DISCRETE SINGULARITY METHOD AND ITS APPLICATIONS TO INCOMPRESSIBLE FLOWS

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Somehow I too must find a way of making things;
not plastic, written things, but realities
that arise from the craft itself. Somehow I too must discover
the smallest constituent element, the cell of my art,
the tangible immaterial means of expressing everything ...

– Rainer Maria Rilke

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Chapter 1

INTRODUCTION

The smooth flow of a fluid has sprung many surprises. A flow which at an instant of time is quite regular and orderly could produce on the slightest of disturbance a complex bewildering varieties of flows, broadly termed as *turbulence*. Direct numerical simulation of the Navier-Stokes equations have shown that it is quite possible that these turbulent flows are solutions of the Navier-Stokes equations. In fact it is by now well recognized that many non-linear systems produce chaos quite similar to turbulence. However the large number of scales and their complex interactions involved make turbulence difficult to understand. Direct numerical simulation of the 3D Navier-Stokes equations has been achieved for only low Reynolds number (Kim et. al. (1987)) and the prospects in the near future of a full scale simulation for a flow problem of practical interest seems quite bleak (Reynolds(1989)). Moreover the initial promise that *chaos theory* (Ruelle(1989)) could explain turbulence has faded away. The strong churning action and bursts which are found in turbulence are generally missing in the models of chaos.

Most of the efforts in turbulence have been made towards obtaining better quantitative models. However the purpose of modeling is not only to predict the technologically relevant quantities but also to gain insight into the problem. It is difficult to define the word *insight* here. To quote Jimenez et. al. (1981),

> Apart from a certain amount of esthetic appeal that makes you feel you have *understood* a certain situation in a certain case, insight has something to do with minimum complexity and with the facility of including new features ...

The study of turbulence as a history of flow events and patterns has yielded important insights. The importance of organized structures in the study of turbulence was realized in 1950's and later on, the flow visualization experiments by S.J.Kline and his co-workers (Runstadler et.al.(1963), Kline et.al.(1967), Kim et.al.(1971)) established the importance of these structures for turbulent boundary layers. Brown and Roshko(1974) established the significance of organized structures in mixing layers. Recently Aubry et.al.(1988) have made an attempt to study the evolution of certain spatial modes obtained from Galerkin projection of the Navier-Stokes equation onto certain coherent structure functions obtained by what is known as the method of *Proper Orthogonal Decomposition* (Lumley(1970)). This study has established a bridge between Navier-Stokes equations and the dynamical systems theory. The motivation behind the present thesis work is the same as that of Aubry et.al.(1988), but our approach is quite different. Unlike Aubry et.al.(1988) we do not use the Fourier decomposition, instead we use the decomposition in terms of short waves, i.e., functions with rapidly vanishing tails.

The study of the solutions of the non-linear equations in terms of short waves or wavelets has certain advantages over their study in terms of long waves that are obtained by Fourier decomposition. Some of the coherent phenomena in one dimension like solitons and shock waves form such short waves with rapidly vanishing tails, as we shall point out in Chapter 3 of this thesis. It is known that these coherent structures, once created, get convected in space. If we visualize the coherent structures as composed of many small elements, then we can follow such elements individually and the problem can be thought of as a many-body problem involving the elements. The process will be simplified if these elements are solutions by themselves. Such an element generates a singularity on its reduction to a point, in the limiting process.

Singularities occur in the solutions of the linear and non-linear problems, and they create mathematical inconveniences. In fluid flow problems, singularities are assigned to points inside the solid body present in the flow field. Sometimes singularities occur in the physical domain, e.g., in cosmology the big-bang and black-hole are associated with singularities (Hawking and Penrose (1970), Kriele(1992)). The self-energy problem in particle physics is another example in which the singularity seems to occur in the physical domain and despite many theoretical attempts, the physicists could not get rid of it (Raju(1980)). Singularities arise for some compelling mathematical reasons. In Chapter 2, we shall study irrotational flows and their singularities. Such singularities can be derived from the fundamental solution of the problem, usually termed as the Green's function. Further in this chapter, the relation between these distribution solutions(Green's functions) and certain class of functions having rapidly vanishing tails is discussed.

The singularities of two-dimensional harmonic functions play a major role in the the study of dynamical systems. The well known Painleve analysis and the pole decomposition method, provide a tool for studying the singularities of the ordinary and partial differential equations. An extensive and interesting survey on this topic has been given by Ramani et. al. (1989). We however are interested in considering the singular decomposition procedure for reducing certain partial differential equations to a set of ordinary differential equations in time i.e., a many body problem of the discrete singularities. In Chapter 3 we deal with certain one dimensional partial differential equations, like the Burgers equation, K-dV equation etc. Difficulties arise when solutions of a partial differential equation blow-up, which however can easily be tackled with the discrete singularity method. Besides, this method is of minimal complexity and whenever necessary new components can be incorporated easily. We shall demonstrate these features for a wide variety of one-dimensional evolutionary systems (in Chapter 3).

The connection between singularities and turbulence has been explored in a series of papers by U. Frisch and his co-workers (Frisch and Morf (1981), Frisch(1983,6)). As pointed out by Aref(1983), there is a close similarity between the pole decomposition method and the Discrete Vortex Method(DVM). So as a natural continuation of the ideas discussed in Chapter 3, we consider the discrete vorticity method in Chapter 4 as a model for the solution of the 2D Euler's equations. Hald(1979), Beale and Majda(1982) have studied the convergence of the vortex methods and showed the way of obtaining accurate solutions of the Euler's equations using discrete vorticity method. Here we invoke infinite reflections of the discrete vortices across the channel and arrive at vortex strings in two-dimension, which are used to simulate a 2D channel flow. Such a string configuration has an exponentially vanishing velocity field in the streamwise direction and is helpful in making efficient simulations. These strings have close similarity with the shock wave solutions in one dimensional systems, but their mode of interaction is quite different. The strings can also be thought of as spatial modes from the outer flow as suggested by Narasimha(1989). In contrast, Aubry et.al.(1988) in their work on 3D flows used modes from flow near the wall. Here we obtain results only for the two-dimensional flow in a channel. Similar results have been obtained earlier by Jimenez(1990) for a two-dimensional channel flow using the Fourier decomposition method. However the technique developed here is simple and requires less computational effort. The main reason is that unlike conventional discrete vortices, the velocity field of the vortices (in fact strings) used here vanish exponentially in the stream-wise direction. Using an sixth-order vortex scheme proposed by Beale and Majda(1985), we made simulations of a Poiseulle flow in a flat 2D channel. We observed the formation of inertial waves. These moving waves are quite similar to the Tollmien-Schlichting waves observed in viscous flows, but they are extremely sensitive to the initial conditions. Moreover they breakdown rather quickly. One of the drawbacks of the simulations considered here is that the discrete vorticity method as applied to inviscid flows corresponds to the infinite Reynolds number limit of the solutions of the Navier-Stokes equations. However, it seems that the importance of Reynolds number in fluid flows is generally overemphasized (Frisch(1989)). Except for its role as a fine filter of small scales, the diffusion process may not be considered very important in the dynamics of fluid flows . In fact this role of the diffusion process as a filter of noise makes observation of coherent structures like spots and Λ-vortices (Herbert(1990)) during the transition regime simple. The dynamics of fluid flows at high Reynolds numbers is dominated by inertial processes and in particular, the advection of vorticity by the flow is much stronger than viscous diffusion. Contrary to popular belief, in an inviscid flow it is possible to satisfy the no-slip condition at the boundary, if we consider a thin vortex sheet as being generated at the boundary. It is to be mentioned that in the limiting case of vanishing viscosity, this thin sheet cannot diffuse from the boundary (Morton(1984)) and so the recalculation of the strength of the vortices near the wall may not be required. The crucial point is that in a flow which is more or less steady, if the vorticity field satisfies a no-slip condition initially then it would continue to do the same for some time. The problem is discussed in detail in Chapter 4, where we consider flows over boundaries. Although the 2D flows are useful for predicting the initial phase of transition to turbulence, the break down of the Tollmien-Schlichting waves would lead to a rapid increase in the three-dimensional effects and the formation of spots and Λ-vortices. To understand these features we must consider three-dimensional aspects of the problem. It has been suggested by a few authors that a singularity may appear in three-dimensional flows (Pumir and Siggia(1987), Chefranov(1987)). We have noticed certain inaccuracies in the work of Chefranov(1987). In case the flow field is initially smooth we found that there is no clear evidence of the formation of a singularity. Also it is found that there is no upper bound on the maximum velocity, which is strongly dependent on the initial conditions. In Chapter 5 we derive distribution solutions, termed as *micro-vortons*, of the three-dimensional Euler's equations. Further combining these micro-vortons with the macro-vortical structures, we study various characteristics of 3D flow fields. Saffman(1981) proposed the name *vorton* for a particular three-dimensional vortex singularity and Novikov(1984) studied the dynamics of these vortex singularities. Similar studies were made earlier by Chorin(1981), who considered the vortons as pieces of vortex lines. In the fifth chapter we extend such ideas to higher dimensions. Saffman and Meiron(1986) pointed out that there are some difficulties in using these vortons; unlike the discrete vortex in two-dimension such vortons are not solutions of the Euler's equations. However it is possible (Beale and Majda(1982),Hald(1987)) to regularize this discrete vorton (distribution) by convoluting it with a functions with rapidly decaying tails and by using the smoothed version of these vortons we can devise accurate numerical simulations. An alternative approach may be to consider the stream function formulation for axisymmetric flows, and consider the vortex ring solutions. Oseledets(1988) showed that an Hamiltonian formulation is possible for the Euler's equations, using a limiting case of a vortex ring; a micro-vorton. We show here that the macro-vortex elements like straight filaments or the rings and the micro-vortons can be used to give a dynamical two-scale description of three-dimensional flows.

The materials in this thesis are organized as follows.

In Chapter 2 we consider the irrotational flows. In the first section,the classical harmonic potentials and their properties are studied. An unsolved problem regarding the uniqueness of the potentials is addressed briefly, the details of which can be found in Appendix A. In the second section we consider the singularities of the potential flows and the weak solutions. In the final section we briefly consider the irrotational flows with circulation.

In Chapter 3, we consider the singularities appearing in the solution of the differential equations. The Painleve' technique is discussed in the first section. The pole decomposition for certain partial differential equations are discussed in the next section. Here we study the shocks and solitons occurring in the Burger's and K-dV equations. Finally, we show how the coherent structures like shocks and solitons can be used to develop efficient computational methods.

In Chapter 4, we discuss the Discrete Vortex Methods(DVM) as applied to two-dimensional flows. Here in the first section we consider briefly the mathematical results obtained for incompressible 2D flows, so as to provide an introduction. In the second section we consider the DVM, as applied to flows without boundary. Here the problems of mixing layer and the vortex merger are discussed. In the third section, the flows with simple boundary are considered. We discuss, the vorticity transfer mechanism near the wall and show how in an inviscid flow it is possible to satisfy the no-slip condition. In the final section we consider the inviscid flow in a flat 2D channel. Here starting with the laminar Poiseulle flow, we show how the DVM, can be applied to study the flow in a channel. Using a vortex string obtained by taking infinite mirror reflections at the walls, the simulations are made. We obtain smooth flow fields which satisfies the normal boundary condition at the wall exactly and the no-slip condition at the wall approximately. The velocity field of these vortex strings decays rapidly making the simulations efficient. The formation of inertial waves and their subsequent break down was observed.

In the fifth chapter we consider flows in higher dimensions, mainly with a view to understand the three-dimensional flows. In the first section we consider some of the mathematical results obtained by V.I.Arnold and others. In the second section we consider the the discrete vortex singularities or the *vortons* and the various vortex stretching models. In the third and the last section we show how the vortex line elements and the small vortex rings can be used to obtain a two scale macro-micro description of turbulence.

Finally in the sixth chapter we make some concluding remarks.

Chapter 2

IRROTATIONAL FLOWS

Our interest here is in fluid flows in which the effects due to viscosity and compressibility are negligible. Such flows for mathematical convenience can be assumed to be inviscid and incompressible. It is known that fluid flows in general may have regions where strong rotational components are present. However, in many flows such strong rotational components are confined to a small region (like the *boundary layers*), so to obtain an overall flow pattern as a first approximation we can assume that the flow is irrotational.

2.1 POTENTIAL FLOWS : CLASSICAL SOLUTIONS

Let the fluid region M^n be an open subset of $I\!R^n$, with a smooth boundary ∂M^{n-1}, which is assumed to be compact,unless otherwise stated explicitly. For an irrotational,incompressible flow, the velocity field can be expressed as the gradient of a local potential ϕ, which satisfies,

$$\Delta(\phi) = \sum_k \frac{\partial^2 \phi}{\partial {x_k}^2} = 0 \quad in \ M - \partial M \tag{2.1}$$

It is to be mentioned that from such local potentials, ϕ a global single-valued potential cannot be obtained by analytic continuation, unless the integral of the velocity field around each closed loop in ∂M^{n-1} is zero. In this section, by a solution we shall mean an harmonic potential ϕ in M^n, which is continuous in M^n and has partial derivatives up to order two in the interior of M^n and satisfies equation (2.1). The boundary conditions on ∂M^{n-1} should also be prescribed for a complete description of the problem. Many types of boundary conditions are possible. For the case of free surface flows, it is found that as the fluid region M^n changes with time, even non-linear boundary conditions may be prescribed.

Here we shall confine ourselves to the linear boundary conditions of the form,

$$\alpha\phi + \beta\frac{\partial\phi}{\partial n} = \gamma \quad \text{in} \quad \partial M^{n-1} \tag{2.2}$$

where α, β, γ are continuous functions defined on the boundary and the normal derivative is taken away from the boundary,in the direction of the fluid.

Subtle limit procedure is required for defining the normal derivatives at the non-smooth points of the boundary (in two dimension such a problem can be avoided by considering the stream function instead of the velocity potential: Shinbrot(1972)). For the cases when $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$, the above boundary condition (2.2) reduces respectively to the Dirichlet and Neumann boundary conditions. The existence and uniqueness of the potential ϕ has been established (Kellogg (1929)) for the case when ∂M^{n-1} satisfies certain regularity criteria. General proofs of existence and uniqueness are available for the Robin boundary condition in Kellogg(1929) and for the mixed boundary conditions in Tsuji(1959). The Robin boundary condition corresponds to the case, $\alpha < 0, \beta = 1$ in (2.2). The mixed boundary conditions are obtained when both the Neumann and the Cauchy boundary conditions are applied, but at different parts of the boundary. For the above cases, the uniqueness can be easily proved by invoking the Green's identity,

$$\int_M |\nabla\phi|^2 = -\int_{\partial M} \phi\frac{\partial\phi}{\partial n}\, ds \tag{2.3}$$

However a general proof is lacking for the case corresponding to the linearized free surface boundary condition,i.e., the boundary condition obtained by putting $\alpha > 0, \beta = 1$ in (2.2). This is derived in connection with the small amplitude oscillations of fluids at the free surface. For this case, Kreisel(1949), Ursell(1950), Maz'ja(1977), Simon and Ursell(1984) have proved uniqueness of the solution ϕ, for flows over some simple geometries. In Appendix A of this thesis we provide a proof of the uniqueness of the solution ϕ, corresponding to the linearized boundary condition on the free surface for the two dimensional flows over arbitrary boundaries. The problem however remains open in three-dimension.

The uniqueness of ϕ (upto a constant) indicates that an irrotational incompressible flow can be brought to rest instantaneously by arresting the motion of all the boundaries (including the point at infinity) of the flow. An additional assumption is also required in this case ; the circulation around the rigid boundaries must remain constant (cf.Lighthill(1986)). Thus we see that an incompressible irrotational flow contrary to our intuition is quite like a rigid motion and such a flow is unphysical. However, we will show in section 2.3, that the physical characteristics of the flow can be restored by introducing certain distribution of singularities, known as vortices in the flow. But this does not imply that a simple irrotational model is useless. In fact, for certain problems of unsteady motion of bodies in fluids, like the motion of marine structures in the sea, the forces on these bodies due to the fluid motion is mainly due to the normal stresses, and an irrotational model yields quite good results (Newman(1977,78)).

Now we shall consider certain other properties of the harmonic potentials, the details of which can be found in Stein and Weiss(1971).

Mean Value Property : If u is an harmonic potential in M^n, an open subset of $\mathbb{R}^n$ and if $S^n \subset M^n$ is a sphere contained in it,then the average value of u in S^n is equal to the value at the centre of the sphere S^n.

This is one of the important properties of a harmonic potential. This can be used to prove the following results.

(A) An harmonic potential has all partial derivatives upto any given order.

(B) Liouville's theorem : A function which is bounded and harmonic in $\mathbb{R}^n$ must be a constant.

As a consequence of property (B) we see that any non-trivial harmonic function must have a singularity somewhere in the extended Euclidean plane (infinity being included). Such singularities have been known for a long time and they can be derived from the corresponding Green's function of the problem (i.e., the source potential). However the Green's function is not a solution in the classical sense (it is actually a distribution). We shall consider such solutions when we consider the *weak* solutions in the next section.

Finally we would like to mention that although we have dealt only with the properties of the solutions of Laplace equation, such properties can also be derived for solutions of general elliptic equations using Hopf's maximum principle and the theory of subharmonic functions(Rauch(1991)).

2.2 POTENTIAL FLOWS : WEAK SOLUTIONS

The Liouville's theorem for harmonic potentials shows that any potential flow in the $\mathbb{R}^n$, would contain a singularity in the extended Euclidean plane. In fact such singularities could be considered as the spine of potential flows. They can be derived from the fundamental solution (also known as the Green's function) of the Laplace equation in $\mathbb{R}^n$,

$$\begin{aligned} \phi_0 &= \frac{1}{2\pi} \log|x| & \text{for } n = 2 \\ &= \frac{1}{(n-2)\Pi} |x|^{2-n} & \text{for } n > 2 \end{aligned} \tag{2.4}$$

where Π is the surface area of the unit sphere in $\mathbb{R}^n$.
However the function (2.4) does not satisfy the Laplace equation at $x = 0$, and in fact it satisfies,

$$\Delta(\phi_0) = \delta(x) \tag{2.5}$$

where $\delta(x)$ is the Dirac's delta distribution.
The Dirac delta distribution, used first by the physicist P.A.M.Dirac, was introduced to the applied mathematics community by Lighthill(1958) using the theory of *good functions*,i.e.,the functions with rapidly vanishing tails. The space of such rapidly decreasing functions (or the Schwartz space), S is defined as,

$$S = \{f \in E(R^n) / \lim_{|x|\to\infty} |x^\alpha D^\beta f(x)| = 0 \text{ for all multi} - \text{indices } \alpha, \beta\} \tag{2.6}$$

where $E(\mathbb{R}^n)$ denotes the space of smooth (C^∞) functions and by multi-indices here we mean that there are n-indices $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$, corresponding to the n-independent variables $x_1, x_2, \ldots, x_n$.
Let any function $f \in S$ satisfy,

$$\int_{R^n} f(x)dx = 1 \tag{2.7}$$

then $\delta(x)$ can be considered as the generalized function defined by,

$$\lim_{\epsilon\to 0} \epsilon^{-n} f(\frac{x}{\epsilon}) \overset{def}{=} \delta(x) \tag{2.8}$$

With the discovery of the theory of distributions by L.Schwartz (Schwartz(1966)), proper functional analytical setting was laid, by which the objects such as the delta distributions could be considered. Moreover this also led to the development of powerful analytical tools. We refer to Kesavan(1989), Rauch(1991) for an introduction to theory of distributions and their applications.

In order to frame a problem mentioned above, we begin with the equations,

$$\begin{array}{lcll} \Delta(u) & = & f & in \;\; M \\ u & = & 0 & in \;\; \partial M \end{array} \tag{2.9}$$

Let u be a classical solution of (2.9),i.e., u has spatial derivatives of order up to two in M^n and it satisfies (2.9) point-wise. Let ϕ be a test function, i.e., a function which has derivatives of all orders and has compact support in the interior of M^n (the space of such functions is denoted by $D(M)$), then from (2.9) we get,

$$\int \Delta(u) \cdot \phi = \int f \, \phi \tag{2.10}$$

Applying Green's theorem,we have,

$$\int \nabla u \cdot \nabla\phi = \int f \, \phi \tag{2.11}$$

Equation (2.11) is equivalent to equation (2.9),but however it doesn't involve second derivatives. We shall say that u is a *weak solution* of (2.9), if u satisfies (2.11) for every $\phi \in D(M)$. The appropriate function space for considering the weak solutions is the Sobolev space $H_0^1(M)$. In such a space it is easy to prove uniqueness using the Lax-Milgram theorem and further the solution is characterized by the fact that it minimizes the energy,

$$\mathrm{E}(u) = \frac{1}{2}\int \nabla u \cdot \nabla \phi - \int f \, \phi \tag{2.12}$$

Posing physical problems as a minimization of certain quantities like the the energy is well known to physicists and in fact, the numerical techniques such as the finite element method(Kesavan(1989)) uses this formulation as the corner stone. However, to solve such a problem numerically , another method is available, which seems more efficient. We shall consider it now.

Using the fundamental solution (2.4) it is possible to write the solution of (2.9) as,

$$u(x) = \int_{\partial M} \sigma(y) \; \phi_0(x-y)dy \tag{2.13}$$

Let us show how this can be used to solve the problem of a potential flow around an object such as an aerofoil. If $u(x)$ satisfies the Laplace equation in M^n, the fluid region exterior to the aerofoil ∂M^{n-1}, then using Green's identity we can derive an integral equation,

$$u(x) = -\int_{\partial M} [\phi_0(x-y)\frac{\partial u}{\partial n(y)} - u(y)\frac{\partial \phi_0(x-y)}{\partial n(y)}] \; dy \tag{2.14}$$

The above form was used by Hess and Smith (1964) to obtain numerical solutions of the problem. Since (2.14) is a Fredholm integral equation of the first kind, it need not have a unique solution (Courant and Hilbert(1953)). However it is possible to convert (2.14) into a Fredholm integral equation of the second kind of the form,

$$\sigma(x) - \int_{\partial M} \frac{\partial \phi_0(x-y)}{\partial n}\, \sigma(y)\, dy = -\boldsymbol{n} \cdot V \tag{2.15}$$

where $\boldsymbol{V}$ is the flow velocity at infinity and here $\sigma(x)$ is the source distribution on the boundary, ∂M.

For the Fredholm integral equation of second kind, constructive methods are available which are useful for not only numerical computations but also for proving the existence and uniqueness of the solutions (Courant and Hilbert(1953)). The potential at a point $x \in M$ in the fluid can be evaluated using, the relation (2.13). In this method,we need to perform computations only at the boundary points. So this method not only reduces the dimension of the problem by one, it also reduces the computational domain to a finite region, although the flow region may be infinite. More details about this technique and its applications can be found in Hess and Smith(1966) and Jawson and Symn(1972).

In effect the methods mentioned above consider a discrete distribution of singularities like sources,sinks,dipoles and multipoles,to simulate the flow. It is well known that a distribution whose support is a point must be a linear combination of the delta distribution and its derivatives (Kesavan(1989), Rauch(1991)) . So any discrete singularity which is harmonic,except at a point, can be written as the linear combination of the fundamental singularity (2.4) and its derivatives. Such a linear combination encompasses the multipole expansion or the Laurent expansion in two-dimension. The so called essential singularities, are the ones which cannot be expressed as a finite number of linear combination of the fundamental singularity and its derivatives.

There are also other type of singularities e.g.,the fractional derivatives (Oldham and Spanier(1974)) of the fundamental singularity (2.4). Such singularities can be considered as solutions of algebraic or transcendental equations and they occur in clusters. To show this, we consider the equation,

$$z^c = z_0 \tag{2.16}$$

where $z = x + iy, z_0 = x_0 + iy_0$ are complex variables.

If c is a rational number,then the above equation admits a finite number of solutions, whereas if c is irrational or complex, then the above equation has an infinite number of solutions. In case, c is irrational, the solutions lie on a circle and when c is complex (say $c = c_r + ic_i$), the solutions lies on a logarithmic spiral:

$$\begin{aligned} c_r log\ r - c_i(\theta + 2k\pi) &= x_0 \\ c_i log\ r + c_r(\theta + 2k\pi) &= y_0 \\ & \text{where } k \text{ is any integer} \end{aligned} \tag{2.17}$$

It is possible to construct more complicated clusters than this. To exhibit this , we consider a function of n-complex variables, $f(z_1, z_2, z_3, \ldots, z_n)$, whose singular points are given by,

$$g(z_1, z_2, z_3, \ldots, z_n) = 0 \tag{2.18}$$

This would be the case if,

$$f(z_1, z_2, z_3, \ldots, z_n) = \frac{1}{g(z_1, z_2, z_3, \ldots, z_n)} \tag{2.19}$$

Now if we assume an implicit relation between the n-variables of the form, $z_1 = z^{c_1}, z_2 = z^{c_2}, z_3 = z^{c_3}, ..., z_n = z^{c_n}$, then we can write (2.18),in the form,

$$g(z^{c_1}, z^{c_2}, z^{c_3}, ..., z^{c_n}) = 0 \tag{2.20}$$

If some of the co-efficients, $c_1, c_2, c_3, ..., c_n$ are either irrational or complex, then we see that the solutions of the above equation, would lie in a cluster of spirals forming a fractal domain. A similar situation can possibly occur in higher dimensions. To see that we have to consider the spherical harmonics with fractional or even complex powers of the radius and use Legendre functions with complex index (Morse and Feshbach(1953)).

We would like to note here that,the above classification of the discrete and clustered singularities could be used in general to analyse contours and surfaces. They facilitate the study of both qualitative and quantitative aspects. Frisch(1983) showed that using low-pass filters, it is possible to isolate the high-frequency components from the turbulent time signals. Comparing this high-frequency component with the high frequency contribution obtained from complex singularities, he showed that we can estimate the order of the leading singularities. However this technique does not work well for the case in which the singularities are not isolated but form clusters like the case discussed above. A comprehensive method termed as the multi-resolution analysis (Rosenfeld(1984)), has to be used in this case. The method is similar to the technique used by Frisch(1983), but it uses low-pass filters at several levels. This method is quite suitable for locating and analyzing the singularities. Earlier in this section we found that using the functions of the Schwartz Space we can study the singularities. Recently certain functions of the Schwartz Space known as *wavelets*, have been used to produce a multi-resolution analysis of arbitrary functions(Daubechies(1989)). This resolution has the advantage that it can analyse simultaneously, the local structures and the scales. Such a method is especially useful for studying functions whose singularities form self-similar clusters or fractals.

2.3 FLOWS WITH CIRCULATION

We have noticed in the earlier section, that if the potential exists and is single-valued then the circulation,i.e., the integral of the velocity field around each closed loop in ∂M is zero. However in real flows the circulation is not zero. In fact due to constant shedding of vortices from the body, an asymmetry in the flow profile would induce a net non-zero circulation around the body. To consider flows with circulation, it is necessary to introduce certain other singularities known as point vortices inside the body.

In two dimensional flows, the conjugate of the velocity potential, the stream function (usually denoted by ψ) always exists and is single valued. In that case, we could consider the singularity given by,

$$\Delta(\psi_0) = \frac{\partial^2 \psi_0}{\partial {x_1}^2} + \frac{\partial^2 \psi_0}{\partial {x_2}^2} = \Gamma \delta(\boldsymbol{x}) \tag{2.21}$$

the solution of which is,

$$\psi_0 = \frac{\Gamma}{2\pi} \log|\boldsymbol{x}| \tag{2.22}$$

The circular cylinder is a prototype for an obstacle in two dimensions. In fact in two dimension the Riemann mapping theorem guarantees the existence of a conformal mapping which transforms flow past any given regular obstacle(s) onto the flow past circular cylinder(s). So the addition of the above singularity to the flow, and then mapping the resulting solution from the exterior of the circle on to the exterior of a given arbitrary 2D configuration(body) completes the picture as far as the irrotational flow is concerned.

