Quantum Stochastic Dilation of Completely Positive Semigroups and Flows

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To

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Contents

Notations

$C_c(X)$	Space of all continuous functions on a locally compact Hausdorff topological space X having compact support
$C_0(X)$	C^*-algebra of all continuous functions on X vanishing at infinity
$C_b(X)$	C^*-algebra of all bounded continuous functions on X
$C(X)$	The space of all continuous functions on a compact Hausdorff topological space X
$\mathcal{H}, \mathcal{K}$ etc.	Hilbert spaces
$\mathcal{A}, \mathcal{B}$ etc.	C^* or von Neumann algebras
$M(\mathcal{A})$	Multiplier algebra of $\mathcal{A}$
$a \,\eta\, \mathcal{A}$	The element a is affiliated to the von Neumann algebra $\mathcal{A}$
$\mathcal{B}(\mathcal{H})$	C^* or von Neumann algebra of all bounded operators on $\mathcal{H}$
$\mathcal{B}_0(\mathcal{H})$	C^*-algebra of all compact operators on $\mathcal{H}$
S, T etc.	Operators on Hilbert spaces
$\mathrm{Dom}\, T, \mathcal{D}(T)$	Domain of the operator T
$\sigma(T)$	Spectrum of the operator T
$u, v,$ etc.	Elements of a Hilbert space
$\|u\rangle\langle v\|$	The operator $w \mapsto \langle v, w\rangle u$
$\mathcal{M}_n$	The set of $n \times n$ complex matrices
spV	The complex linear span of a subset V of a vector space.

Introduction

The central theme of the present thesis is quantum stochastic dilation of semigroups of completely positive maps on operator algebras. It is the aim of all mathematical, or even all scientific theories, to understand a given class of objects through a canonical and simpler subclass of it. For example, abstract C^*-algebras are studied through their concrete realisation as algebra of operators, contractions on a Hilbert space by unitaries, Hilbert modules by the factorizable ones, to mention only a few. In most of these cases, a general object of the relevant class is associated with a canonical candidate of the simpler subclass, in which the former is "embedded in some natural way" and obtained back by some canonical operation like restriction or projection. Such an association of larger objects having simpler structure is known as dilation. Typical examples include the Sz Nagy's unitary dilation of contarctions and the Stinespring's dilation of completely positive maps. On the other hand, in many physical theories. a dilation corresponds to viewing a physical phenomenon in an enlarged system containing the original system as a subsystem. Let us now restrict ourselves to physical models which have some relevance to the marhematical theories developed in the thesis. It is customary to model the dynamics of a conservative physical system by an appropriate *Hamiltonian mechanism* described by a group of unitaries (in the Hilbert space framework) or more abstractly a group of automorphisms (in the operator algebra framework), representing the reversible time evolution of the system. However, in many real physical systems, the evolution is irreversible and this is attributed to the interaction with the environment or the so-called heat-bath. The evolution, when seen in the bigger system consisting of the original one as well as the the environment, will be reversible but due to our inability to observe or lack of interest in the dynamics of the total system the phenomenon of irreversibility or dissipativity in the system takes place.

In the context of dissipative systems arising in classical mechanics, it is often reasonable to model the environment by the space of a suitable Markov process (typically Brownian motion) so that the behaviour of the total system is described by a *stochastic differential flow equation* and the evolution within the original subsystem at a given time is obtained by taking conditional expectation with respect to the filtration of the above stochastic process upto that time point. This may be physically interpreted as washing out environmental noises to recover the original evolution, by the positive (or Markov) semigroup associated with the process carrying the noise. This is the case of classical stochastic dilation.

The same idea extends to quantum mechanical systems, but there are considerable conceptual and technical difficulties, some of which are addressed and partially solved in this thesis. The time evolution of an irreversible quantum mechanical system can be represented by a semigroup of linear maps acting on an appropriate operator algebra. In classical mechanics, this algebra is taken to be some suitable commutative algebra of functions on the state space, which is naturally replaced by more general, possibly noncommutative operator algebra, in accordance with the philosophy of quantum theory. However, the rôle of positivity is taken over by a stronger version, namely complete positivity, which is justifiable from some physical intuition if one demands a mild consistency of the dynamical theory with respect to composition of independent systems. Since the model for a reversible dynamics will be given by suitable families of $*$-automorphisms, or at least invertible $*$-homomorphisms of the algebra of observables, the problem of dilation mathematically translates as follows :

given a semigroup T_t of completely positive maps on an operator algebra $\mathcal{A}$, can one construct a semigroup of $1-1$ $*$-homomorphisms η_t acting on a bigger algebra $\mathcal{B}$ equipped with a conditional expectation E from $\mathcal{B}$ to $\mathcal{A}$, such that $T_t = E \circ \eta_t$?

To pose it in a more rigorous manner one puts various topological properties on both T_t and $\mathcal{A}$. If one intends to follow the classical path, it is natural to look for a quantum analogue of classical probability theory and then obtain a time-indexed family j_t of $*$-homomorphisms of $\mathcal{A}$ into a bigger albebra modelling the total system consisting of the original system and some "quantum noise" so that j_t will satisfy a suitable differential equation using which η_t may be constructed.

A natural candidate for modelling noise or heat-bath in quantum theory is the

Fock space over $L^2(\mathbb{R}_+)\otimes k_0$ for some Hilbert space k_0, called the multiplicity space. The choice between various kinds of Fock spaces (i.e. symmetric, antisymmetric and free) is guided by the underlying physical model. However, we shall confine ourselves to the case of symmetric Fock space, a choice motivated by the classical theories as well. With a well developed theory of quantum stochastic calculus ([26], [45]) which was achieved by the pioneering works of Hudson and Parthasarathy and followed by a number of authors, a notion of quantum stochastic differential flow was formulated by Evans and Hudson ([15], [18]) and subsequently studied by many authors ([18], [39] etc.). In this formulation an Evans-Hudson (E-H) flow is a family j_t of $*$-homomorphisms form the initial observable algebra $\mathcal{A}$ to a bigger algebra of the form $\mathcal{A}\otimes\mathcal{B}(\Gamma(L^2(\mathbb{R}_+,k_0)))$ for some suitable multiplicity space k_0, satisfying a quantum stochastic differntial equation $dj_t(x) = j_t(\theta^\alpha_\beta(x))d\Lambda^\beta_\alpha(t)$ with $j_0 = id$, with respect to the canonical quantum stochastic differentials $d\Lambda^\beta_\alpha(t)$ defined in [45], [26] etc. The maps θ^α_β, called the structure maps, are from $\mathcal{A}$ (or a dense subalgebra of it) to itself where θ^0_0 is the generator of the semigroup corresponding to the given dynamical system. However, one may encounter all sorts of technical obstacles, arising from possible unboundedness of the structure maps or the possible infinite dimension of the multiplicity space. There is a considerable amount of literature related to the existence, uniqueness and characterization of E-H flows under various analytic conditions. But the question which seems to be more important both from physical and mathematical points of view, namely how to construct a canonical E-H flow starting from a given semigroup, was not answered except for the relatively simple case of a uniformly continuous semigroup on the full algebra $\mathcal{B}(h)$ for some Hilbert space h ([28]). One of the main achievements of the present work is the complete solution of the above problem for any arbitrary normal uniformly continuous completely positive semigroup on an arbitrary von Neumann algebra as well as any arbitrary uniformly continuous completely positive semigroup on a separable unital C^* algebra. It may be noted here that another general dilation theory was developed by [5] where given any completely positive strongly continuous semigroup it is possible to obtain a time-indexed family j_t of $*$-homomorphisms of $\mathcal{A}$ into a bigger algebra, but in this case there is no further structure of the bigger algebra, and thus there is no differential equation enjoyed by j_t. This theory is too abstract and cannot give as much information about the given semigroup as can be given by an E-H type dilation. However, we have shown that a canonical construction of Bhat-Parthasarathy

Bhat-Parthasarathy (B-P) type dilation is always possible in the framework of Fock space for semigroups to which our theory applies.

Not only we perform the general algebraic construction of canonical structure maps to obtain an E-H flow, we also devise a new and elegant language for describing quantum stochastic differential equations both at the level of operator processes as well as the map-valued processes. This is done by introducing appropriate Hilbert modules and module valued processes. We reformulate all the existing theory and also extend its scope by accomodating any nonseparable initial or multiplicity space without any artifice or extra effort.

Next, we extend our ideas to more general completely positive flows. Typically, such flows can arise by partial washing out of environmental noises and we prove that under suitable boundedness condition this is the only way it arises. Mathematically, the result tells that every completely positive contractive flow admits an E-H type dilation ([22]).

Towards the end of the thesis we consider some class of strongly continuous (not uniformly continuous) dynamical semigroups for which the generator is given only as a form and we carry out the construction of the minimal semigroup along the line of [12], [31], [20] etc. with some applications.

Let us summarise the major new results contained in the thesis (references mentioned in the paranthesis):

1. Development of a module-based approach to quantum stochastic calculus. [25]
2. An extension of existing results to nonseparable spaces. [25]
3. Construction of an E-H dilation for an arbitrary uniformly continuous quantum dynamical semigroup on a von Neumann algebra. [25]
4. A transparent proof of the homomorphism property of the above flow. [25]
5. Implementation of the above flow by a partial-isometry-valued process. [25]
6. Identification of a B-P type dilation in the same Fock space where the above E-H flow is constructed. [25]
7. An extension of the result 3 to the case of a separable unital C^* algebra. [23]
8. Proof of the result that every completely positive contractive flow admits an E-H dilation. [22]
9. Construction of the minimal semigroup on a von Neumann algebra starting from a given formal unbounded generator. [24]

Let us conclude with some relevant remarks. The emphasis of the entire work is given on the semigroup out of which everything else has been constructed. Some motivation for this point of view was in [46], though in a rudimentary form. Furthermore, the achievability of a canonical dilation in the Fock space kindles hope for an interplay between noncommutative geometry and quantum probability, although to reach that ambitious target it would be necessary to extend our theories to semigroups with unbounded generator arising in any typical geometrical set-up.

Plan of the thesis :

The materials of the thesis are organised as follows :

Chapter 1.

In the first section we intruduce all the basic technical materials needed for our work. We assume the reader's familiarity with basics of functional analysis, in particular the theory of bounded and unbounded linear operators on Hilbert spaces. All the main results and concepts from theory of operator algebras, including the structure of normal representation of a von Neumann algebra, predual of a von Neumann algebra, universal enveloping von Neumann algebra etc. . are mentioned without proofs. The section 2 is devoted to a survey of basic results on completely positive maps and semigroups, encompassing Stinespring's theorem. The Hille-Yosida theorem for semigroups on locally convex spaces and a theorem on convergence of semigroups are stated without proofs. Finally the characterization of the generator of a uniformly continuous completely positive semigroup due to Christensen and Evans ([8]) is given. In section 3 of the chapter a brief account of Hilbert C^* and von Neumann modules is presented, including the statements of KSGNS and Kasparov's theorems, which is followed by a survey of basic facts about Fock spaces and Weyl algebras in the last section.

Chapter 2.

In this chapter we develop a coordinate free theory of quantum stochastic calculus, which works nicely for any arbitrary dimensional initial or multiplicity space. In first section, we define quantum stochastic processes $a_R(.), a_S^\dagger(.), \Lambda_T(.)$ for $R, S \in \mathcal{B}(h, h \otimes k_0)$ and $T \in \mathcal{B}(h \otimes k_0)$, where h and k_0 denote the initial and the multiplicity space respectively. Thus we allow an interplay between the initial and multiplicity spaces, leading to the presentation of quantum Ito formulae in a nice algebraic language, for example :

$$a_R(dt)(x \otimes I_{h\otimes\Gamma})a_S^\dagger(dt) = R^*(x \otimes I_{k_0})Sdt,$$

where $\Gamma = \Gamma(L^2(\mathbb{R}_+, k_0))$.

In the second section of this chapter, we extend the theory of quantum stochastic calculus to map-valued processes in von Neumann Fock modules of the form $\mathcal{A} \otimes \Gamma$.

Given $\delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes k_0), \sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0)), \mathcal{L} \in \mathcal{B}(\mathcal{A})$, we introduce map-valued processes of the forms $a_\delta, a_\delta^\dagger, \Lambda_\sigma$ and $\mathcal{I}_\mathcal{L}$ on the elements of the form $x \otimes e(f)$ where $x \in \mathcal{A}, f \in L^2(I\!R_+, k_0)$. the definitions of these processes are given naturally in terms of the analogous processes in $h \otimes \Gamma$, for example, we set

$$a_\delta^\dagger(.)(x \otimes e(f))u \equiv a_{\delta(x)}^\dagger(.)(ue(f)), \quad u \in h.$$

We obtain Ito formulae for such map-valued processes also.

Chapter 3.

Here we take up the problem of solving a class of quantum stochastic differential equations (q.s.d.e.) with bounded coefficients and proving various characterizations of the solutions. In the first section we study Hudson-Parthasarathy type q.s.d.e. of the forms

$$dU_t = U_t(a_R^\dagger(dt) + a_S(dt) + \Lambda_T(dt) + Kdt),$$

and

$$dV_t = (a_R^\dagger(dt) + a_S(dt) + \Lambda_T(dt) + Kdt)V_t$$

with prescribed initial values. We also prove the conditions for contractivity, isometry and unitarity of their solutions. This is followed in the second section by an account of q.s.d.e. of Evans-Hudson type, cast in the coordinate-free language. We establish here the existence and uniqueness of a map-valued q.s.d.e. of the form

$$dJ_t = J_t \circ (a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_\mathcal{L})dt,$$

with $J_0 \equiv id$, where $(\mathcal{L}, \delta, \sigma)$ is a triple of structure maps, that is, $\mathcal{L} \in \mathcal{B}(\mathcal{A}), \delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes k_0), \sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0))$ for some Hilbert space k_0 and satisfying some algebraic relations among them. We also construct a canonical $*$-homomorphism $j_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(I\!R_+), k_0))$ defined in terms of J_t by,

$$j_t(x)(ue(f)) = J_t(x \otimes e(f))u.$$

The proof that j_t is a $*$-homomorphism becomes quite transparent in our module language in contrast to the proof of similar result given by Mohari and Sinha ([42], [39]) in the case of countable infinite multiplicity. Actually, it is seen that the original proof of homomorphism property given in case of one dimensional multiplicity space ([18]) extends verbatim if we choose the right language.

Chapter 4.

We start with an arbitrary uniformly continuous, completely positive, normal semigroup T_t on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(h)$ and recall the canonical structure of the generator $\mathcal{L}$ given by [8]. First we indicate the construction of Hudson-Parthasarathy type dilation, namely a unitary operator-valued process U_t in $h \otimes \Gamma$ for an appropriate Fock space Γ, such that U_t satisfies an appropriate q.s.d.e. and the vacuum expectation of $U_t(x \otimes 1)U_t^*$ gives $T_t(x)$ for all x. However, we explain that such a dilation need not be an Evans-Hudson dilation in general, if the algebra is not $\mathcal{B}(h)$. For a general algebra, we construct a canonical $*$- homomorphism of $\mathcal{A}'$ (commutant of $\mathcal{A}$) and applying the structure theorem for normal $*$-homomorphism of von Neumann algebras we obtain a Hilbert space k_0 and structure maps $(\mathcal{L}, \delta, \sigma)$ by a suitable "rotation " of the set-up given by [8]. Thus, we prove that there exists an Evans-Hudson dilation for T_t. Furthermore, we prove that it is possible to implement the Evans-Hudson dilation j_t constructed by us by a partial-isometry-valued process (not necessarily unitary) V_t, such that the projection on the initial space of V_t belongs to $\mathcal{A}' \otimes \mathcal{B}(\Gamma(L^2([0,t] \otimes k_0)))$ and that on the final space of V_t belongs to $\mathcal{A} \otimes \mathcal{B}(\Gamma(L^2([0,t] \otimes k_0)))$. We also identify a weak Markov flow (in the sense of [5]) constructed from j_t in a canonical manner, such that its filtration is subordinate to that of the Fock filtration. In the last section of this chapter we prove the existence of Evans-Hudson dilation for an arbitrary uniformly continuous completely positive semigroup on a unital separable C^* algebra, using the theory of Hilbert C^* modules.

Chapter 5

In this chapter we generalize the results of chapter 4 to completely positive contractive flows. By such a flow, we mean a time-indexed family $\eta_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_0)))$ for some multiplicity space k_0, such that η_t satisfies a flow equation of the form $d\eta_t(x) = \eta_t(\theta^\alpha_\beta(x))d\Lambda^\beta_\alpha(t)$ and each η_t is completely positive and contractive. We prove that any such η_t can be realized as $\eta_t = E_0 \circ j_t$ for some $*$-homomorphic flow $j_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_0 \oplus k_1)))$ where k_1 is a Hilbert space and E_0 denotes the conditional expectation from $\mathcal{B}(h \otimes \Gamma(L^2(\mathbb{R}_+, k_0 \oplus k_1)))$ to $\mathcal{B}(h \otimes \Gamma(L^2(\mathbb{R}_+, k_0)))$ which " washes out " noises associated with k_1. To obtain this characterization of completely positive contractive flows, we combine the results on the structure of such flows obtained by [35] and [36], with the techniques developed in [25].

Chapter 6

In this final chapter we take up the study of a class of semigroups with unbounded generator. First we concentrate on the problem of constructing the minimal semigroup starting from a formal unbounded generator and generalize the techniques of Davies, Kato et al ([12], [31]) for an arbitrary von Neumann algebra. Under suitable hypotheses on the formal generator, we prove the existence of minimal semigroup and obtain results on its conservativity. We also apply our theory to a large class of classical Markov semigroups as well as some canonical semigroup on a type II_1 von Neumann algebra.

Chapter 1

Preliminaries

In this chapter we shall introduce all the basic materials needed for this thesis.

1.1 C^* and von Neumann algebras :

For the material of this section, the reader may be referred to [51], [14] and [30].

1.1.1 C^* algebras

An abstract normed $*$-algebra $\mathcal{A}$ is said to be a pre-C^* algebra if it satisfies the C^* property : $\|x^*x\| = \|x\|^2$. If $\mathcal{A}$ is furthermore complete under the norm topology, one says that $\mathcal{A}$ is a C^* algebra. The famous structure theorem due to Gelfand, Naimark and Segal (GNS) asserts that every abstract C^* algebra can be imbedded as a norm-closed $*$-subalgebra of $\mathcal{B}(h)$ (the set of all bounded linear operators on some Hilbert space h). In view of this, we shall fix a complex Hilbert space h and consider a concrete C^* algebra $\mathcal{A}$ inside $\mathcal{B}(h)$. The algebra $\mathcal{A}$ is said to be unital or non-unital depending on whether it has an identity or not.

We briefly discuss some of the important aspects of C^* algebra theory. First of all, let us mention the following remarkable characterization of commutative C^* algebras :

Theorem 1.1.1 *Every commutative C^* algebra $\mathcal{A}$ is isometrically isomorphic to the C^* algebra $C_0(X)$ consisting of complex valued functions on a locally compact Hausdorff space X vanishing at infinity. In case $\mathcal{A}$ is unital, X is compact.*

If $\mathcal{A}$ is non-unital. there is a canonical method of adjoining an identity so that $\mathcal{A}$ is imbedded as an ideal in a bigger unital C^* algebra $\tilde{\mathcal{A}}$. In view of this. let us assume $\mathcal{A}$ to be unital for the rest of the subsection. For $x \in \mathcal{A}$. the spectrum of x. denoted by $\sigma(x)$. is defined as the complement of the set $\{z \in \mathbb{C} : (z1-x)^{-1} \in \mathcal{A}\}$. It is known that for a self-adjoint element x, $\sigma(x) \subseteq \mathbb{R}$, and moreover. a self-adjoint element x is positive (that is. x is of the form y^*y for some y) if and only if $\sigma(x) \subseteq [0, \infty)$. There is a rich functional calculus which enables one to form functions of elements of the C^* algebra. For any complex function which is holomorphic in some domain containing $\sigma(x)$. one obtains an element $f(x) \in \mathcal{A}$ by the holomorphic functional calculus. Furthermore. for any normal element x. there is a continuous functional calculus sending $f \in C(\sigma(x))$ to $f(x) \in \mathcal{A}$ where $f \mapsto f(x)$ is a $*$-isometric isomorphism from $C(\sigma(x))$ onto $C^*(x)$, the sub C^*-algebra of $\mathcal{A}$ generated by x. In particular. for any positive element x, we can form a positive square root $\sqrt{x} \in \mathcal{A}$ satisfying $\sqrt{x}^2 = x$. For any element $x \in \mathcal{A}$, we define its absolute value. denoted by $|x|$, to be the element $\sqrt{x^*x}$. For a self-adjoint element x, we define two positive elements x^+ and x^-. called respectively the positive and negative parts of x. by setting $x^+ = \frac{x+|x|}{2}$, $x^- = \frac{|x|-x}{2}$. Clearly. x can be decomposed as $x = x^+ - x^-$ and furthermore $x^+x^- = 0$.

A linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is said to be positive if $\phi(x^*x) \geq 0$ for all x. It can be shown that the algebraic property of positivity implies the boundedness of ϕ, in particular $\|\phi\| = \phi(1)$. Any positive linear functional ϕ with $\phi(1) = 1$ is called a state on $\mathcal{A}$. It is said to be faithful if $\phi(x^*x) = 0$ implies $x = 0$. It is clear that the set of all states on $\mathcal{A}$ is convex. and it is compact in the weak-$*$ topology of $\mathcal{A}^*$. The extreme points of this compact convex set are called pure states. The following theorem. known as the GNS construction for a state. is worthy of mention :

Theorem 1.1.2 *Given a state ϕ on $\mathcal{A}$, there exists a triple (called the GNS triple) $(\mathcal{H}_\phi, \pi_\xi, \psi_\phi)$, consisting of a Hilbert space $\mathcal{H}_\phi$, a $*$-representation π_ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\phi)$ and a vector $\xi_\phi \in \mathcal{H}_\phi$ which is cyclic in the sense that $\{\pi_\phi(x)\xi_\phi : x \in \mathcal{A}\}$ is total in $\mathcal{H}_\phi$, satisfying*

$$\phi(x) = \langle \xi_\phi, \pi_\phi(x)\xi_\phi \rangle.$$

Moreover, ϕ is pure if and only if π_ϕ is irreducible.

1.1.2 von Neumann algebras

As a Banach space, $\mathcal{B}(h)$ is equipped with the operator-norm topology, but there are other important and interesting topologies that can be given to it, making it a locally convex (but not normable in general) topological space. The most useful ones are weak, strong, ultra-weak and ultra-strong topologies. However, although $\mathcal{B}(h)$ is complete in each of these topologies, a general C^* subalgebra $\mathcal{A}$ of $\mathcal{B}(h)$ need not be so. It is easily provable that $\mathcal{A}$ is complete in all of the above four locally convex topologies if and only if it is complete in any one of them, and in such a case $\mathcal{A}$ is said to be a von Neumann algebra. Furthermore, the strong (respectively, weak) and ultra-strong (respectively, ultra-weak) topologies coincide on norm-bounded convex subsets of $\mathcal{A}$. It is known that if h is separable, then any norm-bounded subset of $\mathcal{A}$ is metrizable in each of the ultra-weak and ultra-strong topologies. The following theorem, known as the *Double commutant theorem* due to von Neumann is of fundamental importance in the study of von Neumann algebras. Note that for any subset $\mathcal{B}$ of $\mathcal{B}(h)$, we denote by $\mathcal{B}'$ the commutant of $\mathcal{B}$, i.e. $\{x \in \mathcal{B}(h) : xb = bx \ \forall \ b \in \mathcal{B}\}$.

Theorem 1.1.3 *A unital C^* algebra $\mathcal{A}$ is also von Neumann if and only if $\mathcal{A} = \mathcal{A}''(\equiv (\mathcal{A}')')$.*

For the rest of this subsection, let us denote by $\mathcal{A}$ a unital von Neumann subalgebra of $\mathcal{B}(h)$. $\mathcal{A}$ is said to be σ-finite if there does not exist any uncountable family of mutually orthogonal projections in $\mathcal{A}$. We say that $\mathcal{A}$ is a factor if the centre is trivial, i.e. $\mathcal{A} \bigcap \mathcal{A}' = \{\lambda 1, \lambda \in \mathbb{C}\}$. There is a profound classification theory of factors and also decomposability of any von Neumann algebra into factors, but it is not relevant to us.

Any von Neumann algebra has enough projections and unitaries, in the sense that $\mathcal{A}$ is the strong closure of the $*$-subalgebra generated by all projections (respectively unitaries) in $\mathcal{A}$. Furthermore, if $x \in \mathcal{A}$ and $E_x(.)$ denotes the family of spectral measures of x, then $E_x(\Delta) \in \mathcal{A}$ for all Borel set Δ. We remark that this fact is not true for a general C^* algebra which is not von Neumann. A state ϕ on $\mathcal{A}$ is said to be normal if $\phi(x_\alpha)$ increases to $\phi(x)$ whenever $0 \leq x_\alpha \uparrow x$ for a net $\{x_\alpha\} \subseteq \mathcal{A}$. More generally, we call a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ (where $\mathcal{B}$ is a von Neumann algebra) to be positive if it takes positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. Φ is called normal if whenever $0 \leq x_\alpha \uparrow x$ in $\mathcal{A}$, one has $\Phi(x_\alpha) \uparrow \Phi(x)$ in $\mathcal{B}$. It is known that a positive linear map is normal if and only if it is continuous with respect to the ultra-weak topology mentioned earlier. In view of this fact, we shall say that a bounded linear map between two von Neumann algebras is normal if it is continuous with respect to the respective ultra-weak topologies. Normal states, and more generally normal positive linera maps (in particular, normal $*$- homomorphisms) play a major role in the study of von Neumann algebras. The following result describes the structure of a normal state .

Theorem 1.1.4 *ϕ is a normal state on $\mathcal{A}$ if and only if there is a trace-class operator ρ on h such that $\phi(x) = tr(\rho x)$ for all $x \in \mathcal{A}$.*

$\mathcal{A}$ is said to be maximal abelian if both $\mathcal{A}$ and $\mathcal{A}'$ are abelian. It is known that

Theorem 1.1.5 *A maximal abelian von Neumann algebra is isometrically isomorphic with $L^\infty(\Omega, \mathcal{F}, \mu)$ for some measure space $(\Omega, \mathcal{F}, \mu)$.*

In view of this result, the theory of von Neumann algebras can be looked upon as a noncommutative measure or probability theory. Many of the well-known theorems, such as Radon-Nikodym theorem, martingale convergence theorem etc. have their appropriate generalizations in the set-up of von Neumann algebras. However, we shall not go into that direction.

A remarkable property of von Neumann algebras is the beautiful and particularly simple structure of its normal $*$-homomorphisms. This plays a canonical role in a major portion of the present work. There are three basic and natural ways in which a normal $*$-homomorphism π of $\mathcal{A}$ can arise :

(i) *Reduction :* $\pi(x) = PxP$, where P is a projection. It is easily seen that (see Lemma 4.5.2 for a proof) P necessarily belongs to $\mathcal{A}'$.

(ii) *Dilation :* $\pi(x) = x \otimes 1_k$ for some Hilbert space k.

(iii) *Unitary conjugation :* $\pi(x) = \Gamma^* x \Gamma$ where Γ is a unitary in $\mathcal{B}(h)$.

The following theorem asserts that every normal $*$-homomorphisms of $\mathcal{A}$ is a composition of the above three types.

Theorem 1.1.6 *Given a normal $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$, there exists a pair (Γ, k) where k is a Hilbert space and Γ is a partial isometry from $\mathcal{K}$ to $h \otimes k$ such that $\pi(x) = \Gamma^*(x \otimes 1_k)\Gamma$, and the projection $\Gamma\Gamma^*$ commutes with $x \otimes 1_k$ for all $x \in \mathcal{A}$. Moreover, if π is unital, Γ is an isometry. In case h is separable, one can choose k to be separable as well.*

We conclude our brief account on C^* and von Neumann algebras with some discussions on the enveloping von Neumann algebra and the predual of a von Neumann algebra. Given a unital C^* algebra $\mathcal{B}$, denote the set of all *states* by Ω. For $\phi \in \Omega$, we denote by $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ the associated GNS triple. Let $\mathcal{H} = \oplus_{\phi\in\Omega}\mathcal{H}_\phi$ and $\pi = \oplus_{\phi\in\Omega}\pi_\phi$. We call π to be the universal representation of $\mathcal{B}$ and the weak closure of $\pi(\mathcal{B})$, i.e. $\pi(\mathcal{B})''$ in $\mathcal{B}(\mathcal{H})$, is known as the *universal enveloping* von Neumann algebra of $\mathcal{B}$. We denote it by $\tilde{\mathcal{B}}$. Indeed, it has the following universal property.

Theorem 1.1.7 *Given any $*$-homomorphism ρ of $\mathcal{B}$ in some Hilbert space $\mathcal{K}$, there exists a unique normal $*$-homomorphism $\tilde{\rho} : \tilde{\mathcal{B}} \to \mathcal{B}(\mathcal{K})$ such that $\tilde{\rho} \circ \pi = \rho$. Moreover, the image of $\tilde{\rho}(\tilde{\mathcal{B}})$ is the weak closure of $\rho(\mathcal{B})$ in $\mathcal{B}(\mathcal{K})$.*

For a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(h)$, there is a Banach space $\mathcal{A}_*$, called the *predual* of $\mathcal{A}$, such that the Banach dual of $\mathcal{A}_*$ coincides with $\mathcal{A}$ with norm topology, whereas the weak $*$ topology coincides with the ultra weak topology of $\mathcal{A}$. Let us give an explicit description of the predual.

