SOME PROBLEMS IN EQUIVARIANT TOPOLOGY

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Contents

Chapter 0 INTRODUCTION

This thesis has grown out of efforts to understand in an equivariant set up two old problems which were revived and developed extensively by Quillen, and also by others. The first problem concerns a de Rham type theorem over the field of rationals $\mathbb{Q}$ for a G-simplicial set where G is a finite group. The second problem deals with equivariant plus construction, classification of equivariant acyclic maps, and equivariant G-homotopy type of a G-space where G is a compact Lie group. More specifically, the content of the thesis is governed by the following two theorems, and our main objective is to look for their generalizations in suitable equivariant categories.

Theorem A (Cartan [7]). *If A is a cohomology theory over a commutative ring R with 1, and $\mathcal{A}^{\cdot}(K)$ is the associated differential algebra of a simplicial set K, then there is a natural isomorphism of graded R-modules*

$$H^{\cdot}(\mathcal{A}^{\cdot}(K)) \cong H^{\cdot}(K; R(A)),$$

where $R(A)$ is an R-module functorially determined by A.

Here a differential algebra means a differential graded algebra, which is a graded R-module $\oplus_{p\geq 0} M^p$ having a differential $\delta : M^p \longrightarrow M^{p+1}$ with $\delta^2 = 0$, and a multiplication $M^p \otimes M^q \longrightarrow M^{p+q}$ satisfying the Liebnitz rule $\delta(\alpha\beta) = (\delta\alpha)\beta + (-1)^p \alpha(\delta\beta)$. A cohomology theory is a contravariant functor from the indexing category Δ for simplicial objects to the category **DGA** of differential algebras over R satisfying certain axioms (see Chapter 6).

Theorem B (Kan-Thurston [15]). *If X is a path connected CW-space, then there exists a group π with a perfect normal subgroup N such that X has the homotopy type of $K(\pi, 1)^+_N$.*

Here $K(\pi, 1)^+_N$ is the space obtained from the Eilenberg-MacLane space $K(\pi, 1)$ by applying the plus construction of Quillen [25] with respect to N.

Theorem A has its origin in the commutative cochain problem which was posed by Thom in 1957. A solution to this problem entails in constructing a contravariant functor $A^\cdot : \mathbf{TOP} \longrightarrow \mathbf{DGA}$ (commutative) so as to yield a de Rham type theorem which asserts that there is an isomorphism

$$H^\cdot(A^\cdot(X)) \cong H^\cdot(X; R)$$

for every topological space X, where the cohomology on the right is the singular cohomology. For example, the classical de Rham theorem provides a solution for the subcategory of smooth manifolds where $A^\cdot(X)$ is the differential algebra over the field of reals $\mathbb{R}$ of smooth differential forms on a manifold X. On the other hand, there does not exist a differential algebra over the integers $\mathbb{Z}$ (the commutativity fails) and, as realized by Steenrod some 50 years ago, this accounts for the existence of cohomology operations, such as Steenrod squares, etc.

In [24], Quillen solved the rational commutative cochain problem in an abstract setting and the solution obtained is rather complicated. Later Sullivan [27] gave another proof using his theory of minimal models and the de Rham complex $A^\cdot(K)$ of rational polynomial forms on a simplicial set K. An independent proof, which is based on an earlier proof by Thom in the real case, was given by Swan [28] when the coefficient ring R is a field of characteristic zero. Finally, Cartan [7] formulated the main ideas of Swan in the form of axioms for a cohomology theory, and proved Theorem A. This work of Cartan generalizes Sullivan's theory to arbitrary coefficient ring R. The main feature of both [28] and [7] is that they avoid integration of forms which is standard to proofs of de Rham type theorems. It may be noted that de Rham theorem for arbitrary coefficient was also proved by Miller in [20], and that McCleary [19] proved a local coefficients version of Theorem A. It is also of interest to note that a method of calculating integral cohomology of manifolds using differential forms appeared earlier in [1].

The equivariant rational cochain problem which we present here preserves all the features of Theorem A. Let G be a finite group, and O_G the category of canonical orbits whose objects are left coset spaces G/H and morphisms are equivariant maps $G/H \longrightarrow G/H'$ with respect to left translation. Let $\mathcal{C}_{\mathbb{Q}}$ be the category of cohomology theories over $\mathbb{Q}$ in the sense of Cartan. Then a G-cohomology theory over $\mathbb{Q}$ is a contravariant functor $A : O_G \longrightarrow \mathcal{C}_{\mathbb{Q}}$. On the other hand, given a G-simplicial set K, which is a simplicial set together with a given action of G on K by simplicial maps, and a coefficients system λ, which is a contravariant functor from O_G to the category of R-modules, we can construct equivariant cohomology groups $H^{\cdot}_G(K\,;\lambda)$ in a natural way. That this is a usable cohomology follows form the fact that the Bredon-Illman cohomology groups [4], [14] of a G-space X with coefficients system λ are isomorphic to the groups $H^{\cdot}_G(SX\,;\lambda)$ where SX is the singular G-simplicial set associated to X. Then our first main theorem is

Theorem (6.4.3). *Given a G-cohomology theory A over $\mathbb{Q}$ and a G-simplicial set K, there is a differential algebra $\mathcal{A}^{\cdot}(K)$ over $\mathbb{Q}$, and a coefficients system λ_A from the category O_G to the category of rational vector spaces such that*

$$H^{\cdot}(\mathcal{A}^{\cdot}(K)) \cong H^{\cdot}_G(K\,;\lambda_A).$$

The theorem can be translated easily to the category of G-spaces using the geometric realization functor and the Bredon-Illman cohomology.

The origin of Theorem B may be traced back to Kervaire who in 1969 used in his study of homology spheres and their fundamental groups a special case of a general theory which says that by attaching 2-cells and 3-cells to a path connected *CW*-complex X one can kill a part of its fundamental group $\pi_1(X)$ without changing the homology.

This technique was rediscovered and developed by Quillen [25] with the name 'plus construction'. Given X and a perfect normal subgroup N of $\pi_1(X)$, there is a space X_N^+, obtained by attaching 2-cells and 3-cells to X, and an acyclic map $f: X \longrightarrow X_N^+$ with ker $\pi_1(f) = N$. Note that a subgroup is perfect if it equals its commutator subgroup, and that a space is acyclic if its reduced integral homology is trivial, and a map $f : X \longrightarrow Y$ is acyclic if its homotopy fibre is acyclic, or equivalently, the induced map $f_* : H_*(X; f^*L) \longrightarrow H_*(Y; L)$ is an isomorphism for every local coefficients system L on Y. If f is an acyclic map on X, then $\pi_1(f)$ is always an epimorphism with kernel a perfect normal subgroup, and these subgroups of $\pi_1(X)$ classify acyclic maps on X in the following way. Two acyclic maps $f: X \longrightarrow Y$ and $f' : X \longrightarrow Y'$ are equivalent if there is a homotopy equivalence $h : Y \longrightarrow Y'$ such that $hf \simeq f'$. Then the set of equivalence classes of acyclic maps on X correspond bijectively with the set of perfect normal subgroups of $\pi_1(X)$ (see [13]).

In a remarkable paper [15], Kan and Thurston showed that for any path connected *CW*-space X, there is a group π with a perfect normal subgroup N and a fibration $p : K(\pi, 1) \longrightarrow X$ with domain an Eilenberg-MacLane space of type $(\pi, 1)$ such that p is acyclic and ker $\pi_1(f) = N$. This result together with the classification of acyclic maps imply Theorem B.

Our approach to obtain equivariant analogues of these results uses G-spaces X, where G is a compact Lie group (and we agree to consider only closed subgroups of G), with base point $x_0 \in X^G$ such that X has the G-homotopy type of a G-*CW*-complex, and such that, for each subgroup H of G, the H-fixed point set X^H has the homotopy type of a connected *CW*-complex. We define a G-space X to be G-acyclic if each X^H is acyclic, and a G-map $f : X \longrightarrow Y$ is G-acyclic if its G-homotopy fibre is G-acyclic.

An O_G-group is a contravariant functor from O_G to the category **Grp** of groups. An O_G-group N is a perfect normal O_G-subgroup of an O_G-group λ if each $N(G/H)$ is a perfect normal subgroup of $\lambda(G/H)$. Similarly, we have the notion of O_G-homotopy group $\underline{\pi}_n(X)$ of a G-space X, where $\underline{\pi}_n(X)(G/H)$ is just $\pi_n(X^H, x_0)$. Also a G-map $f : X \longrightarrow Y$ induces a morphism $\underline{\pi}_n(f)$ of O_G-groups given by $\underline{\pi}_n(f)(G/H) = \pi_n(f^H)$, where $f^H = f|X^H$.

Given an O_G-group λ and an integer $n \geq 1$, there is a G-space X such that $\underline{\pi}_n(X) = \lambda$ and $\underline{\pi}_j(X) = \underline{0}$ if $j \neq n$. The G-space X is actually a *G-CW*-complex, and is called an Eilenberg-MacLane G-space $K(\lambda, n)$ of type (λ, n) (see [10]). These are classifying spaces for the Bredon-Illman cohomology [4], [14].

Theorem (8.1.1). *For any G-space X there exists an O_G-group λ with a perfect normal O_G-subgroup N, and a G-acyclic map*

$$p : K(\lambda, 1) \longrightarrow X,$$

such that $\ker \underline{\pi}_1(p) = N$.

For a G-space X and a perfect normal O_G-subgroup N of $\underline{\pi}_1(X)$, one can construct a G-space X_N^+ by applying the plus construction of Quillen to each X^H with respect to the group $N(G/H)$, and then piecing the resulting spaces together by means of a functorial bar construction. Then our second main theorem is

Theorem (8.1.2). *Given a G-space X, there exists an O_G-group λ with a perfect normal O_G-subgroup N such that X has the G-homotopy type of* $K(\lambda, 1)_N^+$.

The thesis is organized as follows. The proof of the first main theorem depends on results that we develop in Chapters 1 through 6. On the way to this goal we make a number of detours in Chapter 3, each with the purpose of expounding a particular point of interest, but none are irrelevant to the main theme of the thesis.

In Chapter 1, we start with the ideas and language of simplicial sets, and then introduce the notion of G-simplicial sets, G-Kan complex, etc, with some of their properties. These concepts formalize a point of view which has to dominate the development of the proof of the first main theorem.

In Chapter 2, we define equivariant cohomology groups $H^*_G(K;\lambda)$ of a G-simplicial set K with a coefficients system λ, and study their relation with the Bredon-Illman cohomology groups of a G-space.

In Chapter 3, we first review a normalization theorem for non-equivariant simplicial cohomology, and then generalize it to the equivariant case. We also prove that under certain conditions on K and G the groups $H^*_G(K;\lambda)$ are finitely generated where λ is a suitable coefficients system. The remaining part of the chapter deals with equivariant Euler characteristic. Brown [5], [6], (see also Serre [26]) defined the Euler characteristic $\chi(G)$ of a discrete group of finite homological type. If an arbitrary group G acts on a simplicial set K in such a way that the simplicial set K/G has only finitely many non-degenerate simplexes, and the isotropy group G_x of every simplex x of K has finite homological type, then one can choose a finite set Σ of representatives for non-degenerate simplexes of K/G and define the equivariant Euler characteristic of K by

$$\chi_G(K) = \sum_{x\in\Sigma}(-1)^{dim\ x}\chi(G_x).$$

We show that for a free action of G on K

$$\chi_G(K) = \chi(G)\sum_i(-1)^i \dim H^i_G(K;\lambda_{\mathbb{Q}}),$$

where $\lambda_{\mathbb{Q}}$ is a suitably defined coefficients system from O_G to the category of rational vector spaces. This formula also holds if G is a free group of finite rank, or, in general, if G has finite cohomological dimension and it is of finite homological type.

In Chapter 4, we study the closed model structure in the sense Quillen [22] of the category $\boldsymbol{GS}$ of G-simplicial sets, and also of the category of simplicial objects in $\boldsymbol{Vec}_G$, which is the category of contravariant functors from O_G to the category of rational vector spaces. We also prove the Whitehead theorem for G-simplicial sets which says that $f\colon K \longrightarrow L$ is a G-homotopy equivalence if and only if each f^H is a homotopy equivalence.

In Chapter 5, we study O_G-Eilenberg-MacLane complexes $K(\lambda, n)$ of type (λ, n) where n is a non-negative integer, and λ is an O_G-group which is abelian if $n > 1$. Such a complex is a contravariant functor $T\colon O_G \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is the category of simplicial sets, so that for each subgroup H of G, $T(G/H)$ is an ordinary Eilenberg-MacLane complex $K(\lambda(G/H), n)$. The principal result of this chapter is a classification theorem which says that there is a bijection between $H^n_G(K;\lambda)$ and the set of homotopy classes of natural transformations $\Phi K \longrightarrow K(\lambda, n)$, where $\Phi \mathbb{Q} K\colon O_G \longrightarrow \mathcal{S}$ is given by $\Phi K(G/H) = K^H$.

In Chapter 6, we define G-cohomology theory, and prove the first main theorem (Theorem 6.4.3). The proof uses more deeper properties of simplicial sets, namely, the $\overline{W}$- and W-constructions for a group complex and the uniqueness of universal principal twisted cartesian product (PTCP) of type (W).

Last two chapters (Chapter 7 and 8) deal with the second problem of the thesis. In Chapter 7, we generalize the Kan-Thurston theorem [15] partially. We prove that if K is a reduced G-simplicial set, then there is a G-fibration $tK : TK \longrightarrow K$ such that tK induces isomorphism on equivariant cohomology $H^*_G(K;\lambda) \longrightarrow H^*_G(TK;\lambda)$ for any O_G-abelian group λ (Theorem 7.4.1). This theorem appears in a more general form in Chapter 8 (Theorem 8.1.2). In this last chapter, our basic assumption is that G is a compact Lie group. Here we introduce equivariant local coefficients system on a G-space, homology O_G-groups, G-acyclic maps, and equivariant plus construction, and finally prove our second main theorem (Theorem 8.1.2).

Chapter 1 PRELIMINARIES

1.1 Introduction

This chapter, except for Section 1.5, is meant for a review of simplicial theory. Over the next three sections we recall some facts about simplicial sets, their geometric realization and homotopy theory. This serves to set up notations and state results which will be used throughout. More deeper properties about simplicial sets will be recalled later when we have the opportunity to use them. The standard references for these three sections are [18], [8], and [12]. Finally in Section 1.5, we define and study some properties of simplicial sets with group action.

1.2 Simplicial objects in a category

Let Δ be the category whose objects are ordered sets $[n] = \{0 < 1 < \ldots < n\}$, $n \geq 0$, and morphisms are non-decreasing maps $f : [n] \longrightarrow [m]$. It is important to focus attention on some distinguished morphisms, the faces $\partial_i : [n-1] \longrightarrow [n]$ and the degeneracies $\sigma_i : [n+1] \longrightarrow [n]$, defined as follows :

$$\partial_i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases} \qquad (n > 0,\ 0 \leq i \leq n)$$

$$\sigma_i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i. \end{cases} \qquad (n \geq 0,\ 0 \leq i \leq n)$$

The faces and degeneracies together generate the category Δ, and they verify the simplicial (actually, cosimplicial) identities :

$$
\begin{aligned}
\partial_j\partial_i &= \partial_i\partial_{j-1} \quad \text{if} \quad i<j, \quad \sigma_j\sigma_i=\sigma_i\sigma_{j+1} \qquad \text{if} \quad i\leq j \\
\sigma_j\partial_i &= \begin{cases} \partial_i\sigma_{j-1} & \text{if} \quad i<j \\ id & \text{if} \quad i=j, \text{ or } \quad i=j+1 \\ \partial_{i-1}\sigma_j & \text{if} \quad i>j+1. \end{cases}
\end{aligned} \tag{1.1}
$$

A simplicial object X in a category $\mathcal{C}$ is a contravariant functor $X : \Delta \longrightarrow \mathcal{C}$. Equivalently, a simplicial object is a sequence $\{X_n\}_{n\geq 0}$ of $\mathcal{C}$-objects, together with $\mathcal{C}$-morphisms

$$d_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_i : X_n \longrightarrow X_{n+1}, \quad 0 \leq i \leq n,$$

verifying identities which are dual to the identities in (1.1). A simplicial map $f : X \longrightarrow Y$ between two simplicial objects in a category $\mathcal{C}$ is a collection of $\mathcal{C}$-morphisms $f_n : X_n \longrightarrow Y_n$ commuting with the d_i and the s_i.

A simplicial object K in the category **SETS** of sets is called a simplicial set (or a complex). Elements of K_n are called n-simplexes, and a simplex $x \in K_n$ is degenerate if $x = s_i x'$, for some $x' \in K_{n-1}$ and $0 \leq i \leq n-1$, and is non-degenerate otherwise. Throughout $\mathcal{S}$ will denote the category of simplicial sets and simplicial maps.

The simplicial set $\Delta[n]$, $n \geq 0$, is given by the contravariant functor

$$\Delta[n]([p]) = \mathrm{Hom}_{\Delta}([p],[n]),$$

where $\mathrm{Hom}_{\Delta}([p],[n])$ denotes the Δ-morphisms from $[p]$ to $[n]$. The face and degeneracy maps are clear. The only non-degenerate n-simplex of $\Delta[n]$ given by $id : [n] \longrightarrow [n]$ will be denoted by Δ_n. The morphisms ∂_i and σ_i in Δ give rise to simplicial maps by composition which we again denote by

$$\partial_i : \Delta[n-1] \longrightarrow \Delta[n], \quad \sigma_i : \Delta[n+1] \longrightarrow \Delta[n]. \tag{1.2}$$

If K is a simplicial set and $x \in K_n$, then there is a unique simplicial map $\bar{x} : \Delta[n] \longrightarrow K$ with $\bar{x}(\Delta_n) = x$. The n-skeleton $K^{(n)}$ of K is by definition the simplicial set generated by simplexes of dimension less than or equal to n. The $(n-1)$-skeleton of $\Delta[n]$ will be denoted by $\dot{\Delta}[n]$.

1.3 Geometric realization

To every simplicial set K we can associate a topological space $|K|$, called the geometric realization of K, as follows. Give K the discrete topology. On the disjoint union

$$\overline{K} = \coprod_{n \geq o} K_n \times \Delta^n$$

define an equivalence relation

$$\begin{aligned} (d_i k_n, u_{n-1}) &\sim (k_n, \partial_i u_{n-1}) \\ (s_i k_n, u_{n+1}) &\sim (k_n, \sigma_i u_{n+1}), \end{aligned}$$

where $k_n \in K_n$, $u_{n-1} \in \Delta^{n-1}$, and $u_{n+1} \in \Delta^{n+1}$. Here Δ^n denotes the standard Euclidean n-simplex, and $\partial_i : \Delta^{n-1} \longrightarrow \Delta^n$ and $\sigma_i : \Delta^{n+1} \longrightarrow \Delta^n$ are the maps

$$\begin{aligned} \partial_i(t_0, \ldots, t_{n-1}) &= (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \\ \sigma_i(t_0, \ldots, t_{n+1}) &= (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}). \end{aligned}$$

Then $|K| = K/\sim$ is a CW-complex with one n-cell for every non-degenerate n-simplex of K. There is a homeomorphism $|K \times L| \longrightarrow |K| \times |L|$ if $|K| \times |L|$ is a CW-complex, where $K \times L$ is the simplicial set with $(K \times L)_n = K_n \times L_n$, and coordinatewise face and degeneracy maps. The Euclidean n-simplex Δ^n is the geometric realization of the simplicial set $\Delta[n]$.

If $f : K \longrightarrow L$ is a simplicial map, then f induces a continuous map $|f| : |K| \longrightarrow |L|$ defined by $|f|[k_n, u_n] = [f(k_n), u_n]$, where $[k_n, u_n]$ denotes the equivalence class of (k_n, u_n). Let **TOP** denote the category of topological spaces, then the geometric realization functor $|\ \ | : \boldsymbol{S} \longrightarrow \mathbf{TOP}$ has a right adjoint $S : \mathbf{TOP} \longrightarrow S$, the total singular complex functor. If X is a topological space, then SX_n is the set of singular n-simplexes $f : \Delta^n \longrightarrow X$ of X. The face and degeneracy maps are defined by compositions with ∂_i and σ_i respectively.

1.4 Homotopy theory of simplicial sets

Let K be a simplicial set. Then its homology and cohomology groups with coefficients in an abelian group G are defined via the chain complex $C(K)$, where $C_n(K)$ is the free abelian group generated by the n-simplexes of K, and the boundary homomorphism $\partial : C_n(K) \longrightarrow C_{n-1}(K)$ is $\sum_{i=0}^{n}(-1)^i d_i$, the alternating sum of the face maps. Then

$$\begin{aligned} H_*(K;G) &= H_*(C(K)\otimes G) \\ H^*(K;G) &= H^*(\mathrm{Hom}(C(K),G)). \end{aligned}$$

Two simplicial maps $f, g : K \longrightarrow L$ between simplicial sets are homotopic if there exists a simplicial map $F : K \times \Delta[1] \longrightarrow L$ with $F(x,(0)) = f(x)$ and $F(x,(1)) = g(x)$. Alternatively, we may define homotopy of maps using functions $h_i : K_n \longrightarrow L_{n+1}$, $0 \le i \le n$, which satisfy the following identities

$$\begin{aligned} d_0 h_0 &= f, \qquad d_{n+1}h_n = g \\ d_i h_j &= h_{j-1} d_i, \quad i < j \\ d_{j+1}h_{j+1} &= d_{j+1} h_j \\ d_i h_j &= h_j d_{i-1}, \quad i > j+1 \\ s_i h_j &= h_{j+1} s_i, \quad i \le j \\ s_i h_j &= h_j s_{i-1}, \quad i > j. \end{aligned} \tag{1.3}$$

We then have the notions of homotopy equivalence, contractibility etc. of simplicial sets.

Definition 1.4.1 *A simplicial set K is a Kan complex if for every collection $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ of $n+1$ n-simplexes satisfying the compatibility conditions*

$$d_i x_j = d_{j-1} x_i \qquad (i < j, \quad i \ne k, \quad j \ne k)$$

there exists $x \in K_{n+1}$ with $d_i x = x_i$, $\ i \ne k$.

If X is a topological space, then its singular complex SX is a Kan complex.

Homotopy of maps in general is not an equivalence relation. However one has ([18] p.20)

Theorem 1.4.2 *If L is a Kan complex then homotopy is an equivalence relation on the set of simplicial maps $K \longrightarrow L$ for every simplicial set K.* ■

The homotopy groups $\pi_n(K)$ of a Kan complex K are defined combinatorially as the quotient set of certain subset $\widetilde{K}_n$ of K_n by certain equivalence relation (see Section 4.4, p.52). For a general simplicial set K, the homotopy groups are defined by setting $\pi_n(K) = \pi_n(S|K|)$. It follows then that $\pi_n(K) = \pi_n(|K|)$. There is yet another description of $\pi_n(K)$ for a simplicial group K (which is a simplicial object in the category **Grp** of groups), where $\pi_n(K)$ appear as the homology groups of certain chain complex (see [18]). A simplicial set K is connected if the set $\pi_0(K)$ is a singleton.

Definition 1.4.3 *A simplicial map $f : K \longrightarrow L$ is a Kan fibration if for every collection $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ of $n+1$ n-simplexes of K satisfying the compatibility condition*

$$d_i x_j = d_{j-1} x_i \qquad (i < j, \quad i \neq k, \quad j \neq k),$$

and for every $y \in L_{n+1}$ with $d_i y = f(x_i)$, $i \neq k$, there exists $x \in K_{n+1}$ with $d_i x = x_i$, $i \neq k$, and $f(x) = y$.

If y_0 is the simplicial set generated by $y_0 \in L_0$, then the simplicial subset $f^{-1}(y_0)$ of K is called the fibre of f, and we then have a long exact sequence of homotopy groups of the fibration. We also mention in passing that according to Quillen [23] the geometric realization of a Kan fibration is a Serre fibration. A simplicial set K is a Kan complex if and only if the constant map $K \longrightarrow *$ is a Kan fibration, where $*$ is the simplicial set generated by a point.

Definition 1.4.4 *A simplicial map $f : K \longrightarrow L$ between two simplicial sets is a weak homotopy equivalence if it induces isomorphism on homotopy groups.*

If K is a simplicial set then the simplicial map $\psi(K) : K \longrightarrow S|K|$ given by

$$\psi(K)(x)(t) = [x, t],$$

where $x \in K_n$, $t \in \Delta^n$, is a weak homotopy equivalence. We then have the Whitehead theorem

Theorem 1.4.5 *A weak homotopy equivalence $f : K \longrightarrow L$ between Kan complexes is a homotopy equivalence.* ∎

Definition 1.4.6 *A Kan complex K is said to be minimal if for any two simplexes x, y, $d_i x = d_i y$, $i \neq k$, implies $d_k x = d_k y$.*

Minimality is a very strong condition on simplicial sets. For example we have the following result (see [8] p. 112)

Theorem 1.4.7 *A homotopy equivalence $f : K \longrightarrow L$ between minimal Kan complexes is an isomorphism.* ∎

1.5 Preliminaries on G-simplicial sets

Every group G determines a constant simplicial group $\underline{G}$ with $\underline{G}_n = G$ for all $n \geq 0$, and all the face and degeneracy maps the identity map of G. A G-simplicial set (or a G-complex) K is a simplicial set together with an action of $\underline{G}$ on K by simplicial maps. This makes each K_n a G-set and the face and degeneracy maps commute with the action of $\underline{G}$. Let $\boldsymbol{GS}$ denote the category of G-simplicial sets and simplicial maps commuting with the G action. Morphisms in $\boldsymbol{GS}$ will be called G-simplicial maps or simply G-maps.

For each $x \in K_n$, there is a unique G-map $\tilde{x} : \Delta[n] \times \underline{G/H} \longrightarrow K$ such that $\tilde{x}(\Delta_n, eH) = x$, where the action of G on $\Delta[n]$ is trivial, H is the isotropy subgroup G_x at x, and $\underline{G/H}$ is the constant G-simplicial set determined by G/H.