The singularity (2.22) is a discrete vortex, which when added to the irrotational flow gets physical significance. Such singularities can be used to derive weak solutions of the Euler's equation in two dimensions, a problem which we shall consider in Chapter 4.

Any asymmetry in three dimensional flow profiles could result in a net overall build up of vorticity around the body. In three dimension the equivalent of the vortex line is the axi-symmetric vortex element, the *vorton* (Saffman(1981)). Inclusion of a vortex element like the vorton inside the body would be sufficient to take care of the shedding of the vortices. The *vorton* is a vortex monopole whose velocity field is given by,

$$\boldsymbol{u} = \frac{\boldsymbol{k} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \tag{2.23}$$

From far this acts like a piece of vortex line, thus taking into account the excess vorticity. However unlike the two dimensional vortex singularity, this is not a distribution solution of the 3D Euler's equations, a fact which we shall consider in more detail in the Chapter 5.

In three dimensional case, the prototype (like circular cylinder in 2D case) may be imagined to be a geometrical configuration having an axis of symmetry. In this case, there exists no mapping concept like the conformal mapping in 2D (Kellogg(1932)). Even no mapping technique is known, which can be used to transform the exterior of a given geometry onto the exterior of the axi-symmetric configuration (the prototypes).

Chapter 3

THE POLE DECOMPOSITION METHOD FOR CONTINUOUS SYSTEMS

3.1 ORDINARY DIFFERENTIAL EQUATIONS

The importance of classifying ordinary differential equation by the singularities of their solutions was realized by B.Riemann in the last century. During that period, L.Fuchs and P.Painleve (Ince (1944),Hille (1976)) analysed the type of singularities occurring in first and second order differential equations. In general two types of singularities occur in ordinary differential equations: fixed singularity, whose location (and order) is determined by the equation itself and movable singularity,whose location depends on the initial conditions. Only fixed singularities can occur in the case of linear ODE's.

Paul Painleve (Ince (1944)) determined the first and second order ODE's with the property that their solutions contain only movable poles. Such a property is now known as the Painleve' property. The motivation behind studying Painleve property was to find whether certain ODE's are integrable in terms of known functions. The connection between integrability and Painleve property was realized by Kowalevskaya (1889), while studying the motion of a rigid body. After several decades, the study of truncated non-linear evolution equations has revived interest in the study of ordinary differential equations. In an interesting and extensive survey, Ramani et.al. (1989) have surveyed all the recent developments. Here to be specific let us consider a differential equation,

$$\frac{d^nW}{dz^n} = P(W, \frac{dW}{dz}, \frac{d^2W}{dz^2}, \ldots, W^{(n-1)}) \tag{3.1}$$

where P is any polynomial in n-variables with co-efficients which are analytic functions. Let us introduce a function I as,

$$I(W, \frac{dW}{dz}, \frac{d^2W}{dz^2}, \ldots, W^{(n)}) = W^{(n)} - P(W, \frac{dW}{dz}, \frac{d^2W}{dz^2}, \ldots, W^{(n-1)}) \tag{3.2}$$

To see whether (3.1) is of the Painleve' type, we set,

$$W = c\,(z - z_0)^{\alpha} \tag{3.3}$$

and find the minimum α which satisfies the condition,

$$Max_c \;\; O\{I(W, \frac{dW}{dz}, \frac{d^2W}{dz^2}, ..., W^{(n)})\} = Min_c \;\; O\{I(W, \frac{dW}{dz}, \frac{d^2W}{dz^2}, ..., W^{(n)})\} \tag{3.4}$$

where O denotes the order of a function and is the lowest power in its expansion, defined as,

$$\begin{array}{lcl} O(c_0(z - z_0)^-d + c_1(z - z_0)^{-d+1} + \ldots) & = & -d \\ O(0) & = & \infty \end{array} \tag{3.5}$$

If such a minimum α is a negative integer (say,$-d; d < 0$) then it satisfies the necessary condition for Painleve' property. Now substituting,

$$W = a_0(z - z_0)^{-d} + a_1(z - z_0)^{-d+1} + \ldots \tag{3.6}$$

we get,

$$I(W, \frac{dW}{dz}, \frac{d^2W}{dz^2}, \ldots, W^{(n)}) = b_0(z - z_0)^{-m} + b_1(z - z_0)^{-m+1} + \ldots \tag{3.7}$$

where,

$$\begin{array}{lcl} m & = & O(I(W)) \\ b_0 & = & b_0(a_0) \\ b_k & = & Q(k, a_0)a_k + P_k(a_{k-1}, \ldots, a_0) \\ & & \text{for } k \; = \; 1, 2, 3 \ldots \end{array} \tag{3.8}$$

$Q(k, a_0)$ is a n-th degree polynomial in k and P_k is a polynomial in the k-variables $a_{k-1}, \ldots, a_0$. The zeros of Q are called the resonances ($k = -1$ is always a resonance for autonomous systems). If $Q(k, c) \neq 0$ we can determine a_k's recursively so that $b_k = 0$. For such a procedure to be possible at a resonance, say $k = r$,given by $Q(r, a_0)$=0, we must have the consistency relation that, $P_r(a_{r-1}, \ldots, a_0) = 0$, where the co-efficients a_k's are determined such that b_k=0 for all $k < r$. Ishii (1990) has shown that the consistent resonances are given by the negative of the order of the algebraic first integrals of equation (3.1). This more or less proves that the Painleve property is necessary for algebraic integrability. However there exist certain integrable systems, with rational integrals, whose analytic structure permits solutions with singularities which are rational powers of $z - z_0$. To study such systems one may use the concept of weak Painleve property (Ramani et.al. (1989)).

Regarding non-integrability,Yoshida (1983) has obtained the important result that for homogeneous differential equations, if certain resonances known as the Kowalevski exponents are either irrational or imaginary, then the system does not posses algebraic first integrals. Yoshida's (1983) work establishes the connection between non-integrability, non-analyticity and self-similar clusters (or fractals). For certain homogeneous systems,the solution can be written of the form,

$$W(z) = z^{-d}[c + P(z^{c_1}, z^{c_2}, \dots z^{c_n})] \tag{3.9}$$

where $c_1, c_2, \dots, c_n$ are Kowalevski exponents (or the resonances), which can be obtained as the solution of certain n-th degree polynomial.
If any of the Kowalevski exponents happens to be irrational or complex, then we see that (*vide* section 2.2 of chapter 2) the singularities of $W(z)$ occur in clusters. The geometry of these clusters consists of certain spirals, which can be derived from a set of self-similar triangles (Yoshida (1983)). However these so called non-integrable phenomena are yet to be understood thoroughly. In an interesting on the Duffings system Fournier et.al. (1988) studied the recursive formation of singularities. However they have not been able to provide necessary condition for the formation of the clustered singularities. So each case has to be tested using the numerical integration procedure introduced by Chang et.al. (1983) to see if they really do form complex clusters.

The importance of the study of the analytic structures of turbulent flows was realized by Frisch and Morf (1981). Tabor and Weiss (1981) studied the analytic structure of the Lorenz system. These and other studies (Frisch (1983,85)) have emphasized the crucial role that singularities play in the structure of turbulence. Most of the afore mentioned works however, use a finite dimensional framework, whereas the fluid flows are described better in an infinite dimensional framework. So before studying the fluid flow systems here, as a first step, we shall consider the infinite dimensional systems given by the evolutionary equations in $(1 + 1)$ dimension.

3.2 POLE DECOMPOSITION FOR PARTIAL DIFFERENTIAL EQUATIONS

Kruskal (1974) and Moser (1975) first reported a study on the motion of poles of the Korteweg-de Vries (K-dV) equation. Choodnovsky and Choodnovsky (1977) later showed that pole decomposition solutions are possible for a general class of partial differential equations which includes,the Burgers and the K-dV equations. The pole decomposition method reduces the evolutionary equations to a many-body problem of the poles. Just as in Newtonian mechanics, the concept of attraction and repulsion between the poles can be introduced to take into account, the dynamic interactions between the coherent structures.

Consider a PDE of the form,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \gamma\,\frac{\partial^m u}{\partial x^m} \tag{3.10}$$

where m is any positive integer.
We assume a solution of the form,

$$u(x,t) = -\gamma(-1)^m \frac{(2m-2)!}{(m-1)!} \sum_k (x - a_k(t))^{-m+1} \tag{3.11}$$

with,

$$\frac{da_i}{dt} = -\gamma\,(-1)^m\,\frac{(2m-2)!}{(m-1)!} \sum_{k(\neq i)} (a_i - a_k)^{-m+1} \tag{3.12}$$

Some global constraints on the movements of these poles, may also have to be considered along with the above equations.

The case $m = 2$ and $m = 3$ correspond respectively to the Burgers equation and the *K-dV* equation for which pole decomposition solutions have been provided by Choodnovsky and Choodnovsky (1977). They are the generic cases,i.e.,the solutions obtained for even values of m,resemble solutions of the Burgers equation, while solutions obtained for all odd values of m,resemble solutions of the *K-dV* equation. For the case, $m = 2$,there are no global constraints,while for the case $m = 3$, there are infinite global constraints:

$$\sum_{k(\neq i)} (a_i - a_k)^{-3} = 0 \quad \text{for all} \quad i = 1, 2, \ldots \tag{3.13}$$

However for a qualitative study such global constraints are not important.

We see that,

$$u(x) = -\gamma(-1)^m \frac{(2m-2)!}{(m-1)!} (x - x_0)^{-m+1} \tag{3.14}$$

(where x_0 is in general any complex number with non-zero imaginary part) is a singular solution of (3.10). If we plot the real part of this function,then we see that for even values of m, this has the shape of a triangular shock. Tatsumi and Kida (1972) demonstrated that such triangular shocks do appear in solutions of Burgers equation. We have found from numerical experiments that such shocks occur for $m = 4, 6, \ldots$ and that the shocks become steeper as m increases. From (3.12) it follows that for $(-1)^{\frac{m}{2}}\gamma < 0$, two poles located on a line parallel to the imaginary axis repel each other (or attract each other if $(-1)^{\frac{m}{2}}\gamma > 0$). In case the poles are located on a line parallel to real line then, they attract each other (or repel each other if $(-1)^{\frac{m}{2}}\gamma > 0$). As a result for $(-1)^{\frac{m}{2}}\gamma < 0$, there is a tendency for the poles to line up perpendicular to the real axis. The stable equilibrium corresponds to a string of equidistant poles located along an axis perpendicular to the real axis. For the Burgers equation this corresponds to a tan-hyperbolic solution,

$$u(x) = -tanh[\ \frac{(x - x_0)}{2\,\gamma}\ ] \tag{3.15}$$

where x_0 is a real value.
However for the case, $(-1)^{\frac{m}{2}}\gamma > 0$, the above solution will be unstable and the poles now line up parallel to the real axis. So it is quite possible that the $u(x,t)$ would blow-up for some initial conditions producing numerical overflow in simulations. This is one of the reasons why proper care is required to ensure that the effective viscosity never becomes negative in finite deference methods. In finite difference methods, simply increasing the order of the method, without studying the stability is usually not advisable since the negative hyperviscosity (corresponding to $m \geq 4$) is even more destabilising.

If m is odd, then we see that the real part of the function given by (3.14) is quite like the soliton pulses one gets in the numerical simulation of the K-dV equation. If m is odd,then from (3.12) it follows that all the poles move towards the positive real axis: the ones which are closer together move faster, while the ones which are further apart move slower. Again if we consider a string of equidistant poles we get the standard traveling wave solution,

$$u(x,t) = 3V\ sech^2[\ \frac{1}{2}\ (\frac{V}{\gamma})^{\frac{1}{2}}\ (x - Vt)\ ] \tag{3.16}$$

Two such strings travel with different speeds if the separation distance between the consecutive poles in one of them differs from the other. The string in which the separation distance is smaller (and hence steeper pulse) will move faster and soon overtake one with larger separation (which has a smaller pulse). The solitons speed up and distort as they come nearer but soon pass over each other and regain the original shapes. Vast literature is now available regarding solitons and other integrable systems (Bullough and Caudrey (1980),Lamb (1980),Ablowitz and Segur (1981)). Most of them however, pertain to the Inverse Scattering Transforms.

Although here we have basically studied only two types of equations,i.e., the Burgers equation and the K-dV equation, we would like to point out that these are generic cases for general integrable equations. The solutions (3.15), (3.16) and their derivatives can be used to model the spatial structures of general coherent features occurring in continuous one-dimensional systems. The interactions between the coherent structures is also important. For the Burgers case the coherent structures (the shocks) merge, causing a loss of information, in a process which is irreversible. Where as in the K-dV case the coherent structures (the solitons) which were initially close together after a lapse of time separate out, in reversible process. This causes an increase in information, since the states which were originally almost indistinguishable, could at a latter time become distinguishable. Apart from these two contrasting cases, we may envisage a process in which the information increases so rapidly that it becomes difficult to comprehend. Such systems are said to exhibit genuine *chaos*.

Yamada and Kuramoto (1976) did numerical simulations of the equation,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0 \tag{3.17}$$

and found that it posses chaotic solutions. Here the negative viscosity due the third term in the left hand side of (3.17) is (over)compensated by fourth term,which gives the effect of positive hyper-viscosity. Since near the real line,the repulsion (due to the fourth term) between the conjugate poles,is greater than the attraction (due to the second term)between them, no blow-up occurs. However this only gives a heuristic proof for well- posedness of (3.17). For a rigorous study, an extension of the Painleve test has to be undertaken. Weiss et.al. (1983) have introduced Painleve property for partial differential equations (see also Ramani et.al. (1989)). Thual and Frisch (1983),studied the Painleve property of the Kuramoto-Sivashinsky equation, (3.17) and found two complex resonances (Frisch (1983)) in the spatial variables, indicating the possibility of fractal natural boundaries and non- integrability. However one should be cautious in applying the Painleve test. If the PDE passes the Painleve test, then it need not be integrable. To see this consider the model for flame instability (Michelson and Sivashinsky (1977)) and plasma turbulence (Lee and Chen (1982)) :

$$\frac{\partial u}{\partial t} + \mu\ H[\frac{\partial u}{\partial x}] + \beta\ H[\frac{\partial^2 u}{\partial x^2}] - \nu\ \frac{\partial^2 u}{\partial x^2} - 2u\ \frac{\partial u}{\partial x} = 0 \tag{3.18}$$

This equation passes the Painleve test, and has a pole decomposition solution,

$$u(x,t) = -2\ Real\ \{\sum_j \frac{\nu - i\beta}{(x - x_j(t))}\} \tag{3.19}$$

where,

$$\frac{dx_j}{dt} = -i\mu - \frac{\mu\beta}{\nu} - 2(i\beta+\nu) \sum_{k=1\ k\neq j}^{N} (x_j - x_k)^{-1} + 2(i\beta-\nu) \sum_{k=1\ k\neq j}^{N} (x_j - x_k)^{-1} \tag{3.20}$$

Lee and Chen (1982) have shown that for certain values of the parameters, the three-pole solution corresponding to the case $N = 3$ in the above equations, exhibits chaotic behaviour. Thus the system is non-integrable although the solution has a meromorphic dependence in spatial variables. So it is important to distinguish between the analyticity of spatial and temporal variables in studying the integrability of the solutions.

3.3 A DECOMPOSITION METHOD FOR COMPUTATIONS

We have seen in the previous section how the method of pole decomposition helps in gaining insight into the problems. Here we shall see how such a method can be used to make numerical studies also. One of the difficulties encountered in the pole decomposition is that, long range interactions exist between the poles. So some adhoc far field approximations are required, to make computations in infinite regions. Earlier,we have noticed that the evolutionary equations has solutions correspond to certain coherent structures. It seems that an expansion in terms of such structures may be useful. To be specific let us consider the Burgers equation. The coherent structures are the shocks and the structure function is given by the steady state solution (3.15). If we try to expand in terms of such function, then we would get the solution given by Choodnovsky and Choodnovsky (1977),

$$u(x,t) = -\sum_k coth[\ \frac{(x - a_k(t))}{2\ \gamma}\ ] \tag{3.21}$$

where a_k's follow the relation,

$$\frac{da_i}{dt} = -\sum_{k \neq i} coth[\frac{(a_i - a_k)}{2\gamma}] \tag{3.22}$$

This solution can be used to study the evolution of the shocks. As the coth-function reaches an asymptotic value rapidly, the summation in (3.21) could be restricted at a finite number without incurring large errors. Thus an efficient algorithm could be developed. Using the Cole-Hopf transformation, Parker (1991) obtained the periodic solutions of the Burgers equations. By considering an infinite strings a_i's arranged periodically along the x-axis, we can recover from (3.21), the periodic solutions obtained by Parker (1991). Such solutions can also be obtained by replacing the coth-function in (3.21-3.22) by a doubly periodic function (Choodnovsky and Choodnovsky (1977)). However we found that, the solution (3.21-3.22) is easier to simulate and by considering a few periods of coth-pulses, the effect of an infinite string can easily be realized. Thus (3.21-3.22) can be used to study the dynamics of shock pulses numerically.

Solution similar to (3.21-3.22) can also be derived for the K-dV equation , using the $Sech^2$ (or $Cosech^2$) soliton pulses:

$$u(x,t) = 3 \sum_k Sech^2[\frac{1}{2} \gamma^{-\frac{1}{2}} (x - a_k)] \tag{3.23}$$

where,

$$\frac{da_i}{dt} = 3 \sum_k Sech^2[\frac{1}{2} \gamma^{-\frac{1}{2}} (a_i - a_k)] \tag{3.24}$$

However it is important to realize that the global constraints, (3.13) must be satisfied at all times. But for numerical simulations,it is enough if the initial values satisfy this constraint. The conversion of a given initial condition into the short pulses (i.e...*Coth* and $Sech^2$ functions) however, is non-trivial. These short pulses are related to the functions of the Schwartz space we discussed in the previous chapter. The decomposition of the functions in terms of certain Schwartz class functions, i.e., the *wavelets* is termed as the wavelet decomposition and is a very convenient tool for analyzing non-linear signals. Such a tool has already been used by Meneveau (1991) for analyzing turbulent signals. This decomposition of signals, in terms of *wavelets* (or solitons), instead of the usual sine and cosine functions (or waves), is in fact equivalent to considering a distribution of particles instead of waves. Particle methods effect localization in space and allow the study of movements and patterns (say on graphics monitor). We shall see later how a solution quite similar to (3.21) can be used to simulate the Euler's equation in two (space) dimensions, where such discerning of patterns and movements would be advantageous for studying the coherent structures in the flow.

Chapter 4

VORTEX ALGORITHMS FOR 2D FLOWS

In incompressible fluid flows the pressure waves travel instantaneously and hence, pressure plays no direct role (Thomas and Bull(1982)) in the intricate mechanism of the flow, even though it is possible that the initiation of the flow itself might be due to the existence of a pressure difference. Pressure is a isotropic tensor, which tends to disperse the local disturbances. However it is very difficult to ascertain the cause and effects in the complex formation of structures in fluids using conventional measurements of velocity and pressure. S.J.Kline and his co-workers in the 1960's made studies in the boundary layer flows, with help of flow visualization. Later Brown and Roshko(1974) made similar studies in the mixing layers. These studies show that flow visualization offers important clues. But one has to be rather careful in associating causal relationships. It is better, as a first step to study the fluid flows as a history of flow events and patterns. The flow visualization studies have shown that the vorticity field is important for understanding fluid flows. Moreover, unlike the velocity field, the vorticity field is a Galilean invariant (i.e.,its measured value remains same even if the measurement device is traveling at a constant velocity, instead of being fixed). We found in the first chapter, that certain potential solutions may be imagined to provide the spine for the flow, the introduction of vorticity provides such a flow, flexibility and muscle. In this connection an important phrase due to Kücheman(1964),

vorticity forms the sinews and muscles of fluid flows is *often* quoted. Although the study of vorticity provides deep insights into the formation of flow patterns , but it is difficult to measure vorticity in practice. However numerical simulations are considered strong tool for studying the vorticity field. The Discrete Vortex Method(DVM) is such a tool by which we are able to trace the flow movements and patterns. But before considering the DVM, we recapitulate some properties of inviscid flows in two-dimensions.

4.1 THE TWO DIMENSIONAL FLOWS : AN INTRODUCTION

The *curl* of the velocity field,$\boldsymbol{u}(\boldsymbol{x},t)$,i.e...,the vorticity, $\Omega(\boldsymbol{x},t)$ is a scalar in two dimension and its evolution is given by,

$$\frac{D\Omega}{Dt} = \nu\Delta\Omega \tag{4.1}$$

where $\frac{D}{Dt}$ stands for the total derivative, $\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$.
For flows in which the diffusive effects are negligible, the right hand side of (4.1) can be dropped, and hence we would have basically an inviscid flow. If $\boldsymbol{u}(\boldsymbol{x},t)$, is the velocity of the flow, then the Lagrangian flow function $\phi(\boldsymbol{x},t)$,is described by,

$$\begin{aligned} \frac{d\phi(\boldsymbol{x},t)}{dt} &= \boldsymbol{u}(t,\phi(\boldsymbol{x},t)) \\ \phi(\boldsymbol{x},0) &= \boldsymbol{x} \end{aligned} \tag{4.2}$$

For incompressible flows, the map $\phi(.,t)$ is area preserving (since divergence of $\boldsymbol{u}$ is zero) and the vorticity Ω,is a flow invariant,i.e.,

$$\Omega(\phi(\boldsymbol{x},t),t) = \Omega(\boldsymbol{x},0) \tag{4.3}$$

It is now well known (Youdovitch(1963) and Kato(1967)) , that if $\Omega(\boldsymbol{x},0)$ is a regular function,then there exists a classical solution to the Euler's equations in two dimension.

There exist infinitely many constants of motion in two-dimensional flows. The important ones being the energy,

$$E = \frac{1}{2}\int_M u^2 dxdy = \frac{1}{2}\int_M \psi\Omega dxdy \tag{4.4}$$

and the (generalized enstrophy) integrals of the form,

$$I_f(\Omega) = \int_M f(\Omega)dxdy \tag{4.5}$$

where f is any regular(C^∞) function and ψ is the stream function of the flow. Depending on the symmetries of the flow geometry, the linear and (or) angular momentum may also be conserved. These constants of motions are very important, since they provide a way of checking and controlling the simulations of 2D flows. The existence of infinite number of constants of motion in this case does not imply that the Euler's equations are integrable (Arnold and Khesin(1992)) . We also know that the existence of chaos does not neccesarily imply that the system has a meromorphic solution (*vide.*section 3.2). So the existence of chaos (Novikov(1978), Arnold(1966)) in 2D flows is not sufficient to show that the inviscid 2D flows do not have simple solutions. Ebin(1983) has shown that 2D Euler's equations are weakly integrable and Hald(1987) has shown that the Lagrangian flow is infinitely differentiable. Kwak(1991) proved the existence of a *centre manifold* for the 2D Navier-Stokes equation. Murometz and Razboynich(1990) have shown that sufficiently large finite dimensional reductions of the 2D Euler flows exists, which are integrable Hamiltonian systems. Bardos et.al.(1976) have shown that if the velocity field is initially analytic in a strip around the real line,then as $t \to \infty$, the width of this strip does not go to zero faster than an exponential of an exponential (Frisch (1983)) . But as yet no explicit time dependent solution like the solutions for the Burgers and the K-dV equations (see the last Chapter), has been derived for the 2D Euler's equations. However,the steady state solutions to Euler's equation can be derived explicitly. If ψ denotes the stream function in two dimension and if ψ satisfies,

$$\triangle\ \psi = f(\psi) \tag{4.6}$$

for any regular function f, then ψ,yields a steady state solution to the Euler's equations (Lamb(1932)) .

The equation (4.6) looks simple, but it has a variety of solutions, including the solutions of the Klein-Gordon equations and the Liouville equations. Some of these solutions could also be considered as the fixed point solutions to which flow gets attracted. Among these steady state solutions, Robert and Sommerlia (1990) have studied the ones which maximizes certain entropy functions. This study is a sophisticated version of the Reynolds two-scale description of turbulence. However such interesting steady solutions could be of any practical utility only if they are stable solutions which evolves out of a smooth initial flow field. Moreover the dynamics of the flow itself could be crucial for an understanding the complex flows. In an important study on the dynamics of the 2D flows, Weiss(1981) found that the growth of the gradients of the vorticity is related to the transfer of enstrophy (of vortices) to the smaller scales. He found that if in a region, the squared magnitude of the rate of strain exceeds the squared magnitude of the rate of rotation, then in that region the gradients of vorticity will tend to grow exponentially fast, while in the opposite case the vorticity gradients will behave in a periodic manner in that region. Essentially when the strain rate exceeds vorticity, the fluid is in a hyperbolic mode of motion that strongly shears the passively advected vorticity and conversely when vorticity exceeds the strain, the fluid is in a elliptic mode that advects the vorticity smoothly. As a consequence the vorticity gradients will tend to concentrate in regions of hyperbolicity,i.e., in between the large scales. Kida and Yamada (1984) has also pointed out that for inviscid flows the gradients of the vorticity may increase rather sharply. Thus we see that it is not convenient to compute gradients of the vorticity. It is convenient to use the DVM for simulating inviscid 2D flows, since computation of the gradients are not required with this approach. Finally before we introduce the Discrete Vortex Method (DVM), we would like to point out that the qualitative aspects of the flow depends crucially on the topological aspects of the flow domain,especially on its boundaries (Weiss(1981)) .

4.2 THE DISCRETE VORTEX METHOD : FREE FLOWS

The DVM was first used by Rosenhead(1932) to study the evolution of a vortex sheet. The calculations in fact were so simple, that he computed them by hand! Vast amount of literature is now available on vortex methods. Surveys by Saffman and Baker(1979),Aref(1983),Leonard(1985),Sarpkaya(1989) and the collected works of A.J.Chorin (Chorin(1989)), provide an adequate introduction to the subject. DVM is automatically adaptive since the vortices concentrate in the region of interest. Being a Lagrangian method, there are no inherent errors in DVM, which act like the numerical viscosity in the conventional finite difference methods. In the conventional finite difference methods,the numerical viscosity, tend to dominate the effects of viscous diffusion, making it difficult to make simulations at high Reynolds numbers. DVM is highly robust, since new elements can be easily added (*vide.*Ghoniem et.al.(1982), who considers a complex flow problem involving combustion). However the method has to be used with sufficient care. A critical review of the DVM, is given in Sarpkaya(1991).

The DVM basically considers a discrete distribution of point vortices and by tracking the movements of these vortices the flow is resolved. The motion of such point vortices is governed by,

$$
\begin{aligned}
u &= -\sum_i \gamma_i \frac{(y-y_i)}{(x-x_i)^2+(y-y_i)^2} \\
v &= \sum_i \gamma_i \frac{(x-x_i)}{(x-x_i)^2+(y-y_i)^2}
\end{aligned} \tag{4.7}
$$

and,

$$
\begin{aligned}
\frac{dx_i}{dt} &= -\sum_{j(\neq i)} \gamma_i \frac{(y_j-y_i)}{(x_j-x_i)^2+(y_j-y_i)^2} \\
\frac{dy_i}{dt} &= \sum_{j(\neq i)} \gamma_i \frac{(x_j-x_i)}{(x_j-x_i)^2+(y-j-y_i)^2}
\end{aligned} \tag{4.8}
$$

where γ_i's are the strengths of the discrete vortices.
Aref(1983) observed that there is a close similarity between (4.7) and the solution of the Burgers equation (3.11). However unlike (3.11), the solution (4.7) is a singular (distribution) solution of the two dimensional Euler's equations. So he conjectured that a regular solution may also be possible for the Euler's equations just like the Burgers equation. But as the solution of the one-dimensional Burgers equation consists of singularities in two-dimension, one would expect the corresponding solution for the two-dimensional Euler's equations would admit of three dimensional singularities. In Venkatesan(1992a) we investigated such a possibility (the details can be found in Chapter 5) and it was found that such an approach of considering singularities in three dimension is equivalent to considering a vortex blob in the physical two-dimensional plane.

