing slopes* and $n_{c_1'}(1-G(c_1'))$ is strictly increasing function of c_1' the right hand side of (36) is easily shown to be positive and hence $\Delta_{c_1'} N(c_1', c_1^*) \geqslant 0$ implying that the point of intersection is an increasing function of c_1'. This completes the proof that the solution is a unique one.

These optimal plans can be systematically tabulated as follows :

Step 1 : Take some arbitrary value of c_1 and obtain a plan say $S(c_1)$ belonging to S by using (21) and (22).

Step 2 : For the plan $S(c_1)$ so obtained, compute the value of N_{c_1} and N_{c_1-1} uing (28).

Step 3 : Choose $c_1 = 0, 1, 2, 3\dots$ systematically and proceed as in steps 1–2 and tabulate the sampling plans and the corresponding bounds for the lot sizes.

Step 4 : For two plans with overlapping N-interval use (30)-(32) to select the optimal plans.

Thus from tabulated plans an optimal plan for a given lot size can be obtained against the lot range in which the given lot size falls. For illustration let us consider the following example.

Example 1 : Obtain single sampling plan by attributes with three-decision criteria minimising the total average cost per lot when the following information is given :

The four cost functions be $k_s(p) = 23+35p$, $k_a(p) = 720p$, $k_t(p) = 10+85p$, $k_r(p) = 20$, the coefficients denote costs per item in money units i.e.

cost of sampling and testing is 23 money units per item in the sample, cost of accepting a defective item is 720 money units and the cost involved in rejecting an item out right is 20 money units etc. (see Section 4).

Let us further assume that lots are generated with probability $w_1 = 0.93$ from a binomially controlled process with $p' = 0.01$ and with probability $w_2 = 0.07$ from the process with $p'' = 0.15$. Also, suppose it is given that the probability of misclassification as a superior grade lot is 0.07 for a lot containing 5% defective items and it is 0.10 for a lot containing 20% defective items.

The break-even qualities work out as $p_u = 0.0158$ and $p_v = 0.1176$ From (13) and (14) $k_s = 23.693$, $k_m = 8.096$, $k_s - k_m = 15.597$ and substituting the relevant values in the expression of ν_{ij} as defined in (20) we obtain $\nu_{11} = 0.217638$, $\nu_{12} = 0.382606$, $\nu_{21} = 0.545586$ and $= 0.012342$. To indicate the computational procedure let us choose $c_1 = 1$ in the step 1. From (21) and (22) we get $c_2 = 12$ and $n = 86$. To compute the bounds for the specified lot sizes for which (86, 1, 12) is the optimal plan we shall use (28).
The necessary quantities to compute N_{c_1} for $c_1 = 1$ are

$$\Delta B_{c_1}(p') = 0.093020,\ \Delta B_{c_1}(p'') = -0.000010,\ \Delta B_{c_1}(c_2, p') = 0,$$

$$\Delta B_{c_1}(c_2, p'') = 0.096490,\ U = 1.544128,\ B_{c_1+1}(p') = 0.880360$$

$$B_{c_1+1}(c_2, p') = 1,\ B_{c_1+1}(p'') = 0,\ B_{c_1+1}(c_2, p'') = 0.562450.$$

Substituting these values in (28) we get

$$N_{c_1}' = 86 + (78.351966) \times 34/1.544128 = 1811.224103.$$

Similarly $N_{c_1-1} = 847$.

Thus for any given lot size N in the range $847 \leqslant N \leqslant 1811$ the optimal plan is given by (86, 1, 12).

Optimal single sampling plans with three-decision criteria for the same set of values of the cost parameters as in the above example and $p_1 = 0.05$, $p_2 = 0.20$, $\beta_1 = 0.07$, $\beta_2 = 0.10$ and double binomial prior distribution with $p' = 0.01$ and $p'' = 0.15$ with $w_1 = 0.93$ and $w_2 = 0.07$ for c_1 taking values 0 to 6 have been provided in Table 1.

Average cost of acceptance without inspection per item is 14.26 money units. The percentage saving increases with the lost size and for $c_1 = 6$ it is as high as 41 percent.

7. Double binomial unrestricted Bayes solution

To determine (n, c_1, c_2) minimising $R(N, n, c_1, c_2)$ given by (20) is the problem of unrestricted Bayesian solution. These plans are unrestricted in the sense that they are not required to satisfy any restriction on their operating characteristic function i.e. there is no limitation in terms of risks. The plans providing minimum R is to be used if the minimum R is less than the costs of accepting or rejecting all lots without inspection. As mentioned earlier a necessary condition for such a sampling plan to exist is that $\nu_{11} > 0$, $\nu_{12} > 0$ $\nu_{21} > 0$ and $\nu_{22} > 0$ i.e. $p' < p_u < p_v < p''$.

According to Hald (1967a) the value of (n, c_1, c_2) minimising R must satisfy the following three conditions

$$\Delta c_1 R(N, n, c_1-1, c_2) \leqslant 0 < \Delta c_1 R(N, n, c_1, c_2) \qquad 0 \leqslant c_1, c_2 \leqslant n, c_1 < c_2 \quad \dots \quad (37)$$

$$\Delta c_2 R(N, n, c_1, c_2-1) \leqslant 0 < \Delta c_2 R(N, n, c_1, c_2) \qquad 0 \leqslant c_1, c_2 \leqslant n, c_1 < c_2 \dots \quad (38)$$

$$\Delta_n R(N, n-1, c_1, c_2) \leqslant 0 < \Delta_n R(N, n, c_1, c_2) \qquad c_1 < c_2 \quad \leqslant n \leqslant N \quad \dots \quad (39)$$

Since $\Delta_{c_1} B(c_1, n, p') = b(c_1+1, n, p')$ and $\Delta_n B(c_1, n, p') = -p'b(c_1, n, p')$ we get from (20)

$$\Delta_{c_1} R(N, n, c_1, c_2) = (N-n)(-\nu_{11} b(c_1+1, n, p')+\nu_{12} b(c_1+1, n, p'')) \qquad \dots \quad (40)$$

$$\Delta_{c_2} R(N, n, c_1, c_2) = (N-n)(-\nu_{21} b(c_2+1, n, p')+\nu_{22} b(c_2+1, n, p'')) \qquad \dots \quad (41)$$

$$\begin{aligned}\Delta_n R(N, n, c_1, c_2) = 1-(\nu_{11}(1-B(c_1, n, p'))+\nu_{12} B(c_1, n, p'')+\nu_{21}(1-B(c_2, n, p'))\\ +\nu_{22} B(c_2, n, p''))+(N-n-1)(\nu_{11} p' b(c_1, n, p')-\nu_{12} p''(c_1, n, p'')\\ +\nu_{21} p' b(c_2, n, p')-\nu_{22} p'' b(c_2, n\, p'')) \qquad \dots \quad (42)\end{aligned}$$

