Some problems of continuum percolation

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Contents

Chapter 1

Introduction

1.1 Introduction

The model of continuum percolation can be described as follows. We start with a homogeneous Poisson point process X. At each point of X we centre a ball with a random radius such that the radii corresponding to different points are independent of each other and also independent of the Poisson process X. In this way, the space is divided into two regions, the *covered region* or the *occupied region* consisting of the region which is covered by at least one ball, and the *uncovered region* or the *vacant region* which is complement of the covered region. In this dissertation we study various properties of this covered region.

Percolation theory first found its application in solid-state physics, but in the later years it has been applied in many more diverse fields like geophysics, astrophysics, chemistry of polymers etc. In physics, the phenomenon of phase transition as observed in stirred mixtures of immiscible liquids is modelled by the continuum percolation model. Consider the following experiment of adding oil slowly in water and stirring it constantly. If the amount of oil added, i.e. fraction of oil to water, is very small, droplets of random size of oil are formed in the background of water. If we keep on adding more oil, the system goes into phase change to reach a situation where water droplets are dispersed in oil. The physical literature on this subject is primarily a study based on Monte Carlo simulations although heuristic arguments are provided in some of the works (see, for example, Scher and Zallen [1970], Pike and Seager [1974], Kertesz and Vicsek [1982], Gawlinski and Redner [1983], Phani and Dhar [1984]).

The mathematical study of continuum percolation was initiated by Hall [1985, 1986]. This model which is known as the Boolean model in stochastic geometry, has been studied extensively by geometers, albeit with a view of solving problems of a geometric nature. Hall [1988] is an excellent book devoted to the study of the geometric and statistical aspects of the Boolean model. The model of continuum percolation was first introduced in a study of communication networks by Gilbert [1961] as a model for the growth and structure of random networks. Menshikov, Molchanov and Sidorenko [1985], Zuev and Sidorenko [1985], Menshikov [1986], Roy [1990], Alexander [1993], Meester and Roy [1994] studied the model to obtain various results.

The other model of continuous percolation that we study is the random connection model. Given a homogeneous Poisson point process X, another way of constructing random objects is to connect the pair of points according to a given rule. In the random connection model, we connect a pair of points x_1, x_2 with the probability $g(|x_1 - x_2|)$ where g is a given function known as the *connection function* and $|\cdot|$ is the Euclidean distance. The components here are defined in the usual way. The transmission of disease among trees in a forest can be modelled by such a process. Penrose [1991], Burton and Meester [1993], Meester [1994] studied this model to obtain various results.

In the next section we introduce the models and give the necessary definitions and results.

1.2 The Poisson Boolean Model

Mathematically, the model of continuum percolation can be described as follows. Consider a homogeneous Poisson point process $X = \{x_1, x_2, \ldots\}$ of intensity λ in the d-dimensional Euclidean plane. At each point x_i of this process we place a closed ball $S(x_i)$ of radius ρ_i. The random variables $\rho_1, \rho_2, \ldots$ are i.i.d. and are independent of the Poisson process X. Let ρ be a positive random variable having the same distribution as that of ρ_1. The random variable ρ is called the *radius* *random variable.* We call this a Poisson Boolean model driven by X and denote this model by the triplet (X, λ, ρ).

Often to study the model we include the origin as a point in the process. Let the process $X \cup \{0\}$ be denoted by X' and we construct a percolation model driven by X'. From the theory of Palm measures this process X' is equivalent to a Poisson process except the fact that there is a point at origin. In other words, we may say that X' is Poisson point process with an arbitrary point of the process declared to be the origin (see Hall [1988]).

The mathematically precise construction of the Poisson Boolean model is not always needed to describe the problems of this model. However for completeness we present the mathematical description of the model. Let Ω_1 be the collection of all sets of countable points in $\mathbb{R}^d$ which have a finite number of points inside every bounded subset of $\mathbb{R}^d$. For any point $\omega_1 \in \Omega_1$, let $N(A)(\omega_1)$ be the number of points of ω_1 in the set $A \subseteq \mathbb{R}^d$. Define a σ-algebra $\mathcal{F}_1 = \sigma(N(A) : A \in \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d))$ is the Borel σ-algebra on $\mathbb{R}^d$. Now assign a probability measure P_1 on $(\Omega_1, \mathcal{F}_1)$ as follows: for disjoint $A_1, A_2, \ldots, A_m$ with $A_i \in \mathcal{B}(\mathbb{R}^d)$ for every $1 \leq i \leq m, m \geq 1, N(A_i)$ are independent random variables with $N(A_i)$ having a Poisson distribution given by

$$P_1(N(A_i) = n) = \frac{(\lambda \ell(A_i))^n}{n!} \exp(-\lambda \ell(A_i)),$$

for $i = 1, 2, \ldots, m$ and $n = 0, 1, \ldots$ and $\lambda > 0$. Here, and subsequently in this dissertation, ℓ denotes the Lebesgue measure. So $\omega_1 \in \Omega_1$ will represent the points of the process X. Consider a second probability space $(\Omega_2, \mathcal{A}_2, P_2) := ((\mathbb{R}^+)^{\mathbb{R}^d}, \mathcal{B}^{\mathbb{R}^d}, \mu^{\mathbb{R}^d})$, where μ is the probability measure on $\mathbb{R}^+$ induced by the random variable ρ. Then we take the product space $\Omega := \Omega_1 \times \Omega_2$ and equip Ω with product σ-algebra $\mathcal{F} := \mathcal{F}_1 \times \mathcal{F}_2$ and the product measure $P := P_1 \times P_2$. An element of Ω is denoted by $\omega = (\omega_1, \omega_2)$, where $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. In the realization corresponding to $\omega = (\omega_1, \omega_2) \in \Omega$, the positions of the points are precisely the points in ω_1 i.e., $X(\omega_1) = \{x_1, x_2, \ldots\}$ and the radii of the balls centred at those points are $(\omega_2)_{x_1}, (\omega_2)_{x_2}, \ldots$. The product measure P ensures that the radii are independent of the point process. The product measure on Ω_2 implies that different points have balls with independent radii but they are i.i.d. as all of them have the common distribution μ. Thus the random variable ρ governs the radii of the balls. To emphasize the dependence of P on λ and the distribution of ρ we shall often write $P_{(\lambda,\rho)}$ for P.

Definition 1.1 *The covered region or the occupied region is the set of all points in $\mathbb{R}^d$ which are covered by at least one of the balls. Thus the covered region $C = \bigcup_{i\geq 1} S(x_i)$ is a random set. The complement of the covered region is $V = \mathbb{R}^d \setminus C$ which is called the uncovered region or the vacant region.*

For $a, b \in \mathbb{R}^d$ we say that a and b are connected (denoted by $a \rightsquigarrow b$) if there is a continuous curve γ such that $a, b \in \gamma$ and $\gamma \in C$. For sets $A, B \subseteq \mathbb{R}^d$ we say that there is an occupied connection from A to B if there exist $a \in A$ and $b \in B$ such that $a \rightsquigarrow b$. With a slight abuse of notation we denote this also by $A \rightsquigarrow B$. For $A \subset \mathbb{R}^d$ we denote by $W(A)$ the set of all points in $\mathbb{R}^d$ which are connected to some point in A, i.e., $W(A) = \{x \in \mathbb{R}^d$: there exists $a \in A$ such that $a \rightsquigarrow x\}$ When $A = \{\mathbf{0}\}$, we write $W(\mathbf{0}) := W(\{\mathbf{0}\})$. We call $W(\mathbf{0})$ the occupied cluster of the origin. The occupied cluster of origin $\mathbf{0}$ is thus

$$W(\mathbf{0}) = \left\{x \in \mathbb{R}^d : \mathbf{0} \rightsquigarrow x\right\}.$$

Percolation theory is concerned with unbounded objects. Hence, the basic question one asks is: given a Poisson Boolean model (X, λ, ρ), is there a positive probability that the occupied cluster of the origin is unbounded? To study that various notions of the size of the cluster have been defined. The definition that we use is the diameter of the random set $W(\mathbf{0})$. Thus we define the size of $W(\mathbf{0})$ by

$$d(W) = \sup\left\{d(x,y) \in \mathbb{R}^d : x, y \in W(\mathbf{0})\right\}$$

called the *diameter* of the cluster. We denote by $\theta_\rho(\lambda) = \theta(\lambda)$ the probability that the origin is an element of an unbounded occupied component. In other words,

$$\theta(\lambda) = P_{(\lambda,\rho)}\left\{d(W(\mathbf{0})) = \infty\right\}.$$

The function θ is called the *percolation function.* Note that θ is non-decreasing in λ (a rigorous proof of this fact can be given by using coupling methods, described in the next section). Hence, we can define the *critical intensity* $\lambda_H = \lambda_H(\rho)$ as follows:

$$\lambda_H(\rho) = \inf\Big\{\lambda \geq 0 : \theta_\rho(\lambda) > 0\Big\}. \tag{1.2}$$

In addition we define another critical intensity. Consider the quantity $E_{(\lambda,\rho)}d(W)$ where $E_{(\lambda,\rho)}$ is the expectation operator. $E_{(\lambda,\rho)}d(W)$ is also non-decreasing in λ. Thus we may define,

$$\lambda_T = \lambda_T(\rho) = \inf\Big\{\lambda \geq 0 : E_{(\lambda,\rho)}d(W) = \infty\Big\}. \tag{1.3}$$

It can easily be seen that, $\lambda_H(\rho) \geq \lambda_T(\rho)$.

Another critical intensity can be defined through the crossing probabilities of a box. Let B be the d-dimensional box defined by $B := [0, l_1] \times \cdots \times [0, l_d]$ and let $B_0(i) := [0, l_1] \times \cdots \times [0, l_{i-1}] \times \{0\} \times [0, l_{i+1}] \times \cdots \times [0, l_d]$ and $B_1(i) := [0, l_1] \times \cdots \times [0, l_{i-1}] \times \{l_i\} \times [0, l_{i+1}] \times \cdots \times [0, l_d]$ be two faces of the box B. For $1 \leq i \leq d$, we define the *occupied crossing probability in the* i-th *direction* as

$$\begin{aligned}
&\sigma((l_1, \ldots, l_d); i, \lambda) := \\
&\qquad P_\lambda\{\text{there is a continuous curve } \gamma \text{ in } B \text{ such that} \\
&\qquad (i)\ \gamma \subseteq C \cap B \\
&\qquad (ii)\ \gamma \cap B_0(i) \neq \emptyset \text{ and } \gamma \cap B_1(i) \neq \emptyset\}.
\end{aligned} \tag{1.4}$$

Note here we have not included the random variable ρ in the notation as there is no chance of confusion. But, whenever, we work with more than one radius random variable we shall mention the radius random variable in the notation to avoid any confusion.

Coupling methods can be applied to prove that $\limsup_{n\to\infty} \sigma((n, 3n, \ldots, 3n); 1, \lambda)$ is a non-decreasing function of λ. This enables us to define the critical intensity corresponding to the crossing probabilities.

$$\lambda_S = \lambda_S(\rho) = \inf\Big\{\lambda \geq 0 : \limsup_{n\to\infty} \sigma((n, 3n, \ldots, 3n); 1, \lambda) > 0\Big\}. \tag{1.5}$$

Hall [1985] has shown that if $E_{(\lambda,\rho)}(\rho^{(2d-1)}) < \infty$ and $E_{(\lambda,\rho)}(\rho^{2d}) = \infty$, then $\lambda_H > 0$ and $\lambda_T = 0$, i.e., the critical densities do not coincide.

However, if we assume that,

$$0 < \rho < R \text{ a.s. for some } R > 0, \tag{1.6}$$

Menshikov [1986] has shown that the above two critical intensities agree, i.e.,

Theorem 1.1 *In a Poisson Boolean model (X, λ, ρ) where ρ satisfies (1.6), we have*

$$\lambda_H(\rho) = \lambda_T(\rho) = \lambda_S(\rho). \tag{1.7}$$

In this dissertation we assume that the radius random variable obeys the condition (1.6) and we denote the value $\lambda_H(\rho) = \lambda_T(\rho) = \lambda_S(\rho)$ by $\lambda_c(\rho)$.

By a simple use of the Kolmogorov's $0-1$ law we may show that if $\lambda > \lambda_c(\rho)$ there is an infinite occupied cluster with probability 1. So the natural question that arises is: how many unbounded occupied clusters are there? Meester and Roy [1994] have shown that there is exactly one unbounded occupied cluster in the supercritical regime.

Theorem 1.2 *In a Poisson Boolean model (X, λ, ρ) where ρ satisfies the condition (1.6) and $\lambda > \lambda_c$ there is exactly one unbounded occupied cluster.*

A similar study can be carried out for the vacant region. For $a, b \in \mathbb{R}^d$ we say that a and b are connected by a vacant path (denoted by $a \overset{v}{\rightsquigarrow} b$) if there is a continuous curve γ such that $a, b \in \gamma$ and $\gamma \in V$. The vacant cluster $W^*(\mathbf{0})$ of origin $\mathbf{0}$ is defined as

$$W^*(\mathbf{0}) = \left\{x \in \mathbb{R}^d : \mathbf{0} \overset{v}{\rightsquigarrow} x\right\}.$$

The same question as in the occupied case may be asked for the vacant cluster: given a Poisson Boolean model (X, λ, ρ), is there a positive probability that the vacant cluster of the origin is unbounded? We denote by $\theta^*_\rho(\lambda) = \theta^*(\lambda)$ the probability that the origin is an element of an unbounded vacant component, i.e.,

$$\theta^*(\lambda) = P_{(\lambda,\rho)}\left\{d(W^*(\mathbf{0})) = \infty\right\}. \tag{1.8}$$

Here the function $\theta^*(\lambda)$ is non-increasing in λ. Hence, *vacant critical intensity* $\lambda_H^* = \lambda_H^*(\rho)$ is defined as follows:

$$\lambda_H^*(\rho) = \sup\Big\{\lambda \geq 0 : \theta_\rho^*(\lambda) > 0\Big\}. \tag{1.9}$$

The critical intensity corresponding to the expected size of $W^*(0)$ is defined similarly. Here again, $E_{(\lambda,\rho)}d(W^*(0))$ is non-increasing in λ. Hence,

$$\lambda_T^* = \lambda_T^*(\rho) = \sup\Big\{\lambda \geq 0 : E_{(\lambda,\rho)}d(W^*(0)) = \infty\Big\}. \tag{1.10}$$

The vacant crossing of the box B is defined by considering the continuous curves connecting two faces of the box which lie entirely in the vacant region V. Thus, for $i \geq 1$, we define the *vacant crossing probability in the* i-th *direction* as

$$\begin{aligned} &\sigma^*((l_1,\ldots,l_d);i,\lambda) := \\ &\qquad P_\lambda\{\text{ there is a continuous curve } \gamma^* \text{ in } B \text{ such that} \\ &\qquad (i)\ \gamma^* \subseteq V \cap B \\ &\qquad (ii)\ \gamma^* \cap B_0(i) \neq \emptyset \text{ and } \gamma^* \cap B_1(i) \neq \emptyset\}. \end{aligned} \tag{1.11}$$

Note that $\limsup_{n\to\infty} \sigma^*((n,3n,\ldots,3n);1,\ \lambda)$ is a non-increasing function of λ. As in the occupied case we define a critical intensity corresponding to the vacant crossings.

$$\lambda_S^*(\rho) = \sup\Big\{\lambda \geq 0 : \limsup_{n\to\infty} \sigma^*((n,3n,\ldots,3n);1,\lambda) > 0\Big\}. \tag{1.12}$$

In two dimensions, all the critical intensities (1.9), (1.10) and (1.12) coincides with the critical intensities defined through the occupancy structure (see Roy [1990]). In other words,

Theorem 1.3 *In a Poisson Boolean model in 2-dimensions where ρ satisfies the condition (1.6), we have*

$$\lambda_H^*(\rho) = \lambda_T^*(\rho) = \lambda_S^*(\rho). \tag{1.13}$$

Again by Kolomogorov's $0-1$ law for $\lambda < \lambda_H^*(\rho)$ there exist at least one unbounded vacant cluster. In this case too Meester and Roy [1994] shows that there is exactly one unbounded vacant cluster in the region $\lambda < \lambda_H^*(\rho)$.

Theorem 1.4 *In a Poisson Boolean model (X, λ, ρ) where ρ satisfies the condition (1.6) and $\lambda < \lambda_H^*(\rho)$ there is exactly one unbounded vacant cluster.*

1.3 Coupling and scaling

In this section we describe briefly two important techniques in the theory of continuum percolation. Both the methods are very useful and we will be using them several times in this dissertation.

First we discuss some properties of Poisson process. The *superposition* of two independent Poisson processes X_1 and X_2 of intensities λ_1 and λ_2 respectively, forms again a Poisson process with intensity $\lambda_1 + \lambda_2$. Also a Poisson process X with intensity λ may be *thinned* with probability p of retaining a point and the residual points form once again a Poisson point process with intensity λp, i.e., each point of the Poisson process X of intensity λ is either retained with probability p or removed with probability $1-p$ independent of other point of X.

Secondly, if we have a Poisson point process $X = \{x_1, x_2, \ldots\}$ with intensity λ and we make a change of scale, then also we get a Poisson point process. More precisely, for any $\alpha > 0$, we consider the process $X_\alpha = \{\alpha x_1, \alpha x_2, \ldots\}$. The resulting process X_α is a Poisson point process with intensity $\alpha^{-d}\lambda$ where d is the dimension of the underlying space.

By coupling we mean that on a single probability space we can construct various models of percolation so that we can compare them without much difficulty. Suppose we take two independent Poisson Boolean models (X_1, λ_1, ρ) and (X_2, λ_2, ρ). Then if we superpose them, i.e., if we look at the union of the two processes, we obtain that the superposed model will once again be a Poisson Boolean model with intensity $(\lambda_1 + \lambda_2)$ and radius random variable ρ. Thus we are able to compare the models which have a common radius random variable but have different intensities. This will readily prove that the percolation function is non-decreasing.

The coupling technique will allow us to compare models with different radius random variables. Let us look at a simple example. Suppose that we have two random variables ρ_1 and ρ_2, where each $\rho_i, i = 1, 2$ assume two values a and $b, a < b$ (say). Suppose ρ_1 assumes the value a with probability p_1 whereas ρ_2 assumes the value a with probability $p_2, p_1 < p_2$ (say). Here we take three independent Poisson processes X_1, X_2 and X_3 with intensities $p_1\lambda, (1-p_2)\lambda$ and $(p_2-p_1)\lambda$ respectively. We centre balls of radius a and b around each point of the processes X_1 and X_2 respectively. For the process X_3, we make two cases: i) at each point of X_3 we centre balls of radius a and ii) at each point of X_3 we centre balls of radius b. Now we superpose the three processes. In case (i), we obtain a Poisson Boolean model $(X_1 \cup X_2 \cup X_3 = X, \lambda, \rho_2)$, while in the case (ii) we have a Poisson Boolean model $(X_1 \cup X_2 \cup X_3 = X, \lambda, \rho_1)$. Thus we are now in a position to compare the two Boolean models with different radius random variables. We will return to this later.

As is obvious from the name, scaling is to change the length of the unit. This is often used in combination with coupling. It is basically the property of the Poisson process. In order to compare between two Poisson Boolean models, we effect a change of scale to get to the required model. Clearly, if (X, λ, ρ) is a Poisson Boolean model, then $(\alpha X, \alpha^{-d}\lambda, \alpha\rho)$ is scaled version of the previous model. Thus the properties like having unbounded clusters will carry over from one model to the other and vice versa. This will be used crucially to prove the continuity of θ in Chapter 3.

1.4 The Random Connection Model

Given a homogeneous Poisson point process X, we construct a different model of percolation by connecting a pair of points according to a given rule and thereby obtaining *random connected sets*. As in the Boolean model, we study unbounded connected sets of the origin.

In a Boolean model, with each point, we associate a random radius which governs the model. Here we shall take a pair of points and connect them in a given manner. In a Random Connection Model

(RCM), we are given a function, called the *connection function*, from the positive reals into $[0, 1]$. Thus starting with a homogeneous Poisson process $X = \{x_1, x_2, \ldots\}$ of intensity λ in the d-dimensional Euclidean space and a given connection function g, we connect a pair of points x_i and x_j, for $i > j$, of the process X with probability $g(|x_i - x_j|)$, independently of all other pairs of points and the process itself. In other words, for every pair of points $(x_i, x_j), i > j$, we define a random variable $b(x_i, x_j)$ taking values in $\{0, 1\}$ with $P(b(x_i, x_j) = 1) = g(|x_i - x_j|)$ and in such a case call the pair (x_i, x_j) bonded. For $j > i$, the pair (x_i, x_j) will be called bonded if the pair (x_j, x_i) is bonded. The random variable $b(x_i, x_j)$ is independent of the process X and also of the random variables corresponding to other such pair of points.

Two points x and y of the process are said to be *connected* (denoted by $x \rightsquigarrow y$) if there exists a sequence $(x =: x_1, x_2, \ldots, x_n := y)$, such that the pair (x_i, x_{i+1}) is bonded for all $i = 1, \ldots, n-1$. Now we define the connected cluster of the origin in the usual way, $W = \{x \in X : 0 \rightsquigarrow x\}$. We say that the RCM is *driven* by X, and the model is denoted by (X, g).

We note here that the Boolean model with fixed radius $\rho = r$ can easily be seen to be the random connection model with connection function $g(x) = I_{\{x \leq 2r\}}$.

Mathematical construction of the random connection model (X, g) is quite similar to that of a Boolean model. First we construct the Poisson process X on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$. On a second probability space $(\Omega_2, \mathcal{F}_2, P_2)$, we construct a set of uniform $[0, 1]$-valued random variables $\{U(x, y) : (x, y) \in \mathcal{C}\}$ where $\mathcal{C}$ is the indexing set of unordered pairs of distinct elements from $x, y \in \mathbb{R}^d$ and the random variables $\{U(x, y) : (x, y) \in \mathcal{C}\}$ are independent of each other. Such a construction is possible by the Kolmogorov consistency theorem. Next we define a set of Bernoulli ($\{0, 1\}$-valued) random variables $\{b(x, y) : (x, y) \in \mathcal{C}\}$ such that $b(x, y) = 1$ if and only if $U(x, y) \leq g(|x - y|)$ and $b(x, y) = 0$ otherwise for all $(x, y) \in \mathcal{C}$. Hence $P_2(b(x, y) = 1) = g(|x - y|)$ and the random variables $\{b(x, y) : (x, y) \in \mathcal{C}\}$ are independent of each other. Then we take the product space, $\Omega := \Omega_1 \times \Omega_2$, and equip Ω with product measure $P := P_1 \times P_2$. An element of Ω is denoted by $\omega = (\omega_1, \omega_2)$, where $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. In the realization corresponding to $\omega = (\omega_1, \omega_2) \in \Omega$, the positions of the points are the occurrences of $X(\omega_1) = \{x_1, x_2, \ldots\}$ and the bonds between the points are given by the random variables $\{b(x_i, x_j) : x_i, x_j \in X\}$. The product structure of the probability measure ensures the independence of the random variables and the given process. To emphasize the dependence of P on λ and g we occasionally denote it by $P_{(\lambda,g)}$.

Here also the basic question that is addressed is: given a random connection model, when is the origin a part of the unbounded cluster? As before, we define the percolation function $\theta_g(\lambda) = \theta(\lambda)$ as the probability that the origin has an unbounded cluster, i.e.,

$$\theta(\lambda) = P_{(\lambda,g)}\Big\{\#(W) = \infty\Big\}, \tag{1.14}$$

where $\#(\cdot)$ is the cardinality of the set. Clearly θ is non-decreasing in λ. Thus the *critical intensity* $\lambda_H = \lambda_H(g)$ is defined as

$$\lambda_H(g) = \inf\Big\{\lambda \geq 0 : \theta_g(\lambda) > 0\Big\}. \tag{1.15}$$

The other critical intensity can be defined likewise :

$$\lambda_T = \lambda_T(g) = \inf\Big\{\lambda \geq 0 : E_{(\lambda,g)}\#(W) = \infty\Big\}, \tag{1.16}$$

where $E_{(\lambda,g)}$ is the corresponding expectation operator. Clearly, $\lambda_H(g) \geq \lambda_T(g)$.

In the random connection model, it can be easily seen that, if

$$\int_{\mathbb{R}^+} x^{d-1} g(x)dx = \infty$$

then both the critical intensities are trivially 0. Penrose [1991] has shown that if

$$0 < \int_{\mathbb{R}^+} x^{d-1} g(x)dx < \infty \tag{1.17}$$

then both the critical intensities are nontrivial, i.e.,

$$0 < \lambda_T(g) \leq \lambda_H(g) < \infty. \tag{1.18}$$

We show that under certain conditions on the function g the two critical intensities coincide. Subsequent to this work Meester [1994] has shown that,

Theorem 1.5 *For every g which satisfies (1.17), we have*

$$\lambda_H(g) = \lambda_T(g).$$

1.5 Structure of chapters

This dissertation consists of five chapters. In the second chapter, we develop a correlation inequality which is similar in spirit to the BK inequality for the disctrete percolation model. We apply this inequality to obtain some results regarding the growth of the occupied cluster of the origin in the subcritical region. In addition, we use the BK inequality to answer some questions of Hartigan [1981] in cluster analysis.

In Chapter 3, we study the covered volume fraction of a Poisson Boolean model. First we disprove a conjecture by Kersetz and Vicsek [1982] about the universality of the critical covered volume fraction. Then we prove that the critical covered volume fraction is a continuous function of the radius random variable. Further we show that the percolation function is a continuous function of the intensity of the underlying Poisson process except perhaps at the critical intensity. Furthermore, we show that the percolation functions converge when the radius random variables converge weakly except at the critical intensity of the limiting Boolean model.

Chapter 4 is devoted to the study of rare events. Alexander [1993] obtained compression results in a high density Poisson Boolean model with balls of fixed size for the occurrence of finite cluster. We show that when the balls are of varying sizes, rarefaction is observed instead of compression. This and the previous chapter about covered volume fraction underline the differences between the continuum percolation models with a fixed ball size and that with varying ball sizes.

Finally, in Chapter 5 we study the random connection model. We show that the critical intensities defined for this model are same when the connection function g satisfies certain conditions. Further we study the mosaic random connection model and obtain some asymptotic results about finite clusters.

Chapter 2

The BK Inequality

2.1 Introduction

In the mathematical literature of percolation theory on lattices two correlation inequalities play a very useful role and may be said to be the only tools available for the subject. These are the FKG inequality obtained by Fortuin, Kasteleyn and Ginibre [1971], and the BK inequality obtained by van den Berg and Kesten [1985].

While the FKG inequality has been generalised considerably (see for instance Kemperman [1977]) not much has been done in the BK inequality. In this section, we develop a version of the BK inequality for continuum percolation on $\mathbb{R}^d$. The inequality we obtain is quite unsatisfactory in the sense that it cannot handle arbitrary increasing events, however it suffices for the purposes of continuum percolation models.

This inequality, together with the version of the FKG inequality for continuum percolation (Roy [1990]) is expected to yield results on the power laws and scaling relations as have been obtained in the discrete percolation set-up. As an example of this we have an application of the BK inequality in Section 2.7 which shows that the probability that the size of the cluster is larger than n decreases exponentially in n. This exponential decay yields that at criticality the expected size of the cluster is infinite.

Finally, we study a problem of cluster analysis. Consider a unit d-

dimensional cube and place n cubes each of which have sides of length αn^{-d} where $\alpha > 0$ is used to parametrize the model. Hartigan [1981], in an article on cluster analysis, raised some questions concerning the clustering property of these cubes. We answer those questions for the two dimensional case and obtain some results in the higher dimensional case.

We remark here that BK inequality we obtain is also valid in the random connection models. Furthermore, our results are valid in the PIAsin model, i.e. the percolation model which consists of 1-dimensional sticks placed at points of a Poisson process on $\mathbb{R}^2$ (see Ambartzumian [1990] and Roy [1991] for details of the model). Since the FKG inequality is also available for this model, our results of Sections 2.7 about the size of the cluster would go through for this model.

In the next two sections we briefly state the related results that we need from the discrete percolation case and introduce a lattice approximation of the model. Then we obtain the BK inequality for "lattice approximable" events and present an important example of a lattice approximable event.

2.2 The discrete BK inequality

Here we briefly describe the BK inequality in the case of discrete percolation. Let $\mathbb{L}$ be a regular lattice with a given adjacency relation: $v_i \stackrel{adj}{\longleftrightarrow} v_j$ between vertices $v_i, v_j \in \mathcal{V}$, where the set of all vertices is denoted by $\mathcal{V}$. Each vertex of this lattice $\mathbb{L}$ is called a site. Every site is either open or closed. A *path* is a sequence $(x_1, x_2, \ldots)$ of sites x_i such that $x_i \stackrel{adj}{\longleftrightarrow} x_{i+1}$, for all $i \geq 1$. An *open path* is a path whose sites are all open. Two sites are said to be *connected* (denoted by $x\mathbb{L}y$) if there is a finite open path from one to the other. Two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{V}$ are connected if there is an open path starting from a point in A and ends in a point in B, i.e., there exists $x \in A$ and $y \in B$ such that $x\mathbb{L}y$.

Next we introduce a probability structure on the lattice. For $0 \leq p \leq 1$, we equip the space $\Omega = \{0,1\}^{\mathcal{V}}$ with the product σ-algebra. On this we assign probability in such a way that the site x is open with probability $f_x(p)$ independently of all other sites where $f_x(\cdot)$ is a given non-decreasing function taking values in $[0,1]$, for each $x \in \mathcal{V}$. The natural way to do this is to equip the space with the product measure $P_p = \prod_{x\in\mathcal{V}} P_x$ where P_x is defined as $P_x(\omega(x) = 1) = f_x(p)$ for all $x \in \mathcal{V}$. For any realisation $\omega \in \Omega$, the site x is said to be open if $\omega(x) = 1$ and closed if $\omega(x) = 0$. Thus we have set up an independent site percolation problem on the lattice with a parameter p.

Now, for any $\omega \in \Omega$, the set $\{x \in \mathcal{V} : \omega(x) = 1\}$ is the set of all vertices which are open. This map which takes a point in Ω to the set of all open vertices is a one to one and onto mapping. Thus any configuration $\mathcal{C}$ of open vertices corresponds to the configuration $\omega \in \Omega$ given by $\mathcal{C} = \{x : \omega(x) = 1\}$. Let us denote this inverse map by $\Phi(\cdot)$.

Now we define a partial ordering on $\Omega = \{0,1\}^{\mathcal{V}}$. We say that $\omega \overset{\mathcal{V}}{\leq} \omega'$ if and only if $\omega(x) \leq \omega'(x)$ for all $x \in \mathcal{V}$. An event A is called increasing, if for all $\omega \overset{\mathcal{V}}{\leq} \omega'$, $\omega' \in A$ whenever $\omega \in A$.

Let A and B be increasing events which depend only on finitely many vertices of the lattice $\mathbb{L}$. We define $A \Box B$ to be the set of all configurations ω for which there exist two disjoint sets of open vertices with the property that the one such set guarantees the occurrence of A and the other guarantees the occurrence of B. More precisely, $A \Box B$ is the set of all configurations ω for which there exist finite and disjoint sets of vertices K_A and K_B such that if ω' is such that $\omega'(x) = 1$, for all $x \in K_A$, then $\omega' \in A$ and if ω'' is such that $\omega''(x) = 1$ for all $x \in K_B$, then $\omega'' \in B$ (see Grimmett [1988], page 29). It has been shown by van-den Berg and Kesten [1985] that

Theorem 2.1 BK inequality *For two increasing events A and B which depend only on finitely many vertices,*

$$P_p(A \Box B) \leq P_p(A) P_p(B).$$

2.3 Preliminary results

Let $(\Omega, \mathcal{F}, P)$ be defined as in Section 1.2 of Chapter 1. Recall that $\Omega = \Omega_1 \times \Omega_2$ where Ω_1 is the collection of all sets of countable points in $\mathbb{R}^d$ which have a finite number of points inside every bounded subset of $\mathbb{R}^d$ and $\Omega_2 = (\mathbb{R}^+)^{\mathbb{R}^d}$. On $\Omega = \Omega_1 \times \Omega_2$ we define a partial ordering as follows: $\omega = (\omega_1, \omega_2) \leq \omega' = (\omega_1', \omega_2')$ if and only if $\omega_1 \subseteq \omega_1'$ and $\omega_2(x) \leq \omega_2'(x)$ for all $x \in \mathbb{R}^d$. An event is said to be *increasing* if its indicator function satisfies $I_A(\omega) \leq I_A(\omega')$ whenever $\omega \leq \omega'$. An event A is said to be a decreasing event if A^c is an increasing event.

Roy [1990] has shown that

Theorem 2.2 FKG Inequality: *If A_1 and A_2 are both increasing or both decreasing events in $\mathcal{F}$, then $P(A_1 \cap A_2) \geq P(A_1)P(A_2)$.*

Roy [1990] has also shown that in a Boolean model if the crossing probabilities are very small, the cluster size of the origin decays exponentially. More precisely,

Lemma 2.1 *Consider a Boolean model where condition (1.6) holds.*
(a) If for some $\overline{n} = (n_1, n_2, \ldots, n_d)$ with $n_j \geq R$ for all $j = 1, \ldots, d$ where R is as in (1.6), we have,

$$\sigma((3n_1, \ldots, 3n_{i-1}, n_i, 3n_{i+1}, \ldots, 3n_d), i, \lambda) \leq \kappa \tag{2.1}$$

for some $\kappa < (1/2d)(e5^d)^{-11^d}$ and for all $1 \leq i \leq d$, then we have

$$P\Big\{d(W(0)) \geq a\Big\} \leq C_1 e^{-C_2 a}, \tag{2.2}$$

for all $a > 0$, where C_1 and C_2 are positive constants independent of a, R is as in (1.6) and $W(0)$ is the occupied cluster of the origin in the Poisson Boolean model (X, λ, ρ).
(b) If for some $\overline{n} = (n_1, n_2, \ldots, n_d)$ with $n_j \geq R$ for all $j = 1, \ldots, d$ where R is as in (1.6), we have,

$$\sigma^*((3n_1, \ldots, 3n_{i-1}, n_i, 3n_{i+1}, \ldots, 3n_d), i, \lambda) \leq \kappa \tag{2.3}$$

for some $\kappa < (1/2d)(e5^d)^{-11^d}$ and for all $1 \leq i \leq d$, then we have

$$P\Big\{d(W^*(0)) \geq a\Big\} \leq C_3 e^{-C_4 a}, \tag{2.4}$$

for all $a > 0$, where C_3 and C_4 are positive constants independent of a, R is as in (1.6) and $W^*(0)$ is the occupied cluster of the origin in the Poisson Boolean model (X, λ, ρ).

Before we end the section we give an application of the FKG inequality. This will be used in later chapters. Let $A \subseteq I\!R^d$ be a bounded region containing the origin. For $m \geq 0$ consider the events $E := \left\{d(W(A)) \geq m\right\}$ and $F := \Big\{$for every $a \in A$, there exists a Poisson point $x \in X$ with $a \in S(x)\Big\}$, in the Poisson Boolean model (X, ρ, λ). It can be easily verified that both E and F are increasing events. By the FKG inequality we have $P_\lambda(E \cap F) \geq P_\lambda(E)P_\lambda(F)$. Thus

$$
\begin{aligned}
P_\lambda\{d(W(0)) \geq m\} &\geq P_\lambda(F \text{ occurs and } d(W(A)) \geq m\} \\
&= P_\lambda(E \cap F) \\
&\geq P_\lambda(E)P_\lambda(F) \\
&= C(\lambda, A)P_\lambda\{d(W(A) \geq m\},
\end{aligned}
$$

where $C(\lambda, A) = P_\lambda(F) > 0$ because A is a bounded region. Thus we have the inequality

$$P_\lambda\Big\{d(W(A)) \geq m\Big\} \leq K(\lambda, A)P_\lambda\Big\{d(W(0)) \geq m\Big\} \tag{2.5}$$

for any bounded region A containing the origin and a positive constant $K(\lambda, A)$. Similar calculations can be done for the vacant clusters to yield

$$P_\lambda\Big\{d(W^*(A)) \geq m\Big\} \leq K^*(\lambda, A)P_\lambda\Big\{d(W^*(0)) \geq m\Big\} \tag{2.6}$$

where $W^*(A)$ is the vacant cluster of the set A.

2.4 Disjoint occurrence of events

In this section we define the disjoint occurrence of two increasing events and prove an important proposition about increasing events. In the discrete case, in the definition of disjoint occurrence of events A and B we required that the set of all open vertices, that is used for the occurrence of event A, must be disjoint from the set of open vertices required for the occurrence of event B. In the continuum case, we want the set of Poisson points that is used by one event A to be disjoint from the set of Poisson points that is used by the event B. More precisely,

Definition 2.1 *Let A and B be two increasing events. The disjoint occurrence $A\Box B$ of the events A and B is defined as follows :*

$$\begin{aligned} A\Box B &= \Big\{\omega = (\omega_1,\omega_2) : \textit{ there exist disjoint subsets } H_1(\omega_1) \\ &\quad \textit{and } H_2(\omega_1) \textit{ of } \omega_1 \textit{ such that} \\ &\quad \textit{(a) } (\omega',\omega_2) \in A \textit{ for all } (\omega',\omega_2) \geq (H_1(\omega),\omega_2) \\ &\quad \textit{and (b) } (\omega',\omega_2) \in B \textit{ for all } (\omega',\omega_2) \geq (H_2(\omega),\omega_2)\Big\} \end{aligned}$$

Now we define a class of events for which we shall prove the BK inequality.

Definition 2.2 *An increasing event A in the Poisson system (X,λ,ρ) where ρ satisfies the condition (1.6) in Chapter 1, is said to be finitely approximable if for each $n \geq 1$, there exists random variables ρ_n taking only finitely many values and increasing events A_n in the Poisson system (X,λ,ρ_n) such that $\rho_n \geq \rho$ and $A \subseteq A_n$ and $P_{(\lambda,\rho)}(A) = \lim_{n\to\infty} P_{(\lambda,\rho_n)}(A_n)$.*

As an example of a finite approximable event let A, B and K be bounded regions with $A, B \subseteq K$. Define the event $E_K(A,B)$ by

$$\begin{aligned} E_K(A,B) &= \{A \rightsquigarrow B \text{ in } K\} \\ &= \{\text{there exist } a \in A, b \in B \text{ and a continuous curve} \\ &\quad \gamma \text{ such that } a \in \gamma, b \in \gamma \text{ and } \gamma \subseteq K \cap C\}, \end{aligned} \tag{2.7}$$

where C is the occupied region. We want to show that the event $E_K(A,B)$ is finitely approximable. To do this we shall define a sequence of random variables which take only finitely many values.

Lemma 2.2 *In the Poisson Boolean model (X,λ,ρ) where ρ satisfies the condition (1.6) of Chapter 1, the event $E_K(A,B)$ is finitely approximable.*

Proof : Let F be the distribution funtion of ρ. For $i = 1, 2, \ldots, R2^n - 1$, where R is as in (1.6), define random variables U_n and V_n as follows:

$$\begin{aligned} U_n = & \tfrac{i+1}{2^n} \quad \text{if } \frac{i}{2^n} < \rho \leq \frac{i+1}{2^n}, \\ V_n = & \tfrac{i}{2^n} \quad \text{if } \frac{i}{2^n} < \rho \leq \frac{i+1}{2^n}. \end{aligned}$$

Clearly, for every $i \geq 0$,

$$\begin{aligned} P(U_n = \frac{i+1}{2^n}) &= P(V_n = \frac{i}{2^n}) \\ &= P(U_n = \frac{i+1}{2^n}, V_n = \frac{i}{2^n}) \\ &= F(\frac{i+1}{2^n}) - F(\frac{i}{2^n}) \\ &= p_i \text{ (say)}. \end{aligned}$$

U_n is the "upper" approximation of ρ while V_n is the "lower" approximation of ρ. Thus, for each $n \geq 1$, we have two Poisson Boolean model, (X, λ, U_n) and (X, λ, V_n). We construct them on the same probability space $(\Omega, \mathcal{F}, P)$. Then, for any increasing event E, it is easily seen that *whenever the event occurs with radius random variable ρ, it will continue to occur with radius random variable U_n.* This motivates the definition of the approximation events.