As the density of the point vortices increases, the flow given by (4.7) does not converge to a solution of the Euler's equations in the classical sense(point-wise convegence), but they may converge in in a weak sense(convergence in probability) (Hou(1991)) . It is very costly to make simulations using random methods and take ensemble averages of the simulations. Moreover the velocity field generated by the point vortices is not smooth and so it is not suitable for generating flow patterns. Hence it is necessary to use vortex blobs instead of point vortices. A vortex blob can be generated by convoluting the vorticity in the blob $\xi(x,y)$, with the fundamental vortex solution. The velocity field of the vortex blob then is given by,

$$
\begin{aligned}
u &= -\frac{1}{2\pi}\int \frac{y-y'}{(x-x')^2+(y-y')^2}\xi(x',y')dx'dy' \\
v &= \frac{1}{2\pi}\int \frac{x-x'}{(x-x')^2+(y-y')^2}\xi(x',y')dx'dy'
\end{aligned} \tag{4.9}
$$

Here the cut-off function $\xi(x,y)$ is also the vorticity in the blob. The point vortex corresponds to the case, $\xi(x,y) = \delta(x,y)$, where $\delta(x,y)$ is the 2D Dirac delta distribution. Hald(1979), has shown that the methods which use certain class of vortex blobs instead of point vortices, converge with second order accuracy within an interval of time. Beale and Majda(1982) have proposed an alternative method, which has higher order of accuracy. They showed that by considering a uniform space lattice, very high order of convergence can be achieved. If the blob has vorticity, $\xi(x,y)$, which satisfies the conditions,

(i) $\xi(x,y)$ belongs to $C^2(R^2)$

(ii) $\int \xi(x,y)dxdy = 1$ and $\int x^m y^n \xi(x,y)dxdy = 0$ for all $m, n \leq p-1$

(iii) ξ (α,β),the Fourier transform of $\xi(x,y)$ belongs to the Schwartz class, $S_{0,1}^{-L}$ for some number L.

then (we say $\xi \in FeS^{-L,p}$) it is possible to construct vortex schemes, which has a convergence of order, h^{pq}. Here h is spacing of the vortex lattice considered and q lies between 0 and $\dfrac{(L-1)}{(p+L)}$. One of the simplest function which satisfies (iii) is an entire function of the form, $exp(-\{x^2+y^2\}/\sigma)$. Beale and Majda(1985) showed that it is possible to construct such entire functions, which belong to the class $FeS^{-\infty,p}$ for large values of p. An infinite order method was devised by Hald(1987) in terms of the cylindrical Bessel functions, who also studied accuracy in time discretisation. Nordmark(1991) has computed the $\xi(x,y)$ corresponding to the infinite order method in terms of simple functions. The proper choice of the blob-function $\xi(x,y)$ and the scheme used for discretisation in time are important in using the vortex blob methods.

It should be emphasized that in principle there is very little to choose between a deterministic method which uses vortex blobs and a random method method which uses point vortices. Although it doesn't make much sense to use a deterministic method which uses point vortices, it has been argued (Saffman(1991)) that the solutions with point vortices are better (however unphysical they may be) since they are atleast weak solutions of the Euler's equations, while the solutions with vortex blobs are not. To counter this argument we would like to mention the presence of vorticity does not necessarily imply that fluid particles at that point are rotating. In fact in the laminar Poïseulle flow through a flat 2D channel, although vorticity is present everywhere the fluid particles flow straight! To approximate such a smooth flow we have to use vortex blobs instead of point vortices. It should be mentioned that the vortex blob method can also be thought off as a probabilistic method, in which the probability distribution of the existence of a point vortex at a point in space is given by the blob-function $\xi(x, y)$. At present we could envisage only one problem with this interpretation; we have to restrict ourselves to blob-functions $\xi(x, y)$ which are non-negative. However the probability methods require large number of numerical experiments and we shall not pursue this approach further here.

The underlying difficulty behind the deterministic DVM which uses the blob, is the assumption that vorticity can discretized into circular vorticity patches, each of which is assumed to undergo no deformation during the flow. However strigent this assumption might seem, it should mentioned the use of vortex blobs of the kind mentioned earlier, acts more like coarse scale filter on velocity. It is possible to capture the finer scale patterns in the flow by working with vortex clusters, rather than with rigid vortex blobs. This is actually quite similar to the multi-grids used in finite difference schemes, where one works with several levels of grids. In this multi-level vortex method, the computational effort required to compute the velocity of the flow field can be reduced form an order of N^2 (where N is the number of discrete vortices used) to an order of $O(NlogN)$ or even $O(N)$. There are many procedures to reduce the computational effort, like the clustering, convolution and multipole algorithms (**Greengard and Rokhlin(1987)**). We shall consider the multi-level vortex method in more detail in the last section, where we shall consider the simulations of a flow in a 2D channel.

The discovery of large quasi-2D structures in a plane turbulent mixing layer by Brown and Roshko(1974), has renewed the interest in the study of vorticity. In their study of the mixing layers, **Browand and Winant(1974)** observed that the vortex pairing mechanism is responsible for the formation of coherent structures. Later Jimenez et.al.(1991) also made an interesting study on the pairing mechanism in mixing layers. The study of two-dimensional flows is undertaken as quite often, it helps in the understanding of the intrinsic properties of the flows and in distinguishing 2D structures from 3D structures. In 2D inviscid flows it can be shown that strictly speaking there can be no merger (Majda(1986)) . This has been verified by Baden(1987) using the DVM with 10^5 vortices. He observed that although the vortices form a pair , they do not merge into one. But in calculations done with lower accuracy (especially in time discretisation) it seems we achieve a merger. Since the method yields results which are very similar to experimental results, it has been suggested by Kawahara and Takami(1984) that errors involved in time discretisation might provide a good model for turbulence. In fact it is interesting to study the similarities and differences between the two phenomena-shocks and vortices. Just as shocks merge , loosely speaking vortices do merge. However to be precise, in inviscid flows vortices can only come close together and cannot merge (by Kelvin-Helmholtz theorem : Majda(1986)) . Although the inclusion of a viscous diffusion in the flow would make it difficult to have clear cut conclusions, one would expect that these premises would still be valid at high Reynolds numbers. In fact it is possible to study the effects of viscous diffusion using a vortex algorithm such as the one proposed by Cottet and Gallic(1985,90), Lu and Ross(1991).

In an earlier study (Venkatesan(1991)), it has been found that two vortex patches, with the same orientation, rotate together and forms a spiral structure. Vortex splitting seems to occur sometimes; the tail portion of the spiral detaches itself. Such an event was more probable if the vortex patches differ very much in strengths. It was also found that if the two patches were of the opposite orientation, then they get elongated in one direction and come closer. Here the merger results in the cancellation of vorticity. This process has been studied in detail by Buntine and Pullin(1989), who have also considered the influences of stretching and straining(three dimensional effects) on merger and cancellation. They found that these three dimensional effects accelerate considerably the merging and cancellation process. This process has been studied analytically by Kambe(1984), using an exact solution of a one-dimensional (for an axi-symmetric version, see Lundgren(1982)) model. Unfortunately the real physical flows are three dimensional and the vorticity possibly gets twisted and tangled. Thus in two dimensional projections, it is difficult to asses, whether the vortices actually merge or not. We will consider the three dimensional interactions in the Chapter 5.

The problem of mixing layers is usually studied by considering the evolution of vortex sheet. While applying the DVM, the number of vortices in this case increase with time. So it is difficult to compute the flow after certain time. Since the velocity field of individual vortex decays slowly (inversely proportional to the distance), if we discard the down stream vortices at a stage, it would produce large computation errors. Instead of the DVM, if we apply Fourier techniques, it is necessary to compute the flow field by considering certain periodicity in the streamwise direction. This periodicity could lead to spurious reflections at the artificial boundary. Recently Basu et.al.(1988) have resolved this difficulty by introducing certain far field conical structure for the vortices. They also monitored the errors in the constants of motions.

Finally, we would like to mention a drawback of all time marching methods, which do not use constants of motion as global constraints. Even the conventional methods like the finite difference methods, have to be used with care, to quote Lesieur(1990),

> For the point of view of strictly two-dimensional turbulence in the context of 2D Navier-Stokes, the problem of sub-grid modeling is far from being solved...

Basdevant and Sadourny(1983) studied various sub-grid scale diffusion operators and found that higher order diffusion operators work better as small scale filters. However care is required in such sub-grid models. We found in the earlier chapter how in one-dimensional systems the addition of even dimensional derivatives stabilizes even an unstable system. Schreiber and Keller(1983), showed that in some cases the solutions do not converge to the correct results even if we use higher order schemes. As a remedy one can consider considered a simplectic algorithm which uses the Hamiltonian as a constraint to calculate the evolution on the appropriate manifolds (Pullin and Saffman(1991)). But such methods are useful only for cases with point vortices, for which the Hamiltonian is easy to calculate. In real situations,vorticity is continuously distributed and the Hamiltonian is too difficult to calculate. But it is not difficult to study some of the constants of motion like the linear moments and/or angular moments, as a check on the results. However if one is interested only in a short time study or in simple qualitative results, then these methods seem to be quite adequate.

4.3 THE DISCRETE VORTEX METHOD : REGION WITH SIMPLE BOUNDARY

Lighthill (1963) studied the vorticity transfer mechanism of a incompressible fluid flow near a boundary. However at that time powerful computers were not available for computations. Later Chorin (1972) introduced the operator splitting algorithm for studying the viscous diffusion process in a flow and the vorticity creation algorithm to satisfy no-slip condition. This greatly improved the applicability of the DVM. However some care is required in using such algorithms. For real time simulations the random walk method may not be very useful but, the operator splitting algorithm can still be effective, if a diffusion algorithm such as the one used by Cottet and Gallic (1985,90), Lu and Ross (1991) can be implemented. But it is important to realize that diffusion acts at fine scales and that for coarse simulations the effect of numerical error in simulating the inviscid flow itself could be more dominant than the effects of diffusion. Diffusion itself can be thought of as a model of the very fine chaotic motion occurring at the molecular level (quite like the turbulent diffusion models), then since at the molecular level the laws of physics are reversible, it may be better to stick to a reversible model. The inadequacy of the diffusion model is rather clear atleast in the case of the super-fluids. However it should be noted that we are in no way doubting the practical utility of viscous diffusion models, but only that we are suggesting that there may be alternative ways of looking at the so called diffusion phenomena. Moreover there are certain ways of guiding and checking the simulations of the inviscid flows (like the constants of motion), which are not available for the Navier-Stokes simulations. So we have neglected diffusion completely. But then how does one satisfy the no-slip boundary condition at the wall? We shall answer this a little later, but before that we will consider the vorticity transfer mechanism at the wall. Lighthill (1963) studied in detail the vorticity transfer at the wall. But there remained some confusion regarding the exact process of creation and loss (if any) of vorticity at the boundary. The study by Morton (1984) has cleared some of the misconceptions. His conclusions are the following :

(i) Vorticity is created at the wall due to acceleration of the fluid near the wall.

(ii) Vorticity once generated cannot be destroyed by diffusion at the wall.

Since the absolute velocity at the wall is zero, the only process by which vorticity can be transferred from the boundary is through viscous diffusion, a slow process in the time scale of turbulence. In our study here since we neglect diffusion, we also assume that vorticity is neither created nor destroyed at the wall. Such an assumption of course would not be suitable for unsteady flows, but for flows more or less steady in the Eulerian frame, this method can be used with a bit of caution. Finally regarding the question that how can an inviscid flow satisfy no-slip boundary condition,we would like mention that it is possible, if we allow a tangential discontinuity in velocity,i.e., the presence of a singular vortex sheet at the wall. In viscous flows such a vortex sheet spreads into a thin layer of vorticity and then diffuses into the flow, creating the so called laminar (?) sublayer. If we work with a multi-level vortex method which has a large number of fine scale vortices, then the above process can be simulated using a diffusion algorithm similar to the one introduced by Cottet and Gallic (1985,90), Lu and Ross (1991). However any such thin layer of vorticity occurs along with its reflection on the wall and it will have little influence on the vortices away from the wall , unless the sheet diffuses for a sufficient period of time.

To satisfy the normal boundary condition, Chorin (1973) proposed a source distribution along the boundary. But it led to large errors in the flow near the boundary and so he later considered the mirror reflection of the vortices (Chorin (1980)), for a flow over a flat plate. However strictly speaking this applies only to an infinite flat plate and not for a flat plate with finite width. Care must be taken for an arbitrary boundary to deduce the reflection principle. The correct fundamental vortex function could be obtained by from a conformal map of the fluid domain on to a half-plane or a circle. Such a conformal map is important since we have to also consider the distortions caused by the conformal maps on the vorticity distributions of the cores while considering the reflection principles. In Appendix B of this thesis we consider details about such conformal maps and on how they can be derived.

In the pioneering studies on the use of vortex methods, Chorin (1978,80) studied the flow over a semi-infinite flat plate, using the boundary-layer approximations,random walk diffusion and a creation (or loss; which was considered equivalent to creation of vortices of opposite orientation) algorithm at the boundary. But for real time deterministic studies such an approach is not appropriate, so in Venkatesan (1991) we studied the same problem but we dropped the diffusion altogether and assumed a fixed vorticity distribution which closely follows the laminar profile for a particular span and later allowed the flow to evolve using the discrete vorticity algorithm (it is also important to realize that angle of incidence at the tip of the plate has to be exactly zero). No creation or loss was allowed at the boundary. Instabilities quite reminiscent of the initial period of the transition stage were observed. For large time steps, ejection of vortices have been observed, but such instabilities disappear once the time step is reduced. However as the flow evolves the number of vortices become very large and the computational effort increases enormously as the flow evolves. So we have not been able to study these transitional instabilities further. To reduce computational effort for such a study, a method similar to that adopted by Basu et.al. (1988) may have to be adopted. It is important to realize that no-slip condition would be satisfied at least for some interval of time quite accurately if the initial vorticity distribution already satisfies such a condition. Anyway Rashid and Bandopadyaya (1990) have shown using experiments that the no-slip boundary condition is not quite important as it is generally believed. Narasimha (1989) has pointed out that experimental evidence suggests that in the boundary layer, the general flow characteristics are not sensitive to the wall roughness and hence what happens very near the wall may not be terribly important for the rest of the boundary layer. Although it has not yet mathematically proved that the solution of the Navier-Stokes equation tends to the solution of the Euler's equation with the same initial conditions. For a flow past a flat plate, Asano (1989) is supposed to have made some progress in that direction (Heywood (1991)) .

Cheer (1989) has used the discrete vorticity method to study an unsteady flow past a circular cylinder. Although the method has yielded satisfactory results for the large scales, it allows the vortices to be generated and destroyed at the boundary and hence one could expect the spatial and temporal variations of vorticity near the boundary to be rather too coarse than what it should be. The vorticity field has to be rather smooth to yield quite reliable results near the boundary. Moreover Cheer (1989) do not consider the core distortions due to conformal maps. A refinement could be obtained by applying the diffusion algorithms of Lu and Ross (1989) and strictly following conclusions of Morton (1984). All these results are for two-dimensional flows and hence an exact confirmation with experiments is not quite necessary, but a comparison with experiments could be useful to locate the possible three dimensional effects. T.Sarpkaya has been studying the flow past bluff bodies for over many years and his views on the subject can be found in an extensive survey (Sarpkaya (1989)) on the vortex algorithms.

4.4 THE DISCRETE VORTEX METHOD : CHANNEL FLOWS

For boundary free flow problems and problems of flow over a simple boundary, the computational cost becomes prohibitively high with the increase in number of vortices. We shall show in this section, that the computational costs for flows bounded by more than one boundary could be lesser. Here we shall deal with the flow in a flat 2D channel of infinite extent.

It is known that in clean wind tunnels, laminarity of the flows could be maintained even at very high Reynolds numbers. So to get an appropriate initial distribution of vorticity for the inviscid flow, we use the laminar velocity profile of a flow in the channel for some low Reynolds numbers. Discretizing the initial vorticity distribution thus obtained, the appropriate infinite Reynolds number flows could be simulated.

The complex velocity field of a point vortex is given by,

$$u - iv = \frac{\gamma}{2\pi i}(z - z_0)^{-1} \tag{4.10}$$

where $z = x + iy$ and $z_0 = x_0 + iy_0$ are complex numbers.
For a problem with a single plane boundary at $y = 0$, to satisfy the normal boundary conditions at all times, we include the mirror reflection of the point vortex at the wall, so that,

$$u - iv = \frac{\gamma}{2\pi i}(z - z_0)^{-1} + \frac{\gamma}{2\pi i}(z - z_0^*)^{-1} \tag{4.11}$$

where z_0^* is the complex conjugate of z_0.
However if there were two plane boundaries ; at $y = 0$ and $y = 1$, then we may consider an infinite sequence of mirror reflections and so the complex velocity field in this case will be a periodic function z, with period $2i$. The Mittag-Leffler expansion of *Coth z* is given by,

$$coth\ z = z^{-1} + \sum_m (z - m\pi i)^{-1} \tag{4.12}$$

So for the case with two boundaries we obtain,

$$u - iv = \frac{\gamma}{4i}\{coth[\frac{\pi}{2}(z - z_0)] + coth[\frac{\pi}{2}(z - z_0^*)]\} \tag{4.13}$$

If we have many such vortices,then we may write,

$$u - iv = \sum_k \frac{\gamma_k}{4i}\ \{\ coth[\frac{\pi}{2}(z - z_k)]\ +\ coth[\frac{\pi}{2}(z - z_k^*)]\ \} \tag{4.14}$$

Simulations with equation (4.14), for a 2D channel flow leads to some difficulty. Since the channel extends infinitely in the streamwise direction finite number of vortices are not sufficient (especially since the vortices would soon flow out and more have to be added). To avoid this difficulty we assume that the flow is periodic in the streamwise direction with a period d. So in effect we will be considering an array of doubly periodic vortices. The complex velocity field,is given by,

$$u - iv = \sum_{k=1}^{N} \sum_{l=-\infty}^{l=\infty} \frac{\gamma_k}{4i} \{ \, coth[\frac{\pi}{2}(z - z_k - ld)] \; + \; coth[\frac{\pi}{2}(z - z_k^* - ld)] \, \} \tag{4.15}$$

Such doubly periodic vortices were considered in another context by Weiss and McWilliams (1991). The Hamiltonian (or the energy of the interactions) of the doubly periodic vortices (with a period $d \times 1$) can be written as,

$$H = -\sum \frac{\gamma_i \gamma_j}{2} h(x_{ij}, y_{ij}) \tag{4.16}$$

where γ_i's are the circulation associated with each point vortex, $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$ and $h(x, y)$ is given by,

$$h(x,y) = \sum ln[cosh(\frac{\pi x}{2} - 2dm) - \frac{cos(\frac{\pi y}{2})}{cosh(dm)}] - \frac{\pi x^2}{8} \tag{4.17}$$

The motion of the vortices being governed by the Hamilton-Jacobi equations,

$$\begin{aligned} \gamma_i \frac{dx_i}{dt} &= \frac{\partial H}{\partial y_i} \\ \gamma_i \frac{dy_i}{dt} &= -\frac{\partial H}{\partial x_i} \end{aligned} \tag{4.18}$$

Although the method with point vortices is easy to apply, the velocity field of a point vortex is singular, so the flow pattern generated by it is inappropriate for calculations. In Venkatesan (1992a) we considered a desingularization procedure similar to the one used by Krasny (1989); introducing a small positive constant, ϵ in the denominator of the velocity field given by (4.15). The simulations for a straight 2D channel was reported in Venkatesan (1992a). The flow was studied in the periodic box of size 5×1. Since the function *Coth z* reaches an asymptotic value rapidly as $x \to \infty$, (4.15), was simplified to,

$$u - iv = \sum_{k=1}^{N} \sum_{l=-1}^{l=1} \frac{\gamma_k}{4i} \{ \, coth[\frac{\pi}{2}(z - z_k - ld)] \; + \; coth[\frac{\pi}{2}(z - z_k^* - ld)] \, \} \tag{4.19}$$

The vortices were assumed to have equal strengths and were initially distributed such that the parabolic velocity profile was approximately produced. The vortices were then perturbed slightly and subsequently their movements were tracked. As soon as the vortices cross the downstream or upstream boundary of the box, they were shifted back by a period d ($= 5$), so that the points remained within the box (note that this leaves the velocity field unchanged, since we consider periodic strings).

The mean stream velocity U (in the x-direction) and the r.m.s values of the fluctuating velocity components u', v' were calculated by averaging over narrow strips extending in the x-direction:

$$\begin{aligned} U &= \frac{1}{M_y} \sum_{\{i:\ y-\delta y<y_i<y+\delta y\}} \frac{dx_i}{dt} \\ u'^2 &= \frac{1}{M_y} \sum_{\{i:\ y-\delta y<y_i<y+\delta y\}} \left(\frac{dx_i}{dt}\right)^2 - U^2 \\ v'^2 &= \frac{1}{M_y} \sum_{\{i:\ y-\delta y<y_i<y+\delta y\}} \left(\frac{dy_i}{dt}\right)^2 \end{aligned} \tag{4.20}$$

The evolution (figure 4.1) is studied in a frame moving with average velocity of the flow and here the fact that vorticity is a Galilean invariant turns out useful. However, the desingularization procedure produced some unphysical effects. For example it induced a mild attraction between vortex strings with opposite orientation and a repulsion between strings with same orientation. Also the vortex core associated with such a desingularization process was found to be too large and this in turn we would expect would cause difficulty when we study the convergence of the scheme. Although in the limit as $\epsilon \rightarrow 0$, the desingularised vortices reduce to point vortices, we found that as the number of point vortices were increased, the results was found to be highly sensitive to the precision of the calculations, which we believe is an undesirable property for any scheme. But despite all these drawbacks, the results, which are reported in figure 4.1, resemble the characteristics of real flow in some respect. The initial waves introduced in the flow overturn and breakdown into turbulence (see figure 4.1). This overturning of waves is in agreement with study by Pullin (1981) on the evolution of a vortex sheet over a flat plate. The mean and r.m.s values of the velocities were found to be reasonable agreement with reality. An increase in the value of the desingularization parameter, ϵ provided greater stability to the flow, which resembles the viscous effects to some extent. However to gain insight into the real flow mechanisms it is necessary to produce faithful simulations, so we consider better schemes.

To develop better schemes we use vortex blobs instead of point vortices. Hald (1979) has shown that using vortex blobs, one can develop methods which produces quadratic

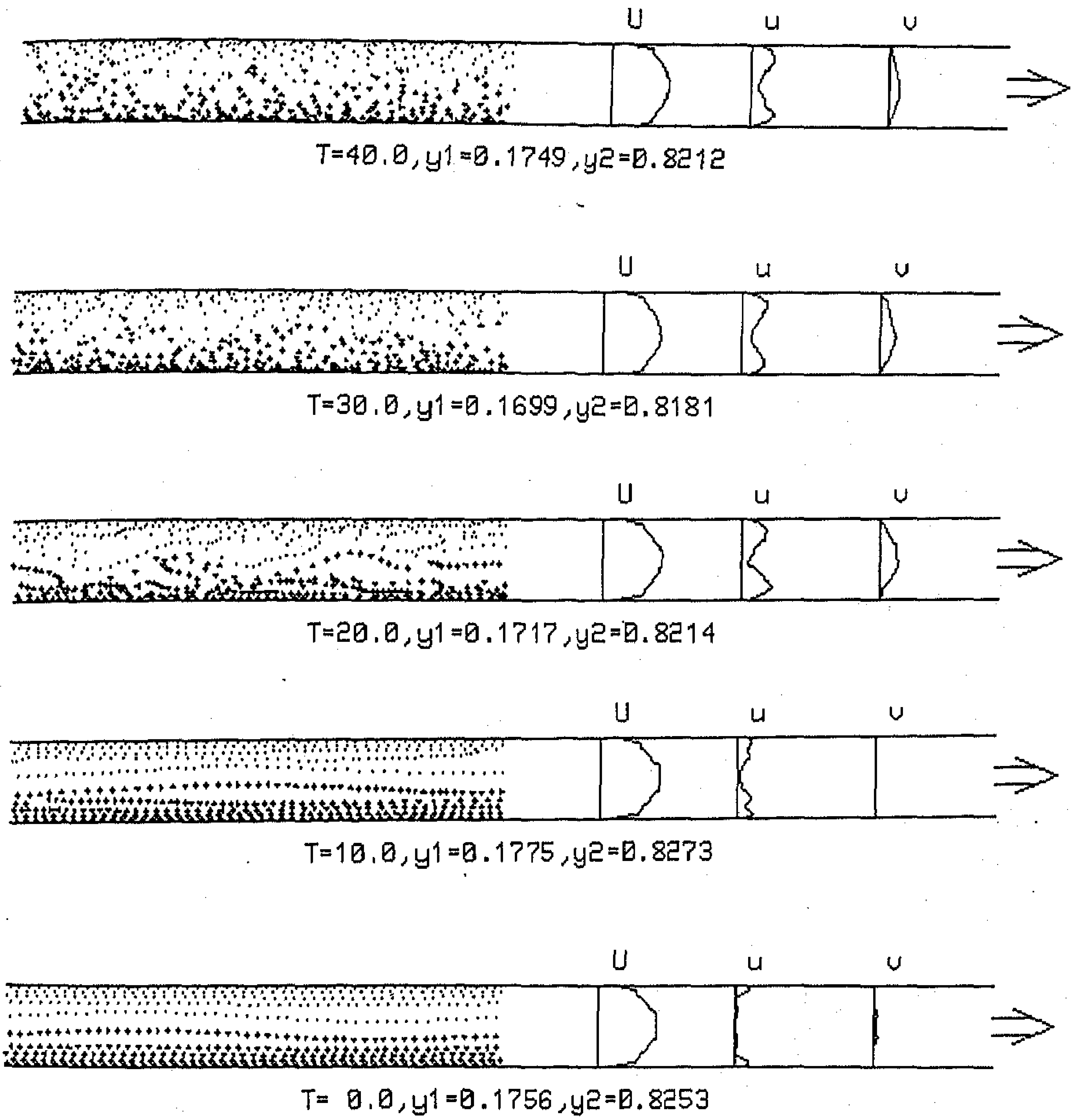


Figure 4.1: Evolution of vortices in a 2D channel-desingularization method

convergence to the solution of the Euler's equation, while with point vortices, the convergence (point-wise) may not even be guaranteed (Hou (1991)). Beale and Majda (1982) have shown how very high order of convergence can be achieved with little numerical overheads. Here we shall use that sixth-order velocity kernel given by,

$$\gamma(x,y) \;=\; 1.0 \;-\; \frac{1}{3}[exp(-\frac{r^2}{4h^2}) + 6exp(-\frac{r^2}{2h^2}) - 8exp(-\frac{r^2}{h^2})] \qquad (4.21)$$

where $\gamma(x,y)$ is the total circulation inside a circle of radius, $r = (x^2+y^2)^{\frac{1}{2}}$ and, is related to the vorticity $\xi(x,y)$, by the relation,

$$\gamma(x,y) = \int_0^r 2\pi r \xi(x,y)dr \qquad (4.22)$$

With this cut-off function, it is possible to obtain convergence of high order (Beale and Majda (1985)). To incorporate the core (4.21) to the string of point vortices we subtract the nearest three singular contributions from the expansion of (4.13), and add the corresponding vortex blob at the respective points. The string configuration so obtained is described in figure 4.2.

