EXTENSIONS OF GROMOV THEORY

BY
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CONTENTS

SOME CONVENTIONS AND NOTATIONS

We adopt the following conventions and notations throughout the thesis (though some inconsistencies may persist).

All our manifolds are finite dimensional, smooth (C^{∞}), paracompact, Hausdorff spaces with or without boundary. A manifold X is closed if it is compact and without boundary ($\partial X = \varphi$), and X is non-closed (also referred to in the literature as open) if no connected component of X is closed. Maps between manifolds are always smooth, and the topology of a space of mappings is the C^{∞} topology of compact convergence unless explicitly stated otherwise.

If X and Y are manifolds, $C^{\infty}(X,Y)$ will denote the space of smooth maps $X \longrightarrow Y$, $Emb(X,Y)$ will be the subspace of embeddings, $Imm(X,Y)$ will be the subspace of immersions, $Sub(X,Y)$ will be the subspace of submersions, $Diff(X,Y)$ will be the subspace of diffeomorphisms, $Mer_k(X,Y)$ will be the subspace of k-mersions, $k \leq \min(\dim X, \dim Y)$, which are maps of rank at least k everywhere, $Subimm_k(X,Y)$ will be the subspace of k-subimmersions, k as above, which are maps of constant rank k. Two immersions from X to Y are regularly homotopic if one can be deformed into the other through immersions or, in other words, if they belong to the same path component of $Imm(X,Y)$.

The following spaces will have the compact-open topology : the space of tangent bundle monomorphisms $TX \longrightarrow TY$, denoted $Mono(TX,TY)$, the space of tangent bundle epimorphisms $Epi(TX,TY)$, the space of tangent bundle homomorphisms of rank at least k *everywhere*, denoted $Hom_k(TX,TY)$, and the space of tangent bundle homomorphisms of constant rank k, denoted $Lin_k(TX,TY)$.

Weak homotopy equivalence will often be abbreviated as w.h.e., neighbourhood as nbd. and the symbol ||| will denote the end of a proof. The symbol $|$, used as in $f|$, will denote the restriction of the function f to a subspace ~~evident~~ from the context.

CHAPTER 0

GENERAL INTRODUCTION

In [10], Gromov formulated and proved a very general Smale-Hirsch-Phillips type theorem. The theorem concerns the classification of a class of cross-sections σ of a smooth fibre bundle $E \to X$ over a non-closed manifold X such that each r-jet $j^r(\sigma)$ satisfies an openness condition as well as a stability condition. Briefly, the openness condition is that each $j^r(\sigma)$ is a cross-section of some prescribed open subbundle of the bundle of r-jets of local cross-sections of $E \to X$, and the stability condition is that this subbundle remain invariant under the action of the pseudogroup of local diffeomorphisms of X.

In this thesis we study extensions of Gromov's theory in two directions. In Part I, we present a Gromov-type theorem which holds for closed manifolds as well, and illustrate its applicability. In Part II, we examine a particular Gromov-type theorem for a class of maps defined on a non-closed manifold, but not satisfying the openness condition and, after deriving some results in algebraic topology, apply it to prove a classification theorem for certain smooth maps between manifolds.

In retrospect, Gromov's celebrated theorem represented the culmination of work that could be said to have started with the paper of Whitney [35] in 1937 wherein he proved, following an idea of Graustein, that the set of regular homotopy classes of immersions of

S^1 in $\mathbb{R}^2$ (both oriented) is in a 1-1 correspondence with the set of integers $\mathbb{Z}$. During 1958-59, Smale [29], [30], [31] clarified the Whitney-Graustein theorem and proved his famous theorem : the set of based regular homotopy classes of based immersions of S^n into $\mathbb{R}^m$, $n < m$, corresponds bijectively with the n th homotopy group of the Stiefel manifold of n-frames in $\mathbb{R}^m$. Here a based immersion f is an immersion such that both f and df take prescribed values at a given base point. A more geometric and conceptual proof of the Smale theorem was found by Thom [33]. Hirsch [16] generalized the Smale theorem for arbitrary manifold pairs proving that, if $\dim X < \dim Y$, the correspondence $f \mapsto df$ sets up a bijection between the path components of the space $\mathrm{Imm}(X,Y)$ and the path components of the space $\mathrm{Mono}(TX, TY)$. A further generalization of the immersion theorem states that the differential map $d : \mathrm{Imm}(X,Y) \to \mathrm{Mono}(TX,TY)$ is, in fact, a weak homotopy equivalence (w.h.e.). This theorem was first proved by Hirsch and Palais [17]. A proof of the theorem may be found in Poénaru, [25] or [26], for the differentiable case, and in Haefliger and Poénaru [14] for the combinatorial case.

The technique, mainly geometric, as it evolved with these theorems lay essentially in showing certain maps to be Serre fibrations. In the next important exposition of this technique Phillips [23] proved an analogous result for submersions : if $\dim X \geq \dim Y$ and X is non-closed, then the differential map $d : \mathrm{Sub}(X,Y) \to \mathrm{Epi}(TX,TY)$ is a w.h.e.

Finally, it was Gromov who in his thesis [10] described the intrinsic geometry in its most natural setting and placed the existing theory into proper perspective. The following is a short outline of Gromov's main result as formulated by Haefliger [13] and Poénaru [27].

Let E and X be manifolds and $p : E \to X$ a smooth fibre bundle. Denote by $\mathcal{D}(X)$ the pseudogroup of local diffeomorphisms of X which is, to be precise, the category of open subsets of X and their diffeomorphisms. Consider a functor Φ from $\mathcal{D}(X)$ to $\mathcal{D}(E)$ which associates with each open subset U of X the open subset $E|U$ $(= p^{-1}(U))$ of E, and to each diffeomorphism $\lambda : U \to V$ between open subsets U, V of X a diffeomorphism $\Phi(\lambda) : E|U \to E|V$ so that the following diagram is commutative :

$$\begin{array}{ccc} E|U & \xrightarrow{\Phi(\lambda)} & E|V \\ {\scriptstyle p|U}\downarrow & & \downarrow{\scriptstyle p|V} \\ U & \xrightarrow[\lambda]{} & V \end{array}$$

Further suppose that $\Phi : \mathrm{Diff}\,(U,V) \to \mathrm{Diff}\,(E|U, E|V)$ is continuous for every pair of open subsets U, V of X.

Let E^r be the space (with the usual topology) of r-jets of local sections of E and $p^r : E^r \to X$ be the smooth fibre bundle where the projection p^r maps each r-jet onto its source. Then the functor Φ induces a functor Φ^r from $\mathcal{D}(X)$ to $\mathcal{D}(E^r)$ in the following way : if $\lambda : U \longrightarrow V$ is a diffeomorphism in $\mathcal{D}(X)$ and $\psi : V \longrightarrow E$ is a germ of section at some point $\lambda(x)$, $x \in U$, representing an r-jet $j^r_{\lambda(x)}\,\psi$ in $E^r \mid V$, then $\Phi^r(\lambda)(j^r_{\lambda(x)}\,\psi)$ is the r-jet in $E^r \mid U$ represented by the germ of section $\Phi(\lambda)^{-1} \circ \psi \circ \lambda$ at x, so

$$\Phi^r(\lambda)(j^r_{\lambda(x)}\,\psi) = j^r_x(\Phi(\lambda)^{-1} \circ \psi \circ \lambda).$$

Let E^r_ω be a subbundle of E^r which is invariant under the action of Φ^r, that is, $\Phi^r(\lambda)(j^r_{\lambda(x)}\,\psi)$ is in $E^r_\omega \mid U$ for every diffeomorphism $\lambda : U \longrightarrow V$ in $\mathcal{D}(X)$, $x \in U$ and $j^r_{\lambda(x)}\,\psi$ in $E^r_\omega \mid V$. Denote by $\Gamma^0 E^r_\omega$ the space of continuous sections of E^r_ω, with the compact open topology, and by $\Gamma^\infty_\omega E$ the space of smooth sections σ of E, with the C^∞ topology, whose r-jets $j^r(\sigma)$ lie in $\Gamma^0 E^r_\omega$.

Then Gromov's theorem states that if (1) X is non-closed, and (2) E^r_ω is open in E^r, the r-jet map $j^r : \Gamma^\infty_\omega E \longrightarrow \Gamma^0 E^r_\omega$ is a w.h.e.

One can look upon this remarkable theorem in at least two ways. Firstly, it is a classification theorem. It translates the problem of classifying the smooth sections of E which satisfy certain differential inequalities (defined by the subbundle E^r_ω) to the classical problem in algebraic topology of classifying the continuous sections of a bundle, namely E^r_ω. Secondly, it is an integrability theorem. The inclusion of the space of integrable sections of E^r_ω (sections that are j^r-images of sections of E) in the space of all continuous sections of E^r_ω is a w.h.e.

The applications of Gromov's theorem are many (see [10]). In particular, the Smale-Hirsch theorem on immersions, the Phillips theorem on submersions, and, in general, the k-mersion theorem of Feit ([7], in the case where the source manifold is non-closed) are all easy consequences. For the k-mersion theorem, for example, take E to be the trivial bundle $X \times Y$, Φ the trivial functor defined by $\Phi(\lambda)(x,y) = (\lambda(x), y)$, $r = 1$, and E^1_ω the subspace of $J^1(X,Y)$ of 1-jets of maps $X \to Y$ of rank at least k. Then $\Gamma^\infty_\omega E$ represents the space of k-mersions $Mer_k(X,Y)$, $\Gamma^0 E^1_\omega$ may be identified with the space $Hom_k(TX,TY)$ and the jet map j^1 with the differential map d. So Gromov's theorem asserts that the differential map $d : Mer_k(X,Y) \to Hom_k(TX,TY)$ is a w.h.e. in the case where X is non-closed.

Next observe that both assumptions (1) and (2) are vital to Gromov's theorem in the general setting formulated above. The geometric constructions in the proof make essential use of both assumptions and counter-examples are available if we drop either of them. If we drop assumption (1) we may refute the conclusion in the case of submersions by noting that S^1 is parallelizable but does not submerge in $\mathbb{R}^1$. On the other hand, if Gromov's theorem were true without assumption (2), it would imply that the differential map $d : Subimm_k(X,Y) \to Lin_k(TX,TY)$ is a w.h.e. when $k \leq \min(\dim X, \dim Y)$ and X is non-closed. Note that the subspace

$\mathrm{Subimm}_k(X,Y)$ is not open in $C^\infty(X,Y)$ unless $k = \min(\dim X, \dim Y)$. Now, if $X = Y = S^{k+1} \times \mathbb{R}$, there exists a $\varphi \in \mathrm{Lin}_k(TX,TY)$ covering the identity map, but such a φ cannot be homotopic to a df for some $f \in \mathrm{Subimm}_k(X,Y)$ for then f, by Sard's theorem, would be null homotopic. So, without assumption (2) Gromov's statement fails.

The theme of this thesis is to look for Gromov-type theorems even after relaxing, alternately, either of Gromov's assumptions and so, of necessity, imposing some others.

In Part I, we alter Gromov's setting somewhat and, additionally, impose a certain 'local stability' condition on the subbundle E^r_ω to prove a Gromov-type theorem true for closed manifolds as well. Our formulation of the problem is detailed in Chapter 1 and the main theorem proposed. In Chapter 2 we prove this theorem through a sequence of propositions and Chapter 3 describes some of its applications. In particular, we deduce Feit's k-mersion theorem for closed source manifolds [7], Feldman's theorem for immersions with non-vanishing mean curvature vector [9], the theorem of Gromov and Eliashberg on higher order non-degenerate immersions [12], and further, another theorem, first proved by Gromov, classifying immersions transverse to a field of planes [10].

Now, it should be mentioned that du Plessis in [5] described a somewhat similar generalization of Gromov theory. In fact, his paper suggested some problems that we attempt to solve here and we do borrow some techniques. However, our development of the theory diverges significantly from his and we believe that we meet with greater success as our main theorem is more powerful than that proved in [5]. This is evident on comparing an application. The case in point is the Gromov-Eliashberg theorem which we deduce in its full generality, a form which du Plessis is unable to derive. In fact, du Plessis manages only to reduce the problem to a complicated problem in algebra. Further, it is unclear as to how to retrieve the k-mersion theorem using du Plessis' theorem.

In Part II, we start by examining a Gromov-type theorem of Phillips [24] (see also Gromov [11]) which states that the differential map $d : \mathrm{Subimm}_k(X,Y) \rightarrow \mathrm{Lin}_k(TX,TY)$ is a w.h.e. if X is non-closed and admits a proper Morse function with no critical points of index greater than k. Firstly, in Chapter 4, we try and elucidate Phillips' proof of the theorem - his own paper is uncomfortably terse - and bring out the elegant geometry involved. Chapter 5 is a study purely in algebraic topology of certain spaces which are related, by Phillips' theorem, to subimmersions in the same manner as the Stiefel manifolds are related to immersions in the classical theorem of Smale classifying immersions of spheres in Euclidean space [31]. We calculate the homotopy groups and cohomology algebras of these spaces which are termed generalized Steifel manifolds. It should be noted that, though the cohomology algebra has not been further used in this thesis, its calculation may be treated as an interesting digression to our primary discussion. Finally, in Chapter 6, we employ an obstruction theory in applying the preceding results to prove a classification theorem for subimmersions. This extends the theorem of Smale on immersions of spheres in Euclidean space.

Though they both share the common aim of generalizing Gromov theory, Part I and Part II of this thesis are independent of each other.

PART I

CHAPTER 1

FORMULATION OF THE MAIN THEOREM

As indicated in the general introduction, Gromov's theorem in its full generality holds for non-closed manifolds but not necessarily for closed ones. The reason is that it is the subclass of non-closed manifolds (of any given dimension, say n) that possess the special property of having a handle decomposition into handles $D^k \times D^{n-k}$ of index $k < n$ (see [21], [23] for details). This fact is essential to Gromov's theory at a crucial step : the non-trivial transverse discs D^{n-k} then provide room for deformations inside the handle that are necessary to prove that the restriction map $\Gamma^{\infty}_{w} E \mid X_2 \to \Gamma^{\infty}_{w} E \mid X_1$ is a Serre fibration where X_1 and X_2 are submanifolds of X, X_2 being X_1 with a k-handle attached.

§ 1. We describe in this section a formulation, different from Gromov's, with a view to extending the theory to closed manifolds.

If X is a manifold, let $\mathcal{E}(X)$ denote the category whose objects are submanifolds of X and morphisms are embeddings $Y \to Z$ where Y, Z are submanifolds with $\dim Y \leq \dim Z$. Then the pseudo-group $\mathcal{D}(X)$ of local diffeomorphisms of X is a subcategory of $\mathcal{E}(X)$.

Assume that to each manifold X there is associated a smooth fibre bundle $E(X) \to X$ such that, if Y is a submanifold of X, then $E(Y) = E(X) \mid Y$ if $\dim Y = \dim X$, and $E(Y)$ is a subbundle of $E(X) \mid Y$ if $\dim Y < \dim X$.

Assume further that there is a functor $\Psi : \mathcal{E}(X) \to \mathcal{E}(E(X))$ which associates to each submanifold Y in $\mathcal{E}(X)$ the bundle space $E(Y)$, and to each embedding $\lambda : Y \to Z$ in $\mathcal{E}(X)$ an embedding $\Psi(\lambda) : E(Y) \to E(Z)$ which is also a bundle morphism covering λ such that, for every pair of submanifolds Y, Z of X with $\dim Y \leq \dim Z$, the map $\Psi : \mathrm{Emb}\,(Y, Z) \to \mathrm{Emb}\,(E(Y), E(Z))$ is continuous. Note that $\Psi(\lambda)$ defines maps $\Psi_1(\lambda) : E(Y) \to \lambda^* E(Z)$ and $\Psi_2(\lambda) : \lambda^* E(Z) \to E(Z)$ such that $\Psi(\lambda) = \Psi_2(\lambda) \circ \Psi_1(\lambda)$ and the following diagram is commutative :

$$\begin{array}{ccccc} E(Y) & \xrightarrow{\Psi_1(\lambda)} & \lambda^* E(Z) & \xrightarrow{\Psi_2(\lambda)} & E(Z) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow[1_Y]{} & Y & \xrightarrow[\lambda]{} & Z \end{array}$$

Here the right-hand rectangle represents the canonical pull-back over λ. We suppose that the restriction of Ψ to $\mathcal{D}(X)$ is the functor $\Phi : \mathcal{D}(X) \to \mathcal{D}(E(X))$ of Gromov's theory.

Moreover, we assume that, to each embedding $\lambda : Y \to Z$ in $\mathcal{E}(X)$, there corresponds a bundle morphism $\pi(\lambda) : \lambda^* E(Z) \to E(Y)$ over 1_Y such that

(1) $\pi(1_Y) = 1_{E(Y)}$

(2) $\pi(\lambda) \circ \Psi_1(\lambda) = 1_{E(Y)}$

and (3) if $\lambda : Y \to Z$ and $\mu : Z \to W$

are embeddings in $\mathcal{E}(X)$, then $\pi(\lambda)_{\circ}\lambda^*\pi(\mu) = \pi(\mu \circ \lambda)$ or, in other words, the following diagram over Y is commutative :

$$\begin{array}{ccc}
\lambda^*\mu^*E(W) & \xrightarrow{\simeq} & (\mu\lambda)^*E(W) \\
\lambda^*\pi(\mu)\downarrow & & \downarrow \pi(\mu\lambda) \\
\lambda^*E(Z) & \xrightarrow[\pi(\lambda)]{} & E(Y) .
\end{array}$$

Note that, in the case of Gromov's theory, if λ is a diffeomorphism in $\mathcal{D}(X)$, then $\pi(\lambda)$ may be realized as $\Psi(\lambda)^{-1}$.

Let $\Gamma^{\infty}E(X)$ denote the space of smooth sections of $E(X)$. Then $\mathcal{E}(X)$ acts on $\Gamma^{\infty}E(X)$ in the following sense. Let $\lambda : Y \to Z$ be an embedding in $\mathcal{E}(X)$ and $f : Z \to E(Z)$ a smooth section. Then $\pi(\lambda) \circ \lambda^*(f) : Y \to E(Y)$, where $\lambda^*(f) : Y \to \lambda^*E(Z)$ is the pull-back of f by λ, is a smooth section of $E(Y)$. We shall write $\Psi(\lambda)(f) = \pi(\lambda) \circ \lambda^*(f)$,

$$\begin{array}{ccccc}
E(Y) & \xleftarrow{\pi(\lambda)} & \lambda^*E(Z) & \xrightarrow{\Psi_2(\lambda)} & E(Z) \\
\uparrow \Psi(\lambda)(f) & & \uparrow \lambda^*(f) & & \uparrow f \\
Y & \xrightarrow[1_Y]{} & Y & \xrightarrow[\lambda]{} & Z .
\end{array}$$

Note that, restricting Ψ to $\mathcal{D}(X)$, we retrieve the natural action of $\mathcal{D}(X)$ on $\Gamma^{\infty}E(X)$ as considered in Gromov's theory. More precisely, if $\lambda : U \to V$ is a diffeomorphism between open subsets U, V of X and if $f : V \to E(V)$ is a smooth section, then $\Psi(\lambda)(f) = \Phi(\lambda)^{-1} \circ f \circ \lambda$. The naturality of the action of $\mathcal{E}(X)$ may be checked easily with the help of (1) - (3) above. In fact,

$$(4) \quad \Psi(1_W)(f) = f$$

$$\text{and} \quad (5) \quad \Psi(\mu \circ \lambda)(f) = \Psi(\lambda)(\Psi(\mu)(f))$$

where $f : W \to E(W)$ is a smooth section and $\lambda : Y \to Z$, $\mu : Z \to W$ are embeddings in $\mathcal{E}(X)$.

Again, if $E^r(X) \to X$ is the bundle of r-jets of local sections of $E(X)$, then we have an action Ψ^r of $\mathcal{E}(X)$ on $E^r(X)$ defined as follows : if $\lambda : Y \to Z$ is an embedding in $\mathcal{E}(X)$, $x \in Y$ and f is a germ of section of $E(Z)$ at $\lambda(x)$, then $\Psi(\lambda)(f)$ is a germ of section of $E(Y)$ at x. Define

$$\Psi^r(\lambda)(j^r_{\lambda(x)}f) = j^r_x(\Psi(\lambda)(f)).$$

Clearly this action induces the action Φ^r of $\mathcal{D}(X)$ on $E^r(X)$ of Gromov's theory.

As in Gromov's setting, suppose $E^r_\omega(X) \to X$ to be an open subbundle of $E^r(X)$ which is invariant under the action of $\mathcal{D}(X)$. Let $\Gamma^0 E^r_\omega(X)$ be the space of continuous sections of $E^r_\omega(X)$, with the compact - open topology, and $\Gamma^{\infty}_\omega E(X) \subset \Gamma^{\infty}E(X)$ be the subspace of sections $f : X \to E(X)$ such that $j^r f(x) \in E^r_\omega(X)$ for all $x \in X$. Then j^r induces a continuous function $j^r : \Gamma^{\infty}_\omega E(X) \to \Gamma^0 E^r_\omega(X)$.

§ 2. In this section we motivate and formulate the main theorem. First, we describe an additional condition required to extend Gromov-type results to closed manifolds.

For a manifold X, the space $\Gamma_w^\infty E(X)$ is said to be locally stable if, for each section $f \in \Gamma_w^\infty E(X)$ and each $x \in X$, there exists

(i) an open neighbourhood U of x in X,

(ii) a manifold N with $\dim N > \dim X$ and $U \subset \operatorname{Int} N$,

and (iii) an open subbundle $E_o^r(N) \to N$ of $E^r(N)$ which is invariant under the action of $\mathcal{D}(N)$, such that

(a) there exists a section $f' \in \Gamma_o^\infty E(N)$ with $\mathfrak{g}(i)(f') = f|U$,

and (b) $\mathfrak{g}(i)(g') \in \Gamma_w^\infty E(U)$ for every $g' \in \Gamma_o^\infty E(N)$,

where $i : U \to N$ is the inclusion. Of course, $\Gamma_o^\infty E(N)$ is the subspace of sections of $E(N)$ whose r-jets lie in $E_o^r(N)$.

Observe that, if $\Gamma_w^\infty E(X)$ is locally stable and Y is a submanifold of X with $\dim Y = \dim X$, then $\Gamma_w^\infty E(Y)$ is locally stable too.

Before further discussion, let us at this point formulate the theorem that is our objective :

<u>Main Theorem</u> : If X is a manifold where $\Gamma_w^\infty E(X)$ is locally stable, then the map,

$$j^r : \Gamma_w^\infty E(X) \to \Gamma^0 E_w^r(X)$$

is a w.h.e.