If K' is a G-simplicial subset of K, and $f: K' \longrightarrow L$ is a G-map, then the adjunction L' is a G-simplicial set, where L'_n is obtained from the disjoint union $K_n \coprod L_n$ by identifying $x \in K'_n$ with $f(x) \in L_n$. Clearly, L' is the push out of the diagram $K \xleftarrow{i} K' \xrightarrow{f} L$, and L' is obtained from L by attaching K via f. In particular, if $K^{(n)}$ is the n-skeleton of K (which is a G-simplicial set), then $K^{(n)}$ is obtained from $K^{(n-1)}$ by attaching $\coprod_{H \subseteq G} \Delta[n] \times \underline{G/H}$ via certain G-map $\coprod_{H \subseteq G} \dot{\Delta}[n] \times \underline{G/H} \longrightarrow K^{(n-1)}$, where H is a representative of the conjugacy class of the isotropy subgroup at a non-degenerate n-simplex of K. The resulting G-map $\coprod_{H \subseteq G} \Delta[n] \times \underline{G/H} \longrightarrow K^{(n)}$ is the characteristic map of the attaching (cf [11], p.145).

The notion of homotopy between two G-maps is given as follows

Definition 1.5.1 *Two G-maps $f, g : K \longrightarrow L$ between two G-simplicial sets are G-homotopic if there exists a G-map $F : K \times \Delta[1] \longrightarrow L$ with $F(x, (0)) = f(x)$ and $F(x, (1)) = g(x)$. Here we consider $K \times \Delta[1]$ as a G-simplicial set with trivial action of G on $\Delta[1]$.*

Alternatively, we may define G-homotopy using G-functions $h_i : K_n \longrightarrow L_{n+1}$, $0 \leq i \leq n$, satisfying the identities (1.3). One then has the notion of G-homotopy equivalence, G-contractibility etc. of G-simplicial sets.

Definition 1.5.2 *A G-simplicial set K is a G-Kan complex if the H-fixed point subcomplex K^H is a Kan complex for every subgroup H of G. Here $(K^H)_n = (K_n)^H$.*

Clearly, if X is a G-space then its singular complex SX is a G-Kan complex. As in the non-equivariant case (Theorem 1.4.2), we have

Theorem 1.5.3 *If L is a G-Kan complex, then G-homotopy is an equivalence relation on the set of G-maps $K \longrightarrow L$, for every G-simplicial set K.*

First note that if K and L are G-simplicial sets, then we have another G-simplicial set L^K defined as follows. The n-simplexes of L^K are simplicial maps $f : K \times \Delta[n] \longrightarrow L$ with the G action given by

$$(gf)(x,t) = gf(g^{-1}x,t).$$

The face and degeneracy maps are

$$d_i(f) = f \circ (id \times \partial_i), \quad s_i(f) = f \circ (id \times \sigma_i)$$

where ∂_i and σ_i are as in (1.2). Then the n-simplexes of the G-fixed point set $(L^K)^G$ are precisely the G-maps $f : K \times \Delta[n] \longrightarrow L$. In particular, the 0-simplexes are G-maps $K \longrightarrow L$ and 1-simplexes are G-homotopies.

The proof of Theorem 1.4.2 is adaptable to the equivariant case. To see this, recall from ([18] p.17) that if p, q are non-negative integers, then a (p,q)-shuffle (μ,ν) is a partition of the set $[p+q-1]$ into two disjoint subsets $\mu_1 < \ldots < \mu_p$ and $\nu_1 < \ldots < \nu_q$. A (p,q)-shuffle (μ,ν) is completely determined by μ or ν, and it may be one of the following three types with respect to an integer $i \in [p+q]$:

- Type I. If $i < \mu_1$, or $i, i-1 \in \{\nu_1, \ldots, \nu_q\}$, or $i = p+q, i-1 = \nu_q$.
- Type II. If $i < \nu_1$, or $i, i-1 \in \{\mu_1, \ldots, \mu_p\}$, or $i = p+q, i-1 = \mu_p$.
- Type III. Otherwise.

To each (p,q)-shuffle (μ,ν) and each i, one can associate a new $(\overline{p},\overline{q})$-shuffle $(\overline{\mu},\overline{\nu})$ and an index r such that

(1) $\overline{p} = p, \overline{q} = q-1$ if (μ,ν) is of type I.

(2) $\overline{p} = p-1, \overline{q} = q$ if (μ,ν) is of type II.

(3) $\overline{p} = p, \overline{q} = q$ if (μ,ν) is of type III.

In case (1), $\overline{\mu}_j = \mu_j$ for $j \leq r$ and $\overline{\mu}_j = \mu_j - 1$ for $r < j \leq p$. In case (2), $\overline{\nu}_j = \nu_j$ for $j \leq r$ and $\overline{\nu}_j = \nu_j - 1$ for $r < j \leq q$. In case (3), $\overline{\mu}_j = \mu_j$ for $j \neq r$ and $\overline{\mu}_r = i$. Also, to each (p,q)-shuffle (μ, ν) and each i, we can associate a $(p+1,q)$-shuffle $(\widetilde{\mu}, \widetilde{\nu})$ and a second index t.

Let $F: K \times \Delta[q] \longrightarrow L$ be a G-map, and (μ,ν) be a (p,q)-shuffle. Then, we can define a G-function $h_{(\mu,\nu)} : K_p \longrightarrow L_{p+q}$ by

$$h_{(\mu,\nu)}(y) = F(s_{\nu_q} \cdots s_{\nu_1} y, s_{\mu_p} \cdots s_{\mu_1} \Delta_q). \tag{1.4}$$

It can be checked that the family of G-functions $\{h_{(\mu,\nu)}\}$ satisfy the following four conditions :

(a) $d_i h_{(\mu,\nu)} = h^{i-r}_{(\overline{\mu},\overline{\nu})}$ if (μ, ν) is a (p,q)-shuffle of type I and index r with respect to i.

(b) $d_i h_{(\mu,\nu)} = h_{(\overline{\mu},\overline{\nu})} d_{i-r}$ if (μ, ν) is a (p,q)-shuffle of type II and index r with respect to i.

(c) $d_i h_{(\mu,\nu)} = d_i h_{(\overline{\mu},\overline{\nu})}$ if (μ, ν) is a (p,q)-shuffle of type III with respect to i.

(d) $s_i h_{(\mu,\nu)} = h_{(\widetilde{\mu},\widetilde{\nu})} s_{i-t}$ if (μ, ν) is a (p,q)-shuffle and t is the second index of (μ, ν) with respect to i.

Conversely, given a family of G-functions $h_{(\mu,\nu)} : K_p \longrightarrow L_{p+q}$ indexed by the (p,q)-shuffles (q fixed) (μ, ν) satisfying the conditions (a)-(d) above, we may define a G-map $F: K \times \Delta[q] \longrightarrow L$ by setting

$$F(y, d_{\nu_q+1} \cdots d_{\nu_1+1} s_{\mu_p} \cdots s_{\mu_1} \Delta_q) = d_{\nu_q+1} \cdots d_{\nu_1+1} h_{(\mu,\nu)}(y). \tag{1.5}$$

Theorem 1.5.3 follows directly from

Theorem 1.5.4 *If L is a G-Kan complex, then the G-fixed point set $(L^K)^G$ is a Kan complex for every G-simplicial set K.*

Proof. Let $f_0, f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_q$ be a collection of q $(q-1)$-simplexes of $(L^K)^G$ satisfying the compatibility condition

$$d_i f_j = d_{j-1} f_i \quad \text{whenever} \quad i < j, \;\; i \neq k, \;\; j \neq k.$$

Then $f_i : K \times \Delta[q-1] \longrightarrow L$, $i \neq k$, are G-maps and we need to find a G-map $f : K \times \Delta[q] \longrightarrow L$ with $d_i f = f_i$, $i \neq k$.

The G-maps f_i determine by (1.4) a family of G-functions $\{h^i_{(\mu,\nu)}\}$ indexed by the $(p, q-1)$-shuffles (μ, ν). We will define G-functions $h_{(\mu,\nu)}$ indexed by the (p,q)-shuffles (μ,ν) satisfying the conditions (a)-(d) above. Then the corresponding G-map $f : K \times \Delta[q] \longrightarrow L$ determined by (1.5) will be the required extension.

The construction of the G-functions $h_{(\mu,\nu)}$ is similar to the non-equivariant case ([18], p.20), and is based on an induction scheme with respect to an ordering of the (p,q)-shuffles (q fixed) which is defined as follows: $(\mu,\nu) < (\mu',\nu')$ if $\mu_i = \mu'_i$ for $i < j$ and $\mu_j < \mu'_j$. The first shuffle of this ordering is the $(0,q)$-shuffle, and we define

$$h_{(0,q)} : K_0 \longrightarrow L_q$$

as follows. Take a simplex y in K_0 and consider it as a simplex of the fixed point set K^H where $H = G_y$. Then the simplexes $h^i_{(0,q-1)}(y)$, $i \neq k$, form a compatible collection of $(q-1)$-simplexes of L^H. As L^H is a Kan complex, we may find a q-simplex z with $d_i z = h^i_{(0,q-1)}(y)$. Now set $h_{(0,q)} = z$, and extend $h_{(0,q)}$ equivariantly on the orbit of y. In this way we may complete the induction proceeding as in [18]. ∎

Remark 1.5.5 *If K is a G-simplicial set, then its geometric realization $|K|$ is a CW-complex with a cellular action of G in the sense of tom Dieck* [9] *(note that G has discrete topology). The action is cellular, because non-degenerate n-simplexes of K correspond bijectively with n-cells of $|K|$. Therefore $|K|$ is actually a G-CW-complex, by a result of tom Dieck* [9], p.101.

We conclude this section with a basic property of G-simplicial sets, satisfying a connectivity condition.

Definition 1.5.6 *A G-simplicial set K is G-connected if each fixed point simplicial set K^H is connected for every subgroup H of G.*

Theorem 1.5.7 *Let K be a G-connected G-Kan complex with a G-fixed 0-simplex. Then K has the G-homotopy type of G-Kan complex with a single 0-simplex.*

The proof will appear in Section 4.4 after we prove the Whitehead theorem for the category ***GS***.

Chapter 2 COHOMOLOGY GROUPS

2.1 Introduction

To every G-simplicial set K and a suitably defined coefficients system λ, we associate certain cohomology groups $H_G^*(K;\lambda)$. We then show that the equivariant singular Illman cohomology groups $\overline{H}_G^*(X;\lambda)$ [14] of a G-space X (where G is discrete) are isomorphic to the groups $H_G^*(SX;\lambda)$, where SX is the singular G-simplicial set of X.

2.2 Definition of cohomology groups

Throughout R will denote a commutative ring with 1. For any group G, O_G will denote the category of canonical orbits of G.

For future use we record some standard facts about O_G. The category O_G has as objects orbits G/H and as morphisms G-maps $G/H \longrightarrow G/K$. Any G-map $\widehat{a} : G/H \longrightarrow G/K$ is of the form $\widehat{a}(gH) = gaK$, for some $a \in G$ with $a^{-1}Ha \subseteq K$. Moreover $\widehat{a} = \widehat{b}$ if and only if $ab^{-1} \in K$. There is a bijection $Hom_{O_G}(G/H, G/K) \leftrightarrow (G/K)^H$ given by $\widehat{a} \mapsto aK$. Let **R-mod** denote the category of R-modules. Then an O_G-R-module, or a coefficients system, λ is a contravariant functor $\lambda : O_G \longrightarrow$ **R-mod.**

For any G-simplicial set K and an O_G-R-module λ, let $C^n(K;\lambda)$ denote the R-module of functions c defined on n-simplexes $x \in K_n$ such that $c(x) \in \lambda(G/G_x)$, where G_x is the isotropy subgroup at x. The inclusion $G_x \subseteq G_{d_ix}$ gives rise to a morphism $\widehat{e} : G/G_x \longrightarrow G/G_{d_ix}$ in O_G. Denote by $\lambda(d_ix \to x)$ the induced homomorphism of R-modules $\lambda(G/G_{d_ix}) \longrightarrow \lambda(G/G_x)$. Then the homomorphism $\delta^n : C^n(K;\lambda) \longrightarrow C^{n+1}(K;\lambda)$ defined by

$$\delta^n(c)(x) = \sum_{i=0}^{n+1}(-1)^i\lambda(d_ix \to x)c(d_ix) \ .$$

verifies

Lemma 2.2.1 $\delta^{n+1} \circ \delta^n = 0$. ∎

We define an action of the group G on $C^n(K;\lambda)$ by

$$(gc)(x) = \lambda(\widehat{g})(c(g^{-1}x))$$

where $c \in C^n(K;\lambda)$, $x \in K_n$, and $\lambda(\widehat{g}) : \lambda(G/G_{g^{-1}x}) \longrightarrow \lambda(G/G_x)$ is the isomorphism induced by the conjugacy relation $g^{-1}G_x g = G_{g^{-1}x}$.

Lemma 2.2.2 *If $C^n_G(K;\lambda)$ denotes the submodule of G-invariant cochains $(C^n(K;\lambda))^G$, then $\delta^n(C^n_G(K;\lambda)) \subseteq C^{n+1}_G(K;\lambda)$.*

Proof. Let $c \in C^n_G(K;\lambda)$ and $g \in G$. Then for $x \in K_{n+1}$,

$$\begin{aligned} g(\delta c)(x) &= \lambda(\widehat{g})(\delta c)(g^{-1}x) \\ &= \lambda(\widehat{g})\sum_{i=0}^{n+1}(-1)^i\lambda(d_i g^{-1}x \to g^{-1}x)c(d_i g^{-1}x) \\ &= \sum_{i=0}^{n+1}(-1)^i\lambda(d_i x \to x)\lambda(\widehat{g})c(g^{-1}d_i x). \end{aligned}$$

The lemma now follows as $c \in C^n_G(K;\lambda)$. ∎

We then define

$$H^n_G(K;\lambda) = H^n(C^{*}_G(K;\lambda)).$$

Any G-map $f : K \longrightarrow L$ between G-simplicial sets induces a cochain map $f^\# : C^*_G(L;\lambda) \longrightarrow C^*_G(K;\lambda)$ defined by

$$f^\#(c)(x) = \lambda(fx \to x)c(fx).$$

Here $\lambda(fx \to x) : \lambda(G/G_{fx}) \longrightarrow \lambda(G/G_x)$ is the homomorphism induced by the inclusion $G_x \subseteq G_{fx}$. Hence we have a homomorphism

$$f^* : H^*_G(L;\lambda) \longrightarrow H^*_G(K;\lambda)$$

in cohomology. Given a G-simplicial subset K' of K, we may now define the relative groups $H^*_G(K, K';\lambda)$ in the usual way, and obtain a long exact sequence in cohomology of the pair (K, K').

Theorem 2.2.3 *If $f, g : K \longrightarrow L$ are G-homotopic G-maps, then*

$$f^* = g^* : H^*_G(L;\lambda) \longrightarrow H^*_G(K;\lambda).$$

Proof. We shall show that $f^\sharp, g^\sharp : C^*_G(L;\lambda) \longrightarrow C^*_G(K;\lambda)$ are cochain homotopic. Choose a G-homotopy

$$h_i : K_n \longrightarrow L_{n+1} \qquad (0 \leq i \leq n)$$

where h_i are G-functions verifying the relations (1.3) of Section 1.4. Define $h : C^n_G(L;\lambda) \longrightarrow C^{n-1}_G(K;\lambda)$ by

$$h(c)(x) = \sum_{i=0}^{n-1}(-1)^j \lambda(h_j x \to x) c(h_j x).$$

Then some messy calculations will show that $\delta h + h\delta = f^\sharp - g^\sharp$. ∎

Remark 2.2.4 *If the action of G on K is free, then we have $H^*_G(K;\lambda) \cong H^*(K/G;\lambda(G/\{e\})$ for every O_G-R-module λ.*

2.3 Another description of cohomology groups

We now present an alternative description of the cohomology groups. Given a G-simplicial set K, define an O_G-R-module

$$\underline{C}_n(K) : O_G \longrightarrow \mathbf{R\text{-}mod}$$

for each integer $n \geq 0$ in the following way

$$\underline{C}_n(K)(G/H) = C_n(K^H; R),$$

where $C_n(K^H;R)$ denotes the free R-module generated by the n-simplexes of K^H, and, for $\widehat{a} : G/H \longrightarrow G/H'$, $a^{-1}Ha \subseteq H'$,

$$\underline{C}_n(K)(\widehat{a}) = a_*,$$

where a_* is the chain map induced by the left translation $a : K^{H'} \longrightarrow K^H$. Clearly this gives a chain complex $\underline{C}_*(K)$ (where the boundary $\partial : \underline{C}_n(K;R) \longrightarrow \underline{C}_{n-1}(K : R)$ is defined by $\partial(G/H) = \partial_H : C_n(K^H) \longrightarrow C_{n-1}(K^H))$ in the abelian category of O_G-R-modules, and if λ is an O_G-R-module, then $\mathrm{Hom}(\underline{C}_*(K), \lambda)$, which is the R-module of natural transformations $\underline{C}_*(K) \longrightarrow \lambda$, becomes a cochain complex of R-modules.

Theorem 2.3.1 *There is an isomorphism*

$$\Psi : C_G^*(K;\lambda) \longrightarrow Hom(\underline{C}_*(K), \lambda)$$

of cochain complexes.

Proof. Let $c \in C_G^n(K;\lambda)$. Define a natural transformation

$$\Psi(c) : \underline{C}_n(K) \longrightarrow \lambda$$

as follows. If $x \in K_n^H$, then $H \subseteq G_x$, and this induces a homomorphism $\lambda(G_x \to H) : \lambda(G/G_x) \longrightarrow \lambda(G/H)$. Then $\Psi(c)(G/H) : C_n(K^H;R) \longrightarrow \lambda(G/H)$ is the homomorphism

$$\Psi(c)(G/H)(x) = \lambda(G_x \to H)c(x).$$

We must check the following points :

(i) $\Psi(c)$ is indeed a natural transformation.

(ii) Ψ is cochain map.

Proof of (i). Let $a^{-1}Ha \subseteq H'$, then $\hat{a} : G/H \longrightarrow G/H'$ and $\lambda(\hat{a}) : \lambda(G/H') \longrightarrow \lambda(G/H)$. It is required to check the commutativity of the following diagram

$$\begin{array}{ccc} C_n(K^{H'};R) & \xrightarrow{\Psi(c)(G/H')} & \lambda(G/H') \\ \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle \lambda(\widehat{a})} \\ C_n(K^{H};R) & \xrightarrow{\Psi(c)(G/H)} & \lambda(G/H) \end{array}$$

For this purpose, take $x \in K_n^{H'}$. Then, since $H' \subseteq G_x$ and $H \subseteq G_{ax}$, we have a commutative diagram

$$\begin{array}{ccc} G/H & \longrightarrow & G/G_{ax} \\ \downarrow{\scriptstyle \widehat{a}} & & \downarrow{\scriptstyle \widehat{a}} \\ G/H' & \longrightarrow & G/G_x \end{array}$$

Consequently

$$\begin{aligned} \lambda(\widehat{a})\Psi(c)(G/H')(x) &= \lambda(\widehat{a})\lambda(G_x \to H')c(x) \\ &= \lambda(G_{ax} \to H)\lambda(\widehat{a})c(x) \\ &= \lambda(G_{ax} \to H)(ac)(ax) = \Psi(ac)(G/H)(ax) \\ &= \Psi(ac)(G/H)a_*(x) = \Psi(c)(G/H)a_*(x). \end{aligned}$$

Because $c \in C_G^n(K;\lambda)$, this proves naturality of $\Psi(c)$.
Proof of (ii). If $x \in K_{n+1}^H$, then

$$\begin{aligned} \Psi(\delta c)(G/H)(x) &= \lambda(G_x \to H)(\delta c)(x) \\ &= \lambda(G_x \to H)\sum_{i=0}^{n+1}(-1)^i\lambda(d_i x \to x)c(d_i x) \\ &= \sum_{i=0}^{n+1}(-1)^i\lambda(G_{d_i x} \to x)c(d_i x) = \delta(\Psi(c))(x). \end{aligned}$$

This shows that Ψ is a cochain map.

Next, we define $\mu : Hom(\underline{C}_*(K), \lambda) \longrightarrow C^*_G(K;\lambda)$ as follows. If $T : \underline{C}_*(K) \longrightarrow \lambda$ is a natural transformation and $x \in K_n$, then

$$\mu(T)(x) = T(G/G_x)(x).$$

Then, as before, we check the following points

(iii) $\mu(T) \in C^n_G(K;\lambda)$.

(iv) μ is a cochain map.

Proof of (iii). Let $g \in G$, and $x \in K_n$. Then, since T is a natural transformation, there is a commutative diagram

$$\begin{array}{ccc} C_n(K^{G_{g^{-1}x}};R) & \xrightarrow{T(G/G_{g^{-1}x})} & \lambda(G/G_{g^{-1}x}) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle \lambda(\widehat{g})} \\ C_n(K^{G_x};R) & \xrightarrow{T(G/G_x)} & \lambda(G/G_x) \end{array}$$

Therefore

$$\begin{aligned} (g\mu(T))(x) &= \lambda(\widehat{g})(\mu(T))(g^{-1}x) \\ &= \lambda(\widehat{g})T(G/G_{g^{-1}x})(g^{-1}x) \\ &= T(G/G_x)(x) = \mu(T)(x). \end{aligned}$$

This completes the proof of (iii).

Proof of (iv). We compute

$$\begin{aligned} \delta\mu(T)(x) &= \sum_{i=0}^{n+1}(-1)^i\lambda(d_ix \to x)\mu(T)(d_ix) \\ &= \sum_{i=0}^{n+1}(-1)^i\lambda(d_ix \to x)T(G/G_{d_ix})(d_ix). \end{aligned}$$

Once we observe that $G_x \subseteq G_{d_i x}$, the commutativity of

$$\begin{array}{ccc} C_n(K^{G_{d_i x}};R) & \xrightarrow{T(G/G_{d_i x})} & \lambda(G/G_{d_i x}) \\ \downarrow & & \downarrow \\ C_n(K^{G_x};R) & \xrightarrow{T(G/G_x)} & \lambda(G/G_x) \end{array}$$

shows that μ is a cochain map. This completes the proof of (iv).

Finally we observe that

$$\begin{aligned} \Psi(\mu(T))(G/H)(x) &= \lambda(G_x \to H)\mu(T)(x) \\ &= \lambda(G_x \to H)T(G/G_x)(x) = T(G/H)(x) \end{aligned}$$

and

$$\begin{aligned} \mu(\Psi(c))(x) &= \Psi(c)(G/G_x)(x) \\ &= \lambda(G_x \to G_x)c(x) = c(x). \end{aligned}$$

This completes the proof of the theorem. ∎

The category of O_G-R-modules is an abelian category (see [16] p. 258), and has sufficiently many injectives (cf. [21]). The coefficients system $\underline{C}_*(K)$ is projective in this category. Let λ^* be an injective resolution of λ. We have then a double complex $\operatorname{Hom}(\underline{C}_*(K), \lambda^*)$. The homological algebra applied to this double complex yields an universal coefficients spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^s(\underline{H}_t(K), \lambda) \Longrightarrow H_G^{s+t}(K;\lambda).$$

where $\underline{H}_t(K) : O_G \longrightarrow$ **R-mod** is given by

$$\underline{H}_t(K)(G/H) = H_t(K^H;R) \quad \text{and} \quad \underline{H}_t(K)(\hat{a}) = H_t(a).$$

For any G-simplicial set K, let RK denote the G-simplicial set with $(RK)_n = RK_n$, where RK_n is the free R-module with basis K_n, and the face and degeneracy maps are the linear extensions of the corresponding maps of K. The G-action on RK is also obtained as the linear extension of the G-action on K.

Proposition 2.3.2 *For any O_G-R-module $\lambda : O_G \longrightarrow$* **R-mod**, *there is an isomorphism*

$$H_G^*(K;\lambda) \cong H_G^*(RK;\lambda)$$

for every G-simplicial set K.

Proof. The groups $H_G^*(K;\lambda)$ and $H_G^*(RK;\lambda)$ arise from the cochain complexes $\mathrm{Hom}(\underline{C}_*(K),\lambda)$ and $\mathrm{Hom}(\underline{C}_*(RK),\lambda)$ respectively. These cochain complexes are isomorphic by the cochain map $\theta : Hom(\underline{C}_*(K),\lambda) \longrightarrow Hom(\underline{C}_*(RK),\lambda)$ defined by

$$\theta(T)(G/H)(\sum n_i x_i) = T(G/H)\sum n_i x_i. \blacksquare$$

Theorem 2.3.3 *Let $f : K \longrightarrow L$ be G-map such that $f^H : K^H \longrightarrow L^H$ induces isomorphism in the classical homology with R coefficients for every $H \subseteq G$. Then*

$$f^* : H_G^*(L;\lambda) \longrightarrow H_G^*(K;\lambda)$$

is an isomorphism for every O_G-R-module λ.

Proof. The G-map f gives a natural transformation $f_* : \underline{H}_t(K) \longrightarrow \underline{H}_t(L)$ of O_G-R-modules defined by

$$f_*(G/H) = f_*^H.$$

Observe that $f_*(G/H)$ is an isomorphism for every subgroup H of G. As the category of O_G-R-modules is an abelian category, and f_* is both an epi as well as a mono in this category, f_* is an isomorphism. This natural transformation extends to a morphism f_* between the universal coefficients spectral sequences corresponding to K and L. As f_* is an isomorphism at the E_2 level, the theorem follows. ■

Remark 2.3.4 *A pre-simplicial set is a sequence of sets $\{K_n\}$ equipped only with the face maps $d_i : K_n \longrightarrow K_{n-1}$ satisfying $d_i d_j = d_{j-1} d_i$, whenever $i < j$. A G-pre-simplicial set is a pre-simplicial set together with an action of G which commutes with the face maps. We remark that if K is a G-pre-simplicial set then also the groups $H^*_G(K;\lambda)$ can be defined. All the properties that we have mentioned above hold good for these groups, except for the homotopy invariance (Theorem 2.2.3) where the degeneracy maps are essential.*

2.4 Relation with Illman cohomology

In this section we shall prove a theorem relating the equivariant singular cohomology groups of a G-space X and the cohomology groups of the associated singular G-simplicial set SX, where G is discrete.

We briefly recall the construction of the equivariant singular cohomology groups of Illman [14].