We follow then closely the theory of Majda and Beale (1982) and use vortex blobs arranged in a regular lattice. In a Lagrangian simulation the vortices do not remain in such a lattice, but move to a new locations. We however found that the two linear moments (which are constants of motion),

$$\begin{aligned} K_x &= \sum_{k=1}^{N} \gamma_k x_k \\ K_y &= \sum_{k=1}^{N} \gamma_k y_k \end{aligned} \qquad (4.23)$$

remain more or less constant for some interval time (see the values of K_x, K_y displayed in the plates in 4.3 and 4.4).
In fact, as long as the vortices continue to be uniformly distributed no large errors occur. The computations produced (in a frame moving with the average velocity of flow) for the short boxes are displayed in figure 4.3. The width to length ratio for short boxes is $1:5(i.e., d=5)$.

The above results (figure 4.3) were obtained with a spatial resolution : $h=0.1$, using 500 vortices. Results were also obtained with a spatial resolution: $h=0.05$, where 2000 vorticies were used. They are reported in figure 4.4. The initial mean velocity

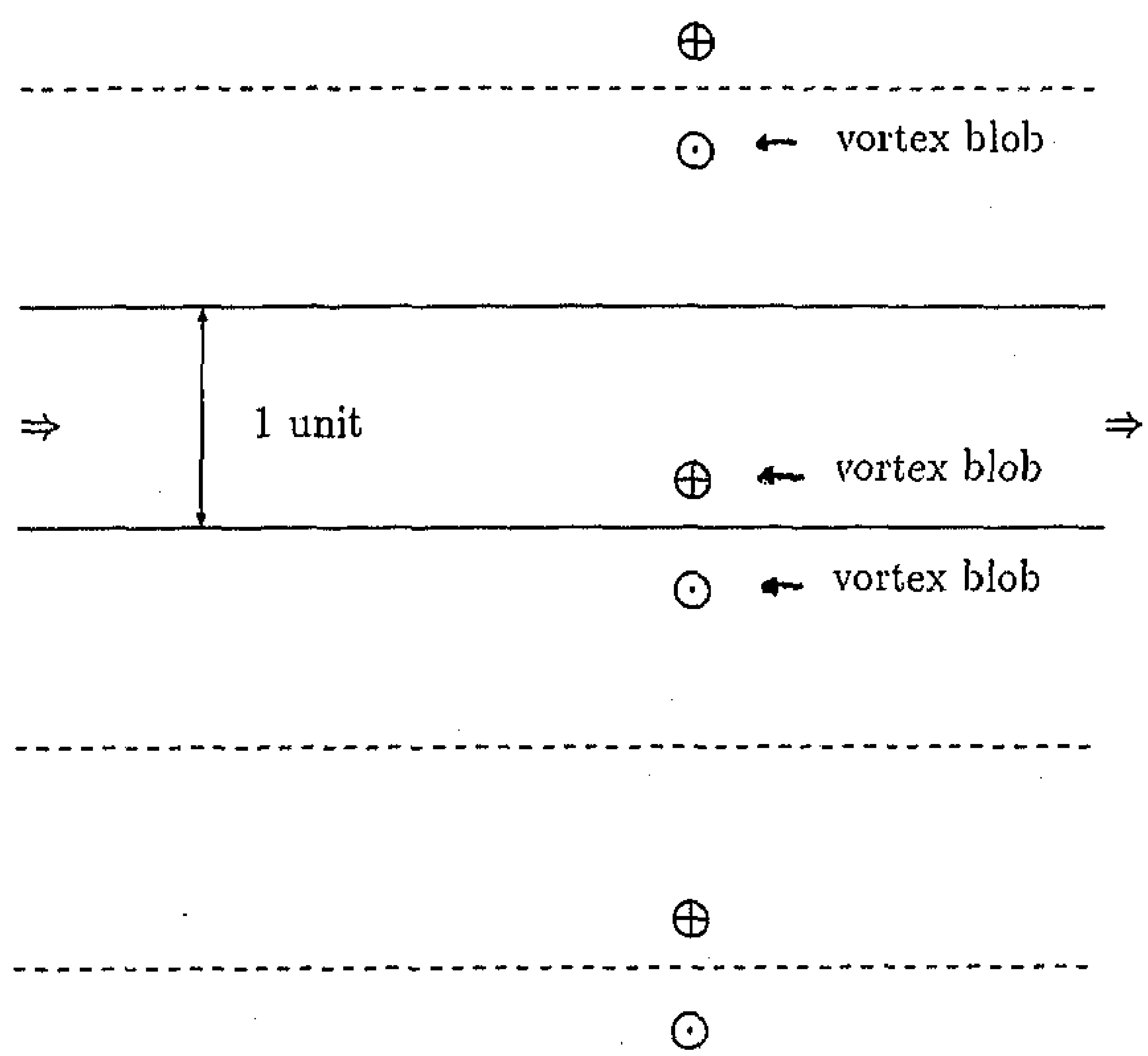


Figure 4.2: Mirror images of vortices-the string configuration

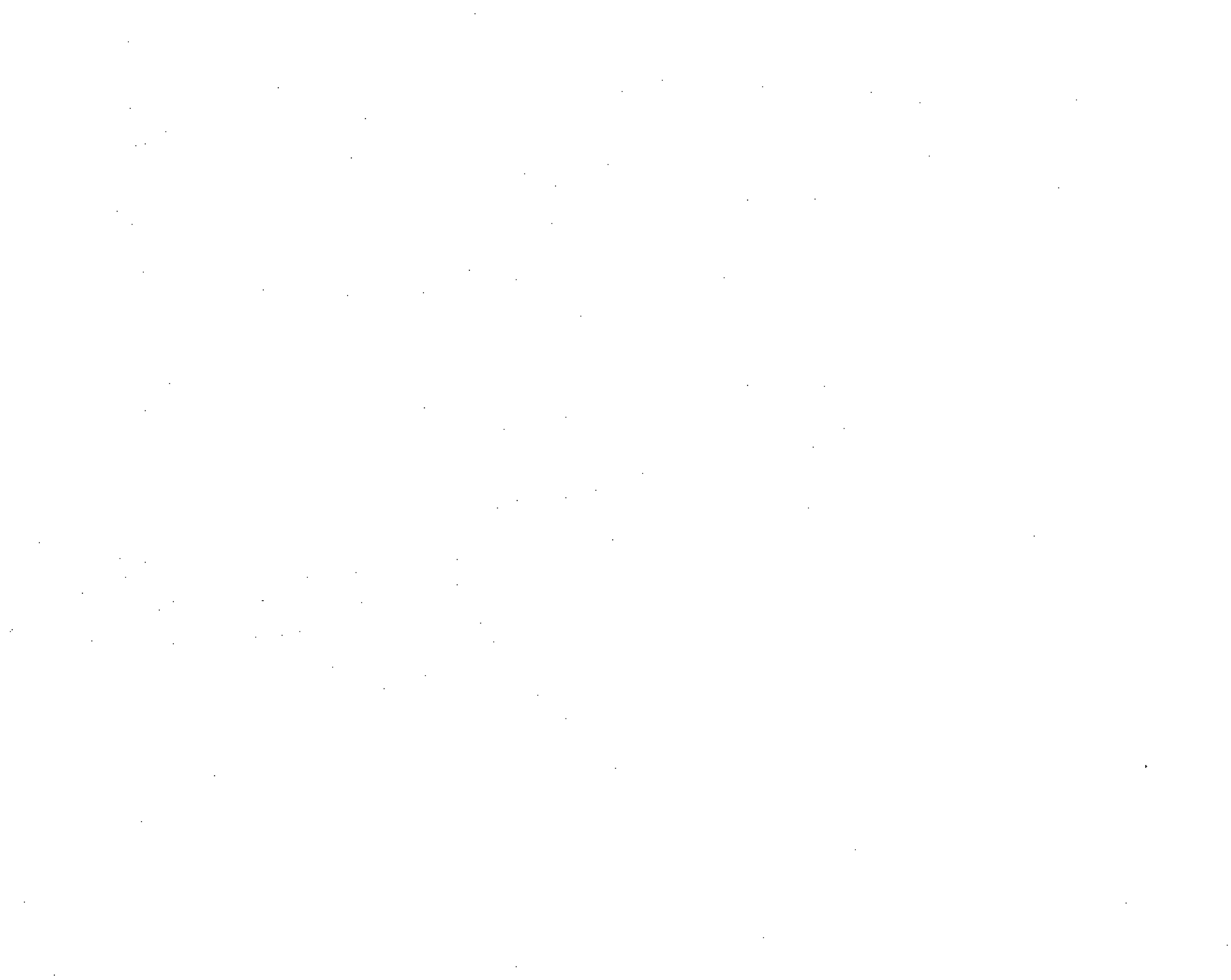
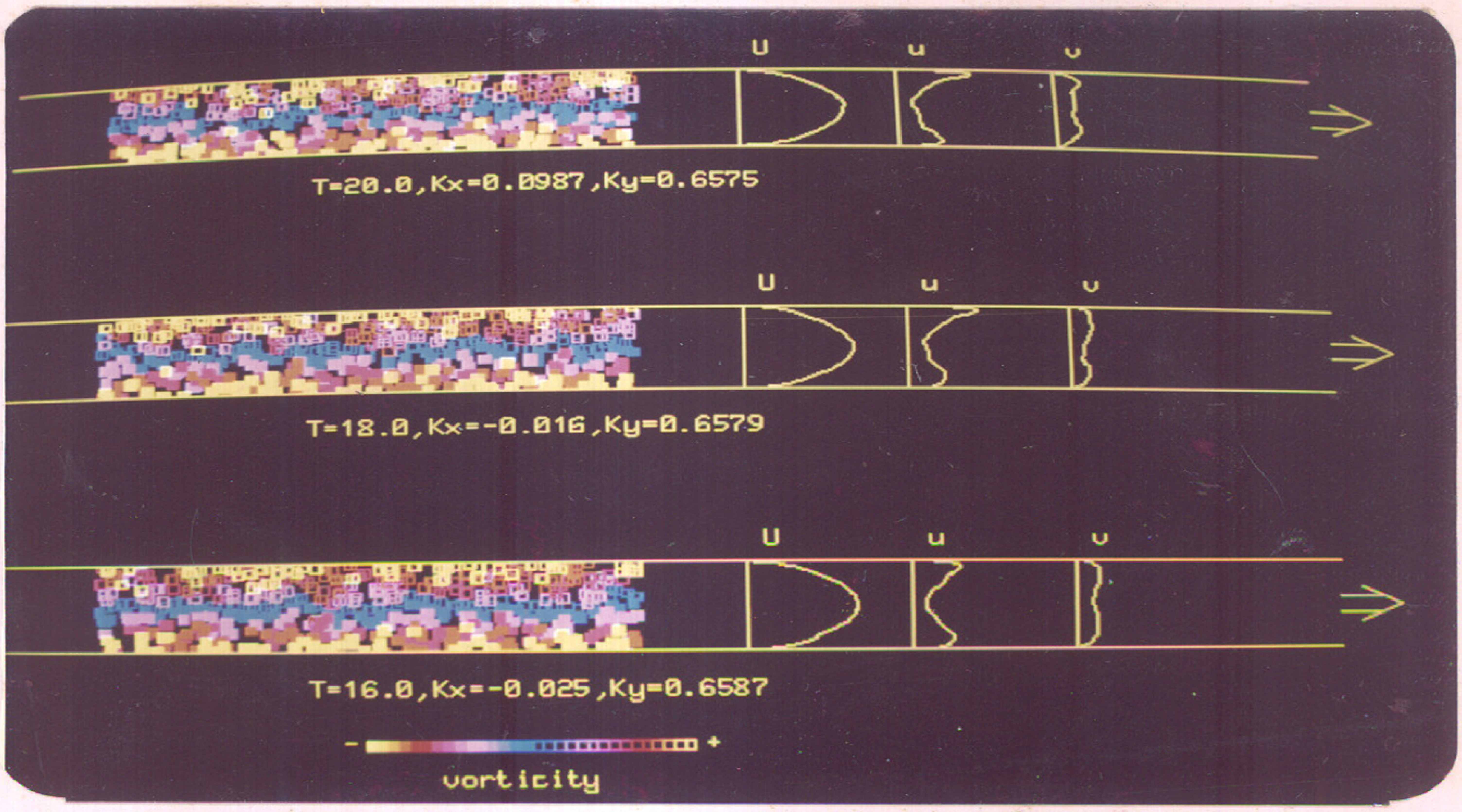
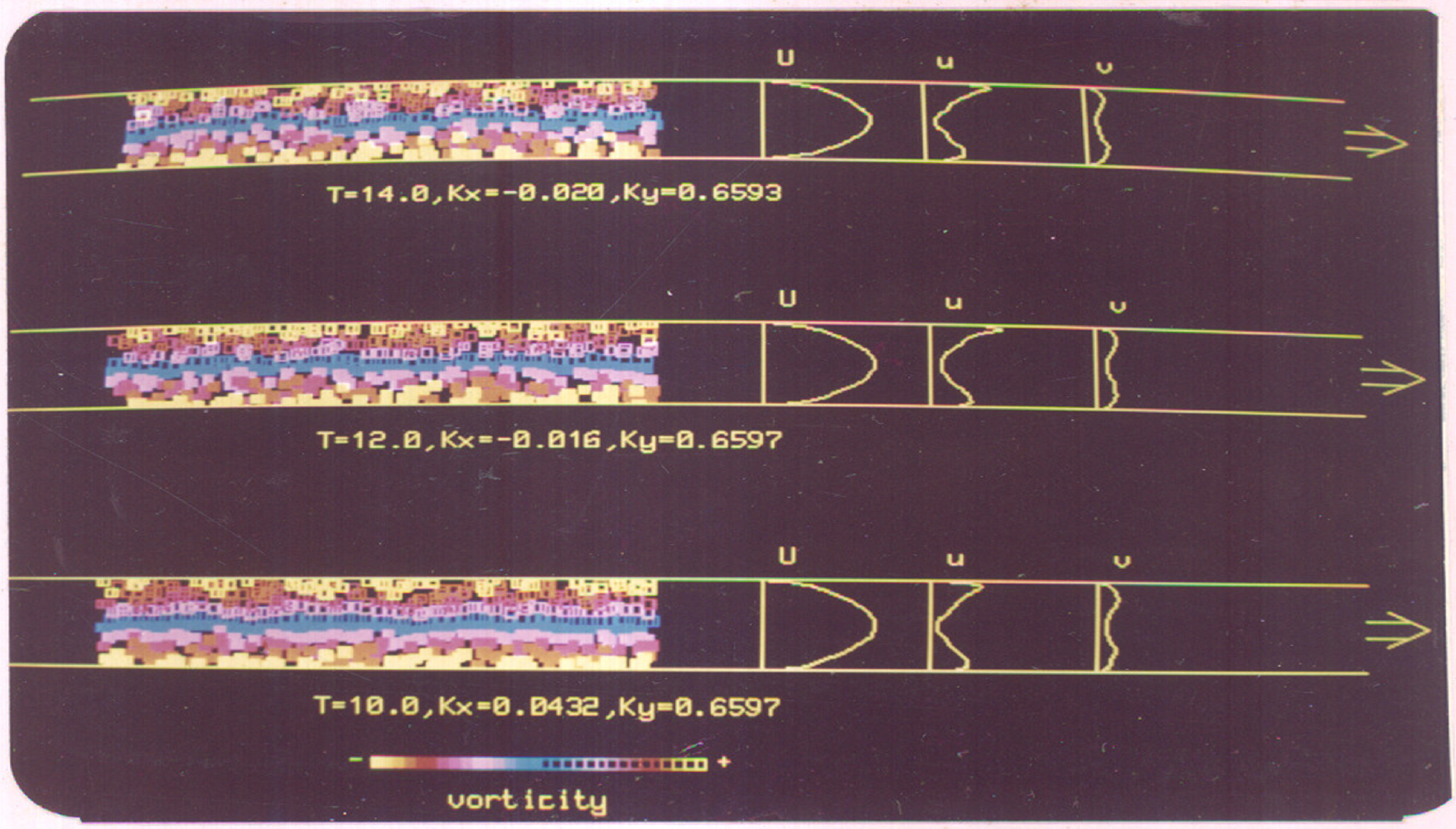


Figure 4.3: The evolution of vorticity in a 2D channel-Simulations in a short box ($d = 5$) with a coarse grid (500 vortices)


U
u
v
T=20.0,Kx=0.0987,Ky=0.6575
U
u
v
T=18.0,Kx=-0.016,Ky=0.6579
U
u
v
T=16.0,Kx=-0.025,Ky=0.6587
-
+
vorticity


U
u
v
T=14.0,Kx=-0.020,Ky=0.6593
U
u
v
T=12.0,Kx=-0.016,Ky=0.6597
U
u
v
T=10.0,Kx=0.0432,Ky=0.6597
-
+
vorticity

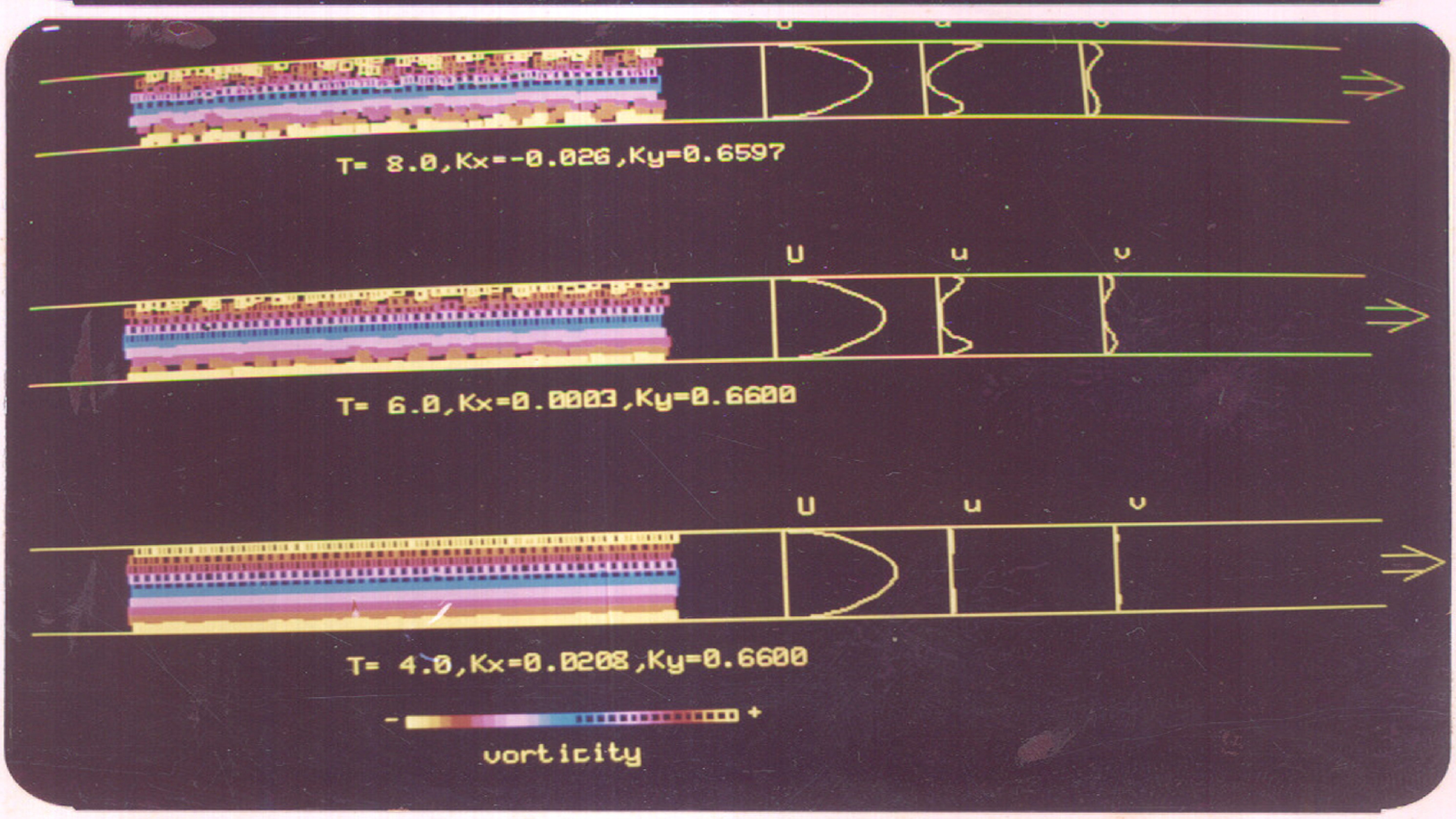
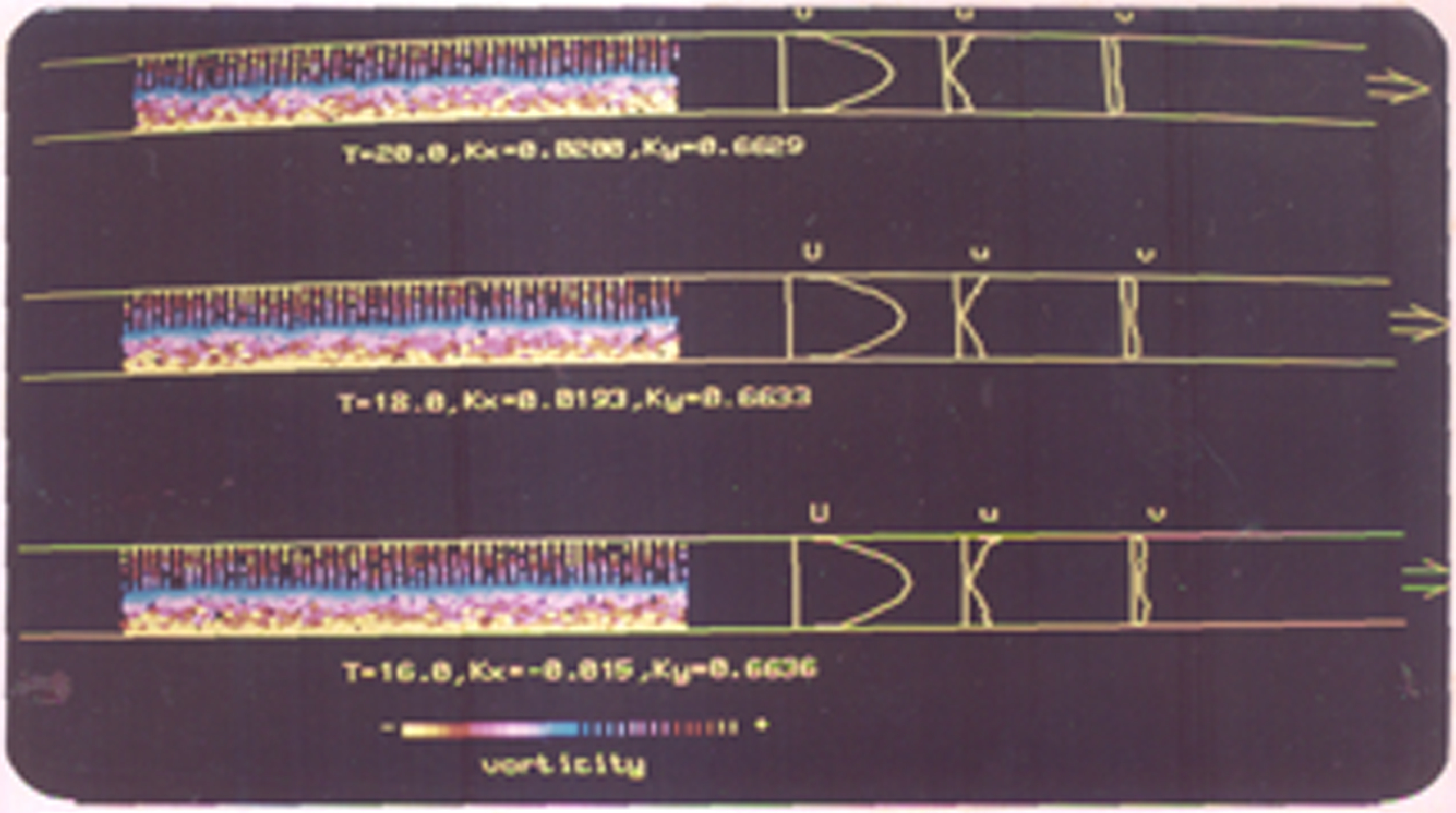
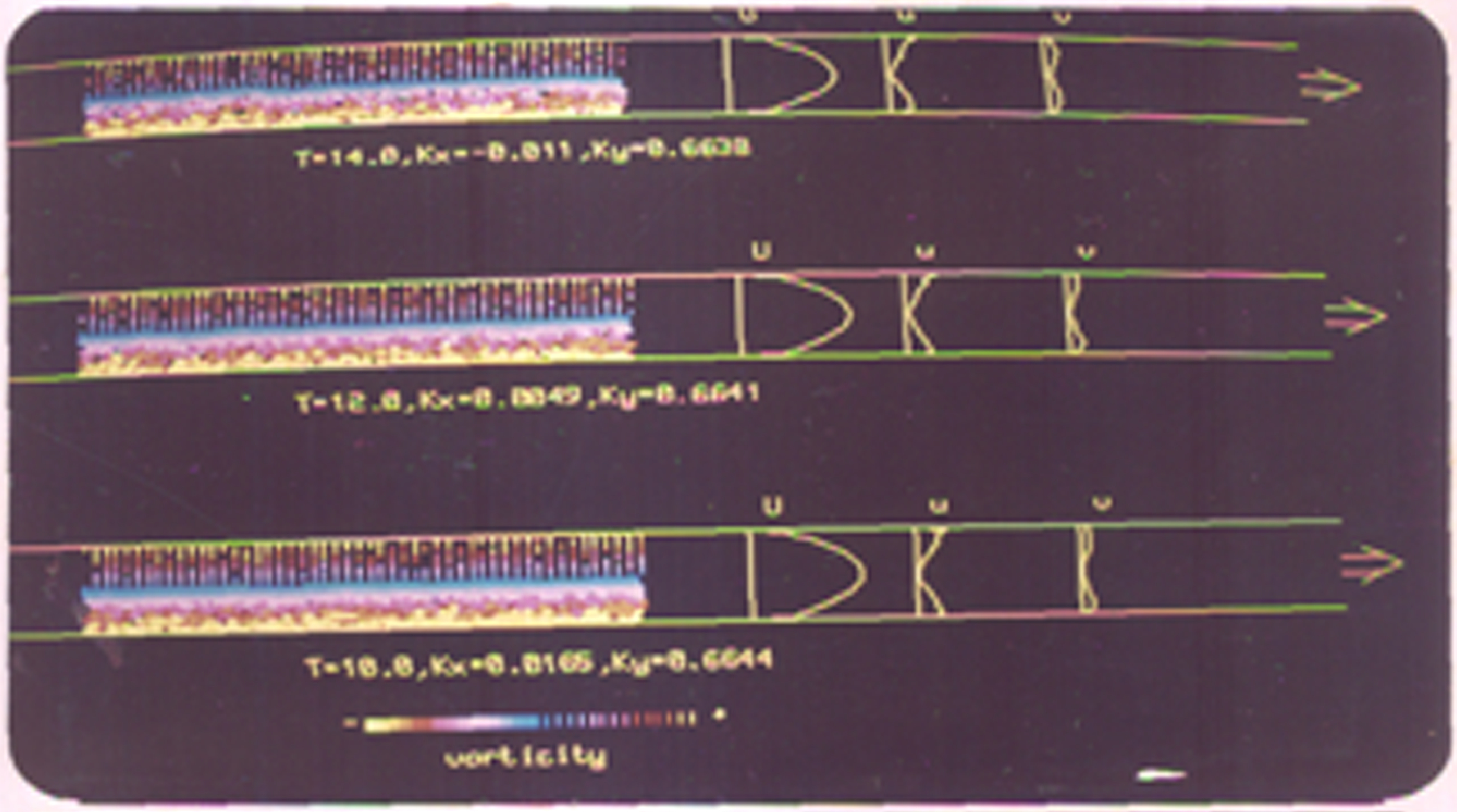
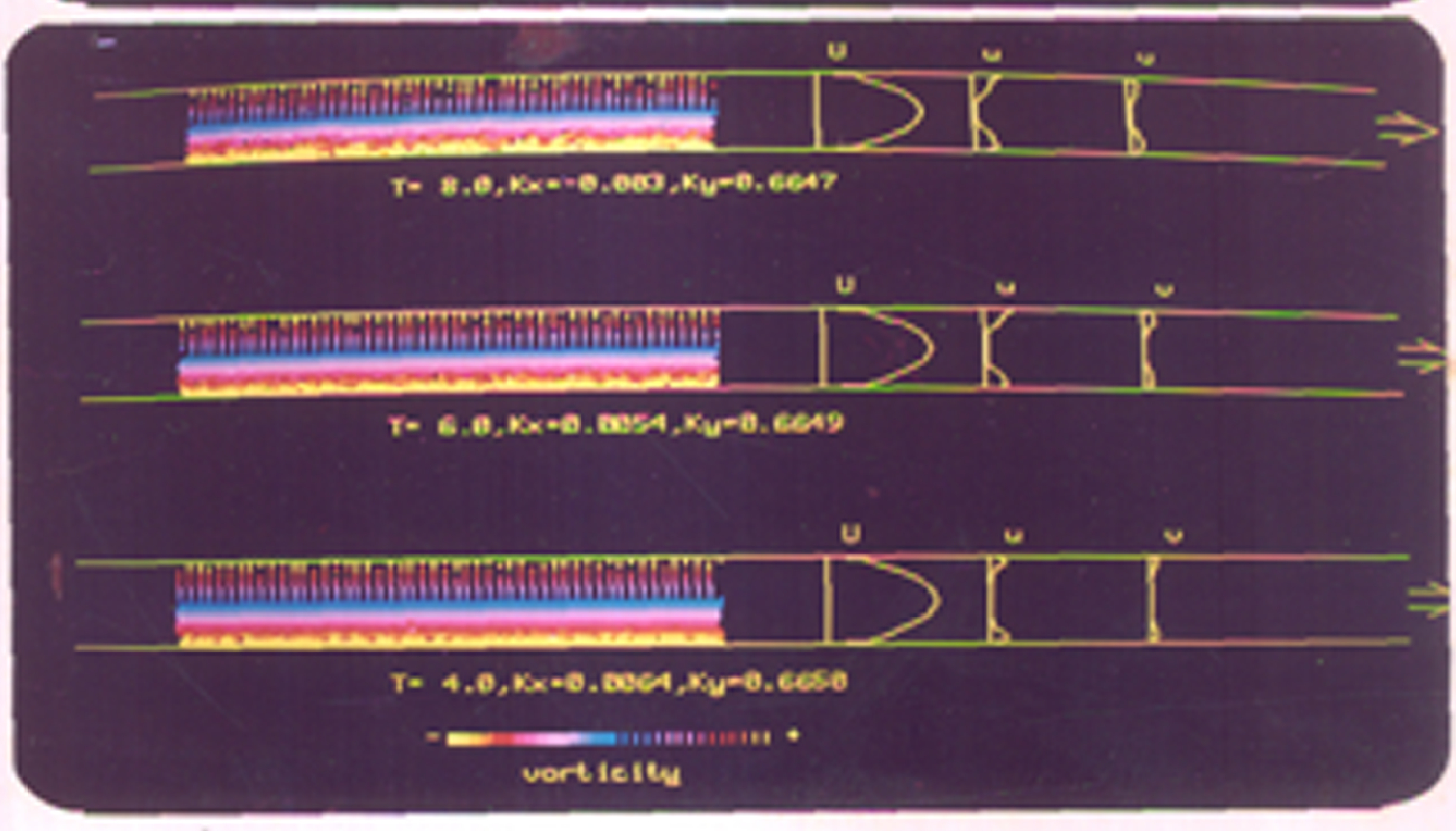