For a real linear space we shall consider as its dual the space of all real linear functionals on it. We denote by $\mathcal{B}_1(h)$ and $\mathcal{B}_2(h)$ the set of all trace-class operators and of all Hilbert-Schmidt operators on h respectively. Let $\mathcal{B}^{s.a.}(h)$ and $\mathcal{B}_1^{s.a.}(h)$ stand for the real linear spaces of all bounded self-adjoint operators and all trace-class self-adjoint operators on h respectively. For a von Neumann algebra $\mathcal{A}$ contained in $\mathcal{B}(h)$, we denote by $\mathcal{A}_h$ the subset of all self-adjoint elements in $\mathcal{A}$, and $\mathcal{A}_{h,*}$ be the predual of $\mathcal{A}_h$. We define an equivalence relation $\sim$ on $\mathcal{B}(\mathcal{H})$ by saying $\rho_1 \sim \rho_2$ if and only if $tr(\rho_1 x) = tr(\rho_2 x)\ \forall\ x \in \mathcal{A}$. We denote by $\mathcal{A}^\perp$ the closed subspace $\{\rho \in \mathcal{B}_1(h) : \rho \sim 0\}$. For $\rho \in \mathcal{B}_1(h)$, we denote by $[\rho]$ its equivalence class with respect to $\sim$, and $|||[\rho]||| = inf_{\eta\sim\rho}\|\eta\|_1$, where $\|.\|_1$ denotes trace-class norm. By $\mathcal{A}_h^\perp$ we shall denote the set of all self-adjoint elements in $\mathcal{A}^\perp$. Clearly, $\mathcal{A}_h^\perp$ is a closed subspace of the real Banach space $\mathcal{B}_1^{s.a.}(h)$ and hence one can consider the quotient space $\mathcal{B}_1^{s.a.}(h)/\mathcal{A}_h^\perp$. For $\rho \in \mathcal{B}_1^{s.a.}(h)$, let us denote by $[\rho]_h$ the equivalence class corresponding to ρ in the above quotient. It is easy to observe that the quotient norm of $[\rho]_h$, say $\|[\rho]_h\|$, coincides with $|||[\rho]|||$ defined earlier. To see this, it is enough to note that whenever $\eta \sim \rho$ and ρ is self-adjoint, then $\eta^* \sim \rho$, and thus $(\eta+\eta^*)/2 \in [\rho]_h$ and $\|(\eta+\eta^*)/2\|_1 \leq \|\eta\|_1$. This implies that $\|[\rho]_h\| \leq |||[\rho]|||$ and hence they are equal.

We now describe the structure of $\mathcal{A}_*$ and $\mathcal{A}_{h,*}$ as follows :

Proposition 1.1.8 *(i) $\mathcal{A}_* \cong \mathcal{B}_1(h)/\mathcal{A}^\perp \cong \Omega_{\mathcal{A}}$, where $\cong$ denotes isometric isomorphism and $\Omega_{\mathcal{A}}$ denotes the space of all normal complex linear bounded functionals on $\mathcal{A}$. The canonical identification between $\mathcal{A}$ and $(\mathcal{B}_1(h)/\mathcal{A}^-)^*$ is given by, $\mathcal{A} \ni x \mapsto \zeta_x \in (\mathcal{B}_1(h)/\mathcal{A}^\perp)^*$ where $\zeta_x([\rho]) \equiv tr(\rho x)$. Moreover, an element $[\rho]$ of $\mathcal{B}_1(h)/\mathcal{A}^\perp$ is canonically associated with $\varphi_{[\rho]}$ in Ω_A where $\varphi_{[\rho]}(x) \equiv tr(\rho x)$, $x \in \mathcal{A}$.*

(ii) $\mathcal{A}_{h,*} \cong \mathcal{B}_1^{s.a.}(h)/\mathcal{A}_h^\perp \cong \Omega_{\mathcal{A}_h}$, where $\Omega_{\mathcal{A}_h}$ denotes the space of all real linear normal bounded functionals on $\mathcal{A}_h$. The identification between $[\rho]_h$ and its counterpart $\varphi_{[\rho]_h}$ (say) in $\Omega_{\mathcal{A}_h}$ is given by, $\varphi_{[\rho]_h} = tr(\rho x)$, $x \in \mathcal{A}_h$.

PROOF : (i) is contained in Proposition 2.4.18 of [6]. We prove (ii) as an easy application of (i). Let us consider $\psi : \mathcal{A}_h \to (\mathcal{B}_1^{s.a.}(h)/\mathcal{A}_h^\perp)^*$ defined by, $\psi(x)([\rho]_h) = tr(\rho x)$, $\rho \in \mathcal{B}_1^{s.a.}(h)/\mathcal{A}_h^\perp$, $x \in \mathcal{A}_h$; which is clearly well-defined, linear and one-to-one. To see the ontoness of ψ, it is enough to note that given $\vartheta \in (\mathcal{B}_1^{s.a.}(h)/\mathcal{A}_h^\perp)^*$, we can extend it to $\hat{\vartheta}$ by defining $\hat{\vartheta}([\rho]) = \vartheta([Re\rho]_h) + i\vartheta([Im\rho]_h)$ and by (i), there is an $x \in \mathcal{A}$ such that $\hat{\vartheta}([\rho]) = tr(\rho x)\ \forall\ \rho$. Thus, $\vartheta([\rho]_h) = \hat{\vartheta}([\rho]) = tr(\rho Re(x)) + i\ tr(\rho Im(x))$ for $\rho \in \mathcal{B}_1^{s.a.}(h)$; and since $\vartheta([\rho]_h)$ is real, we must have that $\vartheta([\rho]_h) = tr(\rho Re(x)) = \psi(Re(x))([\rho]_h)$. Now we observe that for a positive $\rho \in \mathcal{B}_1(h)$, $\varphi_{[\rho]}$ (as defined in the statement of (i)) is a positive linear normal functional on $\mathcal{A}$, and hence $\|[\rho]_h\| = \|[\rho]\| = \|\varphi_{[\rho]}\| = \varphi_{[\rho]}(1) = tr(\rho)$. We have $|\psi(x)([\rho]_h)| \leq \|x\|\|[\rho]_h\|$, hence $\|\psi(x)\| \leq \|x\|$. On the other hand, for any self-adjoint x, $\|x\| = sup_{u \in h, \|u\|=1} | < u, xu > | = sup_{u\in h,\|u\|=1} |tr(\rho_u x)| = sup_{u \in h, \|u\|=1} |\psi(x)([\rho_u]_h)| \leq sup_{u\in h, \|u\|=1} \|\psi(x)\|.\|[\rho_u]_h\| = \|\psi(x)\|$ (where $< ... >$ denotes the inner product in h and ρ_u denotes the rank-one operator $|u >< u|$). Thus, $\|x\| \leq \|\psi(x)\|$ also, proving that $\|x\| = \|\psi(x)\|$.

The assertion that $[\rho]_h \mapsto \varphi_{[\rho]_h}$ is an isometric isomorphism is a straightforward consequence of (i), after noting that any $\alpha \in \Omega_{\mathcal{A}_h}$ can be extended to $\hat{\alpha}$ defined by $\hat{\alpha}(x) = \alpha(Re(x)) + i\ \alpha(Im(x))$ for $x \in \mathcal{A}$ and it is easy to prove that $\|\hat{\alpha}\| = \|\alpha\|$. In fact, $\hat{\varphi}_{[\rho]_h} = \varphi_{[\rho]}$, and $\|[\rho]_h\| = \|[\rho]\|$, which completes the proof by invoking (i).

□

Let us fix some more notational convention which will be useful particularly in chapter 6. For Hilbert spaces h_i and von Neumann algebras $\mathcal{A}_i$ contained in $\mathcal{B}(h_i)$ ($i = 1, 2$), and a linear map $T : \mathcal{B}_1^{s.a.}(h_i) \to \mathcal{B}_1^{s.a.}(h_j)$ ($i, j \in (1, 2)$), we shall say that T induces a map $\tilde{T} : \mathcal{B}_1^{s.a.}(h_i)/\mathcal{A}_{ih}^\perp \to \mathcal{B}_1^{s.a.}(h_j)/\mathcal{A}_{jh}^\perp$ if $T\mathcal{A}_{ih}^\perp \subseteq \mathcal{A}_{jh}^\perp$; and in such a case we define $\tilde{T}$ by $\tilde{T}([\rho]_h) = [T\rho]_h$. Upto canonical identification, $\tilde{T}$ will give rise to a map from $\Omega_{\mathcal{A}_{ih}}$ to $\Omega_{\mathcal{A}_{jh}}$ which we shall denote by the same notation as $\tilde{T}$. For

$[\rho]_h \in \mathcal{B}_1^{s.a.}(h)$ and $x \in \mathcal{A}_h$, $[\rho]_h \mapsto tr([\rho]_h x) \equiv tr(\rho x)$ is a class-map. We say $[\rho]_h$ to be positive if $tr([\rho]_h x) \geq 0$ $\forall$ positive $x \in \mathcal{A}_h$. It has already been noted in the proof of the previous proposition that for positive ρ, $[\rho]_h$ is also positive, and $\|[\rho]_h\|$ $= tr(\rho) = \|\rho\|_1$. Conversely, if $[\rho]_h$ is positive, then the associated functional $\varphi_{[\rho]}$ ($\in \Omega_{\mathcal{A}}$) is positive and hence $\exists$ a positive ρ_0 such that $tr(\rho x) \equiv \varphi_{[\rho]}(x) = tr(\rho_0 x)$ $\forall$ x $\in \mathcal{A}$: which in particular implies $[\rho]_h = [\rho_0]_h$. Thus, $\|[\rho]_h\| = \|[\rho_0]_h\| = tr(\rho_0) = tr(\rho)$. This observation will be useful and let us summarise it as follows :

Lemma 1.1.9 *$[\rho]_h$ is positive if and only if $\exists$ a positive $\rho_0 \in [\rho]_h$; and in such a case, $\|[\rho]_h\| = tr(\rho) = tr(\rho_0)$.*

We also have.

Lemma 1.1.10 *For two positive elements $[\rho]_h$ and $[\sigma]_h \in \mathcal{B}_1^{s.a.}(h)/\mathcal{A}_h^{\perp}$, $\|[\rho]_h + [\sigma]_h\|$ $= \|[\rho]_h\| + \|[\sigma]_h\|$.*

PROOF : It is immediate from the previous lemma since $[\rho+\sigma]_h = [\rho]_h + [\sigma]_h$, which is clearly positive: and thus $\|[\rho+\sigma]_h\| = tr(\rho+\sigma)$. □

1.2 Completely positive maps and semigroups

1.2.1 Complete positivity

Let us consider two unital $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ and a linear map $T : \mathcal{A} \to \mathcal{B}$. Recall that T is said to be positive if it takes positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. It is clear that a positive map is "real" in the sense that it takes a self-adjoint element into a self-adjoint element. Given any such positive map T, it is natural to consider $T_n \equiv T \otimes id : \mathcal{A} \otimes \mathcal{M}_n \to \mathcal{B} \otimes \mathcal{M}_n$ where $\mathcal{M}_n$ denotes the algebra of $n \times n$ complex matrices. A natural question which arises is the following :
is T_n a positive map for each n ?
The answer to this question is negative, as the following simple example illustrates.

Example 1.2.1 *Let $\mathcal{A}$ be $\mathcal{M}_2$, the algebra of 2×2 complex matrices, and $T : \mathcal{A} \to \mathcal{A}$ be the map given by $T(X) = X'$, where $'$ denotes transpose. That is, (i,j)-th element of $T(X)$ is the (j,i)-th element of X. Clearly, $T(X^*X) = \overline{X}^*\overline{X} \geq 0$, where the (i,j)-th element of $\overline{X}$ is the complex conjugate of (i,j)-th element of X. Hence*

T is positive. We claim that T is not 2-positive. Take $X_1 = \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. *Consider the element of* $\mathcal{M}_2 \otimes \mathcal{M}_2$ *given by the block matrix form* $X = \begin{pmatrix} X_1^*X_1 & X_1^*X_2 \\ X_2^*X_1 & X_2^*X_1 \end{pmatrix}$. *Clearly X is positive in* $\mathcal{M}_2 \otimes \mathcal{M}_2 \cong \mathcal{M}_4$. *But by a simple computation it can be verified that* $T(X)$ *is the* 4×4 *matrix*

$$\begin{pmatrix} 2 & 0 & -i & 0 \\ 0 & 0 & -i & 0 \\ i & i & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

which is not positive since its determinant $= -2 < 0$.

We say that T is n-positive if T_k is positive for all $k \le n$, and not for $k = n+1$. T is said to be completely positive (CP for short) if it is n- positive for each n. The role of positivity in classical probability is played by complete positivity in the quantum theory.

Let us now formulate the notion of complete positivity in a slightly different but convenient language, namely that of positive definite kernels. For this purpose, we first need a few definitions and facts. For a set X and a Hilbert space $\mathcal{H}$, a map $K : X \times X \to \mathcal{B}(\mathcal{H})$ is called a kernel. The set of all kernels is a vector space, denoted by $K(X; \mathcal{H})$.

Definition 1.2.2 *A kernel K in* $K(X;\mathcal{H})$ *is said to be +ve definite if for each positive integer n and each choice of vectors* $u_1, \ldots, u_n$ *in* $\mathcal{H}$ *and elements* $x_1, \ldots, x_n \in X$, *one has* $\sum_{i,j=1}^n \langle K(x_i, x_j)u_j, u_i\rangle \ge 0$.

Definition 1.2.3 Kolmogorov decomposition :
Let $K \in K(X;\mathcal{H})$. *Let* $\mathcal{H}_V$ *be a Hilbert space and* $V : X \to \mathcal{B}(\mathcal{H}, \mathcal{H}_V)$ *be a map such that* $K(x,y) = V(x)^*V(y)\ \forall\, x, y \in X$. *Then* $(V, \mathcal{H}_V)$ *is said to be a Kolmogorov decomposition of K. It is said to be minimal if the set* $\{V(x)u : x \in X, u \in \mathcal{H}\}$ *is total in* $\mathcal{H}_V$. *Two Kolmogorov decompositions* $(V, \mathcal{H}_V)$ *and* $(V', \mathcal{H}_{V'})$ *are said to be equivalent if there exists a unitary* $U : \mathcal{H}_V \to \mathcal{H}_{V'}$ *such that* $V'(x) = UV(x)\ \forall x \in X$.

Let us now prove that any positive definite kernel admits a canonical minimal Kolmogorov decomposition. Let $F_0 = F_0(X;\mathcal{H})$ denote the vector space of $\mathcal{H}$-valued functions on X having finite support and let $F = F(X:\mathcal{H})$ denote the vector space of all $\mathcal{H}$-valued functions on X. We identify F with a subspace of the algebraic dual F_0' of F_0 by defining for $p \in F$ the functional $\langle p, . \rangle$ on F_0 given by,

$$\langle p, f \rangle = \sum_{x \in X} \langle p(x), f(x) \rangle$$

for $f \in F_0$; where the summation is actually over a finite set since f has finite support. Given $K \in K(X:\mathcal{H})$ we define an associated operator $\tilde{K} : F_0(X:\mathcal{H}) \to F(X:\mathcal{H})$ by,

$$(\tilde{K}f)(x) = \sum_{y \in X} K(x,y)f(y).$$

Then it is easy to verify that K is +ve if and only if $\langle \tilde{K}f, f \rangle \geq 0 \ \forall \ f \in F_0(X;\mathcal{H})$.

Lemma 1.2.4 *Let V be a vector space, V' be its algebraic dual, with the pairing $V' \times V \to \mathbb{C}$ written as $v'.v \mapsto \langle v', v \rangle$. Let $A : V \to V'$ be a linear map such that $\langle Av, v \rangle \geq 0 \ \forall v \in V$. Then there exists a well defined inner product on the image space AV given by, $\langle Av_1, Av_2 \rangle = \langle Av_1, v_2 \rangle$.*

Proof: The sesquilinear form $v_1, v_2 \mapsto a(v_1, v_2) \equiv \langle Av_1, v_2 \rangle$ is nonnegative, so that by Schwarz's inequality one obtains

$$|\langle Av_1, v_2 \rangle|^2 \leq \langle Av_1, v_1 \rangle \langle Av_2, v_2 \rangle.$$

It follows that the set $V_A \equiv \{v \in V : \langle Av, v \rangle = 0\}$ coincides with $KerA$ and the natural projection $\pi : V \to V/KerA$ carries the form $a(.,.)$ into an inner product $\langle .,. \rangle_A$ on $V/kerA$ given by, $\langle \pi(v_1), \pi(v_2) \rangle_A = a(v_1, v_2)$. the vector space isomorphism $A' : V/KerA \to AV$ given by $A'\pi = A$ carries the inner product $\langle .,. \rangle_A$ into an inner product $\langle .,. \rangle$ on AV given by, $\langle Av_1, Av_2 \rangle = \langle A'\pi(v_1), A'\pi(v_2) \rangle = \langle \pi(v_1), \pi(v_2) \rangle_A = \langle Av_1, v_2 \rangle$. □

We now prove the existence-uniqueness for a minimal Kolmogorov decomposition in a few steps.

Theorem 1.2.5 *Given a +ve definite kernel $K \in K(X;\mathcal{H})$, there exists a unique Hilbert space $\mathcal{R}(K)$ of $\mathcal{H}$-valued functions on X such that*
(i) $\mathcal{R}(K)$ is the closed linear span of $K(.,x)u; x \in X, u \in \mathcal{H}$.
(ii) $\langle f(x), u \rangle = \langle f, K(.,x)u \rangle \ \forall f \in \mathcal{R}(K)$, $x \in X$ and $u \in \mathcal{H}$.

Proof : Since K is +definite, the associated operator $\tilde{K}$ satisfies the hypothesis of 1.2.4 , so that we obtain an inner product $\langle .,. \rangle$ on $\tilde{K}F_0$ and let us denote by $\overline{\tilde{K}F_0}$ the completion of $\tilde{K}F_0$ with respect to the norm inherited from this inner product, and identify $\tilde{K}F_0$ with a dense subset of $\overline{\tilde{K}F_0}$. For each $x \in X$ and $u \in \mathcal{H}$, define the function u_x in F_0 by setting $u_x(y) = u$ if $y = x$ and 0 otherwise. Clrearly, $(\tilde{K}u_x)(y) = K(y,x)u$. Define K_x on $\mathcal{H}$ by setting $K_x u = \tilde{K}u_x$ for all $x \in X$, $u \in \mathcal{H}$. Then $\|K_x u\| \leq \|K(x,x)\|^{\frac{1}{2}}.\|u\|$; and hence K_x is a bounded linear map. A straightforward calculation shows that on $\tilde{K}F_0$ we have $K_x^* f = f(x)$. The mapping from $\overline{\tilde{K}F_0}$ into the space of all $\mathcal{H}$-valued functions on X which sends f into the function $x \mapsto K_x^* f$ is linear, injective and compatible with the identification of $\tilde{K}F_0$ with a dense subset of $\overline{\tilde{K}F_0}$, thus we regard $\overline{\tilde{K}F_0}$ as a Hilbert space $\mathcal{R}(K)$ consisting of $\mathcal{H}$-valued functions on X. We have already proven that $\mathcal{R}(K)$ satisfies (i) and (ii). Uniqueness of $\mathcal{R}(K)$ is easy to see. □

Definition 1.2.6 *$\mathcal{R}(K)$ in the above theorem is called the reproducing kernel Hilbert space for K.*

It is now easy to prove the following :

Theorem 1.2.7 *A kernel K is +ve definite if and only if it admits a minimal Kolmogorov decomposition. Moreover, any two minimal Kolmogorov decompositions for the same kernel are equivalent.*

Proof : The if part of the first statement is trivial; for the only if part we take $\mathcal{H}_V = \mathcal{R}(K)$ and $V(x) = K_x : \mathcal{H} \to \mathcal{R}(K)$ as in the proof of theorem 1.2.4, noting that $(V, \mathcal{H}_V)$ is minimal by (i) of that theorem. To prove the second part of the present theorem, let us assume that $(V_1, \mathcal{H}_{V_1})$ and $(V_2, \mathcal{H}_{V_2})$ are two minimal Kolmogorov decompositions for the same kernel K. Define a unitary $U : \mathcal{H}_{V_1} \to \mathcal{H}_{V_2}$ by setting $U(V_1(x)u) = V_2(x)u$ and extend it by linearity and density to the whole of $\mathcal{H}_{V_1}$. It is clear that U is well defined and unitary. □

Let us now come back to complete positivity and deduce the fundamental theorem of Stinespring. Let us fix a Hilbert space h and a unital $*$-subalgebra $\mathcal{A}$ of $\mathcal{B}(h)$.

Proposition 1.2.8 *Let T be a linear map from $\mathcal{A}$ into $\mathcal{B}(h)$. Define an associated kernel $K_T : \mathcal{A} \times \mathcal{A} \to \mathcal{B}(h)$ given by, $K_T(x, y) = T(x^*y)$ for $x, y \in \mathcal{A}$. Then T is completely positive (CP) if and only if K_T is positive definite.*

Proof : For $u_1, \ldots, u_n \in h$ and $x_1, \ldots, x_n \in \mathcal{A}$, $\sum_{i,j} \langle K_T(x_i, x_j) u_j, u_i \rangle = \langle T_n(X)\tilde{u}, \tilde{u} \rangle$, where X denotes the element in $\mathcal{A} \otimes \mathcal{M}_n \equiv \mathcal{M}_n(\mathcal{A})$ given by the $\mathcal{A}$-valued $n \times n$ matrix $((x_i^* x_j))$, T_n denotes the map $T \otimes I_{\mathcal{M}_n}$ and $\tilde{u}$ denotes the vector $u_1 \oplus u_2 \oplus \ldots \oplus u_n$ in $h \oplus \ldots \oplus h$. Since by our choice $X \geq 0$ as an element of $\mathcal{A} \otimes \mathcal{M}_n$, it is clear that positivity of T_n for each n is equivalent to the positive definiteness of K_T. □

Theorem 1.2.9 (Stinespring's theorem .)
A linear map $T : \mathcal{A} \to \mathcal{B}(h)$ is CP if and only if there is a triple $(\mathcal{K}, \pi, V)$ consisting of a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and $V \in \mathcal{B}(h, \mathcal{K})$ such that $T(x) = V^* \pi(x) V$ for all $x \in \mathcal{A}$, and $\{\pi(x)Vu : u \in h, x \in \mathcal{A}\}$ is total in $\mathcal{K}$. Such a triple, to be called the 'Stinespring triple' associated with T, is unique in the sense that if $(\mathcal{K}', \pi', V')$ is another such triple, then there is a unitary operator $\Gamma : \mathcal{K} \to \mathcal{K}'$ such that $\pi'(x) = \Gamma \pi(x) \Gamma^*$ and $V' = \Gamma V$.

Furthermore, if $\mathcal{A}$ is a von Neumann algebra and T is normal, π can be chosen to be normal.

Proof : Let $(\lambda, \mathcal{K})$ be the minimal Kolmogorov decomposition for the kernel K_T defined in the statement of 1.2.8. For $x \in \mathcal{A}$, define a map $\pi(x)$ on the linear span of vectors $\lambda(y)u$ by setting $\pi(x)(\lambda(y)u) = \lambda(xy)u$, and by extending linearly. The complete positivity of T enables us to verify that indeed $\pi(x)$ is well defined and one has,

$$\|\pi(x)(\sum_{i=1}^{n} \lambda(x_i)u_i)\|^2 \leq \|x\|^2 \|\sum_{i=1}^{n} \lambda(x_i)u_i\|^2$$

for any finite collection $x_1, \ldots, x_n$ of elements of $\mathcal{A}$ and $u_1, \ldots, u_n$ in h. Thus, $\pi(x)$ extends as a bounded linear map on the whole of $\mathcal{K}$ and it is also clear that $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a $*$-homomorphism. To complete the proof of the existence part, we choose $V = \lambda(1)$ and note that $T(x) = \lambda(1)^* \lambda(x) = V^* \pi(x) V$. The proof of uniqueness is straightforward and omitted.
In case $\mathcal{A}$ is a von Neumann algebra and T is normal, let us prove the normality of π. Let $0 \leq x_\alpha \uparrow x$ where x_α is a net of elements in $\mathcal{A}$ and $x \in \mathcal{A}$. Note that normality of T implies its ultra-weak continuity, which coincides with the weak continuity on norm-bounded convex sets. Thus, for $y, z \in \mathcal{A}$, $u, v \in h$, we have,

$$\langle \pi(x_\alpha)(\lambda(y)u), \lambda(z)v \rangle = \langle u, T(y^* x_\alpha z)v \rangle$$

$$\to \langle u, T(y^* x z)v \rangle = \langle \pi(x)(\lambda(y)u), \lambda(z)v \rangle.$$

Here, we have used the fact that x_α is a bounded net and hence so is the net $y^* x_\alpha z$. Thus, for any vector ξ which is a finite linear combination of vectors of the form $\lambda(y)u$, $y \in \mathcal{A}$, $u \in h$, we have that $\langle \pi(x_\alpha)\xi, \xi \rangle$ converges to $\langle \pi(x)\xi, \xi \rangle$. Since the net $\pi(x_\alpha)$ is norm-bounded, this holds for all ξ in $\mathcal{K}$. Hence π is normal. □

We now mention a result (see [17] for a proof) which shows that the distinction between positivity and complete positivity appears only in noncommutative algebras.

Theorem 1.2.10 *If $\mathcal{A}$ is a commutative C^* algebra and $\mathcal{B}$ is any C^* algebra, then any positive map from $\mathcal{A}$ to $\mathcal{B}$ is automatically CP. Similar statement holds for any positive map from $\mathcal{B}$ to $\mathcal{A}$.*

1.2.2 Semigroups of linear maps in locally convex spaces

In this brief subsection, we mention without proofs a few standard and useful results from the theory of one-parameter semigroups of continuous linear maps acting on a locally convex topological vector space. For a more elaborate account and proofs, the reader may be referred to [52].

Let X be a locally convex, sequentially complete, linear topological space and $(T_t)_{t \geq 0}$ be a 1-parameter family of continuous linear operators from X to itself satisfying $T_t T_s = T_{t+s}$, $T_0 = I$, and $\lim_{t \to t_0} T_t x = T_{t_0} x$ for all $x \in X$ and $t \geq 0$. Such a family is called a 1-parameter semigroup of class C_0 (or strongly continuous) of operators on X. The family T_t is called equi-continuous if given any continuous seminorm p on X, there exists a continuous seminorm q on X such that $p(T_t x) \leq q(x)$ for all $t \geq 0$ and $x \in X$. If X is Banach space and $t \mapsto T_t$ is continuous in map-norm, that is, $\lim_{t \to t_0} \|T_t - T_{t_0}\| = 0$, then we say that T_t is uniformly continuous or norm-continuous. It is easy to show that T_t is uniformly continuous if and only if there exists $L \in \mathcal{B}(X)$ such that $T_t = e^{tL}$ for all t.

Given an equi-continuous semigroup of class C_0 on X, we define a linear operator A on X, called the generator of T_t, with the domain $\mathcal{D}(A) = \{x \in X : \lim_{t \to 0+} \frac{T_t x - x}{t} \text{ exists}\}$, given by $Ax = \lim_{t \to 0+} \frac{T_t x - x}{t}$ for $x \in \mathcal{D}(A)$. It is a remarkable fact that $\mathcal{D}(A)$ is dense and A is closed. Furthermore, for any $x \in \mathcal{D}(A)$, $T_t x$ also belongs to $\mathcal{D}(A)$ for all $t \geq 0$ and $AT_t x = T_t Ax$. The following beautiful and useful theorem due to Hille and Yosida characterizes generators of equi-continuous semigroups of class C_0.

Theorem 1.2.11 *A closed linear operator A on X with dense domain is the generator of an equi-continuous semigroup of class C_0 if and only if $\forall n = 1, 2, \ldots$ $(nI - A)^{-1}$ exists as a bounded operator and the family $\{(I - n^{-1}A)^{-m}\}_{n=1,2,\ldots;m=0,1,2,\ldots}$ is equi-continuous.*

Specializing to Banach spaces, the above equi-continuity translates into the existence of a positive constant C satisfying $\|(I - n^{-1}A)^{-m}\| \leq C$ $\forall n, m$; and furthermore, A is the generator of a contractive semigroup (that is, each T_t is a contraction) if and only if $\|(I - n^{-1}A)^{-1}\| \leq 1$ $\forall n = 1, 2, \ldots$.

Convergence of semigroups is an important issue in the theory of semigroups and in this context, we state the following theorem (due to Trotter and Kato) which will be utilized by us in the last chapter of the thesis.

Theorem 1.2.12 *Let X be a locally convex, sequentially complete, complex linear space and for each $n = 1, 2, \ldots$ let $(T_t^{(n)})_{t \geq 0}$ be an equi-continuous semigroup of class C_0 with generator A_n. Assume furthermore that*
(i) For any continuous seminorm p on X, there exists a continuous seminorm q on X such that $p(T_t^{(n)}x) \leq q(x)$ $\forall t \geq 0, n = 1, 2, \ldots$ and $x \in X$.
(ii) For some λ_0 with $Re\lambda_0 > 0$, there exists an invertible operator $J : X \to X$ such that the range of J is dense in X and $J(x) = \lim_{n\to\infty}(\lambda_0 - A_n)^{-1}(x)$ $\forall x \in X$.
Then $(\lambda_0 - J^{-1})$ is the generator of an equi-continuous semigroup $(T_t)_{t \geq 0}$ of class C_0 satisfying $T_t(x) = \lim_{n\to\infty} T_t^{(n)}(x)$ $\forall x \in X$ and the above convergence is uniform over compact subintervals of $[0, \infty)$.

