OPTIMUM CONTINUOUS SAMPLING PLANS AND A FEW OTHER SQC PROBLEMS

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Dedicated
To
Roys of Bolepur,
Santiniketan.

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CONTENTS

CHAPTER 1
INTRODUCTION

1.0 General Introduction

The wide acceptance of Statistics as a basic tool in technological growth, later recognised as a 'Key technology', gained ground with the pioneering work of Shewhart in 20's, introducing Statistical Quality Control (SQC) in manufacturing industry. Around the same time a solid statistical basis was being worked out for the ageold concepts of sampling inspection for industrial products. By 1930, acceptance sampling for lot by lot inspection was being applied in Western Electric Company and elsewhere. Since then statistical tools have been the major technical inputs of Total Quality Management which has spread far and wide as a movement for better customer satisfaction as well as reduced loss and cost for manufacturing industries. Statistical techniques may be broadly used in industry for (i) Process Control (ii) Acceptance Sampling for Quality Evaluation and Assurance (iii) Special in-depth studies with a view to achieving Process Improvement and (iv) Design and Planning of Industrial Experimentation for optimal determination of factor combinations for a breakthrough.

In our present work we have mainly dealt with (a) selection of Optimum Continuous Sampling plans that minimise the amount of inspection when the incoming process quality is maintained at a desirable level.

We have also studied a few topics of interest such as (b) computation of Tolerance Factor λ (c) institution of Gauge Control chart (d) determination of Optimum Inspection Interval for controlling wastes of a high speed printing machine and (e) assessment of the effect of Inspection Error on lot by lot acceptance rectification inspection, covering ranges of application in the domains of quality evaluation, process control and quality assurance.

1.1 Introduction to CSP Plans Several authors have enriched the field of

Continuous Sampling Plan since the publication of CSP - 1 by Dodge (1943). A first comprehensive review of the work was made by Bowker (1956). This has been updated properly from time to time by various authors including Lieberman (1965), Phillips (1969), Banzhof and Brugger (1970), Wetherill (1977) and Stephens (1979). Close on the heels along with Dodge, Wald and Wolfowitz (1945) introduced SPA plans which, unlike CSP - 1, start with sampling inspection first and change to cent per cent inspection whenever required. The authors also modified SPA to SPB; several other related modifications were later introduced by Sahani (1979).

Dodge and Torrey (1951) in their plans, CSP - 2 and CSP - 3 suggested some modifications over CSP - 1 regarding return to cent per cent inspection from the sampling inspection phase. Derman, Johns and Lieberman (1959) considered further modifications and proposed CSP - 4 and CSP - 5. Lieberman and Solomon (1955) introduced a major change in the operating rules of CSP's by suggesting Multilevel Sampling Plans (MLP) introducing two to an infinite number of levels of inspection. Read and Beattie (1961) introduced the concept of variable lot size in the same context.

In industry, however, Dodge type CSP plans became popular largely due to its adoption by US Army Ordnance Corps in the early fifties for inspection of ammunition components, aircraft engines and propellers. Various standards, viz., NAVORD OSTD - 81 (1952), ORDM 608 II (1954), AMC Manual (1956), H 106, H 107, MIL - STD 1235 ORD (1962), MIL - STD - 1235 A (1974) were developed, modified and finally revised to MIL - STD - 1235 C (1988).

While several authors have suggested important and useful modifications to CSP plans, many have worked in the area of Multilevel Sampling Plans. Appreciable work on the use of Continuous Sampling Plan in the field of trouble shooting and adaptive control has been done by a large number of research workers. The idea of CSP leads to the development of some special purpose plans also. An attempt has been made to list the notable contributors in all these areas in section A of the reference. The range of contributions gives a broad idea about the main directions of theoretical development and widening areas of application of what can be generally characterised as Continuous Sampling Plans originally introduced by Dodge (1943).

1.2 Scope of the Present Work on CSP (in chapters 2,3,4 and 5)

A common feature of Dodge's CSP - 1, CSP - 2, CSP - 3 as well as Multilevel CSP plan is that several combinations of plan parameters are possible which will ensure a desired AOQL. This makes the choice of a CSP plan extremely difficult.
Stephens (1981) provided a procedure of selecting the plan parameters (i,f) for CSP - 1 for which the consumer's risk is 0.10 for a stipulated value of incoming process quality called the Limiting Quality Level (LQL). The plan is somewhat analogous to Dodge's LTPD stipulated plan for lot-by-lot acceptance rectification inspection. Though this restricts the choice of (i,f) in a narrower region it does not lead to a unique choice of (i, f). Govinda Raju (1989) provided methods to find plan parameters of CSP - 1 which will satisfy a given combination of (LQL, AOQL) or (Acceptable quality level (AQL), AOQL). However, all combinations were not compatible, moreover the values of i were found to be exceedingly large in many cases, limiting its practical use. Agarwal (1980) studied the optimal determination of CSP - 1 under uncertainty using decision theoretic approaches of Wald's minimax criterion, Laplace's criterion, Savage's regret criterion and Hurwic's criterion. However, choice of any one of these criteria is subjective, adding fresh controversy over the choice of plan parameters. Further, all such modifications have been tried on CSP - 1 and no attempts are observable in developing practically useful methods for determination of plan parameters in case of CSP - 2 and CSP - 3 or Multi level CSP plans.
In the present work, we are primarily concerned with the unique selection of plan parameters of CSP - 1, CSP - 2, CSP - 3 and Multilevel plan with k = 2 (MLP - 2), herein after referred to as Two level CSP Plan, by incorporating some practically meaningful and useful criteria along with the specified AOQL. In this we follow the classical approach of Dodge and Romig (1941) as in the case of AOQL stipulated lot-by-lot inspection scheme.
For the case of continuous inspection, in Chapter 2, we have developed procedures to find plan parameters (i,f) in case of CSP - 1 and to find (i,f), given **k** for plan

parameters (i,f,k) in case of CSP - 2 and CSP - 3 uniquely which will minimise the average fraction inspected (AFI) for a given process average $\overline{p}$, while ensuring a desired AOQL. A major hurdle in studying CSP - 2 and CSP - 3 plans was the absence of a precise analytical procedure to find the combinations of plan parameters (i,f,k) which will ensure a given AOQL. We have succeeded in developing a neat procedure for the purpose.

In Chapter 3, Stephens' (1981) work on LQL - based continuous sampling plan (CSP - 1) has been extended to find unique optimum combination of (i,f) which will (a) ensure a stipulated LQL with associated consumer's risk of 0.10, (b) ensure an AOQL which is less than or equal to some given value p_L and (c) minimise the AFI for a given process average $\overline{p}$ among those plans satisfying (a) and (b).

In Chapter 4, we have also developed a procedure for finding the optimum plan parameters (i,f) for CSP - 1 which will minimise the AFI when incoming quality p follows any known distribution. This has been extended to CSP - 2 and CSP - 3 plans also.

The minimum AFI for the given process average $\overline{p}$ is found to be same for all the optimum CSP plans (1,2 and 3). So, in Chapter 5, the performances of optimum CSP plans are compared with respect to other performance criteria like p_t (%), the resulting AOQ curve and the average fraction inspected curve over the entire range of p.

Since expressions for p_t (%) for CSP - 2 and CSP - 3 are not available in literature we have worked out the expressions for these two plans using Markov Chain formulation of the plans. The scope of comparisons has also been extended to Two level CSP plan (MLP (k = 2)) as the operational difficulty for this plan is more or less comparable to CSP - 2 and CSP - 3 plans.

To make the comparison valid, we have found out, empirically, the unique optimum plan parameters for the Two level CSP plan and also obtained expression for p_t (%) in this case.

The comparison of the performances of different optimum plans facilitates the selection of the most appropriate plan under a wide variety of situations.

1.3 Summary of Results in Different Areas of Statistical Quality Control in Chapter 6

(a) Section 6.1 deals with the Tolerance Factor λ. Wald and Wolfowitz (1946) provide us with a formula for the Tolerance Factor to obtain two sided tolerance limits for a quality characteristic which is distributed normally and whose mean and variance are both unknown. Bowker (1946) proposed an approximate formula for λ which performs well for large samples. We have developed two formulas for λ which perform well for both large and small samples.

(b) In order to improve the performance of 'c - a' and 'c + a' gauge charts introduced by Stevens (1948), we propose in section 6.2 a Median Gauge chart which is a combination of the usual Median ($\tilde{X}$) chart and Steven's 'c + a' chart in a manner that no measurement is required, ensuring the retention of the benefit of gauging and yet we get better efficiency as compared to Stevens' chart.

(c) In section 6.3 we determine the Optimum Inspection Interval for the control of bulk damage for a high speed printing machine in Textile industry. Extending Duncan (1956) we develop a procedure to minimise the total cost per cycle comprising of (a) the cost of unnecessary stoppages plus (b) the cost of avoidable waste, by proper choice of the inspection interval.

(d) In section 6.4 we study the effect of two kinds of inspection errors on acceptance rectification sampling scheme. We observe that if the incoming quality p exceeds some value, p^u which depends only on the magnitude of inspection errors and the desired AOQL, then the AOQL stipulation can not be met. If, however, the inspection errors are of the same order and both less than the desired AOQL, then it is shown that the value of p^u exceeds 0.50. Since the actual p is usually expected to be less than 0.50, the application of Dodge - Romig plans to ensure a stipulated AOQL can be safely recommended in the above case where inspection errors are present.

CHAPTER 2

OPTIMUM CSP PLANS MINIMISING AMOUNT OF INSPECTION FOR PROCESS AVERAGE $\bar{p}$

2.0 Introduction

Continuous sampling plans were devised for processes involving a continuous flow of products. CSP plans provide for corrective inspection with a view to having a limiting average outgoing quality called AOQL, no matter what quality is submitted for inspection. The basic assumptions are that (i) all defectives found during inspection are rectified or replaced by good items and (ii) the process is statistically controlled at some p i.e. the process is producing defectives with probability p [Wetherill (1991)].

The pioneering work in the field of continuous sampling plans is due to Dodge (1943). His original plan designated as CSP - 1 is characterised by two parameters (i,f) and suggests 100 percent inspection of products at the outset till i successive units are found to be non-defective. There after only a fraction f of the items is to be inspected. As soon as a single defective item is observed, immediate reversion to 100 percent inspection takes place, which continues until again i non-defective items are found in succession. The plan parameters (i,f) are chosen to attain a desired AOQL.

Dodge & Torrey (1951) offered two modifications of CSP - 1 plan which are designated as CSP - 2 and CSP - 3.

In CSP - 2 the decision criterion for changing over to cent percent inspection is relaxed. Thus the plan CSP - 2 essentially differs from CSP - 1 in that once sampling inspection is started, 100 percent inspection is not invoked immediately when a defect is found but is invoked only if a second defect occurs in the next k or less sample units. The factor k may be theoretically assigned any value. It is difficult to find analytically (i,f,k) that would ensure a given AOQL. Dodge &

Torrey studied CSP - 2 plans with k = i and obtained several combinations of (i,f,i) by trials to ensure a desired OQL.

CSP - 3 introduces a simple and effective refinement of CSP - 2 and aims at providing extra protection against sudden deterioration of incoming quality. Following the occurrence of a defective unit during a sampling phase, the next four consecutive units are inspected and are required to be free of defects, for sampling to continue on the CSP - 2 basis. Otherwise, the 100 percent inspection phase is invoked immediately. The parameters of CSP - 3 plans are also (i,f,k) with the compulsory inspection 'rule of four'. The flow charts of operation for CSP - 1, CSP - 2 and CSP - 3 are given in Fig. 2.0.1.

The basic problem with all the CSP plans is that there will be many combinations of plan parameters, namely, (i,f) for CSP - 1 and (i,f,k) for CSP - 2 and CSP - 3 that would ensure a given AOQL. Yet, there is no well defined criterion of choosing a particular combination for a given AOQL. Coupled with this, the CSP - 2 and CSP - 3 plans have not been studied adequately in literature in the absence of well defined procedures to find (i,f,k) for them. In the present work we are primarily concerned with these two problems.

For a formal presentation of the problems in Mathematical terms we need to introduce a few symbols. For all the plans to be considered AOQL is given and the given value is denoted by p_L. The process is producing defectives with probability p which is taken as the incoming quality. $\overline{p}$ is the desired process average (in the Dodge Romig sense) at which the process should be ideally controlled and is given. i, f and k are the plan parameters. Let AOQ represent in general the average out going quality and F the average fraction inspected (AFI) for a given p. p_1 denotes the quality level at which the AOQL is reached.

All these quantities are also functions of plan parameters. Thus the symbols used are :

$$\left.\begin{array}{r} p_1^1(i,f),\ F^1(i,f,p),\ AOQ^1(i,f,p) \text{ for } CSP-1 \\ p_1^2(i,f,k),\ F^2(i,f,k,p),\ AOQ^2(i,f,k,p) \text{ for } CSP-2 \\ p_1^3(i,f,k),\ F^3(i,f,k,p),\ AOQ^3(i,f,k,p) \text{ for } CSP-3 \end{array}\right\} \qquad (2.0.1)$$

Let us define the following sets :

N = Set of all natural numbers,

R = Set of real numbers,

$\Omega = \{p \in R, o \leq p \leq 1\}$.

Then (i,f) $\in N \times \Omega$ and $(i,f,k) \in N \times \Omega \times N$.

Thus the functions introduced above can be appropriately defined with their domains and ranges specified as follows :

$F^1(i,f,p) : N \times \Omega \times \Omega \to \Omega$

$F^2(i,f,k,p) : N \times \Omega \times N \times \Omega \to \Omega$

$F^3(i,f,k,p) : N \times \Omega \times N \times \Omega \to \Omega$

$AOQ^1(i,f,p) : N \times \Omega \times \Omega \to \Omega$

$AOQ^2(i,f,k,p) : N \times \Omega \times N \times \Omega \to \Omega$

$AOQ^3(i,f,k,p) : N \times \Omega \times N \times \Omega \to \Omega$

$p_1^1(i,f) : N \times \Omega \to \Omega$

$p_1^2(i,f,k) : N \times \Omega \times N \to \Omega$

$p_1^3(i,f,k) : N \times \Omega \times N \to \Omega$

It may be noted that mathematically

AOQL = Maximum of AOQ function with respect to p $\in \Omega$, given the plan parameters (i, f) in case of CSP - 1 and (i, f, k) in case of CSP - 2 and CSP - 3. (2.0.2)

For a plan with AOQL specified to be p_L, f can be written in terms of (i) i in case of CSP - 1 and (ii) (i, k) in case of CSP - 2 and CSP - 3.

Thus for plans with AOQL = p_L given, we will use the corresponding symbols as

$$\left.\begin{array}{r} p_1^{1L}(i),\ F^{1L}(i,p),\ AOQ^{1L}(i,p) \text{ forCSP} - 1 \\ p_1^{2L}(i,k),\ F^{2L}(i,k,p),\ AOQ^{2L}(i,k,p) \text{ for CSP} - 2 \\ p_1^{3L}(i,k),\ F^{3L}(i,k,p),\ AOQ^{3L}(i,k,p) \text{ for CSP} - 3 \end{array}\right\} \tag{2.0.3}$$

The domains and ranges of the functions are obvious from preceding discussions. The parameters $\overline{p}$, p, f, p_L, i and k will always be used in the present work in chapters 2 through 5 in the same sense as explained. When multiple, say m values, of AOQL are to be considered simultaneously, we may also use the symbols p_{Li}, i = 1, 2, ... m for them.

In all subsequent discussions and statements in chapters 2 through 5, whenever the aforementioned symbols are used it is tacitly assumed that

$$\overline{p}, p, f, p_L, p_{Li}, \in \Omega \text{ and } i, k \in N$$

This may not always be stated explicitly.

Now we are in a position to state the problems formally and they are presented in the following lines :

(i) Given AOQL = $p_L \in \Omega$, find the parameters (i, f, k) $\in$ N $\times\ \Omega \times N$ that will ensure the given AOQL, in case of CSP - 2 and CSP - 3.

(ii) Given AOQL = $p_L \in \Omega$ and $\overline{p}$ = process average $\in \Omega$, find a CSP - 1 with unique (i, f) $\in N \times \Omega$ which while ensuring the AOQL at p_L, will minimize the AFI, viz., F^{1L} at $\overline{p}$, among all the CSP - 1 plans with AOQL = p_L.

(iii) Given AOQL = $p_L \in \Omega$, $\overline{p}$ = process average $\in \Omega$, and k $\in$ N, find a CSP - 2 (CSP - 3)with unique (i, f) $\in N \times \Omega$ which, while ensuring the AOQL at p_L will minimize the AFI, viz., F^{2L} (F^{3L}) at $\overline{p}$ among all CSP - 2 (CSP - 3) plans with AOQL = p_L.

The problems (i) and (iii) in case of CSP - 2 under the additional condition k = i have been completely solved Mathematically in the present chapter, where as practical solutions have been provided for (i) and (iii) in the general situation (k not necesarily equal to i) for both CSP - 2 and CSP - 3 although no exact analytical solution could be provided. In the latter case the approach is partly Mathematical and partly empirical based on numerical computations in the normal region of interest. The CSP - 2 and CSP - 3 plans so established are found to be unique in all numerical examples covered in our computation (presented in Appendix - 1,2 and 3).

All relevant results and observations in this connection are presented in Section 2.1 through 2.4.

Some of the results presented here in Sec. 2.1, 2.2 and 2.3 have been published in Ghosh (1988, 1989, 1990 and 1990).

The problem of finding the combination of (i, f) for CSP - 1 which will minimise the amount of inspection for a given $\bar{p}$ was also considered by Resnikoff (1960). It may be mentioned that (i) our approach is different, (ii) done independently and already published and (iii) provides results which are stronger and more general (with some overlap).

2.1 Optimum CSP - 1 Plan

2.1.1. Introduction

Since several combinations of (i, f) are possible for a stipulated AOQL in CSP - 1, it is difficult to select a particular combination of (i, f).

One procedure would be to select f not too low such that the spotty quality p_t (%)as defined by Dodge is not high. [p_t (%) = the percent defective in a consecutive run of N = 1000 units for which the probability of acceptance under sampling phase is 0.10.] However, p_t (%) may not always be a suitable criterion, firstly because the incoming quality may never be as bad as p_t (%) and secondly because it depends on the value of N which has been choosen somewhat arbitrarily as 1000. The same value 1000 of N was used by Stephens (1979) and others. We also follow the same convention for the sake of parity but it may be noted from table 2.1.1 that a change in N can substantially alter the p_t (%) for a given f.

Table 2.1.1 : p_t (%) For Different N.

	N				
f	1000	2000	3000	4000	5000
.0020	100.0	57.5	38.4	28.8	23.0
.0266	8.6	4.3	2.9	2.2	1.7

It is also not true that for a specified AOQL selection of a higher value of f to protect against spotty quality will necessarily amount to larger inspection. This can be seen from the amount of inspection for a few plans given below.

Table 2.1.2 : Average Fraction Inspected (%) For a Few Selected Plans, Each Ensuring an AOQL of 5%.

	AFI (%)		
	Plan 1	Plan 2	Plan 3
Process	$f = .20$	$f = .10$	$f = .05$
Average $\bar{p}$	$i = 13$	$i = 21$	$i = 29$
.01	23.2	12.2	6.7
.02	25.7	14.5	8.7
.03	28.7	17.4	11.5
.04	31.3	20.7	14.9
.05	34.1	24.5	19.2
.06	37.3	28.9	24.4
.07	40.6	33.7	30.5
.08	44.0	38.9	37.6
.09	47.5	44.5	45.2
.10	51.5	50.3	53.2
.12	58.3	61.9	68.6

It would appear that for plans with $\bar{p}$ upto 0.08, lower value of f will result in lesser inspection. But for $\bar{p}$ between 0.09 and 0.12 higher f, which ensures better protection against spotty quality, may also result in lower inspection.

Let us consider two plans, one with i = 10 and f = .2851 and the other with i = 50 and f = .0105. Both the plans ensure an AOQL of 5%. The AFI (%) for the two plans for $\overline{p} = 0.06$ is shown below :

Plan parameters	AFI (%) at $\overline{p} = 0.06$
i = 10, f = .2851	42.54
i = 50, f = .0105	19.02

Thus the choice of (i, f) in a particular situation greatly affects the amount of inspection.

For all the lot by lot acceptance rectification sampling plans developed by Dodge and Romig the criterion to minimise the amount of inspection if the process is controlled at a certain proces average $\overline{p}$ was followed.

It is, therefore, natural to look for a combination of (i, f) that would not only ensure AOQL but would also require the minimum amount of inspection for a given process average $\overline{p}$.

2.1.2 Basic Formulas

The symbols and the formulas used here are either those originally adopted by Dodge or others given in the introduction 2.0.

In a typical cycle of complete and sampling inspections in a CSP plan let

u stand for the expected number of items inspected under complete enumeration phase and

v stand for expected number of items that are produced during the sampling phase of the cycle.

We have, in CSP - 1,

$$u = \frac{1 - q^i}{pq^i} \tag{2.1.1}$$

$$v = \frac{1}{fp}, \qquad (2.1.2)$$

where $q = 1 - p$.

The Average Fraction Inspected as defined by $F^1(i, f, p)$ in CSP - 1 in (2.0.1), is given by

$$F^1(i, f, p) = \frac{u + fv}{u + v}$$

$$= \frac{f}{f + (1 - f)(1 - p)^i} \qquad (2.1.3)$$

and the Average Outgoing Quality function as defined by AOQ^1 (i, f, p) in (2.0.1) is given by

$$AOQ^1(i, f, p) = p(1 - F^1) = p\left(1 - \frac{u + fv}{u + v}\right) \qquad (2.1.4)$$

p_L is the specified AOQL as mentioned before. Now the plan identified by the plan parameters (i, f) *must satisfy* this AOQL specification. One can easily see that

$$p_L = p_1^{1L}(i)(1 - \frac{f}{f + (1 - f)(1 - p_1^{1L}(i))^i}, \qquad (2.1.5)$$

giving

$$p_1^{1L}(i) = \frac{ip_L + 1}{i + 1} \qquad (2.1.6)$$

$$\text{and } f = \frac{(1 - p_1^{1L}(i))^{i+1}}{ip_L + (1 - p_1^{1L}(i))^{i+1}} \qquad (2.1.7)$$

Here it may be recalled that p_1^{1L} (i) is the value of p where the specified AOQL is attained. Also, it may be repeated here that given p_L, f of the plan is simply a function of i and hence *in the argument of* p_1^{1L} *only one element* i is used.

2.1.3 On Some Properties of p_1^{1L}, i, f and F^{1L} for CSP - 1

Lemma 2.1.1 : *For a given p_L, the value of proportion defective p_1^{1L} (i) for which the AOQL value is attained decreases as i increases.*

$$\text{Proof}: p_1^{1L}(i) = \frac{ip_L + 1}{i+1} = \frac{(i+1)p_L}{(i+1)} + \frac{1-p_L}{i+1} = p_L + \frac{1-p_L}{i+1}$$

Since $\frac{1-p_L}{i+1}$ decreases as i increases, the result follows.

Lemma 2.1.2 : *For a given p_L, the sampling fraction f decreases as i increases.*

$$\text{Proof}: f = \frac{(1-p_1^{1L}(i))^{i+1}}{ip_L + (1-p_1^{1L}(i))^{i+1}} \text{ from (2.1.7)}$$

$$\text{or } \frac{1}{f} = \frac{ip_L}{(1-p_1^{1L}(i))^{i+1}} + 1$$

$$\begin{aligned} Again 1 - p_1^{1L}(i) &= 1 - (p_L + \frac{1-p_L}{i+1}) \\ &= (1-p_L)[1 - \frac{1}{i+1}] \\ &= (1-p_L).\frac{i}{i+1} \end{aligned}$$

$$\begin{aligned} \frac{1}{f} &= \frac{ip_L}{(1-p_L)^{i+1}.(\frac{i}{i+1})^{i+1}} + 1 \\ &= \{p_L.\frac{(i+1)^{i+1}}{i^i}\} \{(\frac{1}{1-p_L})^{i+1}\} + 1 \\ &= g_1(i)\ g_2(i) + 1 \end{aligned}$$

where g_1(i), g_2(i) are both > 0 for all i > 0 and have positive differential coefficients for all i > 0.

Thus, $\frac{1}{f}$ increases with i.

Hence the result.

Lemma 2.1.3 *: For a given p_L, the average fraction inspected as measured by F^{1L} (i, p) is given by*

$$F^{1L}(i,p) = \frac{1}{1+\frac{(i+1)^{i+1}}{i^i}.\frac{p_L}{1-p_L}.(\frac{1-p}{1-p_L})^i} \tag{2.1.8}$$

Proof. Using (2.1.3) and (2.1.7) we have

$$\begin{aligned} F^{1L}(i,p) &= \frac{f}{f+(1-f)(1-p)^i} \\ &= \frac{1}{1+(\frac{1}{f}-1)(1-p)^i} = \frac{1}{1+\frac{ip_L}{(1-p_1^{1L}(i))^{i+1}}.(1-p)^i} \end{aligned}$$

Using (2.1.6) we have

$$\begin{aligned} \frac{ip_L}{(1-p_1^{1L}(i))^{i+1}} &= \frac{ip_L}{1}.\frac{(i+1)^{i+1}}{i^{i+1}(1-p_L)^{i+1}} \\ &= \frac{(i+1)^{i+1}}{i^i}.\frac{p_L}{1-p_L}.\frac{1}{(1-p_L)^i} \end{aligned}$$

Hence the result.

Thus for a given p_L, AFI is a function of i alone as mentioned before and hence in the notation the argument f is dropped. Treating i as a continuous variable and p fixed along with p_L we rewrite F^{1L} (i, p) as F(x), where

$$F(x) = \frac{1}{1+cb^x.\frac{(x+1)^{x+1}}{x^x}} \tag{2.1.9}$$

It may be noted that c = $\frac{p_L}{1-p_L}$ and b = $\frac{1-p}{1-p_L}$ are here constants. It may be noted that b < 1 for all p > p_L.

We define $\Psi(x) = cb^x.\frac{(x+1)^{x+1}}{x^x}$ and

$$\phi(x) = ln\ \Psi(x)$$

$$\text{and obtain } \phi'(x) = lnb + ln(x+1) - lnx \tag{2.1.10}$$

$$\begin{aligned} \text{and } \phi''(x) &= \frac{1}{x+1} - \frac{1}{x} \\ &= -\frac{1}{x(x+1)} \end{aligned} \tag{2.1.11}$$

which is less than zero for all x > 0.

Lemma 2.1.4 : $F'(x) \lesseqgtr 0$ according as x $\lesseqgtr x_0$ where $\frac{x_0}{x_0+1} = b$

Proof : We have F(x) $= \frac{1}{\Psi(x)+1} = \frac{1}{e^{\phi(x)}+1}$

$$\begin{aligned} \text{and } F'(x) &= -\frac{e^{\phi(x)}\phi'(x)}{(e^{\phi(x)}+1)^2} \\ &= \frac{e^{\phi(x)}}{(e^{\phi(x)}+1)^2}[ln\frac{x}{x+1} - lnb] \end{aligned}$$

Hence the result.

Lemma 2.1.5 : *The function* $h(x) = \frac{1}{x(x+1)} - \{ln\frac{b(x+1)}{x}\}^2$ *is positive for all positive integer* $x \leq x_0$ *and for all* $b < 1$.

Proof : We have

$$\begin{aligned} h'(x) &= -\frac{2x+1}{x^2(x+1)^2} + 2\{lnb + ln(x+1) - lnx\}.\frac{1}{x(x+1)} \\ &= \frac{1}{x(x+1)}\{-\frac{2x+1}{x(x+1)} + 2lnb + 2ln(1+\frac{1}{x})\} \\ &= \frac{1}{x(x+1)}[-\frac{1}{x} - \frac{1}{x+1} + 2lnb + 2\{\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4(x+\theta)^4}\}] \end{aligned}$$

where $0 < \theta < 1$.

$$= \frac{1}{x(x+1)}[2lnb + \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x^2} + \frac{2}{3x^3} - \frac{1}{2(x+\theta)^4}]$$
< 0 for $x \geq 2$.

At x_0, $b = \frac{x_0}{x_0+1}$ and hence $ln\ \frac{b(x_0+1)}{x_0} = 0$.

Thus $h(x_0) = \frac{1}{x_0(x_0+1)} - 0 > 0$.

It can be easily verified that h(x) > 0 for x = 1 and for all b < 1.
So, h(x) > 0 for $1 \leq x \leq x_0$ and hence the result follows.
Lemma 2.1.6 : *$F''(x) > 0$ for all $x \leq x_0$ and $b < 1$.*
Proof : We have

$$F''(x) = -[(e^{\phi(x)}+1)^2\{e^{\phi(x)}(\phi'(x))^2 + e^{\phi(x)}\phi''(x)\} -$$

$$2e^{\phi(x)}\phi'(x)(e^{\phi(x)}+1)e^{\phi(x)}\phi'(x)]/(e^{\phi(x)}+1)^4$$

$$= -\frac{(e^{\phi(x)}+1)e^{\phi(x)}\{(\phi'(x))^2 + \phi''(x)\} - 2e^{2\phi(x)}(\phi'(x))^2}{(e^{\phi(x)}+1)^3}$$

$$= -\frac{e^{\phi(x)}}{(e^{\phi(x)}+1)^3}[e^{\phi(x)}(\phi'(x))^2 + e^{\phi(x)}\phi''(x) + (\phi'(x))^2 + \phi''(x) - 2e^{\phi(x)}(\phi'(x))^2]$$

$$= -\frac{e^{\phi(x)}}{(e^{\phi(x)}+1)^3}[\phi''(x)(1+e^{\phi(x)}) + (\phi'(x))^2(1-e^{\phi(x)})]$$

Using (2.1.10) and (2.1.11), the sufficient condition for F'' (x) > 0 is that

$$\frac{1+e^{\phi(x)}}{x(x+1)} - (\phi'(x))^2(1-e^{\phi(x)}) > 0$$

$$or \ \frac{1}{x(x+1)} > (\phi'(x))^2$$

$$or \frac{1}{x(x+1)} > \{ln\frac{b(x+1)}{x}\}^2$$

a condition which is true in view of **lemma** 2.1.5. This completes the proof.

2.1.4 Optimum Combination of (i, f) that Minimises AFI for a Given Process Average $\overline{p}$ and Ensures a Desired AOQL.

The problem has been formally introduced in Section 2.0. The procedure is developed on the basis of the following theorems.

Theorem 2.1.1 : *For a given p_L, $F^{1L}(i,\overline{p})$ for a given incoming quality $\overline{p}$ decreases monotonically for all i as long as $\frac{i}{i+1} < \frac{1-\overline{p}}{1-p_L}$ and then increases monotonically for all i for which $\frac{i}{i+1} > \frac{1-\overline{p}}{1-p_L}$ i.e. attains its minimum for the value of i for which $\frac{i}{i+1} = \frac{1-\overline{p}}{1-p_L}$ for all $\overline{p} > p_L$.*

Proof : Consider $F^{1L}(i,\overline{p})$ as given in (2.1.8)
The above result follows from Lemmas 2.1.3 and 2.1.4 once we note that b = $\frac{1-\overline{p}}{1-p_L} < 1$

$$\text{Thus } F^{1L}(i,\overline{p}) \downarrow \text{ in i for } \frac{i}{i+1} < \frac{1-\overline{p}}{1-p_L}$$

$$\text{and } F^{1L}(i,\overline{p}) \uparrow \text{ in i for } \frac{i}{i+1} > \frac{1-\overline{p}}{1-p_L}$$

and attains its minimum for $\frac{i}{i+1} = \frac{1-\overline{p}}{1-p_L}$

$$\text{i.e. } i = \frac{1-\overline{p}}{\overline{p}-p_L} \qquad (2.1.12)$$

Theorem 2.1.2 : *For a given p_L and the process average $\overline{p} > p_L$, $F^{1L}(i, \overline{p})$ goes on decreasing with increase in i and corresponding decrease in p_1^{1L} (i) as long as p_1^{1L} (i) $> \overline{p}$ and becomes minimum when p_1^{1L} (i) $= \overline{p}$.*

Proof : Let $\overline{p}$ be less than or equal to p_1^{1L} (i)

$$\text{Then } \overline{p} \leq p_1^{1L}(i) = \frac{ip_L+1}{i+1} = p_L + \frac{1-p_L}{i+1}$$

$$\iff \overline{p} - p_L \leq \frac{1-p_L}{i+1}$$

$$\iff \frac{\overline{p}-p_L}{1-p_L} \leq \frac{1}{i+1}$$

$$\iff 1 - \frac{\overline{p}-p_L}{1-p_L} \geq 1 - \frac{1}{i+1}$$

$$\iff \frac{i}{i+1} \leq \frac{1-\overline{p}}{1-p_L}$$

Hence the result.

Thus we have, $\overline{p} \overset{\leq}{\underset{>}{=}} p_1^{1L}(i) \iff \frac{i}{i+1} \overset{\leq}{\underset{>}{=}} \frac{1-\overline{p}}{1-p_L}$.

We thus have a unique combination of (i, f), say (i_0, f_0), which minimises the AFI for a given process average $\overline{p}$ provided $\overline{p} > p_L$.
The procedure for finding (i_0, f_0) is given in the following algorithm.

Algorithm : MNFCS1 $(\overline{p}, p_L, i, f)$
(* input parameters : $\overline{p}, p_L$
output parameters : i, f *)
(* Given $\overline{p}, p_L$, the algorithm gives (i, f) that ensures AOQL $= p_L$ and minimise AFI at $\overline{p}$ under CSP - 1 inspection *)

```
Begin
        i ← (1−p̄)/(p̄−p_L);
        if i is not an integer then
        begin i ← ⌊ i ⌋;
           if (F^{1L}(i,p̄) > F^{1L}(i+1,p̄)) then
           i ← i + 1
        (* F^{1L}(i,p̄) is calculated using (2.1.8) *)
        end;
        p_1 ← p_L + (1−p_L)/(i+1); f ← (1−p_1)^{i+1} / (i*p_L+(1−p_1)^i);
        (* See formula (2.1.7) *)
end.
```

The following points may be highlighted at this stage.

N.B. 2.1.1

It may be noted that there my be atmost two consecutive i's for which the same minimum inspection is satisfied given p_L and $\overline{p}$. However, given $\overline{p}$ and p_L, the large number of examples worked out by us show that (i_0, f_0) turns out to be unique.

N.B. 2.1.2

It is interesting to note that given $\overline{p}$ and p_L, the optimum plan is one for which the specified AOQL as p_L is attained at p_1^{1L} (i) $= \overline{p}$. In view of this the optimum combination (i_0, f_0) also leads to the minimum amount of inspection to achieve the AOQL considering the class of all CSP plans (not merely CSP - 1 plan and not necessarily of Dodge's type) with AOQL specified to be p_L. This is so because, given any CSP plan, AOQL $= p_L$ will be attained at some p, say p_1, which may or may not be same as $\overline{p}$. Now assuming unimodality of AOQ curves (for these plans), if p_1 is different from $\overline{p}$, AFI of the plan at $\overline{p}$ will be greater than or equal to the AFI for the given CSP - 1 plan for which AOQL of p_L is attained at $\overline{p}$. This is so because of the following relationship *being satisfied for all CSP - Plans. For incoming quality p for any given CSP plan, if F(p) represents the AFI and AOQ (p) the AOQ, then AOQ (p) = p (1 - F(p)).*

The optimum inspection plans for a wide range of AOQL and process average $\overline{p}$ is given in Appendix I. The table gives (i_0, f_0) for a given p_L and $\overline{p}$ for $\overline{p}$ varying from 0.005 to 0.20 and p_L ranging from 0.005 to 0.10 and $\overline{p} > p_L$. It is not meaningful to devise a plan minimising AFI at $\overline{p}$ where $\overline{p}$ happens to be less than p_L. However, in a practical situation if needed, it is recommended that the plan for $\overline{p}$ which is just greater than p_L in the table should be used.

2.1.5 Optimum (i, f) under the Restriction That p_t (%) Should Not Exceed a Given Value

The customer may stipulate some value of p_t (%) say p_t° (%) to protect against any sudden deterioration of quality and this chosen value may differ from what results from the optimum plan given in Appendix I. Under such a situation the procedure given in See 2.1.4 is to be modified to accommodate the additional restriction relating to $p_t(\%)$.
The problem can be stated formally as follows :

Given p_L, $\overline{p}$ and $p_t^\circ \in \Omega$, find (i, f) for a CSP - 1 plan, which will (i) ensure AOQL $= p_L$ and $p_t \leq p_t^\circ$, and in addition will (ii) minimize F^{1L} (i, $\overline{p}$) among all CSP - 1 plans satisfying (i).
For CSP - 1, it can be easily seen that $p_t = \frac{1}{f}$ (1 - $(.10)^{.001}$). Stephens (1979) used an approximate formula for the same as $p_t \simeq 2.3 \times 10^{-3}/f$. The alogorithm modified to accommodate the stipulation on p_t (%) is presented below.

Algorithm : MNPTCS1 ($\overline{p}, p_L, p_t^\circ$, i, f)
(* input parameters : $\overline{p}, p_L, p_t^\circ$

output parameters : i, f *)
(* Finding optimum (i, f) for CSP - 1 minimising AFI at $\overline{p}$ satisfying AOQL = p_L and $p_i \leq p_i^\circ$ *)
Begin

$f_1 \leftarrow \frac{1-(.10)^{.001}}{p_i^\circ}; i \leftarrow 1;$

repeat

$p_1 \leftarrow \frac{i*p_L+1}{i+1}; f \leftarrow \frac{(1-p_1)^{i+1}}{i*p_L+(1-p_1)^{i+1}};$

if (f > f_1) then begin $i \leftarrow i+1; f_2 \leftarrow f$; end;

until (f < f_1);

i ← i - 1; f ← f_2;

MNFCS1 ($\overline{p}, p_L, i_0, f_0$);

if (f ≤ f_0) then begin $f \leftarrow f_0; i \leftarrow i_0$; end;

end.

It is easy to see that the plan so selected is the minimum inspection plan under the additional condition imposed.

N.B. 2.1.3

The problem discussed in section 2.1.1 was also considered by Resnikoff (1960) in his paper entitled 'Minimum Average Fraction Inspected for a Continuous Sampling Plan'. The results appearing in the above paper coincide with a part of our result enumerated. However, it is the difference in approaches which is worth noting. While Resnikoff has made a crucial assumption that F^{1L} (i, $\overline{p}$) is minimum when AOQ^{1L} (i, $\overline{p}$) i,e average out going quality at $\overline{p}$ equals to p_L, we have proved it. That makes a fundamental difference in approach. Moreover, to make a blanket assumption of this type may not always be valid e.g. we have seen that the same is not true for Multilevel CSP plan proposed by Lieberman and Solomon (1955).

Besides, we have other results on the behaviour of F^{1L} (i, $\overline{p}$) for different i, *interrelationship between $\bar{p}$, p_1^{1L} (i) and p_L and uniqueness of optimum solution. These results are stronger as they lead to (a) development of optimum plans when p follows a probability distribution as demonstrated in Chapter 4 and (b) modified optimum plan in the presence of additioal restriction on p_t (%), as already discussed in the present section.*

2.2 Optimum CSP - 2 Plan with k = i

2.2.1 Introduction

Dodge and Torrey (1951) introduced CSP - 2 plans with parameters (i, f, k). The CSP - 2 plan with k = i is easier to handle and some exact results have been obtained in this case. Accordingly we treat this case separately.
Dodge and Torrey too considered CSP - 2 with k = i only. For a whole range of AOQL they obtained several combinations of (i, f, i) that would ensure a desired AOQL following a trial and error method. Abraham (1971) developed a graphical method of selecting (i, f, i) CSP - 2 plans.
We have obtained algebraic relations, for the first time, to determine plan parameters (i, f, i) and also developed a procedure for obtaining optimum CSP - 2 (k = i) plan which will minimise AFI at the process average $\bar{p}$ among all CSP - 2 (k = i) plans which ensure a desired AOQL. Our results are presented below.

2.2.2 Notation and Formulas

The symbols u and v have already been introduced in Section 2.1.2. From the results of Dodge and Torrey we have in case of CSP - 2 in general

$$u = \frac{1-q^i}{pq^i} \qquad (2.2.1)$$

$$fv = \frac{1}{p} + \frac{q^k}{1-q^k}(\frac{1}{p} + k) + \frac{1}{p} - \frac{kq^k}{1-q^k}$$

$$= \frac{2-q^k}{p(1-q^k)} = \frac{2-q^i}{p(1-q^i)} \text{ for } k=i \tag{2.2.2}$$

Now,

$$F^2(i,f,i,p) = \frac{u+fv}{u+v}$$

$$= \frac{f}{f+(1-f)q^i(2-q^i)} \tag{2.2.3}$$

$$\text{and} AOQ^2(i,f,i,p) = p(1-F^2(i,f,i,p)) \tag{2.2.4}$$

The symbols used in (2.2.3) and (2.2.4) are introduced in (2.0.1).

2.2.3 Determination of (i, f, i) for a Desired AOQL under CSP - 2 (k = i) Scheme of Inspection.

The problem has already been introduced in Sec 2.0. Using (2.2.3) and (2.2.4) we have

$$AOQ^2(i,f,i,p) = p[1 - \frac{f}{f+(1-f)q^i(2-q^i)}] \tag{2.2.5}$$

Hence, given AOQL $= p_L$, using (2.0.3) we can also write

$$p_L = p_1^{2L}(i,i)[1 - \frac{f}{f+(1-f)(q_1^{2L}(i,i))^2(2-(q_1^{2L}(i,i))^i)}]$$

where it may be recalled $p_1^{2L}(i,i) = 1 - q_1^{2L}(i,i)$ is the value of of p for which the AOQL is reached.

To determine AOQL, we differentiate (2.2.5) with respect to p, equate it to zero and this equation will obviously yield the solution p_1^2 (i, f, i), which we for simplicity denote by p_1 in the subsequent expressions. It is to be noted that our problem is to obtain the value of p_1 for AOQL given to be p_L . Thus, we have the equation :

$$1 - \frac{A_1 - fp_1(B_1+C_1)}{D_1} = 0$$

$$\text{where } A_1 = f^2 + f(1-f)(1-p_1)^i(2-(1-p_1)^i)$$

$$B_1 = (1-f).-i(1-p_1)^{i-1}(2-(1-p_1)^i)$$

$$C_1 = (1-f)(1-p_1)^i.-i(1-p_1)^{i-1}.-1$$

$$\text{and } D_1 = \{f+(1-f)(1-p_1)^i.(2-(1-p_1)^i)\}^2$$

$$\text{or } 1 - \frac{f^2 + f(1-f)q_1^i(2-q_1^i) + ip_1 f(1-f)q_1^{i-1}(2-2q_1^i)}{\{f+(1-f)(1-p_1)^i.(2-(1-p_1)^i)\}^2} = 0$$

$$\text{or } f^2 + f(1-f)q_1^i(2-q_1^i) + ip_1 f(1-f)q_1^{i-1}(2-2q_1^i)$$

$$= f^2 + (1-f)^2 q_1^{2i}(2-q_1^i)^2 + 2f(1-f)q_1^i(2-q_1^i)$$

which implies that

$$ip_1 = \frac{q_1(2-q_1^i)}{2-2q_1^i} + \frac{1-f}{f}\frac{q_1^{i+1}(2-q_1^i)^2}{2-2q_1^i} \tag{2.2.6}$$

Writing 2 - $2q_1^i = s_1$ and 2 - $q_1^i = r_1$ we have

$$\frac{s_1}{r_1}.\frac{ip_1}{1-p_1} = 1 + \frac{1-f}{f}(1-p_1)^i.r_1 \tag{2.2.7}$$

$$\text{or} \frac{1-p_1}{ip_1}.\frac{r_1}{s_1} = \frac{f}{f+(1-f)(1-p_1)^i r_1}$$

$$\text{or } 1 - \frac{1-p_1}{ip_1}.\frac{r_1}{s_1} = 1 - \frac{f}{f+(1-f)(1-p_1)^i r_1}$$

It follows from (2.2.6) that

$$(is_1 + r_1)p_1 - r_1 = is_1 p_1[1 - \frac{f}{f+(1-f)(1-p_1)^i r_1}]$$

$$= is_1\, p_L \tag{2.2.8}$$

$$\text{and hence } p_1 = \frac{ip_L + \frac{r_1}{s_1}}{i + \frac{r_1}{s_1}} \tag{2.2.9}$$

From (2.2.7) and (2.2.8) we have

$$\frac{s_1}{r_1}.ip_1 + p_1 - 1 = \frac{1-f}{f}(1-p_1)^{i+1}.r_1$$

$$\text{or } \frac{p_1(is_1 + r_1) - r_1}{r_1} = \frac{1-f}{f}(1-p_1)^{i+1}.r_1$$

$$\text{or } \frac{is_1 p_L}{r_1} = \frac{1-f}{f}(1-p_1)^{i+1}.r_1$$

$$\text{or } fi\, p_L \frac{s_1}{r_1} = (1-f)(1-p_1)^{i+1}.r_1$$

$$\text{or } fi\, p_L.\frac{s_1}{r_1} + fq_1^{i+1}.r_1 = q_1^{i+1}.r_1$$

$$\text{and hence } f = \frac{q_1^{i+1}}{ip_L.\frac{s_1}{r_1^2} + q_1^{i+1}} \tag{2.2.10}$$

Thus for a given i and p_L, p_1 can be obtained by solving (2.2.9) numerically. For this we express (2.2.9) as a function of p_1 and solve numerically the equation $g(p_1)$ = 0,

$$\text{where } g(p_1) = ip_1 + \frac{2-q_1^i}{2-2q_1^i}(p_1 - 1) - ip_L$$

$$= p_1(i + \frac{r_1}{s_1}) - ip_L - \frac{r_1}{s_1}.$$

For finding a solution to $g(p_1) = 0$, we differentiate $g(p_1)$ w.r.t. p_1 and get

$$g'(p_1) = i + \frac{2-q_1^i}{2-2q_1^i} + (p_1-1).\frac{-2i(1-p_1)^{i-1}}{\{2-2(1-p_1)^i\}^2}$$
$$= i + \frac{r_1}{s_1} + \frac{i(2-s_1)}{s_1^2}$$

Using Newton - Raphson method and writing as usual $p_1(n)$ as the value of p_1 at the nth iteration, $q_1(n) = 1 - p_1(n)$, and $r_1(n)(s_1(n))$ denoting $r_1(s_1)$ where p_1 is replaced by $p_1(n)$ in the corresponding expressions,

$$p_1(n+1) = p_1(n) - \frac{g(p_1(n))}{g'(p_1(n))}$$
$$= p_1(n) - \frac{p_1(n)(i + \frac{r_1(n)}{s_1(n)}) - ip_L - \frac{r_1(n)}{s_1(n)}}{(i + \frac{r_1(n)}{s_1(n)}) + \frac{i(2-s_1(n))}{s_1^2(n)}}$$
$$= \frac{s_1^2(n)(ip_L + \frac{r_1(n)}{s_1(n)}) + ip_1(n)(2-s_1(n))}{s_1^2(n)(i + \frac{r_1(n)}{s_1(n)}) + i(2-s_1(n))}, \qquad (2.2.11)$$

$p_1 = \frac{ip_L+1}{i+1}$ provides a reasonbly good approximate solution to (2.2.9) when p_1 is not too small and i is moderately large.