T= 8.0,Kx=-0.026,Ky=0.6597
U
u
v
T= 6.0,Kx=0.0003,Ky=0.6600
U
u
v
T= 4.0,Kx=0.0208,Ky=0.6600
-
+
vorticity

Figure 4.4: The evolution of vorticity in a 2D channel-Simulations in a short box ($d = 5$) with a fine grid (2000 vortices),continued...


T=20.0,Kx=0.0200,Ky=0.6629
T=18.0,Kx=0.0193,Ky=0.6633
T=16.0,Kx=-0.015,Ky=0.6636
vorticity


T=14.0,Kx=-0.011,Ky=0.6638
T=12.0,Kx=0.0049,Ky=0.6641
T=10.0,Kx=0.0105,Ky=0.6644
vorticity


T= 8.0,Kx=-0.003,Ky=0.6647
T= 6.0,Kx=0.0054,Ky=0.6649
T= 4.0,Kx=0.0064,Ky=0.6650
vorticity

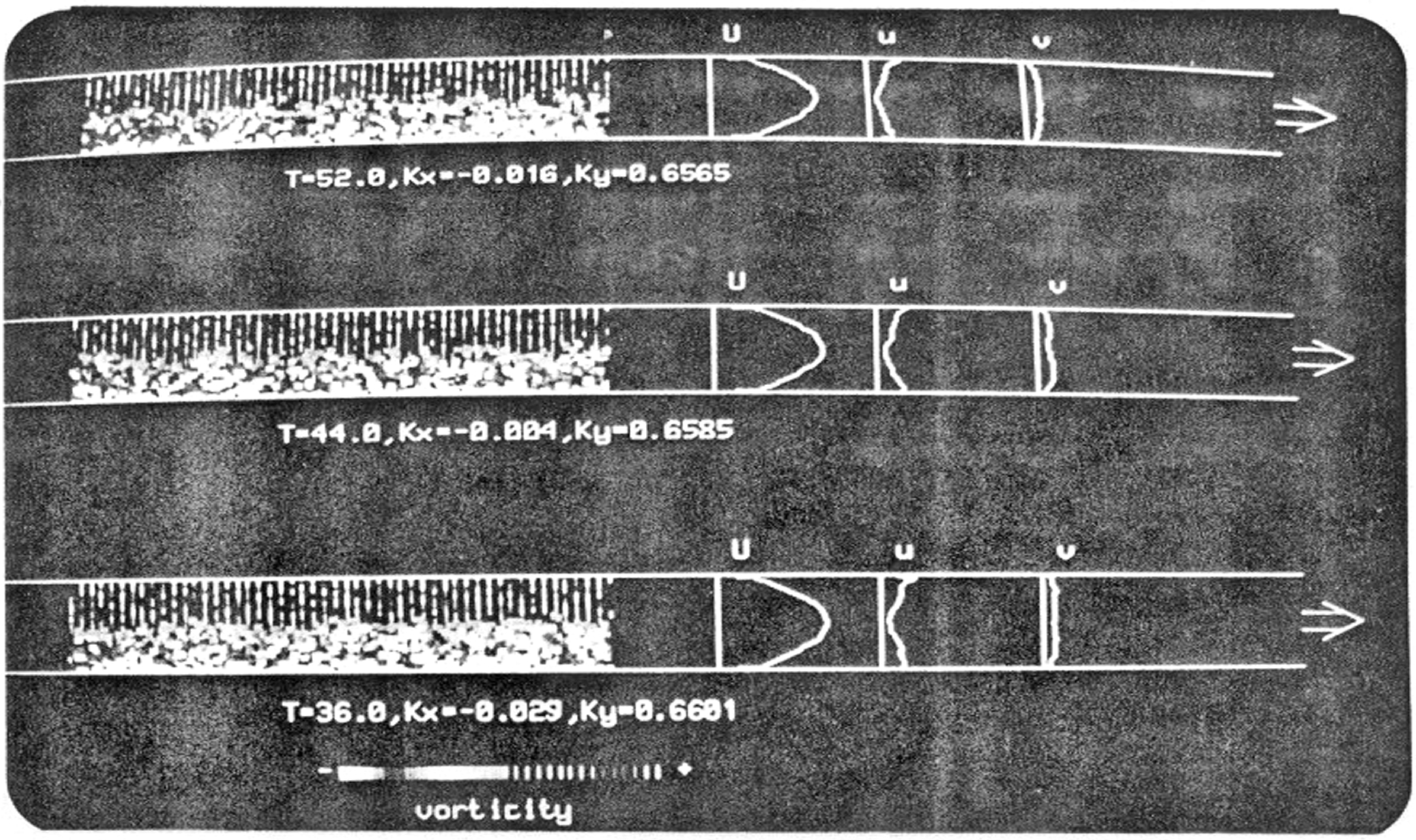


Figure 4.4: The evolution of vorticity in a 2D channel-Simulations in a short box ($d = 5$) with a fine grid (2000 vortices)

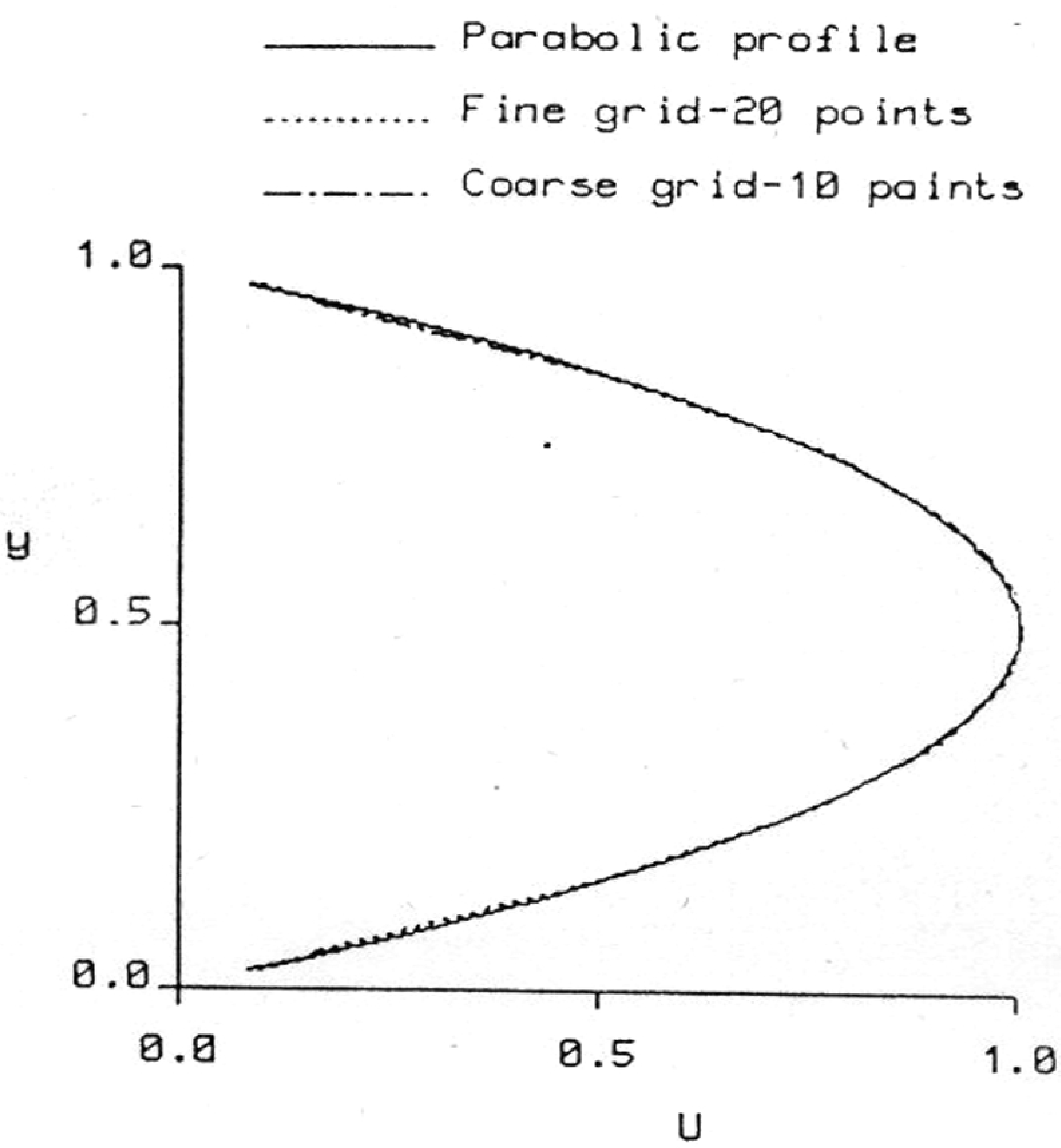


Figure 4.5: The initial mean velocity profiles

profile generated by the discrete vorticity method (for the *sixth-order vortex core*) is shown in figure 4.5.

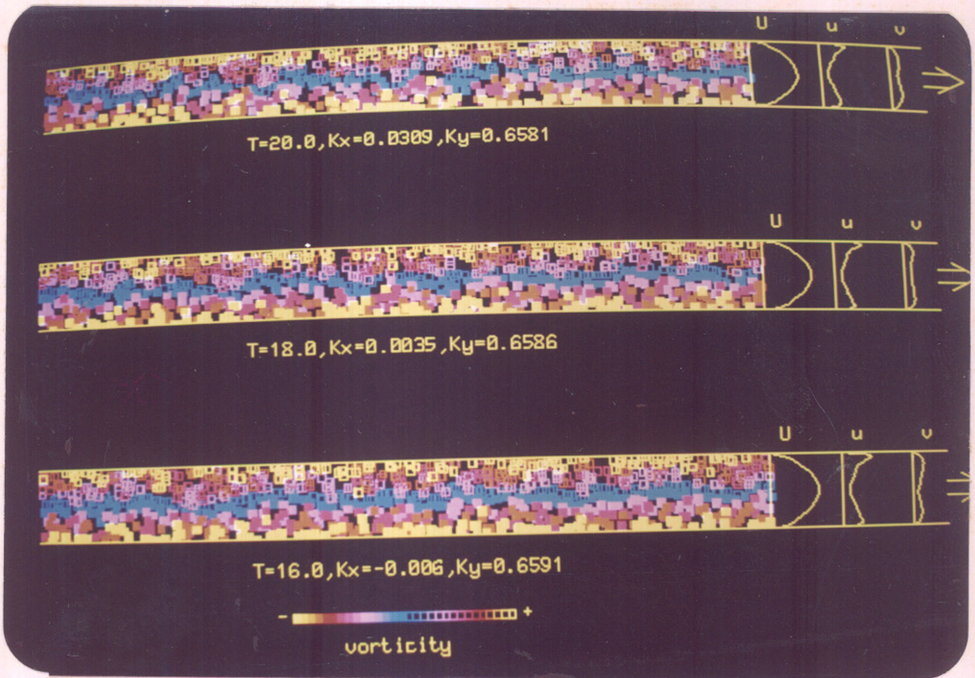
We tested schemes of various orders for time marching and found that the second order scheme is approriate. However some caution is required in using higher order methods in time. Although it is true that, if the velocity field is initially analytic in space then for all time it continues to be analytic in space and analytic in time (see Bardos et.al. (1976) for a proof for the periodic case), the radius of convergence of Taylor series expansions in time, is found to decrease faster than an exponential of a exponential (Bardos (1994) private communications). However, it is approriate to mention here that Hald (1987) has shown that the path taken by a fluid particle is regular (C^∞).

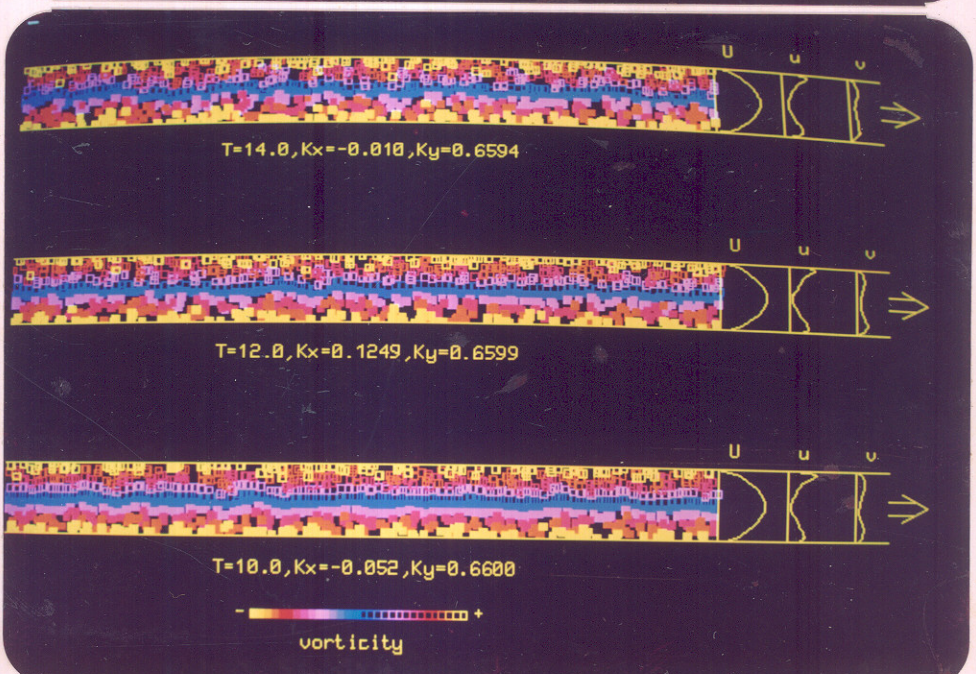
Arnold (1966) showed that the solutions of the Euler's equations develop exponential instability on a torus (which is exactly the case considered here). So a numerical study of the exact path taken by a particle for a long time seems to be not fruitful. However we may expect that the overall flow pattern will not vary much. The initial distribution of vortices develops instability a little away from the wall and soon the inertial waves are produced. As discussed in the first section of this chapter, these are area preserving flows and the motion could develop waves. The wavy layers overturn and soon break down, producing complicated motions. Such complicated motions develop due to the shearing of the wavy layers. Weiss (1981) has shown how in a hyperbolic mode, vorticity could be sheared rather strongly. We see such action here, especially a bit away from the wall. There was the possibility that the instability produced here could be a peculiarity of the discretisation considered here, so we carried out computations taking strengths of vortices equal. In this approach the vortices are not uniformly spaced. We found that the results obtained were consistent with our results. However it is to be mentioned that in considering vortices with equal strength, an inhomogeity is introduced in the computations, since certain zones were there are less number of vortices the results the flow field is quite smooth, while in zones with more vortices, the flow field is not smooth enough. To offset this it was found necessary to constantly change the core width depending on the nearest neighbor-a cumbersome procedure. Moreover the method with equal strengths do not treat all the scales uniformly. The method we have used here, of considering a vortex blobs in a regular lattice, would be quite approriate in simulating fluid flows which are known to produce a large range of scales.

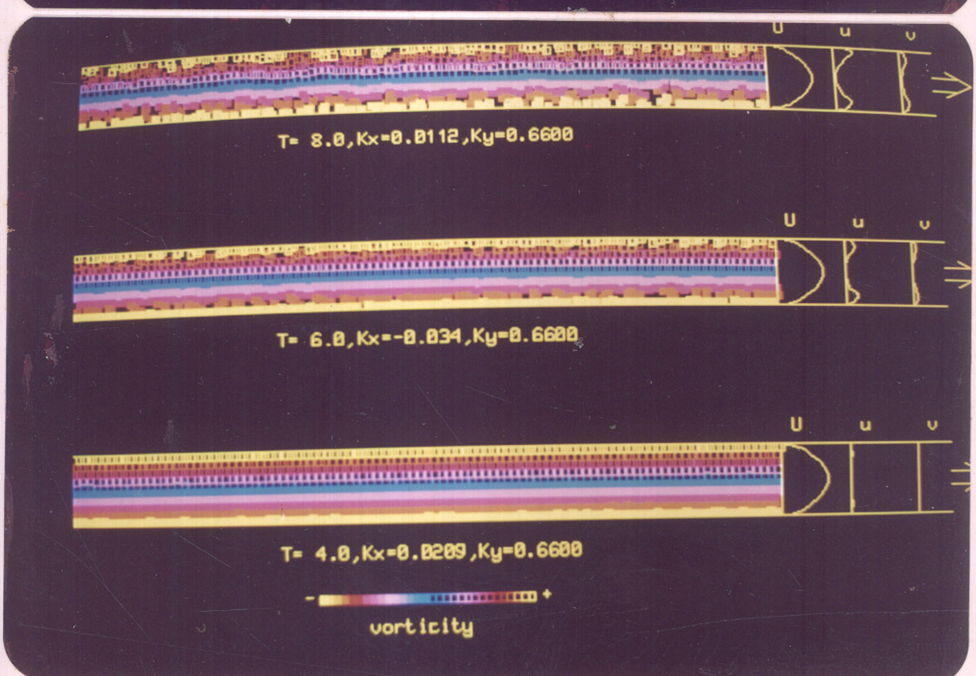
In the simulations of the 2D channel flow reported above, we have been able to capture some of the moving wave patterns quite similar to that reported in Jimenez (1990). The results obtained by using both coarse and fine resolutions (figures 4.3 & 4.4), indicate the development of the instability of the flow field; the process being quicker in the case with coarser resolution. We find that the evolution of the flow field depends strongly on its initial state. As the initial deviation from the parabolic profile is larger in the case with coarse resolution, the development of instability is found to occur quickly. The chaotic features observed in the above results cannot be attributed to the errors involved in the numerical scheme since the time steps used were sufficiently small (the time scale used here is constructed from the maximum flow velocity and the width of the channel) . The qualitative pictures, such as the formation and break-down of wavy layers, remain the same in all the simulations (figure 4.3 & 4.4). So far, we have discussed the results obtained using the computational box of size $d = 5$. It is to be mentioned that simulations were also carried out with boxes of size, $d < 5$, but it was found that the results were not satisfactory since the streamwise variations could not develop sufficiently within such a short box. In figure 4.6, the results of the simulations done in a box of size $d = 10$, with a coarse resolution are displayed.

For short time studies,the qualitative agreement between figures 4.6 and 4.3 suggests that the box of size $d = 5$ is quite sufficient to capture the important mechanisms of the flow. In this context we must discuss, the simulations of 2D channel flows reported by Jimenez (1990). In his work, simulations were carried out in short and long boxes, by using the Fourier expansions (Spectral method) . He found that the waves are sustained in short boxes,while they break-down quickly in longer boxes. So the results they obtained with the shorter boxes differed from that obtained with longer boxes. It is worth mentioning here that in the simulations of Jimenez (1990), the boxes are fixed with respect to the wall, whereas in our case, the boxes are considered as moving with the constant average velocity of the flow. Since the periodicity considered here is a moving wave periodicity, it takes rather long for the waves to feed on the periodicity. The simulations considered by Jimenez (1990) is accurate at the wall, whereas the simulations considered here are accurate away from the wall. Narasimha (1989) suggested that an outer-flow expansion perhaps should be used, since it is well known that the overall flow characteristics are insensitive to wall roughness. The expansions in terms of the vortex strings which used here, is an outer-flow expansion. It is to be mentioned here that Jimenez (1990), carried out his simulations on a very powerful computers, whereas the computing resources used here is quite limited. All the computations reported here were done on a single processor VAX 8650 machine. The results reported in figure 4.3 took roughly 2 hours of CPU time. So it was difficult to raise the accuracy by including more number of vortices,atleast to check the consistency of our results.

Figure 4.6: The evolution of vorticity in a 2D channel-Simulations in a long box ($d = 10$) with a coarse grid (1000 vortices)


U u v
T=20.0,Kx=0.0309,Ky=0.6581
U u v
T=18.0,Kx=0.0035,Ky=0.6586
U u v
T=16.0,Kx=-0.006,Ky=0.6591
- +
vorticity


U u v
T=14.0,Kx=-0.010,Ky=0.6594
U u v
T=12.0,Kx=0.1249,Ky=0.6599
U u v
T=10.0,Kx=-0.052,Ky=0.6600
- +
vorticity


U u v
T= 8.0,Kx=0.0112,Ky=0.6600
U u v
T= 6.0,Kx=-0.034,Ky=0.6600
U u v
T= 4.0,Kx=0.0209,Ky=0.6600
- +
vorticity

The same calculations as reported in figure 4.3,with 2000 vortices would for example require 16 hours of CPU time. However there was one way to make a reduction in CPU time. This was at the algorithm level. Instead of working with single precision arithmetic we worked with a lesser precision. This can for example done by working with integers rather than real numbers. In figure 4.4, the calculations done with integer variables is reported for 2000 vortices. It was found that working with integers rather than single precision arithmetic does not seem to affect the results for a few initial time steps. We also found that calculating the velocity values using look up tables reduces the computational effort. This technique of using look-up tables for velocity influence function seems to produce quite consistent results. The results produced seems to be insensitive to any finer interpolations of the influence-functions, indicating that there may be further scope for reduction in computational effort.

