ON THE GEOMETRISABILITY OF SOME STRONGLY REGULAR GRAPHS RELATED TO POLAR SPACES

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By

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To my parents

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Chapter 1

Introduction

1.1 Graphs

A *graph* $G = (V, E)$ consists of a finite set V and a subset E of $\binom{V}{2}$. (Here $\binom{V}{2}$ denotes the set of all 2-subsets of V.) Elements of V are called the vertices and the elements of E are called the edges of the graph. So V is the vertex set and E is the edge set of the graph G. Two vertices x, y are said to be adjacent if the pair $\{x, y\}$ is an edge; otherwise they are non-adjacent. If two vertices are adjacent then each is called a neighbour of the other vertex.

Sometimes the edges of a graph are ordered pairs of vertices and in this case the graph is called a directed graph. If both ends of an edge terminate at the same vertex then the edge is said to be a "loop". If two edges of a graph join the same two vertices then the graph is said to have multiple edges. But in this thesis the graphs we consider are undirected, loop free and with no multiple edges.

A graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is said to be a *subgraph* of a graph $G = (V, E)$ if $\widetilde{V} \subseteq V$

and $\tilde{E} \subseteq E$. If $\tilde{E} = E \cap \binom{\tilde{V}}{2}$ then $\tilde{G}$ is said to be an *induced subgraph* of G. It is also called the *induced subgraph on* $\tilde{V}$. The *complement* of a graph $G = (V, E)$ is the graph $G^* = (V, \binom{V}{2} \setminus E)$.

An *isomorphism* of a graph G onto a graph G' is a one to one correspondence between the vertices in G and the vertices in G' such that a pair of vertices are adjacent in G iff the corresponding pair of vertices are adjacent in G'. Two graphs are said to be *isomorphic* if there exists an isomorphism between them. In this thesis we identify two graphs if they are isomorphic. An isomorphism of a graph to itself is called an *automorphism*. The automorphisms of a graph form a group under composition; it is called the *automorphism group* of the graph.

The *degree* of a vertex x in a graph G is the number of vertices in G which are adjacent to x. A graph is *regular of degree* k if all its vertices are of degree k.

The unique connected regular graph of degree two on n vertices is called the *n-cycle*. A graph on n vertices in which each vertex is adjacent to all other vertices is called the *complete graph* on n vertices, denoted by K_n. A graph whose edge set is empty is called a *null graph*. In other words, a null graph is the complement of a complete graph. If $n_1, ..., n_k$ are positive integers, the *complete multi-partite graph* $K_{n_1,...,n_k}$ has its vertex set partitioned into k sets of size $n_1, ..., n_k$ such that two vertices are adjacent iff they belong to different parts. The *line graph* $L(G)$ of a graph G has the edges of G as its vertices; two vertices of $L(G)$ are adjacent iff the corresponding edges of G intersect. The *triangular graph* T_n is by definition the line graph of K_n. The line graph of the complete bipartite graph $K_{n,n}$ is called the *$n \times n$ grid*.

A *clique* of a graph is a complete subgraph. A *co-clique* of a graph is an induced null subgraph. A clique in a regular graph is called a *regular clique* if there exists a constant α such that each vertex outside the clique is adjacent to exactly α vertices inside the clique. In this case, α is called the *nexus* of the clique.

An *1-factor* of a graph (on an even number of vertices) is a partition of the vertex set into edges. An *1-factorisation* of the graph is a partition of the graph into 1-factors. Note that the union of any two 1-factors in an 1-factorisation is a union of disjoint cycles. An 1-factorisation is called a *Kotzig 1-factorisation* if the union of any two 1-factors in it is a single cycle.

1.2 Strongly Regular Graphs

A *strongly regular graph* with parameters v, k, λ, μ is a regular graph of degree k on v vertices such that given any two vertices the number of their common neighbours is λ or μ according as the given vertices are adjacent or not. Clearly the complement of a strongly regular graph is also strongly regular. The notion of strongly regular graphs was introduced by Bose [1].

Examples: The triangular graph T_n is a strongly regular graph with parameters $(n(n-1)/2, 2n-4, n-2, 4)$. The $n \times n$ grid is a strongly regular graph with parameters $(n^2, 2n-2, n-2, 2)$.

1.3 Partial Geometry

An *incidence system* is a triple (X, B, I) where X, B are sets and $I \subseteq X \times B$ is a binary relation between X and B. Elements of X are called points and elements of B are called blocks. If $x \in X$ and $\beta \in B$ are such that $x \; I \; \beta$ (i.e., $(x, \beta) \in I$), then we say that x is incident with β and the pair (x, β) is called a flag. If $(x, \beta) \notin I$ then (x, β) is called an *anti-flag.*

The *dual* of an incidence system (X, B, I) is the incidence system (B, X, I^{-1}) where $I^{-1} \subseteq B \times X$, is given by $I^{-1} = \{(\beta, x) \in B \times X : (x, \beta) \in I\}$.

An incidence system in which any two distinct points are together in a unique (respectively at most one) block is called a *linear* (respectively *partial linear*) incidence system. In a linear or partial linear incidence system the blocks are usually called *lines* . Two points are called collinear if there is a line containing both.

A *partial geometry* with parameters s, t, α (in short a $pg(s, t, \alpha)$) is a partial linear incidence system with $s + 1$ points on each line and $t + 1$ lines through each point such that, given any anti-flag (x, l), exactly α points on l are collinear with x. The parameter α of a partial geometry is called the *nexus* of the partial geometry.

The *point graph* (or collinearity graph) of a partial geometry is the graph with the points of the partial geometry as vertices, where two vertices are adjacent iff the corresponding points lie on a line of the partial geometry. Dually, the *line graph* of a partial geometry has the lines of the partial geometry as its vertices, two vertices are adjacent in this graph iff the corresponding lines intersect. Clearly the incidence system dual to a $pg(s,t,\alpha)$ is a $pg(t,s,\alpha)$, so the line graph of a partial geometry is just the point graph of its dual.

The notion of partial geometry was introduced by Bose [1]. It was shown by Bose in [1] that the point graph of a $pg(s,t,\alpha)$ is a strongly regular graph with parameters:

$$\left.\begin{array}{l} v=(s+1)(\frac{st}{\alpha}+1), \quad k=s(t+1), \\ \lambda=s-1+t(\alpha-1), \quad \mu=(t+1)\alpha. \end{array}\right\} \qquad (*)$$

1.4 Geometrisability

Any strongly regular graph with parameters given by the formulae (*) for some positive integers s, t, α is said to be a *pseudo geometric graph* with parameters s, t and α. We say that it is *geometrisable* if it is actually the point graph of some partial geometry.

Because of equality in Hoffman bound, any clique of size $s+1$ in a pseudo geometric graph with parameters s, t, α is automatically regular of nexus α (see, e.g., [2], p.10). Note that each line in a partial geometry is a regular clique in its point graph. It follows that a pseudo geometric graph is geometrisable iff there is a family L of regular cliques in the graph such that any two adjacent vertices lie in a unique member of L; in this case L serves as the set of lines of a partial geometry with the given graph as its point graph.

Example 1: The complement of the $n \times n$ grid is pseudo-geometric with parameters

$s = n-1,\ t = n-2,\ \alpha = n-3$ for every $n \geq 4$. It is geometrisable iff there is an affine plane of order n.

Example 2: The complement T_n^* of the traingular graph T_n is pseudo geometric with parameters $s = \frac{n}{2} - 1,\ t = n-4,\ \alpha = \frac{n}{2} - 2$, for every even integer $n \geq 6$. This graph is geometrisable whenever there is a hyperoval in a projective plane of order $n-2$ (in particular, whenever $n = 2^e + 2$ for some $e \geq 2$). In this case the incidence system whose points are the secant lines to the hyperoval and lines are the points of the plane outside the hyperoval (with induced incidence) is a partial geometry with point graph T_n^*. It is known (and easy to verify) that T_8^* is not geometrisable. See, for instance [7], where it is shown that there is no partial geometry with parameters $s = 3,\ t = 4,\ \alpha = 2$.

Example 3: Cohen [5], Haemers and Van Lint [13] and De Clerck, Dye and Thas [8] gave constructions of partial geometries $pg(7, 8, 4)$ which later turned out to be isomorphic [27].

For a survey of the known partial geometries, see Brouwer and Van Lint [3].

1.5 Question of Uniqueness

Of course, even when a pseudo geometric graph is geometrisable, there may be nothing unique about the geometry. In other words, the set L of lines may not be determined upto automorphisms of the graph, so that there could exist non isomorphic partial geometries with the given point graph. For instance, in [19], Mathon showed that T_{10}^* is geometrisable in two non isomorphic ways.

Even when the point graph of a partial geometry is uniquely geometrisable there is no guarantee that the line graph of the geometry is uniquely geometrisable. For instance, if δ is one of the partial geometries $pg(4,2,2)$ (i.e., nets of order five and degree three) then one can show that the point graph of δ is uniquely geometrisable (in fact, the point graph of any $pg(s,t,\alpha)$ with $s > (\alpha-1)(t+1)$ is uniquely geometrisable in the strong sense that the only regular cliques of the point graph are the lines of the partial geometry) but its line graph $K^{*}_{5,5,5}$ is geometrisable in non-isomorphic ways.

In this thesis, we deal with the geometrisabilty of some graphs derived from orthogonal polar spaces.

1.6 Orthogonal Polar Spaces

All vector spaces in this thesis are finite dimensional vector spaces over finite fields. In the following, V is such a vector space over the finite field $F = F_q$ of order q (thus q is a prime power).

A *symmetric bilinear form* is a function $f: V \times V \to F$ such that

(i) $x \mapsto f(x,y)$ is linear for each fixed $y \in V$

and

(ii) $f(x,y) = f(y,x)$ for all $x,\ y \in V$.

(It follows, of course, that $y \mapsto f(x,y)$ is also linear for each fixed $x \in V$). The *kernel* *ker* f of f is the subspace of V consisting of all $x \in V$ for which $f(x,y) = 0$ for all $y \in V$. f is called non-degenerate if its kernel is trivial. Two vectors $x, y \in V$ are called orthogonal (in symbols, $x \perp y$) with respect to the bilinear form f if $f(x,y) = 0$.

Note that $x \perp y$ implies $\lambda x \perp \mu y$ for any two scalars λ, μ. So it makes sense to say that two points in the projective space $P(V)$ are (or are not) orthogonal with respect to f.

A *quadratic form* g on V is a function $g : V \rightarrow F$ for which there is an associated bilinear form $f : V \times V \rightarrow F$ such that:

$$g(\lambda x + \mu y) = \lambda^2 g(x) + \mu^2 g(y) + \lambda\mu f(x, y) \qquad (**)$$

for all x, $y \in V$ and all λ, $\mu \in F$. (Note that (**) implies that f is determined by g.) The *kernel ker g* of g is the subspace of ker f consisting of all $x \in$ ker f for which $g(x) = 0$. The quadratic form g is called non-degenerate if ker g is trivial.

Let V be an n-dimensional vector space over the finite field of order q. Given a non degenerate quadratic form on V, the corresponding *quadric* is the set of totally isotropic one dimensional subspaces (i.e., subspaces on which the quadratic form takes the value zero) of V, viewed as a set of projective points in the associated projective space $P(V) = PG(n-1, q)$. The associated *orthogonal polar space* is the lattice of all projective flats contained in the quadric (i.e., the totally isotropic subspaces of V, viewed projectively) with set inclusion as the partial order. The *rank* of the polar space (also called the Witt index of the quadric) is the maximum of the ranks (i.e., vector space dimensions) of the flats of the polar space. When n is odd, there is, upto isomorphism, a unique non degenerate quadratic form on V, and the rank of the corresponding polar space is $(n-1)/2$; but when n is even, there are two of them, with ranks $n/2-1$ (the elliptic case) and $n/2$ (the hyperbolic case); see [25]. These polar spaces (as well as the simple cores of their automorphism groups) are denoted by $O(n, q)$ for odd n and by $O^{\pm}(n, q)$ for even n, where the plus sign is for the hyperbolic case and minus for the elliptic case.

1.7 Other Polar Spaces

In this thesis we are almost exclusively concerned with orthogonal polar spaces. But, for completeness, let us briefly recall the other kinds of polar spaces. A symmetric bilinear form $f: V \times V \to F$ is called *symplectic* if $f(x, x) = 0$ for all $x \in V$. Given a non-degenerate symplectic form f on V, a subspace M of V is called totally isotropic (with respect to f) if $f(x, y) = 0$ for all $x, y \in M$. The lattice of totally isotropic subspaces (viewed projectively) constitute the associated symplectic polar space. When V admits a non-degenerate symplectic form, dim V is necessarily even. When dim $V = 2r$ and q is the order of the ground field F, the associated polar space (which is uniquely determined upto isomorphism) is denoted by $Sp(2r, q)$. When $q = s^2$ (for a prime power s), $x \mapsto x^s$ is a field automorphism of order 2 on F, which we shall denote by an overline. In this case, a function $f : V \times V \to F$ is said to be a sesquilinear form if f is linear in the first argument and $f(y, x) = \overline{f(x, y)}$ for all $x, y \in V$. One defines kernel, nondegeneracy and totally isotropic subspaces of a sesquilinear form f exactly as before. If n =dim V, the lattice of totally isotropic subspaces of V (viewed projectively) with respect to a non-degenerate sesquilinear form on V is called a unitary polar space (which is again unique upto isomorphism) and is denoted by $U(n, s)$.

Taking $\lambda = \mu = 1$ and $x = y$ in (**), one sees that when q is even, the bilinear form associated with a (non-degenerate) quadratic form is necessarily symplectic. When q is even and dim $V = 2r+1$ is odd, the symplectic form associated with $O(2r+1, q)$ is degenerate with a one-dimensional kernel. Viewed projectively, this kernel is a distinguished point outside the quadric, called the *nucleus* of the polar space. Fix a hyperplane W which does not pass through the nucleus. Then restriction of the symplectic form to W defines an $\mathrm{Sp}(2r,q)$ polar space on W. Further, projection from the nucleus to W defines an isomorphism between $O(2r+1,q)$ and $\mathrm{Sp}(2r,q)$. Thus we have $O(2r+1,q)=\mathrm{Sp}(2r,q)$ for q even. In this thesis, we use the special case $q=2,\ r=3$ of this isomorphism.

1.8 Graphs in this Thesis

The *collinearity graph* of a polar space is the graph whose vertices are the points of the polar space and two vertices are adjacent if the (projective) line joining them is contained in the quadric (equivalently, adjacency is orthogonality with respect to the bilinear form). (In the rank one case, this is the null graph, so we assume $r \geq 2$ in what follows.) We shall use the same symbol as the one for the polar space (viz., $O(2r+1,q)$ or $O^{\pm}(2r,q)$) to denote this graph. This graph is pseudo geometric with parameters

$$s=\frac{q^{r}-1}{q-1}-1,\ t=q^{r-\epsilon},\ \alpha=\frac{q^{r-1}-1}{q-1}$$

.

Here r is the rank, and $\epsilon=0$ in the elliptic case, $\epsilon=1$ for odd n, while $\epsilon=2$ in the hyperbolic case. In all these graphs the only maximal cliques are the regular ones, and these are precisely the maximal flats of the polar space. (To see this, note that given any clique, the flat generated by the points in the clique is totally isotropic.) So we have a plentiful supply of regular cliques, so that the question of geometrisability of these graphs is an interesting one. When $r=2$, these graphs are clearly uniquely geometrisable: all the lines of the polar space must be chosen as lines, leading to classical examples of generalised quadrangles (i.e., partial geometries of nexus one). In the rank three hyperbolic case, the Plücker correspondence implies that the collinearity graph is geometrisable in two distinct but isomorphic ways: leading to the partial geometry whose dual is (the point-line geometry of) $PG(3,q)$. In the hyperbolic cases of higher rank geometrisability is trivially ruled out since $\alpha > t+1$ for these graphs.

Geometrisability for odd n and rank $r \geq 3$ is open except that the cases q even and $r = 3$ or 4 were settled negatively by De Clerck, Gevaert and Thas in [9]. (An independent proof for the non geometrisability of $O(7,2)$ is given in Chapter 3.)

The elliptic cases of rank ≥ 3 were all open. But the smallest of these cases ($r = 3$, $q = 2$) is settled negatively in chapter 5 of this thesis. In [10] De Clerck and Tonchev describe a computer aided project to decide the geometrisability or otherwise of the graph $O^-(8,2)$. They also prove that if the geometry were to exist, the orders of its non-trivial automorphisms could only be 2 or 3. They conjectured that the partial geometry (with $O^-(8,2)$ as point graph) does not exist. The referee of the paper [21] has kindly informed us that this conjecture has been established by L. Soicher (unpublished) using a computer. In chapter 5 we give a computer free proof of this conjecture: $O^-(8,2)$ is not geometrisable.

In the hyperbolic case, the non collinearity graph $O^+(2r,q)^*$ (i.e., the complement of the collinearity graph $O^+(2r,q)$) is also pseudo geometric with parameters $s = q^{r-1}$, $t = q^{r-1} - 1$ and $\alpha = q^{r-2}(q-1)$. When $r = 2$, this is the complement of the $(q+1) \times (q+1)$ grid, so that it is geometrisable iff there is a projective plane of order $q+1$. Even for $r = 3$, its geometrisability is open except in the cases $q = 2$ and $q = 3$. Non geometrisability for $q = 2$, $r = 3$ may readily be established using the combinatorial description of $O^+(6,2)$ given in section 2.2 of this thesis.