In case X is a Banach space and $T_t^{(n)}, T_t$ are contractive semigroups, it is easy to see that the followings are equivalent :
(a) $T_t^{(n)}(x)$ converges (uniformly on compacts) to $T_t(x)$.
(b) $(\lambda - A_n)^{-1}(x) \to (\lambda - A)^{-1}(x)$ $\forall x$ as $n \to \infty$ for some λ with $Re\lambda > 0$, where A_n and A denote the generators of $T_t^{(n)}$ and T_t respectively.
(c) $(\lambda - A_n)^{-1}(x) \to (\lambda - A)^{-1}(x)$ $\forall x$ as $n \to \infty$ uniformly in λ over compact subsets of the right half plane $\{\lambda : Re\lambda > 0\}$.

For a locally convex, sequentially complete, linear topological space X, let us denote by X^* its dual, viewed naturally as a locally convex space. Given an equi-continuous semigroup T_t of class C_0 on X, it is natural to consider the dual semigroup T_t^* on X^*. T_t^* will be equi-continuous but not in general of class C_0. It is of class C_0 when X^* is also sequentially complete.

1.2.3 Generators of uniformly continuous quantum dynamical semigroups

Let us now restrict ourselves to the case when the general locally convex space X is replaced by a topological algebra $\mathcal{A}$, to be precise, a C^* or a von Neumann algebra. and study the implications of the complete positivity of T_t.

Definition 1.2.13 *A quantum dynamical semigroup on a C^* algebra $\mathcal{A}$ is a contractive semigroup T_t of class C_0 such that each T_t is completely positive map from $\mathcal{A}$ to itself. T_t is said to be conservative if $T_t(1) = 1$ for all $t \geq 0$.*

For a uniformly continuous semigroup on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(h)$, we have the following result :

Lemma 1.2.14 *Let $T_t = e^{t\mathcal{L}}$ be a uniformly continuous contractive semigroup acting on $\mathcal{A}$ with $\mathcal{L}$ as the generator. Then T_t is normal for each t if and only if $\mathcal{L}$ is ultra-strongly (and ultra-weakly) continuous on any norm-bounded subset of $\mathcal{A}$.*

Proof : If $\mathcal{L}$ is ultra-strongly continuous continuous on bounded sets. then clearly $e^{t\mathcal{L}}$ is ultra-strongly continuous on bounded sets for each t, and hence normal. For the converse. first note that for any $t \geq 0$ and $x \in \mathcal{A}$, we have

$$\|(T_t(x) - x)\| \leq \int_0^t \|T_s(\mathcal{L}(x))\| ds \leq \|\mathcal{L}\| \|x\| t.$$

Hence it is easy to see that

$$\begin{aligned} & \|\mathcal{L}(x) - \frac{T_t(x) - x}{t}\| \\ = & \|\frac{1}{t}\int_0^t \{\mathcal{L}(x) - T_s(\mathcal{L}(x))\} ds\| \\ \leq & \frac{1}{t}\int_0^t \|\mathcal{L}\| \|\mathcal{L}(x)\| s ds \\ \leq & \|\mathcal{L}\|^2 \|x\| \frac{t}{2}. \end{aligned}$$

Now suppose that x_α is a bounded net of elements in $\mathcal{A}$ such that x_α strongly converges to $x \in \mathcal{A}$ and let $\|x_\alpha\| \leq M \ \forall \alpha$. Fix $u \in h$ and $\epsilon > 0$. Choose t_0 small enough so that $\|\mathcal{L}\|^2 M \|u\| t_0 < \frac{1}{2}\epsilon$. Clearly,

$$\begin{aligned} & \|\mathcal{L}(x_\alpha - x)u\| \\ \leq & \frac{1}{2}\epsilon + \|\{\frac{T_{t_0}(x_\alpha - x) - (x_\alpha - x)}{t_0}\}u\|, \end{aligned}$$

which proves that $\mathcal{L}(x_\alpha - x)u \to 0$ since T_t is normal. This shows the continuity of $\mathcal{L}$ with respect to the strong, or equivalently ultra-strong topology on bounded sets. Similarly, one can prove the weak (and equivalently ultra-weak) continuity. □

In view of the above result, we shall make the following definition :

Definition 1.2.15 *We define a quantum dynamical semigroup on a von Neumann algebra to be a semigroup T_t of completely positive, contractive maps on the von Neumann algebra such that T_t is normal for each t. In case when T_t is uniformly continuous, its norm-bounded generator is ultra-strongly (and ultra-weakly) continuous on bounded sets.*

It is to be noted that although each T_t acts on an algebra, the domain of the generator need not be an algebra, nor it may contain any $*$-subalgbra which is sufficiently large in any reasonable sense. This is a fundamental difficulty in translating the complete positivity of the semigroup into some property of its generator. However, when T_t is uniformly continuous, its generator is defined as a bounded map on the whole of $\mathcal{A}$, facilitating the analysis of complete positivity. Thus, for the sake of convenience, we assume for the rest of the present subsection that T_t is uniformly continuous. To understand the implication of complete positivity, let us first note some definitions and results.

Definition 1.2.16 *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^* algebras such that the former is a subalgebra of the latter, and $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ be a bounded linear map with the property that $\mathcal{L}$ is real, that is, $\mathcal{L}(x^*) = \mathcal{L}(x)^*$ for all $x \in \mathcal{A}$. We call $\mathcal{L}$ conditionally completely positive (CCP) if*

$$\sum_{i,j=1}^{n} b_i^* \mathcal{L}(a_i^* a_j) b_j \geq 0$$

for all $a_1, \ldots, a_n$ in $\mathcal{A}$ and $b_1, \ldots, b_n$ in $\mathcal{B}$ satisfying $\sum_{i,j=1}^{n} a_i b_i = 0$.

We first give a few characterizations of conditional complete positivity in the language of positive definite kernels. We state the result without proof, which can be found in [17] (page 70-71, Lemma 14.5).

Lemma 1.2.17 *In the notation of the above definition (1.2.16), the followings are equivalent :*

(i) For all $a \in \mathcal{A}$, the kernel $\mathcal{A} \times \mathcal{A} \ni (b, c) \mapsto K_a(b, c) \equiv \mathcal{L}(b^* a^* a c) + b^* \mathcal{L}(a^* a) c - \mathcal{L}(b^* a^* a) c - b^* \mathcal{L}(a^* a c)$ is positive definite.

(ii) The kernel $(\mathcal{A}\times\mathcal{A})\times(\mathcal{A}\times\mathcal{A})\ni(b_1,b_2),(c_1,c_2)\mapsto\mathcal{L}(b_1^*b_2^*c_2c_1)+b_1^*\mathcal{L}(b_2^*c_2)c_1-\mathcal{L}(b_1^*b_2^*c_2)c_1-b_1^*\mathcal{L}(b_2^*c_2c_1)$ *is positive definite.*

Lemma 1.2.18 *Let* $\mathcal{L}$ *be a bounded linear real map from a unital* C^* *algebra* $\mathcal{A}$ *to itself such that* $e^{t\mathcal{L}}$ *is a contraction for each* $t\geq 0$*. Then the followings are equivalent:*
(i) $e^{t\mathcal{L}}$ *is positive for all positive* t*.*
(ii) $(\lambda-\mathcal{L})^{-1}$ *is positive for all sufficiently large positive* λ*.*
(iii) For $y,a\in\mathcal{A}$ *with the property that* $ya=0$*, one has* $a^*\mathcal{L}(y^*y)a\geq 0$*.*

Proof :
(i) $\Rightarrow$ (iii) :
First assume that $\mathcal{L}(1)=0$. If y,a satisfies the hypothesis of (iii), we have that $\frac{(a^*e^{t\mathcal{L}}(y^*y)a-a^*y^*ya)}{t}=\frac{a^*e^{t\mathcal{L}}(y^*y)a}{t}\geq 0$ for all positive t, and hence taking limit as $t\to 0+$, we obtain $a^*\mathcal{L}(y^*y)a\geq 0$. Now, for the general case when $\mathcal{L}(1)$ is not 0, we consider an enlargement of the original algebra $\mathcal{A}$ to the bigger algebra $\mathcal{A}\oplus\mathbb{C}$ and an appropriate extension of $T_t=e^{t\mathcal{L}}$ given by $\tilde{T}_t(x\oplus c)\equiv(T_t(x)+c.(1-T_t(1)))\oplus c$. It is clear that $\tilde{T}_t$ is a conservative quantum dynamical semigroup on $\mathcal{A}\oplus\mathbb{C}$ with the generator $\tilde{\mathcal{L}}$ given by $\tilde{\mathcal{L}}(x\oplus c)=(\mathcal{L}(x)-c\mathcal{L}(1))\oplus 0$. By what we have proven for the conservative case, we have that if $ya=0$ for $y,a\in\mathcal{A}$, then
$0\geq(a\oplus 0)^*\tilde{\mathcal{L}}((y\oplus 0)^*(y\oplus 0))(a\oplus 0)\ =a^*\mathcal{L}(y^*y)a.$
(iii) $\Rightarrow$ (ii) :
Let λ be greater than $\|\mathcal{L}\|$. Let x be such that $(\lambda-\mathcal{L})(x)\geq 0$. Since $\mathcal{L}$ is real, $Im(\lambda-\mathcal{L})(x)=(\lambda-\mathcal{L})(Im\ x)$, and positivity of $(\lambda-\mathcal{L})(x)$ implies in particular that $Im(\lambda-\mathcal{L})(x)=0$. Hence we have $(\lambda-\mathcal{L})(x)=(\lambda-\mathcal{L})(Re\ x)$, and thus we may assume without loss of generality that x is self-adjoint. We want to show that x is positive. Let $x=x^+-x^-$, with x^+ and x^- positive and $x^+x^-=0$. Then, by (iii), we have $x^-\mathcal{L}(x^+)x^-\geq 0$, so that $0\leq x^-[x-\lambda^{-1}\mathcal{L}(x)]x^-\ =-(x^-)^3-\lambda^{-1}x^-\mathcal{L}(x^+)x^-+\lambda^{-1}x^-\mathcal{L}(x^-)x^-$. Thus $0\leq(x^-)^3\leq\lambda^{-1}x^-\mathcal{L}(x^-)x^-$, and hence $\|x^-\|^3\leq\lambda^{-1}\|\mathcal{L}\|\|x^-\|^3$, which implies $\|x^-\|=0$, since $\|\mathcal{L}\|<\lambda$.
(ii) $\Rightarrow$ (i) :
This follows from the identity $e^{t\mathcal{L}}=\lim_{n\to\infty}(1-\frac{t}{n}\mathcal{L})^{-n}$. □

As a simple application of the above lemma, we obtain the following useful result :

Theorem 1.2.19 *A bounded linear adjoint-preserving map* $\mathcal{L}$ *from a unital* C^* *algebra* $\mathcal{A}$ *to itself is CCP if and only if* $e^{t\mathcal{L}}$ *is CP for all positive* t*.*

Proof : It is enough to observe that $\mathcal{L}$ is CCP if and only if $(\mathcal{L}\otimes I) : \mathcal{A}\otimes\mathcal{M}_n \to \mathcal{A}\otimes\mathcal{M}_n$ satisfies the hypothesis of (iii) in Lemma 1.2.18 with $\mathcal{A}$ replaced by $\mathcal{A}\otimes\mathcal{M}_n$. □

We shall now prove a structure theorem for normal CCP maps acting on a von Neumann algebra. For this, we first quote a result without proof. (see [8] for a proof.)

Theorem 1.2.20 *Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{B}(h)$ for some Hilbert space h and $W : \mathcal{A} \to \mathcal{B}(h)$ be a derivation, that is, $W(ab) = W(a)b + aW(b)$. Then there exists an operator $T \in \mathcal{B}(h)$ such that $W(a) = Ta - aT$ for all $a \in \mathcal{A}$.*

As an application of this theorem, the canonical structure theorem for normal CCP maps is established.

Theorem 1.2.21 *(Christensen-Evans)*
Let $(T_t)_{t\geq 0}$ be a uniformly continuous quantum dynamical semigroup (q.d.s.) on a unital von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(h)$ with $\mathcal{L}$ as its ultra-weakly continuous generator. Then there is a quintuple $(\rho, \mathcal{K}, \alpha, H, R)$ where ρ is a unital normal $*$-representation of $\mathcal{A}$ in a Hilbert space $\mathcal{K}$ and a ρ-derivation $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that the set $\mathcal{D} \equiv \{\alpha(x)u | x \in \mathcal{A}, u \in h\}$ is total in $\mathcal{K}$, H is a self-adjoint element of $\mathcal{A}$. and $R \in \mathcal{B}(h, \mathcal{K})$ such that $\alpha(x) = Rx - \rho(x)R$, and $\mathcal{L}(x) = R^*\rho(x)R - \frac{1}{2}(R^*R - \mathcal{L}(1))x - \frac{1}{2}x(R^*R - \mathcal{L}(1)) + i[H,x]$ $\forall x \in \mathcal{A}$. Furthermore, $\mathcal{L}$ satisfies the cocycle relation with α as coboundary, namely,

$$\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y) + x^*\mathcal{L}(1)y = \alpha(x)^*\alpha(y).$$

Moreover, R can be chosen from the ultraweak closure of $\mathrm{sp}\{\alpha(x)y : x, y \in \mathcal{A}\}$ and hence in particular $R^*\rho(x)R \in \mathcal{A}$.

Proof : We briefly sketch only the main ideas behind the proof and refer the reader to [17] and [8] for details.

Consider the trilinear map D on $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ defined by, $D(x,y,z) = \mathcal{L}(xyz) + x\mathcal{L}(y)z - \mathcal{L}(xy)z - x\mathcal{L}(yz)$. It is easy to verify that the kernel $(a_1, a_2), (b_1, b_2) \mapsto D(a_1^*, a_2^*b_2, b_1)$ is positive definite. By the theorem 1.2.7,we obtain a Hilbert space $\mathcal{K}$ and $\lambda : \mathcal{A}\times\mathcal{A} \to \mathcal{B}(h, \mathcal{K})$ such that $(\lambda, \mathcal{K})$ is the minimal Kolmogorov decomposition for the above kernel. As in the proof of 1.2.9, it is easy to verify that $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ defined by $\rho(x)(\lambda(a,b)u) = \lambda(a, xb)u$ extends to a normal $*$-homomorphism of $\mathcal{A}$. The proof of normality is similar to that of the representation π in 1.2.9, using the ultra-weak continuity of $\mathcal{L}$ on norm-bounded sets. Denote by $\alpha(x)$ the operator $\lambda(x,1) \in \mathcal{B}(h,\mathcal{K})$. Then, it is easy to verify that $\lambda(x^*,y^*)^*[\alpha(ab)-\rho(a)\alpha(b)-\alpha(a)b] = D(x,y,ab) - D(x,ya,b) - D(x,y,a)b = 0$. By minimality of $(\lambda,\mathcal{K})$ we conclude that $\alpha(ab) = \rho(a)\alpha(b) + \alpha(a)b$, that is, α is a ρ-derivation. Now, to obtain R, consider the faithful normal representation $\pi : \mathcal{A} \to \mathcal{B}(h \oplus \mathcal{K})$ given by, $\pi(a) = \begin{pmatrix} a & 0 \\ 0 & \rho(a) \end{pmatrix}$. Let $W : \pi(\mathcal{A}) \to \mathcal{B}(h \oplus \mathcal{K})$ given by, $W(\pi(a)) = \begin{pmatrix} 0 & 0 \\ \alpha(a) & 0 \end{pmatrix}$. Then $W(\pi(a)\pi(b)) = W(\pi(a))\pi(b) + \pi(a)W(\pi(b))$. By the theorem 1.2.20, there exists $T \in \mathcal{B}(h \oplus \mathcal{K})$ such that $W(\pi(a)) = T\pi(a) - \pi(a)T$. Writing $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ with respect to the canonical decomposition of $\mathcal{B}(h \oplus \mathcal{K})$, we obtain $\alpha(a) = Ra - \rho(a)R$. Consider the map $\Psi(x) \equiv \mathcal{L}(x) - R^*\rho(x)R$. A simple algebraic calculation will show that $x \mapsto \Psi(x) - \frac{1}{2}(\Psi(1)x + x\Psi(1))$ is a derivation, and hence by 1.2.20 and also noting the fact that Ψ is adjoint-preserving, we obtain a self-adjoint H in $\mathcal{B}(h)$ such that $\Psi(x) - \frac{1}{2}(\Psi(1)x + x\Psi(1)) = i.[H,x]$. The proof that R can be chosen from the ultraweak closure of $\{\alpha(x)y : x,y \in \mathcal{A}\}$ (thus $R^*\rho(x)R \in \mathcal{A}$) and H can be chosen from $\mathcal{A}$ is omitted; referring the reader to the original paper by Christensen and Evans [8]. □

Remark 1.2.22 *The above theorem also applies to the generator of a uniformly continuous q.d.s. acting on a C^* algebra $\mathcal{A}$, with the only essential modification that $R^*\rho(x)R$ and H will belong to the ultraweak closure of $\mathcal{A}$ instead of $\mathcal{A}$ itself.*

Remark 1.2.23 *From the structure obtained by the above theorem (1.2.21) it is clear (since ρ is normal) that $\mathcal{L}$ is normal, i.e. ultra-weakly continuous.*

Let us complete this section by introducing a few notations which will be useful later. For a unital C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(h)$, a representation $(\pi,\mathcal{H})$ of $\mathcal{A}$, and operators $R \in \mathcal{B}(h;\mathcal{H})$ and $H \in \mathcal{B}(h)$ we write $\delta_{R,\pi}$ and $\mathcal{L}_{R,\pi,H}$ for the operators given by

$$\delta_{R,\pi}(a) = Ra - \pi(a)R, \quad \mathcal{L}_{R,\pi,H}(a) = R^*\pi(a)R - \tfrac{1}{2}\{R^*R,a\} + i[a,H],$$

where $\{b,c\} \equiv bc + cb$. Thus $\delta_{R,\pi} : \mathcal{A} \to \mathcal{B}(h;\mathcal{H})$ is a π-derivation and $\mathcal{L}_{R,\pi,H} : \mathcal{A} \to \mathcal{B}(h)$ satisfies

$$\partial\mathcal{L}_{R,\pi,H}(a,b) = \delta_{R,\pi}(a)^*\delta_{R,\pi}(b) + a^*R^*\pi(1)^{\perp}Rb \tag{1.1}$$

where, given $\tau : \mathcal{A} \to \mathcal{B}(h)$, the map $\partial\tau : \mathcal{A} \times \mathcal{A} \to \mathcal{B}(h)$ is defined by

$$\partial\tau(a, b) = \tau(a^*b) - a^*\tau(b) - \tau(a^*)b + a^*\tau(1)b.$$

The following result in contained in [8].

Lemma 1.2.24 *Let $(\tau, \rho, \mathcal{H}, \delta)$ consist of a map $\tau \in \mathcal{B}(\mathcal{A})$, a $*$-representation $(\rho, \mathcal{H})$ of $\mathcal{A}$ and a ρ-derivation $\delta : \mathcal{A} \to \mathcal{B}(h; \mathcal{H})$ satisfying $\partial\tau(a, b) = \delta(a)^*\delta(b)$ and $\delta(1) = 0$. Then there is an operator $R \in \mathcal{B}(h; \mathcal{H})$ which lies in the ultraweak closure of $\mathrm{sp}\{\delta(a)b : a, b \in \mathcal{A}\}$ and an element $H \in \mathcal{A}''$ such that*

$$\delta(\cdot) = \delta_{R,\rho}(\cdot) \text{ and } \tau(\cdot) = \mathcal{L}_{R,\rho,H}(\cdot) + \tfrac{1}{2}\{\tau(1), \cdot\}.$$

If τ is real then H may be chosen so that $H = H^*$.

1.3 Hilbert modules

In this section we briefly discuss some useful results on Hilbert modules, and recommend the book by E. C. Lance [33] for a comprehensive account.

1.3.1 Hilbert C^* modules

A Hilbert space is a complex vector space equipped with a complex valued inner product. A natural generalization of this is the concept of Hilbert module, which has become quite an important tool of analysis and mathematical physics in recent times.

Definition 1.3.1 *Given a C^* algebra $\mathcal{A}$, a semi-Hilbert $\mathcal{A}$-module E is a right $\mathcal{A}$-module equipped with a sesquilinear map $\langle .,. \rangle : E \times E \to \mathcal{A}$ satisfying $\langle x, y\rangle^* = \langle y, x\rangle, \langle x, ya\rangle = \langle x, y\rangle a$ and $\langle x, x\rangle \geq 0$ for $x, y \in E$ and $a \in \mathcal{A}$. A semi-Hilbert module E is called a pre-Hilbert module if $\langle x, x\rangle = 0$ if and only if $x = 0$; and it is called a Hilbert module if furthermore E is complete in the norm $x \mapsto \|\langle x, x\rangle\|^{\frac{1}{2}}$ where $\|.\|$ the C^* norm of $\mathcal{A}$.*

It is clear that any semi-Hilbert $\mathcal{A}$-module can be made into a Hilbert module in a canonical way : first quotienting it by the ideal $\{x : \langle x, x\rangle = 0\}$ and then completing the quotient.

The $\mathcal{A}$-valued inner product $\langle ., . \rangle$ of a Hilbert module shares some of the important properties of usual complex valued inner product of a Hilbert space, such as the Cauchy-Schwarz inequality. However, some of the crucial properties of Hilbert spaces do not extend to general Hilbert modules : the most remarkable ones are the projection theorem and self-duality. Closed submodules of a Hilbert module need not be orthocomplemented, that is, given a closed submodule F of E, there need not exist any closed submodule G such that $E = F \oplus G$. Furthermore, the Banach space of all $\mathcal{A}$-valued, $\mathcal{A}$-linear, bounded maps on a Hilbert $\mathcal{A}$-module E may not be isometrically anti-isomorphic to E, in contrast to the Riesz' theorem for complex Hilbert space. These unpleasant features make the study of Hilbert modules considerably difficult and challenging as opposed to that of a Hilbert space. For example, a bounded $\mathcal{A}$-linear map from one Hilbert $\mathcal{A}$-module to another may not have an adjoint. For this reason, the role played by the set of bounded linear maps between Hilbert spaces is taken over by the set of adjointable $\mathcal{A}$-linear maps. To be more precise, let us make the following definition :

Definition 1.3.2 *Let E and F be two Hilbert $\mathcal{A}$-modules. We say that an $\mathcal{A}$-linear map L from E to F is adjointable if there exists a bounded $\mathcal{A}$-linear map L^* from F to E such that $\langle L(x), y\rangle = \langle x, L^*(y)\rangle$ for all $x \in E$, $y \in F$. We call L^* the adjoint of L. The set of all adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$. In case $E = F$, we write $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$.*

It may be noted that an adjointable map is automatically bounded.

Let us fix two Hilbert $\mathcal{A}$-modules E and F. For $t \in \mathcal{L}(E, F)$ and $x \in E$, it is easy to prove that $\langle tx, tx\rangle \leq \|t\|^2\langle x, x\rangle$, where $\|t\|$ denotes the map-norm of t. The topology on $\mathcal{L}(E, F)$ given by the family of seminorms $\{\|.\|_x, \|.\|_y : x \in E, y \in F\}$ where $\|t\|_x = \langle tx, tx\rangle^{\frac{1}{2}}$ and $\|t\|_y = \langle t^*y, t^*y\rangle^{\frac{1}{2}}$, is known as the strict topology. For $x \in E$, $y \in F$, we denote by $\theta_{x,y}$ the element of $\mathcal{L}(E, F)$ defined by $\theta_{x,y}(z) = y\langle x, z\rangle$ $(z \in F)$. The norm-closed subset generated by $\mathcal{A}$-linear span of $\{\theta_{x,y} : x \in E, y \in F\}$ is called the set of compact operators and denoted by $\mathcal{K}(E, F)$. It should be noted that these objects need not be compact in the sense of compact operators between two Banach spaces, though this abuse of terminology has become standard. It is known that $\mathcal{K}(E, F)$ is dense in $\mathcal{L}(E, F)$ in the strict topology. In case $F = E$, we denote $\mathcal{K}(E, F)$ by $\mathcal{K}(E)$. Note that both $\mathcal{L}(E)$ and $\mathcal{K}(E)$ are C^* algebras.

For a C^* algebra $\mathcal{A}$ (possibly nonunital), its multiplier algebra, denoted by $\mathcal{M}(\mathcal{A})$, is defined as the maximal C^* algebra which contains $\mathcal{A}$ as an essential two-sided ideal. In case $\mathcal{A}$ is unital, one has that $\mathcal{M}(\mathcal{A}) = \mathcal{A}$ and for $\mathcal{A} = C_0(X)$ where X is a noncompact, locally compact Hausdorff space. $\mathcal{M}(\mathcal{A}) = C(\hat{X})$, where $\hat{X}$ denotes the Stone-Čech compactification of X. Let $\tilde{\mathcal{A}}$ denote the universal enveloping von Neumann algebra of $\mathcal{A}$, and let $\mathcal{A}$ be identified as a sub-C^* algebra of $\tilde{\mathcal{A}}$. Then, upto $*$-isomorphism, $\mathcal{M}(\mathcal{A})$ can be described as the C^* algebra $\{x \in \tilde{\mathcal{A}} : xa, ax \in \mathcal{A} \, \forall a \in \mathcal{A}\}$, equipped with the norm given by $\|x\|_m = \sup_{\|a\|=1, a \in \mathcal{A}} \{\|xa\|, \|ax\|\}$. One has (see [33]) the following result :

Proposition 1.3.3 *$\mathcal{M}(\mathcal{K}(E))$ is isomorphic with $\mathcal{L}(E)$ for any Hilbert module E.*

Let us now give a few concrete examples of Hilbert modules. For any Hilbert space $\mathcal{H}$ and C^* algebra $\mathcal{A}$, one may consider the algebraic tensor product $\mathcal{A} \otimes_{\text{alg}} \mathcal{H}$ as a pre-Hilbert module by putting an inner product given by

$$\langle \sum_i x_i \otimes \eta_i, \sum_j x_j' \otimes \eta_j' \rangle = \sum_{i,j} x_i^* x_j' \langle \eta_i, \eta_j' \rangle,$$

which is easily seen to be a valid candidate for inner product. The completion of this pre-Hilbert module under the norm inherited from the above inner product is denoted by $\mathcal{H}_\mathcal{A}$ or $\mathcal{A} \otimes_{C^*} \mathcal{H}$. These relatively simple Hilbert modules are a kind of universal objects, as the following remarkable theorem due to Kasparov asserts.

Theorem 1.3.4 (Kasparov's Stabilisation Theorem)
Let E be a countably generated Hilbert $\mathcal{A}$-module, that is, there is a countable set $B = \{y_1, y_2, \ldots\}$ in E such that the norm closure of the $\mathcal{A}$-linear span of B is the whole of E. Then there exists a unitary element t in $\mathcal{L}(E \oplus \mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{A})$, where $\mathcal{H}$ is a separable infinite dimensional Hilbert space. In other words, $E \oplus \mathcal{H}_\mathcal{A}$ is isomorphic as a Hilbert module with $\mathcal{H}_\mathcal{A}$, and in particular E is imbedded (by the map $t|_E$) in $\mathcal{H}_\mathcal{A}$ as a complemented closed submodule.

Let us mention an important consequence of the above theorem, which will be useful in chapter 4.

Theorem 1.3.5 *Let $\mathcal{B}_0(\mathcal{K})$ denote the C^* algebra of compact operators on a separable Hilbert space $\mathcal{K}$. Then, for any C^* algebra $\mathcal{A}$, $\mathcal{L}(\mathcal{A} \otimes_{C^*} \mathcal{K}) \cong \mathcal{M}(\mathcal{A} \otimes \mathcal{B}_0(\mathcal{K}))$. Thus, in the notation of theorem 1.3.4, we have that for every $s \in \mathcal{L}(E)$, $tst^* \in \mathcal{L}(\mathcal{A} \otimes_{C^*} \mathcal{H}) \cong \mathcal{M}(\mathcal{A} \otimes \mathcal{B}_0(\mathcal{H}))$; hence $\mathcal{L}(E)$ can be imbedded as a C^*-subalgebra in $\mathcal{M}(\mathcal{A} \otimes \mathcal{B}_0(\mathcal{H}))$.*

We shall conclude our discussion on Hilbert C^* modules with the following beautiful unification of Stinespring and GNS constructions in the framework of Hilbert modules.

Theorem 1.3.6 (KSGNS construction)

Let $\mathcal{A}$, $\mathcal{B}$ be C^* algebras, F be a Hilbert $\mathcal{B}$-module and $\rho : \mathcal{A} \to \mathcal{L}(F)$ be continuous with respect to the strict topology on $\mathcal{L}(F)$. Furthermore, assume that ρ is completely positive. Then we have :

(i) There exists a Hilbert $\mathcal{B}$-module F_ρ, $*$-homomorphism $\pi_\rho : \mathcal{A} \to \mathcal{L}(F_\rho)$ and an element v_ρ in $\mathcal{L}(F, F_\rho)$ such that $\rho(a) = v_\rho^* \pi_\rho(a) v_\rho$ for all $a \in \mathcal{A}$ and the $\mathcal{B}$-linear span of $\{\pi_\rho(a) v_\rho f : a \in \mathcal{A}, f \in F\}$ is dense in F_ρ.