Solving the inequalities with respect to n and N we find that a Bayesian sampling plan must satisfy the following three inequalities given by (43), (44) and (45) below :

$$\alpha_1+\beta c_1 \leqslant n < \alpha_1+\beta(c_1+1) \qquad \dots \quad (43)$$

$$\alpha_2+\beta c_2 \leqslant n < \alpha_2+\beta(c_2+1) \qquad \dots \quad (44)$$

where $c_1 < c_2$

$$\alpha_i = \log \frac{\nu_{i2}}{\nu_{i1}} \Big/ \log \frac{q'}{q''} \qquad i = 1, 2, \qquad \dots \quad (45)$$

$$\beta = \log \frac{p''q'}{p'q''} \Big/ \log \frac{q'}{q''} \qquad \dots \quad (46)$$

and $$F(n-1, c_1, c_2) \leqslant N < F(n, c_1, c_2) \qquad \dots \quad (47)$$

where $F(n, c_1, c_2)$ is defined by

$$F(n, c_1, c_2) = (n+1) + \frac{1-\nu_{11}+\nu_{11}B(c_1,n,p')-\nu_{12}B(c_1,n,p'')-\nu_{21}+\nu_{21}B(c_2,n,p')-\nu_{22}B(c_2,n,p'')}{-\nu_{11}p'b(c_1,n,p')+\nu_{12}p''b(c_1,n,p'')-\nu_{21}p'b(c_2,n,p')+\nu_{22}p''b(c_2,n,p'')} \qquad \dots \quad (48)$$

For two plans (n_1, c_1', c_2') and (n_2, c_1'', c_2'') satisfying (43) and (44) and having overlapping N-intervals according to (47) then solving $R(N, n_1, c_1', c_2') = R(N, n_2, c_1'', c_2'')$ for N we get expression for N_{12} as given in (30) and using (32) we can select one of the two plans (n_1, c_1', c_2') and (n_2, c_1'', c_2'') as an optimal plan.

To discuss the uniqueness of the above solution write $R(N, n, c_1, c_2)$ as follows :

$$R(N, n, c_1, c_2) = \begin{cases} N & 0 \leqslant N \leqslant n \\ n+(N-n)\, D(n, c_1, c_2) & n < N \end{cases} \qquad \dots \quad (49)$$

where $D(n, c_1, c_2)$ denotes the expected standardised decision loss and is given as

$$D(n, c_1, c_2) = \nu_{11}(1-B(c_1, n, p'))+\nu_{12}B(c_1, n, p'')+\nu_{21}(1-B(c_2, n, p')) + \nu_{22}B(c_2, n, p'') \qquad \dots \quad (50)$$

The graph of $R(N, n, c_1, c_2)$ as given by (49) is, for a fixed (n, c_1, c_2) a polygonal line consisting of two segments with slopes 1 and $D(n, c_1, c_2)$. Consider the nature of the infimum R function i.e.

$$R_0(N) = \inf_{\psi = \{n, c_1, c_2\}} R(N, n, c_1, c_2) \qquad \dots \quad (51)$$

where ψ is the set of all plans (n, c_1, c_2), $0 \leqslant c_1 < c_2 \leqslant n$.
Define a set of plans $S^* \subset \psi$ as

$$S^* = \{(n, c_1, c_2);\; D(n, c_1, c_2) < 1\}. \qquad \dots \quad (52)$$

From (49) we note that $R(N, n, c_1, c_2)$ is a concave function of N for $(n, c_1, c_2) \epsilon S^*$ and

$$R(N, n, c_1, c_2) \geqslant N \text{ for } (n, c_1, c_2)\epsilon\psi - S^*. \qquad \dots \quad (53)$$

It follows that the minimum in (51) may be attained for $(n, c_1, c_2)\epsilon S^*$ only. In fact for $(n, c_1, c_2)\epsilon\psi - S^*$ the minimum value of $R(N, n, c_1, c_2)$ is N and the optimal procedure is as good as inspecting all the items (i.e. $n = N$). We shall assume that $n < N$. Thus to discuss the optimal plans we need consider only the plans $(n, c_1, c_2)\epsilon S^*$. Arguing as Hald (1967a), the inequalities (43) and (44) show that for each n there exists one and only one minimum R with respect to c_1 and c_2 as it can be easily seen that $1 < \frac{1}{p''} < \beta < \frac{1}{p'}$ and the optimal value of c_1 and c_2 are given by $c_i = [(n-\alpha_i)/\beta]$ for $i = 1$ and $i = 2$ respectively. Thus to find the global or absolute minimum of R with respect to c_1, c_2 and n we need, therefore, consider the values of $(n, c_1, c_2)\epsilon S^*$ and satisfying (43) and (44).

For each c_1 and c_2 the inequality (47) defines N uniquely as a function of n if $F(n, c_1, c_2)$ as defined in (48), is an increasing function of n. From (48) write $F(n, c_1, c_2)$ as

$$F(n, c_1, c_2) = n+1+\frac{1-D(n, c_1, c_2)}{-\Delta D(n, c_1, c_2)} \quad \dots \quad (54)$$

Since $D(n, c_1, c_2) < 1$ for all plans in S^* and observation shows that $D(n, c_1, c_2)$ is non-increasing function of n with increasing slopes (see fig. 6), we have

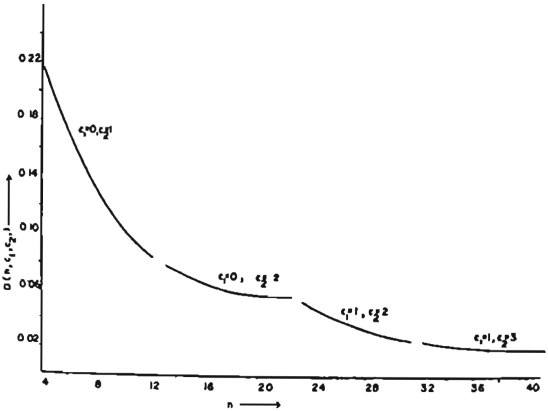


Fig. 6. The expected decision loss (standardised) as function of the Sample Size for double binomial prior distribution with $p' = 0.01$, $p'' = 0.15$ and $w_1 = 0.93$, $w_2 = 1-w_1$.

$\Delta D(n, c_1, c_2) < 0$ and $F(n, c_1, c_2) > n+1$ for all n satisfying (43) and (44). To show that $F(n, c_1, c_2)$ is an increasing function of n we need show $\Delta_n F(n-1, c_1, c_2) \geqslant 0$.

Note that, taking all differences with respect to n

$$\Delta F(n-1, c_1, c_2)\Delta D(n, c_1, c_2)\Delta D(n-1, c_1, c_2)$$

$$= 2\Delta D(n, c_1, c_2)\Delta D(n-1, c_1, c_2)+(1-D(n, c_1, c_2))\Delta^2 D(n-1, c_1, c_2) \quad \ldots \quad (55)$$

and since $\Delta D(n, c_1, c_2) < 0$ and $\Delta D(n-1, c_1, c_2) < 0$ (fig. 6) the first term on the right hand side of (55) is positive and it is sufficient to show that $\Delta^2 D(n-1, c_1, c_2) \geqslant 0$. Since

$$\Delta^2 D(n-1, c_1, c_2) = \Delta D(n, c_1, c_2)-\Delta D(n-1, c_1, c_2) \quad \ldots \quad (56)$$

and $\Delta D(n, c_1, c_2) < 0, \Delta D(n-1, c_1, c_2) < 0$ and $|\Delta D(n, c_1, c_2)| < |\Delta D(n-1, c_1, c_2)|$ for the values of n considered (see figure 6)

$$\Delta^2 D(n-1, c_1, c_2) \geqslant 0 \text{ and hence } \Delta_n F(n-1), c_1, c_2) \geqslant 0.$$

