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ON A CONJECTURE OF GEISSER'S

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In this paper, we establish a conjecture made in S. Geisser (1973), namely,

Theorem : *Let X and Y be i.i.d.r.v.'s and $W = (X+Y)/\sqrt{2}$. Then $W^2 \sim \chi_1^2$ iff X and Y are standard normal.*

Proof : The 'if' part is common knowledge. For the results used below, from the theory of analytic and of entire characteristic functions (c.f.'s), to prove the 'only if' part, one may refer, for instance, to Ramachandran (1967), Chapters 2 and 3. Let F and f be the d.f. and c.f. respectively of X (and Y). $W^2 \sim \chi_1^2$ implies that

$$P[W < -x]+P[W > x] = 1-\Phi(x)+\Phi(-x) \quad \text{for all } x > 0 \qquad \ldots \quad (1)$$

where Φ is the standard normal d.f. It is easy to see from (1), also noting that W has a continuous d.f., that, for all real x,

$$P[W \leqslant x]+1-P[W < -x] = \Phi(x)+1-\Phi(-x) = 2\Phi(x),$$

or, $\widetilde{F}$ denoting the d.f. conjugate to F,

$$F_W(x)+\widetilde{F}_W(x) = 2\Phi(x) \quad \text{for all real } x,$$

or, in terms of c.f.'s,

$$f_W(t)+f_W(-t) = 2\exp\left(-\frac{1}{2}\,t^2\right) \text{ for all real } t. \qquad \ldots \quad (2)$$

Now, $f_W(t) = [f(t/\sqrt{2})]^2$, so that we have from (2) that

$$2\ \mathrm{Re}\{[f(t)]^2\} = [f(t)]^2+[f(-t)]^2 = 2\exp(-t^2). \qquad \ldots \quad (3)$$

Let now $G = F * F$ ($*$ denoting the convolution operation as usual), so that g, its c.f., equals f^2. Then (3) gives :

$$\int \cos tx\, dG(x) = \exp(-t^2) \quad \text{for all real } t. \qquad \ldots \quad (4)$$

Familiar arguments then show that G has moments of all even orders (and so of all orders) and that if α_k be its moment of order k, then

$$\alpha_{2n} = (-1)^n \frac{d^{2n}}{dt^{2n}} e^{-t^2}\bigg|_{t=0}$$

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and

$$\sum_{n=0}^{\infty} (-1)^n \alpha_{2n} t^{2n}/(2n)! = e^{-t^2} \quad \text{for all real } t.$$

In particular, the series $\Sigma \alpha_{2n}(it)^{2n}/(2n)!$ has infinite radius of convergence, and the same is then true of the series $\Sigma\alpha_{2n-1}(it)^{2n-1}/(2n-1)!$ as well, in view of the obvious relation $\alpha_{2n}+\alpha_{2n-2} \geqslant 2|\alpha_{2n-1}|$. In other words, $g = f^2$ is an entire c.f., and hence so is f(as a 'factor' thereof). Further, (4) then holds for all complex t by analytic extension and we have in particular, for $t = ir$,

$$\int \cosh rx\, dG(x) = \exp(r^2) \quad \text{for all } r > 0,$$

so that

$$\begin{aligned} M(r; g) &= \max\{|g(z)| : |z| \leqslant r\} = \max[g(ir), g(-ir)] \\ &\leqslant g(ir)+g(-ir) \\ &= 2\int \cosh rx\, dG(x) = 2\exp(r^2), \end{aligned}$$

showing that g, as an entire function, is of order two at most. The same is then true of f(as a factor of g) and hence also of the two entire functions given by the integrals $\int \cos zx\, dF(x)$ and $\int \sin zx\, dF(x)$.

Let us now define, for real t,

$$u(t) = \int (\cos tx+\sin tx)\, dF(x);\ v(t) = \int (\cos tx-\sin tx) dF(x).$$

Then, in view of what has been proved above, u and v can be extended to the complex plane, by means of the same relations, as entire functions of order two at most. Also, the relation

$$\text{Re}\{[f(t)]^2\} = \exp(-t^2) \text{ for all real } t,$$

is equivalent to

$$u(t)v(t) = \exp(-t^2) \quad \text{for all real } t \qquad \ldots (5)$$

since $\text{Re}\{[f(t)]^2\} = [\text{Re} f(t)]^2-[\text{Im} f(t)]^2$. Thus (5) holds for all complex t as well, showing that the entire functions u and v are non-vanishing as well; since they have already been shown to be of order two at most, it follows that each is necessarily of the form $\exp Q$, where Q is a quadratic polynomial. Since $u(0) = v(0) = 1$, and u and v are real and bounded for real t, it then easily follows that

$$u(t) = \exp(-at^2+bt),\ v(t) = \exp(-ct^2+dt),$$

with $a, c \geqslant 0$ and b, d real. Now, from the definition of u and v, we have $u(t) = v(-t)$ for all t, or, $\exp(-at^2+bt) = \exp(-ct^2-dt)$ for all t, so that $a = c$ and $b+d = 0$. Hence

$$f(t) = (\cosh bt+i \sinh bt) \exp(-at^2).$$

Such an f being absolutely integrable, the corresponding probability density function is given by the familiar inversion formula : $\int f(t)e^{-itx}\,dt/(2\pi)$ for all real x; it can be explicitly evaluated and the condition that the above integral be non-negative for such x leads to $b = 0$. A "quick" proof of this fact may be obtained as follows. Suppose $b \neq 0$ and f as above is a c.f. Then, Re $f(t)$ $= \cosh bt.\ \exp(-at^2)$ would be a c.f. But, as we know, $(\cosh bt)^{-1}$ is a c.f. also, and it would follows that the normal c.f. $\exp(-at^2)$ has non-normal factors, contrary to the well-known theorem of Lévy-Cramér which asserts that a normal c.f. has only normal factors. Thus $b = 0$ and $f(t) = \exp(-at^2)$. Relation (3) then implies that $a = \frac{1}{2}$, proving our theorem.