For any point $\omega = (\omega_1, \omega_2) \in \Omega$ and every fixed $n \geq 1$, we define, for all $x \in I\!\!R^d$,

$$\begin{aligned} \omega_2^n(x) = & \tfrac{i+1}{2^n} \quad \text{if } \frac{i}{2^n} < \omega_2(x) \leq \frac{i+1}{2^n}, \\ \omega_{2n}(x) = & \tfrac{i}{2^n} \quad \text{if } \frac{i}{2^n} < \omega_2(x) \leq \frac{i+1}{2^n}. \end{aligned}$$

This way for every ω we define $\omega^n = (\omega_1, \omega_2^n)$ where ω_2^n is defined by $\omega_2^n = \{\omega_2^n(x) : x \in I\!\!R^d\}$ and $\omega_n = (\omega_1, \omega_{2n})$ where ω_{2n} is defined by $\omega_{2n} = \{\omega_{2n}(x) : x \in I\!\!R^d\}$. Note these two mappings will define two Poisson Boolean models, namely (X, λ, U_n) and (X, λ, V_n). So define now,

$$\begin{aligned} E_K^n(A, B) &= \{A \rightsquigarrow B \text{ in } K \text{ in the model } (X, \lambda, U_n)\} \\ &= \{\omega \in \Omega : \omega^n \in A \rightsquigarrow B \text{ in } K\}. \end{aligned}$$

Similarly, we define the corresponding event for the other percolation model, i.e.,

$$\begin{aligned} F^n_K(A,B) &= \{A \rightsquigarrow B \text{ in } K \text{ in the model } (X,\lambda,V_n)\} \\ &= \{\omega \in \Omega : \omega_n \in A \rightsquigarrow B \text{ in } K\}. \end{aligned}$$

Clearly, from the very construction of the events, we have for all $n \geq 1$

$$F^n_K(A,B) \subseteq A \subseteq E^n_K(A,B).$$

Hence, we obtain $P_{(\lambda,V_n)}(F^n_K(A,B)) \leq P_{(\lambda,\rho)}(E_K(A,B)) \leq P_{(\lambda,V_n)}(E^n_K(A,B))$ for all $n \geq 1$.

Note that $U_n \geq U_{n+1}$ and $V_n \leq V_{n+1}$, for all $n \geq 1$. Hence by the previous argument, we have $P_{(\lambda,U_n)}(E^n_K(A,B)) \geq P_{(\lambda,U_{n+1})}(E^{n+1}_K(A,B))$ and $P_{(\lambda,V_n)}(F^n_K(A,B)) \leq P_{(\lambda,V_{n+1})}(F^{n+1}_K(A,B))$ for all n. So, this monotonicity implies that,

$$\begin{aligned} \lim_{n\to\infty} P_{(\lambda,V_n)}(F^n_K(A,B)) &\leq P_{(\lambda,\rho)}(E_K(A,B)) \\ &\leq \lim_{n\to\infty} P_{(\lambda,U_n)}(E^n_K(A,B)). \end{aligned} \tag{2.8}$$

We shall prove that the limits in the equation (2.8) are actually equal to $P_{(\lambda,\rho)}(E_K(A,B))$. Consider the difference $P_{(\lambda,U_n)}(E^n_K(A,B)) - P_{(\lambda,V_n)}(F^n_K(A,B))$ and note that this is the probability of the event that the event $A \rightsquigarrow B$ in K occurs in the Poisson Boolean model (X,λ,U_n), but the event does not occur in the Poisson Boolean model (X,λ,V_n). Using the notation $N(K)$, as defined in Section 1.2 of Chapter 1, the number of Poisson points in the set K, we have

$$\begin{aligned} &P(E^n_K(A,B))) - P(F^n_K(A,B)) \\ &= P\Big\{A \rightsquigarrow B \text{ in } K \text{ with radius random variable } U_n \text{ but } A \text{ is} \\ &\quad \text{not connected to } B \text{ in } K \text{ with radius random variable } V_n\Big\} \\ &= \sum_{m=0}^{\infty} P\Big\{N(K)=m\Big\} P\Big\{A \text{ occurs in } K \text{ with radius random} \\ &\quad \text{variable } U_n \text{ but } A \text{ does not occur in } K \text{ with radius} \\ &\quad \text{random variable } V_n \mid N(K)=m\Big\} \end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{\exp(-\lambda \ell(K))}{m!} (\lambda \ell(K))^m P\Big\{A \rightsquigarrow B \text{ in } K \text{ with radius random variable } U_n \text{ but } A \text{ is not connected to } B \text{ in } K \text{ with radius random variable } V_n \mid N(K) = m\Big\}.$$

Given that there are m Poisson points in K, the positions $x_1, \ldots, x_m$ of these points are uniformly distributed in K. Hence we must have a pair of points in the set K such that the points are connected with radius U_n but not with V_n. Thus we have,

$$P\Big\{A \rightsquigarrow B \text{ in } K \text{ with radius random variable } U_n \text{ but } A \text{ is not connected to } B \text{ in } K \text{ with radius random variable } V_n \mid N(K) = m\Big\}$$

$$\leq P\Big\{\text{there exist at least two Poisson points } x_i, x_j,\ 1 \leq i,\ j \leq m \text{ in } K \text{ such that } 2V_n < d(x_i, x_j) \leq 2U_n \mid N(K) = m\Big\}$$

$$\leq \binom{m}{2} \int_K \cdots \int_K P\{2V_n < d(x_1, x_2) \leq 2U_n\} \frac{1}{(\ell(K))^m} dx_1 \cdots dx_m$$

$$= \binom{m}{2} \int_K \cdots \int_K \sum_{i=0}^{R2^n-1} P\{\frac{2i}{2^n} < d(x_1, x_2) \leq \frac{2(i+1)}{2^n}\} \times \frac{p_i}{(\ell(K))^k} dx_1 \cdots dx_m$$

$$= \binom{m}{2} \int_K \cdots \int_K \sum_{i=0}^{R2^n-1} 1_{\{i2^{-(n-1)} < d(x_1, x_2) \leq (i+1)2^{-(n-1)}\}} \times \frac{p_i}{(\ell(K))^m} dx_1 \cdots dx_m$$

$$= \binom{m}{2} \sum_{i=0}^{R2^n-1} \frac{p_i}{(\ell(K))^2} \int_K \int_K 1_{\{i2^{-(n-1)} < d(x_1, x_2) \leq (i+1)2^{-(n-1)}\}} dx_1 dx_2$$

$$= \binom{m}{2} \sum_{i=0}^{R2^n-1} \frac{p_i}{(\ell(K))^2} \int_K \int_{\{x_2 : i2^{-(n-1)} < d(x_1, x_2) \leq (i+1)2^{-(n-1)}\}} dx_1 dx_2$$

$$\leq \binom{m}{2} \frac{1}{(\ell(K))^2} \sum_{i=0}^{R2^n-1} p_i \int_K c_d((\frac{2(i+1)}{2^n})^d - (\frac{2i}{2^n})^d) dx$$

$$= \binom{m}{2}\frac{k_d}{\ell(K)}\sum_{i=0}^{R2^n-1}\left(\left(\frac{2(i+1)}{2^n}\right)^d p_i - \left(\frac{2i}{2^n}\right)^d p_i\right)$$
$$= \binom{m}{2}\frac{k_d}{\ell(K)}E(U_n^d - V_n^d).$$

Here and in the subsequent inequality c_d, k_d, K_d are positive constants depending only on d. Thus,

$$\begin{aligned} & P(E_K^n(A,B)) - P(E_K^n(A,B)') \\ \leq & \sum_{m=0}^{\infty}\frac{\exp(-\lambda\ell(K))}{m!}(-\lambda\ell(K))^m\binom{m}{2}\frac{k_d}{\ell(K)}E(U_n^d - V_n^d) \\ = & K_d\frac{\lambda^2\ell(K)}{2}E_{(\lambda,\rho)}(U_n^d - V_n^d). \end{aligned}$$

Now, $| U_n(\omega) - V_n(\omega) | \leq \frac{1}{2^n}$ and $| U_n(\omega) + V_n(\omega) | \leq 2(R+1)$ for all $\omega \in \Omega$. Hence, by Lebesgue's dominated convergence theorem,

$$E(U_n^d - V_n^d) \to 0 \text{ as } n \to \infty.$$

This completes the proof of the lemma. ∎

It should be noted here that this choice of ρ_k is not universal. Consider the Poisson system (X, λ, ρ) where ρ is the random variable which has the following distribution function:

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x}{2} & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

Consider the event A that the unit box has at least k points which have radii > 1. It is clear that if we take the above approximation, then

$$P_{(\lambda,U_n)}(A) - P_{(\lambda,\rho)}(A) \geq \exp(-\lambda)\frac{(\lambda)^k}{k!}(1 - 2^{-k}) > 0.$$

Hence this choice of U_n will not work. However, a different choice can be taken. Define, the random variable ρ_n by,

$$\rho_n = \begin{cases} 1 & \text{if } \rho = 1 \\ 1 + \frac{i}{2^n} & \text{if } \frac{i}{2^n} < \rho \leq \frac{i+1}{2^n} \quad \text{for } i = 1, \ldots 2^n \end{cases}$$

Easy calculations will now prove that the event A is finitely approximable.

Suppose that $\omega \in A\Box B$ for some $\omega = (\omega_1, \omega_2) \in \Omega$, then we must have disjoint subsets $H_1(\omega_1)$ and $H_2(\omega_1)$ of ω_1 (remember that ω_1 is set of points) such that $(H_1(\omega_1), \omega_2) \in A$ and $(H_2(\omega_1), \omega_2) \in B$. Since $A \subseteq A_n$ and $B \subseteq B_n$, $(H_1(\omega_1), \omega_2) \in A_n$ and $(H_2(\omega_1), \omega_2) \in B_n$. Thus $\omega \in A_n \Box B_n$ for every n. So, we have $A\Box B \subseteq A_n \Box B_n$. Hence, if we have a BK inequality for the events A_n which come from Poisson system where the radius takes finitely many values, we can get BK inequality for the events A and B as follows:

$$\begin{aligned} P_{(\lambda,\rho)}(A\Box B)) &\leq P_{(\lambda,\rho)}(\liminf_{n\to\infty} A_n \Box B_n) \\ &\leq \liminf_{n\to\infty} P_{(\lambda,\rho)}(A_n \Box B_n) \\ &\leq \liminf_{n\to\infty} P_{(\lambda,\rho)}(A_n) P_{(\lambda,\rho)}(B_n) \\ &= P_{(\lambda,\rho)}(A) P_{(\lambda,\rho)}(B). \end{aligned}$$

So now, we shall be concerned with Boolean model where ρ assumes only finitely many values.

2.5 Lattice approximable events

We describe now a lattice percolation model which we use in the next section to prove the continuum BK inequality. Consider a continuum percolation model with the radius random variable ρ taking only finitely many values, $r_1, r_2, \ldots, r_k$ with probabilities $p_1, p_2, \ldots, p_k$ respectively.

As we have noted in the first chapter, a continuum percolation model (X, λ, ρ) where ρ is as above can be constructed by superposing k independent continuum percolation models each with a degenerate radius random variable. We take independent continuum percolation models $(X_i, \lambda p_i, \rho_i)$ where $\rho_i = r_i$ a.s. for $i = 1, 2, \ldots, k$ and then superpose them to obtain the model (X, λ, ρ). The idea is to take k different lattices, connected in a suitable way so that on the ith lattice we can approximate the ith continuum model for $i = 1, 2, \ldots, k$.

Consider k copies of $\mathbb{R}$, say $G_1, G_2, \ldots, G_k$. Let $H^d = \bigcup_{i=1}^k G_i^d$. On G_i^d we place a Poisson point process X_i with intensity λp_i and centre, at each Poisson point of G_i^d, a d-dimensional sphere of radius r_i for

$i = 1, 2, \ldots, k$. So instead of looking at them together, we are viewing them as k different slices of $\mathbb{R}^d$.

For each $m \geq 1$, let $G_i^d(m), i = 1, 2 \ldots, k$, be k copies of the lattice $(\frac{1}{2^m}Z)^d$. The ith lattice $G_i^d(m)$ should be thought of as the lattice corresponding to the space G_i^d for each $i = 1, 2, \ldots, k$. Let

$$H_m^d = \bigcup_{i=1}^{k} G_i^d(m).$$

and for $a = (a_1, a_2, \ldots, a_d) \in G_i^d(m)$, consider the box

$$B_m^i(a) = \prod_{j=1}^{d} \left(a_j - \frac{1}{2^{m+1}}, a_j + \frac{1}{2^{m+1}}\right].$$

$B_m^i(a)$ will be called the *cell* containing the vertex $a \in G_i^d(m)$. Clearly $H^d = \bigcup_{i=1}^{k} \bigcup_{a \in G_i^d(m)} B_m^i(a)$. On the vertex set of H_m^d we define the adjacency relation $\overset{adj}{\longleftrightarrow}$ between vertices s_1 and s_2 as follows:

a) for $s_1, s_2 \in G_i^d(m), s_1 \overset{adj}{\longleftrightarrow} s_2$ if $d(B_m^i(s_1), B_m^j(s_2)) \leq 2r_i$ for $i = 1, 2, \ldots, k$,

b) for $s_1 \in G_i^d(m), s_2 \in G_j^d(m)$ for $1 \leq i, j \leq k$ with $i \neq j, s_1 \overset{adj}{\longleftrightarrow} s_2$ if $d(B_m^i(s_1), B_m^j(s_2)) \leq r_i + r_j$,

Here $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$ for A, B bounded sets in $\mathbb{R}^d$. Note that in defining $d(B_m^i(s_1), B_m^j(s_2))$ as in (b) we view both $B_m^i(s_1)$ and $B_m^j(s_2)$ as bounded subsets of $\mathbb{R}^d$. With this adjacency relation we turn the set H_m^d into a lattice $\mathbb{L}_m$.

On H_m^d we construct an independent site percolation model, given by the following occupancy rule:

> for any $i = 1, 2, \ldots, k$ a vertex $s \in G_i^d(m)$ is occupied
> if and only if there is at least one Poisson point
> from the process $(X_i, \lambda p_i, \rho_i)$ inside the cell $B_m^i(s)$.

So,

$$P(\text{ a vertex } s \text{ is occupied }) = 1 - \exp(-\lambda p_i \ell(B_m^i(s)) \text{ if } s \in G_i^d(m)$$

where $\ell(\cdot)$ is the Lebesgue measure on $I\!R^d$.

For each $i \geq 1$ and for each Poisson point $x \in G_i'^d$ from the process $(X_i, \lambda p_i, \rho_i)$, there is a unique cell $B_m^i(a)$ of the lattice $G_i'^d(m)$ in which the Poisson point x lies. Define a mapping π_m^i on the set of the Poisson points by $\pi_m^i(x) = a$. This is a many to one mapping, as all Poisson points inside the cube $B_m^i(a)$ will correspond to the same point.

So given a realization ω, the mappings $\pi_m^1, \pi_m^2, \ldots, \pi_m^k$ will define all the sites which are open in the lattice H_m^d. Denote the set of all such sites of the lattice H_m^d which are open by $\Pi_m(\omega)$. Since $\Pi_m(\omega)$ is a collection of sites of the lattice H_m^d, $\Phi(\Pi_m(\omega)) \in \{0,1\}^{\mathcal{V}}$ where $\mathcal{V} = \cup_{i=1}^k G_i'^d(m)$ and where Φ is as defined in Section 2.2. Clearly, if $\omega \leq \omega'$, then $\Pi_m(\omega) \subseteq \Pi_m(\omega')$ and hence $\Phi(\Pi_m(\omega)) \overset{\mathcal{V}}{\leq} \Phi(\Pi_m(\omega'))$. This way we construct a site percolation model on the lattice H_m^d.

Now we define a class of events, which we call the *lattice approximable* events.

Definition 2.3 *An increasing event A is called lattice approximable if there exist increasing events A_m in the lattice $I\!L_m$ for every $m \geq 1$, such that whenever $\omega \in A, \omega \in A_m$ and $\lim_{m\to\infty} P(A_m) = P(A)$.*

We have not been able to give any general description of the events that are lattice approximable. This will depend on the specific problems. But before we go about proving the theorem, we give an example of a lattice approximable event. This example will be used later to obtain results about the size of the occupied cluster.

We show that the events $E_K(A, B)$ are lattice approximable where A, B and K are as earlier. For ease of notation, we take the case where ρ assumes only two values r_1 and r_2 with probabilities p and q. The space corresponding to radius r_1 is denoted by R_d and corresponding lattice by R_m^d while the space corresponding to the radius r_2 is denoted by G_d and the corresponding latiice by G_m^d.

For $A \subseteq K$ and $r > 0$, let

$$A^r = \{x \in I\!R^d : d(x, a) \leq r \text{ for some } a \in A\} \tag{2.9}$$

be the 'r-fattening' of the region A. Let $\mathcal{C}$ denote the class of all Borel sets $A \subseteq I\!R^d$ such that

$$\ell(\partial A^r) = 0, \tag{2.10}$$

for all $r > 0$, where ∂A^r is the boundary of A^r and ℓ the Lebesgue measure on $\mathbb{R}^d$. Clearly spheres, half-planes, cubes and boxes are in this class $\mathcal{C}$.

Proposition 2.1 *In a Poisson system (X, λ, ρ) where ρ assumes two values with probabilities p and q respectively, the event $E_K(A, B)$ where K is a bounded set in $\mathbb{R}^d$ and $A, B \subseteq K$ and $A, B, K \in \mathcal{C}$ is lattice approximable.*

Proof : Let A, B and $K \in \mathcal{C}$. We approximate $E_K(A, B)$ in the lattice $\mathbb{L}_m = R^d_m \cup G^d_m$. For a region $A \subseteq \mathbb{R}^d$, let $\tilde{A} \subseteq H^d$ be defined by

$$\tilde{A} = \{r \in R^d : r \in A\} \cup \{g \in G^d : g \in A\},$$

i.e., $\tilde{A}$ is the union of the space corresponding to A in the space R^d and the space corresponging to A in the space G^d. Given two points $a, b \in \mathbb{R}^d$ we say that there is an occupied H^d_m path between a and b inside K if there exist occupied vertices $s_1, s_2, \ldots, s_n \in \tilde{K}$ such that

$$s_i \in H^d_m \text{ and } s_i \overset{adj}{\longleftrightarrow} s_{i+1}, \text{ for all } i = 1, 2, \ldots, n-1, \tag{2.11}$$

$$d(s_1, a) \leq \begin{cases} r_1 & \text{if } s_1 \in R^d_m \\ r_2 & \text{if } s_1 \in G^d_m, \end{cases} \tag{2.12}$$

and

$$d(s_n, b) \leq \begin{cases} r_1 & \text{if } s_n \in R^d_m \\ r_2 & \text{if } s_n \in G^d_m. \end{cases} \tag{2.13}$$

Observe that if there is an occupied H^d_m path between a and b then for every $k \geq m$ there is an occupied H^d_k path between a and b. We also note that if there is an occupied H^d_m path between a and b which 'goes through' the occupied H^d_m vertices $s_1, s_2, \ldots, s_n$ (as in (2.11), (2.12) and (2.13)) it does not necessarily imply that $a \rightsquigarrow b$ in the original Boolean model (X, ρ, λ). Indeed it is possible that s and t are two adjacent occupied vertices in H^d_m, but no Poisson point in the cell containing s is connected to any Poisson point in the cell containing t. However, if $a \rightsquigarrow b$ in the original Boolean model, there must be a sequence of Poisson points $x_{i_1}, \ldots, x_{i_n}$ for some $n = n(\omega)$ such that

$$S(x_{i_j}) \cap S(x_{i_{j+1}}) \neq \emptyset \text{ for all } j = 1, \ldots n-1, \tag{2.14}$$

$$a \in S(x_{i_1}) \text{ and } b \in S(x_{i_n}). \tag{2.15}$$

Thus if $s_1, \ldots, s_n$ are the vertices of H_m^d corresponding to the cells containing $x_{i_1}, \ldots, x_{i_n}$ respectively, i.e., $\Pi_m(x_{i_j}) = s_j$ for $j = 1, 2, \ldots, n$, with

$$s_j \in \begin{cases} R_m^d & \text{if } S(x_{i_j}) \text{ has radius } r_1 \\ G_m^d & \text{if } S(x_{i_j}) \text{ has radius } r_2, \end{cases}$$

then $s_1, \ldots, s_n$ satisfy (2.11), (2.12) and (2.13), i.e., there is an occupied H_m^d path between a and b.

For a bounded region K define

$$\begin{aligned} E_K^m(A,B) \quad = \quad & \{\text{there is an occupied } H_m^d \text{ path between} \\ & a \text{ and } b \text{ for some } a \in A, b \in B \text{ in } K\}. \end{aligned}$$

The preceding discussion makes it is clear that if $\omega \in E_K(A,B)$, then $\omega \in E_K^m(A,B)$ for all $m \geq 1$, i.e., $\omega \in \bigcap_{m\geq 1} E_K^m(A,B)$. This proves our first condition of the lattice approximable events.

Now, we want to show that $\lim_{n\to\infty} P(E_K^m(A,B)) = P(E_K(A,B))$. From our observation immediately after (2.13),

$$E_K^{m+1}(A,B) \subseteq E_K^m(A,B). \tag{2.16}$$

Thus

$$P(E_K(A,B)) \leq \lim_{m\to\infty} P(E_K^m(A,B). \tag{2.17}$$

We now show that $P(E_K^m(A,B)) \to P(E_K(A,B))$ as $m \to \infty$.

To this end we observe that if there is an occupied H_m^d path between a and b and no occupied path in the Boolean model between a and b, then there must exist two cells in H_m^d corresponding to two occupied vertices s and t such that $s \overset{adj}{\longleftrightarrow} t$ and no Poisson point in the cell containing s is connected to any Poisson point in the cell containing t. However, the distance between the Poisson points in these cells can differ from the distance between the two vertices s and t by at most

$2\sqrt{d}/2^m$. Thus if, *for every* $m \geq n$, there is an occupied H^d_m path between a and b and no occupied path in the Boolean model between a and b, then there must be two Poisson points in the Boolean model which are separated from each other by a distance equalling exactly the sum of the radii of the balls associated with these points.

Now let

$N_1 :=$ { there are an infinite number of Poisson points in $\tilde{K}$},

$N_2 :=$ {there is either a Poisson point at a distance exactly r_1 from ∂A in R^d or a Poisson point at a distance exactly r_2 from ∂A in G^d},

$N_3 :=$ {there is either a Poisson point at a distance exactly r_1 from ∂B in R^d or a Poisson point at a distance exactly r_2 from ∂B in G^d},

$N_4 :=$ {there exist two Poisson points in X_K such that

(i) both these points are in R^d and they are at a distance exactly $2r_1$ from each other

(ii) both these points are in G^d and they are at a distance exactly $2r_2$ from each other

(iii) one of these points is in R^d, the other is in G^d and they are at a distance exactly $r_1 + r_2$ from each other}.

Clearly $P(N_1) = P(N_2) = P(N_3) = P(N_4) = 0$ and, for $N := N_1 \cup N_2 \cup N_3 \cup N_4$, $P(N) = 0$. Here we have used that $A, B \in C$ and hence satisfy (2.10).

The preceding discussion shows that if $\omega \notin N \cup E_K(A,B)$, then $\omega \notin E^m_K(A,B)$ for all m sufficiently large. Since $E^m_K(A,B) \supseteq E_K(A,B)$ we have from (2.17),

$$P(E^m_K(A,B)) \to P(E_K(A,B)) \text{ as } m \to \infty. \tag{2.18}$$

Thus the proposition is proved. ■

Note the particular choice of the lattice is not mandatory. One may come up with a different choice for a different problem. In Chapter 5, for the the Random Connection Model, a similar choice of a lattice approximation will be used.

2.6 BK inequality

In this section we develop a correlation inequality which is, in spirit, the same as the BK inequality for discrete percolation in Section 2.2. Unlike in the discrete case, the inequality we obtain is more restricted and holds only for a special class of increasing events.

We first concentrate on a continuum percolation model where the radius random variable ρ assumes only finitely many values. We shall prove the inequality for two events which are lattice approximable.

Proposition 2.2 *In a Poisson Boolean model (X, λ, ρ) where ρ assumes only finitely many values, for any two lattice approximable increasing events A and B which depend only on a bounded subset K of $\mathbb{R}^d$, we have*

$$P_{(\lambda,\rho)}(A \Box B) \leq P_{(\lambda,\rho)}(A)P_{(\lambda,\rho)}(B). \tag{2.19}$$

Proof: The proof is rather simple. First we note that, if $\omega = (\omega_1, \omega_2) \in A \Box B$, then there exist two disjoint sets $H_1(\omega_1)$ and $H_1(\omega_1)$ such that $\omega' = (H_1(\omega_1), \omega_2) \in A$ and $\omega'' = (H_2(\omega), \omega_2) \in B$. By definition of the lattice approximable events we have increasing sets A_m, B_m in the lattice $\mathbb{L}_m$, for every $m \geq 1$, such that $\omega' \in A_m$ and $\omega'' \in B_m$. Since the mapping Π_m is a many to one mapping, we may have $\Pi_m(\omega') \cap \Pi_m(\omega'') \neq \emptyset$. But if we choose m large so that the Poisson points belong to different cells, then for each Poisson point we can associate an unique site in the lattice $\mathbb{L}_m$. So, for $m \geq M(\omega)$, the sets $\Pi_m(\omega')$ and $\Pi_m(\omega'')$ are disjoint. Hence by the definition of disjoint occurrence of events, we have, $\omega \in A_m \Box B_m$, for all $m \geq M(\omega)$, i.e., $\omega \in \liminf_{m\to\infty} A_m \Box B_m$. In other words, $A \Box B \subseteq \liminf_{m\to\infty} A_m \Box B_m$. Also, note that K is a bounded region, hence the events A_m and B_m depend only on the states of finitely many sites of the lattice $\mathbb{L}_m$.

Thus

$$\begin{aligned}
P_{(\lambda,\rho)}(A \Box B)) &\leq P_{(\lambda,\rho)}(\liminf_{m\to\infty} A_m \Box B_m) \\
&\leq \liminf_{m\to\infty} P_{(\lambda,\rho)}(A_m \Box B_m) \\
&\leq \liminf_{m\to\infty} P_{(\lambda,\rho)}(A_m)P_{(\lambda,\rho)}(B_m) \\
&= P_{(\lambda,\rho)}(A)P_{(\lambda,\rho)}(B),
\end{aligned}$$

where the second inequality follows from Fatou's lemma, the third inequality follows from the BK inequality Theorem 2.1 and the last equality follows from the fact that both A and B are lattice approximable. ■

Now we define a class of events, called *approximable* events, for which we prove the BK inequality.

Definition 2.4 *In a Poisson Boolean model (X, λ, ρ) where ρ satisfies the condition (1.6), an increasing event A is called an approximable event if for each $n \geq 1$, there exists random variables ρ_n taking finitely many values and increasing events A_n in the Poisson Poisson Boolean model (X, λ, ρ_n) such that $\rho_n \geq \rho$ and $A \subseteq A_n$ and $P_{(\lambda,\rho)}(A) = \lim_{n\to\infty} P_{(\lambda,\rho_n)}(A_n)$ and for each n the events A_n are lattice approximable.*

The BK inequality can now be stated and proved for the Poisson Boolean model (X, λ, ρ) where ρ satisfies the condition (1.6) in Chapter 1.

Theorem 2.3 BK Inequality *In a Poisson Boolean model (X, λ, ρ) where ρ satisfies the boundedness condition (1.6) of Chapter 1, for any two increasing approximable events A and B, we have*

$$P_{(\lambda,\rho)}(A \Box B) \leq P_{(\lambda,\rho)}(A) P_{(\lambda,\rho)}(B). \tag{2.20}$$

Proof of BK Inequality: The proof follows from the definition of the approximable events, Proposition 2.2 and Lemma 2.2. ■

2.7 Application of BK inequality

As an example of the application of the BK inequality, we establish the continuum version of a result well-known in the discrete lattice models (see, e.g., Kesten [1982], Menshikov [1986]). We use this result in the next chapter. We obtain that if the expected size of the occupied component containing the origin, $W(0)$, is finite, then the probability that $W(0)$ extends a distance at least m away from the origin decays exponentially in m. To this end we first introduce some notation. For any region $A \subseteq \mathbb{R}^d$ let $d(A) := \sup\{d(x, y) : x, y \in A\}$ and let $E_m(0)$

denote the event $E_{B_m}(\{0\}, \partial B_m) = \{0 \rightsquigarrow \partial B_m \text{ in } B_m\}$, as defined in (2.7) and $B_m = [-m, m]^d$.

Theorem 2.4 *Consider a Poisson Boolean model (X, ρ, λ) where ρ satisfies (1.6). There exist constants $C_1, C_2 > 0$ and $M \geq 1$, depending on λ and the dimension d, such that if $E(d(W(0)) < \infty$, then*

$$P_{(\lambda,\rho)}(E_m(0))) \leq C_1 \exp(-C_2 m)$$

for all $m \geq M$.

Proof: Consider the lattice

$$\mathbb{L}_R = (2R\mathbf{Z})^d = \{(2Rx_1, 2Rx_2, \ldots, 2Rx_d) : x_i \in \mathbf{Z}\},$$

where R is as in (1.6) of Chapter 1. Let the cells of $\mathbb{L}_R$ be

$$B_R^d(x) = \prod_{i=1}^{d}(2Rx_i - R, 2Rx_i + R],$$

for all $x = (2Rx_1, 2Rx_2, \ldots, 2Rx_d) \in \mathbb{L}_R$. We partition $\mathbb{R}^d$ with such cubes. Since $E(d(W(0))) < \infty$ we can choose M_1 large so that $\mathbb{E}_\lambda(\#W_{M_1}(0)) < 2^{-1} \cdot 3^{-d}$, where $\#W_{M_1}(0)$ is the number of cubes $B_R^d(x)$, $x \in \mathbb{L}_R$ such that $B_R^d(x) \cap W(0) \neq \emptyset$ and B_R^d lies outside the box $(-2RM_1, 2RM_1]^d$. For any $x = (2Rx_1, 2Rx_2, \ldots, 2Rx_d) \in \mathbb{L}_R$, define $I(x)$ to be the set $\{y = (2Ry_1, \ldots, 2Ry_d) \in \mathbb{L}_R : \mid y_i - x_i \mid \leq 1$, for all $1 \leq i \leq d$ and $x \neq y\}$. For any $k \geq M_1 + 2$ define

$$L_k(x) = \{y = (2Ry_1, 2Ry_2, \ldots, 2Ry_d) \in \mathbb{L}_R : -k \leq y_i - x_i \leq k, \\ i = 1, 2, \ldots, d \text{ and at least one } y_i \text{ is such that } \mid y_i - x_i \mid = k\}.$$

$L_k(x)$ is the set of vertices which lie on the perimeter of the box of size $2Rk$ around x.

Now fix any $M \geq M_1 + 2$. For any $m > M$ let

$$\begin{aligned} E_m(x) &:= \left\{B_R^d(x) \rightsquigarrow \partial B_m \text{ in } B_m\right\} \\ &= E_{B_m}(B_R^d(x), \partial B_m) \end{aligned}$$

Observe that any path from $B_R^d(0)$ to ∂B_m must have two disjoint segments; one connecting $B_R^d(0)$ to some cell on the boundary of the box $[-2RM, 2RM]^d$ and the other connecting that cell on the boundary to ∂B_m. We have, from this observation,

$$\begin{aligned}
&P_{(\lambda,\rho)}(E_m(0)) \\
&= P_{(\lambda,\rho)}\Big\{B_R^d(0) \rightsquigarrow \partial B_m \text{ in } B_m\Big\} \\
&\leq P_{(\lambda,\rho)}\Big\{\text{there are two disjoint connections in } B_m, \text{one} \\
&\quad \text{connects } B_R^d(0) \text{ to } \cup_{y\in I(x)} B_R^d(y), \text{ for some} \\
&\quad x \in L_M(0), \text{ and the other connecting } B_R^d(x) \text{ to } \partial B_m\Big\} \\
&\leq P_{(\lambda,\rho)}\Big(\cup_{x\in L_M(0)}\Big\{\text{there are two disjoint connections} \\
&\quad \text{in } B_m, \text{ one connecting } B_R^d(0) \text{ to } \cup_{y\in I(x)} B_R^d(y), \\
&\quad \text{and the other connecting } B_R^d(x) \text{ to } \partial B_m\Big\}\Big) \\
&\leq \sum_{x\in L_M(0)} P_{(\lambda,\rho)}\Big\{\text{ there are two disjoint connections} \\
&\quad \text{in } B_m, \text{ one connecting } B_R^d(0) \text{ to } \cup_{y\in I(x)} B_R^d(y), \\
&\quad \text{and the other connecting } B_R^d(x) \text{ to } \partial B_m\Big\}.
\end{aligned}$$

The BK inequality yields,

$$\begin{aligned}
&P_{(\lambda,\rho)}(E_m(0)) \\
&\leq \sum_{x\in L_M(0)} P_{(\lambda,\rho)}(E_{B_m}(B_R^d(0), \cup_{y\in I(x)} B_R^d(y)) \square E_{B_m}(B_R^d(x), \partial B_m)) \\
&\leq \sum_{x\in L_M(0)} P_{(\lambda,\rho)}(E_{B_m}(B_R^d(0), \cup_{y\in I(x)} B_R^d(y))) P(E_{B_m}(B_R^d(x), \partial B_m)) \\
&\leq \max_{x\in L_M(0)} P_{(\lambda,\rho)}(E_m(x)) \sum_{x\in L_M(0)} P_{(\lambda,\rho)}(E_{B_m}(B_R^d(0), \cup_{y\in I(x)} B_R^d(y))) \\
&\leq \max_{x\in L_M(0)} P_{(\lambda,\rho)}(E_m(x)) \sum_{x\in L_M(0)} P_{(\lambda,\rho)}(\cup_{y\in I(x)}(E_{B_m}(B_R^d(0), B_R^d(y)))) \\
&\leq \max_{x\in L_M(0)} P_{(\lambda,\rho)}(E_m(x)) \sum_{x\in L_M(0)} \sum_{y\in I(x)} P_{(\lambda,\rho)}(E_{B_m}(B_R^d(0), B_R^d(y))) \\
&\leq 3^d \max_{x\in L_M(0)} P_{(\lambda,\rho)}(E_m(x)) \sum_{y\in \cup_{i=-1}^{1} L_{M-i}(0)} P(E_{B_m}(B_R^d(0), B_R^d(y)))
\end{aligned}$$

$$\begin{aligned}
&\leq 3^d \max_{x\in L_M(0)} P_{(\lambda,\rho)}(E_m(x)) \sum_{y\in\cup_{i=M-1}^{\infty} L_k(0)} P_{(\lambda,\rho)}(E_K(B_R^d(0), B_R^d(y))) \\
&= 3^d \max_{x\in L_M(0)} P_{(\lambda,\rho)}(E_m(x)) E_{(\lambda,\rho)}(\# W_{M-1}(0)) \\
&\leq \frac{1}{2} \max_{x\in L_M(0)} P_{(\lambda,\rho)}(E_m(x)). \qquad (2.21)
\end{aligned}$$

Now to estimate $P_{(\lambda,\rho)}(E_m(x))$ we observe that if $m \geq 2M$ then, for a fixed x, a path from x to ∂B_m must intersect the boundary of the box $L_M(x)$ centred at x. Since the Poisson Boolean model is stationary, we have from (2.21),

$$P_{(\lambda,\rho)}(E_m(x)) \leq \frac{1}{2} \max_{y\in L_M(x)} P_{(\lambda,\rho)}(E_m(y)).$$

This argument can be repeated to yield

$$P(E_m(0)) \leq 2^{-\lfloor m/M\rfloor} \max_{x_1\in L_M(0)} \cdots \max_{x_{\lfloor m/M\rfloor}\in L_M(x_{\lfloor m/M\rfloor -1})} P(E_m(x_{\lfloor m/M\rfloor})),$$

where $\lfloor a\rfloor$ denotes largest integer smaller than a. Thus we obtain,

$$P_{(\lambda,\rho)}(E_m(0)) \leq 2^{-\lfloor m/M\rfloor}.$$

This completes the proof of the theorem. ∎

As a corollary we get the following result.

Corollary 2.1 *Suppose $E_{(\lambda,\rho)}[d(W(0))] < \infty$. Then there exists positive constants C_3, C_4 and $M \geq 1$ such that*

$$P_{(\lambda,\rho)}\Big(d(W(0)) \geq m\Big) \leq C_3 \exp(-C_4 m) \qquad (2.22)$$

Proof of the corollary : We note that $\{d(W(0)) \geq m\} \subseteq F_{\lfloor \frac{m}{2\sqrt{d}}\rfloor}(0)$, hence from the previous theorem, our result follows. ∎

As another application of the BK inequality we obtain :

Theorem 2.5 *In a Poisson Boolean model (X, ρ, λ) with ρ bounded as in (1.6) it is the case that $\{\lambda > 0 : E[(d(W(0)))] < \infty\}$ is an open interval.*

REMARK: Theorem 2.5 is a version of Corollary 5.1 of Kesten [1982] obtained for discrete site percolation.

Proof: We prove this for the 2-dimensional case. For higher dimensions the proof is similar.

Suppose the conclusion of the theorem is false. Then Theorem 2.4 yields

$$P_{(\lambda_c,\rho)}(E_m(0)) \leq 2^{-\lfloor m/M \rfloor},$$

where $E_m(0)$ is as in Theorem 2.4 and $m > M$. Consider the rectangle $S_1 = [0, 2RN] \times [0, 6RN]$. Let $A_N = \{(0, 2Rn) : 0 \leq n \leq N\}$, and $B_R(x) = x + [-R, R]^2$. Then

$$\begin{aligned}
&\sigma((6RN, 2RN), 1, \lambda_c) \\
&= P_{(\lambda_c,\rho)}\Big\{\text{there exists a L-R crossing of } [0, 6RN] \times [0, 2RN]\Big\} \\
&\leq P_{(\lambda_c,\rho)}\Big(\bigcup_{x \in A_N}\Big\{B_R(x) \rightsquigarrow \Big(\{6RN\} \times [0, 2RN]\Big)\Big\}\Big) \\
&\leq \sum_{x \in A_N} P_{(\lambda_c,\rho)}\Big\{B_R(x) \rightsquigarrow \Big(\{6RN\} \times [0, 2RN]\Big)\Big\} \\
&\leq \sum_{x \in A_N} P_{(\lambda_c,\rho)}\Big\{B_R(x) \rightsquigarrow \delta([-2RN, 2RN]^2)\Big\} \\
&\leq \sum_{x \in A_N} 2^{-\lfloor N/M \rfloor} \\
&= (N+1)2^{-\lfloor N/M \rfloor}.
\end{aligned}$$

Now as $N \longrightarrow \infty, \sigma((6RN, 2RN), 1, \lambda_c) \longrightarrow 0$. So, choose N large enough such that $\sigma((6RN, 2RN), 1, \lambda_c) < \kappa/2$, where κ is as in the Lemma 2.1 of Section 2.3.