An equivariant singular n-simplex in X is a G-map

$$T\colon \Delta^n \times G/H \longrightarrow X,$$

where G is acting trivially on Δ^n. For $0 \leq i \leq n$, the i-th face $T^{(i)}$ of T is the equivariant singular $(n-1)$-simplex given by the composition

$$\Delta^{n-1} \times G/H \xrightarrow{\partial_i \times id} \Delta^n \times G/H \xrightarrow{T} X.$$

Let λ be an O_G-R-module, and $\widehat{S}^n_G(X;\lambda)$ denote the R-module of functions c defined on equivariant singular n-simplexes $T : \Delta^n \times G/H \longrightarrow X$ such that $c(T) \in \lambda(G/H)$. Then, $\delta : \widehat{S}^n_G(X;\lambda) \longrightarrow \widehat{S}^{n+1}_G(X;\lambda)$ defined by

$$\delta(c)(T) = \sum_{i=o}^{n+1} (-1)^i c(T^{(i)})$$

verifies $\delta^2 = 0$, and we have a cochain complex $\widehat{S}^{\bullet}_G(X;\lambda)$. Say that two equivariant singular n-simplexes in X

$$T\colon \Delta^n \times G/K \longrightarrow X \quad \text{and} \quad T'\colon \Delta^n \times G/K' \longrightarrow X$$

are compatible under a G-map $h : \Delta^n \times G/K \longrightarrow \Delta^n \times G/K'$, if $T' \circ h = T$. Since h covers the identity map of Δ^n and G is discrete, h is of the form $(id, \widehat{a})$ for some G-map $\widehat{a} : G/K \longrightarrow G/K'$. Let $h^{\bullet}$ denote the homomorphism $\lambda(\widehat{a}) : \lambda(G/K') \longrightarrow \lambda(G/K)$. Let $S^n_G(X;\lambda)$ be the submodule of $\widehat{S}^n_G(X;\lambda)$ consisting of those $c \in \widehat{S}^n_G(X;\lambda)$ which satisfy the condition that $h^{\bullet}c(T') = c(T)$ whenever $T\colon \Delta^n \times G/K \longrightarrow X$ and $T' : \Delta^n \times G/K' \longrightarrow X$ are compatible under a G-map $h : \Delta^n \times G/K \longrightarrow \Delta^n \times G/K'$. Then $S^{\bullet}_G(X;\lambda)$ is a cochain subcomplex of $\widehat{S}^{\bullet}_G(X;\lambda)$, and the equivariant singular Illman cohomology groups $\bar{H}^{\bullet}_G(X;\lambda)$ of X are by definition the homology groups of the cochain complex $S^{\bullet}_G(X;\lambda)$.

Theorem 2.4.1 *Let X be a G-space with G discrete and λ an O_G-R-module. Then there is an isomorphism*

$$\bar{H}^{\bullet}_G(X;\lambda) \cong H^{\bullet}_G(SX;\lambda)$$

which is functorial with respect to X.

Proof. We shall exhibit an isomorphism

$$C^{\bullet}_G(SX;\lambda) \longrightarrow S^{\bullet}_G(X;\lambda)$$

of cochain complexes.

Let $T\colon \Delta^n \times G/H \longrightarrow X$ be an equivariant singular n-simplex in X. Then $\sigma_T : \Delta^n \longrightarrow X$ with $\sigma_T(x) = T(x, eH)$ is a singular n-simplex in X, that is, a simplex of the singular G-simplicial set SX. Moreover $H \subseteq G_{\sigma_T}$, for if $h \in H$

$$\begin{aligned}(h\sigma_T)(x) &= h\sigma_T(x) = hT(x, eH) \\ &= T(x, eH) = \sigma_T(x).\end{aligned}$$

Thus we have a homomorphism $\lambda(G_{\sigma_T} \to H) : \lambda(G/G_{\sigma_T}) \longrightarrow \lambda(G/H)$. Define $\mu : C^n(SX;\lambda) \longrightarrow \widehat{S}^n_G(X;\lambda)$ by setting $\mu(c)(T) = \lambda(G_{\sigma_T} \to H)c(\sigma_T)$. We now check the following

(i) If $c \in C^n_G(SX;\lambda)$ then $\mu(c) \in S^n_G(X;\lambda)$.

(ii) μ is a cochain map.

Proof of (i). Let $c \in C^n_G(SX;\lambda)$, and

$$T\colon \Delta^n \times G/K \longrightarrow X, \quad T' : \Delta^n \times G/K' \longrightarrow X$$

be two equivariant singular n-simplexes in X compatible under a G-map $h = id \times \widehat{a} : \Delta^n \times G/K \longrightarrow \Delta^n \times G/K'$, where $\widehat{a} : G/K \longrightarrow G/K'$ is given by a subconjugacy relation $a^{-1}Ka \subseteq K'$. We need to check that $h^*\mu(c)(T') = \mu(c)(T)$. As before, we have that $K \subseteq G_{\sigma_T}$ and $K' \subseteq G_{\sigma_{T'}}$. Also note that

$$\begin{aligned} \sigma_{T'}(x) &= T'(x, eK') = a^{-1}T'h(x, eK) \\ &= a^{-1}T(x, eK) = a^{-1}\sigma_T(x). \end{aligned}$$

Therefore referring to the commutative diagram

$$\begin{array}{ccc} G/K & \longrightarrow & G/G_{\sigma_T} \\ \widehat{a}\big\downarrow & & \big\downarrow\widehat{a} \\ G/K' & \longrightarrow & G/G_{\sigma_{T'}} \end{array}$$

we have

$$\begin{aligned} h^*\mu(c)(T') &= \lambda(\widehat{a})\lambda(G_{\sigma_{T'}} \to K')c(\sigma_{T'}) \\ &= \lambda(G_{\sigma_T} \to K)\lambda(\widehat{a})c(a^{-1}\sigma_T) \\ &= \lambda(G_{\sigma_T} \to K)(ac)(\sigma_T) \\ &= \lambda(G_{\sigma_T} \to K)c(\sigma_T) = \mu(c)(T). \end{aligned}$$

This proves (i).

Proof of (ii). Let $c \in C^n_G(SX;\lambda)$ and $T\colon \Delta^{n+1} \times G/H \longrightarrow X$ be an equivariant singular $(n+1)$-simplex in X. Then

$$\begin{aligned} \delta(\mu(c))(T) &= \sum_{i=0}^{n+1}(-1)^i(\mu(c))(T^{(i)}) \\ &= \sum_{i=0}^{n+1}(-1)^i\lambda(G_{\sigma_{T^{(i)}}} \to H)c(\sigma_{T^{(i)}}). \end{aligned} \tag{2.1}$$

On the other hand

$$\begin{aligned} \mu(\delta(c))(T) &= \lambda(G_{\sigma_T} \to H)(\delta c)(\sigma_T) \\ &= \sum_{i=0}^{n+1}(-1)^i\lambda(G_{\sigma_T} \to H)\lambda(d_i\sigma_T \to \sigma_T)c(d_i\sigma_T). \end{aligned} \tag{2.2}$$

Then $d_i\sigma_T = \sigma_{T^{(i)}}$, and the commutativity of orbit maps prove the equality of (2.1) and (2.2).

Next, we define a homomorphism

$$\Psi : \widehat{S}^n_G(X;\lambda) \longrightarrow C^n(SX;\lambda)$$

as follows. Let $\sigma : \Delta^n \longrightarrow X$ be a singular n-simplex in X. Then $T_\sigma : \Delta^n \times G/G_\sigma \longrightarrow X$, given by

$$T_\sigma(x, gG_\sigma) = g\sigma(x)$$

is an equivariant singular n-simplex in X. We set $\Psi(c)(\sigma) = c(T_\sigma)$. As before, we check the following points.

(iii) If $c \in S^n_G(X;\lambda)$ then $\Psi(c) \in C^n_G(SX;\lambda)$.

(iv) Ψ is a cochain map.

Proof of (iii). Let $c \in S^n_G(X;\lambda)$. Then we have for $g \in G$

$$(g\Psi(c))(\sigma) = \lambda(\widehat{g})\Psi(c)(g^{-1}\sigma) = \lambda(\widehat{g})c(T_{g^{-1}\sigma}),$$

where the homomorphism $\lambda(\widehat{g}) : \lambda(G/G_{g^{-1}\sigma}) \longrightarrow \lambda(G/G_\sigma)$ arises from the conjugacy relation $g^{-1}G_\sigma g = G_{g^{-1}\sigma}$. Now observe that

$$T_\sigma : \Delta^n \times G/G_\sigma \longrightarrow X, \quad \text{and} \quad T_{g^{-1}\sigma} : \Delta^n \times G_{g^{-1}\sigma} \longrightarrow X$$

are equivariant singular n-simplexes in X which are compatible under

$$h = (id, \widehat{g}) : \Delta^n \times G/G_\sigma \longrightarrow \Delta^n \times G/G_{g^{-1}\sigma}.$$

Therefore

$$\begin{aligned} \lambda(\widehat{g})c(T_{g^{-1}\sigma}) &= c(T_\sigma) \\ &= \Psi(c)(\sigma), \end{aligned}$$

and hence $g\Psi(c) = \Psi(c)$ showing that $\Psi(c) \in C^n_G(SX;\lambda)$. This completes proof of (iii).

Proof of (iv). Let $c \in S^n_G(X;\lambda)$ and $\sigma : \Delta^{n+1} \longrightarrow X$ be a singular $(n+1)$-simplex in X. Then

$$\begin{aligned} \delta(\Psi(c))(\sigma) &= \sum_{i=0}^{n+1}(-1)^i\lambda(d_i\sigma \to \sigma)\Psi(c)(d_i\sigma) \\ &= \sum_{i=0}^{n+1}(-1)^i\lambda(d_i\sigma \to \sigma)c(T_{d_i\sigma}). \end{aligned}$$

On the other hand

$$\begin{aligned} \Psi(\delta c)(\sigma) &= (\delta c)(T_\sigma) \\ &= \sum_{i=0}^{n+1}(-1)^i c(T_\sigma^{(i)}). \end{aligned}$$

As the singular simplexes $T_\sigma^{(i)}$ and $T_{d_i\sigma}$ are compatible under the G-map $\Delta^n \times G/G_\sigma \xrightarrow{id \times \widehat{e}} \Delta^n \times G/G_{d_i\sigma}$, we have

$$\lambda(\widehat{e})c(T_{d_i\sigma}) = \lambda(d_i\sigma \to \sigma)c(T_{d_i\sigma}) = c(T_\sigma^{(i)}),$$

and therefore, the two expressions above are equal. This shows that Ψ is a cochain map.

Finally we compute the two compositions $\Psi \circ \mu$ and $\mu \circ \Psi$. For $c \in C^n(SX;\lambda)$ and σ a singular n-simplex in X we have

$$\begin{aligned} (\Psi \circ \mu)(c)(\sigma) &= \Psi(\mu(c))(\sigma) = \mu(c)(T_\sigma) \\ &= \lambda(G_{\sigma_{T_\sigma}} \to G_\sigma)c(\sigma_{T_\sigma}) = c(\sigma), \end{aligned}$$

since $\sigma_{T_\sigma} = \sigma$ and as $\lambda(G_{\sigma_{T_\sigma}} \to G_\sigma)) = id$. On the other hand , if $T : \Delta^n \times G/H \longrightarrow X$, then

$$\begin{aligned}(\mu \circ \Psi)(c)(T) &= \mu(\Psi(c))(T) \\ &= \lambda(G_{\sigma_T} \to H)\Psi(c)(\sigma_T) \\ &= \lambda(G_{\sigma_T} \to H)c(T_{\sigma_T}) = c(T),\end{aligned}$$

since the n-simplexes T and T_{σ_T} are compatible under the G-map $id \times \widehat{\varepsilon} : \Delta^n \times G/H \longrightarrow \Delta^n \times G/G_{\sigma_T}$ where $\widehat{\varepsilon}$ is given by the inclusion $H \subseteq G_{\sigma_T}$. ∎

Chapter 3 Some Applications

3.1 Introduction

In this chapter we prove a normalization theorem for cohomology in analogy with the non equivariant case. We then prove that under certain conditions the cohomology groups $H_G^*(K; \lambda_R)$ of a G-simplicial set K with suitable coefficients λ_R are finitely generated. We also discuss the equivariant Euler characteristic of Brown vis-a-vis the 'equivariant Euler characteristic' arising out of the finite generation of the groups H_G^*. We start by recalling the normalization theorem in the non-equivariant case.

3.2 Normalization

If A is a simplicial abelian group, let C_*A denote the chain complex with $C_nA = A_n$, and

$$d = \sum_{i=0}^{n}(-1)^i d_i : C_nA \longrightarrow C_{n-1}A,$$

the alternating sum of the face maps, as the boundary. Let NA denote the graded group

$$NA_n = \{x \in A_n : d_i x = 0, 0 \leq i < n\},$$

and C_*NA denote the chain complex with $C_nNA = NA_n$ and

$$d = (-1)^n d_n : C_nNA \longrightarrow C_{n-1}NA$$

as the boundary. We then have the normalization theorem [18].

Theorem 3.2.1 *The inclusion chain map $i : C_*NA \longrightarrow C_*A$ induces an isomorphism in homology.* ∎

There is a direct sum decomposition $C_*A \cong C_*NA \oplus C_*DA$, where C_*DA is the chain subcomplex of C_*A generated by the degenerate simplexes of A. We also note that NA is a pre-simplicial abelian group (see (2.3.4)).

3.3 G-Normalization

Let K be a G-simplicial set. For any commutative ring R with 1, let RK denote the free simplicial R-module as considered in Proposition 2.3.2. Let NRK denote the G-pre-simplicial R-module

$$NRK_n = \{x \in RK_n : d_i x = 0, 0 \leq i < n\}.$$

Then we have the contravariant functors

$$\begin{aligned} \underline{H}_*(RK) &: O_G \longrightarrow \textbf{R-mod} \\ \underline{H}_*(NRK) &: O_G \longrightarrow \textbf{R-mod} \end{aligned}$$

defined respectively by

$$\begin{aligned} \underline{H}_*(RK)(G/H) &= H_*(RK^H; R), \quad \underline{H}_*(RK)(\hat{g}) = H_*(g) \\ \underline{H}_*(NRK)(G/H) &= H_*(NRK^H; R), \quad \underline{H}_*(NRK)(\hat{g}) = H_*(g). \end{aligned}$$

Theorem 3.3.1 (G-Normalization) *For any G-simplicial set K, there is an isomorphism*

$$H^*_G(K; \lambda) \cong H^*_G(NRK; \lambda)$$

for any O_G-R-module λ.

Proof. Observe that we have coefficients systems

$$\begin{aligned} \underline{C}_*(RK)(G/H) &= C_*(RK^H; R) \\ \underline{C}_*(NRK)(G/H) &= C_*(NRK^H; R) \end{aligned}$$

and the inclusion chain map

$$i : \underline{C}_*(NRK) \longrightarrow \underline{C}_*(RK)$$

(note that $\underline{C}_*(NRK)$ and $\underline{C}_*(RK)$ are chain complexes in the abelian category of O_G-R-modules). By Theorem 3.2.1, $i(G/H) : C_*(NRK^H; R) \longrightarrow C_*(RK^H; R)$ induces isomorphism on homology with R-coefficients. Consequently the natural transformation

$$i_* : \underline{H}_*(NRK) \longrightarrow \underline{H}_*(RK),$$

given by $i_*(G/H) = i(G/H)_*$, is an isomorphism of coefficients systems. The natural transformation $i_* : \underline{H}_*(NRK) \longrightarrow \underline{H}_*(RK)$ induces a morphism of respective universal coefficients spectral sequences. The theorem now follows by an argument similar to the one given in Theorem 2.3.3. ∎

The splitting mentioned at the end of Section 3.2, and Theorem 3.2.1 together imply

Corollary 3.3.2 *If K is a G-simplicial set with no non-degenerate simplexes in dimension above some N, then $H^n_G(K;\lambda) = 0$ whenever $n > N$.* ∎

Remark 3.3.3 *The corollary also follows from the corresponding fact in the non-equivariant case coupled with the fact that for $t > N$ the coefficients system $\underline{H}_t(RK)$ is zero, and a standard argument involving the universal coefficients spectral sequence.*

Let λ_R denote the O_G-R-module defined by $\lambda_R(G/H) = Hom(R(G/H), R)$ where $R(G/H)$ denotes the free R-module with basis G/H. If $\widehat{g} : G/H \longrightarrow G/H'$, then $\lambda_R(\widehat{g}) = Hom(R(\widehat{g}), id)$.

Theorem 3.3.4 *Suppose that the action of G on the G-simplicial set K is such that*

- *K/G has only finitely many non-degenerate simplexes.*
- *The isotropy subgroup of every non-degenerate simplex of K has finite index in G.*

Then the cohomology groups $H^*_G(K;\lambda_{\mathbb{Q}})$ are finitely generated.

Proof. According to Theorem 3.3.1, the groups $H^*_G(K;\lambda_{\mathbb{Q}})$ are the same as the groups $H^*_G(NRK;\lambda_{\mathbb{Q}})$. Consequently it suffices to show that the cochain groups $C^*_G(NRK;\lambda_{\mathbb{Q}})$ are finitely generated. Let $x_1,\ldots,x_k$ denote the representatives of the orbit classes of the non-degenerate n-simplexes which lie in NRK. Suppose that for $1\le l\le k$, the isotropy group G_{x_l} has index m_l in G. Fix a coset representation

$$G/G_{x_l}=\{a_{l_1}G_{x_l},\ldots,a_{l_{m_l}}G_{x_l}\},\quad 1\le l\le k,\quad a_{l_i}\in G.$$

Then define cochains c_{ij} by

$$c_{ij}(x_l)=\begin{cases}0 & j\neq l\\ (a_{l_i}G_{x_l})^* & j=l,\ 1\le i\le m_l\\ 0 & j=l, i>m_l\end{cases}$$

where $(a_{l_i}G_{x_l})^*$ are basis dual to $a^*_{l_i}G_{x_l}$. There is an unique way to define c_{ij} on the orbit of x_l so that $c_{ij}\in C^n_G(NRK;\lambda_{\mathbb{Q}})$. It is also clear that the set $\{c_{ij}\}$ is a linearly independent set. For, if $\sum\alpha_{ij}c_{ij}=0$, $\alpha_{ij}\in\mathbb{Q}$, then

$$\begin{aligned}\sum\alpha_{ij}c_{ij}(x_l) &= \sum\alpha_{il}c_{il}(x_l)\\ &= \sum_{i=1}^{m_l}\alpha_{il}a_{l_i}G_{x_l}=0.\end{aligned}$$

This implies that for $1\le i\le m_l$, $\alpha_{il}=0$. Repeating the argument for various x_l, we see that all $\alpha_{ij}=0$. It is now clear that any invariant cochain can be written in terms of the c_{ij}'s. This proves the theorem. ■

As a simple consequence we get

Corollary 3.3.5 *If the action of G on K satisfies the conditions of the above theorem then dim $C^n_G(K;\lambda_{\mathbb{Q}})=\sum_{l=1}^{k}[G:G_{x_l}]$, where $[G:G_{x_l}]$ is the index of G_{x_l} in G.* ■

3.4 Equivariant Euler characteristic

In this section we indicate how, under certain conditions, the equivariant Euler characteristic of a G-simplicial set may be obtained using the groups H^*_G.

Serre [26] and Brown [5] defined (see also [2]) Euler characteristics of groups which satisfy certain homological finiteness condition. We quickly recall some definitions.

Definition 3.4.1 *A group G is said to have finite cohomological dimension if the $\mathbb{Z}G$-module $\mathbb{Z}$ (with trivial G action) admits a resolution*

$$0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where each P_i is a projective $\mathbb{Z}G$-module.

The group G is said to have virtually finite cohomological dimension (we write $vcd\, G < \infty$) if it has a subgroup of finite index with finite cohomological dimension.

Definition 3.4.2 *A group G is said to be of finite homological type if the following conditions are satisfied :*

- *$vcd\, G < \infty$, and*
- *every torsion free subgroup of finite index has finitely generated rational homology.*

If G is a group such that its rational homology is finitely generated, then its naive Euler characteristic is the rational number

$$\tilde{\chi}(G) = \sum_i (-1)^i \dim H_i(G; \mathbb{Q}),$$

where the homology on the right is the group homology (see [6]).

Definition 3.4.3 *If G is a group of finite homological type, then its Euler characteristic is the number*

$$\chi(G) = \frac{\tilde{\chi}(G')}{[G : G']},$$

where G' is a torsion free subgroup of finite index $[G : G']$.

That this definition is independent of the choice of G' follows from Theorem 3 of ([5], p.237). The Euler characteristic need not be an integer, unlike the naive Euler characteristic which is always an integer. Clearly, if G is a finite group then it has finite homological type and its Euler characteristic equals the reciprocal of the order $|G|$ of the group G.

Suppose that K is a G-simplicial set and the action satisfies

- K/G has only finitely many non-degenerate simplexes.
- The isotropy subgroups G_x have finite homological type.

Definition 3.4.4 *Under the above conditions the equivariant Euler characteristic $\chi_G(K)$ of K is defined to be the rational number*

$$\chi_G(K) = \sum (-1)^{\dim x} \chi(G_x),$$

where the sum is over the set of representatives of non degenerate simplexes of K/G.

Now, if the isotropy subgroups have finite index in G, then by Theorem 3.3.4 we might consider the alternating sum of dimensions of the groups $H^*_G(K; \lambda_{\mathbb{Q}})$. It would be interesting to know how this alternating sum is related to $\chi_G(K)$.

Theorem 3.4.5 *Let G be a group acting freely on the simplicial set K such that*

- *K/G has only finitely many non-degenerate simplexes.*
- *the isotropy subgroups G_x have finite index in G.*

Then $\chi_G(K) = \chi(G)\sum(-1)^i \dim H^i_G(K;\lambda_{\mathbb{Q}})$.

Proof. The group G is necessarily finite. Therefore $\chi_G(K)$ is defined and, by Theorem 3.3.4, the groups $H^{*}_G(K;\lambda_{\mathbb{Q}})$ are finitely generated. As the action is free we have

$$\chi_G(K) = \sum_i (-1)^i N_i = \sum_i (-1)^i \dim H^i(K/G;\mathbb{Q}),$$

where N_i denotes the number of non-degenerate i-simplexes K modulo the action. Consequently, $\chi_G(K) = \chi(K/G)$, the Euler characteristic of K/G. On the other hand the nature of the action implies

$$H^{*}_G(K;\lambda_{\mathbb{Q}}) \cong H^{*}(K/G;\mathbb{Q}(G))$$

(see Remark 2.2.4), and, as

$$\dim H^{*}(K/G;\mathbb{Q}(G)) = |G| \dim H^{*}(K/G;\mathbb{Q}),$$

the theorem follows. ∎

More generally, we have

Theorem 3.4.6 *Let G be a group of finite homological type and finite cohomological dimension. Let K be a G-simplicial set where the action satisfies the conditions*

- *K/G has only finitely many non-degenerate simplexes.*
- *the isotropy subgroups G_x have finite index in G,*

Then $\chi_G(K) = \chi(G)\sum(-1)^i \dim H^i_G(K;\lambda_{\mathbb{Q}})$.

Proof. Since G has finite cohomological dimension, it is torsion free ([6], p.187). As G is of finite homological type, by a result of ([5], p.237), the isotropy groups G_x also have finite homological type and hence $\chi(G_x)$ is defined. Then we may write $\chi(G_x) = \chi(G)[G : G_x]$ ([6], p.248). Therefore

$$\begin{aligned}
\chi_G(K) &= \sum(-1)^{\dim x}\chi(G_x) \\
&= \sum(-1)^{\dim x}\chi(G)[G : G_x] \\
&= \chi(G)\sum(-1)^i [G : G_x] \\
&= \chi(G)\sum_{i=0}^{\dim (K/G)} (-1)^i \dim C^i_G(K;\lambda_{\mathbb{Q}}),
\end{aligned}$$

by Corollary 3.3.5. As we are dealing with vector spaces, it is not difficult to show that

$$\sum_{i=0}^{\dim (K/G)} (-1)^i \dim C^i_G(K;\lambda_{\mathbb{Q}}) = \sum_{i=0}^{\dim (K/G)} (-1)^i \dim H^i_G(K;\lambda_{\mathbb{Q}}).$$

This proves the theorem. ∎

Remark 3.4.7 *In particular, if G is free of rank n then it is of finite homological type, because its virtual cohomological dimension is n. Moreover its rational homology is finitely generated, because there exists a $K(G,1)$ with one 0-cell and n 1-cells [6]. Therefore $\chi(G) = 1 - n$, and*

$$\chi_G(K) = (1-n)\sum_i (-1)^i dim\ H^i_G(K;\lambda_{\mathbb{Q}}),$$

where K is a G-simplicial set and the action satisfies the conditions of Theorem 3.4.6.

Chapter 4 CLOSED MODEL STRUCTURES

4.1 Introduction

In this chapter we study the closed model structure on the category $\boldsymbol{GS}$ of G-simplicial sets and prove the Whitehead theorem in this category. Let $\boldsymbol{Vect_{\mathbb{Q}}}$ denote the category of rational vector spaces and $\boldsymbol{Vec_G}$ the category of contravariant functors $O_G \longrightarrow \boldsymbol{Vect_{\mathbb{Q}}}$. We shall investigate the closed model structure of the category of simplicial objects over $\boldsymbol{Vec_G}$. The results of this chapter are crucial to the proof of the first main theorem in Chapter 6, and that of a generalization of the Kan-Thurston theorem in Chapter 7.

4.2 Preliminaries on model categories

We recall some definitions and results about closed model categories, all of which may be found in Quillen [22].

Definition 4.2.1 *A category $\boldsymbol{C}$ with three distinguished classes of morphisms called cofibrations, fibrations, and weak equivalences is a closed model category if the following conditions are satisfied. A fibration (resp. cofibration) which is also a weak equivalence is called a trivial fibration (resp. trivial cofibration).*

- *$\boldsymbol{C}$ is closed under finite projective and inductive limits.*
- *The following problem, which is called a left lifting problem (LLP),*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

has a solution $B \longrightarrow X$ whenever i is a cofibration, p a fibration, and either i or p a weak equivalence.