To motivate the definition preceding the theorem, observe that the difficulty in Gromov's theory with a closed n-dimensional manifold X lies in showing that the restriction map $\Gamma_W^{\infty} E(X_2) \rightarrow \Gamma_W^{\infty} E(X_1)$ is a Serre fibration when $X_1 \subset X_2 \subset X$ and X_2 is X_1 plus a handle of top index n attached. Then there is no room for the transverse deformations of Gromov. However if $\Gamma_W^{\infty} E(X)$ is locally stable then a section $f \in \Gamma_W^{\infty} E(X)$ can be dominated locally by sections over a higher dimensional manifold. That, at least locally, this may be done continuously is the content of the following crucial lemma which we prove in the next chapter :

<u>Main Lemma</u> : If $\Gamma_W^{\infty} E(X)$ is locally stable, for each section $f \in \Gamma_W^{\infty} E(X)$ and each $x \in X$, there exists

(i) an open neighbourhood W of x in X,

(ii) an open neighbourhood Ω of f in $\Gamma_W^{\infty} E(X)$,

(iii) an open subbundle $E_0^r(W \times D^k)$ of $E^r(W \times D^k)$,

for some $k > 0$, which is invariant under the action of $\mathcal{D}(W \times D^k)$,

and (iv) a map $\rho : \Omega \rightarrow \Gamma_0^{\infty} E(W \times D^k)$,

such that

(a) $\Psi(i)(\rho(g)) = g \,|\, W$, for every $g \in \Omega$,

and (b) $\Psi(i)(g') \in \Gamma_W^{\infty} E(W)$, for every $g' \in \Gamma_0^{\infty} E(W \times D^k)$,

where $i = 1 \times 0 : W \rightarrow W \times D^k$.

With this lemma, the technique of proving that the restriction map $\Gamma_W^{\infty} E(X_2) \rightarrow \Gamma_W^{\infty} E(X_1)$ is a Serre fibration, when X_2 is

X_1 plus a handle of top index attached, will, roughly speaking, be to split a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & \Gamma_\omega^\infty E(X_2) \\ \Big\downarrow{\scriptstyle 1 \times 0} & & \Big\downarrow{\scriptstyle \text{restriction}} \\ Q \times I & \xrightarrow[G]{} & \Gamma_\omega^\infty E(X_1), \end{array}$$

where Q is a compact polyhedron, into a family of local lifting problems and extending each such problem to a higher dimensional manifold. The extended problems may then be individually solved by Gromov-type methods as there is room in the added dimensions for transverse deformations. The local solutions are then pulled back to the original manifold.

It should be remarked that the local stability of $\Gamma_\omega^\infty E(X)$ is the only hypothesis we shall assume in addition to the usual ones of Gromov theory and that the Main Lemma contains precisely the additional information required to extend the theory to closed manifolds.

A further remark is that, if X is compact, the jet map j^r of the Main Theorem is, in fact, a homotopy equivalence as its domain and range are metrizable (see R. Palais [22]).

CHAPTER 2

PROOF OF THE MAIN THEOREM

Our exposition is based upon the expositions of Gromov theory by Haefliger [13] and Poénaru [27].

§ 1. Firstly, without going into proofs (available in Milnor [21] and Phillips [23]), we record certain facts about the structure of smooth manifolds :

A smooth n-manifold X may be represented as the union of an increasing sequence,

$$X_1^1 \subset X_2^1 \subset X_1^2 \subset \cdots \subset X_1^i \subset X_2^i \subset X_1^{i+1} \subset \cdots$$

of compact manifolds with boundary where X_1^1 is an n-disc ; X_2^i is the union of X_1^i and a collarlike neighbourhood (so, $X_2^i \simeq X_1^i \cup \partial X_1^i \times [0,1]$); X_1^{i+1} is the union of X_2^i and A where A is diffeomorphic to $D^k \times D^{n-k}$ for some k, $0 \leq k \leq n$, and $X_2^i \cap A$ is diffeomorphic to a collarlike neighbourhood B of $\partial D^k \times D^{n-k}$ (so, $(A, A \cap X_2^i) \simeq (2D^k \times D^{n-k}, S^{k-1} \times [1,2] \times D^{n-k})$). We say that X_1^{i+1} is X_1^i with a handle of index k attached. X is non-closed if and only if it has a representation where all the attached handles are of index $< n$.

Next note the following propositions which shall be proved further on :

<u>Proposition 2.1</u> : If D^n is the n-disc, then the map,

$$j^r : \Gamma_w^\infty E(D^n) \to \Gamma^0 E_w^r(D^n)$$

is a w.h.e.

Proposition 2.2 : If X_2 is X_1 plus a collarlike neighbourhood, then the restriction maps,

$$J : \Gamma^{\infty}_{w} E(X_2) \to \Gamma^{\infty}_{w} E(X_1)$$

$$\text{and} \quad K : \Gamma^{o} E^{r}_{w}(X_2) \to \Gamma^{o} E^{r}_{w}(X_1)$$

are both w.h.e. and Serre fibrations.

Proposition 2.3 : If Y is a submanifold of X of the same dimension as X, then the restriction map,

$$K : \Gamma^{o} E^{r}_{w}(X) \to \Gamma^{o} E^{r}_{w}(Y)$$

is a Serre fibration.

Proposition 2.4 : If $A = 2D^{k} \times D^{n-k}$ and $B = S^{k-1} \times [1,2] \times D^{n-k}$, where $k < n$, then the restriction map,

$$J : \Gamma^{\infty}_{w} E(A) \to \Gamma^{\infty}_{w} E(B)$$

is a Serre fibration.

Proposition 2.5 : If $A = 2D^{n}$ and $B = S^{n-1} \times [1,2]$ and $\Gamma^{\infty}_{w} E(A)$ is locally stable, then the restriction map,

$$J : \Gamma^{\infty}_{w} E(A) \to \Gamma^{\infty}_{w} E(B)$$

is a Serre fibration.

It is to prove this last proposition that we shall require the Main Lemma formulated in the previous chapter.

Further we shall need the following two lemmas.

<u>Lemma 2.1</u> : If in the commutative diagram of bundle maps

$$\begin{array}{ccc} E & \xrightarrow{\bar{g}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[g]{} & B', \end{array}$$

p and p' are Serre fibrations and g is a w.h.e., then $\bar{g}$ is a w.h.e. if and only if its restriction to each fibre of E is a w.h.e.

<u>Proof</u> : This follows from the homotopy exact sequences of the fibrations and the five lemma. |||

The next lemma is from Phillips [23].

<u>Lemma 2.2</u> : If in the commutative diagram of continuous maps

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_{i+1} & \longrightarrow & A_i & \longrightarrow \cdots \longrightarrow & A_1 \\ & & \downarrow{\scriptstyle j_{i+1}} & & \downarrow{\scriptstyle j_i} & & \downarrow{\scriptstyle j_1} \\ \cdots & \longrightarrow & B_{i+1} & \longrightarrow & B_i & \longrightarrow \cdots \longrightarrow & B_1, \end{array}$$

all the horizontal maps are Serre fibrations and all the j_i are w.h.e., then

$$\varprojlim j_i : \varprojlim A_i \longrightarrow \varprojlim B_i$$

is also a w.h.e. |||

If, for the present, we assume the five propositions, the main theorem follows by inductive arguments.

<u>Main Theorem</u> : If X is a manifold where $\Gamma_w^{\infty} E(X)$ is locally stable,

then the map,

$$j^r : \Gamma_w^\infty E(X) \to \Gamma^0 E_w^r(X)$$

is a w.h.e.

Proof : Suppose X is of dimension n. Inductively assume that $j^r : \Gamma_w^\infty E(Y) \to \Gamma^0 E_w^r(Y)$ is a w.h.e. if Y is a compact manifold (we assume that each manifold Y considered in this proof is a sub-n-manifold of X so that $\Gamma_w^\infty E(Y)$ is also locally stable) which is a union of handles of index $< k$, where $k \leq n$. Of course, induction begins with Proposition 2.1. To make the induction step we shall show that, if $j^r : \Gamma_w^\infty E(Y) \to \Gamma^0 E_w^r(Y)$ is a w.h.e., then so is $j^r : \Gamma_w^\infty E(Y'') \to \Gamma^0 E_w^r(Y'')$ where Y'' is obtained from Y by attaching a handle of index k. Suppose Y' is Y with a collarlike neighbourhood attached so that $Y'' = Y' \cup A$ and $Y' \cap A = B$ where we have $(A, B) \simeq (2D^k \times D^{n-k}, S^{k-1} \times [1,2] \times D^{n-k})$.

Consider the commutative diagram

$$(1) \qquad \begin{array}{ccc} \Gamma_w^\infty E(A) & \xrightarrow{j^r} & \Gamma^0 E_w^r(A) \\ J \downarrow & & \downarrow K \\ \Gamma_w^\infty E(B) & \xrightarrow[j^r]{} & \Gamma^0 E_w^r(B), \end{array}$$

where j^r is an r-jet map and J, K are restriction maps.

Now K is a Serre fibration by Proposition 2.3 while J is a Serre fibration by Proposition 2.4, if $k < n$, or by Proposition 2.5, if $k = n$. By Proposition 2.1, $j^r : \Gamma^\infty_w E(A) \to \Gamma^0 E^r_w(A)$ is a w.h.e. Further, by the induction hypothesis (as B is a 0-handle with a $(k-1)$-handle attached), $j^r : \Gamma^\infty_w E(B) \to \Gamma^0 E^r_w(B)$ is a w.h.e. Hence, by Lemma 2.1, j^r is a w.h.e. on each fibre.

Next consider the similar diagram

$$(2) \qquad \begin{array}{ccc} \Gamma^\infty_w E(Y'') & \xrightarrow{j^r} & \Gamma^0 E^r_w(Y'') \\ \Big\downarrow J & & \Big\downarrow K \\ \Gamma^\infty_w E(Y') & \xrightarrow[j^r]{} & \Gamma^0 E^r_w(Y'). \end{array}$$

The pair (Y'', Y') restricts to the pair (A,B) and the restriction maps diagram (2) into diagram (1) so that the vertical maps of (2) are pull-backs of the fibrations J,K in (1). It follows that the maps J,K in (2) are also Serre fibrations and that the restriction of j^r to each fibre is a w.h.e. Further, $j^r : \Gamma^\infty_w E(Y') \to \Gamma^0 E^r_w(Y')$ is a w.h.e. by the initial assumption and Proposition 2.2. So, again by Lemma 2.1, we conclude that $j^r : \Gamma^\infty_w E(Y'') \to \Gamma^0 E^r_w(Y'')$ is a w.h.e. completing the induction step.

This proves the theorem if X is compact. If X is non-compact and so not a union of finitely many handles, represent X as a union of an increasing sequence of compact submanifolds,

$$X^1_1 \subset X^1_2 \subset X^2_1 \subset \cdots \subset X^i_1 \subset X^i_2 \subset X^{i+1}_1 \subset \cdots .$$

Then $\qquad \Gamma^\infty_w E(X) = \varprojlim \Gamma^\infty_w (X^i_1),$

$$\Gamma^o E^r_w(X) = \varprojlim \Gamma^o E^r_w(X^i_1)$$

and applying Lemma 2.2 completes the proof. |||

It should be remarked that if we restrict to non-closed n-manifolds X (which are representable as unions of handles of index $< n$) we do not require Proposition 2.5 or, consequently, local stability and the above method proves Gromov's theorem.

§ 2. In this section we shall prove the preliminary propositions.

Proposition 2.1 : If D^n is the n-disc, then the map,

$$j^r : \Gamma^\infty_w E(D^n) \to \Gamma^o E^r_w(D^n)$$

is a w.h.e.

Proof : As D^n is contractible, $E(D^n)$ is a product bundle $D^n \times F$ with fibre F. Hence $\Gamma^\infty E(D^n)$ may be identified with $C^\infty(D^n,F)$, $\Gamma^\infty_w E(D^n)$ with an open subspace $C^\infty_w(D^n,F) \subset C^\infty(D^n,F)$, the fibre of $E^r(D^n)$ over $0 \in D^n$ with $J^r_0(D^n,F)$, and the fibre of $E^r_w(D^n)$ over 0 with an open subspace $J^r_{ow}(D^n,F) \subset J^r_0(D^n,F)$.

We first show that the evaluation map,

$$e : \Gamma^o E^r_w(D^n) \to J^r_{ow}(D^n,F)$$

defined by $e(f) = f(0)$ is a homotopy equivalence. If $x \in \mathbb{R}^n$, define the diffeomorphism $T_x : \mathbb{R}^n \to \mathbb{R}^n$ by $T_x(y) = y - x$. Then define

$$c : J^r_{ow}(D^n,F) \to \Gamma^o E^r_w(D^n)$$

by $c(g)(x) = \Psi^r(T_x)(g)$, assuming $D^n \subset \mathbb{R}^n$, where $g \in J^r_{ow}(D^n,F)$ and $x \in D^n$ (the definition is proper by continuity of Ψ and the invariance of E^r_w under the action of Ψ^r). Clearly $e \circ c$ is the identity on $J^r_{ow}(D^n,F)$ and a homotopy H between $c \circ e$ and the identity on $\Gamma^o E^r_w(D^n)$ is given by

$$H(f,t)(x) = \begin{cases} f(x), & \text{if } \|x\| \leq t \\ \Psi^r(T_{x - t\frac{x}{\|x\|}})(f(t\frac{x}{\|x\|})), & \text{otherwise,} \end{cases}$$

where $0 \leq t \leq 1$, $f \in \Gamma^o E^r_w(D^n)$, $x \in D^n$. So the map e is indeed a homotopy equivalence.

It is now sufficient to prove that $e \circ j^r : \Gamma^\infty_w E(D^n) \to J^r_{ow}(D^n,F)$ is a *w.h.e.*

Consider F as a submanifold of some $\mathbb{R}^m$ and suppose $q : W \to F$ is a smooth retraction of a tubular nbd. W of F in $\mathbb{R}^m$.

Let $g : S^i \to J^r_{ow}(D^n,F) \subset J^r_o(D^n,\mathbb{R}^m)$ be a continuous map from a sphere. Choosing the polynomial representative G_s of degree r of each $g(s) \in J^r_o(D^n,\mathbb{R}^m)$, $s \in S^i$, we have a continuous map $G : S^i \times D^n \to \mathbb{R}^m$ defined by $G(s,x) = G_s(x)$, $s \in S^i$, $x \in D^n$. Note that $G(S^i \times \{0\}) \subset F$ as $G(s,0) = g(s)(0) \in F$, $s \in S^i$, so that by the compactness of S^i there exists a nbd. V of 0 in D^n such that $G(S^i \times V) \subset W$ — in other words, $G_s(V) \subset W$, $s \in S^i$.

Now the r-jet at 0 of $qG_s | V$ is $g(s) \in J^r_{ow}(D^n,F)$ for $s \in S^i$ so that by the openness of $E^r_w(D^n)$ it follows that

$qG_s | U \in \Gamma_w^\infty E(U)$ for some nbd. U of 0 in V and all $s \in S^i$. Let w be an embedding of D^n in U which is identity on a nbd. of 0. Define $\bar{g} : S^i \to \Gamma_w^\infty E(D^n)$ by $\bar{g}(s) = \Psi(w)(qG_s | U)$. It follows that $(e \circ j^r)(\bar{g}(s)) = g(s)$, $s \in S^i$, proving that $(e \circ j^r)_* : \pi_i(\Gamma_w^\infty E(D^n)) \to \pi_i(J_{0w}^r(D^n, F))$ is surjective.

Next let $f_0, f_1 : S^i \to \Gamma_w^\infty E(D^n)$ be continuous maps such that there exists a homotopy $h : S^i \times I \to J_{0w}^r(D^n, F)$ where $h(s,0) = e \circ j^r \circ f_0(s)$ and $h(s,1) = e \circ j^r \circ f_1(s)$, $s \in S^i$. Choosing the polynomial representative $H_{s,t}$ of degree r of each $h(s,t)$ we have a continuous map $H : S^i \times I \times D^n \to \mathbb{R}^m$ defined by $H(s,t,x) = H_{s,t}(x)$. Hence define $\bar{H} : S^i \times I \times D^n \to \mathbb{R}^m$ by

$$\bar{H}(s,t,x) = \begin{cases} (1-3t)f_0(s)(x) + 3tH(s,0,x) & , 0 \le t \le \frac{1}{3} \\ H(s,3t-1,x) & , \frac{1}{3} \le t \le \frac{2}{3} \\ (3t-2)f_1(s)(x) + (3-3t)H(s,1,x), & \frac{2}{3} \le t \le 1 \end{cases}$$

for $s \in S^i$, $t \in [0,1]$, $x \in D^n$. Note that $\bar{H}(S^i \times \{0,1\} \times D^n \cup S^i \times I \times \{0\}) \subset F$ so that there exists a nbd. V of 0 in D^n and a number $\nu \in (0,1)$ such that

$$\bar{H}(S^i \times ([0,\nu] \cup [1-\nu, 1]) \times D^n \cup S^i \times I \times V) \subset W.$$

Further, from the openness of $E_w^r(D^n)$ and the fact that $f_0, f_1 \in \Gamma_w^\infty E(D^n)$, there exists a nbd. U of 0 in V and a number $\mu \in (0,\nu)$ such that $q\bar{H}_{s,t} | U \in \Gamma_w^\infty E(U)$, for $(s,t) \in S^i \times I$, and $q\bar{H}_{s,t} \in \Gamma_w^\infty E(D^n)$, for $(s,t) \in S^i \times ([0,\mu] \cup [1-\mu,1])$ where, of course, $\bar{H}_{s,t} : D^n \to \mathbb{R}^m$ is defined by $\bar{H}_{s,t}(x) = \bar{H}(s,t,x)$.

Let w be an isotopy of embeddings of D^n in itself such that $w(0) = w(1) = 1_{D^n}$ and $w(t)(D^n) \subset U$ for $\mu \leq t \leq 1-\mu$. Define the homotopy $\bar{h} : S^i \times I \to \Gamma^{\infty}_w E(D^n)$ between f_0 and f_1 by $\bar{h}(s,t) = \Psi(w(t))(q\bar{H}_{s,t} \mid w(t)(D^n))$. It follows that $(e \circ j^r)_* : \pi_i(\Gamma^{\infty}_w E(D^n)) \to \pi_i(J^r_{ow}(D^n,F))$ is injective.

Hence $e \circ j^r$ is a w.h.e. |||

<u>*Proposition 2.2*</u> : If X_2 is X_1 plus a collarlike nbd., then the restriction maps,

$$J : \Gamma^{\infty}_w E(X_2) \to \Gamma^{\infty}_w E(X_1)$$

and $$K : \Gamma^{0} E^r_w(X_2) \to \Gamma^{0} E^r_w(X_1)$$

are both w.h.e. and Serre fibrations.

<u>Proof</u> : Any continuous map $g : S^i \to \Gamma^{\infty}_w E(X_1)$ may be extended to a continuous map $g' : S^i \to \Gamma^{\infty} E(V)$ where V is a nbd. of X_1 in X_2 (in fact, continuously choose smooth extensions across the boundary) so that $g'(s) \mid X_1 = g(s)$, $s \in S^i$. Since E^r_w is an open subbundle on manifolds and as $g'(s) \mid X_1 \in \Gamma^{\infty}_w E(X_1)$, there is a nbd. U of X_1 in V such that $g'(s) \mid U \in \Gamma^{\infty}_w E(U)$, $s \in S^i$. Let w be an embedding of X_2 in U which is identity in a nbd. of X_1. Then for $\bar{g} : S^i \to \Gamma^{\infty}_w E(X_2)$ defined by $\bar{g}(s) = \Psi(w)(g'(s) \mid U)$ we have $J(\bar{g}(s)) = g(s)$, $s \in S^i$, so that $J_* : \pi_i(\Gamma^{\infty}_w E(X_2)) \to \pi_i(\Gamma^{\infty}_w E(X_1))$ is surjective.

Next, let $f_0, f_1 : S^i \to \Gamma_w^\infty E(X_2)$ be continuous maps such that there exists a homotopy $h : S^i \times I \to \Gamma_w^\infty E(X_1)$ where $h(s,0) = f_0(s) \mid X_1$ and $h(s,1) = f_1(s) \mid X_1$, $s \in S^i$.

Let $\pi : S^i \times I \times X_2 \to X_2$ be the natural projection and $\bar{E} = \pi^* E(X_2)$ the pull-back. Then h, f_0, f_1 together give a section of $\bar{E} \mid S^i \times I \times X_1 \cup S^i \times \{0,1\} \times X_2$. This may be extended to a section (see Husemoller [19]), say h', of $\bar{E} \mid S^i \times I \times V \cup S^i \times ([0,\nu] \cup [1-\nu,1]) \times X_2$ where V is a nbd. of X_1 in X_2 and $\nu \in (0,1)$. By openness of the subbundle E_w^r on manifolds, there exists a nbd. U of X_1 in V and $\mu \in (0,\nu)$ such that $h'_{s,t} \mid U \in \Gamma_w^\infty E(U)$, for $(s,t) \in S^i \times I$, and $h'_{s,t} \in \Gamma_w^\infty E(X_2)$, for $(s,t) \in S^i \times ([0,\mu] \cup [1-\mu,1])$, where, of course, $h'_{s,t}(x) = h'(s,t,x)$.

Let w be an isotopy of embeddings of X_2 in itself such that $w(0) = w(1) = 1_{X_2}$ and $w(t)(X_2) \subset U$ for $\mu \leq t \leq 1-\mu$. Define the homotopy $\bar{h} : S^i \times I \to \Gamma_w^\infty E(X_2)$ between f_0 and f_1 by $\bar{h}(s,t) = \Psi(w(t))(h'_{s,t} \mid w(t)(X_2))$. It follows that $J_* : \pi_i(\Gamma_w^\infty E(X_2)) \to \pi_i(\Gamma_w^\infty E(X_1))$ is injective. Hence J is a w.h.e.

Next, consider the lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & \Gamma_w^\infty E(X_2) \\ \Big\downarrow{\scriptstyle 1\times 0} & & \Big\downarrow{\scriptstyle J} \\ Q \times I & \xrightarrow{G} & \Gamma_w^\infty E(X_1), \end{array}$$

where Q is a compact polyhedron.