The (spatially averaged) r.m.s values of u, v and uv, are reported for different times in figure 4.7 and 4.8 for the coarse and fine grids respectively. It is found that although features like Tollmien-Schlichting waves are present, the spatially averaged values of the velocitites seem to become more or less steady. But the same cannot be said about their temporal averages, since these doubly periodic vortices do not follow ergodicity, i.e.,time averages and space averages are not the same (Weiss and McWilliams (1991)). So it may not be possible to discern the final equilibrium state using a statistical theory such as the one introduced by Roberts and Somerlia (1990). To show this conclusively here, we have to run the simulations for long time. However the above simulations has certain drawbacks. Despite the fact that initially we use equally spaced vortices, in a Lagrangian scheme the vortices tend percolate (quite similar to the viscous fingering process) and as soon as the distribution becomes less uniform, the scheme tends to loose its accuracy. So it seems necessary to redistribute the vorticity strengths to the points of the lattices. This can be done in many number of ways. One way is to determine the vorticity by calculating the circulation around each square, considering the velocity at the corners of the squares ; here the no-slip condition at the wall is taken care by assuming that velocity is zero at the wall. However this results in large errors, since it is grid based. Another, but more efficient way is to consider two-levels of grids, one which is very fine is just used for tracing the movement of the vorticity and one coarse grid which is used for calculating the velocity field. The tracers are also used to determine the new strength of the vortex blobs. Unlike the single grid method in this two-level method the realignment of the strength of the vortices does not kill the small scales. Since the vorticity is linearly related to the velocity field (Biot-Savart's law), the computational effort is also reduced enormously by this technique. Using a

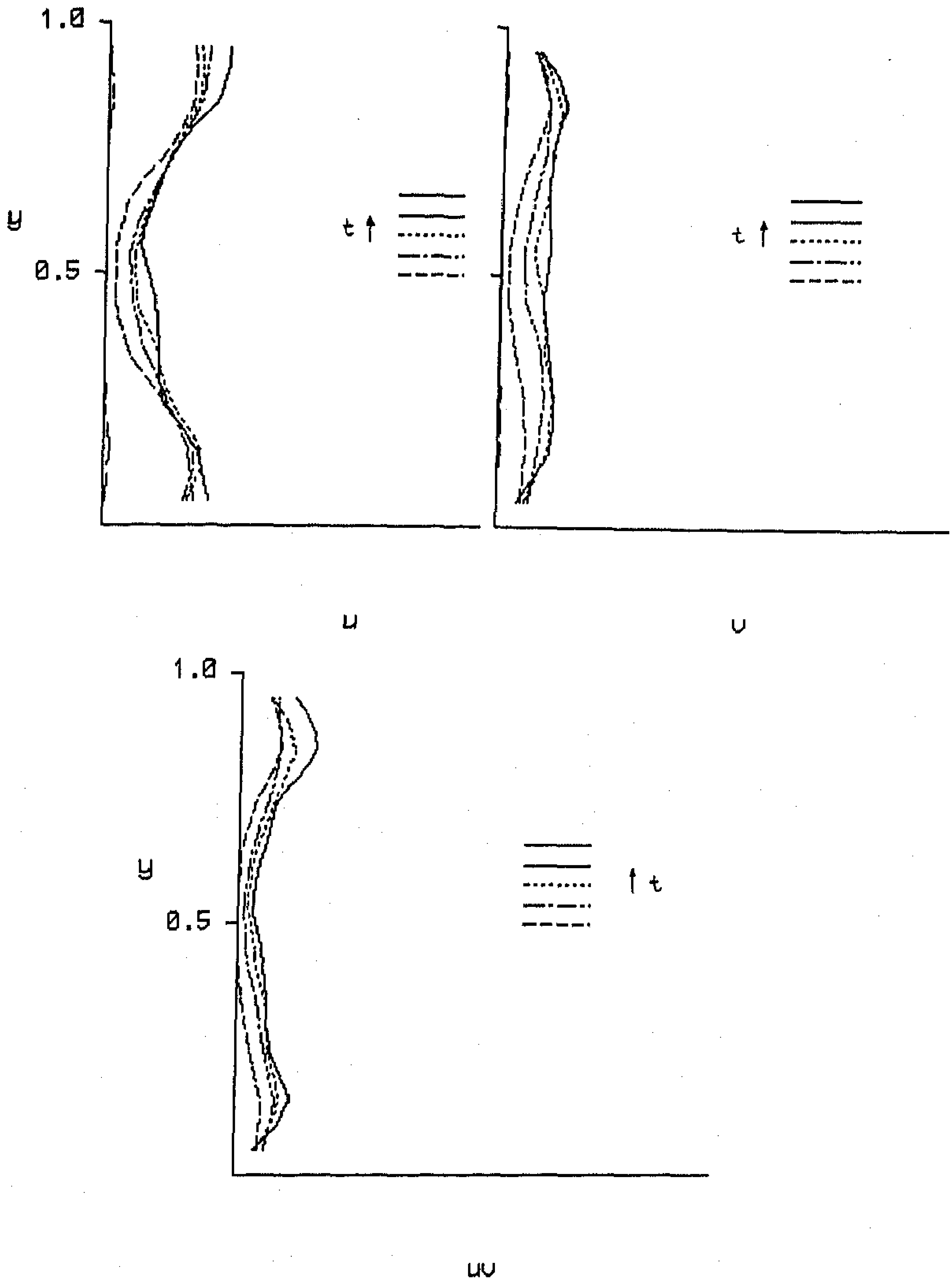


Figure 4.7: The evolution of r.m.s.values in a coarse grid

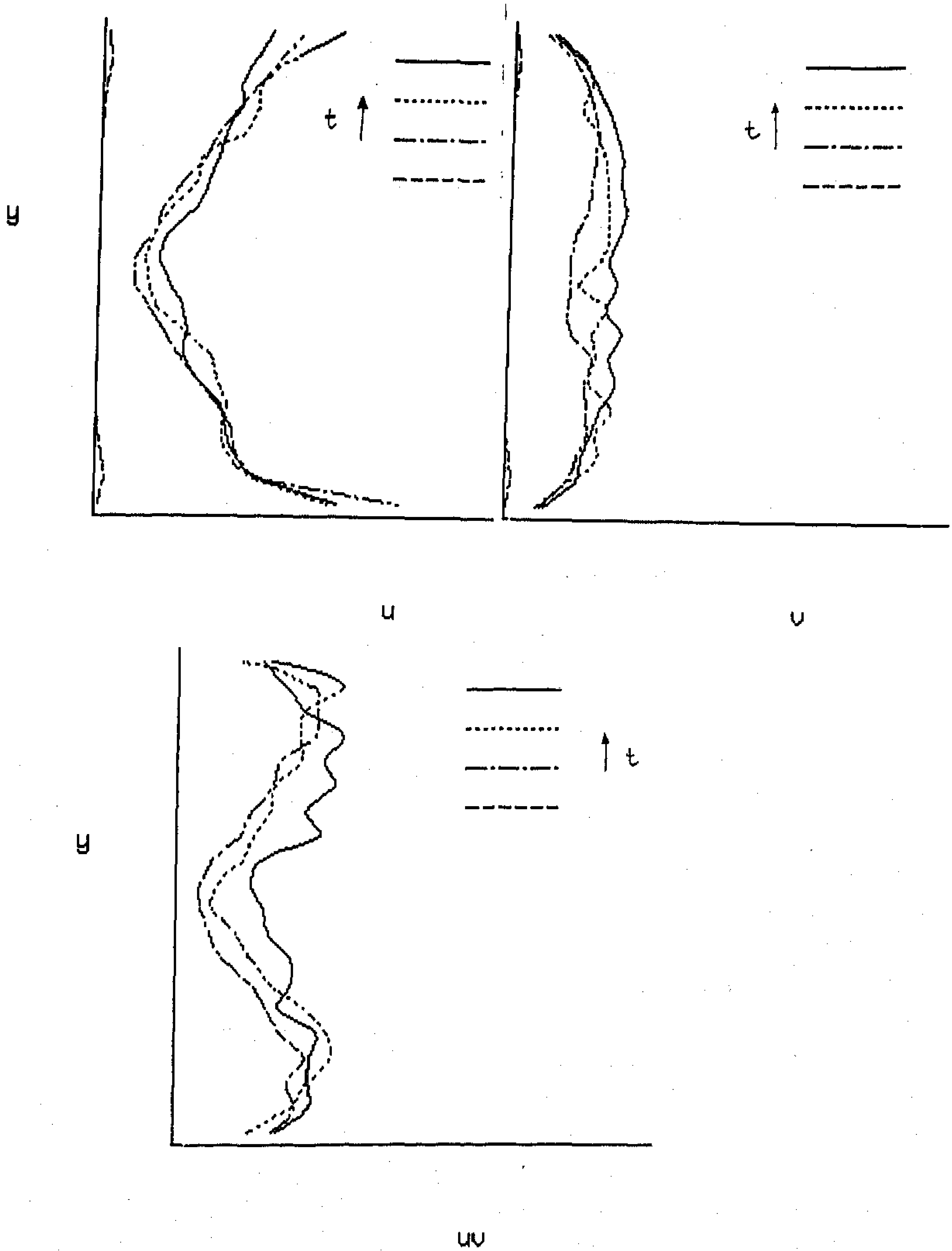
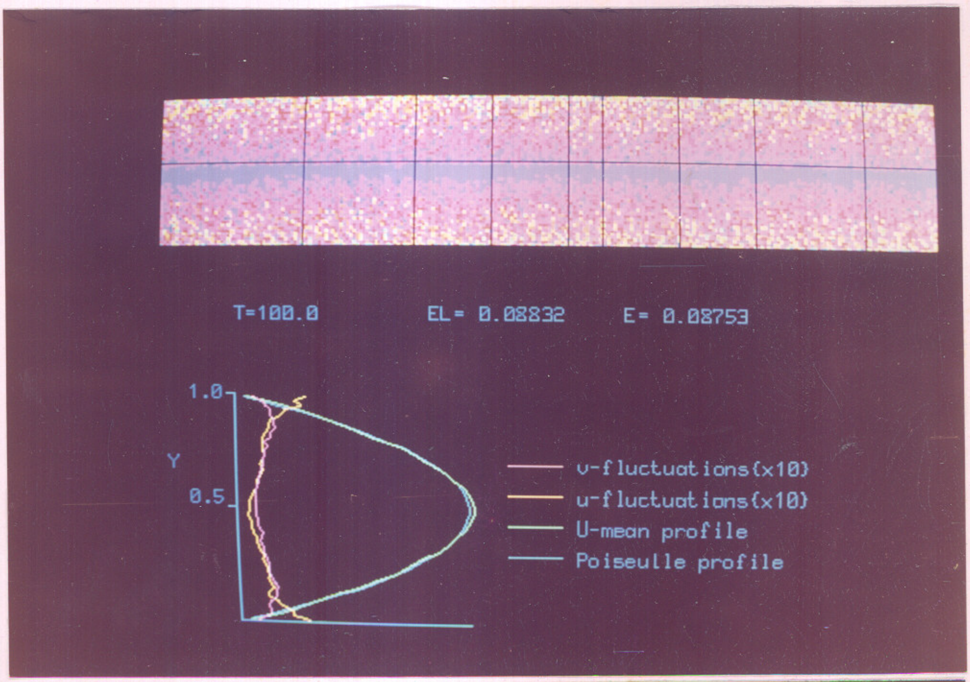


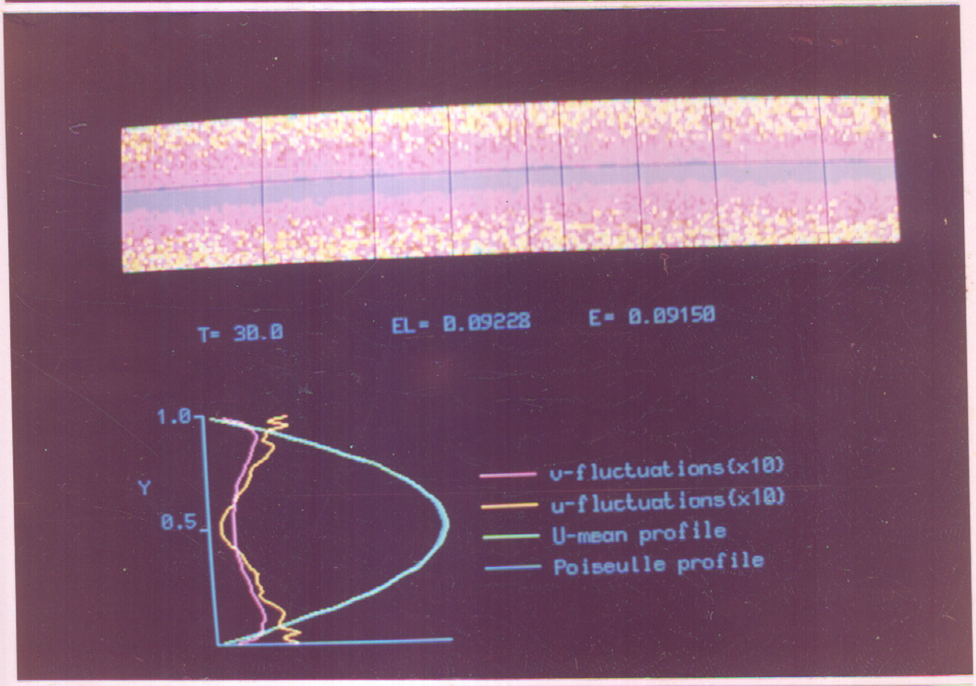
Figure 4.8: The evolution of r.m.s.values in a fine grid

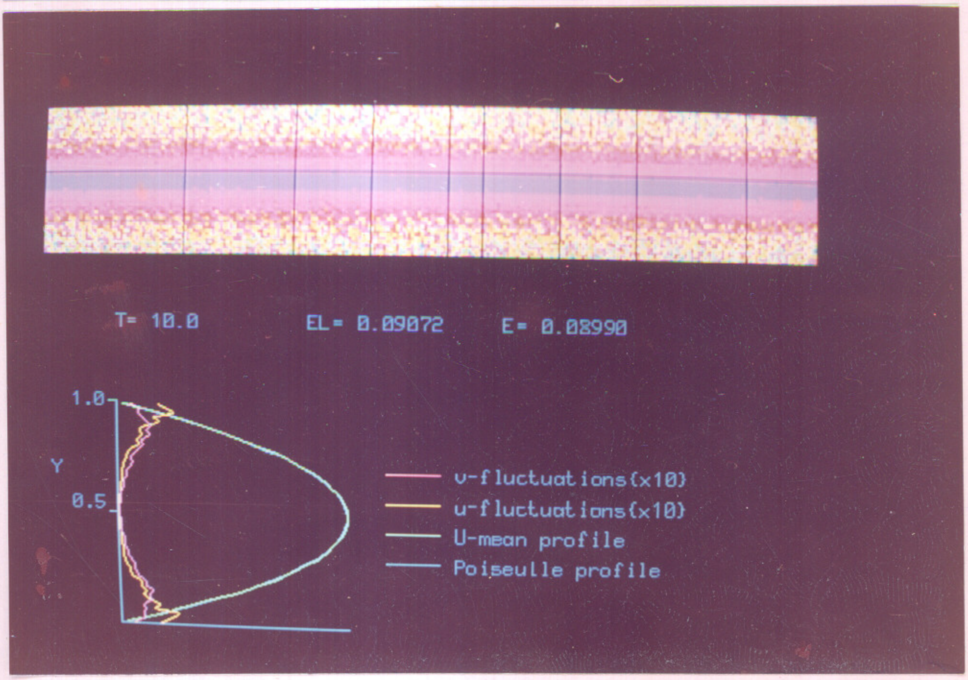
look-up table for the individual vortex blobs (of unit strength) on very fine grid, we can calculate the velocities of the small tracers. It should be noted that only the influence functions should be interpolated to the finer vortices and not the velocity itself. In figure 4.9 we report some preliminary results obtained for a $(50 \times 5) \times (10 \times 5)$ multi-grid with 12,500 vortices.

Now we shall discuss some theoretical results obtained from the simulations done here and elsewhere, about the 2D channel flow. The vortex dipoles seems to play important role in the the dynamics of the 2D channel flows. Weiss and McWilliams (1991) found that since the formation of vortex couples in some cases lowers the kinetic energy (i.e.,the dipoles are negative energy states), they have more chance of being created in such a flow. But, since they move fast they collide with other vortices and break down rather quickly on the average than the vortex pairs, which although is a positive energy state, it survives on the long run. Orlandi (1990) found that when a vortex couple moves towards the wall, secondary vortices are created (due to the no-slip boundary condition) at the wall and, as soon as the secondary vortices created at the wall form a vortex couple of the opposite orientation,it gathers in strength and ejects away from the wall. Although the random motion of these dipoles acts as agents, which perpetuates chaos, there could still be some deterministic patterns left. It has been suspected for a long time that a strong dependence on the initial states might always persist. Rozhdestvensky and Simakin (1984), Orszag and Patera (1981) found that in a inertial process steady waves are formed, in which the vorticity essentially becomes constant along the streamlines. Jimenez (1990) has found that there was even a possibility that soliton like waves might persist at high Reynolds numbers. Jimenez (1990) found that the ejection of the fluid near the wall introduces a strong vortices at the centre of the channel, which induces further ejections from the opposite wall. This seems to introduce a cycle of events. We found evidence for the formation of such moving waves in all the cases. However the process seems to be highly sensitive to the initial conditions and they seem to breakdown after sometime. However it is quite possible that by that time the inaccuracies in the simulations could have grown. To show conclusively the existence of such moving waves, it is necessary to study the simulations on a grid finer than that reported here. The fine scales could be constantly removed by some large scale redistribution, so that the underlying flow patterns could be discerned. As pointed out by Robert and Somerlia (1990), the viscosity probably acts as a fine scale filter. Thess et.al. (1994) have applied the ideas developed in Robert and Somerlia (1990) to inviscid flow in a channel. They found that the most probable macro-states are two rows of periodic vortex streets placed

Figure 4.9: The evolution of vorticity in a 2D channel-Simulations in short box ($d =$
with a multi-grid (12500 vortices)


T=100.0
EL= 0.08832
E= 0.08753
1.0
Y
0.5
v-fluctuations(x10)
u-fluctuations(x10)
U-mean profile
Poiseulle profile


T= 30.0
EL= 0.09228
E= 0.09150
1.0
Y
0.5
v-fluctuations(x10)
u-fluctuations(x10)
U-mean profile
Poiseulle profile


T= 10.0
EL= 0.09072
E= 0.08990
1.0
Y
0.5
v-fluctuations(x10)
u-fluctuations(x10)
U-mean profile
Poiseulle profile

in a staggered arrangement. However as we mentioned earlier their description is a two-scale description of 2D flows, which neglects the role of small scales. Before we neglect small scales one has to be sure what role they play. It is generally assumed that the small scales act just as a eddy viscosity on large scales. On the contrary in 2D flows the small scales are known to come together to form large scales (inverse cascade). So neglecting them as being unimportant could lead to serious errors. But even if assuming that the two-dimensional small scales do not have much effect on the large scales, the process of creation of such small scales may be important for an over all understanding of turbulence. Our simulations here (figure 4.4 & 4.9) point out a possibility that in 2D flows the Tollmien-Schlichting waves may breakdown into certain fractal like structure. In fact a strikingly similar structure has been reported by Mandelbrot (1983), in what he describes as "From a flat sea to a great wave" (figure.2 there in). However Mandelbrot (1983)'s structures are *Lagrangian turbulence*, in which the corresponding Eulerian flow is steady, where as features observed here have strong shearing effects and the Eulerian flow is unsteady, so it is more appropriately called two-dimensional turbulence. These flows (see figure 4.4 & 4.9) give rise to complex spatial patterns. Such features may be related to the existence of clustered complex singularities in spatial and temporal variables. However it should be made clear that the path taken by a fluid particle is always smooth,i.e., an element in C^{∞} and in fact using the results of Bardos et.al. (1977), it can be shown that for these doubly periodic flow fields we consider here, since the initial flow field is analytic in space co-ordinates, it will continue to be analytic in both space and time variables for all time. However the width of the analyticity strip around the real line is found (Bardos (1994) personnel communications) to decrease faster than an exponetial of an exponential (see also Frisch (1983)). This makes one suspect the existence of clustered singularities in complex t-plane. A Painleve' analysis for the Euler's equations could probably yield important insights about such analytic structure of the flows. However as we mentioned earlier these Painleve analysis by themselves are not enough. They only indicate the possibility of formation of complex structures. So numerical simulations could be of help in locating the flow domains for which such complex structures are found. However it is necessary to have certain democracy of scales. It may be necessary to work with several levels of grids (multi-grids) rather than a single level as we have done here. A pyramid structure of grids is considered efficient for capturing features with variety of scales. The method is quite similar to the multi-grid methods used in finite difference schemes. In this method, the space is discretized into a very fine lattice, but the velocity calculations are done first on a coarse grid, which is used to calculate the velocities on the next level, using local corrections (see Puckett (1993) for details). This process can be continued into finer and finer grids. This yields a multi-resolution decomposition of the vorticity field. Such a decomposition is also related to the wavelet transforms (Farge (1992)) and here the vorticity blob-function, could actually be considered as the *father wavelets.* The structures like, the vortex dipoles could then be considered as the *(mother or daughter) wavelets.* On a single processor machine, like the one used here, such multi-level methods may not be of much use, but we found that a two-level method with a large and small scales works very efficiently. By this method we were able to consider a $(50 \times 5) \times (10 \times 5)$ multi-grid with 12,500 vortices (see figure 4.9). The velocity values are interpolated linearly on to an even finer tracers, which are used to track the movement of the vorticity field. However to keep the CPU time within bounds it was found necessary to use look-up tables for the velocity field. This dropping of accuracy although may reduce the accuracy of these simulations, they could be considered as balance between working with the noisy pseudo-random lattice algorithms and that of producing over-conditioned direct numerical simulations.

Finally we would like to point out, that although we have derived a method for simulating flow in only a flat 2D channel, the same technique may be extended to arbitrary channels also, by using a conformal mapping which maps it to a flat channel. Such conformal maps are discussed in Appendix B of this thesis. But it should be noted that in effect such technique converts the inhomogeneity in the geometry into a inhomogeneity in the velocity distribution.

Chapter 5

VORTEX SINGULARITIES AND FLUID FLOWS IN HIGHER (≥ 3) DIMENSIONS

5.1 FLOWS IN HIGHER DIMENSIONS

Numerical simulations of higher dimensional flows are generally difficult to perform. Arnold and Khesin (1992) however, showed that the description of various features of inviscid flows in higher dimensions is possible through a qualitative analysis. The basic questions, regarding the well-posedness of the Euler's and the Navier-Stokes equations, is however yet to be settled. In fact one is not even sure in what function space the solutions of these equations exist. Diperna and Majda (1987) have considered the space of Young measures on the solutions of the Navier-Stokes equations. They consider convergence in probability (instead of the usual point-wise convergence) and use Young measure to define probability on infinite dimensional space (the solutions being considered in the L^∞ space). However such a space may be too large for carrying out analysis. In fact the classical space (C^∞) might itself be enough if no singularities appear in the solutions. We shall show in the next section, that the possibility of a singularity appearing in the solutions is quite remoté. From now onwards in this section, we shall carry out our analysis in classical space, wherein no singularity is considered.

Unlike two dimensional flows, for flows in higher dimensions, vorticity is not an invariant of flow. But in any inviscid incompressible flow the total energy of the flow given by

$$E = \frac{1}{2}\int_M u^2 dV \tag{5.1}$$

is an invariant.

A volume preserving diffeomorphism can be ascribed to an incompressible flow. Now if we consider the set of all such volume preserving diffeomorphisms, then the solution of the Euler's equation can be characterized as the one which minimizes the energy given by (5.1). Arnold (1966) studied the diffeomorphism groups and showed using the right-invariant Riemannian metric induced by the kinetic energy,that the motion of a fluid is given by the geodesics on a particular diffeomorphism group. For some recent results in differential geometry and their applications to inviscid flows, work of Arnold and Khesin (1992) can be consulted. In case of inviscid flows in three dimension, the total *helicity* (Moffat (1969)) given by

$$H = \int_M u \cdot curl(u) dS \tag{5.2}$$

is an invariant.

Arnold and Khesin. (1992) have shown that for odd dimensions, the integral,

$$I(u) = \int u \wedge (du)^n \tag{5.3}$$

and for even dimensions an infinite class of integrals,

$$I_f(u) = \int f(\frac{(du)^n}{\mu})\mu \tag{5.4}$$

(where μ is the differential n-form) are invariants of any inviscid flow.

Describing the flow field by a distribution of vortex lines, Moffatt (1969) showed that the helicity given by (5.2), can be related to the the *linking number* of the vortex lines. However, it should be mentioned here that an incompressible flow which preserves E and H, need not be a solution of the Euler's equation. The total helicity H is after all an average linking number (Arnold and Khesin (1992)) and even the topology of the vortex lines is not completely characterized by the linking number. There are also other invariants associated with the *Isotopy group* of the knot, which may be as important as the helicity, but not so easily expressible in terms of the flow field variables. The invariance of (5.2) is just a consequence of the more general fact that the vortex lines (or sheets) can never cut across each other; a fact which follows from the Kelvin-Helmholtz theorem. This being the case, one is not sure how helicity can be created away from the boundary. Although strictly speaking these results are only valid for inviscid flows, but it seems that even for viscous flows such contraints remain, inspite of the fact that the picture becomes clouded because of the action of diffusion. These intricate topological constraints cannot be easily settled either by numerical simulations or by experiments, since even a small error could easily confuse the picture. However recent numerical studies by Kida et.al. (1991), **Melander and Hussain (1994)*** have provided some insights into these constraints. Their results seem to support the possibility that helicity might tend to zero in the limit of vanishing viscosity. This result which may be true even for flow with boundaries, if no helicity was initially present. Recently Asano (1989) has made some progress in proving that the solution of the Navier-Stokes equations for a flow past an infinite flat plate, in the inviscid limit, tends to the solution of the Euler's equation with the same initial conditions. If such a result is proved then it would imply that the helicity remains an invariant in the limit of vanishing viscosity, independent of the fact whether there is a boundary or not.

Arnold (1966,78) has provided a complete classification of the streamlines of stationary solutions of the Euler's equations, in terms of fiber bundles over torii and infinite tubes (which can be considered locally as cartesian products). However his results are valid only for non-Beltrami flows. Beltrami flows are flows, in which the velocity vector and the vorticity vector are collinear everywhere,i.e.,$u \times curl u = 0$. Daumbre et.al. (1987) have studied the streamlines of certain Beltrami flows, termed as the,ABC (Arnold-Beltrami-Childress) flows and found that in general such flows exhibit clustered singularities in the complex plane, and are non-integrable. Such flows, which are steady in the Eulerian frame but which gives rise to complicated fluid paths (however it should be noted that the path taken by a particle is smooth,i.e., an element in C^{∞}) are termed as *Lagrangian turbulence*. Moffatt (1985) earlier made a topological classification of the stationary solutions of the Euler's equations and found that there exist solutions which are not smooth. Although these exact solutions are interesting, it should be noted that some of these stationary solutions are unstable at high Reynolds numbers (e.g., the ABC flows). For real physical flows, only the stationary solutions which are quite stable and dynamical accessible are important.

Finally we would like to point out that to produce reliable real time simulations, it is necessary to discretize some where along the line. It is not a very easy task to study continuous elements like vortex lines and vortex sheets, since they get distorted and quickly reach a state beyond recognition (Chorin (1982)) . So it is more convenient to adopt methods which consider discrete vortices.

* *J. Fluid Mech.* , 260, p. 57

5.2 VORTEX SINGULARITIES

For the calculation of the n-dimensional singularities we start with the fundamental solution ϕ_0, of the Laplace equation in n-dimension, given by,

$$\Delta\phi_0 = \delta(x) \tag{5.5}$$

where $\delta(x)$ is the dirac delta distribution in n-dimension. The solution of (5.5) for $n > 2$ is given by,

$$\phi = \frac{1}{(n-2)\Pi_n|\boldsymbol{r}|^{n-2}} \tag{5.6}$$

where Π_n is the surface area of the unit sphere in n-dimension.
The velocity field for a point source,dipole or a multipole can be derived by taking derivatives of (5.6). However it is to be mentioned that the divergence free velocity field cannot be obtained from (5.6). Such a velocity field can be obtained with the aid of a skew-symmetric tensor K_{ij},which is in fact the generalized vorticity strength. The fundamental singularity associated with a divergence free velocity field is given by,

$$u_i = \frac{K_{ij}x_j}{(n-2)\Pi_n|\boldsymbol{r}|^{n-2}} \tag{5.7}$$

for $i = 1, 2, 3, ..., n$.
u_i is divergence free since K_{ij} is skew-symmetric,i.e.,

$$K_{ij} = -K_{ji} \quad \text{for all } \text{i, j} = 1, 2, 3, \dots, \text{n}. \tag{5.8}$$

Saffman (1980) proposed the name *vorton* to the singularity obtained by letting $n = 3$, in (5.7). If we consider the Jacobian,$\{ \frac{\partial u_i}{\partial x_j} \}$,we see that the skew-symmetric part of it gives the vorticity components. Thus in n-dimension, the vorticity will have $\frac{n(n-1)}{2}$ components and its evolution can be obtained by differentiating the n-dimensional Euler's equations. Using the Biot-Savart's relation in three-dimension, Novikov (1983) described the dynamics of N-vortons ($i = 1, 2, 3, ..., N$) using the relations

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{1}{4\pi} \sum_{j(\neq i)} \frac{\boldsymbol{K}_j \times \boldsymbol{r}_{ij}}{r_{ij}^3} \tag{5.9}$$

where, $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ and $r_{ij} = (\boldsymbol{r}_{ij} \bullet \boldsymbol{r}_{ij})^{1/2}$ and ,

$$\frac{d\boldsymbol{K}_i}{dt} = \frac{1}{4\pi} \sum_{j(\neq i)} \frac{\boldsymbol{K}_j \times \boldsymbol{K}_i}{r_{ij}^3} - \frac{3}{4\pi} \sum_{j \neq i} \frac{\boldsymbol{K}_i \times \boldsymbol{r}_{ij}}{r_{ij}^5} \tag{5.10}$$

for $i = 1, 2, 3 ..., N$.
From the above equations,Novikov (1983,85), found that any two vortons may interact in two different modes depending on their orientation. One is the hyperbolic mode, wherein the vorticity increases exponentially and the other is an elliptic mode in which the strength of the vortices oscillates. Considering the above equations we studied the dynamics of many vortons. It was found that there is always the possibility that any one of these vortons may increase exponentially in strength. But from the equations (5.10), it is clear that no finite time blow-up can occur in the vorton strength. An exponential blow-up however, results in a rapid increase of the total energy. Since in an inviscid flow, the total energy is supposed to remain constant, the above vorton model seems to overpredict the stretching effects. Saffman and Meiron (1986), noted that unlike the two dimensional discrete vortices, these vortons are not distribution (weak?) solutions of the n-dimensional Euler's equations. However it seems that a large collection of such vortons arranged appropriately might yield satisfactory results. We found that (Venkatesan (1992b)), if we consider a row of identical vortons, arranged on straight line, then on appropriate cross-sectional planes, we get a 2D vortex blob. Also, we can recover a straight vortex filament in the limit as the separation between two consecutive vortons goes to zero. But, for non-zero separations, the vorticity field generated by a row of vortons, decays rather slowly. However certain local corrections can be introduced, which will ensure that the vorticity field decays rapidly. This can be achieved by using small pieces of vortex line. Chorin (1982) earlier produced simulations involving vortex line elements and found that they get stretched rather rapidly. Leonard (1980,84) made simulations of 3D channel flows, super-posing vortex elements, which captures the formation of spots. Ashurst and Meiburg (1984) produced simulations of the shear flow using the vortex line elements. However it should be realized that for simulating smooth flows accurately, it is neccessary to use vortex blobs which are obtained by convoluting the vorton with a smooth rapidly decreasing function as for example suggested by Beale and Majda (1982a), or Hald (1987). Puckett (1993) has recently surveyed the current developments in the vortex methods for three dimensional flows. We study here only the deterministic methods but, as we mentioned earlier in the earlier chapter, there is very little to choose between random methods which uses point vortices and deterministic vortex blob methods. To get meaningful results using the random methods we require large data storage and retrival facilities, while the deterministic methods do not require the handling of such large data.