The iterative procedure can therefore be started with

$$p_1(0) = \frac{ip_L+1}{i+1}, r_1(0) = 2 - q_1^i(0), s_1(0) = 2 - 2q_1^i(0)$$

and the iteration may be terminated when $\mid p_1(n+1) - p_1(n) \mid < \epsilon$, for some ϵ, a preassigned small positive number, say 0.005 or smaller depending on the desired degree of accuracy. Once p_1 is obtained f can be obtained from (2.2.10).

It is noted that every choice of i leads to a particular value of p_1 and hence that of f. Thus there will be many combinations of (i, f, i) ensuring the same AOQL. The p_1 thus obtained for AOQL given to be p_L has been denoted by the symbol p_1^{2L} (i, i).

2.2.4 The Nature of p_1^{2L} (i, i) of CSP - 2 (k = i) Plan

We are now interested in studying how p_1^{2L} (i, i) is affected by different choices of i for a given p_L. Treating p_1^{2L} (i, i) as a variable depending on i and writing p for it (consequently writing symbols r & s for r_1 and s_1 respectively introduced in Section 2.2.3), we have the following Lemmas

Lemma 2.2.1 : *For a given i, $\frac{r}{s}$ which is a function of p decreases as p increases.*

$$Proof: \frac{d(\frac{r}{s})}{dp} = \frac{d}{dp}[\frac{2-(1-p)^i}{2-2(1-p)^i}]$$
$$= \frac{-2i(1-p)^{i-1}}{\{2-2(1-p)^i\}^2}$$

< 0 for all p in the range $0 < p < 1$and for all $i > 0$.

Hence the result.

Lemma 2.2.2 : *For a given i the expression $\frac{ip_L+\frac{r}{s}}{i+\frac{r}{s}}$ is a decreasing function of p bounded by 1 and $\frac{ip_L+1}{i+1}$ for $0 < p < 1$.*

$$Proof: We have \frac{ip_L+\frac{r}{s}}{i+\frac{r}{s}} = \frac{(i+\frac{r}{s}p_L)}{(i+\frac{r}{s})} + \frac{\frac{r}{s}(1-p_L)}{(i+\frac{r}{s})}$$
$$= p_L + (1-p_L)[1 - \frac{i}{i+\frac{r}{s}}]$$

For $p \to 0$, $1 - \frac{i}{i+\frac{r}{s}} \to 1$ and for $p = 1$, $\frac{r}{s} = 1$

In view of Lemma 2.2.1, for a given i, $[1 - \frac{i}{i+\frac{r}{s}}]$ decreases with increase of p.

Hence the whole expression is a decreasing function of p lying between 1 and $\frac{ip_L+1}{i+1}$ for $0 < p < 1$.

Lemma 2.2.3 : *For a given i and p_L, there exists a unique p, called p_1^{2L} (i, i) for which the equation (2.2.9) is satisfied.*

Proof : We are considering the solution for the following equation

$$p = \frac{ip_L + \frac{r}{s}}{i + \frac{r}{s}}$$

The left hand side is a monotonically increasing function of p over o < p < 1. The right hand side is a monotonically decreasing function of p ranging from 1 to $\frac{ip_L+1}{i+1}$ for $0 < p < 1$.

Since, $p_L < 1$, $\frac{ip_L+1}{i+1} < 1$.

Hence as p increases from 0 to 1, there is a unique value of p for which the left hand side and the right hand side will coincide.

Lemma 2.2.4 : *For a given p_L, the value of proportion defective p_1^{2L} (i, i) at which the AOQL is attained decreases as i increases.*

$$\begin{aligned} Proof: We have \frac{ip_L + \frac{r}{s}}{i + \frac{r}{s}} &= p_L + (1 - p_L)\ G(i,p) \\ &= H(i,p), say. \end{aligned}$$

$$\text{where } G(i,p) = \frac{\frac{2-q^i}{2-2q^i}}{i + \frac{2-q^i}{2-2q^i}}$$

Treating G as a function of i alone for a given p we have

$$\frac{d\,G}{d\,i} = \frac{\frac{2iq^i \log q}{(2-2q^i)^2} - \frac{2-q^i}{2-2q^i}}{\{i + \frac{2-q^i}{2-2q^i}\}^2} < 0 \text{ for all p}$$

as $0 < q < 1$ and $\frac{2-q^i}{2-2q^i} > 0$

Hence $\frac{ip_L+\frac{r}{s}}{i+\frac{r}{s}}$ is a decreasing function of i for a given p and

G (i + 1, p) < G (i, p) and hence
$H(i+1,p) < H(i,p)$ for all p.

Since p_1^{2L} (i, i) satisfies equation (2.2.9), p_1^{2L} (i, i) is the abscissa of the intersection of y = H (i, p) and y = p. Hence p_1^{2L} (i + 1, i + 1) < p_1^{2L} (i, i) and the result follows.

The value of p_1^{2L} (i, i) for different values of i are given in Table 2.2.1 for a few choices of p_L.

Table 2.2.1

Values of p_1^{2L} (i, i) for different AOQL

i	AOQL (%)									
	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
1	0.58069	0.58405	0.58743	0.59083	0.59425	0.59769	0.60115	0.60463	0.60812	0.61164
2	0397309	0.40210	0.40695	0.41184	0.41677	0.42174	0.42675	0.43181	0.43690	0.44203
3	0.30233	0.30790	0.31352	0.31922	0.32497	0.33079	0.33667	0.34260	0.34860	0.35465
4	0.24450	0.25053	0.25666	0.26287	0.26916	0.27554	0.28199	0.28851	0.29510	0.30176
5	0.20562	0.21199	0.21846	0.22504	0.23173	0.23850	0.24537	0.25232	0.25936	0.26648
6	0.17770	0.18432	0.19106	0.19792	0.20491	0.21200	0.21919	0.22648	0.23387	0.24133
7	0.15669	0.16350	0.17045	0.17755	0.18477	0.19212	0.19958	0.20715	0.21481	0.22256
8	0.14030	0.14727	0.15440	0.16169	0.16912	0.17668	0.18436	0.19216	0.20006	0.20805
9	0.12717	0.13426	0.14155	0.14900	0.15661	0.16436	0.17223	0.18022	0.18832	0.19651
10	0.11640	0.12361	0.13103	0.13863	0.14639	0.15430	0.16234	0.17051	0.17877	0.18714
11	0.10742	0.11473	0.12226	0.12999	0.13789	0.14595	0.15414	0.16245	0.17087	0.17938
12	0.09982	0.10722	0.11485	0.12270	0.13072	0.13890	0.14723	0.15567	0.16422	0.17286
13	0.09330	0.10078	0.10851	0.11645	0.12459	0.13289	0.14133	0.14989	0.15856	0.16731
14	0.08764	0.09519	0.10301	0.11105	0.11929	0.12770	0.13625	0.14491	0.15368	0.16253
15	0.08269	0.09031	0.09821	0.10634	0.11467	0.12318	0.13182	0.14058	0.14944	0.15838
16	0.07832	0.08600	0.09397	0.10219	0.11061	0.11920	0.12793	0.13677	0.14572	0.15475
17	0.07443	0.08217	0.09022	0.09851	0.10701	0.11568	0.12449	0.13342	0.14243	0.15153
18	0.07095	0.07875	0.08686	0.09523	0.10380	0.11255	0.12143	0.13043	0.13951	0.14867
19	0.06782	0.07567	0.08385	0.09228	0.10093	0.10974	0.11869	0.12775	0.13690	0.14611
20	0.06499	0.07289	0.08112	0.08962	0.09833	0.10721	0.11622	0.12534	0.13454	0.14381
21	0.06242	0.07036	0.07865	0.08721	0.09598	0.10492	0.11399	0.12131	0.13241	0.14172
22	0.06007	0.06806	0.07640	0.08502	0.09384	0.10283	0.11196	0.12118	0.13047	0.13983
23	0.05791	0.06595	0.07434	0.08301	0.09189	0.10093	0.11010	0.11937	0.12871	0.13810
24	0.05593	0.06400	0.07245	0.08117	0.09010	0.09919	0.10841	0.11771	0.12709	0.13652
25	0.05410	0.06222	0.07070	0.07947	0.08845	0.09759	0.10685	0.11619	0.12560	0.13506

2.2.5 Determination of Optimum (i, f, i) that Minimises F^{2L} (i, i, $\bar{p}$) Given p_L.

The problem has been formally introduced in Section 2.0

Theorem 2.2.1 : *For given p_L, and $\overline{p}\epsilon\Omega$, there exists a combination (i, f, i) $\epsilon N\times\Omega\times N$ which minimises F^{2L} $(i, i, \overline{p})$ at $\overline{p}$ among all CSP - 2 $(k = i)$ plans with AOQL $= p_L$, provided $\overline{p} > p_L$. Furthermore, the minimum is attained for that value of i for which p_1^{2L} $(i, i) = \overline{p}$.*

Proof : Let $p_1^{2L}(i_1, i_1), p_1^{2L}(i, i)$ and $p_1^{2L}(i_2, i_2)$ be respectively the values of p at which the desired AOQL $= p_L$ is attained for $i_1 < i < i_2$. Let $p_1^{2L}(i, i)$ be equal to $\overline{p}$.

It is known that $p_1^{2L}(i_1, i_1) > p_1^{2L}(i, i) > p_1^{2L}(i_2, i_2)$. Hence we have from the unimodal property of AOQ curves (as in (2.1.4) for CSP - 1)

$$\overline{p}(1 - F^{2L}(i_1, i_1, \overline{p})) \;<\; p_L$$

$$\overline{p}(1 - F^{2L}(i, i, \overline{p})) \;=\; p_L$$

$$\overline{p}(1 - F^{2L}(i_2, i_2, \overline{p})) \;<\; p_L$$

These imply that $F^{2L}(i, i, \overline{p})$ is the minimum of the three. Since this is true for all $i_1 < i < i_2, F^{2L}(i, i, \overline{p})$ is the minimum over all values of i.

Since for an optimum plan $p_1^{2L}(i, i) = \overline{p}$ it follows from equation (2.2.9) that

$$\overline{p}(i + \frac{2 - \overline{q}^{\,i}}{2 - 2\,\overline{q}^{\,i}}) - ip_L - \frac{2 - \overline{q}^{\,i}}{2 - 2\overline{q}^{\,i}} = 0$$

$$\Rightarrow \quad \frac{i}{i + \frac{2-\overline{q}^i}{2-2\overline{q}^i}} \;=\; \frac{1 - \overline{p}}{1 - p_L}.$$

For $\overline{p} < p_L$, $\frac{1-\overline{p}}{1-p_L} > 1$ but the l.h.s. is less than 1 since $\frac{2-\overline{q}^i}{2-2\overline{q}^i} > 1$.

Thus for $\overline{p} < p_L$, we cannot find any optimum plan as it can be easily verified that the amount of inspection goes on decreasing as i increass. So, we impose the restriction $\overline{p} > p_L$, which seems to be very natural for a practical problem. Hence the theorem is proved.

The algorithm to find the optimum combination (i_0, f_0, i_0) can be stated as follows

Algorithm : MNCS2A ($\overline{p}, p_L$, i, f)
(* input parameters : $\overline{p}, p_L$
output parameters : i, f *)
(* Finding optimum (i, f) for CSP - 2 (k = i) minimising AFI at $\overline{p}$ and satisfying AOQL = p_L *)
Begin

$i \leftarrow \lfloor \frac{1-\overline{p}}{\overline{p}-p_L} \rfloor - 1; p_{12} \leftarrow 0; i_2 \leftarrow 0;$

(* the optimum i for CSP - 2 (k = i) is greater than optimum i for CSP - 1. See lemma 2.2.6 *)

repeat
$i \leftarrow i + 1;$
P1CS2A (i, p_L, p_1);
$p_{11} \leftarrow p_{12}; i_1 \leftarrow i_2;$
$p_{12} \leftarrow p_1; i_2 \leftarrow i;$
(* this gives two consecutive values of i, one of which is optimum *)
until ($p_1 < \overline{p}$);
if ($| p_{11} - \overline{p} | < | p_{12} - \overline{p} |$)
then begin $i \leftarrow i_1; p_1 \leftarrow p_{11};$ end
else begin $i \leftarrow i_2; p_1 \leftarrow p_{12};$ end;
$q_1 \leftarrow 1 - p_1; r_1 \leftarrow 2 - q_1^i; s_1 \leftarrow 2 - 2 * q_1^i;$
$f \leftarrow \frac{q_1^{i+1}}{i * p_L * \frac{s_1}{r_1^i} + q_1^{i+1}}$; (* see (2.2.10) *)

end.

Procedure : P1CS2A (i, p_L, p_1)
(* input : i, p_L; output : p_1, computes p_1^{2L} (i, i) given i, p_L *)
Begin

$p_{14} \leftarrow \frac{i * p_L + 1}{i+1}$; (* starting value for p_1^{2L} (i, i) *)
repeat
$q_{14} \leftarrow 1 - p_{14}; r_1 \leftarrow 2 - q_{14}^i; s_1 \leftarrow 2 - 2 * q_{14}^i;$

$p_1 \leftarrow \frac{s_1^2(i*p_L+\frac{r_1}{s_1})+i*p_{14}*(2-s_1)}{s_1^2(i+\frac{r_1}{s_1})+i*(2-s_1)}$;

$p_{14} \leftarrow p_1$;

until $(| p_1 - p_{14} | < \epsilon)$;

(* $\in$ a preassigned small positive number, say, .005. Also see (2.2.11) *)

end.

N.B. 2.2.1

The search effort for optimum i can be reduced considerably by determining p_1^{2L} *(i, i) only for* $i > \frac{1-\overline{p}}{\overline{p}-p_L}$ *as it has already been pointed out that* $\overline{p} \simeq p_1^{2L}$ *(i, i)* $> \frac{ip_L+1}{i+1}$.

2.2.6 The Nature of F^{2L} *(i, i,* $\overline{p}$*) for CSP - 2 (k = i)*

Let us recall F^{2L} (i, i, $\overline{p}$) refers to CSP - 2 with k = i and F^{1L} (i, $\overline{p}$) refers to CSP - 1 with the same specified AOQL = p_L and same process average $\overline{p}$. Using equations (2.2.3), (2.2.9) and (2.2.10) we have

$$F^{2L}(i,i,\overline{p}) = [1 + (\frac{1}{f} - 1)\overline{q}^{\,i}(2 - \overline{q}^{\,i})]^{-1}$$

$$= [1 + \frac{(i+\frac{r_1}{s_1})^{i+1}}{i^i}.\frac{p_L}{q_L}.(\frac{\overline{q}}{q_L})^i.\frac{s_1}{r_1}.\frac{2-\overline{q}^{\,i}}{2-q_1^{\,i}}]^{-1}$$

where p_1, r_1 and s_1 are as defined in section 2.2.3.

$$= [1 + D_2(i)]^{-1} \text{ say} \tag{2.2.12}$$

We have from equation (2.1.8) in Section 2.1.

$$F^{1L}(i,\overline{p}) = [1 + \frac{(i+1)^{i+1}}{i^i}.\frac{p_L}{q_L}.(\frac{\overline{q}}{q_L})^i]^{-1}$$

$$= [1 + D_1(i)]^{-1}, \text{ say}$$

$$\text{Thus } \frac{D_2(i)}{D_1(i)} = \frac{(i+\frac{r_1}{s_1})^{i+1}}{(i+1)^{i+1}}.\frac{s_1}{r_1}.\frac{2-\overline{q}^i}{2-q_1^i} \quad (2.2.13)$$

Lemma 2.2.5 : *For given AOQL $= p_L$ and process average $\overline{p}$, the minimum of F^{2L} $(i, i, \overline{p})$ over i for CSP - 2 (k = i) plans is same as the minimum of F^{1L} $(i, \overline{p})$ over i for CSP - 1 plans.*

Proof : Let the respective minimum values be attained at i_1 and i_2 for CSP - 1 and CSP - 2 (k = i). The result follows trivially from the fact

$$p_1^{1L}(i_1) = p_1^{2L}(i_2, i_2) = \overline{p}.$$

This result has already been pointed out in N.B. 2.1.2.

Lemma 2.2.6 : *If the minimum AFI referred to in Lemma 2.2.5 is attained at i_1 and i_2 for CSP - 1 and CSP - 2 (k = i) respectively then $i_2 \geq i_1$.*

Proof : For CSP - 1, we have from (2.1.6),

$$\overline{p} = p_1^{1L}(i_1) = \frac{i_1 p_L + 1}{i_1 + 1}$$

From Lemma 2.2.2 we have

$$p_1^{2L}(i_2, i_2) \geq \frac{i_2 p_L + 1}{i_2 + 1}$$

$$\text{Therefore, } \tfrac{i_2 p_L + 1}{i_2 + 1} \leq p_1^{2L}(i_2, i_2) = \overline{p} = \tfrac{i_1 p_L + 1}{i_1 + 1}$$

$$\Rightarrow i_2 \geq i_1.$$

Lemma 2.2.7 : *If for process average $\overline{p} = p_{(1)}$, F^{2L} $(i, i, p_{(1)})$ is minimum at $i = i_1$ and for process average $\overline{p} = p_{(2)}$, F^{2L} $(i, i, p_{(2)})$ is minimum at $i = i_2$, and moreover $p_{(2)} \geq p_{(1)}$, then $i_2 \leq i_1$.*

Proof : As has been pointed out before p_1^{2L} $(i_1, i_1) = p_{(1)}$ and $p_1^{2L}(i_2, i_2) = p_{(2)}$

Hence p_1^{2L} $(i_2, i_2) \geq p_1^{2L}(i_1, i_1)$ and $i_2 \leq i_1$, since p_1^{2L} (i, i) is a decreasing function of i.

Remark 2.2.1

Assuming AOQL to be fixed at p_L, it has been observed that the amount of inspection at $\overline{p}$ for CSP - 2 ($k = i$) plan is greater than that for CSP - 1 for all $i < i_2$ and smaller than that for CSP - 1 for all $i > i_2$ where i_2 is the value of i at which F^{2L} (i, i, $\overline{p}$) is minimum.

This may be intuitively explained as follows :

Recalling the expression for D_2 (i) and D_1 (i) in (2.2.13).

$$\frac{D_2(i)}{D_1(i)} \simeq \frac{2-\overline{q}^i}{2-q_1^i}, \text{ because}$$

it is easy to see that both

$\frac{r_1}{s_1} \rightarrow 1$ and $\frac{s_1}{r_1} \rightarrow 1$ as i $\rightarrow \infty$.

This can be verified from Fig. 2.2.1 which shows the values of

$$\frac{2-\overline{q}^i}{2-q_1^i}, \frac{(i+\frac{r_1}{s_1})^{i+1}}{(i+1)^{i+1}}.\frac{s_1}{r_1} \text{ and} \frac{D_2(i)}{D_1(i)} \text{ plotted against}$$

different values of i for $\overline{p} = .05$ and $p_L = .01$

As expected $\frac{D_2(i)}{D_1(i)}$ is approximately represented by $\frac{2-\overline{q}^i}{2-q_1^i}$ in both form and magnitude.

$$\text{Since } \frac{2-\overline{q}^i}{2-q_1^i} \lesseqgtr 1 \text{ for } i \lesseqgtr i_2$$

$D_2(i) \lesseqgtr D_1(i)$ according as i $\lesseqgtr i_2$ which implies that $F^{2L}(i, i, \overline{p}) \gtreqless F^{1L}(i, \overline{p}$ according as i $\lesseqgtr i_2$.

Theorem 2.2.2 : *For given AOQL = p_L and a process average $\overline{p} > p_L$, the AFI under CSP - 2 ($k = i$), viz., $F^{2L}(i, i, \overline{p})$ decreases monotonically for all i* *as long as p_1^{2L} $(i, i) > \overline{p}$ and then increases monotonically for which p_1^{2L} (i, i) $< \overline{p}$ i.e. attains a unique minimum for the value of i for which p_1^{2L} $(i, i) = \overline{p}$*

Proof : For notational simplicity let us write p_1 for p_1^{2L} (i, i); q_1 = 1 - p_1 and use r_1, s_1 and D_2 (i) as defined in section 2.2.3.

$$q_1 = \frac{q_L}{1+\frac{r_1}{is_1}}, \; q_L = 1-p_L$$

$$\text{or } 1+\frac{r_1}{is_1} = \frac{q_L}{q_1};$$

$$\text{or } \frac{is_1}{r_1}+1 = \frac{q_L}{q_L-q_1};$$

$$q_1(1-\frac{r_1}{s_1}) = q_1(i+1)-iq_L \tag{2.2.14}$$

$$\text{and } \frac{iq_1^{i-1}}{2-q_1^i} = \frac{q_1^i}{s_1(q_L-q_1)} \tag{2.2.15}$$

From (2.2.12) we have

$$F^{2L}(i,i,\overline{p}) = \frac{1}{1+D_2(i)}$$

$$\text{where } D_2(i) = \frac{(i+\frac{r_1}{s_1})^{i+1}}{i^i}.\frac{s_1}{r_1}.\frac{2-\overline{q}^i}{2-q_1^i}.\frac{p_L}{q_L}(\frac{\overline{q}}{q_L})^i$$

$$= (1+\frac{r_1}{is_1})^i(\frac{is_1}{r_1}+1).\frac{2-\overline{q}^i}{2-q_1^i}.\frac{p_L}{q_L}.(\frac{\overline{q}}{q_L})^i$$

$$= (\frac{q_L}{q_1})^i.(\frac{q_L}{q_L-q_1}).\frac{2-\overline{q}^i}{2-q_1^i}.\frac{p_L}{q_L}.(\frac{\overline{q}}{q_L})^i \tag{2.2.16}$$

Let F^{2L} (i, i, $\overline{p}$) attain its minimum at i_0 for which $p_1^{2L}(i_0,i_0) = \overline{p}$. Then, consequently D_2 (i) has a maximum at i_0.

We define $\phi(i) \;=\; logD_2(i)$

$= \; i(logq_L - logq_1) + logq_L - log(q_L - q_1) + log(2-\overline{q}^i) - log(2-q_1^i) + log\frac{p_L}{q_L} + \; ilog(\frac{\overline{q}}{q_L})$

Treating i as a continuous variable and differentiating $\phi(i)$ w.r.t. i,

$$\phi'(i) \;=\; log\frac{q_L}{q_1} - \frac{im(i)}{q_1} + \frac{m(i)}{q_L - q_1} - \frac{\overline{q}^i log\overline{q}}{2-\overline{q}^i} + \frac{q_1^i logq_1}{2-q_1^i} + \frac{iq_1^{i-1}m(i)}{2-q_1^i} + log\frac{\overline{q}}{q_L} \quad (2.2.17)$$

$$\text{where } m(i) \;=\; \frac{dq_1}{di}$$

Using (2.2.14) and (2.2.15), the sum of the terms containing m(i) in (2.2.17) is given as

$$m(i)[\frac{1}{q_L - q_1} - \frac{i}{q_1} + \frac{iq_1^{i-1}}{2-q_1^i}]$$

$$= m(i)[\frac{(i+1)q_1 - iq_L}{(q_L - q_1)q_1} \; + \; \frac{q_1^i}{s_1(q_L - q_1)}]$$

$$= m(i)[\frac{q_1(1-\frac{r_1}{s_1})}{(q_L - q_1)q_1} \; + \; \frac{q_1^i}{s_1(q_L - q_1)}]$$

$$= m(i)[\frac{s_1 - r_1 + q_1^i}{(q_L - q_1)s_1}]$$

$$= m(i)[\frac{2 - 2q_1^i - 2 + q_1^i + q_1^i}{(q_L - q_1)s_1}]$$

$$= m(i).0 \;=\; 0$$

$$\text{Thus, } \phi'(i) \;=\; a_1 b_1 - a_2 b_2$$

$$\text{where } a_1 = \frac{q_1^i}{2-q_1^i} - 1, a_2 = \frac{\overline{q}^i}{2-\overline{q}^i} - 1$$

$$\text{and } b_1 = log q_1, \;\; b_2 = log \bar{q}$$

We have, ϕ' (i) $= \mid a_1 \mid \mid b_1 \mid - \mid a_2 \mid \mid b_2 \mid$, as a_1, a_2, b_1 and b_2 are all negative. Moreover, for $i < i_0$, we have,

$$p_1 > \bar{p} \text{ and } q_1 < \bar{q}$$

$$a_1 < a_2 \text{ and } b_1 < b_2$$

$$\mid a_1 \mid > \mid a_2 \mid \;\text{ and }\; \mid b_1 \mid > \mid b_2 \mid$$

$$\text{and hence } \;\; \mid a_1 \mid b_1 \mid > \mid a_2 \mid b_2 \mid$$

Similarly for i $> i_0$, $\mid a_1 \mid b_1 \mid < \mid a_2 \mid b_2 \mid$; the equality sign holds only when i $= i_0$.

Thus for all i $< i_0$, ϕ (i) and therefore D_2 (i) increases with i, reaches a maximum at i_0 and then decreases. We also note that there is only one solution to $\phi'(i) = 0$. Hence, $F^{QL}(i, i, \bar{p})$ has a unique minimum at i $= i_0$. This completes the proof of the theorem.

2.3 Optimum CSP - 2 Plan with Any k

2.3.1 Introduction

Dodge and Torrey (1951) introduced CSP - 2 plan with parameters (i, f, k). We have already dealt with CSP - 2 plan with parameters (i, f, i) as exact results were obtained for this special case. We now consider the general case.

The general CSP - 2 plan has not been studied elaborately as no procedure is available so far to determine plan parameters (i, f, k) which will ensure a desired AOQL. We develop here, for the first time, a procedure to determine (i, f, k) for a desired

AOQL under CSP - 2 inspection procedure. As before, this is followed by selection of a particular combination of i and f, given k, which while satisfying the AOQL stipulation will in addition, minimise the AFI for a given process average $\bar{p}$ within the restricted class. A method of search for the selection of the unique optimum plan within the restricted class in the context is provided and this procedure is justified mathematically only in part. But it is verified empirically for a large number of cases.

2.3.2 Notation and Formulas

The relevant formulas due to Dodge and Torrey as can be recalled from section 2.2 are as follows :

$$u = \frac{1-q^i}{pq^i} \tag{2.3.1}$$

$$fv = \frac{2-q^k}{p(1-q^k)} \tag{2.3.2}$$

$$F^2(i,f,k,p) = 1 - \frac{(1-f)q^i(2-q^k)}{f(1-q^i)(1-q^k)+q^i(2-q^k)} \tag{2.3.3}$$

$$\text{and } AOQ^2(i,f,k,p) = p[\frac{(1-f)q^i(2-q^k)}{f(1-q^i)(1-q^k)+q^i(2-q^k)}] \tag{2.3.4}$$

2.3.3. Determination of (i, f, k) for a Desired AOQL for CSP - 2 Plan with Any k.

The problem has been formally introduced in Sec 2.0. Given AOQL = p_L, writing p_1 for p_1^{2L} (i, k), it follows from (2.3.4) that

$$p_L = p_1[\frac{(1-f)q_1^i(2-q_1^k)}{f(1-q_1^i)(1-q_1^k)+q_1^i(2-q_1^k)}] \tag{2.3.5}$$

To determine p_1 for a given plan, we differentiate (2.3.4) w.r.t. p, equate it to zero and solve for p.

$$\text{Thus } \frac{dAOQ^2(i,f,k,p)}{dp} = \frac{(1-f)q^i(2-q^k)}{f(1-q^i)(1-q^k)+q^i(2-q^k)}$$

$$+p\frac{(1-f)[(i+k)q^{i+k-1}-2iq^{i-1}][f(1-q^i)(1-q^k)+q^i(2-q^k)]}{[f(1-q^i)(1-q^k)+q^i(2-q^k)]^2}$$

$$-p\frac{(1-f)q^i(2-q^k)[fiq^{i-1}(1-q^k)+kq^{k-1}(1-q^i)+(i+k)q^{i+k-1}-2iq^{i-1}]}{[f(1-q^i)(1-q^k)+q^i(2-q^k)]^2}$$

$$= \frac{(1-f)D}{fC+D} + p\frac{(1-f)A(fC+D)-(1-f)D[fB+A]}{(fC+D)^2} \qquad (2.3.6)$$

where $A = (i+k)q^{i+k-1} - 2iq^{i-1}$

$B = iq^{i-1}(1-q^k) + kq^{k-1}(1-q^i)$

$C = (1-q^i)(1-q^k)$

$D = q^i(2-q^k)$

Using the subscript 1 to denote the value of A, B, C and D at p_1 and setting $\frac{dAOQ^2(i,f,k,p)}{dp} = 0$, we have

$$\frac{(1-f)D_1}{fC_1+D_1} + p_1\frac{(1-f)A_1(fC_1+D_1)-(1-f)D_1[fB_1+A_1]}{(fC_1+D_1)^2} = 0$$

Then, from (2.3.5) given AOQL $= p_L$, this equation reduces to

$$\frac{p_L}{p_1} + p_1\frac{(1-f)A_1(fC_1+D_1)-(1-f)D_1[fB_1+A_1]}{(fC_1+D_1)^2} = 0$$

$$\text{or } \frac{p_L}{p_1} + \frac{p_1(1-f)A_1}{fC_1+D_1} - \frac{(1-f)D_1p_1[fB_1+A_1]}{(fC_1+D_1)^2} = 0$$

$$\text{or } \frac{p_L}{p_1} + \frac{p_1(1-f)A_1}{fC_1+D_1} - \frac{p_L[fB_1+A_1]}{fC_1+D_1} = 0$$

$$\text{or } \frac{p_L}{p_1} + \frac{p_1A_1 - p_1fA_1 - p_LfB_1 - p_LA_1}{fC_1+D_1} = 0$$

$$\text{or } \frac{p_L}{p_1} = \frac{f(p_1A_1 + p_LB_1) + p_LA_1 - p_1A_1}{fC_1+D_1} \tag{2.3.7}$$

It follows from (2.3.5)

$$(fC_1 + D_1)p_L = p_1(1-f)D_1$$

$$\text{or } f = \frac{p_1D_1 - p_LD_1}{p_LC_1 + p_1D_1} \tag{2.3.8}$$

Substituting this value of f in (2.3.7) we have

$$\frac{p_L}{p_1} = \frac{\frac{p_1D_1-p_LD_1}{p_LC_1+p_1D_1}(p_1A_1 + p_LB_1) + p_LA_1 - p_1A_1}{\frac{p_1D_1-p_LD_1}{p_LC_1+p_1D_1}.C_1 + D_1}$$

$$= \frac{D_1p_1p_L[A_1+B_1] - p_1p_LA_1[C_1+D_1] + p_L^2[C_1A_1 - D_1B_1]}{D_1p_1[C_1+D_1]}$$

$$\text{or } p_1 = \frac{D_1p_1[C_1+D_1]}{D_1p_1[A_1+B_1] - p_1A_1[C_1+D_1] + p_L[C_1A_1 - D_1B_1]}$$

$$\text{or } D_1p_1[A_1+B_1] - p_1A_1[C_1+D_1] + p_L[C_1A_1 - D_1B_1] = D_1[C_1+D_1]$$

$$\text{or } p_1 = \frac{D_1[C_1+D_1] - p_L[C_1A_1 - D_1B_1]}{D_1[A_1+B_1] - A_1[C_1+D_1]}$$

$$= p_L + \frac{D_1[C_1+D_1[}{D_1B_1 - A_1C_1} = p_L + G(i,k,p_1) \tag{2.3.9}$$

where $G(i,k,p_1) \;=\; \frac{D_1[C_1+D_1]}{D_1B_1-A_1C_1}$

$$= \frac{q_1^i(2-q_1^k)[(1-q_1^i)(1-q_1^k)+q_1^i(2-q_1^k)]}{\{q_1^i(2-q_1^k)[iq_1^{i-1}(1-q_1^k)+kq_1^{k-1}(1-q_1^i)]}$$

$$-\;[(i+k)q_1^{i+k-1}-2iq_1^{i-1}](1-q_1^i)(1-q_1^k)\} \qquad (2.3.10)$$

The value of p_1 for any choice of i and k can be obtained by solving equation (2.3.9) with a suitable iterative procedure to any desired degree of accuracy. f is obtained by substituting this value of p_1 in equation (2.3.8). This combination of (i, f, k) will ensure the desired AOQL.

This is illustrated with an example. It is required to find f for i = 10 and k = 8 so that the combination (10, f, 8) ensures an AOQL of 0.05, no matter what quality is submitted. From (2.3.9) we find the value of p_1 to be 0.14600. Equation (2.3.8) gives the value of f as 0.42418. AOQ^2 (i, f, k, p) values for different p as obtained from (2.3.4) are plotted against p in Fig. 2.3.1.

Since i and k can be chosen arbitrarily there wll be many combinations of (i, f, k) ensuring the same AOQL.

2.3.4 On the Nature of $p_1^{2L}(i,k)$ for General CSP - 2

The determination of p_1^{2L} (i, k) and the study of its relationship with i and k for given AOQL $= p_L$ are of vital importance from the view point of optimal CSP - 2 plans. We write as in the preceding section, simply p_1 for p_1^{2L} (i, k) and then we have from equation (2.3.9),

$$p_1 \;=\; p_L + \frac{q_1^{2i}(2-q_1^k)+q_1^i(q_1^{2k}-3q_1^k+2)}{q_1^{2i-1}q_1^k(-k)+iq_1^{i-1}q_1^{2k}+2iq_1^{i-1}+q_1^{i-1}q_1^k(k-3i)} \qquad (2.3.11)$$

However, given p_L, i and k, the value of p_1 can be computed by numerically finding the point of intersection of the curves :

$$
\begin{aligned}
y &= p \text{ and} \\
y &= p_L + G(i,k,p) \qquad (2.3.12)
\end{aligned}
$$

By putting k = i, it can be easily verified that we get the same expression for p_1^{2L} (i, i) as in the simpler case of CSP - 2 with k = i. For this special case it was noted that p_1^{2L} (i, i) is a decreasing function of i and we should expect a similar behaviour of p_1^{2L} (i, k) for the general CSP - 2 as well.

It has not been possible to establish the expected results analytically. However, the function G (i, k, p) was evaluated for a large number of combinations of (i, k). It was found in all cases that

(i) G (i, k, p) is a decreasing function of p in the range $0 < p < 1$ for given i and k.

(ii) For given i and k, G (i, k, p) intersects y = p at one point only, and

(iii) G (i, k, p) is a decreasing function of i for given p and k.

Following the same line of arguments as in proving Lemmas 2.2.1 through 2.2.4, we conclude from the nature of the curve G (i, k, p), viz., as demonstrated by the behaviour (i), (ii) and (iii) outlined above that for general CSP - 2 also p_1^{2L} (i, k) decreases as i increases for given k and p_L.

This is illustrated in Fig. 2.3.2.

It will be noted from table 2.3.1 that, for a fixed i, p_1^{2L} (i, k) for a given p_L does not always decreases with increase in k. Hence, subsequently we will study the behaviour of general CSP - 2 plan by considering the behaviour of p_1^{2L} (i, k) for different i, keeping k fixed.

In this context, assuming the monotonically decreasing property of p_1^{2L} (i, k) w r t i, for any given k as demonstrated by a large volume of numerical computations from which the table 2.3.1 is selected for the purpose of illustration, there will always exist an i_0 such that $p_1^{2L}(i_0, k) \simeq \overline{p}$ for some given $\overline{p}$ in the normal range, and of course given k.

Table 2.3.1

Values of p_1^{2L} (i, k) for some choices of (i, k) to ensure an AOQL of 0.05

	k							
i	5	6	7	8	9	10	11	12
5	0.2317	0.2322	0.2329	0.2337	0.2345	0.2354	0.2362	0.2369
6	0.2049	0.2049	0.2053	0.2059	0.2065	0.2072	0.2079	0.2085
7	0.1849	0.1849	0.1848	0.1851	0.1857	0.1862	0.1867	0.1872
8	0.1693	0.1689	0.1689	0.1691	0.1694	0.1698	0.1703	0.1708
9	0.1569	0.1563	0.1562	0.1564	0.1566	0.1569	0.1573	0.1576
10	0.1467	0.1461	0.1460	0.1460	0.1462	0.1464	0.1467	0.1470
•								
•								
14	0.1195	0.1189	0.1186	0.1185	0.1185	0.1186	0.1187	0.1189
15	0.1148	0.1143	0.1139	0.1138	0.1138	0.1139	0.1140	0.1142

2.3.5 On the Determination of i and f (Given k) That Minimise the AFI for a Given Process Average $\overline{p}$.

The problem has been introduced in Section 2.0. The findings based on extensive numerical studies undertaken in the usual region of interest are enumerated below as observations.

Observation 2.3.1 : *For given p_L and k, there exists a particular combination (i, f, k) which minimises the AFI at $\overline{p}$ under CSP - 2 inspection provided $\overline{p} > p_L$. Furthermore, as explained in section 2.3.4 the minimum is attained for the value of i for which $p_1^{2L}(i,k) \simeq \overline{p}$*

Observation 2.3.2 : *For given p_L and k, the AFI under CSP - 2, viz., F^{2L} (i, k, $\overline{p}$) for a given proces average $\overline{p}$ decreases monotonically for all i as long as p_1^{2L} (i, k) $> \overline{p}$ and then increases monotonically for all i for which p_1^{2L} (i, k)*

$< \overline{p}$ *i,e attains a unique minimum for the value of i for which* $p_1^{2L}(i,k) \simeq \overline{p}$ *for all* $\overline{p} > p_L$.

The above observations are based on empirical studies. Though we have not exact mathematical proof for them, it is not difficult to explain the phenomena at least intuitively. The results rest on two properties of p_1^{2L} (i, k), namely,

(a) Uniqueness property of p_1^{2L} (i, k) for given p_L and k, and (b) decreasing property of p_1^{2L} (i, k) with increase of i for given k.

As i increases we find a unique value i_0 for which

$$p_1^{2L}(i_0,k) \simeq \overline{p} \text{ and } p_1^{2L}(i_0,k)(1-F^{2L}(i_0,k,p_1^{2L}(i_0,k))) = p_L$$

$$\text{i.e. } \overline{p}(1-F^{2L}(i_0,k,\overline{p})) = p_L \tag{2.3.13}$$

For all other i different from i_0, $\overline{p}$ (1 - F^{2L} (i, k, $\overline{p}$)) will be less then p_L, because of the unimodal property of AOQ curve.

The restriction $\overline{p} > p_L$ comes because of the fact that p_1^{2L} (i, k) $> p_L$ for all i because of (2.3.9). Thus we cannot find an i_0 for which $p_1^{2L}(i_0, k) = \overline{p}$ when $\overline{p} < p_L$. For $\overline{p} < p_L$ no optimum plan exists.

It may be recalled that given any k ϵ N, the unique values of i and f, so determined (i) ensures the value of AOQL to be p_L and (ii) minimize the AFI at $\overline{p}$ among all such plans satisfying (i). What is interesting is that this minimum AFI at $\overline{p}$ given by equation (2.3.13) remains same for all k started with. Hence, as far as the criteria of the AOQL stipulation and minimization of AFI at $\overline{p}$ are concerned, all these plans for different k happen to be equivalent and are not distinguishable from the optimum (i, f, i) CSP - 2 plan dealt with in section 2.2. However, in the matter of the choice for the best k one can be guided by some other physically meaningful and important criterion additionally to be discussed in Chapter 5.

2.3.6 An Algorithm to Find Optimum CSP - 2 General Plan for a Given

AOQL.

Algorithm : MNCS2B ($\overline{p}, p_L$, k, i, f)
(* input parameters : $\overline{p}, p_L$, k
output parameters; i, f *)
(* Finding optimum (i, f) for CSP - 2 (k $\neq$ i) minimising AFI at $\overline{p}$ and satisfying AOQL = p_L, given k, $\overline{p}, p_L$ *)
Begin

$i \leftarrow 0; p_{12} \leftarrow 0; i_2 \leftarrow 0;$
repeat
$i \leftarrow i + 1;$
$q_1 \leftarrow 1 - \overline{p}; d_1 \leftarrow q_1^i * (2 - q_1^k);$
$c_1 \leftarrow (1 - q_1^i) * (1 - q_1^k);$
$b_1 \leftarrow i * q_1^{i-1} * (1 - q_1^k) + k * q_1^{k-1} * (1 - q_1^i);$
$a_1 \leftarrow (i + k) * q_1^{i+k-1} - 2 * i * q_1^{i-1};$
$p_1 \leftarrow p_L + \frac{d_1 * (c_1 + d_1)}{d_1 * b_1 - a_1 * c_1};$
$p_{11} \leftarrow p_{12}; i_1 \leftarrow i_2;$
$p_{12} \leftarrow p_1; i_2 \leftarrow i;$
until $(p_1 < \overline{p})$;
(* solves (2.3.9) with $p_1 = \overline{p}$ in the r.h.s. to give two consecutive values of i one of which is optimum *)
P1CS2B (p_{11}, p_L, k, i_1, p_1); $p_{11} \leftarrow p_1$;
P1CS2B (p_{12}, p_L, k, i_2, p_1); $p_{12} \leftarrow p_1$;
if ($| p_{11} - \overline{p} | < | p_{12} - \overline{p} |$)
then begin i $\leftarrow i_1; p_1 \leftarrow p_{11}$; end
else begin i $\leftarrow i_2; p_1 \leftarrow p_{12}$; end;
$q_1 \leftarrow 1 - p_1; d_1 \leftarrow q_1^i * (2 - q_1^k);$
$c_1 \leftarrow (1 - q_1^i) * (1 - q_1^k);$
$f \leftarrow \frac{p_1 * d_1 - p_L * d_1}{p_L * c_1 - p_1 * d_1}$; (* see (2.3.8) *)
end.

Procedure : P1CS2B (p, p_L, k, i, p_1)
(* input : p, p_L, k, i, output : p_1, computes p_1^{2L} (i, k) given p_L, k, i and p, a suitable initial value for p_1. In extreme case it may be taken as 1 *)

Begin

$p_1 \leftarrow p$

repeat

$q_1 \leftarrow 1 - p_1, d_1 \leftarrow q_1^i * (2 - q_1^k);$

$c_1 \leftarrow (1 - q_1^i) * (1 - q_1^k);$

$b_1 \leftarrow i * q_1^{i-1} * (1 - q_1^k) + k * q_1^{k-1} * (1 - q_1^i);$

$a_1 \leftarrow (i + k) * q_1^{i+k-1} - 2 * i * q_1^{i-1};$

$p_{13} \leftarrow p_L + \frac{d_1 * (c_1 + d_1)}{d_1 * b_1 - a_1 * c_1};$

$p_1 \leftarrow p_{13}$

until $(| p_{13} - p_1 | < .0005;$

(* solves equation (2.3.9) for p_1^{2L} (i, k) *)

end.

2.4 Optimum CSP - 3 plans

2.4.1. Introduction

In this section is considered the logical extension of the procedures dealt with in Section 2.3 for CSP - 2 plans. The CSP - 3 plans have not been studied in detail in literature largely because there is no procedure, graphical or otherwise, to determine the plan parameters (i, f, k) which will ensure a desired AOQL. In this seciton we first propose to develop a procedure to determine the plan parameters for a plan with AOQL given to be p_L and then search for that (i, f, k) which will additionally minimise the AFI for a given $\overline{p}$ among the class of plans satisfying the AOQL stipulation.

2.4.2 Notation and Formulas

We will use frequently the following formula which was obtained by Dodge and

Torrey :

$$AOQ^3(i,f,k,p) = p(1-F^3(i,f,k,p))$$

$$= p[\frac{(1-f)q^i(2-q^{k+4})}{f(1-q^i)(1-q^{k+4})+q^i(2-q^{k+4})+4fpq^iq^k}] \quad (2.4.1)$$

2.4.3 Determination of (i, f, k) for a Desired AOQL Under CSP - 3

The problem is formally stated in Section 2.0. It follows from (2.4.1) that given AOQL = p_L,

$$p_L = p_1[\frac{(1-f)q_1^i(2-q_1^{k+4})}{f(1-q_1^i)(1-q_1^{k+4})+q_1^i(2-q_1^{k+4})+4fp_1q_1^iq_1^k}], \quad (2.4.2)$$

writing p_1 for $p_1^{3L}(i,k)$

For the determination of p_1 for a given plan with parameters (i, f, k) we proceed in the same manner as we did in the general case of CSP - 2. We first differentiate (2.4.1), given (i, f, k) w.r.t. p, equate to zero and solve for p.

$$Thus \frac{dAOQ^3(i,f,k,p)}{dp} = \frac{(1-f)D}{fC+D} + p\frac{(1-f)A(fC+D)-(1-f)D[fB+A]}{(fC+D)^2} \quad (2.4.3)$$

where $A = (i+k+4)q^{i+k+3} - 2iq^{i-1}$

$$B = iq^{i-1}(1-q^{k+4}) + (k+4)q^{k+3}(1-q^i) + 4(q^{i+k}-(i+k)pq^{i+k-1})$$

$C = (1-q^i)(1-q^{k+4}) + 4pq^{i+k}$ and

$$D = q^i(2-q^{k+4})$$

Using the subscript 1 to denote the values of A, B, C and D at p_1 and setting $\frac{dAOQ^3(i,f,k,p)}{dp} = 0$, we ultimately have

$$f = \frac{p_1 D_1 - p_L D_1}{p_L C_1 + p_1 D_1} \tag{2.4.4}$$

$$\text{and } p_1 = p_L + H(i, k, p_1), \tag{2.4.5}$$

$$\text{where } H(i, k, p_1) = \frac{D_1[C_1 + D_1]}{D_1 B_1 - A_1 C_1}$$

The value of p_1 can be obtained numerically to any degree of accuracy by finding the intersection of the curves :

$$y = p$$

$$\text{and} \quad y = p_L + H(i, k, p) \tag{2.4.6}$$

This is illustrated in Fig. 2.4.1. For $p_L = 0.05$, i = 31 and k = 8 we have $p_1 = 0.07997$ and f = 0.09302 as obtained from (2.4.6) and (2.4.4) respectively. It will be seen from the AOQ^3 curve that at p = 0.08 the AOQL of 0.05 is attained. Since i and k can be chosen arbitrarily there will be many combinations of (i, f, k) satisfying a given AOQL requirement.