- *Any morphism f may be factored as $f = p \circ i$, where p is a trivial fibration and i a cofibration, and also as $f = p \circ i$ where p is a fibration and i is a trivial cofibration.*
- *Fibrations (resp. cofibrations) are stable under pull backs (resp. push outs), compositions, and any isomorphism is both a trivial fibration and a trivial cofibration.*
- *The pull back (resp. push out) of a trivial fibration (resp. trivial cofibration) is a weak equivalence.*
- *For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, if any two of $f, g, g \circ f$ are weak equivalences, so is the third.*
- *Fibrations, cofibrations, and weak equivalences are closed under retracts in the category of diagrams in $\mathcal{C}$.*

If $\mathcal{C}$ is a closed model category, the first condition guarantees the existence of an initial object $\emptyset$ and a terminal object $*$. An object X in $\mathcal{C}$ is fibrant (resp. cofibrant) if the unique morphism $X \longrightarrow *$ (resp. $\emptyset \longrightarrow X$) is a fibration (resp. cofibration).

Example 4.2.2 *The category $\mathcal{S}$ of simplicial sets is a closed model category with the following structure* (see [22])

fibrations	:	*Kan fibrations*
cofibrations	:	*dimensionwise injective maps*
weak equivalences	:	*maps inducing isomorphism in homotopy*
fibrant objects	:	*Kan complexes*
cofibrant objects	:	*simplicial sets.*

Example 4.2.3 *The category $\mathcal{G}$ of simplicial groups is a closed model category with the following structure* (see [22])

fibrations	:	*Kan fibrations*
weak equivalences	:	*maps inducing isomorphism of homotopy groups*
cofibrations	:	*maps having LLP with respect to trivial fibrations*
fibrant objects	:	*every object is fibrant*
cofibrant objects	:	*every object is cofibrant.*

The category $s\,Vect_{\mathbb{Q}}$ of simplicial rational vector spaces is also a closed model category with structure similar to (4.2.3). We now quote two results which we shall need later.

Proposition 4.2.4 *A map $f: A \longrightarrow B$ in $s\,\mathbf{Vect}_{\mathbb{Q}}$ is surjective if and only if f is a fibration and $\pi_0(f)$ is surjective.* ∎

Proposition 4.2.5 ([3], p.16). *Every trivial cofibration $i : A \longrightarrow B$ in a closed model category $\boldsymbol{C}$ with A fibrant admits a retraction.* ∎

We continue with more definitions. In what follows $\boldsymbol{C}$ will always denote a category closed under finite projective and inductive limits.

Definition 4.2.6 *A morphism $f : X \longrightarrow Y$ in $\boldsymbol{C}$ is an effective epimorphism if for every object T in $\boldsymbol{C}$, the sequence of sets*

$$Hom(Y,T) \xrightarrow{f^*} Hom(X,T) \overset{pr_1,pr_2}{\rightrightarrows} Hom(X\times_X X, T)$$

is exact in the sense that $pr_1(a) = pr_2(a)$ implies $f^*(b) = a$ for some b.

Here $X\times_X X$ is the obvious pull back.

Remark 4.2.7 *It is easy to see that if $\boldsymbol{C}$ is the category* **GSETS** *of G-sets, then a morphism in $\boldsymbol{C}$ is an effective epimorphism if and only if it is a genuine set theoretic surjection.*

Definition 4.2.8 *An object P of $\boldsymbol{C}$ is called projective if whenever $X \longrightarrow Y$ is an effective epimorphism the induced map of sets $Hom(P, X) \longrightarrow Hom(P, Y)$ is surjective . Also, $\boldsymbol{C}$ is said to have sufficiently many projectives if for every object X, there exists a projective object P and an effective epimorphism $P \longrightarrow X$.*

Definition 4.2.9 *If $\boldsymbol{C}$ is closed under arbitrary inductive limits, then an object A in $\boldsymbol{C}$ is small if the functor $Hom(A, -)$ commutes with filtered limits (which are inductive limits with countable indexing sets).*

Definition 4.2.10 *A class $\mathcal{U}$ of objects of $\boldsymbol{C}$ is a class of generators, if for every object X there is an effective epimorphism $T \longrightarrow X$ where T is the direct sum of some members of $\mathcal{U}$.*

For any category $\boldsymbol{C}$, $s\boldsymbol{C}$ will denote the category of simplicial objects over $\boldsymbol{C}$. Any object P of $\boldsymbol{C}$ gives rise to a constant simplicial object $\underline{P}$ in $s\boldsymbol{C}$ where $\underline{P}_n = P$ for all n, and all the face and degeneracy maps are the identity morphism of P.

Let X and Y be simplicial objects over $\boldsymbol{C}$ and K a simplicial set. Then an indexed family $f = (f(\sigma)) : X \times K \longrightarrow Y$ is defined to be a collection of $\boldsymbol{C}$-morphisms

$$f(\sigma) : X_q \longrightarrow Y_q,$$

one for each $q \geq 0$ and $\sigma \in K_q$ such that the following diagrams commute

$$\begin{array}{ccc} X_q & \xrightarrow{f(\sigma)} & Y_q \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i} \\ X_{q-1} & \xrightarrow{f(d_i\sigma)} & Y_{q-1} \end{array} \qquad\qquad \begin{array}{ccc} X_q & \xrightarrow{f(\sigma)} & Y_q \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_i} \\ X_{q+1} & \xrightarrow{f(s_i\sigma)} & Y_{q+1} \end{array}$$

Let $Map(X \times K, Y)$ denote the set of indexed families $f : X \times K \longrightarrow Y$. We then have a functor $\underline{Hom}(\ ,\ ) : s\boldsymbol{C}^{op} \times s\boldsymbol{C} \longrightarrow \boldsymbol{S}$ given by

$$\underline{Hom}(X, Y)_n = Map(X \times \Delta[n], Y).$$

We now state a fundamental theorem which is due to Quillen.

Theorem 4.2.11 [22]. *Let $\mathcal{C}$ be a category closed under finite limits and having sufficiently many projectives. Define a map $f : X \longrightarrow Y$ in $s\mathcal{C}$ to be a fibration (resp. weak equivalence) if the simplicial map*

$$\underline{Hom}(\underline{P}, f) : \underline{Hom}(\underline{P}, X) \longrightarrow \underline{Hom}(\underline{P}, Y)$$

is a fibration (resp. weak equivalence) in $\mathcal{S}$ for every projective object P in $\mathcal{C}$, and a cofibration if it has LLP with respect to trivial fibrations. Then $s\mathcal{C}$ is a closed model category, if any of the following two conditions holds :

- *Every object is fibrant.*
- *$\mathcal{C}$ is closed with respect to arbitrary inductive limits and has a set of small projective generators.* ∎

4.3 Closed model structure on $G\mathcal{S}$

Recall that **GSETS** denotes the category of G-sets, and $G\mathcal{S}$, the category of G-simplicial sets, is the category of simplicial objects over **GSETS**. In this section we shall show that $G\mathcal{S}$ is a closed model category and determine its structure. We also prove the Whitehead theorem for G-simplicial sets. Standard reference for the categorical notions we use is [17]. We begin with a

Lemma 4.3.1 **GSETS** *is closed under finite limits and every G-set is projective.*

Proof. Straightforward. ∎

Lemma 4.3.2 **GSETS** *is closed under inductive limits.*

Proof. It suffices to check that **GSETS** has coequalizer of any pair of G-maps, and coproducts. Indeed **GSETS** has arbitrary coproducts (disjoint union), and the coequalizer of a pair of G-maps $f, g : A \longrightarrow B$ is the quotient map $B \longrightarrow B/\sim$, where $\sim$ is the smallest equivalence relation which contains all pairs $(f(a), g(a))$, $a \in A$. ∎

Lemma 4.3.3 **GSETS** *has a set of small projective generators.*

Proof. Every object in **GSETS** is small. Also the coproduct in **GSETS** is disjoint union. We may write any G-set B as

$$B = \coprod_{b \in B} G/G_b$$

where the disjoint union is over one element from each orbit class of B. Then $\mathcal{U} = \{G/H : H \text{ a subgroup of } G\}$ is a set of small projective generators for GSETS. ∎

Lemmas 4.3.1-4.3.3 together with Theorem 4.2.11 now imply

Theorem 4.3.4 *$\mathcal{GS}$ is a closed model category.* ∎

In what follows, we study the closed model structure of $\mathcal{GS}$. It is clear that the functor $\underline{Hom}(\ ,\ ) : \mathcal{GS}^{op} \times \mathcal{GS} \longrightarrow \mathcal{S}$ takes the form

$$\underline{Hom}(K, L) = (L^K)^G,$$

where $(L^K)^G$ is the simplicial set defined in Section 1.5. Since we may regard a G-set G/H as a constant G-simplicial set $\underline{G/H}$, $\mathcal{O}_G^{op}$ may be considered as a subcategory of $\mathcal{GS}^{op}$. Then there is another functor of two variables

$$\Phi(\ ,\ ) : \mathcal{O}_G{}^{op} \times \mathcal{GS} \longrightarrow \mathcal{S}$$

defined by $\Phi(\underline{G/H}, K) = K^H$.

Lemma 4.3.5 *The functors $\underline{Hom}(\ ,\ ), \Phi(\ ,\ ) : \mathcal{O}_G{}^{op} \times \mathcal{GS} \longrightarrow \mathcal{S}$ are naturally equivalent.*

Proof. Define a simplicial map

$$\eta_{(\underline{G/H},K)} : \underline{Hom}(\underline{G/H}, K) \longrightarrow K^H$$

as follows. If $f : \underline{G/H} \times \Delta[n] \longrightarrow K$ is a G-map, that is, an n-simplex of $\underline{Hom}(\underline{G/H}, K)$, then $\eta_{(\underline{G/H},K)}(f) = f(eH, \Delta_n)$. It is easy to see that η is an equivalence. ∎

Proposition 4.3.6 *A G-simplicial set K is fibrant if and only if K is a G-Kan-complex.*

Proof. If $f : K \longrightarrow *$ is a fibration in $\boldsymbol{GS}$, then $\underline{Hom}(\underline{P}, f)$ is a Kan fibration in $\boldsymbol{S}$ for every G-set P. Specializing to $P = G/H$, consider the commutative diagram

$$\begin{array}{ccc} \underline{Hom}(\underline{G/H}, K) & \longrightarrow & K^H \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

In view of Lemma 4.3.5, the top horizontal arrow is an isomorphism. Since the left vertical arrow is a Kan fibration, the right vertical arrow $K^H \longrightarrow *$ a Kan fibration. Therefore, K^H is a Kan-complex, and hence K is a G-Kan complex.

Conversely, if K is a G-Kan-complex, then Theorem 1.5.4 implies that $\underline{Hom}(\underline{P}, K)$ is a Kan-complex for every G-set P. Consequently

$$\underline{Hom}(\underline{P}, f) : \underline{Hom}(\underline{P}, K) \longrightarrow \underline{Hom}(\underline{P}, *) = *$$

is a Kan fibration for every G-set P. This shows that $K \longrightarrow *$ is a fibration in $\boldsymbol{GS}$. ∎

Theorem 4.3.7 *A G-map $f : K \longrightarrow L$ is a fibration if and only if each $f^H : K^H \longrightarrow L^H$ is a Kan fibration.*

Proof. Assuming that $f : K \longrightarrow L$ is a fibration, the method of Proposition 4.3.6 shows that each f^H is a Kan fibration.

Conversely, assume that the G-map $f : K \longrightarrow L$ is such that the simplicial map $f^H : K^H \longrightarrow L^H$ is a Kan fibration for each subgroup H of G. For any G-set P choose a coproduct representation $\coprod_p G/G_p$. Then clearly $\underline{Hom}(\underline{P}, K) \cong \prod_p K^{G_p}$, and we have a commutative diagram

$$\begin{array}{ccc} \underline{Hom}(\underline{P}, K) & \longrightarrow & \prod_p K^{G_p} \\ \downarrow & & \downarrow \\ \underline{Hom}(\underline{P}, L) & \longrightarrow & \prod_p L^{G_p} \end{array}$$

As the horizontal maps are isomorphisms, and the right vertical map $\prod_p f^{G_p}$ a Kan fibration, the left vertical map $\underline{Hom}(\underline{P}, f)$ is a Kan fibration for every G-set P. This proves the theorem. ∎

Along similar lines we may prove

Theorem 4.3.8 *A G-map $f : K \longrightarrow L$ between G-simplicial sets is a weak equivalence if and only if each $f^H : K^H \longrightarrow L^H$ is a weak homotopy equivalence.* ∎

A weak equivalence in ***GS*** will be called a weak G-equivalence.

Lemma 4.3.9 *If $f : K \longrightarrow L$ is a trivial fibration in **GS**, then every LLP*

$$\begin{array}{ccc} \dot{\Delta}[n] \times \underline{G/H} & \xrightarrow{\alpha} & K \\ {\scriptstyle (i\times id)}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta[n] \times \underline{G/H} & \xrightarrow{\beta} & L \end{array}$$

has a solution $h : \Delta[n] \times \underline{G/H} \longrightarrow K$ *with* $h \circ (i \times id) = \alpha$ *and* $f \circ h = \beta$, *where* $i : \dot{\Delta}[n] \longrightarrow \Delta[n]$ *is the inclusion.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \dot{\Delta}[n] & \xrightarrow{\alpha'} & K^H \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f^H} \\ \Delta[n] & \xrightarrow{\beta'} & L^H \end{array}$$

where $\alpha'(x) = \alpha(x, eH)$ and $\beta'(x) = \beta(x, eH)$. By Theorem 4.3.7 and 4.3.8, the map f^H is a Kan fibration and a weak equivalence. Therefore, since i is a cofibration, there is a map $h' : \Delta[n] \longrightarrow K^H$ such that $h' \circ i = \alpha'$ and $f^H \circ h' = \beta'$. Then the G-map $h : \Delta[n] \times \underline{G/H} \longrightarrow K$ defined by $h(x, gH) = gh'(x)$ is the required solution. ∎

Theorem 4.3.10 *Every (dimensionwise) injective map* $f : K \longrightarrow L$ *in* ***GS*** *is a cofibration.*

Proof. Recall from Theorem 4.2.11 that a G-map f is a cofibration if it has LLP with respect to trivial fibrations p. Consider such a LLP,

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & K' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{\beta} & L' \end{array}$$

We may suppose without loss of generality that f is an inclusion. Define G-simplicial subsets $L(n)$ of L in the following way:

$$L(-1) = K, \quad \text{and} \quad L(n) = L^{(n)} \cup K, \ n \geq 0.$$

Then $L = \cup L(n)$, and $L(n)$ is obtained from $L(n-1)$ by attaching $\coprod_{H\subseteq G} \Delta[n] \times \underline{G/H}$ via a G-map $k_n : \coprod_{H\subseteq G} \dot{\Delta}[n] \times \underline{G/H} \longrightarrow L(n-1)$. We shall define inductively G-maps $h_n : L(n) \longrightarrow K'$ such that $h_n \circ f = \alpha$ and $p \circ h_n = \beta$. Then the G-map $h : L \longrightarrow K'$ defined by setting $h|L(n) = h_n$ will be the solution to the LLP.

Take $h_{-1} = \alpha$ and assume that $h_{n-1} : L(n-1) \longrightarrow K'$ is already defined such that $h_{n-1} \circ f = \alpha$ and $p \circ h_{n-1} = \beta$. Now, since the following diagram commutes

$$\begin{array}{ccc} \coprod_{H\subseteq G} \dot{\Delta}[n] \times \underline{G/H} & \xrightarrow{h_{n-1}\circ k_n} & K' \\ {\scriptstyle i\times id}\downarrow & & \downarrow{\scriptstyle p} \\ \coprod_{H\subseteq G} \Delta[n] \times \underline{G/H} & \xrightarrow{\beta\circ\overline{k_n}} & L' \end{array}$$

where $\overline{k_n}$ is the characteristic map corresponding to k_n, and since p is a trivial fibration, we may use Lemma 4.3.9 to complete the above diagram by a G-map $\widetilde{h_n} : \coprod_{H\subseteq G} \Delta[n] \times \underline{G/H} \longrightarrow K'$ so that the resulting triangles also become commutative. Now define $h_n : L(n) \longrightarrow K'$ by $h_n(x) = h_{n-1}(x)$ if $x \in L(n-1)$ and $h_n(x) = \widetilde{h_n}(x)$ otherwise. It may be checked using a push out diagram that h_n is indeed the G-map we are looking for. This completes the proof. ∎

Remark 4.3.11 *The above theorem shows that every G-simplicial K set is cofibrant, and the canonical G-map $\psi(K) : K \longrightarrow S|K|$ is a trivial cofibration. If K is a G-Kan-complex, Proposition 4.2.5 guarantees a retraction $r_K : S|K| \longrightarrow K$.*

By a G-Kan pair we mean a pair (K, L) of G-simplicial sets where both K and L are G-Kan-complexes. As in the topological case, a result basic to proving the Whitehead theorem is the following result on extension of simplicial maps.

Proposition 4.3.12 *Suppose (K, L) and (K', L') are G-Kan pairs with $f : L' \longrightarrow K'$ a weak G-equivalence. If*

$$k : L \times \{1\} \longrightarrow L' \quad \textit{and} \quad F : K \times \{0\} \cup L \times \Delta[1] \longrightarrow K'$$

are G-simplicial maps with $f \circ k = F|L \times \{1\}$, then there exist G-simplicial maps

$$\tilde{F} : K \times \Delta[1] \longrightarrow K' \quad \textit{and} \quad \tilde{k} : K \times \{1\} \longrightarrow L'$$

such that $\tilde{F}|(K \times \{0\} \cup L \times \Delta[1]) = F$ and $f \circ \tilde{k} = \tilde{F}|K \times \{1\}$.

Proof. The G-simplicial maps k and F give rise to G-maps between G-CW-complexes

$$|k| : |L| \times \{1\} \longrightarrow |L'| \quad \text{and} \quad |F| : |K| \times \{0\} \cup |L| \times I \longrightarrow K'$$

by means of the geometric realization functor. The corresponding extension problem for $|F|, |k|, |f|$ has a solution (see [9] p.106)

$$F' : |K| \times I \longrightarrow |K'|, \quad k' : |K| \times \{1\} \longrightarrow |L'|.$$

These give G-simplicial maps

$$SF' : S|K| \times \Delta[1] \longrightarrow S|K'| \quad \text{and} \quad Sk' : S|K| \times \{1\} \longrightarrow S|L'|.$$

Then the G-simplicial maps $\tilde{F} = r_{K'} \circ SF' \circ \psi(K \times \Delta[1])$ and $\tilde{k} = Sk'$ give the required extension, where $r_{K'}$ is the retraction of Remark 4.3.11. ∎

We may now easily prove the Whitehead theorem in $\mathcal{GS}$ following exactly the steps of the corresponding proof for G-spaces [9].

Theorem 4.3.13 *A G-simplicial map $f : K \longrightarrow L$ between G-Kan-complexes is a weak G-equivalence if and only if it is a G-homotopy equivalence.* ∎

4.4 Proof of Theorem 1.5.7

Recall from [18] that two n-simplexes x and y in a simplicial set K are homotopic if $d_ix = d_iy$, $0 \leq i \leq n$, and there is a $z \in K_{n+1}$ (called a **homotopy** from x to y) such that

$$d_nz = x, \quad d_{n+1}z = y, \quad \text{and} \quad d_iz = s_{n-1}d_ix = s_{n-1}d_iy, \quad 0 \leq i < n.$$

If K is a Kan-complex, then homotopy of simplexes is an equivalence relation.

Let K be a Kan-complex with a zero-simplex x_0. Any degeneracy of x_0 will again be denoted by x_0. Let $\widetilde{K}_n$ denote the set of n-simplexes of K having all its faces at x_0. Then the n^{th}-homotopy group $\pi_n(K)$ of K is by definition the set $\widetilde{K}_n$ modulo the relation of homotopy.

We shall modify the above relation to obtain a proof of Theorem 1.5.7. Let K be a G-Kan-complex with a vertex $x_0 \in K_0^G$. Define the sets

$$\begin{aligned} F_0 &= \{x_0\} \\ F_n &= \bigcup_{H \subseteq G} \{\alpha : G/H \longrightarrow K_n \,|\alpha \text{ is a } G\text{-map}, \quad d_i\alpha \in F_{n-1}, \quad 0 \leq i \leq n\}, n \geq 1, \end{aligned}$$

and a relation $\sim$ in F_n as follows :

$$\alpha \sim \beta \iff \begin{cases} (1) \ d_i\alpha = d_i\beta \text{ where } d_i : K_n \longrightarrow K_{n-1}, 0 \leq i \leq n. \\ (2) \ \text{There exists a } G\text{-map } \gamma : G/H \longrightarrow K_{n+1} \text{ with } d_n\gamma = \alpha, \\ \quad\ d_{n+1}\gamma = \beta, \text{ and } d_i\gamma = s_{n-1}d_i\alpha = s_{n-1}d_i\beta, \ 0 \leq i < n. \end{cases}$$

We shall then say that α and β are related via γ.

Lemma 4.4.1 *$\sim$ is an equivalence relation.*

Proof. **(i) (Reflexivity).** Let $\alpha : G/H \longrightarrow K_n$ be a G-map. Then the G-map $\gamma : G/H \longrightarrow K_{n+1}$, where $\gamma = s_n\alpha$, has the property that

$$d_n\gamma = \alpha, \quad d_{n+1}\gamma = \alpha, \quad \text{and} \quad d_i\gamma = s_{n-1}d_i\alpha = s_{n-1}d_i\beta, \quad 0 \leq i < n.$$

(ii) Symmetry. Let α and β be related via γ. It is then clear that the $n+1$-simplex $\gamma(eH)$ is a homotopy from the n-simplex $\alpha(eH)$ to $\beta(eH)$ of K^H. As K^H is a Kan-complex, we can find a $(n+1)$-simplex z which gives a homotopy from $\beta(eH)$ to $\alpha(eH)$. We may now easily check that β and α are related via the G-map $\gamma' : G/H \longrightarrow K_{n+1}$ defined by $\gamma'(gH) = gz$.
(iii) Transitivity. Let α and β be related via γ, and β and θ be related via γ'. Proceeding as in (ii), we see that $\alpha(eH)$ and $\theta(eH)$ are homotopic. We may find, as above, a G-map $\tilde{\gamma}$ which relates α and θ. ∎

Let $\widetilde{K}$ denote the G-simplicial set generated by the G-set

$$K' = \cup_n K'_n \quad \text{where} \quad K'_n = \bigcup_{\alpha \in \Sigma_n} \{\text{orbit}(\alpha(eH)) | \quad \alpha : G/H \longrightarrow K_n\}.$$

Here Σ_n denotes the set consisting of one element from each equivalence class in $F_n/\sim$. Clearly, $\widetilde{K}$ has only one zero simplex. We then show that

Proposition 4.4.2 *$\widetilde{K}$ is a G-Kan-complex.*

Proof. We check that $\widetilde{K}^H$ is a Kan-complex for every subgroup H of G. Let

$$x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$$

be a compatible collection of $(n-1)$-simplexes of $\widetilde{K}^H$, that is,

$$d_i x_j = d_{j-1} x_i \quad (i < j, \quad i \neq k, \quad j \neq k).$$

These x_i are also compatible as simplexes in K^H, and as K^H is a Kan-complex, there exists an $x \in K^H_n$ with $d_i x = x_i$, $i \neq k$. If $x \in \widetilde{K}^H_n$ we are through. So suppose that $x \notin \widetilde{K}^H$. Then we have a G-map $\alpha_x : G/H \longrightarrow K_n$ defined by $\alpha_x(gH) = gx$. We may now find a G-map $\beta : G/H \longrightarrow K_n$ such that $\alpha \sim \beta$ and $\beta \in \Sigma_n$. Then the n-simplex $\tilde{x} = \beta(eH)$ has the property that $d_i\tilde{x} = x_i$ and $\tilde{x} \in \widetilde{K}^H_n$. This completes the proof. ∎

The following proposition now completes the proof of Theorem 1.5.7.

Proposition 4.4.3 *The inclusion map $i : \widetilde{K} \longrightarrow K$ is a G-homotopy equivalence.*

Proof. In view of Theorem 4.3.13, it suffices to check that each homomorphism

$$i_*^H : \pi_n(\widetilde{K}^H) \longrightarrow \pi_n(K^H)$$

is an isomorphism for each $n \geq 0$.
(i) i_*^H *is one-one.*
Suppose that $i_*^H([x]) = i_*^H([y])$. Then the simplexes $i^H(x)$ and $i^H(y)$ are homotopic in K^H. If z is a homotopy from $i^H(x)$ to $i^H(y)$, then the G-maps $\alpha_x, \alpha_y : G/H \longrightarrow K_n$ defined by

$$\alpha_x(gH) = gi^H(x), \quad \text{and} \quad \alpha_y(gH) = gi^H(y)$$

are related via the G-map $\gamma_z : G/H \longrightarrow K_{n+1}$ defined by $\gamma_z(gH) = gz$. If $\gamma_z \in \Sigma_{n+1}$, we are through. If not we can always find a G-map γ' with $\gamma_z \sim \gamma'$ and $\gamma' \in \Sigma_{n+1}$. Then the $(n+1)$-simplex $\gamma'(eH) \in \widetilde{K}^H$ gives a homotopy from x to y. This shows that i_*^H is one-one.
(ii) i_*^H *is onto.*
Let $[x] \in \pi_n(K^H)$. We have a G-map $\alpha_x : G/H \longrightarrow K_n$ defined by $\alpha_x(gH) = gx$. Let γ denote the G-map which is related to α_x and $\gamma \in \Sigma_n$. Then clearly $\tilde{x} = \gamma(eH) \in \widetilde{K}^H$ and $i_*^H(\tilde{x}) = [x]$. ■

4.5 Closed model structure on $sVec_G$

Recall that $\boldsymbol{Vect_{\mathbb{Q}}}$ denotes the category of rational vector spaces, and Vec_G the category of contravariant functors $O_G \longrightarrow \boldsymbol{Vect_{\mathbb{Q}}}$. Then Vec_G is an abelian category (see [17] p.258), and therefore a morphism $s : T \longrightarrow S$ in Vec_G is epi (resp. mono) if and only if $s(G/H)$ is epi (resp. mono) in $\boldsymbol{Vect_{\mathbb{Q}}}$ for each subgroup H of G. In this section G will consistently denote a finite group. It is then a result of Triantafillou that

Theorem 4.5.1 (Triantafillou [29]). *The category Vec_G has sufficiently many projectives.* ■

Let $s\,Vec_G$ denote, as usual, the category of simplicial objects over Vec_G, and fibrations, cofibrations, weak equivalences in $s\,Vec_G$ be defined as in Theorem 4.2.11. It will be seen in Proposition 4.5.8 below that every object in $s\,Vec_G$ is fibrant. Consequently Quillen's theorem will imply

Theorem 4.5.2 *$s\,Vec_G$ is a closed model category.* ■

Before studying the model structure of $s\,Vec_G$, let us briefly recall from [29] the description of projective objects of Vec_G.