Let $\pi : Q \times I \times X_2 \to X_2$ be the natural projection and $\bar{E} = \pi^* E(X_2)$ the pull-back. Then G_0 and G together give a section of $\bar{E} \mid Q \times I \times X_1 \cup Q \times \{0\} \times X_2$. This may be extended to a section, say G', of $\bar{E} \mid Q \times I \times U \cup Q \times [0,\mu] \times X_2$ where U is a nbd. of X_1 in X_2 and $\mu \in (0,1)$. In fact, choosing U and μ sufficiently small, we may suppose that $G'_{q,t} \mid U \in \Gamma^{\infty}_{w} E(U)$, for $(q,t) \in Q \times I$, and $G'_{q,t} \in \Gamma^{\infty}_{w} E(X_2)$, for $(q,t) \in Q \times [0,\mu]$, where $G'_{q,t}(x) = G'(q,t,x)$.

Let w be an isotopy of embeddings of X_2 in itself such that $w(0) = 1_{X_2}$, $w(t)(X_2) \subset U$, for $\mu \leq t \leq 1$, and $w(t)$ is identity in some nbd. of X_1, for all t. Then $\bar{G} : Q \times I \to \Gamma^{\infty}_{w} E(X_2)$ defined by $\bar{G}(q,t) = \Psi(w(t))(G'_{q,t} \mid w(t)(X_2))$ is the required lift for (G, G_0). Hence J is a Serre fibration.

The corresponding proofs for K are similar. |||

<u>Proposition 2.3</u> : If Y is a submanifold of X, of the same dimension as X, then the restriction map,

$$K : \Gamma^0 E^r_w(X) \to \Gamma^0 E^r_w(Y)$$

is a Serre fibration.

<u>Proof</u> : Consider a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & \Gamma^0 E^r_w(X) \\ \downarrow 1\times 0 & & \downarrow K \\ Q \times I & \xrightarrow{G} & \Gamma^0 E^r_w(Y), \end{array}$$

where Q is a compact polyhedron.

This may be translated to the lifting problem

$$\begin{array}{ccc} Q\times I\times Y \cup Q\times \{0\} \times X & \xrightarrow{\;G\,\cup\, G_0\;} & E^r_w(X) \\ \Big\downarrow \text{inclusion} & & \Big\downarrow \text{bundle map} \\ Q\times I\times X & \xrightarrow{\;\text{natural projection}\;} & X\,. \end{array}$$

Since $E^r_w(X) \longrightarrow X$ is a locally trivial bundle, it has the polyhedral covering homotopy extension property (PCHEP, see Hu [18]). Now $(Q\times X,\; Q\times Y)$ is a polyhedral pair and so the PCHEP provides a lift $G' : Q\times I\times X \longrightarrow E^r_w(X)$ for the second lifting problem above. Then $\bar{G} : Q\times I \longrightarrow \Gamma^0 E^r_w(X)$ defined by $\bar{G}(q,t)(x) = G'(q,t,x)$ is a lift for the first lifting problem. Hence, K is a Serre fibration. |||

<u>Proposition 2.4</u> : If $A = 2D^k \times D^{n-k}$ and $B = S^{k-1} \times [1,2] \times D^{n-k}$, where $k < n$, then the restriction map,

$$J : \Gamma^{\infty}_w E(A) \longrightarrow \Gamma^{\infty}_w E(B)$$

is a Serre fibration.

<u>Proof</u> : Consider the lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{\;G_0\;} & \Gamma^{\infty}_w E(A) \\ \Big\downarrow 1\times 0 & & \Big\downarrow J \\ Q\times I & \xrightarrow{\;G\;} & \Gamma^{\infty}_w E(B)\,, \end{array}$$

where Q is a compact polyhedron. There are three stages in the construction of a lift.

(1) By the openness of E^r_w in E^r we may extend G to $G' : Q \times I \longrightarrow \Gamma^\infty_w E(S^{k-1} \times [a,2] \times D^{n-k})$, for some $a \in (0,1)$, so that $G(q,t) = G'(q,t) \mid$ and $G'(q,0) = G_o(q) \mid$ when $q \in Q$, $t \in I$.

(2) We require the constructs provided by the following :

<u>Lemma 2.3</u> : There exists

(a) a partition $0 = t_0 < t_1 < \ldots < t_p = 1$ of I,

(b) numbers b,c such that $a < b < c < 1$, and

(c) maps $\mu_i : Q \times [t_i, t_{i+1}] \longrightarrow \Gamma^\infty_w E(S^{k-1} \times [a,2] \times D^{n-k})$,

for $0 \leq i \leq p-1$, such that

$$\mu_i(q,t)(x,s,y) = \begin{cases} G'(q,t)(x,s,y), & t = t_i \text{ or } c \leq s \leq 2 \\ G'(q,t_i)(x,s,y), & a \leq s \leq b. \end{cases}$$

<u>Proof</u> : As A is contractible, $E(A)$ is a product bundle $A \times F$ with fibre F so that $E(S^{k-1} \times [a,2] \times D^{n-k}) \simeq S^{k-1} \times [a,2] \times D^{n-k} \times F$. Hence $\Gamma^\infty E(S^{k-1} \times [a,2] \times D^{n-k})$ may be identified with $C^\infty(S^{k-1} \times [a,2] \times D^{n-k}, F)$, $\Gamma^\infty_w E(S^{k-1} \times [a,2] \times D^{n-k})$ with an open subspace $\Omega \subset C^\infty(S^{k-1} \times [a,2] \times D^{n-k}, F)$ and we may suppose the map $G' : Q \times I \longrightarrow \Omega$. Choose numbers b,c as in (b) and a smooth map $h : [a,2] \longrightarrow [0,1]$ such that $h \mid [a,b] = 0$ and $h \mid [c,2] = 1$. If $t' \leq t''$ are real numbers define $h_{t',t''} : [a,2] \longrightarrow \mathbb{R}$ by $h_{t',t''} = L \circ h$ where $L : \mathbb{R} \longrightarrow \mathbb{R}$ is the unique linear map with $L(0) = t'$, $L(1) = t''$. Choose a Riemannian metric on F. By the compactness of $Q \times S^{k-1} \times [a,2] \times D^{n-k}$, there exists $\varepsilon_1, \varepsilon_2 > 0$ such that if $t', t'' \in I$ with $t' \leq t'' < t' + \varepsilon_1$ then there is a unique

geodesic arc $\lambda^{q,t',t''}_{x,s,y} : [t',t''] \to F$ of length less than ε_2 from $G'(q,t')(x,s,y)$ to $G'(q,t'')(x,s,y)$. Hence define the smooth maps
$\mu^{q,t',t''} : S^{k-1} \times [a,2] \times D^{n-k} \to F$ by
$\mu^{q,t',t''}(x,s,y) = \lambda^{q,t',t''}_{x,s,y}(h_{t',t''}(s))$.

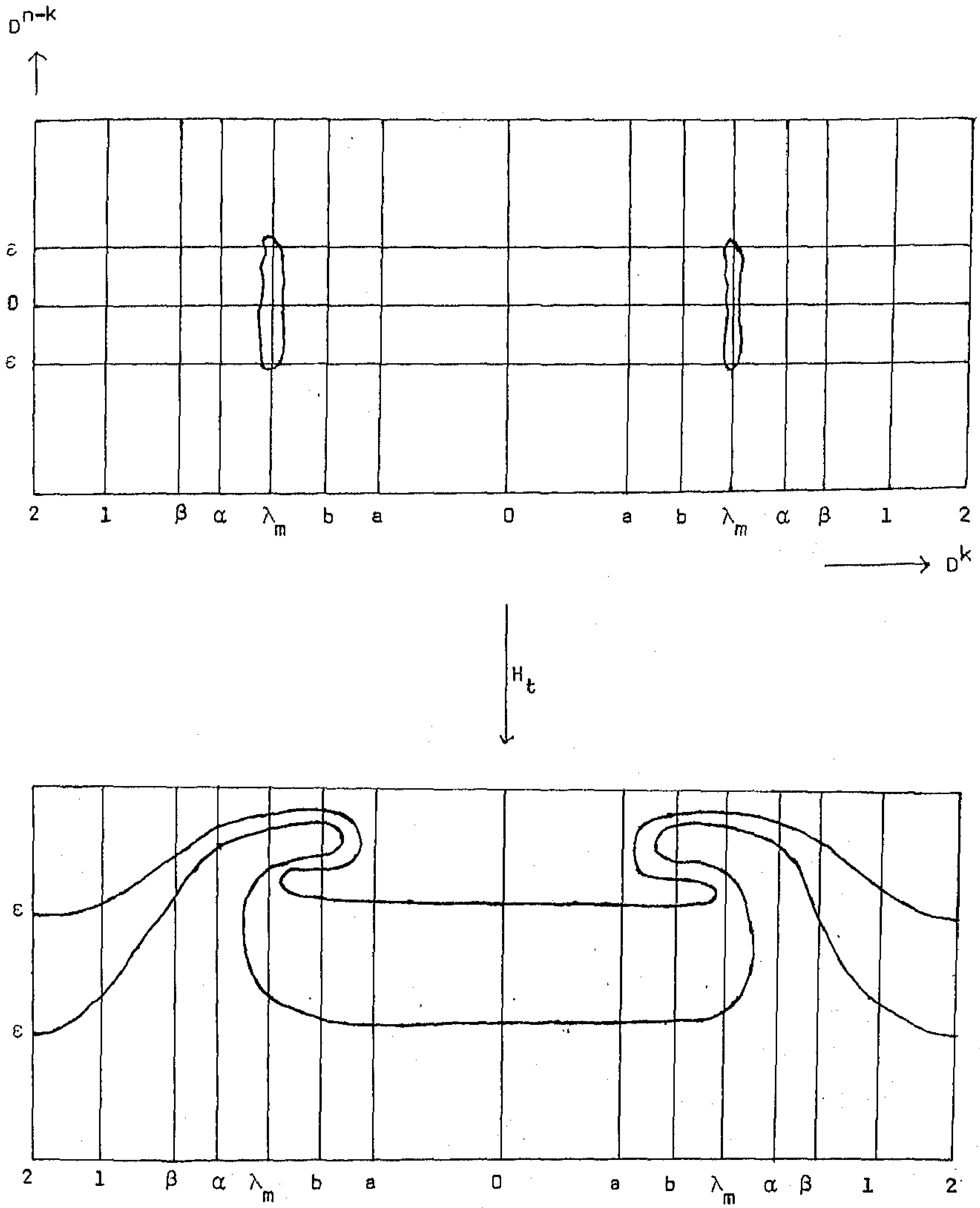
As $\mu^{q,t',t'} = G'(q,t') \in \Omega$ which is open, for all $t' \in I$, there exists $\varepsilon > 0$ such that $\mu^{q,t',t''} \in \Omega$ if $|t' - t''| < \varepsilon$. Now choose a partition as in (a) with the t_i satisfying $t_{i+1} - t_i < \varepsilon$, $0 \leq i \leq p-1$, and define $\mu_i : Q \times [t_i, t_{i+1}] \to \Omega$ by
$\mu_i(q,t)(x,s,y) = \mu^{q,t_i,t}(x,s,y)$. |||

(3) With the maps μ_i, $0 \leq i \leq p-1$, provided by the preceding lemma in hand, we prove :

<u>Lemma 2.4 (Inductive Lemma)</u> : Suppose for some m, $1 \leq m \leq p-1$, we have a lift $G_m : Q \times [0,t_m] \to \Gamma^{\infty}_{w} E(A)$ such that $G_m(q,t) = G'(q,t)$ on $S^{k-1} \times [\lambda_m, 2] \times D^{n-k}$, for some λ_m with $b < \lambda_m < 1$, and such that $G_m(q,0) = G_0(q)$. Let $\varepsilon \in (0,1)$ be given. Then there is a λ_{m+1} with $\lambda_m < \lambda_{m+1} < 1$ and a lift
$G_{m+1} : Q \times [0,t_{m+1}] \to \Gamma^{\infty}_{w} E(2D^k \times \varepsilon D^{n-k})$ such that $G_{m+1}(q,t) = G'(q,t)$ on $S^{k-1} \times [\lambda_{m+1}, 2] \times \varepsilon D^{n-k}$ and $G_{m+1}(q,0) = G_0(q)$ |.

<u>Proof</u> : Choose numbers α, β such that $\lambda_m < \alpha < \beta < 1$ and using the fact that $k < n$ define an isotopy $H_t : A \to A$, $0 \leq t \leq t_m$, such that

(i) H_t is the identity on a nbd. of $B \cup \partial A$ and on a nbd. of $S^{k-1} \times \{\lambda_m\} \times \varepsilon D^{n-k}$, $0 \leq t \leq t_m$,



D^{n-k}
ε
0
ε
2
1
β
α
λ_m
b
a
0
a
b
λ_m
α
β
1
2
D^k
H_t
ε
ε
2
1
β
α
λ_m
b
a
0
a
b
λ_m
α
β
1
2

Figure 2.1

(ii) $H_t(S^{k-1} \times [\lambda_m, 2] \times D^{n-k}) \subset S^{k-1} \times [a,2] \times D^{n-k}$, $0 \leq t \leq t_m$, and

(iii) $H_{t_m}(S^{k-1} \times [\alpha,\beta] \times \varepsilon D^{n-k}) \subset S^{k-1} \times [a,b] \times D^{n-k}$.

Figure 2.1 indicates the construction of H_t.

Next define

$$G_{m+1} : Q \times [0,t_{m+1}] \longrightarrow \Gamma_w^{\infty} E(2D^k \times \varepsilon D^{n-k}) \quad \text{by}$$

$$G_{m+1}(q,t)(x,y) = \begin{cases} G_m(q,t)(x,y), & \|x\| \leq \lambda_m, \ 0 \leq t \leq t_m \\ \Psi(H_t|)(G'(q,t))(x,y), & \lambda_m \leq \|x\| \leq 2, \ 0 \leq t \leq t_m \\ \Psi(H_{t_m}|)(\mu_m(q,t))(x,y), & \alpha \leq \|x\| \leq 2, \ t_m \leq t \leq t_{m+1} \\ G_{m+1}(q,t_m)(x,y), & \|x\| \leq \alpha, \ t_m \leq t \leq t_{m+1} \end{cases}$$

where $(x,y) \varepsilon 2D^k \times \varepsilon D^{n-k}$ and $(q,t) \varepsilon Q \times [0,t_{m+1}]$.

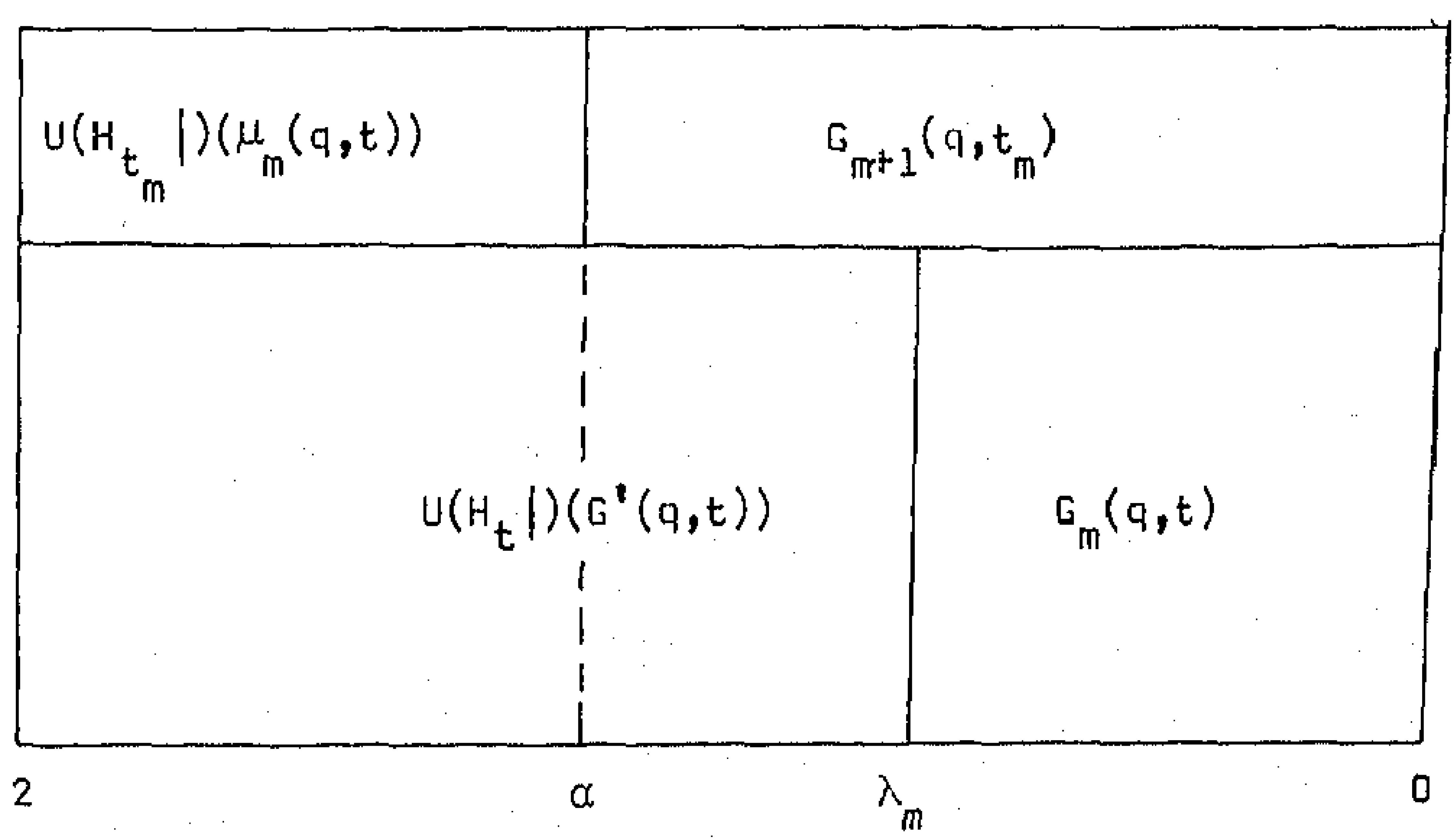


Figure 2.2

See figure 2.2. We observe that the second line is well-defined because of (ii), and that the first and second lines agree smoothly when $\| x \| = \lambda_m$ because of (i) and the fact that $G_m(q,t) = G'(q,t)$ when $\| x \| \geq \lambda_m$. The third line is again well-defined because of (ii), and the second and third lines agree when $t = t_m$ because $\mu_m(q,t_m) = G'(q,t_m)$. By (iii) and the fact that $\mu_m(q,t)(x,y) = G'(q,t_m)(x,y)$ for $a \leq \| x \| \leq b$ and $t_m \leq t \leq t_{m+1}$, we have that $G_{m+1}(q,t) = G_{m+1}(q,t_m)$ for $\alpha \leq \| x \| \leq \beta$ and $t_m \leq t \leq t_{m+1}$, so that the fourth line fits smoothly.

Noting that E^r_ω is invariant under diffeomorphisms and that $\Psi(H_t)$ varies continuously with t we see that G_{m+1} is indeed a continuous map $Q \times [0, t_{m+1}] \to \Gamma^\infty_\omega E(2D^k \times \varepsilon D^{n-k})$, finally (i) permits us to choose a suitable λ_{m+1} to complete the proof of the lemma. |||

To complete the proof of the proposition, observe that, considering A as embedded in $\mathbb{R}^n$, we may, by the openness of E^r_ω, extend the original lifting problem to the problem

$$
\begin{array}{ccc}
Q & \xrightarrow{\;H_o\;} & \Gamma^\infty_\omega E(2D^k \times \eta D^{n-k}) \\
\big\downarrow{\scriptstyle 1 \times 0} & & \big\downarrow{\scriptstyle J} \\
Q \times I & \xrightarrow{\;H\;} & \Gamma^\infty_\omega E(S^{k-1} \times [1,2] \times \eta D^{n-k}) ,
\end{array}
$$

where $\eta > 1$ and $G_o(q) = H_o(q) \mid$, $G(q,t) = H(q,t) \mid$. Obtain maps μ_i, $0 \leq i \leq p-1$, by applying Lemma 2.3 to this extended problem. Let $\eta = \eta_o > \eta_1 > \ldots > \eta_{p-1} = 1$ be an arbitrary sequence. We have a lift H_1 on $Q \times [0,t_1]$ for the extended problem by defining

$$H_1(q,t)(x,y) = \begin{cases} \mu_0(q,t)(x,y), & \|x\| \geq a \\ H_0(q)(x,y), & \|x\| \leq a \end{cases},$$

where $(x,y) \in 2D^k \times \eta D^{n-k}$. Next, repeatedly apply the Inductive Lemma to the extended problem so that after the m th application, $1 \leq m \leq p-1$, we have a lift $H_{m+1} : Q \times [0,t_{m+1}] \longrightarrow \Gamma_w^\infty E(2D^k \times \eta_m D^{n-k})$ satisfying $H_{m+1}(q,0) = H_0|$ and $H_{m+1}(q,t) = H(q,t)$ on $S^{k-1} \times [1,2] \times \eta_m D^{n-k}$ (note that with each application we need only restrict the *domain of* the $\mu_i(q,t)$ without altering the parameters t_i or a,b,c). Finally, it is $H_p : Q \times [0,1] \longrightarrow \Gamma_w^\infty E(2D^k \times D^{n-k})$ which solves the original lifting problem and completes the proof of the proposition. ///

It might not be out of place to point out here an inaccuracy in the corresponding proof provided by du Plessis [5] (following Haefliger [13]). He attempts to complete the proof, after deriving the Inductive Lemma, in the following manner :

First define

$G'_{m+1} : Q \times [0,t_{m+1}] \longrightarrow \Gamma_w^\infty E(S^{k-1} \times [\lambda_{m+1},2] \times D^{n-k} \cup 2D^k \times \varepsilon D^{n-k})$ by

$$G'_{m+1}(q,t)(x,y) = \begin{cases} G'(q,t)(x,y), & (x,y) \in S^{k-1} \times [\lambda_{m+1},2] \times D^{n-k} \\ G_{m+1}(q,t)(x,y), & (x,y) \in 2D^k \times \varepsilon D^{n-k} \end{cases}.$$

Then define $G''_{m+1} : Q \times [0,t_{m+1}] \longrightarrow \Gamma_w^\infty E(A)$ by $G''_{m+1}(q,t) = \Psi(h_t)(G'_{m+1}(q,t))$, where h_t, $0 \leq t \leq t_{m+1}$, is an isotopy of embeddings of $2D^k \times D^{n-k}$ into itself such that h_0 is the identity, h_t is an identity in a nbd. of B in A, for all t, and, $h_t(A) \subset S^{k-1} \times [\lambda_{m+1}, 2] \times D^{n-k} \cup 2D^k \times \varepsilon D^{n-k}$, for $t \geq t_m/2$.