Non geometrisability in the case $q = 3,\ r = 3$ may be read off from the computer aided results of M. Hall, jr. and R. Roth in [15]. In this paper, the authors showed that there is no projective plane of order 12 containing the point-line geometry of $PG(3,3)$ as a subgeometry. If such a plane existed then the lines of the subgeometry and the points off the subgeometry would yield a partial geometry with $O^+(6,3)^*$ as its line graph. The arguments in [15] actually show that such a partial geometry does not exist. By Theorem 1 of Dye [12], the clique size of $O^+(2r,2)^*$ is at most $2r+1$ (with equality only when r is a multiple of 4). Hence $O^+(2r,2)^*$ does not have any regular clique (and hence is not geometrisable) for $r > 4$. In view of these remarks, it is a surprise that the graph $O^+(8,q)^*$ is indeed geometrisable for $q = 2$ and 3. Indeed, in [8], the authors construct an infinite sequence of partial geometries whose line graphs have the same parameters as $O^+(2r,2)^*$ for even r. However, as Kantor shows in Proposition 4.2 and Corollary 4.5 of [18], the line graph is actually isomorphic to $O^+(2r,2)^*$ only for $r = 4$. Thus the line graph of the partial geometry $pg(7,8,4)$ in Example 3 (section 1.4) is $O^+(8,2)^*$. This is also more or less apparent from the construction of Haemers and Van Lint [13]. (To keep this discussion simple, we have deliberately blurred the distinction between a partial geometry and its dual.) Later on, in [26], Thas generalised the above mentioned construction to get a partial geometry $pg(26,27,18)$ whose line graph is (again by Kantor's result) the graph $O^+(8,3)^*$. For the remaining parameters, the geometrisability of these non-collinearity graphs is wide open.

Let $\Gamma(r)$ denote the graph whose vertices are the non-isotropic points in the ambient projective space of the polar space $O^+(2r,2)$, adjacency being orthogonality with respect to the associated bilinear form. This graph is pseudo geometric (and, in fact, the action of $O^+(2r,2)$ on its vertices is rank three) with parameters $s = 2^{r-1}-1,\ t = 2^{r-1},\ \alpha = 2^{r-2}$. $\Gamma(3)$ is easily seen to be isomorphic to the co-triangular graph T_8^*, hence it is not geometrisable. (This isomorphism probably explains the appearance of T_8^* in the thesis.) However, $\Gamma(r)$ is actually geometrisable for even values of r since it is the point graph of a partial geometry in the De Clerck-Dye-Thas series, as is apparent from their construction in [8]. In this connection, it may be noted that the action of $O^+(2r,3)$ on each half of the set of non-isotropic points is again rank three. This yields a pseudo geometric graph (with non-orthogonality as adjacency) with parameters $s = 3^{r-1} - 1$, $t = 3^{r-1}$, $\alpha = 2.3^{r-2}$. When $r = 4$, this is the point graph of the partial geometry of Thas mentioned above. More generally, Thas shows in [26] that, for even values of r, this graph is geometrisable whenever there is a spread of $O^+(2r,3)$.

By a *diameter* of a root system we shall mean the line joining an antipodal pair of roots. The 240 roots of the root system E_8 determine 120 diameters. Any two of these diameters make an angle of sixty degree or ninety degree. Define a graph Γ with the diameters of E_8 as vertices and with orthogonality of the diameters as adjacency. We call it the *diameters graph* of E_8. The Weyl group of E_8 acts in a rank three way on the 120 diameters (see, e.g., [17]). So Γ is a strongly regular graph. A computation shows that this graph is pseudo geometric with parameters $s = 7$, $t = 8$ and $\alpha = 4$. Clearly its regular cliques correspond to the orthogonal bases (modulo sign), of the ambient euclidean space, contained in the root system.

It is wellknown that the Weyl group of E_8 is isomorphic to the group $2O^+(8,2).2$, in the notation of the Atlas [6]. The central involution acts trivially on the diameters so that $O^+(8,2).2$ is the full automorphism group of the diameters graph Γ. From the description of the isomorphism between the Weyl group of E_8 with $2O^+(8,2).2$ given in the Atlas, it is immediate that Γ is isomorphic to $\Gamma(4)$ and hence it is geometrisable.

Similar constructions yield pseudo-geometric graphs from symplectic and unitary polar spaces as well. See the survey article of Hubaut [16] on strongly regular graphs for further details.

1.9 Chapterisation

The main results of this thesis are (0) the polar space $O(7,2)$ = Sp(6,2) has a unique spread upto isomorphism [22], (i) the graph $O^-(8,2)$ is not geometrisable [21], (ii) the diameters graph Γ of the root system E_8 is uniquely geometrisable [22] and (iii) the graph $\Lambda = O^+(8,2)^*$ is uniquely geometrisable [23]. We have seen above that Γ and Λ are the point graph and line graph of the partial geometry $pg(7,8,4)$ of Cohen, Haemers-Van Lint and De Clerck-Dye-Thas. Thus the results (ii) and (iii) characterise this partial geometry in terms of its point graph and line graph, respectively. Notice that both graphs have the same automorphism group, viz. $O^+(8,2):2$.

The present chapter, Chapter 1, contains the prerequisites for the thesis and gives an overall idea of the problems we deal with. In Chapter 2, we present combinatorial models of the graphs $O^+(6,2)$, $O(7,2)$, $O^-(8,2)$, $O^+(8,2)^*$, $\Gamma(4)$ and $O^-(10,2)$. These descriptions amount to looking at the vertex - orbits under the action of a suitable subgroup (which is Sym(8), Sym(9) or Sym(10)) of the automorphism group. The description of $O^-(10,2)$ identifies it with a graph attributed by Brouwer and Van Lint [3] to Mathon. Our description of $O^-(8,2)$ is derived from that of $O^-(10,2)$. The description of $O(7,2)$ and $\Gamma(4)$ show that $\Gamma(4)$ is locally $O(7,2)$, i.e., the neighbours of any vertex in $\Gamma(4)$ induce a copy of $O(7,2)$. The description of $O^-(8,2)$ and $\Gamma(4)$ further show that $O^-(8,2)$ may be obtained from

$\Gamma(4)$ by isolating any fixed vertex by switching.

To prove the geometrisability or otherwise of a pseudo geometric graph, the first step is to have a convenient description of the regular cliques of the graph. So in Chapter 3, we find all the regular cliques of the relevant graphs and classify them in terms of the action of a suitable subgroup (Sym(8) or Sym(9)) of the automorphism group.

A *spread* in a pseudo geometric graph is a partition of the vertex set into regular cliques. In Chapter 4, we show that the graph $O(7,2)$ = Sp(6,2) has a unique spread upto isomorphism. For comparison, note that it was only relatively recently that Brouwer and Wilbrink [4] classified the spreads of the (much smaller) graph $O^-(4,2)$ (=Schlafli graph): upto isomorphism there are two non-isomorphic spreads in this case. The unique spread in $O(7,2)$ is an example of the desarguesian spreads discussed in Example 5.1 of [18]. Our uniqueness proof for this spread proceeds by reducing the problem to that of classifying the Kotzig 1-factorisations of the complete graph K_8: we show that K_8 has a unique Kotzig 1-factorisation upto isomorphism, and its automorphism group is the affine group $PGL(1,7)$.

In Chapter 5, we prove that the graph $O^-(8,2)$ is not geometrisable. The assumption of the geometrisability of $O^-(8,2)$ implies that exactly six of the lines of the putative $pg(6,8,3)$ (having $O^-(8,2)$ as its point graph) are contained in the vertex set of each induced subgraph isomorphic to $O^+(6,2)$. We find that, up to isomorphism, there are only seven possible configurations for these six lines and then prove that the occurence of any of these configurations leads to a contradiction.

In Chapter 6, we prove that the diameters graph $\Gamma = \Gamma(4)$ of the root system

E_8 is uniquely geometrisable. The description in Chapter 2 implies that the graph Γ is locally $O(7,2)$, i.e., the neighbours of each vertex in Γ induce a copy of $O(7,2)$. In view of the result in Chapter 4, this implies that given any fixed vertex of Γ, there is an essentially unique way to choose the lines of the partial geometry through this vertex. It will finally turn out that, once this choice is made, there are only two (isomorphic) ways to choose the rest of the lines. This readily implies that the full automorphism group of the partial geometry $pg(7,8,4)$ is Alt(9), acting transitively on the flags (i.e.,incident point-line pairs) of the geometry. The point stabiliser in Alt(9) is $P\Gamma L(2,8)$ (= $L_2(8)$: 3 in Atlas notation) while the line stabiliser is $AGL(3,2)$ (= $2^3 : L_3(2)$ in Atlas notation).

As we have already pointed out (in section 1.5), although the point graph Γ is uniquely geometrisable, this does not by itself imply that the line graph Λ is also uniquely geometrisable. In Chapter 7, we prove that $O^+(8,2)^*$ is indeed uniquely geometrisable. Our uniqueness proof yeilds a new description of the partial geometry in terms of the affine plane of order three. Indeed, it is shown that if N is any regular clique of the line graph which does not correspond to any point of the partial geometry, then N is the point set of a subgeometry isomorphic to $EG(2,3)$ and the structure of the partial geometry is essentially uniquely determined by the structure of this affine plane.

Chapter 2

Combinatorial Models of The Graphs

2.1 Bisections and Trisections

By a *bisection* of an 8-set we mean an unordered partition of the 8-set into two subsets of size 4 each. Thus there are $\frac{1}{2} \times \binom{8}{4} = 35$ bisections.

By a *trisection* of a 9-set, we mean a circularly ordered partition of the set into three 3-subsets. (Thus we do not distinguish between the trisections (N_1, N_2, N_3), (N_2, N_3, N_1) and (N_3, N_1, N_2). However, e.g., (N_1, N_2, N_3) and (N_2, N_1, N_3) are considered distinct.) So we have $\frac{1}{2} \times \binom{9}{3} \times \binom{6}{3} \times \binom{3}{3} = 560$ trisections.

2.2 Combinatorial Model of $O^+(6,2)$

Proposition 2.2.1 *The graph $O^+(6,2)$ is isomorphic to the graph Γ_0 whose vertices are the bisections of an 8-set and adjacency is even intersection.*

Proof: Consider the 7-dimensional vector space W over F_2 whose elements are the even subsets of an 8-set E with symmetric difference as vector addition. Define a quadratic form q^0 and a bilinear form $<. \; , .>$ on W as follows. For $w, w_1, w_2 \in W$, $q^0(w) = \frac{1}{2}|w| \pmod 2$ and $< w_1, w_2 >= |w_1 \cap w_2| \pmod 2$. It is easy to see that $<. \; , .>$ is the bilinear form associated with q^0. But q^0 is degenerate with kernel $\{\phi, E\}$. So q^0 descends to a non-degenerate quadratic form q on $V = W/\{\phi, E\}$, which is a 6-dimensional vector space. The elements of V are complementary pairs of even subsets of E. Let us identify the projective points with the nonzero vectors in V. Under this identification the totally isotropic projective points correspond to the 4-subsets of E and complements. So they may be identified with the bisections of E. Under this identification, the collinearity graph of the quadric of q is identified with the graph Γ_0 defined above. But the maximal cliques of Γ_0 are easily seen to be of size 7. Hence q is of plus (or hyperbolic) type and so $O^+(6,2)$ is isomorphic to the graph Γ_0. □

Remark 2.2.2 From [14], one sees that the graph Γ_0 is isomorphic to the line graph of $PG(3,2)$. So by Proposition 2.2.1, $O^+(6,2)$ is isomorphic to the line graph of PG(3,2). This is also a consequence of the Klein correspondence between the lines of $PG(3,q)$ and the totally isotropic points of $O^+(6,q)$.

2.3 Combinatorial Model of $O(7,2)$

Proposition 2.3.1 $O(7,2)$ *is isomorphic to the graph* Γ_1 *which is the vertex disjoint union of two induced subgraphs* T_8^* *and* Γ_0 *(of Proposition 2.2.1), where adjacency between* T_8^* *and* Γ_0 *is even intersection.*

Proof: Let the notations be as in the proof of Proposition 2.2.1. Then $<.,.>$ is a non-degenerate symplectic form on the 6-dimensional vector space V. The points of the associated symplectic space Sp(6,2) are naturally identified (as before) with bisections and 2-subsets of the 8-set E. This identifies the graph Sp(6,2) = $O(7,2)$ with Γ_1. □

2.4 Combinatorial Model of $O^+(8,2)^*$

Proposition 2.4.1 *The graph $O^+(8,2)^*$ is isomorphic to the graph Λ with vertex set $N \cup \binom{N}{4}$, where N is a set of 9 symbols and $\binom{N}{4}$ denotes the set of all 4-subsets of N; N induces a clique, adjacency in $\binom{N}{4}$ is odd intersection and $x \in N$ is adjacent with $f \in \binom{N}{4}$ if $x \in f$.*

Proof: Let V be the 8-dimensional vector space over F_2 whose vectors are the even subsets of a 9-set N, vector addition is set theoretic symmetric difference. Now, if q is the quadratic form defined on V by $q(x) = \frac{1}{2}|x| \pmod 2$ (where $|x|$ is the Hamming weight of x), then one readily checks that q is a non-degenerate quadratic form of Witt index 4 on V, and the associated bilinear form is the standard inner product. The totally isotropic points are the subsets of N of size 4 and 8. If we identify each subset of size 8 with the unique element of N in its complement, then the above description of Λ results. □

Remark 2.4.2 This is essentially the description of the point graph of $pg(8,7,4)$ given by Haemers and Van Lint in [13] (except that they use the language of binary codes and Hamming weights). They proceed to use the action of $P\Gamma L(2,8)$ on the co-ordinate positions to choose 120 regular cliques of this graph.

2.5 Combinatorial Model of the Graph $\Gamma = \Gamma(4)$

Notation 2.5.1 Let E be a set of 8 symbols. Take $E' = \{a' : a \in E\}$ and $E'' = \{a'' : a \in E\}$ to be two disjoint "copies" of E. Here $a \to a'$, $a \to a''$ are bijections from E onto E' and E'' respectively.

Proposition 2.5.2 *Consider the graph Γ whose vertex set is the disjoint union of the four sets A, B, C, S and an extra vertex ∞, where S is the set of all bisections of E and the sets A, B, and C are the sets of all 2-subsets of E, E' and E'' respectively. The vertex ∞ is adjacent with all the vertices of $S \cup A$. Inside each of the four sets S, A, B, C, adjacency is even intersection. Adjacency is odd intersection between B and C, S and B and between S and C while adjacency between A and $B \cup C \cup S$ is even intersection. Then Γ is isomorphic to the diameters graph $\Gamma(4)$ of the root system E_8.*

Proof : Let e_i, $1 \leq i \leq 8$, be the standard basis of R^8. One of the standard descriptions of the root system E_8 is as follows. It is the (disjoint) union of two sets U and V where $U = \{\pm e_i \pm e_j : 1 \leq i \neq j \leq 8\}$ and $V = \{\frac{1}{2}\sum_1^8 \pm e_i$ with an even number of minus signs $\}$. For any antipodal pair x, $y \in E_8$ $(y = -x)$, we represent the diameter of E_8 joining x and y by the pair $\{x, y\}$. Let

$A = \{\{x, -x\},\ x \in U :$ the two components of x have opposite signs $\}$

$B = \{\{x, -x\},\ x \in U :$ both the components of x have the same sign $\}$

$C = \{\{x, -x\},\ x \in V :$ only two or six components of x have plus signs $\}$

$S = \{\{x, -x\},\ x \in V :$ only four components of x have plus signs $\}$

$\infty = \{x, -x\},\ x \in V$ and all the components of x have the same sign.