The G-set G/H may be written as a disjoint union

$$G/H = (G/H)^{H_0} \cup \ldots\ldots \cup (G/H)^{H_m}$$

where $H_0 = H, H_1, \ldots, H_m$ are distinct subgroups of G conjugate to H. As NH/H acts on $(G/H)^{H_0}$ on the right (NH is the normalizer of H), and as the sets $(G/H)^{H_i}$ are all isomorphic, NH/H acts on $(G/H)^{H_i}$ in the same way. Each of these action is free and transitive, and is given by

$$(gH)(aH) = gaH.$$

If H' is a subgroup of G, then $(G/H)^{H'}$ contains those $(G/H)^{H_i}$ for which $H' \subseteq H_i$. Consequently, there is an action of NH/H on $(G/H)^{H'}$ which is trivial outside those $(G/H)^{H_i}$ which are contained in $(G/H)^{H'}$. Hence the rational vector space $\mathbb{Q}((G/H)^{H'})$ with basis $(G/H)^{H'}$ becomes a free right $\mathbb{Q}(NH/H)$-module with a basis consisting of one element from each $(G/H)^{H_i}$ for which $H' \subseteq H_i$.

Now, if V_H is a left $\mathbb{Q}(NH/H)$-module, then we have a contravariant functor $\underline{V}_H : O_G \longrightarrow Vect_{\mathbb{Q}}$ defined by setting

$$\underline{V}_H(G/H') = \mathbb{Q}((G/H)^{H'}) \otimes_{\mathbb{Q}(NH/H)} V_H$$

on objects G/H' of O_G, and, if $\widehat{a} : G/H' \longrightarrow G/H''$ is a morphism in O_G, then $\underline{V}_H(\widehat{a}) : \underline{V}_H(G/H'') \longrightarrow \underline{V}_H(G/H')$ is given by

$$\underline{V}_H(\widehat{a})(\widehat{b} \otimes v) = (\widehat{b} \circ \widehat{a}) \otimes v,$$

where $\widehat{b} \in (G/H)^{H''} = Hom_{O_G}(G/H'', G/H)$ and $v \in \underline{V}_H$. It is then not very difficult to see that $\underline{V}_H$ is projective as an object of $\boldsymbol{Vec_G}$. Also observe that $\mathbb{Q}(NH/H)$ is semi-simple, and hence every left $\mathbb{Q}(NH/H)$-module is projective.

Again if A is a G-set, define an object $\underline{\mathbb{Q}}(A) \in \boldsymbol{Vec_G}$ by $\underline{\mathbb{Q}}(A)(G/H) = \mathbb{Q}(A^H)$, where $\mathbb{Q}(A^H)$ denotes the $\mathbb{Q}$-vector space with basis A^H. Therefore, if we take $V_H = \mathbb{Q}(NH/H)$, we have

$$\begin{aligned} \underline{V}_H(G/H') &= \mathbb{Q}((G/H)^{H'}) \otimes_{\mathbb{Q}(NH/H)} \mathbb{Q}(NH/H) \\ &= \mathbb{Q}((G/H)^{H'}) \\ &= \underline{\mathbb{Q}}(G/H)(G/H'). \end{aligned}$$

Thus $\underline{V}_H = \underline{\mathbb{Q}}(G/H)$, and this implies $\underline{\mathbb{Q}}(G/H)$ is projective as an object in $\boldsymbol{Vec_G}$. As every G-set A is the disjoint union of its orbits, the object $\underline{\mathbb{Q}}(A)$ in $\boldsymbol{Vec_G}$ is also projective.

We now proceed to the study of closed model structure of $\boldsymbol{s\,Vec_G}$. First, note that the category $\boldsymbol{s\,Vec_G}$ of simplicial objects over $\boldsymbol{Vec_G}$ is same as the category of contravariant functors from O_G to the category $\boldsymbol{s\,Vect_{\mathbb{Q}}}$ of simplicial vector spaces. Any such contravariant functor $T\colon O_G \longrightarrow \boldsymbol{s\,Vect_{\mathbb{Q}}}$ may be identified with a contravariant functor $\beta T\colon \Delta \longrightarrow \boldsymbol{Vec_G}$ by means of the bijection

$$\beta : \mathrm{Contra}(O_G, \boldsymbol{s\,Vect_{\mathbb{Q}}}) \longrightarrow \mathrm{Contra}(\Delta, \boldsymbol{Vec_G}) = \boldsymbol{s\,Vec_G}$$

given by $\beta T([n])(G/H) = T(G/H)_n$. Now look at the following diagram of categories and functors.

$$\begin{array}{ccc} O_G{}^{op} \times \mathrm{Contra}(O_G, \boldsymbol{s\,Vect_{\mathbb{Q}}}) & \xrightarrow{\underline{\mathbb{Q}} \times \beta} & \boldsymbol{Vec_G}{}^{op} \times \boldsymbol{s\,Vec_G} \\ & {\scriptstyle Ev} \searrow \quad \swarrow {\scriptstyle \underline{Hom}} & \\ & \mathcal{S} & \end{array}$$

Here $\underline{\mathbb{Q}} : O_G \longrightarrow \boldsymbol{Vec}_G$ is the covariant functor $G/H \mapsto \underline{\mathbb{Q}}(G/H)$, $\underline{Hom}$ is the functor explained in Section 4.2, and Ev is the evaluation functor

$$Ev(G/H,T) = T(G/H).$$

Observe that for $M \in \boldsymbol{Vec}_G$ and $T \in \boldsymbol{sVec}_G$, $\underline{Hom}(\underline{M},T)$ is a simplicial $\mathbb{Q}$-vector space. Henceforth we shall identify $\mathrm{Contra}(O_G, \boldsymbol{sVect}_{\mathbb{Q}})$ with $\boldsymbol{sVec}_G$.

For G/H in O_G and $T \in \mathrm{Contra}(O_G, \boldsymbol{sVect}_{\mathbb{Q}}) = \boldsymbol{sVec}_G$, we define a map

$$\eta_{(T,H)} : T(G/H) \longrightarrow \underline{Hom}(\underline{\mathbb{Q}}(G/H), T)$$

as follows. Let $x \in T(G/H)_0$ be a zero simplex. Then $\eta_{(T,H)}(x)$ is a collection of natural transformations

$$\eta_{(T,H)}(x)(\sigma) : \underline{\mathbb{Q}}(G/H) \longrightarrow T_q$$

one for each $\sigma \in \Delta[0]_q$ and each $q \geq 0$. For $q = 0$ and $\sigma = (0)$, the natural transformation $\eta_{(T,H)}(x)(\sigma)$ is given by the linear transformation

$$\eta_{(T,H)}(x)(\sigma)(G/K) : \mathbb{Q}((G/H)^K) \longrightarrow T(G/K)_0$$

where

$$(\eta_{(T,H)}(x)(\sigma))(G/K)(\widehat{a}) = T(\widehat{a})(x).$$

This definition makes sense, because we have a bijection $(G/H)^K \longrightarrow Hom_{O_G}(G/K, G/H)$ which provides the vector space $\mathbb{Q}((G/H)^K)$ with a basis consisting of G-maps $\widehat{a} : G/K \longrightarrow G/H$. Now extend $\eta_{(T,H)}(x)(\sigma)$ simplicially to all degeneracies of σ and x. In general, if $x \in T(G/H)_n$ is non-degenerate, we define the natural transformation

$$\eta_{(T,H)}(x)(\Delta_n) : \underline{\mathbb{Q}}(G/H) \longrightarrow T_n$$

by setting

$$((\eta_{(T,H)}(x)(\Delta_n))(G/K))(\widehat{a}) = T(\widehat{a})(x),$$

and then extend it simplicially over all faces and degeneracies of x and Δ_n. This process defines $\eta_{(T,H)}$ on the whole of $T(G/H)$. It is now clear that

Lemma 4.5.3 *The map* $\eta_{(T,H)} : T(G/H) \longrightarrow \underline{Hom}(\underline{\mathbb{Q}}(G/H), T)$ *as defined above is a simplicial map.* ∎

Next, we define a map

$$\xi_{(T,H)} : \underline{Hom}(\underline{\mathbb{Q}}(G/H), T) \longrightarrow T(G/H)$$

(which will be the inverse of $\eta_{(T,H)}$) as follows. Let $f = (f(\sigma))$ be an n-simplex of $\underline{Hom}(\underline{\mathbb{Q}}(G/H), T)$. Then $f(\Delta_n) : \underline{\mathbb{Q}}(G/H) \longrightarrow T_n$ is a natural transformation. This gives a linear transformation

$$f(\Delta_n)(G/H) : \mathbb{Q}((G/H)^H) \longrightarrow T(G/H)_n.$$

We then set

$$\xi_{(T,H)}(f) = f(\Delta_n)(G/H)(\widehat{e}),$$

where $\widehat{e} : G/H \longrightarrow G/H$ is the identity map. Then

Lemma 4.5.4 $\xi_{(T,H)} : \underline{Hom}(\underline{\mathbb{Q}}(G/H), T) \longrightarrow T(G/H)$ *is a simplicial map.*

Proof. Let $f = (f(\sigma)) \in \underline{Hom}(\underline{\mathbb{Q}}(G/H), T_n)$ be an n-simplex. Then,

$$d_i \xi_{(T,H)}(f) = d_i((f(\Delta_n)(G/H))(\widehat{e})).$$

Let $d_i f = (h(\sigma'))$. Then, by definition, the $(n-1)$-simplex $d_i f$ is given by natural transformations

$$h(\sigma') : \underline{\mathbb{Q}}(G/H) \longrightarrow T_q$$

one for each $q \geq 0$ and $\sigma' \in \Delta[n-1]_q$ such that

$$h(\sigma') = f(\partial_i \sigma'),$$

where $\partial_i : \Delta[n-1] \longrightarrow \Delta[n]$ is the usual simplicial map. Therefore

$$\begin{aligned} \xi_{(T,H)}(d_i f) &= h(\Delta_{n-1})(G/H)(\widehat{e}) \\ &= f(\partial_i \Delta_{n-1})(G/H)(\widehat{e}) \\ &= f(d_i \Delta_n)(G/H)(\widehat{e}), \end{aligned}$$

because $\partial_i \triangle_{n-1} = d_i \triangle_n$. Now since f is an n-simplex, we have a commutative diagram

$$\begin{array}{ccc} \underline{\mathbb{Q}}(G/H) & \xrightarrow{f(\triangle_n)} & T_n \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle d_i} \\ \underline{\mathbb{Q}}(G/H) & \xrightarrow{f(d_i\triangle_n)} & T_{n-1} \end{array}$$

Hence $\xi_{(T,H)}(d_i f) = d_i f(\triangle_n)(G/H)(\hat{e}) = d_i \xi_{(T,H)}(f)$. We may similarly check with the degeneracies. This completes the proof. ∎

Lemma 4.5.5 $\xi_{(T,H)} \circ \eta_{(T,H)} = id$.

Proof. It is enough to check this on the non-degenerate simplexes of $T(G/H)$. If $x \in T(G/H)_n$ is non-degenerate, then

$$\begin{aligned} \xi_{(T,H)} \circ \eta_{(T,H)}(x) &= \xi_{(T,H)}(\eta_{(T,H)}(x)(\sigma)) \\ &= \eta_{(T,H)}(x)(\triangle_n)(G/H)(\hat{e}) \\ &= T(\hat{e})(x) = x \end{aligned}$$

as $T(\hat{e})$ is the identity. ∎

Lemma 4.5.6 $\eta_{(T,H)} \circ \xi_{(T,H)} = id$

Proof. Let $f = (f(\sigma))$ be an n-simplex of $\underline{Hom}(\underline{\mathbb{Q}}(G/H), T)$. Then denote

$$\eta_{(T,H)} \circ \xi_{(T,H)}(f) = \eta_{(T,H)}(f(\triangle_n)(G/H)(\hat{e})) \tag{4.1}$$

by h, where $h = (h(\sigma))$, say. To show $h = f$, it suffices to check that $h(\triangle_n) = f(\triangle_n)$. Now these are natural transformations

$$f(\triangle_n), h(\triangle_n) : \underline{\mathbb{Q}}(G/H) \longrightarrow T_n,$$

and hence there are linear transformations

$$h(\Delta_n)(G/K), f(\Delta_n)(G/K) : \mathbb{Q}((G/H)^K) \longrightarrow T(G/K)_n.$$

where in view of (4.1), h has the form

$$h(\Delta_n)(G/K)(\widehat{a}) = T(\widehat{a})(f(\Delta_n)(G/H)(\widehat{e})).$$

Now, since $f(\Delta_n)$ is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Q}(G/H)^H & \xrightarrow{f(\Delta_n)(G/H)} & T(G/H)_n \\
\big\downarrow {\scriptstyle \widehat{a}\circ -} & & \big\downarrow {\scriptstyle T(\widehat{a})} \\
\mathbb{Q}(G/H)^K & \xrightarrow{f(\Delta_n)(G/K)} & T(G/K)_n
\end{array}$$

where $\widehat{a} : G/K \longrightarrow G/H$ is considered as an element of $(G/H)^K$ via the identification $(G/H)^K \longrightarrow Hom_{O_G}(G/K, G/H)$. Therefore $h(\Delta_n)(G/K)(\widehat{a}) = f(\Delta_n)(G/K)(\widehat{a})$, and the lemma is proved. ∎

Now suppose that $s : T \longrightarrow T'$ is a natural transformation of O_G-simplicial vector spaces. Then the following diagrams commute:

$$\begin{array}{ccc}
T(G/H) & \xrightarrow{\eta(T,H)} & \underline{Hom}(\mathbb{Q}(G/H), T) \\
\big\downarrow {\scriptstyle s(G/H)} & & \big\downarrow {\scriptstyle \underline{Hom}(\mathrm{id}, s)} \\
T'(G/H) & \xrightarrow{\eta(T',H)} & \underline{Hom}(\mathbb{Q}(G/H), T')
\end{array} \qquad (4.2)$$

$$\begin{array}{ccc}
T(G/H) & \xleftarrow{\xi(T,H)} & \underline{Hom}(\mathbb{Q}(G/H), T) \\
\big\downarrow {\scriptstyle s(G/H)} & & \big\downarrow {\scriptstyle \underline{Hom}(\mathrm{id}, s)} \\
T'(G/H) & \xleftarrow{\xi(T',H)} & \underline{Hom}(\mathbb{Q}(G/H), T')
\end{array} \qquad (4.3)$$

To check the commutativity of diagram (4.2), choose a non-degenerate simplex $x \in T(G/H)_n$. Then $(\underline{Hom}(id, s) \circ \eta_{(T,H)})(x)$ has the form $(s \circ \eta_{(T,H)}(x)(\sigma)$. Hence

$$(s \circ \eta_{(T,H)}(x)(\sigma))(G/K)(\widehat{a}) = s(G/K)\,T(\widehat{a})(x). \tag{4.4}$$

On the other hand

$$\begin{aligned} \eta_{(T',H)}s(G/H)(x) &= \eta_{(T',H)}(s(G/H)(x)) \\ &= (\eta_{(T',H)}(s(G/H)(x))(\sigma)), \end{aligned}$$

so that

$$\eta_{(T',H)}(s(G/H)(x))(\sigma)(G/K)(\widehat{a}) = T'(\widehat{a})(s(G/H)(x)) \tag{4.5}$$

The expressions of (4.4) and (4.5) are clearly equal since s is a natural transformation. To check the commutativity of diagram (4.3), we compute

$$s(G/H) \circ \xi_{(T,H)}(f) = s(G/H)f(\Delta_n)(G/H)(\widehat{e}).$$

On the other hand

$$\begin{aligned} \underline{Hom}(id, s) \circ \xi_{(T',H)}(f) &= \xi_{(T',H)}(s \circ f(\sigma)) \\ &= s(G/H)f(\Delta_n)(G/H)(\widehat{e}). \end{aligned}$$

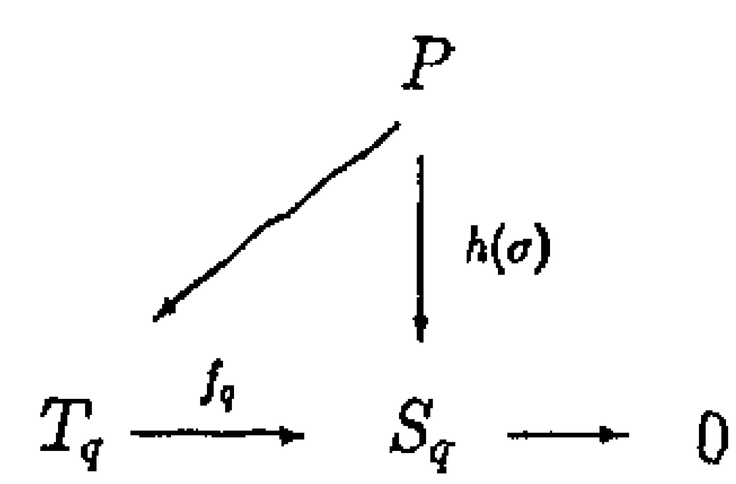
Keeping Quillen's theorem (Theorem 4.2.11) in mind, we now study the closed model structure of $s\,Vec_G$.

Theorem 4.5.7 *A morphism $f : T \longrightarrow S$ in $s\,Vec_G$ is surjective if and only if it is a fibration and $\pi_0(f(G/H))$ is surjective.*

Proof. Assume that $f : T \longrightarrow S$ is a surjection, that is each $f(G/H)$ is surjective as a map of simplicial vector spaces. Proposition 4.2.4 then implies that $\pi_0 f(G/H)$ is surjective. If P is any projective object of Vec_G, we show that the simplicial map

$$\underline{Hom}(\underline{P}, f) : \underline{Hom}(\underline{P}, T) \longrightarrow \underline{Hom}(\underline{P}, S)$$

of simplicial vector spaces is a Kan fibration in $\mathcal{S}$. It suffices to show that the map $\underline{Hom}(\underline{P}, f)$ is surjective. Let $h = (h(\sigma)) \in \underline{Hom}(\underline{P}, S)_n$ be an n-simplex. Define $\tilde{h} = (\tilde{h}(\sigma)) \in \underline{Hom}(\underline{P}, T)$ as follows. If $q \geq 0$ and $\sigma \in \Delta[n]_q$, then $\tilde{h}(\sigma) : P \longrightarrow T_q$ is defined to be the solution of the problem



which exists since f_q is surjective and P projective. Clearly, we have $\underline{Hom}(\underline{P}, f)(\tilde{h}) = h$. To prove the converse, we look at the commutative diagram (4.3). As $\underline{\mathbb{Q}}(G/H)$ is projective in Vec_G, $\underline{Hom}(id, f)$ is a fibration in $\mathcal{S}$ and hence in the category of simplicial rational vector spaces. As $\pi_0(f(G/H))$ is onto, the homomorphism $\pi_0(\underline{Hom}(id, f))$ is onto, and hence $\underline{Hom}(id, f)$ is surjective. As $\xi_{(T,H)}$ and $\xi_{(S,H)}$ are isomorphisms, $f(G/H)$ is surjective. This proves the surjectivity of f. ∎

Proposition 4.5.8 *Every object in $sVec_G$ is fibrant.* ∎

Proposition 4.5.9 *If $f : T \longrightarrow S$ in $sVec_G$ is a weak equivalence, so is the map $f(G/H) : T(G/H) \longrightarrow S(G/H)$ of simplicial vector spaces.*

Proof. We make use of the fact that $\underline{\mathbb{Q}}(G/H)$ is projective and the simplicial map $\underline{Hom}(id, f)$ in diagram (4.2) and (4.3) is a homotopy equivalence. These commutative diagrams and a simple computation now proves the proposition. ∎

Proposition 4.5.10 *Every object in* $s\,\mathbf{Vec}_G$ *is cofibrant.*

Proof. It suffices to show that every trivial fibration $p : T \longrightarrow S$ in $s\,\mathbf{Vec}_G$ admits a section $s : S \longrightarrow T$, because an LLP of a map $0 \longrightarrow U$ with respect to p has a solution.

As the fibration $p : T \longrightarrow S$ is trivial, $p(G/H) : T(G/H) \longrightarrow S(G/H)$ is a weak equivalence (Proposition 4.5.9), and hence a homotopy equivalence since $T(G/H)$ and $S(G/H)$ are Kan-complexes. Consequently $\pi_0(f(G/H))$ is surjective and hence, by Theorem 4.5.7, $p : T \longrightarrow S$ is surjective. We therefore have a split exact sequence

$$0 \longrightarrow \ker\ p(G/H) \longrightarrow T(G/H) \overset{p(G/H)}{\longrightarrow} S(G/H) \longrightarrow 0.$$

with isomorphism $T(G/H) \cong S(G/H) \oplus \ker p(G/H)$ which is natural because of the following commutative diagram arising from a morphism $\hat{a} : G/K \longrightarrow G/H$ in $\mathcal{O}_G$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\ p(G/H) & \longrightarrow & T(G/H) & \longrightarrow & S(G/H) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \ker\ p(G/K) & \longrightarrow & T(G/K) & \longrightarrow & S(G/K) & \longrightarrow & 0
\end{array}$$

This means that we have a section $s(G/H) : S(G/H) \longrightarrow T(G/H)$ for every $H \subseteq G$, and these make up the section s by naturality.

4.6 Homotopy in $s\mathit{Vec}_G$

There are two notions of homotopy in $s\,\mathbf{Vec}_G$:

(i) the left homotopy coming from its closed model structure, and

(ii) the abstract homotopy coming from combinatorial considerations as described by the relations (1.3) of Section 1.4.

We shall show that the two notions are essentially the same.

First let us look at abstract notion of homotopy in $s\,Vec_G$. Let $\underline{\mathbb{Q}I}: O_G \longrightarrow$ $\boldsymbol{s\,Vect_{\mathbb{Q}}}$ be the contravariant functor defined by $\underline{\mathbb{Q}I}(G/H) = \mathbb{Q}I$ and $\underline{\mathbb{Q}I}(\widehat{a}) = id$, where $\mathbb{Q}I$ is the rational simplicial vector space generated by $I = \Delta[1]$. Note that if $\{e_0, e_1, \ldots, e_{n+1}\}$ is the basis of the vector space $(\mathbb{Q}I)_n$, where $e_k = (0, 0, \ldots, 0, 1, \ldots, 1)$ (with $n-k+1$ zeros and k ones) $\in \Delta[1]_n$, then

$$(\mathbb{Q}I)_n = \mathbb{Q}e_0 \oplus \cdots \oplus \mathbb{Q}e_{n+1}.$$

If $T: O_G \longrightarrow \boldsymbol{s\,Vect_{\mathbb{Q}}}$ is another contravariant functor, define a contravariant functor $T \otimes \underline{\mathbb{Q}I}: O_G \longrightarrow \boldsymbol{s\,Vect_{\mathbb{Q}}}$ by $T \otimes \underline{\mathbb{Q}I}(G/H) = T(G/H) \otimes \mathbb{Q}I$. Also, define natural transformations $i_0, i_1 : T \longrightarrow T \otimes \underline{\mathbb{Q}I}$ by

$$i_0(G/H)(x) = x \otimes (e_0, 0, \ldots, 0) \quad \text{and} \quad i_1(G/H)(x) = x \otimes (0, \ldots, e_{n+1}). \tag{4.6}$$

Lemma 4.6.1 *Two simplicial maps $f, g : T \longrightarrow S$ in $s\,Vec_G$ are homotopic (in the abstract sense) if and only if there exists a simplicial map*

$$F: T \otimes \underline{\mathbb{Q}I} \longrightarrow S$$

with $F \circ i_0 = f$ and $F \circ i_1 = g$. ∎

Lemma 4.6.2 *Every homotopy equivalence in $s\,Vec_G$ is a weak equivalence.*

Proof. It suffices to prove that if $f, g : T \longrightarrow S$ in $s\,Vec_G$ are homotopic, then

$$\underline{Hom}(\underline{P}, f), \underline{Hom}(\underline{P}, g) : \underline{Hom}(\underline{P}, T) \longrightarrow \underline{Hom}(\underline{P}, S)$$

are homotopic simplicial maps (P being a projective object in Vec_G). Let the morphisms $h_i : T_n \longrightarrow S_{n+1}$, $0 \le i \le n$, be a homotopy from f to g. Then $\widetilde{h_i} : \underline{Hom}(\underline{P}, T)_n \longrightarrow \underline{Hom}(\underline{P}, S)_{n+1}$ defined by

$$\widetilde{h_i}(\alpha)(\Delta_{n+1}) = h_i \circ \alpha(\Delta_n), \ \alpha \in \underline{Hom}(\underline{P}, T)_n$$

is the required homotopy. ∎

Now we turn to the notion of left homotopy in a closed model category $\boldsymbol{C}$. Let $A \vee_{\emptyset} A$ be the push out of the diagram $A \longleftarrow \emptyset \longrightarrow A$ in $\boldsymbol{C}$ and

$$\nabla_A : A \vee A \longrightarrow A$$

be the corresponding folding map. Recall from Quillen [22] that in a closed model category $\boldsymbol{C}$, a cylinder of an object A is an object IA together with morphisms $i_0, i_1 : A \longrightarrow IA$ and $p : IA \longrightarrow A$ such that $i_0 + i_1 : A \vee_{\emptyset} A \longrightarrow IA$ is a cofibration, and p is a weak equivalence, such that $p \circ (i_0 + i_1) = \nabla_A$. Two morphisms $f_0, f_1 : A \longrightarrow B$ in $\boldsymbol{C}$ are called left homotopic ($f_0 \sim_C f_1$) if there is a morphism $H : IA \longrightarrow B$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$. Quillen proved that if A is cofibrant then i_0 and i_1 are trivial cofibrations, and the left homotopy relation $\sim_C$ is an equivalence relation.