However, this definition fails as, for small t, $h_t(x,y)$ lies outside $S^{k-1} \times [\lambda_{m+1}, 2] \times D^{n-k} \cup 2D^k \times \varepsilon D^{n-k}$, the domain of $G'_{m+1}(q,t)$, for some $(x,y) \in A$.

Main Lemma : If $\Gamma^{\infty}_{\omega} E(X)$ is locally stable, for each section $f \in \Gamma^{\infty}_{\omega} E(X)$ and each $x \in X$, there exists

(i) an open neighbourhood W of x in X,

(ii) an open neighbourhood Ω of f in $\Gamma^{\infty}_{\omega} E(X)$,

(iii) an open subbundle $E^r_0(W \times D^k)$ of $E^r(W \times D^k)$,

for some $k > 0$, which is invariant under the action of $\mathcal{D}(W \times D^k)$,

and (iv) a map $\rho : \Omega \to \Gamma^{\infty}_0 E(W \times D^k)$,

such that

(a) $\Psi(i)(\rho(g)) = g \mid W$, for every $g \in \Omega$,

and (b) $\Psi(i)(g') \in \Gamma^{\infty}_{\omega} E(W)$, for every $g' \in \Gamma^{\infty}_0 E(W \times D^k)$,

where $i = 1 \times 0 : W \to W \times D^k$.

Proof : Suppose $\Gamma^{\infty}_{\omega} E(X)$ is locally stable and choose $x \in X$ and $f \in \Gamma^{\infty}_{\omega} E(X)$. From the definition of local stability we have the following :

an open nbd. U of x in X,

a manifold N with $\dim N > \dim X$ and $U \subset \operatorname{Int} N$,

an open subbundle $E_0^r(N)$ of $E^r(N)$ which is invariant under the action of $\mathcal{D}(N)$,

a section $f' \in \Gamma_0^\infty E(N)$ such that $\Psi(i)(f') = f \mid U$, and the fact that

$\Psi(i)(g') \in \Gamma_W^\infty E(U)$ for every $g' \in \Gamma_0^\infty E(N)$,

where $i : U \longrightarrow N$ is the inclusion.

Consider the diagram

$$\begin{array}{ccccc}
E(U) & \underset{\pi(i)}{\overset{\Psi_1(i)}{\rightleftarrows}} & E(N) \mid U & \xrightarrow{\Psi_2(i)} & E(N) \\
\downarrow \text{proj.} & & \downarrow \text{proj.} & & \downarrow \text{proj.} \\
U & \xrightarrow[1_U]{} & U & \xrightarrow[i]{} & N ,
\end{array}$$

where $\pi(i) \circ \Psi_1(i) = 1_{E(U)}$ and $\Psi_2(i) \circ \Psi_1(i) = \Psi(i)$. Since $\pi(i) \circ \Psi_1(i)$ is the identity, $\pi(i)$ is a surjective submersion on fibres. By restricting to a tubular nbd. of U in N we may suppose that $N = U \times D^k$, where $k = \dim N - \dim X > 0$, and that $i : U \longrightarrow N$ is given by $i(u) = (u,0)$. Further choose so small a nbd. $\widetilde{W}$ of x in U and an $\varepsilon > 0$, that

$E(U) \mid \widetilde{W} \simeq \widetilde{W} \times F$

$E(N) \mid \widetilde{W} \times \varepsilon D^k \simeq \widetilde{W} \times \varepsilon D^k \times F'$

$E(N) \mid \widetilde{W} \simeq \widetilde{W} \times F'$, identifying $\widetilde{W}$ with $\widetilde{W} \times \{0\}$, and where F and F' are the fibres of $E(X)$ and $E(N)$, respectively. The following commutative diagram arises :

$$\begin{array}{ccc} E(N)\,|\,\widetilde{W} \simeq \widetilde{W} \times F' & \xrightarrow{\pi(i)} & \widetilde{W} \times F \simeq E(U)\,|\,\widetilde{W} \\ \downarrow \text{proj.} & & \downarrow \text{proj.} \\ \widetilde{W} & \xrightarrow[1_{\widetilde{W}}]{} & \widetilde{W}\ , \end{array}$$

where $i = 1 \times 0 : \widetilde{W} \to \widetilde{W} \times \varepsilon D^k$.

Since $\pi(i)(f'(x)) = f(x)$ and, as observed, $\pi(i)$ is a surjective submersion on fibres, we may choose local coordinates

$(w_1,\dots,w_n)$ in a nbd. W of x in $\widetilde{W}$,

$(w_1,\dots,w_n,\ y_1,\dots,y_p)$ in a nbd. $W \times Y$ of $f'(x)$ in $\widetilde{W} \times F'$,

$(w_1,\dots,w_n,\ z_1,\dots,z_q)$ in a nbd. $W \times Z$ of $f(x)$ in $\widetilde{W} \times F$,

where $\dim F' = p \geq q = \dim F$, such that

$$\pi(i)(w_1,\dots,w_n,\ y_1,\dots,y_p) = (w_1,\dots,w_n,\ y_1,\dots,y_q).$$

Assume that Y and Z are identified with $\mathbb{R}^p$ and $\mathbb{R}^q$ via the coordinates $(y_1,\dots,y_p)$ and $(z_1,\dots,z_q)$, respectively. Define a local right inverse $k : W \times Z \to W \times Y$ of $\pi(i)$ by

$$k(w_1,\dots,w_n,\ z_1,\dots,z_q) = (w_1,\dots,w_n,\ z_1,\dots,z_q,\ 0,\dots,0).$$

Since the restrictions of $E(X)$ over W and $E(N)$ over $W \times \varepsilon D^k$ are trivial we may consider $f\,|\,W$ as a map from W to F

and $f' | W \times \varepsilon D^k$ as a map from $W \times \varepsilon D^k$ to F' with $f(x) \varepsilon Z$ and $f'(x) \varepsilon Y$. Choosing smaller W and ε, if necessary, we may suppose $f(W) \subset Z$ and $f'(W \times \varepsilon D^k) \subset Y$. In fact, we may further suppose there is a nbd. $\tilde{\Omega}$ of f in $\Gamma_W^\infty E(X)$ such that $g(W) \subset Z$ for $g \varepsilon \tilde{\Omega}$. Then define the map $\rho : \tilde{\Omega} \to \Gamma^\infty E(W \times \varepsilon D^k)$ by

$$\rho(g)(w_1,\dots,w_n,d) = f'(w_1,\dots,w_n,d) - kf(w_1,\dots,w_n) + kg(w_1,\dots,w_n)$$

where $(w_1,\dots,w_n) \varepsilon W$, $d \varepsilon D^k$ (use the smooth identifications $Y \simeq \mathbb{R}^p$, $Z \simeq \mathbb{R}^q$ for arithmetic).

Observe that $\rho(f) = f' | W \times \varepsilon D^k$ and that

$$\begin{aligned} &\pi(i) \circ \rho(g) \circ i(w_1,\dots,w_n) \\ = \; &\pi(i) \circ \rho(g)(w_1,\dots,w_n,0) \\ = \; &\pi(i)\{ f'(w_1,\dots,w_n,0) - kf(w_1,\dots,w_n) + kg(w_1,\dots,w_n)\} \\ = \; &g(w_1,\dots,w_n), \end{aligned}$$

as $\quad \pi(i)\{f'(w_1,\dots,w_n,0)\} = f(w_1,\dots,w_n)$,

proving that $\pi(i) \circ \rho(g) \circ i = g | W$ or, in other words, $\Psi(i)(\rho(g)) = g | W$ for $g \varepsilon \tilde{\Omega}$.

Since $\rho(f) = f' | W \times \varepsilon D^k \in \Gamma_0^\infty E(W \times \varepsilon D^k)$, there is a nbd. Ω of f in $\tilde{\Omega}$ such that $\rho(\Omega) \subset \Gamma_0^\infty (W \times \varepsilon D^k)$ recalling that $E_0^r(N)$ is open in $E^r(N)$. The lemma is finally obtained by identifying εD^k with D^k. Observe that condition (b) is directly a consequence of the definition of local stability. ///

<u>Proposition 2.5</u> : If $A = 2D^n$ and $B = S^{n-1} \times [1,2]$ and $\Gamma^\infty_w E(A)$ is locally stable, then the restriction map,

$$J : \Gamma^\infty_w E(A) \to \Gamma^\infty_w E(B)$$

is a Serre fibration.

<u>Proof</u> : It is sufficient to show that J has the local PCHP (polyhedral covering homotopy property). So suppose $f \in \Gamma^\infty_w E(B)$. From the local stability of $\Gamma^\infty_w E(B)$ and by the Main Lemma, we have for each $x \in S^{n-1}$ $(= S^{n-1} \times \{1\})$:

a nbd. $N \times [1, 1+d]$ of x in $S^{n-1} \times [1,2]$ where N is a nbd. of x in S^{n-1},

a nbd. Ω of f in $\Gamma^\infty_w E(S^{n-1} \times [1,2])$,

an open subbundle $E^r_o(N \times [1, 1+d] \times D^k)$ of $E^r(N \times [1,1+d] \times D^k)$, where $k > 0$, which is invariant under the action of $\mathcal{D}(N \times [1,1+d] \times D^k)$,

a map $\rho : \Omega \to \Gamma^\infty_o E(N \times [1,1+d] \times D^k)$, such that

$\Psi(i)(\rho(g)) = g \mid N \times [1,1+d]$, for each $g \in \Omega$,

$\Psi(i)(g') \in \Gamma^\infty_w(N \times [1,1+d])$, for each $g' \in \Gamma^\infty_o(N \times [1,1+d] \times D^k)$,

where $i = 1 \times 1 \times 0 : N \times [1,1+d] \to N \times [1,1+d] \times D^k$.

As S^{n-1} is compact there is a finite subset $\{x_i\}$ of S^{n-1} so that the corresponding $\{N_i\}$ cover S^{n-1}. Let $\Omega = \bigcap_i \Omega_i$

and $d = \min_i d_i$. Now, consider the lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & J^{-1}\Omega \\ \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle J} \\ Q\times I & \xrightarrow{G} & \Omega, \end{array}$$

where Q is a compact polyhedron.

Smoothly extend G to a family

$$\bar{G} : Q\times I \to \Gamma^{\infty} E(S^{n-1}\times [1-a, 2]),\ a \varepsilon (0,1),$$

so that $\bar{G}(q,t)\,|\,S^{n-1}\times[1,2] = G(q,t)$ and $\bar{G}(q,0) = G_0(q)\,|\,S^{n-1}\times[1-a,2]$.

For $0 \le s \le a$, define $L_s : S^{n-1}\times[1,2] \to S^{n-1}\times[1-s,2]$ to be the diffeomorphism given by linear expansion along radial segments.

Define $A : Q\times I\times[0,a] \to \Gamma^{\infty}E(S^{n-1}\times[1,2])$ by $A(q,t,s) = \Psi(L_s)(\bar{G}(q,t)\,|\,S^{n-1}\times[1-s,2])$. As $A(q,t,0) = G(q,t) \varepsilon \Omega$, for $(q,t) \varepsilon Q\times I$, by the continuity of A and compactness of $Q\times I$, there is $b > 0$ such that $\Psi(L_s)(\bar{G}(q,t)\,|\,S^{n-1}\times[1-s,2])$ is in Ω for $0 \le s \le 2b$, $(q,t) \varepsilon Q\times I$. Now, identify $S^{n-1}\times[1,2]$ with $S^{n-1}\times[1-2b,2]$ by L_{2b}, and correspondingly identify $E(S^{n-1}\times[1,2])$ and $E^r(S^{n-1}\times[1,2])$ with $E(S^{n-1}\times[1-2b,2])$ and $E^r(S^{n-1}\times[1-2b,2])$, respectively, by $\Psi(L_{2b})$. By these identifications, there correspond to the families

$$N_i\times[1,1+d],\ \Omega\ (\subset \Gamma^{\infty}_{w}E(S^{n-1}\times[1,2])),$$

$E^r_i(N_i\times[1,1+d]\times D^{k_i})$ a subbundle of $E^r(N_i\times[1,1+d]\times D^{k_i})$, and $\rho_i : \Omega \to \Gamma^{\infty}_i E(N_i\times[1,1+d]\times D^{k_i})$ the following families, respectively,

$$N_i \times [\,1\text{-}2b,\ 1\text{+}c\,],\ \widetilde{\Omega}\ (\subset \Gamma^{\infty}_{w}E(S^{n-1}\times[\,1\text{-}2b,2\,]),$$

$E^r_i(N_i \times [\,1\text{-}2b,\ 1\text{+}c\,]\times D^{k_i})$ a subbundle of $E^r(N_i \times [\,1\text{-}2b,\ 1\text{+}c\,]\times D^{k_i})$, and $\widetilde{\rho}_i : \widetilde{\Omega} \to \Gamma^{\infty}_i E(N_i \times [\,1\text{-}2b,\ 1\text{+}c\,]\times D^{k_i})$.

Choosing b sufficiently small we ensure that $1\text{+}c = L_{2b}(1+d) > 1$.

Most importantly, observe that $\overline{G}(q,t)\,|\,S^{n-1}\times[1\text{-}2b,2\,] \in \widetilde{\Omega}$ for $(q,t) \in Q\times I$. This was, in fact, the object of the identifications.

Henceforth, suppose the domain of each $\overline{G}(q,t)$ to be $S^{n-1}\times[\,1\text{-}2b,2\,]$. Define $G' : Q\times I \to \Gamma^{\infty}_{w}E(S^{n-1}\times[1,2\,]\cup(1\text{-}b)D^n)$ by

$$G'(q,t)(x) = \begin{cases} G(q,t)(x), & x \in S^{n-1}\times[\,1,2\,] \\ G_o(q)(x), & x \in (1\text{-}b)D^n, \end{cases} \qquad \text{for } (q,t) \in Q\times I.$$

Putting $X = S^{n-1}\times\{[\,1\text{-}2b,\ 1\text{-}b\,]\cup[\,1,1\text{+}c\,]\}$ and $Y = S^{n-1}\times[\,1\text{-}2b,\ 1\text{+}c\,]$, it is sufficient now to lift $(G'\,|\,X,\ G_o\,|\,Y)$.

Note, $\quad G'(q,t)\,|\,S^{n-1}\times[\,1,1\text{+}c\,] = \overline{G}(q,t)\,|$,

$\quad G'(q,t)\,|\,S^{n-1}\times[\,1\text{-}2b,\ 1\text{-}b\,] = \overline{G}(q,0)\,|$,

and $\quad G_o(q)\,|\,Y = \overline{G}(q,0)\,|$.

Let K be a triangulation of S^{n-1} each of whose $(n\text{-}1)$-simplexes $|A|$ lies in exactly one of the nbds. N_i, say $N_{i(A)}$. For each simplex C of K, choose an open nbd. $N(C)$ of $|C|\times[\,1\text{-}2b,\ 1\text{+}c\,]$ such that $N(C) \subset N_{i(A)}\times[\,1\text{-}2b,\ 1\text{+}c\,]$ for each $(n\text{-}1)$-simplex A such that $C < A$.

Define $\quad {}^{A}H^{C}_{o} : Q \to \Gamma^{\infty}_{i(A)}E(N(C)\times D^{k_{i(A)}})$

by $\quad {}^{A}H^{C}_{o}(q) = \widetilde{\rho}_{i(A)}(\overline{G}(q,0))\,|\,N(C)\times D^{k_{i(A)}}$, and $^{A}H^{C} : Q \times I \to \Gamma^{\infty}_{i(A)} E((N(C) \cap X) \times D^{k_{i(A)}})$

by $$^{A}H^{C}(q,t) \mid N(C) \cap (S^{n-1} \times [1, 1+c\,]) \times D^{k_{i(A)}}$$

$$= \tilde{\rho}_{i(A)}(\overline{G}(q,t)) \mid N(C) \cap (S^{n-1} \times [1,1+c\,]) \times D^{k_{i(A)}}$$

and $$^{A}H^{C}(q,t) \mid N(C) \cap (S^{n-1} \times [\,1-2b, 1-b\,]) \times D^{k_{i(A)}}$$

$$= \tilde{\rho}_{i(A)}(\overline{G}(q,0)) \mid N(C) \cap (S^{n-1} \times [\,1-2b,\ 1-b\,]) \times D^{k_{i(A)}}$$

for each $C \in K$ and each $(n-1)$-simplex A such that $C < A$ (all well-defined as $\overline{G}(q,t)$ is in $\tilde{\Omega}$!).

Denote by K^{j} the j-skeleton of K and suppose inductively that we have constructed, for some $j \geq 0$, the following :

(i) a nbd. $\overline{N}(C)$ of $|C| \times [\,1-2b,\ 1+c\,]$ in $N(C)$ for each $(j-1)$-simplex C so that

$$N(K^{j-1}) = X \cup \bigcup \{\overline{N}(C) : C \text{ is a } (j-1)\text{-simplex of } K\}$$

is a union of X and nbd. of $|K^{j-1}| \times [\,1-2b,\ 1+c\,]$ in Y,

(ii) $G^{j-1} : Q \times I \to \Gamma^{\infty}_{\omega} E(N(K^{j-1}))$ lifting $(G' \mid X,\ G_{0} \mid N(K^{j-1}))$,

and (iii) $^{A}H^{C} : Q \times I \to \Gamma^{\infty}_{i(A)} E(\overline{N}(C) \times D^{k_{i(A)}})$ lifting $(^{A}H^{C} \mid (\overline{N}(C) \cap X) \times D^{k_{i(A)}},\ ^{A}H^{C}_{0} \mid \overline{N}(C) \times D^{k_{i(A)}})$, such that

(a) $\Psi(i)(^{A}H^{C}(q,t)) = G^{j-1}(q,t) \mid$, for each $(j-1)$-simplex C and each $(n-1)$-simplex A such that $C < A$, where $i = 1 \times 0 :$ $\overline{N}(C) \hookrightarrow \overline{N}(C) \times D^{k_{i(A)}}$.

Further, if both $(j-1)$-simplexes $C, C' < A$, an $(n-1)$-simplex, we should have

(b) ${}^{A}H^{C}(q,t) = {}^{A}H^{C'}(q,t)$ on $(\overline{N}(C) \cap \overline{N}(C')) \times D^{k_{i(A)}}$.

We may begin induction with $j = 0$ when $K^{-1} = \varphi$. Choose a nbd. $N'(K^{j-1})$ of $X \cup |K^{j-1}| \times [1-2b, 1+c]$ in $N(K^{j-1})$ and for each j-simplex E choose a nbd. $\overline{N}(E)$ of $E \times [1-2b, 1+c]$ in $N(E)$ such that

$$\overline{N}(E) \cap N'(K^{j-1}) \subset \cup \{\overline{N}(C) : C \text{ is a } (j-1)\text{-simplex, } C < E\},$$

and there is a diffeomorphism

$$(\overline{N}(E), \overline{N}(E) \cap N'(K^{j-1})) \simeq (2D^{j+1} \times D^{n-j-1}, S^{j} \times [1,2] \times D^{n-j-1}).$$

Then, for each j-simplex E we have a $(j+1)$-handle lifting problem

$$\begin{array}{ccc}
Q & \xrightarrow{G_o|} & \Gamma^{\infty}_{\omega} E(\overline{N}(E)) \\
\Big\downarrow 1 \times 0 & & \Big\downarrow J \\
Q \times I & \xrightarrow{G_{j-1}|} & \Gamma^{\infty}_{\omega} E(\overline{N}(E) \cap N'(K^{j-1}))
\end{array}$$

and for each j-simplex E and each $(n-1)$-simplex A such that $E < A$ we have a $(j+1)$-handle lifting problem

$$\begin{array}{ccc}
Q & \xrightarrow{{}^{A}H^{E}_{o}|} & \Gamma^{\infty}_{i(A)} E(\overline{N}(E) \times D^{k_{i(A)}}) \\
\Big\downarrow 1 \times 0 & & \Big\downarrow J \\
Q \times I & \xrightarrow{{}^{A}\overline{H}^{E}} & \Gamma^{\infty}_{i(A)} E((\overline{N}(E) \cap N'(K^{j-1})) \times D^{k_{i(A)}})
\end{array}$$

where we define ${}^{A}\overline{H}^{E} = {}^{A}H^{C}(q,t)$ on $\overline{N}(C) \times D^{k_{i(A)}}$ — well-defined by (b) above. Observe, for each $(n-1)$-simplex A such that $E < A$, $\Psi(i)\,{}^{A}H^{E}_{o}(q) = G_{o}(q)|$ and $\Psi(i)\,{}^{A}\overline{H}^{E}(q,t) = G^{j-1}(q,t)|$ by (a) above.

To make the induction step consider two cases :

Firstly, if $j < n-1$, use Proposition 2.6 described below to find lifts ${}^AH^E$ for $({}^A\tilde{H}^E, {}^AH_0^E|)$ such that $\Psi(i)({}^AH^E) = \Psi(i)({}^{A'}H^E)$ for $(n-1)$-simplexes A, A' such that $E < A, A'$. Then define G^j on $N(K^j) = X \cup \bigcup\{\overline{N}(E) : E$ is a j-simplex of $K\}$ by

$$G^j(q,t)(x) = \begin{cases} \Psi(i)({}^AH^E(q,t))(x), & x \in \overline{N}(E),\ E \text{ a } j\text{-simplex of } K \\ G^{j-1}(q,t)(x), & x \in N'(K^{j-1}). \end{cases}$$

Secondly, if $j = n-1$, use Proposition 2.4 to find lifts ${}^AH^A$ for $({}^A\tilde{H}^A, {}^AH_0^A|)$. Then define G^{n-1} on

$N(K^{n-1}) = \bigcup\{\overline{N}(A) : A$ is an $(n-1)$-simplex of $K\} = Y$ by

$G^{n-1}(q,t)(x) = \Psi(i)({}^AH^A(q,t))(x)$ for $x \in A$, A an $(n-1)$-simplex of K. Then, G^{n-1} is a lift for $(G'|X, G_0|Y)$. |||

<u>Proposition 2.6</u> : Suppose $A = 2D^k \times D^{n-k}$, $B = S^{k-1} \times [1,2] \times D^{n-k}$, where $k < n$, and suppose, for $j = 1,\dots,r$, we have open subbundles $E_j^r(A \times D^{k_j})$ of $E^r(A \times D^{k_j})$, each $k_j > 0$, which are invariant under the action of $\mathcal{D}(A \times D^{k_j})$.