2.4.4. On the Nature of p_1^{3L} (i, f, k) for CSP - 3

It was found mathematically in case of CSP - 1and in the special case (k = i) of CSP - 2 that p_1 at which the AOQL is reached is a decreasing function of i for a given AOQL. The same result for p_1 was obtained empirically for all given k in the general case of CSP - 2. For the CSP - 3 also, we proceed in the same line to study the nature of p_1 which, as mentioned in section 2.0 is denoted by p_1^{3L} (i, k).
The function H (i, k, p) as in (2.4.6) was evaluated for a large number of combinatins of i, k and p. It was found in all cases that

(i) H (i, k, p) is a decreasing function of p in the range of $0 < p < 1$ for given i and k, for all $i > 2$ when k is small and for all $i \geq 1$ for k large.
(ii) H (i, k, p) is a decreasing function of i for all given k and p.
The observations made above have been amply verified by exhaustive numerical computations. As an illustration we present below in table 2.4.1 the values of H (i, k, p) for a few combinations of (i, k, p).
The salient points made above are clearly exhibited by the table. So, we note as in sec. 2.3.4 empirically that p_1^{3L} (i, k) decreases as i increases, for a given k.
By following the arguments as in the general CSP - 2, it is clear from observation (ii) above that given a choice of k and p_L, we can find for every $\bar{p} > p_L$ a unique i such that

$$p = p_L + H(i, k, p).$$

Thus given k, for any choice of $\bar{p} > p_L$ we can find an i_0 such that p_1^{3L} (i_0, k) $\simeq \bar{p}$, this p_1 leads to the AOQL whose value is given to be p_L.

Table 2.4.1
Values of H (i, k, p) for Some Choices of i, k and p

	p = .1			p = .3			p = .5		
i	k = 2	4	8	k = 2	4	8	k = 2	4	8
1	0.714	1.057	2.088	1.428	2.050	1.656	1.162	0.956	0.772
2	0.530	0.666	0.896	0.597	0.634	0.581	0.363	0.334	0.315
3	0.412	0.472	0.552	0.346	0.347	0.329	0.200	0.192	0.188
4	0.330	0.358	0.389	0.232	0.229	0.222	0.136	0.134	0.133
5	0.271	0.283	0.295	0.170	0.168	0.165	0.104	0.103	0.103
10	0.126	0.123	0.121	0.071	0.071	0.072	0.050	0.050	0.050
15	0.074	0.072	0.070	0.046	0.046	0.047	0.033	0.033	0.033
20	0.049	0.049	0.048	0.034	0.035	0.035	0.025	0.025	0.025

2.4.5 On the Determination of (i, f, k) that Minimises the F^{8L} (i, k, $\overline{p}$), for a Given Process Average $\overline{p}$.

The problem is formaly stated in Section 2.0. Based on empirical studies over the usual range of AOQL and $\overline{p}$ we have arrived at the following observations.

Observation 2.4.1 : *For a given p_L and an arbitrarily chosen k, there exists a unique combination (i, f, k) which minimises the AFI, F^{8L} (i, k, $\overline{p}$) at $\overline{p}$ given under CSP - 3 inspection provided $\overline{p} > p_L$. Furthermore, the minimum is attained for the value of i for which p_1^{3L} (i, k) $= \overline{p}$. In case p_1^{3L} (i, k) is not exactly equal to $\overline{p}$, the value of i for which p_1^{3L} (i, k) is nearest to $\overline{p}$ is the optimal choice.*

Observation 2.4.2 : *For given p_L, $\overline{p}$ k, the AFI, F^{8L} (i, k, $\overline{p}$) under CSP - 3 decreases monotonically for all i as long as p_1^{3L} (i, k) $> \overline{p}$ and then increases monotonically for all i for which p_1^{3L} (i, k) $< \overline{p}$ i.e. attains a unique minimum for the value of i for which $p_1^{3L}(i,k) = \overline{p}$* for all $\overline{p} > p_L$

The above observation is an outcome of the decreasing nature of p_1^{3L} (i, k) with i given p_L and k. The explanation provided in case of CSP - 2 in Sec. 2.3.5 is also relevant here. To illustrate this point we present the AOQ values for 5 plans each ensuring the same AOQL = 0.05 and having i values within a close range of 29 - 33 and k = 8, in Table 2.4.2. Given k = 8, $p_L = 0.05$ and $\overline{p} = 0.08$, the plan (31, 8) minimises F^{8L} (i, k, $\overline{p}$) and the choice of i is obviously unique. The same feature is observed in all the problems of the type, given k, p_L and $\overline{p}$.

Table 2.4.2

AOQ^3 For a Few Plans With Some Consecutive Values Of i, Each Ensuring $p_L = 0.05$ (The Plan with i = 31, k = 8 minimises F^{3L} (i, k, $\overline{p}$), where $\overline{p} = 0.08$)

p	k = 8				
	i = 29	i = 30	i = 31	i = 32	i = 33
.060	.044931	.045251	.045550	.045830	.046092
.062	.045976	.046282	.046566	.046831	.047077
.065	.046910	.047196	.047460	.047704	.047928
.067	.047727	.047989	.048226	.048444	.048641
.070	.048425	.048655	.048860	.049045	.049208
.072	.048998	.049191	.049358	.049503	.049625
.075	.049444	.049593	.049715	.049814	.049889
.077	.049760	.049859	.049929	.049976	.049997
.080	.049945	.049987	.050000	.049986	.049948
.082	.049999	.049977	.049926	.049847	.049742
.085	.049920	.049830	.049708	.049558	.049380
.087	.049711	.049546	.049349	.049123	.048867
.090	.049373	.049130	.048852	.048545	.048207

2.4.6 An Algorithm to Find Optimum CSP - 3 Plan for a Desired AOQL.

The algorithm can be stated as follows :
Algorithm : MNCS3 ($\overline{p}, p_L$, k, i, f)
(* input parameters : $\overline{p}, p_L$, k;
output parameters : i, f; *)
(* Finding optimum (i, f) for CSP - 3 minimising AFI at $\overline{p}$ and satisfying AOQL = p_L, given k, $\overline{p}, p_L$ *)

Begin

$i \leftarrow 0; p_{12} \leftarrow 0; i_2 \leftarrow 0;$

repeat

$i \leftarrow i+1; q_1 \leftarrow 1-\overline{p}; d_1 \leftarrow q_1^i * (2-q_1^{k+4});$

$c_1 \leftarrow (1-q_1^i) * (1-q_1^{k+4}) + 4p_1 q_1^{i+k};$

$b_1 \leftarrow i * q_1^{i-1} * (1-q_1^{k+4})$

$+(k+4)q_1^{k+3}(1-q_1^i) + 4(q_1^{i+k} - (i+k)p_1 q_1^{i+k-1})$

$a_1 \leftarrow (i+k+4) * q_1^{i+k+3} - 2iq_1^{i-1};$

$p_1 \leftarrow p_L + \frac{d_1*(c_1+d_1)}{d_1*b_1-a_1*c_1};$

$p_{11} \leftarrow p_{12}; i_1 \leftarrow i_2;$

$p_{12} \leftarrow p_1; i_2 \leftarrow i;$

until $(p_1 < \overline{p})$

(* solves (2.4.5) with $p_1 = \overline{p}$ in r.h.s. to give two consecutive values of i one of which is optimum *)

P1CS3 $(p_{11}, p_L, k, i_1, p_1)$; $p_{11} \leftarrow p_1;$

P1CS3 $(p_{12}, p_L, k, i_2, p_1)$; $p_{12} \leftarrow p_1;$

if $(| p_{11} - \overline{p} | < | p_{12} - \overline{p} |)$

then begin $p_1 \leftarrow p_{11}; i \leftarrow i_1;$ end

else begin $p_1 \leftarrow p_{12}; i \leftarrow i_2;$ end;

$q_1 \leftarrow 1-p_1; d_1 \leftarrow q_1^i * (2-q_1^{k+4});$

$c_1 \leftarrow (1-q_1^i) * (1-q_1^{k+4}) + 4p_1 q_1^{i+k};$

$f \leftarrow \frac{p_1*d_1-p_L*d_1}{p_L*c_1-p_1*d_1};$ (* see (2.4) *)

end.

Procedure : P1CS3 (p, p_L, k, i, p_1)

(* input : p, p_L, k, i; output : p_1, computes p_1^{3L} (i, k) given p_L, k, i and a suitable p as intitial value of p_1. In extreme case p may be taken as 1 *)

Begin

$p_1 \leftarrow p;$

repeat

$q_1 \leftarrow 1-p_1; d_1 \leftarrow q_1^i * (2-q_1^{k+4});$

$c_1 \leftarrow (1-q_1^i)*(1-q_1^{k+4})+4p_1q_1^{i+k};$
$b_1 \leftarrow i*q_1^{i-1}*(1-q_1^{k+4})+(k-4)q_1^{k+3}(1-q_1^i)+4(q_1^{i+k}-(i+k)p_1q_1^{i+k-1})$
$a_1 \leftarrow (i+k+4)*q_1^{i+k+3}-2*i*q_1^{i-1};$
$p_{13} \leftarrow p_L + \frac{d_1*(c_1+d_1)}{d_1*b_1-a_1*c_1};$
$p_1 \leftarrow p_{13};$
until $(|p_{13}-p_1|< .0005)$;
(* Solves (2.4.5) iteratively for p_1 *)
end.

2.5 Some Concluding Remarks

Remark 2.5.1

The optimum plans for CSP - 1 are given in Appendix - I for different values of AOQL ranging from .05% to 10.0% and $\overline{p}$ from .05% to 20.0%. The optimum plans for CSP - 2 with k = i are given in Appendix - II for AOQL from 1.0% - 10.0% and $\overline{p}$ from 1% - 10.0%. The optimum plans for general CSP - 2 are given in Appendix - II for the same range of AOQL and $\overline{p}$ and for two values of k, namely, k =i - 10 and k = i + 10. The optimum plans for CSP - 3 are given in Appendix - III for k = i - 10, k = i and k = i + 10 for AOQL ranging from 1.0% to 5.0% and $\overline{p}$ from 1.0% to 10.0%. It may be noted that for all the plans supplied in cases of general CSP - 2 and CSP - 3 the observations on and properties of the optimum plans which were established empirically have been found to hold.

Remark 2.5.2

The optimum CSP plans can be chosen only when $\overline{p} > p_L$. In the relevant sections we have discussed how this condition arises. For $\overline{p} < p_L$, we do not have any optimum plan. In fact in such a situation no inspection is needed. But inspection cannot be altogether avoided as incoming quality may deterio-

But inspection cannot be altogether avoided as incoming quality may deteriorate and if not detected and controlled quickly different segments of outgoing products may have inferior quality, worse than the AOQL. Thus for $\overline{p} < p_L$, we recommend the plan for which the tabulated value of $\overline{p}$ is just greater than the required AOQL.

Remark 2.5.3

The problem of finding (i_0, f_0) for a CSP - 1 which ensures a given p_t (%) and an $AOQL = p_L$ and moreover minimizes AFI at a given $\overline{p}$ among all such plans satisfying the AOQL and $p_t(\%)$ stipulation has been solved in section 2.1.5. We have developed procedures in Chapter 5 for finding the corresponding plan parameters for CSP - 2 and CSP - 3 ensuring a given $p_t(\%)$. Thus if the same $p_t(\%)$ restriction is imposed on CSP - 2 and CSP - 3, optimum plans in those cases under exactly the same stipulations can be easily obtained.

Remark 2.5.4

It has been shown in sec. 2.1 that in CSP - 1, for given p_L and $p_{(2)} > p_{(1)}$, $i_1 > i_2$ where i_1 and i_2 are respectively the optimum choice of i for process averages $p_{(1)}$ and $p_{(2)}$ respectively.
The same is true for CSP - 2 and CSP - 3 plans also, given p_L and k. This is again a direct outcome of decreasing nature of the p_1^{2L} or p_1^{3L} functions with increase in i, given k.

Remark 2.5.5

From a comparison of optimum plans for CSP - 1, CSP - 2 and CSP - 3 it will be seen that the optimum value of i in all the cases lies within a close range whereas the variation in values of f is quite high. This is found to be so within the range of values of k chosen for our computation in cases of CSP

- 2 and CSP - 3.

Remark 2.5.6

The minimum AFI's, F^{1L}, F^{2L} and F^{3L} at $\overline{p}$ for the optimum CSP - 1, CSP - 2 and CSP - 3 plans, respectively are all equal and this minimum is also the barest minimum AFI required at $\overline{p}$ within the class of all CSP plans satisfying the desired AOQL $= p_L$. Thus all these optimum CSP plans are also globally optimum as elaborated in connection with CSP - 1 plans in section 2.1.4. (NB 2.1.2).

FLOW CHART FOR CSP PLANS

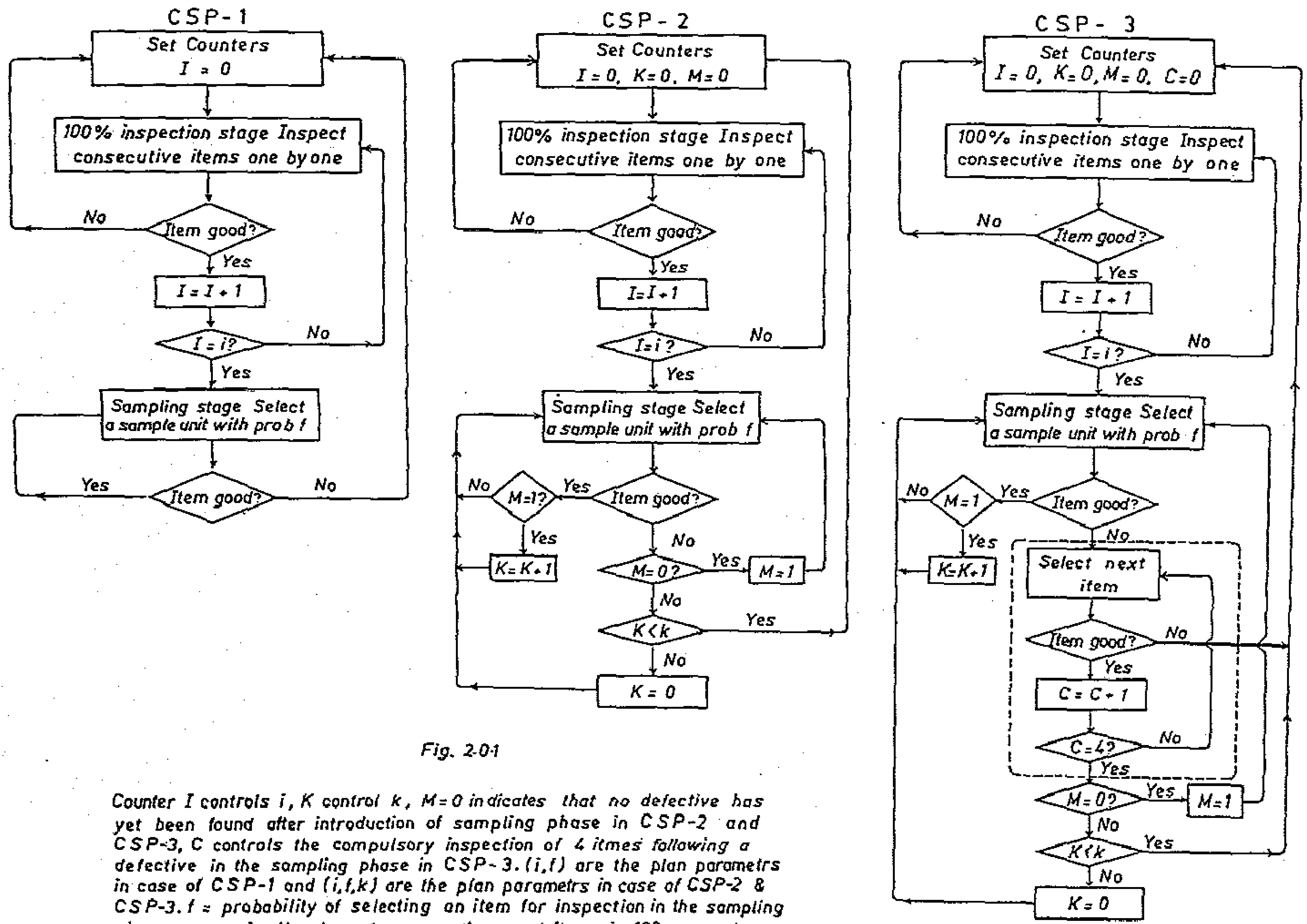


Fig. 2.0.1

Counter I controls i, K control k, M = 0 indicates that no defective has yet been found after introduction of sampling phase in CSP-2 and CSP-3, C controls the compulsory inspection of 4 itmes following a defective in the sampling phase in CSP-3. (i, f) are the plan parametrs in case of CSP-1 and (i, f, k) are the plan parametrs in case of CSP-2 & CSP-3. f = probability of selecting an item for inspection in the sampling phase. i = Number of consecutive good items in 100 percent inspection stage leading to sampling phase (clearance number) k = minimum number of good items between two defectives in the sampling phase of inspection allowing sampling phase to continue.

Values of $\frac{D_2}{D_1}$ for different i

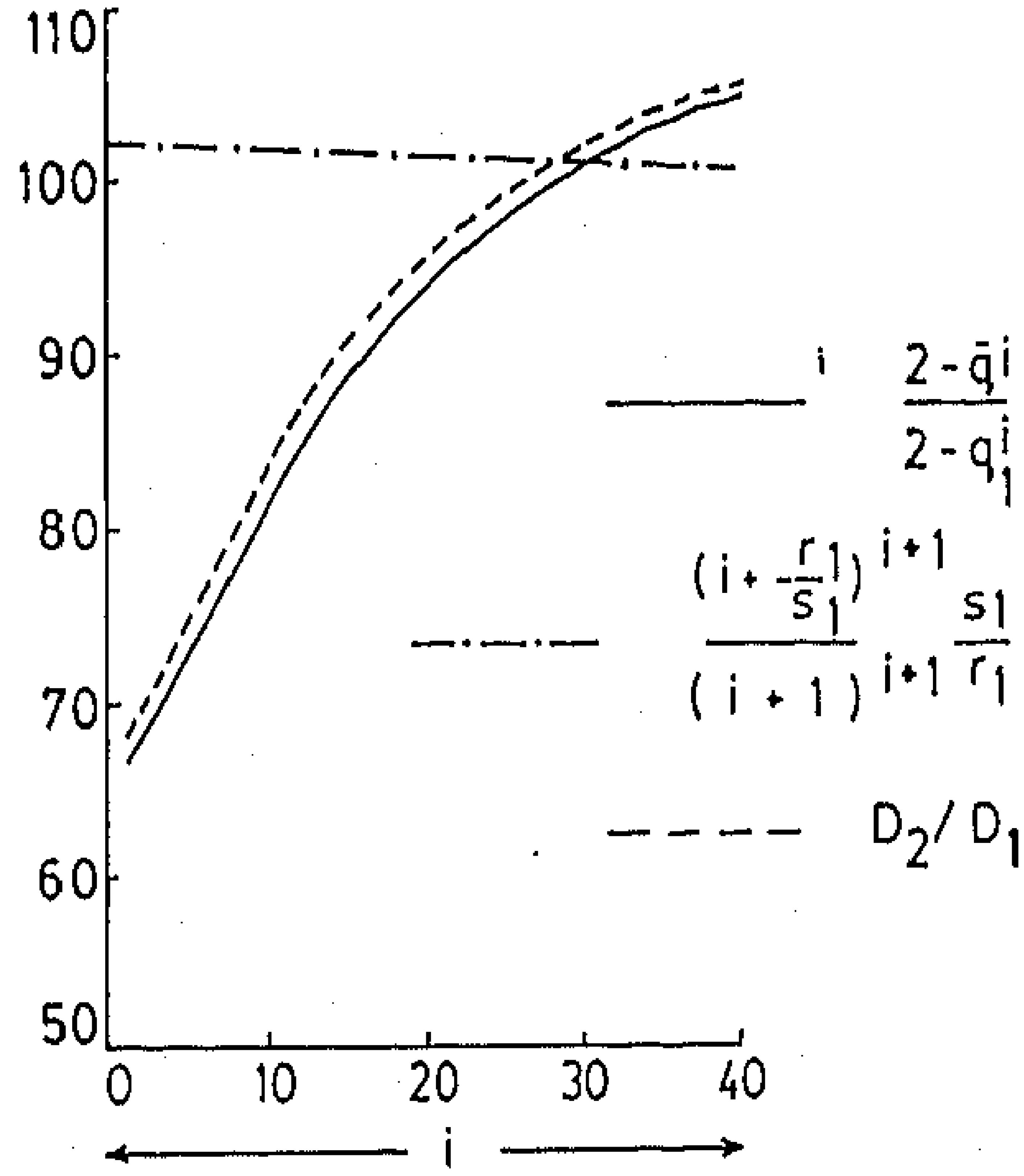


Fig : - 2.2.1

AOQ Curve for CSP - 2 with K = 8, i = 10,
f = 424185 PL= 05

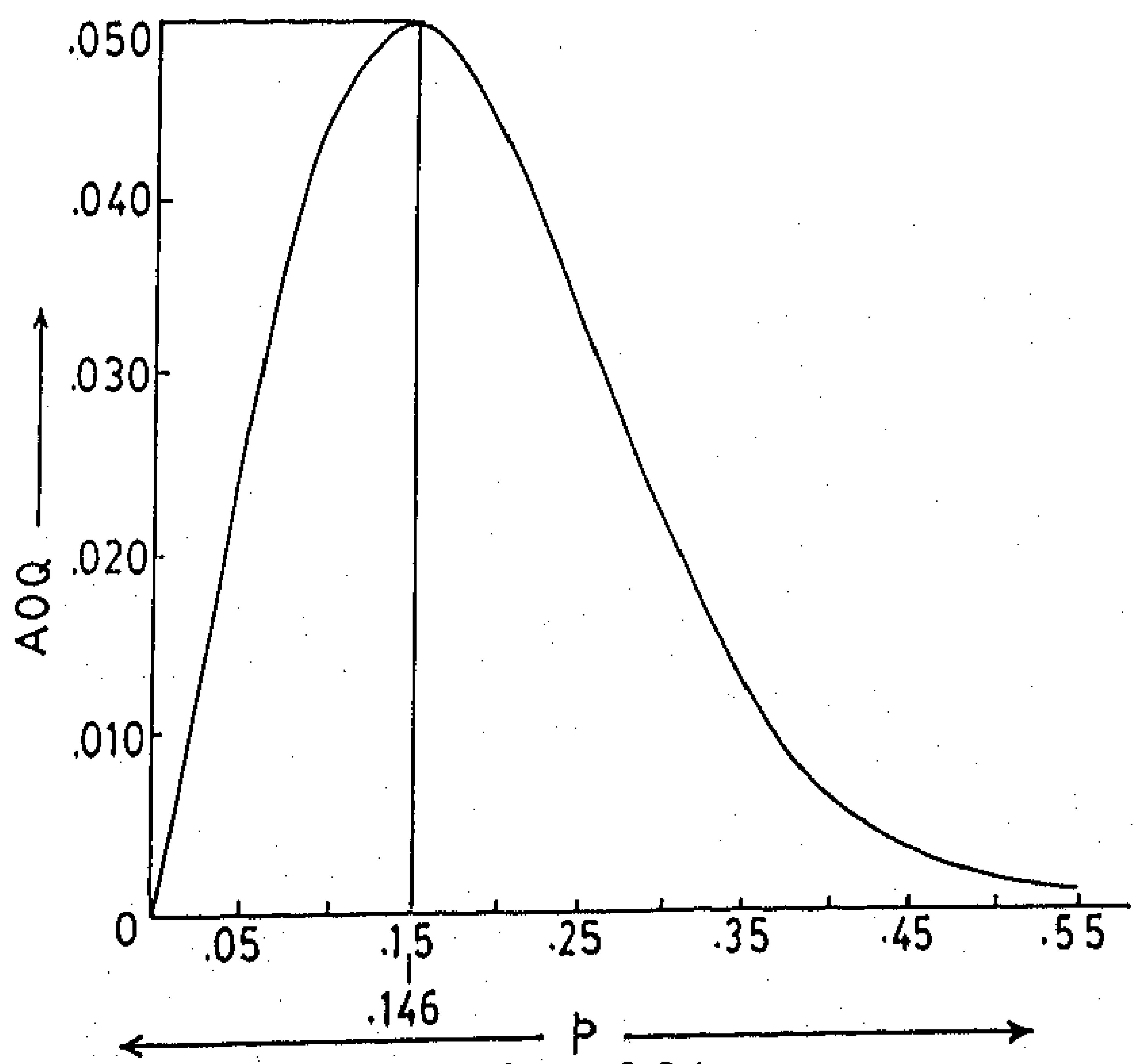


Fig :- 2.3.1

Intersection of Y = p and Y = G (i, k, p) for different values of k and i

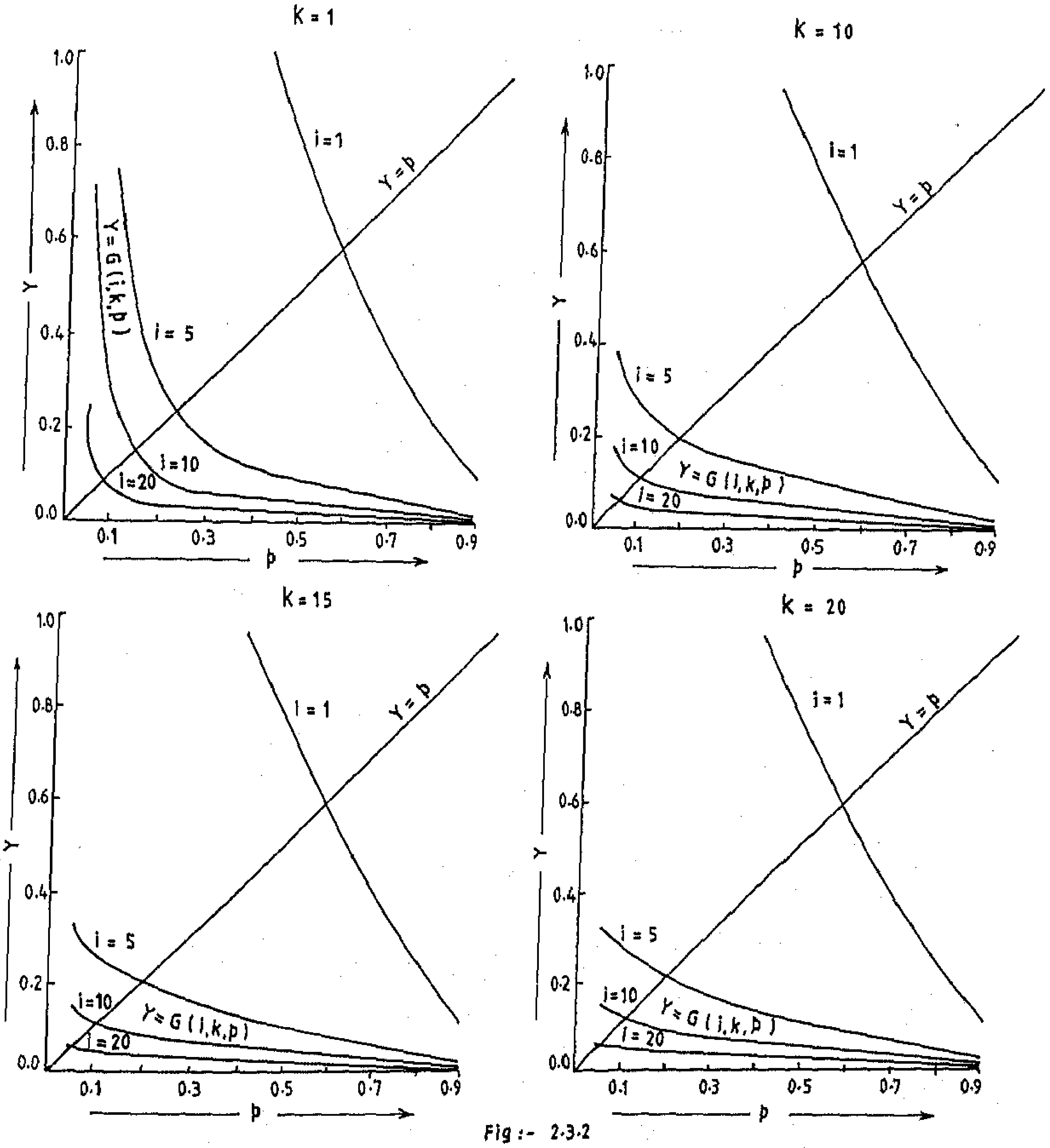


Fig :- 2.3.2

AOQ Curve for CSP - 3 Plan

K = 8, L = 31, f = 09302, P_L = .05

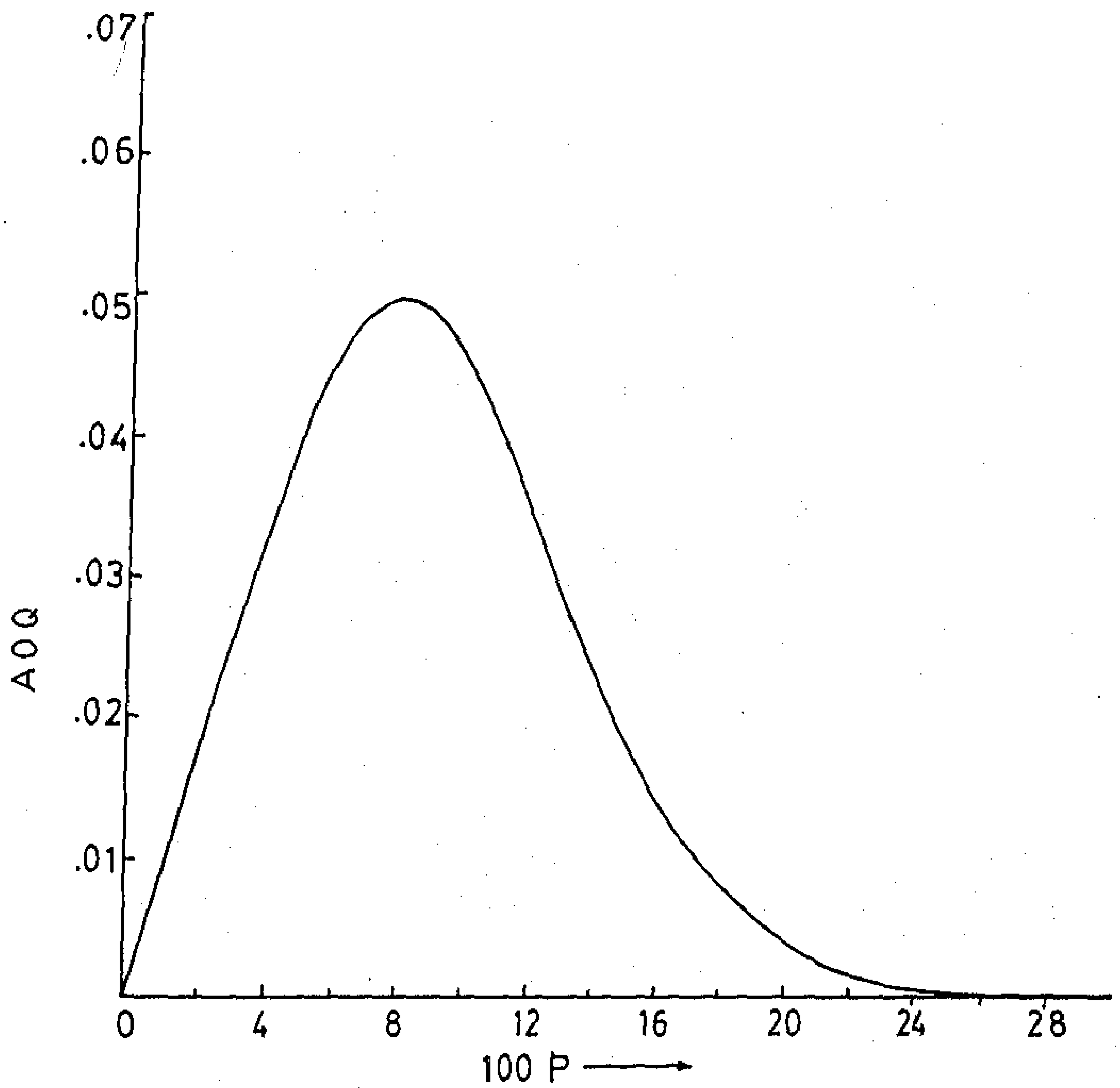


Fig. 2.4.1

CHAPTER - 3

LQL INDEXED OPTIMUM CSP - 1 PLAN THAT MINIMISES AMOUNT OF INSPECTION

3.1 Introduction

The acceptance rectification scheme for lot-by-lot inspection introduced by Dodge and Romig was developed to ensure either (i) a desired AOQL or (ii) a given LTPD as a means of consumer protection. To safeguard the producer's interest plan parameters were so chosen as to minimise the average fraction inspected for a given process average $\overline{p}$ in addition to satisfying the AOQL or LTPD restrictions.

The equivalent of lot-by-lot LTPD plans in continuous sampling inspection was developed by Stephens (1981) by selecting plan parameters (i, f) of CSP - 1 in such a manner that the plan provided customer protection based on LQL i,e Limiting Quality level.

LQL as defined by Stephens is as follows :

This is the incoming quality p for which the 'percent of total production units accepted on a sampling basis is 0.10 i,e the value of p for which

$$L(p) = \frac{v}{u+v} = \frac{q^i}{f+(1-f)q^i} = 0.10$$

However, neither Dodge nor Stephens considered the problem of minimising the AFI at $\overline{p}$ in case of continuous sampling plan to safeguard producer's interest. In this chapter we present a procedure of selecting plan parameters (i, f) of CSP - 1 such that the plan ensures a given LQL $= p^*$, say, along with a desired AOQL, say, $\leq p_L$ and additionally, minimise the amount of inspection for a given process average $\bar{p}$ among all the CSP - 1 plans satisfying the LQL and AOQL stipulations.

Thus the work presented here together, with those in chapter 2 supplies a complete analogue for continuous sampling plans of Dodge - Romig lot-by-lot sampling plans based on AOQL and LTPD approaches.

3.2 Definitions and Notation

LQL has been defined in section 3.1. So, LQL indexed plan will have a fixed consumer's risk of 0.10. Thus LQL corresponds to LTPD of Lot-by-Lot sampling inspection. It may be pointed out here that LQL is not to be confused with p_t (%) which has been used in CSP plans to designate spotty quality and already introduced in section 2.1.1.

As the restriction is on LTPD in the plans considered in the present chapter, it will be needed to consider plans with different AOQL values simultaneously and compare them. We will use the notation F^{L_k} (i, p) to denote the AFI, F^{1L} (i, p) when $p_L = p_{L_k}$ (given) for all i ϵ N and all p $\epsilon\Omega$. This is done for simplicity of notation, as we are considering only CSP - 1 in the present chapter.

Using Stephens (1979), let us write for a given (i, f) CSP - 1,

$$L(p) = \frac{v}{u+v} = q^i/(f+(1-f)q^i) \qquad (3.2.1)$$

Let p^* denote the LQL

Then it follows from (3.2.1) that

$$f = \frac{9q^{*^i}}{1-q^{*^i}} \qquad (3.2.2)$$

$$\text{where} \quad q^* = 1 - p^*.$$

From Stephens (1981) we have equations for i (in terms of f and p^*) and p^* (in terms of f and i).

They are as follows :

$$i = [\ln f - \ln(9+f)]/\ln(1-p^*) \tag{3.2.3}$$

$$\text{and} \quad p^* = 1 - [f/(9+f)]^{\frac{1}{i}} \tag{3.2.4}$$

3.3 Determination of Optimum (i, f) for CSP - 1 Ensuring a Given LQL.

Stephens (1981) provided nomograph and tables for finding plan parameters (i, f) for LQL stipulated CSP - 1 plan. The general feature of these plans is that there are a large number of combinations of (i, f) which ensure the same LQL (as can be seen from equation (3.2.2)). We note from the table provided by Stephens that for every such (i, f) the AOQL will be different though the resulting LQL is same for all the plans.

We consider two plans from Table 1 and Table 2 of Stephens (1981) such that both ensure the same LQL of 32.5%. The AFI at a process average of 10% for the two plans is shown below :

LQL = 32.5%

Plan	(i, f)	AFI(%) at $\bar{p} = 0.1$	Resulting AOQL (%)
1	(9, .333)	52.3	4.76
2	(18, .010)	5.5	13.24

From the viewpoint of AFI at $\overline{p} = 0.1$, certainly plan 2 will be preferred, but the AOQL of this plan is considerably high, viz., 13.24% , making it not so useful for practical purpose. One can in this case with given LQL, find a CSP - 1 plan for which the AFI at the process average is nearly 0 and the AOQL in that case will become arbitrarily large. Thus the need for a judicious choice of (i, f) to meet a given LQL, taking into account the other criterion of AOQL, can hardly be overemphasised.

We, therefore, given $\overline{p}$, p_L and $p' \epsilon\ \Omega$, set out to find a combination of (i, f) for a CSP - 1 plan which will ensure.

(a) LQL $= p'$

(b) AOQL $\leq p_L$

and also (c) minimization of AFI at $\overline{p}$ viz., $F'(i, f, \overline{p})$

among all the CSP - 1 plans satisfying (a) and (b). The need for the inclusion of (b) in the present set of criteria is quite clear from the preceding discussion.

The main results of this section are presented in the following theorems.

Theorem 3.3.1 *Let there be n CSP - 1 plans each ensuring the same LQL. Let the plan parameters for the m^{th} plan be (i_m, f_m) having corresponding AOQL as p_{L_m} for $1 \leq m \leq n$. Then, $i_1 < i_2 < \ldots < i_m$ implies $p_{L_1} < p_{L_2} \ldots\ldots < p_{L_m}$.*

Proof : We have from Lieberman and Solomon (1955), for CSP - 1 plan with AOQL $= p_L$.

$$f = \frac{(1-p_L)^{i+1}}{(1-p_L)^{i+1} + p_L.\frac{(i+1)^{i+1}}{i^i}} \tag{3.3.1}$$

Let us assume that the given value of LQL is $p^* = 1 - q^*$. Combining (3.3.1) with (3.2.2), we have

$$\frac{9q^{*^i}}{1-q^{*^i}} = \frac{(1-p_L)^{i+1}}{(1-p_L)^{i+1} + p_L.\frac{(i+1)^{i+1}}{i^i}}$$

$$\text{or } 9q^{*^i}\left\{(1-p_L)^{i+1} + p_L.\frac{(i+1)^{i+1}}{i^i}\right\} = (1-q^{*^i})(1-p_L)^{i+1}$$

$$\text{or } (1-q^{*^i} - 9q^{*^i})(1-p_L)^{i+1} = 9p_L.q^{*^i}\frac{(i+1)^{i+1}}{i^i}$$

$$\text{or } ln(1-10q^{*^i}) + (i+1)ln(1-p_L) = ln9 + lnp_L + ilnq^*$$

$$+(i+1)ln(i+1) - ilni \qquad (3.3.2)$$

Given LQL $= q^*$, there can be different CSP - 1 plans with different values of i and the corresponding AOQL values denoted by p_L here will be diferent for these different plans. Treating i as a continuous variable the implicit function (3.3.2) in i and p_L is continuous. Differentiating with respect to i.

$$\frac{1}{1-10q^{*^i}}[-10q^{*^i}lnq^*] + \frac{i+1}{1-p_L}(-\frac{dp_L}{di})$$

$$= \frac{1}{p_L}.\frac{dp_L}{di} + lnq^* + ln(i+1) + \frac{i+1}{i+1} - lni - \frac{i}{i}$$

$$\text{or } (\frac{1}{p_L} + \frac{i+1}{1-p_L})\frac{dp_L}{di} = -lnq^* - ln\frac{i+1}{i} - \frac{10q^{*^i}lnq^*}{1-10q^{*^i}}$$

$$= -[1 + \frac{10q^{*^i}}{1-10q^{*^i}}]lnq^* - ln(1+\frac{1}{i})$$

$$= -\frac{1}{1-10q^{*^i}}lnq^* - ln(1+\frac{1}{i})$$

we will show that the last expression > 0

$$\text{i, e } \frac{1}{1-10q^{*^i}}(-lnq^*) > ln(1+\frac{1}{i})$$

$$\text{i, e } -lnq^* > (1-10q^{*^i})ln(1+\frac{1}{i}) \tag{3.3.3}$$

Since ln $(1+\frac{1}{i}) < \frac{1}{i}$ for i ≥ 1 a sufficient condition for (3.3.3) is that

$$-lnq^* > \frac{1-10q^{*^i}}{i} \tag{3.3.4}$$

Writing - ln $q^* =$ x, x is positive

$$\text{or } q^* = e^{-x}$$

It is, therefore, necessary to show that

$$ix > 1-10e^{-ix}$$

or $1-ix < 10e^{-ix}$, for x positive. (3.3.5)

which is trivially true.

Thus $\frac{dp_L}{di} > 0$. Hence the result

Lemma 3.3.1 *Given $\overline{p}$, let there be two CSP - 1 plans with same i and different values of f, given AOQL values p_{L_1} and p_{L_2} with $p_{L_2} > p_{L_1}$. Then F^{L_2} $(i, \overline{p}) < F^{L_1}$ $(i, \overline{p})$ for all $i \in N$.*

Proof : For any given i and $\overline{p}$, F^{L_k} (i, $\overline{p}$) considered as a continuous function of p_{Lk} = the value of AOQL (eqn. 2.1.8) can be easily shown to be strictly monotonically decreasing. Hence the proof.

Lemma 3.3.2 *Let i_{01} and i_{02} be respectively the values of i for which $F^{L_1}(i,\overline{p})$ and $F^{L_2}(i,\overline{p})$ attain their minimum. Then $i_{01} < i_{02}$ if $p_{L_1} < p_{L_2}$.*

Proof : we have from Theorem 2.1.1.

$$\frac{i_{01}}{i_{01}+1} = \frac{1-\overline{p}}{1-p_{L1}} \text{ and } \frac{i_{02}}{i_{02}+1} = \frac{1-\overline{p}}{1-p_{L2}}$$

$$\text{Since } p_{L_1} < p_{L_2}, \frac{1-\overline{p}}{1-p_{L_1}} < \frac{1-\overline{p}}{1-p_{L_2}}$$

$$\text{Thus } \frac{i_{01}}{i_{01}+1} < \frac{i_{02}}{i_{02}+1} \text{ and hence } i_{01} < i_{02}.$$

Theorem 3.3.2 *Let there be k CSP -1 plans, each ensuring the same $LQL = p^*$ such that the prarameters of the m^{th} plan are (i_m, f_m) with corresponding AOQL as p_{L_m}, $1 \leq m \leq k$. Then $i_1 < i_2 < \ldots < i_m$ implies*

$$\min_{1 \leq m \leq k} F^{L_m}(i_m, \overline{p}) = F^{L_k}(i_k, \overline{p}) \text{ for all } \overline{p} < p_{L_K}$$

Proof : : It follows from Theorem 3.3.1 that

$$p_{L_1} < p_{L_2} < \ldots < p_{L_k} \text{ as}$$

$$i_1 < i_2 < \ldots < i_K$$

Now, given p_{L_k} and $\overline{p}$, $F^{L_k}(i, \overline{p})$ is monotonically decreassing for all i as $\overline{p} < p_{L_k}$ (from Theorem 2.1.1). Also $F^{L_j}(i,\overline{p}) > F^{L_m}(i,\overline{p})$ for m > j and for all i (Lemma 3.3.1).

Hence the theorem.

This result stated in Theorem 3.3.2 is illustrated in Fig. 3.3.1.

Theorem 3.3.3 *Let $i_1 < i_2 < \ldots < i_k$ be the clearance numbers for k CSP - 1 plans, each ensuring the same $LQL = p^*$. Let the corresponding AOQL's be $p_{L_1}, p_{L_2}, \ldots, p_{L_k}$. Let p_c be the process average such* *that F^{L_k} $(i_k,\, p_c)$ is minimum over all i ensuring the same AOQL as p_{L_K}. Then for $p_{L_k} < \overline{p} < p_c$, $F^{L_k}(i_k, \overline{p}) = \min_{i \le m \le k} F^{L_m}(i_m, \overline{p})$*

Proof : It is to be noted that for the CSP - 1 plan with $i = i_k$ and AOQL $= p_{L_k}$, $F^{L_k}(i_k, p) = \min_{i \epsilon N} F^{L_k}$ (i, p) for some p and this value of p is denoted by p_c.

Now, in view of Theorem 3.3.1

$p_{L_1} < p_{L_2} < \ldots < p_{L_k} < \overline{p} < p_c$ where p_c is obtained by solving

$$\frac{i_k}{i_k + 1} = \frac{1 - p_c}{1 - p_{L_k}} \tag{3.3.6}$$

(See Theorem 2.1.1)

Given p_L and $\overline{p}$, let $F^{L_k}(i, \overline{p})$ be minimum at i $= i_k'$. Then it will be shown in Lemma 4.1.2 that $i_k' \ge i_k$.

Hence given $\overline{p} \le p_c$ and p_{L_k}, $F^{L_k}(i, \overline{p})$ is monotonically decreasing for all i $\le i_k$. (See Lemma 2.1.4 and Theorem 2.1.1). Then essentially the same arguments as in Theorem 3.3.2 implies the results.

Remark 3.3.1 *It is interesting to note here that with all other conditions same as in Theorem 3.3.3, but $\overline{p} > p_c$, $F^{L_k}(ik, \overline{p})$ is not necessarily the minimum over all the k plans. The arguments provided below clarify the point. Since $\overline{p} > p_c$, the optimum i_0 that minimises $F^{L_k}(i, \overline{p})$ over all i will be such that $i_0 < i_k$. Thus $F^{L_k}(i, \overline{p})$ is an increasing function of i for all $i > i_0$ and $\overline{p} > p_c$. If in any case $i_{k-1} > i_0$ (there will be many situations where it will be so) then*

$$F^{L_k}(i_{k-1},\overline{p}) < F^{L_k}(i_k,\overline{p}) \text{ as } i_0 < i_{k-1} < i_k.$$

But in view of Lemma 3.3.1,

$p_{L_{k-1}} < p_{L_k} \rightarrow F^{L_k-1}(i_{k-1},\overline{p}) > F^{L_k}(i_{k-1},\overline{p})$ *so,* $F^{L_k-1}(i_{k-1},\overline{p})$ *may not be greater than* $F^{L_k}(ik,\overline{p})$.

In view of the above results, the essence of the procedure to be adopted to find a solution to the problem can be described as :

Given p_L and p^* find the CSP - 1 plan with largest integer i satisfying AOQL $\leq p_L$ and LQL $= p^*$, by using equations (3.2.2) and (3.3.2). Denote it by i_k. Then find p_c for this plan using equation (3.3.6).

If $\overline{p} \leq p_c$, the CSP - 1 plan obtained above (i_k, f_k) is the required optimum plan. If $\overline{p} > p_c$, consider all CSP - 1 plans with AOQL $\leq p_L$ and LQL $= p^*$ (Let there be k such plans (i_m, f_m), i $\leq m \leq k$ with $i_1 < i_2 .... < i_k$ such that $p_{L_1} < p_{L_2} .... \leq p_{L_k}$). Compute the AFI for each one of the plans. The plan with minimum AFI amongst them is the requred CSP - 1 plan. The algorithm giving the formal scheme of computerization is presented below. The optimum (i, f) combination is given in Appendix 4 for LQL ranging from 5.0 - 32.0%, AOQL ranging from 1.00% - 15.00% and the process average ranging from 1.0% - 20.0%.