In order to confirm the insights gained from the vortex methods, other approaches e.g., finite difference techniques may also be used. Bell and Marcus (1992) recently studied the stretching of vortex filament using finite difference method. They extrapolated their results in time to suggest a formation of a singularity. They however, themselves admit that such a formation of singularity, may occur due to the under-resolution of the flow. The existence of hyperbolic modes in which the vorticity increases exponentially do not suggest the formation of a singularity. It is known that in two dimensional flows, such hyperbolic modes exist (Weiss(1981), Kida and Yamada(1984)), in which the vorticity gradients increases exponentially. But, no singularity appears in the two-dimensional flows. So it seems quite possible that the solutions of the Euler's equations could remain regular for all time, although the absolute bound on the vorticity strength may strongly depend on the initial vorticity distribution. Beale et.al.(1984) have shown that a singularity can form only if the vorticity field blows-up at some point. Constantin et.al.(1985) proposed a model of stretching of vortex filaments, in a Lagrangian reference frame. They considered the simple one-dimensional equation,

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega \tag{5.11}$$

where ω,is the vorticity and $H(\omega)$ is the Hilbert transform of ω defined by,

$$H(\omega) = \int_{-\infty}^{\infty} \frac{\omega(x')}{x - x'} dx' \tag{5.12}$$

The velocity is given by the integral,

$$v(x,t) = \int_{-\infty}^{x} \omega(y,t) dy \tag{5.13}$$

Using the auxiliary complex variable,

$$Q = H(\omega) + i\omega \tag{5.14}$$

the equation can be transformed to,

$$\frac{\partial Q}{\partial t} = \frac{1}{2} Q^{*2} \tag{5.15}$$

where Q^* is the complex conjugate of Q.
The solution of (5.15) is easily obtained as,

$$H(\omega) + i\omega = Q(x,t) = \frac{Q_0(x)}{(1 - \frac{1}{2} t Q_0(x))} \tag{5.16}$$

Equating the imaginary parts from both sides of (5.16),we obtain,

$$\omega(x,t) = \frac{4\omega_0}{[2 - tH(\omega_0)]^2 + t^2\omega_0^2} \tag{5.17}$$

As shown in Constantin et.al. (1985) this last solution satisfies certain scaling laws and invariance principles. Here a finite time blow-up can occur if and only if the set,

$$Z = \{x/\omega_0(x) = 0 \text{ and } H(\omega_0(x)) > 0\} \tag{5.18}$$

is not empty.

Schochet (1986) included the effects due to the viscosity and considered the equation,

$$\frac{\partial Q}{\partial t} = \frac{1}{2}Q^{*2} + \nu\frac{\partial^2 Q}{\partial t^2} \tag{5.19}$$

The above equation has a solution of the form,

$$Q = -\frac{12i\nu}{(z - z_0)^2} \tag{5.20}$$

This indicates that a pole decomposition solution (*vide.* Section 3.2) is possible for the equation (5.19). Schochet (1986) studied the interaction between the poles and found that viscosity hastens the formation of the singularity. This reflects the drawbacks of such a simple one-dimensional model. Therefore, attempts were made to describe the stretching process by more complex models (see Majda (1991) for a survey of the recent results on the complex stretching process). Klein and Majda (1991,91,93) have recently modeled the vortex stretching process using a Non-linear Schrodinger equations and they were able to show that the existence of the quadratic stretching term in the vorticity equation does not imply that a singularity will occur (Klein and Ting (1991)).

We have seen in the earlier chapters, that the expansion in terms of rapidly decreasing functions has certain advantages. Truesdell (1951,1954) showed that if the vorticity field has rapidly vanishing tail, then $\frac{(n+3)(n+2)}{2}$ linear combinations of the n-th moments of the vorticity must vanish. Klein and Ting (1990) reconsidered this and other moment relations, in an attempt to derive expressions for the three dimensional vortex balls with a rapidly vanishing tail. They found that the vortex source (the vorton) term, vanishes. Since a piece of vortex element can never end inside the fluid, it is conceivable why the source term must vanish. However it is possible consider certain axi-symmetric elements like the vortex ring or the Hill's vortex. In fact the three-dimensional vortex dipole considered by Klein and Ting (1990) in their moment relations is the limiting case of the vortex ring or the Hill's vortex. We now briefly consider certain axi-symmetric configurations.

The quaternionic fields and functions in four dimension when applied to fluid problems in three dimensions, reduces to the class of axi-symmetric flows. The conjugate potential in a quaternionic field gives rise to the well known axi-symmetric stream function,Ψ. In terms of the axi-symmetric co-ordinates (z, r), the condition the steady state axi-symmetric solution, Ψ of the 3D Euler's equation is given by (Lamb(1932)),

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} = r^2 \, f(\Psi) \tag{5.21}$$

where f is any regular function.
The solutions of (5.21) are well known and if we let,

$$\Psi(z, r) = e^{ikz}\psi(r) \tag{5.22}$$

then separating the variables,we obtain an ODE in r, whose fundamental (distribution) solution is given by,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - k^2\psi = \frac{1}{2\pi r}\delta(r) \tag{5.23}$$

The solution of which is the well known modified Bessel function, $K_0(kr)$. This solution,

$$\Psi(z, r) = e^{ikz}K_0(kr) \tag{5.24}$$

is also known as the geostrophic or the Bessel vortex, which describes a straight vortex filament, with a periodically varying vortex strength. The general straight vortex filament in 3D is then is given by,

$$\Psi(z, r) = \int_{-\infty}^{\infty} G(k)e^{ikz}K_0(kr)dk \tag{5.25}$$

Arnold(1966), showed that the three dimensional flow geometry can be described in terms of fiber bundles over tubes and torii. So along with the straight filaments considered here we shall consider the vortex ring. The study of vortex rings is quite old, dating back to Helmholtz(1867). For a circular vortex ring filament (of zero thickness) located in the (y,z)-plane, having its centre at the origin, the stream function is given by (Lamb(1932)),

$$\Psi = -\frac{\gamma}{2\pi}(r_1 + r_2)\{(K(\lambda) - E(\lambda)\} \tag{5.26}$$

where

$$\lambda = \frac{r_2 - r_1}{r_2 + r_1} \tag{5.27}$$

here $r_1^2 = x^2 + (a-r)^2, r_2^2 = x^2 + (a+r)^2$ and $K(k), E(k)$ are the complete elliptic integrals of the first and second kind respectively and a is the radius of the vortex ring. Unfortunately unlike the straight vortex filament (of zero thickness), these structures just cannot exist as such, since they would move with an infinite velocity. So it is just cannot exist as such, since they would move with an infinite velocity. So it is neccessary to consider filaments of finite thickness. Such a filament can be obtained by convoluting it with certain function (say $\xi(x,r)$) with rapidly vanishing tails and is given by (Lamb (1932)) ,

$$\Psi = -\gamma \int\int (r_1 + r_2)\{(K(\lambda) - E(\lambda)\} \; \xi \; dx \; dr \tag{5.28}$$

The vortex rings of finite cross-section, quite like solitons move with constant speed and pass through each other and come out unaffected. In fact the term *vorton* has been used by Moffatt (1986) to denote such axi-symmetric vortical structures. He has also considered, general axi-symmetric configurations. Way back,in the last century, Helmholtz (1867), Kelvin (1869), Thompson (1883) studied the interaction between vortex rings and found that they interact strongly over a very short range. Thompson (1883) tried to establish that these vortex rings are stable and, it was probably thought at that time, that these vortex rings may be the ideal building block for describing the structure of atoms. But later Widnall and Tsai (1977) established that these structures are unstable. However it seems that a description in terms of such vortex rings might be quite useful for describing turbulence. Recently Leonard and Shariff (1992) have surveyed the literature on the vortex rings. Here, we shall be concerned with the use of small vortex ring as a computational tool.

5.3 INVISCID FLOW IN THREE DIMENSION - A MICRO-MACRO DESCRIPTION

Beale and Majda (1982), Beale (1986) have described vortex methods on three dimensional space lattice with very high order of accuracy. The method uses a three dimensional vortex monopole blobs as a building block. Chorin (1993a) has recently tried to simulate flows using vortex filaments and found that the rapid distortion of vortex filaments lead to inaccuracies. Just like the vorticity gradients in two dimensional flows (see section 4.1), the vorticity has tendency to increase exponentially (see equation (5.10)) in some regions. In such regions the energy is tranfered rapidly to the small scales. But during the stretching process, the vorticity varies rather too rapidly, affecting the accuracy of the simulation (Chorin (1993a)) . In such regions it would be better if some lower order quantity like the velocity can be studied rather than vorticity. Oseledets (1989) found that Hamiltonian formulation is possible for the Euler's equations in terms of a vector potential of the vorticity. Chorin and Buttke (1992) have studied this formulation in terms of, what they call, *magnetization variables.* Moffatt (1989) earlier proposed such a description in terms of generalized vortex rings, which he calls as *vortons.* To avoid confusion we shall call the small vortex structures, *micro-vortons* and the large vortex structures (which can be either 2D or 3D structures) *macro-vortons.* It should be mentioned that these *micro-vortons* have been used in the past in a quite different context in the super-fluid literature. The quantised circular vortices known as *rotons* were proposed by **Onsager (1949)** and their interactions were studied by Roberts (1972) (see also Williams (1987,92) for recent results).

The *micro-vorton* singularity is given by,

$$u = \gamma\delta(x) - (\gamma \cdot \nabla)\nabla(\frac{1}{r}) \tag{5.29}$$

where $\delta(x)$ is the Dirac's delta distribution in three dimension and $r = |\boldsymbol{r}|$ is the radius vector.

Chefranov (1987) showed that the above singularity can be considered to be the limit as the vortex ring or a Hill's spherical vortex reduces to a point. He considered the above singularity without $\delta(\boldsymbol{x})$ distribution term (it seems to avoid the self-energy contribution). He then derived the Hamiltonian as, the finite part of the total energy (or the kinetic energy of the interactions between the micro-vortons),given by,

$$H = -\frac{1}{8\pi}\sum_{ij}\sum_{\alpha\beta}\gamma_i^\alpha\gamma_j^\beta \left(\frac{\delta_{ij}}{|\boldsymbol{x}^\beta - \boldsymbol{x}^\alpha|^3} - 3\frac{(x_i^\beta - x_j^\alpha)(x_i^\beta - x_j^\alpha)}{|\boldsymbol{x}^\beta - \boldsymbol{x}^\alpha|^5} \right) \tag{5.30}$$

where $i, j = 1,2,3$ & $\alpha, \beta = 1,2,3...M(\alpha \neq \beta)$.

The evolution is given by the Hamilton-Jacobi equations,

$$\begin{aligned} \frac{dx_i^\alpha}{dt} &= -\frac{\partial H}{\partial \gamma_i^\alpha} \\ \frac{d\gamma_i^\alpha}{dt} &= \frac{\partial H}{\partial x_i^\alpha} \end{aligned} \tag{5.31}$$

for all $i = 1,2,3$ & $\alpha = 1,2,3...M$.

Chefranov (1987) pointed out that if we consider a system of two dipoles with equal and opposite strength, then they collapse to a point, an irreversible state. We found certain inaccuracies in Chefranov (1987)'s work. According to him,

$$H = \frac{5\gamma^2}{4\pi l^3}(1 - \frac{3(\gamma.l)}{(\gamma l)^2}) \tag{5.32}$$

is an invariant.

He used this invariant to obtained solution for the case with two dipoles as,

$$\gamma^2 = 4\pi H l^3 + \frac{3}{l^2}(5Ht + (\gamma_0 . l_0))^2 \tag{5.33}$$

$$l = [l_0^5 - \frac{5}{\pi}(t(\gamma_0 . l_0) + \frac{5}{2}Ht^2)]^{1/5} \tag{5.34}$$

$$\phi = \phi(0) + \frac{M}{25\rho H(t - t_-)} ln|\frac{(t - t_+)}{t - t_-}| \tag{5.35}$$

where ϕ is the polar angle made by the separation vector, $\boldsymbol{l} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and t_+ and t_- are the two real roots obtained by setting $l = 0$ in equation (5.34).

From (5.34) he also noted that for certain range of values of H and t, l becomes negative, and hence concluded that the solution is unphysical and the system will continue to remain in the collapsed state. Here we note that to obtain the physical value of l, it is necessary to take the appropriate branch of the square root of l^2, so that l could remain always positive. As Chefranov (1987) neglects the $\delta(x)$ contribution, the velocity field obtained by him violates the continuity equation in the weak formulation.

The corrected version of the Hamiltonian of a many micro-vorton system is given by ,

$$H = -\frac{1}{8\pi}\sum_{ij}\sum_{\alpha\beta}\gamma_i^\alpha\gamma_j^\beta \; (\; \delta(\boldsymbol{x}^\beta - \boldsymbol{x}^\alpha) + \frac{\delta_{ij}}{|\boldsymbol{x}^\beta - \boldsymbol{x}^\alpha|^3} - 3\frac{(x_i^\beta - x_j^\alpha)(x_i^\beta - x_j^\alpha)}{|\boldsymbol{x}^\beta - \boldsymbol{x}^\alpha|^5} \;) \tag{5.36}$$

where $i, j = 1, 2, 3$ & $\alpha, \beta = 1, 2, 3 ... M(\alpha \neq \beta)$.

But even here the self-energy term (which is infinite for a point micro-vorton) is neglected. To include such a self-energy term, we later consider a micro-vorton blob.

Using the Hamiltonian (5.36) and the Hamilton-Jacobi equations (5.31), the dynamics of the system of micro-vortons can be described. We find that although the corrected version is more or less similar to the solution (5.32 -5.35),but there can be no absolute collapse, since infinitely strong repulsion may occur when the two micro-vortons come very close to each other. To make the picture clear, we shall consider a blob-version of the singularity (5.29) (Chorin and Buttke (1992)) :

$$\boldsymbol{u} = \gamma\xi(\boldsymbol{x}) - (\gamma \cdot \nabla)\nabla\psi(\boldsymbol{x}) \tag{5.37}$$

where $\Delta\psi(\boldsymbol{x}) = \xi(\boldsymbol{x})$ and $\xi(\boldsymbol{x})$ is a function of the Schwartz class,which approximates the Dirac's delta distribution.

The system of two small micro-vorton blobs of equal and opposite strengths can be represented by the Hamiltonian,

$$H = \frac{\gamma^2}{2\epsilon^2} - \frac{\gamma^2}{16l^3}(1 - \frac{3(\gamma . l)}{\gamma^2 l^2}) \tag{5.38}$$

where ϵ is the core diameter of the dipoles. Now the motion of the two micro-vorton blobs of opposite strengths, can be studied using the Hamilton-Jacobi equations (5.31). It is well known fact that when two vortex rings are oriented in the same direction, they pass through each other in a leap frogging motion (Helmholtz (1867)). But such cases correspond to elliptic modes, with oscillating strengths. Gurzhi and Konstantinov (1989) studied the collision of two co-axial vortex rings. They found that when two vortex rings oriented in the opposite direction, come close to each other their size increase rapidly and the filaments gets stretched further and further. They considered only perfectly symmetrical case of co-axial rings. In general in these hyperbolic modes, the maximum strength that the vortex filaments can attain depends crucialy on the angle of the approach and on the core size of the filaments. This maximum strength is in fact inversely proportional to the core sizes or radii of the filaments. These results are not in contradiction with the conjecture of Frisch (1986) that a real singularity is never produced in the Taylor-Green vortex flow. Pumir and Siggia (1987) considered slender vortex filaments in an inviscid flow and reported that a singularity may appear in the flow. We would like to point out, that as the the core size of the filament decreases, the maximum vorticity strength that can be attained during the flow increases, which could have been construed as the appearance of a singularity. When two vortex rings of opposite strength, come close to each other, they begin to stretch each other and their vortex strengths increase rapidly. Intricate small scale distrubances start forming at this stage. At this stage even small asymmetry or disturbance in the flow can move them suddenly apart. A cartoon of the close interaction of two micro-vortex blobs is sketched in figure 5.1 (ice skater's somersaults). However the above cartoon is a too simplistic picture of the asymmetric collision of micro-vortex blobs. The exact process requires much more intricate analysis (see Majda (1991)). This process is related to the problem of bouncing of a vortex ring from a wall, in an inviscid flow wherein the mirror reflection at the wall is considered. It however, differs from the case we considered above, in that the vortex ring and its image are perfect copies of each other since the presence of the wall forces this symmetry on the flow. Walker et.al. (1987) have studied the problem of bouncing of a vortex ring from the wall. They considered the no-slip condition at the wall, which in effect is responsible for the creation of new secondary vortex rings. Orlandi (1990) has considered in detail the analogue in two dimension of a vortex dipole bouncing-off from the wall. He showed that the case with no-slip boundary condition and the free-slip condition differs substantially. To study such a process using the inviscid methods considered here it is neccesssary to include the secondary vortices created at the wall, in the system. This requires the study of the

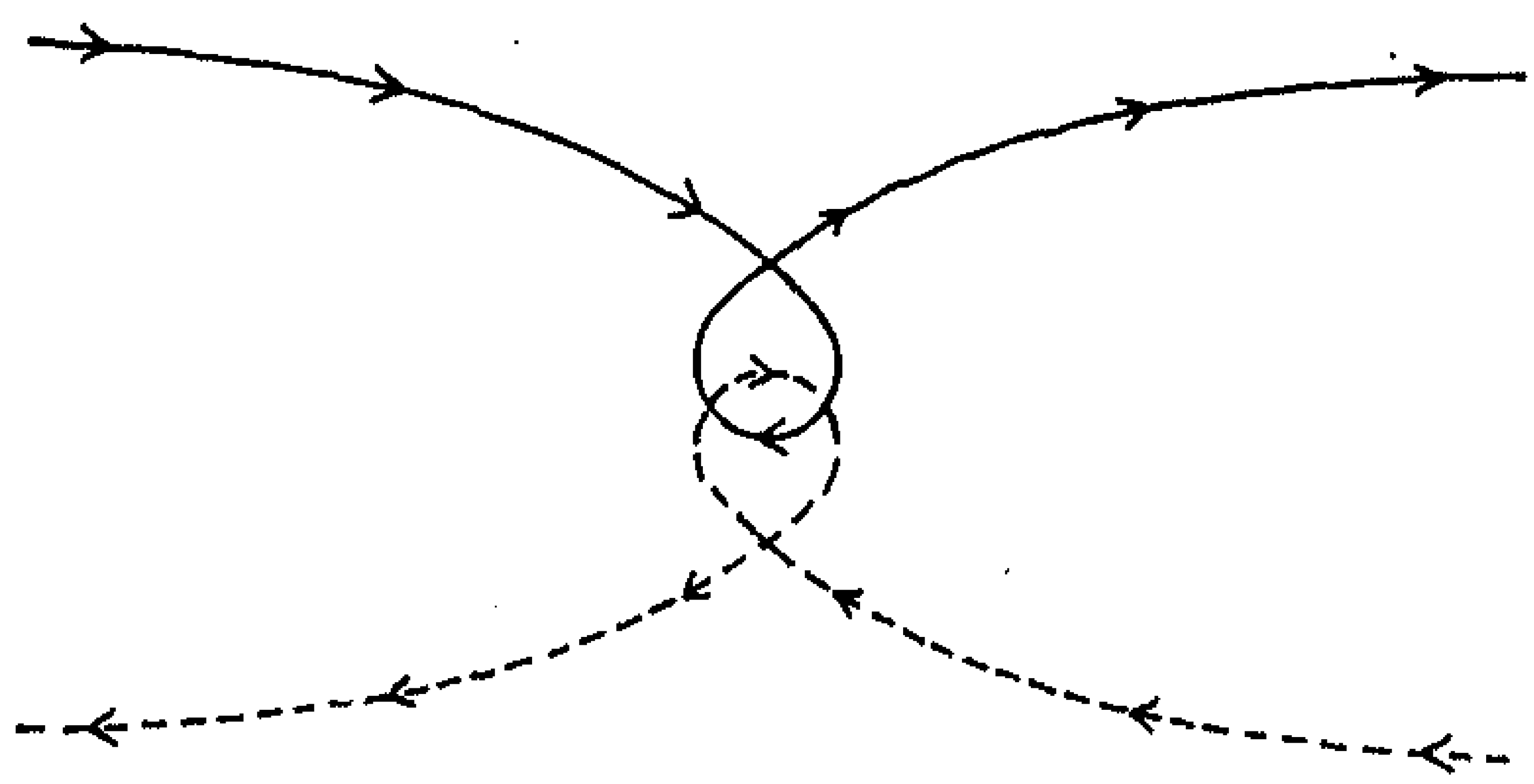


Figure 5.1: The path taken by two micro-vortons of opposite strength

dynamics of more than two vortex rings. Konstantinov (1994) studied the interaction of more than two co-axial vortex rings. He found that under certain conditions a set of four or more vortex rings move rather chaoticaly. He also found that sometimes they also tend to form quasi-periodic sub-systems which behave quite like the leap-frogging motion observed in the study of two vortex rings.

The vortex rings by themselves however, are not quite satisfactory for representing the macro vortex structures like large vortex rings. As mentioned by Chorin (1993b), the representation of the flow using the micro-vortex blobs soon swell in size causing difficulties. So, along with these micro-vortons we also consider other macro-vortex structures, like the vortex filaments, and vortex sheets. We now discuss how a macro-micro vorton description of the flow is possible. Such a macro-micro description can be traced to Chorin (1993a)'s work, regarding the removal of hairpin shaped vortices. To be specific, we now consider certain addition operations (see figure 5.2) among vortex filaments and sheets. A straight vortex filament tapered slightly around a point can be considered as the super-position of the straight filament and a small vortex ring (see figure 5.2). Since the velocity field of the tapered filament may now be considered as the sum of the velocity of a small vortex ring and the straight vortex line element, the tapered portion gets twisted. Such a twisted filament is usually obtained in filament calculations (Klein and Ting (1991)). If instead of a single tapering, there be two tap-

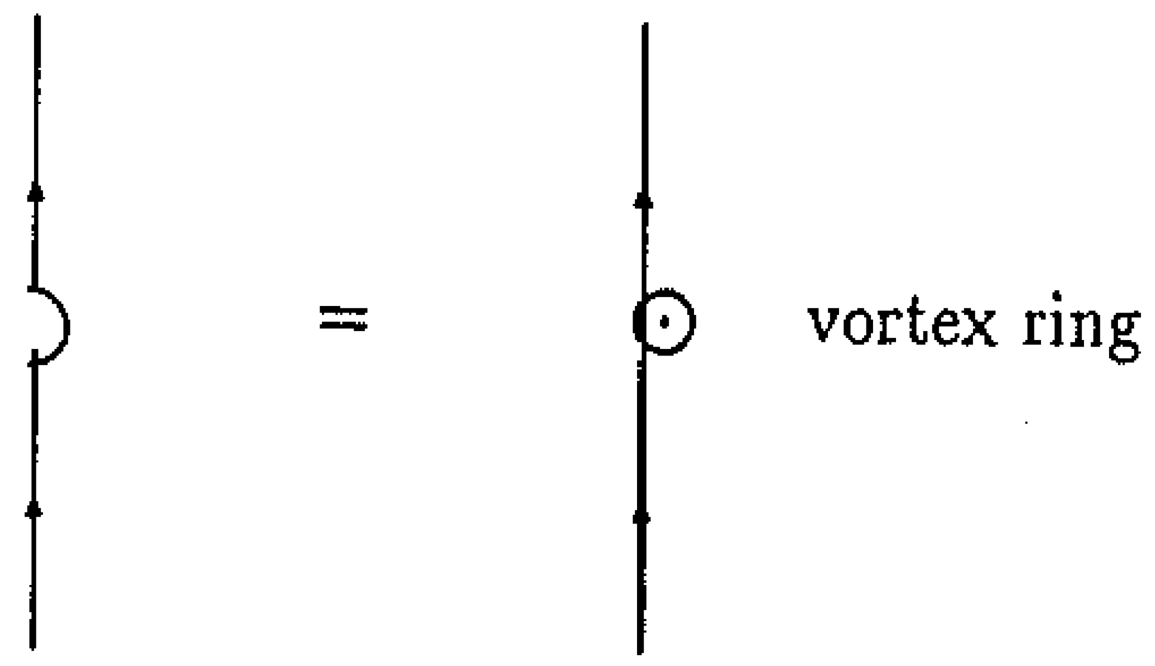


Figure 5.2: The super-position of a vortex line and a vortex ring

perings, then we may consider the situation as super-position of two small vortex rings on the straight filament. If these vortex rings orient themselves in the opposite direction, then, as we observed earlier, when they come together their filament strengths increase rapidly, leading to the so called *filament breakdown* obtained in calculations (Ting and Klein (1991)). Here we considered a vortex line for computational convenience. However a better idealization would be to consider a vortex sheets instead of line elements. The localized swellings (or bumps) on such a vortex sheet can be considered as the superposition of a small Hill's spherical vortex on a flat vortex sheet. We can again imagine such a small Hill's vortex as a micro-vorton blob. In the case of two bumps projecting into the opposite sides, if the two micro-vorton blobs oriented in the opposite direction, come together, it results in a bubble like breakdown (Krause (1990)) of the vortex sheet. Important insight into the formation of the turbulence spots, can be gained from the study of the interaction among the micro-vorton blobs associated with the large scales structures. In this regard it should be realized that these micro-vortons are negative energy states and hence they have a high probability of getting created. The formation of a sea of such micro-vortons could probably explain the formation of the turbulent spots and their spreading. Moffatt (1989) proposed such a scheme in terms of generalized vortex rings, which he terms as *vortons*. To avoid confusion it may be better to call the small vortex rings, *micro-vortons* and the large scale vortex structures (which can be either 2D or 3D structures), as *macro-vortons.*

The micro-vortons act as if they are the carriers of helicity. The introduction of a sphere in a flow (which is equivalent to the introduction of a small Hill's vortex (Lamb (1932)), generates an helicity proportional to $\Omega \times \gamma$ in the flow (Hunt and Hussain (1991)). Here Ω is the vorticity at that point and γ is the dipole strength of the sphere. So the movement of these micro-vortons and any change in the strength of these micro-vortons causes helicity transport in the flow. However as the equations demand the total helicity should remain constant. Thus we see that the helicity density plays a crucial role in the small scale structures. It is interesting to note here that Widnall (1983) found that the turbulent spots behave as if they are small blockages in the flow. So we can think of these turbulent spots as if they are small intense spherical Hill's vortices.