Suppose we have a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & \Gamma_w^\infty E(A) \\ \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle J} \\ Q \times I & \xrightarrow{G} & \Gamma_w^\infty E(B) \end{array}$$

and lifting problems

$$\begin{array}{ccc} Q & \xrightarrow{\ {}_jG_o\ } & \Gamma^{\infty}_j E(A \times D^{k_j}) \\ \Big\downarrow 1\times 0 & & \Big\downarrow J \\ Q \times I & \xrightarrow{\ {}_jG\ } & \Gamma^{\infty}_j E(B \times D^{k_j}), \end{array}$$

Q is a compact polyhedron, $1 \leq j \leq r$, such that

$$\Psi(i)({}_jG_o(q)) = G_o(q) \quad \text{and} \quad \Psi(i)({}_jG(q,t)) = G(q,t)$$

for $q \in Q$, $t \in I$, $1 \leq j \leq r$.

Then there is a lift $\overline{G}$ of (G, G_o) and lifts ${}_j\overline{G}$ of $({}_jG,\ {}_jG_o)$ *such that*

$$\Psi(i)({}_j\overline{G}(q,t)) = \overline{G}(q,t),\ (q,t) \in Q \times I,\ 1 \leq j \leq r.$$

<u>Proof</u> : The constructions parallel that of Proposition 2.4.

(1) Extend G to $G' : Q \times I \longrightarrow \Gamma^{\infty}_{\omega} E(U)$ where U is a nbd. of B in A so that $G(q,t) = G'(q,t)|$ and $G'(q,0) = G_o(q)|$ for $q \in Q$, $t \in I$ (use the openness of $E^r_{\omega}(A)$).

Then extend each ${}_jG$, $1 \leq j \leq r$, to ${}_jG' : Q \times I \longrightarrow \Gamma^{\infty} E(U_j \times D^{k_j})$ where U_j is a nbd. of B in U so that ${}_jG(q,t) = {}_jG'(q,t)|$, ${}_jG'(q,0) = {}_jG_o(q)|$ for $q \in Q$, $t \in I$, and $\Psi(i)({}_jG'(q,t)) = G'(q,t)\,|\,U_j$, where $i = 1 \times 0 : U_j \longrightarrow U_j \times D^{k_j}$. These extensions are obtained as follows :

Fix j, $1 \leq j \leq r$. Consider the bundles $E(A) \longrightarrow A$ and

$E(A \times D^{k_j}) \,|\, A \longrightarrow A$ $(A = A \times \{0\})$. Let $p : A \times Q \times I \longrightarrow A$ be the natural projection. Since $\pi(i)$ in

$$\begin{array}{ccc} E(A \times D^{k_j}) \,|\, A & \xrightarrow{\pi(i)} & E(A) \\ & \text{bundle proj.} \searrow \quad \swarrow \text{bundle proj.} & \\ & A & \end{array}$$

is a surjective submersion on fibres (compare Main Lemma), so is $p^*\pi(i)$ in

$$\begin{array}{ccc} p^*(E(A \times D^{k_j}) \,|\, A) & \xrightarrow{p^*\pi(i)} & p^*E(A) \\ & \text{pulled-back proj.} \searrow \quad \swarrow \text{pulled-back proj.} & \\ & A \times Q \times I\,. & \end{array}$$

Now, G' and ${}_jG \cup {}_jG_o \,|$ provide partial sections of the two pulled-back bundles in a commutative diagram

$$\begin{array}{ccc} p^*(E(A \times D^{k_j}) \,|\, A) & \xrightarrow{p^*\pi(i)} & p^*E(A) \\ \Big\downarrow {}_jG \cup {}_jG_o \,| & & \Big\downarrow G' \\ B \times Q \times I \cup U \times Q \times \{0\} & \xhookrightarrow{\text{incl.}} & U \times Q \times I\,. \end{array}$$

For each $s \in S^{k-1} \times \{1\} \times D^{n-k}$ (a point on the inside boundary of B), $q \in Q$, $t \in I$, we may set up local co-ordinates in a nbd. of ${}_jG(q,t)(s)$ in $p^*(E(A \times D^{k_j}) \,|\, A)$ and in a nbd. of $G'(q,t)(s)$ in $p^*E(A)$ so that $p^*\pi(i)$ is the natural projection. By the local triviality of $p^*\pi(i)$, we may extend ${}_jG \cup {}_jG_o$ to cover G' over a nbd. of (s,q,t) in $U \times Q \times I$. Finally, using the compactness of $S^{k-1} \times D^{n-k} \times Q \times I$ to piece together the local extensions, it is possible to extend ${}_jG \cup {}_jG_o$ to a section ${}_j\bar{G}$, covering G', defined on $U_j \times Q \times I$ where U_j is a nbd. of B in U. Next, considering ${}_jG \cup {}_j\bar{G}$ as a $(Q \times I)$-family of smooth sections of $E(A \times D^{k_j}) \mid B \times D^{k_j} \cup U_j \times \{0\}$, we may extend smoothly to ${}_jG' : Q \times I \to \Gamma^\infty E(U_j \times D^{k_j})$.

Since ${}_jG'(q,t) \mid B \times D^{k_j} \varepsilon \Gamma_j^\infty(B \times D^{k_j})$, $(q,t) \varepsilon Q \times I$, we may (by continuity, openness, etc.) choose $a \varepsilon (0,1)$ such that ${}_jG'(q,t) \mid S^{k-1} \times [a,2] \times D^{n-k} \times D^{k_j}$ is E_j^r-regular (a section $f : X \to E(X)$ is E_j^r-regular if $f \varepsilon \Gamma_j^\infty E(X)$), and $G'(q,t) \mid S^{k-1} \times [a,2] \times D^{n-k}$ is E_w^r-regular, for each j and (q,t). Henceforth, suppose ${}_jG'$ and G' as defined on these restricted domains.

(2) As in step (2) of Proposition 2.4, construct

(a) a partition $0 = t_o < t_1 < \ldots < t_p = 1$ of I

(b) numbers b,c such that $a < b < c < 1$, and

(c) maps $\mu_i : Q \times [t_i, t_{i+1}] \to \Gamma_w^\infty E(S^{k-1} \times [a,2] \times D^{n-k})$,

$${}_j\mu_i : Q \times [t_i, t_{i+1}] \to \Gamma_j^\infty E(S^{k-1} \times [a,2] \times D^{n-k} \times D^{k_j})$$

such that

$$\mu_i(q,t)(x,s,y) = \begin{cases} G'(q,t)(x,s,y), & t = t_i \quad \text{or} \quad c \le s \le 2 \\ G'(q,t_i)(x,s,y), & a \le s \le b, \end{cases}$$

$$_j\mu_i(q,t)(x,s,y,z) = \begin{cases} _jG'(q,t)(x,s,y,z), & t = t_i \quad \text{or} \quad c \le s \le 2 \\ _jG'(q,t_i)(x,s,y,z), & a \le s \le b, \end{cases}$$

and $\Psi(i)(_j\mu_i(q,t)) = \mu_i(q,t)$, for $0 \le i \le p-1$, $1 \le j \le r$.

We outline the construction. Henceforth, denote $S^{k-1} \times [a,2] \times D^{n-k}$ by C_a.

Find an $\varepsilon > 0$ and maps $\mu^{q,t',t''}$ for $t' < t'' < t' + \varepsilon$ exactly as in step (2) of Proposition 2.4. Define $\mu^{t',t''} : C_a \times Q \times [t',t''] \to F$ by $\mu^{t',t''}(x,s,y,q,t) = \mu^{q,t',t}(x,s,y)$ where $(x,s,y) \in C_a$, so that $\mu^{t',t''}$ is smooth with respect to x,s,y. Consider also G' and $_jG'$ as maps from $C_a \times Q \times I$ to F and $C_a \times D^{k_j} \times Q \times I$ to F', respectively, where F and F' are the respective fibres. Observe that $\mu^{t',t''}$ is homotopic to $G' \mid C_a \times Q \times [t', t'']$ (move along the unique geodesic arcs - see Proposition 2.4). Let $p : C_a \times Q \times [t',t''] \to C_a$ be the natural projection. Pulling back the diagram of bundle maps

$$\begin{array}{ccc} E(C_a \times D^{k_j}) \mid C_a & \xrightarrow{\pi(i)} & E(C_a) \\ & \searrow \quad \swarrow & \\ & C_a & \end{array}$$

by p we have the diagram of hundle maps

$$\begin{array}{ccc} p^*(E(C_a \times D^{k_j}) \mid C_a) & \xrightarrow{p^*\pi(i)} & p^*E(C_a) \\ & \searrow \quad \swarrow & \\ & C_a \times Q \times [t',t''] & \end{array}$$

where $p^*\pi(i)$ is a surjective submersion on fibres. Now we have a diagram of sections of the preceding bundle maps so that the unbroken lines form a commutative diagram :

$$\begin{array}{ccc} p^*(E(C_a \times D^{k_j}) \mid C_a) & \xrightarrow{p^*\pi(i)} & p^*E(C_a) \\ {}_jG' \mid \nwarrow \; \nwarrow & \mu^{t',t''} & \nearrow \; \nearrow \; G' \mid \\ & C_a \times Q \times [t',t'']. & \end{array}$$

As in the previous step we may set up local coordinates in a nbd. of each point of the total spaces so that $p^*\pi(i)$ is locally a natural projection. If we took care to choose $\varepsilon > 0$ so small that the paths, above each point of the base space, of the homotopy between $\mu^{t',t''}$ and $G' \mid$ lay within such nbds. we could lift the homotopy (ensuring that paths that are stationary with the homotopy lift to stationary paths) and so find ${}_j\mu^{t',t''}$ (the unmarked line in the diagram) such that $p^*\pi(i) \circ {}_j\mu^{t',t''} = \mu^{t',t''}$. The partial sections defined by

$${}_j\mu^{t',t''} \text{ on } C_a \times \{0\} \times Q \times [t',t''],$$

$${}_jG' \text{ on } C_a \times D^{k_j} \times Q \times \{t'\} \cup C_c \times D^{k_j} \times Q \times [t',t'']$$

and $_jG'(x,y,q,t')$ at (x,y,q,t) when $x \in S^{k-1} \times [a,b] \times D^{n-k}$, may be smoothly extended to all of $C_a \times D^{k_j} \times Q \times [t',t'']$. Choosing t',t'' close enough to ensure E_j^r-regularity of such extensions, we obtain the required maps $_j\mu_i$.

(3) The following lemma holds (compare Inductive Lemma of Proposition 2.4) :

<u>Lemma 2.5 (Extended Inductive Lemma)</u> : Suppose for some m, $1 \leq m \leq p-1$, we have a lift

$$G_m : Q \times [0,t_m] \longrightarrow \Gamma_w^\infty E(A) \text{ and lifts}$$

$$_jG_m : Q \times [0,t_m] \longrightarrow \Gamma_j^\infty E(A \times D^{k_j}),\ 1 \leq j \leq r,$$

such that for some λ_m, where $b < \lambda_m < 1$,

$G_m(q,t) = G'(q,t)$ on C_{λ_m}, $G_m(q,0) = G_o(q)$,

$_jG_m(q,t) = {_jG'}(q,t)$ on $C_{\lambda_m} \times D^{k_j}$, $_jG_m(q,0) = {_jG_o}(q)$, and

$\Psi(i)({_jG_m}(q,t)) = G_m(q,t)$, for $(q,t) \in Q \times [0,t_m]$ and $1 \leq j \leq r$.

Further, let $\varepsilon \in (0,1)$. Then there is a λ_{m+1}, where $\lambda_m < \lambda_{m+1} < 1$, and a lift $G_{m+1} : Q \times [0,t_{m+1}] \longrightarrow \Gamma_w^\infty E(A)$ and lifts

$$_jG_{m+1} : Q \times [0,t_{m+1}] \longrightarrow \Gamma_j^\infty E(A \times D^{k_j}),\ 1 \leq j \leq r,$$

such that $G_{m+1}(q,t) = G'(q,t)$ on $C_{\lambda_{m+1}}$, $G_{m+1}(q,0) = G_o(q)$,

$_jG_{m+1}(q,t) = {_jG'}(q,t)$ on $C_{\lambda_{m+1}} \times D^{k_j}$, $_jG_{m+1}(q,0) = {_jG_o}(q)$, and

$\Psi(i)({_jG_{m+1}}(q,t)) = G_{m+1}(q,t)$, for $(q,t) \in Q \times [0,t_{m+1}]$, $1 \leq j \leq r$.

Proof : This lemma may be proved exactly as the Inductive Lemma of Proposition 2.4, provided we use the isotopy $H_t \times 1_{D^{k_j}}$: $A \times D^{k_j} \longrightarrow A \times D^{k_j}$ simultaneously with the prescribed isotopy $H_t : A \longrightarrow A$. |||

We proceed with the induction, again as in Proposition 2.4, first extending the lifting problem by replacing D^{n-k} by ηD^{n-k}, $1 < \eta$, in the statement of the proposition. Obtain the maps μ_i and ${}_j\mu_i$, $0 \leq i \leq p-1$, $1 \leq j \leq r$, of step (2) for these extended problems. Define a lift H_1 and lifts ${}_jH_1$ on $Q \times [0, t_1]$ for the extended problems by

$$H_1(q,t)(x,y) = \begin{cases} \mu_0(q,t)(x,y), & \|x\| \geq a \\ G_0(q,t)(x,y), & \|x\| \leq a \end{cases},$$

$${}_jH_1(q,t)(x,y) = \begin{cases} {}_j\mu_0(q,t)(x,y), & \|x\| \geq a \\ {}_jG_0(q,t)(x,y), & \|x\| \leq a \end{cases}.$$

Of course, G_0 and ${}_jG_0$ correspond to the extended problem.

This starts the induction and repeated application of the Extended Inductive Lemma leads to a solution of our original problem. |||

CHAPTER 3

APPLICATIONS

Throughout all the sections of this chapter X and Y will denote Riemannian manifolds of dimension n and m respectively. If Z is a submanifold of X we shall take for $E(Z)$ the trivial bundle $Z \times Y \longrightarrow Z$ with the functor $\Psi : \mathcal{E}(X) \longrightarrow \mathcal{E}(X \times Y)$ defined by $\Psi(\lambda)(x,y) = (\lambda(x),y)$. Then $\Gamma^{\infty} E(X)$ may be identified with the space $C^{\infty}(X,Y)$ and $E^{r}(X)$ becomes the r-jet bundle $J^{r}(X,Y) \longrightarrow X$, and $\mathcal{E}(X)$ acts on the right of $\Gamma^{\infty} E(X)$ (or $E^{r}(X)$) by the composition of maps (or jets). Further, if $\lambda : Z \longrightarrow W$ is an embedding in $\mathcal{E}(X)$, the map $\pi(\lambda) : \lambda^{*}E(W) \longrightarrow E(Z)$ is defined by $(z,\lambda(z),y) \mapsto (z,y)$.

§ 1. <u>k-mersions</u> : Let $E^{1}_{w}(X) \subset E^{1}(X)$ be the subspace of 1-jets of maps $X \longrightarrow Y$ of rank at least k where $k \leq \min(n,m)$. Then $\Gamma^{\infty}_{w} E(X)$ is the space $\mathrm{Mer}_{k}(X,Y)$ of k-mersions of X into Y and $\Gamma^{o} E^{1}_{w}(X)$ is the space $\mathrm{Hom}_{k}(TX,TY)$ of tangent bundle homomorphism of TX into TY whose restrictions to each fibre has rank at least k.

Local stability may be realized for $\mathrm{Mer}_{k}(X,Y)$ if $k < m$. In this case, for any k-mersion $f : X \longrightarrow Y$ and any point $x_{0} \in X$, there is an open nbd. V of x_{0} in X, and k vector fields $v_{1},\ldots,v_{k}$ on V such that $df|_{x}(v_{1}(x)),\ldots,df|_{x}(v_{k}(x))$ are linearly independent for $x \in V$. Let B be a closed n-ball in V containing x_{0}. Fixing a Riemannian metric on Y, we may split the pull-back bundle $(f|B)^{*}TY$ of TY by $f|B$ so that $(f|B)^{*}TY = \mu \oplus \eta$ where the fibre of μ over $x \in B$ is spanned by $df|_x(v_1(x)),\ldots, df|_x(v_k(x))$ and η is orthogonal to μ. Let $\bar{f} : (f|B)^*TY \longrightarrow TY$ be the canonical map of the pull-back. Let G be a nbd. of the zero-section of TY on which the map $\mathrm{Exp} : TY \longrightarrow Y$ given by $\mathrm{Exp}(x,v) = \exp_x v$, exp being the exponential map of the Riemannian connection of Y, is defined. Since B is compact we can choose $\varepsilon > 0$ so that the ε-disc bundle N associated to η is mapped into G by $\bar{f}$. We check easily that the map $f' = \mathrm{Exp}\circ(\bar{f}|N)$ is a submersion $N \longrightarrow Y$ and that $f'|B = f|B$. Moreover, since $\dim N = m+n-k$, for any submersion $g' : N \longrightarrow Y$ and any embedding $\lambda : B \longrightarrow N$, rank $g' \circ \lambda \geq$ rank g' + rank $\lambda - (m+n-k) = m+n-(m+n-k) = k$, by Sylvester's law. In particular, any submersion $g' : N \longrightarrow Y$ restricts to a k-mersion $B \longrightarrow Y$. Thus, if we allow $E(N)$ to be the trivial bundle $N \times Y$ and $E^1_o(N)$ the subspace of 1-jets of local submersions $N \longrightarrow Y$, then the conditions of local stability may be seen to be satisfied by taking $U = \mathrm{Int}\, B$. Consequently we obtain the k-mersion theorem of Feit [7] :

<u>Theorem 3.1 (Feit)</u> : If $k < m$, then the differential map,

$$d : \mathrm{Mer}_k(X,Y) \longrightarrow \mathrm{Hom}_k(TX,TY)$$

is a w.h.e. |||

§ 2. <u>Non-degenerate immersions</u> : For a fixed integer $r \geq 1$, the r th order tangent bundle $T_r(X)$ of X is the vector bundle of linear differential operators of order $\leq r$ on smooth real valued functions on X. If $x_1,\ldots,x_n$ are local co-ordinates around $x \in X$, then the fibre $T_r(X)_x$ is the real vector space spanned by the linear functionals $\partial^k/\partial x_{i_1} \ldots \partial x_{i_k}\big|_x$, $1 \leq k \leq r$, $1 \leq i_1 \leq \ldots \leq i_k \leq n$. The fibre dimension of $T_r(X)$ may then be seen to be

$$\nu(n,r) = \sum_{i=1}^{r} \binom{n+i-1}{i}.$$

If $f : X \longrightarrow Y$ is smooth, there is induced a vector bundle homomorphism $T_r(f) : T_r(X) \longrightarrow T_r(Y)$, covering f, which is defined by $T_r(f)(u) = u \circ f^*$ where $f^* : C^{\infty}(Y,\mathbb{R}) \longrightarrow C^{\infty}(X,\mathbb{R})$ is the function composition on the right by f. Clearly T_r is a functor from the category of smooth manifolds to the category of smooth vector bundles and we have $T_1(X) = TX$, the usual tangent bundle, and $T_1(f) = df$, the usual differential map. Generally, $T_r(f)$ is called the r th order differential of f.

For each $r \geq 2$ we have a natural exact sequence of vector bundles

$$0 \longrightarrow T_{r-1}(X) \xrightarrow{I_{r-1}} T_r(X) \xrightarrow{P_{r-1}} O^r(TX) \longrightarrow 0 \qquad \ldots(1)$$

where $O^r(TX)$ is the r-fold symmetric tensor product of TX, I_{r-1} is the canonical inclusion, and $P_{r-1} = m_{r-1}^{-1} \circ \pi_{r-1}$ where $\pi_{r-1} : T_r(X) \longrightarrow T_r(X)/T_{r-1}(X)$ is the canonical epimorphism and $m_{r-1} : O^r(TX) \longrightarrow T_r(X)/T_{r-1}(X)$ is the isomorphism defined in terms of local co-ordinates by

$$m_{r-1}(\partial/\partial x_{i_1}|_x \circ \dots \circ \partial/\partial x_{i_r}|_x) = \pi_{r-1}(\partial^r/\partial x_{i_1} \dots \partial x_{i_r}|_x).$$

The exact sequence is natural in the sense that the following diagram commutes for any smooth map $f : X \to Y$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{r-1}(X) & \longrightarrow & T_r(X) & \longrightarrow & 0^r(TX) & \longrightarrow & 0 \\ & & \downarrow T_{r-1}(f) & & \downarrow T_r(f) & & \downarrow 0^r(Tf) & & \\ 0 & \longrightarrow & T_{r-1}(Y) & \longrightarrow & T_r(Y) & \longrightarrow & 0^r(TY) & \longrightarrow & 0 \end{array} \qquad \dots(2).$$

A splitting of the exact sequence (1) is called an $(r-1)$th order dissection on X, $D_{(r-1)} : T_r(X) \to T_{r-1}(X)$. A classical theorem of Ambrose, Palais and Singer [1] says that the 1st order dissections on X are in 1-1 correspondence with symmetric linear connections on X. Moreover, by a result of Pohl [28] (see also Feldman [8]) a 1st order dissection on X induces an r th order dissection on X for all r.

Let $D_{(i)}$ be the sequence of dissections induced on Y by its Riemannian structure and let

$$D_r = D_{(1)} \circ \dots \circ D_{(r-1)} : T_r(Y) \to TY .$$

For any smooth map $f : X \to Y$, the bundle homomorphism $D_r \circ T_r(f) : T_r(X) \to TY$ covering f is called the r th order osculating map of f with respect to the dissections $D_{(i)}$ on Y. The map f is called r th order nondegenerate (with respect to the given dissections on Y) if $D_r \circ T_r(f)$ is of maximal rank everywhere on X. It follows from diagram (2) that if $\nu(n,r) \leq m$ and if f is r th order nondegenerate, then it is s th order nondegenerate for all $s \leq r$. Thus, if $\nu(n,r) \leq m$, all r th order nondegenerate maps are immersions.

Let $Imm_r(X,Y)$ be the space of r th order nondegenerate immersions. Clearly $Imm_r(X,Y)$ is open in $C^\infty(X,Y)$. Let $HOM(T_r(X), TY)$ be the bundle over $X \times Y$ whose fibre over $(x,y) \in X \times Y$ is the space of linear maps $T_r(X)_x \to (TY)_y$. Then the correspondence $j_x^r f \mapsto (D_r \circ T_r(f))_x$, where f is a germ at x of a map $X \to Y$ with $f(x) = y$, defines a bundle isomorphism $J : J^r(X,Y) \to HOM(T_r(X), TY)$ and, therefore, we may identify $J^r(X,Y)$ with $HOM(T_r(X), TY)$ and $\Gamma^0 J^r(X,Y)$ with the space of bundle maps $T_r(X) \to TY$. Let $Mono(T_r(X), TY)$ be the space of bundle maps which are fibrewise injective. The subbundle $MONO(T_r(X), TY)$ of $HOM(T_r(X), TY)$, whose space of continuous sections is $Mono(T_r(X), TY)$, corresponds under the isomorphism J to an open subbundle $E_\omega^r(X)$ of $J^r(X,Y)$ so that $\Gamma_\omega^\infty E(X) = Imm_r(X,Y)$ and $\Gamma^0 E_\omega^r(X) = Mono(T_r(X), TY)$.