Algorithm : MNCS1 $p^*(p_0, p_L, \overline{p}, i, f)$
(* input parameters : $p_0, p_L, \overline{p}$;
output parameters : i, f; *)
(* Given $p_0, p_L, \overline{p}$ the algorithm gives (i, f) for CSP - 1

which will satisfy LQL = p_0, AOQL $\leq p_L$ and minimise AFI at $\overline{p}$ *)

(* ii, pp_L, ff are array variables each of dimension 150 *)

Begin

```
        i ← 0; k ← 0; q_0 ← 1 − p_0;
        repeat
          i ← i + 1; k ← k + 1;
          const ← ln(9) + i * ln(q_0) + (i + 1) * ln(i + 1) − i * ln(i)
              −ln(1 − 10 * q_0^i);
          p_1 ← p_L;
          repeat
            q_1 ← 1 − p_1; x ← (const + ln(p_1))/(i + 1);
            p_11 ← 1 − e^x; p_1 ← p_11;
          until (| p_11 − p_1 |< ε);
            (* ε is a preassigned small positive number say .0005.
            For a given i corresponding AOQL is obtd by solving
               (3.3.2) given p_0 *)
            ii(k) ← i; pp_L(k) ← p_1; ff(k) ← (9*q_0^i)/(1−q_0^i);
          until (pp_L(k) > p_L);
          (* for a given i corresponding AOQL and f ensuring p_L
          and p_0 as given by (3.2.2) are stored *)
          k ← k − 1; p_c ← (ii(k)*pp_L(k)+1)/(ii(k)+1);
          if (p̄ < p_c) then
          begin i ← ii(k); f ← ff(k); end
          else begin
            val ← 1.0;
            for j = 1 to k do
```

```
        if F^{1L}(ii(j), p̄) < val then
        begin val ← F^{1L}(ii(j), p̄);
            store ← j;
        end;
    end;
    (* F^{1L}(ii(j), p̄) is calculated
    using (2.1.8) *)
    i ← ii (store); f ← ff(store);
end.
```

ILLUSTRATION OF THEOREM 3.3.2

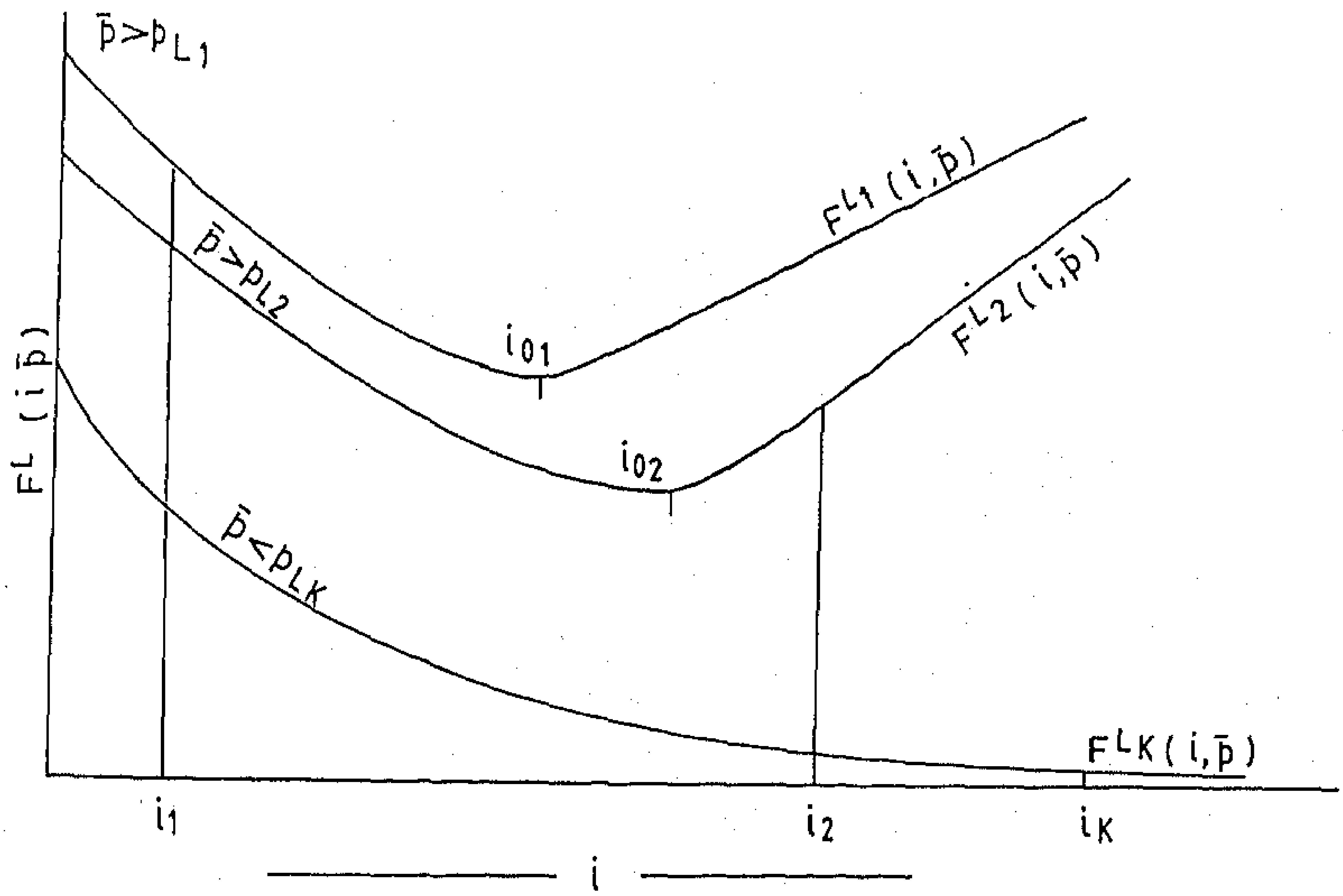


Fig. 3.3.1

i_{01} and i_{02} are the values of i for which $F^{L_1}(i,\bar{p})$ and $F^{L_2}(i,\bar{p})$ are minimum. i_1, i_2, i_K are the values of i for which three plans have the same LQL.

$$\underset{i \leq j \leq K}{\text{Min}} \; F^{L_j}(i_j,\bar{p}) = F^{L_K}(i_K,\bar{p})$$

CHAPTER 4
AOQL LINKED OPTIMUM CSP PLANS THAT MINIMISE AMOUNT OF INSPECTION WHEN INCOMING QUALITY FOLLOWS A DISTRIBUTION.

4.0 Introduction

The underlying assumption in obtaining optimum combination of plan parameters of CSP plans is that incoming quality is controlled all the time at a single value. However, it is more realistic to assume that incoming quality follows a distribution. For example, to start with the simplest case, we may assume that the desirable incoming quality which is normally expected follows a two point Binomial distribution. For the sake of uniformity with the preceding chapters we call this desirable quality also, process average and denote it by the same $\overline{p}$. But here $\overline{p}$ is not fixed. It is a random variable. It may be quite logical and practical too to assume that the process is such that most of the time $\overline{p}$ assumes the value $p_{(1)}$ and occasionally deteriorates to $p_{(2)}$ ($> p_{(1)}$), which when detected is brought back to the usual level $p_{(1)}$. In section 4.1 we continue with AOQL plans and work out the optimum CSP plans under the assumption that $\overline{p}$ is $p_{(1)}$ with probability w_1 and $p_{(2)}$ with probability w_2 so that $0 < p_L < p_{(1)} < p_{(2)} < 1$ and $w_1 + w_2 = 1$, $w_1, w_2 \geq 0$.
We choose the plan parameters in such a way that the AOQL is p_L (given) and the expected AFI, viz., $E_{\overline{p}}$ (F^{tL}) is minimum among all the CSP - t plans satisfying the AOQL stipulation, t = 1, 2, 3. Here, $E_{\overline{p}}$ (f ($\overline{p}$)) denotes the expectation of the function $f(\overline{p})$ w.r.t. the probability distribution of $\overline{p}$.
The assumption that $\overline{p}$ follows a three point Binomial distribution may also seem appropriate in some cases. For example, $\overline{p}$ may be considered to be controlled at $p_{(2)}$ most of times. Yet incoming quality may deteriorate to $p_{(3)}$ for some assignable

cause and continue at that level till it is brought back to $p_{(2)}$ following diagnosis and trouble shooting exercises. Again, $\overline{p}$ may also improve to $p_{(1)}$ from $p_{(2)}$. However, this desirable situation may not prevail for long and $\overline{p}$ again attains the usual level $p_{(2)}$. In section 4.2 we work out the optimum CSP - 1, k = i and k $\neq$ i cases of CSP - 2 and the general case of CSP - 3 plans under the assumption that the $\overline{p}$ is $p_{(i)}$ with probability w_i, $1 \leq i \leq 3$, so that $0 < p_L < p_{(1)} < p_{(2)} < p_{(3)} < 1$ and $w_1 + w_2 + w_3 = 1$. We seek to minimise, as before, $E_{\overline{p}}$ (F^{tL}) subject to the AOQL requirement, t = 1, 2, 3.

In section 4.3 we consider the most general case where $\overline{p}$ follows any discrete (finite) probability distribution.

In section 4.4 we make some observations relevant to CSP plans.

In this chapter we retain, by and large, the symbols introduced in chapter 2.

Some of the results presented in this chapter have appeared in Ghosh (1988, 89, 90 and 91).

4.1 Optimum CSP plans When $\overline{p}$ Follows A Two point Binomial Distribution.

4.1.1. Determination of (i, f) For Optimum CSP - 1 plan

As already stated, we assume that $\overline{p}$ takes values $p_{(1)}$ and $p_{(2)}$ with probability w_1 and w_2 respectively so that $0 < p_L < p_{(1)} < p_{(2)} < 1$ and $w_1 + w_2 = 1$. Our aim is to choose (i, f) for CSP - 1 such that AOQL = max AOQ' (i, f, p) = p_L and $E_{\overline{p}}$ (F^{1L} (i, $\overline{p}$)) is minimum among all CSP - 1 plans having AOQL = p_L. Under the given situation the expected AFI is

$$F^{1L}(i) \;=\; w_1 F^{1L}(i, p_{(1)}) \;+\; w_2 F^{1L}(i, p_{(2)}) \qquad (4.1.1)$$

$$\text{where} F^{1L}(i,p_{(j)}) = \frac{1}{1+\frac{(i+1)^{i+1}}{i^i}.\frac{p_L}{1-p_L}.(\frac{1-p_{(j)}}{1-p_L})^i} \tag{4.1.2}$$

The optimum i is obtained by solving the equation

$$\frac{dF^{1L}(i)}{di} = 0.$$

$$\text{i,e } w_1A_1\left[log\frac{i}{i+1} - log\frac{1-p_{(1)}}{1-p_L}\right] + w_2A_2\left[log\frac{i}{i+1} - log\frac{1-p_{(2)}}{1-p_L}\right] = 0 \tag{4.1.3}$$

$$\text{where } \mathrm{A_j} = \frac{\frac{(i+1)^{i+1}}{i^i}.\frac{p_L}{1-p_L}.(\frac{1-p_{(j)}}{1-p_L})^i}{\{1+\frac{(i+1)^{i+1}}{i^i}.\frac{p_L}{1-p_L}.(\frac{1-p_{(j)}}{1-p_L})^i\}^2}, \mathrm{j} = 1, 2.$$

[See Lemma 2.1.4]

The optimum plan is based on the following Lemmas and Theorem.

Lemma 4.1.1 *$F^{1L}(i,p_{(2)})$ is greater than $F^{1L}(i,p_{(1)})$ fpr all $0 < p_{(1)} < p_{(2)} < 1$ and for all integer i.*

Proof The result follows immediately once we compare $F^{1L}(i,p_{(1)})$ with $F^{1L}(i,p_{(2)})$ using (4.1.2).

Lemma 4.1.2 *Let i_j be the value of i which minimizes $F^{1L}(i,p_{(j)})$, $j = 1,2$. Then $i_1 > i_2$ if $p_{(1)} < p_{(2)}$.*

Proof We have $\frac{i_1}{i_1+1} = \frac{1-p_{(1)}}{1-p_L}; \frac{i_2}{i_2+1} = \frac{1-p_{(2)}}{1-p_L}$ and $p_{(1)} < p_{(2)}$.

Hence the result.

Lemma 4.1.3 *The second derivative of $F^{1L}(i,p_{(j)})$ with respect to i is positive for all $i \leq i_j$, $j = 1, 2$.*

Proof : Since $p_L < p_{(j)}$ we have $b = \frac{1-p_{(j)}}{1-p_L} < 1$ of Lemma 2.1.6 and hence the result follows from the lemma.

Theorem 4.1.1 *There exists a unique $i_0 (i_2 \leq i_0 \leq i_1)$ for which the amount of inspection $F^{1L}(i)$ is minimum provided $0 < p_L < p_{(1)} < p_{(2)} < 1$.*

Proof : We note the following from Lemma 2.1.4 and Lemma 4.1.3 treating i as a continuous variable and using the symbols :

$$a(i) = \frac{dF^{1L}(i, p_{(1)})}{di} \text{ and } b(i) = \frac{dF^{1L}(i, p_{(2)})}{di}$$

(a) for $i < i_2$, $a(i) < 0, b(i) < 0$

(b) for $i > i_1$, $a(i) > 0, b(i) > 0$

(c) $a(i_1) = 0, \quad b(i_2) = 0$

(d) for $i_2 < i < i_1$, $a(i) < 0 \quad a'(i) > 0$ and $b(i) > 0$

(e) $w_1 > 0, \; w_2 > 0$.

Thus, $\frac{dF^{1L}(i)}{di} = a_{(i)} + b_{(i)}$ changes sign from -ve to +ve from i_2 to i_1. Also the solution of a(i) + b(i) = 0 is unique. Hence the result follows.

The optimum integer i_0 can be obtained by evaluating $F^{1L}(i)$ for integral values of i in the range (i_2, i_1) till i = i_0 such that $F^{1L}(i_0 + 1) \geq F^{1L}(i_0)$ and fixing i at i_0. f_0 the corresponding value of f in the CSP - 1, can then be obtained from equations (2.1.6) and (2.1.7).

The AFI (%) for a given p_L for different values of i when $\bar{p}$ follows a two point

Binomial distribution is given in the accompanying Fig. 4.4.2 to illustrate the choice of i_0.

4.1.2 Determination of Optimum (i, f, i) for CSP - 2 (k = i) Plan.

We consider here the problem of choosing (i, f, i) for CSP - 2 (k = i) such that AOQL $= p_L$ and $E_{\overline{p}}(F^{2L}(i,i,\overline{p})$ is minimum among all CSP - 2 (k = i) plans with AOQL $= p_L$. for $0 < p_L < p_{(1)} < p_{(2)} < 1, w_1 > 0, w_2 > 0$ and $w_1 + w_2 = 1$.

Let i_1 and i_2 be respectively, the values of i which minimise $F^{2L}(i,i,p_{(1)})$ and $F^{2L}(i,i,p_{(2)})$. Then in view of Lemma 2.2.7 we have $i_1 > i_2$. The optimum (i, f, i) can be obtained by using the following theorem.

Theorem 4.1.2 *For a given p_L, there exists an i_0 ($i_2 < i_0 < i_1$) for which the expected AFI, $E_{\overline{p}}(F^{2L}(i,i,\overline{p}))$ under CSP - 2 (k = i) is minimum for the two point Binomial distribution of $\overline{p}$ [$p_{(1)} : w_1, p_{(2)} : w_2$] provided 0 $< p_L < p_{(1)} <$ $p_{(2)} < 1$.*

Proof Let us write $F^{2L}(i,i)$ for $E_{\overline{p}}(F^{2L}(i,i,\overline{p}))$. Then

$$F^{2L}(i,i) = w_1F^{2L}(i,i,p_{(1)}) + w_2F^{2L}(i,i,p_{(2)}) \tag{4.1.1}$$

Treating i as a continuous variable, i_0 is the solution of $\frac{dF^{2L}(i,i)}{di} = 0$, i.e. the solution of $w_1c(i) + w_2d(i) = 0$ where c(i) $= \frac{dF^{2L}(i,i,p_{(1)})}{di}$ and $d(i) = \frac{dF^{2L}(i,i,p_{(2)})}{di}$.

We note from various results in sec. 2.2.4 - 2.2.6 that

(a) $i_2 < i_1$

(b) For $i < i_2$, $c(i) < 0$, $d(i) < 0$

(c) For $i > i_1$, $c(i) > 0$, $d(i) > 0$

(d) $c(i_1) = 0, \quad d(i_2) = 0$

(e) For $i_2 < i < i_1, \quad c(i) < 0, \quad d(i) > 0$

(f) $w_1 > 0, \quad w_2 > 0, \quad w_1 + w_2 = 1.$

Thus $F^{2L}(i,i)$ changes sign from -ve to +ve as i changes from i_2 to i_1. So, there exists an i_0, $i_2 < i_0 < i_1$ for which $\frac{dF^{2L}(i,i)}{di}\big|_{i=i_0} = 0$. Hence the theorem.

N.B. 4.1.1 :

In the case of CSP - 1, the uniqueness of this i_0 could be established as $\frac{d^2F^{1L}(i,p_{(1)})}{di^2} > 0$ for $i_2 < i < i_1$. But in case of CSP 2, it has not been possible to prove the same. It is clear that the optimum integer i_0 lies between i_2 and i_1 and can be obtained by evaluating $F^{2L}(i,i)$ as in (4.1.4) for all integers in the range (i_2, i_1) and choosing that i as i_0 for which $F^{2L}(i_0, i_0)$ is minimum of $F^{2L}(i,i)$ in the range $i_2 < i < i_1$. The value of f_0, i.e. the corresponding f of the CSP - 2 plan can be obtained from equations (2.2.9) and (2.2.10).

4.1.3 Determination of Optimum (i, f, k) for General CSP - 2 Plan, Given k.

We consider here the problem of choosing (i, f) given k for CSP - 2 (general case) such that AOQL $= p_L$ and $E_{\overline{p}}(F^{2L}(i,k,\overline{p})$ is minimum among all CSP - 2 plans given k with AOQL $= p_L$ for $0 < p_L < p_{(1)} < p_{(2)} < 1$; $w_1 > 0, w_2 > 0$ and $w_1 + w_2 = 1$. The procedure can be developed in view of the following obseravation which is based on empirical study of a large number of cases.

Observation 4.1.1 : *For given p_L and k, there exists an i_0 $(i_2 < i_0 < i_1)$ for which the expected AFI, viz. $F^{2L}(i,k) = E_{\overline{p}}(F^{2L}(i,k,\overline{p}))$ under general CSP - 2 is minimum for the two point Binomial distribution of $\overline{p}[p_{(1)} : w_1; p_{(2)} : w_2]$,* *provided* $0 < p_L < p_{(1)} < p_{(2)} < 1$.

The result follows similarly if we accept the properties of $p_1^{2L}(i, k)$ established empirically in Section 2.3.4. Thus we have, assuming i_j to be the value of i for which $F^{2L}(i, k, p_{(j)})$ is minimum, given k, $j = 1, 2$.

(a) $i_1 > i_2$ for $p_{(2)} > p_{(1)}$ [Since (a) optimum is reacheed when $p_1^{2L}(i, k)$ equals $p_{(1)}$ or $p_{(2)}$ depending upon the problem and (b) $p_1^{2L}(i, k)$ is a decreasing function of i for a given k].

(b) Given k, $F^{2L}(i, k, p_{(j)})$ is a *decreasing function of* i *for* $i < i_j$ and *increasing* funciton of i for i $> i_j$ and assumes a unique minimum for i $= i_j$ for $j = 1, 2$.

(c) Thus given k, there must exists an i in the range (i_2, i_1) for which $F^{2L}(i, k) = w_1 F^{2L}(i, k, p_{(1)}) + w_2 F_2^{2L}(i, k, p_{(2)})$ is minimum.

The value of i_0 is obtained by evaluating $F^{2L}(i, k)$ in the range (i_2, i_1) and choosing that i for which $F^{2L}(i, k)$ is minimum. The value of f_0 is obtained by using equations (2.3.8) and (2.3.9).

4.1.4. Determination of Optimum (i, f, k) for General CSP - 3 Plan.

Given p_L and k, we consider here the problem of choosing (i, f) for CSP - 3 such that AOQL $= p_L$ and $E_{\bar{p}}(F^{3L}(i, k, \bar{p}))$ is minimum among all CSP - 3 plans with AOQL $= p_L$, when $\bar{p}$ follows the two point Binomial distribution as before.

The procedure is based on the following observation arrived at empirically.

Observation 4.1.2 *For given* p_L *and* k, *there exists an* i_0 $(i_2 < i_0 < i_1)$ *for which the expected AFI, viz.,* $F^{3L}(i, k) = E_{\bar{p}}(F^{3L}(i, k, \bar{p}))$ *under general CSP - 3 is minimum for the two point Binomial distribution of* $\bar{p}[p_{(1)} : w_1; p_{(2)} : w_2]$, *provided* $0 < p_L < p_{(1)} < p_{(2)} < 1$.

The above observation can be justified by using similar arguments as in Sec. 4.1.3 in case of general CSP - 2, once we accept the properties of $p_1^{3L}(i,k)$ esstablished empirically for CSP - 3 in Sec. 2.4.4.

The value of i_0 can be determined as before by evaluating $F^{6L}(i,k)$ for all integer i in the range (i_2, i_1). The value of f_0 can be obtained from equation (2.4.4) and (2.4.5).

4.2 Optimum CSP plans when $\overline{p}$ Follows A Three-point Binomial Distribution

4.2.1 Determination of Optimum (i, f) For CSP - 1 Plan

This is a logical extension of the two-point Binomial case. Here we consider the problem of choosing (i, f) for CSP - 1 such that AOQL $= p_L$ and $E_{\overline{p}}(F^{1L}(i,\overline{p}))$ is minimum among all CSP - 1 plans with AOQL $= p_L$, for $0 < p_L < p_{(1)} < p_{(2)} < p_{(3)} < 1$, $w_1 > 0$, $w_2 > 0$, $w_3 > 0$ and $w_1 + w_2 + w_3 = 1$. The solution is based on the following theorem.

Theorem 4.2.1 : *Let the underlying distribution of $\overline{p}$ be three-point Binomial such that $p_{(j)}$ occurs with probability w_j and $0 < p_L < p_{(1)} < p_{(2)} < p_{(3)} < 1$ holds. Then, given p_L there exists a unique i_0 under CSP - 1 for which $F^{1L}(i) = w_1 F^{1L}(i,p_{(1)}) + w_2 F^{1L}(i,p_{(2)}) + w_3 F^{1L}(i,p_{(3)})$ is minimum.*

Proof : Let i_j be the optimum value of i for which $F^{1L}(i,p_{(j)})$ is minimized, j = 1, 2, 3.

Then $i_3 < i_2 < i_1$

Let us write G(i) $= w_2 F^{1L}(i,p_{(2)}) + w_3 F^{1L}(i,p_{(3)})$

Then, $F^{1L}(i) = w_1 F^{1L}(i,p_{(1)}) + G(i)$.

In view of Theorem 4.1.1, there exists a unique i_m $(i_3 < i_m < i_2)$ for which G(i) attains its minimum value. For i < i_m, G(i) decreases monotonically and for i > i_m, G(i) increases monotonically.

We note that $i_m < i_1$.

Writing $\frac{dF^{1L}(i,p_{(1)})}{di} = a(i)$ and $\frac{dG(i)}{di} = e(i)$, we further note that for

$$\begin{aligned}
&\text{(a) } i < i_m, \quad a(i) < 0, \quad e(i) < 0 \\
&\text{(b) } i > i_1, \quad a(i) > 0, \quad e(i) > 0 \\
&\text{(c) } \quad a(i_1) = 0, \quad e(i_m) = 0 \\
&\text{(d) } i_m < i < i_1, \quad a(i) < 0, \quad a'(i) > 0 \text{ and } e(i) > 0 \\
&\text{(e) } w_1 > 0
\end{aligned}$$

Thus as in Theorem 4.1.1, there exists a unique i_0 in the range (i_3, i_1) for which $\frac{dF^{1L}(i)}{di} = 0$. Hence the result.

The optimum i_0 can be easily obtained by evaluating $F^{1L}(i)$ for all integral i in the range (i_3, i_1), till $F^{1L}(i_0+1) \geq F^{1L}(i_0)$ and taking i as i_0. The value of f_0 corresponding to i_0 can be obtained from equations (2.1.6) and (2.1.7).

4.2.2 Determination of Parameters of Optimum CSP - 2 (k = i), CSP - 2 (general) and CSP - 3 (general) Plans.

Here given p_L and k when k $\neq i$ (for CSP - 2 and CSP - 3 plans) we consider the problem of determining plan parameters for the above plans such that AOQL = p_L and $E_{\bar{p}}(F^L(i,k,\bar{p}))$ is minimum among all CSP - t plans with AOQL = p_L, t = 2, 3 and $\bar{p}$ follows the three point Binomial distribution considered in Section 4.2.1. This is based on the following theorem.

Theorem 4.2.2 *Let the underlying distribution of $\bar{p}$ be three point Binomial such that $\bar{p}$ takes the value $p_{(j)}$ with probability w_j and $0 < p_L < p_{(1)} < p_{(2)} < p_{(3)} < 1$ holds. Then, given k (when $\neq i$ for CSP - 2 and CSP - 3 plans)* *there exists a value of i for which*

(i) $E_{\overline{p}}(F^{2L}(i,i,\overline{p}))$ is minimum for CSP - 2 (k = i) plans,

(ii) $E_{\overline{p}}(F^{2L}(i,k,\overline{p}))$ is minimum for CSP - 2 (k $\neq$ i) plans,

(iii) $E_{\overline{p}}(F^{3L}(i,k,\overline{p}))$ is minimum for CSP - 3 plans.

The above theorem can be proved analytically for the special case of CSP - 2 (k = i) plan by extending the arguments of Theorem 4.1.2 as we have just done in case of Theorem 4.2.1.

The truth of the above theorem can be similarly established for the general case of CSP - 2 and CSP - 3 once we accept the properties of $p_1^{2L}(i,k)$ and $p_1^{3L}(i,k)$ which have been established empirically.

The exact determination of i_0 and f_0 can be made by following a procedure similar to that in case of two point Binomial case as discussed in Sections 4.1.2, 4.1.3 and 4.1.4.

4.3 Optimum CSP Plans When $\overline{p}$ Follows An m-point Probability Distribution.

4.3.1 Determination of Optimum (i, f) For CSP - 1 Plans

We now extend the procedure developed earlier to find optimum CSP - 1 plan when $\overline{p}$ follows either a discrete distribution or a continuous distribution which can be approximated by discrete probability masses being distributed over a finite number of values.

We assume that the distribution of $\overline{p}$ is such that $\overline{p}$ assumes the values $p_{(j)}$ with probability w_j, $1 \leq j \leq m$ and that $0 < p_L < p_{(1)} < p_{(2)} \cdots < p_{(m)} < 1, w_j > 0$, $\Sigma w_j = 1$. We are interested in choosing (i, f) such that AOQL = p_L, $E_{\overline{p}}(F^{1L}(i,\overline{p}))$

is minimum among all CSP - 1 plans with AOQL = p_L.

The procedure is based on the following theorem.

Theorem 4.3.1 : *Let $\overline{p}$ follow a distribution whch can be represented by discrete probabilities w_j assigned to a finite number of points $p_{(j)}$, j = 1, 2, ... m such that $\sum_{j=1}^{m} w_j = 1$. Let i_j be the values of i for which $F^{1L}(i, p_{(j)})$ is minimum, j = 1, 2, ... m. Then there exists a unique i_0 in the range (i_m, i_1) for which $F^{1L}(i) = \sum_{j=1}^{m} w_j F^{1L}(i, p_{(j)})$ is minimum, assuming $0 < p_L < p_{(1)} < p_{(2)} \ldots < p_{(m)}$.*

Proof : The proof follows easily as in 2 or 3 point Binomial case.

The operative part is, as before, to evaluate $F^{1L}(i)$ for successive values of i in the range (i_m, i_1) till i = i_0 such that $F^{1L}(i_0 + 1) \geq F^{1L}(i_0)$ and fixing i as i_0. The value of f_0 can be obtained by using equations (2.1.6) and (2.1.7).

4.3.2 Determination of Parameters of Optimum CSP - 2 ($k = i$ or $k \neq i$) and CSP - 3 Plans.

Assuming a distribution of $\overline{p}$ as stated in section 4.3.1, given p_L and k, when not equal to i, we attempt to choose the parameters i and f for CSP - t plans in such a manner that AOQL = p_L and $E_{\overline{p}}(F^{tL}(i, k, \overline{p})$ is minimum among all CSP - t plans with AOQL = p_L, t = 2, 3.

The procedure is based on the following theorem.

Theorem 4.3.2 *Let the underlying distribution for $\overline{p}$ be approximated by discrete probabilities w_j for a finite number of points, $p_{(j)}$, j = 1, 2, ... m such that $\sum_{j=1}^{m} w_j = 1$. Let $0 < p_L < p_{(1)} < \ldots\ldots < p_{(m)}$. Given p_L and k (when $\neq i$), let i_j be the value of i for which $F^{tL}(i, k, p_{(j)})$ is minimum j = 1, 2, ... m. Then there exists an i_0 in the range (i_m, i_1) for which $E_{\overline{p}}(F^{tL}(i, k, \overline{p})$*

is minimum, t = 2, 3.

The above theorem can be proved by repeated application of Theorem 4.1.2 for the special case of CSP - 2 where k = i.

For the general case of CSP - 2 and CSP - 3, the theorem is observed to hold from the empirical study of a large number of numerical problems as discussed in case of two point and three point Binomial distribution of $\overline{p}$.

The value of i_0 in case of CSP - t, can be obtained by evaluating $E_{\overline{p}}(F^{tL}(i,k,\overline{p})$, for all integer i in the range (i_m, i_1) and taking that value of i as i_0 for which this quantity is minimum, t = 2, 3. The value of f_0 can be obtained by using relevant formulas for f developed earlier in sections 2.2.3, 2.3.3, and 2.4.3.

4.4 Some Remarks

We conclude our discussion with the following remarks.

Remark 4.4.1 : *The procedures for finding optimum parameters of the CSP - plans when the process average $\overline{p}$ follows a distribution fail, if any one of $p_{(j)}$'s is found to be smaller than p_L. If all $p_{(j)}$'s are smaller than p_L we treat the problem as one with $\overline{p}$ assuming the value $p_{(m)}$ with probability 1. Since this fixed value $p_{(m)} < p_L$, we consider the CSP plan from chapters 2 for which the fixed $\overline{p}$ is just greater than the specified p_L. If only a few of the $p_{(j)}$'s are less than p_L then we can use the following heuristic procedure :*

Compute $\overline{\overline{p}} = \Sigma w_j p_{(j)}$ and obtain the optimum CSP plan having process average at this computed $\overline{\overline{p}}$, following the procedures described in sections 2.1, 2.2, 2.3 or 2.4 as the case may be.

Remark 4.4.2 *It may be noted that the optimum i_0 has been proved to be unique in case of CSP - 1 only when $\overline{p}$ follows a distribution. For other types of CSP plans, uniqueness of i_0 has not been established. As a result,* *computational effort is more for CSP plans other than CSP - 1. However, numerical examples show that i_0 is unique in all cases of CSP plans considered by us.*

In any case, the search for i_0 in the range (i_m, i_1) can be reduced as i_0 has been found to be close to $\bar{i}$ where $\bar{i}$ is the value of i which minimises $F^{1L}(i, \bar{\bar{p}})$ in case of CSP - 1, $F^{tL}(i, k, \bar{\bar{p}})$ in case of CSP - t, t = 2, 3 and $\bar{\bar{p}} = \sum_{j=1}^{m} p_{(j)} w_j$. This is in agreement with the general result that $E_{\overline{p}}(g(\overline{p})) \simeq g(E_{\overline{p}}(\overline{p}))$ for all continuous functions g.

Remark 4.4.3 *All along the emphasis has been in obtaining a plan that minimises the expected AFI for a given probability distribution of $\overline{p}$. If, however, there is a stipulation on p_t (%) in addition to AOQL, it should be examined whether the optimal plan (i_0, f_0) in case of CSP - 1 [(i_0, f_0, k) in case of CSP - t, t = 2, 3] for the given AOQL also satisfies the stipulation on p_t (%). If not, the plan parameters are to be worked out by trial and a procedure has been suggested in Section 2.1.5 for CSP - 1. This can also be extended to other CSP plans also using expressions for p_t (%) as developed in the next chapter. It is obvious that the expected AFI in such a case will be more than that of the optimum plan without restriction on p_t (%).*

Remark 4.4.4 : *The expected AFI in case of CSP - 1, 2 and 3 plans for different values of i for some given p_L values and a few arbitrarily selected distribution of $\overline{p}$ are shown in Figs. 4.4.1 through 4.4.3.*

AFI (%), FOR A GIVEN p_L AND $\bar{p}$.

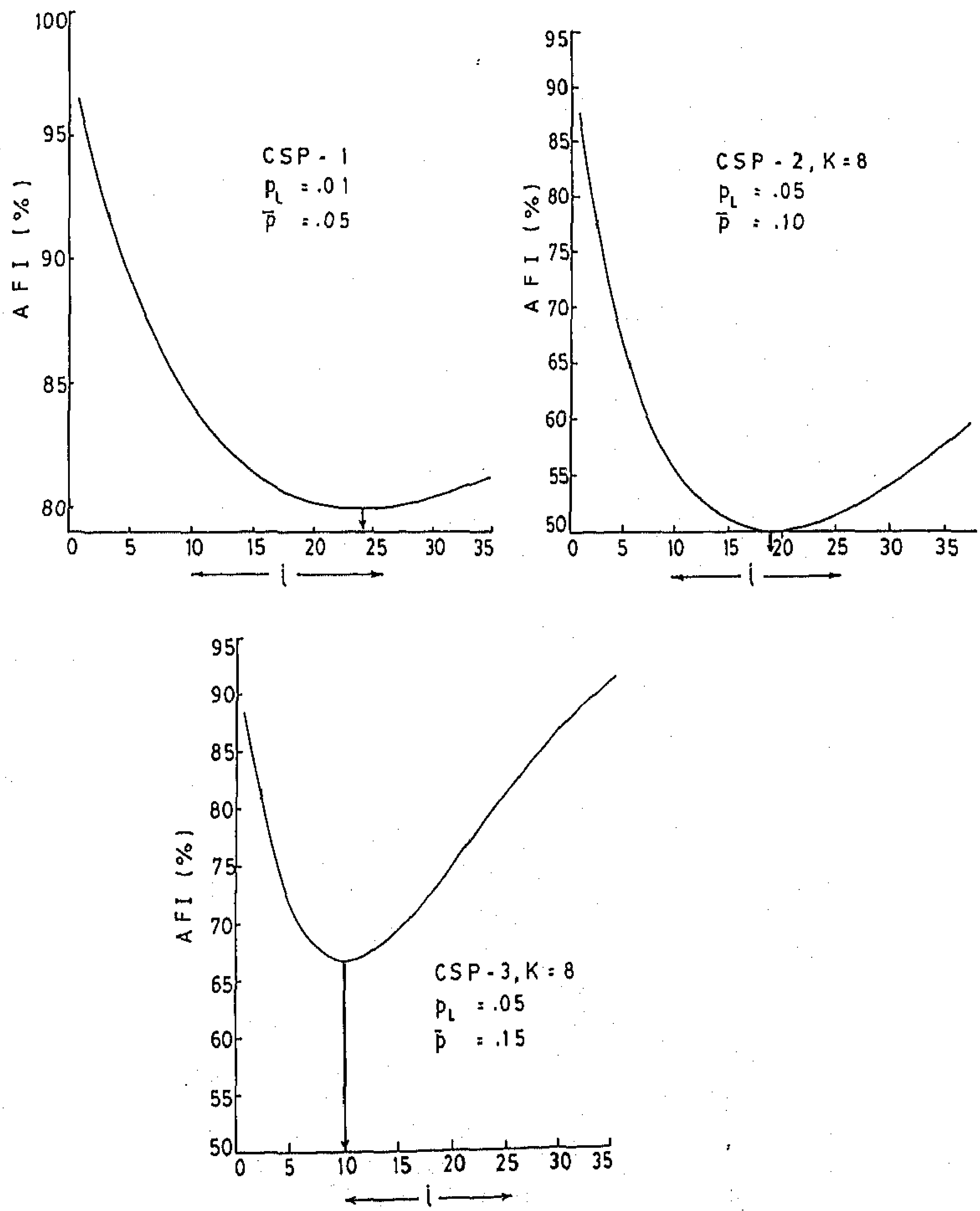


Fig: 4.4.1

EXPECTED AFI (%), FOR A GIVEN p_L AND TWO POINT DISTRIBUTION OF p.

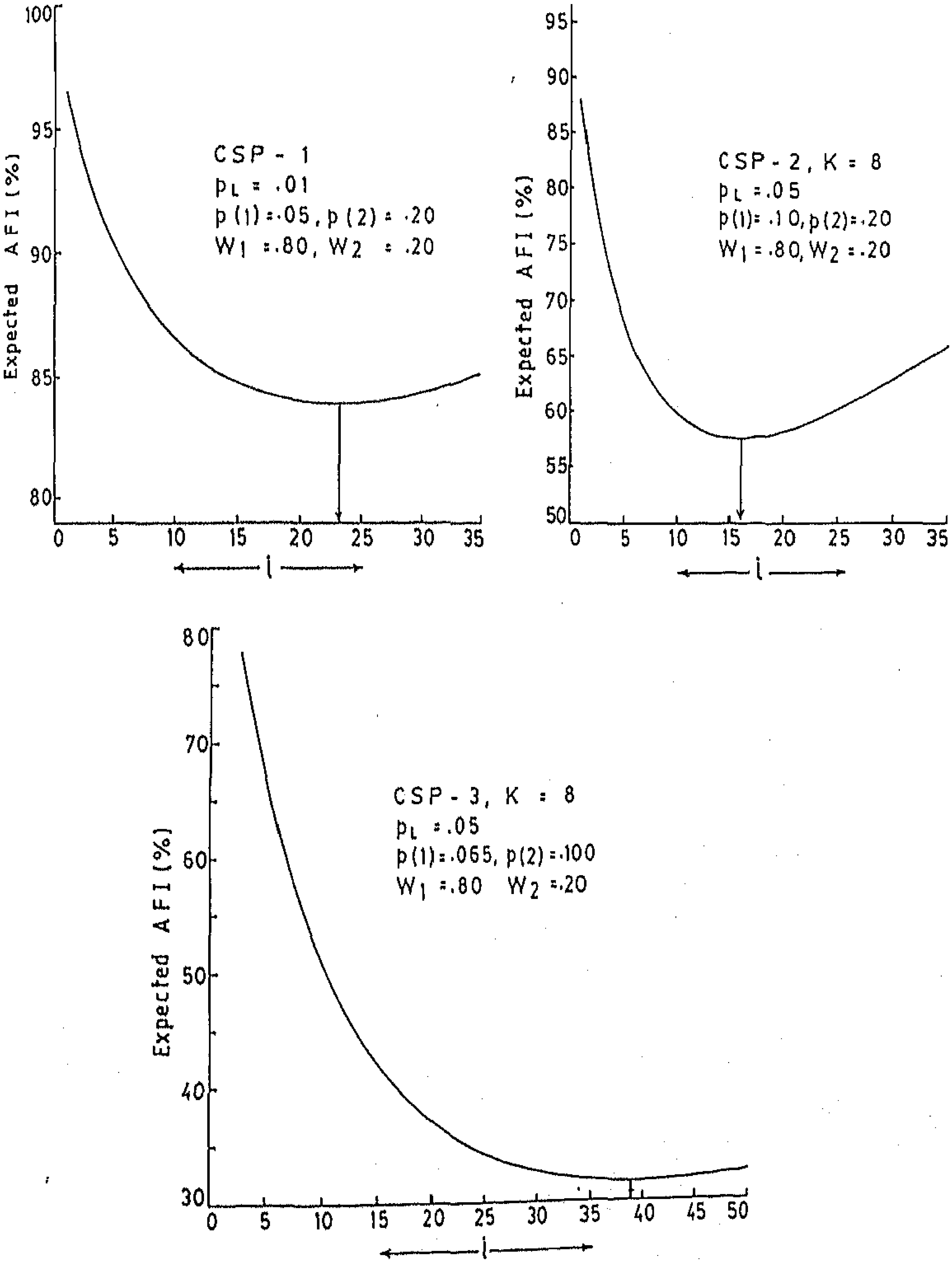


Fig: 4.4.2

EXPECTED A F I (%), FOR A GIVEN p_L AND THREE POINT DISTRIBUTION OF p.

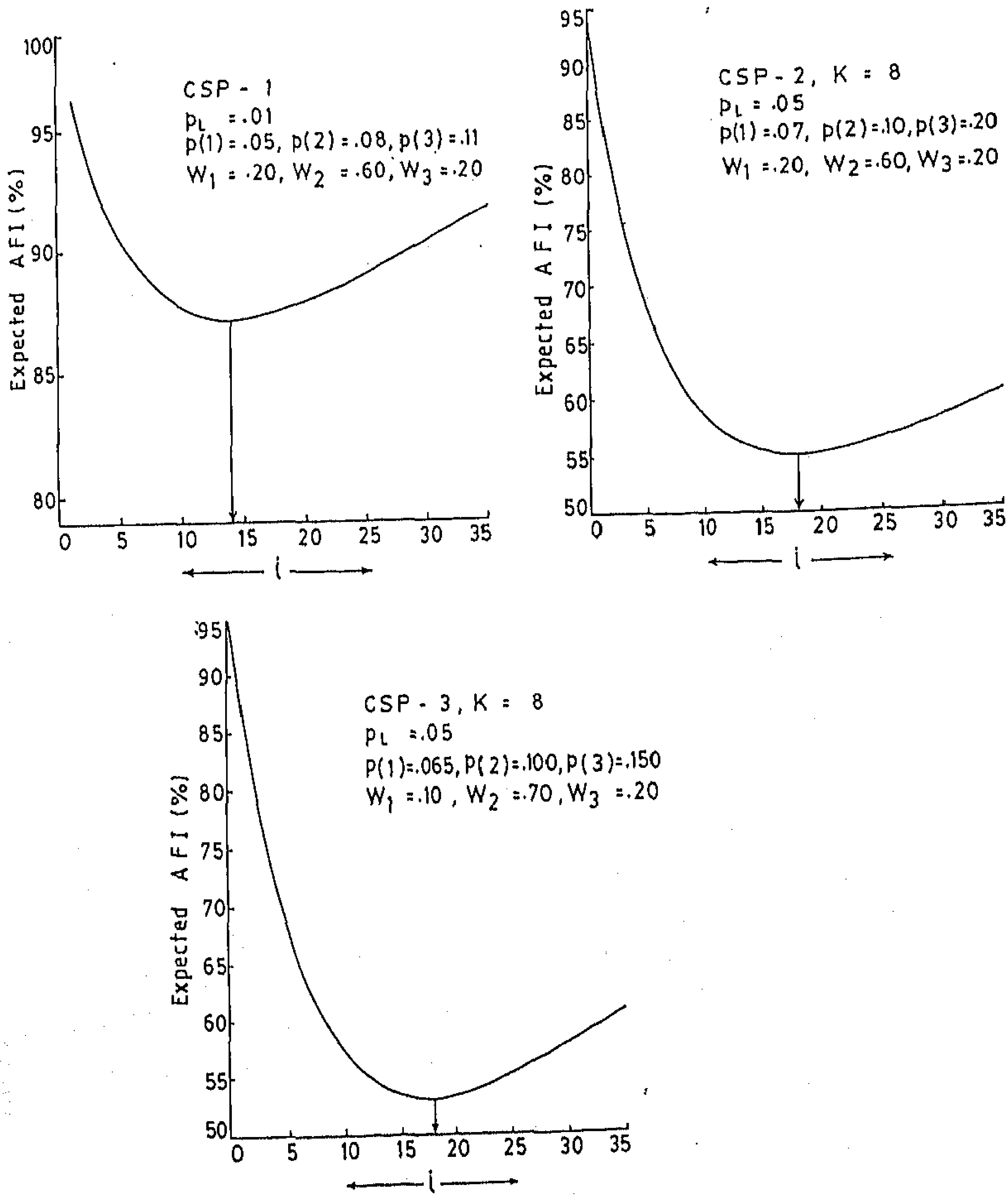


Fig: 4.4.3

CHAPTER 5
A COMPARATIVE EVALUATION OF DODGE TYPE OPTIMUM CSP PLANS.

5.0 Introduction

We have seen that the AFI at process average $\overline{p}$ for all the optimum CSP plans considered are same. This is so because the optimum plan parameters (i_0, f_0) for CSP - 1 or plan parameters (i_0, f_0) given k for CSP - 2 or CSP - 3 are such that the corresponding $p_1^{1L}(i_0)$, $p_1^{2L}(i_0, k)$ or $p_1^{3L}(i_0, k)$ as the case may be, at which AOQL = p_L is attained is approximately $\overline{p}$.

It is, therefore, necessary to compare the performances of different types of optimum CSP plans with regard to some other performance criteria. The criteria selected for the purpose of comparisons are (i) p_t points (ii) AFI at some selected values of incoming quality in relation to $\overline{p}$ say .25, .50, .75, 1.00, 1.25, 1.50, 1.75 and 2.00 times given $\overline{p}$ and (iii) the nature of the resulting AOQ curve and not just the AOQL only and (iv) the nature of the AFI curve for the whole range of incoming quality and not just its value at $\overline{p}$.

However, we have enlarged the scope of comparison to another Dodge type CSP plan, viz., two level continuous sampling plan, a special case of Multilevel continuous sampling plans developed by Lieberman and Solomon (1955). As pointed out earlier, two aspects of CSP - 1 plan, viz., (a) an abrupt change between partial inspection and 100% inspection and (b) extent of protection against spotty quality lead Dodge to modify it. The primary purpose of Lieberman and Solomon (1955) was to extend Dodge's approach in such a way that (a) it allows for smoother transition between sampling inspection and 100% inspection (b) requires 100% inspection only when the quality submitted is quite inferior and (c) allows for a very low AFI when quality is definitely good. This was achieved by Multilevel (k) CSP plan. The CSP-1 plan can be easily recognised as a special case of the above plan with k=1.

The two level CSP plan (k = 2) is of special interest to us as it is comparable to CSP - 2 and CSP - 3 in the sense that this too delays the return to 100% inspection from sampling and from the operational point of view the level of difficulty is more or less same. For two level CSP plans too, there will be a large number of combinations of (i, f) which will ensure a desired AOQL. It is interesting to study if there exists a combination of (i, f) for two level CSP plan which minimises the AFI at $\overline{p}$. Our enquiry affirms this to be true.