So the vertex set of Γ is the disjoint union of the sets A, B, C, S and $\{\infty\}$. The vertex ∞ is adjacent in the diameters graph with all the vertices of $S \cup A$. Let E be the set of the eight co-ordinate positions. Identify each element $\{x, -x\}$ of $A \cup B$ with the set of co-ordinate positions where x is non-zero. Identify each element $\{x, -x\}$ of C with the set of co-ordinate positions where x has the 'minority' sign. Thus each of the sets A, B, C is identified with the set of all 2-subsets of E. To emphasize that these three sets are mutually disjoint, think of them as the set of all 2-subsets of three disjoint copies of E. Finally, identify each element $\{x, -x\}$ of S with the bisection $\{a, b\}$ of E, where a (respectively b) is the set of all co-ordinate positions where x is positive (respectively negative). Thus S is identified with the set of all bisections of E. Then it is easy to check that under these identifications, adjacency in the diameters graph (defined by perpendicularity of diameters) carries over to the description of adjacency for the graph Γ as given in the statement of this theorem. $\square$

2.6 Combinatorial Model of $O^-(10,2)$

Proposition 2.6.1 *The graph $O^-(10,2)$ is isomorphic to the graph Λ_0 whose vertices are the 4-subsets of a 12-set and adjacency is even intersection.*

Proof: Consider the 11-dimensional vector space W over F_2 whose elements are the even subsets of a set T of size 12, with symmetric difference as vector addition. Define a quadratic form q^0 and a bilinear form $<. \ , .>$ on W as in Proposition 2.2.1. Then q^0 is degenerate with kernel $\{\phi, T\}$. So q^0 descends to a non-degenerate quadratic form q on $V = W/\{\phi, T\}$, which is a 10-dimensional vector space. The elements of V are complementary pairs of even subsets of T. Let us identify the projective points with the nonzero vectors in V. Under this identification the totally isotropic projective points correspond to the 4-subsets of T and complements. So they may be identified with the 4-subsets. Under this identification, the collinearity graph of the quadric of q is identified with the graph Λ_0 defined above. But the maximal cliques of Λ_0 are easily seen to be of size 15 (for instance, given any partition of T into 2-subsets, the pairwise unions of the cells of this partition form a maximal clique of Λ_0). Hence q is of minus (or elliptic) type and so $O^-(10,2)$ is isomorphic to the graph Λ_0. □

Remark 2.6.2 In [3] Brouwer and Van Lint attribute the graph Λ_0 of Proposition 2.6.1 to Mathon. But the above theorem shows that this is not really a new strongly regular graph. The construction in Proposition 2.6.1 exhibits the sporadic maximal subgroup $Sym(12)$ of the group $O^-(10,2)$; see the Atlas [6]. The following is an amusing way to see that, despite appearance, $Sym(12)$ is not the full automorphism group of Λ_0, and indeed, the full group is rank three on its vertices. Take D to be the point set of a dodecad in the extended binary Golay code. Let D' be the complementary dodecad. An easy counting shows that, for any 4-subset F of D, there is a unique 4-subset F' of D' such that $F \cup F'$ is an octad. If one identifies D' with D (via any fixed bijection) then $F \mapsto F'$ is a permutation J of the vertex set of Λ_0. Since the Golay code is self orthogonal, J, thus defined, is actually an automorphism of Λ_0. It is easy to see that J mixes up the $Sym(12)$-orbitals.

2.7 Combinatorial Model of $O^-(8,2)$

Proposition 2.7.1 *The graph $O^-(8,2)$ is isomorphic to the graph Ω which is the vertex-disjoint union of four induced subgraphs, three of which are copies* *of T_8^* and the fourth one is the graph Γ_0 of Proposition 2.2.1. Adjacency between each copy of T_8^* and Γ_0 is even intersection, while adjacency between two distinct copies of T_8^* is odd intersection.*

(A more detailed description of Ω is included in the proof of this Proposition.)

Proof: The graph $O^-(8,2)$ is the induced subgraph of $O^-(10,2)$ on the set of common neighbours of any pair of non-adjacent vertices w_1 and w_2 of $O^-(10,2)$. In view of Proposition 2.6.1, take $O^-(10,2) = \Lambda_0$ and without loss of generality let w_1, w_2 be the 4-subsets $\{1,2,3,4\}$ and $\{1,2,3,5\}$ of $T = \{1,2,...,12\}$. Let $T_1 = T \setminus (w_1 \cup w_2)$. So $|T_1| = 7$. Then the set of all common neighbours of w_1 and w_2 in $O^-(10,2)$ is $S \cup A_1 \cup A_2 \cup A_3$, where $S = \binom{T_1}{4}$ and for i = 1, 2, 3, $A_i = \{\{4,5,i,b\} : b \in T_1\} \cup \{\{c,d,e,f\} : c,d \in \{1,2,3\} \setminus \{i\} : e,f \in T_1\}$. Introducing a new symbol ∞, identify an element w of S with the bisection $(w, T_1 \cup \{\infty\} \setminus w)$ of the 8-set $E = T_1 \cup \{\infty\}$ and identify the elements $\{4,5,i,b\}$ and $\{c,d,e,f\}$ of A_i with the 2-subsets $\{\infty^{(i)}, b^{(i)}\}$ and $\{e^{(i)}, f^{(i)}\}$ of the 8-set $E^{(i)} = \{x^{(i)} : \ x \in E\}$. Under this identification, the graph $O^-(8,2)$ is seen to have the following description. Take an 8-set E and three pairwise disjoint 'copies' $E^{(i)}$, $1 \le i \le 3$ of E. For $1 \le i \le 3$, let $x \mapsto x^{(i)}$ be a bijection from E onto $E^{(i)}$. Let S denote the set of all bisections of E and, for $1 \le i \le 3$, let $A_i = \binom{E^{(i)}}{2}$ be the set of all 2-subsets of $E^{(i)}$. The vertex set of the graph is $S \cup A_1 \cup A_2 \cup A_3$. Each A_i induces T_8^* (i.e., two vertices in A_i are adjacent iff they are disjoint). Adjacency within S is even intersection of bisections (i.e., two bisections of E are adjacent iff their common refinement is a partition of E into pairs). For $i \ne j$, adjacency between A_i and A_j is odd intersection (i.e., $\{a^{(i)}, b^{(i)}\} \in A_i$ is adjacent with $\{c^{(j)}, d^{(j)}\} \in A_j$ if $\{a,b\}$ and $\{c,d\}$ meet oddly). Adjacency between the vertices in S and the vertices outside S is even intersection (i.e., the vertex $\{a^{(i)}, b^{(i)}\} \in A_i$ is adjacent with the vertex $x = \{x_1, x_2\} \in S$ iff the 2-subset $\{a, b\}$ of E is contained in one of the two cells x_1, x_2 of the bisection x). This is precisely the graph Ω described above. □

Notation 2.7.2 We shall continue to use the notation S, A_1, A_2, A_3 introduced in the proof of Proposition 2.7.1. Thus the vertex set of $O^-(8,2)$ is the disjoint union of these four sets, where S induces a (fixed but arbitrary) copy of $O^+(6,2)$ and each A_i induces a copy of $T_8{}^*$.

COROLLARY 2.7.3 *The induced subgraph of $O^-(8,2)$ on $S \cup A_i$ is a copy of $O(7,2)$ for each i.*

Proof: Immediate from Propositions 2.3.1 and 2.7.1. □

Remark 2.7.4 By Proposition 2.7.1, Γ_0 is an induced subgraph $O^+(6,2)$ of $O^-(8,2)$, and the significance of Proposition 2.7.1 is that it illuminates the structure of the complementary induced subgraph. Since the automorphism group of $O^-(8,2)$ acts transitively on the induced subgraphs isomorphic to $O^+(6,2)$, this description applies equally well to all these subgraphs.

Remark 2.7.5 Here is yet another description of the graph $\Gamma = \Gamma(4)$. (This description, however, will not be used in this thesis.) Propositions 2.5.2 and 2.7.1 readily imply that Seidel Switching of Γ with respect to the neighbourhood of ∞ yields the graph $O^-(8,2) \cup \{\infty\}$ where ∞ is an isolated vertex. This shows that Γ is a strongly regular graph in the regular two-graph corresponding to the switching class of $O^-(8,2) \cup \{\infty\}$.

Chapter 3

Regular Cliques in The Graphs

3.1 Regular Cliques in $O^+(6,2)$

The regular cliques of $O^+(6,2)$ are of size 7. Since the graph $O^+(6,2)$ is isomorphic to the line graph of PG(3,2) (see Remark 2.2.2) the regular cliques of $O^+(6,2)$ are naturally divided into two equal classes (with 15 cliques in each class) so that the cliques in the same class meet in a single vertex, while cliques from different classes are either disjoint or meet at three vertices. We shall arbitrarily name these classes as class 1 and class 2. For $k = 1,2$, the cliques belonging to class k will be called the class k cliques.

Definition 3.1.1 A clique of size 3 in $O^+(6,2)$ will be called a *claw* if it is a totally isotropic projective line. In other words, a clique of size 3 in $O^+(6,2)$ is a claw iff it is the intersection of two regular cliques.

(Via the isomorphism between $O^+(6,2)$ and the line graph of PG(3,2), any totally isotropic line of $O^+(6,2)$ is identified with a set of three coplanar lines through a point of PG(3,2), which does look like a three-fingered claw!)

Since the claws in S are the totally isotropic (projective) lines of $O^+(6,2)$, the following lemma is immediate.

LEMMA 3.1.2 *(a) Each vertex of $O^+(6,2)$ outside a claw has one or three neighbours in the claw.*

(b) If α is a regular clique of $O^+(6,2)$ and x is a vertex of $O^+(6,2)$ outside α then the three neighbours of x in α form a claw.

(c) Each regular clique of $O^+(6,2)$ contains seven claws and any two of them meet exactly at one vertex.

(d) Each claw of $O^+(6,2)$ is contained in exactly two regular cliques of $O^+(6,2)$, one from each class.

3.2 Regular Cliques in $O^-(8,2)$

By an obvious extension of Definition 3.1.1, we shall refer to the lines of the polar space $O^-(8,2)$ as claws.

LEMMA 3.2.1 *(a) All regular cliques of $O^-(8,2)$ have size 7; there are 765 regular cliques. Each vertex is in 45 of them and each claw is in five of them.*

(b) Given any regular clique in $O^-(8,2)$, there are 28, 224 and 512 regular cliques in $O^-(8,2)$ which meet the given clique in 3, 1 and 0 vertices respectively.

(c) Given an induced copy of $O^+(6,2)$ in $O^-(8,2)$ with vertex set S, any regular clique is either contained in S or meets S in a claw or in a single vertex.

(d) Any two disjoint regular cliques of $O^-(8,2)$ are together contained in a

unique copy of $O^+(6,2)$ *in* $O^-(8,2)$. *Any two regular cliques of* $O^-(8,2)$ *intersecting in a single vertex are together contained in 4 copies of* $O^+(6,2)$.

Proof: (a), (b) and (c) follow from easy counting arguments.

(d): Let W_1 and W_2 be two disjoint maximal cliques of $O^-(8,2)$, V and q be the vector space and the quadratic form associated with $O^-(8,2)$. If one thinks of W_1, W_2 as 3-dimensional subspaces of V, then $W_1 \oplus W_2$ is a 6-dimensional subspace of V. Let q_1 be the restriction of the quadratic form q to $W_1 \oplus W_2$. It is easy to see that q_1 is a non-degenerate quadratic form on $W_1 \oplus W_2$. The quadric on $W_1 \oplus W_2$ corresponding to q_1 contains both W_1 and W_2. But both W_1 and W_2 are totally isotropic subspaces of dimension 3. So the collinearity graph of the above quadric is $O^+(6,2)$ and is uniquely determined by W_1 and W_2. This proves the first part of (d).

If b is the total number of copies of $O^+(6,2)$ in $O^-(8,2)$, then by counting in two ways the ordered triples (Θ, α, β), where Θ is a copy of $O^+(6,2)$, α and β are disjoint maximal cliques of $O^-(8,2)$ and $\alpha, \beta \subset \Theta$, we get $b = 1632$. Then counting in two ways the ordered triples $(\Theta, \alpha_1, \beta_1)$, where Θ is as above, α_1 and β_1 are maximal cliques of $O^-(8,2)$ intersecting in a single vertex and $\alpha_1, \beta_1 \subset \Theta$, we get the second part of (d). □

Definition 3.2.2 For a fixed copy of $O^+(6,2)$ in $O^-(8,2)$, a maximal clique of $O^-(8,2)$ is a type 1, type 2 or type 3 clique if it meets this copy of $O^+(6,2)$ in 7, 3 or 1 vertices respectively.

The type 1 cliques of $O^-(8,2)$ are actually the regular cliques of $O^+(6,2)$ on the vertex set S. So type 1 cliques are divided into two equal classes (see Section 3.1).

In the followong lemma S, A_1, A_2, A_3 are as in Notation 2.7.2.

LEMMA **3.2.3** *Each type 2 clique is contained in* $S \cup A_i$ *for some* i, $1 \leq i \leq 3$. *Also, each type 3 clique meets each* A_i *in two vertices.*

Proof: For adjacent vertices x, y with $x \in A_i$ and $y \in A_j$, $i \neq j$, the set of common neighbours of x and y in S (five in number) induces a co-clique. So any maximal clique of $O^-(8,2)$ meeting both A_i and A_j can contain at the most one vertex in S. So a type 2 clique has to be contained in $A_i \cup S$ for some i. Similarly , for x, y as above, the set of common neighbours of x and y in A_i (or in A_j) (again five in number) also induces a co-clique and so a type 3 clique has to meet each A_i ($i = 1, 2, 3$) in (at most, hence exactly) two vertices. □

3.3 Regular Cliques in $O(7,2)$

By Corollary 2.7.3 $O(7,2)$ is contained as an induced subgraph of the graph $O^-(8,2)$. Both these graphs are pseudo-geometric with the same size (=7) of regular cliques. It follows that the regular cliques of $O(7,2)$ are precisely those of $O^-(8,2)$ contained in the vertex set of $O(7,2)$. Therefore, from the classification of the regular cliques of $O^-(8,2)$, we get, in particular, the following classification of the regular cliques of $O(7,2)$.

Definition 3.3.1 Let us say that a regular clique of $O(7,2)$ is of type 1 or type 2 (with respect to the partition $S \cup A$ of its vertex set given in Proposition 2.3.1) according as it is contained in S (that is, a regular clique of $O^+(6,2)$ induced on S) or meets S and A in 3 and 4 vertices respectively.

Remark 3.3.2 *The non-empty intersections with the vertex set A of T_8^* of the* regular cliques of $O(7,2)$ are precisely the regular cliques of T_8^*. Therefore, if $O(7,2)$ were geometrisable then T_8^* would also be geometrisable. But T_8^* is not geometrisable. Thus we have an easy proof that $O(7,2)$ is not geometrisable. This is a special case of the result in [9].

LEMMA **3.3.3** *Each regular clique of $O(7,2)$ is of type 1 or type 2. There are 30 regular cliques of type 1 and 105 regular cliques of type 2 in $O(7,2)$.*

Remark 3.3.4 The 105 type 2 cliques of $O(7,2)$ have the following description. Note that there are 105 regular cliques in T_8^*, each of size 4. If $\alpha = \{a,b,c,d\}$ is one of these regular cliques, then $\hat{\alpha} = \{\{a \cup b, c \cup d\}, \{a \cup c, b \cup d\}, \{a \cup d, b \cup c\}\}$ is a claw in (the copy of $O^+(6,2)$ on) S. Then $\alpha \cup \hat{\alpha}$ is a regular clique of type 2 in $O(7,2)$, and all the regular cliques of type 2 arise this way. Notice that the number of claws in $O^+(6,2)$ is also 105. Since $\alpha \rightarrow \hat{\alpha}$ is clearly one to one, we get the following lemma.

LEMMA **3.3.5** *The map $\alpha \rightarrow \hat{\alpha}$ defined above provides a natural bijection from the set of regular cliques of T_8^* to the set of claws in $O^+(6,2)$.*

3.4 Regular Cliques in The Diametres Graph of E_8

LEMMA **3.4.1** *The diameters graph Γ is locally $O(7,2)$. All the maximal cliques of Γ are regular cliques (of size 8) and there are 2025 of them. The automorphism group $O^+(8,2):2$ of Γ is transitive on these cliques.*

Proof: In view of Propositions 2.3.1 and 2.5.2 the neighbourhood $S \cup A$ of ∞ induces a copy of $O(7,2)$. Since the automorphism group of Γ acts transitively on the vertices, the neighbourhood of each of the 120 vertices of Γ induces a copy of $O(7,2)$. By Lemma 3.5.1, the graph $O(7,2)$ has 135 regular cliques (which are of size 7). So the total number of regular cliques in Γ is $\frac{(120 \times 135)}{8} = 2025$. Since all the maximal cliques of $O(7,2)$ are regular and the automorphism group of $O(7,2)$ is transitive on them, the analogous statement holds for Γ as well. □

Notation 3.4.2 In view of Proposition 2.5.2 we shall identify the graph Γ with the diameters graph of E_8 and freely use its description (including Notation 2.5.1) in what follows. Thus the vertex set of this graph is the disjoint union of the four sets S, A, B, C and the vertex ∞. By [14], S induces the graph $O^+(6,2)$ while each of the sets A, B, and C induces a copy of $T_8{}'$.
The bijections $'$ and $''$ of Notation 2.5.1 induce isomorphisms from the copy of $T_8{}'$ induced on A onto the copies of $T_8{}'$ induced on B and C respectively. We shall use $'$ and $''$ to denote these induced isomorphisms as well. For any vertex set γ contained in A, $\gamma\,'$ and $\gamma\,''$ will denote the images of γ (contained in B and C, respectively), under these two graph isomorphisms.

In this section we classify the regular cliques of the graph Γ in terms of its description in Proposition 2.5.2. (In view of Lemma 3.4.1 this is an artificial classification.) The regular cliques of the graph Γ are of size 8. All the regular cliques through the vertex ∞ are of the form $\{\infty\} \cup \alpha$, where α is a regular clique of $O(7,2)$ contained in $S \cup A$.

Definition 3.4.3 A regular clique of Γ not passing through ∞ will be called (i) *type 1* if it meets S at 4 vertices and is contained in $S \cup B$ or $S \cup C$, (ii) *type 2* if it meets A in 4 vertices and is contained in $A \cup B$ or $A \cup C$, and (iii) *type 3* if it meets each of the sets S, A, B, C in two vertices. The following lemma shows that these are the only possibilities.