In the proof of the Proposition 2.1, we have seen that $P_{(\lambda,\rho)}E_K(A, B)$ is a decreasing limit of upper semi continuous functions of λ, hence itself a upper semi continuous function. Now we note that $P_{(\lambda,\rho)}E_K(A, B)$ is an increasing function of λ. Hence it a right continuous function of λ. So choose, $\lambda > \lambda_c$ such that

$$\sigma((6RN, 2RN), 1, \lambda) < \sigma((6RN, 2RN), 1, \lambda_c) + \frac{\kappa}{2} < \kappa.$$

So by Lemma 2.1, we get

$$P_{(\lambda,\rho)}(d(W(0)) \geq b) \leq C_1 \exp(-C_2 b)$$

which implies, for some $\lambda > \lambda_c$, $E_{(\lambda,\rho)}(d(W(0))) < \infty$, contradicting Theorem 1.1 of Chapter 1. ■

2.8 Hartigan Conjecture

In a study of single linkage clusters Hartigan proposed a model which consists of n d-dimensional cubes each of which have sides of length $(\alpha/n)^{1/d}$ placed according to a uniform distribution in a d-dimensional unit cube. He posed some problems about the connectivity of these cubes and its asymptotics in n. In this study, we place n spheres of radius $(\alpha/n)^{1/d}$, instead of n cubes, in the unit cube.

All our results in this section and the subsequent sections are valid only for 2-dimensions. Penrose [1992] has studied this model for higher dimensions.

The Hartigan model and the statement of results :

For simplicity of notation, we describe the 2-dimensional model; higher dimensional analogue can be easily understood from this description. Let x be a random vector on some probability space $(\Omega, \mathcal{A}, P)$ such that $P(x \in A) = \ell(A)$ for any Borel $A \subseteq [0,1] \times [0,1]$, where $\ell(\cdot)$ denotes the Lebesgue measure in the two dimensional Euclidean space.

Let $x_1, x_2, \cdots$ be a sequence of i.i.d. copies of the random vector x. The $n-$th stage $(n \geq 1)$ of the Hartigan model consists of the points $\Xi_n = (x_1, x_2, \cdots, x_n)$ in $[0,1] \times [0,1]$ and at each point $x_i, 1 \leq i \leq n$, we centre a disc $S_n(x_i)$, of radius $\sqrt{\alpha/n}$, where $\alpha > 0$ is some quantity which will be used to parametrize the model. Thus, for any sample point $\omega \in \Omega$, the n-th stage Hartigan model $(n \geq 1)$ consists of discs each of radius $\sqrt{\alpha/n}$ centred at the n points $x_1(\omega), x_2(\omega), \cdots, x_n(\omega)$. Note the $(n+1)$-th stage of the Hartigan model has the extra point $x_{n+1}(\omega)$ along with the points $x_1(\omega), \cdots, x_n(\omega)$ of the n-th stage model, but the discs centred at these points each have radius $\sqrt{\alpha/(n+1)}$.

The *occupied* region can be defined as earlier, i.e., the n-th-stage occupied region C_n, is $C_n = \bigcup_{i=1}^n S_n(x_i)$. The complement of the set C_n in $[0,1] \times [0,1]$ is defined as the n-th stage *vacant* region, denoted by V_n.

Definition 2.5 *An n-th stage L-R (respectively T-B) occupied crossing of* $[0,1] \times [0,1]$ *is a continuous curve* γ *in* $[0,1] \times [0,1]$ *such that*

(i) $\gamma \in C_n$ *and (ii)* $\gamma \cap \{0\} \times [0,1] \neq \emptyset, \gamma \cap \{1\} \times [0,1] \neq \emptyset$ *(respectively* $\gamma \cap [0,1] \times \{0\} \neq \emptyset, \gamma \cap [0,1] \times \{1\} \neq \emptyset$.*)*

Let $L_n = \{$there exists an n-th stage L-R occupied crossing of $[0,1] \times [0,1]\}$ and $A = \limsup_{n\to\infty} L_n$. Define

$$\alpha_c = \inf\{\alpha : P(A) > 0\}. \tag{2.23}$$

The questions asked by Hartigan are

1) For $\alpha < \alpha_c$, is it true that the maximum diameter of clusters approaches zero asymptotically ?

2) For $\alpha > \alpha_c$, is it true that the maximum diameter of all clusters except the "big" cluster, approaches zero asymptotically ? Does the distance of any point in the cube from the "big' cluster approaches zero as $n \to \infty$?

3) Asymptotically, is the "big" occupied cluster connected ?

Before we interpret these questions formally, we introduce some more notations.

Definition 2.6 *An n-th stage L-R (respectively T-B) vacant crossing of* $[0,1] \times [0,1]$ *in a continuous curve* γ *such that*

(i) $\gamma \in V_n$ *and (ii)* $\gamma \cap \{0\} \times [0,1] \neq \emptyset, \gamma \cap \{1\} \times [0,1] \neq \emptyset$ *(respectively* $\gamma \cap [0,1] \times \{0\} \neq \emptyset, \gamma \cap [0,1] \times \{1\} \neq \emptyset$.*)*

Let $L_n^* = \{$there exists an n-th stage L-R vacant crossing of $[0,1] \times [0,1]\}$ and $A^* = \limsup L_n^*$. Define

$$\alpha_c^* = \sup\{\alpha : P(A^*) > 0\}. \tag{2.24}$$

Note that in 2-dimensions, using the rotation invariance of the model $P(L_n) + P(L_n^*) = 1$. Thus, if for some $\alpha > 0$, $P(A) = 0$ then $P(A^*) = 1$. Hence

$$\alpha_c \leq \alpha_c^*. \tag{2.25}$$

Now, we consider the cluster of all points in $[0,1]\times[0,1]$ which are connected to the left edge of the box and as well as the right edge of the box. Formally speaking,

$$\begin{aligned} R_n \;=\; \Big\{ & x \in [0,1]\times[0,1] : \text{ there exist two continuous curves } \gamma_L \\ & \text{and } \gamma_R \text{ in } [0,1]\times[0,1], \text{ not necessarily disjoint such that} \\ & i)\;\; x \in \gamma_L \text{ and } \gamma_L \cap \{0\}\times[0,1] \neq \emptyset \text{ and } \gamma_L \in C_n \\ & ii)\;\; x \in \gamma_R \text{ and } \gamma_R \cap \{1\}\times[0,1] \neq \emptyset \text{ and and } \gamma_R \in C_n \Big\}. \end{aligned}$$

and

$$\begin{aligned} R_n^* \;=\; \Big\{ & x \in [0,1]\times[0,1] : \text{ there exist two continuous curves } \gamma_L^* \\ & \text{and } \gamma_R^* \text{ in } [0,1]\times[0,1] \text{ not necessarily disjoint such that} \\ & i)\;\; x \in \gamma_L^* \text{ and } \gamma_L^* \cap \{0\}\times[0,1] \neq \emptyset \text{ and } \gamma_L^* \in V_n \\ & ii)\;\; x \in \gamma_R^* \text{ and } \gamma_R^* \cap \{1\}\times[0,1] \neq \emptyset \text{ and and } \gamma_R^* \in U_n \Big\}. \end{aligned}$$

Thus R_n is the set of all points which are connected to both the left and the right edges of the box. We show that in two dimensions,

Theorem 2.6

$$\alpha_c = \alpha_c^*, \tag{2.26}$$

and

Theorem 2.7 *(i) For $\alpha > \alpha_c$,*

$$P\Big(\lim_{n\to\infty} \{\sup d(x, R_n) : x \in [0,1]\times[0,1]\} = 0\Big) = 1;$$

(ii) for $\alpha < \alpha_c$

$$P\Big(\lim_{n\to\infty} \{\sup d(x, R_n^*) : x \in [0,1]\times[0,1]\} \;=\; 0\Big) =\; 1.$$

Let η_n be the total number of connected components of R_n and $d_n = \sup\{d(x,y) : x, y \in C$ and C a connected component of $[0,1]\times[0,1]\setminus R_n\}$ where the supremum is taken over all connected components C of $[0,1]\times[0,1]\setminus R_n$.

Note d_n measures the size of the largest connected component outside R_n and so the size of the largest occupied cluster outside R_n can be at most d_n. Similarly we define η_n^* and d_n^*.

Theorem 2.8 *If* $\alpha > \alpha_c$

$$P\left[\lim_{n\to\infty} \eta_n = 1 \text{ and } \lim_{n\to\infty} d_n = 0\right] = 1$$

and if $\alpha < \alpha_c$

$$P_\alpha\left[\lim_{n\to\infty} \eta_n^* = 1 \text{ and } \lim_{n\to\infty} d_n^* = 0\right] = 1.$$

These theorems along with some corollaries will answer the questions asked by Hartigan.

2.9 The modified Hartigan model

In this section we introduce a slightly modified model, called the *modified Hartigan model* and prove the theorems for the modified Hartigan model. Later we connect the Hartigan model and the modified model and show that Theorems 2.6, 2.7 and 2.8 follow from the results of the modified Hartigan model.

Let $N_1, N_2, \ldots$ be a sequence of i.i.d. Poisson random variables with $E(N_1) = 1$ on $(\Omega, \mathcal{A}, \mathcal{P})$. Let $Y_n = N_1 + N_2 + \ldots + N_n$; the $n-$th stage $(n \geq 1)$ of the modified Hartigan model consists of the points $\Xi_n = (x_1, x_2, \cdots, x_{Y_n})$ in $[0,1] \times [0,1]$ and at each point $x_i, 1 \leq i \leq Y_n$, we centre a disc of radius $\sqrt{\alpha/n}$, where $\alpha > 0$ is as in the Hartigan model. In other words, here at the n-th stage we are considering a Poisson process of intensity n, and the $(n+1)-$th stage is obtained by superposing the realisation of a Poisson point process of unit intensity on an existing realisation of a Posson point process of intensity n.

Note the same notation Ξ_n has been used in the definition of the Hartigan model in Section 2.8. We do this intentionally so as to define $L_n, A, L_n^*, A^*, \alpha_c, \alpha_c^*, R_n, R_n^*$ and $\eta_n, d_n, \eta_n^*, d_n^*$ for our modified Hartigan model exactly as we had defined in Section 2.8 for the Hartigan model.

For this model we show

Theorem 2.9

$$\alpha_c = \alpha_c^* = \lambda_c(1) \tag{2.27}$$

where $\lambda_c(1)$ is the criticality for continuum percolation model where the associated discs are of radius 1.

This theorem yields the following corollary which can be proved easily.

Corollary 2.2 *For $\alpha > \alpha_c, P(\liminf_{n\to\infty} L_n) = 1$ and for $\alpha < \alpha_c$, $P(\liminf_{n\to\infty} L_n^*) = 1$.*

Theorem 2.10 *(a) If $\alpha > \alpha_c$ then*

$$P\left[\lim_{n\to\infty} \sup_{x\in[0,1]\times[0,1]} d(x, R_n) = 0\right] = 1$$

and (b) if $\alpha < \alpha_c$ then

$$P\left[\lim_{n\to\infty} \sup_{x\in[0,1]\times[0,1]} d(x, R_n^*) = 0\right] = 1.$$

Theorem 2.11 *If $\alpha > \alpha_c$ then*

$$P\left[\lim_{n\to\infty} \eta_n = 1 \text{ and } \lim_{n\to\infty} d_n = 0\right] = 1$$

and if $\alpha < \alpha_c$ then

$$P\left[\lim_{n\to\infty} \eta_n^* = 1 \text{ and } \lim_{n\to\infty} d_n^* = 0\right] = 1.$$

We first prove a lemma which is used to prove these theorems. Fix any $\delta > 0$. Consider the rectangle $R = [0, \delta] \times [0, 1]$. Let

$$\begin{aligned} A_n(\delta) &= \{\text{there exists a L-R occupied crossing of the rectangle R} \\ &\quad \text{at the } n\text{-th stage}\} \\ A_n^*(\delta) &= \{\text{there exists a L-R vacant crossing of the rectangle R} \\ &\quad \text{at the } n\text{-th stage}\}. \end{aligned}$$

Note $A_n(\delta)$ (respectively $A_n^*(\delta)$) is the event that there is a occupied (respectively vacant) crossing in the longer direction rather than in the shorter direction.

Lemma 2.3 *(a) For $\alpha < \lambda_c$ there exist constants $C_1, C_2 > 0$ and $M_1 \geq 1$ such that if $n \geq M_1$*

$$P(A_n(\delta)) \leq C_1\left[\sqrt{\frac{n}{\alpha}}\right]\exp(-C_2\delta\sqrt{n})$$

and (b) for $\alpha > \lambda_c$ there exist constants $C_3, C_4 > 0$ and $M_2 \geq 1$ such that if $n \geq M_2$

$$P(A_n^*(\delta)) \leq C_3\left[\sqrt{\frac{n}{\alpha}}\right]\exp(-C_4\delta\sqrt{n}).$$

Proof : (a) From the scaling property of the Poisson point process (see Chapter 1, Section 1.3) we know that the cluster shapes in the Poisson Boolean model (X, λ, ρ) have the same distribution as that of a Poisson Boolean model $(\beta X, \beta^{-2}\lambda, \beta^{-2}\rho)$ except for a change in the length scale as β varies over $(0, \infty)$. Here we take the radius random variable $\rho = 1$ a.s. So, if we make a transformation

$$(x_1, x_2) \longrightarrow (\sqrt{\frac{n}{\alpha}}x_1, \sqrt{\frac{n}{\alpha}}x_2),$$

we get a Poisson point process of intensity α and around each Poisson point we have a disc of radius 1. Thus we get,

$$P(A_n(\delta)) = \sigma\Big((\delta\sqrt{\frac{n}{\alpha}}, \sqrt{\frac{n}{\alpha}}), 1, \alpha\Big)$$
$$P(A_n^*(\delta)) = \sigma^*\Big((\delta\sqrt{\frac{n}{\alpha}}, \sqrt{\frac{n}{\alpha}}), 1, \alpha\Big).$$

Let $K_n = \{(0,k) : 0 \leq k \leq \lfloor\sqrt{n/\alpha}\rfloor + 1\}$ and, for $(0,k) \in K_n$ let $B(k) = (-1/2, 1/2] \times (k - 1/2, k + 1/2]$. For $\alpha < \lambda_c$ we have

$$\begin{aligned}
&\sigma((\sqrt{n/\alpha}, \delta\sqrt{n/\alpha}), 1, \alpha) \\
&\leq P\Big(\bigcup_{k=0}^{\lfloor\sqrt{n/\alpha}\rfloor+1}\Big\{B(k) \rightsquigarrow \Big(\{\delta\sqrt{n/\alpha}\} \times [0, \sqrt{n/\alpha}]\Big)\Big\}\Big) \\
&\leq \sum_{k=0}^{\lfloor\sqrt{n/\alpha}\rfloor+1} P\Big\{B(k) \rightsquigarrow \Big(\{\delta\sqrt{n/\alpha}\} \times [0, \sqrt{n/\alpha}]\Big)\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{[\sqrt{n/\alpha}]+1} P\Big(d(W(B(k))) \geq \delta/2\sqrt{n/\alpha}\Big)\\
&\leq \sum_{k=0}^{[\sqrt{n/\alpha}]+1} C_1 \exp(-C_2\delta\sqrt{n})\\
&\leq C_1\left[\sqrt{\frac{n}{\alpha}}\right]\exp(-C_2\delta\sqrt{n}),
\end{aligned}$$

where the fourth inequality is obtained by an application of the FKG inequality (see (2.5)) and Theorem 2.1 of Section 2.7.

To show (b) we first note from Lemma 2.1 of Section 2.3 and (1.12) and Theorem 1.3 of Chapter 1 that for $\alpha > \lambda_c$

$$P(d(W^*(0)) \geq b) \leq C_3 \exp(-C_4 b) \qquad (2.28)$$

for $b > 0$.

Now replacing "occupied" by "vacant" in the proof of part (a) and using (2.6), we obtain (b). ∎

REMARK: In higher dimensions we get (i) for $\alpha < \lambda_c$ there exist constants $C_1, C_2 > 0$ and $M_1 \geq 1$ such that if $n \geq M_1$

$$P(A_n(\delta)) \leq C_1\left[\sqrt{\frac{n}{\alpha}}\right]\exp(-C_2\delta\sqrt{n})$$

and (ii) for $\alpha > \lambda_s^*$ there exist constants $C_3, C_4 > 0$ and $M_2 \geq 1$ such that if $n \geq M_2$

$$P(A_n^*(\delta)) \leq C_3\left[\sqrt{\frac{n}{\alpha}}\right]\exp(-C_4\delta\sqrt{n})$$

where $A_n(\delta)$ $(A_n^*(\delta))$ is the event that there is an occupied (vacant) crossing in the cube $[0,1] \times [0,1] \ldots \times [0,1] \times [0,\delta]$ along the shortest direction at the n-th stage and λ_s^* is the critical intensity corresponding to crossing probabilities (see (2.5) in Chapter 1).

Proof of Theorem 2.9: By (2.25) we have $\alpha_c \leq \alpha_c^*$. To complete the theorem we prove $\alpha_c^* \leq \lambda_c(1) \leq \alpha_c$, where $\lambda_c(1)$ is as defined critical intensity for the continuum percolation model with radius random variable taking value 1 a.s.. More precisely, we show $P(A) = 0$ for

$\alpha < \lambda_c(1)$ and $P(A^*) = 0$ for $\alpha > \lambda_c(1)$. From part a) of Lemma (2.3) with $\delta = 1$, we have for $\alpha < \lambda_c(1)$,

$$P(L_n) \le C_1 \left[\sqrt{\frac{n}{\alpha}}\right] \exp(-C_2\sqrt{n}).$$

So,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(L_n) &\le C_1 \sum_{n=1}^{\infty} \left[\sqrt{\frac{n}{\alpha}}\right] \exp(-C_2\sqrt{n}) \\
&\le C_1 \sum_{l=0}^{\infty} \sum_{n=[\alpha l^2]+1}^{[\alpha(l+1)^2]} \left[\sqrt{\frac{n}{\alpha}}\right] \exp(-C_2\sqrt{n}) \\
&\le C_1 \sum_{l=0}^{\infty} \sum_{n=[\alpha l^2]+1}^{[\alpha(l+1)^2]} l \exp(-C_2\sqrt{\alpha} l) \\
&\le C_1 \sum_{l=0}^{\infty} l(2\alpha l + \alpha + 1) \exp(-C_2\sqrt{\alpha} l) \\
&< \infty
\end{aligned}$$

and hence $P(A) = 0$ by Borel Cantelli lemma.

To show $P(A^*) = 0$ for $\alpha > \lambda_c(1)$ we apply Lemma 2.3(b). ■

REMARK: In higher dimensions ($d \ge 3$), we get

$$\lambda_c(1) \le \alpha_c \le \alpha_c^* \le \lambda_s^*(1).$$

Proof of Theorem 2.10: (a) Fix $m \ge 1$ and divide the unit square into m rectangles each with length 1 and breadth $1/m$. The idea is to place an occupied crossing along the length in each of this squares.

Let $B_k = \{$there is a L-R occupied crossing in the k-th rectangle $\}$ for $k = 1, 2, \ldots, m$. Note by translation and rotation invariance of the model, $P(B_k) = P(B_1) = 1 - P(A_n^*(1/m))$.

Hence,

$$\begin{aligned}
P(\sup_{x\in[0,1]\times[0,1]} d(x, R_n) \le 1/m) &\ge P(\bigcap_{k=1}^{m} B_k) \\
&\ge \prod_{k=1}^{m} P(B_k) \\
&= \left(1 - P(A_n^*(1/m))\right)^m
\end{aligned}$$

where the second inequality is obtained by using the FKG inequality. Hence,

$$P(\sup_{x\in[0,1]\times[0,1]} d(x,R_n) \geq 1/m) \leq 1 - \left(1 - P(A_n^*(1/m))\right)^m.$$

For $\alpha > \lambda_c(1)$, we obtain from Lemma 2.3 with $\delta = \frac{1}{m}$, $\sum_{n=1}^{\infty} P(A_n^*(\frac{1}{m}))$ $< \infty$ and hence, $\sum_{n=1}^{\infty} 1-(1-P(A_n^*(\frac{1}{m})))^m < \infty$. So, by Borel-Cantelli lemma the result follows.

(b) follows similarly. ■

REMARK: In higher dimensions ($d \geq 3$), we get (a) if $\alpha > \lambda_s^*$

$$P\Big[\lim_{n\to\infty} \sup_{x\in[0,1]\times[0,1]} d(x,R_n) = 0\Big] = 1$$

and (b) if $\alpha < \lambda_c(1)$

$$P\Big[\lim_{n\to\infty} \sup_{x\in[0,1]\times[0,1]} d(x,R_n^*) = 0\Big] = 1.$$

Proof of Theorem 2.11: (a) The idea of the proof of this theorem is similar to that of the proof of Theorem 2.10. Here we consider the rectangles $[0,1] \times (i/m,(i+1)/m]$ and $(j/m,(j+1)/m] \times [0,1]$ for all $1 \leq i \leq m, 1 \leq j \leq m$. As before we place occupied crossings in the longer direction of each of these rectangles. Then, using the FKG inequality and the invariance of the model under translation and rotation we obtain,

$$P(\eta_n = 1 \text{ and } d_n \leq (2\sqrt{2})/m) \geq \left(1 - P(A_n^*(1/m))\right)^{2m}.$$

Again using the Borel-Cantelli lemma we conclude the result.

(b) follows similarly. ■

REMARK: In higher dimensions ($d \geq 3$), we have,

$$P\Big[\lim_{n\to\infty} \eta_n = 1 \text{ and } \lim_{n\to\infty} d_n = 0\Big] = 1 \text{ for } \alpha > \lambda_s^*$$

and

$$P\Big[\lim_{n\to\infty} \eta_n^* = 1 \text{ and } \lim_{n\to\infty} d_n^* = 0\Big] = 1 \text{ for } \alpha < \lambda_c(1).$$

Now, we connect the Hartigan model with the modified Hartigan model we have defined and show that all the results which are true for the modified Harigan model can be lifted to the Hartigan model. We start with the proof of Theorem 2.6.

Proof of Theorem 2.6 We simply note that if the event L_n occurs in the Hartigan model and $Y_n > n$ then the event L_n occurs in the modified Hartigan model. So using the independence of Y_n and $x_i, i \geq 1$, we obtain,

$$\begin{aligned} &P(L_n \text{ occurs in Hartigan model }) \\ &\quad \leq \; P(L_n \text{ occurs in the modified Hartigan model })/P(Y_n > n). \end{aligned}$$

Now using the fact that $P(Y_n > n) \geq 1 - \Phi(1)$ (Johnson and Kotz [1969]) where

$$\Phi(1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} \exp(-x^2/2)dx,$$

we obtain,

$$\begin{aligned} &P(L_n \text{ occurs in the Hartigan model }) \\ &\quad \leq \; P(L_n \text{ occurs in the modified Hartigan model})/(1 - \Phi(1)). \end{aligned}$$

Hence we get α_c(Hartigan) $\leq \alpha_c$(modified Hartigan) where the notation is self-explanatory.

For the converse we note that

$$\begin{aligned} &P(L_n^* \text{ occurs in Hartigan model }) \\ &\quad \leq \; P(L_n^* \text{ occurs in the modified Hartigan model })/P(Y_n \leq n). \end{aligned}$$

Now using $P(Y_n \leq n) \geq 1/2$ (Johnson and Kotz [1969]), we obtain,

$$\begin{aligned} &P(L_n^* \text{ occurs in the Hartigan model }) \\ &\quad \leq \; 2P(L_n^* \text{ occurs in the modified Hartigan model}), \end{aligned}$$

and this proves the theorem. ∎

This technique of connecting the Hartigan and the modified Hartigan model can be used to obtain all the remaining theorems.

Chapter 3

The Covered Volume Fraction

3.1 Introduction

In the literature on continuum percolation, two related parameters have been studied. The first is the *covered volume fraction* (CVF) which has been studied primarily by physicists (Scher and Zallen [1970], Pike and Seager [1974], Gawlinski and Redner [1983], Phani and Dhar [1984]), while the other is the *intensity* of the underlying point process, studied primarily by mathematicians (Hall [1985,1986], Menshikov [1986], Roy [1990]). The results obtained in the first set of work is limited in that the results are primarily based on Monte Carlo simulations, while the latter set of work is limited in that the results primarily pertain to the existence of the percolating regime in a setting where the balls are random but of a given fixed distribution. In this chapter we settle a question raised in the first set of work regarding the universality of the critical CVF by methods established in the second set of work. Secondly, we obtain a continuity result concerning the critical CVF when the radii converge weakly. Finally, we prove that, the percolation function as defined by (1.1) in Chapter 1, is a continuous function except perhaps at the critical point. This has been used to obtain a stronger result of continuity. We show that when the radii converge weakly the percolation functions converge except perhaps at the critical point.

3.2 Covered Volume Fraction

In a realisation of the model let $x_1, \ldots, x_n$ be all the points in the unit box $[0,1]^d$ and $r_1, \ldots, r_n$ the associated radii of these balls at these points. Consider the quantity $\sum_{1\leq i\leq n} \pi_d r_i^d$, where π_d denotes the d-dimensional volume of a ball of unit radius. This corresponds to the sum of the volumes of each of the balls centred in the box $[0,1]^d$. It can be easily seen that the expected sum of the volumes of each of the balls centred in the unit box $[0,1]^d$ is $\lambda\pi_d E_{(\lambda,\rho)}\rho^d$. This quantity is called the *volume density* of (X,λ,ρ). By the invariance properties of the model it is obvious that the volume density is unaffected, if instead of $[0,1]^d$, we chose a different unit box in $I\!R^d$. The CVF is the quantity $A_\rho(\lambda) := 1 - \exp(-\lambda\pi_d E_{(\lambda,\rho)}\rho^d)$, which corresponds to the expected volume in a unit box covered by balls (see Hall [1988], page 128). A simple argument using ergodic theorem yields that if $B_n = [-n,n]^d$, then $\lim_{n\to\infty} \frac{1}{(2n)^d}\ell(B_n \cap C)$ exists and equals to $A_\rho(\lambda)$, where $\ell(A)$ denotes the d-dimensional volume of a region $A \subseteq I\!R^d$. Note here that as a function of λ, $A_\rho(\lambda)$ is a continuous non-decreasing function. Moreover a simple calculation (see Hall [1985]) shows that

$$P_{(\lambda,\rho)}(0 \text{ is covered by a ball}) = 1 - \exp(-\lambda\pi_d E_{(\lambda,\rho)}\rho^d),$$

thus,

$$\text{CVF of } (X,\lambda,\rho) = P_{(\lambda,\rho)}(0 \text{ is covered by a ball}). \tag{3.1}$$

Note that if the radius random variable ρ is such that $E_{(\lambda,\rho)}\rho^d = \infty$ then for every $\lambda > 0$, the CVF equals 1. To rule out such instances and to be able to apply the known mathematical results of this model we restrict ourselves to the case where ρ has bounded support i.e., ρ satisfies condition (1.6) in Chapter 1.

Clearly, for a fixed ρ, coupling methods, as explained in Section 1.3 of Chapter 1, will allow us to show that, if $\lambda_1 \leq \lambda_2$ then

$$P_{(\lambda_1,\rho)}(W(0) \text{ is unbounded}) \leq P_{(\lambda_2,\rho)}(W(0) \text{ is unbounded}),$$

where $W(0)$ is the occupied cluster of the origin.

The *critical volume density* and the *critical CVF* are defined as $\lambda_c \pi_d E_{(\lambda,\rho)} \rho^d$ and $A_c(\rho) := 1 - \exp(-\lambda_c \pi_d E_{(\lambda,\rho)} \rho^d)$ respectively.

It is obvious that if ρ_1 and ρ_2 are such that $\rho_1 \equiv r_1$ and $\rho_2 \equiv r_2$ for some fixed reals $0 < r_1 < r_2 < \infty$, then

$$\lambda_c(r_1) := \lambda_c(\rho_2) \geq \lambda_c(\rho_1) =: \lambda_c(r_2). \tag{3.2}$$

Now if we have a Boolean model (X, λ, ρ_1) and we effect a change of scale $x \to (r_2/r_1)x$, we obtain a Boolean model $((r_2/r_1)\, X,\ (r_1/r_2)^d\, \lambda,\ (r_2/r_1)\rho_1)$. This clearly tells us that (X, λ, ρ_1) is equivalent in law to $((r_2/r_1)X, (r_1/r_2)^d\lambda, \rho_2)$. Thus, their critical intensities are related by the obvious relation (see Zuev and Sidorenko [1985])

$$\lambda_c(r_1) r_1^d = \lambda_c(r_2) r_2^d. \tag{3.3}$$

Clearly, (3.3) implies that

$$A_c(r_1) := A_c(\rho_1) = A_c(\rho_2) =: A_c(r_2) =: A_c \text{ (say)}. \tag{3.4}$$

The equality in (3.4) suggested the conjecture (Kersetz and Vicsek [1982]) that, for all random variables ρ with bounded support, $A_c(\rho)$ is a constant independent of ρ. Phani and Dhar [1984] gave a heuristic argument which showed that the conjecture is false, and supported their argument with Monte Carlo simulations.

In this chapter, we prove the following:

Theorem 3.1 *There exists a random variable ρ taking values a and b with probability p and $1-p$ respectively, where $a \neq b$, $a > 0$, $b > 0$ and $0 < p < 1$ such that*

$$A_c(\rho) > A_c, \tag{3.5}$$

where, as in (3.4), A_c denotes the critical CVF of a model with balls of fixed radius.

Our second result is concerned with the continuity of $A_c(\rho_k)$ when the sequence $\{\rho_k\}$ converges weakly:

Theorem 3.2 *Let ρ_k and ρ be random variables such that for some $R > 0$ we have $0 \leq \rho \leq R$ and $0 \leq \rho_k \leq R$ a.s. for all $k \geq 1$. If $\rho_k \Rightarrow \rho$ then $A_c(\rho_k) \to A_c(\rho)$, where '$\Rightarrow$' denotes weak convergence.*

In combination with Theorem 3.1, this result shows that for a whole class of distributions of ρ, inequality (3.5) is valid. Also, it states that simulation methods cannot distinguish between two models very 'close' to each other. In Section 3.3 we obtain further results on bounds on the rate of convergence of the critical intensities.

Finally, we use the uniqueness of the occupied unbounded cluster of the origin in the supercritical region to prove that the percolation function is continuous except perhaps at the critical point. We show

Theorem 3.3 *In a Poisson Boolean model (X, λ, ρ), where ρ satisfies condition (1.6) of Chapter 1, the percolation function is a continuous function of λ for all $\lambda \neq \lambda_c(\rho)$.*

Viewing the percolation function as a function of the radius random variable for a fixed intensity, we have

Theorem 3.4 *Let ρ_k and ρ be random variables such that for some $R > 0$ we have $0 \leq \rho \leq R$ and $0 \leq \rho_k \leq R$ a.s. for all $k \geq 1$. If $\rho_k \Rightarrow \rho$ then $\theta_{\rho_k}(\lambda) \to \theta_\rho(\lambda)$ for all $\lambda \neq \lambda_c(\rho)$.*

Our proof of Theorem 3.1 in Section 3.3 strongly suggests that whenever ρ is not a constant a.s., then $A_c(\rho) > A_c$. However, we do not have a proof of this inequality.

Finally, denote the critical volume density of a model with fixed size balls by VD_c. Our proof of Theorem 3.1 also shows that for any $\epsilon > 0$, it is possible to construct a model such that the critical volume density of this model is between $2VD_c - \epsilon$ and $2VD_c$. This justifies a claim made by Phani and Dhar [1984].

3.3 Proof of Theorems 3.1 and 3.2

For ease of description we present the proofs in the two-dimensional case; all our arguments, however, are valid for higher dimensions.

Proof of Theorem 3.1 Let (X_1, λ_1, ρ_1) and (X_2, λ_2, ρ_2) be two independent Poisson Boolean models with their respective CVF's being $A_1(\lambda_1)$ and $A_2(\lambda_2)$. The CVF of the process obtained by the superposition of these two Poisson Boolean models is just $A(\lambda_1 + \lambda_2) =$

$A_1(\lambda_1) + A_2(\lambda_2) - A_1(\lambda_1)A_2(\lambda_2)$. To observe this, we use (3.1) and the standard inclusion-exclusion formula.

Let $0 < r_1 < r_2 < \infty$ be arbitrary positive numbers. Fix $\epsilon, \delta > 0$ such that

$$(2 - \epsilon - \delta)A_c - (1 - \epsilon)(1 - \delta)A_c^2 > A_c, \qquad (3.6)$$

where A_c is given by (3.4).

Now, as noted earlier, the CVF is a continuous non-decreasing function of λ, so we may choose $\lambda_1 < \lambda_c(r_1)$ and $\lambda_2 < \lambda_c(r_2)$ such that the CVFs of the Poisson Boolean models (X_1, λ_1, r_1) and (X_2, λ_2, r_2) are equal to $(1 - \epsilon)A_c$ and $(1 - \delta)A_c$ respectively, where $\lambda_c(r_1)$ and $\lambda_c(r_1)$ are as defined in (3.2). By our choice of λ_1 and λ_2, (X_1, λ_1, ρ_1) and (X_2, λ_2, ρ_2) are subcritical. *We fix this choice of* λ_1 *and* λ_2 *throughout this proof.*

Next we consider the superposition of the processes. We note that the CVF of this superposition is given by the left hand side of the expression (3.6) and hence is strictly larger than A_c. We intend to show that the superposed process is in its subcritical regime. Since the CVF $A_\rho(\lambda)$ is a non-decreasing function of λ, we can thereby conclude that the CVF of the superposed model is *strictly larger* than A_c.

First note that scaling does not change the CVF. Consider the process (X_1, λ_1, r_1) and scale it by a factor $\alpha < 1$ to obtain a process which is equivalent in law to $(\alpha X_1, \alpha^{-2}\lambda_1, \alpha r_1)$. In other words, if a realisation of (X_1, λ_1, r_1) are the points $\{x_1, x_2, \ldots\}$ with associated balls of radius $r_1, r_2, \ldots$, respectively then the realisation of the scaled model consists of the points $\{\alpha x_1, \alpha x_2, \ldots\}$ with associated balls of radius $\alpha r_1, \alpha r_2, \ldots$ respectively. (Note that in this way, we couple all processes together for $\alpha < 1$.) The CVF, as it is the probability of the origin being covered by at least one ball, remains the same, i.e.,

$$\text{CVF of } (X_1, \lambda_1, r_1) = \text{ CVF of } (\alpha X_1, (\alpha)^{-2}\lambda_1, \alpha r_1) \qquad (3.7)$$

So now our idea is to scale down one process, keeping the other fixed.

Superpose the scaled process $(\alpha X_1, (\alpha)^{-2}\lambda_1, \alpha r_1)$ with (X_2, λ_2, r_2). It follows from (3.6) and (3.7) that the CVF of the superposition of (X_2, λ_2, r_2) and $(\alpha X_1, \alpha^{-2}\lambda_1, \alpha r_1)$ is strictly larger than A_c. Our goal now is to show that the superposed model is subcritical for α small enough.

For this we are going to employ Lemma 2.1 of Chapter 2. We note from the definition in (1.5) in Chapter 1, that for $\lambda < \lambda_S(\rho) = \lambda_c(\rho)$,

$$\limsup_{n\to\infty} \sigma((n,3n),1,\lambda,\rho) = 0. \tag{3.8}$$

Fix $\kappa > 0$ as in Lemma 2.1 of Chapter 2. Since $\lambda_2 < \lambda_c(r_2)$, (3.8) implies that for $\lambda_2 < \lambda_S(r_2)$, we can find a number N so large that

$$\sigma((N,3N),1,\lambda_2,r_2) < \frac{1}{3}\kappa,$$

where $\sigma(N,3N),1,\lambda_2,r_2)$ is the occupied crossing probability as defined in (1.4) of Chapter 1 for the Poisson Boolean model (X_2,λ_2,ρ_2).

If there is no occupied left-right (L-R) crossing of $[0,N]\times[0,3N]$, i.e., there is no occupied crossing in the first direction, then there must be vacant top-bottom (T-B) crossing, i.e., a vacant crossing in the second direction. In other words, there is at least one component in $([0,N]\times[0,3N])\cap V$ intersecting the top and bottom side of the rectangle, where V is the uncovered region as introduced Chapter 1. We can order these components from left to right, say, and the leftmost component is called U. Only finitely many balls intersect $[0,N]\times[0,3N]$ a.s. and hence the boundary ∂U of U has only finitely many components a.s. Hence, for n large enough, the event $E_n := \{U$ exists and all components of $\partial U \cap (0,N)\times(0,3N)$ are at a distance at least n^{-1} from each other$\}$ has probability at least $1-\frac{1}{2}\kappa$. We fix n_0 such that

$$P_{(\lambda_2,r_2)}(E_{n_0}) > 1 - \frac{1}{2}\kappa. \tag{3.9}$$

Next we turn again to (X_1,λ_1,r_1). Since $\lambda_1 < \lambda_c(r_1)$, it follows from (2.4) of Section 2.7 in Chapter 2, and an application of the FKG-inequality that for $B_1 = [-1,1]^2$, for all $b > 0$ and positive constants C_3 and C_4 independent of b,

$$P_{(\lambda_1,r_1)}(d(W(B_1)) \geq b) \leq C_3 e^{-C_4 b},$$

where $W_1(B_1)$ is the occupied cluster of the box B_1 in the model (X_1,λ_1,r_1) (see (2.5) of Section 2.3 in Chapter 2) . Scaling down by a factor $\alpha < 1$, yields

$$P_{(\alpha^{-2}\lambda_1,\alpha r_1)}(d(W_\alpha(B_\alpha)) \geq \alpha b) \leq C_3 e^{-C_4 b},$$

where $B_\alpha = [-\alpha, \alpha]^2$ and $W_\alpha(B_\alpha)$ is the occupied cluster of the box B_α in the model $(\alpha X_1, (\alpha)^{-2}\lambda_1, \alpha r_1)$. Taking $\alpha = m^{-1}$ for some large integer m, and $b = (2\alpha n_0)^{-1}$ (with n_0 as in (3.9)), we obtain

$$P_{(m^2\lambda_1, m^{-1}r_1)}\left(d(W_{m^{-1}}(B_{m^{-1}})) \geq (2n_0)^{-1}\right) \leq C_3 e^{\frac{-C_4 m}{2n_0}}. \qquad (3.10)$$

Now we combine the conclusions obtained in (3.9) and (3.10). Divide $[0, N]\times[0, 3N]$ into $3N^2m^2$ boxes with side length m^{-1}, and denote these boxes by $B^1, B^2, \ldots, B^{3N^2m^2}$. Then, from (3.10), the probability that in the model $(m^{-1}X_1, m^2\lambda_1, m^{-1}r_1)$ the event

$$F^m_{n_0} := \bigcup_{i=1}^{3N^2m^2} \{d(W_{m^{-1}}(B^i)) \geq (2n_0)^{-1}\}$$

occurs has probability at most $3N^2m^2C_3e^{\frac{-C_4 m}{2n_0}}$, which tends to zero as $m \to \infty$. We now fix an m_0 such that this probability is at most $\frac{1}{3}\kappa$. If the event E_{n_0} occurs in (X_2, λ_2, r_2) and the event $F^{m_0}_{n_0}$ does not occur in $(m^{-1}X_1, m_0^2\lambda_1, m_0^{-1}r_1)$, then it follows that there is no occupied L-R crossing in $[0, N] \times [0, 3N]$ in the superposition of the two processes. This superposition is in fact the model $(X, \lambda_2 + m_0^2\lambda_1, \rho)$, where ρ is a random variable taking values r_2 and $m_0^{-1}r_1$ with probability $\lambda_2(m_0^2\lambda_1 + \lambda_2)^{-1}$ and $m_0^2\lambda_1(m_0^2\lambda_1 + \lambda_2)^{-1}$ respectively and $X = X_1 \cup X_2$. Hence, the probability of an occupied L-R crossing of $[0, N]\times[0, 3N]$ in $(X, \lambda_2 + m_0^2\lambda_1, \rho)$ is at most $\frac{1}{2}\kappa + \frac{1}{3}\kappa < \kappa$. By Lemma 2.1 of Chapter 2, this implies

$$P_{(\lambda_2 + m_0^2\lambda_1, \rho)}\left\{d(W(\mathbf{0})) \geq a\right\} \leq C_5 e^{-C_6 a},$$

for all $a > 0$ where $W(\mathbf{0})$ is the occupied cluster of the origin $\mathbf{0}$ in $(X, \lambda_2 + m_0^2\lambda_1, \rho)$. Hence $E_{(\lambda_2 + m_0^2\lambda_1, \rho)}[d(W(\mathbf{0}))] < \infty$. Thus the model is subcritical and this proves the theorem. ∎

Proof of Theorem 3.2: If $\rho_k \Rightarrow \rho$, then the boundedness of the radii implies that $E_{(\lambda,\rho_k)}\rho_k^d \to E_{(\lambda,\rho)}\rho^d$. It is therefore enough to prove that $\lambda_c(\rho_k) \to \lambda_c(\rho)$ when $k \to \infty$.