(ii) If G is any Hilbert $\mathcal{B}$-module, $\pi : \mathcal{A} \to \mathcal{L}(G)$ is a $*$-homomorphism, $w \in \mathcal{L}(F, G)$ such that $\rho(a) = w^* \pi(a) w \;\forall a \in \mathcal{A}$, and furthermore the $\mathcal{B}$-linear span of $\{\pi(a) w f : a \in \mathcal{A}, f \in F\}$ is dense in G, then there exists a unitary $u \in \mathcal{L}(F_\rho, G)$ such that $\pi(a) = u \pi_\rho(a) u^*$ and $w = u v_\rho$.

The triple $(F_\rho, \pi_\rho, v_\rho)$ is called the KSGNS triple associated with ρ. In case $F = \mathcal{B} = \mathbb{C}$, we recover the GNS theorem, whereas the Stinespring's theorem is obtained by putting $\mathcal{B} = \mathbb{C}$.

1.3.2 Hilbert von Neumann modules

If $\mathcal{A}$ is a concrete C^* algebra in $\mathcal{B}(h)$ for some Hilbert space h, then for any Hilbert space $\mathcal{H}$, the pre-Hilbert module $\mathcal{A} \otimes_{\text{alg}} \mathcal{H}$ may be viewed as a subset of $\mathcal{B}(h, h \otimes \mathcal{H})$ and $\mathcal{A} \otimes_{C^*} \mathcal{H}$ is the closure of this subset under the operator-norm inherited from $\mathcal{B}(h, h \otimes \mathcal{H})$. Instead, we may inherit one of the locally convex topologies from $\mathcal{B}(h, h \otimes \mathcal{H})$, e.g. the topology of strong convergence, and close $\mathcal{A} \otimes_{\text{alg}} \mathcal{H}$ under that topology. This will lead to another topological module, in general bigger than $\mathcal{A} \otimes_{C^*} \mathcal{H}$. To be precise, let us consider the closure under the topology of strong convergence, that is, given by the seminorms $X \mapsto \|Xu\|$ for $u \in h$. We denote the closure by $\mathcal{A} \otimes_s \mathcal{H}$ or simply by $\mathcal{A} \otimes \mathcal{H}$ when there is no possiblity of confusion. $\mathcal{A} \otimes_s \mathcal{H}$ has a natural $\mathcal{A}''$ module action from both sides and has a natural $\mathcal{A}''$ -valued inner product. In view of this, we assume that $\mathcal{A}$ itself is a unital von Neumann algebra in $\mathcal{B}(h)$. We note a few simple but useful facts about the Hilbert von Neumann module $\mathcal{A} \otimes \mathcal{H}$. For this, let us first introduce some notations, which will be very useful in subsequent chapters also.

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and A be a (possibly unbounded) linear operator from $\mathcal{H}_1$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with domain $\mathcal{D}$. For each $f \in \mathcal{H}_2$, we define a linear operator $\langle f, A\rangle$ with domain $\mathcal{D}$ and taking value in $\mathcal{H}_1$ such that.

$$\langle\langle f, A\rangle u, v\rangle = \langle Au, v \otimes f\rangle \tag{1.2}$$

for $u \in \mathcal{D}$, $v \in \mathcal{H}_1$. This definition makes sense because we have. $|\langle Au, v \otimes f\rangle| \leq \|Au\| \ \|f\| \ \|v\|$, and thus $\mathcal{H}_1 \ni v \to \langle Au, v \otimes f\rangle$ is a bounded linear functional. Moreover. $\|\langle f, A\rangle u\| \leq \|Au\| \ \|f\|$, for all $u \in \mathcal{D}$, $f \in \mathcal{H}_2$. Similarly, for each fixed $u \in \mathcal{D}, v \in \mathcal{H}_1$, $\mathcal{H}_2 \ni f \to \langle Au, v \otimes f\rangle$ is bounded linear functional. and hence there exists a unique element of $\mathcal{H}_2$. to be denoted by $A_{v,u}$, satisfying

$$\langle A_{v,u}, f\rangle = \langle Au, v \otimes f\rangle = \langle\langle f, A\rangle u, v\rangle. \tag{1.3}$$

We shall denote by $\langle A, f\rangle$ the adjoint of $\langle f, A\rangle$, whenever it exists. Clearly, if A is bounded. then so is $\langle f, A\rangle$ and $\|\langle f, A\rangle\| \leq \|A\| \ \|f\|$. Similarly, for any $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $f \in \mathcal{H}_2$, one can define $T_f \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ by setting $T_f u = T(u \otimes f)$.

With the above notations at our disposal, let us give a brief sketch of some properties of $\mathcal{A} \otimes \mathcal{H}$.

Lemma 1.3.7 *Any element X of $\mathcal{A} \otimes \mathcal{H}$ can be written as, $X = \sum_{\alpha \in J} x_\alpha \otimes \gamma_\alpha$, where $\{\gamma_\alpha\}_{\alpha \in J}$ is an orthonormal basis of $\mathcal{H}$ and $x_\alpha \in \mathcal{A}$. The above sum over a possibly uncountable index set J makes sense in the usual way: it is strongly convergent and $\forall u \in h$, there exists an at most countable subset J_u of J such that $Xu = \sum_{\alpha \in J_u} (x_\alpha u) \otimes \gamma_\alpha$. Moreover, once $\{\gamma_\alpha\}$ is fixed, x_α's are uniquely determined by X.*

Proof : Set $x_\alpha = \langle \gamma_\alpha, X\rangle$. Clearly, if $X \in \mathcal{A} \otimes_{alg} \mathcal{H}$, $x_\alpha \in \mathcal{A}$ for all α. Since any element of $\mathcal{A} \otimes \mathcal{H}$ is a strong limit of elements from $\mathcal{A} \otimes_{alg} \mathcal{H}$; and since $\mathcal{A}$ is strongly closed, if follows that $x_\alpha \in \mathcal{A}$ for an arbitrary $X \in \mathcal{A} \otimes \mathcal{H}$. Now, for a fixed $u \in h$, let J_u be the (at most countable) set of indices such that $\forall \alpha \in J_u$, $\exists\, v_\alpha \in h$ with $\langle Xu, v_\alpha \otimes \gamma_\alpha\rangle \neq 0$. Then for any $v \in h$ and $\gamma \in \mathcal{H}$, we have with $c_\alpha^\gamma = \langle \gamma_\alpha, \gamma\rangle$,

$$\begin{aligned} \langle Xu, v \otimes \gamma\rangle &= \sum_{\alpha \in J_u} c_\alpha^\gamma \langle Xu, v \otimes \gamma_\alpha\rangle = \sum_{\alpha \in J_u} c_\alpha^\gamma \langle\langle \gamma_\alpha, X\rangle u, v\rangle \\ &= \sum_{\alpha \in J_u} \langle x_\alpha u, v\rangle \langle \gamma_\alpha, \gamma\rangle = \langle \sum_{\alpha \in J_u} (x_\alpha \otimes \gamma_\alpha) u, \ v \otimes \gamma\rangle; \end{aligned}$$

that is, $X = \sum_{\alpha \in J} x_\alpha \otimes \gamma_\alpha$ in the sense described in the statment of the lemma. Given $\{\gamma_\alpha\}$, the choice of x_α's is unique, because for any fixed α_0. $\langle \gamma_{\alpha_0}, X\rangle = x_{\alpha_0}$, which follows from the previous computation if we take γ to be γ_{α_0}. □

Corollary 1.3.8 *Let $X, Y \in \mathcal{A} \otimes \mathcal{H}$ be given by $X = \sum_{\alpha \in J} x_\alpha \otimes \gamma_\alpha$ and $Y = \sum_{\alpha \in J} y_\alpha \otimes \gamma_\alpha$ as in the lemma above. For any finite subset I of J, if we denote by X_I and Y_I the elements $\sum_{\alpha \in I} x_\alpha \otimes \gamma_\alpha$ and $\sum_{\alpha \in I} y_\alpha \otimes \gamma_\alpha$ respectively, then $\lim_I \langle X_I, Y_I \rangle = \langle X, Y \rangle$ where the limit is taken over the directed family of finite subsets of J with usual partial ordering by inclusion.*

Proof : The proof is an easy adaptation of Lemma 27.7 in [45]. □

We give below a convenient necessary and sufficient criterion for verifying whether an element of $\mathcal{B}(h, h \otimes \mathcal{H})$ belongs to $\mathcal{A} \otimes \mathcal{H}$.

Lemma 1.3.9 *Let $X \in \mathcal{B}(h, h \otimes \mathcal{H})$. Then X belongs to $\mathcal{A} \otimes \mathcal{H}$ if and only if $\langle \gamma, X \rangle \in \mathcal{A}$ for all γ in some dense subset $\mathcal{E}$ of $\mathcal{H}$.*

Proof : That $X \in \mathcal{A} \otimes \mathcal{H}$ implies $\langle \gamma, X \rangle \in \mathcal{A}\ \forall\ \gamma \in \mathcal{H}$ has already been observed in the proof of the previous lemma. For the converse, first we claim that $\langle \gamma, X \rangle \in \mathcal{A}$ for all γ in $\mathcal{E}$ (where $\mathcal{E}$ is dense in $\mathcal{H}$) will imply $\langle \gamma, X \rangle \in \mathcal{A}$ for all $\gamma \in \mathcal{H}$. Indeed, for any $\gamma \in \mathcal{H}$ there exists a net $\gamma_\alpha \in \mathcal{E}$ such that $\gamma_\alpha \to \gamma$, and hence $\|\langle \gamma, X \rangle - \langle \gamma_\alpha, X \rangle\| \leq \|\gamma_\alpha - \gamma\| \|X\| \to 0$. Now let us fix an orthonormal basis $\{\gamma_\alpha\}_{\alpha \in J}$ of $\mathcal{H}$ and write $X = \sum_{\alpha \in J} \langle \gamma_\alpha, X \rangle \otimes \gamma_\alpha$ by lemma 1.3.7. Clearly, the net $X_{\mathcal{I}}$ indexed by finite subsets $\mathcal{I}$ of J (partially ordered by inclusion) converges strongly to X. Since $X_{\mathcal{I}} \in \mathcal{A} \otimes_{alg} \mathcal{H}$ for any such finite subset $\mathcal{I}$ (as $\langle \gamma_\alpha, X \rangle \in \mathcal{A}\ \forall\ \alpha$), the proof follows by noting that $\mathcal{A}$ is strongly closed. □

In case $\mathcal{H} = \Gamma(k)$, we call the module $\mathcal{A} \otimes \Gamma(k)$ as the right Fock $\mathcal{A}$-module over $\Gamma(k)$, for short the *Fock module*, and denote it by $\mathcal{A} \otimes \Gamma$.
For various applications of Hilbert modules in quantum probability and related fields, we refer the reader to [2] and [49].

1.4 Fock spaces and Weyl operators

In this final and brief section of the first chapter we recall some well-known facts about Fock spaces. For a Hilbert space $\mathcal{H}$ and positive integer n, let $\mathcal{H}_n \equiv \mathcal{H}^{\otimes^n}$ denote the n-fold tensor product of h, and $\mathcal{H}_0$ denote the one-dimensional Hilbert space $\mathbb{C}$. The free Fock space $\Gamma^f(\mathcal{H})$ is defined as ,

$$\Gamma^f(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

The distinguished vector $\Omega \equiv 1 \oplus 0 \oplus 0 \oplus \ldots$ is called the vacuum. For two Hilbert spaces $\mathcal{H}, \mathcal{K}$ and a contraction $T : \mathcal{H} \to \mathcal{K}$, we denote by T_n the n-fold tensor product of T and set $T_0 = I$. Let us define $\Gamma^f(T) \equiv \oplus_{n=0}^{\infty} T_n : \Gamma^f(\mathcal{H}) \to \Gamma^f(\mathcal{K})$. Then, it is easy to verify the following.

Lemma 1.4.1 *Γ^f is a functor on the category whose objects are Hilbert spaces and morphisms are contractions, that is, $\Gamma^f(ST) = \Gamma^f(S)\Gamma^f(T)$, $\Gamma^f(I) = I$. Furthermore, $\Gamma^f(0)$ is the projection on the Fock vacuum vector and $\Gamma^f(T^*) = (\Gamma^f(T))^*$.*

The proof of the lemma is straightforward and hence omitted.

Let us now discuss symmetric and antisymmetric Fock spaces. Let $\mathcal{H}_n^s$ and $\mathcal{H}_n^a$ denote respectively the symmetric and antisymmetric n-fold tensor products of $\mathcal{H}$ for any positive integer n, and $\mathcal{H}_0^s = \mathcal{H}_0^a = \mathcal{H}_0$. Then the symmetric (or Boson) and antisymmetric (or Fermion) Fock spaces over $\mathcal{H}$, denoted respectively by $\Gamma^s(\mathcal{H})$ and $\Gamma^a(\mathcal{H})$, are defined as.

$$\Gamma^s(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}_n^s,$$

$$\Gamma^a(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}_n^a.$$

We shall be mostly concerned with the symmetric Fock spaces in the present work, and hence for simplicity of notation, we shall use the notation $\Gamma(\mathcal{H})$ for the symmetric Fock space. Let us mention the basic factorization property of $\Gamma(\mathcal{H})$.

Theorem 1.4.2 *Consider the map $\mathcal{H} \ni u \mapsto e(u) \in \Gamma(\mathcal{H})$ given by $e(u) = \oplus_{n=0}^{\infty} (n!)^{\frac{1}{2}} u^n$, where u^n is the n-fold tensor product of u for positive n and $u^0 = 1$. Then the map $e(.)$ is the minimal Kolmogorov decomposition for the positive definite kernel $\mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$ given by $u, v \mapsto exp(\langle u, v \rangle)$. Furthermore, $\{e(u) : u \in \mathcal{H}\}$ is a linearly independent total set of vectors in $\Gamma(\mathcal{H})$.*

Proof : That $e(.)$ is a Kolmogorov decomposition for the above-mentioned kernel is verified by the relation $\langle e(u), e(v) \rangle = exp(\langle u, v \rangle)$. Furthermore, the relation

$$\frac{d^n}{dt^n} e(tu)|_{t=0} = (n!)^{\frac{1}{2}} u^n$$

shows that for every $u \in \mathcal{H}$, u^n belongs to the closed linear span of $e(u)$. Since the vectors of the form u^n where n varies over $\{0, 1, 2, \ldots\}$ are total in $\Gamma(\mathcal{H})$, the assertion about minimality follows. To prove the linear independence, suppose that $u_1, u_2, \ldots, u_n$ are distinct vectors in $\mathcal{H}$ and $z_1, \ldots, z_n$ are complex numbers such that $\sum_{j=1}^{n} z_j e(u_j) = 0$. Then we have, for all $t \in I\!R$, $\sum_{j=1}^{n} z_j exp(t\langle u_j, v\rangle) = 0$ for all $v \in \mathcal{H}$. Since $u_1, u_2, \ldots, u_n$ are distinct, there exists $v \in \mathcal{H}$ such that the scalers $\langle u_j, v\rangle$ are distinct and hence the functions $\{e^{t\langle u_j, v\rangle}\}$ are linearly independent, which implies that $z_j = 0$ for all j. □

Corollary 1.4.3 *For any dense subset S of $\mathcal{H}$, the set $\{e(u) : u \in S\}$ is total in $\Gamma(\mathcal{H})$.*

The proof is easy and we refer the reader to [45] (corollary 19.5, page 127).

Corollary 1.4.4 *There is a natural identification of $\Gamma(\mathcal{H} \oplus \mathcal{K})$ with $\Gamma(\mathcal{H}) \otimes \Gamma(\mathcal{K})$ under which $e(u \oplus v) \mapsto e(u) \otimes e(v)$.*

Proof : The proof is a strtaightforward consequence of the minimality of the Kolmogorov decomposition mentioned in the theorem (1.4.2). □

Let us conclude this section by mentioning the second quantization and Weyl operators. For a contraction C on $\mathcal{H}$, we difine the second quantization $\Gamma(C)$ on $\Gamma(\mathcal{H})$ by

$$\Gamma(C)e(u) = e(Cu).$$

It follows that for $u_1, u_2, \ldots, u_n \in \mathcal{H}$, $\|\Gamma(C)(\sum_{i=1}^{n} e(u_i))\|^2 = \sum_{i,j=1}^{n} e^{\langle Cu_i, Cu_j\rangle} \leq \sum_{i,j=1}^{n} e^{\langle u_i, u_j\rangle} = \|\sum_{i=1}^{n} e(u_i)\|^2$. This shows that $\Gamma(C)$ is well defined and extends to a bounded operator. Clearly, if C is isometry (respectively unitary), then so is $\Gamma(C)$. For $u \in \mathcal{H}$, we define the Weyl operators $W(u)$ by setting

$$W(u)e(v) = exp(-\frac{1}{2}\|u\|^2 - \langle u, v\rangle)e(u + v).$$

It is known that the von Neumann algebra generated by the family $\{W(u) : u \in S\}$ is the whole of $\mathcal{B}(\Gamma(\mathcal{H}))$ whenever S is a dense subspace of $\mathcal{H}$.
We refer the reader to [45] for a further detailed description of Fock spaces and related topics with applications to quantum probability.

Chapter 2

A coordinate-free quantum stochastic calculus

2.1 Basic processes

Let us recall the notations introduced in the subsection 1.3.2 and section 1.4 of chapter 1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces (possibly nonseparable). Now,we define a map $S : \Gamma^f(\mathcal{H}_2) \to \Gamma(\mathcal{H}_2)$ by setting,

$$S(g_1 \otimes g_2 \otimes \cdots \otimes g_n) = \frac{1}{(n-1)!} \sum_{\sigma \in S_n} g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)}, \tag{2.1}$$

and linearly extending it to $\mathcal{H}_2^{\otimes^n}$, where S_n is the group of permutations of n objects. Clearly, $\|S|_{\mathcal{H}_2^{\otimes^n}}\| \leq n$. We denote by $\tilde{S}$ the operator $1_{\mathcal{H}_1} \otimes S$.

Let A be a linear map from $\mathcal{H}_1$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with domain $\mathcal{D}$. Let us now define the creation operator $a^\dagger(A)$ abstractly which will act on the linear span of vectors of the form $vg^{\otimes^n}$ and $ve(g)$ (where $g^{\otimes^n}$ denotes $\underbrace{g \otimes \cdots \otimes g}_{n \text{ times}}$), $n \geq 0$, with $v \in \mathcal{D}$, $g \in \mathcal{H}_2$. It is to be noted that we shall often omit the tensor product symbol $\otimes$ between two or more vectors when there is no confusion. We define,

$$a^\dagger(A)(vg^{\otimes^n}) = \frac{1}{\sqrt{n+1}} \tilde{S}((Av) \otimes g^{\otimes^n}). \tag{2.2}$$

It is easy to observe that $\sum_{n \geq 0} \frac{1}{n!} \|a^\dagger(A)(vg^{\otimes^n})\|^2 < \infty$, which allows us to define $a^\dagger(A)(ve(g))$ as the direct sum $\bigoplus_{n \geq 0} \frac{1}{(n!)^{\frac{1}{2}}} a^\dagger(A)(vg^{\otimes^n})$. We have the following simple but useful observation :

Lemma 2.1.1 *For $v \in \mathcal{D},\ u \in \mathcal{H}_1, g, h \in \mathcal{H}_2$,*

$$\langle a^\dagger(A)(ve(g)),\ ue(h)\rangle = \langle A_{u,v}, h\rangle\langle e(g),\ e(h)\rangle = \frac{d}{d\varepsilon}\langle e(g+\varepsilon A_{u,v}), e(h)\rangle|_{\varepsilon=0}. \quad (2.3)$$

Proof : First observe that

$$\langle a^\dagger(A)(vg^{\otimes^n}), ue(h)\rangle = \langle \frac{1}{\sqrt{n+1}}\tilde{S}((Av)\otimes g^{\otimes^n}), \frac{1}{\sqrt{(n+1)!}}u\otimes h^{\otimes^{n+1}}\rangle.$$

It is clear that the adjoint S^* of the operator S is given by $S^*(f^{\otimes^n}) = nf^{\otimes^n}$. Thus, we have that $\langle \tilde{S}((Av)\otimes g^{\otimes^n}), u\otimes h^{\otimes^{n+1}}\rangle = \langle ((Av)\otimes g^{\otimes^n}), u\otimes S^*(h^{\otimes^{n+1}})\rangle = (n+1)\langle Av, u\otimes h\rangle\langle g, h\rangle^n = (n+1)\sqrt{n!}\langle A_{u,v}, h\rangle\langle g^{\otimes^n}, e(h)\rangle$. Hence $\langle a^\dagger(A)(vg^{\otimes^n}), ue(h)\rangle = \frac{(n+1)\sqrt{n!}}{\sqrt{(n+1)!}\sqrt{n+1}}\langle A_{u,v}, h\rangle\langle g^{\otimes^n}, e(h)\rangle = \langle A_{u,v}, h\rangle\langle g^{\otimes^n}, e(h)\rangle$. From this the result follows. □

In the same way, one can define annihilation and number operators in $\mathcal{H}_1\otimes\Gamma(\mathcal{H}_2)$ for $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1\otimes\mathcal{H}_2)$ and $T \in \mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$ as :

$$a(A)ue(h) = < A, h > ue(h),$$

$$\Lambda(T)ue(h) = a^\dagger(T_h)ue(h).$$

One can also verify that in this case $a^\dagger(A)$ is the adjoint of $a(A)$ on $\mathcal{H}_1\otimes\mathcal{E}(\mathcal{H}_2)$, where $\mathcal{E}(\mathcal{H}_2)$ is the linear span of exponential vectors $e(g), g\in\mathcal{H}_2$. Next, to define the basic processes, we need some more notations. Let k_0 be a Hilbert space, $k = L^2(\mathbb{R}_+, k_0),\ k_t = L^2([0,t])\otimes k_0, k^t = L^2((t,\infty))\otimes k_0,\ \Gamma_t = \Gamma(k_t),\ \Gamma^t = \Gamma(k^t),\ \Gamma = \Gamma(k)$. We assume that $R \in \mathcal{B}(h, h\otimes k_0)$ and define $R_t^\Delta : h\otimes\Gamma_t \to h\otimes\Gamma_t\otimes k^t$ for $t\geq 0$ and a bounded interval Δ in (t,∞) by,

$$R_t^\Delta(u\psi) = P((1_h\otimes\chi_\Delta)(Ru)\otimes\psi)$$

where $\chi_\Delta : k_0 \to k^t$ is the operator which takes α to $\chi_\Delta(\cdot)\alpha$ for $\alpha\in k_0$, and P is the canonical unitary isomorphism from $h\otimes k\otimes\Gamma$ to $h\otimes\Gamma\otimes k$. We define the creation field $a_R^\dagger(\Delta)$ on either of the domains consisting of the finite linear combinations of vectors of the form $u_t\otimes f^{t\otimes^n}$ or of $u_t\otimes e(f^t)$ for $u_t\in h\otimes\Gamma_t,\ f^t\in\Gamma^t,\ n\geq 0$, as :

$$a_R^\dagger(\Delta) = a^\dagger(R_t^\Delta), \quad (2.4)$$

where $a^\dagger(R_t^\Delta)$ carries the meaning discussed before lemma 2.1.1, with $\mathcal{H}_1 = h\otimes\Gamma_t,\ \mathcal{H}_2 = k^t$. Similarly the other two fields $a_R(\Delta)$ and $\Lambda_T(\Delta)$ can be defined by

$$a_R(\Delta)(u_t e(f^t)) = ((\int_\Delta \langle R.f(s)\rangle ds)u_t)e(f^t). \tag{2.5}$$

and for $T \in \mathcal{B}(h \otimes k_0)$,

$$\Lambda_T(\Delta)(u_t e(f^t)) = a^\dagger(T_{ft}^\Delta)(u_t e(f^t)). \tag{2.6}$$

In the above. $T_{ft}^\Delta : h \otimes \Gamma_t \to h \otimes \Gamma_t \otimes k^t$ is defined as,

$$T_{ft}^\Delta(u\alpha_t) = P(1 \otimes \hat{\chi}_\Delta)(\hat{T}(uf^t) \otimes \alpha_t), \tag{2.7}$$

and $\hat{T} \in \mathcal{B}(h \otimes L^2((t,\infty),k_0))$ is given by, $\hat{T}(u)(s) = T(u(s))$, $s > t$, and $\hat{\chi}_\Delta$ is the multiplication by $\chi_\Delta(\cdot)$ on $L^2((t,\infty),k_0)$. Clearly, $\|\hat{T}\| \leq \|T\|$, which makes T_{ft}^Δ bounded. We note here that objects similar to $a_R(.), a_R^\dagger(.)$ and $\Lambda_T(.)$ were used in [27], however in a coordinatized form. In what follows, we shall assume that $(H_t)_{t\geq 0}$ and $(H_t')_{t\geq 0}$ are two operator-valued Fock-adapted processes (in the sense of [45]). having all vectors of the form $ve(f_t)\psi^t$ in their domains. where $v \in h$, $f_t \in k_t$, $\psi^t \in \Gamma^t$. We also assume that there exist constants $c(t,f)$ and $c'(t,f)$ such that for $t \geq 0$.

$$\sup_{0\leq s\leq t} \|H_s(ue(f))\| \leq c(t,f)\|u\|, \quad \sup_{0\leq s\leq t} \|H_s'(ue(f))\| \leq c'(t,f)\|u\| \tag{2.8}$$

We shall often denote an operator B and its trivial extension $B \otimes I$ to some bigger space by the same notation, unless there is any confusion in doing so. We also denote the unitary isomorphism from $h \otimes k_0 \otimes \Gamma(k)$ onto $h \otimes \Gamma(k) \otimes k_0$ and that from $h \otimes k \otimes \Gamma(k)$ onto $h \otimes \Gamma(k) \otimes k$ by the same letter P. Clearly, $H_t P$ acts on any vector of the form $w \otimes e(g)$ where $w \in h \otimes_{alg} k_0, g \in k$ and $\sup_{0\leq s\leq t} \|H_s P(we(g))\| \leq c(t,g)\|w\|$. This allows one to extend $H_t P$ on the whole of the domain containing vectors of the form $\tilde{w}e(g)$. $\tilde{w} \in h \otimes k_0$. $g \in k$. We denote this extension again by $H_t P$. Similarly we define $H_t' P$. When P is taken to be the isomorphism from $h \otimes k \otimes \Gamma(k)$ onto $h \otimes \Gamma(k) \otimes k$. we define $H_t P$ and $H_t' P$ in an exactly parallel manner.

Next we prove a few preliminary results which will be needed for establishing quantum Ito formula in the next subsection.