In sec 5.1 we introduce two level continuous sampling plan and develop an empirical procedure of finding optimum (i, f) which will ensure a given AOQL, say p_L, and in addition minimise the AFI at a given $\overline{p}$ within the class of all two level CSP's with AOQL $= p_L$.

The comparison of CSP plans with respect to p_t (%) available in literature has so far been incomplete, largely because the expressions for p_t (%) were not known before except in case of CSP - 1. In section 5.2 we develop expressions for p_t (%) analytically for CSP - 1 and extend it to CSP - 2, CSP - 3 and two level CSP plans.

In section 5.3 we undertake the comparisons of these CSP plans with regard to the criteria already stated.

5.1 Optimum Two Level Continuous Sampling Plan.

5.1.1. Introduction

The Multilevel Inspection Plan (M.L.P) as proposed by Lieberman and Solomon is as follows :

(0) At the outset inspect 100 percent of the units consecutively until i units in succession are found clear of defects.

(1) When i units in succession are found clear of defects, discontinue 100 percent inspection and inspect only a fraction f of the units. If the next i inspected units are non-defective proceed to the next level; if a defective occurs, revert immediately to 100 percent inspection.

(2) When at rate f, i inspected units are found clear of defects, discontinue sampling at rate f and proceed to sampling at rate f^2. If the next i inspected units are non defective, proceed to the next level; if a defective occurs, revert immediately to sampling at rate f.

(3) When at rate f^2, i inspected units are found clear of defects, discontinue sampling at rate f^2 and proceed to sampling at rate f^3. If the next i inspected units are non defective proceed to the next level; if a defective occurs, revert immediately to sampling at rate f^2.

(k) When at rate f^{k-1}, i inspected uits are found clear of defects discontinue sampling at rate f^{k-1} and proceed to sampling at rate f^k. If a defective occurs, revert immediately to sampling at rate f^{k-1}, otherwise continue sampling at rate f^k.

We note that for k = 1, the MLP reduces to first Dodge plan, viz. CSP - 1. A flow chart for the two level CSP plan (k = 2) is given in Fig (5.1.1).

5.1.2. Notation and Formulas

The (i, f) are the parameters of a two level CSP. The symbols u and v used in the chapters 2 though 4 mean the same thing here also.

For the average fraction inspected or AFI at the incoming quality level, p in case of a two level CSP, we will use the notation $F^{2*}(i,f,p)$. Similarly for AOQ we write $AOQ^{2*}(i,f,p)$ and for the value of p, called p_1 where AOQL is reached we write $p_1^{2*}(i,f)$. For the class of two level CSP's given AOQL = p_L, f can be treated as a function of i and so for this class of plans, the notation can be simplified as :

$$F^{2L*}(i,p) \text{ for AFI,}$$
$$AOQ^{2L*}(i,p) \text{ for AOQ.}$$

$p_1^{2L*}(i)$ for p_1, where the maximum of the AOQ curve i.e. AOQL is attained.

The domain and range of the functions are obvious from the definitions and follow similar lines as explained in connection with other CSP plans in Chapter 2.

Now, we have from Lieberman and Solomon (1955),

$$AOQ^{2*}(i,f,p) = pq^i(1-f)\left\{\frac{f+q^i}{f^2-q^i(2f^2-f)+q^{2i}(1-f+f^2)}\right\} \qquad (5.1.1)$$

The combination (i, f) which ensure a desirecd AOQL = p_L will satisfy the equation

$$q_1^{3i+1}[f^2 - f + 1] + q_1^{2i+1}[(1-i)f^3 - f^2(3+i) + 2f]$$
$$+q_1^{2i}[if^3 + if^2] + q_1^{i+1}[f^2(2+2i) - 2f^3] + q_1^{i}[-2if^2]$$
$$+q_1[f^3(1+i)] - if^3 = 0, \qquad (5.1.2)$$

where p_1 = 1- q_1 stands for $p_1^{2L*}(i)$ = the abscissa fo the AOQ curve where the ordinate is equal to p_L.

For given i, f and p, the following relation is known to hold between AFI and AOQ, viz.,

$$AOQ^{2*}(i, f, p) = p(1 - F^{2*}(i, f, p))$$

and given $AOQL = p_L$,

$$AOQ^{2L*}(i, p) = p(1 - F^{2L*}(i, p)) \qquad (5.1.3)$$

5.1.3 Determination of the Optimum Plan Parameter (i, f) Which Minimises Average Fraction Inspected For a Given Process Average $\overline{p}$ and Ensures a Desired AOQL.

The optimum combination of (i, f) is to be obtained empirically following the procedure outlined below.

(1) Given p_L one can use figure 2 in Lieberman and Solomon (1955) to find (i, f) for a two level CSP. Lieberman and Solomon consider the range of values of AOQL from .01 percent to 10 percent and the range of values of i from 10 to 40,000. In all cases, given AOQL = p_L, i decreases monotonically as f increases. Using this figure the maximum value of i which corresponds to f = .01 can be easily determined. This value of i is denoted by i_{max} and for

investigation of the optimum plan, the value of i $\leq i_{max}$ are only considered. This means given p_L, the choice of f is restricted to be $\geq .01$ as implicitly suggested by Lieberman and Solomon (1955) from practical considerations, e.g. for $p_L = 0.02$, i can take values upto $i_{max} = 170$.

Thus given an AOQL, i_{max} will be obtained by use of the graph as mentioned above and this will provide a required input to the further calculations that follow.

(2) An algorithm is presented below for finding an optimal (i, f) which minimises the AFI at process average $\overline{p}$ among all two level CSP plans satisfying AOQL $= p_L$.

(i) The exact value of f for a selected $i \leq i_{max}$ can be obtained by iteratively solving equations (5.1.2) and (5.1.1) simultaneously. For a given i and the approximate value of f, equation (5.1.2) is solved for q_1. These values of (i, f, q_1) are substituted in (5.1.1) to find $AOQ^{2*}(i, f, p_1)$. If the resulting $AOQ^{2*}(i, f, p_1)$ is found to be equal to p_L, the values of i, f and p_1 = 1 - q_1 are recorded. Otherwise, the value of f is suitably adjusted till $AOQ^{2*}(i, f, p_1)$ becomes equal to p_L.

(ii) For a given $\overline{p}$ and p_L ($\overline{p} > p_L$), the value of $F^{2*}(i, f, \overline{p})$ is computed for all the plans in (i) with i $\leq i_{max}$ to find i_0 for which $F^{2*}(i, f, \overline{p})$ is minimum. The corresponding value of f_0 and p_1 are recorded.

Algorithm : MN2LCS ($\overline{p}, p_L, I_{max}$, i, f)
(* input parameters : $\overline{p}, p_L, I_{max}$;
output parameters : i, f)

(* The algorithm finds (i, f) for two level CSP plan which ensures AOQL = p_L and minimise AFI at $\overline{p}$ given $\overline{p}, p_L, I_{max}$ *)

(* ii, ff, AAF are array variable each of dimension 300 *)

Begin

$k \leftarrow 0; i \leftarrow 0; f_1 \leftarrow 1; p_1 \leftarrow 1; q_1 \leftarrow 1 - p_1, \overline{q} \leftarrow 1 - \overline{p};$

repeat

$i \leftarrow i + 1;\ k \leftarrow k + 1$

repeat

repeat

$q_{11} \leftarrow (i * f_1^3 - q_1^i * (-2 * i * f_1^2)$

$- q_1^{i+1} * (f_1^2 * (2 + 2 * i) - 2 * f_1^3)$

$- q_1^{2*i}(i * f_1^3 + i * f_1^2)$

$- q_1^{2*i+1} * ((1 - i) * f_1^3 - f_1^2 * (3 + i) + 2 * f)$

$- q_1^{3*i+1} * (f_1^2 - f_1 + 1))/(f_1^3 * (i + 1))$

$q_1 \leftarrow q_{11};$

until $(\mid q_1 - q_{11} \mid < \epsilon;$

(* solves (5.1.2) for q_1 given i, f_1, ϵ is small preassigned positive no, say, .005 *)

$p_1 \leftarrow 1 - q_1;$

$AQ \leftarrow p_1^{*}\ q_1^{i} * (1 - f)\{\frac{f_1 + q_1^i}{f_1^2 - q_1^i * (2 * f_1^2 - f_1) + q_1^{2*i} * (1 - f_1 + f_1^3)}\};$

if (AQ $< p_L$) then $f_1 \leftarrow f_1 - \epsilon_1$ else

begin

$\epsilon_1 \leftarrow \frac{\epsilon_1}{2};$

$f_1 \leftarrow f_1 + \epsilon_1;$

end;

until $(\mid AQ - p_L \mid < \epsilon)$;

(* for a given i and p_L solve (5.1.1) for f*)

AQPB $\leftarrow \overline{p} * \overline{q}^i * (1 - f_1)\{\frac{f_1 + \overline{q}^i}{f_1^2 - \overline{q}^i * (2 * f_1^2 - f_1) + \overline{q}^{2i} * (1 - f_1 + f_2^2)}\}$;

AF $\leftarrow 1 - (AQPB/\overline{p})$;

ii(k) $\leftarrow i; ff(k) \leftarrow f_1; AAF(k) \leftarrow$ AF;

until (i $= I_{max}$);

val $\leftarrow$ 1

for k = 1 to I_{max} do -

if AAF (k) < val then

begin val $\leftarrow$ AAF (k); store $\leftarrow$ k; end;

i $\leftarrow$ ii (sotre); f $\leftarrow$ ff (store);

end;

The optimum combination of (i_0, f_0) as obtained by using the above procedure is shown in Tables 5.1.1 and 5.1.2 for two values of AOQL.

N.B. 5.1.1 : *It was observed that for a given $\overline{p}$ and p_L, $F^{2L*}(i, \overline{p})$ decreases with i initially, attains a minimum at i_0 and then increases with increase in i. However, the value of $p_1^{2L*}(i)$ at which AOQL is reached is not always equal to $\overline{p}$. For $\overline{p}$ very large compared to p_L, the difference between $p_1^{2L*}(i_0)$ and $\overline{p}$ is noticeable. Thus for the two level CSP plan, the minimum AFI at $\overline{p}$ will be greater than or equal to the minimum AFI at $\overline{p}$ for other plans, viz., CSP - 1, CSP - 2 and CSP - 3 plans for all of which $\overline{p}$ happens to coincide with p_1.*

N.B. 5.1.2 : *In view of the nature of $F^{2L*}(i, \overline{p})$ for different i's, it will be possible to find optimum two level CSP plans when $\overline{p}$ follows a probability distribution by following procedures similar to those in Chapter 4.*

Table 5.1.1

Optimum Combination of (i, f) for Two Level CSP Plan Ensuring an AOQL of 5.0%

Process average $\bar{p}$	i	f	p_1	AFI at $\bar{p}$	AOQ at $\bar{p}$
0.06	56	0.0176	0.0600	0.1667	0.05
0.07	28	0.1132	0.0702	0.2857	0.05
0.08	19	0.2209	0.0798	0.3750	0.05
0.09	14	0.3258	0.0905	0.4444	0.05
0.10	11	0.4132	0.1018	0.5001	0.05

Table 5.1.2

Optimum Combination of (i, f) for Two Level CSP Plan Ensuring an AOQL of 2.0%

Process average $\bar{p}$	i	f	p_1	AFI at $\bar{p}$	AOQ at $\bar{p}$
0.03	59	0.1626	0.0301	0.3333	0.0200
0.04	30	0.3932	0.0403	0.5000	0.0200
0.05	21	0.5214	0.0495	0.6003	0.0200
0.06	16	0.6095	0.0591	0.6667	0.0200
0.07	13	0.6690	0.0684	0.7144	0.0200
0.08	11	0.7115	0.0774	0.7502	0.0200
0.09	10	0.7336	0.0831	0.7789	0.0200
0.10	10	0.7336	0.0831	0.8054	0.0195
0.11	10	0.7336	0.0831	0.8296	0.0187
0.12	10	0.7336	0.0831	0.8514	0.0178

5.2 Expressions For $p_t(\%)$ of CSP - 1, CSP - 2 and CSP - 3 Plans.

5.2.1 Introduction

$p_t(\%)$ for continuous sampling plans is defined as the value of incoming quality (in percent defective) in a consecutive run of N = 1000 product units for which the probability of acceptance, P_a, is 0.10 during the sampling phase. Let us note that during the sampling phase of inspection, an item is either inspected or not inspected. When the item is inspected, it may be found to be defective or non - defective. If it is non defective on inspection it is accepted and if it is defective it is rejected. An item which is not inspected is naturally always accepted, irrespective of whether it is defective or not. Thus acceptance of an item implies that in the sampling phase the item is either not inspected or found to be good on inspection. Thus 0.10 probability associated with $p_t(\%)$ is actually the probability that each of the N = 1000 consecutive items in the sampling phase is either non - inspected or found good on inspection.

Thus, we can define $p_t(\%)$ as follows :
[In the notation below P (A / B) denotes the conditional probability of the event A, given the event B.]
Let 1 - p_s = P (an item is inspected and found defective / the item is in sampling phase of inspection.)
Then p_s = P (an item is accepted / the item is in sampling phase of inspection)
and p_s^{1000} = P (consecutive 1000 items are accepted / all the items are in sampling phase of inspection.)
Setting $p_s^{1000} = 0.10$, the corresponding value of p, the incoming quality is calculated. This value of p so determined is called the p_t of the plan :

Stephens (1979) obtained an expression for $p_t(\%)$ for CSP - 1 using following argument.

During the sampling phase,

(i) the unit is sampled with probability f and is found on inspection to be good with conditional probability $q = 1 - p$,

(ii) the unit is not sampled (and hence not inspected) with probability 1 - f.

Thus For CSP - 1, $p_s = fq + (1 - f) = 1 - f(1 - q) = 1 - fp$

$$p_s^{1000} = (1 - fp)^{1000} \qquad (5.2.1)$$

Thus solving the equation $(1 - fp_t)^{1000} = 0.10$

$$\text{we have } p_t = (1 - (0.10)^{.001})/f = (2.2999 \times 10^{-3})/f \qquad (5.2.2)$$

However, it is difficult to extend the above logic to CSP - 2, CSP - 3 or 2 levels CSP where operational procedures are quite complicated. In fact no explicit expressions for CSP - 2, CSP - 3 and 2 level CSP are given in literature. Defining the states of a system under a CSP plan appropriately, a Markov Chain model can be developed. Roberts (1965) has defined the states and presented flow diagrams of transitions from and to the states for CSP - 1, CSP - 2 and CSP - 3 and this is shown in Fig. 5.2.1. In the figure to avoid the clumsiness in appearance some dummy stages denoted by $D_1, D_2, DS_1 .... DS_K$ are intorduced. All transition probabilities indicated by arcs leading to dummy stages are exactly 1.0. However in writing the steady state probabilities these dummy stages can be completely ignored.

The steady state probabilities for the different states of the Markov Chain were obtained by Roberts (1965) for CSP - 1. The same for CSP - 2 and CSP - 3 were solved by Stephens (1979). The Markov Chains are found to be irreducible in all these cases.

The solutions lead to the expressions for AOQ and other performance characteristics for the CSP plans in the steady state, which have been used earlier. Using these solutions, we obtain expressions for $p_t(\%)$ for CSP - 1, CSP - 2 and CSP - 3 plans by using the precise definition given above. In the developments which follow for CSP - 1, CSP - 2 and CSP - 3, we follow the notation and state designations as provided by Roberts (1965) and Stephens (1979).

5.2.2 *Expression for $P_t(\%)$ for CSP - 1*

The equilibrium probability for a state, say T is denoted by P_T. They are as follows.

$$\left.\begin{aligned} P_{A_0} &= \tfrac{fp(1-q^i)}{D} \\ P_{A_j} &= \tfrac{fpq^j}{D};\ j=1,2,\ldots..i \\ P_{Id} &= \tfrac{fpq^i}{D}; \\ P_{In} &= \tfrac{fq^{i+1}}{D} \\ P_N &= \tfrac{(1-f)q^i}{D}; \end{aligned}\right\} \tag{5.2.3}$$

$$\text{where } D = f + (1-f)q^i$$

For further details the original paper by Stephens (1979) may be referred to.
Obviously, in this case of CSP - 1

$$p_s = \frac{P_N + P_{In}}{P_{Id} + P_{In} + P_N}$$

$$= \frac{\frac{(1-f)q^i}{D} + \frac{fq^{i+1}}{D}}{\frac{fpq^i}{D} + \frac{fq^{i+1}}{D} + \frac{(1-f)q^i}{D}} \quad .$$

This can be simplified to

$$p_s \;=\; 1 - fp \tag{5.2.4}$$

This expression for p_s is same as (5.2.1) obtained by Stephens (1979) and $p_t(\%)$ can be taken as the solution of

$$(1 - fp_t)^{1000} \;=\; 0.10 \tag{5.2.5}$$

i.e $p_t = (2.2999 \times 10^{-3}) \;/\; f$ - which is same as the expression obtained by Stephens (5.2.2)

5.2.3 Expression for $p_t(\%)$ for CSP - 2

The equilibrium probabilities for the different states of a CSP - 2 plan have been worked out by Stephens (1979) as follows.

$$\left.\begin{aligned}
P_{A_0} &= \tfrac{fp(1-q^k)(1-q^i)}{D}, \\
P_{A_j} &= \tfrac{(1-q^k)fpq^j}{D};\; j = 1, 2, 3, ...i \\
P_{Id} &= \tfrac{fpq^i}{D}, \\
P_{Iu} &= \tfrac{fq^{i+1}}{D}, \\
P_N &= \tfrac{(1-f)q^i}{D}, \\
P_{Id_n} &= \tfrac{fp^2q^{i+n-1}}{D},\; n = 1, 2, 3....k \\
P_{Iu_n} &= \tfrac{fpq^{i+n}}{D},\; n = 1, 2, 3....k \\
\text{and } P_{N_n} &= \tfrac{(1-f)pq^{i+n-1}}{D},\; n = 1, 2, 3.....k \\
\text{where } D &= f(1-q^k)(1-q^i) + q^i(2-q^k)
\end{aligned}\right\} \tag{5.2.6}$$

For further details we refer to Fig. 5.2.1 and also to Stephens (1979).

In this case of CSP - 2 again, obviously

$$p_* = \frac{P_{I_u} + P_N + P_{I_{u_n}} + P_{N_n}}{P_{I_d} + P_{I_u} + P_N + P_{I_{d_n}} + P_{I_{u_n}} + P_{N_n}}$$

$$= \frac{\frac{q^i(2-q^k)}{D} - \sum_{n=1}^{k} \frac{fp^2q^{i+n-1}}{D} - \frac{fpq^i}{D}}{\frac{q^i(2-q^k)}{D}}$$

which can be simplified to

$$p_* = (1 - fp) \tag{5.2.7}$$

Hence p_t is given by the solution of

$$(1 - fp_t)^{1000} = 0.10 \tag{5.2.8}$$

5.2.4 Expression for p_t (%) for CSP - 3

Here again, the equilibrium probabilities for various states of CSP - 3 as shown in Fig. 5.2.1 are taken from Stephens (1979) and are presented below.

$$\text{Thus } P_{A_0} = \frac{fp(1-q^{k+1})(1-q^i)}{D}$$

$$P_{A_j} = \frac{(1-q^{k+1})fpq^i}{D},\ j = 1, 2, \ldots i$$

$$P_{I_d} = \frac{fpq^i}{D},$$

$$P_{I_n} = \frac{fq^{i+1}}{D},$$

$$P_N = \frac{(1-f)q^i}{D} \tag{5.2.9}$$

$$P_{d_m} = \frac{fpq^i(1-q^m)}{D}, \; m = 1,2,3,4$$

$$P_{i_m} = \frac{fpq^{i+m}}{D}, \; m = 1,2,3,4$$

$$P_{Id_n} = \frac{fp^2q^{i+n+3}}{D}, \; n = 1,2.....k$$

$$P_{Ii_n} = \frac{fpq^{i+n+4}}{D}, \; n = 1,2.....k$$

$$P_{N_n} = \frac{(1-f)q^{i+n+3}}{D}, \; n = 1,2.....k$$

$$\text{where } D = f(1-q^{k+4})(1-q^i) + q^i(1+q^4-q^{k+4}) + 4fpq^i$$

In this case

$$p_s = \frac{(1-P_{100}) - \sum_m P_{d_m} - \sum_n P_{Id_n} - P_{Id}}{1-P_{100}},$$

where P_{100} gives the probability that the item is in 100% inspection stage. We have the denominator as $\frac{q^i(1+q^k-q^{k+4})+4fpq^i}{D}$ and the numerator as

$$\frac{q^i(1+q^k-q^{k+4})+4fpq^i}{D} - \sum_{m=1}^{4}\frac{fpq^i(1-q^m)}{D} - \frac{\sum_{n=1}^{k} fp^2q^{i+n+3}}{D} - \frac{fpq^i}{D}$$

$$= \frac{q^i(1+q^k-q^{k+4})+4fpq^i}{D} - \frac{fq^i\{4p-q(1-q^4)\}}{D} - \frac{fpq^{i+4}(1-q^k)}{D} - \frac{fpq^i}{D}$$

$$= \frac{q^i(1+q^k-q^{k+4})+4fpq^i}{D} - \frac{fq^i}{D}[4p-q(1-q^4)+pq^4(1-q^k)+p]$$

$$p_s = \frac{q^i(1+q^k-q^{k+4}+4fp) - fq^i[5p-q(1-q^4)+pq^4(1-q^k)]}{q^i(1+q^k-q^{k+4}+4fp)}$$

$$= 1 - \frac{fp[5-\frac{q(1-q^4)}{p}+q^4(1-q^k)]}{1+q^k-q^{k+4}+4fp} \qquad (5.2.10)$$

Hence p_t is the value of p for which

$$\left(1 - \frac{fp\{5-\frac{q(1-q^4)}{p}+q^4(1-q^k)\}}{1+q^k-q^{k+4}+4fp}\right)^{1000} = 0.10 \qquad (5.2.11)$$

5.2.5 *Expression For $p_t(\%)$ For Two Level Continuous Sampling Plan.*

For this, we first define the states and present flow diagrams of transitions from and to the states for Two Level Continuous Sampling Plans in Fig. 5.2.2. The states defined have more or less the same rationale as in other CSP plans for which similar procedures were developed by Roberts (1965) and Stephens (1979). The states have their obvious meanings and are quite clear from the symbols used for them. However, in the figure to avoid clumsiness in appearance some dummy states are introduced, denoted by D_1, D_2 ..... D_i and DS_1. The transition probabilities denoted by arcs leading to dummy states are all exactly 1.0. This introduction of so called dummy states does not affect the transition probability table given in table 5.2.1. In computing the steady state probabilities and all subsequent expressions depending on them the dummy states can be completely ignored. Equations for the steady state probabilities of the above states are obtained from the successive columns of the transition matrix given in Table 5.2.1 yielding $4i + 4$ equations and one additional equation stipulating the sum of probabilities over all states equating to 1. We are interested in obtaining a non-negative solution to the system of equations given next page.

$$
\begin{aligned}
P_{A_0} &= p(P_{A_0} + P_{A_1} + \ldots\ldots + P_{A_{i-1}}) + p(P_{1d_1} + \ldots . P_{1d_i}) \\
P_{A_1} &= qP_{A_0} + q(P_{1d_1} + \ldots\ldots + P_{1d_i}) \\
P_{A_2} &= qP_{A_1} \\
&\cdots \\
&\cdots \\
&\cdots \\
P_{A_i} &= qP_{A_{i-1}}
\end{aligned}
$$

$$
\begin{aligned}
P_{1d_1} &= fpP_{A_i} + fpP_{N_1} + fpP_{2d_1} \\
P_{1n_1} &= fqP_{A_i} + fqP_{N_1} + fqP_{2d_1} \\
P_{N_1} &= (1-f)P_{A_i} + (1-f)P_{N_1} + (1-f)P_{2d_1}
\end{aligned}
\tag{5.2.12}
$$

$$
\begin{aligned}
P_{1d_2} &= fp(P_{1n_1} + P_{N_2}) \\
P_{1n_2} &= fq(P_{1n_1} + P_{N_2}) \\
P_{N_2} &= (1-f)(P_{1n_1} + P_{N_2}) \\
P_{1d_i} &= fp(P_{1n_{i-1}} + P_{N_i}) \\
P_{1n_i} &= fq(P_{1n_{i-1}} + P_{N_i}) \\
P_{N_i} &= (1-f)(P_{1n_{i-1}} + P_{N_2}) \\
P_{2d_1} &= f^2 p(P_{1n_i} + P_{2n_1} + P_{2N_1}) \\
P_{2n_1} &= f^2 q(P_{1n_i} + P_{2n_1} + P_{2N_1}) \\
P_{2N_1} &= (1-f^2)(P_{1n_i} + P_{2n_1} + P_{2N_1}), \text{ with} \\
\sum_{j=0}^{i} P_{A_j} + \sum_{j=1}^{i} P_{1d_j} + \sum_{j=1}^{i} P_{1n_j} + \sum_{j=1}^{i} P_{N_j} + P_{2d_1} + P_{2n_1} + P_{2N_1} &= 1
\end{aligned}
$$

It may be noted that a unique non-negative solution to (5.2.12) will imply the Markov Chain irreducible.

It can be seen that

$$P_{2d_1} + P_{2n_1} + P_{2N_1} = P_{1n_i} + P_{2n_1} + P_{2N_1}$$

$$\text{or } P_{2d_1} = P_{1n_i}$$

writing $P_{2d_1} = P_{1n_i} = D$ and $P_{A_1} = x$ we arrive at the solution in stages I, II, III and IV.

Stage - I,

$$P_{2d_1} = D$$

$$P_{2n_1} = \frac{q}{p}.D$$

$$P_{2N_1} = \frac{1-f}{fp}.D$$

$$P_{A_k} = q^{k-1}x; \text{ for } k = 1, 2, 3, \ldots..i$$

Stage - II

$$\text{we have } P_{1d_k} = fp(P_{1n_{k-1}} + P_{N_k})$$

$$P_{N_k} = (1-f)(P_{1n_{k-1}} + P_{N_k})$$

$$\text{which lead to } P_{N_k} = \frac{1-f}{f} P_{1n_{k-1}}$$

$$\text{and } P_{1d_k} = \frac{fp}{f} P_{1n_{k-1}} = pP_{1n_{k-1}} \text{ for } k = 2, 3, \ldots..i$$

Stage - III

$$\begin{aligned} \text{we have } P_{1n_k} &= fq(P_{1n_{k-1}} + P_{N_k}) \\ &= fq(P_{1n_{k-1}} + \frac{1-f}{f}.P_{1n_{k-1}}) \\ &= qP_{1n_{k-1}} \text{ for } k = 2, 3, \ldots.i \end{aligned}$$

Thus knowing P_{1n_1}, we can find P_{1n_k}, P_{N_k} and P_{1d_k} for all k = 2, 3, ... i.

Stage - IV

We have $fP_{N_1} = (1-f)(q^{i-1}x + D)$ giving $P_{N_1} = \frac{1-f}{f}(q^{i-1}x + D)$
Thus $P_{1n_1} = fq(q^{i-1}x + D) + q(1-f)(q^{i-1}x + D) = q(q^{i-1}x + D)$.
and $p_{1d_1} = fp(q^{i-1}x + D) + p(1-f)(q^{i-1}x + D) = p(q^{i-1}x + D)$.
Thus we can rewrite equations (5.2.12) as follows

$$\left.\begin{array}{rcl}
P_{1d_k} &=& pq^{k-1}(q^{i-1}x + D) \;=\; pq^{k-1}\frac{D}{q^i},\; k = 2, 3, \ldots i \\
P_{1n_k} &=& q^k(q^{i-1}x + D) \;=\; q^k\frac{D}{q^i},\; k = 2, 3, \ldots i \\
P_{N_k} &=& \frac{1-f}{f}q^{k-1}(q^{i-1}x + D) \;=\; \frac{1-f}{f}.q^{k-1}.\frac{D}{q^i}, k = 2, 3, \ldots i \\
\hline
P_{1d_1} &=& p(q^{i-1}x + D) \;=\; p.\frac{D}{q^i} \\
P_{1n_1} &=& q(q^{i-1}x + D) \;=\; q.\frac{D}{q^i} \\
P_{N_1} &=& \frac{1-f}{f}(q^{i-1}x + D) \;=\; \frac{1-f}{f}.\frac{D}{q^i} \\
\hline
P_{2d_1} &=& D \\
P_{2n_1} &=& \frac{q}{p}.D \\
P_{2N_1} &=& \frac{1-f^2}{f^2 p}.D \\
\hline
P_{A_0} &=& \frac{x(1-q^{i-1}) + p(q^{i-1}x + D)(1-q^i)}{q} \;=\; \frac{(1-q^i)^2}{q^{2i}}.D \\
P_{A_k} &=& \frac{1-q^i}{q^{i-k}}.\frac{D}{q^i} \;=\; q^{k-1}x, k = 1, 2, \ldots, i.
\end{array}\right\} \quad (5.2.13)$$

Finally we have the equations in a compact form as :

$$\left.\begin{aligned}
P_{1d_k} &= pq^{k-1}.\tfrac{D}{q^i};\, k = 1, 2, ... i \\
P_{1n_k} &= q^k.\tfrac{D}{q^i};\, k = 1, 2, ... i \\
P_{N_k} &= \tfrac{1-f}{f}.q^{k-1}.\tfrac{D}{q^i};\, k = 1, 2, ... i \\
P_{2d_1} &= D. \\
P_{2n_1} &= \tfrac{q}{p}.D \\
P_{2N_1} &= \tfrac{1-f^2}{f^2 p}.D \\
P_{A_0} &= \tfrac{(1-q^i)^2}{q^{2i}}.D \\
P_{A_k} &= \tfrac{1-q^i}{q^{i-k}}.\tfrac{D}{q^i},\, k = 1, 2, ..., i.
\end{aligned}\right\} \qquad (5.2.14)$$

The value of D can be obtained by solving the equation that sum of all the probabilities in (5.2.14) is 1. However, the exact value of D is not required for our purpose.

Hence, for the 2 level CSP, if P_s denotes the probability that an item chosen at random is in sampling phase of inspection.

$$\begin{aligned}
\text{then } P_s &= \frac{D}{q^i}.p\sum_{k=1}^{i} q^{k-1} + \frac{D}{q^i}\sum_{k=1}^{i} q^k + \frac{1-f}{f}.\frac{D}{q^i}\sum_{k=1}^{i} q^{k-1} \\
&\quad + D + \frac{Dq}{p} + D.\frac{1-f^2}{f^2 p} \\
&= D\{\frac{1-q^i}{q^i} + \frac{q(1-q^i)}{q^i p} + \frac{1-f}{f}.\frac{1}{q^i}.\frac{1-q^i}{p} + 1 + \frac{q}{p} + \frac{1-f^2}{f^2 p}\} \\
&= D[\frac{1-q^i}{q^i}\{1 + \frac{q}{p} + \frac{1-f}{fp}\} + \frac{1}{f^2 p}] \\
&= D\{\frac{1-q^i}{q^i}.\frac{1}{fp} + \frac{1}{f^2 p}\} \\
&= \frac{D}{q^i}\{\frac{1-q^i}{fp} + \frac{q^i}{f^2 p}\} \qquad (5.2.15)
\end{aligned}$$

We have finally,

$$
\begin{aligned}
p_s &= \frac{P_s - \sum_{k=1}^{i} P_{1d_k} - P_{2d_i}}{P_s} \\
&= 1 - \frac{\sum P_{1d_k} + P_{2d_i}}{P_s} \\
&= 1 - \frac{\frac{D.p}{q^i}.\frac{1-q^i}{p} + D}{P_s} \\
&= 1 - \frac{\frac{D}{q^i}(1-q^i) + D}{\frac{D}{q^i}\{\frac{1-q^i}{fp} + \frac{q^i}{f^2p}\}} \\
&= 1 - \frac{\frac{D}{q^i}(1-q^i+q^i)}{\frac{D}{q^i}\{\frac{1-q^i}{fp} + \frac{q^i}{f^2p}\}} \\
&= 1 - \frac{1}{\frac{f(1-q^i)+q^i}{f^2p}} \\
&= 1 - \frac{f^2p}{f+q^i(1-f)} \qquad (5.2.16)
\end{aligned}
$$

Thus p_t is taken as that value of p for which

$$\left(1 - \frac{f^2p}{f+q^i(1-f)}\right)^{1000} = 0.10 \qquad (5.2.17)$$

5.3 Comparison of Optimum CSP Plans in Respect of Different Criteria.

5.3.1 Introduction

For the sake of comparison, only those plans having same AOQL and the minimum of inspection at the process average $\bar{p}$ are considered. The criteria considered relevant for the purpose of comparison are as follows :

(i) $p_t(\%)$ points

(ii) Average Fraction Inspected, AFI and average outgoing quality, AOQ at some selected values of incoming quality in relation to $\bar{p}$, say, $.25\bar{p}$, $.50\bar{p}$, $.75\bar{p}$, $1.00\bar{p}$, $1.25\bar{p}$, $1.50\bar{p}$, $1.75\bar{p}$ and $2.00\bar{p}$. and

(iii) the nature of the resulting AFI curve and not just the value of AFI at $\overline{p}$ and the nature of AOQ curve and not just AOQL for the whole range of incoming quality p from 0 to 1.

Generally we compare all the other plans against the corresponding CSP - 1. For the purpose of comparison we select two values of AOQL namely $p_L = 0.02$ and 0.05. For ready reference we present in Table 5.3.1 the parameters of the different optimum CSP plans taken from the relevant appendices giving the details of different optimum plans.

5.3.2 Comparison with respect to $p_t(\%)$.

Using the expression fro $p_t(\%)$ as developed in section 5.2 for different plans we compute the $p_t(\%)$ for different optimum plans for different values of $\overline{p}$ ($> p_L$) for two values of p_L, viz., 0.02 and 0.05. The values are shown in Table 5.3.2.

It will be seen that $p_t(\%)$ for CSP - 2 plans is smaller than that of CSP - 1 plans. Similarly $p_t(\%)$ for CSP - 3 plans is smaller than that of the corresponding CSP - 2 plans for smaller values of $\overline{p}$. Again among CSP - 2 and CSP - 3 plans, those with smaller values of k perform better in detecting sudden deterioration of quality as revealed by the $p_t(\%)$. This is possibly due to the fact that the value of f, the sampling frequency, for CSP - 2 and CSP - 3 plans are higher as compared to CSP - 1. Though the value of f is smaller for CSP - 3 as compared to CSP - 2, the improved performance of CSP - 3, particularly when $\overline{p}$ is small i.e. close to p_L, may be attributed to the beneficial effect of 'rule of 4' as applied in CSP - 3.

The performance of two level CSP plans in respect of $p_t(\%)$ is better than that of CSP - 2 particularly when $\overline{p}$ is small, close to p_L. For larger value of $\overline{p}$, however, CSP - 2 performs better though in numerical terms the difference is not of much consequence. Broadly speaking, CSP - 3 generally performs the best in this respect. But it does not perform uniformly better than the other plans for all p_L and $\overline{p}$. It is found for instance from Table 5.3.2 that for $p_L = 0.05$ and $\overline{p} = 0.06$ the performance of the two level CSP is the best. The reason for this difference is not obvious from the table and may be a subject for further investigation.

5.3.3 Comparison in respect of AFI and AOQ for Some Selected Values of p in relation to $\overline{p}$.

In table 5.3.3 (a) - 5.3.3 (d), we present the values of AFI and AOQ for different optimum plans against different values of p in relation to $\overline{p}$ for two levels of AOQL, viz., 0.05 and 0.02.

It may be pointed out that for a given optimum CSP, AFI increases as AOQ decreases. Hence the two criteria can be discussed simultaneously.

It may be broadly concluded by studying the tables that

(a) for optimum plans with $\overline{p}$ close to p_L, AFI (AOQ) for CSP - 1 > (<) AFI (AOQ) for Two level CSP plans for all values of $p \neq \overline{p}$.

(b) for optimum plans with $\overline{p}$ large compared p_L, AFI (AOQ) for CSP - 1 < (>) AFI (AOQ) for Two level CSP for smaller values of p.

And AFI (AOQ) for CSP - 1 > (<) AFI (AOQ) for two level CSP for higher values of p.

(c) AFI (AOQ) for CSP - 1 < (>) AFI (AOQ) for CSP - 2 for all $p \neq \overline{p}$.

(d) AFI (AOQ) for CSP - 1 < (>) AFI (AOQ) for CSP - 3 for all $p \neq \overline{p}$.

For CSP - 2 and CSP - 3, no regular pattern is observed between k = i - 10, k = i and k = i + 10 plans.

The behaviour of Two level CSP plan is interesting in the sense that even though it has a smaller value of $p_t(\%)$ as compared to CSP - 1 this is achieved without increasing the amount of inspection particularly for larger value of p indicating deterioration of quality. For smaller values of p the increase in AFI over CSP - 1 is comparatively smaller. In CSP - 2 and CSP - 3 plans also there is a reduction in $p_t(\%)$ in comparison with CSP - 1 plans, but at the cost of increased amount of inspection for all $p \neq \bar{p}$.

5.3.4 Comparison in respect of AFI and AOQ Curves.

To study this we trace the AFI and AOQ curves for the different optimum plans with $\bar{p}$ = 0.08 and AOQL = 0.02 in Figures 5.3.1 through 5.3.6. Since no regular pattern has been observed in section 5.3.3 in CSP - 2 and in CSP - 3 for different values of k, we have considered only CSP - 2 (k = i) and CSP - 3 (k = i) plans. The figures more or less confirm the findings in section 5.3.3. For the purpose of a ready reckoner, we use the following codes for the different criterion.

Smaller $p_t(\%)$ - Code 1
Better AFI curve - Code 2
Better AOQ curve - Code 3

The meanings of the terms smaller and better are clear from the context in which they are used. From the study of figures and different tables we may broadly follow a course of action with regard to an appropriate choice of the sampling plan in a given situation, as indicated in the table below :

Code	Appropriate plans
1	CSP - 3, Two level CSP, CSP - 2
2	CSP - 1, Two level CSP
3	CSP - 2, CSP - 3
1 and 2	Two level CSP
1 and 3	CSP - 3, CSP - 2
2 and 3	CSP - 1

It may be noted that all the plans in the table above are optimum plans given a value of AOQL and a value of $\bar{p}$. Hence, the plans have the same AOQL value and the same value of AFI at $\bar{p}$.

To further pin point the choice of a plan additional numerical computations leading to the AFI, AOQ curves and $p_t(\%)$ in the list may be carried out and one can arrive at an appropriate compromise solution, based on this partial investigation.

Flow Chart for Two Level CSP Plan

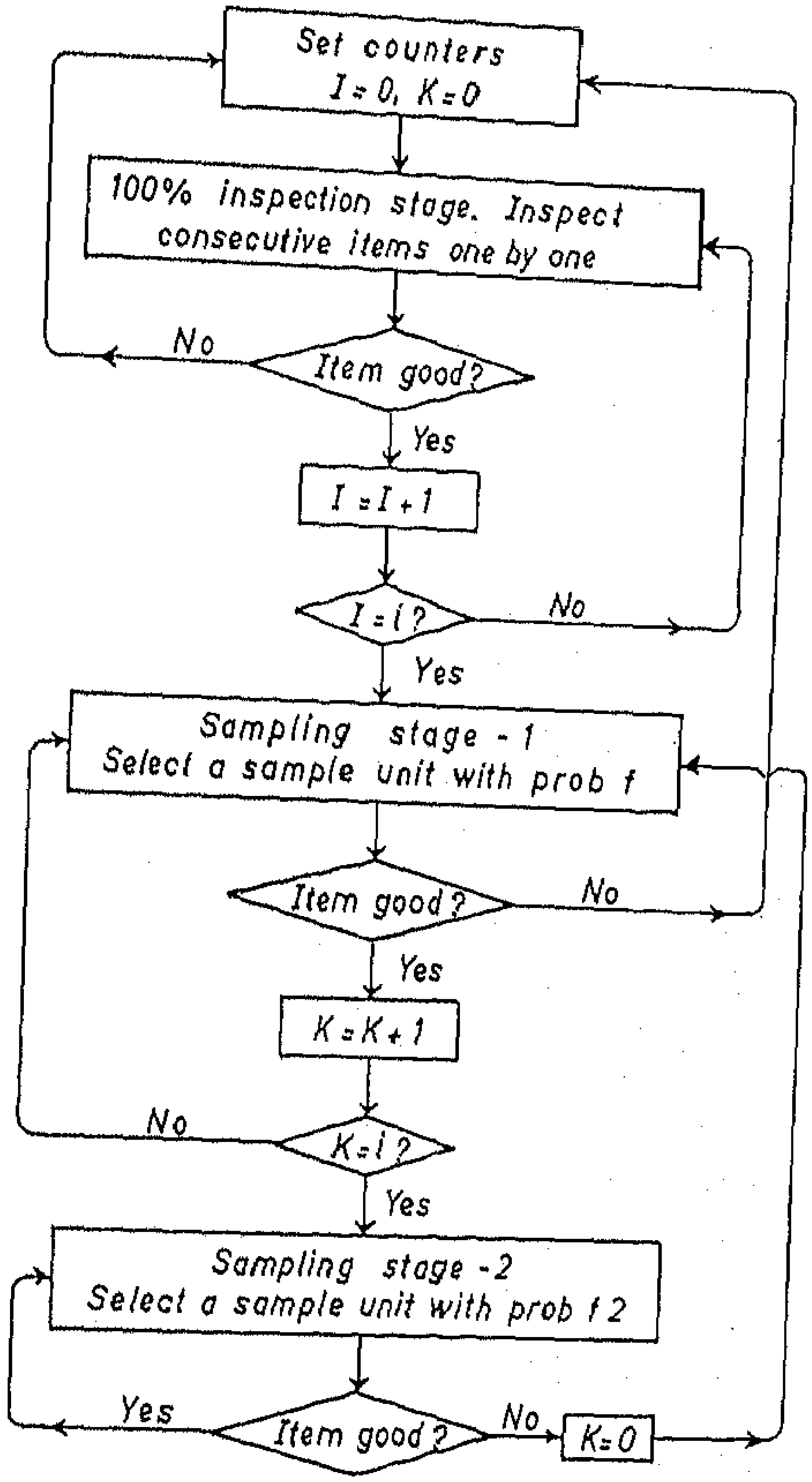


Fig. 5.1.1

Counter I controls i during 100% inspection.
Counter K controls i during sampling stage - 1
i (100% stage) = clearance number, number of consecutive good items leading to sampling stage - 2.
i (Sampling stage 1) = number of good items in succession leading to sampling stage - 1.

f = the prob of selection of an item in sampling stage - 1.
f^2 = the prob of selection of an item in sampling stage 2.
(i, f) are the plan parameters.

FLOW CHART OF STATES AND TRANSITIONS FOR
CSP-1, CSP-2, AND CSP-3.