The regret function R for a given value of c_1(and c_2), and n satisfying (43) and (44) would be according to (49) a non-decreasing piecewise linear function of N with decreasing non-negative slope. The proof to uniqueness of the minimum is complete if we further show that $R(N, n_1, c_1, c_2)$ and $R(N, n_2, c_1+1, c_2')$ have only one point in common, and that this point increases with c_1(and c_2). If the slopes of the two regret functions are equal i.e. $D(n_1, c_1, c_2) = D(n_2, c_1+1, c_2')$, they will intersect at only one point and that is infinity because the numerator in (30) is finite. Suppose one of the intersection point is $N(c_1, c_1+1)$ given by

$$N(c_1, c_1+1) = \frac{(n_2-n_1)+n_1 D(n_1, c_1, c_2)-n_2 D(n_2, c_1+1, c_2')}{D(n_1, c_1, c_2)-D(n_2, c_1+1, c_2')} \quad \ldots \quad (57)$$

and $D(n_1, c_1, c_2) \neq D(n_2, c_1+1, c_2')$. Since $D(n, c_1, c_2)$ is non-increasing function of n with decreasing slope at $N(c_1, c_1+1)$ we have

$$D(n_1, c_1, c_2) > D(n_2, c_1+1, c_2'). \quad \ldots \quad (58)$$

Suppose the plans (n_1, c_1, c_2) and (n_2, c_1+1, c_2') intersect at another point $N'(c_1, c_1+1)$ also where $N'(c_1, c_1+1) > N(c_1, c_1+1)$. At $N'(c_1, c_1+1)$ we must have $D(n_1, c_1, c_2) < D(n_2, c_1+1, c_2')$ which contradicts (58). Hence the two plans can intersect at one point only. The point of intersection can be easily shown to be an increasing function of c_1 (and c_2) as in Section 6. This completes the proof to the uniqueness of the minimum.

The plans discussed in this section can be tabulated as follows :

Step 1 : Compute v_{ij}, $i, j = 1, 2$ from the expression given in (20) and hence compute α_1, α_2 from (45) and β from (46).

Step 2 : Choose some arbitrary c_1 (say $c_1 = 0$) and using the value of α_1, α_2 and β as obtained in the step 1 compute the lower limit $l(c_1)$ and the upper limit $u(c_1)$ for n from (43) such that $l(c_1) < n < u(c_1)$.

Step 3 : Take c_2 as c_1+1 and compute the lower limit $l(c_2)$ and the upper limit $u(c_2)$ for n from (44) such that $l(c_2) < n < u(c_2)$.

Step 4 : In case $l(c_2) \leqslant u(c_1)$ choose c_2 as c_1+1 and the values of n satisfying both (43) and (44) would lie in the closed interval $l(c_1, c_2) < n < u(c_1, c_2)$ where $l(c_1, c_2) = \max\ \{l(c_1), l(c_2)\}$ and $u(c_1, c_2) = \min\ \{u(c_1), u(c_2)\}$. In case $l(c_2) > u(c_1)$ increase c_2 systematically till we get $l(c_2) < u(c_1)$.

Step 5 : Choose $c_1 = 0, 1, 2, \ldots$ systematically and proceed as in steps 2-4 and list the values of n satisfying (43) and (44) corresponding to each pair of c_1 and c_2.

Step 6 : For the plans listed in the step 5 using (47) compute the bounds (N-intervals) of the specified lot sizes for which these plans are optimal.

Step 7 : For two plans (n_1, c_1', c_2') and (n_2, c_1'', c_2'') having overlapping N-intervals select one of these plans as an optimal plan by using (30)—(32).

Example 2. Using the costs as given in the example 1 of section 6 and assuming that the lots are generated with probability $w_1 = 0.93$ from a binomially controlled process with $p' = 0.01$ and with probability $w_2 = 0.07$ from the process with $p'' = 0.15$ the steps 1-7 gave the values of c_1, c_2, n and N for the optimal plans as given in table 2.

where $\alpha_1 = +3.696568$, $\alpha_2 = -24.850612$ and $\beta = 18.761415$.

The detailed calculations along with intermediate tables are given in Pandey (1974). The values of δ_1 and δ_2 corresponding to each combination of c_1, c_2, n and the N-interval are also given in the table 2.

The values of cost per lot of the size $\bar{N}$ (the geometric mean of the limits of the lot size range) divided by the value of the geometric mean have been computed and is given in Table 2. The percentage saving effected by the use of the plan with the three decision criteria as compared with the average cost involved in accepting an item without inspection is found to be increasing for the increasing lot sizes. It can be seen from Table 2 that the above percentage saving in cost is at least 40 percent for the lot of size 2115 and above. The computational results given in Table 2 show numerically that the optimal

TABLE 2. OPTIMAL SAMPLING PLANS WITH THREE DECISION FOR $w_1 = .93$ AND $w_3 = .07$

N		n	c_1	c_3	$\overline{N}$	$K/\overline{N}$	(%) saving	100 $P_a(p')$	100 $P_d(p')$	100 $P_r(p'')$	100 $P_d(p'')$
27–	32	4	0	1	29	13.20	7.43	96.08	3.88	10.95	36.85
33–	39	5	0	1	35	12.88	9.68	95.10	4.80	16.48	39.15
40–	48	6	0	1	43	12.52	12.20	94.15	5.71	22.35	39.93
49–	58	7	0	1	53	12.15	14.80	93.21	6.59	28.34	39.60
59–	71	8	0	1	64	11.83	17.04	92.27	7.46	34.28	38.47
72–	86	9	0	1	78	11.50	19.35	91.35	8.30	40.05	36.79
87–	105	10	0	1	95	11.20	21.46	90.44	9.14	45.57	34.74
106–	129	11	0	1	116	10.92	23.42	89.53	9.95	50.78	32.48
130–	155	12	0	1	141	10.68	25.10	88.64	10.74	55.65	30.12
156–	179	13	0	2	167	10.48	26.50	87.75	12.22	30.80	57.11
180–	220	14	0	2	198	10.30	27.77	86.87	13.09	35.21	54.51
221–	272	15	0	2	245	10.10	29.17	86.01	13.95	39.58	51.69
273–	342	16	0	2	305	9.92	30.43	85.15	14.80	43.86	48.71
343–	366	17	0	2	354	9.81	31.20	84.29	15.64	48.02	45.60
367–	369	28	1	2	367	9.79	31.35	96.82	2.91	81.29	12.43
370–	371	29	1	2	370	9.78	31.42	96.60	3.10	83.10	11.35
372–	381	30	1	2	376	9.77	31.49	96.39	3.28	84.86	10.34
382–	607	31	1	2	481	9.51	33.31	96.16	3.47	86.41	9.39
608–	636	32	1	3	621	9.29	34.85	95.93	4.04	72.79	23.54