Lemma 2.1.2 *Let* $\Delta, \Delta' \subseteq (t,\infty)$ *be intervals of finite length,* $R, S \in \mathcal{B}(h, h \otimes k_0)$; $u, v \in h$; $g, f \in k$. *Then we have,*

$$\langle H_t a_R^\dagger(\Delta)(ve(g)),\ H_t' a_S^\dagger(\Delta')(ue(f))\rangle = e^{\langle g^t, f^t\rangle}\{\langle H_t R_t^\Delta(ve(g_t)),\ H_t' S_t^{\Delta'}(ue(f_t))\rangle$$

$$+\langle\langle f^t, H_t R_t^{\Delta}\rangle ve(g_t), \langle g^t, H_t' S_t^{\Delta'}\rangle ue(f_t)\rangle\} = \int\limits_{\Delta\cap\Delta'} \langle (H_t PR)(ve(g)), (H_t' PS)(ue(f))\rangle ds$$

$$+\int\limits_{\Delta}\int\limits_{\Delta'} \langle\langle f(s), H_t PR\rangle(ve(g)), \langle g(s'), H_t' PS\rangle(ue(f))\rangle ds\, ds'. \qquad (2.9)$$

Proof. For the present proof, we make the convention of writing $\frac{df(\varepsilon)}{d\varepsilon}|_{\varepsilon=\varepsilon_0}$ for the limit $\lim_{n\to\infty} n(f(\varepsilon_0+\frac{1}{n}) - f(\varepsilon_0))$ whenever it exists, and R_Δ will denote $(1\otimes\chi_\Delta)R \in \mathcal{B}(h, h\otimes k)$ for $R\in\mathcal{B}(h, h\otimes k_0)$. Let us now choose and fix orthonormal bases $\{e_\nu\}_{\nu\in J}$ and $\{k_\alpha\}_{\alpha\in I}$ of $h\otimes\Gamma_t$ and Γ^t respectively $(t\geq 0)$. We also choose subsets J_0 and I_0, which are at most countable, of J and I respectively as follows. Let J_0 be such that
$\langle H_t PR_\Delta(ve(g)), e_\nu\otimes k_\alpha\rangle = 0 = \langle e_\nu\otimes k_\alpha, H_t' PS_{\Delta'}(ue(f))\rangle$ for all $\alpha\in I$ whenever $\nu\notin J_0$. Fixing this J_0, we choose I_0 to be the union of $I_{\nu,n}$, $\nu\in J_0$, $n = 1, 2, \ldots, \infty$, such that

$$\langle e(g^t + \frac{1}{n}(H_t PR_\Delta)_{e_\nu, ve(g_t)}), k_\alpha\rangle = 0 = \langle k_\alpha, e(f^t + \frac{1}{n}(H_t' PS_{\Delta'})_{e_\nu, ue(f_t)})\rangle$$

for all $\alpha\neq I_{\nu,n}$ when $n<\infty$, and

$$\langle e(g^t), k_\alpha\rangle = 0 = \langle k_\alpha, e(f^t)\rangle \quad \text{for} \quad \alpha\neq I_{\nu,\infty}.$$

We have now,

$$\begin{aligned}
&\langle H_t a_R^\dagger(\Delta)(ve(g)), H_t' a_S^\dagger(\Delta')(ue(f))\rangle \\
&= \sum_{\substack{\nu\in J_0\\ \alpha\in I_0}} \langle H_t a_R^\dagger(\Delta)(ve(g)), e_\nu\otimes k_\alpha\rangle\langle e_\nu\otimes k_\alpha, H_t' a_S^\dagger(\Delta')(ue(f))\rangle \\
&= \sum_{\substack{\nu\in J_0\\ \alpha\in I_0}} (\frac{d}{d\varepsilon}\langle e(g^t + \varepsilon(H_t PR_\Delta)_{e_\nu, ve(g_t)}), k_\alpha\rangle|_{\varepsilon=0}) \times (\frac{d}{d\eta}\langle k_\alpha, e(f^t + \eta(H_t' PS_{\Delta'})_{e_\nu, ue(f_t)})\rangle|_{\eta=0}) \\
&= \sum_{\nu\in J_0} \frac{\partial^2}{\partial\varepsilon\partial\eta}(\sum_{\alpha\in I_0}\langle e(g^t + \varepsilon(H_t PR_\Delta)_{e_\nu, ve(g_t)}), k_\alpha\rangle\langle k_\alpha, e(f^t + \eta(H_t' PS_{\Delta'})_{e_\nu, ue(f_t)})\rangle|_{\varepsilon=0=\eta}) \\
&= \sum_{\nu\in J_0} \frac{\partial^2}{\partial\varepsilon\partial\eta}(\langle e(g^t + \varepsilon(H_t PR_\Delta)_{e_\nu, ve(g_t)}), e(f^t + \eta(H_t' PS_{\Delta'})_{(e_\nu, ue(f_t))})\rangle|_{\varepsilon=0=\eta}) \\
&= \sum_{\nu\in J_0} e^{\langle g^t, f^t\rangle}(\langle (H_t PR_\Delta)_{e_\nu, ve(g_t)}, (H_t' PS_{\Delta'})_{e_\nu, ue(f_t)}\rangle \\
&\quad + \langle (H_t PR_\Delta)_{e_\nu, ve(g_t)}, f^t\rangle\langle g^t, (H_t' PS_{\Delta'})_{e_\nu, ue(f_t)}\rangle)
\end{aligned}$$

Before proceeding further, let us justify the intermediate step in the above calculations, which involves an interchange of summation and limit, by appealing to the dominated convergence theorem. Indeed, for any fixed $\alpha \in I_0$, ψ, $\psi' \in k^t$, if we write $k_\alpha^{(n)}$ for the projection of k_α on $k^{t^{\otimes n}}$ $(n \geq 0)$, then $\langle e(g^t + \varepsilon\psi), k_\alpha\rangle$ can be expressed as $\sum_{i\geq 0} c_i^{(\alpha)}\varepsilon^i$, where $c_i^{(\alpha)} = \sum_{n\geq i} \frac{1}{\sqrt{n!}}\binom{n}{i}\langle g^{t\otimes^{(n-i)}} \otimes \psi^{\otimes^{(i)}}, k_\alpha^{(n)}\rangle$, where $g^{t\otimes^{(n-i)}} \equiv \underbrace{g^t \otimes \cdots \otimes g^t}_{(n-i)-\text{times}}$, and $\psi^{\otimes^{(i)}} \equiv \underbrace{\psi \otimes \cdots \otimes \psi}_{i-\text{times}}$. It can be easily verified that the above is an absolutely summable power series in ε, converging uniformly for $\varepsilon \in [0, M]$, say, for any fixed $M > 0$. Similar analysis can be done for $\langle k_\alpha, e(f^t + \eta\psi')\rangle$. By Mean Value Theorem and some straightforward estimate, we have that for $\varepsilon, \eta, \varepsilon', \eta'$ in $[0, M]$,

$$\frac{1}{|(\varepsilon-\varepsilon')(\eta-\eta')|}\sum_{\alpha\in I_0}|(\langle e(g^t+\varepsilon\psi),k_\alpha\rangle - \langle e(g^t+\varepsilon'\psi),k_\alpha\rangle)$$

$$\times(\langle k_\alpha, e(f^t+\eta\psi')\rangle - \langle k_\alpha, e(f^t+\eta'\psi')\rangle)|$$

$$\leq \sum_{\substack{n\geq 0, m\geq 0,\\ 0\leq i\leq n, 0\leq j\leq m}} \frac{i.j.M^{i+j-2}}{\sqrt{n!m!}}\binom{n}{i}\binom{m}{j}\sum_{\alpha\in I_0}|\langle g^{t\otimes^{(n-i)}}\otimes\psi^{\otimes^{(i)}}, k_\alpha^{(n)}\rangle\,\langle k_\alpha^{(n)}, f^{t\otimes^{(m-j)}}\otimes\psi'^{\otimes^{(j)}}\rangle|$$

$$\leq \sum_{n,m,i,j}\frac{ijM^{i+j-2}}{\sqrt{n!m!}}\binom{n}{i}\binom{m}{j}\|g^{t\otimes^{(n-i)}}\otimes\psi^{\otimes^{(i)}}\|\;\|f^{t\otimes^{(m-j)}}\otimes\psi'^{\otimes^{(j)}}\|$$

[since $\{k_\alpha^{(n)}\}_{\alpha\in I_0}$ are mutually orthogonal for any fixed n, with $\|k_\alpha^{(n)}\| \leq 1\ \forall\alpha$]

$$\leq \sum_{\substack{n\geq 0\\ m\geq 0}} \frac{mn\|\psi\|\,\|\psi'\|\,(\|g^t\|+M\|\psi\|)^{n-1}(\|f^t\|+M\|\psi'\|)^{m-1}}{\sqrt{m!n!}} < \infty$$

This allows us to apply dominated convergence theorem.

Let us now choose a countable subset I_0' of I so that $0 = \langle (H_tPR_\Delta)_{e_\nu, ve(g_t)}, k_\alpha\rangle = \langle k_\alpha, (H_t'PS_{\Delta'})_{e_\nu, ue(f_t)}\rangle$ for α not in I_0', for all $\nu \in J_0$.
Clearly, we have

$$\begin{aligned}&\textstyle\sum_{\nu\in J_0}\langle (H_tPR_\Delta)_{e_\nu, ve(g_t)}, (H_t'PS_{\Delta'})_{e_\nu, ue(f_t)}\rangle\\ &= \sum_{\nu\in J_0, \alpha\in I_0'}\langle (H_tPR_\Delta)(ve(g_t)), e_\nu\otimes k_\alpha\rangle\,\langle e_\nu\otimes k_\alpha, (H_t'PS_{\Delta'})(ue(f_t))\rangle\\ &= \langle (H_tPR_\Delta)(ve(g_t)), (H_t'PS_{\Delta'})(ue(f_t))\rangle.\end{aligned}$$

We choose sequences $\omega^{(n)}$, $\omega'^{(n)}$ of vectors which can be written as finite sums of the form, $\omega^{(n)} = \sum v_i^{(n)} \otimes \beta_i^{(n)}$, $\omega'^{(n)} = \sum_i u_i^{(n)} \otimes \alpha_i^{(n)}$, where $u_i^{(n)}, v_i^{(n)} \in h$, $\beta_i^{(n)}, \alpha_i^{(n)} \in k_0$, and $\omega^{(n)} \to Rv$, $\omega'^{(n)} \to Su$ as $n \to \infty$.

Then we have,

$$\|H_tP(1 \otimes \chi_\Delta)\ (\omega^{(n)} \otimes e(g_t)) - H_tPR_\Delta(ve(g_t))\|$$

$$\leq c(t,g)\|\omega^{(n)} - (Rv)\|\ |\Delta|\ \to 0 \text{ as } n \to \infty,$$

where $|\Delta|$ denotes the Lebesgue measure of Δ. Similarly, $\|H_t'P(1 \otimes \chi_{\Delta'})\ (\omega'^{(n)} \otimes e(f_t)) - (H_t'PS_{\Delta'})(ue(f_t))\| \to 0$ as $n \to \infty$. Hence we obtain

$$\begin{aligned}
&\langle (H_tPR_\Delta)(ve(g_t)), (H_t'PS_{\Delta'}(ue(f_t))\rangle \\
&= \lim_{n\to\infty} \langle H_tP(1 \otimes \chi_\Delta)(\omega^{(n)} e(g_t)), H_t'P(1 \otimes \chi_{\Delta'})(\omega'^{(n)} e(f_t))\rangle \\
&= \lim_{n\to\infty} \int \langle H_t(\sum_i v_i^{(n)} \otimes e(g_t) \otimes \beta_i^{(n)}), H_t'(\sum_i u_i^{(n)} \otimes e(f_t) \otimes \alpha_i^{(n)})\rangle \chi_{\Delta\cap\Delta'}(s)ds \\
&= \lim_{n\to\infty} |\Delta \cap \Delta'| \langle (H_tP)(\omega^{(n)} e(g_t)), (H_t'P)(\omega'^{(n)} e(f_t))\rangle \\
&= |\Delta \cap \Delta'| \langle (H_tPR)(ve(g_t)), (H_t'PS)(ue(f_t))\rangle \\
&= \int_{\Delta\cap\Delta'} \langle H_tPR(ve(g_t)), H_t'PS(ue(f_t))\rangle ds.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sum_{\nu\in J_0} \langle (H_tPR_\Delta)_{e_\nu, ve(g_t)}, f^t\rangle\ \langle g^t, (H_t'PS_{\Delta'})_{e_\nu, ue(f_t)}\rangle \\
&= \sum_{\nu\in J_0} \langle \langle f^t, H_tPR_\Delta\rangle(ve(g_t)), e_\nu\rangle\ \langle e_\nu, \langle g^t, H_t'PS_{\Delta'}\rangle\ (ue(f_t))\rangle \\
&= \langle \langle f^t, H_tPR_\Delta\rangle(ve(g_t)), \langle g^t, (H_t'PS_{\Delta'})\rangle(ue(f_t))\rangle,
\end{aligned}$$

where the last step follows by Parseval's identity, noting the fact that for $\nu \notin J_0$, $\langle \langle f^t, H_tPR_\Delta\rangle(ve(g_t)), e_\nu\rangle = 0$ because for such ν, $\langle (H_tPR_\Delta)(ve(g_t)), e_\nu \otimes k_\alpha\rangle = 0$ for all $\alpha \in I$; and similarly $\langle e_\nu, \langle g^t, (H_t'PS_{\Delta'})\rangle(ue(f_t))\rangle = 0\ \forall \nu \notin J_0$.

We complete the proof by observing that

$$\begin{aligned}
\langle f^t, H_tR_t^\Delta\rangle &= \int_\Delta \langle f(s), H_tPR\rangle ds, \text{ and} \\
\langle g^t, H_t'S_t^{\Delta'}\rangle &= \int_{\Delta'} \langle g(s'), H_t'PS\rangle ds'.
\end{aligned}$$

To see this, it is enough to note that for $\omega \in h$, $h_t \in k_t$, we have,

$$\begin{aligned}&\langle\langle f^t, H_t R_t^{\Delta}\rangle(ve(g_t)), \omega e(h_t)\rangle \\ &= \int_\Delta \langle (H_t PR)(ve(g_t)), \omega e(h_t)\otimes f^t(s)\rangle ds,\end{aligned}$$ which can be justified by considering $\omega^{(n)}$ as before and applying dominated convergence theorem.

Since $\Delta \subseteq (t,\infty)$ and hence $f^t(s) = f(s)$ for $s \in \Delta$, the above expression can now be written as

$$\begin{aligned}&\int_\Delta \langle (H_t PR)(ve(g_t)), we(h_t) f(s)\rangle ds \\ &= \int_\Delta \langle\langle f(s), H_t PR\rangle(ve(g_t)), we(h_t)\rangle ds.\end{aligned}$$

This completes the proof. □

Remark 2.1.3 *If H_t and H_t' are bounded, then (2.9) of Lemma 2.1.2 holds with u, v replaced by arbitrary vectors in $h\otimes\Gamma_t$ and f, g by the same in k^t.*

Lemma 2.1.4 *Let $T, T' \in \mathcal{B}(h\otimes k_0)$. Then we have,*

$$\begin{aligned}&\langle (H_t T_{g^t}^{\Delta})(ve(g)), (H_t' T_{f^t}^{\Delta'})(ue(f))\rangle \\ &= \int_{\Delta\cap\Delta'} \langle H_t PTP^*(ve(g)g(s)), H_t' PT'P^*(ue(f)f(s))\rangle ds,\end{aligned}$$

and

$$\langle g^t, H_t' T_{f^t}^{\Delta'}\rangle = \int_{\Delta'} \langle g(s), H_t' T'_{f(s)}\rangle ds.$$

Proof : We choose sequences $w^{(n)}, w'^{(n)}$ in $h\otimes L^2((t,\infty),k_0)$, which can be written as finite sums, $w^{(n)} = \sum_i v_i^{(n)}\otimes g_i^{(n)}$ and $w'^{(n)} = \sum_j u_j^{(n)}\otimes f_j^{(n)}$; $v_i^{(n)}, u_j^{(n)} \in h$, $g_i^{(n)}, f_j^{(n)} \in L^2((t,\infty),k_0)$; and $w^{(n)} \to \hat{T}(vg^t)$, $w'^{(n)} \to \hat{T}(uf^t)$ in L^2 as $n\to\infty$. Clearly we have,

$$\begin{aligned}&\langle H_t T_{g^t}^{\Delta}(ve(g)), H_t' T'^{\Delta'}_{f^t}(ue(f))\rangle \\ &= \lim_{n\to\infty}\int \langle \sum_i H_t(v_i e(g))\chi_\Delta(s) g_i(s), \sum_j H_t'(u_j e(f))\chi_{\Delta'}(s) f_j(s)\rangle ds \\ &= \lim_{n\to\infty}\int_{\Delta\cap\Delta'} \langle H_t P(w^{(n)}(s)e(g)), H_t' P(w'^{(n)}(s)e(f))\rangle ds \\ &= \int_{\Delta\cap\Delta'} \langle H_t P(\hat{T}(vg^t)(s)e(g)), H_t' P(\hat{T}'(uf^t)(s)e(f))\rangle ds.\end{aligned}$$

The last step follows because $\int_{\Delta\cap\Delta'} \|H_tP(w^{(n)}(s)e(g)) - H_tP(\hat{T}(vg^t)(s)e(g))\|^2 ds$ is majorized by a constant times $\|w^{(n)} - \hat{T}(vg^t)\|^2$ which goes to 0, and similar statement holds for H_t' and $w'^{(n)}$. Since for $s \in \Delta \bigcap \Delta' \subseteq (t,\infty)$, $H_tP(\hat{T}(vg^t)(s)e(g)) = H_tP(T(vg(s))e(g)) = H_tPTP^*(ve(g)g(s))$, and similarly $H_t'P(\hat{T'}(uf^t)(s)e(f)) = H_t'PTP^*(ue(f)f(s))$, the proof of the first part of the lemma is complete. The other part is similar. □

Lemma 2.1.5 *For $\eta \in k_0$, $\langle \eta, H_tPR\rangle ve(g) = H_t((\langle \eta, R\rangle v)e(g))$, where $v \in h$, $g \in k$.*

Proof: It is easy to see that by virtue of 2.8, for every fixed $g,\ f \in k, \eta \in k_0, t \geq 0$, $\langle H_t(ve(g)), ue(f)\rangle = \langle v, M_t u\rangle$ defines an operator $M_t \in \mathcal{B}(h)$. Let $\tilde{M}_t = M_t \otimes 1_{k_0}$. Then we have, for $w = v \otimes \alpha,\ w' = u \otimes \beta; \alpha, \beta \in k_0,\ u, v \in h$,

$$\langle H_tP(we(g)),\ P(w'e(f))\rangle = \langle w, \tilde{M}_t w'\rangle.$$

By the density of $h \otimes_{alg} k_0$ in $h \otimes k_0$, we have that $\langle H_tP(we(g)),\ P(w'e(f))\rangle = \langle w, \tilde{M}_t w'\rangle$ for all $w, w' \in h \otimes k_0$. Thus

$$\langle \langle \eta, H_tPR\rangle ve(g)\ ue(f)\rangle = \langle H_tP((Rv)e(g)),\ ue(f)\eta\rangle$$

$$= \langle H_tP((Rv)e(g)),\ P(u\eta e(f))\rangle = \langle Rv,\ \tilde{M}_t(u\eta)\rangle = \langle Rv,\ (M_tu) \otimes \eta\rangle$$

$$= \langle \langle \eta, R\rangle v,\ M_tu\rangle = \langle H_t((\langle \eta, R\rangle v)e(g)), ue(f)\rangle.$$

This completes the proof, since the vectors of the form $ue(f)$ are total in $h \otimes \Gamma(k)$. □

2.2 Stochastic integrals and Quantum Ito formulae.

Following [26] and [45], we call an adapted process $(H_t)_{t\geq 0}$ satisfying $\sup_{0\leq s\leq t} \|H_s ve(g)\| \leq c(t,g)\|v\|$ (for all $v \in h,\ f \in k$), to be *simple* if H_t is of the form.

$$H_t = \sum_{i=0}^{m} H_{t_i} \chi_{[t_i, t_{i+1})}(t)$$

where m is an integer (≥ 1), and $0 \equiv t_0 < t_1 < \cdots < t_m < t_{m+1} \equiv \infty$. If M denotes one of the four basic processes $a_R, a_R^\dagger$ and Λ_T and tI, and if (H_t) is simple, then we define the left and right integrals $\int_0^t H_s M(ds)$ and $\int_0^t M(ds)H_s$ respectively in the natural manner :

$$\int_0^t H_s M(ds) = \sum_{i=0}^m H_{t_i} M([t_i, t_{i+1}) \cap [0,t]),$$
$$\int_0^t M(ds) H_s = \sum_{i=0}^m M([t_i, t_{i+1}) \cap [0,t]) H_{t_i}.$$

We call H_t to be *regular* if $t \mapsto H_t(ue(f))$ is continuous for all fixed $u \in h$ and $f \in k$. Also note that if H_t is regular, then so is the extension $H_t P$. The next proposition gives the quantum Ito formulae for simple integrands.

Proposition 2.2.1 *Let $u, v \in h$; $f, g \in L^2(\mathbb{R}_+, k_0)$; $R, S, R', S' \in \mathcal{B}(h, h \otimes k_0)$ and let $T, T' \in \mathcal{B}(h \otimes k_0)$. Furthermore, assume that E, F, G, H and E', F', G', H' are adapted simple processes satisfying the bound given at the beginning of this subsection, and that*

$$X_t = \int_0^t \left(E_s \Lambda_T(ds) + F_s a_R(ds) + G_s a_S^\dagger(ds) + H_s ds \right),$$
$$X'_t = \int_0^t \left(E'_s \Lambda_{T'}(ds) + F'_s a_{R'}(ds) + G'_s a_{S'}^\dagger(ds) + H' ds \right).$$

Then we have,
(i) (first fundamental formula)

$$\begin{aligned} &< X_t ve(g), ue(f) > \\ &= \int_0^t ds < \{< f(s), E_s P T_{g(s)} > + F_s < R, g(s) > + G_s < f(s), S > + H_s\}(ve(g)), ue(f) > \end{aligned} \tag{2.10}$$

(ii)(second fundamental formula or Quantum Ito formula)
For this part suppose that $f, g \in k \cap L^\infty(\mathbb{R}_+, k_0)$. Then

$$\begin{aligned} &< X_t ve(g), X'_t ue(f) > \\ &= \int_0^t ds \left[< X_s ve(g), \{< g(s), E'_s P T'_{f(s)} > + F'_s < R', f(s) > + G'_s < g(s), S' > + H'_s\}(ue(f)) >\right] \\ &+ \int_0^t ds \left[< \{< f(s), E_s P T_{g(s)} > + F_s < R, g(s) > + G_s < f(s), S > + H_s\}(ve(g)), X'_s ue(f) >\right] \end{aligned}$$

$$+ \int_0^t ds\Big[< E_sPT_{g(s)}(ve(g)), E_s'PT_{f(s)}'(ue(f)) > + < E_sPT_{g(s)}(ve(g)), G_s'PS'(ue(f)) >$$
$$+ < G_sPS(ve(g)), E_s'PT_{f(s)}'(ue(f)) > + < G_sPS(ve(g)), G_s'PS'(ue(f)) >\Big]. \qquad (2.11)$$

Proof : The proof is very similar in spirit to the proof in [26], [45]. First, a comment with regard to the notation used above is in order. For example, for almost all $s \in I\!R_+$, the expression $E_sPT_{g(s)}(ve(g))$ is to be understood as $(E_s \otimes I_{k_0})P(T_{g(s)}v \otimes e(g)) = (E_s \otimes I_{k_0})P(T(v \otimes g(s)) \otimes e(g)) \in h \otimes \Gamma \otimes k_0$. Thus the operator $E_sPT_{g(s)}$ maps $h \otimes \Gamma$ into $h \otimes \Gamma \otimes k_0$ and therefore by the discussion in subsection 2.1, $< f(s), E_sPT_{g(s)} >$ maps $h \otimes \Gamma$ into $h \otimes \Gamma$.

We denote by $\|f\|_\infty$ and $\|g\|_\infty$ the essential supremums of f and g respectively. We fix $t \geq 0$, and without loss of generality, take $\mathcal{P} \equiv \{t_i\}_{i=0,1,\dots,m+1}$ to be a partition of $[0,t]$ such that $0 = t_0 < t_1 < \dots < t_{m+1} = t$; and $L_s = \sum_{i=o}^m L_{t_i}\chi_{[t_i,t_{i+1})}(s)$, $L_s' = \sum_{i=0}^m L_{t_i}'\chi_{[t_i,t_{i+1})}(s)$, where L is one of the four coefficient processes E, F, G, H and L' is one of the processes E', F', G', H'. Observe that the definition of stochastic integrals for simple adapted processes does not depend on the choice of the partition as long as L_s and L_s' take constant values in any subinterval of the partition; and this allows us to refine $\mathcal{P}$ arbitrarily.

By definitions of the basic processes as given in the previous section, we have that

$$\begin{aligned}
&\langle X_t ve(g), ue(f)\rangle \\
&= \sum_{i=0}^m \langle\{E_{t_i}\Lambda_T([t_i,t_{i+1})) + F_{t_i}a_R([t_i,t_{i+1})) + G_{t_i}a_S^\dagger([t_i,t_{i+1})) + H_{t_i}(t_{i+1}-t_i)\}, ue(f)\rangle \\
&= \sum_{i=0}^m \langle\{E_{t_i}a^\dagger_{T_{g^{t_i}}^{[t_i,t_{i+1})}} + G_{t_i}a^\dagger_{S_{t_i}^{[t_i,t_{i+1})}}\}(ve(g)), ue(f)\rangle \\
&+ \int_{[t_i,t_{i+1})} \langle\{F_{t_i}\langle R, g(s)\rangle + H_{t_i}\}(ve(g)), ue(f)\rangle ds.
\end{aligned}$$

Now, note that by 2.1.1, $\langle G_{t_i}a^\dagger_{S_{t_i}^{[t_i,t_{i+1})}}(ve(g)), ue(f)\rangle = \langle G_{t_i}\langle f, S_{t_i}^{[t_i,t_{i+1})}\rangle(ve(g)), ue(f)\rangle = \int_{[t_i,t_{i+1})} \langle G_{t_i}\langle f(s), S\rangle ve(g), ue(f)\rangle ds$. Similarly, $\langle E_{t_i}a^\dagger_{T_{g^{t_i}}^{[t_i,t_{i+1})}}(ve(g)), ue(f)\rangle = \int_{[t_i,t_{i+1})} \langle E_{t_i}\langle f(s), T_{g^{t_i}(s)}\rangle(ve(g)), ue(f)\rangle ds = \int_{[t_i,t_{i+1})} \langle f(s), E_sPT_{g(s)}\rangle(ve(g)), ue(f)\rangle$. From this (i) follows clearly.

Let us now prove (ii). We briefly sketch the arguments very similar to that of [45]. It is clear that the left hand side of (2.11) can be written as $S_1^{\mathcal{P}} + S_2^{\mathcal{P}} + S_3^{\mathcal{P}}$, where

$$S_1^{\mathcal{P}} = \sum_{i=0}^{m} \langle X_{t_i}(ve(g)), \{E'_{t_i}\Lambda_{T'}([t_i,t_{i+1}))+F'_{t_i}a_{R'}([t_i,t_{i+1}))+G'_{t_i}a^\dagger_{S'}([t_i,t_{i+1}))+H'_{t_i}(t_{i+1}-t_i)\}(ue(f))\rangle,$$

$$S_2^{\mathcal{P}} = \sum_{i=0}^{m} \langle \{E_{t_i}\Lambda_T([t_i,t_{i+1}))+F_{t_i}a_R([t_i,t_{i+1}))+G_{t_i}a^\dagger_S([t_i,t_{i+1}))+H_{t_i}(t_{i+1}-t_i)\}(ve(g)), X'_{t_i}(ue(f))\rangle,$$

$$S_3^{\mathcal{P}} = \sum_{i=0}^{m} \langle \{E_{t_i}\Lambda_T([t_i,t_{i+1})) + F_{t_i}a_R([t_i,t_{i+1})) + G_{t_i}a^\dagger_S([t_i,t_{i+1})) + H_{t_i}(t_{i+1}-t_i)\}(ve(g)),$$
$$\{E'_{t_i}\Lambda_{T'}([t_i,t_{i+1})) + F'_{t_i}a_{R'}([t_i,t_{i+1})) + G'_{t_i}a^\dagger_{S'}([t_i,t_{i+1})) + H'_{t_i}(t_{i+1}-t_i)\}(ue(f))\rangle.$$

Similarly. denote by $S_1. S_2, S_3$ respectively the first, second and third summand in the right hand side of (2.11). We want to show $S_i^{\mathcal{P}} \to S_i$ as $\mathcal{P}$ is made of arbitrarily small norm . for $i = 1, 2, 3$. Let $\|\mathcal{P}\|$ denote the norm of the partition $\mathcal{P}$. Suppose that c_0 is a constant such that $\|E_s ve(g)\| + \|F_s ve(g)\| + \|G_s ve(g)\| + \|H_s ve(g)\| + \|E'_s ue(f)\| + \|F'_s ue(f)\| + \|G'_s ue(f)\| + \|H'_s ue(f)\| \leq c_0 \ \forall s \leq t$. Now.

$$\begin{aligned} &|S_1^{\mathcal{P}} - S_1| \\ &\leq \sum_{i=0}^{m} \Big|\int_{t_i}^{t_{i+1}} \langle (X_{t_i} - X_s)(ve(g)), B_s(ue(f))\rangle\Big| \\ &\text{(where } \mathrm{B_s = \langle g(s), E'_s P T'_{f(s)}\rangle + F'_s\langle R', f(s)\rangle + G'_s\langle g(s), S'\rangle + H'_s}) \end{aligned}$$

$$\leq \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \|(X_{t_i} - X_s)(ve(g))\| c_0 \|u\| \{\|g\|_\infty \|f\|_\infty \|T'\| + \|f\|_\infty \|R'\| + \|g\|_\infty \|S'\| + 1\}. \tag{2.12}$$

But for $s \in [t_i, t_{i+1})$,

$$\begin{aligned} &\|(X_{t_i} - X_s)(ve(g))\| \\ &\leq \|E_{t_i}\Lambda_T([t_i,s))(ve(g))\| + \|F_{t_i}a_R([t_i,s))(ve(g))\| + \|G_{t_i}a^\dagger_S([t_i,s))(ve(g))\| \\ &+ \|H_{t_i}(s-t_i)(ve(g))\|. \end{aligned}$$

Clearly, by the Lemma 2.1.2,

$$\begin{aligned} &\|G_{t_i}a^\dagger_S([t_i,s))(ve(g))\|^2 \\ &= (s-t_i)\|G_{t_i}PS(ve(g))\|^2 \\ &+ \int_{[t_i,s)\times[t_i,s)} \langle\langle g(\tau), G_{t_i}PS\rangle(ve(g)), \langle g(\tau'), G_{t_i}PS\rangle(ve(g))\rangle d\tau d\tau' \\ &\leq (t_{i+1}-t_i)\|v\|^2 c_0^2 \|S\|^2 \{1 + (t_{i+1}-t_i)\|g\|_\infty^2\}. \end{aligned}$$

Similar estimates for other terms will allow us to majorize the last expression in 2.12 by $C\|\mathcal{P}\|^{\frac{1}{2}}$ when $\|\mathcal{P}\| \leq 1$, where C is some constant independent of $\mathcal{P}$. This shows that $S_1^{\mathcal{P}} \to S_1$ as $\|\mathcal{P}\| \to 0$. Similar analysis can be carried out for $S_2^{\mathcal{P}}$ and $S_3^{\mathcal{P}}$. For example, let us consider one typical term in the expansion of $S_3^{\mathcal{P}}$, say $\sum_i \langle G_{t_i} a_S^\dagger([t_i, t_{i+1}))(ve(g)), G'_{t_i} a_{S'}^\dagger([t_i, t_{i+1}))(ue(f))\rangle$. By the Lemma 2.1.2, this is equal to

$$\begin{aligned} &\sum_i \langle G_{t_i} a_S^\dagger([t_i, t_{i+1}))(ve(g)), G'_{t_i} a_{S'}^\dagger([t_i, t_{i+1}))(ue(f))\rangle \\ &= \int_0^t \langle G_s PS(ve(g)), G'_s PS'(ue(f))\rangle ds \\ &+ \sum_{i=0}^{m} \int_{[t_i,t_{i+1})\times[t_i,t_{i+1})} \langle\langle f(s), G_{t_i} PS\rangle(ve(g)), \langle g(s'), G'_{t_i} PS'\rangle(ue(f))\rangle ds ds'. \end{aligned}$$

the second term of which can be majorized (by a reasoning similar to the one given earlier) by some constant multiplied by $\|\mathcal{P}\|$, and hence goes to 0 as $\|\mathcal{P}\| \to 0$. A similar analysis can be done for other terms which will show $S_3^{\mathcal{P}} \to S_3$. This completes the proof. □

Let us state the well-known and useful lemma due to Gronwall.