If $\nu(n,r) < m$, then $Imm_r(X,Y)$ becomes locally stable. In this case, for any $f \in Imm_r(X,Y)$, we have the notion of the r th order normal bundle $N_r(f)$ of f so that $f^* TY = T_r(X) \oplus N_r(f)$ (see Feldman [8]). Therefore there is a suitable disc bundle N' over X associated to $N_r(f)$ so that f extends to an immersion (of order 1) $\varphi' : N' \to Y$ by the exponential map of the Riemannian connection on Y. Again, by the same argument, there exists a small disc bundle N'' over N' associated to the 1st order normal bundle $N_1(\varphi')$ of φ' and an immersion $\varphi'' : N'' \longrightarrow Y$ such that $\varphi''|N' = \varphi'$. Note that $\dim N'' = m$. Now, since rank $D_r \circ T_r(\varphi'') = \nu(n,r)$ on X, it has rank $\geq \nu(n,r)$ on a neighbourhood N of x in N''. Set $f' = \varphi''|N$ noting that $f'|X = f$. So, now, let $E(N)$ be the trivial bundle $N \times Y$ and $E_0^r(N)$ be the subspace of $J^r(N,Y)$ which corresponds (under an isomorphism similar to J above) to the space of bundle homomorphisms $T_r(N) \longrightarrow TY$ whose restriction to each fibre has rank $\geq \nu(n,r)$. Then $E_0^r(N)$ is invariant under the action of local diffeomorphisms of N and is an open subbundle of $J^r(N,Y)$. Moreover, if $\lambda : X \longrightarrow N$ is an embedding and $g' \in \Gamma_0^{\infty} E(N)$, then rank $D_r \circ T_r(g \circ \lambda) = \text{rank } D_r \circ T_r(g') \circ T_r(\lambda) \geq \nu(n,r)$ because each of the matrices $D_r \circ T_r(g')$ and $T_r(\lambda)$ has a non-zero minor of order $\nu(n,r)$. This means $g' \circ \lambda \in Imm_r(X,Y)$. Thus $Imm_r(X,Y)$ is indeed locally stable and we therefore have the following theorem [12]:

<u>Theorem 3.2 (Gromov-Eliashberg)</u> : If $\nu(n,r) < m$, then the rth order osculating map,

$$D_r \circ T_r : Imm_r(X,Y) \longrightarrow Mono(T_r(X),TY)$$

is a w.h.e. |||

§ 3. <u>Immersions with non-vanishing mean curvature</u> :

As in the previous section, a smooth map $f : X \longrightarrow Y$ gives rise to a commutative diagram with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & TX & \xrightarrow{i_X} & T_2(X) & \longrightarrow & TX \circ TX & \longrightarrow & 0 \\
 & & \big\downarrow df & & \big\downarrow T_2(f) & & \big\downarrow df \circ df & & \\
0 & \longrightarrow & TY & \xrightarrow{i_Y} & T_2(Y) & \longrightarrow & TY \circ TY & \longrightarrow & 0
\end{array}$$

where $T_2(X)$ is the 2nd order tangent bundle and $TX \circ TX$ is the symmetric product. The Riemannian metric on Y induces a splitting $D_2 : T_2(Y) \rightarrow TY$ of the lower exact sequence.

If f is an immersion, then it has a normal bundle N_f over X,

$$(N_f)_x = (\text{Im } df_x)^{\perp} \subset (TY)_{f(x)} .$$

Let $\pi_x : (TY)_{f(x)} \rightarrow (N_f)_x$ be the orthogonal projection. Then the bundle map $\pi \circ D_2 \circ T_2(f) : T_2(X) \rightarrow N_f$ vanishes on $i_X(TX)$, since $\pi \circ D_2 \circ T_2(f) \circ i_X = \pi \circ D_2 \circ i_Y \circ df = \pi \circ df = 0$, and, therefore, induces a linear map $B_f : TX \circ TX \rightarrow N_f$. This is, in fact, the second fundamental form of the immersion f.

Let g be the inner product on TY provided by the Riemannian metric on Y and g_f be the induced metric on TX defined by

$$(g_f)_x(u,v) = g_{f(x)}(df(u),df(v)),$$

$x \in X$; $u,v \in (TX)_x$. Then the mean curvature vector H_f at $x \in X$ is defined to be the trace of $(B_f)_x$ with respect to $(g_f)_x$, that is, $H_f(x) = \sum_{i=1}^{n} B_f(e_i , e_i)$ where e_i is an orthonormal basis of $(TX)_x$ with respect to the metric $(g_f)_x$. Thus H_f is a section of the normal bundle N_f .

Let $M(X,Y)$ be the space of immersions $X \to Y$ with nowhere vanishing mean curvature. Now, if α is a 2-jet from X to Y with source x and target y and α is represented by a local immersion $f : X \to Y$, then B_f is defined on a neighbourhood of x and $B_f|_x$ depends only on α. Let $E^2_w(X)$ be the subspace of 2-jets of local immersions $X \to Y$ with nowhere zero mean curvature. Then $\Gamma^\infty_w E(X)$ becomes $M(X,Y)$ and $\Gamma^0 E^2_w(X)$ the space of 2-jet fields whose underlying 1-jet field is a vector bundle monomorphism and whose mean curvature vector is never zero.

Let k be a fixed positive integer $< n$ and G_k be the Grassmannian k-plane bundle over X associated to TX. The fibre $(G_k)_x$ over $x \in X$ is the Grassmann manifold of k-planes through the origin in $(TX)_x$. Let $B : TX \circ TX \to E$ be a linear map where E is a vector bundle over X. Then, for $P \in (G_k)_x$ and an orthonomal basis $e_1,\ldots,e_k$ of P, write $T_k(B)(P) = \sum_{i=1}^{k} B(e_i, e_i)(x)$. Then B is said to have nowhere vanishing k-trace if $T_k(B)(P) \neq 0$ for every $P \in G_k$. Let $M_k(X,Y)$ be the subspace of $Imm(X,Y)$ consisting of immersions $f : X \to Y$, whose second fundamental form B_f has nowhere vanishing k-trace with respect to the Riemannian metric on X induced by f. Clearly $M_k(X,Y) = \Gamma^\infty_0 E(X)$ for an open subbundle $E^2_0(X)$ of $E^2(X)$ which is invariant under $\mathcal{D}(X)$.

We apply the notions of the preceding paragraph to establish the local stability of $M(X,Y)$. Let $m > n+1$ and $f \in M(X,Y)$. The normal bundle N_f inherits an inner product structure and let N'

be the subbundle of N_f orthogonal to the curvature vector H_f so that $N_f = N' \oplus H_f$. Then, as shown by Feldman [9, Proposition 2.1], there exists a tubular neighbourhood N of X in N' and an $f' \in M_n(N,Y)$ such that $f' \mid X = f$. In fact, f' is given by

$$f'(x,v) = \exp_{f(x)}(v + c(x)\| v \|^2 u(x))$$

where exp is the exponential map of the Riemannian connection on Y, $c : X \longrightarrow \mathbb{R}$ is a smooth positive function and $u(x)$ is the unit mean curvature vector $H_f(x)/\| H_f(x) \|$. Note that the map $v \longmapsto v + c(x)\| v \|^2 u(x)$ embeds each fibre N_x as a paraboloid of revolution in $(N')_x$. Again, if $g' \in M_n(N,Y)$ then $g' \mid X \in M(X,Y)$, as may be seen easily. Taking $E_0^2(N) = M_n(N,Y)$, this amounts to saying that the space $M(X,Y)$ is locally stable and therefore we obtain the theorem of [9] :

<u>Theorem 3.3 (Feldman)</u> : If $m > n+1$, then the 2-jet map,

$$j^2 : M(X,Y) \longrightarrow \Gamma^0 E_w^2(X)$$

is a w.h.e. |||

§ 4. <u>Immersions transverse to a field of planes</u> :

Let ξ be a subbundle of TY such that $\dim \xi \leq m-n$ and η be the quotient bundle TY/ξ with natural projection $\pi : TY \longrightarrow \eta$. Then a smooth map $f : X \longrightarrow Y$ is said to be transverse to ξ if $\pi \circ df : TX \longrightarrow TY$ is of maximal rank at each point of X. This implies, by the restriction on the dimension of ξ, that $\pi \circ df$ is fibrewise injective, and so f must be an immersion. The space of such immersions

$X \longrightarrow Y$ transverse to ξ is denoted $\mathrm{Trans}_\xi(X,Y)$. Let $E^1_w(X) \subset J^1(X,Y)$ denote the associated open subbundle such that $\Gamma^0 E^1_w(X)$ becomes the space of bundle maps $\varphi : TX \longrightarrow TY$ which are fibrewise injective and satisfy $\mathrm{Im}\,\varphi \cap \xi = 0$. Then $\mathrm{Trans}_\xi(X,Y) = \Gamma^\infty_w E(X)$.

We shall show that, if $\dim \xi < m-n$, then $\mathrm{Trans}_\xi(X,Y)$ is locally stable. Let $f \in \Gamma^\infty_w E(X)$, $x_0 \in X$ and V be a contractible open neighbourhood of x_0 in X. Since $\dim \xi < m-n$, we can choose a section u of the normal bundle of the immersion $f\,|\,V$ over V which is always outside the pull-back $(f\,|\,V)^*\xi$. This means, for each $x \in V$, $g(u(x))$ does not lie in $\mathrm{Im}\,df_x \oplus \xi_{f(x)}$, where $g : (f\,|\,V)^*TY \longrightarrow TY$ is the canonical map of the pull-back. Then $df\,|\,TV$ extends to a bundle map $\varphi : TV \oplus \mathbb{R} \longrightarrow TY$ defined by $\varphi_x(v,t) = df_x(v) + tg(u(x))$, where $x \in V$, $v \in (TV)_x$ and $t \in \mathbb{R}$. Clearly, rank $\varphi = n+1$ and $\mathrm{Im}\,\varphi \cap \xi = 0$. This implies that there exists an $f' \in \Gamma^\infty(U \times J) = C^\infty(U \times J, Y)$, where U is an open neighbourhood of x_0 in V and J is an open interval about 0 in $\mathbb{R}$, such that $j^1 f' = df' = \varphi\,|\,U \times J$ and $f'\,|\,U$ has the same 1-jet as $f\,|\,U$ on U so that, too, $f'\,|\,U = f\,|\,U$. Therefore, to establish the local stability of $\mathrm{Trans}_\xi(X,Y)$, merely take $N = U \times J$ and $E^1_0(N)$ as the subspace of 1-jets of bundle monomorphisms $TN \longrightarrow TY$ which are transverse to ξ.

The classification theorem implied by the local stability of $\mathrm{Trans}_\xi(X,Y)$ was first proved by Gromov [10] :

<u>Theorem 3.4 (Gromov)</u> : If $\dim \xi < m - n$, then the 1-jet map,

$$j^1 : \mathrm{Trans}_\xi(X,Y) \longrightarrow \Gamma^0\, E^1_\omega(X)$$

is a w.h.e. ///

Note that, if $\xi = 0$, it gives the classical theorem of Smale and Hirsch on immersions [16].

PART II

CHAPTER 4

A THEOREM OF PHILLIPS

In this chapter, the first of the second part of this thesis, we examine a theorem of Phillips [24]*. The reason is that, combined with some results of our own, this theorem will have applications of interest in this thesis. As indicated in the general introduction, this study will constitute an extension of Gromov theory to classes of maps, defined on a non-closed manifold, but not satisfying the openness condition. Our objective is the classification theorem proved in Chapter 6.

§ 1. We, however, do not content ourselves with merely quoting Phillips' theorem. For Phillips' paper is in the nature of an announcement and the proofs provided are telegraphic and, we believe, do not bring to the surface the elegant geometry involved. For such reasons, we hope it will not be considered a digression to elaborate in this section on Phillips' technique as used to prove the following :

Theorem 4.1 (Phillips [24]) : If X is open and has a proper Morse function with no critical points of index greater than k, then the differential map,

$$d : \mathrm{Subimm}_k(X,Y) \rightarrow \mathrm{Lin}_k(TX,TY)$$

is a w.h.e.

*See also Gromov [11].

(Note : A manifold X with a proper Morse function with all critical points of index $\leq k$ has a handle body decomposition with all handles of index $\leq k$ and we denote geo dim $X \leq k$.)

We shall first prove a weaker version of the theorem. Define a function a on integers by :

$$a(0) = a(1) = 0,\ a(2) = 1,\ a(x) = \frac{1}{2}(x-1) \quad \text{if} \quad x \geq 3.$$

Theorem 4.2 : Let $\dim X = n$. If geo dim $X \leq \min(a(n), k)$, then,

$$d : \mathrm{Subimm}_k(X,Y) \longrightarrow \mathrm{Lin}_k(TX,TY)$$

is a w.h.e.

Following Gromov [10], it would have sufficed to show that the restriction map $\mathrm{Subimm}_k(X_2,Y) \longrightarrow \mathrm{Subimm}_k(X_1,Y)$ has the covering homotopy property where $X_1 \subset X_2$ ($\subset X$) are n-dimensional submanifolds and X_2 is X_1 plus a handle of index λ where $\lambda \leq \min(a(n),k)$. However, this is not generally true (see remarks of [24]) but with the results of [4] and [10] in hand, we may reduce the problem to proving the following lemma :

Lemma 4.1 (Weak Micro-covering Homotopy Lemma) :

Suppose we are given X_1, X_2 as above, a compact polyhedron P and a continuous map $F : P \longrightarrow \mathrm{Subimm}_k(X_2,Y)$ and $f : P \times [0,1] \longrightarrow \mathrm{Subimm}_k(X_1,Y)$ with $f_{p,o} = F_p \mid X_1$, for $p \in P$. Then there exists $\varepsilon > 0$ and a continuous $\tilde{F} : P \times [-1,\varepsilon] \longrightarrow \mathrm{Subimm}_k(X_2,Y)$ with $\tilde{F}_{p,-1} = F_p$, for $p \in P$, such that $\tilde{F}_{p,t} \mid X_1 = f_{p,o}$, if $t \leq 0$, and $\tilde{F}_{p,t} \mid X_1 = f_{p,t}$, if $0 \leq t \leq \varepsilon$ for $p \in P$.

Proof : For simplicity, assume P is a point (one can easily generalize to compact P) so that $F \in \text{Subimm}_k(X_2, Y)$ and $f : I \longrightarrow \text{Subimm}_k(X_1, Y)$ with $f_o = F \mid X_1$. By smooth extension, assume f_t, $0 \leq t \leq 1$, is defined on a collar nbd. N of X_1 in X_2.

Now, first consider the simplest case when $k = \lambda = 1$ and we examine the details of the geometry. Choose two points z and z' in the interior of either component of $N - \text{Int } X_1$. Next, fix attention on the component containing z though the procedure is the same for the other component.

There exists a ball B about z in $N - \text{Int } X_1$, and a co-ordinatization $(x_1, \ldots, x_n)$ of B such that

$$f_o(x_1, \ldots, x_n) = \sigma_o(x_1)$$

where $\sigma_o : \mathbb{R} \longrightarrow Y$ is an immersion (this follows from a constant rank theorem).* If $\varepsilon > 0$ is chosen small enough we may assume that in B,

$$f_t(x_1, \ldots, x_n) = \sigma_t(x_1)$$

where $\sigma_t : \mathbb{R} \longrightarrow Y$ is an immersion, $0 \leq t \leq \varepsilon$, and σ_t varies continuously with t. (For if a curve - here the x_1-axis or a parallel - is transverse to the foliation induced in B by f_o, it will also be transverse to the foliations induced in B by 1-subimmersions close enough to f_o.)

*See Milnor [20, (1.9)].

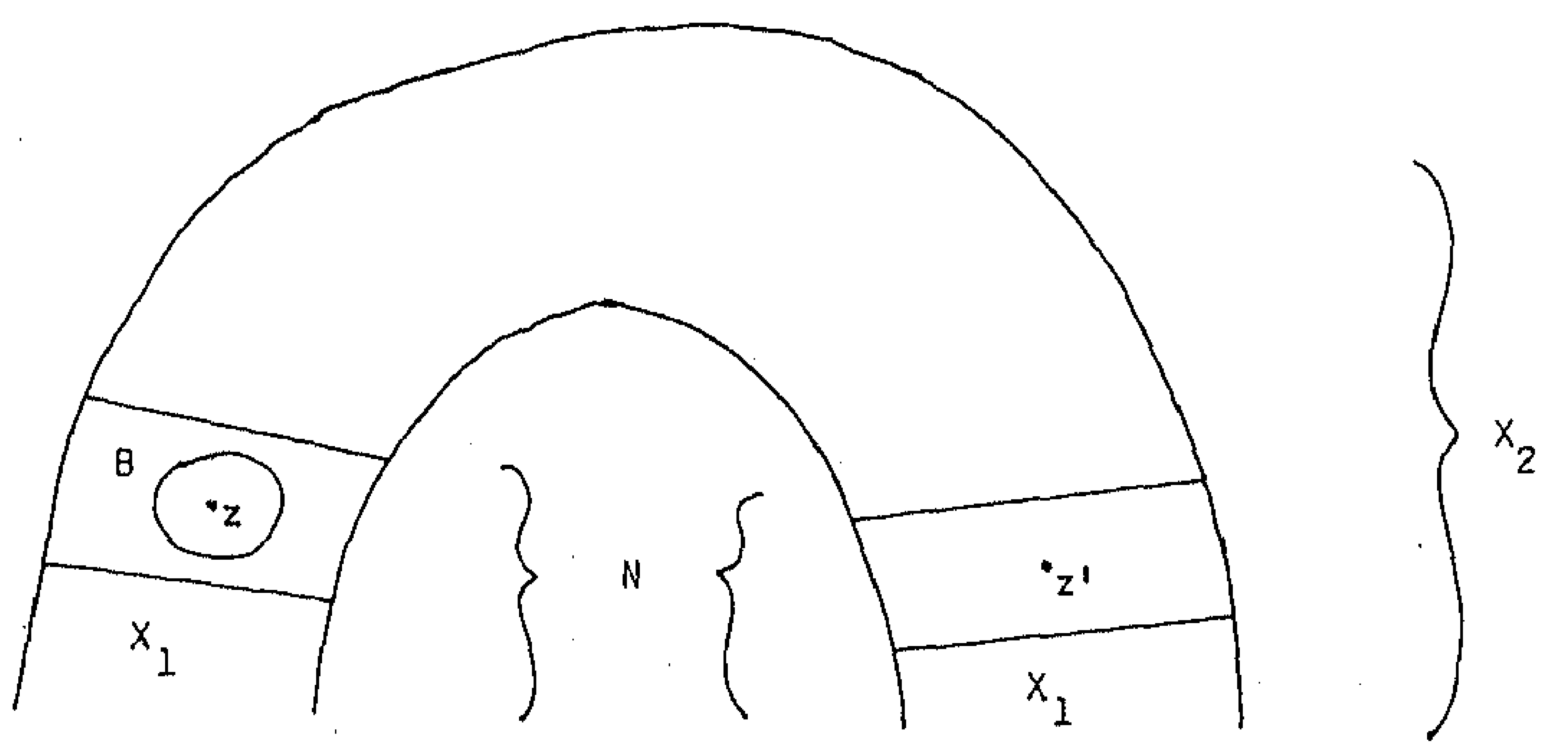


Figure 4.1

Next, construct an isotopy $h_t : N - \text{Int } X_1 \longrightarrow N - \text{Int } X_1$, $0 \le t \le 1$, such that identifying each component of $N - \text{Int } X_1$ with $D^{n-1} \times I$, where $I = [-1,1]$, we have,

$h_o = 1_{N - \text{Int } X_1}$,

h_t is fixed on a nbd. of $D^{n-1} \times \{-1,1\}$,

$h_1(D^{n-1} \times [a,b]) \subset B$ for some $-1 < a < b < 1$,

and, $h_1(y,x) = (x, x_2,\ldots,x_n)$, for some $x_2,\ldots,x_n \in \mathbb{R}$, for every $(y,x) \in D^{n-1} \times [a,b]$. Note that in the last equation the co-ordinates on the left and right hand sides are those of $D^{n-1} \times I$ and B, respectively. For this last condition we may have to reco-ordinatize B if necessary - geometrically, the condition means that $y \times [a,b]$ is moved transverse to the foliation defined by f_o in B, for each $y \in D^{n-1}$.

Define $f'_t : D^{n-1} \times [a,b] \longrightarrow Y$ by

$$f'_t(y,x) = f_t \circ h_1(y,x), \; 0 \leq t \leq \varepsilon.$$

It follows that

$$f'_t(y,x) = \sigma_t(x), \quad 0 \leq t \leq \varepsilon.$$

Next, choose a', b' such that $a < a' < b' < b$.