In the closed model category $\boldsymbol{s\,Vec_G}$, every object is cofibrant (Proposition 4.5.10), the initial object $\emptyset$ is just 0, and, for an object T, $T \vee_{\emptyset} T$ is simply $T \oplus T$ with the folding map $\nabla_T : T \oplus T \longrightarrow T$ given by $\nabla_T(G/H)(x, x') = x + x'$. We define $IT = T \otimes \underline{\mathbb{Q}I}$, and natural transformations $i_0, i_1 : T \longrightarrow IT$ by (4.6), and $p : IT \longrightarrow T$ by $p(G/H)(x \otimes u) = x$. Then the natural transformation $i_0 + i_1 : T \oplus T \longrightarrow IT$ is given by

$$(i_0 + i_1)(G/H)(x, x') = i_0(G/H)(x) + i_1(G/H)(x'),$$

and we have $p \circ (i_0 + i_1) = \nabla_T$. Also, p is a homotopy equivalence in the abstract sense with homotopy inverse i_0. Therefore, by Lemma 4.6.2, p is a weak equivalence in $\boldsymbol{s\,Vec_G}$.

Again $i_0 + i_1$ is a cofibration. To see this, consider a LLP of $i_0 + i_1$ with respect to a trivial fibration $q : U \longrightarrow V$ in $\boldsymbol{s\,Vec_G}$.

$$\begin{array}{ccc} T \oplus T & \xrightarrow{\alpha} & U \\ {\scriptstyle i_0+i_1}\downarrow & & \downarrow{\scriptstyle q} \\ T \otimes \underline{\mathbb{Q}I} & \xrightarrow{\beta} & V \end{array} \qquad (4.7)$$

We can identify $T\otimes \underline{\mathbb{Q}I}$ with $T\oplus T\oplus S$, where $S = \operatorname{coker}(i_0 + i_1)$, by means of a splitting of the exact sequence

$$0 \longrightarrow T\oplus T \longrightarrow T\otimes \underline{\mathbb{Q}I} \longrightarrow S \longrightarrow 0,$$

(note that $i_0 + i_1$ is injective). Also, the LLP of the cofibration $0 \longrightarrow S$ with respect to the trivial fibration $q : U \longrightarrow V$ has a solution $\gamma : S \longrightarrow U$ such that $q\circ\gamma = \beta$. Then a solution to the LLP (4.7) is given by $\alpha+\gamma : T\oplus T\oplus S \longrightarrow U$.

We have proved

Lemma 4.6.3 *In the category* $s\,\mathbf{Vec}_G$, $T\otimes \underline{\mathbb{Q}I}$ *is a cylinder object for* T. ∎

Theorem 4.6.4 *Two morphisms* $f, gT \longrightarrow S$ *in* $s\,\mathbf{Vec}_G$ *are left homotopic if and only if they are homotopic in the abstract sense. Consequently, the homotopy between morphisms in the abstract sense is an equivalence relation.* ∎

Chapter 5 CLASSIFICATION THEOREM

5.1 Introduction

We have proved in Chapter 4 that homotopy is an equivalence relation on the set of simplicial maps between simplicial objects in $\boldsymbol{s}\,Vec_G$. In this chapter we shall show that if G is a finite group and λ is a contravariant functor from O_G to $\boldsymbol{Vect_{\mathbb{Q}}}$, then, for any G-simplicial set K, there is a bijection between the cohomology group $H^n_G(K;\lambda)$ and the set of homotopy classes of simplicial maps $\Phi\mathbb{Q}K \longrightarrow K(\lambda, n)$ in $\boldsymbol{s}\,Vec_G$, where $K(\lambda, n)$ is what we call O_G-Eilenberg-MacLane complex.

In Section 5.2, we recall some basic facts about Eilenberg-MacLane complexes. In Section 5.3, we define O_G-Eilenberg-MacLane complexes $K(\lambda, n)$ and prove their uniqueness. Finally in Section 5.4, we prove the classification theorem.

5.2 Eilenberg-MacLane complexes

Recall that if π is a group, which is to be abelian if $n \geq 2$, then an Eilenberg-MacLane complex of type (π, n) is a Kan complex K with

$$\pi_i K = \begin{cases} \pi & i = n \\ 0 & i \neq n \end{cases}$$

A minimal Eilenberg-MacLane complex of type (π, n) is denoted by $K(\pi, n)$. A standard fact about Eilenberg-MacLane complexes is

Proposition 5.2.1 [18]. *If the groups π, π' are abelian, and $f : \pi \longrightarrow \pi'$ is a homomorphism, then there exists a unique simplicial homomorphism*

$$\varphi : K(\pi, n) \longrightarrow K(\pi', n)$$

with $\varphi_n = f : K(\pi,n)_n \longrightarrow K(\pi',n)_n$. ■

As a consequence, minimal Eilenberg-MacLane complexes are unique upto isomorphism.

Example 5.2.2 *Let* π *be an abelian group and let* $C^*(\Delta[q];\pi)$ *denote the normalized cochain complex. Then for* $n \geq 0$, $L(\pi, n+1)$ *defined by*

$$L(\pi, n+1)_q = C^n(\Delta[q];\pi) \qquad q \geq 0,$$

is a group complex, and the subcomplex $\overline{K}(\pi,n)$ *given by*

$$\overline{K}(\pi,n)_q = Z^n(\Delta[q];\pi)$$

with obvious face and degeneracies is a $K(\pi,n)$ (see [18] p. 101).

Theorem 5.2.3 [18]. *For any simplicial set* K, *there is a bijection between* $H^n(K;\pi)$ *and* $[K, K(\pi,n)]$, *where* $[K, K(\pi,n)]$ *is the homotopy classes of simplicial maps* $K \longrightarrow K(\pi,n)$. ■

5.3 O_G Eilenberg-MacLane complexes

Let $\lambda : O_G \longrightarrow \mathbf{Ab}$ be an O_G-abelian group and $n \geq 0$, and for a G-simplicial set K, $\underline{\pi}_n(K) : O_G \longrightarrow \mathbf{Grp}$ be the O_G-group defined by $\underline{\pi}_n(K)(G/H) = \pi_n(K^H)$.

Definition 5.3.1 *An* O_G*-Eilenberg-MacLane complex of the type* (λ, n) *is an* O_G*-simplicial set* T, *which is a contravariant functor* $O_G \longrightarrow \mathcal{S}$ *such that*

(*i*) $T(G/H)$ *is a* $K(\lambda(G/H), n)$.

(*ii*) $T(\widehat{g}) : T(G/H) \longrightarrow T(G/H')$ *is the unique simplicial homomorphism induced by the homomorphism* $\lambda(\widehat{g}) : \lambda(G/H) \longrightarrow \lambda(G/H')$.

(*iii*) $\underline{\pi}_n \circ T = \lambda$, *and* $\underline{\pi}_i \circ T = 0$ *if* $i \neq n$.

Example 5.3.2 *The contravariant functor* $Z^n(\Delta[\ ];\lambda) : O_G \longrightarrow \mathcal{S}$ *defined by*

$$Z^n(\Delta[\ ];\lambda)(G/H)([q]) = Z^n(\Delta[q];\lambda(G/H))$$

is an O_G*-Eilenberg-MacLane complex of the type* (λ, n).

The following theorem shows that the O_G-Eilenberg-MacLane complex of the above example is uniquely defined upto isomorphism.

Theorem 5.3.3 *Any two* O_G*-Eilenberg-MacLane complexes of type* (λ, n) *are naturally isomorphic.* ■

Proof. Suppose that $T, S : O_G \longrightarrow \mathcal{S}$ are two O_G-Eilenberg-MacLane complexes of type (λ, n). Define a natural transformation $\varphi : T \longrightarrow S$ by setting $\varphi(G/H) : T(G/H) \longrightarrow S(G/H)$ to be the unique simplicial homomorphism induced by $id : \lambda(G/H) \longrightarrow \lambda(G/H)$. Since there is a bijection

$$[T(G/H), S(G/H)] \leftrightarrow Hom(\lambda(G/H), \lambda(G/H)),$$

$\varphi(G/H)$ is a homotopy equivalence, and hence an isomorphism by Theorem 1.4.7. Now the following diagram commutes.

$$\begin{array}{ccc} T(G/H) & \xrightarrow{\varphi(G/H)} & S(G/H) \\ \downarrow{\scriptstyle T(\widehat{g})} & & \downarrow{\scriptstyle S(\widehat{g})} \\ T(G/H') & \xrightarrow{\varphi(G/H')} & S(G/H') \end{array}$$

This is so because $S(\widehat{g}) \circ \varphi(G/H)$ and $\varphi(G/H') \circ T(\widehat{g})$ are the unique simplicial maps induced by $\lambda(\widehat{g}) \circ id_{\lambda(G/H)}$ and $id_{\lambda(G/H')} \circ \lambda(\widehat{g})$ respectively. This proves the theorem. ■

5.4 Classification theorem

Throughout this section G will denote a finite group and λ will denote a contravariant functor $\lambda : O_G \longrightarrow \mathbf{Vect}_{\mathbb{Q}}$. We may then define functors

$$\underline{C}_*(\Delta[q]) \ : \ O_G \longrightarrow \mathbf{Ch}_{\mathbb{Q}}$$
$$C^*(\Delta[q];\lambda) \ : \ O_G \longrightarrow \mathbf{Ch}^c_{\mathbb{Q}}$$

by

$$C^*(\Delta[q];\lambda)(G/H) = C^*(\Delta[q];\lambda(G/H)),$$
$$\underline{C}_*(\Delta[q])(G/H) = C_*(\Delta[q];\mathbb{Q}),$$

where $\mathbf{Ch}_{\mathbb{Q}}$ denotes the category of chain complexes and $\mathbf{Ch}^c_{\mathbb{Q}}$ that of cochain complexes over $\mathbb{Q}$. All chain and cochain complexes are understood to be normalized.

Fix $n \geq 0$, and define functors

$$L(\lambda, n+1), K(\lambda, n) : O_G \longrightarrow \mathcal{S}$$

by setting

$$L(\lambda, n+1)(G/H)([q]) = C^n(\Delta[q];\lambda(G/H))$$
$$K(\lambda, n)(G/H)([q]) = Z^n(\Delta[q];\lambda(G/H)).$$

As we have already observed in Example 5.3.2, $K(\lambda, n)$ is an O_G-Eilenberg-MacLane complex of type (λ, n). Let K be a G-simplicial set. Form the G-simplicial set $\mathbb{Q}K$, where $(\mathbb{Q}K)_n$ is the vector space over $\mathbb{Q}$ generated by the set K_n, and define a contravariant functor $\Phi\mathbb{Q}K : O_G \longrightarrow \mathcal{S}$ by setting $\Phi\mathbb{Q}K(G/H) = \mathbb{Q}K^H$. Then define a map

$$\Sigma : Hom(\Phi\mathbb{Q}K, L(\lambda, n+1)) \longrightarrow Hom(\underline{C}_n(K), \lambda)$$

as follows. Let $f : \Phi\mathbb{Q}K \longrightarrow L(\lambda, n+1)$ be a natural transformation. Then $f(G/H) : \mathbb{Q}K^H \longrightarrow L(\lambda, n+1)(G/H)$ is a simplicial: If $x \in K^H_n$, then

$f(G/H)(x) \in L(\lambda, n+1)(G/H) = C^n(\Delta[n]; \lambda(G/H))$ is a cochain. Since the vector space $C^n(\Delta[n]; \lambda(G/H))$ is the vector space of all linear transformations from the vector space $\mathbb{Q}\Delta_n$ with basis Δ_n to $\lambda(G/H)$, we may identify it with $\lambda(G/H)$. Then define $\Sigma f : \underline{C}_n(K) \longrightarrow \lambda$ by setting

$$(\Sigma f)(G/H)(x) = (f(G/H)(x))(\Delta_n).$$

It is straightforward to check that Σf defined as above is natural with respect to morphisms in O_G.

Next define

$$\Lambda : Hom(\underline{C}_n(K), \lambda) \longrightarrow Hom(\Phi\mathbb{Q}K, L(\lambda, n+1))$$

as follows. Let $T: \underline{C}_n(K) \longrightarrow \lambda$ be a natural transformation. It is sufficient to define simplicial map

$$\Lambda(T)(G/H) : \mathbb{Q}K^H \longrightarrow L(\lambda, n+1)(G/H).$$

Let $x \in \mathbb{Q}K^H_q$. This induces a simplicial map $\bar{x} : \mathbb{Q}\Delta[q] \longrightarrow \mathbb{Q}K^H$ with $\bar{x}(\Delta_q) = x$. Then

$$\bar{x}^* : C^n(K^H; \lambda(G/H)) \longrightarrow C^n(\Delta[q]; \lambda(G/H))$$

is a cochain map. Observe that $C^n(\Delta[q]; \lambda(G/H)) = L(\lambda, n+1)(G/H)([q])$. We then set

$$(\Lambda T)(G/H)(x) = \bar{x}^*(T(G/H)).$$

The definition of Λ will be complete if we show that ΛT is natural. To see this, consider

Lemma 5.4.1 *There is an isomorphism θ_H between the cochain groups $C^n_G(\Delta[n] \times G/H; \lambda)$ and $C^n(\Delta[n]; \lambda(G/H))$ which is natural with respect maps induced from morphisms in O_G.*

Proof. Define $\theta_H : C^n_G(\Delta[n] \times G/H; \lambda) \longrightarrow C^n(\Delta[n]; \lambda(G/H)$ by

$$\theta_H(c)(x) = c(\Delta_n, eH).$$

Clearly, θ_H is the required isomorphism. ∎

Lemma 5.4.2 *ΛT is a natural transformation.*

Proof. Let $\widehat{g} : G/H' \longrightarrow G/H$ be a morphism in O_G. As we are working in the normalized set up, it is sufficient to check the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Q}K^H & \xrightarrow{\Lambda T(G/H)} & L(\lambda, n+1)(G/H) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \lambda(\widehat{g})} \\ \mathbb{Q}K^{H'} & \xrightarrow{\Lambda T(G/H')} & L(\lambda, n+1)(G/H') \end{array} \tag{5.1}$$

only for n-simplexes. First we reinterpret Λ in the following way. As seen in Theorem 2.3.1, we may identify $T \in Hom(\underline{C}_n(K), \lambda)$ with $\widehat{T} \in C^n_G(K;\lambda)$, where $\widehat{T}(x) = T(G/G_x)(x)$. Given $x \in K^H_n$, there is a G-simplicial map $\widetilde{x} : \Delta[n] \times G/G_x \longrightarrow K$ defined by $\widetilde{x}(\Delta_n, eG_x) = x$, and hence a homomorphism

$$\widetilde{x}^{\bullet} : C^n_G(K;\lambda) \longrightarrow C^n_G(\Delta[n] \times G/G_x;\lambda).$$

Now consider the following sequence of homomorphisms:

$$\begin{array}{l} C^n_G(K;\lambda) \xrightarrow{\widetilde{x}^{\bullet}} C^n_G(\Delta[n] \times G/G_x;\lambda) \xrightarrow{(id\times\widehat{e})^{\bullet}} C^n_G(\Delta[n] \times G/H;\lambda) \xrightarrow{\theta_H} \\ \qquad C^n(\Delta[n];\lambda(G/H)), \end{array}$$

where $\widehat{e} : G/H \longrightarrow G/G_x$ is induced by the inclusion $H \subseteq G_x$. Now

$$\begin{aligned} \theta_H(id \times \widehat{e})^{\bullet}\widetilde{x}^{\bullet}(\widehat{T})(\Delta_n) &= (id \times \widehat{e})^{\bullet}\widetilde{x}^{\bullet}(\widehat{T})(\Delta_n, eH) \\ &= \lambda(\widehat{e})\widetilde{x}^{\bullet}(\widehat{T})(\Delta_n, eG_x) \\ &= \lambda(\widehat{e})(\widehat{T})(x) \\ &= \lambda(\widehat{e})T(G/G_x)(x). \end{aligned}$$

As $T : \underline{C}_n(K) \longrightarrow \lambda$ is a natural transformation, the diagram

$$\begin{array}{ccc} C_n(K^{G_x}) & \xrightarrow{T(G/G_x)} & \lambda(G/G_x) \\ \downarrow & & \downarrow{\scriptstyle \lambda(\widehat{e})} \\ C_n(K^H) & \xrightarrow{T(G/H)} & \lambda(G/H) \end{array}$$

commutes, where the left vertical arrow is induced by the inclusion $K^{G_x} \subseteq H$. Therefore,

$$\theta_H(id \times \widehat{e})^{\cdot}\tilde{x}^{\cdot}(\widehat{T})(\Delta_n) = T(G/H)(x).$$

On the other hand,

$$\begin{aligned} \Lambda T(G/H)(x)(\Delta_n) &= \bar{x}^{\cdot}(T(G/H))(\Delta_n) \\ &= T(G/H)(x). \end{aligned}$$

Hence, $\Lambda T(G/H)(x) = \theta_H(id \times \widehat{e})^{\cdot}\tilde{x}^{\cdot}(\widehat{T})$. Now observe that there is a commutative diagram

$$\begin{array}{ccccc} C^n_G(K;\lambda) & \xrightarrow{\widetilde{gx}'} & C^n_G(\Delta[n] \times G/G_{gx};\lambda) & & \\ {\scriptstyle \tilde{x}^{\cdot}}\downarrow & \nearrow{\scriptstyle \cong} & \downarrow{\scriptstyle (id\times\widehat{e})^{\cdot}} & & \\ C^n_G(\Delta[n] \times G/G_x;\lambda) & & C^n_G(\Delta[n] \times G/H';\lambda) & \xrightarrow{\theta_{H'}} & C^n(\Delta[n];\lambda(G/H')) \\ {\scriptstyle (id\times\widehat{e})^{\cdot}}\searrow & & \nearrow{\scriptstyle (id\times\widehat{g})^{\cdot}} & & \uparrow{\scriptstyle \lambda(\widehat{g})} \\ & C^n_G(\Delta[n] \times G/H;\lambda) & & \xrightarrow{\theta_H} & C^n(\Delta[n];\lambda(G/H)) \end{array}$$

The upper triangle and the lower left square commute because the corresponding maps at the simplicial level commute. That the lower right square commutes has already been seen in Lemma 5.4.1. Then, if $x \in K^H$, we have

$$\begin{aligned}\lambda(\widehat{g})(\Lambda T)(G/H)(x) &= \lambda(\widehat{g})\theta_H(id \times \widehat{e})^{\bullet}\widetilde{x}^{\bullet}(\widehat{T}) \\ &= \theta_{H'}(id \times \widehat{e})^{\bullet}(\widetilde{gx}^{\bullet}(\widehat{T}) \\ &= (\Lambda T)(G/H')(gx).\end{aligned}$$

This proves the Lemma. ∎

Next we compute

$$\begin{aligned}(\Sigma \circ \Lambda)(T)(G/H)(x) &= \Sigma(\Lambda T)(G/H)(x) \\ &= (\Lambda T)(G/H)(x)(\Delta_n) \\ &= \overline{x}^{\bullet}(T(G/H))(\Delta_n) \\ &= T(G/H)(x).\end{aligned}$$

On the other hand, if $f : \Phi\mathbb{Q}K \longrightarrow L(\lambda, n+1)$ is a natural transformation, then

$$\begin{aligned}(\Lambda \circ \Sigma)(f)(G/H)(x)(\Delta_n) &= \Lambda(\Sigma f)(G/H)(x)(\Delta_n) \\ &= \theta_H(id \times \widehat{e})^{\bullet}\widetilde{x}^{\bullet}(\widehat{\Sigma f})(\Delta_n) \\ &= (id \times \widehat{e})^{\bullet}\widetilde{x}^{\bullet}(\widehat{\Sigma f})(\Delta_n, eH) \\ &= \lambda(\widehat{e})\overline{x}^{\bullet}(\widehat{\Sigma f})(\Delta_n, eG_x) = \lambda(\widehat{e})(\widehat{\Sigma f})(x) \\ &= \lambda(\widehat{e})(\Sigma f)(G/G_x)(x) = (\Sigma f)(G/H)(x) \\ &= f(G/H)(x)(\Delta_n).\end{aligned}$$

We have thus proved

Proposition 5.4.3 *The map Σ is an isomorphism between the functors $Hom(\Phi\mathbb{Q}K, L(\lambda, n+1))$ and $Hom(\underline{C}_n(K), \lambda)$ with inverse Λ.* ∎

Let $\mu : Hom(\underline{C}_n(K), \lambda) \longrightarrow C^n_G(K; \lambda)$ denote the isomorphism of Theorem 2.3.1, with inverse Ψ. Denote the composition

$$\mu \circ \Sigma : Hom(\Phi\mathbb{Q}K, L(\lambda, n+1)) \longrightarrow C^n_G(K; \lambda)$$

by Γ, then Γ is an isomorphism.

Proposition 5.4.4 *The map Γ is an isomorphism between the cocycles $Z^n_G(K;\lambda)$ and $Hom(\Phi\mathbb{Q}K, K(\lambda,n))$ with inverse $\Gamma' = \Lambda \circ \Psi$.*

Proof. Let $f \in Z^n_G(K;\lambda)$. We need to show that $(\Psi f)(G/H)(x) \in K(\lambda(G/H),n)$, for all $x \in K^H_q$ and $H \subseteq G$, that is $(\Psi f)(G/H)(x) \in Z^n(\Delta[q];\lambda(G/H))$. But this is true, because

$$\begin{aligned}\delta(\Gamma' T(G/H)(x)) &= \delta(\Lambda(\Psi T)(G/H)(x)) \\ &= \delta(\theta_H(id \times \hat{e})^* \tilde{x}^*(\widehat{\Psi T})) \\ &= \delta(\theta_H(id \times \hat{e})^* \tilde{x}^* T) = 0.\end{aligned}$$

Conversely, suppose that $(\Psi f)(G/H)(y) \in Z^n(\Delta[q];\lambda(G/H))$ for all $y \in K^H_q$. Then, if $x \in K_{n+1}$ is non-degenerate, we find that

$$\begin{aligned}\delta(\Gamma f)(x) &= \sum_{i=0}^{n+1}(-1)^i \lambda(d_i x \to x)(\Gamma f)(d_i x) \\ &= \sum_{i=0}^{n+1}(-1)^i \lambda(d_i x \to x)(\Sigma f)(G/G_{d_i x})(d_i x) \\ &= \sum_{i=0}^{n+1}(-1)^i \lambda(d_i x \to x) f(G/G_{d_i x})(d_i x)(\Delta_n) \\ &= \sum_{i=0}^{n+1}(-1)^i f(G/G_x)(d_i x)(\Delta_n) \\ &= \sum_{i=0}^{n+1}(-1)^i (d_i f(G/G_x)(x))(\Delta_n) \\ &= \sum_{i=0}^{n+1}(-1)^i f(G/G_x)(x)(\partial_i \Delta_n) \\ &= \sum_{i=0}^{n+1}(-1)^i f(G/G_x)(x)(d_i \Delta_{n+1}) = 0\end{aligned}$$

This proves the proposition. ∎

Theorem 5.4.5 *Let $f_0, f_1 \in Hom(\Phi\mathbb{Q}K, K(\lambda, n))$. Then $f_0 \sim f_1$ if and only if Γf_0 and Γf_1 are cohomologous.*

Proof. Let $F : \Phi\mathbb{Q}K \otimes \mathbb{Q}I \longrightarrow K(\lambda, n)$ be a homotopy between f_0 and f_1 (see Section 4.6). Therefore for each $H \subseteq G$, the simplicial maps $f_0(G/H)$ and $f_1(G/H)$ are homotopic by the homotopy $F(G/H)$. Let us look at the definition of Γ more closely. Given $f : \Phi\mathbb{Q}K \longrightarrow L(\lambda, n+1)$, $f(G/H) : \mathbb{Q}K^H \longrightarrow L(\lambda, n+1)(G/H)$ is a simplicial map. Now $\Sigma f : \underline{C}_n(K) \longrightarrow \lambda$ is defined by

$$(\Sigma f)(G/H)(x) = f(G/H)(x)(\Delta_n).$$

Therefore by definition of μ (see Theorem 2.3.1), $\Gamma f \in C^n_G(K; \lambda$ is given by

$$(\Gamma f)(x) = f(G/G_x)(x)(\Delta_n).$$

Now we may identify $L(\lambda, n+1)(G/H)([n])$ with $\lambda(G/H)$, the former being the vector space of homomorphisms from $\mathbb{Q}\Delta_n$ into $\lambda(G/H)$. Then define an element u_H of the n-cochain group $C^n(L(\lambda, n+1)(G/H); \lambda(G/H))$ by setting $u_H(c) = c(\Delta_n)$. This gives a homomorphism

$$f(G/G_x)^* : C^n(L(\lambda, n+1)(G/G_x); \lambda(G/G_x)) \longrightarrow C^n(\mathbb{Q}K^{G_x}; \lambda(G/G_x))$$

such that

$$\begin{aligned} f(G/G_x)^*(u_{G_x})(x) &= u_{G_x}(f(G/G_x)(x)) \\ &= f(G/G_x)(x)(\Delta_n) \\ &= \Gamma f(x). \end{aligned}$$

Now as the simplicial maps $f_0(G/G_x)$ and $f_1(G/G_x)$ are homotopic, $f_0(G/G_x)^* = f_1(G/G_x)^*$. Consequently $\Gamma f_0 = \Gamma f_1$.