Lemma 2.2.2 *Let F, G, α be nonnegative continuous functions on $I\!R_+$ and F, G be monotone nondecreasing. Suppose $F(0) = 0$ and*

$$\alpha(t) \leq G(t) + \int_0^t \alpha(s) dF(s) \; \forall t \geq 0.$$

Then we have,

$$\alpha(t) \leq G(t) exp(F(t)) \; \forall t \geq 0.$$

We refer the reader to [45] for a proof (proposition 25.5).

For a simple integrand H_t, one can easily derive the following estimate by Gronwall's Lemma as in [45].

Lemma 2.2.3 *Let v, g, X_t be as in Proposition 2.2.1 (ii). then one has*

$$\begin{aligned} ||X_t ve(g)||^2 \leq e^t \int_0^t ds[||\{E_s PT_{g(s)} + G_s PS\}(ve(g))||^2 \\ +||\{< g(s), E_s PT_{g(s)} > +F_s < R, g(s) > + < g(s), G_s PS > +H_s\}(ve(g))||^2]. \end{aligned} \quad (2.13)$$

Proof : Write $A(s) = \{\langle g(s), E_sPT_{g(s)}\rangle + F_s\langle R, g(s)\rangle + \langle g(s), G_sPS\rangle + H_s\}(ve(g))$, $B(s) = (E_sPT_{g(s)} + G_sPS)(ve(g))$. Then, by the second fundamental formula (2.11) we have,

$$\begin{aligned} &\|X_tve(g)\|^2 \\ &\leq\ 2Re\{\int_0^t \langle X_s(ve(g)), A(s)\rangle ds\} + \int_0^t \|B(s)\|^2 ds \\ &\leq\ \int_0^t \|X_s(ve(g))\|^2 ds + \int_0^t (\|A(s)\|^2 + \|B(s)\|^2)ds. \end{aligned}$$

The last step follows by the inequality $2Re\langle x, y\rangle \leq \|x\|^2 + \|y\|^2$. Now, by Gronwall's lemma (2.2.2) we complete the proof, by taking $\alpha(t) = \|X_t(ve(g)\|^2$, $F(t) = t$ and $G(t) = \int_0^t \|A(s)\|^2 + \|B(s)\|^2 ds$. □

The extension of the definition of X_t to the case when the coefficients (E, F, G, H) are regular is now fairly standard and we have the following result :

Proposition 2.2.4 *The integral X_t with regular coefficients (E, F, G, H) exists as a regular process and the first and second fundamental formulae as well as the estimate 2.13 remain valid in such a case.*

Proof : For $n = 1, 2, \ldots$ define $L_t^{(n)} = L_{\frac{j}{n}}$ if $\frac{j}{n} \leq t < \frac{j+1}{n}$ for $j = 0, 1, 2, \ldots$, where L denotes one of the four processes E, F, G, H. By regularity of (E, F, G, H) $L_t^{(n)}(ve(g))$ converges to $L_t(ve(g))$ uniformly over bounded subintervals of $\mathbb{R}_+$. Clearly, each $L^{(n)}$ is simple adapted process and we can form $X_t^{(n)} = \int_0^t (E_s^{(n)}\Lambda_T(ds) + F_s^{(n)}a_R(ds) + G_s^{(n)}a_S^\dagger(ds) + H_s^{(n)}ds)$. By the estimate obtained in 2.2.3, it is easy to see that $X_t^{(n)}ve(g)$ is a Cauchy sequence, and hence its limit defines the stochastic integral $X_t(ve(g))$. It is straightforward to see that X_t is indeed a regular adapted process and the fundamental formulae and the estimnate of 2.2.3 are valid. See [45] for a detailed proof, which will be applicable here also almost verbatim. □

Corollary 2.2.5 *(i) Assume that in the above proposition E, F, G, H satisfy $C = \sup_{0\leq t\leq t_0}(\|E_s\| + \|F_s\| + \|G_s\| + \|H_s\|) < \infty$. Suppose furthermore that R, S, T are functions of t such that $t \mapsto R(t)u, S(t)u, T(t)\psi$ are strongly continuous for u belonging to a dense subspace $\mathcal{D} \subseteq \mathrm{Dom}(R(t)) \cap \mathrm{Dom}(S(t)) \subseteq h$ and $\psi \equiv u \otimes f(t) \in \mathrm{Dom}(T(t)) \subseteq h \otimes k_0$ for all $t \in [0, t_0]$ with $f \in C$, the set of all bounded continuous* *functions in* $L^2(\mathbb{R}_+, k_0)$. *Then the integral :*

$$X(t) = \int_0^t (E_s \Lambda_T(ds) + F_s a_R(ds) + G_s a_S^\dagger(ds) + H_s ds)$$

defines an adapted regular process satisfying the estimate 2.13 with the constant coefficients T, R, S *replaced by* $T(s), R(s)$ *and* $S(s)$ *respectively.*
(ii)In the first part of the corollary, if we replace $T(t), R(t), S(t)$ *by adapted processes denoted by the same symbols respectively but with* $\mathcal{D}$ *replaced by* $\mathcal{D} \otimes_{\text{alg}} \mathcal{E}(C_t)$, *where* $C_t = C \cap k_t$, *then the conclusions as in (i) remain valid.*

Proof : (i) Clearly we can choose sequences $T^{(n)}(t), R^{(n)}(t), S^{(n)}(t)$ of simple coefficients such that $T^{(n)}(t)\psi, R^{(n)}(t)u$ and $S^{(n)}(t)u$ converge to $T(t)\psi, R(t)u$ and $S(t)u$ respectively for u and ψ as mentioned in the statement of the corollary. With these, we can define the integral $X^{(n)}(t)$ on $u \otimes e(f)$ in a natural way using Proposition 2.2.4. The hypotheses of continuity of the coefficients will allow one to pass to the limit in the integral as well by using the estimate 2.13. For example, $\| \int_0^t E_s(\Lambda_{T^{(n)}}(ds) - \Lambda_{T^{(m)}}(ds))ue(f)\|^2 \leq Ce^t \|e(f)\|^2 \int_0^t (1 + \|f(s)\|^2) \|(T^{(n)}(s) - T^{(m)}(s))(uf(s))\|^2 ds \to 0$ as $m, n \to \infty$. The estimate for $\|X(t)ue(f)\|$ will also follow by continuity.
(ii) This part follows easily from (i) with obvious adaptations. For instance, in the estimate above we shall have instead $\| \int_0^t E_s[\Lambda_{T^{(n)}}(ds) - \Lambda_{T^{(m)}}(ds)]ue(f)\|^2 \leq$
$Ce^t \int_0^t ds (1 + \|f(s)\|^2) \|(T^{(n)}(s) - T^{(m)}(s))(u \otimes e(f_s) \otimes f(s))\|^2 \|e(f^s)\|^2$. □

Remark 2.2.6 *Instead of the left integral, one could as well have dealt with the right integral* $\int_0^t M(ds)H_s$ *and obtained formulae similar to those in Propositions 2.2.1 and 2.2.4*

Remark 2.2.7 *(i)The Ito formulae derived in proposition 2.2.4 can be put in a convenient symbolic form. Let* $\tilde{\pi_0}(x)$ *denote* $x \otimes 1_{\Gamma(k)}$ *and* $\pi_0(x)$ *denote* $x \otimes 1_{k_0}$. *Then the Ito formulae are :*
$a_R(dt)\tilde{\pi_0}(x)a_S^\dagger(dt) = R^*\pi_0(x)Sdt$, $\Lambda_T(dt)\tilde{\pi_0}(x)\Lambda_{T'}(dt) = \Lambda_{T\pi_0(x)T'}(dt)$,
$\Lambda_T(dt)\tilde{\pi_0}(x)a_S^\dagger(dt) = a^\dagger_{T\pi_0(x)S}(dt)$, $a_S(dt)\tilde{\pi_0}(x)\Lambda_T(dt) = a_{T^*\pi_0(x)S}(dt)$, *and the products of all other types are* 0.
(ii) The coordinate-free approach of quantum stochastic calculus developed here includes the old coordinatized version as presented in [45]. Let us consider for example,

for $f \in L^2(\mathbb{R}_+, k_0)$, *the operator* R_f *defined by* $R_f u = u \otimes f$, *for* $u \in h$. *It is easy to see that the creation and annihilation operators* $a^\dagger(R_f)$, $a(R_f)$ *coincide with the creation and annihilation operators* $a^\dagger(f)$ *and* $a(f)$ *(respectively) defined in [45] associated with* f. *Indeed, it is easy to see that* $(R_f)_{u,v} = \langle u, v\rangle f$ *for* $u, v \in h$. *Thus, for* $g, l \in k$, $\langle a^\dagger(R_f)ve(g), ue(l)\rangle = \frac{d}{d\varepsilon}(\langle e(g+\varepsilon\langle u,v\rangle f), e(l)\rangle)|_{\varepsilon=0} = e^{\langle g,l\rangle}\langle v,u\rangle\langle f,l\rangle$ $= \langle v,u\rangle\frac{d}{d\varepsilon}\langle e(g+\varepsilon f), e(l)\rangle|_{\varepsilon=0} = \langle v,u\rangle\langle a^\dagger(f)e(g), e(l)\rangle$. *It is also clear that* $\langle R_f, g\rangle$ $= \langle f, g\rangle$ *and hence* $a(R_f)(ve(g)) = \langle f, g\rangle ve(g) = v(a(f)e(g))$. *Finally, the number operator* $\Lambda(T)$ *in the sense of [45] for* $T \in \mathcal{B}(k)$ *can be identified with* $\Lambda_{1_h \otimes T}$.

Chapter 3

Quantum Stochastic Diffrential Equations

3.1 Hudson-Parthasarathy type Equations

We consider the quantum stochastic differntial equations (q.s.d.e.) of the form,

$$dX_t = X_t(a_R(dt) + a_S^\dagger(dt) + \Lambda_T(dt) + Adt), \tag{3.1}$$

$$dY_t = (a_R(dt) + a_S^\dagger(dt) + \Lambda_T(dt) + Adt)Y_t, \tag{3.2}$$

with prescribed initial values $\tilde{X}_0 \otimes 1$ and $\tilde{Y}_0 \otimes 1$ respectively, with $\tilde{X}_0, \tilde{Y}_0 \in \mathcal{B}(h)$ where $R, S \in \mathcal{B}(h, h \otimes k_0)$, $T \in \mathcal{B}(h \otimes k_0)$, $A \in \mathcal{B}(h)$.

Proposition 3.1.1 *The q.s.d.e.'s (3.1) and (3.2) admit unique solutions as regular processes.*

Proof: The standard proofs of existence and uniqueness of solutions along the lines of that given in [45] (section 26 for the left equation and section 27 for the right equation) work here also. We set up the iteration by taking $X_t^{(0)} = I$ and $X_t^{(n+1)} = \int_0^t X_s^{(n)}(a_R(ds) + a_S^\dagger(ds) + \Lambda_T(ds) + Ads)$ for $n = 0, 1, \ldots$. Fix $t_0 \geq 0$, $g \in L^\infty(\mathbb{R}_+, k_0) \cap L^2(\mathbb{R}_+, k_0)$. By the estimates (2.13) and corollary 2.2.5 we have for $t \leq t_0$,

$$\begin{aligned} ||X_t^{(1)}(ve(g))||^2 &\leq e^t \int_0^t ds[||(T_{g(s)} + S)(ve(g))||^2 \\ &+ \quad ||(< g(s), T_{g(s)} > + < R, g(s) > + < g(s), S > +A)(ve(g))||^2] \end{aligned}$$

$$\leq \; C^2 t\|v\|^2\|e(g)\|^2,$$

where $C = 4e^{t_0/2}(\|T\|(\|g\|_\infty + \|g\|^2_\infty) + \|S\|(1+\|g\|_\infty) + \|R\|\|g\|_\infty + \|A\|)$. Note that here we have used the fact that $\|p+q+r+s\|^2 \leq 4(\|p\|^2 + \|q\|^2 + \|r\|^2 + \|s\|^2)$. Now take the induction hypothesis that $\|X_t^{(n)}(ve(g))\|^2 \leq C^{2n}\frac{t^n}{n!}\|v\|^2\|e(g)\|^2$. Observe that for $W \in h\otimes k_0$, $\|X_t^{(n)}P(We(g))\|^2 \leq C^{2n}\frac{t^n}{n!}\|W\|^2\|e(g)\|^2$, and hence we have

$$\begin{aligned}
&\|X_t^{(n+1)}ve(g)\|^2 \\
\leq\; & e^t\int_0^t ds[\|\{X_s^{(n)}PT_{g(s)} + X_s^{(n)}PS\}(ve(g))\|^2 \\
+\; & \|\{< g(s), X_s^{(n)}PT_{g(s)} > + X_s^{(n)} < R, g(s) > + < g(s), X_s^{(n)}PS > + X_s^{(n)}A\}(ve(g))\|^2] \\
\leq\; & 4e^{t_0}C^{2n}\|v\|^2\|e(g)\|^2(\|T\|(\|g\|_\infty + \|g\|^2_\infty) + \|S\|(1+\|g\|_\infty) + \|R\|\|g\|_\infty + \|A\|)\int_0^t \frac{s^n}{n!}ds \\
=\; & C^{2n+2}\frac{t^{n+1}}{(n+1)!}\|v\|^2\|e(g)\|^2.
\end{aligned}$$

This proves the induction hypothesis for all n. Clearly, the process X_t defined by $X_t(ve(g)) = \sum_{n=0}^\infty X_t^{(n)}(ve(g))$ is well defined, since the sum converges by virtue of the above estimates. The estimates also enable one to show that X_t is an adapted regular process. That X_t is a solution of (3.1) is also clear. This proves the existence of the left equation. To show uniqueness,suppose that X_t' is another solution of (3.1). Since $X_t - X_t' = \int_0^t (a_R(ds) + a_S^\dagger(ds) + \Lambda_T(ds) + Ads)$, a reasoning similar to the above shows that $\|(X_t - X_t')(ve(g))\|^2 \leq C^{2n}\frac{t^n}{n!}\|v\|^2\|e(g)\|^2$ for all n and hence $X_tve(g) = X_t've(g)$.

The proof for the right equation (3.2) is similar. For the iteration process in the case of the right equation to make sense, one has to take into account the remark 2.2.7 while interpreting the right integrals involved. We also obtain the estimates : $\sup_{0\leq s\leq t}\{\|X_sue(f)\| + \|Y_sue(f)\|\} \leq c(t,f)\|u\|$, for $u \in h, f \in C$ and some constant $c(t,f)$. □

We now consider a pair of special q.s.d.e.'s:

$$dU_t = U_t(a_R^\dagger(dt) + \Lambda_{T-1}(dt) - a_{T^*R}(dt) + (iH - \frac{1}{2}R^*R)dt),\; U_0 = I. \tag{3.3}$$

$$dW_t = (a_R(dt) + \Lambda_{T^*-I}(dt) - a_{T^*R}^\dagger(dt) - (iH + \frac{1}{2}R^*R)dt)W_t,\; W_0 = I; \tag{3.4}$$

where T is a contraction in $\mathcal{B}(h\otimes k_0)$, $R \in \mathcal{B}(h, h\otimes k_0)$ and H is a selfadjoint element of $\mathcal{B}(h)$. Then we have :

Proposition 3.1.2 *(see [39] also)*

(i) The solutions of both equations 3.3 and 3.4 exist as regular contraction-valued processes and $W_t = U_t^*$.

(ii) If furthermore T *is a co-isometry, then* W_t *is an isometry, or equivalently* U_t *is a co-isometry.*

(iii) If T *is unitary, then* U_t *is a unitary process.*

Proof: (i) We have already seen the existence and uniqueness of the solutions U_t and W_t in the previous proposition. A simple calculation using the second fundamental formula in Proposition 2.2.4 and the right equation (3.4) give for $u, v \in h$ and $f, g \in C$:

$$< W_t ve(g), W_t ue(f) > - < ve(g), ue(f) >= \int_0^t < W_s ve(g), < g(s), (TT^* - I)_{f(s)} > W_s ue(f) > \tag{3.5}$$

This implies that for vectors $u_1, u_2, \ldots, u_n \in h$ and $f_1, f_2, \ldots, f_n \in C$, we have

$$\begin{aligned}
&\| \sum_{i=1}^n W_t(u_i e(f_i)) \|^2 - \| \sum_{i=1}^n u_i e(f_i) \|^2 \\
&= \sum_{i,j} \int_0^t \langle W_s u_i e(f_i), \langle f_i(s), (TT^* - I)_{f_j(s)} \rangle W_s u_j e(f_j) \rangle ds \\
&= \int_o^t \sum_{i,j} - \langle (I - TT^*)^{\frac{1}{2}}_{f_i(s)} (W_s u_i e(f_i)), (I - TT^*)^{\frac{1}{2}}_{f_j(s)} (W_s u_j e(f_j)) \rangle ds \\
&= - \int_0^t \| \sum_i (I - TT^*)^{\frac{1}{2}}_{f_i(s)} (W_s u_i e(f_i)) \|^2 ds \leq 0;
\end{aligned}$$

where we have used the fact that T is a contraction to take the square root of $I - TT^*$. This clearly implies that W_t is a contraction for each t. Since $W_t \in \mathcal{B}(h \otimes \Gamma)$, an application of the first fundamental formula in Proposition 2.2.4 shows that U_t admits a bounded extension (which we denote also by U_t) to the whole of $h \otimes \Gamma$ and that $U_t^* = W_t$.

(ii) The relation (3.5) shows clearly that W_t is an isometry if and only if T is a co-isometry.

(iii) We note the following simple facts :

(a) For fixed $g, f \in L^2(\mathbb{R}_+, k_0) \cap L^\infty(\mathbb{R}_+, k_0)$ and $t \geq 0$, there exists a unique operator $M_t^{f,g} \in \mathcal{B}(h)$ such that $\langle v, M_t^{f,g} u \rangle = \langle U_t(ve(g)), \ U_t(ue(f)) \rangle$.

(b) Setting $\tilde{M}_t^{f,g} = M_t^{f,g} \otimes 1_{k_0}$, we have for all $w, w' \in h \otimes k_0$, $\langle U_t P(we(g)), \ U_t P(w'e(f)) \rangle = \langle w, \tilde{M}_t^{f,g} w' \rangle$.

It is an easy computation using the Quantum Ito formulae (Proposition 2.2.4) to verify that,

$$\langle v, M_t^{f,g}u\rangle - \langle ve(g), ue(f)\rangle = \int_0^t ds[-\langle v, M_s^{f,g}\langle T^*R, f(s)\rangle u\rangle - \langle v, \langle g(s), T^*R\rangle M_s^{f,g}u\rangle$$

$$+\langle v, M_s^{f,g}\langle g(s), R\rangle u\rangle + \langle vg(s), \tilde{M}_s^{f,g}((T-1)(uf(s)))\rangle + \langle v, M_s^{f,g}(iH - \frac{1}{2}R^*R)u\rangle$$

$$+\langle v, \langle R, f(s)\rangle M_s^{f,g}u\rangle + \langle vg(s), (T^*-1)\tilde{M}_s^{f,g}(uf(s))\rangle + \langle v, (-iH - \frac{1}{2}R^*R)M_s^{f,g}u\rangle$$

$$+\langle Rv, \tilde{M}_s^{f,g}(Ru)\rangle + \langle Rv, \tilde{M}_s^{f,g}(T-1)(uf(s))\rangle + \langle vg(s), (T^*-1)\tilde{M}_s^{f,g}(Ru)\rangle$$

$$+\langle vg(s), (T^*-1)\tilde{M}_s^{f,g}(T-1)(uf(s))\rangle].$$

Let us consider maps $Y_i, i = 1, \ldots, 5$ from $[0,\infty) \times \mathcal{B}(h)$ to $\mathcal{B}(h)$ given by: $Y_1(s,A) = -A\langle T^*R, f(s)\rangle - \langle g(s), T^*R\rangle A + A\langle g(s), R\rangle - \frac{1}{2}(AR^*R + R^*RA) + i[A,H] + \langle R, f(s)\rangle A$, $Y_2(s,A) = R^*\tilde{A}R$, where $\tilde{A} = A \otimes 1_{k_0}$, $Y_3(s,A) = \langle g(s), \{(T^*-1)\tilde{A} + \tilde{A}(T-1) + (T^*-1)\tilde{A}(T-1)\}_{f(s)}\rangle$, $Y_4(s,A) = \langle (T^*-I)\tilde{A}^*R, f(s)\rangle$,
$Y_5(s,A) = \langle (T^*-I)\tilde{A}R, g(s)\rangle^*$.
Then it follows that, $\langle v, M_t^{f,g}u\rangle - \langle ve(g), ue(f)\rangle = \int_s^t \langle v, \sum_{i=1}^5 Y_i(s, M_s^{f,g})u\rangle ds$, i.e.

$$\frac{dM_t^{f,g}}{dt} = \sum_{i=1}^5 Y_i(t, M_t^{f,g}).$$

We also have that $M_0^{f,g} \equiv \langle e(g), e(f)\rangle I$ is a solution since the isometry property of T implies that $Y_i(t,I) = 0 \; \forall \; i$. Moreover, Y_i's are linear and bounded, hence by the uniqueness of the solution of the Banach space valued initial value problem, we conclude that $M_t^{f,g} = M_0^{f,g}$ for all t, or equivalently that U_t is an isometry. □

3.2 Solution of Evans-Hudson type q.s.d.e.

In the previous subsections, we have considered q.s.d.e.'s on the Hilbert space $h \otimes \Gamma$. Now we shall study an associated class of q.s.d.e.'s, but on the Fock module $\mathcal{A} \otimes \Gamma$. This is closely related to the Evans-Hudson type of q.s.d.e.'s ([18], [45]).

For this part of the theory, we assume that we are given the *structure maps*, that is, the triple of normal maps $(\mathcal{L}, \delta, \sigma)$ where $\mathcal{L} \in \mathcal{B}(\mathcal{A}), \delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes k_0)$ and $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0))$ satisfying :

(S1) $\sigma(x) = \pi(x) - x \otimes I_{k_0} \equiv \Sigma^*(x \otimes I_{k_0})\Sigma - x \otimes I_{k_0}$, where Σ is a partial isometry

in $h \otimes k_0$ such that π is a $*$-representation on $\mathcal{A}$.
(S2) $\delta(x) = Rx - \pi(x)R$, where $R \in \mathcal{B}(h, h \otimes k_0)$ so that δ is a π-derivation, i.e. $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$.
(S3) $\mathcal{L}(x) = R^*\pi(x)R + lx + xl^*$, where $l \in \mathcal{A}$ with the condition $\mathcal{L}(1) = 0$ so that $\mathcal{L}$ satisfies the second order cocycle relation with δ as coboundary, i.e.

$$\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x)^*y = \delta(x)^*\delta(y) \ \forall x, y \in \mathcal{A}.$$

Given the generator $\mathcal{L}$ of a q.d.s., that one can choose k_0 and Σ such that the hypotheses (S1)-(S3) are satisfied will be established in the next section.
To describe Evans-Hudson flow in this language, it is convenient to introduce a map Θ encompassing the triple $(\mathcal{L}, \delta, \sigma)$ as follows :

$$\Theta(x) = \begin{pmatrix} \mathcal{L}(x) & \delta^\dagger(x) \\ \delta(x) & \sigma(x) \end{pmatrix}, \tag{3.6}$$

where $x \in \mathcal{A}$, $\delta^\dagger(x) = \delta(x^*)^* : h \otimes k_0 \to h$, so that $\Theta(x)$ can be looked upon as a bounded linear normal map from $h \otimes \hat{k_0} \equiv h \otimes (\mathbb{C} \oplus k_0)$ into itself. It is also clear from (S1)-(S3) that Θ maps $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(\hat{k_0})$. The next lemma sums up important properties of Θ.

Lemma 3.2.1 *Let Θ be as above. Then one has :*

$$(i) \qquad \Theta(x) = \Psi(x) + K(x \otimes 1_{\hat{k_0}}) + (x \otimes 1_{\hat{k_0}})K^*, \tag{3.7}$$

where $\Psi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{k_0})$ is a completely positive map and $K \in \mathcal{B}(h \otimes \hat{k_0})$.
(ii) Θ is conditionally completely positive and satisfies the structure relation :

$$\Theta(xy) = \Theta(x)(y \otimes 1_{\hat{k_0}}) + (x \otimes 1_{\hat{k_0}})\Theta(y) + \Theta(x)\hat{Q}\Theta(y), \tag{3.8}$$

where $\hat{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1_{h \otimes k_0} \end{pmatrix}$.
(iii) There exists $D \in \mathcal{B}(h \otimes \hat{k_0})$ such that $\|\Theta(x)\zeta\| \le \|(x \otimes 1_{\hat{k_0}})D\zeta\| \ \forall \zeta \in h \otimes \hat{k_0}$.

Proof : Define the following maps with respect to the direct sum decomposition $h \otimes \hat{k_0} = h \oplus (h \otimes k_0)$:

$$\tilde{R} = \begin{pmatrix} 0 & 0 \\ R & -I \end{pmatrix}, \tilde{\pi}(x) = \begin{pmatrix} x & 0 \\ 0 & \pi(x) \end{pmatrix}, K = \begin{pmatrix} l & 0 \\ R & -\frac{1}{2}1_{h \otimes k_0} \end{pmatrix},$$

$\tilde{\Sigma} = \begin{pmatrix} 1_h & 0 \\ 0 & \Sigma \end{pmatrix}$.

Then it is easy to see that (i) is verified with $\Psi(x) = \tilde{R}^*\tilde{\Sigma}^*(x \otimes 1_{\hat{k_0}})\tilde{\Sigma}\tilde{R} = \tilde{R}^*\tilde{\pi}(x)\tilde{R}$. Clearly, Ψ is completely positive. That Θ is conditionally completely positive and satisfies the structure relation (3.8) is also an easy consequence of (i) and (S1)-(S3). The estimate in (iii) follows from the structure of Ψ given above with the choice of D as

$$D = ||\tilde{\Sigma}\tilde{R}||\ \tilde{\Sigma}\tilde{R} + ||K||\ 1_{h\otimes\hat{k_0}} + K^*.$$

□

We now introduce the basic processes. Fix $t \geq 0$, a bounded interval $\Delta \subseteq (t, \infty)$, elements $x_1, x_2, \ldots, x_n \in \mathcal{A}$ and vectors $f_1, f_2, \ldots, f_n \in k; u \in h$. We define the followings :

$$\left(a_\delta(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right) u = \sum_{i=1}^n a_{\delta(x_i^*)}(\Delta)(ue(f_i)),$$

$$\left(a_\delta^\dagger(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right) u = \sum_{i=1}^n a_{\delta(x_i)}^\dagger(\Delta)(ue(f_i)),$$

$$\left(\Lambda_\sigma(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right) u = \sum_{i=1}^n \Lambda_{\sigma(x_i)}(\Delta)(ue(f_i)),$$

$$\left(\mathcal{I}_\mathcal{L}(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right) u = \sum_{i=1}^n |\Delta|(\mathcal{L}(x_i)u) \otimes e(f_i)),$$

where $|\Delta|$ denotes the length of Δ.

Lemma 3.2.2 *The above processes are well defined on $\mathcal{A} \otimes_{\text{alg}} \mathcal{E}(k)$ and they take values in $\mathcal{A} \otimes \Gamma(k)$.*

Proof : First note that $e(f_1), \ldots, e(f_n)$ are linearly independent whenever $f_1, \ldots, f_n$ are distinct, from which it is easy to see that $\sum_{i=1}^n x_i \otimes e(f_i) = 0$ implies $x_i = 0\ \forall i$, whenever f_i's are distinct. This will establish that the processes are well defined. The second part of the lemma will follow from lemma 1.3.9 with the choice of the dense set $\mathcal{E}$ to be $\mathcal{E}(k)$ and $\mathcal{H} = \Gamma(k)$ and by some simple computation, noting the fact that $\mathcal{L}, \delta, \sigma$ are structure maps. For example, $\langle e(g), a_\delta^\dagger(\Delta)(x \otimes e(f))\rangle = \langle e(g), e(f)\rangle \int_\Delta \langle g(s), \delta(x)\rangle ds \in \mathcal{A}$, which shows that the range of $a_\Delta^\dagger$ is in $\mathcal{A} \otimes \Gamma(k)$.

Similarly, one verifies that $\langle e(g), \Lambda_\sigma(\Delta)(x \otimes e(f))\rangle = \langle e(g), e(f)\rangle \int_\Delta \langle g(s), \sigma(x)_{f(s)}\rangle ds \in \mathcal{A}$, since $\sigma(x) \in \mathcal{A} \otimes \mathcal{B}(k_0)$. □

Next, we want to consider the solution of an equation of the Evans-Hudson type which in our notation can be written as :

$$J_t = id_{\mathcal{A}\otimes\Gamma} + \int_0^t J_s \circ (a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_\mathcal{L})(dt), \quad 0 \le t \le t_0 \tag{3.9}$$

where the solution is looked for as a map from $\mathcal{A} \otimes \Gamma$ into itself. For this, we first need an abstract lemma which allows us to interpret the above integral on the right hand side and to get an appropriate bound for such integrals.