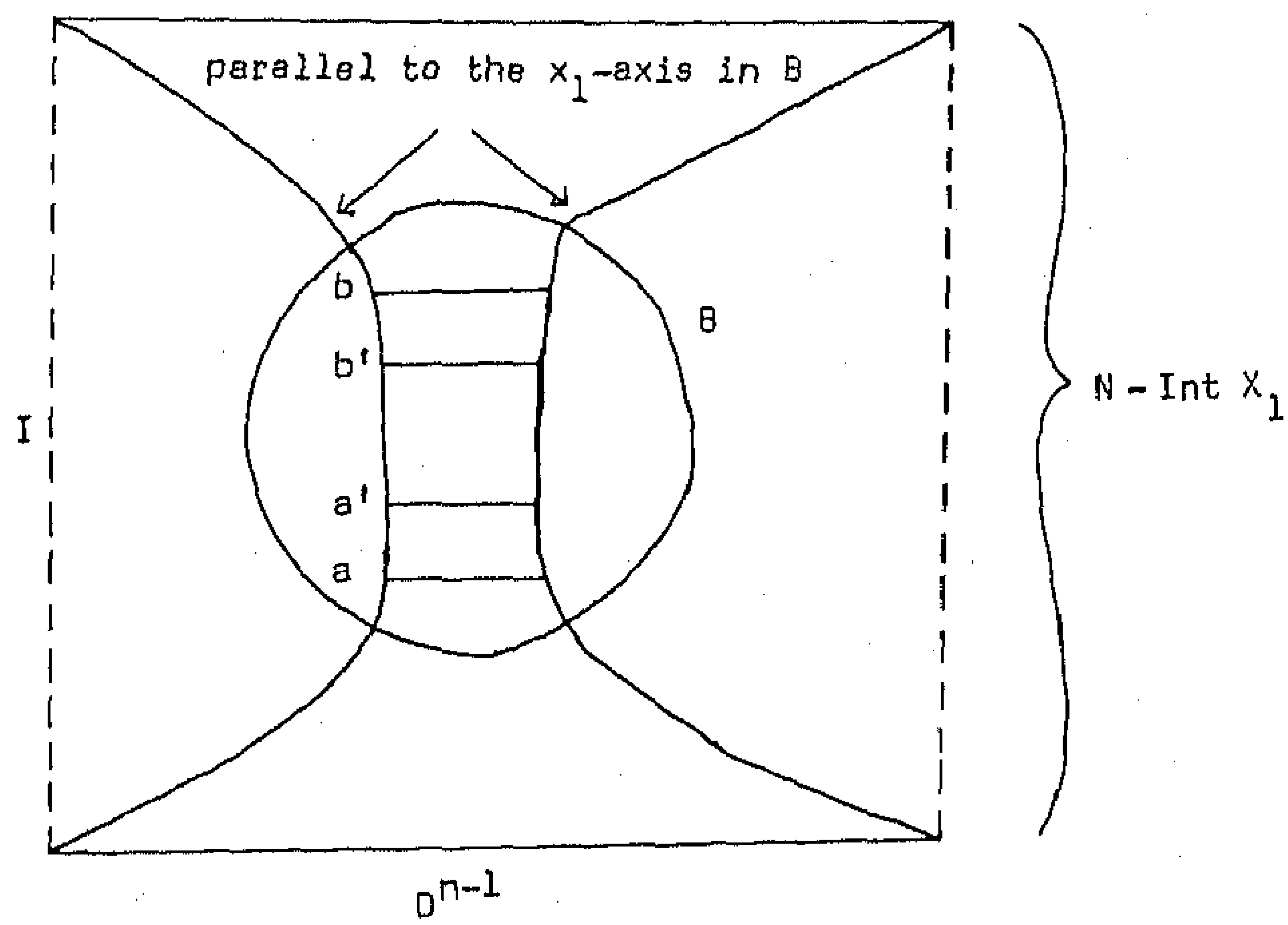


Figure 4.2

Define g_t on $[a,a'] \cup [b,b']$, $0 \leq t \leq \varepsilon$, by

$$g_t(x) = \begin{cases} \sigma_t(x), & a \leq x \leq a' \\ \sigma_o(x), & b' \leq x \leq b \end{cases}$$

and G on $[a,b]$ by $G(x) = \sigma_o(x)$.

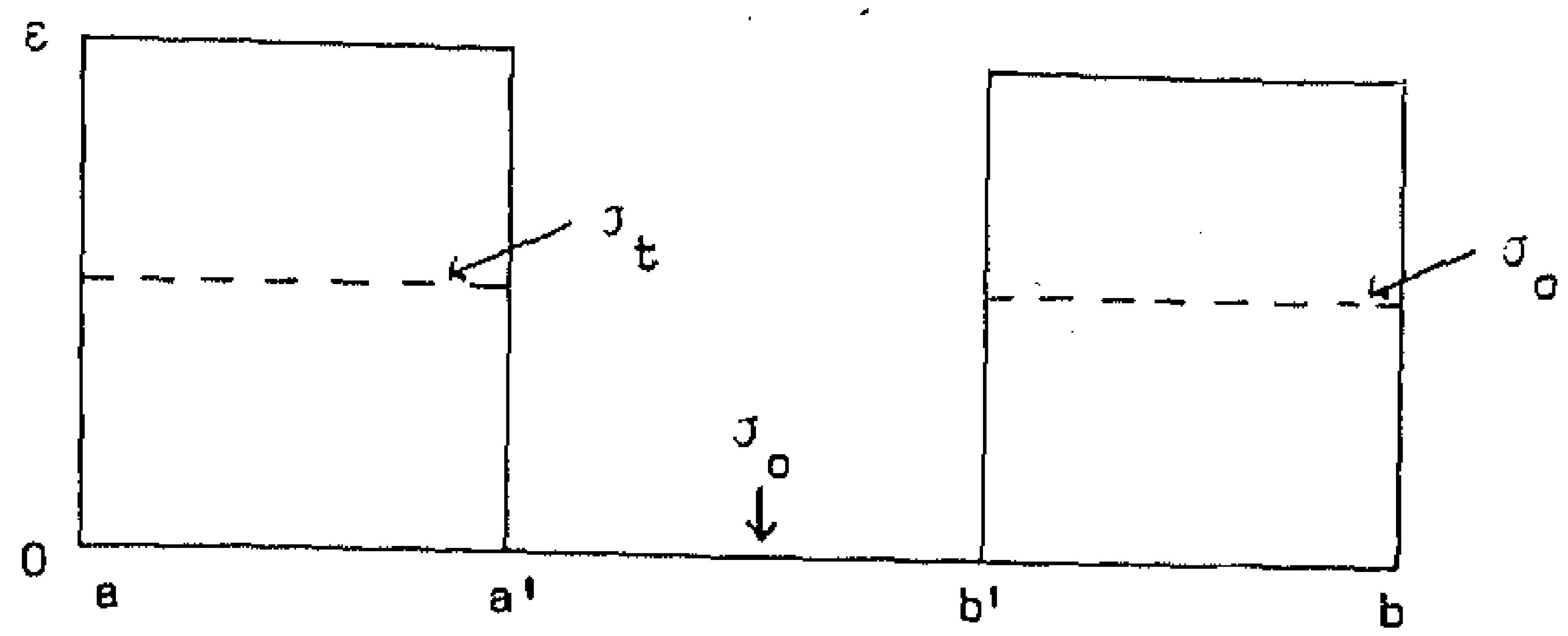


Figure 4.3

Let $G_t' : [a,b] \to Y$, $0 \leq t \leq \varepsilon$, be the solution of the 1-handle lifting problem (G, g_t) of immersions of rank 1, so that $G_t' \mid [a,a'] \cup [b',b] = g_t$ and $G_o' = G$.

Define $F_t' : D^{n-1} \times [a,b] \to Y$, $0 \leq t \leq \varepsilon$, by

$$F_t'(y,x) = G_t'(x), \ (y,x) \in D^{n-1} \times [a,b].$$

Finally, extend the isotopy h_t to an isotopy H_t, $0 \leq t \leq 1$, of X_2, defining

$$H_t = \begin{cases} h_t, & \text{on } N - \text{Int } X_1 \\ \text{identity}, & \text{elsewhere on } X_2. \end{cases}$$

The required $\tilde{F} : [-1,\varepsilon] \to \text{Subimm}_1(X_2,Y)$ is defined by

$$\tilde{F}_t(z) = \begin{cases} F \circ H_t(z), & -1 \leq t \leq 0, \ z \in X_2 \\ f_t \circ H_1(z), & 0 \leq t \leq \varepsilon, \ z \in X_1 \cup (D^{n-1} \times [0,a]) \\ F_t'(z), & 0 \leq t \leq \varepsilon, \ z = (y,x) \in D^{n-1} \times [a,b] \\ F \circ H_1(z), & 0 \leq t \leq \varepsilon, \ z \in (D^{n-1} \times [b,1]) \cup (X_2 - N). \end{cases}$$

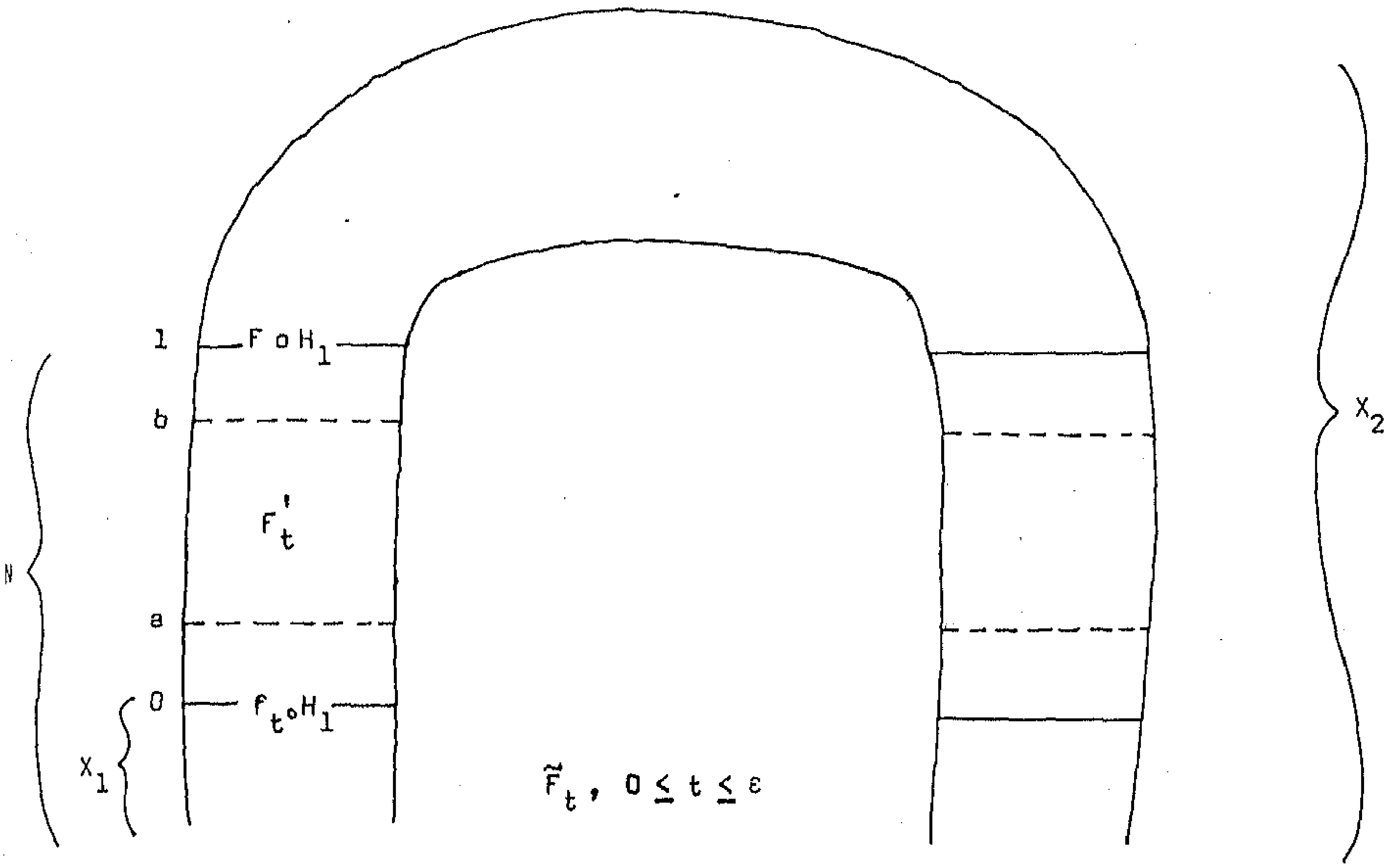


Figure 4.4

Next, consider the case for arbitrary $\lambda \leq \min(a(n),k)$. The method is similar. Generally, $N - \text{Int } X_1 \simeq S^{\lambda-1} \times I \times D^{n-\lambda}$. Consider the compact submanifold

$$S = S^{\lambda-1} \times \{0\} \times \{0\}$$

inside the collar $N - \text{Int } X_1$. Assume first that (we return to this later) $F \mid S$ $(= f_0 \mid S)$ is an immersion. Then, by a constant rank theorem for compact submanifolds, we deduce that there is a tubular nbd. T of S in $N - \text{Int } X_1$ so that
$T \simeq S \times D^{n-\lambda+1} \simeq S \times D^{k-\lambda+1} \times D^{n-k}$, and there exists an $\varepsilon > 0$, such that

$$f_t(z) = (J_t \circ p)(z) \quad \text{for} \quad z \in T,\ 0 \leq t \leq \varepsilon,$$

where $p : S \times D^{k-\lambda+1} \times D^{n-k} \to S \times D^{k-\lambda+1}$ is the natural projection,

and $\quad J_t : S \times D^{k-\lambda+1} \to Y, \quad 0 \leq t \leq \varepsilon,$

are immersions varying continuously with t.

Again, construct an isotopy $h_t : N - \mathrm{Int}\, X_1 \to N - \mathrm{Int}\, X_1$, $0 \leq t \leq 1$, such that,

$$h_o = 1_{N - \mathrm{Int}\, X_1},$$

h_t is fixed on a nbd. of $S^{\lambda-1} \times \{-1,1\} \times D^{n-\lambda}$,

$h_1(S^{\lambda-1} \times [a,b] \times D^{n-\lambda}) \subset T$ for some $-1 < a < b < 1$,

and, identifying $S^{\lambda-1} \times [a,b] \times D^{n-\lambda} \simeq S^{\lambda-1} \times [a,b] \times D^{k-\lambda} \times D^{n-k}$,

$$h_1(s,r,x,y) = (s,(r,x),y') \in T (= S \times D^{k-\lambda+1} \times D^{n-k})$$

for some $y' \in D^{n-k}$, for every $(s,r,x,y) \in S^{\lambda-1} \times [a,b] \times D^{k-\lambda} \times D^{n-k}$. Interpreting geometrically, $S^{\lambda-1} \times [a,b] \times D^{k-\lambda} \times \{y\}$ is moved transverse to the foliation induced by f_o in T, for each $y \in D^{n-k}$.

The essential geometry is complete and, without repeating details, it is seen that, if we proceed as before, we shall have a lifting problem of immersions of rank k over the manifold pair

$$(S^{\lambda-1} \times [a,b] \times D^{k-\lambda},\ S^{\lambda-1} \times [a,a'] \cup [b',b] \times D^{k-\lambda})$$

which is a λ-handle pair, and this problem is solvable.

Returning to our assumption that $F \mid S$ is an immersion, it is evidently sufficient to show that S is isotopic to a sphere immersed by F (for, then we may 'add on' this isotopy before the isotopy h_t). By the result of § 5.2.1 of [10] and the assumption that $\lambda \leq k$, the inclusion $i : S \to N - \mathrm{Int}\, X_1$ is homotopic to an immersion i' transverse to the foliation defined by F so that $F \circ i'$ is also an immersion. Further, by the assumption that $\lambda \leq a(n)$, we may approximate i' by an embedding i''. If the approximation is close enough $F \circ i''$ will still be an immersion and i and i'' will be isotopic.

This completes the proof of Lemma 4.1 and hence that of Theorem 4.2. ||| |||

§ 2. Finally, from Theorem 4.2 to Theorem 4.1 is purely technical and an application of some of Gromov's results [10]. The details may be checked in [24] but, for the sake of completeness, we indicate the method.

Choose q sufficiently large that geo dim $X \leq a(n+q)$. Let $X' = X \times \mathbb{R}^q$ so that X' satisfies the hypotheses of Theorem 4.2. Give X a Riemannian metric and X' the product metric. Let $p : X' \to X$ be the projection and $i : X \hookrightarrow X \times \{0\} \subset X'$ be the inclusion.

To show that $d : \mathrm{Subimm}_k(X,Y) \to \mathrm{Lin}_k(TX,TY)$ is a w.h.e., we have to show that

$$d_* : \pi_i \{\mathrm{Subimm}_k(X,Y)\} \to \pi_i \{\mathrm{Lin}_k(TX,TY)\}$$

is both surjective and injective for each $i \geq 0$.

We shall here show that d_* is surjective for $i = 0$, the technique being similar in other cases. So, given $H \in \mathrm{Lin}_k(TX,TY)$ we must find $F \in \mathrm{Subimm}_k(X,Y)$ such that H is homotopic to dF. Now, by Theorem 4.2, $H' = H \circ dp : TX \to TY$ is homotopic to dF' for some $F' \in \mathrm{Subimm}_k(X',Y)$. It follows that the projection $TX \to (\ker H)^{\perp} = (\ker H')^{\perp} | X$ is homotopic to an epimorphism $TX \to (\ker dF')^{\perp}$ covering i. Hence (see [10]), i is homotopic to a smooth map $\varphi : X \to X'$ transverse to $\ker dF'$. Consequently, H is homotopic to $d(F' \circ \varphi)$. Set $F = F' \circ \varphi$. |||

CHAPTER 5

ON GENERALIZED STIEFEL MANIFOLDS

A generalized Stiefel manifold is the space of $m \times n$ matrices of a fixed rank $k \leq \min(m,n)$. If $k = \min(m,n)$, the space becomes a classical Stiefel manifold. These spaces were introduced by Milnor [20] for the study of immersions of manifolds. In [6], Favaro determined the integral homology groups of generalized Stiefel manifolds of rank 1 by the method of cellular decomposition. In this chapter, we determine the homotopy groups and mod 2 cohomology algebras of arbitrary generalized Stiefel manifolds. There are three kinds of generalized Stiefel manifolds according as the entries of matrices are real, complex or quaternionic numbers. The case of interest to our intended application is that of real entries and it is for this case that we go into calculations though parallel results for the other cases may be deduced similarly.

The technique of calculation will be to exploit the following fact : since the row (or column) space of an $m \times n$ matrix of rank k spans a k-plane, a generalized Stiefel manifold fibres over a Grassmannian with fibre a Stiefel manifold.

§ 1. Homotopy groups

Let $M(m,n;k)$ be the generalized Stiefel manifold of $m \times n$ matrices of rank k with real entries. Then $GL(m, \mathbb{R}) \times GL(n, \mathbb{R})$

acts on the left of $M(m,n\,;k)$ by $(u,v).x = u.x.v^t$, where v^t denotes the transpose of v, and the action is transitive. The subgroup of stability fixing $\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \in M(m,n\,;k)$, I_k being the kth order identity matrix, is the group consisting of elements of the form

$$\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} , \begin{bmatrix} (a^t)^{-1} & d \\ 0 & e \end{bmatrix} \right)$$

where $a \in GL(k, \mathbb{R})$, $c \in GL(m-k, \mathbb{R})$, $e \in GL(n-k, \mathbb{R})$, and 0 denotes a certain null matrix. It follows then, by a standard argument, that $M(m,n\,;k)$ has the homotopy type of the quotient space $(O(m) \times O(n))/G$, where G is the subgroup consisting of elements of the form

$$\left(\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} , \begin{bmatrix} u & 0 \\ 0 & w \end{bmatrix} \right)$$

where $u \in O(k)$, $v \in O(m-k)$, $w \in O(n-k)$.

Let $V_{m,k}$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^m$, and regard an element of $V_{m,k}$ as an $m \times k$ matrix. Then $O(k)$ acts freely on the right of the product space $V_{m,k} \times V_{n,k}$ by the diagonal action $(x,y).g = (xg, yg)$. Let $V^k_{m,n}$ be the quotient space of $V_{m,k} \times V_{n,k}$ by equivalence under $O(k)$, and $p : V_{m,k} \times V_{n,k} \rightarrow V^k_{m,n}$ be the quotient map. Then the group $O(m) \times O(n)$ is a transitive topological transformation group of $V^k_{m,n}$ under the action defined by $(a,b).p(x,y) = p(ax,by)$, and the subgroup of stability leaving $p\left(\begin{bmatrix} I_k \\ 0 \end{bmatrix}, \begin{bmatrix} I_k \\ 0 \end{bmatrix}\right)$ fixed is the group G described above. This amounts to saying that $M(m,n;k)$ has the homotopy type of $V^k_{m,n}$.

Let $G_{m,k}$ be the Grassmann manifold of k-planes in $\mathbb{R}^m$. The construction of $V^k_{m,n}$ gives us a fibre bundle $V^k_{m,n} \rightarrow G_{m,k}$ with fibre $V_{n,k}$ and structure group $O(k)$, which is associated to the principal $O(k)$-bundle $V_{m,k} \rightarrow G_{m,k}$. If $n \geq m$, we have an $O(k)$-equivariant map $s : V_{m,k} \rightarrow V_{n,k}$ given by $s(x) = \begin{bmatrix} x \\ 0 \end{bmatrix}$. Therefore, by the correspondence between equivariant maps and cross-sections of an associated bundle, the fibre bundle $V^k_{m,n} \rightarrow G_{m,k}$ admits a cross-section. Therefore, by Steenrod [32, Theorem 17.7], we deduce :

<u>Theorem 5.1</u> : (i) If $n \geq m$, then

$$\pi_i(V^k_{m,n}) = \pi_i(V_{n,k}) \oplus \pi_i(G_{m,k}) \quad \text{for all } i \geq 1.$$

(ii) If $m \geq n$, then

$$\pi_i(V^k_{m,n}) = \pi_i(V_{m,k}) \oplus \pi_i(G_{n,k}) \quad \text{for all } i \geq 1. \qquad |||$$

Note that (ii) follows from (i) by interchanging the role of m and n. The case for $\pi_1(V^1_{2,2})$ follows because $V^1_{2,2}$ has the homotopy type of $S^1 \times S^1$.

§ 2. <u>Mod 2 cohomology algebra</u>

Recall from Borel [3] that the algebra $H^*(V_{n,k}; Z_2)$ has a simple system of generators $(x_{n-k}, \ldots, x_{n-1})$, where degree $x_j = j$. We denote the simple system of generators of $H^*(V_{n+1,k}; Z_2)$ by

$(y_{n-k+1}, \ldots, y_n)$, and that of $H^*(V_{n+1,k+1} ; Z_2)$ by $(z_{n-k}, \ldots, z_n)$.

Lemma 5.1 : If $i : V_{n,k} \to V_{n+1,k}$ is the natural inclusion, then the elements $x_{n-k}, i^*(y_{n-k+1}), \ldots, i^*(y_{n-k+r})$ generate a subalgebra of $H^*(V_{n,k} ; Z_2)$ containing the elements $x_{n-k}, \ldots, x_{n-k+r}$, where r is a number such that $1 \leq r \leq k-1$.

Proof : First note that the map i is the composition of the inclusion $f : V_{n,k} \to V_{n+1,k+1}$ followed by the projection $g : V_{n+1,k+1} \to V_{n+1,k}$. By the Serre exact sequence of the fibre bundle $V_{n+1,k+1} \to S^n$ with fibre $V_{n,k}$, $f^* : H^q(V_{n+1,k+1} ; Z_2) \to H^q(V_{n,k} ; Z_2)$ is a monomorphism for $q \leq n-1$. But for $q \leq n-1$, $H^q(V_{n+1,k+1} ; Z_2)$ and $H^q(V_{n,k} ; Z_2)$ are isomorphic as finite dimensional vector spaces. Therefore f^* is an isomorphism for $q \leq n-1$. Again, by [3], $g^*(y_j) = z_j$ for $n-k+1 \leq j \leq n-1$. These make the proof of the lemma quite clear. ///

Lemma 5.2 : The system of local coefficients $H^*(V_{n,k} ; Z_2)$ in the fibre bundle $V^k_{m,n} \to G_{m,k}$ is simple.