The vortex method used for simulating two dimensional flows can be generalized to three-dimensional flows by super-imposing the micro-vortons. Since the velocity is additive (by Biot-Savart's law), except for some local deformations (which can be considered as the creation of new micro-vortons), the macro-vortex elements would move more or less as if the flow is two dimensional. Thus in the model presented here the two-dimensional coherent structures never disappears, but the vigorous three-dimensional dynamics of the micro-vortons make the features appear or disappear at any instant of time. But the large scales structures are not rigid structures. They would be constantly interacting with the sea of micro-vortons and at the same time giving birth to new micro-vortons. The micro-vortons also interact actively among themselves.

Finally we would like to point out that although we have considered here only two dimensional flows, this is not quite necessary. We can study flows in wind tunnels or pipes by introducing macro-vortex elements like the thick circular vortex filaments considered in the earlier section and the micro-vorton blobs. Thus we see that a bilateral description involving both large scale vortex filaments and the small scale micro-vortons is possible.

Chapter 6

CONCLUSION

In this thesis just by using the fundamental singularity of the Laplace's equation,it's derivatives and their hybrids we were not only able to describe the coherent features observed in space, but also study rigorously the dynamic interaction between them. We have found that in many cases it is possible to derive exact solutions in terms of the coherent structure functions. However, this is related to the integrability of the relevant system of equations, which can be tested by finding the Painleve' exponents. In case the Painleve exponents are irrational or complex, following the theory of Yoshida (1983), such a system can be shown to be non-integrable. In non-integrable systems,the complex singularities occur in fractal clusters and the geometry of these clusters can be determined by the similarity triangles described by Yoshida (1983). However it is important to distinguish between spatial and temporal singularity structures while analyzing the equations, since there exists equations (Lee and Chen (1982)) whose solutions are at any time meromorphic in spatial variables,but still the system is chaotic in time, indicating non-integrability in time. This difference between the time and space co-ordinates is an essential part of Newtonian mechanics.

In two-dimensional flows the movement of the singularities of the vortex field can be used as a model, but for smooth (C^{∞}) flows it is necessary to consider a vortex blob, instead of a point singularity. If the flow field has a rapidly vanishing tail, then it is easy to carry out computations. Beale and Majda (1982) have shown that very high order convergence can be achieved in this case. We have shown here that the inviscid flows are easier to simulate than viscous flows. Moreover there are certain ways of checking the consistency of the results utilizing the constants of motion in inviscid flows. Such a check on computations is not available for viscous flows. The study of

2D flows in a channel using the vortex method indicates that it is possible to make accurate predictions using them with little computational effort. The simulation of 2D channel flows indicate that the Tollmien-Schlichting like waves starts forming slightly away from the wall and such waves trigger the adjacent layer and the next and so on and then after some time such waves are also formed in the center of the channel. These waves overturn and break down and due to the strong shearing action form rough vortex sheets. The vortex dipoles seem to play an important role in the rapid transport of the fluid in a 2D channel. However,the studies made here were preliminary in nature and the computational resources used is quite limited. Calculations on a finer grid than that reported here can ascertain whether, steady waves really persist in inviscid flows. Such a study could be quite important for understanding channel flows.

It is generally believed that these two dimensional flows are of little importance since real flows are three dimensional. However we found that the two dimensional flows can be used to model the large scales, over which the three dimensional small scale effects can be superimposed. Oseledets (1989) showed that using micro-vorton distribution solution (which is actually the three dimensional vortex dipole) the Euler's equations can be described using an Hamiltonian formulation. Buttke and Chorin (1992) showed how it is possible to make computations using the micro-vortons blobs. We found that by super-imposing these micro-vortons over the two dimensional flows and studying their dynamics, real three dimensional simulations can be made. The motion of a pair of micro-vortons of opposite strength, provides a cartoon for the strong churning action present in small scales. The sudden increase in the strength of these micro-vortons (which happens just after they come together) could probably explain the small scale bursts. The study of the interaction between these micro-vortons and the large scales structures would yield important insight into the formation of the turbulence spots. In this regard it should be realized that these vortons are negative energy states and hence they have a high probability of getting created. The formation of a sea of such micro-vortons could probably explain the formation of the turbulent spots. These micro-vortons also act as carriers of helicity, a quantity considered to be of fundamental importance in turbulence. Thus we see that a macro-micro-vorton study could be of paramount importance for an understanding of turbulence. We would like to point out that Reynolds (1989) emphasized that the turbulent spots probably could be considered as the *molecules of turbulence.* Theoretical results here suggests that,this is possibly the case.

Finally we would like to point out that a multi-level vorticity method, which is quite in spirit of the multi-resolution analysis or the wavelet transform, seems possible. In such a decomposition, the vortex blobs (or vortons) can be considered as the *father wavelets* (Farge (1992)) and the structures like the dipoles or rings (or micro-vortons) would be the corresponding *wavelets*. To develop such a method, the results of Beale and Majda (1982) have to extended to a multi-grid lattice and instead of the non-orthogonal isotropic decomposition, an orthogonal direction sensitive decomposition has to be used (see Daubechies (1988)) . However to study very large number of scales, one probably has to develop a way to reduce computational effort. A vortex method, which makes calculations on very fine lattice using look-up table for the influence functions seems to work very efficiently in two-dimension. Such a 3D method could produce considerable reduction in computational effort. In essence this will be a compromise between using over-conditioned computational methods on the one hand and the noisy lattice gas methods on the other hand.

Appendix A

THE LINEARIZED 2-D WATER WAVE POTENTIAL

The linearized water wave problem is a classical potential problem discussed in Lamb (1932). More recently the problems of radiation and diffraction of water waves by bodies have attracted attention due to the wide application it has in marine hydrodynamics (Newman (1977)). For a complete discussion of these problems one can refer to Wehausen and Laitone (1960) and more recent article by Newman (1978). Although there exists many techniques for deriving the solution of these problems, a general uniqueness proof is lacking, despite the fact that we are looking only for solutions of Laplace equation with linear boundary conditions. Some progress has been made for surface piercing bodies by Beale (1977) and wide class of geometry by Simon and Ursell (1984). The later work is an extension of the results of John (1950). For the case of fully submerged bodies a uniqueness proof has been given by Maz'ja (1978) for some configurations (see Hulme (1984) for a survey of his work).

John (1950) first formulated the problem in terms of an integral equation of the second kind. However this solution is not uniquely solvable at certain discrete frequencies known as the *irregular frequencies.* So Ursell (1953) and many others reformulated the problem using either by modifying the kernels or by using the method of multipoles. However all these studies assumed that the partially immersed body has a smooth boundary which intersects the mean free surface orthogonally. Kleinman (1982) reformulated the problem for more general geometries in terms of certain integral equations which has no irregular frequencies. Wienert (1988) proved the existence and uniqueness of a solution for these integral equations. Although Wienert (1988) allows boundaries non-orthogonal to the free surface, essentially his uniqueness proof is valid only for boundaries restricted by a vertical line as given by John (1950). Simon and Ursell (1984) considered a wider class of geometries in their proof of uniqueness,where they consider regions bounded by slant lines and curves.

Here we shall consider the 2-D linearized wave problem using complex variable techniques. Kreisel (1949) and Ursell (1950) have used the complex variable techniques to prove uniqueness for certain simple geometries. However their methods do not seem to generalize easily to arbitrary geometries. Here we use the linearized boundary condition to analytically continue the solution in upper half-plane.

A.1 THE UNIQUENESS PROBLEM

Let the fluid region M,be a open subset of $\mathbb{R}^2_-$ (the lower-half plane), whose boundary ∂M, is the union of finite number of compact smooth components (which are the obstacles), and the free surface F (which is a open subset of the line $y = 0$). The fluid is irrotational and the circulation around each of the obstacles is zero. So the velocity can be derived from a global harmonic potential, $\Phi(x, y, t)$. Assuming time harmonic solution we write,

$$\Phi(x,y,t) = \phi(x,y,t)exp(i\omega t) \tag{A.1}$$

so that,

$$\Delta\phi = 0 \quad \text{in } M \tag{A.2}$$

The Neumann boundary condition on the boundaries of the obstacle,

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{in } \{\partial M - F\} \tag{A.3}$$

The assumption of small amplitude waves,the Bernoulli equation and the kinematic boundary condition at the free surface yields, the linearized boundary condition on the mean free surface,

$$K\phi - \frac{\partial\phi}{\partial y} = 0 \quad in\ F \tag{A.4}$$

where $K = \dfrac{\omega^2}{g}$ and the y-axis being taken away from the fluid (upwards).
The radiation conditions at the two infinities being,

$$\frac{\partial\phi}{\partial x} \pm iKA\phi \to 0 \quad as\ x \to \pm\infty \tag{A.5}$$

where A is a constant (this actually ensures that the waves travel outwards at the two infinities and hence no energy is supplied to the fluid). The infinite depth condition being,

$$\frac{\partial \phi}{\partial y} \rightarrow 0 \quad \text{as } y \rightarrow -\infty \tag{A.6}$$

The problem (A.1)- (A.6) completes the uniqueness problem for 2-D linearized water wave potential. To prove uniqueness,it is necessary to show that $\phi(x, y)$ is identically zero. Physically this makes sense since if no energy is being generated and all the energy is being radiated from the system, it will reduce to the zero state. For the cases $\omega \rightarrow 0$ or $\omega \rightarrow \infty$, the problem reduces to the simple Neuman or the Cauchy type boundary conditions at the free surface (and at infinity),for which a simple uniqueness proof can be given using the Green's identity,

$$\int_M |\nabla \phi|^2 dx dy = -\int_{\partial M} \phi \frac{\partial \phi}{\partial n} ds \tag{A.7}$$

Here we propose a method for constructing the solution of the above problem, which probably constitutes also an uniqueness proof. Simon and Ursell (1984), have shown that the amplitude at infinity, $A = 0$, so that by (A.5),the real and imaginary part satisfy a similar problem. So without loss of generality we assume that $\phi(x, y)$ is real. Consider the function,

$$H(x, y) = K\phi - \frac{\partial \phi}{\partial y} \tag{A.8}$$

On F, this satisfies $H(x, y) = 0$, so this function defined in the lower half-plane can be continued into the upper-half plane as an odd function of y, i.e..

$$H(x, y) = -H(x, -y) \;\; \text{forall } (x, y) \in M \text{ and } \; y \geq 0 \tag{A.9}$$

It is easy to check that $H(x, y)$ is harmonic in $M \cup M^*$, where M^* is the region obtained by a mirror reflection of M about $y = 0$. Now let us see how one can obtain the value of $\phi(x, y)$ in the region M^*. Here we restrict ourselves to only the two-dimensional problem. Consider the function of complex variable $z = x + iy$,

$$\Gamma(z) = \frac{dW}{dz} + iKW \tag{A.10}$$

where $W(z)$ is the complex potential given by,

$$W(z) = \phi(x, y) + i\psi(x, y) \tag{A.11}$$

Then,

$$H(x, y) = Im\ \{\Gamma(z)\} \tag{A.12}$$

and so $\Gamma(z)$ is a analytic function in $M \cup M^*$,which satisfies the relation,

$$\Gamma(z) = \Gamma^*(z^*) \tag{A.13}$$

for all $z \in M \cup M^*$.
Solving (A.10) for $W(z)$ we get,

$$W(z) = exp(-iKz) \int_{\infty}^{z} exp(iKz)\Gamma(z)dz \tag{A.14}$$

The integrand being analytic in $M \cup M^*$, the path of integration can be conveniently chosen in the interior of $M \cup M^*$. Although at the non-smooth points of the boundary we expect that $\frac{dW}{dz}$ would be unbounded, the singularity at these points are still integrable singularities. So the integral (A.13) still makes sense at these points. Since $\Gamma(z)$ is analytic in $M \cup M^*$, it has a Laurent expansion of the form,

$$\Gamma(z) = \sum_{i=2}^{i=\infty} \frac{a_i}{z^i} \tag{A.15}$$

$a_0 = 0$ and $a_1 = 0$, since as shown in Simon and Ursell (1984),

$$\phi(x,y) = O(\frac{1 - Ky}{x^2 + y^2}) \text{ as } x \rightarrow \pm\infty \tag{A.16}$$

Using (A.14), outside a circle of sufficiently large radius, R (which excludes the boundaries of the obstacles), the values of $W(z)$ can be evaluated along a line parallel to the x-axis and taking absolute values we get,

$$|W(z)| \leq |\int_{\infty+iy}^{z} |\Gamma(z)|dz| \tag{A.17}$$

Since the expansion (A.15) is valid for sufficiently large z, we can see that,

$$|\int_{\infty+iy}^{z} |\Gamma(z)|dz| \leq M(K)/R \tag{A.18}$$

for all z,satisfying $|z| > R$.
To calculate the value of $W(z)$ at the points $M \cup M^*$ in the interior of the circle of radius R, we choose a convenient contour in the interior which avoids the boundaries of the obstacles. Since $\Gamma(z)$ is bounded in the interior of $M \cup M^*$ (if the boundary is smooth as we have assumed here, then $\Gamma(z)$ will also be bounded at the boundary ∂M) , using (A.14) we arrive at the result,

$$|W(z)| \leq \Lambda(K) \tag{A.19}$$

for all z in $M \cup M^*$.
Here the constant $\Lambda(K)$ depends only on the frequency parameter K and the geometry of the obstacles.

The normal boundary condition (A.3), on any of the boundaries of the obstacle, can be used to generate a reflection principle for $W(z)$. We shall describe a reflection principle for only a circle since by Riemann mapping theorem any such boundary can be mapped to a circle (in Appendix B we shall describe how to derive such conformal mappings).If the boundary is a circle of radius a, then the reflection principle is simply given by,

$$W(z) = W^*(\frac{a^2}{z^*}) \tag{A.20}$$

for all z in $M \cup M^*$. Thus we are able to define $W(z)$ on a larger domain than $M \cup M^*$, which includes a large portion of the interior of obstacles also. By (A.20), we see that the inequality (A.18) applies to this extended region also. So now using (A.10), we can define $\Gamma(z)$ also on this extended region. By reflecting this region about $y = 0$, we can again extend the domain of definition, from which one can again find $W(z)$, again one can reflect it in the inside of the obstacles and so on ... Thus we have established a method by which the domain of definition of $W(z)$ can be gradually extended to include the entire (extended)plane. Across the boundary of the obstacle, we continue the potential as an even function and hence the normal derivative is continued as an odd function,while the tangential derivatives are preserved. However at every reflection about the circle, using (A.20), we can show that the bound on the derivatives grow by a factor greater than one, say α,i.e...

$$|(\frac{dW}{dz})_{i+1}| \leq \alpha|(\frac{dW}{dz})_i| \tag{A.21}$$

To obtain the values of $W(z)$ in the region reflected about the free surface, we use the integral (A.14) and the relation (A.13). If we assume that the contour in (A.14) is taken always above the x-line, then we can arrive at the inequality (A.17). Thus we obtain using (A.21) that,

$$|W(z)_{i+1}| \leq \beta|W(z)_i| \leq \beta^i|W(z)_0| \tag{A.22}$$

It can be shown that after every reflection about the boundaries of the obstacles, the images reduce geometrically in size, reducing in the limit to points. If we assume that these discrete points are singularities of W (z),then they are isolated singularities. But then using (A.22) and the fact that the images reduces geometrically in size, we can show that they cannot be essential singularities. So the isolated singularities must be poles. Let us say that at one of the points $z_0, W(z)$, has a singularity of the form,

$$W(z) \sim \frac{c}{(z-z_0)^m} \tag{A.23}$$

By (A.13), at $z = z_0^*$, we get,

$$W(z) \sim \frac{c}{(z-z_0^*)^m} \tag{A.24}$$

Reflecting this singularity about the boundary(without loss of generality we shall assume that the boundary is a circle of radius a, with centre at $z = b$), we get at $z = z_0$,

$$W(z) \sim \frac{c^*}{(\frac{a^2}{z+b^*} - z_0)^m} \cdot \frac{(\frac{-a^2}{z_0^2})^m c}{(z-z_0)^m} \tag{A.25}$$

where z_0, satisfies the relation, $z_0 + b = \frac{a^2}{z_0}$. This then contradicts, the fact that $W(z)$ has a pole at $z = z_0$, of the form (A.22) assumed in the first place. Hence a pole of the form (A.22) is ruled out. So the singularities are removable singularities. Hence we arrive at an bounded entire function, which by Liouville's theorem must be a constant. Thus $W(z)$ is identically zero.

However the proof given here crucially depends on whether the process of infinite reflections can cover the entire plane or not. But this not difficult to show. Without loss of generality if we consider that all the boundaries are circles, then during every reflection the boundaries become strictly smaller than that considered before the reflection. Since the radius of the circles obtained in the next generation ultimately reduces geometrically in size, they will ultimately converge to zero faster than a geometric sequence. So that the above process of taking infinite reflections, ultimately covers the entire plane ♠

A.2 EXAMPLES

To illustrate the above proof, we consider some examples.

EXAMPLE 1 : Half-submerged circular cylinder

Consider the fluid region in the lower-half plane outside a circle whose centre is the origin(see figure A.1). The functions $W(z)$ and $\frac{dW}{dz}$ are well-defined in the fluid region M. Using the relations (A.13) and (A.14), we can analytically continue $W(z)$ in the mirror-image in the upper-half plane M^*. It immediately follows from (A.15), that

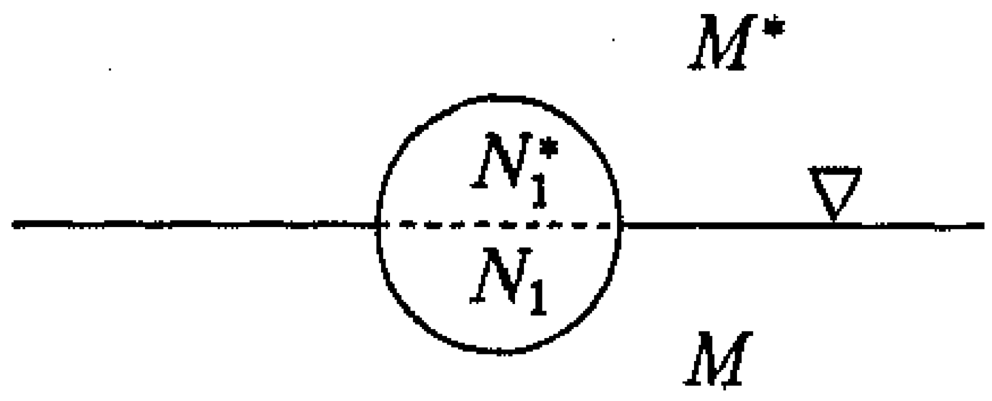


Figure A.1: Half-submerged cylinder

$W(z)$ is bounded in $M \cup M^*$. We now consider the streamline passing through the boundary of the obstacle (this can either be closed or open extending upto infinity). Reflecting the part of the region $M \cup M^*$ below this streamline about this streamline (it should be noted that the normal boundary condition (A.3) is valid in the lower-half of the circle and along this extended streamline), we can define $W(z)$ above this streamline using the relation (A.20). In this analytic continuation, $W(z)$ remains bounded. But now (see figure 1), $M \cup M^* \cup N_1 \cup N_1^* = I\!R^2$. So $W(z)$ is an bounded entire function.

EXAMPLE 2 : Fully-submerged circular cylinder

Again let M be the fluid region outside the circle, whose centre is at $z = -b$, with a radius a ()b). We now get a sequence of regions (see figure 2), $M \cup M^*$,$M \cup M^* \cup N_1 \cup N_1^*$,... The limit point of the converging images is the point,$z = z_0$ inside the circle, where z_0 can be obtained from the relation $z_0 + b = \frac{a^2}{z_0}$. It is easy to show in this case that the region left uncovered,i.e., $I\!R^2 - M \cup M^* \cup N_i \cup N_i^*$ converges like a geometric sequence to the points $z = z_0$ and $z = z_0^*$. Now proceeding as described in the proof given in the last section, we arrive at an bounded entire function.

EXAMPLE 3 : Region with many circular obstacles

In this case the proof is more or less the same, except for the fact that here we will have many images and many limit points. However the limit points are finite and discrete, so the same ideas can be applied.

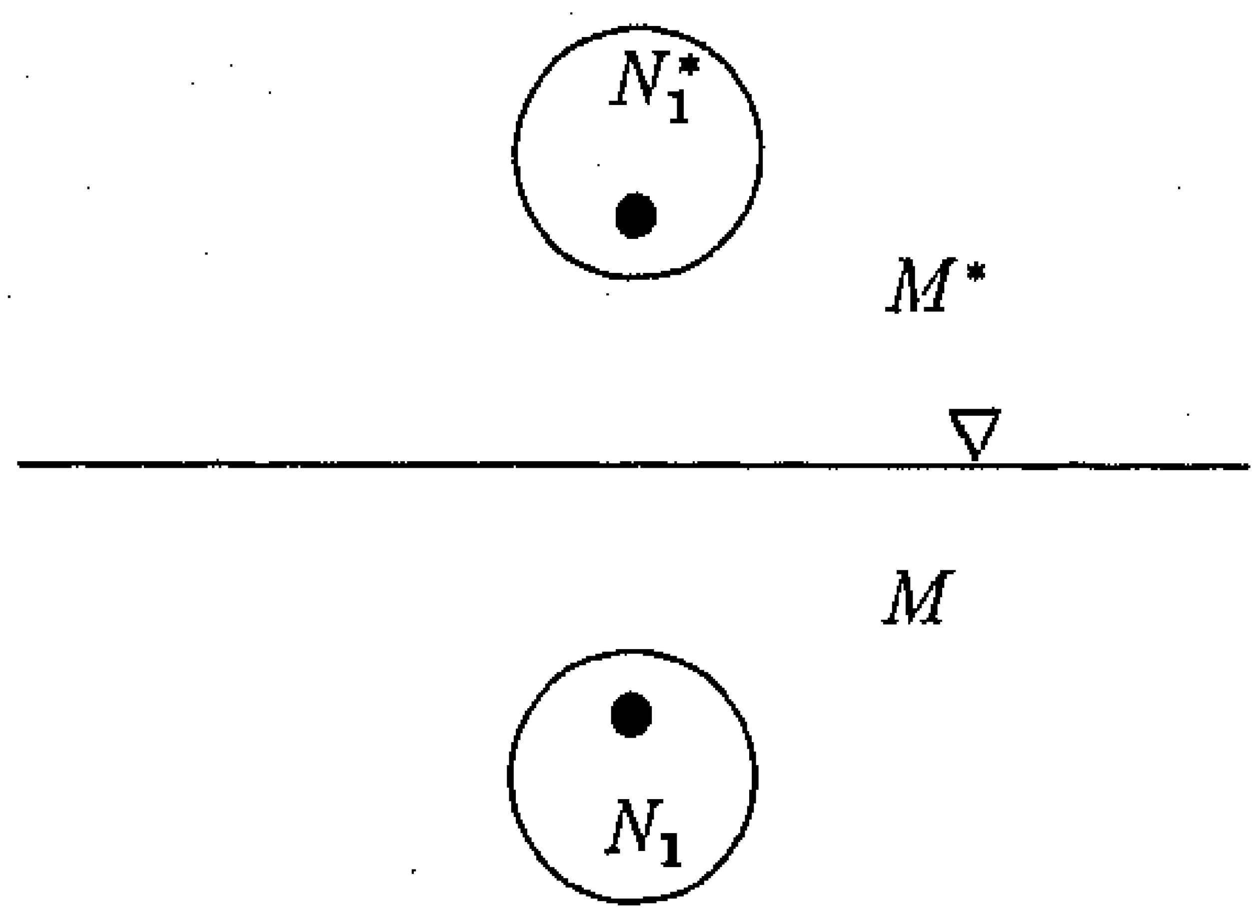


Figure A.2: Fully submerged cylinder

Appendix B

CONFORMAL MAPPING TECHNIQUES FOR 2-D FLOW PROBLEMS

The flow around two dimensional obstacles can conveniently studied by using conformal maps. For 2D potential flows,the flow problem can itself be reduced to the problem of determination of a conformal map. Although for rotational flows,the determination of conformal map is not sufficient to solve the problem, it is useful for simplifying it. Many grid generation procedures have been designed using them (Sridhar and Davis (1985)). By Riemann mapping theorem, any simply connected domain can be mapped to a unit circle (or a half-plane) and for multiply-connected regions similar canonical regions are known (Ahlfors (1953)). Based on this it is possible derive certain form of the mapping functions. Here we shall consider some familiar ones.

B.1 EXTERIOR REGIONS

We now consider regions in $I\!R^2$ which lie exterior to some compact obstacles. As shown in Ahlfors (1953),the canonical region for such exterior region is the region exterior to some circles. The Laurent expansion at infinity of the conformal maps which maps them to circles can be written as,

$$F(z) = c_0 z + \sum_{k=1}^{k=\infty} c_k z^{-k} \tag{B.1}$$

The co-efficients of such maps can be determined by a numerical method. Elementary form of (B.1), has been used in many areas. The Juckowski profiles in aerodynamics (Lighthill (1986)) and the Lewis forms in Naval hydrodynamics (Landweber and Macagno (1957)) are some simpler versions of (B.1), which have been quite useful. Landweber and Macagno (1959) also consider certain extensions of the Lewis forms for ship-like sections. For region exterior to more than one boundary, the expansion (B.1) has to be replaced by a general Mittag-Leffler expansion,

$$F(z) = c_0 z + \sum_{ij} c_{ij}(z - z_i)^{-j} \tag{B.2}$$

For doubly connected region, using (B.2),the author in Venkatesan (1985) found applications for Naval hydrodynamics. The inverse problem of determining the co-efficients of (B.2) for a given geometry is more difficult. Some of the numerical methods, which can used to determine the co-efficients of the conformal maps is described in Trefethen (1986).

B.2 CONFINED REGIONS AND CHANNELS

The canonical region for such domains is the corresponding region with circular or straight obstacles. A polygonal region such as square cavity can be mapped onto a circular region or a half-plane using Schwartz-Christoffel transformation (Milne-Thompson (1968)) . Multiply-connected regions require more intricate analysis. To derive such maps doubly periodic functions, like the elliptic and theta functions, are required. The degenerate cases being the trignometric functions. To illustrate we shall now consider a flow past an arbitrary channel. Channel flows play a very important role: in fact most of the experiments are done only in confined channels. Amick and Frankael (1980) considered the conformal transformation of arbitrary channels onto a flat channels in a theoretical study on the solutions of the Navier-Stokes equations. The maps in general can be written in the form,

$$\frac{df}{dz} = K exp[\frac{(\theta + \delta)\zeta}{2h}]\pi \frac{\{sinh[(\frac{\pi}{2h}(\zeta - b_k)]\}^{\frac{\alpha}{\pi}}}{\{cosh[\frac{\pi}{2h}(zeta - b_m)]\}^{\frac{\alpha_m}{\pi}}} \tag{B.3}$$

where b_k is the location in the transformed flat channel of the m-th corner of the lower channel wall,α_k is the turning angle at the m-th corner in the physical plane, and b_m, α_m are similar quantities for the upper channel wall. A numerical procedure for determination of these co-efficients for a given channel geometry is described in Sridhar and Davis (1985).

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