LEMMA **3.4.4** *The type 1, type 2 and type 3 cliques and the regular cliques through ∞ are the only regular cliques of the graph Γ.*

Proof: Let c be a claw in S and $\alpha \subseteq S$ be a regular clique of $O^+(6,2)$ containing the claw c. If $\gamma \subseteq A$ is the regular clique of $T_8{}^*$ corresponding to the claw $c \subseteq S$ of $O^+(6,2)$ (i.e., $c = \hat{\gamma}$, see Lemma 3.3.5) then $(\alpha \setminus c) \cup \gamma\,'$ and $(\alpha \setminus c) \cup \gamma\,''$ (recall Notation 3.4.2) are two regular cliques of type 1 in Γ. There are 105 claws in S. For each claw c in S there are two regular cliques α in S containing c. So we have found $105 \times 2 \times 2 = 420$ type 1 cliques.

If $\gamma \subseteq A$ is a regular clique of $T_8{}^*$ then $\gamma \cup \gamma\,'$ and $\gamma \cup \gamma\,''$ are two regular cliques of type 2 in Γ. Since there are 105 regular cliques of $T_8{}^*$, we have found $105 \times 2 = 210$ type 2 cliques.

For each of the 210 edges $a \subseteq B$ there are exactly two edges $b \subseteq C$ such that $a \cup b$ is a clique. So we have 420 cliques of size 4 meeting each of B, C in two vertices. For each such clique α, there are 3 edges $c \subseteq A$ such that $\alpha \cup c$ is a clique of size 6. So we have $420 \times 3 = 1260$ cliques of size 6 meeting each of A, B, C in two vertices. For each such clique β, there is unique edge $d \subseteq S$ such that $\beta \cup d$ is clique of Γ. So we have 1260 regular cliques of type 3 in Γ.

Finally, by Lemma 3.3.3 and Lemma 3.4.1, there are 135 regular cliques in Γ ugh ∞. So we have found $135 + 420 + 210 + 1260 = 2025$ regular cliques of ce by Lemma 3.4.1, this is the total number of regular cliques of Γ, we have accounted for all of them. □

3.5 Regular Cliques in $O^+(8,2)^*$

We use the notation Λ for the graph $O^+(8,2)^*$. By Proposition 2.4.1, we know that N induces a regular clique of Λ. Let us call this the *special clique.*

LEMMA 3.5.1 *Every regular clique of Λ (other than the special clique) meets the special clique at zero, one or three vertices.*

Proof: Let α be a regular clique in Λ which meets N at two vertices, x and y. Then $\{f \setminus \{x,y\} : f \in \alpha \setminus \{x,y\}\}$ forms a clique of size 7 in T_7. But there is no clique of size 7 in T_7. So no regular clique in Λ meets N at two vertices.

Obviously for any four vertices in N there is only one vertex in $\binom{N}{4}$ adjacent with each of the four vertices. So no regular clique other than the special clique contains more than three vertices of N. □

Definition 3.5.2 *A regular clique of Λ which meets N at 3, 1 or 0 vertices will be called a clique of type 1, type 2 or type 3 respectively.*

We now proceed to determine all the regular cliques of each type in Λ. In the following, v, b and k will be the generic notation for the number of points, blocks and block size (respectively) of any incidence system with constant block size.

LEMMA 3.5.3 *Any regular clique of type 1 in Λ is of the form*

$$\delta \cup \{f \in \binom{N}{4} : \delta \subset f\} \quad (1)$$

where δ is a 3-subset of N.

Conversely, for each 3-subset δ of N, (1) defines a regular clique of type 1 in Λ. So there are $\binom{9}{3} = 84$ cliques of type 1.

Proof: Trivial. □

Let α be a regular clique of type 2 in Λ. Say $\alpha \cap N = \{x\}$. Then $\{f \setminus \{x\} : f \in \alpha \setminus \{x\}\}$ are the blocks of an incidence system with $v = 8$, $b = 8$ and $k = 3$ in which any two blocks meet evenly.

LEMMA 3.5.4 *There is a unique incidence system with $v = 8$, $b = 8$ and $k = 3$ in which any two blocks meet evenly. It may be described as follows: The point set is partitioned into two 4-subsets F_1, F_2 and the blocks are the 3-subsets contained in either F_1 or F_2.*

Proof : Let I be an incidence system with the given properties. It is easy to see that for any block l of I there are at least two blocks of I disjoint from l. So for any disjoint pair b and c of blocks in I there are blocks $b_1 \neq b$ and $c_1 \neq c$ such that $b_1 \cap c = \phi$ and $c_1 \cap b = \phi$. Since $v = 8$, $|b \cap b_1| = 2$, $|c \cap c_1| = 2$ and $b_1 \cap c_1 = \phi$. Let $F_1 = b \cup b_1$ and $F_2 = c \cup c_1$. Then it is easy to see that no block of I can intersect both F_1 and F_2. □

In view of Lemma 3.5.4 and the preceding comment, we have:

LEMMA 3.5.5 *For any symbol x in N and any bisection $\{F_1, F_2\}$ of $N \setminus \{x\}$,*

$$\{x\} \cup \{\{x\} \cup \delta : \delta \in \binom{N}{3},\ \delta \subseteq F_1 \text{ or } \delta \subseteq F_2\} \tag{2}$$

is a regular clique of type 2 in Λ.

Conversely, every regular clique of type 2 in Λ looks like this for a uniquely determined pair (x, α) where $x \in N$ and $\alpha = \{F_1, F_2\}$ is a bisection of $N \setminus \{x\}$. Thus there are $\frac{1}{2} \times 9 \times \binom{8}{4} = 315$ cliques of type 2 in Λ.

If α is a regular clique of type 3 in Λ then the elements of α are the blocks of an incidence system with $v = 9$, $b = 9$ and $k = 4$ in which any two blocks meet oddly.

LEMMA 3.5.6 *There is a unique incidence system with $v = 9$, $b = 9$ and $k = 4$ in which any two blocks meet oddly. It may be described as follows: The point set is partitioned into three 3-subsets T_1, T_2, T_3 and the blocks are $T_i \cup \{x\}$ with $x \in T_{i+1}$, $1 \le i \le 3$ (addition in suffix is modulo 3).*

Proof: Let I be an incidence system with the given properties. Thus any two blocks of I have one or three points in common. In particular, the blocks are distinct.

Claim: If β_1, β_2, β_3 are distinct blocks of I such that $|\beta_1 \cap \beta_2| = 3 = |\beta_1 \cap \beta_3|$ then $\beta_1 \cap \beta_2 = \beta_2 \cap \beta_3 = \beta_3 \cap \beta_1$.

Suppose, if possible, $\beta_1 \cap \beta_2 \ne \beta_1 \cap \beta_3$. Then $\beta_3 \subseteq \beta_1 \cup \beta_2$. In this case, if α is any 4-set not contained in the 5-set $\beta_1 \cup \beta_2$ such that α meets each of β_1, β_2, β_3 in 1 or 3 points, then α contains only one point in $\beta_1 \cup \beta_2$. If α_1, α_2 are two such 4-sets then they have at least two points outside $\beta_1 \cup \beta_2$ in common. In order to be blocks of I, $\alpha_1 \cap \alpha_2$ contains either no point from $\beta_1 \cup \beta_2$ and three points from outside $\beta_1 \cup \beta_2$ or one point from $\beta_1 \cup \beta_2$ and two points from outside $\beta_1 \cup \beta_2$. In the former case, there are at most two blocks of I not contained in $\beta_1 \cup \beta_2$. In the latter case there are at most $\binom{4}{3} = 4$ blocks of I not contained in $\beta_1 \cup \beta_2$. If all 4-subsets of $\beta_1 \cup \beta_2$ are blocks then no other 4-set can meet all of them in one point. Thus there are at most four blocks contained in $\beta_1 \cup \beta_2$ and at most four blocks not contained in $\beta_1 \cup \beta_2$. This contradicts $b = 9$. So we have proved the claim.

Let us call two blocks of I equivalent if they are equal or have three common points. By the claim, this is an equivalence relation. Clearly there is at least one equivalence class containing two or more blocks (otherwise I would be a projective plane in consequence of the Erdös -de Bruijn theorem (see, e.g., [20], p.11). But the parameters $v = b = 9,\ k = 4$ are impossible for a projective plane.) Take any such class. By the claim, there is a set T_1 of points, $|T_1| = 3$, such that all the blocks in this class contain T_1. Let T_2 be the set of points x such that $T_1 \cup \{x\}$ is a block of I. Let T_3 be the set of points not contained in $T_1 \cup T_2$. Thus $T_1,\ T_2,\ T_3$ is a partition of the point set. $|T_1| = 3,\ |T_2| \geq 2$ (= the number of blocks in the equivalence class under consideration) and so $|T_3| \leq 4$.

By the claim, there are three types of blocks: Type 1 blocks contain T_1 (and are contained in $T_1 \cup T_2$), type 2 blocks are disjoint from T_1 (i.e., are contained in $T_2 \cup T_3$) and type 3 blocks have exactly one point in T_1.

Since trivially $|T_2| \leq 6$, there are at most six blocks of type 1. Since $b = 9$, either there are at least two blocks of type 2 or there are at least two blocks of type 3.

Since a type 2 block must meet all the type 1 blocks oddly (hence in a unique point), it follows that all of them contain T_2 (and are contained in $T_2 \cup T_3$). Therefore, if there are at least two type 2 blocks then (since they have at most three common points) $|T_2| \leq 3$. So there are at most three type 1 blocks in this case.

Since a type 3 block must meet each type 1 block oddly, it follows that each type 3 block contains three points from T_3 (and is contained in $T_3 \cup T_1$). Hence, if there is a type 3 block then $|T_3| \geq 3$ and then $|T_2| \leq 3$. Therefore there are at most three type 1 blocks in this case also. Thus in any case, there are at most three type 1 blocks. Since the type 1 blocks constitute an arbitrary equivalence class with at least two blocks, it follows that each equivalence class contains at most three blocks.

The type 1 blocks are equivalent by choice. Since all the type 2 blocks contain T_2 and $|T_2| \geq 2$, they are all equivalent. Since all type 3 blocks contain three points each from T_3 and $|T_3| \leq 4$, it follows that any two type 3 blocks have (at least two points and hence) three points in common, i.e., they are equivalent as well. So there are at most three equivalence classes and each of them contains at most three blocks. Since $b = 9$, it follows that each equivalence class contains exactly three blocks and there are exactly three classes; namely, the blocks of type i constitute the ith class, $i = 1, 2, 3$. Thus there are three blocks of each type. Hence $|T_1| = |T_2| = |T_3| = 3$, and I has the description given in the statement of this Lemma. □

LEMMA 3.5.7 *For any trisection (of section 2.1) (N_1, N_2, N_3) of N, the set*

$$\{N_i \cup \{x\} : 1 \leq i \leq 3,\ x \in N_{i+1}\} \tag{3}$$

is a type 3 clique of Λ (where addition in the suffix is modulo 3). Conversely, every type 3 clique of Λ looks like this for a uniquely determined trisection of N. Thus there are $\frac{1}{3} \times \binom{9}{3} \times \binom{6}{3} \times \binom{3}{3} = 560$ cliques of type 3 in Λ.

Proof: Immediate from Lemma 3.5.6 and the preceding comment. □

THEOREM 3.5.8 *There are exactly 960 regular cliques in $\Lambda = O^+(8,2)^{\bullet}$. The automorphism group $O^+(8,2):2$ of Λ acts transitively on these cliques. Any two of these cliques have 0, 1 or 3 vertices in common.*

Proof: By Lemma 3.5.1, 3.5.3, 3.5.5 and 3.5.7, the total number of regular cliques of Λ is $1 + 84 + 315 + 560 = 960$. From the description of Λ used above, it is clear that there is a subgroup $Sym(9)$ of $O^+(8,2):2$ which fixes the "special" clique N. Since $Sym(9)$ is a maximal subgroup of $O^+(8,2):2$ (and since the latter group does not have any permutation representation of degree 9), the full stabiliser of N in $O^+(8,2):2$ is $Sym(9)$. Since $Sym(9)$ is of index 960 in $O^+(8,2):2$, the transitivity of $O^+(8,2):2$ on the regular cliques follows. The last statement is now an immediate consequence of Lemma 3.5.1. □

Remark 3.5.9 In Theorem 1 of [12], Dye shows that the automorphism group of $O^+(2r,2)^{\bullet}$ acts transitively on its cliques of largest possible size for any $r \geq 2$.

Chapter 4

The Graph $O(7,2) = \mathrm{Sp}(6,2)$ has a Unique Spread

In this chapter we prove that $O(7,2)$ has a unique spread upto isomorphism. Observe that an 1-factor of K_n is nothing but a regular clique of T_n^*, and an 1-factorisation of K_n is a spread of T_n^*.

LEMMA 4.0.10 *Any spread of $O(7,2)$ contains seven cliques of type 2 and two cliques of type 1 (recall Definition 3.3.1).*

Proof: Let e_7 and e_3 be the number of type 1 and type 2 cliques in a spread of $O(7,2)$. Then $e_7 + e_3 = 9$ and counting pairs (x,l) where x is a vertex in S and l is a clique in the spread through x, we get $7e_7 + 3e_3 = 35$. Solving these equations we get $e_3 = 7$ and $e_7 = 2$. □

LEMMA 4.0.11 *If γ_1 and γ_2 are any two disjoint regular cliques in T_8^* and c_1, c_2 are the corresponding claws in $O^+(6,2)$ (in the sense of Lemma 3.3.5) then $c_1 \cap c_2 = \phi$ iff $\gamma_1 \cup \gamma_2$ is an 8-cycle.*

Proof: It is easy to see that $\gamma_1 \cup \gamma_2$ is either a union of two 4-cycles or is an 8-cycle. Then the proof is immediate from the definition of the correspondence in Remark

3.3.4. □

LEMMA 4.0.12 *For any spread* $\underline{\alpha} = \{\alpha_i : 1 \leq i \leq 7\}$ *of* T_8^*, *the following are equivalent:*

(a) There exist seven mutually disjoint type 2 cliques β_i, $1 \leq i \leq 7$, *of* $O(7,2)$ *with* $\alpha_i \subset \beta_i$, $1 \leq i \leq 7$.

(b) $\underline{\alpha}$ *is a Kotzig 1-factorisation of* K_8.

Further, when (b) holds, the cliques β_i *of (a) are uniquely determind by the spread* $\underline{\alpha}$. *Thus there is a natural bijection between sets of seven pairwise disjoint type 2 cliques in* $O(7,2)$ *and Kotzig 1-factorisations of* K_8.

Proof: (a) $\Rightarrow$ (b): For $1 \leq i \leq 7$, $c_i =: \beta_i \setminus \alpha_i$ are pairwise disjoint claws in $O^+(6,2)$. Since c_i corresponds to α_i for $1 \leq i \leq 7$, Lemma 4.0.11 implies that $\underline{\alpha}$ is a Kotzig 1-factorisation.

(b) $\Rightarrow$ (a): If $\underline{\alpha}$ is a Kotzig 1-factorisation and c_i is the claw in $O^+(6,2)$ corresponding to α_i for $1 \leq i \leq 7$, then by Lemma 4.0.11 $\{\beta_i = \alpha_i \cup c_i : 1 \leq i \leq 7\}$ is a set of seven mutually disjoint type 2 cliques of $O(7,2)$.

The last statement follows since $\alpha_i \cup c_i$ (where c_i is the claw corresponding to α_i) is the only type 2 clique of $O(7,2)$ containing a regular clique α_i of T_8^*. □

The following is a wellknown construction of Kotzig 1-factorisations.

LEMMA 4.0.13 *For any odd prime* p *let* $F_p \cup \{\infty\}$ *be the vertex set of* K_{p+1}, *where* F_p *is the field of order* p. *For* $\alpha \in F_p$, *let* $f_\alpha = \{ \{a,b\} : a,\ b \in F_p : a + b = \alpha, a \neq \frac{\alpha}{2}\} \cup \{\{\frac{\alpha}{2}, \infty\}\}$, *then each* f_α *is an 1-factor and* $\{f_\alpha : \alpha \in F_p\}$ *is a Kotzig 1-factorisation of* K_{p+1}. *The group* $AGL(1,p)$ *of order* $p(p-1)$ *(fixing* ∞*) is an automorphism group of this 1-factorisation.*

(Recall that the affine group $AGL(1,p)$ consists of all the permutations of F_p of the form $x \to ax + b$, $a, b \in F_p$, $a \neq 0$.)

Definition 4.0.14 For a fixed 8-cycle c in K_8, by a *diagonal* of c we mean an edge (of K_8) joining two vertices which are non-adjacent in c. Let us say that a diagonal of c joining the vertices x and y (say) is a *short diagonal, mid diagonal* or *long diagonal* according as the distance in c between x and y is 2,3 or 4. Thus there are 8 short diagonals, 8 mid diagonals and 4 long diagonals for a fixed 8-cycle c. We shall represent the 8-cycle c by a regular octagon in the real euclidean plane. Then the diagonals of c are represented by line segments in the plane, so that the words "parallel" and "perpendicular", when applied to diagonals of an 8-cycle, have their usual school geometry meaning. However, we shall say that two diagonals of c are disjoint if they have no vertex of c in common even though the corresponding line segments may meet elsewhere.