TABLE 2 (*contd.*). OPTIMAL SAMPLING PLANS WITH THREE DECISION FOR $w_1 = .03$ AND $w_2 = .07$

N		n	c_1	c_2	$\overline{N}$	$K/\overline{N}$	(%) saving	100 $P_a(p')$	100 $P_d(p')$	100 $P_r(p'')$	100 $P_d(p'')$
637–	793	33	1	3	710	9.10	35.55	95.70	4.27	75.05	21.76
794–	956	34	1	3	871	9.00	36.46	95.46	4.51	77.15	20.06
957–	1155	35	1	3	1051	8.95	37.23	95.21	4.74	79.12	18.45
1156–	1163	47	2	3	1130	8.91	37.51	98.83	1.04	93.64	4.29
1164–	1165	48	2	3	1164	8.90	37.59	98.76	1.10	94.28	3.89
1166–	1196	49	2	3	1180	8.90	37.59	98.69	1.16	94.87	3.52
1197–	1778	50	2	3	1458	8.78	38.43	98.62	1.22	95.40	3.19
1779–	1883	51	2	4	1830	8.67	39.20	98.54	1.44	89.78	8.97
1884–	2376	52	2	4	2115	8.61	39.62	98.46	1.52	90.69	8.21
2377–	2859	53	2	4	2606	8.54	40.11	98.38	1.60	91.54	7.50
2860–	3281	54	2	4	3063	8.49	40.48	98.30	1.68	92.31	6.84
3282–	3297	66	3	4	3289	8.47	40.60	99.56	0.39	97.74	1.53
3298–	3361	67	3	4	3329	8.47	40.60	99.54	0.41	97.97	1.39
3362–	4977	68	3	4	4090	8.41	41.02	99.51	0.43	98.18	1.25
4978–	5679	70	3	5	5316	8.35	41.44	99.46	0.54	96.16	3.39
5680–	6744	71	3	5	6189	8.32	41.65	99.43	0.56	96.52	3.09
6745–	8129	72	3	5	7404	8.29	41.80	99.40	0.59	96.84	2.81
8130–	8927	73	3	5	8510	8.27	42.00	99.37	0.62	97.14	2.55
8928–	8002	85	4	5	8944	8.27	42.00	99.83	0.15	99.17	0.56

TABLE 2 (contd.). OPTIMAL SAMPLING PLANS WITH THREE DECISION FOR $w_1 = .93$ AND $w_2 = .07$

N	n	c_1	c_2	$\overline{N}$	$K/\overline{N}$	(%) saving	100 $P_a(p')$	100 $P_d(p')$	100 $P_r(p'')$	100 $P_d(p'')$
8963– 9151	86	4	5	9056	8.26	42.07	99.82	0.15	99.26	0.51
9152– 13583	87	4	5	11149	8.24	42.21	99.81	0.16	99.33	0.46
13584– 15687	89	4	6	14597	8.21	42.43	99.79	0.20	98.55	1.29
15688– 18666	90	4	6	17112	8.20	42.50	99.70	0.21	98.69	1.17
18667– 22564	91	4	6	20523	8.18	42.64	99.77	0.22	98.81	1.06
22565– 23699	92	4	6	23125	8.18	42.64	99.76	0.24	98.92	9.96
23700– 23794	104	4	6	23746	8.17	42.70	99.93	0.06	99.69	0.21
23795– 24378	105	5	6	24084	8.17	42.70	99.70	0.06	99.72	0.19
24379– 34571	106	5	6	29031	8.17	42.70	99.93	0.06	99.75	0.17
34572– 35176	107	5	7	34872	8.16	42.78	99.92	0.08	99.39	0.54
35177– 42662	108	5	7	38739	8.15	42.85	99.92	0.08	99.45	0.49
42663– 50891	109	5	7	46595	8.14	42.92	99.91	0.08	99.50	0.45
50892– 61633	110	5	7	56005	8.14	42.92	99.98	0.09	99.55	0.40
61634– 61638	122	6	7	61635	8.13	42.99	99.97	0.02	99.87	0.09
61639– 91016	125	6	7	74900	8.13	42.99	99.97	0.02	99.91	0.06
91017– 97247	126	6	8	91629	8.12	43.06	99.97	0.03	99.77	0.21
97248– 114782	127	6	8	102800	8.12	43.06	99.97	0.03	99.79	0.19
114783– 137310	128	6	8	125542	8.11	43.13	99.97	0.03	99.81	0.17
137311– 160113	129	6	8	148274	8.11	43.13	99.99	0.01	99.83	0.15

TABLE 2 (*contd.*). OPTIMAL SAMPLING PLANS WITH THREE DECISION FOR $w_1 = .93$ AND $w_2 = .07$

N	n	c_1	c_2	$\bar{N}$	$K/\bar{N}$	(%) saving	100 $P_a(p')$	100 $P_r(p')$	100 $P_r(p'')$	100 $P_a(p'')$
160114– 160115	141	7	8	160114	8.11	43.13	99.99	0.01	99.95	0.03
160116– 160117	142	7	8	160116	8.11	43.13	99.99	0.01	99.96	0.03
160118– 164674	143	7	8	162380	8.11	43.13	99.99	0.01	99.96	0.03
164675– 166798	144	7	8	165633	8.11	43.13	99.99	0.01	99.96	0.02
166799– 222350	145	7	9	192581	8.11	43.13	99.99	0.01	99.91	0.08
222351– 306371	146	7	9	261001	8.10	43.20	99.99	0.01	99.92	0.07
306372– 367664	147	7	9	335621	8.10	43.20	99.99	0.01	99.93	0.07
367665– 400133	148	7	9	383555	8.10	43.20	99.99	0.01	99.93	0.06
400134– 400135	160	8	9	400134	8.10	43.20	100.00	0.00	99.98	0.01
400136– 406386	161	8	9	403248	8.10	43.20	100.00	0.00	99.98	0.01
406387– 668222	162	8	9	521111	8.10	43.20	100.00	0.00	99.98	0.01
668223– 685201	164	8	10	676658	8.10	43.20	100.00	0.00	99.97	0.03
685202– 812787	165	8	10	746272	8.10	43.20	100.00	0.00	99.97	0.03
812788– 978829	166	8	10	891953	8.10	43.20	99.99	0.01	99.97	0.03
978830–1000153	167	8	10	989434	8.10	43.20	99.99	0.01	99.97	0.02
1000154–1000169	179	9	10	1000161	8.10	43.20	100.00	0.00	99.99	0.00
1000170–1535285	182	9	11	1239171	8.10	43.20	100.00	0.00	99.98	0.01
1535286–1801106	183	9	11	1664277	8.10	43.20	100.00	0.00	99.99	0.01
1801107–2146163	184	9	11	1967716	8.10	43.20	100.00	0.00	99.99	0.01
2146164–2594620	185	9	11	2359702	8.10	43.20	100.00	0.00	99.99	0.01
2594621–3000168	186	9	11	2700035	8.10	43.20	100.00	0.00	99.99	0.01
3000169–3000172	198	10	11	3000170	8.10	43.20	100.00	0.00	100.00	0.00
3000173–3250170	199	10	11	3122070	8.09	43.27	100.00	0.00	100.00	0.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

sample size increases with the lot size and the optimal decision loss per non-sampled item decreases with the sample size (Fig. 6). These two results were observed by Thyregod (1972) also for plans with two decision criteria.

The probabilities of acceptance and the probabilities of screening are computed for the lot quality $p' = .01$ and the probabilities of rejection and screening are computed for the lot quality $p'' = .15$ for each of the optimal plans in Table 2. The probability of acceptance at p' increases with the optimal sample size whereas the probability of screening decreases with the increasing lot sizes as it should be. The probability of rejecting a lot of quality $p'' = .15$ out right increases with the increasing optimal sample size whereas the probability of screening a lot of this quality increases in the beginning (upto $\bar{N} = 141$) and then decreases for higher lot sizes.