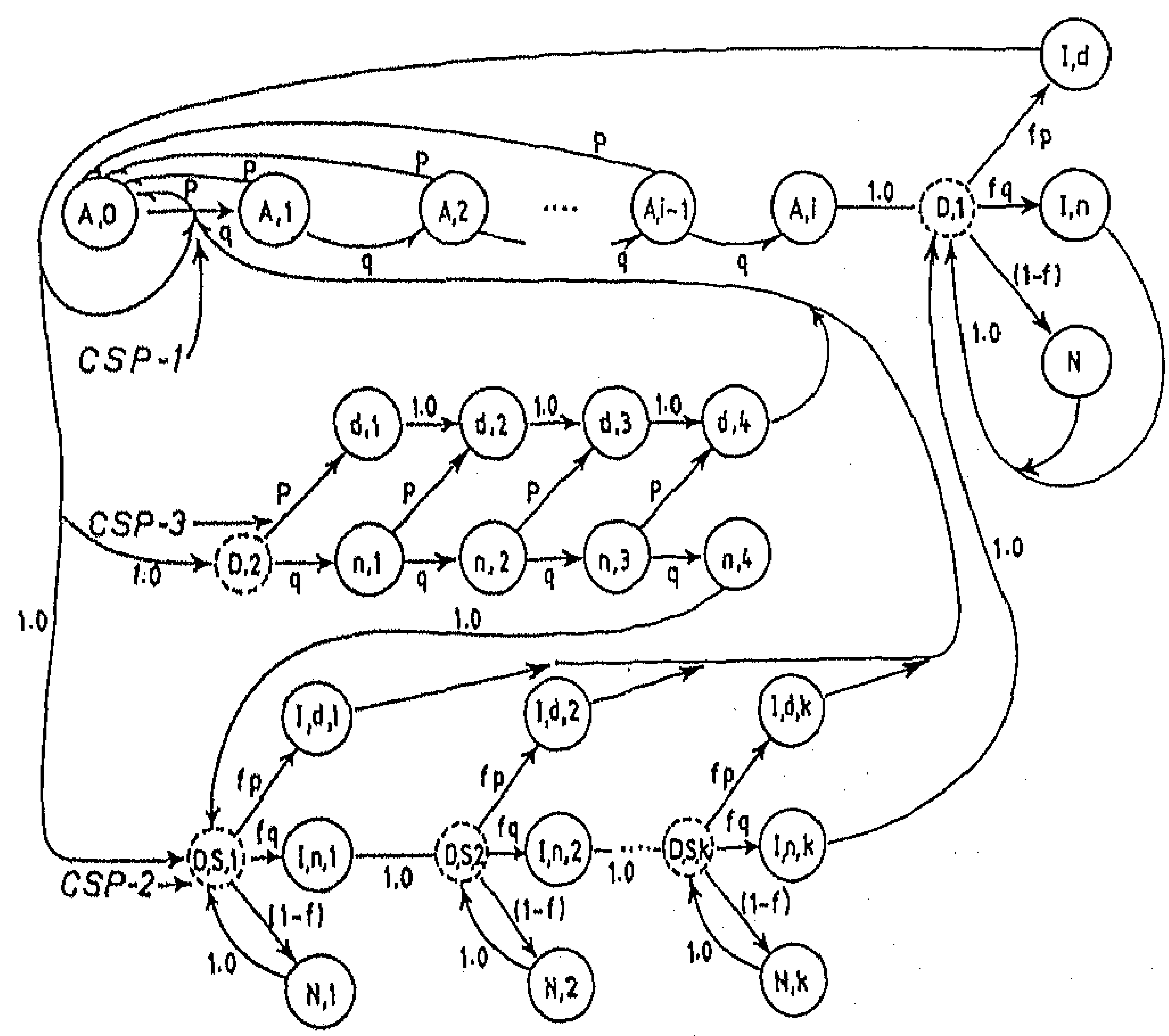


Fig.= 5.2.1

FLOW CHART OF STATES AND TRANSITIONS FOR
TWO LEVEL CONTINUOUS SAMPLING PLAN:

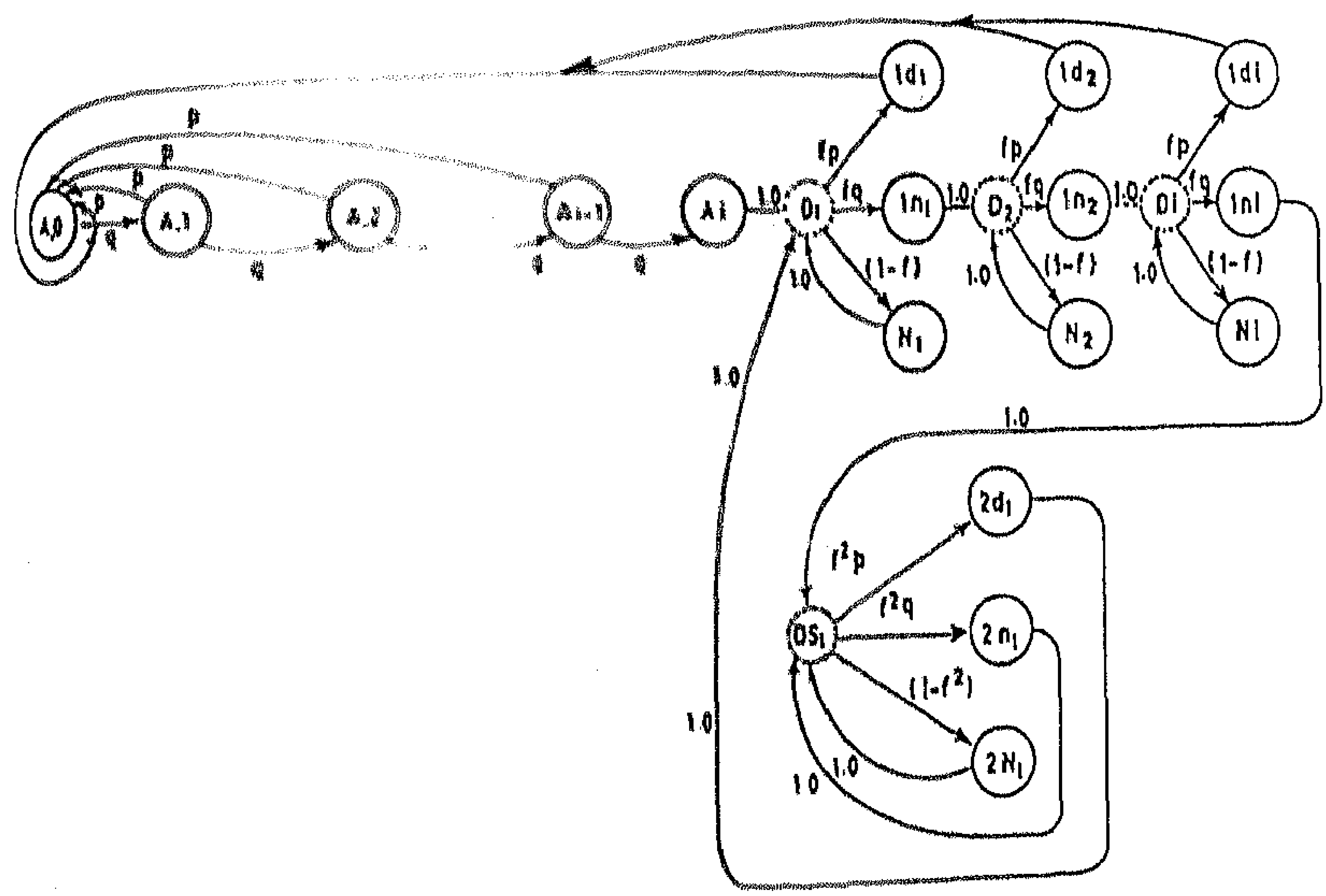


Fig 5.2.2

AFI CURVE FOR OPTIMUM CSP-1 AND OPTIMUM CSP-2

$P_L = 0.02,\ \bar{p} = 0.08.$

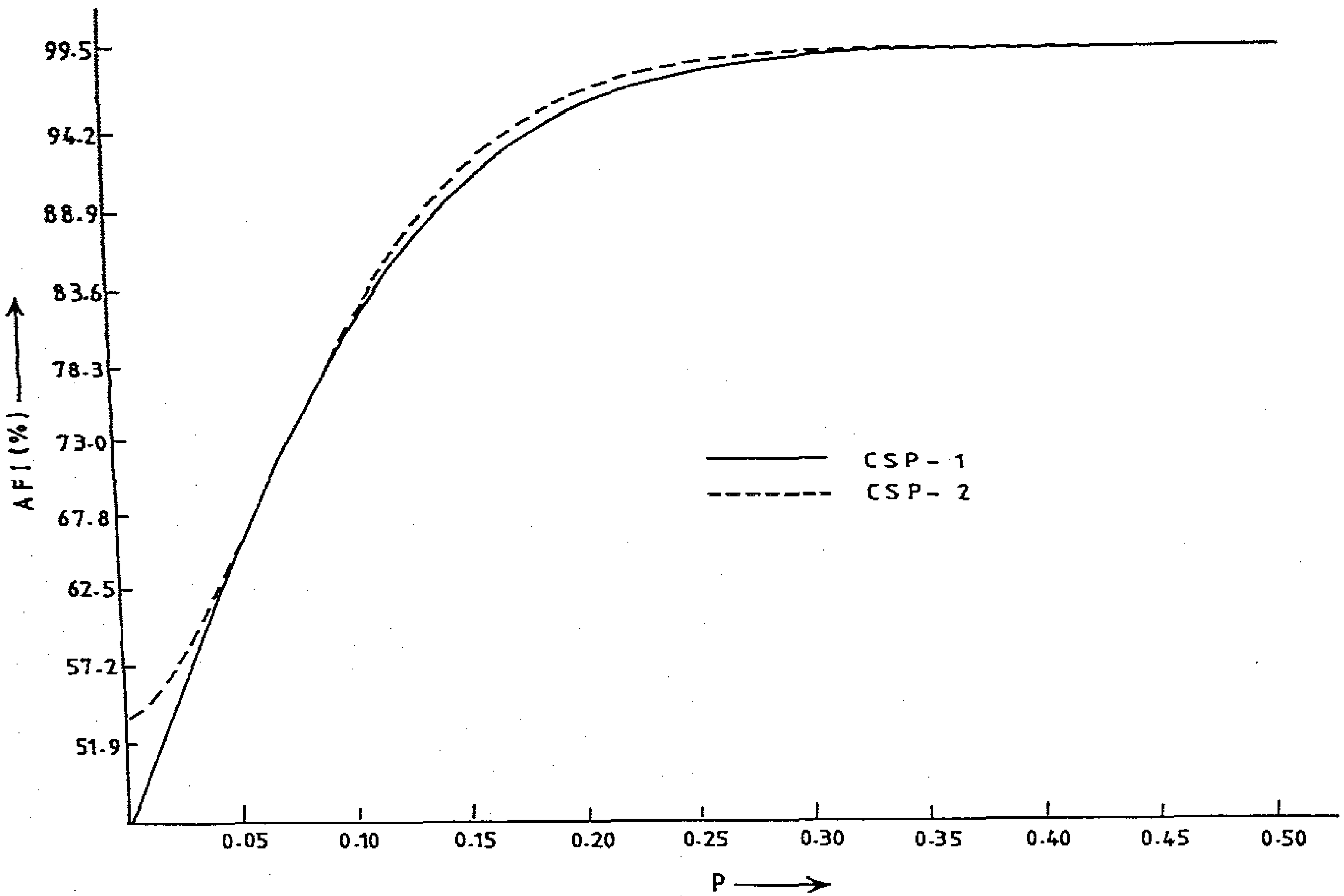


Fig. 5.3.1

AFI CURVE FOR OPTIMUM CSP-1 AND CSP-3

$P_L = 0.02, \bar{p} = 0.08.$

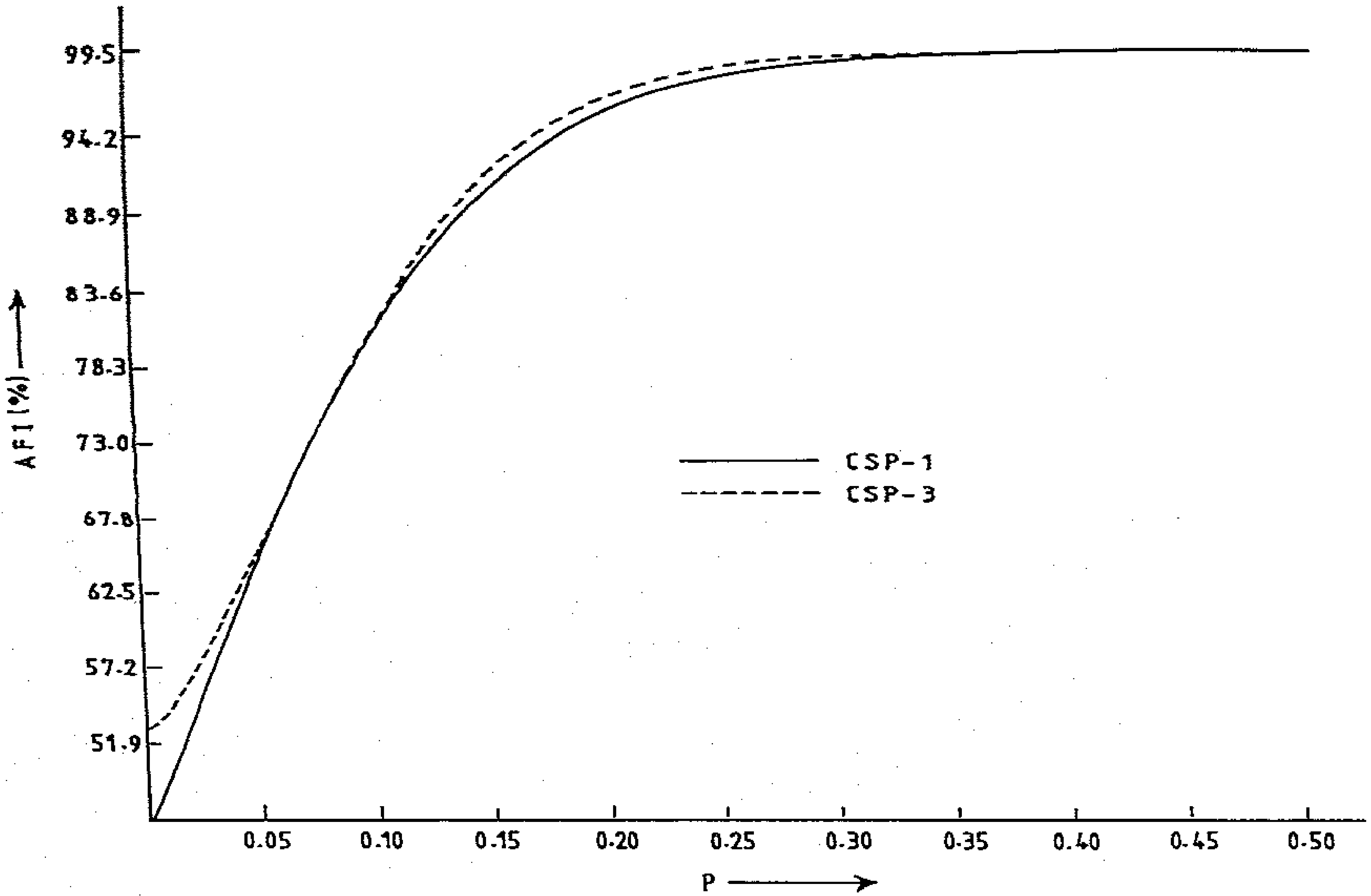


Fig. 5.3.2

AOQ CURVE FOR OPTIMUM CSP-1 AND OPTIMUM CSP-2

$P_L = 0.02$, $\bar{p} = 0.08$.

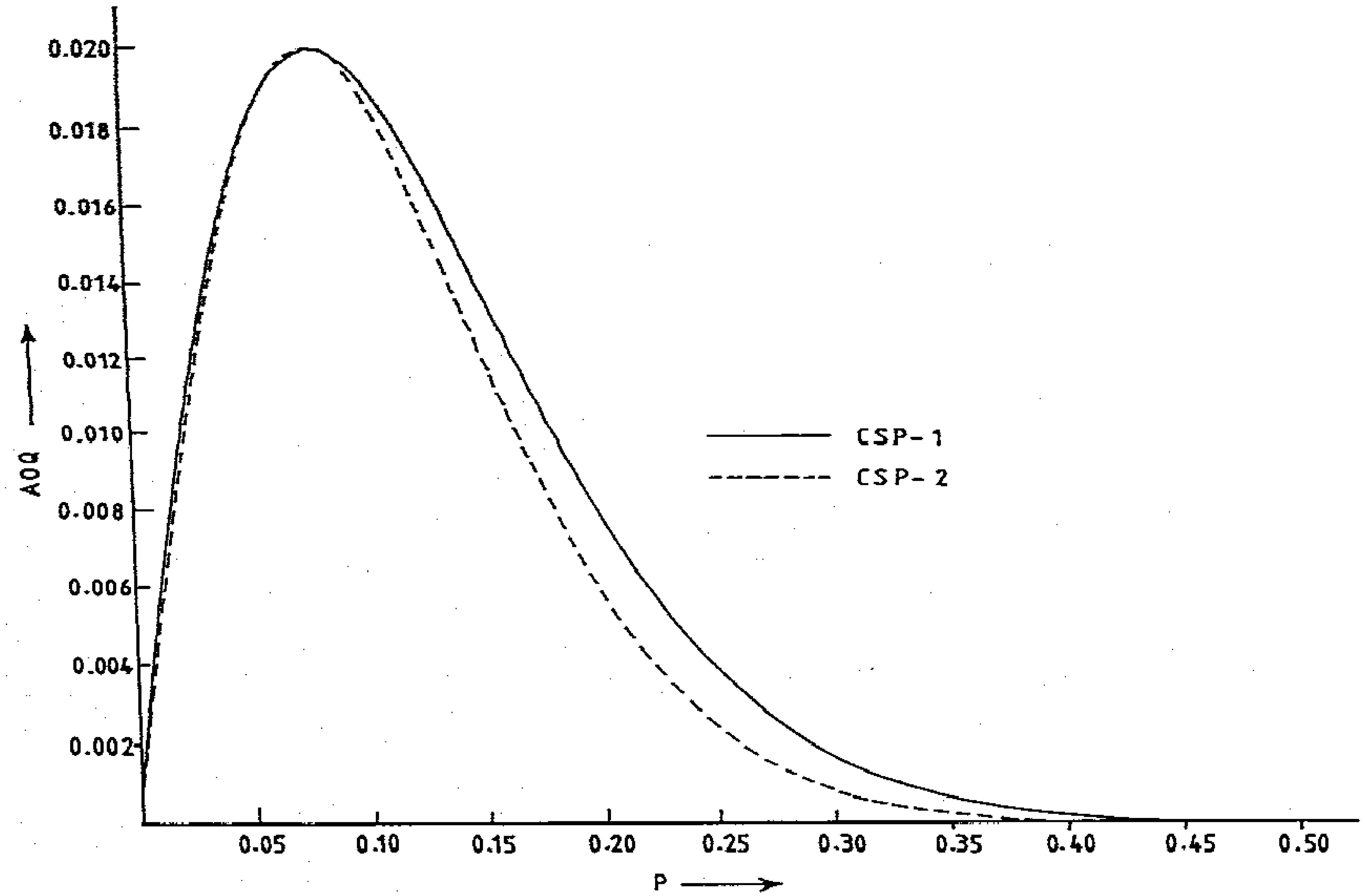


Fig. 5.3.3

AFI CURVE FOR OPTIMUM CSP - 1 AND TWO LEVEL OPTIMUM CSP PLAN

$P_L = 0.02 \quad \bar{P} = 0.08.$

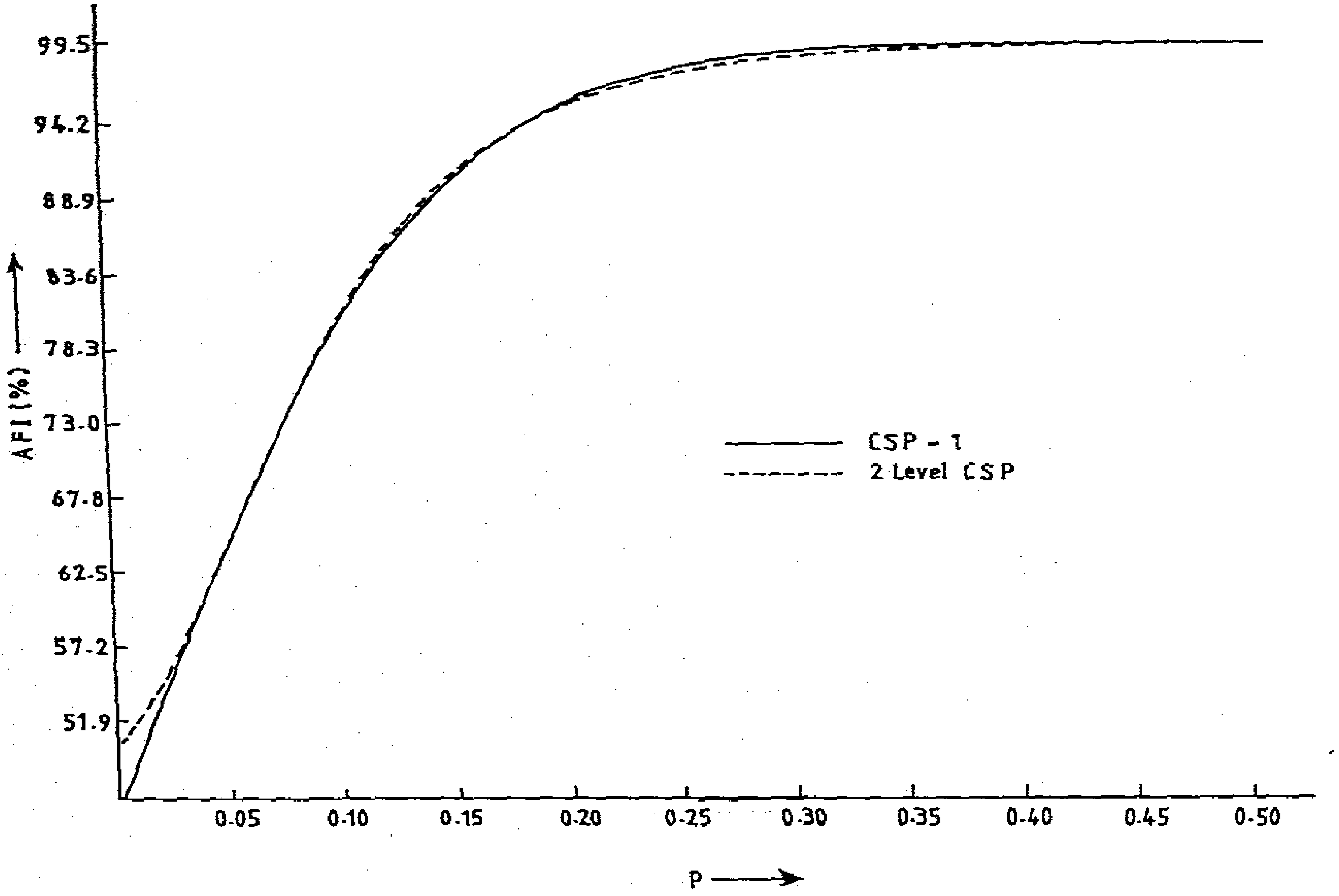


Fig. 5.3.4

AOQ CURVE FOR OPTIMUM CSP-1 AND OPTIMUM CSP-3

$P_L = 0.02$, $\bar{p} = 0.08$.

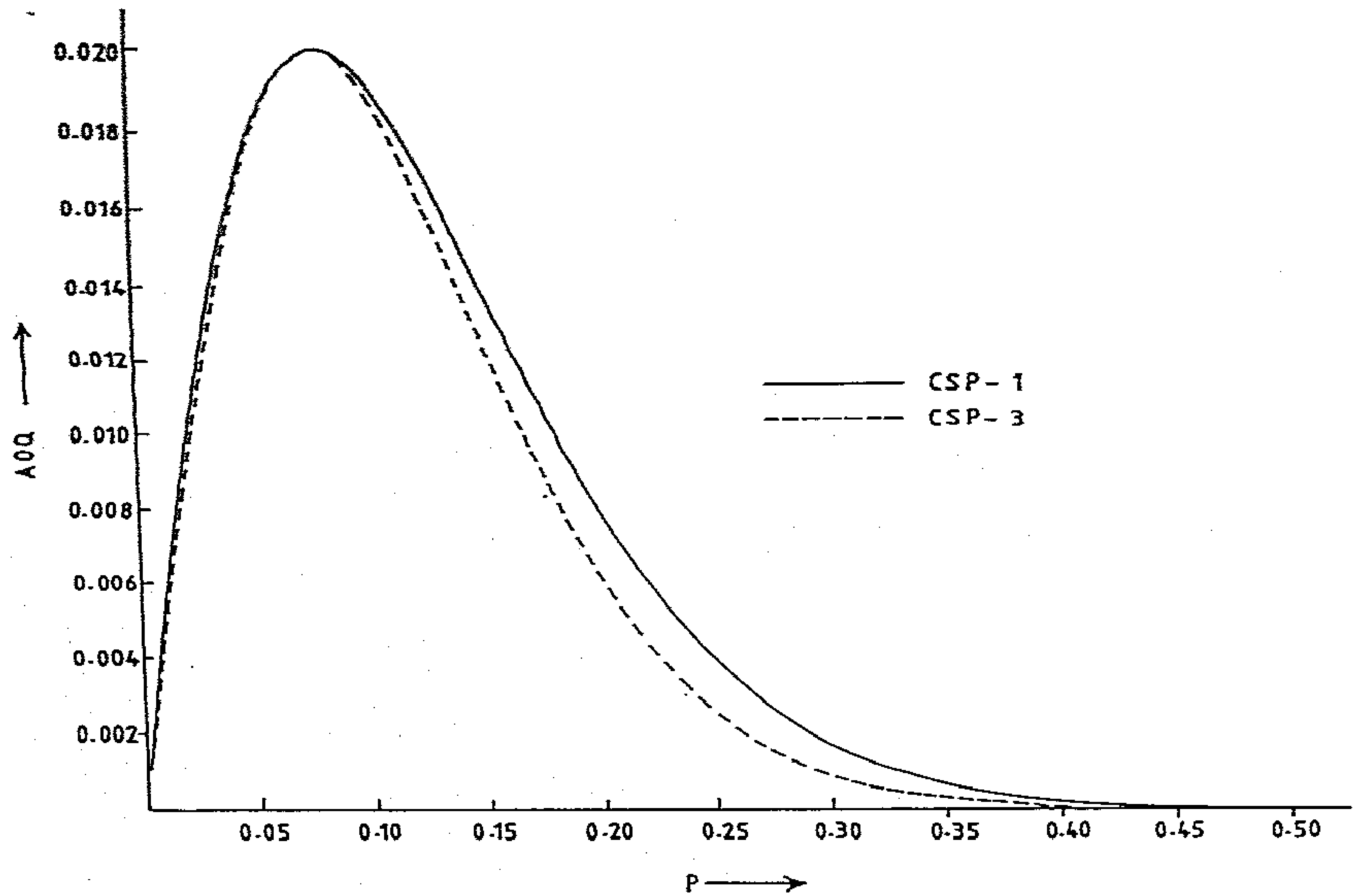


Fig. 5.3.5

AOQ CURVE FOR OPTIMUM CSP-1 AND TWO LEVEL OPTIMUM CSP PLAN

$p_L = 0.02,\ \bar{P} = 0.08.$

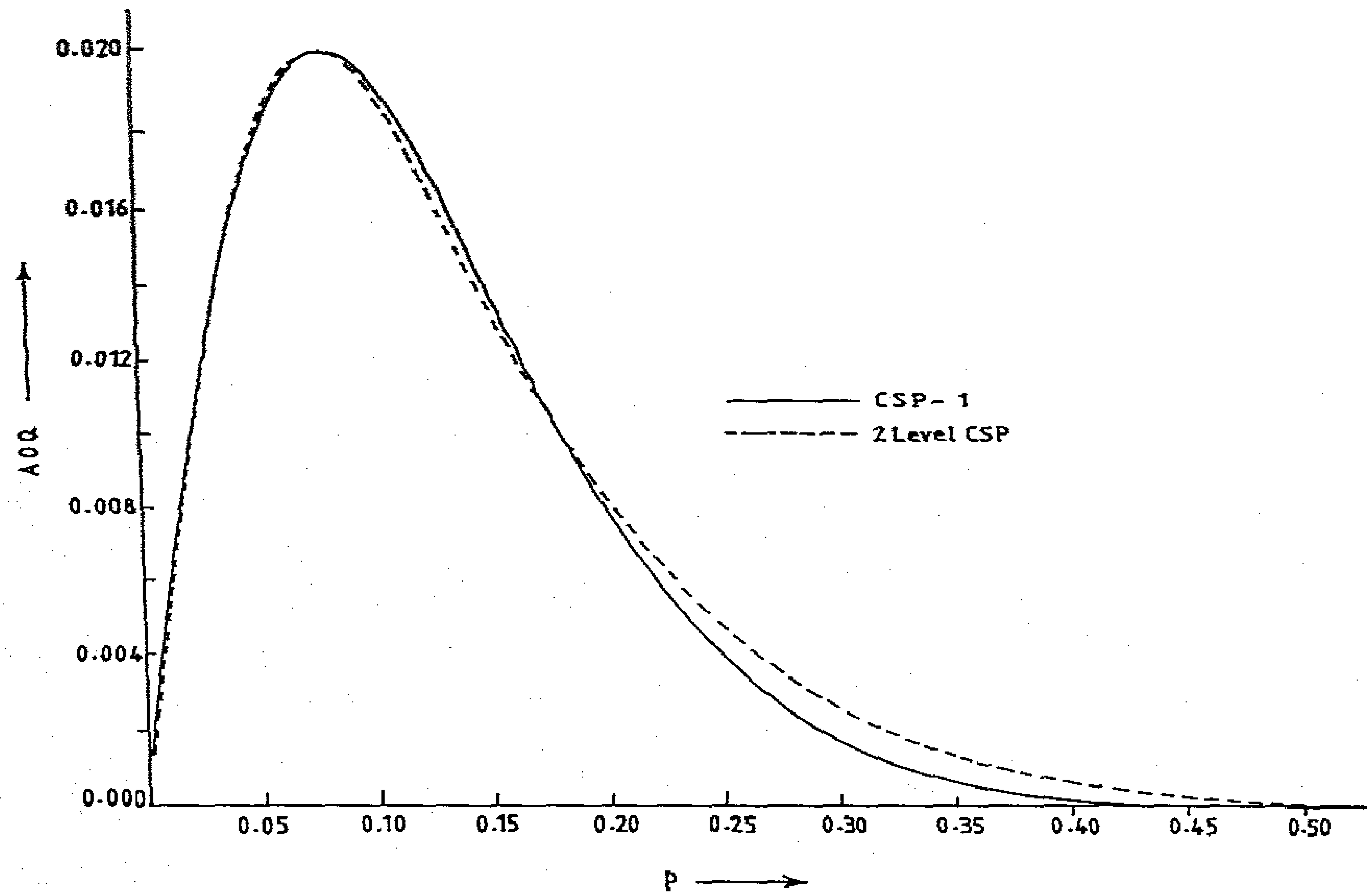


Fig. 5.3.6

Table 5.2.1

Transition Matrix for Two Level CSP Plan

	A_0	A_1	A_2	...	A_{i-1}	A_i	$1d_1$	$1n_1$	N_1	$1d_2$	$1n_2$	N_2	...	$1d_i$	$1n_i$	N_i	$2d_1$	$2n_1$	$2N_1$
A_0	p	q																	
A_1	p		q																
A_2	p																		
⋮																			
A_{i-1}	p					q													
A_i							fp	fq	1-f										
$1d_1$	p	q																	
$1n_1$										fp	fq	1-f							
N_1							fp	fq	1-f										
$1d_2$	p	q																	
$1n_2$													...						
N_2										fp	fq	1-f							
⋮																			
$1d_i$	p	q																	
$1n_i$																	f^2p	f^2q	$1-f^2$
N_i														fp	fq	1-f			
$2d_1$							fp	fq	1-f										
$2n_1$																	f^2p	f^2q	$1-f^2$
$2N_1$																	f^2p	f^2q	$1-f^2$

Table 5.3.1

Plan parameters for different optimum CSP plans

Plan type		$p_L = .02$								$p_L = .05$				
		$\bar{p}$=.03	$\bar{p}$=.04	$\bar{p}$=.05	$\bar{p}$=.06	$\bar{p}$=.07	$\bar{p}$=.08	$\bar{p}$=.09	$\bar{p}$=.10	$\bar{p}$=.06	$\bar{p}$=.07	$\bar{p}$=.08	$\bar{p}$=.09	$\bar{p}$=.10
CSP – 1	i	97	48	32	24	19	15	13	11	94	46	31	23	18
	f	.0254	.1235	.2252	.3118	.3864	.4621	.5067	.5566	.0006	.0140	.0433	.0838	.1305
CSP – 2(k=i-10)	i	99	51	35	26	21	18	16	14	94	47	31	24	19
	f	.0460	.1943	.3256	.4403	.5237	.5835	.6292	.6823	.0012	.0258	.0839	.1469	.2256
CSP – 2(k=i)	i	99	51	35	26	21	18	15	13	94	47	32	24	19
	f	.0456	.1896	.3136	.4188	.4925	.5431	.5991	.6397	.0012	.0253	.0743	.1363	.2013
CSP – 2(k=i+10)	i	100	51	35	27	21	18	15	13	94	48	32	24	20
	f	.0441	.1810	.3074	.3966	.4812	.5306	.5853	.6251	.0012	.0234	.0731	.1329	.1807
CSP – 3(k=i-10)	i	99	51	35	27	22	18	16	14	94	47	32	24	19
	f	.0458	.1915	.3177	.4111	.4834	.5504	.5871	.6260	.0012	.0255	.0757	.1397	.2074
CSP – 3(k=i)	i	100	52	35	27	22	18	16	14	94	47	32	24	20
	f	.0442	.1821	.3095	.3991	.4685	.5328	.5685	.6064	.0012	.0252	.0736	.1341	.1822
CSP – 3(k=i+10)	i	100	52	36	27	22	18	16	14	94	48	32	25	20
	f	.0440	.1800	.2958	.3936	.4624	.5267	.5625	.6010	.0012	.0234	.0728	.1225	.1795
Two level CSP	i	59	30	21	16	13	11	10	10	56	28	19	14	11
	f	.1626	.3931	.5214	.6095	.6690	.7115	.7336	.7396	.0176	.1132	.2209	.3258	.4132

Table 5.3.2

p_t (%) for Different Optimum Plans

	p_L = 2.0%								p_L = 5.0%				
$\bar{p}$ (%)	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	6.0	7.0	8.0	9.0	10.0
CSP - 1	9.06	1.86	1.02	0.74	0.60	0.50	0.45	0.41	100.0	16.43	5.31	2.75	1.76
CSP - 2(k=i-10)	5.00	1.18	0.70	0.52	0.43	0.39	0.36	0.34	100.0	8.92	2.74	1.56	1.02
CSP - 2(k=i)	5.04	1.21	0.73	0.55	0.46	0.42	0.39	0.36	100.0	9.09	3.10	1.69	1.13
CSP - 2(k=i+10)	5.21	1.27	0.75	0.58	0.48	0.43	0.39	0.37	100.0	9.81	3.15	1.73	1.27
CSP - 3(k=i-10)	2.49	0.89	0.61	0.50	0.45	0.40	0.38	0.36	47.41	4.44	2.00	1.34	0.98
CSP - 3(k=i)	2.53	0.90	0.61	0.50	0.45	0.40	0.38	0.36	47.41	4.45	2.00	1.34	1.00
CSP - 3(k=i+10)	2.53	0.90	0.61	0.50	0.45	0.40	0.38	0.36	47.41	4.52	2.00	1.34	1.00
Two level CSP Plan	2.79	1.21	0.78	0.59	0.50	0.45	0.42	0.42	13.31	5.39	3.07	1.83	1.24

Table 5.3.3 (a)

Comparision of AFI(%) and AOQ for Different Optimum Plans at Some Selected Values of p.

$p_L = 0.05$: $\bar{p} = 0.06$

Values of p		CSP - 1	CSP - 2 k=i-10	CSP - 2 k = i	CSP - 2 k = i+10	CSP - 3 k = i-10	CSP - 3 k = i	CSP - 3 k = i+10	Two level CSP
.25 $\bar{p}$	AFI(%)	.25	.28	.28	.29	0.28	0.29	0.29	0.12
	AOQ	.0150	.0150	.0150	.0150	0.0150	0.0150	0.0150	0.0150
.50 $\bar{p}$	AFI(%)	1.03	1.06	1.07	1.08	1.07	1.08	1.08	0.74
	AOQ	.0297	0.0297	0.0297	0.0297	0.0297	0.0297	0.0297	0.0298
.75 $\bar{p}$	AFI(%)	4.32	4.37	4.38	4.39	4.37	4.39	4.40	3.97
	AOQ	0.0431	0.0430	0.0430	0.0430	0.0430	0.0430	0.0430	0.0432
1.00 $\bar{p}$	AFI(%)	16.67	16.79	16.80	16.81	16.79	16.81	16.82	16.67
	AOQ_p	0.0500	0.0499	0.0499	0.0499	0.0499	0.0499	0.0499	0.0500
1.25 $\bar{p}$	AFI(%)	47.57	47.78	47.79	47.79	47.78	47.79	47.79	44.71
	AOQ	0.0393	0.0392	0.0392	0.0392	0.0392	0.0392	0.0392	0.0415
1.50 $\bar{p}$	AFI(%)	80.84	80.96	80.97	80.97	80.97	80.97	80.97	73.11
	AOQ	0.0172	0.0171	0.0171	0.0171	0.0171	0.0171	0.0171	0.0242
1.75 $\bar{p}$	AFI(%)	95.27	95.30	95.30	95.30	95.30	95.30	95.30	88.89
	AOQ	0.0050	0.0049	0.0049	0.0049	0.0049	0.0049	0.0049	0.0117
2.00 $\bar{p}$	AFI(%)	99.00	99.00	99.00	99.00	99.00	99.00	99.00	95.66
	AOQ	0.092	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0052

Table 5.3.3 (b)

Comparison of AFI(%) and AOQ for Different Optimum Plans at Some Selected Values of p

$p_L = 0.05$; $\bar{p} = 0.10$

Values of p		CSP - 1	CSP - 2 k = i-10	CSP - 2 k = i	CSP - 2 k = i+10	CSP - 3 k = i-10	CSP - 3 k = i	CSP - 3 k = i+10	Two level CSP
.25 $\bar{p}$	AFI(%)	19.14	24.34	22.78	21.33	23.86	21.75	21.76	21.14
	AOQ	0.0202	0.0189	0.0193	0.0197	0.0190	0.0196	0.0196	0.0197
.50 $\bar{p}$	AFI(%)	27.42	29.63	29.16	28.27	29.83	28.44	28.62	28.22
	AOQ	0.0363	0.0352	0.0354	0.0359	0.0351	0.0358	0.0357	0.0359
.75 $\bar{p}$	AFI(%)	37.91	38.39	38.48	38.03	38.71	38.04	38.19	38.23
	AOQ	0.0466	0.0462	0.0461	0.0465	0.0460	0.0465	0.0464	0.0463
1.00 $\bar{p}$	AFI(%)	50.00	50.00	50.00	50.00	50.02	50.00	50.00	50.01
	AOQ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
1.25 $\bar{p}$	AFI(%)	62.41	62.79	62.39	62.83	62.40	62.88	62.73	61.78
	AOQ	0.0470	0.0465	0.0470	0.0465	0.0470	0.0464	0.0466	0.0478
1.50 $\bar{p}$	AFI(%)	73.67	74.62	73.88	74.65	73.98	74.75	74.53	72.04
	AOQ	0.0395	0.0381	0.0392	0.0380	0.0390	0.0379	0.0382	0.0419
1.75 $\bar{p}$	AFI(%)	82.72	83.99	83.16	84.06	83.32	84.15	83.96	80.17
	AOQ	0.0302	0.0280	0.0295	0.0279	0.0291	0.0277	0.0281	0.0347
2.00 $\bar{p}$	AFI(%)	89.28	90.51	89.80	90.63	89.97	90.70	90.56	86.21
	AOQ	0.0214	0.0190	0.0204	0.0188	0.0200	0.0186	0.0189	0.0276

Table 5.3.3 (c)

Comparison of AFI(%) and AOQ for Different Optimum Plans at Some Selected Values of p

$p_L = 0.02$; $\bar{p} = 0.05$

Values of p		CSP - 1	CSP - 2 k = i-10	CSP - 2 k = i	CSP - 2 k = i+10	CSP - 3 k = i-10	CSP - 3 k = i	CSP - 3 k = i+10	Two level CSP
.25 $\bar{p}$	AFI(%)	30.30	35.05	34.35	34.12	35.02	34.53	33.54	32.04
	AOQ	0.0087	0.0081	0.0082	0.0082	0.0081	0.0082	0.0083	0.0085
.50 $\bar{p}$	AFI(%)	39.52	41.27	41.11	41.18	41.49	41.38	40.79	39.77
	AOQ	0.0151	0.0147	0.0147	0.0147	0.0146	0.0147	0.0148	0.0151
.75 $\bar{p}$	AFI(%)	49.69	49.99	50.05	50.15	50.17	50.20	49.94	49.59
	AOQ	0.0189	0.0188	0.0187	0.0187	0.0187	0.0187	0.0188	0.0189
1.00 $\bar{p}$	AFI(%)	60.00	60.00	60.00	60.00	60.00	60.00	60.00	60.00
	AOQ	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
1.25 $\bar{p}$	AFI(%)	69.63	69.91	69.76	69.63	69.77	69.65	69.86	69.56
	AOQ	0.0190	0.0188	0.0189	0.0190	0.0189	0.0190	0.0188	0.0190.
1.50 $\bar{p}$	AFI(%)	77.89	78.60	78.34	78.13	78.38	78.17	78.52	77.47
	AOQ	0.0166	0.0160	0.0163	0.0164	0.0162	0.0164	0.0161	0.0169
1.75 $\bar{p}$	AFI(%)	84.48	85.48	85.18	84.96	85.25	85.02	85.42	83.59
	AOQ	0.0136	0.0127	0.0130	0.0132	0.0129	0.0131	0.0128	0.0144
2.00 $\bar{p}$	AFI(%)	89.43	90.51	90.24	90.05	90.31	90.10	90.49	88.13
	AOQ	0.0106	0.0095	0.0098	0.0100	0.0097	0.0099	0.0095	0.0119

Table 5.3.3 (d)

Comparison of AFI(%) and AOQ for Different Optimum Plans at Some Selected Values of p

$p_L = 0.02$; $\bar{p} = 0.07$

Values of p		CSP - 1	CSP - 2 k = i-10	CSP - 2 k = i	CSP - 2 k = i+10	CSP - 3 k = i-10	CSP - 3 k = i	CSP - 3 k = i+10	Two level CSP plan
.25 $\bar{p}$	AFI(%)	46.83	53.99	51.77	51.24	51.60	50.61	50.35	49.30
	AOQ	0.0093	0.0089	0.0084	0.0085	0.0085	0.0086	0.0087	0.0089
.50 $\bar{p}$	AFI(%)	55.34	58.32	57.32	57.29	57.13	56.74	56.74	55.92
	AOQ	0.0156	0.0146	0.0149	0.0150	0.0150	0.0151	0.0151	0.0154
.75 $\bar{p}$	AFI(%)	63.70	64.47	64.22	64.31	64.04	63.98	64.03	63.70
	AOQ	0.0191	0.0187	0.0188	0.0187	0.0189	0.0189	0.0189	0.0191
1.00 $\bar{p}$	AFI(%)	71.43	71.43	71.43	71.43	71.43	71.44	71.43	71.44
	AOQ	0.0200	0.0200	0.0200	0.0200	0.0200	0.0220	0.0200	0.0200
1.25 $\bar{p}$	AFI(%)	78.20	78.23	78.17	78.04	78.45	78.39	78.31	78.27
	AOQ	0.0191	0.0191	0.0191	0.0192	0.0189	0.0189	0.0190	0.0190
1.50 $\bar{p}$	AFI(%)	83.83	84.17	83.97	83.76	84.48	84.33	84.20	83.81
	AOQ	0.0170	0.0166	0.0168	0.0170	0.0163	0.0165	0.0166	0.0170
1.75 $\bar{p}$	AFI(%)	88.29	88.92	88.63	88.40	89.24	89.06	88.92	88.07
	AOQ	0.0143	0.0136	0.0139	0.0142	0.0132	0.0134	0.0136	0.0146
2.00 $\bar{p}$	AFI(%)	91.71	92.47	92.17	91.95	92.78	92.59	92.47	91.25
	AOQ	0.0116	0.0105	0.0110	0.0113	0.0101	0.0104	0.0105	0.0123

CHAPTER 6
A FEW OTHER SQC PROBLEMS

6.0 Introduction

In this chapter we consider four different problems which have a practical relevance and can be expected to be of interest to the users of Statistical Quality Control techniques in industry.

Section 6.1 deals with the computation of λ, the tolerance factor for two sided tolerance interval for a Normal population whose parameters are not known. The formula for λ considered in this section has been adopted by the Bureau of Indian Standards for its publication 'Statistical Tolerance Interval - Methods For Determination : IS 13131 : 1991'.

Section 6.2 introduces Median - Gauge chart for process control using go-no go gauge data generated automatically during production.

Section 6.3 gives a procedure for determination of optimum sampling interval for inspection for a situation where the appearance of an assignable cause makes the entire production till its correction and removal unsuitable for use. The model was developed while working on a live problem of controlling incidence of excessive damages in a printing machine in the processing department of a textile mill.

Section 6.4 studies the effect of inspection error on average outgoing quality of a single sampling acceptance rectification system in which the apparent defectives are replaced whenever they are encountered during inspection.

Contents of sections 6.1, 6.2, 6.3 and 6.4 have been published in Ghosh (1980, 1985, 1981, 1985).

6.1 Computation of Factor for Tolerance Limits for a Normal Distribution.

6.1.1 Introduction

In an industry it is often required to obtain a range for a quality characteristic (which is usually found to be distributed normally) such that a certain percentage of the product from a manufacturing process under control have the quality characteristic falling between the limits. The problem for two sided tolerance limits can be formally stated as follows :

Let $x_1, x_2 .....x_N$ be a random sample from a normal population with mean and variance unknown. Then the tolerance limits L_1 and L_2 defined as the functions of $x_1, x_2, ...x_N$ are to be so obtained that for given β, $\gamma \in \Omega$, and X, a random variable which represents the value of the quality characteristic of a unit chosen at random from the process,

$$P\left(P\left(L_1 < X < L_2\right) \geq \gamma\right) = \beta \qquad (6.1.1)$$

It may be noted that L_1 and L_2, being functions of the random sample are themselves random variables and hence the expression $P(L_1 < X < L_2)$ in (6.1.1) is itself a random variable.

It was shown by Wald and Wolfowitz (1946) that the best two sided tolerance limits, L_1 and L_2 in the above situation are given by $L_1 = \overline{x} - \lambda s, L_2 = \overline{x} + \lambda s$,

$$\text{where } \lambda = r\sqrt{\frac{n}{\chi^2_{n,\beta}}} \qquad (6.1.2)$$

Here N represents sample size,

$\overline{x}$ represents sample mean,

s represents sample standard deviation,

n = N - 1,

$\chi^2_{n,\beta}$ is the value such that $P[\chi^2_n > \chi^2_{n,\beta}] = \beta$, where χ^2_n represents a χ^2 with n d.f. and r is the root of the equation

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{1}{\sqrt{N}}-r}^{\frac{1}{\sqrt{N}}+r} e^{-t^2/2}dt = \gamma \qquad (6.1.3)$$

The determination of λ = tolerance factor, is computationally tedious largely because finding r as the root of the equation (6.1.3) involves an iterative procedure and is quite time consuming even for large N. Further, $\chi^2_{n,\beta}$ is not readily available for all values of N particularly when N is large. The values of λ for selected values of N, β and γ were tabulated by Bowker in Eisenhart et. al (1947).

However, the table may not be readily available for Quality Control practitioners or even if available the interest may be on values of N, β and γ outside the range of tabulation there.

It is, therefore, useful to have an approximate formula for λ which will be satisfactory for practical purposes.

6.1.2 Bowker's Approximate Formula for Large N.

Bowker (1946) proposed the following formula for λ when N is large

$$\lambda = r_\infty(1 - \frac{x_{1-\beta}}{\sqrt{2N}} + \frac{5x^2_{1-\beta}+10}{12N} \qquad (6.1.4)$$

$$\text{where } \frac{1}{\sqrt{2\pi}}\int_{-r_\infty}^{r_\infty} e^{-t^2/2}dt = \gamma$$

and $x_{1-\beta}$ is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_{1-\beta}} e^{-t^2/2}dt = 1-\beta$$

This has the advantage that in computing λ a table for Normal distribution alone is required. But for small values of N the error due to approximation in (6.1.4) is considerable.

It will be noted from Bowker (1946) that his formula (6.1.4) is derived in two stages.

(i) r is expressed as $r \simeq r_\infty(1+\frac{1}{2N})$ retaining terms of order $\frac{1}{N}$;

(ii) The series expansion of χ^2 in terms of the powers of a normal deviate which can be found in Goldberg and Levine (1946) gives

$$\frac{\chi^2_{n,\beta}}{n} \simeq 1+\frac{\sqrt{2}x_{1-\beta}}{\sqrt{n}}+\frac{2}{3}\frac{x^2_{1-\beta}-1}{n}$$

after neglecting terms containing higher powers of $\frac{1}{n}$.

6.1.3 An Approximate Formula for λ for all N.

Lemma 6.1.1 *The tolerance factor λ can be approximately written as*

$$\lambda \simeq r_\infty\sqrt{\frac{N}{\chi^2_{n,\beta}}}$$

Proof : We have $\sqrt{n} = (N-1)^{\frac{1}{2}} = N^{\frac{1}{2}}(1-\frac{1}{N})^{\frac{1}{2}}$

$$= N^{\frac{1}{2}}(1-\frac{1}{2N})+o(\frac{1}{\sqrt{N}})$$

where $o(\frac{1}{\sqrt{N}})$ denotes a term of order less than $\frac{1}{\sqrt{N}}$

Using Bowker's result we have clearly,

$$r\sqrt{n} = r_\infty\sqrt{N}+o(\frac{1}{\sqrt{N}})$$

Hence, we have the result ignoring the term $o(\frac{1}{\sqrt{N}})$.