LEMMA 4.0.15 *Let f_1 and f_2 be any two disjoint 1-factors of K_8 such that $f_1 \cup f_2$ is an 8-cycle. Let f be another 1-factor of K_8 such that f is disjoint from both f_1 and f_2 and both $f \cup f_1$ and $f \cup f_2$ are 8-cycles. Then the edges in f are diagonals of c, and in terms of the classification of the diagonals in Definition 4.0.14, there are only three possibilities for the edges in f. Namely, f must consist of*

(i) two perpendicular long diagonals and a pair of parallel short diagonals both of which are parallel to one of the two long diagonals in f (Figure (i)), or

(ii) one long diagonal, one short diagonal perpendicular to the long diagonal and the unique pair of mid diagonals disjoint from these two (Figure (ii)), or

(iii) two perpendicular mid diagonals and the unique pair of short diagonals

disjoint from these two (Figure (iii)).

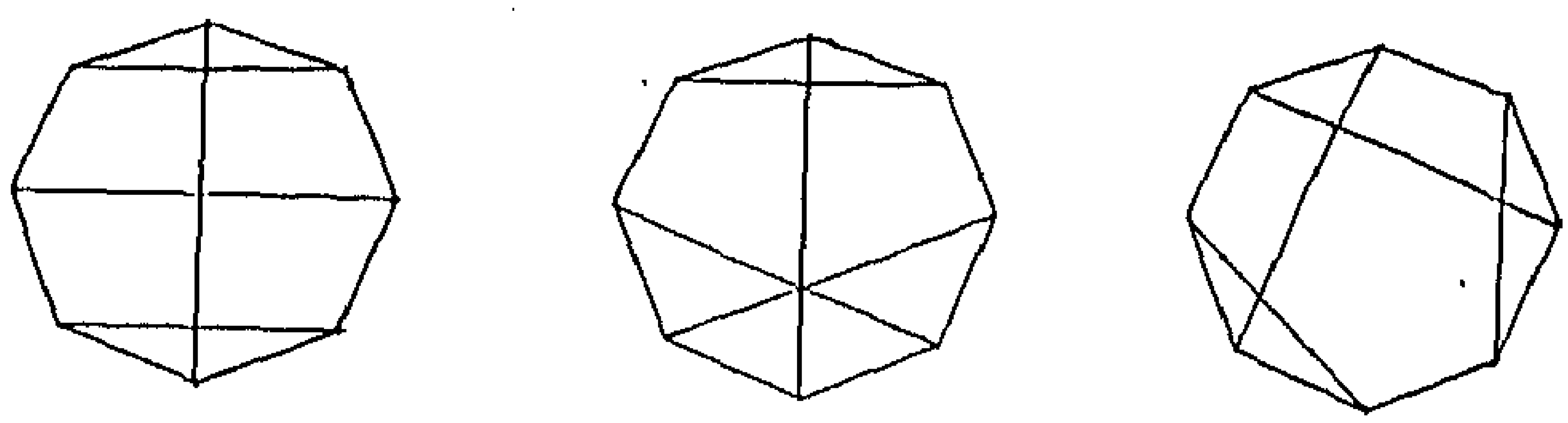

Fig.(I) Fig.(II) Fig.(III)

Proof: Easy. □

LEMMA 4.0.16 *If λ is a Kotzig 1-factorisation of K_8 and f_1, f_2 are any two 1-factors in λ then with respect to the 8-cycle $f_1 \cup f_2$ the number of type (i), type (ii) and type (iii) 1-factors (described in Lemma 4.0.15) in λ is 1, 2 and 2 respectively.*

Proof: Let e_1, e_2, and e_3 be the number of 1-factors in λ (different from f_1 and f_2) of type (i), type (ii) and type (iii) respectively, with respect to $f_1 \cup f_2$. Then $e_1 + e_2 + e_3 = 5$. Counting in two ways the pairs (f, l), (f, s), where f is a 1-factor ($\neq f_1, f_2$) in λ, l is a long diagonal and s is a short diagonal of $f_1 \cup f_2$ such that $l, s \in f$, we get $2e_1 + e_2 = 4$ and $2e_1 + e_2 + 2e_3 = 8$ respectively. Solving these equations we get $e_1 = 1$, $e_2 = 2$ and $e_3 = 2$. □

THEOREM 4.0.17 *Upto isomorphism there is a unique Kotzig 1-factorisation of K_8. Its full automorphism group is the affine group $AGL(1,7)$ of order 42.*

Proof: There are 105 (=index of $2^4 : Sym(4)$ in $Sym(8)$) 1-factors f_1 of K_8 and clearly $Sym(8)$ acts transitively on them. Choose and fix an f_1. Then there are $6 \times 4 \times 2 = 48$ 1-factors f_2 (disjoint from f_1) such that $f_1 \cup f_2$ is an 8-cycle, and the stabiliser $2^4 : Sym(4)$ of f_1 acts transitively on these 48 1-factors. Choose and fix an f_2 and let $c = f_1 \cup f_2$ be the corresponding 8-cycle. Now, as is clear from Figure (iii), there are 8 choices for an 1-factor f_3 (disjoint from f_1 and f_2) such that f_3 is of type (iii) with respect to c. Each such 1-factor f_3 corresponds naturally, to the unique edge of c which meets both the mid diagonals in f_3. Therefore the stabiliser D_{16} of the 8-cycle c acts transitively on these choices. Choose and fix an f_3. Next there are two choices for an 1-factor f_4 (disjoint from f_1, f_2, f_3) which is of type (i) with respect to c (this is because there are two choices for the pair of long diagonals in f_4, and then the requirement that f_4 should be disjoint from f_3 determines the two short diagonals in f_4). Look at the edge of c joining ends of the two mid diagonals in f_3. The reflection across the perpendicular bisector of this edge fixes f_1, f_2, f_3 and interchanges the two choices for f_4. Choose and fix f_4. By Lemma 4.0.16, to complete a Kotzig 1-factorisation, one needs to choose two 1-factors of type (ii) and one of type (iii) with respect to c. But one now sees that there are exactly two 1-factors, say f_5 and f_6, which are disjoint from f_i, $1 \leq i \leq 4$, and are of type (ii) with respect to $f_1 \cup f_2$; again, there is a unique 1-factor, say f_7, which is disjoint from f_i, $1 \leq i \leq 4$, and is of type (i) with respect to $f_1 \cup f_2$. Finally, one verifies that the set $\{f_i : 1 \leq i \leq 7\}$ thus obtained is indeed a Kotzig 1-factorisation. This last verification may be omitted because by the case $p = 7$ of Lemma 4.0.13, a Kotzig 1-factorisation of K_8 does indeed exist.

Thus there are $105 \times 48 \times 8 \times 2$ mutually isomorphic tuples (f_1, f_2, f_3, f_4) of pairwise disjoint 1-factors of K_8 such that f_3 and f_4 are of type (iii) and (i) (respectively) with respect to $f_1 \cup f_2$, and each such tuple extends to a unique Kotzig

1-factorisation. However, Lemma 4.0.16 shows that for each Kotzig 1-factorisation λ, there are $7 \times 6 \times 2 \times 1$ such four-tuples (f_1, f_2, f_3, f_4) with $f_i \in \lambda$ for $1 \leq i \leq 4$. So all the Kotzig 1-factorisations of K_8 are isomorphic, and their number is $\frac{105\times 48\times 8\times 2}{7\times 6\times 2\times 1} = 960$. Since $Sym(8)$ acts transitively on these 960 1-factorisations, the stabiliser of any of them is of order $\frac{8!}{960} = 42$. In other words, the automorphism group of the (essentially) unique Kotzig 1-factorisation of K_8 is of order 42. Since by Lemma 4.0.13, this group contains the affine group of order 42 as a subgroup, it follows that the full automorphism group is $AGL(1,7)$. □

LEMMA 4.0.18 *Any set of seven pairwise disjoint regular cliques of type 2 in $O(7,2)$ can be extended to a unique spread of $O(7,2)$.*

Proof: Since Γ is geometrisable (see the introduction) and locally $O(7,2)$ (see the next section), the lines through any given point of a partial geometry with point graph Γ induce a spread of $O(7,2)$. Therefore $O(7,2)$ does have a spread. (Alternatively, Example 5.1 in [18] gives an explicit example of a spread in $O(7,2)$.) By Lemma 4.0.10, this spread contains a set of seven mutually disjoint type 2 cliques of $O(7,2)$. So there does exist such a set which extends to a spread. But by Lemma 4.0.12 and Theorem 4.0.17, all such sets are isomorphic. So each set of seven pairwise disjoint type 2 cliques extends to at least one spread, and only the uniqueness of this extension remains to be proved.

So fix a set $\underline{\alpha}$ of seven disjoint type 2 cliques, and let F be the set of vertices of $O(7,2)$ not covered by these cliques. Note that $F \subseteq S$ and $|F| = 14$. So we only need to show that the set F of 14 vertices of $O^+(6,2)$ can be written as a (disjoint) union of two regular cliques of $O^+(6,2)$ in at most one way. This is clear since if $\beta_1 \cup \beta_2 = F = \beta_3 \cup \beta_4$ were two such expressions then β_3 would meet one of β_1 and β_2

in at least four vertices. This is impossible since any two regular cliques of $O^+(6,2)$ meet in at most three vertices. □

THEOREM 4.0.19 *Upto isomorphism, $O(7,2)$ has a unique spread. Its full automorphism group is $L_2(8):3 = P\Gamma L(2,8)$ of order 1512.*

Proof: By Lemma 4.0.12 and Lemma 4.0.18, there is a natural bijection between the spreads of $O(7,2)$ and the (unique isomorphism class of) 960 Kotzig 1-factorisation of K_8. Therefore, the spread of $O(7,2)$ is unique upto isomorphism and its automorphism group is of index 960 in the group $O(7,2)$. From the list of maximal subgroups of $O(7,2)$ in the Atlas [6], one can see that the only subgroup of index 960 in $O(7,2)$ is the maximal subgroup $L_2(8):3 = P\Gamma L(2,8)$. □

Chapter 5

The Graph $O^-(8,2)$ is Not Geometrisable

We assume that the graph $O^-(8,2)$ is geometrisable, i.e, it is the point graph of a pg(6,8,3). For brevity, we shall refer to this putative partial geometry simply as "the *pg*". Also, we shall refer to the lines of this *pg* simply as lines.

Definition 5.0.20 *A line of the pg will be called type k if it is a type k clique of $O^-(8,2)$, $k = 1,2,3$. A type 1 line will be called class l if it is a class l clique, $l = 1,2$.*

Notation 5.0.21 For $x \in S$, let $r(x)$ denote the number of type 1 lines through x. For a non-negative integer j, put $S_j = \{x \in S : r(x) = j\}$.

LEMMA 5.0.22 *(a) For $x \in S$, $0 \leq r(x) \leq 3$; also there are $9 - 3r(x)$ type 2 lines and $2r(x)$ type 3 lines through x.*
(b) There are equal number (namely, $3 - r(x)$) of type 2 lines through x contained in $S \cup A_i$ for $i = 1,2,3$.

Proof: (a) For $x \in S$, let $s(x)$ and $t(x)$ be the number of type 2 and type 3 lines through x. Then $r(x) + s(x) + t(x) = 9$ and since x has 18 neighbours in S, $s(x) = 9 - 3r(x)$. This implies that $t(x) = 2r(x)$ and $0 \leq r(x) \leq 3$.
(b) If $s_i(x), i = 1, 2, 3$ is the number of type 2 lines through x contained in $A_i \cup S$, then counting the neighbours of x in A_i in two ways, we get $4s_i(x) + 2t(x) = 12$, so that $s_i(x) = 3 - t(x)/2$ for $1 \leq i \leq 3$. □

LEMMA 5.0.23 *Each copy of $O^+(6,2)$ in $O^-(8,2)$ contains six lines of the pg In other words, there are six type 1 lines.*

Proof : Let e_i, $i = 1, 2, 3$ be the number of type i lines of the *pg* with respect to the given copy of $O^+(6,2)$. Then $e_1 + e_2 + e_3 = 153$. Counting the ordered tuples (l, x) and (l, x, y) in two ways, where l is a line of the *pg* and x, y are distinct vertices of (the fixed copy of) $O^+(6,2)$ contained in the line l, we get the equations $7e_1 + 3e_2 + e_3 = 35 \times 9$ and $\binom{7}{2}e_1 + \binom{3}{2}e_2 = (35 \times 18)/2$ respectively. Solving these equations, we get $e_1 = 6$. □

LEMMA 5.0.24 *If l, l' are type 2 lines intersecting within S such that $l \cup l' \subset S \cup A_i$ for some i, $i = 1, 2, 3$, then $S \cap (l \cup l')$ is a clique in S.*

Proof: Write l, l' as disjoint unions $l = l_1 \cup l_2$, $l' = l'_1 \cup l'_2$ where $l_1, l'_1 \subset S$, l_2, $l'_2 \subset A_i$. Then l_2, l'_2 are disjoint regular cliques (of nexus 2) in A_i. So any vertex $u \in l_2$ is adjacent to two vertices in l'_2 and to the unique vertex z in $l_1 \cap l'_1$, accounting for all the neighbours of u in l'. Since l' is regular of nexus 3, it follows that no $u \in l_2$ is adjacent to any element of $l'_1 \setminus \{z\}$. Therefore, for any $v \in l'_1 \setminus \{z\}$, the three neighbours of v in l belong to l_1. But l_1 has only three vertices on it. So each $v \in l'_1$ is adjacent to all the vertices in l_1. So $(l \cup l') \cap S = l_1 \cup l'_1$ is a clique. □

Notation 5.0.25 For $x \in S_0$ and $1 \leq i \leq 3$, let $l_i(x)$ denote the intersection with S of the three type 2 lines through x contained in $A_i \cup S$. Thus $l_i(x)$ is a set of seven vertices in S. Also Lemma 5.0.24 implies that each $l_i(x)$ is a clique. Thus $l_i(x)$ is a maximal clique in S. Clearly, for each $x \in S_0$, any two of $l_1(x)$, $l_2(x)$, $l_3(x)$ intersect in a singleton. Hence these three are cliques from the same class.

This justifies the following :

Definition 5.0.26 *For $x \in S_0$, we shall say that x is in class j ($j = 1, 2$) if the three cliques $l_i(x)$, $1 \leq i \leq 3$ are from class j. Also for $x \in S \setminus S_0$, we shall say that x is in class j if the type 1 lines through x are in class j ($j = 1,2$).*

LEMMA 5.0.27 *Any two distinct non-adjacent vertices from S_0 belong to different classes. In consequence there are no co-triangles (i.e. co-cliques of size 3) in the induced subgraph on S_0.*

Proof: Suppose $x \neq y$ are non-adjacent vertices in S_0 from the same class. Hence $l_1(x)$ and $l_1(y)$ meet in a unique vertex $z \in S$, $z \neq x, y$. By the definition of $l_1(x)$ (respectively $l_1(y)$) there is a type 2 line l (respectively l') joining x and z (respectively y and z) and lying in $S \cup A_1$. By Lemma 5.0.24 $(l \cup l') \cap S$ is a clique. But x and y belong to this clique and they are non-adjacent. Contradiction. □

LEMMA 5.0.28 *Any two non-adjacent vertices, one of which belongs to S_0 and the other to S_2, belong to the same class.*

Proof : Let $y \in S_0$ and $z \in S_2$ be non-adjacent vertices. Without loss of generality say y is in class 2 and z is in class 1. Since $z \in S_2$, it follows that out of the three type 1 cliques of class 1 through z, exactly two are lines of the pg. Let l be the type 1 clique of class 1 through z which is not a line. Since y is of class 2, $l_i(y)$ $(1 \leq i \leq 3)$ are the three type 1 cliques of class 2 through y. Since the nexus of l is 3 and $y \notin l$ (as y and z are non adjacent and $z \in l$), three vertices on l are collinear with y. As each $l_i(y)$ meets l in 0 or 3 vertices and together they cover all the neighbours of y in S, it follows that exactly one of the cliques $l_i(y)$ meets l (in three vertices). Say $l_1(y)$ meets l. Since $z \in S_2$, there are three type 2 lines of the pg through z, say these are $m_i \subset S \cup A_i$, $i = 1, 2, 3$. For each i, $m_i \cap S$ is a claw containing z, so that $m_i \cap S$ is contained in one of the three cliques of type 1 and class 1 through z. Of these three cliques, the two lines of the pg do not meet m_i except at z (since any two lines of a pg meet in at most one point). So $m_i \cap S$ is contained in l for $i = 1, 2, 3$. Hence $(m_1 \cup m_2 \cup m_3) \cap S = l$. Since y and z are non-adjacent, $z \notin l_1(y)$. But $z \in m_i \cap S$. Hence $m_i \cap S \not\subset l_1(y)$. Since $m_i \cap S$ is a claw in S and $l_1(y)$ is a type 1 clique , it follows that no m_i meets $l_1(y)$ in more than one vertex. Since $(m_1 \cup m_2 \cup m_3) \cap S = l$ and $l_1(y)$ meets l in three vertices, it follows that each of the lines m_i $(i = 1, 2, 3)$ meets $l_1(y)$ in exactly one vertex. Say w is the vertex in common between m_1 and $l_1(y)$. Since $w \in l_1(y)$, there is a type 2 line m joining y and w and lying in $A_1 \cup S$. Since m_1 is a type 2 line lying in $A_1 \cup S$ and meeting m in $w \in S$, so by Lemma 5.0.24 $(m \cup m_1) \cap S$ is a clique. But y and z are non-adjacent vertices belonging to this clique. Contradiction. □

Notation 5.0.29 In order to refer to points, lines and planes of PG(3,2), we shall write "points","lines" and " planes". This is meant to avoid confusion with points and lines of the pg or with points, lines and planes of the projective space PG(7,2) associated with $O^-(8,2)$. Now we shall use the isomorphism between $O^+(6,2)$ and the line graph of PG(3,2). Under this identification, the "points" and "planes" correspond to the two classes of type 1 cliques. By changing the names, if necessary, we may (and do) assume that class 1 cliques correspond to "points" and class 2 cliques correspond to "planes".