Our strategy will be to approximate the radii by radii which take only finitely many values. Thus we first investigate the case in which both ρ_k and ρ take only finitely many values.

Lemma 3.1 *Let $0 < r_1 < r_2 < \cdots < r_n < \infty$ and let ρ and ρ' be random variables taking values r_i with probability p_i and p'_i respectively for $i = 1, 2, \ldots, n$. Suppose there exist $1 \leq j < l \leq n$ such that for all $i \neq j$ or l and $i = 1, \ldots, n, p_i = p'_i$ and p_l and p'_l are both positive. Then,*

$$|\lambda_c(\rho) - \lambda_c(\rho')| \leq \frac{\lambda_c(r_1)}{\min\{p_l, p'_l\}}|p_j - p'_j|.$$

Proof: Suppose first that $p_j > p'_j$. We shall use a coupling argument to prove that

$$\lambda_c(\rho) \geq \lambda_c(\rho'). \tag{3.11}$$

To see this, consider n independent Poisson processes $X_1, X_2, \ldots, X_n$ of intensities $p_1\lambda, p_2\lambda, \ldots, p_{j-1}\lambda, p'_j\lambda, p_{j+1}\lambda, \ldots, p_n\lambda$ respectively. At each point of the process X_i we centre a ball of radius r_i. Now consider another independent Poisson process X' of intensity $\lambda(p_j - p'_j)$. Note that if at each point of this process X' we centre a ball of radius r_j then the superposition of the models $(X_1, p_1\lambda, r_1)$, $(X_2, p_2\lambda, r_2), \ldots, (X_{j-1}, p_{j-1}\lambda, r_{j-1}), (X_j, p'_j\lambda, r_j), (X_{j+1}, p_{j+1}\lambda, r_{j+1}), \ldots, (X_n, p_n\lambda, r_n)$ and $(X', (p_j - p'_j)\lambda, r_j)$ is a Poisson Boolean model (X, λ, ρ) where $X = \cup_{i=1}^n X_i \cup X'$. If, instead, at the points of the process X' we centre a ball of radius r_l and then superpose all the models, we obtain a Poisson Boolean model (X, λ, ρ'). Since $r_j < r_l$, the occupied region in (X, λ, ρ) will be contained in the occupied region in (X, λ, ρ'). Hence the existence of an unbounded cluster in the model (X, λ, ρ) will imply the existence of an unbounded cluster in the model (X, λ, ρ'). Thus we obtain the inequality (3.11). We have explained this in detail because we shall be using this kind of coupling results very often later without going through the details of the proof.

Now choose $\lambda > \lambda_c(\rho')$. First consider a Boolean model $(\tilde{X}, \lambda, \rho')$. Also consider the models $(X_i, \lambda l_i, r_i)$, for $i = 1, \ldots, l-1, l+1, \ldots, n$, where the l_i's are chosen such that

$$\frac{\lambda p'_i + \lambda l_i}{\lambda(1 + L)} = p_i, i = 1, \ldots, l-1, l+1, \ldots, n, \tag{3.12}$$

for $L := l_1 + \cdots + l_{l-1} + l_{l+1} + \cdots + l_n$. The system of linear equations (3.12) can be solved explicitly, to yield $l_i = (p_l)^{-1}(p_i p'_l - p_l p'_i) \geq 0$. Next, consider the superposition of $(\tilde{X}, \lambda, \rho')$ and $(X_i, \lambda l_i, r_i)$, $i = 1, \ldots, l-1, l+1, \ldots, n$ to obtain a model equivalent in law to $(X, \lambda(1+L), \rho)$ where $X = X_1 \cup \cdots \cup X_{l-1} \cup X_{l+1} \cup \cdots \cup X_n \cup \tilde{X}$. (To see that the radius random variable in this superposition is ρ, just use (3.12).) Since $\lambda > \lambda_c(\rho')$, the model $(\tilde{X}, \lambda, \rho')$ is supercritical, the superposition is certainly supercritical and hence

$$\lambda(1+L) > \lambda_c(\rho).$$

The above inequality holds for all $\lambda > \lambda_c(\rho')$, so we have

$$\lambda_c(\rho')(1+L) \geq \lambda_c(\rho).$$

From (3.12) and some elementary calculations one shows that $L = (p_l)^{-1}(p_j - p_j')$ and the result follows, using that $\lambda_c(\rho') \leq \lambda_c(r_1)$. For the case $p_j < p_j'$, we just reverse the roles of ρ and ρ'. ■

REMARK: Note that this lemma says that the rate of growth of the critical intensities is no more than linear.

Lemma 3.2 *Let $0 < r_1 < \cdots < r_n$, and let ρ be a random variable taking values r_i with probability p_i for $i = 1, 2, \ldots, n$. Suppose $p_n > 0$. For all $k = 1, 2, \ldots$, define the random variables ρ_k taking values r_i with probability $p_{k,i}$, for all $i = 1, \ldots, n$. If $p_{k,i} \to p_i$ for all i when $k \to \infty$, then $\lambda_c(\rho_k) \to \lambda_c(\rho)$.*

Proof: We have assumed that $p_n > 0$ so we can pick $0 < \delta < p_n$. Take k_0 so large that $\sum_{i=1}^{n-1} |p_{k,i} - p_i| < \frac{1}{2}\delta$ for all $k \geq k_0$. Then, of course, we have $p_{k,n} > \frac{1}{2}\delta$ for all $k \geq k_0$. For $l = 1, \ldots, n-1$ and $k \geq k_0$ let $\xi_k^{(l)}$ be the random variable defined by

$$P(\xi_k^{(l)} = r_i) = \begin{cases} p_{k,i}, & \text{for } i = 1, \ldots, l, \\ p_i, & \text{for } i = l+1, \ldots, n-1, \\ p_n + \sum_{i=1}^{l}(p_i - p_{k,i}), & \text{for } i = n. \end{cases}$$

Clearly, $\xi_k^{(n-1)}$ has the same distribution as ρ_k and we define $\xi_k^{(0)} := \rho$.

According to Lemma 3.1, for $l = 1, \ldots, n-1$, we have

$$|\lambda_c(\xi_k^{(l)}) - \lambda_c(\xi_k^{(l-1)})| \leq 2\delta^{-1}\lambda_c(r_1)|p_l - p_{k,l}|.$$

Adding the previous inequalities over l, and using the triangle inequality, we obtain

$$|\lambda_c(\rho_k) - \lambda_c(\rho)| \leq 2\delta^{-1}\lambda_c(r_1) \sum_{l=1}^{n-1} |p_l - p_{k,l}|,$$

for all $k \geq k_0$. This proves the lemma. ■

Next we drop the assumption that p_n should be positive:

Lemma 3.3 *Let ρ take values $0 < r_1 < \cdots < r_n$ with probabilities $p_1, \ldots, p_n$ respectively. Suppose ρ_k takes values $r_1, \ldots, r_n$ with probabilities $p_{k,1}, \ldots, p_{k,n}$. If $p_{k,i} \to p_i$ for all $1 \leq i \leq n$ then $\lambda_c(\rho_k) \to \lambda_c(\rho)$.*

Proof: In view of Lemma 3.2, we need to prove this lemma for the case when there exists $1 \leq m \leq n-1$ such that

$$p_m > 0 \text{ and } p_{m+1} = \ldots = p_n = 0. \tag{3.13}$$

First we show that it suffices to prove the lemma for the case $m = n-1$. Indeed, if the random variables ξ_k' and ξ_k'' assume values $r_1, \ldots, r_n$ with probabilities $p_{k,1}, p_{k,2}, \ldots, p_{k,m}, 0, \ldots, 0, \sum_{i=m+1}^{n} p_{k,i}$ and $p_{k,1}, p_{k,2}, \ldots, p_{k,m}, \sum_{i=m+1}^{n} p_{k_i}, 0, \ldots, 0$ respectively, then we clearly have

$$\lambda_c(\xi_k') \leq \lambda_c(\rho_k) \leq \lambda_c(\xi_k'').$$

So it suffices to show that $\lambda_c(\rho_k)$ converges to $\lambda_c(\rho)$ when the ρ_k's take at most one value larger than r_m with positive probability. Thus we henceforth assume that $m = n-1$, i.e. $p_{n-1} > 0$ and $p_n = 0$.

Next let ρ_k' be a random variable taking values $r_1, \ldots, r_n$ with probabilities $p_1, p_2, \ldots, p_{n-2}, p_{k,n-1}', p_{k,n}$ respectively, where $p_{k,n-1}' := p_{n-1} - p_{k,n}$. For k large enough, since $p_{n-1} > 0$ and $p_{k,n} \to 0$ as $k \to \infty$, we have $p_{k,n-1}' \geq 0$.

We shall now prove

$$\lim_{k\to\infty} \lambda_c(\rho_k') = \lambda_c(\rho). \tag{3.14}$$

From our choice of ρ_k', we observe that $\lambda_c(\rho_k') \leq \lambda_c(\rho)$. So to prove (3.14) we need to show that $\liminf_{k\to\infty} \lambda_c(\rho_k') \geq \lambda_c(\rho)$.

Suppose there exists a λ such that $\liminf_{k\to\infty} \lambda_c(\rho'_k) < \lambda < \lambda_c(\rho)$. Since $\lambda < \lambda_c(\rho)$, for κ as in Lemma 2.1 in Chapter 1, we can find an N such that

$$\sigma((N,3N),\lambda,\rho) < \frac{1}{2}\kappa. \tag{3.15}$$

Now we construct independent Poisson Boolean models $(X_i, \lambda l_{k,i}, r_i)$, for $i = 1, 2, \ldots n-2, n$, and another independent Poisson Boolean model (X', λ, ρ) so as to yield the model $(X, \lambda(1+L_k), \rho_k)$ when all the models are superposed, where $X = X' \cup \cup_{i=1}^{n-2} X_i \cup X_n$ and $L_k = l_{k,1} + \cdots + l_{k,n-2} + l_{k,n}$. For this, we choose $l_{k,1}, \ldots, l_{k,n-2}, l_n$ to satisfy

$$\frac{p_i + l_{k,i}}{1+L_k} = p_i, \text{ for } i = 1, \ldots, n-2, \tag{3.16}$$

and

$$\frac{l_{k,n}}{1+L_k} = p_{k,n}. \tag{3.17}$$

The system of linear equations (3.16) and (3.16) can be solved explicitly to yield

$$l_{k,i} = \left(\frac{p_{n-1} - p'_{k,n-1}}{p'_{k,n-1}}\right) p_i \geq 0, \text{ for } i = 1, \ldots, n-2,$$

and

$$l_{k,n} = \frac{p_{n-1}}{p'_{k,n-1}} p_{k,n} \geq 0.$$

Clearly, for every $i = 1, \ldots, n-2$ and $i = n$, $l_{k,i} \to 0$ when $k \to \infty$. Thus, we can choose k large enough such that for all $i = 1, \ldots, n-2$ and $i = n$, we have

$$P_{\lambda l_{k,i}}(X_i \text{ has at least one point in } [-R, N+R] \times [-R, 3N+R]) < \tfrac{1}{2n}\kappa, \tag{3.18}$$

where κ is as chosen before.

The superposition of the Poisson Boolean models $(X_i, \lambda l_{k,i}, r_i)$ for all $i = 1, \ldots, n-2, n$ and (X', λ, ρ) is equivalent in law to the Poisson Boolean model $(X, \lambda(1+L_k), \rho'_k)$. For k large enough, (3.15) and (3.18) imply that $\sigma((N,3N), 1, \lambda(1+L_k), \rho) < \kappa$, and thus it follows from Lemma 2.1 of Chapter 2, that the superposed model is subcritical. However, by the choice of λ, (X, λ, ρ'_k) is supercritical, hence so is $(X, \lambda(1 + L_k), \rho'_k)$ which is the desired contradiction.

Finally, to complete the proof of the lemma, we construct $\xi_k^{(l)}$ as in the previous lemma, where $\xi_k^{(n-1)}$ has the same distribution as ρ_k and $\xi_k^{(0)} = \rho'_k$. This method shows that

$$|\lambda_c(\rho_k) - \lambda_c(\rho'_k)| \leq 2(p_{n-1})^{-1}\lambda_c(r_1) \sum_{i=1}^{n-2} |p_{k,i} - p_i|,$$

and the lemma follows. ∎

Now are in a position to prove Theorem 3.2. First we suppose that the supports of both ρ and $\rho_k, k = 1, 2, \ldots$ are concentrated in an interval $[a, R]$, where $a > 0$. The distribution function of ρ is denoted by F, and the distribution function of ρ_k by F_k. We can assume that both a and R are continuity points of F. Take a sequence $\{\pi_n\}$ of partitions of $[a, R]$, which we write as $\pi_n = \{a = \gamma_0^n < \gamma_1^n < \cdots < \gamma_{k_n}^n = R\}$. The partitions are chosen in such a way that π_{n+1} refines π_n, all points γ_i^n are continuity points of F and such that $|\pi_n| := \max_{1 \leq i \leq k_n}\{\gamma_i^n - \gamma_{i-1}^n\} \to 0$ when $n \to \infty$. Now define, for all $n \geq 1$, the random variables $\rho^{(n)}$ and $\rho_{(n)}$ by the requirement that if $\rho \in (\gamma_{i-1}^n, \gamma_i^n]$, then $\rho^{(n)} = \gamma_i^n$ and $\rho_{(n)} = \gamma_{i-1}^n$. It follows from a simple coupling argument that $\lambda_c(\rho^{(n)}) \leq \lambda_c(\rho) \leq \lambda_c(\rho_{(n)}) \leq \lambda_c(a)$. Also, it is easy to see that $\lambda_c(\rho^{(n)})$ is increasing and $\lambda_c(\rho_{(n)})$ is decreasing in n. Now write

$$\alpha_n := \max_{1 \leq i \leq k_n} \frac{\gamma_i^n}{\gamma_{i-1}^n} \leq 1 + \frac{|\pi_n|}{a},$$

which tends to 1 when $n \to \infty$. Hence $\rho^{(n)} \leq \alpha_n \rho_{(n)}$ which implies that $\lambda_c(\rho^{(n)}) \geq \lambda_c(\alpha_n \rho_{(n)}) = \alpha_n^{-2}\lambda_c(\rho_{(n)})$. Hence

$$\lambda_c(\rho^{(n)}) \leq \lambda_c(\rho) \leq \alpha_n^2 \lambda_c(\rho^{(n)}).$$

We can now write

$$\begin{aligned} \lambda_c(\rho) - \lambda_c(\rho^{(n)}) &\leq \left[\left(1 + \frac{|\pi_n|}{a}\right)^2 - 1\right] \lambda_c(\rho^{(n)}) \\ &\leq \left[\left(1 + \frac{|\pi_n|}{a}\right)^2 - 1\right] \lambda_c(a) =: \beta_n, \text{ (say)}. \qquad (3.19) \end{aligned}$$

The previous calculation can also be done for ρ_k instead of ρ and we obtain, in the obvious notation

$$\lambda_c(\rho_k) - \lambda_c(\rho_k^{(n)}) \leq \beta_n. \tag{3.20}$$

Now given any $\epsilon > 0$, take n so large that $\beta_n < \epsilon$. Observe that $\rho^{(n)}$ takes the value γ_i^n with probability $F(\gamma_i^n) - F(\gamma_{i-1}^n)$ and $\rho_k^{(n)}$ takes the value γ_i^n with probability $F_k(\gamma_i^n) - F_k(\gamma_{i-1}^n)$. Hence by the choice of the partitions, the fact that $\rho_k \Rightarrow \rho$ and Lemma 3.3, we see that $|\lambda_c(\rho^{(n)}) - \lambda_c(\rho_k^{(n)})| < \epsilon$ for k sufficiently large. Together with (3.19) and (3.20) this proves the theorem in this case.

Next we drop the assumption that the supports are bounded from below by some positive number. Let $\delta > 0$ be a continuity point of F and let $\eta > 0$ be such that $P_{(\lambda,\rho)}(\rho > \delta) > \eta$. Since $\rho_k \Rightarrow \rho$, we have $P_{\lambda,\rho_k}(\rho_k > \delta) > \eta$ for k sufficiently large. Certainly, if $(X', \eta\lambda, \delta)$ is supercritical, so is (X, λ, ρ_k) and it follows that if $\eta\lambda > \lambda_c(\delta)$ then $\lambda > \lambda_c(\rho_k)$, or

$$\lambda_c(\rho_k) \leq \frac{1}{\eta}\lambda_c(\delta). \tag{3.21}$$

Now let $\epsilon > 0$ and choose a to be a continuity point of F such that $F(a) < \epsilon$, and choose k_0 so large that $F_k(a) < \epsilon$ for all $k \geq k_0$. Let ρ^a be a random variable with distribution equal to the conditional distribution of ρ, given that $\rho \geq a$. Similarly, let ρ_a be a random variable with distribution equal to the conditional distribution of ρ given $\rho < a$. Then we have $\lambda_c(\rho^a) \leq \lambda_c(\rho)$.

Consider the model (X_1, λ, ρ^a) and $(X_2, \lambda l, \rho_a)$, where l is chosen such that $l(1+l)^{-1} = P_{(\lambda,\rho)}(\rho \leq a)$. This means that

$$l = \frac{F(a)}{1 - F(a)}. \tag{3.22}$$

The superposition of the two models is equivalent in law to a process $(X, \lambda(1+l), \rho)$. Thus if $\lambda > \lambda_c(\rho^a)$, then certainly this superposition is supercritical and hence $\lambda(1+l) > \lambda_c(\rho)$, i.e. $\lambda_c(\rho^a)(1+l) \geq \lambda_c(\rho)$. Hence

$$|\lambda_c(\rho) - \lambda_c(\rho^a)| \leq l\lambda_c(\rho^a) \leq \frac{\epsilon}{1-\epsilon}\lambda_c(\rho), \tag{3.23}$$

where we have used (3.22). In the same way we find, in the obvious notation and using (3.21),

$$|\lambda_c(\rho_k) - \lambda_c(\rho_k^a)| \leq \frac{\epsilon}{1-\epsilon}\lambda_c(\rho_k^a) \leq \frac{\epsilon}{\eta(1-\epsilon)}\lambda_c(\delta). \tag{3.24}$$

When $\rho_k \Rightarrow \rho$, then $\rho_k^a \Rightarrow \rho^a$ and from the case already proved we conclude that

$$|\lambda_c(\rho_k^a) - \lambda(\rho^a)| < \epsilon \tag{3.25}$$

for k large enough. The result now follows by combining (3.23), (3.24) and (3.25). ■

3.4 Convergence of percolation function

In this section we deal with the percolation functions $\theta(\lambda)$. We prove that the percolation function is a continuous function of λ except at the point $\lambda_c(\rho)$. This is a version of the Theorem (6.35) of Grimmett [1988], page 117.

Proof of Theorem 3.3: First note that the θ is a right-continuous function. This proof is similar to that in the discrete case. The event $\{d(W) = \infty\}$ is the decreasing limit of the events $E_n = \{0 \rightsquigarrow \partial(B_n)\}$ where $B_n = [-n,n]^d$. By a lattice approximation as in Chapter 2, it is easy to see that $P(E_n)$ is a decreasing limit of continuous functions of λ, hence it is an upper semi continuous function. Thus we obtain that θ is a decreasing limit of upper semi continuous functions. So it is also an upper semi continuous function. Since $\theta(\lambda)$ is a non-decreasing function of λ, it is right continuous.

Now we prove the left continuity of $\theta(\lambda)$. If $\lambda < \lambda_c(\rho)$ the function θ is identically zero, hence trivially left continuous. Fix $\lambda_0 > \lambda_c(\rho)$. We have to prove that $\theta(\lambda_0) = \lim_{\lambda \uparrow \lambda_0} \theta(\lambda)$.

Fix $\lambda \in (\lambda_c(\rho), \lambda_0)$. By Kolmogorov's $0-1$ law the Poisson Boolean model (X, λ, ρ) admits an unbounded occupied cluster with probability 1. Now we couple the two processes (X, λ, ρ) and (X_0, λ_0, ρ) i.e., we construct (X, λ, ρ) from (X_0, λ_0, ρ) by thinning the process as explained in Section 1.3 in Chapter 1. The uniqueness of the unbounded occupied cluster (Theorem 1.2 of Chapter 1) yields that the unbounded occupied cluster C of (X, λ, ρ) is contained in the unbounded occupied cluster C_0 of (X_0, λ_0, ρ).

We scale the model (X, λ, ρ) suitably to yield a model of intensity λ_0, albeit with a different radius random variable. Consider the scale change:

$$x \mapsto \left(\frac{\lambda}{\lambda_0}\right)^{1/d} x = \alpha x \text{ (say)}.$$

If we apply this change of scale to the process (X, λ, ρ) we obtain a Poisson Boolean model $(\alpha X, \lambda_0, \alpha\rho)$. It is clear that whenever the process (X, λ, ρ) has an unbounded cluster the process $(\alpha X, \lambda_0, \alpha\rho)$ will admit an unbounded cluster and vice versa. So we have

$$\Psi(\alpha) := \theta_{\alpha\rho}(\lambda_0) = \theta_\rho(\lambda). \tag{3.26}$$

Thus the left continuity of θ at λ_0 is equivalent to the left continuity of Ψ at 1.

Suppose a realisation of the process (X_0, λ_0, ρ) consists of the Poisson points $\{x_1, x_2, \ldots, \}$ with associated radii $r_1, r_2, \ldots$ respectively. To get a realisation of the process $(X_0, \lambda_0, \alpha\rho); 0 < \alpha < 1$ we centre a ball of radii αr_i at the point x_i. Thus we have constructed all the Poisson Boolean models $\{(X_0, \lambda_0, \alpha\rho); 0 < \alpha < 1\}$ with a fixed set of Poisson points $X_0 = \{x_1, x_2, \ldots\}$. *For the rest of the proof we fix this construction of the models.*

By the construction of the Poisson process in Section 1.2 of Chapter 1, we have only finitely many points inside every bounded set in $\mathbb{R}^d$. Hence the event $\{d(W(\mathbf{0})) = \infty\}$ occurs if and only if $\{\#(W(\mathbf{0})) = \infty\}$, where $\#(W(\mathbf{0}))$ denotes the number of Poisson points in the cluster $W(\mathbf{0})$. Thus

$$\Psi(\alpha) = P_{(\lambda_0, \alpha\rho)}(\#(W_{(\lambda_0, \alpha\rho)}(\mathbf{0})) = \infty), \tag{3.27}$$

where $W_{(\lambda_0, \alpha\rho)}(\mathbf{0})$ is the occupied cluster of the origin in the model $(X_0, \lambda_0, \alpha\rho)$.

For $0 < \alpha \leq 1$, denote the occupied cluster of origin in the model $(X_0, \lambda_0, \alpha\rho)$ by W_α i.e., $W_\alpha = W_{(\lambda_0, \alpha\rho)}(\mathbf{0})$. Clearly we have $W_{\alpha_1} \subseteq W_{\alpha_2}$ for $0 < \alpha_1 < \alpha_2 \leq 1$. Thus we are required to prove that

$$P\Big(\#(W_1) = \infty, \text{ but } \#(W_\alpha) < \infty \text{ for all } \alpha < 1\Big) = 0. \tag{3.28}$$

Fix $\lambda_1 > \lambda_c(\rho)$ and let $\alpha_0 = (\lambda_1/\lambda_0)^{1/d}$. Let

N_1 = {there are more than one unbounded cluster in the model (X_0, λ_0, ρ)}.

N_2 = {there exist two Poisson points in the model (X_0, λ_0, ρ) whose associated balls just touch each other}.

N_3 = {there is a Poisson point x in the model (X_0, λ_0, ρ) with associated ball of radius r such that $d(x, 0) = r$}.

N_4 = {the Poisson Boolean model $(X_0, \lambda_0, \alpha_0\rho)$ does not admit any unbounded cluster}.

Let $N = N_1 \cup N_2 \cup N_3 \cup N_4$. We claim that

$$P(N) = 0. \tag{3.29}$$

First we prove the theorem assuming the claim.

Let us denote by A the event $\{\#(W_1) = \infty$, but $\#(W_\alpha) < \infty$ for all $\alpha < 1\}$. Fix $\omega \in A \cap N^c$. Since $\omega \in A, \#(W_1)(\omega) = \infty$. Thus, for the configuration ω, there is an infinite sequence of Poisson points $y_1, y_2, \ldots$ with associated radii $r_{i_1}, r_{i_2}, \ldots$ respectively such that $S_{r_j}(y_j) \cap S_{r_{j+1}}(y_{j+1}) \neq \emptyset$ for $j = 1, 2, \ldots$, where $S_r(x)$ is the ball of radius r centred at the point x. Since $\omega \in A \cap N^c$, we have $\#(W_{\alpha_0}(\omega) < \infty$. However the configuration ω admits an unbounded cluster $U_{\alpha_0}(\omega)$ in the model $(X_0, \lambda_0, \alpha\rho)$, so we have $U_{\alpha_0}(\omega) \cap W_{\alpha_0}(\omega) = \emptyset$. The uniqueness of the unbounded occupied cluster in the supercritical regime (Theorem 1.2 in Section 1.2 of Chapter 1) yields that $U_{\alpha_0}(\omega) \subseteq W_1(\omega)$. Hence there must exist Poisson points $\{y_{k_1}, y_{k_2}, \ldots\} \subseteq \{y_1, y_2, \ldots\}$ with associated radii $\alpha_0 r_{k_1}, \alpha_0 r_{k_2}, \ldots$ such that $S_{\alpha_0 r_{k_j}}(y_{k_j}) \cap S_{\alpha_0 r_{k_{j+1}}}(y_{k_{j+1}}) \neq \emptyset$. But $U_{\alpha_0}(\omega) \cap W_{\alpha_0}(\omega) = \emptyset$ implies that there must exist a point $y = y(\omega) \in \{y_{k_1}, y_{k_2}, \ldots\}$ such that for every sequence of Poisson points $\{x_{i_1}, \ldots, x_{i_n} = y\}$ with associated radii $\alpha_0 r_{i_1}, \ldots, \alpha r_{i_n}$, it must be that, either for some $j = 1, \ldots, n-1$

$$d(x_{i_j}, x_{i_{j+1}}) > \alpha_0(r_{i_j} + r_{i_{j+1}})$$

or

$$d(x_{i_1}, 0) > \alpha_0 r_{i_1}.$$

But the point $y \in \{y_1, y_2, \ldots\} \subseteq W_1(\omega)$. Thus there must exist a sequence of Poisson points $\{x_{k_1}, \ldots, x_{k_m} = y\}$ with associated radii $r_{k_1}, \ldots, r_{k_m}$ respectively such that

$$d(x_{k_j}, x_{k_{j+1}}) \leq r_{k_j} + r_{k_{j+1}} \text{ for all } j = 1, \ldots, m-1, \tag{3.30}$$

and

$$d(x_{k_1}, 0) \leq r_{k_1}. \tag{3.31}$$

Further $\omega \in N^c$ implies, from (3.30) and (3.31),

$$d(x_{k_j}, x_{k_{j+1}}) < r_{k_j} + r_{k_{j+1}} \text{ for all } j = 1, \ldots, m-1 \tag{3.32}$$

and

$$d(x_{k_1}, 0) < r_{k_1}. \tag{3.33}$$

So we may choose $\beta = \beta(\omega) \in (\alpha_0, 1)$ such that

$$d(x_{k_j}, x_{k_{j+1}}) \leq \beta(r_{k_j} + r_{k_{j+1}}) \text{ for all } j = 1, \ldots, m-1, \tag{3.34}$$

and

$$d(x_{k_1}, 0) \leq \beta r_{k_1}. \tag{3.35}$$

i.e., the point y is connected to the origin in the model $(X_0, \lambda_0, \beta\rho)$. Since $\beta > \alpha_0$, we have $U_{\alpha_0}(\omega) \subseteq U_\beta(\omega)$ and hence $y \in U_\beta(\omega)$. Thus $U_{\alpha_0}(\omega) \subseteq W_\beta(\omega)$ and unboundedness of $U_{\alpha_0}(\omega)$ implies that $\#\,(W_\beta(\omega)) = \infty$ contradicting the fact that $W_\alpha(\omega) < \infty$ for all $\alpha < 1$. Thus $P(A) = 0$.

Now we prove (3.29). By the uniqueness of the infinite cluster we obtain, $P(N_1) = 0$ and by the uniqueness of the unbounded cluster in the supercritical regime we have $P(N_4) = 0$.

To show $P(N_2) = 0$, it is enough to show that for every $M \geq 1$, $P(N_2(M)) = 0$ for $N_2(M) = \{$there exist two Poisson points in $[-M, M]^d$ in the model (X, λ_0, ρ) whose associated balls just touch each other$\}$. So fix an $M \geq 1$. For every $n \geq 1$ define the events $D_n = \{$there exists a pair of Poisson points x_i, x_j inside $[-M, M]^d$ having balls of radius r_i and r_j respectively such that $0 \leq d(x_i, x_j) - (r_i + r_j) < 2^{-n}\}$ and $E_n = \{$there exists a pair of Poisson points x_i, x_j inside $[-M, M]^d$ having balls of radius r_i and r_j respectively such that

$0 \leq (r_i + r_j) - d(x_i, x_j) < 2^{-n}\}$. Clearly, $F_n = D_n \cup E_n$ decreases to $N_2(m)$. Now we show that $P_{(\lambda_0,\rho)}(D_n) \to 0$ as $n \to \infty$.

Let F be the distribution function of ρ. For $n = 1, 2, \ldots$, define random variables U_n:

$$U_n = \frac{i}{2^n} \quad \text{if} \quad \frac{i}{2^n} < \rho \leq \frac{i+1}{2^n}, \quad i = 1, 2 \ldots, R2^n - 1$$

where R is as in (1.6). Clearly, for every $i \geq 0$,

$$P(U_n = \frac{i}{2^n}) = F(\frac{i+1}{2^n}) - F(\frac{i}{2^n}) = p_i \text{ (say)}.$$

This will define a Poisson Boolean model (X_0, λ_0, U_n). Define, $D'_n = \{$there exists a pair of points (x_i, x_j) in the model (X_0, λ_0, U_n) with associated radii U_n^i and U_n^j respectively inside $[-M, M]^d$ such that $0 \leq (U_n^i + U_n^j) - d(x_i, x_j) < 3 \cdot 2^{-n}\}$. Clealy, $D_n \subseteq D'_n$. Now,

$$\begin{aligned} & P_{(\lambda_0,U_n)}(D'_n)) \\ = & \sum_{m=0}^{\infty} P_{(\lambda_0,U_n)}\Big\{N([-M,M]^d) = m\Big\} P_{(\lambda_0,U_n)}\Big\{\text{there exists a pair of} \\ & \quad \text{Poisson points } x_i, x_j \text{ in the model } (X_0, \lambda_0, U_n), \text{ with associated} \\ & \quad \text{radii } U_n^i \text{ and } U_n^j \text{ respectively inside } [-M,M]^d \text{ such that} \\ & \quad 0 \leq (U_n^1 + U_n^2) - d(x_1, x_2) < 3 \cdot 2^{-n} \mid N([-M,M]^d) = m\Big\} \\ = & \sum_{m=0}^{\infty} \exp(-\lambda_0 (2M)^d) \frac{(\lambda_0 (2M)^d)^m}{m!} P_{(\lambda_0,U_n)}\Big\{\text{there exists a pair} \\ & \quad \text{of Poisson points } x_i, x_j \text{ in the model } (X_0, \lambda_0, U_n) \text{ with} \\ & \quad \text{associated radii } U_n^i \text{ and } U_n^j \text{ respectively inside } [-M,M]^d \text{ such} \\ & \quad \text{that } 0 \leq (U_n^i + U_n^j) - d(x_i, x_j) < 3 \cdot 2^{-n} \mid N([-M,M]^d) = m\Big\} \end{aligned}$$

Now, given that there are m Poisson points in $[-M, M]^d$, the position of these points are uniformly distributed in $[-M, M]^d$, thus

$$\begin{aligned} & P_{(\lambda_0,U_n)}\Big\{\text{there exists a pair of Poisson points } x_i, x_j \text{ in the} \\ & \quad \text{model } (X_0, \lambda_0, U_n) \text{ with associated radii } U_n^i \text{ and } U_n^j \end{aligned}$$

$$\text{respectively inside } [-M,M]^d \text{ such that } 0 \le (U_n^i + U_n^j) - d(x_i,x_j) < 3\cdot 2^{-n} \mid N([-M,M]^d) = m\Big\}$$

$$\le \binom{m}{2} \int_{[-M,M]^d} \cdots \int_{[-M,M]^d} P\{0 \le (U_n^1 + U_n^2) - d(x_1,x_2) < 3\cdot 2^{-n}\} \frac{1}{((2M)^d)^m} dx_1 \cdots dx_m$$

Let $E^n_{(i,j)}$ denote the set $\{(x_1,x_2) : 0 \le ((i/2^n) + (j/2^n)) - d(x_1,x_2) < 3\cdot 2^{-n}\}$. Then

$$P_{(\lambda_0,U_n)}\Big\{\text{there exists a pair of Poisson points } x_i, x_j \text{ in the model } (X_0,\lambda_0,U_n), \text{ with associated radii } U_n^i \text{ and } U_n^j \text{ respectively, inside } [-M,M]^d \text{ such that } 0 \le (U_n^i + U_n^j) - d(x_i,x_j) < 3\cdot 2^{-n} \mid N([-M,M]^d) = m\Big\}$$

$$\le \binom{m}{2} \int_{[-M,M]^d} \cdots \int_{[-M,M]^d} \sum_{i=0}^{R2^n-1}\sum_{j=0}^{R2^n-1} \frac{p_ip_j}{((2M)^d)^m} I_{E^n_{(i,j)}} dx_1 \cdots dx_m$$

$$= \binom{m}{2} \int_{[-M,M]^d}\int_{[-M,M]^d} \sum_{i=0}^{R2^n-1}\sum_{j=0}^{R2^n-1} \frac{p_ip_j}{((2M)^d)^2} I_{E^n_{(i,j)}} dx_1 dx_2$$

$$= \binom{m}{2} \int_{[-M,M]^d} \sum_{i=0}^{R2^n-1}\sum_{j=0}^{R2^n-1} \frac{p_ip_j}{((2M)^d)^2} \int_{\{x_2:(x_1,x_2)\in E^n_{(i,j)}\}} dx_2 dx_1$$

$$\le \binom{m}{2} \int_{[-M,M]^d} \sum_{i=0}^{R2^n-1}\sum_{j=0}^{R2^n-1} \frac{p_ip_j}{((2M)^d)^2} c_d\Big[\Big(\frac{i+j+3}{2^n}\Big)^d - \Big(\frac{i+j}{2^n}\Big)^d\Big] dx_1$$

$$\le \binom{m}{2} \int_{[-M,M]^d} \sum_{i=0}^{R2^n-1}\sum_{j=0}^{R2^n-1} \frac{p_ip_j}{((2M)^d)^2} c_d\Big[\frac{3}{2^n} dR^{(d-1)}\Big] dx_1$$

$$\le \binom{m}{2} \sum_{i=0}^{R2^n-1}\sum_{j=0}^{R2^n-1} \frac{p_ip_j}{((2M)^d)} c_d \frac{3}{2^n} dR^{(d-1)}$$

$$= \binom{m}{2} C_d \frac{3}{2^n}. \tag{3.36}$$

Here, and in the subsequent inequality, c_d, C_d, K_d are positive constants depending only on d. Thus, we have

$$\begin{aligned} & P_{(\lambda_0,U_n)}(D'_n) \\ \leq & \sum_{m=0}^{\infty} \frac{\exp(-\lambda(2M)^d)}{m!}(-\lambda(2M)^d)^m \binom{m}{2} \frac{C_d}{(2M)^d} \frac{3}{2^n} \\ = & K_d \frac{\lambda^2(2M)^d}{2} \frac{3}{2^n}. \end{aligned}$$

Hence,

$$P_{(\lambda_0,U_n)}(D'_n) \to 0 \text{ as } n \to \infty.$$

It can similarly be shown that that $P_{(\lambda,\rho)}(E_n) \to 0$ as $n \to \infty$. Thus $P(N_2) = 0$. Similar calculations can be done to show that $P(N_3) = 0$. This proves our claim. ∎

Proof of Theorem 3.4: We first note that if $\lambda < \lambda_c(\rho)$, then for n large $\lambda < \lambda_c(\rho_n)$ by Theorem 3.2. So $\theta_\rho(\lambda) = \theta_{\rho_n}(\lambda) = 0$ for n large. Thus we may consider $\lambda > \lambda_c(\rho)$.

As in the proof of Theorem 3.2, we shall approximate the radius random variable by random variables which take only finitely many values. The approximation techniques used to prove this theorem are similar to those used to prove Theorem 3.2.

Lemma 3.4 *Let $0 < r_1 < r_2 < \cdots < r_n < \infty$ and let ρ and ρ' be random variables taking values r_i with probability p_i and p'_i respectively for $i = 1,\ldots,n$. Suppose that there exist $1 \leq j < l \leq n$ such that $p_i = p'_i$ for all $i \neq j, l$ and where p_l and p'_l are both positive. Then,*

$$\theta_\rho\left(\frac{\lambda}{1+(p'_l)^{-1}|p_j - p'_j|}\right) \leq \theta_{\rho'}(\lambda) \leq \theta_\rho(\lambda(1+(p_l)^{-1}|p_j - p'_j|)).$$

Proof: Suppose first that $p_j > p'_j$. So by a coupling argument as before we obtain

$$\theta_\rho(\lambda) \leq \theta_{\rho'}(\lambda) \tag{3.37}$$

As in the proof of Lemma 3.1, we consider the models $(X, \lambda l_i, r_i)$, for $i = 1,\ldots,l-1,l+1,\ldots,n$, where the l_i's are chosen as in (3.12). Next, consider the superposition of $(\tilde{X}, \lambda, \rho')$ and $(X, \lambda l_i, r_i)$, $i = 1,\ldots,l-$

$l, l+1, \dots n$ to obtain a model equivalent in law to $(X, \lambda(1+L), \rho)$ where $L = l_1 + \cdots + l_{l-1} + l_{l+1} + \cdots + l_n$ and $X = \tilde{X} \cup X_1 \cup \ldots \cup X_{l-1} \cup X_{l+1} \cup \ldots \cup X_n$. Hence by a coupling argument we obtain

$$\theta_\rho(\lambda(1+L)) \geq \theta_{\rho'}(\lambda). \tag{3.38}$$

Now $L = (p_l)^{-1}(p_j - p'_j)$ and thus we have

$$\theta_\rho(\lambda(1+L)) \geq \theta_{\rho'}(\lambda) \geq \theta_\rho(\lambda) \geq \theta_\rho(\lambda/(1+L')),$$

where $L' = (p'_l)^{-1}|p_j - p'_j|$ and the last inequality follows by the non-decreasing nature of the percolation function.