The approximate formula $\lambda \simeq r_\infty\sqrt{\frac{N}{\chi^2_{n,\beta}}}$ has been called formula F_1 in subsequent discussions.

It is interesting to compare the form of F_1, namely,
$\lambda = r_\infty\sqrt{\frac{N}{\chi^2_{n,\beta}}}$ with the exact formula, $\lambda = r\sqrt{\frac{n}{\chi^2_{n,\beta}}}$ as given by Wald and Wolfowitz. Now keeping the first four terms (i.e. upto accuracy of order $\frac{1}{\sqrt{N}}$) of the formula given by Goldberg and Levine (1946) for χ^2 in terms of a standard Normal deviate we have

$$\chi^2_{n,\beta} = n + G_1(x)\sqrt{n} + G_2(x) + \frac{G_3(x)}{\sqrt{n}},$$

$$\text{where } G_1(x) = \sqrt{2}x_{1-\beta}$$

$$G_2(x) = \frac{2}{3}(x^2_{1-\beta} - 1),\ G_3(x) = \frac{x^3_{1-\beta} - 7x_{1-\beta}}{9\sqrt{2}} \qquad (6.1.5)$$

We note that in order to facilitate expansion of $(\frac{n}{\chi^2_{n,\beta}})^{\frac{1}{2}}$ Bowker used the first 3 terms only in expression (6.1.5)].

The values of $G_1(x)$, $G_2(x)$ and $G_3(x)$, which are independent of n, can be tabulated for different values of β once for all.

Hence, we have the following result which is stated in the form of a lemma.

Lemma 6.1.2 *λ can be approximately written as $\lambda \simeq r_\infty\sqrt{\frac{N}{D}}$, where $D = n + G_1(x)\sqrt{n} + G_2(x) + \frac{G_3(x)}{\sqrt{n}}$.*

The formula given by Lemma 6.1.2 which makes use of the Normal table only has been called F_2 in future discussions. F_2 is expected to work satisfactorily for N not very small.

6.1.4 Performance of The Proposed Formulas F_1 and F_2.

F_1 can be used where $\chi^2_{n,\beta}$ can be obtained readily. F_2 is to be used where only standard Normal table is available.

We present in table 6.1.1 the values of λ obtained by formula F_1 and that by using the exact formula (6.1.2), as given in Table 2.1 of Eisenhart et. al. (1947) for $\beta = .95$ and $\gamma = .95$ and $.99$.

Table 6.1.1

Values of λ as evaluated by F_1 and the exact formula

N	$\beta = .95$			
	$\gamma = .95$		$\gamma = .99$	
	F_1	Exact	F_1	Exact
5	5.199	5.079	6.832	6.634
8	3.766	3.732	4.949	4.891
10	3.399	3.379	4.467	4.433
13	3.091	3.081	4.063	4.044
15	2.961	2.954	3.892	3.878
18	2.824	2.819	3.711	3.702
20	2.756	2.752	3.622	3.615
23	2.677	2.673	3.518	3.512
25	2.633	2.631	3.461	3.457
28	2.581	2.579	3.392	3.388
30	2.551	2.549	3.353	3.350

It will be seen from Table 6.1.1 that F_1 which is in a very elegant form closely resembling the exact formula proposed by Wald and Wolfowitz performs quite well for $N \geq 10$ and for $N \geq 25$ the maximum difference is less than 0.005. This agreement should be considered to be satisfactory particularly when we notice that it eliminates completely the tedious iterative procedure to be followed in the exact case.

Now we compare the performance of F_2 with that of Bowker's formula.

In order to study the performance of these two formulas for large as well as small N the approximate values of λ were computed by both formulas for different N for all combinations of (β, γ) with 0.75, 0.95 and 0.99 for values of β and 0.75, 0.95 and 0.999 for values of γ. These are compared with the exact values of λ given in Table 2.1 of Eisenhart et. al. (1947) and with that in Table 1, page 239 of Bowker (1946) (which gives λ correct upto 5 places of decimals for some selected values of $N \geq 50$).

The maximum and minimum difference over all combinations of (β, γ) in the above range for a given N are shown in table 6.1.2.

Table 6.1.2

Comparative Performance of Approximate Formulas for λ.

N	F_2 error		Bowker error	
	Min	Max	Min	Max
10	0.007	0.063	0.051	1.112
15	0.003	0.020	0.015	0.511
20	0.001	0.010	0.015	0.304
25	0.001	0.006	0.010	0.206
30	0.001	0.004	0.007	0.151
50	0.000	0.001	0.003	0.063
100	0.000	0.000	0.001	0.020
160	0.000	0.000	0.001	0.010
500	0.000	0.000	0.000	0.002
800	0.000	0.000	0.000	0.000

The proposed formula F_2 performs better than the one given by Bowker for all N. F_2 values coincide with exact values upto 3 places of decimals for N > 50 where as Bowker's values coincide with exact values for N > 500.

6.2 Median Gauge Chart - A New Procedure For Process Control By Gauging.

6.2.1 Introduction

The usual $\overline{X}$ - R chart for controlling a variable characteristic requires measurement of items. Taking measurements is costly, time consuming and at times difficult. Tippett had suggested gauging (testing product against a pair of go and no-go gauges) instead of measurements. Gauging is cheap and, what is more, is always done by an operator during production. The enormous volume of information thus generated is hardly utilised in a systematic manner. It did not, however, receive much attention by Quality Control practitioners as earlier efforts indicated that a gauge control chart would be less sensitive than $\overline{X}$ - R chart.

Stevens (1948) made an elaborate study in his celebrated paper 'Control by gauging' and concluded that by a judicious choice of gauging points efficient detection of change in μ (process average) and σ (process variability) was possible.

We introduce the Median Gauge Chart - a variation of Stevens' chart - as a new procedure for control.

6.2.2 Stevens' Gauge Control Chart

If we have a pair of gauges set to the values u_1 and u_2 ($u_1 < u_2$; and u_1 and u_2 not necessarily the specification limits) the observation x for a variable characteristic can be put into one of the three disjoint classes (-), (0) and (+) where

(-) indicates $x < u_1$

(0) indicates $u_1 \leq x \leq u_2$

(+) indicates $x > u_2$

Let a, b and c be the number of observations in (-), (0) and (+) classes respectively. Let p, q and r be the proportion of observations in the respective classes in the population.

Stevens proposed

c - a as an indicator for population mean and

c + a as an indicator for population standard deviation.

Control limits for the two charts are as follows :

	c - a chart	c + a chart
Central line	n (r - p)	n (r + p)
Control limits	$\pm 3\sqrt{n\{(p+r)-(p-r)^2\}}$	$\pm 3\sqrt{n(r+p)(1-r-p)}$

The efficiency of c - a chart which is used as an indicator for μ (process mean) is measured as the ratio of the variance of the estimate of μ, obtained from exact n measurement i.e. $\overline{x}$ to the variance of the estimate of μ obtained by analysis of gauging data of n items.

Similarly, the efficiency of c + a chart which is used as an indicator for σ (process standard deviation) is measured as the ratio of the variance of the estimate of σ obtained from exact measurements to the variance of the estimate of σ, obtained by the analysis of gauging data of n items. It has been observed by Stevens (1948) that these efficiencies depend on gauge points u_1 and u_2 which define gauge proportion p, q and r. A gauge with gauge proportion p, q and r is denoted by (p, r) gauge. For symmetrical gauge proportios i,e p = r or a gauge (p, p), the efficiency of c - a chart increases while that of c + a chart decreases with increase in p. The efficiencies for different (p, p) gauges were tabulated by Stevens (1948).

Any attempt to improve the efficiency of c - a chart will bring down the efficiency of c + a chart. To strike a balance, Stevens suggested gauge proportions p = r = .1041 where both c + a and c - a charts have the same efficiency of 62.74% as compared to control charts based on exact measurements. The gauge points u_1 and u_2 are so chosen that p = .1041 = $\int_{-\infty}^{u_1}$ f (x) dx and p = .1041 = $\int_{u_2}^{\infty}$ f (x) dx where f (x) denotes the p.d.f of the variable X, denoting the quality characteristic under consideration. Assuming Normal distribution, u_1 and u_2 can be easily obtained.

6.2.3 Median Gauge Chart For Odd n.

The proposed Median Gauge chart (MG) is a combination of usual median ($\tilde{X}$) control chart and Stevens' c + a chart (SG). The basis for the MG chart is provided by the following theorem.

Theorem 6.2.1 *Let X be N (μ_0, σ_0) and $\tilde{x}$ be the sample median. Suppose $u_1 = \mu_0 - t\sigma_{\tilde{x}}$ and $u_2 = \mu_0 + t\sigma_{\tilde{x}}$ are the lower and upper gauge points for a specified $t > 0$. Then for a random sample of size n where n is odd*

$$\text{Prob}\,\{a > \frac{n}{2}\} = \text{Prob}\,\{c > \frac{n}{2}\} = 1 - F(t)$$

$$\text{where } F(t) = \text{Prob}\,(\frac{\tilde{x} - \mu_0}{\sigma_{\tilde{x}}} \leq t)$$

$$\simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx (\text{assuming Normal distribution for } \tilde{x}).$$

Proof : Let $x_1, x_2 \ldots x_n$ be arranged in order of increasing magnitude so that $x_{(1)} < x_{(2)} \ldots < x_{(k)} < \ldots x_{(n)}$ where $x_{(k)}$ is the k-th order statistic. Then for odd n, $x_{([\frac{n}{2}]+1)}$ is the sample median. The proof of the theorem is trivial once we note that

$$\{\tilde{x} = x_{([\frac{n}{2}]+1)} < u_1\} \iff a > \frac{n}{2} \text{ and}$$

$$\{\tilde{x} = x_{([\frac{n}{2}]+1)} > u_2\} \iff c > \frac{n}{2}$$

Fixing the type 1 error at α would amount to setting gauge points at u_1 and u_2 as defined above, where $F(t) = 1 - \frac{\alpha}{2}$.

It is thus noted that for a sample of size n from a manufacturing process where n is odd, if c or a exceeds or equals $[\frac{n}{2}] + 1$ then the sample median $\tilde{x}$ must have violated one of the control limits of the Median control chart and hence the population mean may be considered to have changed. This Median Chart combined with Stevens' c + a chart to control variability is termed as Median Gauge chart. Thus, by restricting sample size to odd n we can do away with actual measurements of any of the items.

It is further noted that for n = 7 the 3 - sigma control limits for sample median $(\mu_0 \pm 3\sigma_{\tilde{x}})$ define symmetrical gauge points (u_1, u_2) having (p, p) = (.0844, .0844), this p being smaller than the one (.1041) suggested by Stevens for his c - a and c + a chart.

As a result, the efficiency of Median Gauge chart in detecting μ will be that of the usual Median ($\tilde{x}$) chart where as the efficiency in detecting σ will be more for this chart as p for the gauge (p, p) is smaller here.

The efficiency of the proposed Median Gauge chart (MG) and that of Stevens' Gauge chart (SG) as compared with usual control charts based on exact measurements is shown in Table 6.2.1.

Table - 6.2.1

Approximate Efficiency of MG and SG as compared with Control charts based on exact measurements

chart	sample size	Efficiency (%) MG	Efficiency (%) SG
For control of μ	7	67.90	62.74
For control of σ	7	64.50	62.74

It may be noted that the efficiency of $\tilde{x}$ chart is given by $\frac{\sigma_{\bar{x}}^2}{\sigma_{\tilde{x}}^2} \times 100$. For efficiency of c - a and c + a chart we refer to Stevens (1948).

It is, therefore, natural to expect that if we set the gauge points at u_1 and u_2 as defined for n = 7, the Median Gauge chart will have better discrimination in detecting change in μ and σ than that possessed by Stevens' c - a and c + a charts for the same sample size.

6.2.4 Procedure For Control

The procedure for control by Median Gauge chart can be outlined in the following steps :

(i) Take a random sample of 7 consecutive items.

(ii) Fix the upper gauge point u_2 as $\mu_0 + 3\,\sigma_{\tilde{x}}$ and the lower gauge point u_1 as $\mu_0 - 3\,\sigma_{\tilde{x}}$. The value of $3\,\sigma_{\tilde{x}}$ is tabulated as F for different values of n in Table 11.2 in 'Formulae and Tables for Statistical work' edited by Rao, Mitra, Mathai and Ramamurthy (1966).

(iii) Draw control limits for the chart for control of μ as

$$UCL = [\frac{n}{2}] + 1 = 4$$
$$LCL = -[\frac{n}{2}] - 1 = -4$$
$$CL = 0$$

(iv) Draw control limits for the c + a chart for the control of σ

$$UCL = n.2p + 3\sqrt{n.2p.(1-2p)}$$
$$LCL = n.2p - 3\sqrt{n.2p.(1-2p)}$$

$$CL = n.2p.$$

$$\text{where } p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{u_1-\mu_0}{\sigma_0}} e^{-t^2/2} dt = \int_{\frac{u_2-\mu_0}{\sigma_0}}^{\infty} e^{-t^2/2} dt.$$

(v) Make a go and no-go gauge with dimensions u_2 and u_1 respectively.

(vi) Count c, the number of items above the upper gauge, and a, the number of items below the lower gauge.

(vii) Determine maximum of (c, a). If maximum occurs at c plot it as +c against the subgroup in the chart for control of mean. If maximum occurs at a plot it as -a.

(viii) Lack of control is indicated by a point if it reaches or exceeds the respective control limits.

(ix) If μ_0 and σ_0 are not known decide upon the choice of μ_0 and σ_0 by process capability study using $\overline{X}$ - R chart.

(x) In the c + a chart plot the value of c + a as usual.

6.2.5 Comparison Of Performance

I. Operating Characteristic Curve (OC)

In order to compare the performance of Median Gauge chart (MG) with Stevens' (c-a, c+a) chart (SG) at the fixed gauge points (p,p) = (.1041, .1041) as recommended by him the Operating Characteristic function giving the probability of acceptance by the respective charts for the two systems are evaluated for two values of n, namely, 7 and 9.

The MG chart for n = 9 defines the gauge (p,p) where p is greater than .1041. From a study of efficiency for different values of (p,p) as given in Stevens (1948) it is expected that c+a chart in SG system should perform better than the corresponding chart in MG system. However, the performance of MG chart in controlling μ should be better than that of c-a chart in SG system.

For n=7, the performance of MG system in respect of both μ and σ is expected to be better than SG system. For purpose of comparison we have choosen n as 7 and 9 which appear to be the most convenient sizes.

We compute the probability of acceptance by the respective charts in the two systems when

(i) the process mean changes from μ_0, σ_0 remaining same, and

(ii) the process standard deviation changes from σ_0, μ_0 remaining same.

The probabilities are obtained by evaluating the trinomial frequency function

$$P(a, b, c) = \frac{n!}{a!b!c!} p^a q^b r^c$$

The performance can be compared from the OC curve as shown in Fig. 6.2.1, the top two curves represent case (i) and the two curves below represent case (ii).

The MG system of charts is found to detect a change in μ or σ more quickly than the SG system for both n = 7 and 9.

The performance of c+a in MG system for n = 9 is more or less same as that of c+a chart in SG system when change in σ is not much pronounced. As σ changes more, MG system performs better than SG system, contrary to our expectation as discussed earlier. This suggests that the performance of SG depends not only on (p,p) as suggested by Stevens and moreover the order of accuracy of Stevens' approximation procedure has not been investigated properly as yet.

In Fig. 6.2.2 we study further the performance of c+a chart at gauge points defined by MG system with n=7 and that of c+a chart at Stevens' gauge points (.1041, .1041) with n = 10. It is interesting to note that c+a chart for n = 7 under MG system is more powerful then c+a chart for n = 10 under SG system for $\frac{\sigma}{\sigma_0} \leq 1.5$. However, for $\frac{\sigma}{\sigma_0} > 1.5$, the former is found to be less powerful than the latter, as expected.

II. Joint OC surface

It will be interesting to study how the two systems behave when both (μ, σ) vary. For this we explore the joint OC surface giving the probability of acceptance by both charts in a system for different combinations of (μ, σ).

For MG system this probability is taken as

$$P[c < \frac{n}{2}, a < \frac{n}{2}, LCL_{c+a} < c + a < UCL_{c+a} \mid \mu, \sigma]$$

For SG system this is given by

$$P[LCL_{c-a} < c - a < UCL_{c-a}, LCL_{c+a} < c + a < UCL_{c+a} \mid \mu, \sigma]$$

These are given in Table 6.2.2.

It is evident that MG system is more powerful in detecting a change than the SG system for both n = 7 and 9. Recalling that the efficiency of MG chart for controlling μ and σ as 67.9(%) and 64.50(%) (see Table 6.2.1) respectively for n = 7, the equivalent sample size for $\overline{X}$ - chart is 4.75 and that for R chart is 4.51. Since in $\overline{X}$ - R chart we usually prefer a sample of size 4 or 5, we safely recommend a MG chart with n = 7 for use in practice.

6.2.6 Control of Thickness Of Refractory Ladle Brick With MG Chart - An Example.

The control of thickness of Ladle brick is very important for the life of Ladles which are used in steel plants. For this the operator used a go-no go gauge to check the thickness as soon as a brick is released from the mould by the press. The specification for thickness is 120 mm ± 1.5 mm. Both $\overline{X}$ - R charts and MG charts were tried for this process. The $\overline{X}$ - R chart with n = 5 and MG chart with n = 7 are given in Fig. 6.2.3.

A random sample of size 7 were taken from current production and the first 5 were used as a subgroup in $\overline{X}$ - R chart and all the 7 for MG chart.

It is clear from the accompanying figure that we can conveniently use a MG control chart with n = 7 without any actual measurement of the items in place of $\overline{X}$ - R control charts requiring measurements of 5 items per subgroup.

6.2.7 Control Scheme With Even n.

When the sample size n is even, the efficiency of Median Gauge chart can be still improved if we are prepared to measure a few items in some cases. Assuming we take n = 6, the median is obtained by taking the mean of $x_{(3)}$ and $x_{(4)}$. We take advantage of the fact that $\sigma_{\tilde{x}}$ for n = 6 is smaller than $\sigma_{\tilde{x}}$ for n = 7.

The efficiency of Stevens' system with gauge point (.1041, .1041) is compared with MG system for n = 6 in Table 6.2.3.

Table 6.2.3

Comparative Performance For n = 6

Type	Efficiency	
	μ	σ
SG (c-a / c+a)	62.74	62.74
MG ($\tilde{x}$ / c+a)	77.60	64.90

However, while working with even n, we will have to measure some items for the following two cases

$$\text{(i) } a = \tfrac{n}{2}, c \le \tfrac{n}{2}$$
$$\text{and (ii) } a \le \tfrac{n}{2}, c = \tfrac{n}{2}$$

Let $k = \frac{n}{2}$.

Then the probability of measuring k + i items for $1 \le i \le k-1$ is given by

$$P[a = k, c = k - i] + P[a = k - i, c = k] \tag{6.2.1}$$

$$= \frac{n!}{k!(n-2k+i)!(k-i)!} p^k q^{n-2k+i} r^{k-i}$$

$$+ \frac{n!}{(k-i)!(n-2k+i)!k!} p^{k-i} q^{n-2k+i} r^k$$

$$= \frac{n!}{k!i!(k-i)!} p^k q^i r^k \left(\frac{1}{p^i} + \frac{1}{r^i}\right) \tag{6.2.2}$$

and probability of measuring 2k items

$$= P(a = 0, c = k) + P(a = k, c = 0) + P(a = k, c = k)$$

The average number of items (ANI) which are required to be measured is given by

$$ANI = \sum_{i=1}^{k-1} (k+i).\frac{n!}{k!i!(k-i)!} p^k q^i r^k \left(\frac{1}{p^i} + \frac{1}{r^i}\right)$$
$$+ n\{P(a = \frac{n}{2}, c = \frac{n}{2}) + P(a = 0, c = \frac{n}{2}) + P(a = \frac{n}{2}, c = 0)\}$$
$$= \sum_{i=1}^{k-1} (k+i).\frac{n!}{k!i!(k-i)!} p^k q^i r^k \left(\frac{1}{p^i} + \frac{1}{r^i}\right) + n.\frac{n!}{(k!)^2}.\{q^k(p^k + r^k) + p^k r^k\} \tag{6.2.3}$$

For a given (μ_0, σ_0) and u_1, u_2, sample size n determines p, q, and r uniquely. The average number of items to be meausred depends on μ for a given σ_0 as any change in μ will alter the values of p, q and r, for given u_1 and u_2.

The following table 6.2.4 shows ANI for different values of μ for a MG chart drawn for the control of a N (0, 1) process with n = 6.

Table 6.2.4

ANI For Control OF a N (0, 1) Process (n = 6)

μ	ANI
0.0	0.10
± 1.0	1.39
± 1.5	1.83
± 2.0	0.93
± 3.0	0.02

We note that ANI (μ) does not exceed 2 in the present case.

6.2.8 Procedure For Control

Suppose n = 6.

We consider two cases : (i) $a = \frac{n}{2}$, $c \leq \frac{n}{2}$ and (ii) $a \leq \frac{n}{2}$, $c = \frac{n}{2}$. We measure the requisite number of k + 1 items (which varies between $\frac{n}{2}$ and n) to determine $x_{(3)}$ and $x_{(4)}$ and finally obtain $\tilde{x}$ as $(x_{(3)} + x_{(4)})/2$. For control, the observed value of the median should be between gauge limits u_1 and u_2.

For all other situations, the procedure is same as elaborated for odd n. The control limits for $\tilde{x}$ chart in MG system are $(\frac{n}{2}, -\frac{n}{2})$.

6.3 Optimum Inspection Interval For Control Of Bulk Damage For High Speed Printing Machine In a Textile Mill.

6.3.1 Introduction

There are many industrial situations where it is not possible to inspect the output of a manufacturing system without stopping the machine. Since all such forced stoppages may not reveal a situation requiring remedial measures it becomes important to determine the optimum inspection interval so that the total cost is minimised under the inspection scheme. The total cost comprises of

(a) the cost of unnecessary stoppages, and

(b) the cost of avoidable waste.

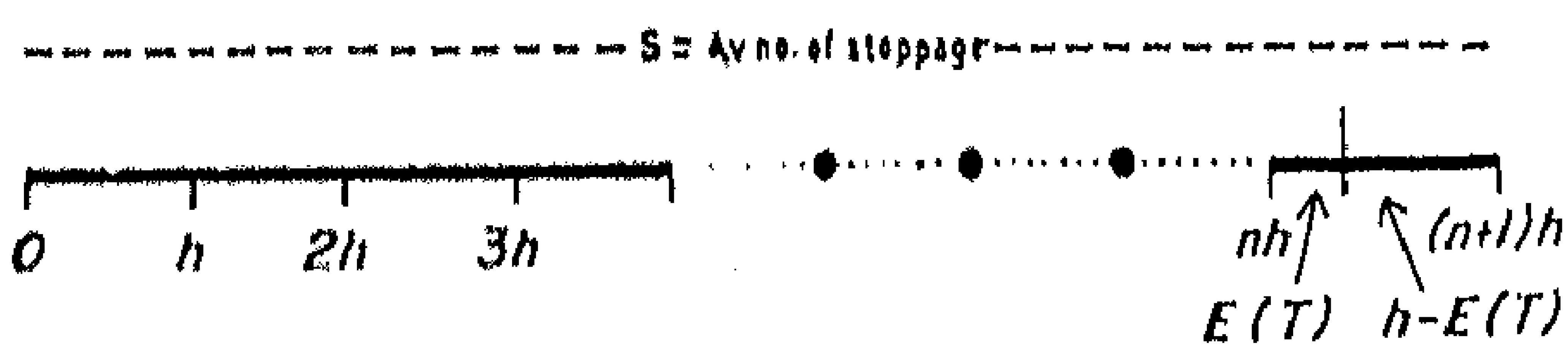
We consider a particular model. We assume that once a defect makes its appearance, the process will continue with it giving rise to waste which makes the entire production unsuitable for use till the assignable cause is eliminated. Such a situation, for example, arises in textile industry, where high speed machines are employed for printing of cloth.

The present model was developed in the context of a textile mill where it was originally applied. However, similar problems are also encountered in rolling, packaging and paper industries.

6.3.2 The Model

We assume that the arrival of the defect follows a Poisson Process with parameter λ, the average number of appearances of the defect per hour.

Let the machine be stopped for inspection at a uniform interval of h hours as shown below.



Inspection scheme at an interval of *h* hours.

Let us assume that the assignable cause for a defect occurs between the nth and $(n+1)$th time points of inspection. Also, let the random variable T denote the time that elapses until the occurrence of the defect within the interval since the beginning of the interval.

As shown by Duncan (1956), the conditional expectation, $E_n(T) = E(T \mid$ a defect occurs for the first time in the interval (nh, (n + 1)h)

$$\frac{\int_0^h x\lambda e^{-\lambda x}dx}{1-e^{-\lambda h}} = \frac{1-\lambda h e^{-\lambda h}-e^{-\lambda h}}{\lambda(1-e^{-\lambda h})} \tag{6.3.1}$$

where h = inspection interval

λ = arrival density.

$E_n(T)$ is clearly independent of n and hence the unconditional expectation E(T) = E_n (T), where E(T) denotes the expected amount of time that elapses in the interval upto the occurrence of the defect (from the start of the interval) in which the defect appears for the first time.

Let N be the random variable representing the number of stoppages until the first defect. Then

$$P\,[N = n] = \text{Prob [no defect in } (0, nh) \text{ and at least one in } (nh, (n+1)\,h)\,] = e^{-n\lambda h}(1-e^{-\lambda h}) \tag{6.3.2}$$

Thus the expected number of unnecessary stoppages

$$= \sum_0^\infty n e^{-n\lambda h}(1-e^{-\lambda h}) = \frac{(1-e^{-\lambda h})e^{-\lambda h}}{(1-e^{-\lambda h})^2} = \frac{e^{-\lambda h}}{1-e^{-\lambda h}} \tag{6.3.3}$$

Therefore, the expected number of stoppages (S) including the one where a defect is detected is given by

$$S = \frac{e^{-\lambda h}}{1-e^{-\lambda h}} + 1 = \frac{1}{1-e^{-\lambda h}} \tag{6.3.4}$$

The expected length of a cycle (a cycle being the period of time from the start of production following a defect to the detection of the next defect) = h.s.

Let us define the following :

A = The cost of stopping a machine (which includes the inspection cost and the cost due to consequent loss of production);

B = The cost of producing a meter of damaged cloth;

P = The production in meter per hour;

Then C, the expected total cost under the inspection scheme for a cycle is given by

$$\begin{aligned} C &= AS + BP(h - E(T)) \\ &= \frac{A}{1-e^{-\lambda h}} + BP\frac{\lambda h - 1 + e^{-\lambda h}}{\lambda(1-e^{-\lambda h})} \qquad (6.3.5) \end{aligned}$$

$$\text{Expected Cost per unit of time} = \frac{C}{hs} = \frac{A}{h} + \frac{BP}{\lambda h}(\lambda h - 1 + e^{-\lambda h}) \qquad (6.3.6)$$

It is easy to see that other things remaining constant expected C.P.U. (cost per unit of time) is a function of inspection interval h. The function is denoted by f(h). We now show that expected C.P.U. has a unique extremum.
Differenting f(h) w.r.t.h. we have,

$$f(h) = -\frac{A}{h^2} + \frac{BP}{\lambda h^2} + \frac{BP}{\lambda h}e^{-\lambda h}(-\lambda) - \frac{BP}{\lambda h^2}.e^{-\lambda h}$$

Equating to o we have,

$$\begin{aligned} -\lambda A + BP - \lambda hBPe^{-\lambda h} - BPe^{-\lambda h} &= 0 \\ \text{or } e^{-\lambda h}(\lambda h + 1) &= \frac{BP - \lambda A}{BP} \\ &= 1 - \frac{\lambda A}{BP} \end{aligned}$$

$$\text{writing } \lambda h = x$$

$$\text{wehave } e^{-x}(x+1) = 1 - \frac{\lambda A}{BP} \tag{6.3.7}$$

The differential coefficient of L.H.S. of (6.3.7) with respect to x gives

$$-xe^{-x} \text{which is } < 0 \text{ for all } x > 0.$$

Thus the l.h.s. which takes value 1 at x = 0 is a steadily decreasing function of x. For o $< \frac{\lambda A}{BP} < 1$, r.h.s. is a fixed const less than 1.

Thus there exists a unique x (> 0) and hence a unique h (> 0) for which f(h) has an extremum.

We note that $\underset{h \to 0}{Lt} f(h) = \infty$

and $\underset{h \to \infty}{Lt} f(h) = BP$, a finite quantity.

Further, at the initial stage as h increases from o, f(h) starts decreasing. Since f(h) has a unique extremum, there exists a unique positive h for which f(h) is minimised provided o $< \frac{\lambda A}{BP} < 1$. The condition on $\frac{\lambda A}{BP}$ is usually met with in practice.

6.3.3 Solution Procedure

The equation (6.3.7) can be solved following Newton - Rapson method. Using (6.3.7) we define

$$g(x) \text{ as } e^{-x}(x+1) + m - 1 \text{ where } m = \frac{\lambda A}{BP}.$$

$$\text{Thus } g(x) = e^{-x}(x+1) + m - 1 = 0 \tag{6.3.8}$$

Let x_n be the approximate solution of g(x) = 0 at the nth stage of iteration. Then

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$= \frac{e^{-x_n}(x_n^2 + x_n + 1) + m - 1}{x_n e^{-x_n}} \quad \text{(on simplification)}$$

The initial solution x_0 can be obtained by replacing e^{-x} by (1 - x) in (6.3.8) and this is given as

$$x_0 = \sqrt{m}$$

Hence the iterative procedure can be given as

$$x_{n+1} = \frac{e^{-x_n}(x_n^2 + x_n + 1) + m - 1}{x_n e^{-x_n}}$$
$$\text{and } x_0 = \sqrt{m}. \qquad (6.3.9)$$

6.3.4 A General Procedure For Finding h^* For a Number Of Varieties with Different Arrival Rates.

The optimum inspection interval h^* can be obtained as above for any given value of m $= \lambda A/BP$. However, even for a small factory there will be a large number of sorts [1] to deal with, having different values of λ, A, B and P. The optimum h^* in each case depends only on λ and $k = \frac{A}{BP}$, Given the values of λ and k, one should be able to find optimum h, called h^*. With that purpose, the table 6.3.1 is prepared and presented for practical use by factory personnel.

6.3.5 A Case Study

In a composite textile mill the bulk damage at printing was found to be 24.0 percent. Using the procedure described in Sections 6.3.3 and 6.3.4, the optimum inspection interval was worked out for all the *sorts* in the factory.

[1]It is the customary practice in textile industry to identify a particular variety of cloth by a *sort* number, which specifies *interalia* the quality of the fabric, the printing design as well as all other technical details related to it.

Along with short time corrective measures long term preventive measures were also planned with the experience gained from the feedback on type of defects. This brought down the values of λ for almost all the *sorts*. The values of A, B and P were not affected by these measures. With each such improvement optimum inspection interval was recalculated.

The printing performance, as reviewed after two months, is shown in Table 6.3.2.

Table - 6.3.2

Recovery Percentage In Different Grades

Category	Before (%)	After (%)
Fresh	19	51
Seconds	57	32
Damage	24	17

The effective use of optimum inspection interval reduced the percentage of damage category significantly and the consequent information feedback from each inspection (even when bulk damage was not present) revealed certain types of defects. Introductions of remedial measures along with the optimum inspection procedure brought down th percentage of 'seconds' category appreciably, resulting in a significant rise of fresh category (good quality) cloth.

The above example is cited to serve as a live demonstration of the impact of following an optimum inspection scheme for control of defects giving rise to bulk damage.

6.4 Effect Of Inspection Error On AOQ Of a Single Sampling Plan : Acceptance Rectification System.

6.4.1 Introduction

In recent times several authors have studied the nature and magnitude of inspection error and its effect on acceptance sampling plans, an account of which may be had in Dorris et. al (1978) and Mittag & Rinne (1993). Different approaches have been used to investigate the effect of inspection errors in different acceptance sampling set ups, viz., acceptance rejection, acceptance rectification and continuous sampling plans, (Case et al 1973). Some have attacked the problem from the point of view of optimization, Bauer (1987) has considered the problem of propagation and estimation of inspection errors and has studied the cost optimal acceptance rejection sampling plans in the sense of Bayes in the presence of these errors. In acceptance rectification scheme an entire lot may have to be inspected at times and hence error in inspection cannot be avoided altogether. Hence the study of effect of inspection error in case of acceptance rectification scheme assumes additional significance.

AOQ expressions for different types of sample and lot dispositions in presence of inspection error are available in Case et. al (1975) and Wortham et. al (1970).

Inspection error was found by Case and others to cause a significant change in the shape of AOQ curve. Given the inspection errors e_1 and e_2 (defined in section 6.4.2), AOQ curve first rises with increase in p, reaches an initial peak, then decreses for a while and afterwards increases monotonically as $p \to 1$. The conventional concept of the Average Outgoint Quality Limit (AOQL) is, thus, not meaningful in the presence of inspection errors.

From the practical point of view we are not interested in the whole AOQ curve. The incoming lot quality p is expected to lie within a certain range and certainly under no circumstances will be allowed to be very large.

Hence we study the effect of inspection error in details for the acceptance rectification scheme S3 - L3 as designated by Case et. al (1975) in which the apparent defectives are replaced whenever they are found either in the sample or in the lot.

What we find is that the AOQL concept is still very much applicable, provided we can assume p' to be $\leq p^u$, some upper bound dependent on e_1 and e_2. AOQ under the influence of inspection errors > stipulated AOQL, if p > p^u. It is established that if both e_1 and e_2 are equal and also less than AOQL, then $p^u > 0.5$. Several other related results are also presented.

6.4.2 Notation

The inspection error may be of two types.

e_1 = the probability of classifying a good item as a defective one;

and e_2 = the probability of classifying a defective item as a good one.

For a given sampling plan, the average outgoing quality without inspection error for an incoming lot quality p is denoted by AOQ (p). The same with inspection error is denoted by $AOQ_e(p)$. The value of the AOQL is indicated as earlier by p_L. L(p) as usual denotes the probability of acceptance of a lot of quality p assuming no inspection error. The same with inspection error is dentoed by L_e (p). P(c, p) will denote the probability of obtaining c defectives in a sample of size n for a lot of quality p, assuming no inspection error.

6.4.3 Effect Of Inspection Errors (e_1, e_2) On Average Outgoing Quality

It has been shown by Lavin (1946) and Collins et. al (1976) that due to error in inspection the probability of acceptance of a lot will be obtained by replacing the true fraction defective p by the apparent fraction defective p_e given by

$$p_e = p(1 - e_2) + (1 - p)e_1 \qquad (6.4.1)$$

where $p_e \gtreqless$ p according as

$$p \lesseqgtr \frac{e_1}{e_1 + e_2}$$

and consequently $L_e(p) = L(p_e) \lesseqgtr L(p)$

$$\text{according as } p \lesseqgtr \frac{e_1}{e_1 + e_2} \tag{6.4.2}$$

For the acceptance rectification S3 - L3 plan, where a defective is replaced as soon as it is found in the sample or lot inspection, we have from Beainy et al (1981)

$$AOQ_e(p) \;=\; pL_e(p) + \frac{p_{e_2}(1 - L_e(p))}{1 - p_e} \tag{6.4.3}$$

We have $\frac{\partial p_e}{\delta e_1} \;=\; 1$ - p and assuming Poisson approximation

$$\begin{aligned}
\frac{\partial L(p_e)}{\partial e_1} &= e^{-np_e}(1-p)\sum_{r=0}^{c}(-\frac{n^{r+1}}{r1}p_e^r + \frac{n^r}{(r-1)!}p_e^{r-1}) \\
&= -n(1-p)P(c, p_e). \\
\frac{\partial AOQ_e(p)}{\partial e_1} &= -np(1-p)P(c, p_e) \\
+p(1-p)e_2\Big[&\frac{n(1-p_e)P(c,p_e) + (1 - L_e(p))}{(1-p_e)^2}\Big] < 0 \\
&\text{for all p if } e_2 \;=\; 0
\end{aligned} \tag{6.4.4}$$

Thus we have the following lemma.

Lemma 6.4.1 *For a given S3 - L3 plan, if $e_2 = 0$ and $e_1 > 0$, AOQ_e (p) $\leq$ AOQ (p) for all $p \in \Omega$.*

Lemma 6.4.1 generalises the findings of Case, Bennett and Schmidt (1975), who establish the same for a particular value of e_1.

It will be interesting to study the behaviour of AOQ_e (p) curve in the presence of both kinds of inspection errors in relation to the AOQ (p) curve for different values of p.

It follows from (6.4.3) that AOQ_e (p) $\gtreqless$ AOQ (p) if

$$L_e(p) - L(p) + \frac{e_2(1 - L_e(p))}{1 - p_e} \gtreqless 0 \tag{6.4.5}$$

We note that for p $\geq \frac{e_1}{e_1+e_2}$, the following hold :

$$\text{(i)} \;\; p \geq p_e$$
$$\text{(ii)} \;\; L(p) \leq L_e(p) \leq 1$$
$$\text{(iii)} \;\; 1 - p_e \leq 1$$

Hence the L.H.S. in (6.4.5) ≥ 0 in this case. Thus, we have the following Lemma.

Lemma 6.4.2 *For a given S3 - L3 plan, for* $p \geq \frac{e_1}{e_1+e_2}$, $AOQ_e(p) \geq AOQ(p)$.

It thus follows from Lemma 6.4.2, that if $e_1 = 0$, then for all $p \geq 0$, AOQ_e (p) will be larger than AOQ (p). Thus the particular case considered in Case et. al (1975) is in agreement with this general result presented in Lemma 6.4.2.

Now we investigate what happens when $p < \frac{e_1}{e_1+e_2}$. $AOQ_e(p)$ can still be greater than AOQ (p) for some values of p in this range. The relevant result is presented in the form of a Lemma.

Lemma 6.4.3 *For a given S3 - L3 plan, for* $0 < p \leq \frac{e_1}{e_1+e_2}$ *and* $e_1 + e_2 \leq 1$

(i) $AOQ_e(p) < AOQ(p) \Rightarrow L(p) > \frac{e_2}{1-p_e}$

and (ii) there exists a unique p_0 *such that* $AOQ_e(p) \geq AOQ(p)$ *for all* $p \geq p_0$.

Proof It follows from (6.4.5) that

$$AOQ_e(p) \lesseqgtr AOQ(p) \text{ for } 0 \leq p \leq \frac{e_1}{e_1 + e_2} \text{ if}$$

$$L_e(p)[1 - \frac{e_2}{1 - p_e}] \lesseqgtr L(p) - \frac{e_2}{1 - p_e} \tag{6.4.6}$$

$e_1 + e_2 \leq 1 \Rightarrow \frac{e_2}{1-p_e} \leq 1$ for all p.

Hence (i) immediately follows. (This is demonstrated in Fig. 6.4.2)

Now $f(p) = L(p) - \frac{e_2}{1-p_e}$ is monotonically decreasing in p for given (e_1, e_2) with the value at p = 0 positive and the value at p = 1 negative. Hence there exists a unique p_0 such that $f(p_0) = 0$ and f(p) < 0 for all p > p_0.

On the other hand the function h(p) = $L_e(p)[1 - \frac{e_2}{1-p_e}]$ remains positive for all p $\in \Omega$. So, we can safely say that for all $p \geq p_0$, AOQ_e (p) $\geq$ AOQ (p) from (6.4.6).

6.4.4 Incoming Quality p For Which AOQ_e (p) is More Than The Specified AOQL.

Given e_1 and e_2, as p $\rightarrow$ 1, AOQ_e (p) increases steadily and as pointed out earlier, the sampling plan will not ensure an AOQL as visualised by Dodge - Romig in the perfect inspection case. However, from a study of AOQ_e (p) curve it appears that if the incoming quality lies within some particular range, the AOQ_e (p) will still not exceed the prescribed AOQL value of the sampling plan. Since in practical situations, the incoming quality p from a controlled process will lie within some known range, an attempt is made to find

(i) a set of values of incoming quality for which a specified AOQL, say p_L is not exceeded given a pair of (e_1, e_2),

(ii) the allowable range of (e_1, e_2) which can be absorbed by the sampling plan without violating the AOQL given that the incoming quality in practice will not exceed a specified value.

(a) It is difficult to determine the maximum value p^0 of p such that $AOQ_e(p) \leq p_L$ = specified value of AOQL, for all $p \leq p^0$, without knowing the elements of the sampling plan and its OC. However, some usuful approximation can be made. We have the folliwng Lemma :

Lemma 6.4.4 *For an S3 - L3 plan with* $e_1 + e_2 \leq 1$, $AOQ_e(p) \geq p_L$ *for all* $p \geq p^u$, *where* $\frac{1}{p^u} = 1 + \frac{e_2}{1-e_1} \cdot \frac{1-p_L}{p_L}$

Proof We have from (6.4.3)

AOQ_e (p) $\gtreqless p_L$ if and only if

$$L_e(p)[1 - \frac{e_2}{1-p_e}] + \frac{e_2}{1-p_e} \gtreqless \frac{p_L}{p} \qquad (6.4.7)$$

Now, $\frac{p_L}{p}$ is a decreasing function of p. The quantity $\frac{e_2}{1-p_e}$ is an increasing function lying between $\frac{e_2}{1-e_1}$ and 1 under the assumption $e_1 + e_2 \leq 1$. Hence $L_e(p)[1 - \frac{e_2}{1-p_e}]$ lies between 1 and 0. Suppose for some p^u the curve $y = \frac{e_2}{1-p_e}$ crosses the curve $y = \frac{p_L}{p}$. Then, $AOQ_e(p) \geq p_L$ for all $p \geq p^u$.

Recalling that $p_e^u = p^u(1-e_2) + (1-p^u)e_1$, we have

$$\frac{p^u}{(1-p^u) + p^u(e_1+e_2) - e_1} = \frac{p_L}{e_2}$$

$$\text{or } p^u = \frac{p_L(1-e_1)}{e_2 + p_L(1-e_1-e_2)} \qquad (6.4.8)$$

(6.4.8) can be simplified as

$$\frac{1}{p^u} = k.\frac{1-p_L}{p_L} + 1, \text{ where } k = \frac{e_2}{1-e_1} \qquad (6.4.9)$$

It is noted from Lemma 6.4.4 that if the incoming quality exceeds p^u (note that p^u does not depend on the parameters of the sampling plan and depends only on p_L and on the inspection errors e_1 and e_2 only through the ratio of e_2 to $1 - e_1$) with inspection errors, AOQL stipulation can never be met.

The exact value of the incoming quality upto which AOQ_e (p) will not exceed p_L i.e.p^0. as defined in Lemma 6.4.3 may be obtained by solving (6.4.7) numerically and obviously $p^0 \leq p^u$. p^u can be taken as the initial value of p^0 for starting the iteration. This is shown in Fig. 6.4.3.

However, we can assume, as adequately demonstrated by numerical computations in resonable circumstances (refer Table 6.4.2) that if the incoming quality p does not exceed p^u, the AOQL stipulation will be met or at least not be disturbed violently. In fact, the departure from the stipulated value of AOQL can be determined by computing the exact value of $AOQ_e(p^u)$, given the parameters of the sampling plan.

Values of p^u for different values of k and p_L are given in table 6.4.1 to illustrate how p^u changes with different values of k for a given AOQL. It is noticed from (6.4.9) that for a given p_L, p^u decreases as k increases indicating that strict control is needed to keep the value of k low during inspection.

(b) We now assume that the incoming quality of a given process does not exceed p^*. The AOQL stipulation will be met or at least not disturbed violently if

$$p^* \leq p^u = \frac{p_L(1-e_1)}{e_2 + p_L(1-e_1-e_2)} = \frac{p_L}{k(1-p_L)+p_L}$$

$$\text{which implies that } k \leq \frac{p_L(1-p^*)}{(1-p_L)p^*} \qquad (6.4.10)$$

The above equation gives us a criterion to judge whether existing inspection error can be tolerated for a manufacturing process whose incoming quality p is not expected to exceed p^*, a known quantity and which is expected to meet certain specified AOQL.

6.4.5 Comparison Of AOQL With Maximum Of AOQ_e (p) For $p \leq p^u$.

In order to study the extent of departure from stipulated AOQL, if the incoming quality is controlled within p^u, AOQ_e (p) is computed for a few single sampling plans as given by Dodge Romig for some selected AOQL. Two groups of sampling plans are considered. Three error levels for each of e_1 and e_2 are considered. They are as follows :

$$e_1 : .01, .05, .10$$
$$e_2 : .01, .05, .10$$

The result of the study is shown in Table 6.4.2. The sampling plans are grouped into two types. Details of the sampling plans under group 1 and 2 are as follows.

Group	lot size	n	c	$\bar{p}$	p_L (%)	Max AOQ_e (p)
1	31-50	30	0	0.00 - 0.01	0.5	0.50 - 1.39
	26-50	22	0	0.00 - 0.02	1.0	1.00 - 2.10
	16-50	14	0	0.00 - 0.04	2.0	2.00 - 3.60
	6-50	6	0	0.00 - 0.10	5.0	5.00 - 9.50
	4-50	3	0	0.00 - 0.20	10.0	10.00 - 20.00
2	801-1000	145	1	0.21 - 0.30	0.5	0.50 - 0.51
	801-1000	80	1	0.21 - 0.40	1.0	1.00 - 1.04
	801-1000	65	2	0.81 - 1.20	2.0	2.00 - 2.05
	801-1000	37	3	2.01 - 3.00	5.0	5.00 - 5.40
	801-1000	25	4	4.01 - 6.00	10.0	10.00 - 10.55

Maximum AOQ_e (p) as presented in the extract above is obtained as follows : Find Max of AOQ_e (p) for $p \leq p^u$ for each of the 9 level combinations of (e_1, e_2). Then the range is indicated as a-b if a is the minimum of these 9 values.

It will be seen from table 6.4.2 that for plans under group 2, the maximum of AOQ_e (p) in presence of inspection error is almost same as respective specified AOQL's

for $p \leq p^u$ and for $e_2 \leq .10$. This is, however, not true for group 1 plans which are characterised by very small lot size and acceptance number c = 0. However, for small lot sizes as in group 1, inspection error should not be a serious problem.
Since in most practical cases lot size is large or around that of group 2 plans, it can be said that if p remains smaller than p^u, the AOQL stipulation will not be seriously disturbed unless e_2 is very large.