Remark 5.0.30 Notice that the claws contained in S correspond to "flags",i.e, incident pairs of "points" and "planes". Namely, if (x, π) is such a pair, then the "lines" through x contained in π is a claw in S, and all the claws contained in S arise in this way (this is why we have chosen the word "claw").

For the case (4) of the next lemma, recall that a hyperoval in a projective plane of even order is a set of "points" which meet every "line" in 0 or 2 "points". The only hyperovals in a "plane" of order 2 are the complements of "lines".

LEMMA 5.0.31 *The following are the only possibilities for the six type 1 lines :*

(1) Six "points" on two disjoint "lines" .

(2) Six "points" on a "plane" .

(3) Five "points" on a "plane" and one "point" outside this "plane".

(4) Six "points" in $\Theta \cup l$ where Θ is a hyperoval (lying in some plane of PG(3,2)) and l is a "line" such that $l \cap \Theta$ is a singleton.

(5) Six "points" in the complement of a "plane".

(6) One "plane" and five "points" outside this plane.

(7) Two "planes" and the four "points" lying outside their union.

Proof: Let A be the set of six type 1 lines of the *pg*. So the elements of A are "points" and "planes" of PG(3,2), with the condition that no "point" in A

belongs to any "plane" in A. Since there is a polarity of PG$(3,2)$ inducing an automorphism of $O^+(6,2)$ and interchanging "points" and "planes", we may assume that the number of "planes" in A is less than or equal to the number of "points" in A. Thus the number of "planes" in A is less than or equal to 3.

First note that the number of "planes" in A can not be equal to 3. This is because there are at most two "points" lying outside the union of any three *distinct* "planes" of PG$(3,2)$. So A contains at most two "planes". If A contains two "planes" π_1, π_2, then there are only four "points" outside $\pi_1 \cup \pi_2$ and A must consist of π_1, π_2 and these four "points". This gives the configuration (7). Next suppose A contains exactly one "plane" π. Then the five "points" of A are from the complement of π, yielding configuration (6). In the remaining cases, A consists of six "points". It is well known that if no three "points" in a set A of six "points" from PG$(3,2)$ are collinear, then A must be contained in the complement of a "plane", yielding the configuration (5). In the remaining cases, there is at least one "line" which is a subset of A. If there are two such "lines" and they are disjoint then A is their union, yielding configuration (1). If there are two such "lines" l_1, l_2 which intersect then A contains the five "points" in $l_1 \cup l_2$ and a sixth "point" outside $l_1 \cup l_2$. This sixth "point" may or may not belong to the "plane" containing $l_1 \cup l_2$, yielding configurations (2) and (3) respectively. In the remaining situation, a unique "line" l is contained as a subset of A. In this case the three "points" in $A \setminus l$ are non collinear and hence they lie in a unique "plane" π. The "plane" π meets l in a unique "point", so that $\pi \cap A$ is a set of four "points" in π, no three collinear. So $\Theta := \pi \cap A$ is a hyperoval in π and $A = \Theta \cup l$, yielding the configuration (4). □

Recall that our classification of the lines of the putative *pg.* into three types refers to a fixed copy of $O^+(6,2)$ in $O^-(8,2)$. We shall say that the configuration (j) occurs in a particular copy of $O^+(6,2)$ if the set of six lines contained in its vertex set is isomorphic to the set (j) in Lemma 5.0.31, $1 \leq j \leq 7$. Clearly exactly one of these configurations must occur. We finish the proof by showing that there is at least one copy of $O^+(6,2)$ in which none of these configurations can occur.

LEMMA 5.0.32 *There exists a copy of $O^+(6,2)$ in $O^-(8,2)$, in which the configurations (6) and (7) do not occur.*

Proof: For any copy Λ of $O^+(6,2)$ in $O^-(8,2)$ and for $i \geq 0$, let $e_i(\Lambda)$ denote the number of vertices of Λ which are in exactly i lines of the pg contained in Λ. By Lemma 5.0.22 , we have $e_i(\Lambda) = 0$ for $i \geq 4$. Also $\sum_{i\geq 0} e_i(\Lambda) = 35$, and, by the proof of Lemma 3.2.1(d), the number of copies of $O^+(6,2)$ in $O^-(8,2)$ is 1632. So we get

$$\sum_{\Lambda}\sum_{i\geq 0} e_i(\Lambda) = 35 \times 1632. \tag{5.0.1}$$

Again $\sum_{i\geq 0} ie_i(\Lambda) = 6 \times 7$ and hence

$$\sum_{\Lambda}\sum_{i\geq 0} ie_i(\Lambda) = 42 \times 1632. \tag{5.0.2}$$

Next let us count in two ways the ordered triples (α, β, Λ), where α and β are two lines of $O^-(8,2)$ meeting in a single vertex and Λ is a copy of $O^+(6,2)$ containing $\alpha \cup \beta$. In view of Lemma 3.2.1(d) and since the total number of lines of the pg is 153, this yields

$$\sum_{\Lambda}\sum_{i\geq 0} i(i-1)e_i(\Lambda) = 153 \times 56 \times 4. \tag{5.0.3}$$

From equations 5.0.1 , 5.0.2 and 5.0.3 we get $\sum_\Lambda \sum_{i\geq 0}(i-1)(i-2)e_i(\Lambda) = 11424$. So $\sum_\Lambda(e_0(\Lambda)+e_3(\Lambda)) = 5712$. Hence $\sum_\Lambda(e_0(\Lambda)+e_3(\Lambda))/\sum_\Lambda 1 > 3$, which implies that there exists at least one Λ such that $e_0(\Lambda)+e_3(\Lambda) > 3$. But in the configuration (6) $e_0(\Lambda)+e_3(\Lambda) = 3$ and in the configuration (7) $e_0(\Lambda)+e_3(\Lambda) = 0$. So, with Λ chosen as above, configurations (6) and (7) do not occur in this Λ. □

THEOREM 5.0.33 *The graph $O^-(8,2)$ is not geometrisable.*

Proof : In view of the Lemma 5.0.32, it remains to prove that the cofigurations (1) to (5) of Lemma 5.0.31 do not occur in any copy of $O^+(6,2)$ in $O^-(8,2)$.

Case 1: *Six " points" in the union of two disjoint "lines".*

In this case S_0 consists of the six "lines" disjoint from both the given "lines". It follows that the induced subgraph on S_0 is the bipartite graph $K_{3,3}$. But this contradicts Lemma 5.0.27.

Case 2: *Six "points" on a "plane" π.*

In this case, let the six type 1 lines be the "points" in $\pi \setminus \{z\}$ where $z \in \pi$ is a fixed "point". Then S_2 consists of the three "lines" through z lying in the "plane" π and S_0 consists of the remaining four "lines" through z. Note that $S_0 \cup S_2$ induces a K_7, but no type 1 line contains more than one vertex in $S_0 \cup S_2$. Therefore the line joining any two vertices in $S_0 \cup S_2$ is a type 2 line.

S_2 consists of the three vertices in the claw corresponding to the "flag" (z, π). Since no line of the *pg* can meet a claw in exactly two vertices, it follows that there is a unique type 2 line l such that $S_2 \subset l$. Let us say that $l \subset S \cup A_3$. Since each vertex in S_2 lies in a unique type 2 line contained in $S \cup A_3$, it follows that any type 2 line which meets S_2 in a singleton must be contained in $S \cup A_1$ or $S \cup A_2$.

The "plane" containing two of the "lines" in S_0 meets π in a "line" through z. That is, the claw containing any two vertices in S_0 meets S_2 in a singleton. Therefore the same is true of the type 2 line joining any two vertices in S_0. For $j = 1, 2$, we define Δ_j to be the graph with S_0 as its set of vertices where two distinct vertices $x, y \in S_0$ are adjacent in Δ_j if the type 2 line joining x and y is contained in $S \cup A_j$. Then, in view of the preceding paragraph, Δ_1 and Δ_2 are complementary graphs. Each claw (and hence each type 2 line) joining a vertex in S_2 to a vertex in S_0, contains two vertices of S_0. Therefore both of the type 2 lines (other than l) through any vertex in S_2 have two vertices both from S_0. One of these two lines is contained in $S \cup A_1$ and the other is in $S \cup A_2$. Therefore both Δ_1 and Δ_2 are non-null graphs. It follows that either Δ_1 or Δ_2 has a vertex of degree two (this is true of any complementary pair of non-null graphs on four vertices). Say, without loss of generality, that $s \in S_0$ has degree two in Δ_1. Then two of the type 2 lines through s contained in $S \cup A_1$ together cover $1 + 2 + 2 = 5$ vertices of $S_0 \cup S_2$ (including s) while the remaining such line meets $S_0 \cup S_2$ in $\{s\}$. Thus the two maximal cliques $l_1(s)$ and $S_0 \cup S_2$ in $O^-(8, 2)$ have exactly five vertices in common. Contradiction.

Case 3: *Five "points" on a "plane" and one "point" outside this plane.*

Let π be the given "plane", $y, z \in \pi$ and w be a "point" outside π so that the type 1 lines consist of the "points" in $(\pi \setminus \{y, z\}) \cup \{w\}$. Then S_0 consists of the "lines" l_i, m_i, $1 \leq i \leq 3$, where l_i (respectively m_i) are the three "lines"not lying in π, passing through y (respectively z) and missing w. It is easy to see that there is an l_i and an m_j which are non-adjacent. Say, without loss of generality, l_1 and m_1 are non-adjacent (i.e, $l_1 \cap m_1 = \phi$) and then by Lemma 5.0.27 l_1 is of class 1 and m_1 is of class 2 (without loss of generality). Take a "line" l through y but not through z such that l lies in π. Then $l \in S_2$ is of class 1, $m_1 \in S_0$ is of class 2 and l, m_1 are non-adjacent. But this contradicts Lemma 5.0.28.

Case 4: *Six "points" in $\Theta \cup l$, where Θ is a hyperoval (in some "plane" π) and l is a "line" meeting Θ in one "point".*

S_0 consists of the "line " $m = \pi \setminus \Theta$ and the six "lines" meeting m but not l and not lying in π. Given any of the six "lines" m' in $S_0 \setminus \{m\}$, one can find a secant $m'' \subset \pi$ to Θ disjoint from m'. Then $m'' \in S_2$ and is of class 1. So by Lemma 5.0.28 all six elements of $S_0 \setminus \{m\}$ are vertices of class 1. Fix $m_1 \in S_0 \setminus \{m\}$. Let x be the "point" in common between m and m_1. Let σ be the "plane" through m which does not contain m_1 and is different from π. Let y be one of the two "points" in $m \setminus m_1$. Let m'' be the "line" through y contained in σ such that $m'' \neq m$ and m'' does not pass through the "point" in $l \cap \sigma$. Then $m_1, m'' \in S_0$ are non-adjacent vertices from the same class, in contradiction to Lemma 5.0.27.

Case 5: *Six "points" in the complement of $\pi \cup \{x, y\}$ where π is a "plane" and x, y are two "points" outside π.*

S_0 consists of all the "lines" from the "plane" π and the "line" l joining x and y. It is easy to see that for every $m \in S_0$, there is a "line" m_1 in S_2 such that m and m_1 are disjoint. Since m_1 is of class 1, by Lemma 5.0.28 m is also of class 1. So all the elements of S_0 are of same class. If $l \cap \pi = \{z\}$ then take a "line" l' contained in π such that $z \notin l'$. Now l and l' are non-adjacent vertices of S_0 and are of same class, in contradiction to Lemma 5.0.27. □

Chapter 6

The Diameters Graph of E_8 is Uniquely Geometrisable

Let us fix an arbitrary partial geometry with point graph Γ. In this chapter, we shall refer to this partial geometry as "the *pg*". Thus the lines of this *pg* are regular cliques of Γ. Eventually we shall show that this geometry is determined uniquely upto isomorphism by the structure of Γ.

Notation 6.0.34 Let S, A, B, C be as in Notation 3.4.2. Let $c \subseteq S$ be a claw of $O^+(6,2)$ and let $\delta \subseteq S$ be a regular clique of $O^+(6,2)$ with $c \subseteq \delta$. Let $\gamma \subseteq A$ be the regular clique of $T_8{}^*$ corresponding to the claw c (i.e., $c = \hat{\gamma}$, see Lemma 3.3.5). Let $\gamma\,' \subseteq B$ and $\gamma\,'' \subseteq C$ be the images of γ under the isomorphisms $'$ and $''$ of Notation 3.4.2. Then we shall use the notation $[c,\delta]_B$ (respectively $[c,\delta]_C$) to denote the type 1 clique $\gamma\,' \cup (\delta \setminus c)$ (respectively $\gamma\,'' \cup (\delta \setminus c)$). We shall also use the notation $[c]_B$ (respectively $[c]_C$) to denote the type 2 clique $\gamma \cup \gamma\,'$ (respectively $\gamma \cup \gamma\,''$). Notice that by Lemma 3.3.5 these cliques depend only on c and δ. Also, the proof of Lemma 3.4.4 shows that these are all the cliques of type 1 and 2 in Γ.

By Lemma 3.4.1, Γ is locally $O(7,2)$. So the nine lines through ∞ of the *pg* induce a spread of the copy of $O(7,2)$ on the neighbourhood $S \cup A$ of ∞. In view of Lemma 4.0.10, it follows that the structure of the lines through ∞ are as in Notation 6.0.35(a).

Notation 6.0.35 (a) Let $\alpha_i \subseteq S \cup A$, $1 \leq i \leq 7$ denote the (pairwise disjoint) type 2 cliques (see Definition 3.3.1) of $O(7,2)$ and let $\alpha \subseteq S$ and $\beta \subseteq S$ be the (mutually disjoint and disjoint from all α_i) regular cliques of $O^+(6,2)$ such that the nine lines of the *pg* through ∞ are

$$\alpha_i \cup \{\infty\},\ 1 \leq i \leq 7,\quad \alpha \cup \{\infty\} \text{ and } \beta \cup \{\infty\}. \tag{4}$$

(b) Let c_i, $1 \leq i \leq 7$ be the claws of $O^+(6,2)$ defined by $c_i = \alpha_i \cap S$, $1 \leq i \leq 7$. Let c_i^* (respectively c_i^{**}), $1 \leq i \leq 7$, be the seven claws of $O^+(6,2)$ contained in α (respectively in β).

(c) Notice that the regular cliques α and β are disjoint and hence they belong to different classes. For any claw $c \subseteq S$, let $\delta(c) \subseteq S$ (respectively $\bar{\delta}(c) \subseteq S$) denote the unique regular clique of $O^+(6,2)$ containing the claw c and in the same class as α (respectively β).

LEMMA 6.0.36 *For $x \in \alpha \cup \beta$, x is adjacent to all three vertices of only one claw c_i, and has only one neighbour in each of the remaining claws c_j, $1 \leq j \leq 7$, $j \neq i$. Further, if $x, y \in \alpha$ (or β) and $x \neq y$ then both x and y can not be adjacent to all three vertices of any claw c_i, $1 \leq i \leq 7$.*

Proof: Without loss of generality let $x, y \in \alpha$. x has $6+3=9$ neighbours in $\alpha \cup \beta$. Since the degree of $O^+(6,2)$ is 18, x has $18-9=9$ neighbours in $\cup_1^7 c_i$. Since c_i's are seven pairwise disjoint claws and x is outside all of them, now the first part follows from Lemma 3.1.2(a).

For the second part, if possible, let both x and y be adjacent to all three vertices in c_i for some i. Since all the the maximal cliques of $O^+(6,2)$ are regular, there is a regular clique $\delta \subseteq S$ containing $\{x,y\} \cup c_i$. δ contains two disjoint claws (namely c_i and $\alpha \cap \delta$), contradicting Lemma 3.1.2(c). □

Definition 6.0.37 A line of the pg (not passing through ∞) will be called *type k* if it is a type k clique of the graph Γ, $k = 1, 2, 3$, (see Definition 3.4.3).