In case, $p_j < p'_j$, we can repeat the whole argument starting with an intensity $\lambda' = \lambda/(1+L')$ with the roles of ρ and ρ' interchanged. ∎

Lemma 3.5 *Let $0 < r_1 < \cdots < r_n$, and let ρ be a random variable taking values r_i with probability p_i for $i = 1, \dots, n$. Suppose that $p_n > 0$. For all $k = 1, 2, \dots$, define the random variables ρ_k taking values r_i with probability $p_{k,i}$, for all $i = 1, \dots, n$. If $p_{k,i} \to p_i$ as $k \to \infty$ for all i, then $\theta_{\rho_k}(\lambda) \to \theta_\rho(\lambda)$ as $k \to \infty$ for all $\lambda > \lambda_c(\rho)$.*

Proof: The proof of this lemma is again similar to that of Lemma 3.2. We first choose $0 < \delta < p_n$ and take k_0 so large that $\sum_{i=1}^{n-1} |p_{k,i} - p_i| < \frac{1}{2}\delta$, for all $k \geq k_0$. Then, of course, we have $p_{k,n} > \frac{1}{2}\delta$, for all $k \geq k_0$. So, by using Lemma 3.4, we obtain,

$$\theta_\rho(\lambda \prod_{i=1}^{n-1}(1 - \frac{2|p_{k,i} - p_i|}{\delta})) \leq \theta_{\rho_k}(\lambda) \leq \theta_\rho(\lambda \prod_{i=1}^{n-1}(1 + \frac{2|p_{k,i} - p_i|}{\delta})).$$

Thus,

$$|\theta_{\rho_k}(\lambda) - \theta_\rho(\lambda)| \leq \theta_\rho(\lambda \prod_{i=1}^{n-1}(1 + \frac{2|p_{k,i} - p_i|}{\delta})) - \theta_\rho(\lambda \prod_{i=1}^{n-1}(1 - \frac{2|p_{k,i} - p_i|}{\delta})). \tag{3.39}$$

Now, by continuity of $\theta_\rho(\lambda)$ for $\lambda > \lambda_c(\rho)$ (Theorem 3.3), we have

$$\theta_\rho(\lambda \prod_{i=1}^{n-1}(1 + \frac{2|p_{k,i} - p_i|}{\delta})) - \theta_\rho(\lambda \prod_{i=1}^{n-1}(1 - \frac{2|p_{k,i} - p_i|}{\delta})) \to 0$$

as $k \to \infty$. ∎

Lemma 3.6 *Let ρ take values $0 < r_1 < \cdots < r_n$ with probabilities $p_1, \ldots, p_n$ respectively. Suppose ρ_k takes values $r_1, \ldots, r_n$ with probabilities $p_{k,1}, \ldots, p_{k,n}$. If $p_{k,i} \to p_i$ as $k \to \infty$ for all $1 \leq i \leq n$ then $\lambda_c(\rho_k) \to \lambda_c(\rho)$ as $k \to \infty$.*

Proof: We need to prove this lemma for the case when there exists $1 \leq m \leq n-1$ such that

$$p_m > 0 \text{ and } p_{m+1} = \ldots = p_n = 0.$$

The same argument as in the proof of Lemma 3.3 will show that we may assume that $m = n-1$, i.e. $p_{n-1} > 0$ and $p_n = 0$ and that it is sufficient to prove the lemma when $p_{k,i} = p_i$ for all $i = 1, 2, \ldots, n-2$ for each $k \geq 1$. Also we may assume that $p_{k,n}$ decreases to zero as $k \to \infty$.

Now, let $B_M = [-M, M]^d$ and $\partial(B_M)$ be the boundary of the B_M. Then, for every $k \geq 1$, we have

$$P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M)) \downarrow \theta_{\rho_k}(\lambda) \text{ as } M \to \infty.$$

Similarly,

$$P_{(\lambda,\rho)}(0 \rightsquigarrow \partial(B_M)) \downarrow \theta_{\rho}(\lambda) \text{ as } M \to \infty.$$

Fix an $M \geq 1$. We claim that

$$\lim_{k\to\infty} P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M)) = P_{(\lambda,\rho)}(0 \rightsquigarrow \partial(B_M)). \tag{3.40}$$

Clearly, for each $k \geq 1$, we have

$$P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M)) \geq P_{(\lambda,\rho_{k+1})}(0 \rightsquigarrow \partial(B_M)) \geq P_{(\lambda,\rho)}(0 \rightsquigarrow \partial(B_M)).$$

Hence,

$$\lim_{k\to\infty} P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M)) \geq P_{(\lambda,\rho)}(0 \rightsquigarrow \partial(B_M)) \tag{3.41}$$

Given $\epsilon > 0$ we choose k large such that $1 - \exp(-\lambda(2M)^d p_{k,n}) < \epsilon$. Now we consider n independent Poisson processes $X_1, X_2, \ldots, X_n$ with intensities $\lambda p_1, \lambda p_2, \ldots, \lambda p_{n-2}, \lambda p_{k,n-1}, \lambda p_{k,n}$ respectively. At each point of X_i, $1 \leq i \leq n-1$, we centre a ball of radius r_i. For the nth process X_n we make two cases: (i) at each point of the process X_n we centre a ball of radius r_n and (ii) at each point of the process X_n we centre a ball of radius r_{n-1}. In case (i) we obtain the Poisson Boolean model (X, λ, ρ_k) where $X = \cup_{i \geq 1} X_i$, while in case (ii) we obtain the Poisson Boolean model (X, λ, ρ_k). Thus by this coupling, we obtain,

$$\begin{aligned} & P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M)) - P_{(\lambda,\rho)}(0 \rightsquigarrow \partial(B_M)) \\ \leq\ & P(X_n \text{ has at least one point inside the box}[-M, M]^d) \\ =\ & 1 - \exp(-\lambda(2M)^d p_{k,n}) \\ <\ & \epsilon. \end{aligned}$$

This proves (3.40).

Now consider the double sequence $\{P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M))\}$ in k and M. Also, note that the sequence is decreasing in both M and k. Hence both the iterated limits exist and are equal. Hence,

$$\begin{aligned} \lim_{k\to\infty} \theta_{\rho_k}(\lambda) &= \lim_{k\to\infty} \lim_{M\to\infty} P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M)) \\ &= \lim_{M\to\infty} \lim_{k\to\infty} P_{(\lambda,\rho_k)}(0 \rightsquigarrow \partial(B_M)) \\ &= \lim_{M\to\infty} P_{(\lambda,\rho)}(0 \rightsquigarrow \partial(B_M)) \\ &= \theta_\rho(\lambda). \end{aligned} \tag{3.42}$$

■

Now we are in a position to prove Theorem 3.4. First we assume that

$$\text{there exists } a > 0 \text{ such that } a < \rho, \rho_k \leq R \text{ for all } k \geq 1 \tag{3.43}$$

where R is as in (1.6) of Chapter 1. Our strategy is to approximate the random variables ρ and ρ_n by random variables which take only finitely many values. Let the distribution functions of ρ and ρ_k be denoted by F and F_k respectively. We can assume that both a and R are continuity points of F. Take a sequence $\{\pi_n\}$ of partitions of $[a, R]$, which we write as $\pi_n = \{a = \gamma_0^n < \gamma_1^n < \cdots < \gamma_{k_n}^n = R\}$. We choose the partitions in such a way that π_{n+1} is a refinement of π_n. Also assume that all points γ_i^n are continuity points of F and $|\pi_n| := \max_{1 \leq i \leq k_n}\{\gamma_i^n - \gamma_{i-1}^n\} \to 0$, as $n \to \infty$. Now define, for all $n \geq 1$, the random variables $\rho^{(n)}$ and $\rho_{(n)}$ by the requirement that if $\rho \in (\gamma_{i-1}^n, \gamma_i^n]$, then $\rho^{(n)} = \gamma_i^n$ and $\rho_{(n)} = \gamma_{i-1}^n$. It follows from a simple coupling argument that $\theta_{\rho_{(n)}}(\lambda) \leq \theta_\rho(\lambda) \leq \theta^{\rho^{(n)}}(\lambda)$. Now for each $k \geq 1$, define the random variables $\rho_{k,(n)}$ and $\rho_k^{(n)}$ as follows: if $\rho_k \in (\gamma_{i-1}^n, \gamma_i^n]$, then $\rho_{k,(n)} = \gamma_{i-1}^n$ and $\rho_k^{(n)} = \gamma_i^n$. Clearly, for each $n \geq 1$ and $k \geq 1$, we have $\theta_{\rho_{k,(n)}}(\lambda) \leq \theta_{\rho_k}(\lambda) \leq \theta^{\rho_k^{(n)}}(\lambda)$.

Now, given $\epsilon > 0$, choose

$$\lambda_1 > \lambda > \lambda_2 > \lambda_c(\rho) \tag{3.44}$$

such that

$$\theta_\rho(\lambda_1) - \theta_\rho(\lambda_2) < \epsilon. \tag{3.45}$$

For each $n \geq 1$, let

$$\alpha_n := \max_{1 \leq i \leq k_n} \frac{\gamma_i^n}{\gamma_{i-1}^n} \leq 1 + \frac{|\pi_n|}{a},$$

which tends to 1 as $n \to \infty$. Hence $\rho^{(n)} \leq \alpha_n \rho_{(n)}$. Now we start with a model $(X, \lambda_0, \rho_{(n)})$, and by a change of scale we obtain the model $(\alpha_n X, (\alpha_n)^{-d}\lambda_0, \alpha_n \rho_{(n)})$. Since $\rho^{(n)} \leq \alpha_n \rho_{(n)}$, we have for any $\lambda_0 > 0$

$$\theta_{\rho_{(n)}}(\lambda_0) = \theta_{\alpha_n \rho_{(n)}}((\alpha_n)^{-d}\lambda_0) \geq \theta_{\rho^{(n)}}((\alpha_n)^{-d}\lambda_0). \tag{3.46}$$

Choose n large such that $(\alpha_n)^{-d}\lambda > \lambda_2$ and $(\alpha_n)^d\lambda < \lambda_1$. For this n we have, $\theta_{\rho_k}(\lambda) - \theta_\rho(\lambda) \leq \theta_{\rho_k^{(n)}}(\lambda) - \theta_\rho(\lambda)$. Now, by the choice of the partitions and the fact that $\rho_k \Rightarrow \rho$, we note that the random variables $\rho_k^{(n)}$ and $\rho^{(n)}$ satisfy the conditions of the Lemma 3.6. Thus applying the lemma, we obtain that there exists K_1 such that for $k \geq K_1$

$$|\theta_{\rho_k^{(n)}}(\lambda) - \theta_{\rho^{(n)}}(\lambda)| < \epsilon.$$

Also, applying (3.46) with $\lambda_0 = \lambda\alpha_n^d$, we get

$$\begin{aligned} \theta_{\rho^{(n)}}(\lambda\alpha_n^d) &\geq \theta_{\rho^{(n)}}((\alpha_n)^{-d}\lambda\alpha_n^d) \\ &= \theta_{\rho^{(n)}}(\lambda) \\ &\geq \theta_\rho(\lambda). \end{aligned}$$

Thus, for $k \geq K_1$,

$$\begin{aligned}
\theta_{\rho_k}(\lambda) - \theta_\rho(\lambda) &\leq \theta_{\rho(n)}(\lambda) + \epsilon - \theta_\rho(\lambda) \\
&\leq \theta_{\rho_{(n)}}(\alpha_n^d \lambda) + \epsilon - \theta_\rho(\lambda) \\
&\leq \theta_\rho(\alpha_n^d \lambda) + \epsilon - \theta_\rho(\lambda) \\
&\leq \theta_\rho(\lambda_1) + \epsilon - \theta_\rho(\lambda) \\
&\leq 2\epsilon,
\end{aligned}$$

where the last inequality follows from (3.44) and (3.45).

Similarly, for fixed n choose K_2 large so that, $|\theta_{\rho_{k,(n)}}(\lambda) - \theta_{\rho_{(n)}}(\lambda)| < \epsilon$ for $k \geq K_2$.

$$\begin{aligned}
\theta_\rho(\lambda) - \theta_{\rho_k}(\lambda) &\leq \theta_\rho(\lambda) - \theta_{\rho_{k,(n)}}(\lambda) \\
&\leq \theta_\rho(\lambda) - \theta_{\rho_{(n)}}(\lambda) + \epsilon \\
&\leq \theta_\rho(\lambda) - \theta_{\rho(n)}(\alpha_n^{-d}\lambda) + \epsilon \\
&\leq \theta_\rho(\lambda) - \theta_\rho(\alpha_n^{-d}\lambda) + \epsilon \\
&\leq \theta_\rho(\lambda) - \theta_\rho(\lambda_2) + \epsilon \\
&\leq 2\epsilon,
\end{aligned}$$

where the last inequality again follows from (3.44) and (3.45). This proves the theorem for the case when (3.43) holds.

Now let ρ have support $(0, R]$. Here again, given $\epsilon > 0$ choose $\lambda_2 < \lambda < \lambda_1$ such that (3.45) holds. Let a be a continuity point of F, the distribution function of ρ. Let ρ^a be a random variable with distribution equal to the conditional distribution of ρ, given that $\rho \geq a$. Similarly, let ρ_a be a random variable with distribution equal to the conditional distribution of ρ given $\rho < a$. Then we have $\theta_\rho(\lambda) \leq \theta_{\rho^a}(\lambda)$. Also define ρ_k^a as the random variable having the distribution function equal to the conditional distribution of ρ_k, given that $\rho_k \geq a$ and $\rho_{k,a}$ as the random variable having the distribution function equal to the conditional distribution of ρ_k, given that $\rho_k < a$. Thus $\theta_{\rho_k}(\lambda) \leq \theta_{\rho^a}(\lambda)$.

For any $\lambda_0 > 0$, consider the models (X_1, λ_0, ρ^a) and $(X_2, \lambda_0 l, \rho_a)$ where l is such that $l(1+l)^{-1} = F(a)$, i.e.,

$$l = \frac{F(a)}{1 - F(a)}. \tag{3.47}$$

The superposition of these two models is equivalent to the model $(X = X_1 \cup X_2, \lambda_0(1+l), \rho)$. Thus, we obtain,

$$\theta_\rho(\lambda_0(1+l)) \geq \theta_{\rho^a}(\lambda_0). \tag{3.48}$$

Similar calculations may be carried out for ρ_k with $l_k = F_k(a)/(1 - F_k(a))$ to yield

$$\theta_{\rho_k}(\lambda_0(1+l_k)) \geq \theta_{\rho_k^a}(\lambda_0). \tag{3.49}$$

Now, as $a \to 0, F(a) \to 0$. Thus we may choose a small enough so that $\lambda(1+l) < \lambda_1$ and $\lambda/(1+l) > \lambda_2$. Now, for this a, we have $F_k(a) \to F(a)$ as $k \to \infty$. Choose K_3 large so that $\lambda(1+l_k) < \lambda_1$ and $\lambda/(1+l_k) > \lambda_2$ for $k \geq K_3$.

Now, the random variables ρ_k^a and ρ^a are bounded below by a. Also, $\rho_k^a \Rightarrow \rho^a$ as $\rho_k \Rightarrow \rho$ and a is a continuity point of F. Hence by the first part of the argument, we choose K_4 large so that for all $k \geq K_4$,

$$|\theta_{\rho_k^a}(\lambda) - \theta_{\rho^a}(\lambda)| < \epsilon \tag{3.50}$$

and

$$|\theta_{\rho_k^a}(\lambda_2) - \theta_{\rho^a}(\lambda_2)| < \epsilon. \tag{3.51}$$

Thus, we have from (3.45), (3.51) and (3.49) with $\lambda_0 = \lambda/(1+l_k)$,

$$\begin{aligned} \theta_\rho(\lambda) - \theta_{\rho_k}(\lambda) &\leq \theta_\rho(\lambda) - \theta_{\rho_k^a}(\lambda/(1+l_k)) \\ &\leq \theta_\rho(\lambda) - \theta_{\rho_k^a}(\lambda_2) \\ &\leq \theta_\rho(\lambda) - \theta_{\rho^a}(\lambda_2) + \epsilon \\ &\leq \theta_\rho(\lambda) - \theta_\rho(\lambda_2) + \epsilon \\ &\leq 2\epsilon, \end{aligned}$$

and using (3.45), (3.50) and (3.48) with $\lambda_0 = \lambda$,

$$\begin{aligned} \theta_{\rho_k}(\lambda) - \theta_\rho(\lambda) &\leq \theta_{\rho_k^a}(\lambda) - \theta_\rho(\lambda)) \\ &\leq \theta_{\rho^a}(\lambda) - \theta_\rho(\lambda) + \epsilon \\ &\leq \theta_\rho(\lambda(1+l)) - \theta_\rho(\lambda) + \epsilon \\ &\leq \theta_\rho(\lambda_1) - \theta_\rho(\lambda) + \epsilon \\ &\leq 2\epsilon. \end{aligned}$$

This proves the theorem. ∎

Chapter 4

Rarefaction phenomenon

4.1 Introduction

In this chapter we study finite clusters in a high density Boolean model. Alexander [1993] studied the geometric structures of the events $E_k = \{\#(W(\mathbf{0})) = k\}$ in a high density Boolean model with fixed sized balls where $\#(W(\mathbf{0}))$ denotes the number of Poisson points in the cluster $W(\mathbf{0})$ of the origin. Clearly, for any fixed $k \geq 1$, $P_\lambda(\#(W(\mathbf{0})) = k)$ is very small for large λ and $P_\lambda(\#(W(\mathbf{0})) = k) \to 0$ as $\lambda \to \infty$. Alexander showed that as $\lambda \to \infty$, it is most likely that such an event occurs when all k Poisson points comprising the cluster $W(\mathbf{0})$ are packed tightly inside a small sphere of radius $\mathrm{O}(k/\lambda)$ centred at the origin and there is an annular region surrounding the cluster which is free of any Poisson points. This gives rise to the phenomenon of compression as the number k of Poisson points in this small sphere of radius $\mathrm{O}(k/\lambda)$ is very large compared to the expected number of points $\lambda \mathrm{O}((k/\lambda)^d)$ (as $\lambda \to \infty$) given by the ambient density λ of the underlying Poisson process.

We consider a Boolean model (X, λ, ρ) where X is a homogeneous Poisson point process with intensity λ and is 'conditioned to have a point at the origin' and ρ assumes two values r_1 and r_2 ($r_1 > r_2$) with probabilities p_1 and p_2 ($p_1 + p_2 = 1$) respectively. Let $P_{(\lambda,\rho)}$ denote the probability measure of this process conditional to have a point at the origin. We call the balls of radius r_1 as *big balls* and the balls of radius r_2 as *small balls*. We consider the event $E = \{W(\mathbf{0})$ consists of k_1 big balls and k_2 small balls}. Clearly, $P_{(\lambda,\rho)}(E) \to 0$ as $\lambda \to \infty$ whenever either of k_1 or k_2 is non-zero. In this chapter, we study the geometric structure of the event E.

In the case when the origin is the centre of a big ball, a possible structure of the event E, is that the centres of all big balls are compressed in a small sphere centred at the origin and the small balls are distributed uniformly inside the region formed by the big balls in such a way that the small balls are totally contained inside the big balls. This requires that an annular region of width r_1 surrounding the region created by the big balls to be free of Poisson points which are centres of big balls and another annular region of width r_2 surrounding the region created by big balls to be free of Poisson points which are centres of small balls. It is clear that the volume of these two regions will determine the probability of E. We show that the probability of the structure described above will be much higher than the probability of other possible structures as the given structure will minimise the volume of the two annular regions just described. Thus, it is most likely that the event E occurs with such a geometric structure.

In the case when the origin is the centre of a small ball, the structure of the event E will be very similar. The possible structure of the event E here, is that the centres of the big balls in $W(0)$ are clustered in a small sphere (which is not necessarily centred at the origin; in fact the centre will be uniformly distributed inside a ball of radius $(r_1 - r_2)$ around the origin) and all the small balls are distributed uniformly inside the region formed by the big balls in such a way that the small balls are totally contained inside the big balls. As before there will be two annular regions one of which will contain no Poisson points which are centres of big balls and the other will be free of Poisson points which are centres of small balls. Once again, we show that the structure has the maximum probability and hence, it is this structure we observe in case the origin is the centre of a small ball. It is clear that this structure is obtained from the previous case (when the origin is the centre of a big ball) by just a change of the position of origin to a random point which is uniformly distributed inside the sphere of radius $(r_1 - r_2)$.

If $W(0)$ consists only of big balls or only of small balls, the scenario observed is similar to the case when we have fixed sized balls. In these two cases, it is most likely that the centres of the balls are tightly packed in a small sphere near the origin and two annular regions are created one of which does not contain any Poisson points which are centres of big balls and the other region is free of Poisson points which are centres of small balls.

In the case when the cluster $W(0)$ admits at least one big ball, it is most likely that the centres of the small balls are distributed uniformly over a shpere of radius $(r_1 - r_2)$. Typically such a region should contain $\lambda\pi_d(r_1 - r_2)^d$ Poisson points whereas the cluster $W(0)$ consists of only $(k_1 + k_2)$ Poisson points. This gives rise to the rarefaction phenomenon as the cluster contains too few points than the expected number of points it should contain. This is the case in Theorems 4.1 to 4.4 below.

However, in the case when $W(0)$ comprises only of small balls, the volume of the annular regions described above is much smaller than the volume of the corresponding regions in the cases when $W(0)$ admits at least one big ball. Hence, the probability that $W(0)$ comprises only of small balls will dominate all other terms in $P_{(\lambda,\rho)}\{\#(W(0)) = k\}$ and thus it is most likely that in a high density Boolean model a finite cluster comprises only of small balls. This is the case in Theorem 4.5 below.

Our results hold for more general varying radius distribution; however for the sake of simplicity we restrict ourselves to the case when there are only two distinct sizes of balls.

4.2 Statement of results

The driving process of the continuum percolation model we consider is a homogeneous Poisson point process with intensity λ and 'conditioned to have a point at the origin'. The independence of the radius random variable and the driving Poisson point process guarantees that the process, consisting of points other than the origin which are centres of big balls, is a homogeneous Poisson point process with intensity λp_1. We denote this process by Y. Similarly, the point process consisting of points other than the origin which are centres of small balls is a homogeneous Poisson point process with intensity λp_2. We denote this process by Z. Moreover, Y and Z are independent point processes. Clearly, the union of the processes Y and Z comprise the original Poisson process of intensity λ *without the point at the origin.* Thus, to arrive at the continuum percolation model, we add one point at the origin to the union of the processes Y and Z and place either a big ball or a small ball at the origin, independently of the processes Y and Z, with probabilities p_1 and p_2 respectively. Our model can be viewed as the union of Y and Z and the point at the origin with a ball as described above. Henceforth we consider the processes Y and Z with the points $y_1, y_2, \ldots$ of Y being centres of big balls and the points $z_1, z_2, \ldots$ of Z being centres of small balls.

Now we encounter two possibilities : a) the origin is the centre of a big ball and b) the origin is the centre of a small ball. The conditional probability measure given that the origin is the centre of a big ball is denoted by P_B while the conditional probability measure given that the origin is the centre of a small ball is denoted by P_S. The original probability measure $P_{(\lambda,\rho)}$ can be recovered from these two measures by setting:

$$P_{(\lambda,\rho)}(\cdot) = p_1 P_B(\cdot) + p_2 P_S(\cdot).$$

Now, we define two events $E(k_1, k_2)$ and $E'(k_1, k_2)$, as follows:
(i) given that the origin is the centre of a big ball, we define,

$$E(k_1, k_2) = \{W(0) \text{ consists of } (k_1 + 1) \text{ big balls} \\ \text{(including the origin) and } k_2 \text{ small balls}\},$$

(ii) given that the origin is the centre of a small ball, we define,

$$E'(k_1, k_2) = \{W(0) \text{ consists of } k_1 \text{ big balls and} \\ k_2 + 1 \text{ small balls (including the origin)}\}.$$

Using a simple marked point process argument, we can derive a relation between $P_B(E(k_1, k_2))$ and $P_S(E'(k_1, k_2))$. We say that a cluster is a *finite* (k_1, k_2)*-cluster* if it consists only of k_1 many Poisson points which are centres of big balls and k_2 many Poisson points which are centres of small balls.

Let us fix $\lambda > 0$ and $k_1 \geq 1$ and $k_2 \geq 1$. Let $B_n = [-n, n]^d$ and define $M_n(B)$ to be the number of Poisson points inside B_n, each of which is the centre of a big ball and is a constituent of a finite (k_1, k_2)-cluster.

We are going to calculate the expectation of $M_n(B)$ using marked point process argument. Let $\mathcal{M}$ be the space of marks, which in our case is just the set $\{0,1\}$ as we shall see shortly. Let M_i be the mark at the point x_i. Campbell's theorem for marked point processes (see Hall [1988], page 200) guarantees that if the marked point process $\{x_i, M_i\}_{i\geq 1}$ is stationary then for any non-negative, measurable function f on $\mathbb{R}^k \times \mathcal{M}$ we have

$$\begin{aligned} E(\sum_i f(x_i, M_i)) &= \lambda E(\int_{\mathbb{R}^k} f(x,M)dx) \\ &= \lambda \int_{\mathbb{R}^k} Ef(x,M)dx, \end{aligned} \tag{4.1}$$

where M is a random mark having the so-called "mark distribution". In our context, to apply Campbell's theorem we take the mark

$$M_i = \begin{cases} 1 & \text{if } x_i \text{ is a centre of a big ball and} \\ & x_i \text{ is a part of finite } (k_1,k_2)\text{-cluster.} \\ 0 & \text{otherwise .} \end{cases}$$

and

$$f(x,M) = \begin{cases} M_i & \text{if } x = x_i \text{ for some } x_i \in B_n \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$M_n(B) = \sum_{i=1}^{\infty} f(x_i, M_i).$$

Hence, from (4.1) we obtain,

$$\begin{aligned} E(M_n(B)) &= E\Big(\sum_{i=1}^{\infty} f(x_i,M_i)\Big) \\ &= \lambda E\Big(\int_{\mathbb{R}^d} f(x,M)dx\Big) \\ &= \lambda \int_{\mathbb{R}^d} E(f(x,M))dx \\ &= \lambda \int_{B_n} p_1 P_B(E(k_1-1,k_2))dx \\ &= \lambda(2n)^d p_1 P_B(E(k_1-1,k_2)). \end{aligned} \tag{4.2}$$

Now let $M_n(S)$ be the number of Poisson points inside B_n each of which is the centre of a small ball and is a constituent of a finite (k_1, k_2)-cluster. Using a similar marked point process argument we obtain,

$$E(M_n(S)) = \lambda(2n)^d p_2 P_S(E'(k_1, k_2 - 1)). \tag{4.3}$$

Let R_n be the number of finite (k_1, k_2)-clusters inside B_n such that all the $(k_1 + k_2)$ points in each of these finite (k_1, k_2)-cluster are contained in B_n. In our definition of $M_n(B)$ and $M_n(S)$ the finite (k_1, k_2) cluster need not be completely contained in B_n, so it is clear that

$$k_1 R_n \leq M_n(B) \text{ and } k_2 R_n \leq M_n(S).$$

Thus we have,

$$p_1 P_B(E(k_1 - 1, k_2)) = \frac{E(M_n(B))}{\lambda(2n)^d} \geq \frac{k_1 E(R_n)}{\lambda(2n)^d}, \tag{4.4}$$

and

$$p_2 P_S(E'(k_1, k_2 - 1)) = \frac{E(M_n(S))}{\lambda(2n)^d} \geq \frac{k_2 E(R_n)}{\lambda(2n)^d}. \tag{4.5}$$

Further, any finite (k_1, k_2)-clusters, at least one point of which is inside B_n, must be totally contained inside $B_{n+(k_1+k_2)2r_1}$. Hence, we also have,

$$k_1 R_{n+(k_1+k_2)2r_1} \geq M_n(B) \text{ and } k_2 R_{n+(k_1+k_2)2r_1} \geq M_n(S). \tag{4.6}$$

Thus, from (4.4) and the second inequality above, we have

$$\begin{aligned}
&\lambda p_1 P_B(E(k_1 - 1, k_2))/k_1 \\
&\geq \limsup_{n\to\infty} \frac{E(R_n)}{(2n)^d} \\
&\geq \limsup_{n\to\infty} \frac{E(R_{n+(k_1+k_2)2r_1})}{(2(n + (k_1 + k_2)2r_1))^d} \\
&\geq \limsup_{n\to\infty} \frac{E(M_n(S))}{k_2(2n)^d} \times \frac{(2n)^d}{(2(n + (k_1 + k_2)2r_1))^d} \\
&= \frac{1}{k_2} \lim_{n\to\infty} \frac{E(M_n(S))}{(2n)^d} \times \lim_{n\to\infty} \frac{(2n)^d}{(2(n + (k_1 + k_2)2r_1))^d} \\
&= \lambda p_2 P_S(E'(k_1, k_2 - 1))/k_2.
\end{aligned}$$

Similarly, using (4.5) and the first inequality in (4.6) we obtain

$$\lambda p_2 P_S(E'(k_1, k_2 - 1))/k_2 \geq \lambda p_1 P_B(E(k_1 - 1, k_2))/k_1.$$

Combining the above two inequalities we obtain

$$p_1 P_B(E(k_1 - 1, k_2))/k_1 = p_2 P_S(E'(k_1, k_2 - 1))/k_2. \qquad (4.7)$$

From the above relation, it follows that the results in the case when the origin is the centre of a small ball can be obtained from the results in the case when the origin is the centre of a big ball. So, unless specified, from now on we will assume that the origin is the centre of a big ball.

Finally, as the measure of the size of the cluster $W(0)$ of the origin, we use

$$d(W(0)) := \max\{d(0, x) : x \text{ is a Poisson point in } W(0)\}.$$

Alexander [1993] used $\text{diam}(W(0))$ as the measure of the size of the cluster where

$$\text{diam}(W(0)) := \max\{d(x_i, x_j) : x_i, x_j \text{ are Poisson points in } W(0)\}.$$

Clearly these two measures are equivalent as

$$d(W(0)) \leq \text{diam}(W(0)) \leq 2d(W(0)).$$

Define the relative density of the occupied cluster $W(0)$ of the origin as

$$\delta(\lambda) = \frac{\#(W(0))}{\lambda \pi_d d(W(0))^d}. \qquad (4.8)$$

Alexander [1993] showed that in the case when balls are of fixed size, for $k \geq 1$ fixed or $k \to \infty$ but $k/\lambda \to 0$,

$$P\Big[\delta(\lambda) \to \infty \mid \#(W(0)) = k\Big] \to 1 \text{ as } \lambda \to \infty. \qquad (4.9)$$

This phenomenon was termed *compression* by Alexander.

In the case of varying sized balls, the results are best understood when we divide them into several cases. We first consider the case when both k_1 and k_2 are fixed.

Theorem 4.1 *Suppose that both k_1 and k_2 are fixed. Then we have, as $\lambda \to \infty$,*

$$\begin{aligned} P_B\Big(E(k_1, k_2)\Big) \\ = \exp\Big(-\lambda\pi_d E(\rho + r_1)^d + (k_2 - k_1(d-1))\log\lambda + O(1)\Big) \end{aligned}$$

and

$$\begin{aligned} i) \quad & P_B\Big(d(W(0)) > a_1(\lambda) \mid E(k_1, k_2)\Big) \to 1, \\ ii) \quad & P_B\Big(\delta(\lambda) \to 0 \mid E(k_1, k_2)\Big) \to 1 \end{aligned}$$

where $a_1(\lambda)$ is function of λ such that $a_1(\lambda) \to 0$ but $\lambda(a_1(\lambda))^d \to \infty$ as $\lambda \to \infty$.

Next we consider the case when k_2 is fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ as $\lambda \to \infty$.

Theorem 4.2 *Suppose that k_2 is fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ as $\lambda \to \infty$. Then, we have, as $\lambda \to \infty$,*

$$\begin{aligned} P_B\Big(E(k_1, k_2)\Big) = \exp\Big(&-\lambda\pi_d E(\rho + r_1)^d \\ &-(d-1)k_1\log(\lambda/k_1) + k_2\log\lambda + O(k_1)\Big) \end{aligned}$$

and

$$\begin{aligned} i) \quad & P_B\Big(d(W(0)) > a_2(\lambda) \mid E(k_1, k_2)\Big) \to 1, \\ ii) \quad & P_B\Big(\delta(\lambda) \to 0 \mid E(k_1, k_2)\Big) \to 1 \end{aligned}$$

where $a_2(\lambda)$ is a function of λ such that $a_2(\lambda) \to 0$ and $\lambda(a_2(\lambda))^d/k_1 \to \infty$ as $\lambda \to \infty$.

Now we suppose that k_1 is fixed and $k_2 \to \infty$ but $k_2/\lambda \to 0$ as $\lambda \to \infty$.

Theorem 4.3 *Suppose that k_1 is fixed and $k_2 \to \infty$ but $k_2/\lambda \to 0$ as $\lambda \to \infty$. Then, we have, as $\lambda \to \infty$,*

$$\begin{aligned} P_B\Big(E(k_1, k_2)\Big) = \exp\Big(&-\lambda\pi_d E(\rho + r_1)^d \\ &-(d-1)k_1\log\lambda + k_2\log(\lambda p_2/k_2) + O(k_2)\Big) \end{aligned}$$

and

$$i) \quad P_B\Big(d(W(\mathbf{0})) > a \mid E(k_1, k_2)\Big) \to 1,$$
$$ii) \quad P_B\Big(\delta(\lambda) \to 0 \mid E(k_1, k_2)\Big) \to 1$$

for every fixed $0 < a < r_1 - r_2$.

Now supose that both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0, k_2/\lambda \to 0$ as $\lambda \to \infty$.

Theorem 4.4 *Suppose that both* $k_1, k_2 \to \infty$ *but* $k_1/\lambda \to 0, k_2/\lambda \to 0$ *as* $\lambda \to \infty$. *Then, we have, as* $\lambda \to \infty$,

$$\begin{aligned} & P_B\Big(E(k_1, k_2)\Big) \\ = \;& \exp\Big(-\lambda\pi_d E(\rho + r_1)^d - (d-1)k_1 \log(\lambda/k_1) \\ & + k_2 \log(\lambda p_2/k_2) + O(k_1) + O(k_2)\Big) \end{aligned}$$

and

$$i) \quad P_B\Big(d(W(\mathbf{0})) > a \mid E(k_1, k_2)\Big) \to 1,$$
$$ii) \quad P_B\Big(\delta(\lambda) \to 0 \mid E(k_1, k_2)\Big) \to 1$$

for every fixed $0 < a < r_1 - r_2$.

Next we consider the case when cluster of the origin $W(\mathbf{0})$ consists only of small balls or only of big balls. Let $E(k, 0)$ be the event that the cluster of the origin consists only of $k+1$ ($k \geq 0$) big balls. Similarly, let $E'(0, k)$ be the event that the cluster of the origin consists only of $k+1$ ($k \geq 0$) small balls.

Theorem 4.5 *Let* k *be fixed or* $k \to \infty$ *but* $k/\lambda \to 0$ *as* $\lambda \to \infty$. *Then, we have as* $\lambda \to \infty$,

$$\begin{aligned} i) \; & P_B\Big(E(k, 0)\Big) \\ = \;& \exp\Big(-\lambda\pi_d E(\rho + r_1)^d - (d-1)k \log(\lambda/k) + O(k)\Big) \end{aligned}$$

and

$$ii)\ P_S\Big(E'(0,k)\Big) = \exp\Big(-\lambda\pi_d E(\rho + r_2)^d - (d-1)k\log(\lambda/k) + O(k)\Big)$$

and

$$P_{(\lambda,\rho)}\Big(E'(0,k) \mid \#(W(0)) = k+1\Big) \to 1. \qquad (4.10)$$

REMARK: Theorem 4.5 justifies the discussion at the end of Section 4.1. The important quantity to note is that the leading term in the exponential in the expression in (ii) is $-\lambda\pi_d E(\rho + r_2)^d$ which makes the exponential significantly larger (for large λ) than $-\lambda\pi_d E(\rho + r_1)^d$ obtained in the exponential of similar quantities in the other theorems.

4.3 Lower bounds

In this section, we obtain the lower bounds of the probabilities of the events we have considered.

Let $N_Y(A)$ and $N_Z(A)$ be the number of Poisson points inside A of the point processes Y and Z respectively. Let rU be the ball of radius r centred at the origin and $x + rU$ be the ball of radius r centred at the point x. We will be using Stirling's formula quite often. So, we state the version of Stirling's formula, which we use (Feller [1978] page 52-54).

Stirling's formula:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi} n^{n+1/2} e^{-n}} = 1.$$

Also, for every $n \geq 1$,

$$\begin{aligned}\sqrt{2\pi} n^{n+1/2} e^{-n} &\leq n! \\ &\leq \sqrt{2\pi} n^{n+1/2} e^{-n} \exp(1/(12n)).\end{aligned}$$

Lemma 4.1 *Let either k_1, k_2 be both fixed or k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or both $k_1, k_2 \to \infty$*

but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then, we have, for all λ large

$$\begin{aligned} &P_B\Big(E(k_1,k_2)\Big) \\ &\geq \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) \\ &\quad +k_2\log(\lambda p_2/k_2) + c_1k_1 + c_2k_2 + g(k_1,k_2,\lambda)\Big), \qquad (4.11) \end{aligned}$$

where c_1 and c_2 are constants and $g(k_1,k_2,\lambda)$ is a function of k_1, k_2 and λ.

Proof: Since the origin is the centre of a big ball, if k_1 points of Y in $W(0)$ are placed in a ball of radius $\alpha(k_1/\lambda)$, they will belong to the cluster $W(0)$ for all large λ where $\alpha = p_1(\pi_d(2r_1)^{d-1})^{-1}$. Note that if we assume that there are only big balls available for the cluster $W(0)$, then the optimal radius inside which all the Poisson points are packed is $\alpha(k_1/\lambda)$ (see Alexander (1993)). Now the small balls can be placed inside a sphere of radius (r_1-r_2) centred at the origin without affecting the region covered by the balls. This creates two annular regions, one free of points of Y and the other free of points of Z. Thus we obtain,

$$\begin{aligned} &P_B\Big(E(k_1,k_2)\Big) \\ &\geq P_B\Big(N_Y((\alpha k_1/\lambda)U) = k_1, N_Z((r_1-r_2)U) = k_2, \\ &\quad N_Y((2r_1+\alpha(k_1/\lambda))U \setminus (\alpha(k_1/\lambda))U) = 0, \\ &\quad N_Z(((r_1+r_2)+(\alpha k_1/\lambda))U \setminus (r_1-r_2)U) = 0\Big) \\ &= \exp(-\lambda\pi_d p_1(\alpha k_1/\lambda)^d)\frac{(\lambda\pi_d p_1(\alpha k_1/\lambda)^d)^{k_1}}{k_1!}\exp(-\lambda\pi_d p_2(r_1-r_2)^d) \\ &\quad \frac{(\lambda\pi_d p_2(r_1-r_2)^d)^{k_2}}{k_2!}\exp(-\lambda\pi_d p_1((2r_1+(\alpha k_1/\lambda))^d-(\alpha k_1/\lambda)^d)) \\ &\quad \exp(-\lambda\pi_d p_2(((r_1+r_2)+(\alpha k_1/\lambda))^d-(r_1-r_2)^d)). \end{aligned}$$

We now use Stirling's approximation for $k_1!$ and $k_2!$ to obtain

$$P_B\Big(E(k_1,k_2)\Big)$$

$$
\begin{aligned}
\geq\ & \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) + k_1\log(e\pi_d p_1\alpha^d) \\
& +k_2\log(\lambda p_2/k_2) + k_2\log(e\pi_d(r_1-r_2)^d) + g_1(k_1,k_2)\Big) \\
& \exp(-\lambda\pi_d\sum_{j=1}^{d}\binom{d}{j}(\alpha k_1/\lambda)^j(p_1(2r_1)^{d-j} + p_2(r_1+r_2)^{d-j})
\end{aligned}
$$

where $g_1(k_1,k_2) = -(1/(12k_1)+1/(12k_2)) - 1/2\log(k_1k_2) + \log(2\pi)$. Now, we choose λ so large that $k_1/\lambda < 1$. Then the last term in the exponential can be written as

$$
\begin{aligned}
& \lambda\pi_d\sum_{j=1}^{d}\binom{d}{j}(\alpha k_1/\lambda)^j(p_1(2r_1)^{d-j} + p_2(r_1+r_2)^{d-j}) \\
=\ & k_1 d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1}) \\
& +\lambda\pi_d\sum_{j=2}^{d}\binom{d}{j}(\alpha k_1/\lambda)^j(p_1(2r_1)^{d-j} + p_2(r_1+r_2)^{d-j}) \\
\leq\ & k_1 d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1}) \\
& +\lambda\pi_d(k_1/\lambda)^2\sum_{j=0}^{d}\binom{d}{j}(\alpha)^j(p_1(2r_1)^{d-j} + p_2(r_1+r_2)^{d-j}) \\
=\ & k_1 d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1}) + C_1k_1^2/\lambda,
\end{aligned}
$$

where $C_1 = \pi_d[p_1(\alpha+(2r_1))^d + p_2(\alpha+(r_1+r_2))^d]$.