Conversely, suppose that $f_0, f_1 : \Phi\mathbb{Q}K \longrightarrow K(\lambda, n)$ are such that Γf_0 and Γf_1 are cohomologous, that is, $\Gamma f_0 = \Gamma f_1 + \delta h$, where $h \in C^{n-1}_G(\mathbb{Q}K; \lambda)$. It suffices to find a $\gamma \in Z^n_G(\mathbb{Q}K \otimes \mathbb{Q}I; \lambda)$ such that $i_0^*(\gamma) = \Gamma f_0$ and $i_1^*(\gamma) = \Gamma f_1$

where $i_0, i_1 : \mathbb{Q}K \longrightarrow \mathbb{Q}K \otimes \mathbb{Q}I$ are the inclusions like (4.6). Then the natural transformation

$$\Gamma'(\gamma) : \Phi\mathbb{Q}K \otimes \underline{\mathbb{Q}I} \longrightarrow K(\lambda, n)$$

will be a homotopy from f_0 to f_1. To get such a γ, write $\gamma_0 = p^*(\Gamma f_0) \in Z^n_G(\mathbb{Q}K \otimes \mathbb{Q}I; \lambda)$, where p is projection $\mathbb{Q}K \otimes \mathbb{Q}I \longrightarrow \mathbb{Q}K$. Then

$$i_0^*(\gamma_0) = i_1^*(\gamma_0) = \Gamma f_0.$$

Further, regarding $h \in C^{n-1}_G(\mathbb{Q}K; \lambda)$ as a cochain defined on $i_1(K)$, we may choose a cochain $\beta \in C^{n-1}_G(\mathbb{Q}K \otimes \mathbb{Q}I; \lambda)$ which extends h and vanishes on $i_0(K)$. Thus $i_0^*(\beta) = 0$ and $i_1^*\beta = h$. Now take $\gamma = \gamma_0 - \delta\beta$. This completes the proof.

■

We have in effect proved

Theorem 5.4.6 (Classification). *For any G-simplicial set K, there is a bijection* $[\Phi\mathbb{Q}K, K(\lambda, n)] \leftrightarrow H^n_G(K; \lambda)$. ■

Note that although this is not the most general classification theorem that one expects, this result suffices to prove a generalization of the Cartan's theorem in Chapter 6.

Chapter 6 FIRST MAIN THEOREM

6.1 Introduction

In this chapter, we prove our first main theorem. This theorem generalizes a result of Cartan [7] (Theorem A of Introduction) for the case when the group is finite.

In Section 6.2, we recall Cartan's theorem in more details. In Section 6.3, we recall the W-construction of a simplicial group, and prove that it is naturally contractible. Finally in Section 6.4, we use our earlier results to prove our first main theorem.

Throughout this chapter G will be a finite group and R a commutative ring with 1.

6.2 Theorem of Cartan

Definition 6.2.1 *A differential graded algebra over R, is a graded R-module $A^{\bullet} = \oplus_{n\geq 0} A^n$ with an associative R-linear multiplication $A^n \otimes_R A^m \longrightarrow A^{n+m}$ and a degree 1 homomorphism $\delta : A^{\bullet} \longrightarrow A^{\bullet}$ satisfying*

$$\begin{aligned} \delta^2 &= 0 \\ \delta(xy) &= (\delta x)y + (-1)^{|x|} x(\delta y) \end{aligned}$$

Let **DGA/R** be the category whose objects are differential graded algebras over R, and morphisms are degree zero maps commuting with the differentials.

Definition 6.2.2 *A simplicial differential graded algebra $A^{\bullet}_{\bullet}$ over R is a contravariant functor $A^{\bullet}_{\bullet} : \Delta \longrightarrow$* **DGA/R**.

Note that the upper degree denotes the algebra index and the lower suffix the simplicial degree. Thus, for each $p \geq 0$, we have a differential graded algebra $A_p^\bullet = \oplus_{n\geq 0} A_p^n$ over R together with face and degeneracy maps

$$d_i : A_p^\bullet \longrightarrow A_{p-1}^\bullet, \quad s_i : A_p^\bullet \longrightarrow A_{p+1}^\bullet$$

which are homomorphisms of differential graded algebras satisfying the usual simplicial identities.

Definition 6.2.3 *A cohomology theory in the sense of Cartan over R is a simplicial differential graded algebra $A_\bullet^\bullet$ over R satisfying the following conditions :*

(*i*) (Homology axiom). *The sequence*

$$\mathcal{A} : A^0 \xrightarrow{\delta} A^1 \longrightarrow \cdots$$

of R-linear simplicial differentials is exact, and the simplicial R-algebra $Z^0\mathcal{A} = Ker(\delta : A^0 \longrightarrow A^1)$ is simplicially trivial, that is, all the face and degeneracy maps are isomorphisms.

(*ii*) (Homotopy axiom). $\pi_i(A^n) = 0$, *whenever* $i \geq 0, \quad n \geq 0$.

Given a cohomology theory $A_\bullet^\bullet$, and a simplicial set K, we may define a differential graded algebra

$$\mathcal{A}^\bullet(K) = \mathrm{Hom}(K, A^\bullet) = \oplus_{n\geq 0}\mathrm{Hom}(K, A^n)$$

over R, where $\mathrm{Hom}(K, A^n)$ is the R-module of simplicial maps $K \longrightarrow A^n$, and the differential in $\mathcal{A}^\bullet(K)$ is induced by that of $A_\bullet^\bullet$.

Theorem 6.2.4 (Cartan [7]) *Let $A^\bullet$ be a cohomology theory. Then there is an isomorphism of graded R-modules*

$$H^\bullet(A^\bullet(K)) \cong H^\bullet(K; R(A)),$$

where $R(A) = (Z^0\mathcal{A})_0$, for every simplicial set K.

Proof. For a cohomology theory A^*, the second part of the homology axiom implies that the simplicial R-module $Z^0\mathcal{A} = \mathrm{Ker}(\delta : A^0 \longrightarrow A^1)$ is a $K(R(A), 0)$. The exact sequence

$$\mathcal{A} : A^0 \xrightarrow{\delta} A^1 \longrightarrow \cdots$$

gives rise to the short exact sequence

$$0 \longrightarrow Z^n\mathcal{A} \longrightarrow A^n \longrightarrow Z^{n+1}\mathcal{A} \longrightarrow 0, \quad n \geq 0,$$

where $Z^n\mathcal{A} = \mathrm{Ker}(\delta : A^n \longrightarrow A^{n+1})$. As the map $A^n \longrightarrow Z^{n+1}\mathcal{A}$ is surjective, it is a Kan fibration with fiber $Z^n\mathcal{A}$. Induction and the homotopy exact sequence of these fibrations imply that $Z^n\mathcal{A}$ is a $K(R(A), n)$.

We may also identify $Z^n\mathcal{A}$ with the set of simplicial maps $K \longrightarrow Z^n\mathcal{A}$. Moreover, a simplicial map $K \longrightarrow A^n$ is in the image of $\delta : A^{n-1}(K) \longrightarrow A^n(K)$ if and only if it factors through $\delta : A^{n-1} \longrightarrow A^n$. The homotopy axiom now implies that a morphism $K \longrightarrow Z^n\mathcal{A}$ is a boundary if and only if it is null homotopic. We may thus identify $H^n(A^{\cdot}(K))$ with $[K, Z^n\mathcal{A}]$, and as $Z^n\mathcal{A}$ is a $K(R(A), n)$, the theorem follows. ∎

Example 6.2.5 .*Let* $C^{\cdot}_p = \oplus_{n\geq 0} C^n(\Delta[p]; R)$ *be the differential graded algebra of cochains of the contractible simplicial set* $\Delta[p]$. *Then* $C^{\cdot}$ *is a cohomology theory. In this case*

$$H^{\cdot}(C^{\cdot}(K)) = H^{\cdot}(K; R).$$

Example 6.2.6 *Let* $R = \mathbb{R}$, *the field of real numbers, and* $\Omega^{\cdot}_p = \Omega^{\cdot}(\Delta^p)$, *the differential graded algebra of smooth differential forms on the standard p-simplex* Δ^p *in* $\mathbb{R}^{p+1}$. *Then* $\Omega^{\cdot}$ *is a cohomology theory, and in this case*

$$H^{\cdot}(\Omega^{\cdot}(K)) = H^{\cdot}_{dR}(K; \mathbb{R}),$$

where $H^{\cdot}_{dR}$ *denotes the de Rham cohomology.*

6.3 $\overline{W}$- and W-construction

Recall that if A is a group complex, then the group complex $\overline{W}A$ is defined by setting $(\overline{W}A)_0 = *$ and for $n > 0$,

$$(\overline{W}A)_n = A_{n-1} \times \cdots \times A_0$$

with the face and degeneracy maps as

$$\begin{aligned}
d_i(g_{n-1}, \ldots, g_0) &= \begin{cases} (g_{n-2}, \ldots, g_0) & i = 0 \\ (d_{i-1}g_{n-1}, \ldots, d_0 g_{n-i} \cdot g_{n-i-1}, g_{n-i-2}, \ldots, g_0) & 0 < i \leq n \end{cases} \\
s_0(g_{n-1}, \ldots, g_0) &= (e_n, g_{n-1}, \ldots, g_0) \\
s_{i+1}(g_{n-1}, \ldots, g_0) &= (s_i g_{n-1}, \ldots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0), \quad i \geq 0
\end{aligned}$$

where $e_i \in A_i$ is the identity. Then $\overline{W}A$ is a Kan complex and acts as the classifying complex for A. The complex WA is given by

$$\begin{aligned}
(WA)_n &= A_n \times \cdots \times A_0, \quad n \geq 0 \\
d_0(g_n, \ldots, g_0) &= (d_0 g_n \cdot g_{n-1}, g_{n-2}, \ldots, g_0) \\
d_i(g_n, \ldots, g_0) &= (d_i g_n, \ldots, d_0 g_{n-i} \cdot g_{n-i-1}, g_{n-i-2}, \ldots, g_0) \quad i > 0 \\
s_0(g_n, \ldots, g_0) &= (s_0 g_n, e_n, g_{n-1}, \ldots, g_0) \\
s_{i+1}(g_n, \ldots, g_0) &= (s_{i+1} g_n, s_i g_{n-i}, \ldots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0). \quad i \geq 0.
\end{aligned}$$

Then WA is contractible (see [18]), and the map $p : WA \longrightarrow \overline{W}A$ which sends $(g_n, \ldots, g_0) \mapsto (g_{n-1}, \ldots, g_0)$ is a *PTCP* (principal twisted cartesian product) of type (W), with group A. Define a function $T : (WA)_n \longrightarrow (WA)_{n+1}$ by $T(g_n, \ldots, g_0) = (e_{n+1}, g_n, \ldots, g_0)$. The contraction in WA is given by the functions

$$h_{n-i}(g_n, \ldots, g_0) = (e_{n+1}, \ldots, e_{i+1}, d_0^{n-i} g_n \cdots d_0 g_{i+1} \cdot g_i, g_{i-1}, \ldots, g_0).$$

$0 \leq i \leq n$. We may write

$$h_{n-i} = s_0^{n-i} \circ T \circ d_0^{n-i},$$

where the exponent indicates the number of times a map is to be iterated. It is then not very difficult to check the following very crucial fact.

Proposition 6.3.1 *The functions h_i defined above give a contraction of WA which is natural with respect to the simplicial homomorphisms $f : A \longrightarrow B$.* ∎

6.4 G-Cohomology theories

Let $\mathcal{C}_R$ denote the category of cohomology theories over R, as defined in (6.2.3).

Definition 6.4.1 *A G-cohomology theory over R is a contravariant functor $A : O_G \longrightarrow \mathcal{C}_R$.*

Given a G-cohomology theory $A : O_G \longrightarrow \mathcal{C}_R$ over R, each $A(G/H)$ is a cohomology theory. We may then define an O_G-R-module $\lambda_A : O_G \longrightarrow$ **R-mod** by $\lambda_A(G/H) = R(A(G/H))$. Recall that $R(A(G/H)) = (Z^0(A(G/H)))_0$, where $Z^0(A(G/H))$ is the kernel of the homomorphism

$$\delta_H : A^0(G/H) \longrightarrow A^1(G/H).$$

We also have functors

$$\begin{aligned} A^n &: O_G \longrightarrow \mathbf{sR\text{-}mod} \\ Z^n A &: O_G \longrightarrow \mathbf{sR\text{-}mod} \end{aligned}$$

defined in the obvious manner. Then

Proposition 6.4.2 *If $A : O_G \longrightarrow \mathcal{C}_\mathbb{Q}$ is a G-cohomology theory over $\mathbb{Q}$, then each $A^n : O_G \longrightarrow$ **sVect**$_\mathbb{Q}$ is contractible as an object of* **sVec**$_G$.

Proof. For every $H \subseteq G$, we have principal fibrations

$$0 \longrightarrow Z^n A(G/H) \longrightarrow A^n(G/H) \longrightarrow Z^{n+1}(G/H) \longrightarrow 0, \quad n \geq 0.$$

This is naturally isomorphic (see [18]) to the universal *PTCP* of type (W) given by the $\overline{W}$- and W-constructions on the appropriate complexes. As the contraction in the universal *PTCP* of type (W) is natural with respect to the simplicial homomorphisms, the result follows. ∎

Starting with a G-cohomology theory $A : O_G \longrightarrow \mathcal{C}_{\mathbb{Q}}$ over $\mathbb{Q}$, and a G-simplicial set K, we can construct a differential graded algebra

$$\mathrm{Hom}(\Phi K, A^{\cdot}) = \oplus_{n\geq 0}\mathrm{Hom}_{sVec_G}(\Phi\mathbb{Q}K, A^n).$$

Since each $A(G/H) : \Delta \longrightarrow \mathbf{DGA/Q}$ is a simplicial differential graded algebra, $\mathrm{Hom}(\Phi K, A^{\cdot})$ is a differential graded algebra in an obvious way. We shall denote $Hom(\Phi K, A^{\cdot})$ by $\mathcal{A}^{\cdot}(K)$.

We now come to the first main theorem.

Theorem 6.4.3 *For any G-cohomology theory $A : O_G \longrightarrow \mathcal{C}_{\mathbb{Q}}$ over $\mathbb{Q}$, there is an isomorphism*

$$H^{\cdot}_G(K; \lambda_A) \cong H^{\cdot}(\mathcal{A}^{\cdot}(K))$$

for every G-simplicial set K.

Proof. Recall that the coefficients system $\lambda_A : O_G \longrightarrow \mathbf{R\text{-}mod}$ is defined by $\lambda_A(G/H) = R(A(G/H)) = (Z^0(A(G/H)))_0$. Clearly $Z^n A : O_G \longrightarrow \mathbf{sVect_{\mathbb{Q}}}$ is an O_G-Eilenberg-MacLane complex of the type (λ_A, n).

Let $n > 0$. We have a short exact sequence

$$0 \longrightarrow Z^n A \longrightarrow A^n \longrightarrow Z^{n+1} \longrightarrow 0,$$

in the category $\boldsymbol{s}\boldsymbol{Vec_G}$. We may therefore identify $Z^n\mathcal{A}^{\cdot}(K) = \mathrm{Ker}(\mathcal{A}^n(K) \longrightarrow \mathcal{A}^{n+1}(K))$ with $\mathrm{Hom}(\Phi\mathbb{Q}K, Z^n A)$, which is the R-module of simplicial maps from $\Phi\mathbb{Q}K$ to $Z^n A$. There is an obvious map

$$\mathrm{Hom}(\Phi\mathbb{Q}K, Z^n A) \longrightarrow [\Phi\mathbb{Q}K, Z^n A] \cong H^n_G(K; \lambda_A),$$

where the isomorphism is as given in the classification theorem (Theorem 5.4.6). We shall show that if $f \in \mathrm{Hom}(\Phi\mathbb{Q}K, Z^n A)$ is homotopic to constant, then it factors through $p : \mathcal{A}^{n-1} \longrightarrow Z^n A$. Consider a commutative diagram:

$$\begin{array}{ccc} \Phi\mathbb{Q}K \otimes 0 & \longrightarrow & A^{n-1} \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ \Phi\mathbb{Q}K \otimes I & \xrightarrow{F} & Z^n A \end{array}$$

where the horizontal map on the top is the constant map, F is a homotopy between f and the constant map, the vertical map i_0 on the left is a trivial cofibration. Since p is surjective, it is a fibration. Consequently, the above LLP for i_0 with respect to p has a solution, by Theorem 4.5.2,

$$\tilde{F} : \Phi\mathbb{Q}K \otimes I \longrightarrow A^{n-1}$$

such that $p \circ \tilde{F}|\Phi\mathbb{Q}K \otimes 1 = f$. This proves the theorem when $n > 0$.

For $n = 0$, we argue as follows. As $Z^0\mathcal{A}(K) = \mathrm{Hom}(\Phi\mathbb{Q}K, Z^0 A)$, two morphisms $f, g \in \mathrm{Hom}(\Phi\mathbb{Q}K, Z^0 A)$ are homotopic if and only if they are equal. This completes the proof of the theorem. ∎

We conclude this chapter with two examples of G-cohomology theory.

Example 6.4.4 *Consider the contravariant functor $A : O_G \longrightarrow \mathcal{C}_{\mathbb{Q}}$ defined by*

$$A(G/H) = C^*(\ ; \lambda(G/H)),$$

*where $\lambda : O_G \longrightarrow$ **Vect**$_{\mathbb{Q}}$ is a coefficients system, and $C^*(\Delta[q]; \lambda(G/H))$ denotes the ordinary singular cochain group. That $A(G/H)$ is a cohomology theory follows from Example 6.2.3, and A is a G-cohomology theory. Observe that we have $\lambda_A = \lambda$, and hence by Theorem 6.4.3*

$$H^*_G(K; \lambda) \cong H^*(\mathcal{A}^*(K)).$$

Example 6.4.5 *If A is a cohomology theory over $\mathbb{Q}$ in the sense of Cartan, and $\lambda_{\mathbb{Q}} : O_G \longrightarrow$ **Vect**$_{\mathbb{Q}}$ is a coefficients system as defined in Section 3.3, p.34, then $A^G : O_G \longrightarrow \mathcal{C}_{\mathbb{Q}}$ defined by $A^G(G/H) = \lambda_{\mathbb{Q}}(G/H) \otimes A$ is a G-cohomology theory, where $\lambda_{\mathbb{Q}}(G/H)$ is considered as a simplicial differential graded algebra concentrated in dimension zero. Observe that $\lambda_A^G = \lambda \otimes R(A)$. Hence*

$$H_G^{\bullet}(K; \lambda_{\mathbb{Q}}) \cong H^{\bullet}(\mathcal{A}^G(K)).$$

Chapter 7 COHOMOLOGY OF A G-SPACE

7.1 Introduction

In this chapter, we shall show that every G-space has the same cohomology of some Eilenberg-MacLane G-space. This result was first proved by Kan-Thurston [15] in the non-equivariant case. Explicitly they proved

Theorem 7.1.1 (Kan-Thurston) *For any path connected space X with base point there exists a Serre fibration*

$$tX : TX \longrightarrow X$$

which is natural with respect to X and

- *The map tX induces isomorphism in homology and cohomology with any local coefficients on X,*
- *$\pi_i(TX) = 0 \quad i > 1$ and $\pi_1(tX)$ is onto.*

Thus TX is an Eilenberg-MacLane space, and X has the same homology and cohomology as TX. It further turns out that ker $\pi_1(tX)$ is a perfect normal subgroup of $\pi_1(TX)$, and that X can be obtained from TX by applying plus construction of Quillen [25] with respect to the perfect normal subgroup ker $\pi_1(tX)$ of $\pi_1(TX)$.

The equivariant version of the above theorem uses equivariant singular Illman cohomology, and Eilenberg-MacLane G-spaces of type $(\lambda, 1)$, which is, roughly speaking, a G-space Y such that $\pi_1(Y^H) = \lambda(G/H)$, and $\pi_i(Y^H) = 0$ for $i > 1$ and every $H \subseteq G$, where $\lambda : O_G \longrightarrow \mathbf{Grp}$ is a contravariant functor (see Chapter 8 for a precise definition). We shall prove

Theorem 7.1.2 *Let X be a G-path connected G-space, with G discrete, and a G-fixed base point. There is a G-fibration*

$$tX : TX \longrightarrow X$$

such that

(i) *tX induces isomorphism in equivariant singular Illman cohomology.*

(ii) *TX is an Eilenberg-MacLane G-space.*

Here "G-path connected" means that each X^H is path connected.

We shall consider Theorem 7.1.2 again in a more general setting in Chapter 8.

7.2 Elmendorf Construction

The main ingredient of the proof of Theorem 7.1.2 is a construction due to Elmendorf [10]. We shall adapt this construction to a G-simplicial setting.

Recall that in [10], Elmendorf described a beautiful method of constructing G-spaces with prescribed fixed point data. Let $O_G\mathcal{S}$ denote the category of contravariant functors $O_G \longrightarrow \mathcal{S}$. Such functors will be called O_G-simplicial sets. In particular, if K is a G-simplicial set, then $\Phi K : O_G \longrightarrow \mathcal{S}$ given by

$$(\Phi K)(G/H) = K^H$$

is an O_G-simplicial set.

Theorem 7.2.1 (Elmendorf). *There is a functor $E : O_G\mathcal{S} \longrightarrow G\mathcal{S}$, and a natural transformation*

$$\eta : \Phi E \longrightarrow id$$

such that for an O_G-simplicial set T and a subgroup H of G,

$$\eta(T)(G/H) : ET^H \longrightarrow T(G/H)$$

is a homotopy equivalence.

Proof. Given an O_G-simplicial set T, we form the bar complex $B_*(T; O_G)$, the n-simplexes of which are $(n+2)$-tuples $(t; f_1, \ldots, f_n; x)$, where

$$G/H_n \xrightarrow{f_n} G/H_{n-1} \longrightarrow \cdots \longrightarrow G/H_1 \xrightarrow{f_1} G/H_0$$

are composable morphisms in O_G, $t \in T(G/H_0)_n$ and $x \in G/H_n$. The face and degeneracy maps are given by

$$\begin{aligned} d_i(t; f_1, \ldots, f_n; x) &= \begin{cases} (d_0 T(f_1)(t); f_2, \ldots, f_n; x) & i = 0 \\ (d_i t; f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n; x) & 0 < i < n \\ (d_n t; f_1, \ldots, f_{n-1}; f_n x) & i = n \end{cases} \\ s_i(t; f_1, \ldots, f_n; x) &= (s_i t; f_1, \ldots, f_i, id, f_{i+1}, \ldots, f_n; x) \quad i = n \end{aligned}$$

where the face and degeneracy maps on the right are those of appropriate complexes. There is an obvious action of G on the last coordinate which makes $B_*(T; O_G)$ a G-simplicial set. Define $ET = B_*(T; O_G)$. Then, since the action is only on the last coordinate, the n-simplexes of the simplicial set ET^H consist of $(n+2)$-tuples $(t; f_1, \ldots, f_n; f)$ where $f : G/H \longrightarrow G/H_n$. There are also obvious simplicial maps

$$\eta_H^T : ET^H \longrightarrow T(G/H) \quad \text{and} \quad \xi_H^T : T(G/H) \longrightarrow ET^H$$

given by

$$\begin{aligned} \eta_H^T(t; f_1, \ldots, f_n; f) &= T(f_1 \circ \cdots \circ f_n)(t), \quad \text{and} \\ \xi_H^T(t) &= (t; id, \ldots, id; id), \end{aligned}$$

where $id : G/H \longrightarrow G/H$. Clearly $\eta_H^T \circ \xi_H^T = id$. We shall show that $\xi_H^T \circ \eta_H^T \sim id$. First note that η is natural, however ξ is not. The homotopy is given by the functions

$$\begin{aligned} h_i &: ET_n^H \longrightarrow ET_{n+1}^H, \qquad 0 \le i \le n, \quad \text{where} \\ h_i(t; f_1, \ldots, f_n; f) &= (t; f_1, \ldots, f_i, f_{i+1} \circ \cdots \circ f_n \circ f, id, \ldots, id; id) \end{aligned}$$

It is straightforward to check that the h_i's constitute the required homotopy. This completes the proof. ∎

Corollary 7.2.2 *If K is a G-simplicial set, then $\eta(\Phi K)(G/\{e\}) : E\Phi K \longrightarrow K$ is a weak G-equivalence.* ∎

In general, for a G-simplicial set T, ET is not a G-Kan complex. The above corollary however shows that

Corollary 7.2.3 *If K is a G-Kan complex then every fixed point simplicial set $E\Phi K^H$ has the homotopy type of a Kan complex.* ∎

These are the results central to the proof of Theorem 7.1.2. Note that a topological version of Theorem 7.2.1 may be obtained by replacing the category $\mathcal{S}$ by **TOP** (see Theorem 8.4.1). This topological version will be used in Chapter 8.

7.3 Kan-Thurston Theorem

In this section, we sketch the proof of the following simplicial version of Theorem 7.1.1. This will be used in the proof of Theorem 7.1.2.

Theorem 7.3.1 *For every reduced simplicial set K, there exists a fibration*

$$tK : TK \longrightarrow K$$

which is natural with respect to K and has the following properties.

(*i*) *tK induces isomorphism on homology and cohomology with local coefficients*

(*ii*) *$\pi_i(TK) = 0$ for all $i > 1$, $\pi_1(tK)$ is onto*

(*iii*) *$\pi_1(TK)$ has the same cardinality as K.*

Proof (Sketch). Associated to every reduced simplicial set K is its loop group LK, which is a free simplicial group given as follows

(1) LK_n is the group which has one generator $\overline{x}$ for every $x \in K_{n+1}$ and one relation $\overline{s_0 x} = e$ for every $x \in K_n$.

(2) The face and degeneracy maps are

$$\begin{aligned} d_0\bar{x} &= \overline{(d_1x)}\overline{(d_0x)} \quad i=0 \\ d_i\bar{x} &= \overline{d_{i+1}x} \quad 0 < i \leq n \\ s_ix &= \overline{s_{i+1}x}. \end{aligned}$$

There is a natural homotopy equivalence

$$i_K : K \longrightarrow \overline{W}LK \tag{7.1}$$

(see [8]). Then Kan-Thurston constructed a natural sequence

$$0 = L^0K \longrightarrow L^1K \longrightarrow \ldots \longrightarrow L^{n-1}K \longrightarrow L^nK \longrightarrow \ldots \tag{7.2}$$

of simplicial groups and homomorphisms, together with compatible maps

$$L^nK \longrightarrow LK^{(n)},$$

where $K^{(n)}$ is the n-skeleton of K, such that in the inductive limit, the induced map

$$\overline{W}L^\infty K = \overline{W}\varinjlim L^nK \longrightarrow \overline{W}\varinjlim LK^n = \overline{W}LK$$

is a fibration natural with respect to K satisfying (i)-(iii). Then fibration $tK : TK \longrightarrow K$ is obtained as the pull back of this fibration by the homotopy equivalence $K \longrightarrow \overline{W}LK$. ∎

Note that all the constructions L^nK, LK, $\overline{W}LK$ considered above are functorial.