LEMMA 6.0.38 *The number of type 1, type 2 and type 3 lines in the pg is 28, 14 and 84 respectively.*

Proof: Let e_i, $i = 1, 2, 3$, be the number of type i lines of the pg. Then $e_1 + e_2 + e_2 = 135 - 9$ (because there are 9 lines through ∞). Count the ordered tuples (ℓ, x) and (ℓ, x, y) in two ways, where ℓ is a line of the pg not passing through ∞ and x and y are distinct points of S contained in ℓ. Each of the 35 points of S lie on nine lines, out of which exactly one line passes through the point ∞. Also, as the degree of the subgraph induced on S is 18, there are $35 \times 9 = 315$ unordered pairs of distinct collinear points in S. Since by Lemma 4.0.10 seven of the lines through ∞ meet S in three points each while the remaining two lines through ∞ meet S in seven points each, it follows that out of these 315 pairs, exactly $7 \times \binom{3}{2} + 2 \times \binom{7}{2} = 63$ pairs are collinear with ∞. Also, each type 1 line meets S in four points, each type three line meets S in two points, while type 2 lines are disjoint from S. Therefore, the suggested counting yields $4e_1 + 2e_3 = 35 \times 8 = 280$ and $6e_1 + e_3 = 315 - 63 = 252$. Solving these equations we get $e_1 = 28$, $e_2 = 14$ and $e_3 = 84$. □

LEMMA 6.0.39 *For any point $x \in \alpha \cup \beta$, there are exactly two type 1 lines through x, one of these is contained in $S \cup B$ and the other one is in $S \cup C$.*

Proof: Without loss of generality let $x \in \alpha$. Let r and s be the number of type 1 and type 3 lines through x, and let M be the set of neighbours of x in $S \setminus \alpha$. So, $|M| = 18 - 6 = 12$. Let r_1 be the number of type 1 lines through x contained in $S \cup B$, and let M_1 be the set of neighbours of x in B. So $|M_1| = 16$. All the lines through x, excepting the one joining x and ∞, are of type 1 or 3. Each type 1 line through x meets M at 3 vertices and if it is contained in $S \cup B$ then it meets M_1 at 4 vertices. Each type 3 line through x meets M in a single vertex and M_1 at two vertices. So we get the equations, $r + s = 8$, $3r + s = 12$ and $4r_1 + 2s = 16$. Solving these equations we get $r = 2$, $s = 6$, $r_1 = 1$. □

The following lemma is immediate from the definition of the correspondence in Remark 3.3.4.

LEMMA 6.0.40 *Let c_1 and c_2 be two distinct claws of $O^+(6,2)$ and let γ_1 and γ_2 be the corresponding regular cliques of $T_8^{\bullet}$. Then*
(a) if $|c_1 \cap c_2| = 1$ and $c_1 \cup c_2$ is not a clique then $|\gamma_1 \cap \gamma_2| = 2$,
(b) if $c_1 \cap c_2 = \phi$ and each vertex of c_1 has only one neighbour in c_2 then $|\gamma_1 \cap \gamma_2| = 1$.
(Note that by Lemma 3.1.2(a), if $c_1 \cap c_2 = \phi$ then either some vertex of c_1 is adjacent to all vertices of c_2 or each vertex of c_1 is adjacent to only one vertex in c_2.)

LEMMA 6.0.41 *Given that the lines through ∞ have been already chosen and fixed, there are exactly two possibilities for the set of type 1 lines of the* *pg. There is an involutional automorphism of* Γ *which fixes the lines through* ∞ *and interchanges these two possibilities.*

Proof: Let $x \in \alpha$ and let ℓ be a type 1 line through x. Let $\delta \subseteq S$ be the unique regular clique of $O^+(6,2)$ containing $\ell \cap S$. Suppose, if possible that the regular cliques δ and α of $O^+(6,2)$ are from different classes. Since δ and α have the point x in common, it follows that $\delta \cap \alpha$ is a claw. Since $\delta \cap \alpha$ and $\delta \setminus \ell$ are two distinct claws contained in δ (the first contains the point x while the second does not), they have a unique point in common. Therfore $\ell \cap \alpha \cap \delta = (\delta \cap \alpha) \setminus (\delta \setminus \ell)$ contains two points. It follows that the lines ℓ and $\alpha \cup \{\infty\}$ of the pg have at least two points in common and hence $\ell = \alpha \cup \{\infty\}$. So ∞ lies on the type 1 line ℓ, a contradiction. So δ and α are in the same class.

Let i be the unique index, $1 \leq i \leq 7$, such that x is adjacent with all the vertices in the claw c_i (see Lemma 6.0.36). Let j be the unique index, $1 \leq j \leq 7$, such that x is adjacent to all the vertices in c_j^{**} (namely, by Lemma 3.1.2(b), c_j^{**} is the set of all neighbours of x in β). Since $\{x\} \cup c_i$ and $\{x\} \cup c_j^{**}$ are cliques of $O^+(6,2)$, they are contained in two regular cliques of $O^+(6,2)$. Let δ_1, δ_2 be the regular cliques of $O^+(6,2)$ such that $\{x\} \cup c_i \subseteq \delta_1$ and $\{x\} \cup c_j^{**} \subseteq \delta_2$. Fix k, $k = 1$ or 2. Since the regular cliques α and δ_k meet in x, if they were from different classes then $\alpha \cap \delta_k$ would be a claw. Then there would be two disjoint claws (namely $\alpha \cap \delta_1$ and c_i if $k = 1$, $\alpha \cap \delta_2$ and c_j^{**} if $k = 2$) contained in the regular clique δ_k of $O^+(6,2)$, which is a contradiction to Lemma 3.1.2(c). So α, δ_1, δ_2 are the only three regular cliques of $O^+(6,2)$, from a given class, passing through x. Since δ is one of these three cliques and $\delta \neq \alpha$ (since δ meets ℓ in three points while α meets ℓ in a unique point), we get $\delta = \delta_1$ or $\delta = \delta_2$. Then either $c_i \subseteq \delta$ or $c_j^{**} \subseteq \delta$. First suppose $c_i \subseteq \delta$. Then $\delta \setminus \ell$ and c_i are two claws contained in the regular clique δ of $O^+(6,2)$. If $\delta \setminus \ell \neq c_i$ then these two claws would have a unique vertex in common and it would follow that c_i and $\ell \cap \delta$ have two vertices in common. Since $c_i \subseteq \alpha_i$ and $\ell \cap \delta \subseteq \ell$, this would imply that the two distinct lines ℓ and $\alpha_i \cup \{\infty\}$ of the pg have two points in common, which is a contradiction. So $\delta \setminus \ell = c_i$ in this case, and hence ℓ is one of the two type 1 cliques $[c_i, \delta(c_i)]_B$ or $[c_i, \delta(c_i)]_C$. Similarly, if $c_j^{**} \subseteq \delta$ then ℓ is one of the two type 1 cliques $[c_j^{**}, \delta(c_j^{**})]_B$ or $[c_j^{**}, \delta(c_j^{**})]_C$. Both of $[c_i, \delta(c_i)]_B$ and $[c_i, \delta(c_i)]_C$ can not be type 1 lines through x because they meet at four vertices in S. Similarly, both of $[c_j^{**}, \delta(c_j^{**})]_B$ and $[c_j^{**}, \delta(c_j^{**})]_C$ can not be type 1 lines through x. Therefore, by Lemma 6.0.39, either the two type 1 lines through x are $[c_i, \delta(c_i)]_B$ and $[c_j^{**}, \delta(c_j^{**})]_C$ or they are $[c_i, \delta(c_i)]_C$ and $[c_j^{**}, \delta(c_j^{**})]_B$. Since there is an involutional automorphism of Γ which fixes ∞ and all its neighbours and interchanges x' and x'' for all x in A (as is obvious from the description of Γ given in Proposition 2.5.2), we may assume that for a fixed $x \in \alpha$, the two type 1 lines through x are $[c_i, \delta(c_i)]_B$ and $[c_j^{**}, \delta(c_j^{**})]_C$.

Let $y \in \alpha$, $y \neq x$. Let k, l be such that y is adjacent to all the vertices of c_k and c_l^{**}. Then by the above argument (with y replacing x), the type 1 lines through y are either $[c_k, \delta(c_k)]_B$ and $[c_l^{**}, \delta(c_l^{**})]_C$ or they are $[c_k, \delta(c_k)]_C$ and $[c_l^{**}, \delta(c_l^{**})]_B$. Next we show that the second possibility can not occur. This is because, if this occured, the lines $[c_i, \delta(c_i)]_B$ and $[c_l^{**}, \delta(c_l^{**})]_B$ would have more than one point in common, forcing $c_i = c_l^{**}$ which is a contradiction since c_i is disjoint from β and c_l^{**} is contained in β. To see that the cliques $[c_i, \delta(c_i)]_B$ and $[c_l^{**}, \delta(c_l^{**})]_B$ have more than one common vertex, note that the claws c_i and c_l^{**} are disjoint, while the cliques $\delta(c_i)$ and $\delta(c_l^{**})$ have a unique vertex in common (since by definition they are in the same class), so the type 1 cliques $[c_i, \delta(c_i)]_B$ and $[c_l^{**}, \delta(c_l^{**})]_B$ have a common vertex in S. Also, since each vertex of c_l^{**} has a neighbour in $\delta(c_i) \setminus c_i$ (namely the vertex common to $\delta(c_i)$ and $\delta(c_l^{**})$) and the nexus of the clique $\delta(c_i)$ is 3, no vertex in c_l^{**} can be adjacent to all three vertices in c_i. Then by Lemma 3.1.2(a) and

Lemma 6.0.40(b), the regular cliques of T_8^+ on B corresponding to these two claws have a common vertex. Thus the two type 1 cliques have a common vertex in B also.

Notice that for each index i, $1 \leq i \leq 7$, there is a vertex $x \in \alpha$ (respectively a vertex $x^* \in \alpha$) such that x is adjacent to all the vertices in c_i (respectively in c_i^{+*}). Therefore, $[c_i, \delta(c_i)]_B$, $[c_i^{+*}, \delta(c_i^{+*})]_C$, $1 \leq i \leq 7$ are the 14 type 1 lines of the *pg* which meet α.
Similarly (interchanging α and β in the above argument) we see that there are just two possibilities for the 14 type 1 lines meeting β. Namely,

(i) $[c_i, \overline{\delta}(c_i)]_C$, $[c_i^+, \overline{\delta}(c_i^+)]_B$, $1 \leq i \leq 7$

or

(ii) $[c_i, \overline{\delta}(c_i)]_B$, $[c_i^+, \overline{\delta}(c_i^+)]_C$, $1 \leq i \leq 7$.

We now show that the second possibility can not occur. Indeed, if it did, then for any fixed i, the "lines" $[c_i, \delta(c_i)]_B$ and $[c_i, \overline{\delta}(c_i)]_B$ would have four common points in B (namely, the four vertices in the regular clique in B corresponding to c_i), forcing $\delta(c_i) = \overline{\delta}(c_i)$, which is a contradiction since these two regular cliques of $O^+(6,2)$ belong to different classes by definition.

By Lemma 6.0.38 there are 28 type 1 lines. So we find that there are just two possibilities for them (given that the lines through ∞ are chosen and fixed). Namely, either they are

$$[c_i, \delta(c_i)]_B,\ [c_i^+, \overline{\delta}(c_i^+)]_B,\ [c_i, \overline{\delta}(c_i)]_C,\ [c_i^{+*}, \delta(c_i^{+*})]_C,\ 1 \leq i \leq 7 \qquad (5)$$

or they are obtained from these by interchanging the suffixes B and C. These two choices are isomorphic since the involutional automorphism of Γ mentioned above fixes the lines in (4) through ∞ and interchanges these two possibilities for the type 1 lines. $\square$

LEMMA 6.0.42 *Once the lines through ∞ and the type 1 lines are chosen and fixed, the type 2 lines of the pg are uniquely determined.*

Proof: Suppose, without loss of generality, that the chosen lines of the pg are as in (4) and (5). Let $c \subseteq S$ be a claw of $O^+(6,2)$ such that $\ell = [c]_B$ is a (type 2) line. Let $\gamma \subseteq A$ be the regular clique of T_8^* corresponding to c in the sense of Lemma 3.3.5. Thus, $\ell = \gamma \cup \gamma\,'$.

If $c \subseteq \alpha$ then the lines ℓ and $[c, \overline{\delta}(c)]_B$ meet in (the four vertices of) $\gamma\,'$, which is a contradiction. If c meets α in a (necessarily unique) vertex x, then there is a claw c^0 through x and contained in α such that $c \cup c^0$ is not a clique. Then by Lemma 6.0.40(a) the lines ℓ and $[c^0, \overline{\delta}(c^0)]_B$ meet in at least two points (in B), which is a contradiction. Thus c is disjoint from α.

If $c = c_i$ for some i $(1 \leq i \leq 7)$ then the lines ℓ and $\alpha_i \cup \{\infty\}$ meet in the four points of γ. Contradiction. Suppose, if possible, $c \subseteq \cup_{i=1}^{7} c_i$. Then (as $c \neq c_i$ for all i) there are three indices i such that c meets c_i (necessarily in a unique point). On the other hand, neither of the two regular cliques of $O^+(6,2)$ through c can contain more than one of the claws c_i (since a regular clique does not contain two disjoint claws). So there are at most two indices i for which $c \cup c_i$ is a clique. Therefore there is an index i, $1 \leq i \leq 7$, such that c and c_i meet in a single point and $c \cup c_i$ is not a clique. For this i, Lemma 6.0.40(a) says that the regular cliques of T_8^* on A corresponding to c and c_i have two points in common. Hence the lines ℓ and $\alpha_i \cup \{\infty\}$ of the pg have two common points. Contradiction.

If c meets $\cup_{i=1}^{7} c_i$ in two points then there is a point $z \in (c \setminus \cup c_i)$. $c_i \cup \{z\}$ is a clique for at most one i. Therefore $c \cup c_i$ is a clique for at most one index i. On the other hand, in this case c meets c_i for two indices i. So there is an index i such that c meets c_i in a singleton and $c \cup c_i$ is not a clique. So we have a contradiction as before.

Thus c is disjoint from α and c meet $\cup c_i$ in at most one point. Hence c has at least two points in common with the regular clique β. Therefore $c \subseteq \beta$. That is, $c = c_i^{**}$ for some i. So the only possible type 2 lines contained in $A \cup B$ are the type 2 cliques $[c_i^{**}]_B$, $1 \leq i \leq 7$. Similarly, the only possible type 2 lines contained in $A \cup C$ are the type 2 cliques $[c_i^{*}]_C$, $1 \leq i \leq 7$. Since by Lemma 6.0.38 there are 14 type 2 lines, the only possibility for the type 2 lines is that they must be

$$[c_i^{*}]_C,\ [c_i^{**}]_B,\ 1 \leq i \leq 7 \tag{6}$$

□

LEMMA 6.0.43 *Once the lines through ∞ and the type 1 lines are chosen and fixed, the type 3 lines are uniquely determined.*

Proof: Assume, without loss of generality, that the cliques listed in (4), (5) and (6) are lines of the *pg*. Let us say that an edge of Γ is *closed* if it is contained in one of these 51 lines. Otherwise let us say that the edge is *open*. More generally, let us say that a clique of Γ is open if all the edges contained in the clique are open edges. Clearly the type 3 lines of the *pg* must be open cliques.
From the lists (4),(5) and (6), one can see that
(i) an edge $e \subseteq A$ is closed iff e is contained in a regular clique of T_8^* corresponding to one of the 21 "special claws" c_i, c_i^{*}, c_i^{**}, $1 \leq i \leq 7$, and

(ii) for any edge $e \subseteq A$, e is closed iff e' is closed iff e'' is closed.

Any $x \in S$ is contained in one or three of the special claws according as $x \in \cup c_i$ or $x \in \alpha \cup \beta$. In view of the characterisation (i) of the closed edges in A, this fact translates as follows. For any 4-subset F of E, either all three edges $\{e, f\} \subset A$ with $e \cup f = F$ are closed or else exactly one such edge is closed (according as the bisection $\{F, E \setminus F\}$ is a vertex in $\alpha \cup \beta$ or not). The number of closed edges $\{x, y\}$ with $x \cup y = F$ equals the number of such edges with $x \cup y = E \setminus F$.

Now take any open edge $\{x', y'\} \subseteq B$ and let l be the type 3 line containing $\{x', y'\}$. We show that l is uniquely determined. By (ii), $\{x', y'\}$ is the image (under $'$) of an open edge $\{x, y\} \subseteq A$ and $\{x'', y''\}$ is also an open edge contained in C. Applying the above observation to the 4-set $F = x \cup y$, one sees that there is a unique open edge $\{u, v\} \neq \{x, y\}$ with $u \cup v = F$. From the description in Proposition 2.5.2 of the adjacencies in Γ, one then sees that $\{u'', v''\}$ is the unique open edge contained in C such that $\theta := \{x', y'\} \cup \{u'', v''\}$ is a clique. θ is an open clique because every edge meeting both B and C at one vertex is open. Since type 3 lines meet each of the sets A, B and C in an edge, l has to contain the open clique θ.

Let m be the type 3 line containing $\{x'', y''\}$. Clearly $m \neq l$. Note that $\{u', v'\}$ is the unique open edge in B such that $\theta_1 := \{x'', y''\} \cup \{u', v'\}$ is a clique. θ_1 is an open clique and $\theta_1 \subset m$.

Since there are two open edges $\{z, w\}$ with $z \cup w = F$, we get that there are also two open edges $\{a, b\}$ such that $a \cup b = E \setminus F$, say these two open edges are $\{a_1, b_1\}$ and $\{a_2, b_2\}$. Note that $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are the only open edges in A such that $\theta \cup \{a_1, b_1\}$ and $\theta \cup \{a_2, b_2\}$ are cliques. Thus $l \supset \theta \cup \{a_1, b_1\}$ or $l \supset \theta \cup \{a_2, b_2\}$.