Now setting $c_1 = \log(e\pi_d p_1\alpha^d) - d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1})$ and $c_2 = \log(e\pi_d(r_1-r_2)^d)$ and $g(k_1,k_2,\lambda) = g_1(k_1,k_2) - C_1k_1^2/\lambda$, the lemma follows. ∎

Note, if both $k_1 \geq 1$ and $k_2 \geq 1$ are fixed, (4.11) can be re-written as

$$
\begin{aligned}
& P_B\Big[E(k_1,k_2)\Big] \\
\geq\ & \exp\Big(-\lambda\pi_d E(\rho+r_1)^d + (k_2-(d-1)k_1)\log\lambda + O(1)\Big),
\end{aligned}
$$

as $\lambda \to \infty$.

In case $k_1 \to \infty, k_1/\lambda \to 0$ and $k_2 \geq 1$ is fixed, (4.11) becomes,

$$
P_B\Big[E(k_1,k_2)\Big]
$$

$$\geq \exp\Big(-\lambda\pi_d E(\rho+r_1)^d + (d-1)k_1\log(k_1/\lambda)$$
$$+k_2\log\lambda + \mathrm{O}(k_1)\Big),$$

as $\lambda \to \infty$.

If $k_1 \geq 1$ is fixed and $k_2 \to \infty, k_2/\lambda \to 0$, (4.11) reduces to

$$P_B\Big[E(k_1,k_2)\Big]$$
$$\geq \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log\lambda$$
$$+k_2\log(\lambda p_2/k_2) + \mathrm{O}(k_2)\Big),$$

as $\lambda \to \infty$.

When both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$, (4.11) reduces to

$$P_B\Big[E(k_1,k_2)\Big]$$
$$\geq \exp\Big(-\lambda\pi_d E(\rho+r_1)^d + (d-1)k_1\log(k_1/\lambda)$$
$$+k_2\log(\lambda p_2/k_2) + \mathrm{O}(k_1) + \mathrm{O}(k_2)\Big).$$

Next we consider the case when $W(\mathbf{0})$ comprises only of big balls or only of small balls. We prove the result only in the case when $W(\mathbf{0})$ comprises only of small balls, the other case being similar.

Lemma 4.2 *Let k be either fixed or $k \to \infty$ but $k/\lambda \to 0$ as $\lambda \to \infty$. Then we have as $\lambda \to \infty$,*

$$P_S\Big(E'(0,k)\Big) \geq \exp\Big(-\lambda\pi_d E(\rho+r_2)^d - (d-1)k\log(\lambda/k) + O(k)\Big).$$

Proof: The proof in the case when $W(\mathbf{0})$ consists of only small balls follows a similar line as in Lemma 4.1. The possible structure of the cluster $W(\mathbf{0})$ is that the centres of all small balls are packed tightly in a small sphere of radius $\alpha_2(k/\lambda)$ where $\alpha_2 = p_2(\pi_d(2r_2)^{d-1})^{-1}$ and there is a spherical region containing no points of Y and an annular region containing no points of Z. Thus, we have

$$
\begin{aligned}
& P_S\Big(E'(0,k)\Big) \\
& \geq \; P_S\Big(N_Z((\alpha_2(k/\lambda))U) = k, N_Y(((r_1+r_2)+(\alpha_2 k/\lambda))U) = 0, \\
& \qquad N_Z((2r_2+\alpha_2(k/\lambda))U \setminus (\alpha_2(k/\lambda))U) = 0\Big) \\
& = \; \exp(-\lambda\pi_d p_2((\alpha_2 k/\lambda)+2r_2)^d)\frac{[\lambda\pi_d p_2(\alpha_2(k/\lambda)^d]}{k!} \\
& \qquad \exp(-\lambda\pi_d p_1((\alpha_2 k/\lambda)+(r_1+r_2))^d).
\end{aligned}
$$

Similar calculations using Stirling's formula, as in previous lemma, yield the result. ■

4.4 Upper bounds

In this section we obtain the upper bounds of the probability of the events we have considered. We divide the event $E(k_1,k_2)$ into several events with restrictions to the size of the cluster and estimate them separately.

In next two lemmas, we obtain the upper bound of the probability of the event when the size of the cluster is very big. For this we define,

$$d_Y(0) := \max\{d(0,y_i) : y_i \in W(0)\cap Y\}.$$

Lemma 4.3 *Let either k_1,k_2 be both fixed or k_1 be fixed, $k_2\to\infty$ but $k_2/\lambda\to 0$ or k_2 be fixed and $k_1\to\infty$ but $k_1/\lambda\to 0$ or both $k_1,k_2\to\infty$ but $k_1/\lambda\to 0$ and $k_2/\lambda\to 0$ (as $\lambda\to\infty$). Then, for all λ large, we have*

$$
\begin{aligned}
& P_B\Big(E(k_1,k_2), d_Y(0) > r_1\Big) \\
& \leq \; \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - \lambda\pi_d p_1(r_1/2)^d/2 + c_3 k_1 + c_4 k_2\Big),
\end{aligned}
$$

where c_3 and c_4 are positive constants not depending on k_1 and k_2.

Proof: Since the cluster has only $(k_1 + k_2)$ Poisson points (besides the origin), the Poisson point in $W(0)$ which has the maximum distance from the origin, can be at most at a distance $2(k_1 + k_2)r_1$ from the origin. So, we have,

$$\begin{aligned} & P_B\Big(E(k_1,k_2), d_Y(0) > r_1\Big) \\ \leq & \sum_{j=1}^{2(k_1+k_2)} P_B\Big(E(k_1,k_2), jr_1 < d_Y(0) \leq (j+1)r_1\Big). \end{aligned}$$

Now, we estimate the summands in the above inequality. Suppose that $jr_1 < d_Y(0) \leq (j+1)r_1$. Then, there is at least one Poisson point in $W(0) \cap Y$ which lies outside the sphere $(jr_1)U$. Let $y_{\max}$ be the Poisson point in $W(0) \cap Y$ which is furthest from the origin and hence $jr_1 < d(y_{\max}, 0) \leq (j+1)r_1$. Now, if we centre a ball of radius $2r_1$ at the point $y_{\max}$, the part of the ball which lies outside the sphere $((j+1)r_1)U$ will contain no points of the process Y. An easy lower bound of the volume of the region of the ball of radius $2r_1$ which lies outside $((j+1)r_1)U$ can be made by noticing that a ball of radius $(r_1/2)$ will always be contained inside such a region. To make this formal, we use a conditioning argument.

Let $C_\star$ be the positions of all Poisson points of Y and Z inside $((j+1)r_1)U$ and the origin, $\{0\}$. Define, for $m, n \geq 0$,

$$\begin{aligned} A_m &:= \{N_Y((2r_1)U) = m\} \\ B_n &:= \{N_Z((r_1+r_2)U) = n\}. \end{aligned}$$

Since the event $E(k_1,k_2)$ occurs and the origin is the centre of a big ball, the ball $(2r_1)U$ may contain besides the origin, at most k_1 Poisson points of Y and hence, the event $\cup_{m=0}^{k_1} A_m$ must occur. Similarly, $\cup_{n=0}^{k_2} B_n$ also occurs. Let $0, y_{i_1}, \ldots, y_{i_{k_1}}$ be the Poisson points in $C_\star \cap W(0)$ which are also centres of big balls. Thus, we have,

$$\begin{aligned} & P_B\Big(E(k_1,k_2), jr_1 < d_Y(0) \leq (j+1)r_1\Big) \\ = \;& E\Big[P_B\Big(E(k_1,k_2), jr_1 < d_Y(0) \leq (j+1)r_1 \mid C_\star\Big)\Big] \\ \leq \;& E\Big[P_B(N_Y((\cup_{i=1}^{k_1}(y_{i_j} + 2r_1U)) \setminus (j+1)r_1U) = 0 \mid C_\star) \end{aligned}$$

$$
\begin{aligned}
&\sum_{m=0}^{k_1}\sum_{n=0}^{k_2} 1_{A_m} 1_{B_n}\Big] \\
\le\ & \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} P_B(A_m) P_B(B_n) \exp(-\lambda\pi_d p_1 (r_1/2)^d) \\
=\ & \exp(-\lambda\pi_d E(\rho + r_1)^d) \exp(-\lambda\pi_d p_1(r_1/2)^d) \\
& \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{(\lambda\pi_d p_1 (2r_1)^d)^m}{m!} \frac{(\lambda\pi_d p_2 (r_1+r_2)^d)^n}{n!}.
\end{aligned}
$$

Choose constants $C_2, C_3 \ge 1$ such that

$$(2r_1)^d < (r_1/2)^d C_2/4$$

and

$$p_2(r_1+r_2)^d < p_1(r_1/2)^d C_3/4.$$

Then, we have

$$
\begin{aligned}
& \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{(\lambda\pi_d p_1 (2r_1)^d)^m}{m!} \frac{(\lambda\pi_d p_2 (r_1+r_2)^d)^n}{n!} \\
\le\ & \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{(\lambda\pi_d p_1 (r_1/2)^d C_2/4)^m}{m!} \frac{(\lambda\pi_d p_1 (r_1/2)^d C_3/4)^n}{n!} \\
\le\ & C_2^{k_1} C_3^{k_2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\lambda\pi_d p_1 (r_1/2)^d/4)^m}{m!} \frac{(\lambda\pi_d p_1 (r_1/2)^d/4)^n}{n!} \\
=\ & C_2^{k_1} C_3^{k_2} \exp(\lambda\pi_d p_1 (r_1/2)^d/2).
\end{aligned}
$$

Now, combining together, we have

$$
\begin{aligned}
& P_B\Big(E(k_1,k_2), d_Y(0) > r_1\Big) \\
\le\ & \exp\Big[-\lambda\pi_d E(\rho+r_1)^d - \lambda\pi_d p_1 (r_1/2)^d/2 \\
& + k_1 \log C_2 + k_2 \log C_3\Big]\Big(2(k_1+k_2)\Big) \\
\le\ & \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - \lambda\pi_d p_1 (r_1/2)^d/2 + c_3 k_1 + c_4 k_2\Big),
\end{aligned}
$$

where $c_3 = \log C_2 + 2$ and $c_4 = \log C_3 + 2$. ■

Lemma 4.3 proves that that centres of big balls cannot be too far apart. Now we look at the case when the centres of the small balls are too far from the origin. For this, we define,

$$d_Z(\mathbf{0}) := \max\{d(\mathbf{0}, z_i) : z_i \in W(\mathbf{0}) \cap Z\}.$$

Lemma 4.4 *Let either k_1, k_2 be both fixed or k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then for all λ large, we have*

$$\begin{aligned} &P_B\Big(E(k_1,k_2), d_Y(\mathbf{0}) \le r_1, d_Z(\mathbf{0}) > r_1\Big) \\ &\le \exp\Big(-\lambda\pi_d E(\rho + r_1)^d - \lambda\pi_d p_2 (r_2/2)^d/2 + c_5 k_1 + c_6 k_2\Big) \end{aligned}$$

where c_5 and c_6 are positive constants not depending on k_1 and k_2.

Proof: The proof of this lemma follows a similar line as in Lemma 4.3.

$$\begin{aligned} &P_B\Big[E(k_1,k_2), d_Y(\mathbf{0}) \le r_1, d_Z(\mathbf{0}) > r_1\Big] \\ &\le \sum_{j=0}^{k_1+k_2} P_B\Big[E(k_1,k_2), d_Y(\mathbf{0}) \le r_1, \\ &\qquad r_1 + jr_2 < d_Z(\mathbf{0}) \le r_1 + (j+1)r_2\Big]. \end{aligned}$$

Now, we follow a similar method as in Lemma 4.3 to estimate the summands in the inequality. Suppose, $r_1 + jr_2 < d_Z(\mathbf{0}) \le r_1 + (j+1)r_2$. Let $z_{\max}$ be the furthest Poisson point from the origin in $W(\mathbf{0})$ which is also centre of a small ball. Thus, $r_1 + jr_2 < d(z_{\max}, \mathbf{0}) \le r_1 + (j+1)r_2$. By a similar argument, a region lying outside $(r_1 + (j+1)r_2)U$ of volume at least $\pi_d(r_2/2)^d$ will contain no Poisson points which are centres of small balls. By a similar conditioning argument as in previous Lemma, we obtain,

$$\begin{aligned} &P_B\Big[E(k_1,k_2), d_Y(\mathbf{0}) \le r_1, \\ &\qquad r_1 + jr_2 < d_Z(\mathbf{0}) \le r_1 + (j+1)r_2\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} P_B(A_m)P_B(B_n)\exp(-\lambda\pi_d p_2(r_2/2)^d) \\
&= \exp(-\lambda\pi_d E(\rho+r_1)^d)\exp(-\lambda\pi_d p_2(r_2/2)^d) \\
&\quad \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{[\lambda\pi_d p_1(2r_1)^d]^m}{m!}\frac{[\lambda\pi_d p_2(r_1+r_2)^d]^n}{n!}.
\end{aligned}
$$

By choosing the constants c_5 and c_6 suitably the lemma follows as earlier. ∎

For a fixed constant $\mu > 1$, we define,

$$\Psi_\mu(y) = 3(3/4)^d p_1\pi_{d-1}(2r_1)^{d-1}y - \log(ep_1\pi_d\mu^d y^d).$$

Note here that $\Psi_\mu(y) \to \infty$ as $y \to \infty$.

Lemma 4.5 *Let either k_1, k_2 be both fixed or k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then for all λ large, we have*

$$
\begin{aligned}
&P_B\Big(E(k_1,k_2), d_Z(0) \leq r_1, yk_1/\lambda < d_Y(0) \leq \mu y k_1/\lambda\Big) \\
&\quad \leq \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) - k_1\Psi_\mu(y) \\
&\qquad k_2\log(\lambda p_2/k_2) + c_7 k_2 + h(k_1)\Big).
\end{aligned}
$$

where c_7 is a constant and $h(k_1)$ is a function of k_1 only.

Proof: We define,

$$
\begin{aligned}
G_* &= \{\text{all points of } Y \text{ inside the ball } (\mu y k_1/\lambda)U\} \cup \{0\}, \\
H_* &= \{\text{all points of } Z \text{ inside the ball } r_1 U\}.
\end{aligned}
$$

The r-fattening of the set E is defined by

$$E^r = \{u \in I\!R^d : \text{ there exists } v \in E \text{ such that } u \in S_r(v)\},$$

where $S_r(v)$ is the ball of radius r at v. Also we define

$$
\begin{aligned}
A &= \{N_Y((\mu y k_1/\lambda)U) = k_1\} \\
B &= \{N_Z(r_1 U) = k_2\}.
\end{aligned}
$$

We have,

$$\begin{aligned}
&P_B\Big(E(k_1,k_2), yk_1/\lambda < d_Y(\mathbf{0}) \le \mu y k_1/\lambda, d_Z(\mathbf{0}) \le r_1\Big) \\
&= E\Big[P_B\Big(E(k_1,k_2), yk_1/\lambda < d_Y(\mathbf{0}) \le \mu y k_1/\lambda, \\
&\quad d_Z(\mathbf{0}) \le r_1 \mid (G_*, H_*)\Big)\Big] \\
&\le E\Big[1_A 1_B P_B\Big(N_Y(G_*^{2r_1} \cup H_*^{r_1+r_2} \setminus (\mu y k_1/\lambda)U) = 0, \\
&\quad N_Z(G_*^{r_1+r_2} \cup H_*^{2r_2} \setminus r_1 U) = 0 \mid (G_*, H_*)\Big)\Big].
\end{aligned}$$

Since the origin is the centre of a big ball, we have $\ell(G_*^{r_1+r_2} \setminus r_1 U) \ge \pi_d((r_1+r_2)^d - r_1^d)$, where $\ell(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$. Now easy calculations yields that (see also Lemma 3.1 of Alexander (1993)), $\ell(G_*^{2r_1} \cup H_*^{r_1+r_2} \setminus (\mu y k_1/\lambda)U) \ge \ell(G_*^{2r_1} \setminus (\mu y k_1/\lambda)U) \ge \pi_d((2r_1)^d - (\mu y k_1/\lambda)^d) + 3(3/4)^d \pi_{d-1}(2r_1)^{d-1} y \frac{k_1}{\lambda}$. Thus, we have, for all λ large,

$$\begin{aligned}
&P_B(E(k_1,k_2), yk_1/\lambda < d_Y(\mathbf{0}) \le \mu y k_1/\lambda, d_Z(\mathbf{0}) \le r_1) \\
&\le \exp(-\lambda \pi_d p_2 r_1^d) \frac{(\lambda \pi_d p_2 r_1^d)^{k_2}}{k_2!} \exp(-\lambda \pi_d p_1 (\mu y k_1/\lambda)^d) \\
&\quad \frac{(\lambda \pi_d p_1 (\mu y k_1/\lambda)^d)^{k_1}}{k_1!} \exp(-\lambda \pi_d p_2 ((r_1+r_2)^d - r_1^d)) \\
&\quad \exp(-\lambda \pi_d p_1 (((2r_1)^d - (\mu y k_1/\lambda)^d) - 3(3/4)^d \pi_{d-1}(2r_1)^{d-1} y k_1/\lambda)) \\
&= \exp(-\lambda \pi_d E(\rho + r_1)^d - (d-1)k_1 \log(\lambda/k_1) \\
&\quad + k_2 \log(\lambda p_2/k_2) - k_1 \Psi_\mu(y) + c_7 k_2 + h(k_1)),
\end{aligned}$$

where $c_7 = \log(e\pi_d r_1^d) + 1$ and $h(k_1) = \log(2\pi) + (\log k_1)/2$. ∎

The function $h(k_1)$ is a constant when k_1 is fixed and is of small order of k_1 when k_1 goes to infinity. The important thing to note here is that the function $h(k_1)$ is independent of μ and y.

Now we want to estimate the probability that all the centres of big balls comprising the cluster $W(\mathbf{0})$ are compressed in the optimal sphere about the origin. Let $\alpha = p_1(\pi_d(2r_1)^{d-1})^{-1}$.

Lemma 4.6 *Let either k_1, k_2 be both fixed or k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then for all λ large and for some constants c_8 and c_9, we have*

$$\begin{aligned}
&P_B\Big(E(k_1,k_2), d_Y(0) \le \alpha k_1/\lambda, d_Z(0) \le r_1\Big) \\
&\quad \le \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) \\
&\qquad + k_2\log(\lambda p_2/k_2) + c_8 k_1 + c_9 k_2 + h(k_1)\Big),
\end{aligned}$$

where $h(k_1)$ is as in previous lemma.

Proof: We have,

$$\begin{aligned}
&P_B(E(k_1,k_2), d_Y(0) \le \alpha k_1/\lambda, d_Z(0) \le r_1) \\
&\le P_B(N_Y((\alpha k_1/\lambda)U) = k_1, N_Y((2r_1)U \setminus (\alpha k_1/\lambda)U) = 0, \\
&\quad N_Z(r_1 U) = k_2, N_Z((r_1+r_2)U \setminus r_1 U) = 0) \\
&= \exp(-\lambda\pi_d p_1(\alpha k_1/\lambda)^d)\frac{(\lambda\pi_d p_1(\alpha k_1/\lambda)^d)^{k_1}}{k_1!}\exp(-\lambda\pi_d p_2 r_1^d) \\
&\quad \frac{(\lambda\pi_d p_2 r_1^d)^{k_2}}{k_2!}\exp(-\lambda\pi_d p_1((2r_1)^d - (\alpha k_1/\lambda)^d)) \\
&\quad \exp(-\lambda\pi_d p_2((r_1+r_2)^d - r_1^d)) \\
&\le \exp(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) \\
&\quad + k_2\log(\lambda p_2/k_2) + c_8 k_1 + c_9 k_2 + h(k_1)),
\end{aligned}$$

where $c_8 = \log(e\pi_d p_1\alpha^d)$ and $c_9 = \log(e\pi_d r_1^d) + 1$. ■

Next we look at the clusters which are of moderate size.

Lemma 4.7 *Let either k_1, k_2 be both fixed or k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then there exists $\beta > 0$ and constants c_{10} and c_{11} so that for all λ large, we have*

$$\begin{aligned}
&P_B\Big(E(k_1,k_2), \beta k_1/\lambda < d_Y(0) \le r_1, d_Z(0) \le r_1\Big) \\
&\quad \le \exp\Big(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) \\
&\qquad + k_2\log(\lambda p_2/k_2) + c_{10}k_1 + c_{11}k_2 + h(k_1)\Big),
\end{aligned}$$

where $h(k_1)$ is as earlier.

Proof: We fix $\mu > 1$. We choose β large so that $\Psi_\mu(\mu^j\beta) \geq j$ for every $j \geq 1$. By the definition of $\Psi_\mu(\cdot)$ this is possible. Now let $M = \min\{j : \beta\mu^j k_1/\lambda > r_1\}$. Hence,

$$\begin{aligned}
&P_B(E(k_1,k_2), \beta k_1/\lambda < d_Y(\mathbf{0}) \leq r_1, d_Z(\mathbf{0}) \leq r_1) \\
&= \sum_{j=0}^{M} P_{(\lambda,\rho)}(E(k_1,k_2), d_Z(\mathbf{0}) \leq r_1, \mu^{j-1}\beta k_1/\lambda < d_Y(\mathbf{0}) \leq \mu^j\beta k_1/\lambda) \\
&\leq \sum_{j=0}^{M} \exp(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) \\
&\quad -k_1\Psi_\mu(\beta\mu^{j-1}) + k_2\log(\lambda p_2/k_2) + c_7k_2 + h(k_1)) \\
&\leq \sum_{j=0}^{\infty} \exp(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) \\
&\quad -k_1 j + k_2\log(\lambda p_2/k_2) + c_7k_2 + h(k_1)) \\
&\leq \exp(-\lambda\pi_d E(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1) \\
&\quad +k_2\log(\lambda p_2/k_2) + c_{10}k_1 + c_7k_2 + h(k_1)),
\end{aligned}$$

where c_{10} is a constant, suitably chosen. ■

Finally, we look at the case when the origin is the centre of a small ball and $W(\mathbf{0})$ comprises only of small balls. Calculations similar to that of previous lemmas yield the next result, whose proof we omit. Let $h_1(k) = (\log(2\pi k))/2$. Then $h_1(k)$ is a constant if k is fixed and is of small order of k when $k \to \infty$.

Lemma 4.8 *Let k be either fixed or $k \to \infty$ but $k/\lambda \to 0$ as $\lambda \to \infty$. Then, for some constants c_{12} and c_{13} and for all λ large, we have*

$$\begin{aligned}
i)\ &P_S\Big(E'(0,k), d_Z(\mathbf{0}) > r_2\Big) \\
&\leq \exp\Big(-\lambda\pi_d E(\rho+r_2)^d - \lambda p_2(r_2/2)^d/2 + c_{12}k\Big),
\end{aligned}$$

$$\begin{aligned}
ii)\ &P_S\Big(E'(0,k), d_Z(\mathbf{0}) \leq \alpha_1 k/\lambda\Big) \\
&\leq \exp\Big(-\lambda\pi_d E(\rho+r_2)^d - (d-1)k\log(\lambda/k) + c_{13}k + h_1(k)\Big).
\end{aligned}$$

iii) For

$$\Phi_\mu(y) = 3(3/4)^d p_2 \pi_{d-1}(2r_2)^{d-1} y - \log(e p_2 \pi_d \mu^d y^d),$$

we have, for $\mu > 1$ and $y > 0$ and for all λ large,

$$\begin{aligned} P_S\Big(E'(0,k), yk/\lambda < d_Z(0) \leq y\mu k/\lambda\Big) \\ \leq \ \exp\Big(-\lambda\pi_d E(\rho + r_2)^d - (d-1)k\log(\lambda/k) \\ +k\Phi_\mu(y) + h_1(k)\Big). \end{aligned}$$

iv) There exists $\beta > 0$ such that for all λ large we have,

$$\begin{aligned} &P_S\Big(E'(0,k), \beta k/\lambda < d_Z(0) \leq r_1\Big) \\ &\leq \ \exp\Big(-\lambda\pi_d E(\rho + r_2)^d - (d-1)k\log(\lambda/k) + c_{14}k + h_1(k)\Big), \end{aligned}$$

where c_{14} is a constant not depending on k.

4.5 Proof of Theorems

The proofs of Theorems 4.1 to 4.4 are similar, so we prove only Theorem 4.1.

Proof of Theorem 4.1: For the first part, we note, for k_1 and k_2 fixed,

$$\begin{aligned} &P_B\Big(E(k_1,k_2)\Big) \\ &= \ P_B(E(k_1,k_2), d_Y(0) > r_1) \\ &\quad +P_B(E(k_1,k_2), d_Y(0) \leq r_1, d_Z(0) > r_1) \\ &\quad +P_B(E(k_1,k_2), d_Z(0) \leq r_1, d_Y(0) \leq \alpha k_1/\lambda) \\ &\quad +P_B(E(k_1,k_2), d_Z(0) \leq r_1, \alpha k_1/\lambda < d_Z(0) \leq r_1) \\ &\quad +P_B(E(k_1,k_2), d_Z(0) \leq r_1, \alpha k_1/\lambda < d_Y(0) \leq \beta k_1/\lambda). \end{aligned}$$

Now, using Lemmas 4.3–4.7 and Lemma 4.5 with $y = \alpha$ and $\mu = \beta/\alpha$ for the last term in the above relation, we obtain,

$$P_B\Big(E(k_1,k_2)\Big) \leq \exp\Big(-\lambda\pi_d E(\rho + r_1)^d + (k_2 - k_1(d-1))\log\lambda + O(1)\Big).$$

This along with (4.11) proves the result.

To show the second part, we see that, for any $0 < a_1(\lambda) < (r_1 - r_2)$,

$$P_B\Big(d(W(0) > a_1(\lambda) \mid E(k_1,k_2)\Big) \geq P_B\Big(d_Z(0) > a_1(\lambda) \mid E(k_1,k_2)\Big). \tag{4.12}$$

If $d_Z(0) \leq r_1$, all points of the process Z are inside the sphere $r_1 U$. As we have discussed earlier, we may place these points uniformly inside the sphere $(r_1 - r_2)$ without changing the region formed by the union of all balls (big and small). This is because any point inside this sphere will be totally contained inside the ball placed at the origin. Thus we have,

$$P_B\Big(d_Z(0) > a_1(\lambda) \mid E(k_1,k_2), d_Z(0) \leq r_1\Big) \geq 1 - \left(\frac{a_1(\lambda)}{r_1 - r_2}\right)^{dk_2}. \tag{4.13}$$

If we take $a_1(\lambda)$ such that $\lambda a_1(\lambda)^d \to \infty$ but $a_1(\lambda) \to 0$ as $\lambda \to \infty$ (one such choice is $a_1(\lambda) = \lambda^{-1/(2d)}$), we obtain from (4.13) and (4.12) as $\lambda \to \infty$,

$$P_B\Big(d(W(0)) > a_1(\lambda) \mid E(k_1,k_2)\Big) \to 1. \tag{4.14}$$

Now, we note that, $\delta(\lambda) = \frac{k_1+k_2+1}{\lambda\pi_d d(W(0))^d} \leq \frac{k_1+k_2+1}{\lambda\pi_d a_1(\lambda)^d} \to 0$ as $\lambda \to \infty$, on the set $\{d(W(0)) > a_1(\lambda)\}$. So, by our choice of the $a_1(\lambda)$, we have,

$$P_B\Big(\delta(\lambda) \to 0 \mid E(k_1,k_2)\Big) \to 1 \tag{4.15}$$

as $\lambda \to \infty$, proving the theorem. ∎

Finally we are left with the proof of Theorem 4.5. The proofs of the first and the second part follow a similar line as in Theorem 4.1. To show the third part we need upper bound of $P_S(E'(k_1,k_2))$ for $k_1 \geq 1$. For this we use the equation (4.7).

Proof of Theorem 5: We have,

$$\begin{aligned} &P_{(\lambda,\rho)}\Big(E'(0,k) \mid \#(W(0)) = k+1\Big) \\ &\geq 1 - \frac{\sum_{k_1=0}^{k} p_1 P_B(E(k_1,k-k_1)) + p_2 \sum_{k_1=1}^{k} P_S(E'(k_1,k-k_1))}{p_2 P_S(E'(0,k))} \\ &\geq 1 - \frac{k+1}{p_2}\Big[\frac{\max_{0\leq k_1\leq k} P_B(E(k_1,k-k_1))}{P_S(E'(0,k))} \\ &\quad + \frac{\max_{1\leq k_1 \leq k} P_S(E'(k_1,k-k_1))}{P_S(E'(0,k))}\Big]. \qquad (4.16) \end{aligned}$$

Define $\eta := E(\rho + r_1)^d - E(\rho + r_2)^d > 0$.
For fixed k, we have,

$$\max_{0\leq k_1 \leq k} P_B(E(k_1, k-k_1)) \leq \exp(-\lambda \pi_d E(\rho + r_1)^d + c_{15}\log\lambda + c_{16}),$$

where c_{15} and c_{16} are fixed positive constants not depending on λ. Further, from (4.7), we have

$$\max_{1\leq k_1\leq k} P_S(E'(k_1, k-k_1)) \leq \exp(-\lambda \pi_d E(\rho + r_1)^d + c_{17}\log\lambda + c_{18})$$

where c_{17}, c_{18} are positive constants not depending on λ. Thus from the lower bound of $P_S(E'(0,k))$ in Lemma 4.2, (4.16) and above upper-bounds, we have for some constant c_{19}

$$\begin{aligned} &P_{(\lambda,\rho)}\Big(E'(0,k) \mid \#(W(0)) = k+1\Big) \\ &\geq 1 - \frac{2}{p_2}\exp(-\lambda\eta + (\max(c_{15},c_{17}) + \log\lambda \\ &\quad + c_{19}k + \log(k+1)) \\ &\to 1 \text{ as } \lambda \to 1, \end{aligned}$$

proving (4.10).

When $k \to \infty$ but $k/\lambda \to 0$, we have to do little more work. We first choose M so that $(\log x)/x \leq \eta/4$ if $x \geq M$, where η is as above. Now we choose λ_0 such that for all $\lambda > \lambda_0$ we have $\lambda p_1/k \geq M$ and $\lambda p_2/k \geq M$. From the upper bounds, whenever $\lambda \geq \lambda_0$, we have for some constant c_{20} (not depending on k and λ),

$$\begin{aligned}
&\max_{0\leq k_1\leq k} P_B(E(k_1, k-k_1)) \\
\leq\ &\max_{0\leq k_2\leq k} \exp(-\lambda\pi_d E(\rho+r_1)^d + k_2\log(\lambda p_2/k_2) + c_{19}k) \\
\leq\ &\exp(-\lambda\pi_d E(\rho+r_1)^d + (\eta/4)\lambda + c_{20}k).
\end{aligned}$$

(notation $0\cdot\infty = 0$) By using the equation (4.7) and the above inequality, we obtain

$$\begin{aligned}
&\max_{1\leq k_1\leq k} P_S(E'(k_1, k-k_1)) \\
\leq\ &\max_{0\leq k_2\leq k} \exp(-\lambda\pi_d E(\rho+r_1)^d + k_2\log(\lambda p_2/k_2) \\
&+c_{20}k + \log(p_1/p_2) + \log k) \\
\leq\ &\exp(-\lambda\pi_d E(\rho+r_1)^d + (\eta/2)\lambda + c_{21}k),
\end{aligned}$$

where c_{21} is a suitable constant not depending on λ. From the lower bound of $P_S(E'(0,k))$ in Lemma 4.2 and the above upper bounds and (4.16), we conclude (4.10). ∎

Chapter 5

The random connection model

5.1 Introduction

Consider a forest of trees and suppose that the mango trees in this forest are susceptible to some virus. It is natural to suppose that the mango trees are distributed uniformly in the forest and that the virus spreads from one infected tree to a non-infected mango tree according to a probabilistic mechanism which depends on the distance between the trees. This and other physical phenomena can be mathematically studied by the random connection percolation model.

The underlying process of this model is a homogeneous Poisson point process of intensity λ on the d-dimensional Euclidean space $\mathbb{R}^d$ for $d \geq 2$. We connect two points x and y of this process by an edge with probability $g(| x - y |)$ where $| \cdot |$ denotes the d-dimensional Euclidean distance. Penrose [1991] parametrised this *random connection model* (RCM) by the intensity λ and introduced two critical intensities $\lambda_H(g)$ and $\lambda_T(g)$, (see (1.15) and (1.16) of Section 1.4 in Chapter 1) which he showed are positive and finite whenever $0 < \int_0^\infty x^{d-1} g(x)dx < \infty$. Burton and Meester [1992] studied this model for $g(x) = \exp(-\beta x), \beta > 0, x > 0$, and an underlying stationary, ergodic point process. They obtained results relating to the existence and uniqueness of the unbounded connected cluster with respect to parameter β.

A related model of interest is the mosaic structure of the random connection model. In this case the Poisson process has intensity λ_n and the connection function is g_n such that $\lambda_n/n^d \longrightarrow \lambda > 0$ and $g_n(y) = g(ny)$. Here we study the asymptotic behaviour of the number of isolated points in a compact set K. Further we obtain a central limit theorem for finite clusters in this model. This model and results here are an extension of similar results in the Boolean mosaic model (see Hall [1988] Sections 4.4 and 4.5 of Chapter 4) for the mosaic RCM. The details are described in Section 5.5.

5.2 The model

Let $X' = \{x_1, x_2, \ldots\}$ be a homogeneous Poisson point process of intensity λ on $\mathbb{R}^d, (d \geq 2)$. Fix $x_0 = 0$, the origin. Let $X = \{x_0, x_1, \ldots\}$; i.e., X is a Poisson point process 'conditioned to have a point at the origin'.

Let $g(x) : (0, \infty) \to [0, 1]$ be a measurable function with

$$0 < \int_0^\infty x^{d-1} g(x) dx < \infty. \tag{5.1}$$

Given two points x_i and x_j of X we connect them by an edge with probability $g(|x_i - x_j|)$ independently of all other pairs of points of X and of the process X. In other words, for each pair $(x_i, x_j), 0 \leq i < j < \infty$, of points in X we define a random variable $b(x_i, x_j)$ such that

(*i*) $P(b(x_i, x_j) = 1) = g(|x_i - x_j|) = 1 - P(b(x_i, x_j) = 0)$,

(*ii*) for $0 \leq i_k < j_k < \infty, k = 1, 2, \cdots$, with $(i_m, j_m) \neq (i_n, j_n)$ for all $m \neq n$, $b(x_{i_1}, x_{j_1}), b(x_{i_2}, x_{j_2}), \cdots$ is an independent sequence of random variables,

(*iii*) for every $0 \leq i < j < \infty$, $b(x_i, x_j)$ is independent of the point process X.

For $0 \leq i < j < \infty$, we define $b(x_i, x_j)$ as identical to $b(x_j, x_i)$. (For a mathematical construction of the random connection model see Section 1.4 of Chapter 1.)

Penrose [1991] has proved that for g satisfying condition (5.1), we have

$$0 < \lambda_T(g) \leq \lambda_H(g) < \infty, \tag{5.2}$$

where the quantities $\lambda_H(g)$ and $\lambda_T(g)$ are as defined by (1.15) and (1.16) in Section 1.4 of Chapter 1.

In this chapter we extend the work of Penrose [1991] and show that if

$$g \text{ is continuous with bounded support} \tag{5.3}$$

then the two quantities λ_H and λ_T are equal.

Theorem 5.1 *In a random connection model where g satisfies the condition (5.3),*

$$\lambda_H(g) = \lambda_T(g). \tag{5.4}$$

The proof of the theorem uses the standard Menshikov [1986] argument to derive an exponential decay for the size of the cluster of origin. The estimates are similar in nature to that of Grimmett [1988] and hence a similar analysis may be carried out.

This work has been generalised by Meester [1994] to prove the equality of the two critical intensities in the case where g is a decreasing function satisfying (5.1).

The advantage of the Menshikov approach is that the BK inequality is available for the discrete approximation model and therefore standard inequalities are available for the random connection model through a limiting argument.

5.3 The lattice approximation

Let $\mathbb{L}_n$ be the lattice $\mathbf{Z}_n^d = (\frac{1}{2^n}\mathbf{Z})^d = \{(\frac{u_1}{2^n}, \ldots, \frac{u_d}{2^n}) : u_i \in \mathbf{Z}\}$, and let E_n denote the collection of all edges between vertices of $\mathbf{Z}_n^d$. Thus, $E_n = \{e_n(u,v) : u, v \in \mathbf{Z}_n^d\}$, where $e_n(u,v)$ is the edge connecting two points u and v.

Now, for each vertex $u \in \mathbb{L}_n$ and edge $e_n(u,v)$, we define $\{0,1\}$-valued random variables $V(u)$ and $b_n(u,v)$ so that

1) $b_n(v,u)$ and $b_n(u,v)$ are identical

2) $\{V(u), b_n(u,v) : u, v \in \mathbb{L}_n\}$ is a collection of independent random variables.

We say that a vertex u (respectively an edge $e_n(u,v)$) is open if $V(u) = 1$ (respectively $b_n(u,v) = 1$). In this lattice we say two vertices u, v are connected by an open path (denoted by $u \stackrel{\mathbb{L}_n}{\rightsquigarrow} v$) if there are vertices $u = s_0, s_1, \ldots, s_m = v \in \mathbb{L}_n$ such that $V(s_i) = 1$ and $b_n(s_{i-1}, s_i) = 1$ for $i = 1, 2, \ldots, m$. Let $C_n(0)$ be the open cluster of the origin, i.e.

$$C_n(0) = \{u \in \mathbb{Z}_n^d : 0 \stackrel{\mathbb{L}_n}{\rightsquigarrow} u\}. \tag{5.5}$$

Now, to relate the random connection model with the lattice percolation model, we need to choose the parameters of the lattice percolation model suitably so that we are able to approximate the random connection model. For this, we require a vertex u to be open if and only if the cube $B_n(u)(:= u + (-2^{-(n+1)}, 2^{-(n+1)}]^d)$ contains at least one Poisson point. Thus,

$$\begin{aligned} p(\lambda) &:= P(V(u) = 1) \\ &= P_\lambda(B_n(u) \text{ contains at least one Poisson point}) \\ &= 1 - \exp(-\lambda 2^{-nd}). \end{aligned} \tag{5.6}$$

We define the edge $e_n(u, v)$ to be open if $U(u, v) \leq g(|u - v|)$, where $U(u, v)$ are the uniform random variables defined as in Chapter 1, in the construction of the random connection model. Hence,

$$\begin{aligned} P(b_n(u, v) = 1) &= P_\lambda\{U(u, v) \leq g(|u - v|)\} \\ &= g(|u - v|). \end{aligned} \tag{5.7}$$

From the independence of the point process and the set of uniform random variables, it is clear that the resulting lattice percolation model is an independent one i.e., the vertices and the edges are open independently of each other. Also this is clearly a finite range model as the connection function g is of bounded support. This model is also translation invariant due to the fact the probability of the edge $e_n(u, v)$ being open is $g(|u - v|)$ which is invariant under translation.