7.4 Proof of Theorem 7.1.2

The proof will be obtained as a corollary to the following

Theorem 7.4.1 *For every reduced G-simplicial set K, there is a G-fibration*

$$tK : TK \longrightarrow K$$

such that

(*i*) $\underline{\pi}_i(TK) = 0$ *whenever* $i > 1$.

(*ii*) $(tK)^* : H^*_G(K;\lambda) \longrightarrow H^*_G(TK;\lambda)$ *is an isomorphism for all* O_G-*abelian groups* λ.

Proof. We consider the simplicial groups L^nK in (7.2) as functors from O_G to $\mathcal{S}$:

$$\Phi K, \overline{W}LK, \overline{W}L^{\infty}K : O_G \longrightarrow \mathcal{S}$$

defined by

$$\begin{aligned} \Phi K(G/H) &= K^H \\ \overline{W}LK(G/H) &= \overline{W}L(K^H) \\ \overline{W}L^{\infty}K(G/H) &= \overline{W}L^{\infty}(K^H). \end{aligned}$$

We also have natural transformations

$$\begin{aligned} i &: \Phi K \longrightarrow \overline{W}LK \\ t &: \overline{W}L^{\infty}K \longrightarrow \overline{W}LK \end{aligned}$$

where $i(G/H)$ is the natural homotopy equivalence (7.1), and $t(G/H)$ is the natural map as constructed in the proof of Theorem 7.3.1. Then, the Elmendorf construction (Theorem 7.2.1) gives a diagram of G-simplicial sets and G-maps.

$$\begin{array}{ccc} & & E\overline{W}L^{\infty}K \\ & & \downarrow{\scriptstyle Et} \\ E\Phi K & \xrightarrow{Ei} & E\overline{W}LK \end{array}$$

Observe that

(1) for each $H \subseteq G$, $(Ei)^H : (E\Phi K)^H \longrightarrow (E\overline{W}LK)^H$ is a homotopy equivalence.

To see this note that there is a commutative diagram by Theorem 7.2.1

$$\begin{array}{ccc} (E\Phi K)^H & \xrightarrow{(Ei)^H} & (E\overline{W}LK)^H \\ \downarrow & & \downarrow \\ K^H & \xrightarrow{i(G/H)} & \overline{W}L(K^H) \end{array}$$

where the vertical maps are homotopy equivalences. Then, since $i(G/H)$ is a homotopy equivalence, $(Ei)^H$ is also so.

It may be noted that

$$\pi_i((E\overline{W}L^{\infty}K)^H) = 0, \quad \text{for} \quad i > 1,$$

by Theorem 7.2.1 and 7.3.1.

(2) the simplicial map $(Et)^H : (E\overline{W}L^{\infty}K)^H \longrightarrow (E\overline{W}LK)^H$ induces isomorphism in cohomology.

This readily follows from the commutative diagram

$$\begin{array}{ccc} (E\overline{W}L^{\infty}K)^H & \longrightarrow & (E\overline{W}LK)^H \\ \downarrow & & \downarrow \\ \overline{W}L^{\infty}(K^H) & \longrightarrow & \overline{W}L(K^H) \end{array}$$

since the vertical maps are homotopy equivalences (Theorem 7.2.1), and the lower horizontal map induces isomorphism in integral homology by Theorem 7.1.1. Again Theorem 2.3.3 (in conjunction with the observations (1) and (2) above) gives the following isomorphisms

$$H^*_G(K;\lambda) \cong H^*_G(E\Phi K;\lambda) \cong H^*_G(E\overline{W}LK;\lambda) \cong H^*_G(E\overline{W}L^{\infty}K;\lambda)$$

induced by appropriate maps.

Let $\tilde{\imath} : \widetilde{K} \longrightarrow E\Phi K$ denote the pullback of $E\bar{W}L^{\infty}K \longrightarrow E\bar{W}LK$ by Ei. Then look at the composition

$$\widetilde{K} \xrightarrow{\tilde{\imath}} E\Phi K \xrightarrow{\eta} K$$

of G-maps. Since $\boldsymbol{GS}$ is a closed model category, this composition may be factored as

$$\widetilde{K} \longrightarrow TK \xrightarrow{tK} K$$

where the first is map a weak equivalence and tK is a G-Kan fibration. Then tK has the required properties (i), (ii). Note that $\pi_i(TK^H) = 0$ for $i > 1$. ∎

Proof of Theorem 7.1.2. Choose a subcomplex $\widetilde{SX}$ of SX of the same G-*homotopy type as* SX *with no other vertex than the base point.* This is possible because of Theorem 1.5.7. Consider the composition

$$|T\widetilde{SX}| \xrightarrow{|t\widetilde{SX}|} |\widetilde{SX}| \longrightarrow |SX| \longrightarrow X.$$

Clearly we have a factorization of this composition into a weak equivalence followed by a G-fibration :

$$|T\widetilde{SX}| \longrightarrow TX \longrightarrow X$$

Then tX is the required G-fibration. ∎

Chapter 8 EQUIVARIANT HOMOTOPY TYPE

8.1 Introduction

In this chapter we shall consider the other aspect of the Kan-Thurston theorem (Theorem 7.1.1). We shall prove an equivariant version of a consequence of this theorem which says that any connected *CW*-space has the homotopy type of a space obtained from an Eilenberg-MacLane space by applying the plus construction of Quillen [25].

Throughout this chapter, we suppose that G is a compact Lie group, and we consider only closed subgroups of G. All G-spaces X are compactly generated weakly Hausdorff with a stationary point $x_0 \in X^G$ as base point such that X has the G-homotopy type of a G-*CW*-complex [9], and, for each subgroup H of G, X^H has the homotopy type of a connected *CW*-complex. For example, X may be a *smooth* G-*manifold* such that each X^H is connected. All G-maps and G-homotopies are base point preserving.

Recall that an O_G group is a contravariant functor $\lambda : O_G \longrightarrow \mathbf{Grp}$. Then a perfect normal O_G-subgroup N of an O_G-group λ is an O_G-group such that each $N(G/H)$ is a perfect normal subgroup of $\lambda(G/H)$. A homotopy O_G-group $\underline{\pi}_n(X)$ of a G-space X is an O_G-group if $n = 1$ and an abelian O_G-group if $n > 1$, and this is defined by $\underline{\pi}_n(X)(G/H) = \pi_n(X^H, x_0)$ and $\underline{\pi}_n(X)(\widehat{g}) = \pi_n(g)$, where $\widehat{g} : G/H \longrightarrow G/K$ is a morphism in O_G, $g^{-1}Hg \subseteq K$, $g : X^K \longrightarrow X^H$ is the left translation by g (cf. Section 5.3). A G-map $f : X \longrightarrow Y$ induces a morphism of O_G-groups $\underline{\pi}_n(f) : \underline{\pi}_n(X) \longrightarrow \underline{\pi}_n(Y)$ defined by $\underline{\pi}_n(f)(G/H) = \pi_n(f^H)$, where $f^H = f|X^H$.

In Section 8.3, we introduce the concept of homology O_G-group $\underline{\widetilde{H}}_n^G(X;\mathbf{Z})$. We say that a G-space is G-acyclic if $\underline{\widetilde{H}}_n^G(X;\mathbf{Z}) = 0$, for all n, and a G-map $f : X \longrightarrow Y$ is G-acyclic if its homotopy fibre is G-acyclic.

Given an O_G-group λ and an integer $n \geq 1$, there is a G-space X such that

$\underline{\pi}_n(X) = \lambda$ and $\underline{\pi}_j(X) = \underline{0}$ if $j \neq n$. The G-space X is actually a G-CW-complex, and is called an Eilenberg-MacLane G-space $K(\lambda, n)$ of type (λ, n) (see [10]).

Then in line of Theorem 7.1.1 we have

Theorem 8.1.1 *For any G-space X there exists an O_G-group λ with a perfect normal O_G-subgroup N, and a G-acyclic map*

$$p : K(\lambda, 1) \longrightarrow X,$$

which is natural with respect to X, such that $Ker\underline{\pi}_1(p) = N$.

In Section 8.4 we construct, for each G-space X and a perfect normal O_G-subgroup N of $\underline{\pi}_1(X)$, a G-space X_N^+ by applying the plus construction of Quillen to each X^H with respect to the group $N(G/H)$, and then piecing the resulting spaces together by means of a functorial bar construction [10] (which is a topological version of Theorem 7.2.1).

Then the main theorem of this chapter is

Theorem 8.1.2 *Given a G-space X there exists an O_G-group λ with a perfect normal O_G-subgroup N such that X has the G-homotopy type of $K(\lambda, 1)_N^+$.*

The proof of this theorem will follow from the following existence and uniqueness theorem of G-acyclic maps from a given G-space (cf. [13]).

Theorem 8.1.3 *If X is a G-space and N a perfect normal O_G-subgroup of $\underline{\pi}_1(X)$, then there exists a G-space X_N^+, and G-acyclic map $f : X \longrightarrow X_N^+$ such that $Ker\underline{\pi}_1(f) = N$.*

Theorem 8.1.4 *If $f : X \longrightarrow Y$ and $f' : X \longrightarrow Y'$ are G-maps, where f is G-acyclic, then there is a G-map $h : Y \longrightarrow Y'$, which is unique up to G-homotopy equivalence, such that $hf \simeq_G f'$ if and only if $Ker\underline{\pi}_1(f) \subseteq Ker\underline{\pi}_1(f')$. In addition, if f' is G-acyclic, then h is also G-acyclic, and h is a G-homotopy equivalence if and only if $Ker\underline{\pi}_1(f) = Ker\underline{\pi}_1(f')$.*

The proofs of these theorems appear in Section 8.4.

8.2 Equivariant Local Coefficients

Recall from tom Dieck [9], that the discrete fundamental group category ΠX of a G-space X is a category whose objects are G-maps $x : G/H \longrightarrow X$, and a morphism from $x : G/H \longrightarrow X$ to $y : G/K \longrightarrow X$ is an equivalence class of pairs $(\widehat{g}, \varphi)$ consisting of a G-map $\widehat{g} : G/H \longrightarrow G/K$ and a G-homotopy $\varphi : G/H \times I \longrightarrow X$ from x to $y \circ \widehat{g}$, where two pairs $(\widehat{g}_1, \varphi_1)$, $(\widehat{g}_2, \varphi_2) : x \longrightarrow y$ are equivalent if there exists a G-homotopy $\Psi : G/H \times I \longrightarrow G/K$ from $\widehat{g}_1$ to $\widehat{g}_2$, and a G-homotopy $\Phi : G/H \times I \times I \longrightarrow X$ from φ_1 to φ_2 such that $\Phi(gH, 0, t) = x(gH)$, and $\Phi(gH, 1, t) = y \circ \Psi(gH, t)$. We shall denote the equivalence class of $(\widehat{g}, \varphi)$ by $[\widehat{g}, \varphi]$.

We have a bijection $a : \mathrm{Map}_G(G/H, X) \longrightarrow X^H$ given by $a(f) = f(eH)$ and $a^{-1}(x)(gH) = gx$. Therefore we may identify an object $x : G/H \longrightarrow X$ with a point $x(eH)$ in X^H. Then a pair $(\widehat{g}, \varphi) : x \longrightarrow y$ corresponds to a path $< \widehat{g}, \varphi >$ from $x(eH)$ to $gy(eK)$ in X^H, and if two pairs $(\widehat{g}_1, \varphi_1)$, $(\widehat{g}_2, \varphi_2) : x \longrightarrow y$ are equivalent, then $< \widehat{g}_1, \varphi_1 >$ is freely homotopic to $< \widehat{g}_2, \varphi_2 >$ along the path $t \mapsto y \circ \Psi(eH, t)$ in X^H (see [30], p.98).

If G is trivial, then ΠX reduces to the opposite of fundamental groupoid $\mathcal{P}X$ of X. Also, for a fixed H, the objects $x : G/H \longrightarrow X$ together with morphisms of the form $[id_{G/H}, \varphi]$ constitute a subcategory of ΠX which is precisely $(\mathcal{P}X^H)^{op}$.

Definition 8.2.1 *An equivariant local coefficients system on X is a contravariant functor M from ΠX to the category* **Ab** *of abelian groups.*

If G is trivial, then M reduces to the classical local coefficients system on X. We define homomorphism and pull back of equivariant local coefficients systems as in [30]

If M is an equivariant local coefficients system, then the restriction $M_H = M|(\mathcal{P}X^H)^{op}$ is an ordinary local coefficients system on X^H. Moreover, a morphism $\widehat{g} : G/H \longrightarrow G/K$ in O_G gives rise to a homomorphism $\underline{M}(\widehat{g}) : M_K \longrightarrow g^* M_H$ of local coefficients system on X^K defined by $\underline{M}(\widehat{g})(x) = M[\widehat{g}, k]$, where $[\widehat{g}, k] : a^{-1}(x) \circ \widehat{g} \longrightarrow a^{-1}(x)$ is the morphism in ΠX given by the constant homotopy k on $a^{-1}(x) \circ \widehat{g}$. Here $g^* M_H$ is the pullback of the local coefficients system M_H by the map $g : X^K \longrightarrow X^H$. Conversely, given local coefficients system M_H on X^H for each $H \subseteq G$ connected by homomorphisms $\underline{M}(\widehat{g}) : M_K \longrightarrow g^* M_H$, where $\widehat{g} : G/H \longrightarrow G/K$, we obtain an equivariant local coefficients system M on X by setting $M(x) = M_H(a(x))$, and $M[\widehat{g}, \varphi] = \underline{M}(\widehat{g}) \circ M_H(< \widehat{g}, \varphi >)$. Clearly the above correspondence is a bijection.

Definition 8.2.2 *An abelian O_G-group λ is a $\underline{\pi}_1(X)$-module if there is a natural transformation $\alpha : \underline{\pi}_1(X) \times \lambda \longrightarrow \lambda$ such that, for each $H \subseteq G$, $\alpha(G/H)$ is an action of the group $\underline{\pi}_1(X)(G/H) = \pi_1(X^H, x_0)$ on $\lambda(G/H)$.*

Suppose λ is a $\underline{\pi}_1(X)$-module. Since X^H is connected, the $\pi_1(X^H, x_0)$-module structure on $\lambda(G/H)$ gives an ordinary local coefficients system L_H on X^H so that $L_H(x) = \lambda(G/H)$, $x \in X^H$ ([30], p.263). It is not difficult to see that if $\widehat{g} : G/H \longrightarrow G/K$, then $\lambda(\widehat{g}) : L_K \longrightarrow g^* L_H$ is a homomorphism of local coefficients on X^K. Thus a $\underline{\pi}_1(X)$-module λ defines an equivariant local coefficients system on X. In particular, the O_G-group $\mathbb{Z}\underline{\pi}_1(X)$, where $\mathbb{Z}\underline{\pi}_1(X)(G/H)$ is the integral group ring $\mathbb{Z}\pi_1(X^H, x_0)$, is the same thing as an equivariant local coefficients system on X.

Let $\mathcal{L}$ be the category whose objects are $(X, A; M)$, where (X, A) is a pair of G-spaces and M an equivariant local coefficients system on X. A morphism $f = (f_1, f_2) : (X, A; M) \longrightarrow (Y, B; N)$ consists of a G-map $f_1 : (X, A) \longrightarrow (Y, B)$ and a homomorphism $f_2 : M \longrightarrow f_1^* N$ of equivariant local coefficients system on X. Clearly f induces a morphism $f^H = (f_1^H, f_2^H) : (X^H, A^H; M_H) \longrightarrow (Y^H, B^H; N_H)$ for each $H \subseteq G$, where $f_1^H = f_1 | X^H$ and f_2^H is the homomorphism induced by f_2 on X^H.

8.3 Homology O_G-groups

Let H_* denote the classical homology with local coefficients system [30]. Then, an object $(X, A; M)$ in $\mathcal{C}$ determines, for each integer $n \geq 0$, an O_G-group $O_G \longrightarrow \mathbf{Ab}$ by sending an orbit G/H to $H_n(X^H, A^H; M_H)$, and sending a G-map $\hat{g} : G/H \longrightarrow G/K$ to $H_n(\underline{g}) : H_n(X^K, A^K; M_K) \longrightarrow H_n(X^H, A^H; M_H)$, where $\underline{g} : (X^K, A^K; M_K) \longrightarrow (X^H, A^H; M_H)$ is the morphism given by the left translation $g : (X^K, A^K) \longrightarrow (X^H, A^H)$, and the homomorphism $\underline{M}(\hat{g}) : M_K \longrightarrow g^* M_H$ defined in Section 8.2. We shall denote this O_G-group by $\underline{H}_n^G(X, A; M)$, and call it the n-th homology O_G-group of (X, A) with equivariant local coefficients system M.

In particular, taking M as a constant coefficients system with $M(x) = \mathbf{Z}$ and $M[\hat{g}, \varphi] = id$, we have the reduced homology O_G-group $\underline{\widetilde{H}}_n^G(X, \mathbf{Z})$.

We now recall some facts about acyclic maps.

Definition 8.3.1 *A space X is acyclic if its reduced integral homology is trivial. A map $f : X \longrightarrow Y$ is acyclic if its homotopy fiber is acyclic.*

Theorem 8.3.2 *The following conditions on a map $f : X \longrightarrow Y$ are equivalent.*

- *f is acyclic.*
- *$f_* : H_*(X; f^* L) \longrightarrow H_*(Y; L)$ is an isomorphism for every local coefficients system L on Y.*
- *$f_* : H_*(X; f^* \mathbf{Z}\pi_1(Y, y_0)) \longrightarrow H_*(Y; \mathbf{Z}\pi_1(Y, y_0))$ is an isomorphism.* ∎

Theorem 8.3.3 *$f : X \longrightarrow Y$ is a homotopy equivalence if and only if f is acyclic and $\pi_1(f)$ is an isomorphism.* ∎

The proofs of these theorems, and other facts about acyclic maps, which we shall use, may be found in [13].

Turning now to the equivariant situation, note that a G-map $f : X \longrightarrow Y$ is a G-fibration if it has equivariant homotopy lifting property. This implies that the fibre $F = f^{-1}(y_0)$ is a G-space, and that each $f^H : X^H \longrightarrow Y^H$ is an ordinary fibration with fibre $F^H = F \cap X^H$. We may replace any G-map $f : X \longrightarrow Y$ by a G-fibration $\overline{f} : \overline{X} \longrightarrow Y$ up to G-homotopy equivalence, i.e., there is a G-homotopy equivalence $i : X \longrightarrow \overline{X}$ such that $\overline{f} \circ i = f$. Also, any G-map may be replaced by an inclusion up to G-homotopy equivalence.

Definition 8.3.4 *A G-space X is G-acyclic if each X^H is acyclic, and a G-map $f : X \longrightarrow Y$ is G-acyclic if its G-homotopy fibre is G-acyclic.*

Clearly a G-map $f : X \longrightarrow Y$ is G-acyclic if and only if for each subgroup $H \subseteq G$, $f^H : X^H \longrightarrow Y^H$ is acyclic. We now have

Proposition 8.3.5 *A G-map $f : X \longrightarrow Y$ is G-acyclic if and only if f induces isomorphisms of homology O_G-groups*

$$\underline{H}^G_*(X; f^*M) \cong \underline{H}^G_*(Y; M),$$

for any equivariant local coefficients system M on Y.

Proof. The direct implication is immediate, since the category of O_G-abelian groups is an abelian category. The reverse implication follows by specializing to the equivariant local coefficients system $M = \mathbf{Z}\underline{\pi}_1(Y)$, and then applying Theorem 8.3.2. ∎

The equivariant Whitehead theorem, says that a G-map f is a G-homotopy equivalence if and only if each f^H is a homotopy equivalence ([9], p.107). It is then immediate that

Proposition 8.3.6 *A G-map f is a G-homotopy equivalence if and only if it is G-acyclic and $\underline{\pi}_1(f)$ is an isomorphism.* ∎

8.4 Proof of the Theorems

The main tool of the proofs is a topological version of Theorem 7.2.1. We record this result here for convenience.

An O_G-space is a contravariant functor $O_G \longrightarrow$ **TOP**. Let O_G − **spaces** be the category of O_G-spaces, and **G-spaces** be the category of G-spaces, and for a G-space X, ΦX be the O_G-space $O_G \longrightarrow$ **TOP** as defined in Section 7.2. We shall use the following topological version of Theorem 7.2.1.

Theorem 8.4.1 (Elmendorf) *There is a functor*

$$E : O_G\textbf{-spaces} \longrightarrow \textbf{G-spaces},$$

and a natural transformation $\eta : \Phi E \longrightarrow id$, *such that*

$$\eta(T)(G/H) : (ET)^H \longrightarrow T(G/H)$$

is a homotopy equivalence. ∎

Proof of Theorem 8.1.1. First note that the Elmendorf functor E gives a G-homotopy equivalence

$$\eta(\Phi X)(G/\{e\}) : E\Phi X \longrightarrow X.$$

Now Theorem 7.1.1 guarantees for each $H \subseteq G$ the existence of a group $\lambda(G/H)$ with a perfect normal subgroup $N(G/H)$, and a fibration

$$p(G/H) : K(\lambda(G/H), 1) \longrightarrow X^H$$

satisfying the conditions :

- $p(G/H)$ is acyclic, and
- $\operatorname{Ker} \pi_1(p(G/H)) = N(G/H)$.

By naturality, these fibrations produce an O_G-fibration $q : B \longrightarrow \Phi X$, where the O_G-space B is given by $B(G/H) = K(\lambda(G/H), 1)$. Applying Elmendorf functor to it, we get a G-map

$$Eq : EB \longrightarrow E\Phi X,$$

where EB is actually an Eilenberg-MacLane G-space. Then the composition

$$EB \xrightarrow{Eq} E\Phi X \xrightarrow{\eta(\Phi X)(G/\{e\})} X$$

is the required G-acyclic map p. ■

Proof of Theorem 8.1.3. If X is a G-space and N is a perfect normal O_G-subgroup of $\underline{\pi}_1(X)$, then applying the plus construction of Quillen [25] to each X^H we get an acyclic map

$$f(G/H) : X^H \longrightarrow (X^H)^+_{N(G/H)}$$

such that $\mathrm{Ker}\,\pi_1(f(G/H)) = N(G/H)$. By naturality of the plus construction [13], these maps give a morphism $\Phi X \longrightarrow (\Phi X)^+_N$ of O_G-spaces, where

$$(\Phi X)^+_N(G/H) = (X^H)^+_{N(G/H)}.$$

Denote $E(\Phi X)^+_N$ by X^+_N. Then the Elmendorf functor gives a G-map $f' : E\Phi X \longrightarrow X^+_N$, and a composition of a G-homotopy equivalence $X \longrightarrow E\Phi X$ with f' gives the required G-acyclic map $f : X \longrightarrow X^+_N$. This completes the proof. ■

Proof of Theorem 8.1.4. If h exists, then $\underline{\pi}_1(f') = \underline{\pi}_1(h) \circ \underline{\pi}_1(f)$, and therefore $\mathrm{Ker}\,\underline{\pi}_1(f) \subseteq \mathrm{Ker}\,\underline{\pi}_1(f')$. Conversely, consider the G-push out diagram, and its restriction to each H-fixed point set

$$\begin{array}{ccc} X & \xrightarrow{f} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{g} & Y \cup_X Y' \end{array} \qquad\qquad \begin{array}{ccc} X^H & \xrightarrow{f^H} & Y^H \\ {\scriptstyle f'^H}\downarrow & & \downarrow{\scriptstyle g'^H} \\ Y'^H & \xrightarrow{g^H} & Y^H \cup_{X^H} Y'^H \end{array}$$

The second diagram, which is also a push out, implies that g^H is acyclic, since f^H is so [13]. Now the van Kampen theorem gives a homomorphism

$$\pi_1(g^H) : \pi_1(Y'^H) \longrightarrow \pi_1(Y^H \cup_{X^H} Y'^H) = \pi_1(Y^H) *_{\pi_1(X^H)} \pi_1(Y'^H),$$

which is an isomorphism, since $\mathrm{Ker}\,\pi_1(f^H) \subseteq \mathrm{Ker}\,\pi_1(f'^H)$. Therefore g^H is a homotopy equivalence, and hence g is a G-homotopy equivalence, by the equivariant Whitehead theorem. Then, if $g^\bullet$ is a G-homotopy inverse of g, $h = g^\bullet \circ g' : Y \longrightarrow Y'$ is the required G-map with $h \circ f \simeq_G f'$. Clearly h is G-acyclic if f' is so, and, since $\underline{\pi}_1(h)$ is an isomorphism if and only if $\mathrm{Ker}\,\underline{\pi}_1(f) = \mathrm{Ker}\,\underline{\pi}_1(f')$, the last assertion follows.

To see that h is unique up to G-homotopy equivalence, suppose that $j : F \longrightarrow X$ is the G-homotopy fibre of $f : X \longrightarrow Y$. Then, since $f \circ j \simeq_G y_0$, f extends to a G-map $k : X \cup_j CF \longrightarrow Y$ over the equivariant mapping cone of j. The G-map k is actually a G-homotopy equivalence, because its restriction to each H-fixed point set $k^H : X^H \cup CF^H \longrightarrow Y^H$ is acyclic and $\pi_1(k^H)$ is an isomorphism. Thus we have an equivariant coexact sequence

$$F \longrightarrow X \longrightarrow Y \longrightarrow \Sigma F,$$

where ΣF is the equivariant suspension of F. Since ΣF^H is simply connected and $\tilde{H}_*(\Sigma F^H; \mathbb{Z}) = 0$, ΣF^H is contractible. This implies that ΣF is G-contractible by the equivariant Whitehead theorem, considering a point as a G-CW-complex with one equivariant 0-cell $G/G \times D^0$. Thus the map $f^\bullet : [Y, Y']^0_G \longrightarrow [X, Y']^0_G$ in the Barratt-Puppe sequence ([9], p.142) is injective, where $[Y, Y']^0_G$ denotes the set of base point preserving G-homotopy classes of G-maps $Y \longrightarrow Y'$. This ensures the uniqueness of h, and the proof of Theorem 8.1.4 is complete. ∎

Proof of Theorem 8.1.2. By Theorem 8.1.1, there exists an O_G-group λ with a perfect normal O_G-subgroup N, and a G-acyclic map $f : K(\lambda, 1) \longrightarrow X$. Then Theorems 8.1.3 and 8.1.4 together give the desired result. ∎

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