Similarly, $m \supset \theta_1 \cup \{a_1, b_1\}$ or $m \supset \theta_1 \cup \{a_2, b_2\}$.

Let $\gamma_i = A \cap [c_i^{**}]_B$, $1 \leq i \leq 7$. Then $\{\gamma_i : 1 \leq i \leq 7\}$ is a spread of T_8^* on A. Since all the edges contained in either of the two sets $\{x, y, u, v\}$ and $\{a_1, b_1, a_2, b_2\}$ are open, no γ_i meets any of these two sets in more than one vertex. Since γ_i's partition A, it follows that there exists an index i such that γ_i contains a vertex from each of these two sets. Then without loss of generality one of the following two cases arises: (a) $x,\ a_1 \in \gamma_i$ or (b) $u,\ a_1 \in \gamma_i$.

In case (a), $x' \in \gamma_i'$. Then the edge $\{x', a_1\} \subset \gamma_i \cup \gamma_i' = [c_i^{**}]_B$. So $\{x', a_1\}$ is not an open edge. So $\theta \cup \{a_1, b_1\}$ is not an open clique and l has to contain $\theta \cup \{a_2, b_2\}$. Then l is determined uniquely since any clique of size six is contained in at most one regular clique.

In case (b), $u' \in \gamma_i'$. Then $\{u', a_1\} \subset \gamma_i \cup \gamma_i'$. So $\theta_1 \cup \{a_1, b_1\}$ is not an open clique and m has to contain $\theta_1 \cup \{a_2, b_2\}$. Then l has to contain $\theta \cup \{a_1, b_1\}$ and is determined uniquely.

Since every type 3 line meets B in an open edge, hence all the type 3 lines are uniquely determined (one can count that there are exactly 84 open edges in B). □

THEOREM 6.0.44 *The diameters graph of the root system E_8 is uniquely geometrisable. The full automorphism group of this partial geometry is the alternating group* Alt(9). *This group acts transitively on the flags of the partial geometry; stabiliser in* Alt(9) *of any point is* PΓL(2,8) *and the stabiliser in* Alt(9) *of any line is* AGL(3,2).

Proof: Since Γ is locally $O(7,2)$, uniqueness of the spread in $O(7,2)$ (Proposition 2.5.2) implies that upto isomorphism there is a unique choice for the lines of the *pg* through any given point. Then Lemma 6.0.41, 6.0.42 and 6.0.43 imply that upto isomorphism there is at most one pg with point graph Γ. In fact, following the above arguments one can show that such a pg must be isomorphic to the one whose lines are the 51 cliques in (4), (5) and (6) together with all the type 3 cliques which meet each of these 51 cliques in at most one vertex. One may prove the geometrisability of Γ by verifying that this actually describes a partial geometry. Alternatively, as we pointed out in the introduction, existing results already imply that Γ is geometrisable, so the uniqueness proof is complete.

Since by Theorem 4.0.17 there are 960 choices for the lines through ∞ and for each of these choices we have seen that there are two choices for the pg, it follows that there are exactly $960 \times 2 = 1920$ partial geometries with point graph Γ and the automorphism group $O^+(8,2) : 2$ of Γ acts transitively on these partial geometries. Thus the automorphism group of the partial geometry is a subgroup of index 1920 in $O^+(8,2) : 2$. From the list of the maximal subgroups of this group given in Atlas [6], one sees that the automorphism group must be Alt(9). Clearly the stabiliser of the point ∞ is contained in the automorphism group $P\Gamma L(2,8)$ of the spread of the graph $O(7,2)$ induced on the neighbours of ∞. On the other hand, $P\Gamma L(2,8)$ acts on the two choices of the partial geometry given the lines through infinity. Since this group has no subgroup of index two, it follows that it acts as an automorphism group of both the geometries. Thus the stabiliser of ∞ in Alt(9) is $P\Gamma L(2,8)$. Since the index of $P\Gamma L(2,8)$ in Alt(9) is 120, which is also the total number of points of the pg, it follows that Alt(9) is transitive on points. Point transitivity, together with the fact that the point stabiliser acts transitively (in fact 3-transitively) on the nine lines through the fixed point, implies transitivity on flags. In particular, this means that Alt(9) is transitive on the 135 lines of the pg. So the stabiliser of any line is a subgroup of index 135 in Alt(9). From the list of the maximal subgroups of Alt(9) in [6], one sees that upto conjugacy the only such subgroup of Alt(9) is

AGL(3, 2). □

Remark 6.0.45 From the Atlas, one sees that $O^+(8,2)$ has three conjugacy classes of subgroups (each maximal) isomorphic to Alt(9). The subgroups in one class are contained in copies of Sym(9) in $O^+(8,2).2$, while the other two classes are merged into a single conjugacy class in $O^+(8,2).2$. Clearly an Alt(9) in the first class can not be the automorphism group of the $pg(7,8,4)$ (or else the full automorphism group of the partial geometry would be Sym(9), contrary to what we saw).

Chapter 7

The Graph $O^+(8,2)^*$ is Uniquely Geometrisable

Throughout this chapter, we fix a partial geometry $pg(8,7,4)$ whose point graph is $\Lambda = O^+(8,2)^*$. We shall refer to it as "the *pg*".

While there are 960 regular cliques of Λ (by Theorem 3.5.8), there are only 120 lines of the *pg*. So, of course, there is a regular clique which is not a line of the *pg*. Since the automorphism group of Λ acts transitively on the regular cliques, we may (and do) assume (without loss of generality) in the following that the **special clique N is not a line of the *pg*.**

Definition 7.0.46 *A line l of the pg will be called type k if it is a type k clique of $O^+(8,2)^*$, k=1,2,3.*

LEMMA 7.0.47 *The type 1 lines of the pg are uniquely determined upto isomorphism.*

Proof: For every pair of vertices in N there is a unique line of the *pg* containing them, and this line must be of type 1. So the elements of the set $\{l \cap N :$

l is a type 1 line} are the blocks a 2-(9,3,1) design, that is, an affine plane of order 3. But the affine plane of order 3 is unique upto isomorphism. By Lemma 3.5.3, a type 1 line l is determined by the triple $l \cap N$. So type 1 lines are uniquely determined upto isomorphism (by the blocks of the affine plane with N as its point set). □

Notation 7.0.48 We fix an affine plane of order 3 with N as its point set and take the type 1 lines of the *pg* to be the type 1 cliques corresponding (via Lemma 3.5.3) to the blocks δ of this affine plane. In the following we refer to this fixed affine plane of order three as "the affine plane".

Remark 7.0.49 Since there are 12 blocks in the affine plane and each point is contained in 4 blocks, the total number of type 1 lines is 12 and each vertex in N is contained in 4 type 1 lines. So through each vertex in N there are 4 type 2 lines also and the total number of type 2 lines is $9 \times 4 = 36$. Hence the total no of type 3 lines is 120 - (12 + 36) = 72.

LEMMA 7.0.50 *(a) A pair of distinct type 2 cliques through a vertex $x \in N$ meet at more than one vertex iff the corresponding bisections (via Lemma 3.5.5) of $N \setminus \{x\}$ meet oddly.*

(b) Let l_1 be the type 1 clique corresponding to the 3-subset $\{x, y, z\}$ (via Lemma 3.5.3) of N. Then a type 2 clique l_2 through x meets l_1 at more than one vertex iff $\{y, z\}$ is contained in some cell of the bisection of $N \setminus \{x\}$ corresponding to l_2.

Proof: Easy. □

Notation 7.0.51 Fix a symbol 1 in N. Put $E = N \setminus \{1\}$. In the following, $\binom{E}{2}$ is taken as the vertex set of T_8^* (with disjointness as adjacency).

(a) For $i \in N$ let $\tilde{\gamma}_i = \{\{x, y\} : \{x, y, i\}$ is a block of the affine plane$\}$. Let $\gamma_i = \tilde{\gamma}_i$ if $i = 1$, and define γ_i to be the image of $\tilde{\gamma}_i$ under the transposition $(1, i)$ of N which interchanges 1 and i (and fixes the remaining symbols in N) in case $i \in E$. So γ_i, $i \in N$, are regular cliques of T_8^*.

(b) Let S denote the set of all the bisections of E. Take any type 2 clique l. If l passes through 1, then let $< l >$ be the bisection of E corresponding to l via Lemma 3.5.5. If l passes through $i \in E$, then let $< l >$ be the image under the transposition $(1, i)$ of the bisection of $N \setminus \{i\}$ corresponding to l. Thus, for each type 2 clique l, $< l > \in S$. Notice that any type 2 clique l is determined by the pair $(x, < l >)$ where $\{x\} = N \cap l$.

LEMMA 7.0.52 *Let l_1 and l_i be type 2 cliques through the vertices 1 and $i \in E$ respectively. Then l_1 and l_i meet at more than one vertex iff $< l_i > = < l_1 >$.*

Proof: Easy. □

Notation 7.0.53 For $i \in N$, let S_i denote the subset of S consisting of the bisections of E which meet each element of γ_i oddly.

LEMMA 7.0.54 *For each $i \in N$, there is a canonical partition of S_i into two 4-subsets α_i and β_i such that any two bisections contained in either α_i or β_i meet evenly while each bisection in α_i meets each bisection in β_i oddly.*

Proof: By Section 2.5, taking even intersection as adjacency defines a copy of $O(7,2)$ on the vertex set $S \cup \binom{E}{2}$. Since γ_i is a regular clique of T_8^*, there exists a claw c (i.e., a totally isotropic line) contained in S such that $\gamma_i \cup c$ is a regular clique of $O(7,2)$ (see Lemma 3.3.5). There are two regular cliques δ_1 and δ_2 contained in S and containing the claw c. Since the nexus of $O(7,2)$ is 3, all the elements in $(\delta_1 \cup \delta_2) \setminus c$ meet every element of γ_i oddly. Every element of $S \setminus (\delta_1 \cup \delta_2)$ has only one neighbour in c and two neighbours in γ_i. So $S_i = (\delta_1 \cup \delta_2) \setminus c$. Now $\alpha_i = \delta_1 \setminus c$ and $\beta_i = \delta_2 \setminus c$ satisfy the requirement of this lemma. □

LEMMA 7.0.55 *For each $i \in E$, S_1 and S_i have exactly two bisections in common and these two bisections meet oddly.*

Proof: It is easy to see that, $\gamma_1 \cap \gamma_i = \{\{i,j\}\}$, where $\{1,i,j\}$ is the unique block of the affine plane containing 1 and i. $(\gamma_1 \cup \gamma_i) \setminus \{\{i,j\}\}$ is the set of edges of a 6-cycle C. There are two sets of alternate vertices in C, say A and B. Then $\{A \cup \{i\}, B \cup \{j\}\}$ and $\{A \cup \{j\}, B \cup \{i\}\}$ are the only bisections contained in $S_1 \cap S_i$. Obviously, these two bisections meet oddly. □

LEMMA 7.0.56 *Once the type 1 lines are chosen and fixed, the type 2 lines of the pg are uniquely determined upto isomorphism.*

Proof: For $i \in N$, let $A_i \subseteq S$ denote the set $\{< l >: l$ is a type 2 line of the *pg* through $i\}$. Note that the type 2 lines are determined once the sets A_i, $i \in N$, are fixed. By Lemma 7.0.50(b), $A_i \subseteq S_i$, for all i. Also, by Lemma 7.0.50(a), any two elements of A_i meet evenly. Hence $A_i \subseteq \alpha_i$ or $A_i \subseteq \beta_i$ by Lemma 7.0.54. Since $|A_i| = |\alpha_i| = |\beta_i| = 4$, it follows that $A_i = \alpha_i$ or $A_i = \beta_i$ for each $i \in N$. By Lemma 7.0.55, for each $i \in E$, $S_i \cap S_1$ is a doubleton, say $S_i \cap S_1 = \{x_i, y_i\}$.

Since by the same lemma x_i and y_i meet oddly, we may arrange the notation in Lemma 7.0.54 so that $x_i \in \alpha_1 \cap \beta_i$ and $y_i \in \beta_1 \cap \alpha_i$ for each $i \in E$. If, now, $A_1 = \alpha_1$ but $A_i = \beta_i$ for some $i \in E$, then by Lemma 7.0.52 the type 2 lines through 1 and i corresponding to the bisection x_i would have more than one point in common. So $A_1 = \alpha_1$ implies $A_i = \alpha_i$ for all $i \in N$. Similarly, $A_1 = \beta_1$ implies $A_i = \beta_i$ for all $i \in N$. Thus either $A_i = \alpha_i$ for all $i \in N$ or $A_i = \beta_i$ for all $i \in N$. Since there is an automorphism of the affine plane interchanging α_1 and β_1, we may take $A_i = \alpha_i$ for all i. This determines all the type 2 lines: for $i \in N$, the type 2 lines through i are the type 2 cliques l through i with $< l > \in \alpha_i$. □

LEMMA 7.0.57 *If x, y, $z \in \binom{N}{4}$ are distinct vertices of Λ such that $|x \cap y \cap z| = 3$ then there is only one type 3 clique of Λ containing x, y and z.*

Proof: Let the trisection (N_1, N_2, N_3) of N represent (via Lemma 3.5.7) a type 3 clique containing the vertices x, y and z. Then $x \cap y \cap z = N_i$, for some $i = 1, 2$ or 3. So without loss of generality let $x \cap y \cap z = N_1$. Then $N_2 = (x \cup y \cup z) \setminus N_1$ and $N_3 = N \setminus (N_1 \cup N_2)$. □

LEMMA 7.0.58 *Let $x \in \binom{N}{4}$ be such that x does not contain any block of the affine plane. Then the type 3 lines through x are uniquely determined once the lines of type 1 and type 2 are chosen and fixed.*

Proof: Since by assumption x is not in any type 1 line, it follows that x is in exactly four type 2 lines l_i, $i \in x$, where l_i denotes the type 2 line joining x and i. Hence x is in four type 3 lines as well.

Claim: For each 3-subset t of x, there are exactly two vertices y in $\binom{N}{4}$ such that $y \supseteq t$, and y is joined to x by a type 3 line.

Let A denote the set of all vertices $y \neq x$ in $\binom{N}{4}$ such that $y \supseteq t$. The description of l_i in terms of a bisection of $N \setminus \{i\}$ (see Lemma 3.5.5) shows that if $i \in t$ then l_i contains exactly one vertex from A, while if $i \in x \setminus t$ then l_i contains no such vertex. Therefore, together, the four type 2 lines through x cover exactly three vertices of A (which must be distinct since otherwise two of these type 2 lines would have two distinct points in common). It follows that the remaining two vertices of A (which are adjacent to x) are joint to x by type 3 lines. This proves the claim.

Now fix a 3-subset t of x, and let y_1, y_2 be the two vertices guaranted by the claim. Let l be the type 3 line joining x and y_1. Since $x \cap y_1 = t$ is a 3-set, the description of l in terms of a trisection of N (see Lemma 3.5.7) shows that t is one of the parts in this trisection. Also, the same description shows that l contains a vertex z other than x, y, such that $z \supseteq t$. Then by the claim, $z = y_2$. Hence $l \supseteq \{x, y_1, y_2\}$ and therefore l is determined by Lemma 7.0.57. Thus each of the four 3-subsets of x determines a type 3 line through x and hence all the type 3 lines through x are determined uniquely. □

LEMMA 7.0.59 *Let $x \in \binom{N}{4}$ be such that x contains a block of the affine plane. Then type 3 lines through x are uniquely determined once the lines of type 1 and type 2 are chosen and fixed.*

Proof: x contains a block t_0 of the affine plane. Thus x lies on a unique type 1 line (namely, the line corresponding to t_0) and a type 2 line (namely, the line joining x and i where $\{i\} = x \setminus t_0$). It follows that x is on six type 3 lines.

Fix a 3-subset t of x such that t is not a block of the affine plane. Define A as in the proof of Lemma 7.0.58. There are three vertices y in A such that y does not contain any block of the affine plane. (Since t is a triangle in the affine plane and each block of the affine plane contains three points, it follows that there are exactly three 4-sets containing t which contain no block of the affine plane.) Since $t \neq t_0$, the type 1 line through x does not intersect A.

Since x is on a unique type 2 line and this type 2 line contains a unique vertex from A (in consequence of Lemma 3.5.5), it follows that there is at least one vertex $y \in A$ such that (i) y does not contain any block of the affine plane and (ii) x and y are joined by a type 3 line. Fix such an y. By Lemma 7.0.58 the type 3 line joining x and y is uniquely determined, and by Lemma 3.5.7, this line contains a third point z from A. Let $w \in A$ be the unique point joined to x by a type 2 line. Put $\{u, v\} = A \setminus \{y, z, w\}$. Then the type 3 line joining x and u must contain v, and hence it is determined by Lemma 7.0.57 as the type 3 clique containing $\{x, u, v\}$. For each of the three 3-subsets t of x which are not blocks of the affine plane, this argument determines two type 3 lines through x. Hence we have determined all six type 3 lines through x. □

THEOREM 7.0.60 *The graph $O^+(8,2)'$ is uniquely geometrisable upto isomorphism.*

Proof: Immediate from Lemma 7.0.47, 7.0.56, 7.0.58 and 7.0.59. □

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