Since g is of bounded support, we may assume that

$$supp(g) \subseteq [0, 1].$$

Let $S_r^n = \{u \in \mathbb{L}_n : |u| \leq r\}$. Let $R_1^n = S_1^n, R_r^n = S_r^n \setminus S_{r-1}^n, r \geq 2$.

Now, we fix an integer $m \geq 1$. Define, $D_m^n = \{C_n(0) \cap R_m^n \neq \emptyset\}$. Let

$$h_m^n(p(\lambda)) = P_{p(\lambda)}(D_m^n), \tag{5.8}$$

where $P_{p(\lambda)}$ is the probability measure governing this independent lattice percolation model.

Definition 5.1 *Let F be an event depending on finitely many vertices and edges. Let $\omega \in \Omega$ be a configuration. A vertex $u \in \mathbb{L}_n$ is said to be pivotal for (F, ω) if $I_F(\omega) \neq I_F(\omega')$ where the configuration ω' is obtained by a change of occupancy of the vertex u, i.e., $\omega'(s) = \omega(s)$ if $s \neq u$ and $\omega'(u) = 1 - \omega(u)$.*

With this definition of pivotal vertices, we may obtain Russo's formula for an increasing event F along the lines of Grimmett [1988].

Lemma 5.1 (Russo's formula) *For an increasing event F and for any vertex $u \in \mathbb{L}_n$, we have*

$$\frac{\partial}{\partial p_u(\lambda)} P(F) = P(u \text{ is pivotal for } (F, \omega)). \tag{5.9}$$

where $p_u(\lambda)$ is the probability that the vertex u is open.

REMARK : We can also give a similar definition for pivotality of edges and obtain a similar formula as in (5.9). But we note that in this model the probability that an edge is open is not a function of $p(\lambda)$. So the pivotality of the edges do not contribute in the rate of growth with respect to $p(\lambda)$ $(d/dp(\lambda)(P_{p(\lambda)}(F))$ of the probability of increasing events F.

Hence, for any increasing event F, we have from (5.9),

$$\begin{aligned} &\frac{1}{P_{p(\lambda)}(F)} \frac{d}{dp(\lambda)} P_{p(\lambda)}(F) \\ &= \frac{1}{p(\lambda)} E_{p(\lambda)}(\text{number of pivotal vertices for } F \mid F). \end{aligned} \tag{5.10}$$

Let $s_1, s_2, \ldots$ be the pivotal vertices for (D^n_m, ω). Let $s_i \in S^n_{k_i}, i = 1, 2, \ldots$. Define $\xi_1 = k_1$ and $\xi_i = \max\{k_i - k_{i-1}, 0\}$ for $i \geq 2$. Now we imitate the proof of Lemma 6.3 of Menshikov, Molchanov and Sidorenko [1986] to obtain

Lemma 5.2 *For $d_1, d_2, \ldots, d_k$ non-negative integers with $\sum_{i=0}^k d_i \leq m$,*

$$P_{p(\lambda)}(\xi_k > d_k \mid D^n_m \cap \{\xi_i \leq d_i, i = 1, 2, \ldots, k-1\}) \leq P_{p(\lambda)}(D^n_{d_k}). \tag{5.11}$$

Once we have Lemma 5.2, we follow the proof of Lemma 3.17 of Grimmett [1988] to obtain,

Lemma 5.3 *For any $0 < p(\lambda) < 1$ we have,*

$$\frac{d}{dp(\lambda)} h_m^n(p(\lambda)) \geq \frac{h_m^n(p(\lambda))}{p(\lambda)} \left[\frac{m}{\sum_{i=0}^m h_i^n(p(\lambda))} - 1 \right], \tag{5.12}$$

where $h_m^n(p(\lambda))$ is as defined in (5.8).

We are going to use inequality (5.12) to obtain a similar inequality for the random connection model through a limiting argument. Once, we have an inequality of this form for the random connection model we may use Lemma 3.24 of Grimmett [1988] to obtain the exponential decay.

5.4 Proof of theorem

The basic step of the proof is to show that $h_m^n(p(\lambda))$ converges in a suitable way to the appropriate limit. We do that in the following lemma.

Fix $m \geq 1$. Let $D_r = \{a \in \mathbb{R}^d : |a| \leq r\}$ and $R_1 = D_1, R_i = D_i \setminus D_{i-1}, i \geq 2$. Clearly, $R_r \subseteq [-r, r]^d$. Define $h_m(\lambda) = P_\lambda\{C(0) \cap R_m \neq \emptyset\}$.

Lemma 5.4 *For $0 < \lambda_0 < \lambda_1 < \infty$ and $m \geq 1$, we have*

$$h_m^n(p(\lambda)) \to h_m(\lambda) \text{ uniformly in } \lambda \in [\lambda_0, \lambda_1] \tag{5.13}$$

as $n \to \infty$.

Proof : It is rather difficult to compare the random connection model and the lattice percolation model we have defined. In order to do that we introduce a dependent lattice percolation model.

Let $\mathbb{L}_n$ be as earlier. We define $u \in \mathbb{L}_n$ to be open if $B_n(u)$ contains at least one Poisson point. Now, for $u, v \in \mathbb{L}_n$, we define the edge $e_n(u, v)$, joining u and v to be open as follows:

a) if both the vertices u and v are open : in this case $e_n(u, v)$ is open if there are Poisson points $x_i, x_j, x_i \in B_n(u), x_j \in B_n(v)$ such that $U(x_i, x_j) \leq g(|u - v|)$.

b) if both the vertices u and v are not open : in such a case the edge $e_n(u,v)$ is open with probability 1.

Having defined this model, we first note that this dependent lattice percolation model stochastically dominates the independent lattice percolation model defined earlier. To show this, it is enough to show that

$$P^{\text{dep}}_{p(\lambda)}\{e_n(u,v) \text{ is open } \mid u,v \text{ both open }\} \geq g(|u-v|),$$

where $P^{\text{dep}}_{p(\lambda)}$ is the probability measure governing this dependent lattice percolation model.

For this we need a conditioning argument. Given that both u and v are open, there is at least one Poisson point in each of their respective boxes. Also, given there are exactly r (≥ 1) Poisson points inside a box, the points are uniformly distributed inside the box. So, if there are k and l Poisson points inside $B_n(u)$ and $B_n(v)$ respectively, the probability that at least one pair of points (one point taken from $B_n(u)$ and other from $B_n(v)$) is connected is exactly $1-(1-g(|u-v|))^{kl}$. Hence, we have,

$$\begin{aligned}
& P^{\text{dep}}_{p(\lambda)}\{e_n(u,v) \text{ is open } \mid u,v \text{ both open }\} \\
= \ & \frac{1}{p(\lambda)^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P^{\text{dep}}_{p(\lambda)}\{e_n(u,v) \text{ is open } \mid N(B_n(u))=k, \\
& N(B_n(v))=l\} P_\lambda\{N(B_n(u))=k\} P_\lambda\{N(B_n(v))=l\} \\
= \ & \frac{1}{p(\lambda)^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \exp(-\lambda 2^{-nd})\frac{(\lambda 2^{-nd})^k}{k!}\exp(-\lambda 2^{-nd})\frac{(\lambda 2^{-nd})^l}{l!} \\
& \frac{1}{(2^{-nd})^{k+l}}\int_{B_n(u)}\cdots\int_{B_n(u)}\int_{B_n(v)}\cdots\int_{B_n(v)}[1-(1-g(|u-v|))^{kl}] \\
& dx_1\ldots dx_k dy_1\ldots dy_l \\
\geq \ & \frac{1}{p(\lambda)^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \exp(-\lambda 2^{-nd})\frac{(\lambda 2^{-nd})^k}{k!}\exp(-\lambda 2^{-nd})\frac{(\lambda 2^{-nd})^l}{l!} \\
& \frac{g(|u-v|)}{(2^{-nd})^{k+l}}\int_{B_n(u)}\cdots\int_{B_n(u)}\int_{B_n(v)}\cdots\int_{B_n(v)} dx_1\ldots dx_k dy_1\ldots dy_l \\
= \ & g(|u-v|).
\end{aligned}$$

Thus, we have

$$\begin{aligned} P^{\text{dep}}_{p(\lambda)}\{C^{\text{dep}}_n(0) \cap R^n_m \neq \emptyset\} &\geq P_{p(\lambda)}\{C_n(0) \cap R^n_m \neq \emptyset\} \\ &= h^n_m(p(\lambda)) \end{aligned} \tag{5.14}$$

where $C^{\text{dep}}_n(0)$ is the open cluster of the origin in the dependent lattice percolation model and $R^n_m = \{u \in \mathbb{L}_n : m-1 < |u| \leq m\}$.

We have observed that the dependent lattice percolation model stochastically dominates the independent lattice percolation model. Now, we prove that on a subset of Ω whose probability is close to 1, two lattice percolation models restricted inside a bounded region are equivalent in law.

Let $A_m = [-(m+1+2^{-(n+1)}), (m+1+2^{-(n+1)})]^d$. Let $\Omega_{m,n} = \{$no box $B_n(u)$ of A_m contains more than one Poisson point$\}$, i.e., $\Omega_{m,n} = \cap_{u\in A_m \cap \mathbb{L}_n}\{B_n(u)$ has at most one Poisson point$\}$. First, we show that, for fixed $m \geq 1$,

$$P_\lambda(\Omega^c_{m,n}) \to 0 \text{ uniformly in } \lambda \in [\lambda_0, \lambda_1] \text{ as } n \to \infty.$$

Clearly, we have,

$$\begin{aligned} P_\lambda(\Omega^c_{m,n}) &\leq 1-(1-\alpha)^{[(2m+3)2^n]^d} \\ &\leq [(2m+3)2^n]^d\alpha \end{aligned}$$

where $\alpha = P_\lambda\{B_n(0)$ contains at least two Poisson points$\}$. A simple estimation gives, $\alpha \leq [\lambda 2^{-nd}]^2$. Thus, we have for all $\lambda \in [\lambda_0, \lambda_1]$,

$$P_\lambda(\Omega^c_{m,n}) \leq [2m+3]^d \lambda_1^2 2^{-nd} \to 0 \text{ as } n \to \infty.$$

Now, we claim that the conditional distribution of $I_{\{C^{\text{dep}}_n(0)\cap R^n_m \neq \emptyset\}}$ given $\Omega_{m,n}$ is the same as that of $I_{\{C_n(0)\cap R^n_m\neq\emptyset\}}$ given $\Omega_{m,n}$.

To see this, we first note that,

$$P^{\text{dep}}_{p(\lambda)}\Big[u \text{ is open} \mid \Omega\Big] = P_{p(\lambda)}\Big[u \text{ is open} \mid \Omega\Big].$$

Also let $\{u_i \in A_m \cap \mathbb{L}_n : 1 \leq i \leq l\}$ be any subset. Then for $\gamma_i = 0$ or 1, $1 \leq i \leq l$, we have,

$$\begin{aligned} &P^{\text{dep}}_{p(\lambda)}\Big[I_{u_i \text{ is open}} = \gamma_i, 1 \leq i \leq l \mid \Omega\Big] \\ &= P_{p(\lambda)}\Big[I_{u_i \text{ is open}} = \gamma_i, 1 \leq i \leq l \mid \Omega\Big]. \end{aligned}$$

Let

$$b_n(u,v) := I_{\{e_n(u,v) \text{ is open in the independent lattice model}\}}$$

and

$$b_n^{\text{dep}}(u,v) := I_{\{e_n(u,v) \text{ is open in the dependent lattice model}\}}.$$

Given $\Omega_{m,n}$ and a set O as the set of all open vertices $\{u_i : 1 \leq i \leq l\}$ in $A_m \cap \mathbb{L}_n$, the joint distribution of $\{b_n(u_i,u_j) : 1 \leq i < j \leq l\}$ is same as that of $\{b_n^{\text{dep}}(u_i,u_j) : 1 \leq i < j \leq l\}$. To prove this, we note that the random variables $\{b_n(u_i,u_j) : 1 \leq i < j \leq l\}$ are independent of $\Omega_{m,n}$ and also of the states of u_i's i.e., on whether or not u_i's are open. Hence for $\eta_{i,j} = 0$ or 1, $1 \leq i < j \leq l$, we have,

$$\begin{aligned}
&P_{p(\lambda)}\Big(b_n(u_i,u_j) = \eta_{i,j}, 1 \leq i < j \leq l \mid \Omega_{m,n} \cap O\Big) \\
&= \prod_{1 \leq i < j \leq l} P_{p(\lambda)}\Big(b_n(u_i,u_j) = \eta_{i,j} \mid \Omega_{m,n} \cap O\Big) \\
&= \prod_{1 \leq i < j \leq l} P_{p(\lambda)}\Big(b_n(u_i,u_j) = \eta_{i,j}\Big) \\
&= \prod_{1 \leq i < j \leq l} g(|u_i - u_j|)^{\eta_{i,j}}(1 - g(|u_i - u_j|))^{1-\eta_{i,j}}.
\end{aligned}$$

Now, for $\eta_{i,j} = 0$ or 1, $1 \leq i < j \leq l$, we have,

$$\begin{aligned}
&P_{p(\lambda)}^{\text{dep}}\Big(b_n(u_i,u_j) = \eta_{i,j}, 1 \leq i < j \leq l \mid \Omega_{m,n} \cap O\Big) \\
&= \frac{1}{(2^{-nd})^l}\int_{B_n(u_1)} \cdots \int_{B_n(u_l)} \prod_{1 \leq i < j \leq l} g(|u_i - u_j|)^{\eta_{i,j}} \\
&\quad (1 - g(|u_i - u_j|))^{1-\eta_{i,j}} dx_1 \ldots dx_l \\
&= \prod_{1 \leq i < j \leq l} g(|u_i - u_j|)^{\eta_{i,j}}(1 - g(|u_i - u_j|))^{1-\eta_{i,j}}.
\end{aligned}$$

Given $\Omega_{m,n}$ and given the set of all open vertices $\{u_i : 1 \leq i \leq l\}$ in $A_m \cap \mathbb{L}_n$, the conditional distribution of $I_{\{C_n^{\text{dep}}(0) \cap R_m^n \neq \emptyset\}}$ and $I_{\{C_n(0) \cap R_m^n \neq \emptyset\}}$ depend only on the random variables $\{b_n^{\text{dep}}(u_i,u_j) : 1 \leq$

$i < j \leq l\}$ and $\{b_n(u_i, u_j) : 1 \leq i < j \leq l\}$ respectively. Hence the conditional distribution of $I_{\{C_n^{\text{dep}}(0)\cap R_m^n \neq \emptyset\}}$ given $\Omega_{m,n}$ is same as that of $I_{\{C_n(0)\cap R_m^n \neq \emptyset\}}$ given $\Omega_{m,n}$. Thus, we have,

$$P_{p(\lambda)}^{\text{dep}}\Big(C_n^{\text{dep}}(0)\cap R_m^n \neq \emptyset \mid \Omega_{m,n}\Big) = P_{p(\lambda)}\Big(C_n(0)\cap R_m^n \neq \emptyset \mid \Omega_{m,n}\Big).$$

Hence we have,

$$\begin{aligned} & P_{p(\lambda)}^{\text{dep}}\Big(C_n^{\text{dep}}(0)\cap R_m^n \neq \emptyset\Big) \\ \leq\ & P_\lambda(\Omega_{m,n})P_{p(\lambda)}^{\text{dep}}\Big(C_n^{\text{dep}}(0)\cap R_m^n \neq \emptyset \mid \Omega_{m,n}\Big) + P_\lambda(\Omega_{m,n}^c) \\ =\ & P_\lambda(\Omega_{m,n})P_{p(\lambda)}\Big(C_n(0)\cap R_m^n \neq \emptyset \mid \Omega_{m,n}\Big) + P_\lambda(\Omega_{m,n}^c) \\ \leq\ & P_{p(\lambda)}\Big(C_n(0)\cap R_m^n \neq \emptyset\Big) + P_\lambda(\Omega_{m,n}^c). \end{aligned} \tag{5.15}$$

Now we compare the random connection model with the dependent percolation model. For this also we work on the event $\Omega_{m,n}$. We have, for $R_m^n = \{u \in \mathbb{L}_n : m-1 < |u| \leq m\}$,

$$\begin{aligned} & P_{p(\lambda)}^{\text{dep}}\{C_n^{\text{dep}}(0)\cap R_m^n \neq \emptyset\} \\ \leq\ & P_{p(\lambda)}^{\text{dep}}\{\{C_n^{\text{dep}}(0)\cap R_m^n \neq \emptyset\}\cap\Omega_{m,n}\} + P_\lambda(\Omega_{m,n}^c) \\ =\ & \sum_{k=0}^{\infty} P_{p(\lambda)}^{\text{dep}}(\{C_n^{\text{dep}}(0)\cap R_m^n \neq \emptyset\}\cap\Omega_{m,n} \mid N(A_m) = k) \\ & \times P_\lambda(N(A_m) = k) + P_\lambda(\Omega_{m,n}^c) \end{aligned} \tag{5.16}$$

where $N(A)$ is number of Poisson points in the set A.

Note that for any configuration in $\Omega_{m,n}$, a box $B_n(u)$ with $u \in A_m \cap \mathbb{L}_n$ may contain at most one Poisson point. Thus each Poisson point of A_m is contained in a unique box $B_n(u)$. So if we condition on the event that there are exactly k Poisson points inside the box A_m, then we have exactly k boxes inside A_m, each of which contain exactly one Poisson point. Let us consider two such boxes, $B_n(u)$ and $B_n(v)$. Let x_i and x_j be the Poisson points such that $x_i \in B_n(u), x_j \in B_n(v)$. Now the Poisson points are connected if $U(x_i, x_j) \leq g(|x_i - x_j|)$ and the vertices u and v are connected (in the dependent percolation model) if $U(x_i, x_j) \leq g(|u-v|)$.

Since, g is continuous with bounded support, g is uniformly continuous. So, given $\epsilon > 0$, we may choose $\delta > 0$ such that $|g(s) - g(t)| < \epsilon$ whenever $|s - t| < \delta$. Choose n large so that $2\sqrt{d}2^{-n} < \delta$. Note that this choice ensures that for any two boxes $B_n(u)$ and $B_n(v)$, we have $|\ |u - v| - |s - t|\ | < \delta$ for any $s \in B_n(u)$ and $t \in B_n(v)$. So, under the condition that $\Omega_{m,n}$ occurs and there are exactly k Poisson points inside the box $[-(m+1), (m+1)]^d$, the event that $C_n^{\text{dep}}(\mathbf{0}) \cap R_m^n \neq \emptyset$ but $C(\mathbf{0}) \cap R_m = \emptyset$, will occur if for at least one pair of Poisson points $x_i, x_j, x_i \in B_n(u)$ and $x_j \in B_n(v)$, we have, $\{g(|x_i - x_j|) < U(x_i, x_j) \leq g(|u-v|)\}$. Clearly the probability of such an event is no larger than $k(k-1)\epsilon$.

Now, to make this formal, we do a conditioning argument. Given there are exactly k Poisson points inside the box A_m, the Poisson points are uniformly distributed. Writing $P_{x_1,\ldots,x_k}(\cdot)$ for this conditional probability measure, we have,

$$\begin{aligned} & P_{p(\lambda)}^{\text{dep}}(\{C_n^{\text{dep}}(\mathbf{0}) \cap R_m^n \neq \emptyset\} \cap \Omega_{m,n} \mid N(A_m) = k)) \\ = \;& \textstyle\int_{A_m} \cdots \int_{A_m} P_{x_1,\ldots,x_k}(\{C_n^{\text{dep}}(\mathbf{0}) \cap R_m^n \neq \emptyset\} \cap \Omega_{m,n}) dx_1 \ldots dx_k. \end{aligned}$$

From our above disscussion it follows, $P_{x_1,\ldots,x_k}(\{C_n^{\text{dep}}(\mathbf{0}) \cap R_m^n \neq \emptyset\} \cap \Omega_{m,n}) \leq P_{x_1,\ldots,x_k}(\{C(\mathbf{0}) \cap R_m \neq \emptyset\} \cap \Omega_{m,n}) + k(k-1)\epsilon$. Thus we obtain from (5.14) and (5.16), for all $\lambda \in (\lambda_0, \lambda_1)$,

$$\begin{aligned} & h_m^n(p(\lambda)) = P_{p(\lambda)}\{C_n(\mathbf{0}) \cap R_m^n \neq \emptyset\} \\ \leq \;& P_{p(\lambda)}^{\text{dep}}\{C_n^{\text{dep}}(\mathbf{0}) \cap R_m^n \neq \emptyset\} \\ \leq \;& P_\lambda(\{C(\mathbf{0}) \cap R_m \neq \emptyset\} \cap \Omega_{m,n}) + P_\lambda(\Omega_{m,n}^c) + [\lambda_1(2m+3)^d]^2\epsilon \\ \leq \;& P_\lambda\{C(\mathbf{0}) \cap R_m \neq \emptyset\} + (2m+3)^d\lambda_1^2 2^{-nd} + [\lambda_1(2m+3)^d]^2\epsilon \\ = \;& h_m(\lambda) + (2m+3)^d\lambda_1^2 2^{-nd} + [\lambda_1(2m+3)^d]^2\epsilon. \end{aligned}$$

From (5.15) and a similar calculation as above, we have,

$$\begin{aligned} & h_m(\lambda) \\ = \;& P_\lambda(C(\mathbf{0}) \cap R_m \neq \phi) \end{aligned}$$

$$
\begin{aligned}
&\le P_{p(\lambda)}^{\text{dep}}\{C_n^{\text{dep}}(\mathbf{0}) \cap R_m^n \neq \emptyset\} + P_\lambda(\Omega_{m,n}^c) + [\lambda_1(2m+3)^d]^2\epsilon \\
&\le P_{p(\lambda)}\{C_n(\mathbf{0}) \cap R_m^n \neq \emptyset\} + 2P_\lambda(\Omega_{m,n}^c) + [\lambda_1(2m+3)^d]^2\epsilon.
\end{aligned}
$$

This proves the lemma. ■

Having obtained this lemma, we prove a similar inequality as in (5.12) for the random connection model. Let us define, for fixed $m \geq 1$ and for all $n \geq 1$,

$$K_m^n(\lambda) := h_m^n(p(\lambda)).$$

By Lemma 5.4, we have $K_m^n(\lambda) \to h_m(\lambda)$ as $n \to \infty$, uniformly for all $\lambda \in [\lambda_0, \lambda_1]$.

Lemma 5.5 *For $0 < \lambda_0 < \lambda_1 < \infty$ and $m \geq 1$, we have,*

$$h_m(\lambda_0) \leq h_m(\lambda_1)\exp\Big(-\frac{\lambda_1 - \lambda_0}{\lambda_1}\Big[\frac{m}{\sum_{k=0}^m h_k(\lambda_1)} - 1\Big]\Big). \tag{5.17}$$

Proof : We just note from the Lemma 5.3,

$$\frac{1}{K_m^n(\lambda)}\frac{d}{d\lambda}K_m^n(\lambda) \geq \frac{\exp(-\lambda 2^{-nd})2^{-nd}}{1-\exp(-\lambda 2^{-nd})}\Big[\frac{m}{\sum_{k=0}^m K_k^n(\lambda)} - 1\Big].$$

Thus integrating between $[\lambda_0, \lambda_1]$, we obtain,

$$K_m^n(\lambda_0) \leq K_m^n(\lambda_1)\exp\Big(-\int_{\lambda_0}^{\lambda_1}\frac{\exp(-\lambda 2^{-nd})2^{-nd}}{1-\exp(-\lambda 2^{-nd})}\Big[\frac{m}{\sum_{k=0}^m K_k^n(\lambda)} - 1\Big]d\lambda\Big).$$

Now using Lemma 5.4 and the continuity of the exponential function, we obtain

$$
\begin{aligned}
h_m(\lambda_0) &\leq h_m(\lambda_1)\exp\Big(-\int_{\lambda_0}^{\lambda_1}\frac{1}{\lambda}\Big[\frac{m}{\sum_{k=0}^m h_k(\lambda)} - 1\Big]d\lambda\Big) \\
&\leq h_m(\lambda_1)\exp\Big(-\frac{\lambda_1 - \lambda_0}{\lambda_1}\Big[\frac{m}{\sum_{k=0}^m h_k(\lambda_1)} - 1\Big]\Big).
\end{aligned}
$$

■

Now we follow Lemma 3.24 of Grimmett [1988] to obtain for $\lambda < \lambda_c$,

$$P_\lambda\{C(\mathbf{0}) \cap R_m \neq \emptyset\} \leq \exp(-\psi(\lambda)m) \tag{5.18}$$

for some $\psi(\lambda) > 0$.

Proof of Theorem 5.1 : For $\lambda < \lambda_c$

$$\begin{aligned}
& P_\lambda(\#(C(0)) \geq 2\lambda(2m+3)^d) \\
\leq\ & P_\lambda(\{\#(C(0)) \geq 2\lambda(2m+3)^d\} \cap \{C(0) \cap R_m = \emptyset\}) \\
& + P_\lambda\{C(0) \cap R_m \neq \emptyset\} \\
\leq\ & P_\lambda(N(A_m) \geq 2\lambda(2m+3)^d) + \exp(-\psi(\lambda)m). \qquad (5.19)
\end{aligned}$$

Now, $N(A_m)$ is a Poisson random variable with expectation $\lambda(2(m+1)+2^{-n})^d$. Thus the first term can be estimated easily. In fact,

$$\begin{aligned}
& P_\lambda(N(A_m) \geq 2\lambda(2m+3)^d) \\
=\ & P_\lambda(\exp(N(A_m)) \geq \exp(2\lambda(2m+3)^d)) \\
\leq\ & \frac{E(\exp(N(A_m)))}{\exp(2\lambda(2m+3)^d)} \\
=\ & \frac{\exp(\lambda(2(m+1)+2^{-n})^d(e-1))}{\exp(2\lambda(2m+3)^d)} \\
\leq\ & \frac{\exp(\lambda(2m+3)^d(e-1))}{\exp(2\lambda(2m+3)^d)} \\
=\ & \exp(-(3-e)\lambda(2m+3)^d),
\end{aligned}$$

where the first inequality follows from the Markov inequality and the moment generating function of Poisson random variables has been used for the next step.

Thus we have,

$$P_\lambda(\#(C(0)) \geq r) \leq \exp(-\phi(\lambda)r^{1/d}) + \exp(-\eta(\lambda)r) \qquad (5.20)$$

for some $\phi(\lambda) > 0$ and some $\eta(\lambda) > 0$. Hence we have the theorem. ■

5.5 RCM mosaic

In this section we study the mosaic structure of the random connection model. The Theorems 5.2, 5.3 and 5.4 are analogues of the Theorems

4.7, 4.8 and 4.9 of Hall [1988] in the RCM. The proof of the theorems also closely follow the proofs of Hall [1988].

Let (X, g) be a random connection model in $\mathbb{R}^d$. Let g be function having bounded support, i.e.,

$$g(r) = 0 \text{ for all } r \geq R \text{ for some } M > 0. \tag{5.21}$$

A Poisson point $x \in X$ is called *isolated* in the RCM (X, g) if for all $y \in X$ x and y are not connected.

Fix a bounded subset K of $\mathbb{R}^d$. Let $N(K)$ be the number of Poisson points inside K and let $\{x_1, \ldots, x_{N(K)}\}$ be a realisation of these Poisson points. Define

$$\begin{aligned} p_1 &= p_1(\lambda, K) \\ &= P(x_1 \text{ is isolated } \mid N(K) \geq 1) \\ &= P(0 \text{ is isolated}). \end{aligned} \tag{5.22}$$

The Poisson points which are connected to the origin form an inhomogeneous Poisson process with intensity function $g(|\cdot|)$. Thus

$$p_1 = \exp(-\lambda \int_{\mathbb{R}^d} g(|y|)dy) = \exp(-\lambda \int_{B_R} g(|y|)dy), \tag{5.23}$$

where B_R is the ball of radius R at the origin. Define also,

$$p_2 = p_2(\lambda, K) = P(x_1 \text{ and } x_2 \text{ are both isolated} \mid N(K) \geq 2). \tag{5.24}$$

A similar calculation may be carried out to find p_2. We note that the Poisson points which are connected to either the point x_1 or x_2 form an inhomogeneous Poisson point process with intensity function $h(y; x_1, x_2) := g(|x_1 - y|) + g(|x_2 - y|) - g(|x_1 - y|)g(|x_2 - y|)$. Thus the probability that two arbitrary points in K are both isolated can be thought as follows: first we put two independent uniform random variables in K. Next we put an independent Poisson process of intensity λ. Connect all the points according to the connection rule given by the function g. Thus the probability that x_1 and x_2 are both isolated is p_2. Clearly,

$$\begin{aligned} &p_2 \\ &= \frac{1}{\ell(K)^2} \int_K \int_K (1 - g(|x_1 - x_2|)) \exp(-\lambda \int_{\mathbb{R}^d} h(y; x_1, x_2)dy)dx_1 dx_2. \end{aligned}$$

Let $M = M(K)$ be the total number of isolated points inside K in the RCM (X,g). The expectation of $M(K)$ can be calculated easily by using a marked point process argument as was used earlier in (4.1). Define a mark l_i at x_i as follows:

$$l_i = \begin{cases} 1 & \text{if } x_i \text{ is isolated} \\ 0 & \text{otherwise} . \end{cases}$$

Then define $f(r,l) = l$ if $r \in K$ and 0 otherwise. Thus,

$$M(K) = \sum_{i=1}^{\infty} f(x_i, l_i).$$

Hence

$$\begin{aligned} E(M(K)) &= E\Big(\sum_{i=1}^{\infty} f(x_i, l_i)\Big) \\ &= \lambda E\Big(\int_{\mathbb{R}^d} f(r,l)dr\Big) \\ &= \lambda \int_{\mathbb{R}^d} E(f(r,l))dr \\ &= \lambda \int_K p_1 dr \\ &= \lambda \ell(K) p_1. \end{aligned} \tag{5.25}$$

This can also be obtained more directly. Define the *R-fattening* of K, K^R as follows:

$$K^R = \{a \in \mathbb{R}^d : |a - b| < R \text{ for some } b \in K\}.$$

Now if we condition on the event that there are exactly m points inside K^R, then the points are uniformly distributed and thus $M(K)$ can be written as

$$M(K) = \sum_{i=1}^{m} I_{\{x_i \text{ is isolated; } x_i \in K\}}. \tag{5.26}$$

So

$E(M(K))$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R)) \frac{[\lambda\ell(K^R)]^m}{m!} \\
&\quad E(\sum_{i=1}^{m} I_{\{x_i \text{ is isolated; } x_i \in K\}} \mid N(K^R) = m) \\
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R)) \frac{[\lambda\ell(K^R)]^m}{m!} \\
&\quad mE(I_{\{x_1 \text{ is isolated; } x_1 \in K\}} \mid N(K^R) = m) \\
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R)) \frac{[\lambda\ell(K^R)]^m}{m!} \\
&\quad mP(x_1 \text{ is isolated; } x_1 \in K \mid N(K^R) = m) \\
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R)) \frac{[\lambda\ell(K^R)]^m}{m!} \\
&\quad m \int_K \int_{K^R} \cdots \int_{K^R} \frac{\prod_{i=2}^{m}(1 - g(|x_1 - x_i|))}{(\ell(K^R))^m} dx_m \ldots dx_2 dx_1 \\
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R)) \frac{[\lambda\ell(K^R)]^m}{m!} \\
&\quad m \int_K \frac{1}{\ell(K^R)} \Big[1 - \frac{1}{\ell(K^R)} \int_{K^R} g(|x_1 - y|) dy\Big]^{m-1} dx_1 \\
&= \lambda\ell(K) \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R)) \frac{[\lambda\ell(K^R)]^m}{m!} \Big[1 - \frac{1}{\ell(K^R)} \int_{B_R} g(|y|) dy\Big]^m \\
&= \lambda\ell(K) \exp\Big(-\lambda\ell(K^R) + \lambda\ell(K^R)[1 - \frac{1}{\ell(K^R)} \int_{B_R} g(|y|) dy]\Big) \\
&= \lambda\ell(K) \exp(-\lambda \int_{B_R} g(|y|) dy) \\
&= \lambda\ell(K) p_1.
\end{aligned}
$$

The variance may also be calculated in a similar fashion.

$$
\begin{aligned}
&E(M^2(K)) \\
&\quad = E\Big[E((\sum_{i=1}^{m} I_{\{x_i \text{ is isolated; } x_i \in K\}})^2 \mid N(K^R) = m)\Big] \\
&\quad = E\Big[E(\sum_{i=1}^{m} I_{\{x_i \text{ is isolated; } x_i \in K\}} \mid N(K^R) = m)\Big]
\end{aligned}
$$

$$+E\Big[E(\sum_{i\neq j} I_{\{x_i \text{ and } x_j \text{ are both isolated}; x_i,x_j\in K\}} \mid N(K^R)=m)\Big].$$

The first term in this sum is $\lambda\ell(K)p_1$. So we are to calculate $E\Big[E(\sum_{i\neq j} I_{\{x_i \text{ and } x_j \text{ are both isolated}; x_i,x_j\in K\}} \mid N(K^R)=m)\Big]$. Thus we have,

$$\begin{aligned}
&E(M^2(K)) - \lambda\ell(K)p_1 \\
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R))\frac{[\lambda\ell(K^R)]^m}{m!} \\
&\quad E(\sum_{i\neq j} I_{\{x_i \text{ and } x_j \text{ are both isolated}; x_i,x_j\in K\}} \mid N(K^R)=m) \\
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R))\frac{[\lambda\ell(K^R)]^m}{m!}m(m-1) \\
&\quad E(I_{\{x_1 \text{ and } x_2 \text{ are both isolated}; x_1,x_2\in K\}} | N(K^R)=m) \\
&= \sum_{m=0}^{\infty} \exp(-\lambda\ell(K^R))\frac{[\lambda\ell(K^R)]^m}{m!}m(m-1) \\
&\quad P(x_1 \text{ and } x_2 \text{ are both isolated}; x_1,x_2\in K \mid N(K^R)=m) \\
&= \sum_{m=2}^{\infty} \exp(-\lambda\ell(K^R))\frac{[\lambda\ell(K^R)]^m}{(m-2)!}\frac{1}{(\ell(K^R))^m}\int_K\int_K\int_{K^R}\cdots\int_{K^R} \\
&\quad (1-g(|x_1-x_2|))\prod_{i=3}^{m}(1-h(x_i;x_1,x_2))dx_m\ldots dx_1 \\
&= \sum_{m=2}^{\infty} \exp(-\lambda\ell(K^R))\frac{[\lambda\ell(K^R)]^m}{(m-2)!}\int_K\int_K(1-g(|x_1-x_2|)) \\
&\quad \frac{1}{(\ell(K^R))^2}\Big[1-\frac{1}{\ell(K^R)}\int_{K^R}h(y;x_1,x_2)dy\Big]^{m-2}dx_1dx_2.
\end{aligned}$$

Now we note that for any $x_1, x_2 \in K$, $h(y;x_1,x_2) = 0$ for all $y \notin K^R$. Thus, $\int_{K^R} h(y;x_1,x_2)dy = \int_{\mathbf{R}^d} h(y;x_1,x_2)dy$ for $x_1,x_2 \in K$. Further all the terms in the above expression are non-negative. Thus using Fubini's Theorem, we obtain,

$$\begin{aligned}
&E(M^2(K)) - \lambda\ell(K)p_1 \\
&= \lambda^2\int_K\int_K(1-g(|x_1-x_2|))\exp(-\lambda\ell(K^R))
\end{aligned}$$

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{[\lambda\ell(K^R)]^m}{m!}\left[1 - \frac{1}{\ell(K^R)}\int_{R^d} h(y;x_1,x_2)dy\right]^m dx_1dx_2 \\
&= \lambda^2 \int_K \int_K (1 - g(|x_1 - x_2|)) \exp\Big(-\lambda\ell(K^R) \\
&\quad +\lambda\ell(K^R)[1 - \frac{1}{\ell(K^R)}\int_{R^d} h(y;x_1,x_2)dy]\Big)dx_1dx_2 \\
&= \lambda^2 \int_K \int_K (1 - g(|x_1 - x_2|)) \exp(-\lambda \int_{R^d} h(y;x_1,x_2)dy)dx_1dx_2 \\
&= (\lambda\ell(K))^2 p_2.
\end{aligned}$$

Hence the variance of $M(K)$ is given by,

$$\text{var}(M(K)) = \lambda\ell(K)p_1 + (\lambda\ell(K))^2(p_2 - p_1^2). \tag{5.27}$$

Let us now consider the mosaic $\{(X_n, g_n) : n \geq 1\}$ where X_n has intensity λ_n and g_n is defined as $g_n(r) = g(nr)$. Let λ_n be such that $\lambda_n/n^d \to \lambda$ as $n \to \infty$.

We now consider the isolated points inside K in this random connection model (X_n, g_n). Let $M_n = M_n(K)$ denote the number of isolated points inside K in (X_n, g_n). We shall prove that the expectation and variance of M_n grow linearly with $\lambda\ell(K)$ when we let n tend to infinity.