The identical values of $K/\bar{N}$ in Table 2 tempt to work out a system of nearly optimum plans containing only one plan for each pair of acceptance numbers c_1 and c_2. Such plans are provided in Table 3. The procedure of simplification used to obtain the plans in the table 3 was to compute n as $(l(c_1, c_2)+u(c_1, c_2))/2$ where $l(c_1, c_2)$ and $u(c_1, c_2)$ are the values as obtained in the step 4 such that the closed interval $l(c_1, c_2) \leqslant n \leqslant u(c_1, c_2)$gives the possible optimal values of sample sizes corresponding to a particular pair c_1 and c_2 The N-intervals were computed by using (30). A comparison between Tables 2 and 3 shows that the system in table 3 provides considerable simplification of the optimal system and the nearly optimum plans for most practical purposes is just as satisfactory as the optimal plans.

Though the figures 1 and 2 in the section 4 clearly brings out the definite advantages of a plan with the three-decision criteria over the one with two-decision criteria in certain cases yet it may be interesting to attempt a numerical assessment of the advantages for an illustration. For this purpose in example 3 we shall consider the situation similar to one in the example 1 but with a plan with two-decision criteria (acceptance/screening). We shall work out average total cost of inspection and decision per item for such a plan with two-decision criteria and will attempt an comparison between these values and the corresponding values for a plan with three-decision criteria as obtained in Table 2.

Example 3 : Let us assume that the prior distribution of the lot quality and the three cost functions viz. cost of inspection, cost of acceptance and the cost of screening be the same as in the example 1 and the following sampling procedure be used.

TABLE 3. NEARLY OPTIMAL PLAN WITH THREE-DECISION CRITERIA FOR $w_1 = .93$ AND $w_2 = .07$

N	n	c_1	c_2	$\overline{N}$	$K/\overline{N}$	(%) saving	$100 \times P_a(p')$	$100 \times P_s(p')$	$100 \times P_r(p')$	$100 \times P_d(p')$
1– 146	8	0	1	12	19.17	*	92.27	7.46	34.28	38.47
147– 363	18	0	2	230	10.22	28.33	83.45	16.48	52.03	42.60
364– 609	27	1	2	470	9.54	33.10	97.03	2.73	79.26	13.58
610– 1134	37	1	3	831	9.10	36.18	94.71	5.24	82.64	15.52
1135– 1782	46	2	3	1422	8.79	38.25	98.90	0.99	92.93	4.73
1783– 3331	55	2	4	2437	8.56	39.97	98.22	1.76	93.02	6.23
3332– 4721	64	3	4	3966	8.42	40.95	99.61	0.35	97.20	1.87
4722– 8950	74	3	5	6500	8.32	41.65	99.34	0.65	97.41	2.32
8951– 13140	83	4	5	10845	8.24	42.21	99.85	0.13	98.98	0.69
13141– 23514	93	4	6	17578	8.19	42.56	99.75	0.25	99.03	0.87
23515– 35801	102	5	6	29014	8.16	42.78	99.94	0.05	99.62	0.20
35802– 60498	112	5	7	46539	8.14	42.92	99.90	0.10	99.63	0.33
60499– 97192	121	6	7	76681	8.12	43.06	99.98	0.02	99.80	0.10
97193– 117769	131	6	8	106987	8.12	43.06	99.96	0.04	99.86	0.13
117770– 409222	141	7	8	210531	8.11	43.13	99.99	0.01	99.95	0.03
409223– 555699	158	8	9	470860	8.10	43.20	100.00	0.00	99.98	0.02
555700–1125149	168	8	10	790724	8.10	43.20	99.99	0.01	99.98	0.02
1125150–1428735	177	9	10	1207888	8.10	43.10	100.00	0.00	99.99	0.01
1428736–3000168	187	9	11	2070373	8.10	43.20	100.00	0.00	99.99	0.01
3000169–5000175	198	10	11	3873160	8.10	43.20	100.00	0.00	100.00	0.00

*Acceptance without inspection is more economic.

Take a sample of size n from a lot of N items and let x denote the number of defectives observed in n then

$$\text{Accept the lot if } x \leqslant c$$
$$\text{Screen the lot if } x > c. \qquad \ldots \quad (59)$$

The cost function $K(N, n, c, p)$ can be written as

$$K(N, n, c, p) = n(S_1+S_2p)+(N-n)((A_1+A_2p)P_a(p)+(T_1+T_2p)P_s(p)) \qquad \ldots \quad (60)$$

where the symbols have the usual meaning and $P_a(p) = B(c, n, p)$ and $P_s(p) = 1-P_a(p)$ denote the probability of acceptance and the probability of screening respectively. The unavoidable minimum cost $k_m(p)$ would be defined as

$$k_m(p) = k_a(p')w_1+k_t(p'')w_2 \qquad \ldots \quad (61)$$

and the standardised cost function is

$$R(N, n, c) = n+(N-n)(\nu_1(1-P_a(p'))+\nu_2P_a(p'')) \qquad \ldots \quad (62)$$

where
$$\nu_1 = w_1(k_t(p')-k_a(p'))/(k_s-k_m)$$
$$\nu_2 = w_2(k_a(p'')-k_t(p''))/(k_s-k_m).$$

It is required to obtain (n, c) which minimises (62). The necessary expressions to determine the optimum plans are

$$\alpha+\beta c \leqslant n < \alpha+\beta(c+1) \qquad \ldots \quad (63)$$

where
$$\alpha = \log\frac{\nu_2}{\nu_1}\Big/\log\frac{q'}{q''}$$
$$\beta = \log\frac{p''q'}{p'q''}\Big/\log\frac{q'}{q''}$$

and
$$F(n-1, c) \leqslant N < F(n, c) \qquad \ldots \quad (64)$$

where $F(n, c)$ is defined by

$$F(n, c) = n+1+\frac{1-\nu_1+\nu_1P_a(p')-\nu_2P_a(p'')}{-\nu_1p'b(c, n, p')+\nu_2p''b(c, n, p'')}$$

The points of intersection for the two plans (n', c') and (n'', c'') having overlapping N-intervals is given by

$$N_{12} = \frac{(n''-n')(1-\nu_1)+n''\delta(n'', c'')-n'\delta(n', c')}{\delta(n'', c'')-\delta(n', c')} \qquad \ldots \quad (65)$$

where $\delta(n, c) = \nu_1B(c, n, p')-\nu_2B(c, n, p'')$

For this example $\nu_1 = 0.217638$, $\nu_2 = 0.382606$, $\alpha = 3.696567$ and $\beta = 18.761415$. Using the expressions (60)-(62) and the values from Tables 5 and 6 in Pandey (1974) and proceeding as Hald (1960) we obtain the optimal plans with two-decision criteria as given in Table 4.