6.4.6 Relationship between p^u and $\frac{e_1}{e_1+e_2}$

We note from (6.4.8) that

$$p^u > \frac{e_1}{e_1+e_2} \Rightarrow \frac{pL(1-e_1)}{e_2(1-pL)+pL(1-e_1)} > \frac{e_1}{e_1+e_2}$$
$$\Rightarrow e_2 pL > e_1 e_2$$
$$\Rightarrow e_1 < pL \qquad (6.4.11)$$

In fine, what are study reveals is that the application of Dodge - Romig plans in presence of inspection errors under S3 - L3 scheme can be safely recommended without causing any serious departure from stipulated AOQL if p^u is reasonably large and p can be safely assumed to be $\leq p^u$.
For a given e_1, $\frac{e_1}{e_1+e_2}$ can be made large by keeping e_2 small and it is noted that if e_1 and e_2 are of the same order and less than pL, the value of p^u exceeds 0.50 which is already too large an upper bound for the incoming quality p of a controlled process. As far as the practical applications of Dodge - Romig single sampling plans are concerned, the AOQL specifications are hardly affected as long as the lot sizes are reasonably large and $e_1 \simeq e_2 < pL$ and the process does not become too erratic resulting in an incoming lot quality $p > 0.5$.
It may be noted that the investigation has been carried out under the S3 - L3 scheme of inspection. The other schemes mentioned by Case et. al (1981)

can also be similarly studied. The general features observed in the present study are expected to be preserved by all the schemes.

OC curve of SG and MG chart

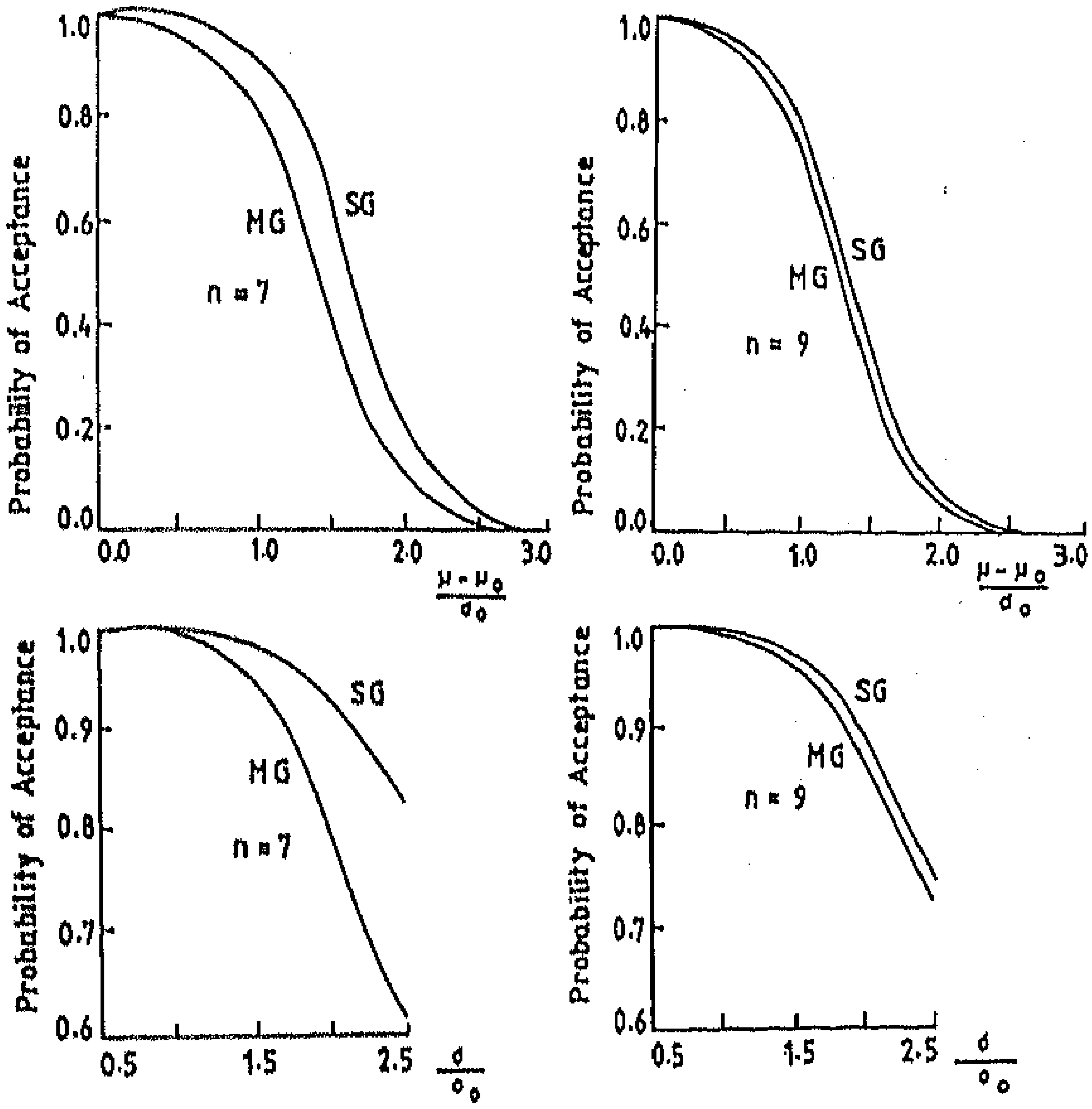


Fig. 6.2.1

OC curve for c + a chart under SG (n = 10) and MG (n = 7) system

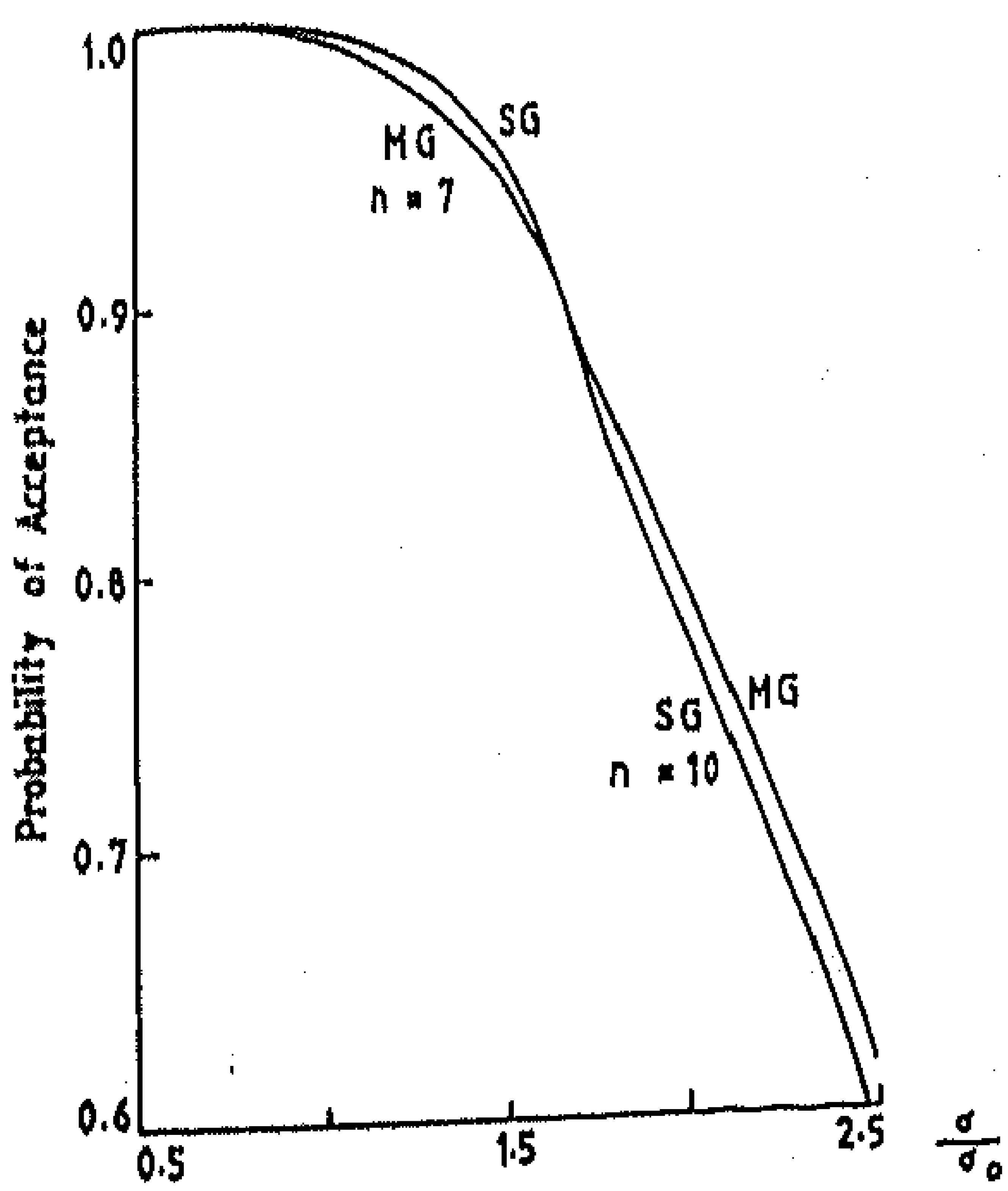
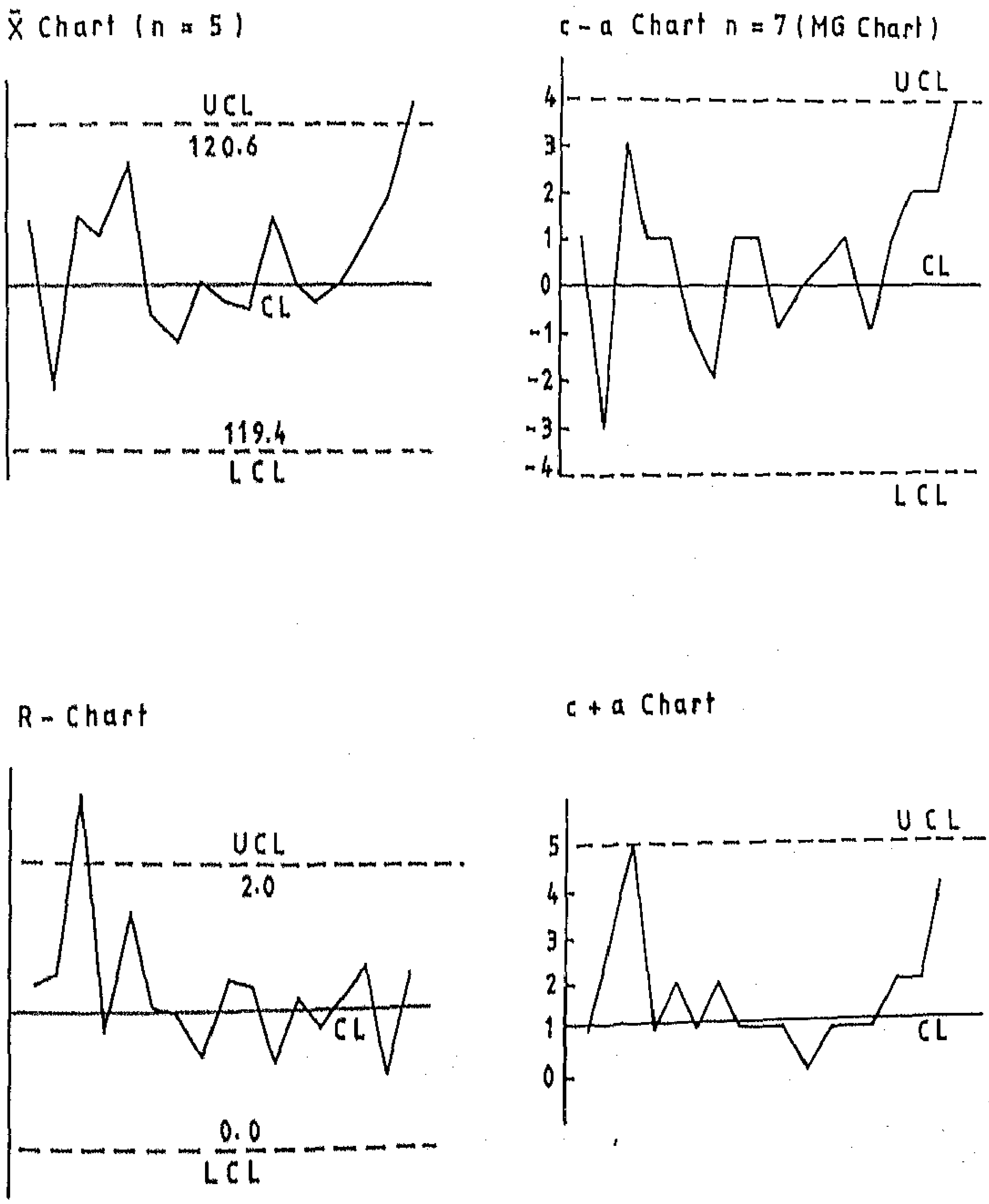


Fig. 6.2.2

Control of thickness:
Comparison of $\bar{X}$ - R charts:
with
MG charts (n = 7)



$$\text{Upper gauge point} = 120.0 + 3\sigma_{\bar{X}} = 120.69$$
$$\text{Lower gauge point} = 120.0 - 3\sigma_{\bar{X}} = 119.31$$

Fig. 6.2.3

Diagram of $\frac{e_2}{1-p_e}$ and $L(p)$

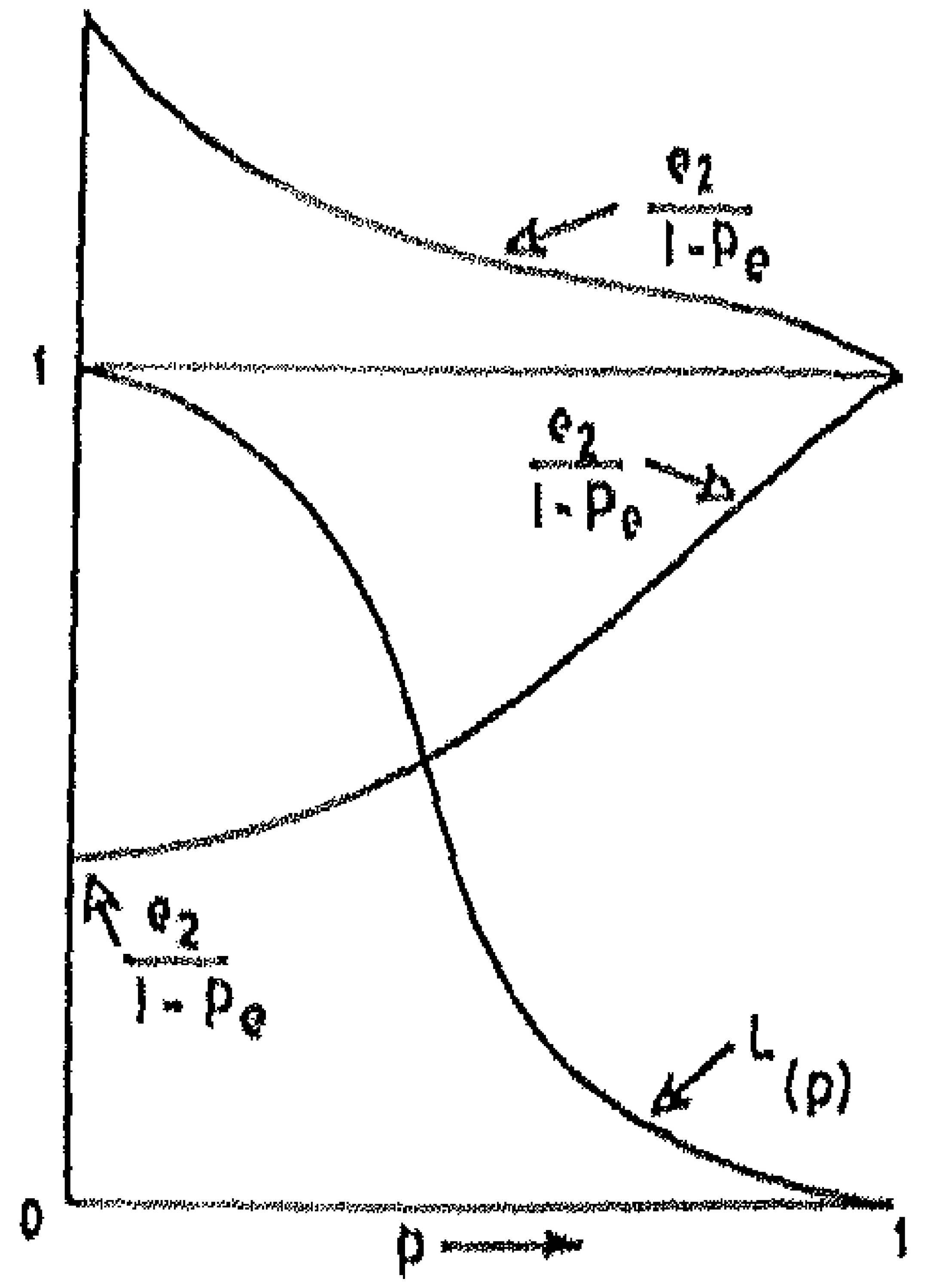


Fig. 6.4.2

Diagram of $\frac{p_L}{p}, L(p_e)\left(1-\frac{e_2}{1-p_e}\right)+\frac{e_2}{1-p_e}, \frac{e_2}{1-p_e}$

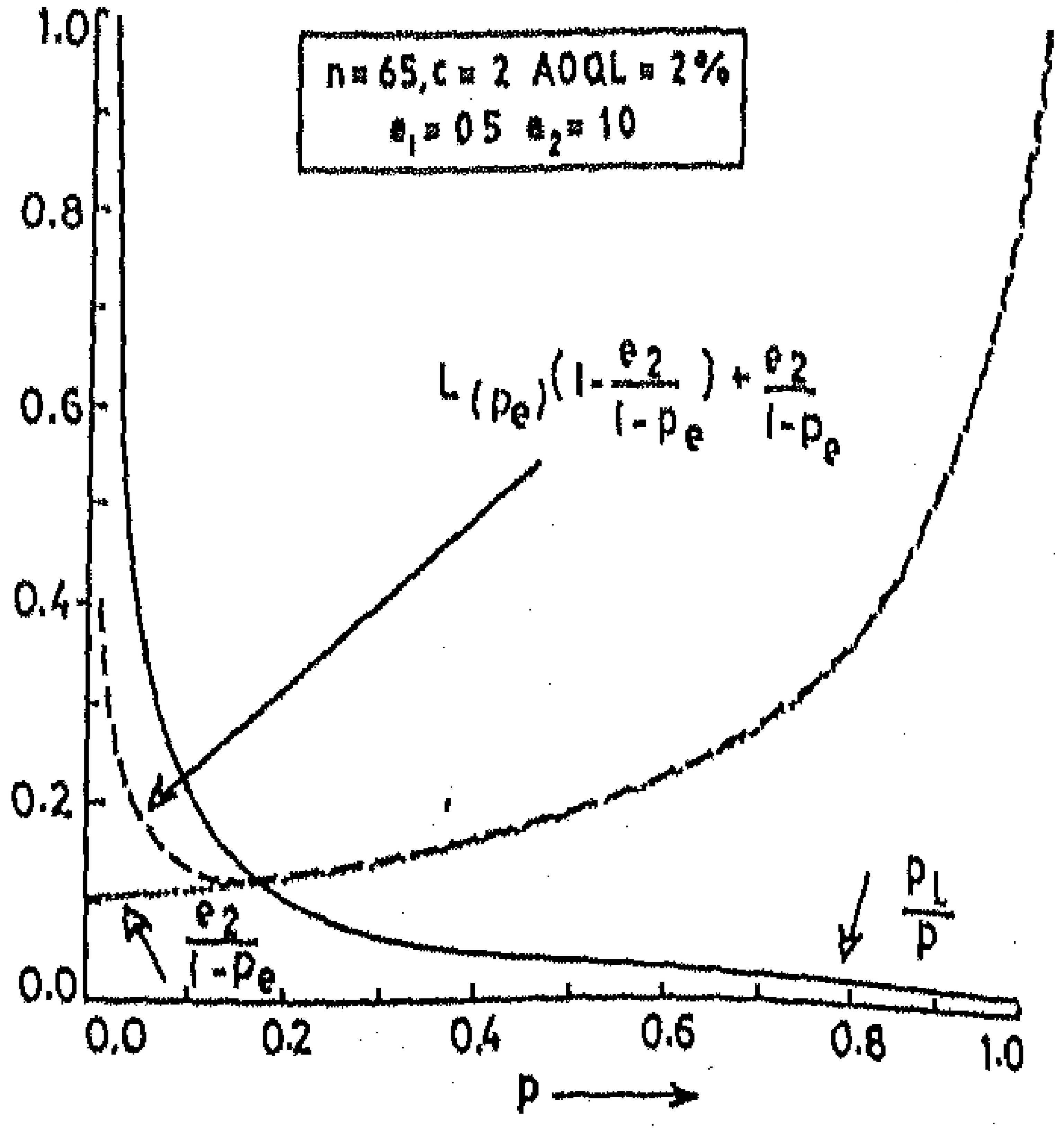


Fig. 6.4.3

TABLE 6.2.2 : JOINT OC SURFACE

n	$\frac{\sigma}{\sigma_0}$ / $\frac{\mu-\mu_0}{\sigma_0}$	Probability of acceptance									
		0.5		1.0		1.5		2.0		2.5	
		SG	MG	SG	MG	SG	MG	SG	MG	SG	MG
7	0.0	1.00000	1.00000	0.99930	0.99582	0.97971	0.91891	0.92120	0.75366	0.84290	0.58133
	0.5	0.99998	0.99992	0.99248	0.97082	0.96323	0.86592	0.90617	0.71073	0.83297	0.55372
	1.0	0.97026	0.94948	0.90885	0.79412	0.88220	0.69269	0.84860	0.58714	0.79746	0.47598
	1.5	0.38695	0.29380	0.59420	0.39308	0.68386	0.42282	0.72695	0.41027	0.72466	0.36435
	2.0	0.00937	0.00339	0.20344	0.08660	0.40510	0.18064	0.54299	0.23387	0.60968	0.24442
	2.5	0.00001	0.00000	0.03123	0.00667	0.17734	0.05109	0.34116	0.10591	0.046465	0.14168
	3.0	0.00000	0.00000	0.00212	0.00023	0.05021	0.00928	0.17667	0.03753	0.31598	0.07794
9	0.0	1.00000	1.00000	0.99829	0.99683	0.96240	0.93869	0.86245	0.80178	0.73657	0.62456
	0.5	0.99989	0.99978	0.97302	0.96174	0.91500	0.86786	0.82405	0.74107	0.71648	0.60524
	1.0	0.89844	0.86436	0.75037	0.70834	0.72767	0.63825	0.69905	0.57268	0.63725	0.49411
	1.5	0.11811	0.08678	0.28376	0.24782	0.41665	0.32351	0.49526	0.35383	0.51212	0.34528
	2.0	0.00015	0.00008	0.03548	0.02837	0.14433	0.10146	0.27491	0.16643	0.35821	0.20244
	2.5	0.00000	0.00000	0.00127	0.00092	0.02944	0.01852	0.11494	0.05796	0.21265	0.09799
	3.0	0.00000	0.00000	0.00000	0.00001	0.00340	0.00191	0.03555	0.01472	0.10551	0.03876

Table - 6.3.1

Values of h^* (in hour) for different k and λ

	λ							
k	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.02	0.414	0.297	0.245	0.215	0.194	0.178	0.166	0.157
0.04	0.594	0.429	0.357	0.314	0.284	0.263	0.246	0.233
0.06	0.736	0.535	0.445	0.394	0.359	0.333	0.313	0.297
0.08	0.859	0.627	0.526	0.466	0.425	0.396	0.373	0.356
0.10	0.969	0.711	0.598	0.532	0.487	0.455	0.431	0.412
0.12	1.070	0.788	0.666	0.594	0.547	0.512	0.487	0.467
0.14	1.165	0.862	0.730	0.654	0.604	0.568	0.542	0.521
0.16	1.254	0.931	0.792	0.712	0.659	0.623	0.596	0.576
0.18	1.340	0.999	0.852	0.769	0.714	0.677	0.650	0.631
0.20	1.421	1.064	0.911	0.824	0.769	0.732	0.706	0.688
0.22	1.500	1.127	0.968	0.869	0.823	0.786	0.762	0.747
0.24	1.577	1.188	1.025	0.934	0.878	0.842	0.820	0.808
0.26	1.651	1.249	1.081	0.988	0.933	0.899	0.879	0.872
0.28	1.723	1.308	1.136	1.043	0.988	0.956	0.941	0.939
0.30	1.794	1.366	1.191	1.097	1.044	1.016	1.005	1.011
0.32	1.863	1.424	1.245	1.152	1.101	1.077	1.073	1.089
0.34	1.931	1.481	1.300	1.207	1.159	1.140	1.145	1.174
0.36	1.977	1.537	1.354	1.263	1.219	1.206	1.221	1.268
0.38	2.063	1.593	1.409	1.319	1.280	1.275	1.304	1.374
0.40	2.127	1.649	1.463	1.376	1.343	1.348	1.394	1.497
0.42	2.191	1.704	1.518	1.434	1.408	1.425	1.493	1.644
0.44	2.254	1.759	1.573	1.493	1.475	1.507	1.604	1.829
0.46	2.316	1.814	1.628	1.554	1.545	1.595	1.730	2.084
0.48	2.377	1.868	1.684	1.615	1.618	1.691	1.879	2.506

TABLE 6.4.1 : VALUES OF $100p^u$, FOR DIFFERENT VALUES OF p_L AND K

$AOQL_k$(%)	.10	.25	.50	.75	1.00	1.50	2.00	2.50	3.00	4.00	5.00	7.00	10.00
.01	9.10	20.04	33.44	43.04	50.25	60.36	67.11	71.94	75.57	80.65	84.03	88.27	91.70
.02	4.77	11.14	20.08	27.42	33.56	43.22	50.50	56.18	60.73	67.57	72.46	79.01	84.75
.03	3.23	7.71	14.35	20.12	25.19	33.67	34.01	46.08	50.76	58.14	63.69	71.50	78.74
.04	2.44	5.90	11.16	15.89	20.16	27.57	33.78	39.06	43.60	51.02	56.82	65.30	73.53
.05	1.96	4.77	9.13	13.13	16.81	23.35	28.99	33.90	38.22	45.45	51.28	60.09	68.96
.06	1.64	4.01	7.73	11.19	14.41	20.24	25.38	29.94	34.01	40.98	46.73	55.64	64.94
.07	1.41	3.46	6.70	9.74	12.61	17.87	22.57	26.81	30.64	37.31	42.91	51.81	61.35
.08	1.24	3.04	5.91	8.63	11.21	15.99	20.32	24.27	27.88	34.25	39.68	48.48	58.14
.09	1.10	2.71	5.29	7.75	10.09	14.47	18.48	22.17	25.58	31.65	36.90	45.54	55.25
.10	0.99	2.45	4.78	7.03	9.17	13.22	16.95	20.41	23.62	29.41	34.48	42.94	52.63
.11	0.90	2.23	4.37	6.43	8.41	12.16	15.56	18.90	21.95	27.47	32.36	40.63	50.25
.12	0.83	2.05	4.02	5.92	7.76	11.26	14.53	17.61	20.49	25.77	30.49	38.49	48.08
.13	0.76	1.89	3.72	5.49	7.21	10.49	13.57	16.47	19.22	24.27	28.82	36.67	46.08
.14	0.71	1.76	3.46	5.12	6.73	9.81	12.72	15.48	18.09	22.94	27.32	34.97	44.25
.15	0.66	1.64	3.24	4.80	6.31	9.22	11.98	14.50	17.09	21.74	25.97	33.41	42.55
.16	0.62	1.54	3.05	4.51	5.94	8.69	11.31	13.81	16.20	20.66	24.75	31.99	40.98
.17	0.59	1.44	2.87	4.26	5.61	8.22	10.72	13.11	15.39	19.68	23.64	30.69	39.53
.18	0.55	1.37	2.72	4.03	5.31	7.80	10.18	12.47	14.66	18.80	22.62	29.49	38.17
.19	0.52	1.30	2.58	3.83	5.05	7.42	9.70	11.89	14.00	17.99	21.69	28.37	36.90
.20	0.50	1.23	2.45	3.64	4.81	7.08	9.26	11.36	13.39	17.24	20.83	27.34	35.71
.21	0.47	1.18	2.34	3.47	4.59	6.76	8.86	10.88	12.84	16.56	20.04	26.38	34.60
.22	0.45	1.13	2.23	3.32	4.39	6.47	8.49	10.44	12.32	15.92	19.30	25.49	33.56
.23	0.43	1.08	2.14	3.18	4.21	6.21	8.15	10.03	11.85	15.34	18.62	24.65	32.57
.24	0.42	1.03	2.05	3.05	4.04	5.97	7.84	9.65	11.41	14.79	17.98	23.87	31.65
.25	0.40	0.99	1.97	2.93	3.88	5.74	7.55	9.30	11.00	14.29	17.39	23.14	30.77

TABLE 6.4.2 : COMPARISON OF MAXIMUM OF AOQ WITH AOQL FOR $p < p^u$ FOR SOME PLANS FOR DIFFERENT ERROR LEVELS.

AOQL (%)	e_1	e_2 = .01 max.of AOQ(%) PL_1	PL_2	$100p^u$		e_2 = .05 max.of AOQ(%) PL_1	PL_2	$100p^u$		e_2 = .10 max.of AOQ(%) PL_1	PL_2	$100p^u$
0.5	.01	0.95	0.50	33.22		1.12	0.50	9.05		1.39	0.51	4.74
1.0	.01	1.40	1.00	50.00		1.62	1.00	16.67		2.10	1.04	9.09
2.0	.01	2.38	2.00	66.89		2.70	2.00	28.18		3.60	2.05	16.81
5.0	.01	6.03	5.00	83.90		7.40	5.00	51.03		9.50	5.08	34.25
10.0	.01	12.50	10.00	91.66		18.00	10.00	68.75		20.00	10.55	52.38
0.5	.05	0.50	0.50	32.31		0.67	0.50	8.72		0.77	0.50	4.56
1.0	.05	1.00	1.00	48.97		1.20	1.00	16.10		1.51	1.00	8.76
2.0	.05	2.00	2.00	65.98		2.36	2.00	27.94		3.03	2.00	16.24
5.0	.05	5.00	5.00	83.33		7.24	5.00	50.00		8.70	5.40	33.33
10.0	.05	15.00	10.00	91.35		18.50	10.00	67.85		19.90	10.00	51.35
0.5	.10	0.50	0.50	31.14		0.52	0.50	8.29		0.56	0.50	4.33
1.0	.10	1.00	1.00	47.62		1.09	1.00	15.38		1.15	1.00	8.33
2.0	.10	2.00	2.00	64.75		2.15	2.00	26.87		2.61	2.00	15.52
5.0	.10	5.00	5.00	82.75		7.00	5.00	48.65		8.20	5.00	32.14
10.0	.10	16.00	10.00	90.91		18.75	10.00	66.67		18.92	10.20	50.00

APPENDIX - 1

Values of optimum (i, f) for CSP - 1 Minimising AFI at $\bar{p}$ and ensuring a given AOQL.

Process average (%)	AOQL % 0.5	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
0.5	198 .1203	*	*	*	*	*	*	*	*	*	*
1.0	198 .1203	98 .1213	*	*	*	*	*	*	*	*	*
2.0	65 .4466	98 .1213	97 .0254	*	*	*	*	*	*	*	*
3.0	39 .6039	48 .3167	97 .0254	96 .0066	*	*	*	*	*	*	*
4.0	27 .6993	32 .4483	48 .1235	96 .0066	95 .0019	*	*	*	*	*	*
5.0	21 .7540	24 .5387	32 .2252	48 .0538	95 .0019	94 .0006	*	*	*	*	*
6.0	17 .7935	19 .6068	24 .3118	31 .1281	47 .0266	94 .0006	93 .0002	*	*	*	*
7.0	14 .8248	16 .6527	19 .3864	23 .2008	31 .0733	46 .0140	93 .0002	92 66×10^{-6}	*	*	*
8.0	12 .8465	13 .7031	15 .4621	18 .2709	23 .1281	31 .0433	46 .0071	92 66×10^{-6}	91 23×10^{-6}	*	*
9.0	11 .9577	11 .7393	13 .5067	15 .3271	18 .1863	23 .0838	30 .0287	46 .0037	91 23×10^{-6}	91 8×10^{-6}	*
10.0	9 .8805	10 .7583	11 .5566	13 .3723	15 .2360	18 .1305	23 .0558	30 .0178	45 .0022	90 8×10^{-6}	98 3×10^{-6}

* Use the first available plan below in the column.

APPENDIX - 1 (Contd.)

Values of optimum (i, f) for CSP - 1 Minimising AFI at $\bar{p}$ and ensuring a given AOQL.

process average (%)	AOQL % 0.5	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
11.0	8	9	10	11	13	15	18	22	30	45	89
	.8922	.7780	.5839	.4253	.2779	.1728	.0928	.0422	.0112	.0012	3×10^{-6}
12.0	8	8	9	10	11	13	15	18	22	29	44
	.8922	.7982	.64128	.4552	.3289	.2100	.1282	.0668	.0292	.0081	.0007
0 13.0	7	7	8	9	10	11	12	15	17	22	29
	.9041	.8192	.6435	.4878	.3587	.2570	.1800	.0960	.0553	.0203	.0053
14.0	6	7	7	8	9	10	11	12	14	17	21
	.9162	.8192	.6762	.5232	.3917	.2851	.2024	.1407	.0833	.0410	.0166
15.0	6	6	7	7	8	9	9	11	12	14	17
	.9162	.8408	.6762	.5619	.4285	.3169	.2580	.1606	.1107	.0641	.0306
16.0	5	6	6	6	7	8	8	9	11	12	14
	.9286	.8408	.7109	.6041	.4696	.3531	.2926	.2113	.1282	.0876	.0497
17.0	5	5	6	6	6	7	8	8	9	10	12
	.9286	..8631	.7109	.6041	.5156	.3944	.2926	.2435	.1738	.1213	.0696
18.0	5	5	5	5	6	6	7	7	8	9	10
	.9286	.8631	.7479	.6503	.5156	.4417	.3328	.2818	.2036	.1436	.0991
19.0	4	5	5	5	5	6	6	7	7	8	9
	.9411	.8631	.7479	.6503	.5672	.4417	.3798	.2818	.2395	.1707	1190
20.0	4	4	4	5	5	5	6	6	7	7	8
	.9411	.8862	.7873	.6503	.5672	.4062	.3798	.3275	.2395	.2041	.1437

Appendix 2

CSP-2

Optimum (i,f) for given values of AOQL (%) and Process Average ($\bar{p}$)

Process Average (%)	AOQL (%)														
	1.0			2.0			3.0			4.0			5.0		
	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K==i+10
1.0	104 .1888	105 .1840	105 .1824	*	*	*	*	*	*	*	*	*	*	*	*
2.0	104 .1888	105 .1840	105 .1824	99 .0460	99 .0456	100 .0441	*	*	*	*	*	*	*	*	*
3.0	54 .4188	54 .4110	55 .3993	99 .0460	99 .0456	100 .0441	97 .0125	97 .0124	97 .0124	*	*	*	*	*	*
4.0	37 .5542	37 .5410	37 .5334	51 .1943	51 .1896	52 .1810	97 .0125	97 .0124	97 .0124	95 .0038	95 .0038	95 .0038	*	*	*
5.0	28 .6446	28 .6264	28 .6173	35 .3256	35 .3136	35 .3074	49 .0962	50 .0898	50 .0886	95 .0038	95 .0038	95 .0038	94 .0012	94 .0012	94 .0012
6.0	22 .7145	22 .6909	22 .6809	26 .4403	26 .4188	27 .3966	33 .2044	34 .1864	34 .1826	48 .0486	48 .0476	49 .0445	94 .0012	94 .0012	94 .0012
7.0	19 .7535	18 .7376	19 .7152	21 .5237	21 .4925	21 .4812	25 .3042	25 .2853	26 .2653	33 .1286	33 .1154	33 .1133	47 .0258	47 .0253	48 .0234
8.0	17 .7814	16 .7621	16 .7516	18 .5835	18 .5431	18 .5306	20 .3949	20 .3630	21 .3365	24 .2169	25 .1891	25 .1844	31 .0839	32 .0743	32 .0731
9.0	15 .8121	14 .7874	14 .7769	16 .6292	15 .5991	15 .5853	17 .4658	17 .4199	17 .4079	20 .2860	20 .2595	20 .2519	24 .1469	24 .1363	24 .1329
10.0	13 .8470	12 .8136	12 .8032	14 .6823	13 .6397	13 .6251	15 .5239	15 .4630	15 .4496	17 .3560	17 .3145	17 .3047	19 .2256	19 .2013	20 .1807

* Use the first available plan below in the column.

Appendix 2 (Contd.)

CSP-2

Optimum (i,f) for given values of AOQL (%) and Process Average ($\bar{p}$)

Process Average (%)	AOQL (%)														
	6.0			7.0			8.0			9.0			10.0		
	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10
1.0	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
2.0	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
3.0	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
4.0	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
5.0	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
6.0	93 .0004	93 .0004	93 .0004	*	*	*	*	*	*	*	*	*	*	*	*
7.0	93 .0004	93 .0004	93 .0004	92 .0001	93 .0001	92 .0001	*	*	*	*	*	*	*	*	*
8.0	46 .0142	46 .0140	47 .0129	92 .0001	93 .0001	92 .0001	91 $47x10^{-6}$	91 $47x10^{-6}$	91 $47x10^{-6}$	*	*	*	*	*	*
9.0	31 .0518	31 .0497	32 .0449	46 .0074	47 .0074	46 .0073	91 $47x10^{-6}$	91 $47x10^{-6}$	91 $47x10^{-6}$	90 $17x10^{-6}$	90 $17x10^{-6}$	90 $17x10^{-6}$	*	*	*
10.0	23 .1100	24 .0927	24 .0907	30 .0357	31 .0311	31 .0307	45 .0044	45 .0043	45 .0043	90 $17x10^{-6}$	90 $17x10^{-6}$	90 $17x10^{-6}$	90 $6x10^{-6}$	89 $6x10^{-6}$	90 $6x10^{-6}$

* Use the first available plan below in the column.

Appendix 3

CSP-3

Optimum (i,f) for given values of AOQL (%) and Process Average ($\bar{p}$)

Process Average (%)	AOQL (%)														
	1.0			2.0			3.0			4.0			5.0		
	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10	K=i-10	K=i	K=i+10
1.0	105 .184817	105 .183111	106 .179018	*	*	*	*	*	*	*	*	*	*	*	*
2.0	105 .184817	105 .183111	106 .179018	99 .045833	100 .044239	100 .044018	*	*	*	*	*	*	*	*	*
3.0	54 .413762	55 .401112	55 .397013	99 .045833	100 .044239	100 .044018	97 .012466	97 .012416	97 .012384	*	*	*	*	*	*
4.0	37 .544909	37 .535597	37 .530091	51 .191483	52 .182146	52 .180019	97 .012466	97 .012416	97 .012384	95 .003802	95 .003794	95 .003789	*	*	*
5.0	28 .630863	28 .619496	28 .613613	35 .317656	35 .309527	36 .295760	49 .094923	50 .089142	50 .088244	95 .003802	95 .003794	95 .003789	94 .001189	94 .001188	94 .001187
6.0	23 .684339	22 .682804	23 .666049	27 .411074	27 .399122	27 .393580	34 .189438	34 .184030	34 .181433	48 .048072	48 .047335	49 .044365	94 .001189	94 .001188	94 .001187
7.0	19 .730224	19 .716904	19 .711575	22 .483425	22 .468456	22 .462351	26 .277186	26 .267375	26 .263289	33 .117501	33 .114148	33 .112686	47 .025504	47 .025211	48 .023394
8.0	16 .766398	16 .752756	16 .748062	18 .550385	18 .532832	18 .526689	21 .352844	21 .338851	21 .333789	25 .193467	25 .185973	25 .183177	32 .075694	32 .073642	32 .072831
9.0	14 .791224	14 .777684	14 .773621	16 .587106	16 .568376	16 .562476	17 .428378	17 .410255	18 .385586	20 .266126	20 .253950	20 .250052	24 .139693	24 .134059	25 .122542
10.0	13 .803777	12 .803475	12 .800264	14 .625955	14 .606371	14 .600986	15 .471879	15 .451648	15 .446167	17 .322817	17 .306823	17 .302348	19 .207354	20 .182242	20 .179526

* Use the first available plan below in the column.

Appendix - 4

Optimum (i, f) Minimising AFI at $\bar{p}$ Ensuring LQL and an AOQL $\le$ a given p_L.

Incoming	LQL = 5.0%						LQL = 10.0%					
Process	$p_L \le 1.0$		$p_L \le 2.0$		$p_L \le 3.0$		$p_L \le 2.0$		$p_L \le 3.0$		$p_L \le 4.0$	
average (%)	i	f	i	f	i	f	i	f	i	f	i	f
1.0	77	.18024	136	.00842	146	.00504	37	.19011	49	.052213	65	.00957
2.0	77	.18024	136	.00842	146	.00504	37	.19011	49	.052213	65	.00957
3.0	77	.18024	136	.00842	146	.00504	37	.19011	49	.052213	65	.00957
4.0	77	.18024	136	.00842	146	.00504	37	.19011	49	.052213	65	.00957
5.0	77	.18024	136	.00842	146	.00504	37	.19011	49	.052213	65	.00957
6.0	77	.18024	81	.14574	81	.14574	37	.19011	49	.052213	65	.00957
7.0	71	.24870	71	.24870	71	.24870	37	.19011	49	.052213	65	.00957
8.0	65	.34499	65	.34499	65	.34498	37	.19011	49	.052213	65	.00957
9.0	62	.40737	62	.40737	62	.40737	37	.19011	49	.052213	65	.00957
10.0	60	.45564	60	.45564	60	.45564	37	.19011	49	.052213	65	.00957
11.0	59	.48207	59	.48207	59	.48207	37	.19011	45	.07994	45	.07994
12.0	59	.48207	59	.48207	59	.48207	37	.19011	39	.15279	39	.15279
13.0	59	.48207	59	.48207	59	.48207	36	.21221	36	.21221	36	.21221
14.0	59	.48207	59	.48207	59	.48207	34	.26484	34	.26484	34	.26484
15.0	59	.48207	59	.48207	59	.48207	33	.29615	33	.29615	33	.29615
16.0	59	.48207	59	.48207	59	.48207	32	.33140	32	.33140	32	.33140
17.0	59	.48207	59	.48207	59	.48207	31	.37115	31	.37115	31	.37115
18.0	59	.48207	59	.48207	59	.48207	30	.41605	30	.41605	30	.41605
19.0	59	.48207	59	.48207	59	.48207	29	.46685	29	.46685	29	.46685
20.0	59	.48207	59	.48207	59	.48207	29	.46685	29	.46685	29	.46685

Appendix - 4 (Contd.)

Optimum (i, f) Minimising AFI at $\bar{p}$ Ensuring LQL and an AOQL ≤ a given p_L.

Incoming Process average (%)	LQL = 10.0%				LQL = 15.0%							
	$p_L \le 5.0$		$p_L \le 2.0$		$p_L \le 3.0$		$p_L \le 4.0$		$p_L \le 5.0$		$p_L \le 6.0$	
	i	f	i	f	i	f	i	f	i	f	i	f
1.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
2.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
3.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
4.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
5.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
6.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
7.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
8.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
9.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
10.0	71	.00508	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
11.0	45	.07994	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
12.0	39	.15279	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
13.0	36	.21221	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
14.0	34	.26484	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
15.0	33	.29615	20	.37753	24	.18969	28	.09709	34	.03614	41	.01152
16.0	32	.33140	20	.37753	24	.18969	28	.09709	31	.05914	31	.05914
17.0	31	.37115	20	.37753	24	.18969	27	.11466	27	.11466	27	.11466
18.0	30	.41605	20	.37753	24	.18969	25	.16024	25	.16024	25	.16024
19.0	29	.46685	20	.37753	24	.18969	24	.18969	24	.18969	24	.18969
20.0	29	.46685	20	.37753	23	.22480	23	.22480	23	.22480	23	.22480

Appendix - 4 (Contd.)

Optimum (i, f) Minimising AFI at $\bar{p}$ Ensuring LQL and an AOQL ≤ a given p_L.

Incoming	LQL = 15.0%							LQL = 20.0%				
Process	$p_L \leq 7.0$		$p_L \leq 3.0$		$p_L \leq 4.0$		$p_L \leq 5.0$		$p_L \leq 6.0$		$p_L \leq 7.0$	
average (%)	i	f	i	f	i	f	i	f	i	f	i	f
1.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
2.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
3.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
4.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
5.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
6.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
7.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
8.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
9.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
10.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
11.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
12.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
13.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
14.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
15.0	46	.00510	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
16.0	31	.05914	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
17.0	27	.11466	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
18.0	25	.16024	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
19.0	24	.18969	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737
20.0	23	.22480	15	.34018	17	.21211	19	.13352	22	.06740	26	.02737

Appendix - 4 (Contd.)

Optimum (i, f) Minimising AFI at $\bar{p}$ Ensuring LQL and an AOQL ≤ a given p_L.

Incoming Process average (%)	LQL = 20.0%				LQL = 25.0%							
	$p_L \leq 8.0$		$p_L \leq 9.0$		$p_L \leq 4.0$		$p_L \leq 5.0$		$p_L \leq 6.0$		$p_L \leq 7.0$	
	i	f	i	f	i	f	i	f	i	f	i	f
1.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
2.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
3.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
4.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
5.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
6.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
7.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
8.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
9.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
10.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
11.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
12.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
13.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
14.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
15.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
16.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
17.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
18.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
19.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204
20.0	29	.01397	33	.00571	12	.30404	13	.22435	14	.16623	16	.09204

Appendix - 4 (Contd.)

Optimum (i, f) Minimising AFI at $\bar{p}$ Ensuring LQL and an AOQL $\leq$ a given p_L.

Incoming Process average (%)	LQL = 25.0%								LQL = 32.0%			
	$p_L \leq 8.0$		$p_L \leq 9.0$		$p_L \leq 10.0$		$p_L \leq 12.5$		$p_L \leq 4.0$		$p_L \leq 5.0$	
	i	f	i	f	i	f	i	f	i	f	i	f
1.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
2.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
3.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
4.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
5.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
6.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
7.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
8.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
9.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
10.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
11.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
12.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
13.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
14.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
15.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
16.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
17.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
18.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
19.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181
20.0	18	.05132	20	.02872	22	.01611	26	.00509	8	.45181	8	.45181

Appendix - 4 (Contd.)

Optimum (i, f) Minimising AFI at $\bar{p}$ Ensuring LQL and an AOQL $\leq$ a given p_L.

Incoming	LQL = 32.0%											
Process	$p_L \leq 6.0$		$p_L \leq 7.0$		$p_L \leq 8.0$		$p_L \leq 10.0$		$p_L \leq 12.5$		$p_L \leq 15.0$	
average (%)	i	f	i	f	i	f	i	f	i	f	i	f
1.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
2.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
3.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
4.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
5.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
6.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
7.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
8.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
9.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
10.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
11.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
12.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
13.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
14.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
15.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
16.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
17.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
18.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
19.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592
20.0	9	.29803	10	.19856	11	.13317	13	.06062	16	.01889	19	.00592

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