Lemma 3.2.3 *(The Lifting lemma) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}$ be a vector space. Let $\beta : \mathcal{A} \otimes_{\text{alg}} \mathcal{V} \to \mathcal{A} \otimes \mathcal{H}$ be a linear map satisfying the estimate*

$$||\beta(x \otimes \eta)u|| \le c_\eta ||(x \otimes 1_{\mathcal{H}''})ru|| \tag{3.10}$$

for some Hilbert space $\mathcal{H}''$ and $r \in \mathcal{B}(h, h \otimes \mathcal{H}'')$ (both independent of η) and for some constant c_η depending on η. Then, for any Hilbert space $\mathcal{H}'$, we can define a map $\tilde{\beta} : (\mathcal{A} \otimes \mathcal{H}') \otimes_{\text{alg}} \mathcal{V} \to \mathcal{A} \otimes (\mathcal{H} \otimes \mathcal{H}')$ by $\tilde{\beta}(x \otimes f \otimes \eta) = \beta(x \otimes \eta) \otimes f$ for $x \in \mathcal{A}, \eta \in \mathcal{V}, f \in \mathcal{H}'$. Moreover, $\tilde{\beta}$ admits the estimate

$$||\tilde{\beta}(X \otimes \eta)u|| \le c_\eta ||(X \otimes 1_{\mathcal{H}''})ru||, \tag{3.11}$$

where $X \in \mathcal{A} \otimes \mathcal{H}'$.

Proof : Let $X \in \mathcal{A} \otimes \mathcal{H}'$ be given by the strongly convergent sum $X = \sum x_\alpha \otimes e_\alpha$, where $x_\alpha \in \mathcal{A}$ and $\{e_\alpha\}$ is an orthonormal basis of $\mathcal{H}'$. It is easy to verify that
$||\tilde{\beta}(\sum x_\alpha \otimes e_\alpha \otimes \eta)u||^2 = \sum ||\beta(x_\alpha \otimes \eta)u||^2 \le c_\eta^2 \sum_\alpha ||(x_\alpha \otimes 1_{\mathcal{H}''})ru||^2$
$= c_\eta^2 ||(X \otimes 1_{\mathcal{H}''})ru||^2$ and thus $\tilde{\beta}$ is well defined and admits the required estimate.
□

Corollary 3.2.4 *If we take $\mathcal{V} = \mathbb{C}$ and identify $\mathcal{A} \otimes_{\text{alg}} \mathcal{V}$ with $\mathcal{A}$, then $\beta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$ is a bounded normal map and $\tilde{\beta}$ is also a bounded normal map from $\mathcal{A} \otimes \mathcal{H}'$ to $\mathcal{A} \otimes \mathcal{H} \otimes \mathcal{H}'$.*

The proof of this corollary is a simple consequence of the estimates.

We now want to define $\int_0^t Y(s)\circ(a_\delta^\dagger+a_\delta+\Lambda_\sigma+\mathcal{I}_\mathcal{L})(ds)$ where $Y(s):\mathcal{A}\otimes_{\text{alg}}\mathcal{E}(k)\to\mathcal{A}\otimes\Gamma(k)$ is an adapted strongly continuous process satisfying the estimate

$$\sup_{o\leq t\leq t_0}\|Y(t)(x\otimes e(f))u\|\leq\|(x\otimes 1_{\mathcal{H}''})ru\|, \tag{3.12}$$

for $x\in\mathcal{A}, f\in C$ and where $\mathcal{H}''$ is a Hilbert space and $r\in\mathcal{B}(h,h\otimes\mathcal{H}'')$. In this the integrals corresponding to a_δ and $\mathcal{I}_\mathcal{L}$ belong to one class while the other two belong to another. In fact, we define $\int_0^t Y(s)\circ(a_\delta+\mathcal{I}_\mathcal{L})(ds)(x\otimes e(f))$ by setting it to be equal to $\int_0^t Y(s)((\mathcal{L}(x)+\langle\delta(x^*),f(s)\rangle)\otimes e(f))ds$. For the integral involving the other two processes, we need to consider $\widetilde{Y(s)}:\mathcal{A}\otimes k_0\otimes\mathcal{E}(k_s)\to\mathcal{A}\otimes\Gamma_s\otimes k_0$ as is given by the previous lemma and fix $x\in\mathcal{A}$ and $g\in C$ (see Corollary 2.2.5). Define two maps $S(s):h\otimes_{\text{alg}}\mathcal{E}(C_s)\to h\otimes\Gamma_s\otimes k_0$ and $T(s):h\otimes_{\text{alg}}\mathcal{E}(C_s)\otimes k_0\to h\otimes\Gamma_s\otimes k_0$ by

$$S(s)(ue(f_s))=\widetilde{Y(s)}(\delta(x)\otimes e(f_s))u,$$

and

$$T(s)(ue(g_s)\otimes f(s))=\widetilde{Y(s)}(\sigma(x)_{f(s)}\otimes e(g_s))u.$$

By virtue of the hypotheses on $Y(s)$, the lifting lemma and the fact that $s\mapsto e(g_s)$ is strongly continuous, the families S and T satisfy the hypotheses of corollary 2.2.5(ii). Therefore we can define the integral $\int_0^t Y(s)\circ(\Lambda_\sigma(ds)+a_\delta^\dagger(ds)(x\otimes e(f))u$ by setting it to be equal to $\left(\int_0^t\Lambda_T(ds)+a_S^\dagger(ds)\right)ue(f)$. Thus we have :

Proposition 3.2.5 *The integral* $Z(t)\equiv\int_0^t Y(s)\circ(a_\delta^\dagger+a_\delta+\Lambda_\sigma+\mathcal{I}_\mathcal{L})(ds)$, *where* $Y(s)$ *satisfies (3.12) is well defined on* $\mathcal{A}\otimes_{\text{alg}}\mathcal{E}(C)$ *as a regular process. Moreover, the integral satisfies an estimate :*

$$\begin{aligned}&\|\{Z(t)(x\otimes e(f)\}u\|^2\\ \leq\ &2e^t\int_0^t exp(\|f^s\|^2)\{\|\hat{Y}(s)(\Theta(x)_{\hat{f}(s)}\otimes e(f_s))u\|^2+\\ &\|\langle f(s),\hat{Y}(s)(\Theta(x)_{\hat{f}(s)}\otimes e(f_s))u\|^2\}ds,\end{aligned} \tag{3.13}$$

where Θ *was as defined earlier,* $\hat{Y}(s)=Y(s)\oplus\widetilde{Y(s)}:\mathcal{A}\otimes\hat{k_0}\otimes_{\text{alg}}\mathcal{E}(C_s)\to\mathcal{A}\otimes\Gamma_s\otimes\hat{k_0}$, $\hat{f}(s)=1\oplus f(s)$ *and* $f(s)$ *is identified with* $0\oplus f(s)$ *in* $\hat{k_0}$.

Proof : We have already seen that the integral is well defined. To obtain the estimate, fix x and define two operator processes $R(s)$ and H_s by setting $R(s)(ue(f_s))=\widetilde{Y(s)}(\delta(x^*)\otimes e(f_s))u$ and $H_s(ue(f_s))=Y(s)(\mathcal{L}(x)\otimes e(f_s))u$ for $u\in h,\ f_s\in C_s$. Then it is clear that

$$Z(t)(x\otimes e(f))u=\left(\int_0^t a_R(ds)+a_S^\dagger(ds)+\Lambda_T(ds)+H_s ds\right)(ue(f)),$$

from which the estimate (3.13) by using (2.13) and Corollary 2.2.5 with $E = F = G = I$ and recalling the definition of Θ. Note that we have also made use of the inequality $\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$. □

Now we are ready to prove the main result of this section .

Theorem 3.2.6 *(i) There exists a unique solution J_t of equation (3.9), which is an adapted regular process mapping $\mathcal{A} \otimes \mathcal{E}(C)$ into $\mathcal{A} \otimes \Gamma$. Furthermore, one has an estimate*

$$\sup_{0 \leq t \leq t_0} \|J_t(x \otimes e(g))u\| \leq C'(g)\|(x \otimes 1_{\Gamma^f(\hat{k})})E_{t_0}u\|,$$

where $g \in C, \hat{k} = L^2([0, t_0], \hat{k_0}), E_t \in \mathcal{B}(h, h \otimes \Gamma^f(\hat{k}))$, $C'(g)$ is a constant and $\Gamma^f(\hat{k})$ is the full Fock space over $\hat{k}$.

(ii) Setting $j_t(x)(ue(g)) = J_t(x \otimes e(g))u$, we have

(a) $\langle j_t(x)ue(g), j_t(y)ve(f)\rangle = \langle ue(g), j_t(x^* y)ve(f)\rangle$ $\forall g, f \in C$, and

(b) j_t extends uniquely to a normal $*$-homomorphism from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(\Gamma)$.

(iii) If $\mathcal{A}$ is commutative, then the algebra generated by $\{j_t(x)|x \in \mathcal{A}, 0 \leq t \leq t_0\}$ is commutative.

(iv) $j_t(1) = 1$ $\forall t \in [0, t_0]$ if and only if $\Sigma^*\Sigma = 1_{h \otimes k_0}$.

Proof : (i) We write for $\Delta \subseteq [0, \infty)$, $M(\Delta) \equiv a_\delta(\Delta) + a_\delta^\dagger(\Delta) + \Lambda_\sigma(\Delta) + \mathcal{I}_{\mathcal{L}}(\Delta)$, and set up an iteration by

$$J_t^{(n+1)}(x \otimes e(f)) = \int_0^t J_s^{(n)} \circ M(ds)(x \otimes e(f)), J_t^{(0)}(x \otimes e(f)) = x \otimes e(f),$$

with $x \in \mathcal{A}$ and $f \in C$ fixed. Since $J_t^{(1)} = M([0, t])$, $J_t^{(1)}$ is adapted regular and has the estimate (by the definition of $M(\Delta)$, estimate (2.13) and lemma 3.2.1(iii)): $\|J_t^{(1)}(x \otimes e(f))u\|^2 \leq 2e^{t_0}\|e(f)\|^2 \int_0^t ds\|\Theta(x)(u \otimes \hat{f}(s))\|^2\|\hat{f}(s)\|^2 \leq 2\|e(f)\|^2 e^{t_0}$ $\int_0^t ds\|\hat{f}(s)\|^2\|(x \otimes 1_{\hat{k_0}})D(u \otimes \hat{f}(s))\|^2$. For the given f, define $E_t^{(1)} : h \to h \otimes \hat{k}$ by $(E_t^{(1)}u)(s) = D(u \otimes \hat{f}(s)\|\hat{f}_t(s)\|)$, where $\hat{f}_t(s) = \chi_{[0,t]}(s)\hat{f}(s)$. Then the above estimate reduces to

$$\|J_t^{(1)}(x \otimes e(f))u\|^2 \leq 2\|e(f)\|^2 e^{t_0}\|(x \otimes 1_{\hat{k}})E_t^{(1)}u\|^2. \tag{3.14}$$

Now, an application of the lifting lemma leads to

$$\|\widehat{J_t^{(1)}}(X \otimes e(f))u\|^2 \leq 2\|e(f)\|^2 e^{t_0}\|(X \otimes 1_{\hat{k}})E_t^{(1)}u\|^2,$$

for $X \in \mathcal{A} \otimes \hat{k_0}$. where as in the previous proposition, $\widehat{J_t^{(1)}} = J_t^{(1)} \oplus \widetilde{J_t^{(1)}}$. As an induction hypothesis, assume that $J_t^{(n)}$ is a regular adapted process having an estimate $||J_t^{(n)}(x \otimes e(f))u||^2 \leq C^n ||e(f)||^2 ||(x \otimes 1_{\hat{k}^{\otimes n}}) E_t^{(n)} u||^2$, where $C = 2e^{t_0}$. $E_t^{(n)} : h \to h \otimes \hat{k}^{\otimes^n}$ defined as :
$(E_t^{(n)}u)(s_1, s_2, \ldots s_n) = (D \otimes 1_{\hat{k}^{\otimes n-1}}) P_n \{(E_{s_1}^{(n-1)}u)(s_2, \ldots s_n) \otimes \hat{f}(s_1) ||\hat{f}_t(s_1)||\}$. Furthermore, $P_n : h \otimes \hat{k}^{\otimes^{(n-1)}} \otimes \hat{k_0} \to h \otimes \hat{k_0} \otimes \hat{k}^{\otimes^{(n-1)}}$ is the operator which interchanges the second and third tensor components and $E_t^{(0)} = 1_h$. Then by an application of the proposition 3.2.5 one can verify that $J_t^{(n+1)}$ also satisfies a similar estimate. Thus, if we put $J_t = \sum_{n=0}^{\infty} J_t^{(n)}$, then

$$
\begin{aligned}
||J_t(x \otimes e(f))u|| &\leq \sum_{n=0}^{\infty} ||J_t^{(n)}(x \otimes e(f))u|| \\
&\leq ||e(f)|| \sum_{n=0}^{\infty} C^{\frac{n}{2}} (n!)^{-\frac{1}{4}} ||(x \otimes 1_{\hat{k}^{\otimes n}})(n!)^{\frac{1}{4}} E_t^{(n)} u|| \\
&\leq ||e(f)|| \left(\sum_{n=0}^{\infty} \frac{C^n}{\sqrt{n!}} \right)^{\frac{1}{2}} ||(x \otimes 1_{\Gamma^f(\hat{k})}) E_t u||,
\end{aligned}
\tag{3.15}
$$

where we have set $E_t : h \to h \otimes \Gamma^f(\hat{k})$ by $E_t u = \bigoplus_{n=0}^{\infty} (n!)^{\frac{1}{4}} E_t^{(n)} u$. It is easy to see that $||E_t u||^2 = \sum_{n=0}^{\infty} (n!)^{\frac{1}{2}} ||E_t^{(n)} u||^2 \leq$
$||u||^2 \sum_{n=0}^{\infty} (n!)^{\frac{1}{2}} ||D||^{2n} \{ \int_{0 < s_n < s_{n-1} < \ldots s_1 < t} ds_n \ldots ds_1 ||\hat{f}(s_n)||^4 \ldots ||\hat{f}(s_1)||^4 \}$
$= ||u||^2 \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} ||D||^{2n} \mu_f(t)^n$, where $\mu_f(t) = \int_0^t ||\hat{f}(s)||^4 ds$. The estimate (3.15) proves the existence of the solution of equation (3.9). as well as its continuity relative to the strong operator topology in $\mathcal{B}(h)$. The uniqueness of the solution follows along standard lines of reasoning.

(ii) First we prove the following identity :

$$
\langle J_t(x \otimes e(f))u, J_t(y \otimes e(g))v \rangle = \langle ue(f), J_t(x^* y \otimes e(g))v \rangle. \tag{3.16}
$$

For this it is convenient to lift the maps J_t to the module $\mathcal{A} \otimes \Gamma^f(\hat{k_0}) \otimes_{\text{alg}} \mathcal{E}(C)$, that is, replace $\mathcal{A}$ by $\mathcal{A} \otimes \Gamma^f(\hat{k_0})$. We define $\hat{J}_t : \mathcal{A} \otimes \Gamma^f(\hat{k_0}) \otimes_{\text{alg}} \mathcal{E}(C) \to \mathcal{A} \otimes \Gamma \otimes \Gamma^f(\hat{k_0})$ by $\hat{J}_t = (J_t \otimes id)P$, where P interchanges the second and third tensor components. Recalling from section 2 that $\Theta(x)_\zeta \in \mathcal{B}(h, h \otimes \hat{k_0})$ for $x \in \mathcal{A}, \zeta \in \hat{k_0}$, we can define $\Theta_\zeta : \mathcal{A} \to \mathcal{A} \otimes \hat{k_0}$ by $\Theta_\zeta(x) = \Theta(x)_\zeta$, and extend as above to $\widehat{\Theta_\zeta} : \mathcal{A} \otimes \Gamma^f(\hat{k_0}) \to \mathcal{A} \otimes \Gamma^f(\hat{k_0})$ by setting $\widehat{\Theta_\zeta}|_{\mathcal{A} \otimes \hat{k_0}^{\otimes n}} = \Theta_\zeta \otimes id_{\hat{k_0}^{\otimes n}}$. By the lifting lemma, both $\hat{J}_t$ and

$\widehat{\Theta_\zeta}$ are well defined and enjoy the estimates as in (i) of the present theorem (3.2.6) and lemma 3.2.1(iii) respectively.

Next, note that for fixed $f, g \in C$ and $x, y \in \mathcal{A}$, one has using the equation (3.9) for J_t and quantum Ito formula (2.2.5) and the structure relation in lemma 3.2.1(ii) :

$$\langle J_t(x \otimes e(f))u, J_t(y \otimes e(g))v\rangle$$

$$\langle xu \otimes e(f), yv \otimes e(g)\rangle + \int_0^t ds\{\langle \hat{J}_s(\Theta_{\hat{f}(s)}(x) \otimes e(f))u, \hat{J}_s(y \otimes \hat{g}(s) \otimes e(g))v\rangle$$

$$+\langle \hat{J}_s(x \otimes \hat{f}(s) \otimes e(f))u, \hat{J}_s(\Theta_{\hat{g}(s)}(y) \otimes e(g))v\rangle + \langle \hat{J}_s(\Theta_{f(s)}(x) \otimes e(f))u, \hat{J}_s(\Theta_{g(s)}(y) \otimes e(g))v\rangle\}, \tag{3.17}$$

where $f(s)$ and $g(s)$ in k_0 are identified with $0 \oplus f(s)$ and $0 \oplus g(s)$ in $\hat{k_0}$ respectively. We claim that the identity above remains valid even when we replace x, y by $X, Y \in \mathcal{A} \otimes \Gamma^f(\hat{k_0})$ and $\Theta_\zeta(x), \Theta_\zeta(y)$ by $\widehat{\Theta}_\zeta(X), \widehat{\Theta}_\zeta(Y)$ respectively, where ζ is one of the vectors $\hat{f}(s), \hat{g(s)}, f(s)$ and $g(s)$. To see this, it suffices to observe that in the resulting identity, both left and right hand sides vanish if $X \in \mathcal{A} \otimes \hat{k_0}^{\otimes n}$ and $Y \in \mathcal{A} \otimes \hat{k_0}^{\otimes m}$ with $m \neq n$, and then use the definition of $\hat{J}_t$ and $\widehat{\Theta}_\zeta$ to prove the identity for $X, Y \in \mathcal{A} \otimes_{\text{alg}} \hat{k_0}^{\otimes m}$. Finally, use corollary 1.3.8 and strong continuity of $\hat{J}_t$ obtained from the estimate in (i) to extend the identity from $X = \sum x_\alpha \otimes e_\alpha$, $Y = \sum y_\alpha \otimes e_\alpha$ (finite sums) to arbitrary X and Y. Thus one has upon setting

$$\Phi_t(X, Y) \equiv \langle \hat{J}_t(X \otimes e(f))u, \hat{J}_t(Y \otimes e(g))v\rangle - \langle ue(f), \hat{J}_t(\langle X, Y\rangle \otimes e(g))v\rangle$$

the equation :

$$\Phi_t(X, Y) = \int_0^t ds\{\Phi_s(\widehat{\Theta}_{\hat{f}(s)}(X), \mathcal{J}_{\hat{g}(s)}(Y)) + \Phi_s(\mathcal{J}_{\hat{f}(s)}(X), \widehat{\Theta}_{\hat{g}(s)}(Y)) + \Phi_s(\widehat{\Theta}_{f(s)}(X), \widehat{\Theta}_{g(s)}(Y))\}. \tag{3.18}$$

where $\langle X, Y\rangle$ is the module inner product in $\mathcal{A} \otimes \Gamma^f(\hat{k_0})$ and we have set for for $\zeta, \eta_1, \ldots, \eta_n \in \hat{k_0}$ the map $\mathcal{J}_\zeta(x \otimes \eta_1 \ldots \otimes \eta_n) = x \otimes \eta_1 \ldots \otimes \eta_n \otimes \zeta$, and extend it naturally as a map from $\mathcal{A} \otimes \Gamma^f(\hat{k_0})$ to itself. It is clear that the estimates in lemma 3.2.1(iii) and theorem 3.2.6(i) extend to

$$||\widehat{\Theta}_\zeta(X)u|| \leq ||X \otimes 1_{\hat{k_0}})D(u \otimes \zeta)||$$

and

$$\sup_{0 \leq t \leq t_0} ||\hat{J}_t(X \otimes e(f))u|| \leq C'(f)||(X \otimes 1_{\Gamma^f(\hat{k})})E_{t_0}u||.$$

From the above estimates and definition of Φ_t, it is clear that $|\Phi_t(X,Y)| \leq ||u||||v||||X||||Y||||E_{t_0}||C'(g)\{||E_{t_0}||C'(f) + ||e(f)||\}$. This implies, by iterating the expression (3.18) sufficient number of times, that $\Phi_t(X,Y) = 0$ which leads to $\Phi_t(x,y) = 0$ for all $x, y \in \mathcal{A}$. Since $\langle ve(g), j_t(x)(\sum_{i=1}^n u_i e(f_i))\rangle = \langle J_t(x^* \otimes e(g))v, \sum u_i e(f_i)\rangle$ by the above identity, it follows that $j_t(x)$ is well defined on $h \otimes_{\text{alg}} \mathcal{E}(C)$, and thus (ii)(a) is proven. The proof of (ii)(b) and (iii) are as in [18] and [45] respectively. For (iv), we note that $j_t(1) = 1$ for all t if and only if $dJ_t(1 \otimes e(f))u = 0 \; \forall \; u \in h, f \in C$; and from equation (3.9) and (S1) it is clear that this can happen if and only if $0 = \int_0^t ds \langle ue(f), J_s \circ (\Lambda_{\Sigma^*\Sigma - I}(ds)(ve(f))\rangle$ $= -\int_0^t ds \langle ue(f), J_s(\langle f(s), (\Sigma^*\Sigma - I)_{f(s)}\rangle \otimes e(f))v\rangle$ for all t, since $\pi(1)\delta = \delta$. But this is possible if and only if $\langle \zeta, (\Sigma^*\Sigma - I)_\zeta\rangle = 0 \; \forall \; \zeta \in k_0$ which is same as $\Sigma^*\Sigma = I$. □

Chapter 4

Dilation of a quantum dynamical semigroup

In this chapter, we start with a uniformly continuous q.d.s T_t on a unital von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(h)$ and consider the problem of its quantum stochastic dilation. We remark that without loss of generality we may assume T_t to be conservative, since the case of a non-conservative T_t can be reduced to that of a conservative one by suitably enlarging the given algebra and extending the given non-conservative semigroup to a conservative semigroup on the bigger algebra (see the proof of the result 1.2.18 for a description of this enlargement).

First, a unitary evolution U_t is constructed in $h \otimes \Gamma$ such that the vacuum expectation of $j_t^0(x) \equiv U_t(x \otimes 1_\Gamma)U_t^*$ gives back the q.d.s. T_t that we have started with. However, $j_t^0(x)$ in general will not satisfy a flow equation of the Evans-Hudson type. It is shown that there exists a suitable choice of a partial isometry in $h \otimes k_0$ such that the Evans-Hudson type of flow equation can be implemented by a partial isometry-valued process in $h \otimes \Gamma$. It is to be noted here that in [28] an Evans- Hudson type dilation was achieved with $\mathcal{A} = \mathcal{B}(h)$ for a countably infinite dimensional h only.

4.1 Hudson-Parthasarathy (H-P) dilation

Let $(\rho, \mathcal{K}, \alpha, H, R)$ be a quintuple associated with T_t obtained from theorem 1.2.21, R is the implementer of α as in 1.2.21, and (k_1, Σ_1) be the pair for the representation ρ as in the theorem 1.1.6. Denote the projection $\Sigma_1 \Sigma_1^*$ by P_1. Now set $\tilde{R} = \Sigma_1 R$. $\tilde{R} \in$

$\mathcal{B}(h, h\otimes k_1)$ so that $\tilde{R}^* = R^*\Sigma_1^*$ and we have

$$\tilde{R}^*(x\otimes 1_{k_1})\tilde{R} = R^*\Sigma_1^*(x\otimes 1_{k_1})\Sigma_1 R = R^*\rho(x)R.$$

Also,

$$\tilde{R}^*\tilde{R} = R^*\Sigma_1^*\Sigma_1 R = R^*R, \text{ as } \Sigma_1^*\Sigma_1 = 1_{\mathcal{K}}.$$

Now, we take the unitary process U_t which satisfies the following q.s.d.e. (as in section 3.1)

$$dU_t = U_t(a^\dagger_{\tilde{R}}(dt) - a_{\tilde{R}}(dt) + (iH - \frac{1}{2}\tilde{R}^*\tilde{R})dt), \quad U_0 = I. \tag{4.1}$$

Let $\tilde{\Gamma}$ denote $\Gamma(L^2(\mathbb{R}_+, k_1))$. Taking $j_t^0(x) = U_t(x\otimes 1_{\tilde{\Gamma}})U_t^*$, we see that for each t, $j_t^0(\cdot)$ is a $*$-homomorphism. We now claim that $\langle ve(0), j_t^0(x)ue(0)\rangle = \langle v, T_t(x)u\rangle$. To prove this, it is enough to show that $\langle ve(0), \frac{d}{dt}j_t^0(x)(ue(0))\rangle = \langle v, T_t(\mathcal{L}(x))u\rangle$, and this follows from the quantum Ito formula for right integrals as mentioned in remark 2.2.6. Indeed we have, $\langle U_t^*(ve(0)), (x\otimes 1_{\tilde{\Gamma}})U_t^*(ue(0))\rangle = \int_0^t ds\langle ve(0), j_s^0(\mathcal{L}(x))(ue(0))\rangle$, where $\mathcal{L}(x) = R^*\rho(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H,x] = \tilde{R}^*(x\otimes 1_{k_1})\tilde{R} - \frac{1}{2}\tilde{R}^*\tilde{R}x - \frac{1}{2}x\tilde{R}^*\tilde{R} + i[H,x]$. Thus, if we denote by $\mathbb{E}_0$ the vacuum expectation map which takes an element G of $\mathcal{B}(h\otimes\tilde{\Gamma})$ to an element $\mathbb{E}_0 G$ in $\mathcal{B}(h)$ satisfying $\langle v, (\mathbb{E}_0 G)u\rangle = \langle ve(0), G(ue(0))\rangle$ (for $u, v\in h$), then

$$\frac{d}{dt}\mathbb{E}_0 j_t^0(x) = \mathbb{E}_0 j_t^0(\mathcal{L}(x)),$$

which implies (since $\mathcal{L}$ is bounded), that $\mathbb{E}_0 j_t^0(x) = T_t(x)$.

A simple calculation using the quantum Ito formula and equation (4.1) shows that

$$dj_t^0(x) = U_t[a^\dagger_{\alpha(x)}(dt) - a_{\alpha(x)}(dt) + \mathcal{L}(x)dt]U_t^*, \tag{4.2}$$

where $\alpha(x) = \tilde{R}x - (x\otimes 1_{k_1})\tilde{R} = \Sigma_1[Rx - \rho(x)R]$, for $x\in\mathcal{A}$. In general, $\alpha(x)$ may not be in $\mathcal{A}\otimes k_1$ and therefore the equation (4.2) is not a flow equation of the Evans-Hudson type. However, in case $\mathcal{A} = \mathcal{B}(h)$, it is trivially a flow equation.

4.2 Existence of structure maps and Evans-Hudson dilation of T_t

In the context of the theorems 1.2.21 and 1.1.6, it should be noted that in general $\mathcal{K}$ need not be of the form $h\otimes k'$ and neither ρ or α be structure maps as defined in chapter 3. that is. ρ need not be in $\mathcal{A} \otimes \mathcal{B}(k')$ nor $\alpha(x)$ be in $\mathcal{A} \otimes k'$. However, the following theorem asserts that one can " rotate " the whole structure appropriately so that the " rotated " ρ and α (denoted π and δ respectively) become structure maps without changing $\mathcal{L}$ (see also [46]).