Theorem 5.2 *Let (X_n, g_n) be a random connection model with intensity λ_n and $g_n(r) = g(nr)$ such that $\lambda_n/n^d \to \lambda$ as $n \to \infty$ where $0 \leq \lambda < \infty$. Then*

$$(\lambda_n\ell(K))^{-1}E(M_n) \to p_1$$

and

$$(\lambda_n\ell(K))^{-1}var(M_n) \to \kappa^2(g)$$

as $n \to \infty$, where $p_1 = p_1(\lambda, K)$ is as in (5.22) and

$$\begin{aligned}
\kappa^2(g) = p_1 + \lambda \int_{B_{2R}} &\left\{P(0 \text{ and } x \text{ are both isolated}) - p_1^2\right\}dx \\
+4\lambda p_1 \int_{B_{2R}} &P(0 \text{ is isolated; } 0 \text{ and } x \text{ are connected})dx.
\end{aligned}$$

Proof of Theorem 5.2: Define

$$\begin{aligned}
&p_1(n, \lambda_n) \\
&= P_{(\lambda_n, g_n)}\{\text{the Poisson point } x \text{ is isolated} \mid N(K) \geq 1\}. \quad (5.28)
\end{aligned}$$

By (5.25), we have $E(M_n(K)) = \lambda\ell(K)p_1(n,\lambda_n)$. Hence we are to show that $p_1(n,\lambda_n) \to p_1$ as $n \to \infty$. Now,

$$\begin{aligned}
&p_1(n,\lambda_n)\\
&= P_{(\lambda_n,g_n)}\{\text{the Poisson point } x \text{ is isolated} \mid N(K) \geq 1\}\\
&= \exp(-\lambda_n \int_{B_{R/n}} g_n(|r|)dr)\\
&= \exp(-\lambda_n \int_{B_{R/n}} g(n|r|)dr)\\
&= \exp(-\frac{\lambda_n}{n^d} \int_{B_R} g(|r|)dr)\\
&\to p_1(\lambda) \quad \text{as } n \to \infty.
\end{aligned}$$

To prove the second part we first note that

$$\begin{aligned}
P(N(K) \geq 2) &= 1 - \exp(-\lambda\ell(K))(1 + \lambda\ell(K))\\
&= 1 + o(\lambda^{-1}). \qquad (5.29)
\end{aligned}$$

Define $p_2(n,\lambda_n)$ as the probability that x_1 and x_2 are both isolated given that there are at least two points inside K in the model (X_n, g_n). Thus

$$\begin{aligned}
&p_2(n,\lambda_n)\\
&= P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \mid N(K) \geq 2\}\\
&= \frac{P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated; } N(K) \geq 2\}}{P_{(\lambda_n,g_n)}(N(K) \geq 2)}\\
&= P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated; } N(K) \geq 2\} + o(\lambda_n^{-1})\\
&= \sum_{m=2}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \mid N(K) = m\}\\
&\quad \times P_{(\lambda_n,g_n)}(N(K) = m) + o(\lambda_n^{-1}). \qquad (5.30)
\end{aligned}$$

As before the last term above can be interpreted as follows: first we put two independent uniform random variables x_1 and x_2 in K. Now we put an independent Poisson process X_n' of intensity λ_n. Let $N'(K)$ be the number of Poisson points inside K. Clearly the distribution of both $N(K)$ and $N'(K)$ are identical. Thus we obtain,

$$P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated } \mid N(K)=m\}$$
$$= P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ both isolated} \mid N'(K)=m-2\}.$$

Hence we have from (5.30)

$$\begin{aligned} & p_2(n,\lambda_n)+o(\lambda_n^{-1}) \\ = & \sum_{m=2}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \\ & \mid N'(K)=m-2\} P_{(\lambda_n,g_n)}(N'(K)=m) \\ = & \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \\ & \mid N'(K)=m\} P_{(\lambda_n,g_n)}(N'(K)=m+2) \\ = & \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \\ & \mid N'(K)=m\} P_{(\lambda_n,g_n)}(N'(K)=m) \\ & + \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \mid N'(K)=m\} \\ & [P_{(\lambda_n,g_n)}(N'(K)=m+2)-P_{(\lambda_n,g_n)}(N'(K)=m)] \\ = & P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated}\}+A(n,\lambda_n), \end{aligned}$$

where $A=A(n,\lambda_n)$ is defined as follows:

$$\begin{aligned} & A(n,\lambda_n) \\ = & \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated } \mid N'(K)=m\} \\ & [P_{(\lambda_n,g_n)}(N'(K)=m+2)-P_{(\lambda_n,g_n)}(N'(K)=m)] \\ = & \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \mid N'(K)=m\} \\ & P_{(\lambda_n,g_n)}(N'(K)=m)\left(\frac{(\lambda_n\ell(K))^2}{(m+1)(m+2)}-1\right). \end{aligned}$$

First we prove that as $n\to\infty$,

$$\lambda_n\ell(K)\Big[P_{(\lambda_n,g_n)}(x_1 \text{ and } x_2 \text{ are both isolated})$$

$$- \prod_{i=1}^{2} P_{(\lambda_n,g_n)}(x_i \text{ is isolated})\Big]$$

$$\to \lambda \int_{B_{2R}} \Big\{ P_{(\lambda,g)}(\mathbf{0} \text{ and } x \text{ are both isolated}) - p_1^2 \Big\} dx. \quad (5.31)$$

We employ the same technique of first putting two independent uniform random variables in K and then putting another independent Poisson process to calculate the left hand side of (5.31). We begin with a change in the length scale, $r \to nr$. This will give us a random connection model (nX_n, g) where nX_n has an intensity λ_n/n^d. Thus we obtain that the left hand side of (5.31) equals

$$\lambda_n \ell(K) \int_K \int_K \frac{1}{\ell(K)^2} \Big\{ P_{(\lambda_n,g_n)}(x_1 \text{ and } x_2 \text{ are both isolated})$$

$$- \prod_{i=1}^{2} P_{(\lambda_n,g_n)}(x_i \text{ is isolated}) \Big\} dx_1 dx_2$$

$$= \lambda_n \ell(K) \int_{nK} \int_{nK} \frac{1}{\ell(nK)^2} \Big\{ P_{(\lambda_n/n^d,g)}(x_1 \text{ and } x_2 \text{ are both isolated})$$

$$- \prod_{i=1}^{2} P_{(\lambda_n/n^d,g)}(x_i \text{ is isolated}) \Big\} dx_1 dx_2.$$

Now by using the translation invariance of the model the above term equals

$$\frac{\lambda_n}{n^d} \int_{K_n} \frac{\ell(nK \cap (nK - x))}{n^d \ell(K)} \Big\{ P_{(\lambda_n/n^d,g)}(\mathbf{0} \text{ and } x \text{ both isolated}) - p_1^2 \Big\} dx,$$

where $K_n = \{y_1 - y_2 : y_1, y_2 \in nK\}$. As $n \to \infty$, $\lambda_n/n^d \to \lambda$. For each fixed $x, \ell(nK \cap (nK - x))/\ell(nK) \to 1$. Also if $|x| > 2R$, the integrand equals zero. Thus by an application of the dominated convergence theorem, we obtain (5.31).

Next we prove

$$\lambda \ell(K) A(n, \lambda) \to$$

$$4\lambda p_1 \int_{B_{2R}} P_{(\lambda,g)}(\mathbf{0} \text{ is isolated; } \mathbf{0} \text{ and } x \text{ are connected}) dx. \quad (5.32)$$

By using Markov inequality we first note that for any $\epsilon, l > 0$,

$$P_{(\lambda_n,g_n)}(|N(K) - \mu| > \mu^{(1/2)+\epsilon}) \leq (E(|N(K) - \mu|/\mu^{(1/2)+\epsilon})^l$$

$$
\begin{aligned}
&= O(\mu^{(1/2)l}/\mu^{((1/2)+\epsilon)l}) \\
&= O(\mu^{-\epsilon l})
\end{aligned}
$$

where $\mu = \lambda_n \ell(K)$. Denote $\nu = \mu^{(1/2)+\epsilon}$. Thus, by choosing l large enough, we obtain

$$
\begin{aligned}
&A(n, \lambda) \\
&\quad = \sum_{m:|m-\mu|\leq\nu} P_{(\lambda_n, g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \mid N(K) = m\} \\
&\qquad \times P_{(\lambda_n, g_n)}(N(K) = m)\left\{\frac{\mu^2}{(m+1)(m+2)} - 1\right\} + o(\mu^{-1}). \quad (5.33)
\end{aligned}
$$

Let $r = m - \mu$. Then for $|r| \leq \nu$ we have

$$
\begin{aligned}
\frac{\mu^2}{(m+1)(m+2)} &= \{1 + (r+1)/\mu\}^{-1}\{1 + (r+2)/\mu\}^{-1} \\
&= 1 - (2r+3)/\mu + 3(r/\mu)^2 \\
&\quad + O(|r|^3/\mu^3 + (|r|+1)/\mu^2).
\end{aligned}
$$

Choose $\epsilon < 1/6$, then $A(n, \lambda_n)$ reduces to

$$
\begin{aligned}
&A(n, \lambda_n) \\
&\quad = \sum_{m:|m-\mu|\leq\nu} P_{(\lambda_n, g_n)}\{x_1 \text{ and } x_2 \text{ both isolated; } N(K) = m\} \\
&\qquad \times\left\{-(2r+3)/\mu + 3(r/\mu)^2\right\} + o(\mu^{-1}) \\
&\quad = \sum_{m=0}^{\infty} P_{(\lambda_n, g_n)}\{x_1 \text{ and } x_2 \text{ both isolated; } N(K) = m\} \\
&\qquad \times\left\{-(2r+3)/\mu + 3(r/\mu)^2\right\} + o(\mu^{-1}) \\
&\quad = -\frac{3}{\mu} P_{(\lambda_n, g_n)}\{x_1 \text{ and } x_2 \text{ both isolated}\} \\
&\qquad -\frac{2}{\mu}\sum_{m=0}^{\infty} P_{(\lambda_n, g_n)}\{x_1 \text{ and } x_2 \text{ both isolated; } N(K) = m\}(m-\mu) \\
&\qquad +\frac{3}{\mu}\sum_{m=0}^{\infty} P_{(\lambda_n, g_n)}\{x_1 \text{ and } x_2 \text{ both isolated; } N(K) = m\}(m-\mu)^2 \\
&\qquad + o(\mu^{-1}).
\end{aligned}
$$

We have shown in (5.31) that $\lambda_n \ell(K)\Big[P_{(\lambda_n,g_n)}\{x_1$ and x_2 are both isolated$\} - P_{(\lambda_n,g_n)}\{\mathbf{0}$ is isolated$\}^2\Big]$ converges to a limit which is non-negative. Thus we obtain that

$$\begin{aligned} & P_{(\lambda_n,g_n)}(x_1 \text{ and } x_2 \text{ both isolated}) \\ = \;& P_{(\lambda_n,g_n)}(\mathbf{0} \text{ is isolated})^2 + O(\lambda_n^{-1}) \\ = \;& P_{(\lambda_n,g_n)}(\mathbf{0} \text{ is isolated})^2 + o(1). \end{aligned}$$

Similar calculations yield

$$\begin{aligned} & \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated; } N(K)=m\}(m-\mu)^2 \\ = \;& \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}(\mathbf{0} \text{ is isolated})^2 P_{(\lambda_n,g_n)}(N(K)=m)(m-\mu)^2 + O(\lambda_n^{-1}) \\ = \;& \mu P_{(\lambda_n,g_n)}(\mathbf{0} \text{ is isolated})^2 + O(\lambda_n^{-1}). \end{aligned}$$

So

$$A(n,\lambda_n) = 2A_1(n,\lambda_n) + o(\mu^{-1}), \tag{5.34}$$

where

$$\begin{aligned} & A_1(n,\lambda_n) \\ = \;& -\frac{1}{\mu 1}\sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ both isolated; } N(K)=m\}(m-\mu) \\ = \;& \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ both isolated} \mid N(K)=m\} \\ & \Big[P_{(\lambda_n,g_n)}(N(K)=m) - P_{(\lambda_n,g_n)}(N(K)=m-1)\Big] \\ = \;& \sum_{m=0}^{\infty} \Big[P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ both isolated} \mid N(K)=m\} \\ & -P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ both isolated} \mid N(K)=m+1\}\Big] \\ & \times P_{(\lambda_n,g_n)}(N(K)=m) \end{aligned}$$

To calculate this we again employ the same technique. Let x_3 be another uniform random variable independent of the all previous random variables and the Poisson process. Thus we obtain,

$$\begin{aligned}
&P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated} \mid N(K)=m+1\}\\
&= P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated; } x_3 \text{ is not}\\
&\quad \text{connected to both } x_1 \text{ and } x_2 \mid N(K)=m\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&A_1(n,\lambda_n)\\
&= \sum_{m=0}^{\infty} P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated; } x_3 \text{ is}\\
&\quad \text{connected to at least one of } x_1 \text{ and } x_2 \mid N(K)=m\}\\
&= P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated; } x_3 \text{ is}\\
&\quad \text{connected to at least one of } x_1 \text{ and } x_2\}.
\end{aligned}$$

Again similar calculations can be carried out to yield

$$\begin{aligned}
&P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated;}\\
&\quad x_3 \text{ is connected to at least one of } x_1 \text{ and } x_2\}\\
&= 2P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated;}\\
&\quad x_3 \text{ is connected to } x_1\} + o(\mu^{-1}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&A_1(n,\lambda_n)\\
&= 2P_{(\lambda_n,g_n)}\{x_1 \text{ and } x_2 \text{ are both isolated;}\\
&\quad x_3 \text{ is connected to } x_1\} + o(\mu^{-1})\\
&= 2P_{(\lambda_n,g_n)}\{x_2 \text{ is isolated}\}P_{(\lambda_n,g_n)}\{x_1 \text{ isolated;}\\
&\quad x_3 \text{ is connected to } x_1\} + o(\mu^{-1})\\
&= 2p_1\frac{1}{\ell(K)^2}\int_K\int_K P_{(\lambda_n,g_n)}\{x_1 \text{ is isolated;}\\
&\quad x_3 \text{ is connected to } x_1\}dx_1dx_3 + o(\mu^{-1})\\
&= 2p_1\frac{1}{\ell(nK)^2}\int_{nK}\int_{nK} P_{(\lambda_n,g_n)}\{x_1 \text{ isolated;}\\
&\quad x_3 \text{ is connected to } x_1\}dx_1dx_3 + o(\mu^{-1})
\end{aligned}$$

$$= 2p_1 \frac{1}{n^d \ell(K)^2} \int_{K_n} \ell(nK \cap (nK - x))$$
$$P_{(\lambda_n, g_n)}\{0 \text{ is isolated; } x \text{ is connected to } 0\} dx + o(\mu^{-1}).$$

Now note that, for each fixed x, $\ell(nK \cap (nK - x))/n^d \ell(K) \to 1$ and the integrand equals zero for $|x| > 2R$. So by an application of the dominated convergence theorem, we obtain

$$[\lambda_n \ell(K)]^{-1} A_1(n, \lambda_n)$$
$$\to 2p_1 \int_{B_{2R}} P_{(\lambda, g)}\{0 \text{ is isolated; } x \text{ is connected to } 0\} dx. \quad (5.35)$$

The theorem follows from (5.35) , (5.34) and (5.31). ■

Now, we consider clusters of finite order $k (k \geq 2)$. A Poisson point x_1 is said to be a part of a finite cluster of order k if there are Poisson points $\{x_2, x_3, \ldots, x_k\}$ such that the set $\{x_1, x_2, \ldots, x_k\}$ form a connected set and no other Poisson point is connected to any of these k points.

Define $p(k)$ to be the probability that an arbitrary Poisson point x is a part of a finite cluster of order k. For $y_1, y_2, \ldots, y_k \in \mathbb{R}^d$, define $f(y_1, y_2, \ldots, y_k)$ to be the probability that the set $\{y_1, y_2, \ldots, y_k\}$ form a connected set. Then

$$f(y_1, y_2, \ldots, y_k) = \sum_G \prod{}' g(|y_i - y_j|) \prod{}'' (1 - g(|y_i - y_j|)), \quad (5.36)$$

where the sum runs over all connected graphs G of $\{1, 2, \ldots k\}$ and the product $\prod'$ runs over all edges $(i, j), (1 \leq i < j \leq k)$ which are in G and the product $\prod''$ runs over all edges $(i, j), (1 \leq i < j \leq k)$ which are not in G.

Now again we consider the mosaic $\{(X_n, g_n) : n \geq 1\}$. Suppose that $\eta_n = \lambda_n / n^d \to 0$ as $n \to \infty$.

Theorem 5.3 *Let $\{(X_n, g_n) : n \geq 1\}$ be the mosaic with $\eta_n \to 0$ as $n \to \infty$. Let $p_n(k)$ be the probability that an arbitrary Poisson point is a part of a finite cluster of order $k (\geq 2)$. Then*

$$p_n(k) = \frac{\eta_n^{k-1}}{(k-1)!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(0, y_1, \ldots, y_{k-1}) dy_1 \ldots dy_{k-1} + o(\eta_n^{k-1}) \quad (5.37)$$

and

$$\sum_{i=k+1}^{\infty} p_n(i) = o(\eta_n^{k-1}), \tag{5.38}$$

as $\eta_n \to 0$.

Proof: Consider the RCM (X_n, g_n) of intensity λ_n. Without any loss of generality we may take the origin to be a point of the Poisson process. Since $g(x) = 0$ for $x > R$, the cluster of order k will be contained in a sphere of radius kR/n. Clearly for the origin $\mathbf{0}$ to be part of a finite cluster of order $(k+1)$ or more, at least k points are needed to be inside the sphere of radius kR/n. Hence we have,

$$\begin{aligned}
&\sum_{i=k+1}^{\infty} p_n(i) \\
&\leq \sum_{i=k}^{\infty} \exp(-\lambda_n \pi_d (kR/n)^d) \frac{(\lambda_n \pi_d (kR/n)^d)^i}{i!} \\
&= \eta_n^k (\pi_d (kR)^d)^k \exp(-\lambda_n \pi_d (kR/n)^d) \sum_{i=0}^{\infty} \frac{(\lambda_n \pi_d (kR/n)^d)^i}{(i+k)!} \\
&\leq \eta_n^k (\pi_d (kR)^d)^k \exp(-\lambda_n \pi_d (kR/n)^d) \sum_{i=0}^{\infty} \frac{(\lambda_n \pi_d (kR/n)^d)^i}{i!} \\
&= \eta_n^k (\pi_d (kR)^d)^k.
\end{aligned}$$

This proves (5.38).

Now, if origin is part of a finite cluster of order k, then the cluster is totally contained inside the sphere S_n of radius $(k-1)R/n$. Now, let $N(S_n)$ be the number of Poisson points inside S_n. Clearly, if $x_1, x_2, \ldots, x_{N(S_n)}$ be the Poisson points inside S_n then the cluster of order k must be subset of these points. Further, given that there are m points inside S_n, the points are uniformly distributed over S_n. Hence

$$\begin{aligned}
&p_n(k) \\
&= \sum_{m=k-1}^{\infty} P_{(\lambda_n, g_n)}(N(S_n) = m) \int_{S_n} \cdots \int_{S_n} \frac{f_k^n(x_1, \ldots, x_m)}{[\ell(S_n)]^m} dx_1 \ldots dx_m \\
&= \sum_{m=k-1}^{\infty} P_{(\lambda_n/n^d, g)}(N(S) = m) \int_{S} \cdots \int_{S} \frac{f_k(x_1, \ldots, x_m)}{[\ell(S)]^m} dx_1 \ldots dx_m
\end{aligned}$$

where S is the sphere of radius $(k-1)R$ and $f_k^n(x_1,\ldots,x_m) = P($the finite cluster of order k formed from the points $\{0,x_1,\ldots,x_m\}$ contains 0 in the model $(X_n,g_n))$ and $f_k(x_1,\ldots,x_m) = P($the finite cluster of order k formed from the points $\{0,x_1,\ldots,x_m\}$ contains 0 in the model $(X,g))$.

Clearly, $f_k(x_1,\ldots,x_{k-1}) = f(0,x_1,\ldots,x_{k-1})$. Now the first term of the series is

$$\begin{aligned}
&P_{(\lambda_n/n^d,g)}(N(S)=k-1)\int_S\cdots\int_S \frac{f(0,x_1,\ldots,x_{k-1})}{[\ell(S)]^{k-1}}dx_1\ldots dx_{k-1}\\
&= \exp(-\eta_n\ell(S))\frac{(\eta_n\ell(S))^{k-1}}{(k-1)!}\int_S\cdots\int_S \frac{f(0,x_1,\ldots,x_{k-1})}{[\ell(S)]^{k-1}}dx_1\ldots dx_{k-1}\\
&= \frac{\eta_n^{k-1}}{(k-1)!}\int_S\cdots\int_S f(0,x_1,\ldots,x_{k-1})dx_1\ldots dx_{k-1}\\
&\quad -\frac{\eta_n^{k-1}}{(k-1)!}\Big(1-\exp(-\eta_n\ell(S))\Big)\\
&\quad \times\int_S\cdots\int_S f(0,x_1,\ldots,x_{k-1})dx_1\ldots dx_{k-1}\\
&= \frac{\eta_n^{k-1}}{(k-1)!}\int_S\cdots\int_S f(0,x_1,\ldots,x_{k-1})dx_1\ldots dx_{k-1}+o(\eta_n^{k-1}).
\end{aligned}$$

The second term is $o(\eta_n^{k-1})$ as $(1-\exp(-\eta_n\pi_d(kR)^d))\to 0$ as $n\to\infty$.

The rest of the terms in the series is dominated by

$$\begin{aligned}
&\sum_{m=k}^{\infty} P_{(\lambda_n/n^d,g_n)}(N(S)=m)\\
&= \sum_{m=k}^{\infty}\exp(-\eta_n\pi_d(kR)^d)\frac{(\eta_n\pi_d(kR))^m}{m!}\\
&= o(\eta_n^{k-1}).
\end{aligned}$$

This proves the theorem. ∎

A finite cluster of order k is a set $\{x_1,x_2,\ldots,x_k\}$ of Poisson points. We define the "centre" of such a cluster as the point x_i whose first co-ordinate is the largest.

Let K be a bounded compact subset of $\mathbb{R}^d$. Let $M_n(k)=M_n(k,K)$ be the number of finite clusters of order k which have centres inside K

in the RCM (X_n, g_n). We may use again a marked process argument to obtain that

$$E(M_n(k)) = \lambda_n \ell(K) k^{-1} p_n(k). \tag{5.39}$$

The factor k^{-1} appears here as all k points in the finite cluster of order k contribute once. Theorem 5.3 implies that

$$\begin{aligned} & (\lambda_n \eta_n^{k-1})^{-1} E(M_n(k)) \\ \to \ & \frac{\ell(K)}{k!} \int_{R^d} \cdots \int_{R^d} f(0, x_1, \ldots, x_{k-1}) dx_1 \ldots dx_{k-1} \\ = \ & \mu(k) \text{ (say).} \end{aligned} \tag{5.40}$$

A similar argument as in Theorem 5.2 may be used to prove that

$$(\lambda_n \eta_n^{k-1})^{-1} \operatorname{var}(M_n(k)) \to \mu(k). \tag{5.41}$$

The equations (5.40) and (5.41) suggests that $M_n(k)$ may have an asymptotic Poisson distribution.

Theorem 5.4 *Let (X_n, g_n) be as in Theorem 5.2. Let $M_n(k, K)$ be the number of finite clusters of order k which have centres inside K in the model (X_n, g_n).*
(i) If $\lambda_n \eta_n^{k-1} \to a (< \infty)$ as $n \to \infty$ then $M_n(k, K)$ has an asymptotic Poisson distribution with mean $a\mu(k)$ where $\mu(k)$ is as given by (5.40).
(ii) If $\lambda_n \eta_n^{k-1} \to \infty$ as $n \to \infty$ then $[var(M_n(k, K)]^{-1/2}(M_n(k, K) - E(M_n(k, K)))$ is asymptotically normal $N(0, 1)$.

Proof: Let m be a fixed large positive number. Define

$$\mathbb{L} = \left(\frac{R(m + 2k)}{n} \mathbf{Z} \right)^d.$$

Let $B = (-Rm/(2n), Rm/(2n)]^d$ and $B_1 = (-R(m + 2k)/(2n), R(m + 2k)/(2n)]^d$. For each point $x \in \mathbb{L}$, define $B(x) = x + B$ and $B_1(x) = x + B_1$. Let $C(x) = B_1(x) - B(x)$.

Let $x_{r_1}, x_{r_2}, \ldots, x_{r_{\alpha_n}}$ be points of $\mathbb{L}$ such that $B(x_i) \subseteq K$ for $i = 1, 2, \ldots, \alpha_n$. Let $A_1 = \cup_{i=1}^{\alpha_n} B(x_{r_i})$. Also suppose that $x_{p_1}, x_{p_2}, \ldots, x_{p_{\beta_n}}$ are points of $\mathbb{L}$ such that $C(x_i) \subseteq K$. Let $A_2 = \cup_{i=1}^{\beta_n} C(x_{p_i})$ and

$A_3 = K \setminus (A_1 \cup A_2)$. Define $M_n(i,k)$ as the number of clusters of order k which have their centres inside $A_i, i = 1,2,3$. Clearly,

$$M_n(k) = M_n(1,k) + M_n(2,k) + M_n(3,k). \tag{5.42}$$

First we prove that $M_n(1,k)$ satisfies the central limit theorem.

Note that any cluster of order k which has its centre inside the box $B(x_{r_i})$ for some $i = 1,2,\ldots,\alpha_n$, must be contained entirely inside the box $B_1(x_{r_i})$. For each $i = 1,2,\ldots\alpha_n$, let $M_n^i(1,k)$ be the number of clusters of order k which has its centre inside the box $B(x_{r_i})$. Thus

$$M_n(1,k) = \sum_{i=1}^{\alpha_n} M_n^i(1,k). \tag{5.43}$$

The random variables $\{M_n^i(1,k) : i = 1,\ldots,\alpha_n\}$ are independent and identically distributed by our choice of the size of boxes. Also we have

$$\begin{aligned}
&E\Big((M_n^1(1,k))^2 I_{\{M_n^1(1,k)\geq 2\}}\Big)\\
&\leq E\Big((N(B_1(x_{r_1})))^2 I_{\{N(B_1(x_{r_1}))\geq 2k\}}\Big)\\
&= \sum_{l=2k}^{\infty} l^2 \frac{(\lambda_n[R(m+2k)/n]^d)^l}{l!} \exp(-\lambda_n[R(m+2k)/n]^d)\\
&\leq \Big(\lambda_n[R(m+2k)/n]^d\Big)^{2k} \sum_{l=2k}^{\infty} \frac{l^2}{l!}\\
&\leq \eta_n^{2k} 2e\Big[R(m+2k)\Big]^{2dk}
\end{aligned}$$

for n so large that $\eta_n[R(m+2k)]^d \leq 1$. Thus we have

$$E\Big((M_n^1(1,k))^2 I_{\{M_n^1(1,k)\geq 2\}}\Big) = O(\eta_n^{2k}). \tag{5.44}$$

Also we have

$$\begin{aligned}
&P(M_n^1(1,k) = 1) - P(M_n^1(1,k) = 1, N(B_1(x_{r_1})) = k)\\
&= P(M_n^1(1,k) = 1, N(B_1(x_{r_1})) \geq k+1)\\
&\leq P(N(B_1(x_{r_1})) \geq k+1)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=k+1}^{\infty} \frac{(\lambda_n[R(m+2k)/n]^d)^l}{l!} \exp(-\lambda_n[R(m+2k)/n]^d) \\
&\leq \eta_n^{k+1} \frac{\left[R(m+2k)\right]^{d(k+1)}}{(k+1)!} \\
&= O(\eta_n^{k+1}). \qquad (5.45)
\end{aligned}
$$

Further we have,

$$
\begin{aligned}
&P(M_n^1(1,k) = 1, N(B_1(x_{r_1})) = k) \\
&\quad = P(N(B_1(x_{r_1})) = k) P(M_N^1(1,k) = 1 \mid N(B_1(x_{r_1})) = k) \\
&\quad = \frac{(\lambda_n[R(m+2k)/n]^d)^k}{k!} \exp(-\lambda_n[R(m+2k)/n]^d) \\
&\qquad \int_{B_1(x_{r_1})} \cdots \int_{B_1(x_{r_1})} \frac{P\Big\{(y_1, \ldots, y_k) \text{ is a connected set}\Big\}}{[R(m+2k)/n]^{dk}} dy_1 \ldots y_k.
\end{aligned}
$$

Now we effect a change of scale $a \mapsto na$. This along with the translation invariance of the model will reduce the above integral to

$$
\frac{1}{n^{dk}} \int_{nB_1} \cdots \int_{nB_1} f(y_1, \ldots, y_k) dy_1 \ldots dy_k.
$$

Thus we have

$$
\begin{aligned}
&P(M_n^1(1,k) = 1, N(B_1(x_{r_1})) = k) \\
&\quad = \frac{\eta_n^k}{k!} \int_{nB_1} \cdots \int_{nB_1} f(y_1, \ldots, y_k) dy_1 \ldots dy_k \\
&\qquad - \frac{\eta_n^k}{k!} (1 - \exp(-\eta_n[R(m+2k)]^d) \\
&\qquad \times \int_{nB_1} \cdots \int_{nB_1} f(y_1, \ldots, y_k) dy_1 \ldots dy_k \\
&\quad = b\eta_n^k + O(\eta_n^{k+1}) \qquad (5.46)
\end{aligned}
$$

where $b = \frac{1}{k!} \int_{nB_1} \cdots \int_{nB_1} f(y_1, \ldots, y_k) dy_1 \ldots dy_k$.

Combining (5.45) and (5.46), we obtain,

$$
P(M_n^1(1,k) = 1) = b\eta_n^k + O(\eta_n^{k+1}). \qquad (5.47)
$$

Thus we obtain from (5.44) and (5.45),

$$\begin{aligned}
&E(M_n^1(1,k)) \\
&= P(M_n^1(1,k)=1) + \sum_{l=2}^{\infty} lP(M_n^1(1,k)=l) \\
&\le P(M_n^1(1,k)=1) + E((M_n^1(1,k))^2 I_{\{M_n^1(1,k)\ge 2\}}) \\
&= b\eta_n^k + O(\eta_n^{k+1}). \qquad (5.48)
\end{aligned}$$

Similarly, we have,

$$\begin{aligned}
&E[M_n^1(1,k)]^2 \\
&= P(M_n^1(1,k)=1) + \sum_{l=2}^{\infty} l^2 P(M_n^1(1,k)=l) \\
&\le P(M_n^1(1,k)=1) + E((M_n^1(1,k))^2 I_{\{M_n^1(1,k)\ge 2\}}) \\
&= b\eta_n^k + O(\eta_n^{k+1}). \qquad (5.49)
\end{aligned}$$

From (5.48) and (5.49), we have,

$$\mathrm{var}(M_n^1(1,k)) = b\eta_n^k + O(\eta_n^{k+1}). \qquad (5.50)$$

Further we have, for any sequence $x(n)$ diverging towards infinity as $n \to \infty$,

$$\begin{aligned}
&E\Big(\Big[M_n^1(1,k) - E(M_n^1(1,k))\Big]^2 I_{\{|M_n^1(1,k)-E(M_n^1(1,k))|\ge x(n)\}}\Big) \\
&\le E\Big((M_n^1(1,k))^2 I_{\{M_n^1(1,k)\ge 2\}}\Big) + \Big(E(M_n^1(1,k))\Big)^2 \\
&= O(\eta_n^{2k}). \qquad (5.51)
\end{aligned}$$

By translation invariance,

$$f(y_1, y_2, \dots, y_k) = f(0, y_2 - y_1, \dots, y_k - y_1).$$

Thus we have,

$$\begin{aligned}
&\int_{nB_1} \cdots \int_{nB_1} f(y_1, \dots, y_k) dy_1 \dots dy_k \\
&\sim \int_{nB_1} \int_{B_{kR}} \cdots \int_{B_{kR}} f(0, y_2, \dots, y_k) dy_1 \dots dy_k \\
&= [R(m+2k)]^d \frac{\mu(k)k!}{\ell(K)},
\end{aligned}$$

where $a \sim b$ implies that $a/b \to 1$ as $n \to \infty$. Thus

$$b \sim \frac{[R(m+2k)]^d \mu(k)}{\ell(K)}. \tag{5.52}$$

Also, we have

$$\begin{aligned} \alpha_n &\sim \frac{\ell(K)}{[R(m+2k)/n]^d} \\ &= \lambda_n \eta_n^{-1} \frac{\ell(K)}{[R(m+2k)]^d} \end{aligned} \tag{5.53}$$

Now we are ready to prove the theorem. Let us first consider the case when $\lambda_n \eta_n^{k-1} \to a < \infty$ as $n \to \infty$. Note that,

$$\begin{aligned} &\sum_{i=1}^{\alpha_n} P(M_n^i(i,k) \geq 2) \\ &\leq \sum_{i=1}^{\alpha_n} E\Big([M_n^i(i,k)]^2 I_{\{M_n^i(1,k) \geq 2\}}\Big) \\ &= \lambda_n \eta_n^{-1} \frac{\ell(K)}{[R(m+2k)]^d} O(\eta_n^{2k}) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Also, we have from (5.52) and (5.53),

$$\begin{aligned} &\sum_{i=1}^{\alpha_n} P(M_n^i(i,k) = 1) \\ &= \alpha_n \eta_n^k b + \alpha_n O(\eta_n^{k+1}) \\ &= \lambda_n \eta_n^{-1} \frac{\ell(K)}{[R(m+2k)]^d} b + O(\lambda_n \eta_n^k) \\ &\to a\mu(k) \quad \text{as } n \to \infty. \end{aligned}$$

Thus, we obtain that $M_n(1,k) = \sum_{i=1}^{\alpha_n} M_n^i(1,k)$ has an asymptotic Poisson distribution with mean $a\mu(k)$ as $n \to \infty$.

Next we show that $M_n(2,k)$ and $M_n(3,k)$ are very small compared to $M_n(1,k)$. First we note from (5.39) and Theorem 5.3 that for $i = 2, 3$

$$\begin{aligned} E(M_n(i,k)) &= \lambda_n k^{-1} p(k) \ell(A_i) \\ &= \lambda_n k^{-1} C_1 \frac{\eta_n^{k-1}}{(k-1)!} \ell(A_i) + \lambda_n o(\eta_n^{k-1}) \end{aligned} \tag{5.54}$$

as $n \to \infty$ and $c_1 = \int_{R^d} \dots \int_{R^d} f(\mathbf{0}, y_1, \dots, y_{k-1}) dy_1 \dots dy_{k-1}$.

Clearly as $n \to \infty, \ell(A_3) \to 0$. Hence we have,

$$E(M_n(3,k)) \to 0 \quad \text{as } n \to \infty. \tag{5.55}$$

For A_2, suppose that K is contained in a big d-dimensional box of length L. Thus total number of disjoint boxes of size $R(m+2k)/n$ which can be placed in this square is at most $2(nL)^d/[R(m+2k)]^d$. Each such box will contribute a maximum volume of $2d(Rm/n)^{d-1}2kR/n$ to $\ell(A_2)$. Thus we have

$$\begin{aligned}\ell(A_2) &\leq \frac{2L^d n^d}{[R(m+2k)]^d} \times \frac{2dR^d m^{d-1}2k}{n^d} \\ &\leq \frac{C_2}{m},\end{aligned}$$

where C_2 is a constant independent of n, m. Thus we have

$$\lim_{m\to\infty} \limsup_{n\to\infty} E(M_n(2,k)) = 0. \tag{5.56}$$

This proves the theorem for the Poisson case.

Now suppose $\lambda_n \eta_n^{k-1} \to \infty$ as $n \to \infty$. By independence of $M_n^i(1,k)$ for $i = 1, 2, \dots, k_n$ and using (5.53) and (5.50), we obtain,

$$\begin{aligned}\text{var}(M_n(1,k)) &= \sum_{i=1}^{\alpha_n} \text{var}(M_n^i(1,k)) \\ &= b\alpha_n \eta_n^k + \alpha_n O(\eta_n^k) \\ &= \lambda_n \eta_n^{k-1}\mu(k) + o(\lambda_n \eta_n^{k-1}).\end{aligned} \tag{5.57}$$

Let $\epsilon > 0$ and $x(n) = \epsilon[\text{var}(M_n(1,k))]^{1/2}$. Then using (5.51), we have

$$\begin{aligned}&\frac{\sum_{i=1}^{\alpha_n} E\Big((M_n^1(1,k) - E(M_n^1(1,k)))^2 I_{\{|M_n^1(1,k) - E(M_n^1(1,k))| \geq x(n)\}}\Big)}{[\text{var}(M_n(1,k))]^{-1}} \\ &= [\lambda_n \eta_n^{k-1}\mu(k) + o(\lambda_n \eta_n^{k-1})]^{-1} \alpha_n O(\eta_n^{2k}) \\ &= O(\eta_n^k) \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence the random variables $\{M_n^i(1,k)\}_{i=1,\ldots,\alpha_n}$ satisfies the conditions for Lindeberg's central limit theorem to hold (see, for example, Chow and Teicher [1978], page 291). Thus, applying Lindeberg's central limit theorem we conclude that, as $n \to \infty$,

$$\text{var}(M_n(1,k))^{-1/2}\Big[M_n(1,k) - E(M_n(1,k))\Big] \Rightarrow Z \tag{5.58}$$

where Z has a $N(0,1)$ distribution.

Now we show that the random variables $M_n(2,k)$ and $M_n(3,k)$ are asymptotically negligible compared to $M_n(1,k)$. First we partition the whole space $\mathbb{R}^d$ into 2^d disjoint sets where each of those sets consists of boxes of size $(\frac{R(m+2k)}{n})$ and two boxes are at a distance of $(\frac{R(m+2k)}{n})$. For fixed i and $j, i = 2,3$ and $j = 1,2,\ldots,2^d$, let A_{ijl} be the region of A_i which is in the l-th box of the j-th partition. Thus the sets $A_i, i = 2,3$ can be written as union of disjoint sets A_{ijl} where j runs over $1,2,\ldots 2^d$ and l runs over all boxes $B_1(x)$ for which $C(x)$ is either contained in K (in case of $i = 2$) or the box $B_1(x)$ is not entirely contained in K but have *non-empty intersection* with K. Let $M_n(ijl)$ be the be number of finite clusters which have their centres inside A_{ijl}. Thus we have

$$M_n(i,k) = \sum_{j=1}^{2^d}\sum_{l=1}^{n_{ij}} M_n(ijl), \tag{5.59}$$

for $i = 2,3$. Note here that for fixed i and j and for $l_1 \neq l_2$, the sets A_{ijl_1} and A_{ijl_2} have a distance more than $2kR/n$ between each other. Hence the random variables $\Big\{M_n(ijl)\Big\}_{l\geq 1}$ are independent for fixed i and j. The total number of terms in the double sum is $O(n^d)$. Now, using the independence of the random variables $M_n(ijl)$ for fixed i and j, we have

$$\begin{aligned}\text{var}(M_n(i,k)) &\leq 2^{2d}\sum_{j=1}^{2^d}\sum_{l=1}^{n_{ij}} \text{var}(M_n(ijl)) \\ &\leq 2^{2d}\sum_{j=1}^{2^d}\sum_{l=1}^{n_{ij}} E(M_n^2(ijl)).\end{aligned} \tag{5.60}$$

Now we have,

$$E(M_n^2(ijl)) \leq E(M_n(ijl)) + E\Big\{M_n^2(ijl)I_{\{M_n(ijl)\geq 2\}}\Big\}. \tag{5.61}$$

Since A_{ijl} is always a part of a box of size $R(m+2k)/n$, we can imitate the argument of (5.44) to conclude that

$$\sup_{i,j,l} E\Big\{M_n^2(ijl)I_{\{M_n(ijl)\geq 2\}}\Big\} = O(\eta_n^{2k}), \tag{5.62}$$

as $n \to \infty$. Combining (5.60) (5.61) and (5.62), we obtain,

$$\begin{aligned} \mathrm{var}(M_n(i,k)) &\leq 2^{2d}\sum_{j=1}^{2^d}\sum_{l=1}^{n_{ij}} E(M_n(ijl)) + C_3\sum_{j=1}^{2^d}\sum_{l=1}^{n_{ij}}\eta_n^{2k} \\ &= 2^{2d}E(M_n(i,k)) + O(n^d\eta_n^{2k}). \end{aligned} \tag{5.63}$$

Now from (5.57) , (5.54) and (5.63), we obtain for $i = 2, 3$,

$$\frac{\mathrm{var}(M_n(i,k))}{\mathrm{var}(M_n(1,k))} \leq C_1 2^{2d}\ell(A_i)/(k\mu(k)) + o(1). \tag{5.64}$$

Thus as in case (i), we have for $i = 2, 3$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{\mathrm{var}(M_n(i,k))}{\mathrm{var}(M_n(1,k))} = 0. \tag{5.65}$$

This along with (5.58) proves the theorem. ∎.

Note the asymptotic variance of the distribution in both the cases is $\lambda_n\eta_n^{k-1}\mu(k)$. For a Poisson random variable X with parameter μ, then

$$\mu^{-1/2}(X-\mu) \Rightarrow Z \text{ as } \mu \to \infty$$

where Z is a normal random variable with mean 0 and variance 1. Thus in the case (ii) of Theorem 5.4, we can say that the Poisson limit becomes a normal as the mean of the Poisson random variables $\lambda_n\eta_n^{k-1}\mu(k) \to \infty$.

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