TABLE 4. OPTIMAL SAMPLING PLAN WITH TWO DECISION CRITERIA FOR $w_1 = .93$ AND $w_2 = .07$

N		n	c	$\bar{N}$	$K/\bar{N}$	(%) saving	$100\times P_a(p')$	$100\times P_r(p'')$
27–	33	4	0	29	13.21	7.36	96.06	47.80
34–	41	5	0	37	12.80	10.24	95.10	55.63
42–	50	6	0	45	12.46	12.62	94.15	62.29
51–	60	7	0	55	12.11	15.08	93.21	68.94
61–	73	8	0	66	11.81	17.18	92.27	72.75
74–	88	9	0	80	11.51	19.28	91.35	76.84
89–	107	10	0	97	11.22	21.32	90.44	80.31
108–	131	11	0	118	10.95	23.21	89.53	83.27
132–	160	12	0	145	10.69	25.03	88.64	85.78
161–	198	13	0	178	10.47	26.58	87.75	87.91
199–	247	14	0	221	10.26	28.05	86.87	89.72
248–	313	15	0	278	10.06	29.45	86.01	91.26
314–	406	16	0	357	9.88	30.71	85.15	92.57
407–		28	1	407	9.80	31.28	96.82	93.73
408–	411	29	1	409	9.79	31.35	96.60	94.51
412–	413	30	1	412	9.79	31.35	96.39	95.20
414–	424	31	1	418	9.78	31.42	96.16	95.80
425–	426	32	1	425	9.78	31.42	95.93	96.34
427–	442	33	1	434	9.77	31.49	95.70	96.80
443–		34	1	443	9.77	31.49	95.46	97.21
444–	463	35	1	453	9.76	31.56	95.21	97.57
464–	485	36	1	474	9.73	31.77	94.97	97.88
486–	487	37	1	486	9.73	31.77	94.71	98.16
488–	511	38	1	499	9.72	31.84	94.45	98.40
512–	1391	39	1	843	9.27	34.99	94.19	98.61
1392–	1454	54	2	1422	8.98	37.03	98.30	99.15
1455–	1422	55	2	1488	8.96	37.17	98.22	99.25

TABLE 4 (*contd.*). OPTIMAL SAMPLING PLAN WITH TWO DECISION CRITERIA FOR $w_1 = .93$ AND $w_2 = .07$

N	n	c	$\bar{N}$	K/N	(%) saving	100× $P_a(p')$	100× $P_r(p'')$
1523– 1524	56	2	1523	8.95	37.24	98.13	99.34
1525– 1602	57	2	1563	8.95	37.24	98.04	99.42
1603– 4128	58	2	2572	8.73	37.38	97.95	99.49
4129– 4324	74	3	4225	8.60	39.69	99.34	99.73
4325– 4534	75	3	4428	8.50	39.76	99.31	99.76
4535– 10795	76	3	6906	8.49	40.46	99.28	99.79
10796– 11254	91	4	11022	8.43	40.88	99.77	99.87
11255– 11749	92	4	11499	8.43	40.88	99.76	99.87
11750– 11782	93	4	11765	8.42	40.95	99.75	99.90
11783– 12361	94	4	12068	8.42	40.95	99.74	99.91
12362– 28655	95	4	18821	8.38	41.23	99.72	99.92
28656– 30502	110	5	29564	8.35	41.44	99.91	99.95
30503– 30938	111	5	30719	8.35	41.44	99.91	99.96
30939– 32517	113	5	31718	8.35	41.44	99.90	99.97
32518– 74730	114	5	49295	8.33	41.58	99.89	99.97
74731– 78152	129	6	76422	8.32	41.65	99.97	99.98
78153– 81833	130	6	79071	8.32	41.65	99.96	99.98
81834– 82229	131	6	82031	8.31	41.73	99.96	99.99
82230– 86228	132	6	84205	8.31	41.73	99.96	99.99
86229– 90579	133	6	88377	8.31	41.73	99.96	99.99
90580– 212623	134	6	138778	8.30	41.80	99.96	99.99
212624– 225122	151	7	218783	8.30	41.80	99.98	100.00
225123– 500142	152	7	335549	8.30	41.80	99.98	100.00
500143– 1250160	168	8	790733	8.29	41.86	99.99	100.00
1250161– 1307850	184	9	1278630	8.29	41.86	100.00	100.00
1307851– 3200180	187	9	2045814	8.29	41.86	100.00	100.00
3200181– 3400180	205	10	3298665	8.29	41.86	100.00	100.00
3400181– 3600180	206	10	3498751	8.29	41.86	100.00	100.00
3600181– 3800180	208	10	3698828	8.29	41.86	100.00	100.00
3800181– 8500200	209	10	5683511	8.29	41.86	100.00	100.00
8500201– 9000200	226	11	8746028	8.29	41.86	100.00	100.00
9000201–15000200	227	11	11610157	8.29	41.86	100.00	100.00
15000201–16000200	260	13	15492134	8.29	41.86	100.00	100.00
16000201–17000200	261	13	16492623	8.29	41.86	100.00	100.00

The figure 7 shows the average total cost of inspection plus the decision cost per item against the lot sizes, $\bar{N}$, for the plans with three decision and two (acceptance/screening) decision criteria (Tables 2 and 4) with $p' = 0.01$, $p'' = 0.15$ $w_1 = .93$ and $w_2 = .07$ as the double binomial prior distribution. The figure 7 clearly shows that the plan with three decision criteria is more economical than the plan with two decision criteria.

The advantages of the plans with three decision criteria as compared with the plans with two decision criteria will be more pronounced for higher values of w_2 i.e. when considerable number of lots with deteriorated quality levels are submitted for inspection the plans with the three decision criteria will have definitely substantial reduction in the values of $K/\bar{N}$ as compared with the plans with two decision criteria. To illustrate it, in example 4 we shall consider the plans with three decision criteria and the plans with two decision criteria for the same set of cost functions as in the example 2 and 3 but with $p' = 0.01$, $p'' = 0.15$ $w_1 = 0.10$ and $w_2 = 0.90$ as the double binomial prior distribution.

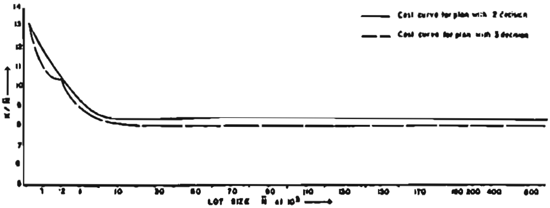


Fig. 7 : Cost curves for plans with two and three decision criteria for $w_1 = 0.93$, $w_2 = 0.07$

Example 4 : The necessary quantities to obtain the plans with three-decision and two-decision criteria (acceptance/screening) for the double binomial prior distribution with $p' = 0.01$, $p'' = 0.15$ and $w_1 = 0.10$ and $w_2 = 0.90$ and for the cost function $k_s(p)$, $k_a(p)$, $k_i(p)$ and $k_r(p)$ same as in examples 2 and 3 are

$k_s = 27.76$; $k_m = 18.72$

$\nu_{11} = 0.403762$; $\nu_{12} = 8.487278$; $\nu_{21} = 0.101217$; $\nu_{22} = 0.273783$

$\alpha_1 = 19.974527$; $\alpha_2 = 6.526389$; $\beta = 18.761415$ and $\alpha = \alpha_1$,

$\nu_1 = \nu_{11}$; $\nu_2 = \nu_{12}$

Using the above values of ν_{ij}'s, α_1, α_2 and β in (43), (44) and (30) we obtain optimum plans with three-decision criteria in Table 5 through Tables 8 and 9 in Pandey (1974). Similarly, using the above values of α, β, ν_1 and ν_2 in (63), (64) and (65) we obtain the optimum plans with two-decision (acceptance/screening) criteria in Table 6 through the Tables 11 and 12 in Pandey (1974). The figure 8 shows the average total cost of inspection plus the decision cost per item against the average lot sizes ($\bar{N}$) for the optimal plans both with three-decision (Table 5) and two-decision (Table 6) criteia. It is clear from the figure 8 that the plans with three-decision criteria is more economical than the

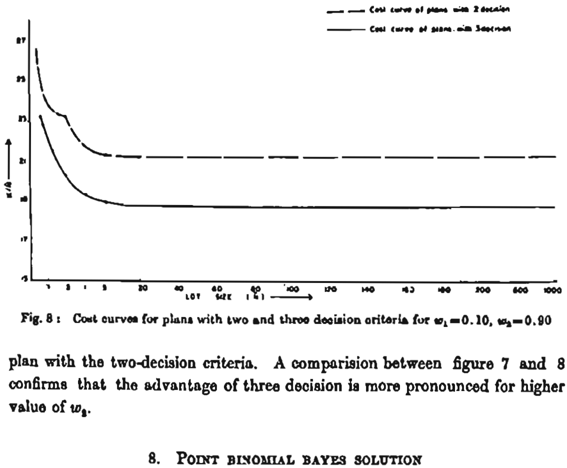


Fig. 8 : Cost curves for plans with two and three decision criteria for $w_1 = 0.10$, $w_2 = 0.90$

plan with the two-decision criteria. A comparision between figure 7 and 8 confirms that the advantage of three decision is more pronounced for higher value of w_2.