Remarks. (1) If X and Y are i.i.d., and $(X-Y)^2/2 \sim \chi_1^2$, then it is almost trivial that X and Y are normal, with unit variance (obviously they may have non-zero expectation). For, $W = (X-Y)/\sqrt{2}$ has necessarily a *symmetric* distribution, and if $W^2 \sim \chi_1^2$, W must be standard normal. Then, by the Lévy-Cramér theorem, X and Y are normal as well—with (the same mean and) unit variance in view of their being identically distributed.

(2) The following comment on Geisser (1973) may be pertinent. The argument immediately following relation (7) there appears to need strengthening. Discussing the result that if X and Y are independent, with X^2 and Y^2 $\sim \chi_1^2$ each, then $W^2 = (aX+bY)^2/c^2 \sim \chi_1^2$ for some $a, b \neq 0$ and $c = (a^2+b^2)^{\frac{1}{2}}$ iff at least one of X and Y is standard normal, the author arrives at the relation

$$[f_Y(bt/c)-f_Y(-bt/c)][f_X(at/c)-f_X(-at/c)] = 0$$

valid for all real t. From this, he concludes that at least one of the above factors, say the first, must be *identically* zero, whereas all that we can claim is that, *for any given real t*, either the first factor or the second vanishes (or both do), leading respectively to

(i) $f_Y(bt/c) = \exp(-b^2t^2/2c^2)$ or (ii) $f_X(at/c) = \exp(-a^2t^2/2c^2)$

in view of the relations : $f_X(u)+f_X(-u) = f_Y(u)+f_Y(-u) = 2\exp\left(-\frac{1}{2}u^2\right)$. The argument will therefore have to be revised somewhat as follows. Either (i) holds for all t in some neighbourhood of the origin, or there is no such neighbourhood; in the latter case, every neighbourhood of the origin contains a t for which (ii) holds. In the former case, $f_Y(t) = \exp\left(-\frac{1}{2}t^2\right)$ identically, as guaranteed by the theory of analytic c.f.'s. In the latter case, there is a sequence of points tending to the origin, at each of which the c.f. f_X coincides with the standard normal c.f. and it follows that f_X coincides with the latter identically (see, for instance, Section 1 of Ramachandran and Rao (1968)). Thus, one of the two r.v.'s X and Y is standard normal, as required to prove.

(3) The arguments for the main theorem of this paper and those made in the course of Remark 1 do not carry over to the case of a general $W = (aX+bY)/c$ with $|a| \neq |b|, a, b \neq 0$ $(c = (a^2+b^2)^{\frac{1}{2}}$ as before). All that we can claim seems to be that if $W^2 \sim \chi_1^2$, then W and hence X and Y (assumed i.i.d. as before) have entire c.f.'s of order two at most. The best result we have at present concerning such W appears to be one due to Geisser (1973), extending an earlier result of Roberts', which asserts that if X and Y are i.i.d., then they are standard normal iff $W_1^2 = (aX+bY)^2/c^2$ and $W_2^2 = (aX-bY)^2/c^2$ are both $\sim \chi_1^2$ for some $a, b \neq 0$, $|a| \neq |b|$: if $|a| = |b|$, we have just seen what the possibilities are.

A result which contains the above result of Geisser's may not be devoid of interest : if X and Y are independent and have the same variance, and W_1^2 and W_2^2 are both $\sim X_1^2$, then at least one of X and Y is standard normal. For, we have

$$f_{W_j}(t)+f_{W_j}(-t) = 2\exp\left(-\frac{1}{2}t^2\right) \quad \text{for} \quad j = 1, 2;$$

adding these two relations and noting that

$$f_{W_1}(t) = f_X(at/c)f_Y(bt/c),\ f_{W_2}(t) = f_X(at/c)\,f_Y(-bt/c),$$

we have

$$[f_X(at/c)+f_X(-at/c)][f_Y(bt/c)+f_Y(-bt/c)] = 4\exp\left(-\frac{1}{2}t^2\right),$$

or,

$$\text{Re}\,f_X(at/c)\,.\;\text{Re}\,f_Y(bt/c) = \exp\left(-\frac{1}{2}t^2\right).$$

The factors on the LHS of the last relation are both c.f.'s (of symmetric d.f.'s) and it follows from the Lévy-Cramér theorem that both correspond to normal d.f.'s with zero mean. Then, the condition that X and Y have the same variance leads to : $\text{Re}\,f_X(t) = \text{Re}\,f_Y(t) = \exp\left(-\frac{1}{2}t^2\right)$, so that X^2 and $Y^2 \sim \chi_1^2$. Then, by the result quoted in Remark 2, at least one of X and Y is standard normal. In particular, if X and Y are i.i.d., both are standard normal, agreeing with the result of Geisser's cited at the beginning of this Remark.

The referee and Professor S. Geisser have pointed out to me that the main result of the present paper has been obtained independently and *inter alia* by G. Funk and R. Rodine, and that the substance of the various remarks above is also to be found in a paper by H. Block. These papers have been added to my original list of references now, and I am grateful to the editors and the referee for pre-publication copies of these two papers.

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