Proof : Assume inductively that (for a fixed k and any n) the generators $x_{n-k}, \ldots, x_{n-k+r}$ of the cohomology of the fibre $V_{n,k}$ over some point $x \in G_{m,k}$ are invariant under the action of the fundamental group $\pi_1(G_{m,k}, x)$, for some $r < k-1$. The induction may be started, as $x_{n-k} \in H^{n-k}(V_{n,k} ; Z_2) \simeq Z_2$ whose only automorphism is the identity. Now we have a bundle map

$$\begin{array}{ccc} V^k_{m,n} & \longrightarrow & V^k_{m,n+1} \\ & \searrow \quad \swarrow & \\ & G_{m,k} & \end{array}$$

induced by the inclusion of the $O(k)$-equivariant subspace $V_{n,k}$ in $V_{n+1,k}$. This gives a homomorphism of the local systems, which at the point x is $i^* : H^*(V_{n+1,k} ; Z_2) \to H^*(V_{n,k} ; Z_2)$. Now, by the inductive hypothesis, the generators $y_{n-k+1}, \dots, y_{n-k+1+r} \in H^*(V_{n+1,k} ; Z_2)$ are invariant under the action of $\pi_1(G_{m,k}, x)$. Therefore the elements $i^*(y_{n-k+1}), \dots, i^*(y_{n-k+1+r})$ are invariant in $H^*(V_{n,k} ; Z_2)$. Hence, by Lemma 5.1, $x_{n-k+1+r}$ is also invariant. This completes the inductive step, and so the proof of the lemma. |||

<u>Lemma 5.3</u> : The generators $x_{n-k}, \dots, x_{n-1} \in H^*(V_{n,k} ; Z_2)$ are transgressive in the fibre bundle $V^k_{m,n} \to G_{m,k}$.

<u>Proof</u> : Noting that the generator x_{n-k} of lowest degree is clearly transgressive, we simply pursue an induction similar to that of Lemma 5.2. |||

<u>Lemma 5.4</u> : All transgressions in the fibre bundle $V^k_{m,n} \to G_{m,k}$ are trivial when $n \geq m$.

<u>Proof</u> : Observe that as the fibre bundle admits a cross-section, p^* is a monomorphism, where p is the bundle projection. |||

With these lemmas in hand, we see that the spectral sequence of the fibre bundle $V^k_{m,n} \longrightarrow G_{m,k}$ collapses when $n \geq m$. This gives :

<u>Theorem 5.2</u> : (i) If $n \geq m$, then

$$H^*(V^k_{m,n} ; Z_2) = H^*(V_{n,k} ; Z_2) \otimes H^*(G_{m,k} ; Z_2).$$

(ii) If $m \geq n$, then

$$H^*(V^k_{m,n} ; Z_2) = H^*(V_{m,k} ; Z_2) \otimes H^*(G_{n,k} ; Z_2).$$ |||

Note that (ii) follows from (i) by interchanging the role of m and n.

CHAPTER 6

AN APPLICATION

§ 1. A classification theorem

Let X and Y be smooth manifolds of dimensions n and m respectively, and fix a constant $k \leq \min(m,n)$. Then, recollect according to Phillips [24], as described in this thesis in Chapter 4, if X is open and has no handles of index $> k$, the differential map $d : \mathrm{Subimm}_k(X,Y) \longrightarrow \mathrm{Lin}_k(TX,TY)$ is a w.h.e.

Henceforth we shall always suppose that X is simply connected and that $Y = \mathbb{R}^m$.

We may identify the space $\mathrm{Lin}_k(TX,TY)$ with the space of cross-sections of a certain fibre bundle $p : E \longrightarrow X$. Indeed, consider the fibre bundle $q : E \longrightarrow X \times Y$ whose fibre over the point $(x,y) \in X \times Y$ is the space of linear maps of rank k between the tangent spaces $(TX)_x \longrightarrow (TY)_y$, and define p to be the composition of q followed by left projection. As the fibre of the bundle $p : E \longrightarrow X$ is homotopic to $V^k_{m,n}$, one would expect, in classifying the cross-sections of E, to meet obstructions that are elements of the cohomology groups $H^i(X ; \pi_{i-1}(V^k_{m,n}))$. As the space $V^k_{m,n}$ is not i-simple, there would further be, in the above cohomology, a system of local groups as coefficients which display the action of $\pi_1(V^k_{m,n})$ on $\pi_{i-1}(V^k_{m,n})$. However, as X is simply connected, this complication of twisted coefficients may be avoided by using the obstruction theory of Barcus [2] as developed by Hermann [15]. This theory is briefly outlined in the next section where we retreive an analogue of a classification theorem for cross-sections of Steenrod [32, § 37.5]. At present, we apply this new classification theorem in the following form : if $E \to X$ admits a cross-section s_o and if $H^i(X ; \pi_i(V^k_{m,n})) = 0$ for $i = 0,1,\dots,n-1$ and $H^{i-1}(X ; \pi_i(V^k_{m,n})) = 0$ for $i = 1,\dots,n-1$, where H^0 is understood to be the reduced cohomology, then the assignment of the cohomology class $\tilde{d}(s,s_o)$ of the difference cocycle $d(s,s_o)$ to each cross-section s sets up a 1-1 correspondence between the based homotopy classes of cross-sections and elements of $H^n(X ; \pi_n(V^k_{m,n}))$.

Now, if $X = S^k \times D^{n-k}$, $k < n$ and $k < m$, the existence of the cross-section s_o follows trivially since S^k immerses in $\mathbb{R}^m$ and, therefore, using the k-skeleton of X (there being no higher obstructions) in the above arguments, we get the following classification theorem :

<u>Theorem 6.1</u> : If $k < n$ and $k < m$, the set of based regular homotopy classes of k-subimmersions of $S^k \times D^{n-k}$ into $\mathbb{R}^m$ correspond bijectively with the elements of the homotopy group $\pi_k(V^k_{m,n})$.

For, if $k < n$, then $S^k \times D^{n-k}$ is open with no handle of index $> k$ so that Phillips' theorem applies - or, to be precise, a based version of Phillips' theorem applies. The corresponding modification of Phillips' proof is not difficult. See, for example the discussion by Poénaru [27, Theorem 1']. |||

Note that, if we set $k = n < m$, the preceding theorem becomes the classical theorem of Smale [31].

§ 2. Obstruction theory

The classical obstruction theory as described in Steenrod [32, Part III] leads to the following classification theorem for cross-sections. We persist throughout this section with the assumption that X is simply connected.

Theorem 6.2 ([32, § 37.5 with modifications of § 36.11]) :

Let (X,Y) be a simplicial pair such that $\dim X = n$. Suppose the bundle $E \to X$ with fibre F admits a cross-section s_o and

(1) $H^i(X,Y ; \pi_i(F)) = 0, \quad i = 0,1,\dots,n-1$

(2) $H^{i-1}(X,Y ; \pi_i(F)) = 0, \quad i = 1,\dots,n-1$

and (3) F is i-simple, $i = 1,\dots,n-1$.

Then the assignment of the cohomology class $\bar{d}(s,s_o)$ of the difference cocycle $d(s,s_o)$ to each cross-section s sets up a 1-1 correspondence between homotopy classes of cross-sections which agree with s_o on Y and elements of $H^n(X,Y ; \pi_n(F))$.

As we shall demonstrate, this theorem is based upon the following propositions each of which is proved by Steenrod. All assumptions are as for the theorem :

<u>Proposition 6.1 [32, § 36.4]</u>: If s_o and s_1 are cross-sections of E which agree on Y, the primary difference $\bar{d}(s_o, s_1)$ is an invariant of the homotopy classes rel Y of s_o and s_1. Its vanishing is necessary and sufficient for $s_o \simeq s_1$ (rel Y). |||

<u>Proposition 6.2 [32, § 36.6]</u>: Let s_o, s_1 and s_2 be three cross-sections of E which agree on Y. Then

$$\bar{d}(s_o, s_2) = \bar{d}(s_o, s_1) + \bar{d}(s_1, s_2). \qquad |||$$

<u>Proposition 6.3 [32, § 33.9]</u>: If s is a cross-section of E and $d \in C^n(X, Y; \pi_n(F))$, then $s \mid Y \cup X^{n-1}$ may be extended to a cross-section s' of E such that $d(s, s') = d$ (X^i is the i-skeleton of X). |||

<u>Proof of Theorem 6.2</u>: If $s \simeq s'$ (rel Y), Proposition 6.1 asserts that $\bar{d}(s, s_o) = \bar{d}(s', s_o)$. Thus each homotopy class corresponds to a single cohomology class. Suppose $\bar{d}(s, s_o) = \bar{d}(s', s_o)$. By Proposition 6.2 we have $\bar{d}(s, s') = 0$ so that, again by Proposition 6.1, $s \simeq s'$ (rel Y). Thus the assignment is 1-1. Next suppose $\bar{d} \in H^n(X, Y; \pi_n(F))$ is given. Choose a cocycle d in the class $\bar{d}$. By Proposition 6.3, $s_o \mid Y \cup X^{n-1}$ extends to a cross-section s' over X such that $d(s_o, s') = -d$. Hence $\bar{d}(s_o, s') = -\bar{d}$ and, by Proposition 6.2, $\bar{d}(s', s_o) = \bar{d}$. The assignment is therefore onto and this completes the proof. |||

To avoid the requirement of i-simplicity of F, we exploit the simple connectedness of X by resorting to the obstruction theory of Barcus [2] as developed by Hermann [15]. The main idea there is to define the obstruction cocycles and difference cochains in terms of the global homotopy properties of the bundle space rather than the local structure as in the classical theory. Following is a brief outline of the theory leading to an analogue of Theorem 6.2.

All spaces considered are path connected and with basepoint. Maps are basepoint preserving.

Let $F \xrightarrow{i} E \xrightarrow{p} X$ be a fibre bundle with e_0 the basepoint of E and F, and $x_0 = p(e_0)$ the basepoint of X.

Let s be a cross-section of E. Then (see Steenrod [32, § 17.7]) there is a canonical isomorphism

$$\pi_j(E,e_0) \simeq \pi_j(X,x_0) \oplus \pi_j(F,e_0),\ j \geq 1.$$

Let $\bar{s} : \pi_j(E,e_0) \rightarrow \pi_j(F,e_0)$ denote the projection. Henceforth we shall suppress the basepoint in our notation as it is fixed. Next assume that $Y \subset X$, $E_Y = p^{-1}(Y)$ and $s : Y \rightarrow E_Y$ is a cross-section of the bundle E_Y. The homotopy obstructions to extending s to a cross-section over X are defined to be the homomorphisms

$$w(s) : \pi_j(X,Y) \rightarrow \pi_{j-1}(F),\ j \geq 2,$$

where each $w(s)$ is the composition

$$\pi_j(X,Y) \xrightarrow{p_*} \pi_j(E,E_Y) \xrightarrow{\partial} \pi_{j-1}(E_Y) \xrightarrow{\bar{s}} \pi_{j-1}(F).$$

Proposition 6.4 : If s is extendable to X, all $w(s) = 0$.

Proof : Suppose s_X denotes an extension of s to X. We have the commutative diagram

$$\begin{array}{ccccccc} & & \pi_j(E,E) & \xrightarrow{\partial_2} & \pi_{j-1}(E) & \xrightarrow{\bar{s}_X} & \pi_{j-1}(F) \\ & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} & & \uparrow{\scriptstyle id} \\ \pi_j(X,Y) & \xrightarrow{\simeq} & \pi_j(E,E_Y) & \xrightarrow{\partial_1} & \pi_{j-1}(E_Y) & \xrightarrow{\bar{s}} & \pi_{j-1}(F) \end{array}$$

so that $\bar{s} \circ \partial_1 = \bar{s}_X \circ \partial_2 \circ i_1 = 0.$ |||

To define difference obstructions, apply the loop functor to construct the fibre bundle $\Omega F \xhookrightarrow{i'} \Omega E \xrightarrow{p'} \Omega X$ (i', p' denote $\Omega i, \Omega p$). If $Y \subset X$ and $s : Y \rightarrow E$ is a cross-section over Y, then $s' : \Omega Y \longrightarrow \Omega E$ is a cross-section of the loop space bundle.

If s_1 and s_2 are cross-sections of E such that $s_1 | Y = s_2 | Y$, construct the map $g : \Omega X \longrightarrow \Omega E$ by defining $g(w) = s_1'(w) \cdot s_2'(w^{-1})$, $w \in \Omega X$, where the inverse and composition operations are as for paths. Now, for $w \in \Omega X$,

$$\begin{aligned} p'g(w) &= p'(s_1'(w) \cdot s_2'(w^{-1})) \\ &= p's_1'(w) \cdot p's_2'(w^{-1}) = w . w^{-1}, \end{aligned}$$

proving that $p'g$ is inessential, so that

$$g_*(\pi_j(\Omega X)) \subset \ker p_*' = \operatorname{im} i_*' .$$

Further, as $s_1 | Y = s_2 | Y$, it follows that $g(\Omega Y)$ is contractible so that we may define $g_1 : \pi_j(\Omega X, \Omega Y) \longrightarrow \pi_j(\Omega E)$ such that the diagram

$$\begin{array}{ccc} \pi_j(\Omega X, \Omega Y) & \xrightarrow{g_1} & \pi_j(\Omega E) \\ \uparrow i_* & \nearrow g_* & \\ \pi_j(\Omega X) & & \end{array}$$

is commutative. Then define

$$d(s_1,s_2) : \pi_j(X,Y) \longrightarrow \pi_j(F), \quad j \geq 1$$

as the composition

$$\pi_j(X,Y) \simeq \pi_{j-1}(\Omega X, \Omega Y) \xrightarrow{g_1} \pi_{j-1}(\Omega E) \xrightarrow{i_*^{-1}} \pi_{j-1}(\Omega F) \simeq \pi_j F.$$

The collection $d(s_1,s_2)$, $j \geq 1$, are called the difference obstructions to deforming s_1 onto s_2 over X.

Having given the relevant definitions in the general case, we restrict attention henceforth to the case where (X,Y) is a CW pair. Let X^n denote the n-skeleton of X, $\hat{X}^n = X^n \cup Y$ and $C_n(X,Y) = H_n(\hat{X}^n, \hat{X}^{n-1})$. Define $\partial : C_n(X,Y) \longrightarrow C_{n-1}(X,Y)$ as the boundary operator of the triple $(\hat{X}^n, \hat{X}^{n-1}, \hat{X}^{n-2})$. It is a standard fact [34] that the n th homology group of the chain complex $(C_n(X,Y), \partial)$ is $H_n(X,Y)$.

Let s be a cross-section of the bundle $E \longrightarrow X$ over $\hat{X}^{n-1}$. Consider the first non-trivial homotopy obstruction (note $\pi_j(\hat{X}^n, \hat{X}^{n-1}) = 0$, $j < n$)

$$w(s) : \pi_n(\hat{X}^n, \hat{X}^{n-1}) \longrightarrow \pi_{n-1}(F)$$

to extending s over $\hat{X}^n$. By the relative Hurewicz theorem,

$$C_n(X,Y) = H_n(\hat{X}^n, \hat{X}^{n-1}) \simeq \pi_n(\hat{X}^n, \hat{X}^{n-1}).$$

Hence $w(s)$ denotes a map

$$w(s) : C_n(X,Y) \longrightarrow \pi_{n-1}(F),$$

that is, $w(s) \varepsilon C^n(X,Y ; \pi_{n-1}(F))$.

Similarly, if $s_1, s_2 : \hat{X}^n \rightarrow E$ are cross-sections that agree on $\hat{X}^{n-1}$, the difference obstruction in dimension n

$$d(s_1, s_2) : \pi_n(\hat{X}^n, \hat{X}^{n-1}) \longrightarrow \pi_n(F)$$

may be interpreted as

$$d(s_1, s_2) : C_n(X,Y) \longrightarrow \pi_n(F),$$

that is, $d(s_1, s_2) \varepsilon C^n(X,Y ; \pi_n(F))$.

In fact [15], $w(s)$ is always a cocycle, $\delta d(s_1,s_2) = w(s_1) - w(s_2)$ and it may be shown that the obstruction and difference cochains (presuming they are interpreted as above as they henceforth will be) are the negatives of the classical ones as defined in [32]. The following proposition proved by Barcus is more than a converse to Proposition 6.4.

Proposition 6.5 (Barcus [2]) : Suppose the base X of the bundle $E \rightarrow X$ is a CW complex with Y a subcomplex. Suppose $s : X^{n-1} \cup Y \rightarrow E$, $n \geq 2$, is a cross-section so that

$w(s) \in C^n(X,Y\,;\,\pi_{n-1}(F))$. Then, $w(s) = 0$ is also sufficient for the extension of s to $X^n \cup Y$. Suppose, secondly, that $s : X^n \cup Y \longrightarrow E$ is a cross-section and $d \in C^n(X,Y\,;\,\pi_n(F))$. Then, $s \mid X^{n-1} \cup Y$ extends to a cross-section $s' : X^n \cup Y \longrightarrow E$ such that $d(s,s') = d$. |||

Without going into details that are just technical, we observe that Proposition 6.5 suffices to prove analogues of Propositions 6.1 and 6.3 in the setting of Barcus's obstruction theory. Further, an analogue of Proposition 6.2 is a consequence of the Hurewicz theorem just as in the classical case. We therefore retrieve the following analogue of Theorem 6.2.

<u>Theorem 6.3</u> : Let (X,Y) be a CW pair such that $\dim X = n$. Suppose the bundle $E \longrightarrow X$ with fibre F admits a cross-section s_o and (1) $H^i(X,Y\,;\,\pi_i(F)) = 0$, $i = 0,\ldots,n-1$

and (2) $H^{i-1}(X,Y\,;\,\pi_i(F)) = 0$, $i = 1,\ldots,n-1$.

Then the assignment of the cohomology class $\bar{d}(s,s_o)$ of the difference cocycle $d(s,s_o) \in C^n(X,Y\,;\,\pi_n(F))$ to each cross-section s sets up a 1-1 correspondence between homotopy classes of cross-sections which agree with s_o on Y and elements of $H^n(X,Y\,;\,\pi_n(F))$. |||

It is important to note that Theorem 6.2 is true even if X is not simply connected (in contrast to Theorem 6.3) in which case we have to replace the group of coefficients $\pi_i(F)$ by a system of local groups $\mathcal{B}(\pi_i(F))$ which display the action of $\pi_1(X)$ on $\pi_i(F)$.

REFERENCES

1. W. Ambrose, R. Palais, I. Singer : Sprays, An. Acad. Brasil. Ci. 32(1960), 163-178.

2. W. Barcus : Note on cross-sections over CW complexes, Quart. J. Math., Oxford, Series (2) 5(1954), 150-160.

3. A. Borel : Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. Math. 57(1953), 115-207.

4. A. Dold : Partitions of unity in the theory of fibrations, Ann. Math. 78(1963), 223-255.

5. A. du Plessis : Homotopy classification of regular sections, Compositio Math. 32(1976), 301-333.

6. L.A. Favaro : Grupos de homologia de M(m,n,1), Proc. Fifth Brazilian Math. Colloq. (1965), Conselho Nacional de Pesquisas, Sãu Paulo, 117-118.

7. S.D. Feit : k-mersions of manifolds, Acta Math. 122(1969), 173-195.

8. E.A. Feldman : The geometry of immersions. I, Trans. AMS 120(1965), 185-224.

9. E.A. Feldman : Immersions with nowhere vanishing mean curvature, Topology 12(1973), 201-227.

10. M.L. Gromov : Stable mappings of foliations into manifolds, Math. USSR-Izvestija 3(1969), 671-694.

11. M.L. Gromov : Singular smooth maps, Mat. Zametki 14(1973), 509-516.

12. M.L. Gromov and J.M. Eliashberg : Removal of singularities of smooth mappings, Math. USSR-Izvestija 5(1971), 615-639.

13. A. Haefliger : Lectures on the theorem of Gromov, Proc. Liverpool Singularities Symposium II, Springer Lecture Notes in Mathematics 209(1971), 128-141.

14. A. Haefliger and V. Poénaru : La classification des immersions combinatoires, IHES Publ. Math. 23(1964), 75-91.

15. R. Hermann : Obstruction theory for fibre spaces, Illinois J. Math. 4(1960), 9-27.

16. M.W. Hirsch : Immersions of manifolds, Trans. AMS 93(1959), 242-278.

17. M.W. Hirsch and R.S. Palais : Unpublished.

18. S.T. Hu : Homotopy Theory, Pure and Applied Mathematics Vol.8, Academic Press (1959).

19. D. Husemoller : Fibre bundles (2nd Ed.), Graduate Texts in Mathematics 20, Springer-Verlag (1966).

20. J. Milnor : Differential topology, mimeographed notes, Princeton University (1958).

21. J. Milnor : Morse theory, Ann. of Math. Studies 51, Princeton University Press (1963).

22. R.S. Palais : The homotopy type of infinite dimensional manifolds, Topology 5(1966), 1-16.

23. A. Phillips : Submersions of open manifolds, Topology 6(1967), 171-206.

24. A. Phillips : Smooth maps of constant rank, Bull. AMS 80(1974), 513-517.

25. V. Poénaru : Sur la theorie des immersions, Topology 1(1962), 81-100.

26. V. Poénaru : Regular homotopy and isotopy, mimeographed notes, Harvard University (1964).

27. V. Poénaru : Homotopy theory and differentiable singularities, Manifolds-Amsterdam 1970, Springer Lecture Notes in Mathematics 197(1970), 106-133.

28. W. Pohl : Connexions in differential geometry of higher order, Trans. AMS 125(1966), 310-325.

29. S. Smale : Regular curves on Riemannian manifolds, Trans. AMS 81(1958), 492-512.

30. S. Smale : A classification of immersions of the two-sphere, Trans. AMS 90(1959), 281-290.

31. S. Smale : The classification of immersions of spheres in Euclidean spaces, Ann. Math. 69(1959), 327-344.

32. N. Steenrod : The topology of fibre bundles, Princeton University Press (1951).

33. R. Thom : La classification des immersions d' apres Smale, Séminaire Bourbaki (Dec. 1957).

34. G.W. Whitehead : Elements of homotopy theory, Graduate Texts in Mathematics 61 , Springer-Verlag (1978).

35. H. Whitney : On regular closed curves in the plane, Compositio Math. 4(1937), 276-284.