8. Point binomial Bayes solution

As a limiting case of the double binomial prior distribution assume that the prior distribution of the lot quality is point binomial with the parameter $\bar{p}$. The cost function, under the assumption 4 (Section 4), has its minimum value when the following decision rule is used

$$\begin{aligned} &\text{Accept all lots from processes with } \bar{p} < p_u \\ &\text{Screen all lots from processes with } \bar{p},\ p_u \leqslant \bar{p} < p_v \qquad \ldots \quad (66) \\ &\text{Reject all lots from processes with } \bar{p} \geqslant p_v \end{aligned}$$

TABLE 5. OPTIMAL SAMPLING PLANS WITH THREE DECISION CRITERIA FOR $w_1 = 0.10$ AND $w_2 = 0.90$

N		n	c_1	c_2	$\overline{N}$	$K/\overline{N}$	(%) saving	$100 \times P_a(p')$	$100 \times P_d(p')$	$100 \times P_r(p'')$	$100 \times P_d(p'')$
60–	68	26	0	1	63	23.29	76.21	77.00	20.22	91.83	6.71
69–	79	27	0	1	73	22.85	76.66	76.23	20.79	92.84	5.92
80–	93	28	0	1	86	22.39	77.13	75.47	21.36	93.73	5.22
94–	109	29	0	1	101	21.99	77.54	74.52	21.89	94.51	4.59
110–	130	30	0	1	119	21.62	77.92	73.97	22.42	95.20	4.04
131–	157	31	0	1	143	21.25	78.30	73.23	22.93	95.80	3.55
158–	174	32	0	1	165	21.00	78.55	72.50	23.43	96.34	3.11
175–	177	45	1	2	175	21.46	78.08	92.54	6.43	97.35	2.06
178		47	1	2	178	21.44	78.10	91.95	6.83	97.93	1.62
179–	182	48	1	2	180	21.41	78.13	91.66	7.10	98.17	1.44
183–	184	49	1	2	183	21.39	78.16	91.36	7.33	98.39	1.27
185–	190	50	1	2	187	21.36	78.19	91.06	7.56	98.58	1.13
191		51	1	2	191	21.34	78.21	90.75	7.79	98.75	1.00
192		52	1	2	192	21.36	78.19	90.44	8.02	98.90	0.88
193–	200	53	1	2	196	21.32	78.22	90.13	8.25	99.03	0.78
201		54	1	2	201	21.30	78.25	89.82	8.49	99.15	0.69
202–	210	55	1	2	205	21.28	78.28	89.50	8.72	99.25	0.61
211–	220	56	1	2	216	21.20	78.35	89.18	8.95	99.34	0.53
221–	524	57	1	2	340	20.36	79.21	88.86	9.18	99.42	0.47

TABLE 5 (contd.). OPTIMAL SAMPLING PLANS WITH THREE DECISION CRITERIA FOR $w_1 = 0.10$ AND $w_2 = 0.00$

N		n	c_1	c_2	$\bar{N}$	$K/\bar{N}$	(%) saving	$100\times P_a(p')$	$100\times P_s(p')$	$100\times P_r(p'')$	$100\times P_s(p'')$
525–	549	74	2	3	536	20.03	79.54	96.16	3.18	99.73	0.21
550–	1284	75	2	3	840	19.59	79.99	96.03	3.27	99.76	0.19
1285–	1346	91	3	4	1315	19.38	80.21	98.66	1.11	99.87	0.10
1347–	1411	02	3	4	1378	19.35	80.24	98.61	1.15	99.89	0.09
1412–	3159	93	3	4	2111	19.14	80.45	98.56	1.19	99.00	0.08
3160–	3302	108	4	5	3230	19.04	80.56	99.52	0.40	99.94	0.05
3303–	3452	109	4	5	3376	19.03	80.57	99.51	0.41	99.95	0.04
3453–	3455	111	4	5	3453	19.02	80.58	99.47	0.44	99.96	0.03
3456–	3628	112	4	5	3540	19.02	80.58	99.45	0.46	99.96	0.03
3629–	3812	113	4	5	3719	19.00	80.60	99.42	0.47	99.97	0.03
3813–	8851	130	5	6	5809	18.93	80.67	99.79	0.17	99.98	0.01
8852–	9313	131	5	6	9079	18.85	80.75	99.78	0.18	99.99	0.01
9314–	21596	147	6	7	14182	18.81	80.79	99.92	0.06	99.99	0.01
21597–	22636	148	6	7	22110	18.78	80.82	99.92	0.06	99.09	0.01
22637–	23772	149	6	7	23197	18.78	80.82	99.92	0.07	99.99	0.00
23773–	52585	164	7	8	33356	17.76	80.84	99.97	0.02	100.00	0.00
52586–	54745	165	7	8	53654	18.75	80.85	99.97	0.02	100.00	0.00
54746–	57184	166	7	8	55951	18.75	80.85	99.97	0.02	100.00	0.00
57185–	57571	167	7	8	57377	81.75	80.85	99.97	0.03	100.00	0.00

TABLE 5 (*contd.*). OPTIMAL SAMPLING PLANS WITH THREE DECISION CRITERIA FOR $w_1 = 0.10$ AND $w_2 = 0.90$

N	n	c_1	c_2	$\overline{N}$	$K/\overline{N}$	(%) saving	$100\times P_a(p')$	$100\times P_d(p')$	$100\times P_r(p'')$	$100\times P_s(p'')$
57572– 60541	168	7	8	59037	18.75	80.85	99.97	0.03	100.00	0.00
60542– 132000	183	8	9	89699	18.74	80.86	99.99	0.01	100.00	0.00
132001– 139287	184	8	9	136056	18.73	80.37	99.99	0.01	100.00	0.00
139288– 145455	185	8	9	142338	18.73	80.87	99.90	0.01	100.00	0.00
145456	186	8	9	145456	18.73	80.87	99.99	0.01	100.00	0.00
145457– 152699	187	8	9	149034	18.73	80.87	99.99	0.01	100.00	0.00
152700– 161173	188	8	9	156879	18.73	80.87	99.99	0.01	100.00	0.00
161174– 361878	205	9	10	241606	18.73	80.87	100.00	0.00	100.00	0.00
361879– 383154	206	9	10	372364	18.72	80.88	99.99	0.00	100.00	0.00
383155– 889083	222	10	11	583657	18.72	80.88	100.00	0.00	100.00	0.00
889084– 941638	223	10	11	916410	18.72	80.88	100.00	0.00	100.00	0.00
941639-1000193	224	10	11	972019	18.72	80.88	100.00	0.00	100.00	0.00

*Cost of acceptance without inspection = 97.02 money units.

TABLE 6. OPTIMAL SAMPLING PLANS WITH TWO-DECISION CRITERIA FOR $w_1 = .10$ AND $w_2 = .90$

N		n	c	$\bar{N}$	$K/\bar{N}$	(%) saving	100× $P_a(p')$	100× $P_r(p'')$
30–	33	20	0	31	26.51	72.92	81.79	96.12
34–	38	21	0	35	26.17	73.27	80.97	96.71
39–	44	22	0	41	25.75	73.70	80.16	97.20
45–	50	23	0	47	25.38	74.08	79.36	97.62
51–	57	24	0	53	25.06	74.41	78.57	97.98
58–	66	25	0	61	24.71	74.76	77.78	98.28
67–	77	26	0	71	24.36	75.12	77.00	98.54
78–	90	27	0	83	24.03	75.46	76.23	98.76
91–	107	28	0	98	23.71	75.78	75.47	98.94
108–	127	29	0	117	23.41	76.09	74.72	99.10
128–	154	30	0	140	23.14	76.37	73.97	99.24
155–	187	31	0	170	22.88	76.63	73.23	99.35
188–	189	46	1	188	23.12	76.39	92.25	99.48
190–	193	47	1	191	23.09	76.42	91.95	99.55
194–		48	1	194	23.07	76.44	91.66	99.61
195–	201	49	1	197	23.04	76.47	91.36	99.66
202–		50	1	202	23.02	76.49	91.06	99.71
203–		51	1	203	23.02	76.49	90.75	99.75
204–	212	52	1	207	22.99	76.52	90.44	99.78
213–		53	1	213	22.97	76.54	90.13	99.81
214–	222	54	1	217	22.95	76.56	89.82	99.84
223–	510	55	1	337	22.39	77.13	89.50	99.86
511–	531	70	2	520	22.16	77.36	96.67	99.90
532–	554	71	2	542	22.12	77.41	96.54	99.91
555–		72	2	555	22.11	77.42	96.42	99.92
556–	582	73	2	568	22.09	77.44	96.29	99.93
583–	609	74	2	595	22.06	77.47	96.16	99.94
610–	1429	75	2	933	21.77	77.77	96.03	99.95
1430–	1500	92	3	1464	21.63	77.91	98.61	99.97
1501–	1575	93	3	1537	21.61	77.93	98.56	90.98

TABLE 6 (*contd.*). OPTIMAL SAMPLING PLANS WITH TWO-DECISION CRITERIA FOR $w_1 = .10$ AND $w_2 = .90$

N		n	c	$\bar{N}$	$K/\bar{N}$	(%) saving	$100\times P_a(p')$	$100\times P_r(p'')$
1576–	3599	94	3	2404	21.47	78.07	99.50	99.98
3600–	3858	111	4	3762	21.40	78.14	99.47	99.99
3859–	8985	112	4	5888	21.33	78.22	99.45	99.99
8986–	9420	128	5	9200	21.29	78.26	99.81	99.99
9421–	9888	129	5	9651	21.29	78.26	99.80	100.00
9889–	22048	130	5	14765	21.26	78.29	99.79	100.00
22049–	23008	145	6	22523	21.24	78.31	99.93	100.00
23009–	24062	146	6	23529	21.24	78.31	99.93	100.00
24063–		147	6	24063	21.24	78.31	99.92	100.00
24064–	25294	148	6	24671	21.24	78.31	99.92	100.00
24295–	58531	149	6	38477	21.22	78.33	99.93	100.00
58532–	61069	165	7	59787	21.21	78.34	99.97	100.00
61070–	63966	166	7	62501	21.21	78.34	99.97	100.00
63967–	64423	167	7	64104	21.21	78.34	99.97	100.00
64424–	67753	168	7	66067	21.21	78.34	99.97	100.00
67754–	154703	169	7	102380	21.20	78.35	99.97	100.00
154604–	162319	186	8	158465	21.20	78.35	99.99	100.00
162320–	372269	187	8	245818	21.20	78.35	99.99	100.00
372270–	395524	203	9	383720	21.20	78.35	100.00	100.00
395525–	409266	204	9	402336	21.20	78.35	100.00	100.00
409267–	937603	205	9	61489	21.20	78.35	100.00	100.00
937604–	941370	220	10	939630	21.20	78.35	100.00	100.00
941371–	1000190	221	10	970334	21.20	78.35	100.00	100.00
1000191–		222	10	1000191	21.20	78.35	100.00	100.00
1000192–	1059020	223	10	1029185	21.20	78.35	100.00	100.00
1059021–	2143070	224	10	1500504	21.20	78.35	100.00	100.00
2143071–	2285930	230	11	2213348	21.20	78.35	100.00	100.00
2285931–	2428780	240	11	2356273	21.19	78.36	100.00	100.00
2428781–	2571640	241	11	2499189	21.19	78.36	100.00	100.00
2571641–	2714500	242	11	2642105	21.19	78.36	100.00	100.00
2714501–	5666900	244	11	3022092	21.19	78.36	100.00	100.00
5666901–	6000230	261	12	5831184	21.19	78.36	100.00	100.00
6000231–	7500280	262	12	6708458	21.19	78.36	100.00	100.00
7500281–	8000280	295	14	7746247	21.19	78.36	100.00	100.00
8000281–	8520280	296	14	8256187	21.19	78.36	100.00	100.00
8520281–	17000300	297	14	12035253	21.19	78.36	100.00	100.00
17000301–	18000300	298	14	17402156	21.19	78.36	100.00	100.00

where p_u and p_v are the breakeven qualities as defined in the Section 4.

The above prior distribution amounts to having a full information of the lot quality. In such a case the optimal procedure is given by (66) resulting in the following decision costs.

$$k_m(\bar{p}) = \begin{cases} k_a(\bar{p}) & \text{if} \quad \bar{p} < p_u \\ k_i(\bar{p}) & \text{if} \quad p_u \leqslant \bar{p} < p_v \\ k_r(\bar{p}) & \text{if} \quad \bar{p} \leqslant p_v. \end{cases} \quad \ldots \quad (67)$$

where we assume, as mentioned earlier, $p_u < p_v$. If $p_u \geqslant p_v$ the three decision criteria reduces to a two-decision criteria with cost function as given in (15).

9. Concluding remarks

Bayesian plans with three decision criteria discussed in this paper have a definite economic advantages over the plans with two decision criteria under the situations as pointed out in the paper. The three decision procedure with its cumbersome cost functions appears some what complicated. But it should not create any difficulty to a user if the optimum plans for most of the practical situations are made available in a standard tabular form (it is intended to publish tables of optimum plans with three-decision criteria for most of the common values of ν_{ij}, $i, j = 1, 2$ with most common double binomial prior distributions soon). A vigilant management with a good feed-back information system will find it rather quite easy to obtain reasonable estimates of ν_{ij}'s.

The economic advantages of such plans, in the absence of standard usable tables, may make it quite tempting to work out these plans with the help of an electronic computer by following the systematic steps given in the paper.

A simple extension of this paper may be done by considering three quality levels i.e. a trinomial prior.

It would be interesting and useful to carry out a sensitivity analysis of the optimum plans for the possible fluctuations in the values of ν_{ij} and to study asymptotic properties of these plans. The asymptotic properties of these plans are going to be discussed in a forthcoming paper. It would be rather for theoretical interest to consider k-decision criteria for even more general cost model and derive the three-decision procedure as a particular case. Some thought has been already given to develop sequential sampling procedure with three-decision criteria and to examine its properties.

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