Theorem 4.2.1 *Let T_t be a conservative norm-continuous q.d.s. with generator $\mathcal{L}$. Then there exist a Hilbert space k_0, a normal $*$-representation $\pi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(k_0)$ and a π-derivation δ of $\mathcal{A}$ into $\mathcal{A} \otimes k_0$ such that the hypotheses (S1)-(S3) in chapter 3 are satisfied.*

Proof : (i) Let $(\rho, \mathcal{K}, \alpha, H, R)$ be a quintuple for T_t as in theorem 1.2.21. We define a map $\rho' : \mathcal{A}' \to \mathcal{B}(\mathcal{K})$, where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$ in $\mathcal{B}(h)$, by

$$\rho'(a)(\alpha(x)u) = \alpha(x)\ au, x \in \mathcal{A},\ \ u \in h,\ \ a \in \mathcal{A}', \tag{4.3}$$

and extend it linearly to the algebraic span of $\mathcal{D} \equiv \{\alpha(x)u\ :\ u \in h, x \in \mathcal{A}\}$.
To show that it is well defined, we need to show that whenever $\sum_{i=1}^{m}\ \alpha(x_i)u_i = 0$ for $x_i \in \mathcal{A}, u_i \in h$. one has $\rho'(a)(\sum_{i=1}^{m}\alpha(x_i)u_i) = 0$. Since $\alpha(x_i)^*\alpha(y) = \mathcal{L}(x_i^*y) - \mathcal{L}(x_i^*)y - x_i^*\mathcal{L}(y) \in \mathcal{A}$ for $y \in \mathcal{A}$, we have for $a \in \mathcal{A}'$, $\langle\rho'(a)(\sum_{i=1}^{m}\alpha(x_i)u_i),\ \alpha(y)v\rangle = \sum_{i=1}^{m}\langle\alpha(x_i)au_i, \alpha(y)v\rangle$

$$= \sum_{i=1}^{m}\langle u_i, a^*\alpha(x_i)^*\alpha(y)v\rangle = \sum_{i=1}^{m}\langle u_i, \alpha(x_i)^*\alpha(y)a^*v\rangle = \langle\sum_{i=1}^{m}\alpha(x_i)u_i,\ \ \alpha(y)a^*v\rangle, \tag{4.4}$$

thereby proving that ρ' is well defined. A similar computation gives,

$$\|\rho'(a)(\sum_{i=1}^{m}\alpha(x_i)u_i)\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{m}\langle u_i, \alpha(x_i)^*\alpha(x_j)a^*au_j\rangle. \tag{4.5}$$

Denoting the operator $\alpha(x_i)^*\alpha(x_j)$ by A_{ij}, and noting that $A \equiv ((A_{ij}))_{i,j=1,\dots,m}$ acts as a positive operator on $\underbrace{h \oplus \ldots \oplus h}_{m\ \text{copies}}$, which commutes with the positive operator $C \otimes I_m$, where $C = \|a\|^2.1 - a^*a$ and I_m denotes the identity matrix of order m, we observe that $A(C \otimes I_m)$ is also a positive operator. Thus, considering $u_1 \oplus u_2 \oplus \ldots \oplus u_m \in \underbrace{h \oplus \ldots \oplus h}_{m\ \text{copies}}$, the right hand side of (4.5) can be estimated as :

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{m}\langle u_i, \alpha(x_i)^*\alpha(x_j)a^*au_j\rangle \\
&\le \ \|a\|^2\sum_{i=1}^{m}\sum_{j=1}^{m}\langle u_i, \alpha(x_i)^*\alpha(x_j)u_j\rangle \\
&= \ \|\sum_{i=1}^{m}\alpha(x_i)u_i\|^2\|a\|^2,
\end{aligned}$$

proving that $\rho'(a)$ extends to a bounded operator on $\mathcal{K}$ satisfying $\|\rho'(a)\| \le \|a\|$ since $\mathcal{D}$ is total in $\mathcal{K}$. It is also easy to see from the definition of ρ' and (4.4) that it is a unital $*$-representation of $\mathcal{A}'$ in $\mathcal{K}$. Next we show that ρ' is normal. For this, take a net $\{a_\alpha\}$ such that $0 \le a_\alpha \uparrow a$ where $a_\alpha, a \in \mathcal{A}'$. It is clear from the definition of ρ' that $\rho'(a_\alpha)\alpha(x)u \to \rho'(a)\alpha(x)u$ for all $x \in \mathcal{A}$, $u \in h$, and thus, $\rho'(a_\alpha) \overset{s}{\to} \rho'(a)$ on $\mathcal{K}$ by totality of $\mathcal{D}$ in $\mathcal{K}$ and since $\|\rho'(a_\alpha)\| \le \|a_\alpha\| \le \|a\| \ \forall \ \alpha$.

(ii) By (i), $\rho' : \mathcal{A}' \to \mathcal{B}(\mathcal{K})$ is a unital normal *- representation. By theorem 1.1.6, there exist a Hilbert space k_2, an isometry $\Sigma_2 : \mathcal{K} \to h \otimes k_2$ with $\mathcal{K}_2 = \mathrm{Ran}\Sigma_2 \cong \mathcal{K}$, satisfying,

$$\rho'(a) = \Sigma_2^*(a \otimes 1_{k_2})\Sigma_2, \tag{4.6}$$

and for all $a \in \mathcal{A}'$, $a \otimes 1_{k_2}$ commutes with $P_2 \equiv \Sigma_2\Sigma_2^*$. Let us now take $\tilde{\delta}(x) = \Sigma_2\alpha(x)$, $\tilde{\pi}(x) = \Sigma_2\rho(x)\Sigma_2^*$. It is clear that $\tilde{\delta}$ is a $\tilde{\pi}$-derivation. Moreover, $\tilde{\delta}(x^*)^*\tilde{\delta}(y) = \alpha(x^*)^*\Sigma_2^*\Sigma_2\alpha(y) = \alpha(x^*)^*\alpha(y)$ and hence $\mathcal{L}(xy) - x\mathcal{L}(y) - \mathcal{L}(x)y = \tilde{\delta}(x^*)^*\tilde{\delta}(y)$ holds. Taking $\tilde{R} = \Sigma_2 R \in \mathcal{B}(h, h\otimes k_2)$, we observe that $\tilde{\delta}(x) = \Sigma_2\alpha(x) = \Sigma_2(Rx - \rho(x)R) = \tilde{R}x - \tilde{\pi}(x)\tilde{R}$. It is also clear that $\mathcal{L}(x) = \tilde{R}^*\tilde{\pi}(x)\tilde{R} - \frac{1}{2}\tilde{R}^*\tilde{R}x - \frac{1}{2}x\tilde{R}^*\tilde{R} + i[H, x] = R^*\rho(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H, x]$.

To show that $\tilde{\delta}(x) \in \mathcal{A}\otimes k_2$ for all $x \in \mathcal{A}$, it is enough (by lemma 1.3.9) to verify that for any $f \in k_2$, $\langle f, \tilde{\delta}(x)\rangle \in \mathcal{A}$, or equivalently that $\langle f, \tilde{\delta}(x)\rangle$ commutes with all $a \in \mathcal{A}'$. For $f \in k_2$, $a \in \mathcal{A}'$, $u, v \in h$, $x \in \mathcal{A}$, since P_2 and $(a \otimes 1_{k_2})$ commute,we have,

$$\langle\langle f, \tilde{\delta}(x)\rangle au, v\rangle = \langle\tilde{\delta}(x)au, v \otimes f\rangle = \langle\Sigma_2\alpha(x)au, v \otimes f\rangle = \langle\Sigma_2\rho'(a)(\alpha(x)u), v \otimes f\rangle$$

$$= \langle\Sigma_2\Sigma_2^*(a\otimes 1_{k_2})\Sigma_2\alpha(x)u, v\otimes f\rangle = \langle P_2(a\otimes 1_{k_2})\Sigma_2\alpha(x)u, v\otimes f\rangle = \langle(a\otimes 1_{k_2})\Sigma_2\Sigma_2^*\Sigma_2\alpha(x)u, v\otimes f\rangle$$

$$= \langle\Sigma_2\alpha(x)u, (a^*v) \otimes f\rangle = \langle\langle f, \tilde{\delta}(x)\rangle u, a^*v\rangle = \langle a\langle f, \tilde{\delta}(x)\rangle u, v\rangle.$$

Next, we want to show that $\tilde{\pi}(x) \in \mathcal{A} \otimes \mathcal{B}(k_2)$ for $x \in \mathcal{A}$; and for this it is enough to verify $\tilde{\pi}(x)(a \otimes 1_{k_2}) = (a \otimes 1_{k_2})\tilde{\pi}(x)$ for all $a \in \mathcal{A}'$. Since $\Sigma_2^* P_2^{\perp} = 0$ and P_2 commutes with $(a \otimes 1_{k_2})$, it is clear that $\tilde{\pi}(x)(a \otimes 1_{k_2})P_2 = \Sigma_2 \rho(x) \Sigma_2^* P_2 (a \otimes 1_{k_2}) = 0$ $= (a \otimes 1_{k_2})\tilde{\pi}(x)P_2$. Thus, it suffices to verify that $\tilde{\pi}(x)(a \otimes 1_{k_2})P_2 = (a \otimes 1_{k_2})\tilde{\pi}(x)P_2$, or equivalently (since $\Sigma_2 \mathcal{D}$ is total in $\mathcal{K}_2$) that $\tilde{\pi}(x)(a \otimes 1_{k_2})\Sigma_2 \alpha(y)u =$ $(a \otimes 1_{k_2})\tilde{\pi}(x)\Sigma_2\alpha(y)u$, for all $y \in \mathcal{A}$, $u \in h$. For this, observe that.

$$\tilde{\pi}(x)(a \otimes 1_{k_2})\Sigma_2\alpha(y)u = \Sigma_2 \rho(x)\Sigma_2^*(a \otimes 1_{k_2})\Sigma_2\alpha(y)u = \Sigma_2\rho(x)\rho'(a)\alpha(y)u$$

$$= \Sigma_2\rho(x)\alpha(y)au = \Sigma_2\alpha(xy)au - \Sigma_2\alpha(x)yau = \Sigma_2\rho'(a)(\alpha(xy) - \alpha(x)y)u$$

$$= \Sigma_2\rho'(a)\rho(x)\alpha(y)u = \Sigma_2\Sigma_2^*(a \otimes 1_{k_2})\Sigma_2\rho(x)\Sigma_2^*(\Sigma_2\alpha(y)u)$$

$$= P_2(a \otimes 1_{k_2})\tilde{\pi}(x)(\Sigma_2\alpha(y)u) = (a \otimes 1_{k_2})\tilde{\pi}(x)(\Sigma_2\alpha(y)u).$$

(iii) Recall that by the theorem 1.1.6, we can write ρ as $\rho(x) = \Sigma_1^*(x \otimes 1_{k_1})\Sigma_1$ as in the previous section. It follows that $\tilde{\pi}(x) = \Sigma_2\Sigma_1^*(x \otimes 1_{k_1})\Sigma_1\Sigma_2^* \equiv \tilde{\Sigma}^*(x \otimes 1_{k_1})\tilde{\Sigma}$ on $h \otimes k_2$ so that $\tilde{\Sigma}$ is a partial isometry with initial set $P_2(h \otimes k_2)$ and final set $P_1(h \otimes k_1)$. Now set $k_0 = k_1 \oplus k_2$ and $\Sigma = \tilde{\Sigma} \oplus 0 : h \otimes k_0 \to h \otimes k_0$ with initial set $(0 \oplus P_2)(h \otimes k_0)$ and final set $(P_1 \oplus 0)(h \otimes k_0)$ and $\pi(x) = \tilde{\pi}(x) \oplus 0$, $\delta(x)u = \tilde{\delta}(x)u$ for $x \in \mathcal{A}, u \in h$. It is clear that $\delta(x) \in \mathcal{A} \otimes k_0$, $\pi(x) \in \mathcal{A} \otimes \mathcal{B}(k_0)$ and (S1)-(S3) are satisfied. □

Remark 4.2.2 *Although ρ was assumed to be unital, π chosen by us is not unital. However, in some cases it may be possible to choose Σ, k_0 in such a manner that π is unital.*

We summarise the main result of this section in form of the following theorem :

Theorem 4.2.3 *Let $(T_t)_{t \geq 0}$ be a conservative norm-continuous q.d.s. with $\mathcal{L}$ as its generator. Then there is a flow $J_t : \mathcal{A} \otimes \mathcal{E}(C) \to \mathcal{A} \otimes \Gamma$ satisfying an Evans-Hudson type q.s.d.e. (3.9) with structure maps $(\mathcal{L}, \delta, \sigma)$ satisfying (S1)-(S3), where $\Gamma \equiv \Gamma(L^2(\mathbb{R}_+, k_0))$ and C consists of bounded continuous functions in $L^2(\mathbb{R}_+, k_0)$, such that $j_t(x)$ defined in theorem 3.2.6 is a (not necessarily unital) $*$-homomorphism of $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(k_0)$ and $\mathbb{E}_0 j_t(x) = T_t(x)\ \forall x \in \mathcal{A}$.*

Proof : The proof is immediate by (i) observing the existence of structure maps δ and σ satisfying (S1),(S2) from theorem 4.2.1,(ii) observing that $\mathcal{L}$ satisfies (S3), and finally (iii) constructing the solution J_t of equation (3.9) with structure maps $(\mathcal{L}, \delta, \sigma)$ as in theorem 3.2.6. That $\mathbb{E}_0 j_t(x) = T_t(x)\ \forall x \in \mathcal{A}$ follows from the q.s.d.e (3.9). □

Remark 4.2.4 *With reference to the last sentence in the statement of 1.1.6, it may be noted that both the Hilbert spaces k_1 and k_2 and hence k_0 can be chosen to be separable if the initial Hilbert space h is separable. In such a case, if we choose an orthonormal basis $\{e_j\}$ in k_0, then the estimate for δ in (S2) is precisely the coordinate-free form of the condition*

$$\sum_{i=0} ||\mu_0^i(x)u||^2 \leq \sum_{i \in \mathcal{I}_0} ||x D_0^i u||^2$$

with $\sum_{i\in\mathcal{I}_0} ||D_0^i u||^2 \leq \alpha_0 ||u||^2$ *as in [42], [39]. The similar conditions on* $\mu_j^i (j \neq 0)$ *as in [42] are trivially satisfied by* $((\mu_j^k))_{k,j=1}^{\infty} \equiv \sigma$ *and for* $\mu_0^0 \equiv \mathcal{L}$ *as can be seen easily from (S1) and (S3). It may also be noted that* j_t *satisfies the E-H equation* $dj_t(x) = \sum_{i,j} j_t(\mu_j^i(x)) d\Lambda_i^j(t)$, with $j_0 = id$, *in the coordinatized form with the appropriate choices of* μ_j^i*'s in terms of* $\mathcal{L}, \delta$ *and* σ *as above. The flow equation (3.9) is in fact a coordinate-free modification of the old coordinatized E-H equation given above.*

4.3 An interesting duality

In the previous section, starting with a $*$-homomorphism ρ of $\mathcal{A}$ and ρ-derivation α, we constructed a $*$-homomorphism ρ' of $\mathcal{A}'$ which satisfies $\rho'(a)\alpha(x) = \alpha(x)a\ \forall a \in \mathcal{A}', x \in \mathcal{A}$. Let us now observe that $\rho(x)$ and $\rho'(a)$ commute for each $x \in \mathcal{A}$ and $a \in \mathcal{A}'$. Due to the totality of vectors of the form $\alpha(y)u,\ y \in \mathcal{A}, u \in h$ in $\mathcal{K}$, it is enough to verify that $\rho'(a)\rho(x)\alpha(y) = \rho(x)\rho'(a)\alpha(y)$. But we have,

$$\rho'(a)\rho(x)\alpha(y) = \rho'(a)\alpha(xy) - \rho'(a)\alpha(x)y = \alpha(xy)a - \alpha(x)ay$$
$$= (\alpha(xy) - \alpha(x)y)a = \rho(x)\alpha(y)a = \rho(x)\rho'(a)\alpha(y).$$

Denote by $\mathcal{E}_\rho$ and $\mathcal{E}_{\rho'}$ respectively the spaces of intertwainers of ρ and ρ', that is,

$$\mathcal{E}_\rho \equiv \{L \in \mathcal{B}(h, \mathcal{K}) : Lx = \rho(x)L\ \forall x \in \mathcal{A}\},$$

$$\mathcal{E}_{\rho'} \equiv \{S \in \mathcal{B}(h, \mathcal{K}) \; : \; Sa = \rho'(a)S \; \forall a \in \mathcal{A}'\}.$$

Cleraly. $\mathcal{E}_\rho$ is a Hilbert von Neumann $\mathcal{A}'$-module and $\mathcal{E}_{\rho'}$ is a Hilbert von neumann $\mathcal{A}$-module. The right module actions are given by $(L, a) \mapsto La$ and $(S, x) \mapsto Sx$ for $a \in \mathcal{A}'$ and $x \in \mathcal{A}$ respectively. Furthermore, it is easy to verify that inner products of $\mathcal{E}_\rho$ and $\mathcal{E}_{\rho'}$, inherited from that of $\mathcal{B}(h, \mathcal{K})$, take values in $\mathcal{A}'$ and $\mathcal{A}$ respectively. To see this, note that $\forall x \in \mathcal{A}$ and $L, M \in \mathcal{E}_\rho$, we have $\langle L, M\rangle x = L^*Mx = L^*\rho(x)M = xL^*M = x\langle L, M\rangle$; and similarly $\forall a \in \mathcal{A}'$, $S, T \in \mathcal{E}_{\rho'}$, $\langle S, T\rangle a = a\langle S, T\rangle$.

Clearly, $\alpha(x) \in \mathcal{E}_{\rho'}$ $\forall x \in \mathcal{A}$, and hence there is an implementer R of α which belongs to $\mathcal{E}_{\rho'}$, since such an R can be chosen from the ultra-weak closure of $\{\alpha(x)y \; : \; x, y \in \mathcal{A}\}$. Now, choose and fix any L from $\mathcal{E}_\rho$ and a self-adjoint element $H' \in \mathcal{A}'$. Consider the ρ'-derivation given by $\beta_L(a) \equiv La - \rho'(a)L$ and the CCP map given by $\mathcal{L}'_{L,H'}(a) = L^*\rho'(a)L - \frac{1}{2}L^*La - \frac{1}{2}aL^*L + i[H', a]$. Since $\rho'(a)$ and $\rho(x)$ commute for all $a \in \mathcal{A}'$ and $x \in \mathcal{A}$, it is clear that $\beta_L(a) \in \mathcal{E}_\rho$ $\forall a \in \mathcal{A}'$. Furthermore. $L^*\rho'(a)Lx = L^*\rho'(a)\rho(x)L = L^*\rho(x)\rho'(a)L = xL^*\rho'(a)L$ for $x \in \mathcal{A}, a \in \mathcal{A}'$. which shows that $L^*\rho'(a)L \in \mathcal{A}'$, and hence the range of the map $\mathcal{L}'_{L,H'}$ is in $\mathcal{A}'$. Thus, given the semigroup $T_t \equiv e^{t\mathcal{L}}$ on $\mathcal{A}$, we are able to construct a family of semigroups $T'^{(L,H')}_t \equiv e^{t\mathcal{L}'_{L,H'}}$ on $\mathcal{A}'$, indexed by L, H', such that each member of this family in "conjugate" or "dual" to T_t in some suitable sense. To make this precise. let us make the following definition :

Definition 4.3.1 *A pair of uniformly continuous q.d.s. (S_t, S'_t) on $\mathcal{A}$ and $\mathcal{A}'$ respectively are said to be conjugate to each other if there exist a Hilbert space $\mathcal{K}$, $*$-homomorphisms η of $\mathcal{A}$ and η' of $\mathcal{A}'$ into $\mathcal{B}(\mathcal{K})$, an η-derivation β of $\mathcal{A}$ and η'-derivation β' of $\mathcal{A}'$ into $\mathcal{B}(h, \mathcal{K})$ and self-adjoint elements $K \in \mathcal{A}$, $K' \in \mathcal{A}'$ such that the followings hold :*

(i) $\eta(x)$ and $\eta'(a)$ commute for each $x \in \mathcal{A}$ and $a \in \mathcal{A}'$.

(ii) There exist $W, W' \in \mathcal{B}(h, \mathcal{K})$ such that $\beta(x) = Wx - \eta(x)W$, $\beta'(a) = W'a - \eta'(a)W'$, $Wa = \eta'(a)W$ and $W'x = \eta(x)W'$ for all $x \in \mathcal{A}$ and $a \in \mathcal{A}'$.

(iii) The generators $\mathcal{L}^S$ and $\mathcal{L}^{S'}$ of S_t and S'_t (respectively) have the forms $\mathcal{L}^S(x) = W^*\eta(x)W - \frac{1}{2}W^*Wx - \frac{1}{2}xW^*W + i[K, x]$ and $\mathcal{L}^{S'}(a) = W'^*\eta'(a)W' - \frac{1}{2}W'^*W'a - \frac{1}{2}aW'^*W' + i[K', a]$.

It is clear that T_t and $T'^{(L,H')}_t$ are conjugate to each other for any L, H' according to the above definition. Note that in this definition, we do not require that either of the sets $\{\beta(x)u : x \in \mathcal{A}, u \in h\}$ and $\{\beta'(a)u : a \in \mathcal{A}', u \in h\}$ is total in $\mathcal{K}$. In fact, in the present context, there need not be any $L \in \mathcal{E}_\rho$ such that $\{\beta_L(a)u : a \in \mathcal{A}', u \in h\}$ is total in $\mathcal{K}$. For example, if $\mathcal{A} = \mathcal{B}(h)$, then $\mathcal{A}'$ is isomorphic to $\mathbb{C}$, and hence for any L, β_L will be identically zero. However, if $\mathcal{A}'$ is not too small compared to $\mathcal{A}$, one may expect that for some L, the above totality will be achieved.

4.4 Appearance of Poisson terms in the dilation

Given the semigroup T_t, we have obtained a quantum stochastic dilation j_t which satisfies Evans-Hudson type q.s.d.e involving the deterministic (time) integrator $\mathcal{I}_\mathcal{L}(dt)$ and non-deterministic integrators $a_\delta(dt), a_\delta^\dagger(dt)$ and $\Lambda_\sigma(dt)$. We shall now investigate the necessity of Λ-coefficient, which we call conservation or Poisson term. We say that the semigroup T_t is *Poisson-free* if there exists an E-H dilation for T_t which has no Poisson or term. It is clear that T_t with the generator $\mathcal{L}$ is Poisson-free if and only if it is possible to obtain a triplet of structure maps $(\mathcal{L}, \delta, \sigma)$ with σ being identically zero. It should be noted here that there may exist some other E-H dilations for a Poisson-free semigroup T_t which involves nonzero Poisson-term. We first state and prove a criterion for Poisson-free nature of a semigroup.

Theorem 4.4.1 *Let T_t be a uniformly continuous conservative q.d.s. on a unital von Neumann algebra $\mathcal{A}$ with the generator $\mathcal{L}$. Denote by $\mathcal{Z}$ the centre of $\mathcal{A}$. Let $D : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be the trilinear form introduced in the proof of the theorem 1.2.21 of chapter 1, that is, $D(a,b,c) = \mathcal{L}(abc) - \mathcal{L}(ab)c - a\mathcal{L}(bc) + a\mathcal{L}(b)c$. Then, the following condition is necessary for T_t to be Poisson-free :*
For all $x, y \in \mathcal{A}$ and $z \in \mathcal{Z}$,

$$D(x^*, z, y) = D(x^*, 1, y)z. \tag{4.7}$$

Furthermore, when either $\mathcal{A}$ or $\mathcal{A}'$ is commutative, then the above condition is also sufficient for T_t to be Poisson-free.

Proof : To prove necessity, assume that T_t is Poisson-free. Hence there will exist a Hilbert space k_0 and a $(x \otimes 1_{k_0})$-derivation $\delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes k_0)$ such that $\mathcal{L}(x^*y) = \mathcal{L}(x^*)y + x^*\mathcal{L}(y) + \delta(x)^*\delta(y)$ $\forall x, y \in \mathcal{A}$. From this cocycle identity it is easily verifiable that $D(a,b,c) = \delta(a^*)^*(b \otimes 1_{k_0})\delta(c)$. But for $x \in \mathcal{A}$, $\delta(x) \in \mathcal{A} \otimes k_0$, which implies that $\delta(x)z = (z \otimes 1_{k_0})\delta(x)$ $\forall z \in \mathcal{Z}$. Hence we have $D(x^*, z, y) = \delta(x)^*\delta(y)z = D(x^*, 1, y)z$.

For the converse (sufficiency) part, recall the notations used in the proof of the theorem 4.2.1. Clearly, $D(a,b,c) = \alpha(a^*)^*\rho(b)\alpha(c)\ \forall a,b,c \in \mathcal{A}$. For $z \in \mathcal{Z}$ and $x \in \mathcal{A}$, we have

$$\begin{aligned}
&(\alpha(x)z - \rho(z)\alpha(x))^*(\alpha(x)z - \rho(z)\alpha(x)) \\
&= z^*D(x^*,1,x)z - D(x^*,z^*,x)z - z^*D(x^*,z,x) + D(x^*,z^*z,x) \\
&= 0,
\end{aligned}$$

using the condition (4.7) and the fact that z and $D(x^*,1,x)$ commute. This shows that $\rho(z)\alpha(x) = \alpha(x)z$. But since $\rho'(z)\alpha(x) = \alpha(x)z$, we obtain that $\rho(z)$ and $\rho'(z)$ agree on the total set of vectors of the form $\alpha(x)u, x \in \mathcal{A}, u \in h$ and hence $\rho(z) = \rho'(z)$ for all $z \in \mathcal{Z}$. As in the proof of 4.2.1, write $\rho'(a) = \Sigma_2{}^*(a \otimes 1_{k_2})\Sigma_2$, for some Hilbert space k_2 and an isometry $\Sigma_2 : \mathcal{K} \to h \otimes k_2$ such that $P_2 = \Sigma_2\Sigma_2{}^* \in \mathcal{A} \otimes \mathcal{B}(k_2)$. It has been shown that $\tilde{\delta}(.) \equiv \Sigma_2\alpha(.) \in \mathcal{A} \otimes k_2$, $\tilde{\pi}(.) \equiv \Sigma_2\rho(.)\Sigma_2{}^* \in \mathcal{A} \otimes \mathcal{B}(k_2)$. But in the present situation, $\rho(z) = \rho'(z) = \Sigma_2{}^*(z \otimes 1_{k_2})\Sigma_2$ for all $z \in \mathcal{Z}$, and hence we have $\tilde{\pi}(z) = (z \otimes 1_{k_2})P_2$. Assume now that $\mathcal{A}$ is commutative, that is, $\mathcal{A} = \mathcal{Z}$. Since $P_2\tilde{\delta} = \tilde{\delta}$ and $\tilde{\delta}$ is a $\tilde{\pi}$-derivation, it is clear that $\tilde{\delta}$ is $(z \otimes 1_{k_2})$-derivation, which proves the existence of a Poisson-free E-H dilation.

In case when $\mathcal{A}' = \mathcal{Z}$, we choose an isometry $\Sigma' : \mathcal{K} \to h \otimes k'$ for some Hilbert space k', such that $\Sigma'\Sigma'^* \in \mathcal{Z} \otimes \mathcal{B}(k')$, and $\rho(x) = \Sigma'^*(x \otimes 1_{k'})\Sigma'$. Then $\rho'(z) = \Sigma'^*(z \otimes 1_{k'})\Sigma'$ for $z \in \mathcal{Z} = \mathcal{A}'$, and it follows that $\Sigma'\alpha(.)$ is a $(x \otimes 1_{k'})$-derivation belonging to $\mathcal{A} \otimes \mathcal{B}(k_2)$. This completes the proof in case when $\mathcal{A}'$ is commutative.

□

Let us now investigate the two extreme cases of von Neumann algebras, namely $\mathcal{B}(h)$ and commutative von Neumann algebras.

Corollary 4.4.2 *Any uniformly continuous conservative q.d.s on $\mathcal{B}(h)$ is Poisson-free. On the other hand, a uniformly continuous conservative q.d.s. T_t on a commutative von Neumann algebra is Poisson-free if and only if T_t is trivial, i.e. $T_t(x) = x$ $\forall x \in \mathcal{A}$.*

Proof : First consider the case $\mathcal{A} = \mathcal{B}(h)$. Since $\mathcal{A}' = \mathcal{Z}$ is isomorphic to $\mathbb{C}$, and $D(x,.,y)$ is clearly $\mathbb{C}$-linear, the condition 4.7 follows trivially.

Now, consider the case when $\mathcal{A}$ is comutative, and hence $\mathcal{Z} = \mathcal{A}$. Assume that 4.7 holds. We claim that this condition forces $\mathcal{L}$ to be a derivation. Take $z = y$ to be a projection in $\mathcal{A}$. By expanding both sides of the condition 4.7, we have :

$$\mathcal{L}(x^*y) - \mathcal{L}(x^*y)y - x^*\mathcal{L}(y) + x^*\mathcal{L}(y)y = (\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x^*)y)y,$$

$$\text{or,}\quad [\mathcal{L}(x^*y) - x^*\mathcal{L}(y)](1-y) = [\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x^*)y]y,$$

$$\text{or,}\quad [\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x^*)y](1-y) = [\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x^*)y]y,$$

where the last step follows because $y(1-y) = 0$. But the above implies

$$[\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y)](2y-1) = 0,$$

which shows (since $(2y-1)^2 = 1$) $\mathcal{L}(x^*y) = \mathcal{L}(x^*)y + x^*\mathcal{L}(y)$ for all $x \in \mathcal{A}$ and all projections $y \in \mathcal{A}$. But the fact that any element of $\mathcal{A}$ can be approximated in the strong topology by a norm-bounded sequence of elements from the linear span of the projections in $\mathcal{A}$ and that $\mathcal{L}$ is continuous in the strong topology on any norm-bounded set imply that the above relation holds for all $x, y \in \mathcal{A}$, i.e. $\mathcal{L}$ is a derivation of $\mathcal{A}$ into itself. But by the general theory of von Neumann algebras (see [14]) there will be a self-adjoint $H \in \mathcal{A}$ such that $\mathcal{L}(x) = [H, x]\ \forall x \in \mathcal{A}$, which shows that $\mathcal{L}(x) = 0$ since $\mathcal{A}$ is commutative, i.e. T_t is trivial. This completes the proof of the corollary. □

However, the statement of the above proposition for commutative von Neumann algebras does not extend to the case when T_t is not uniformly continuous, that is, when its generator is unbounded. The simplest example is provided by the heat semigroup on $\mathcal{A} \equiv L^\infty(\mathbb{R})$. Let $(B_t)_{t\geq 0}$ denote the one-dimensional standard Brownian motion defined on the Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ denotes the one-dimensional Wiener measure. It is well-known (see [45]) that $L^2(\mathbb{P})$ is naturally isomorphic with $\Gamma(L^2(\mathbb{R}_+))$. Let h denote $L^2(\mathbb{R})$. We define a time-indexed family j_t of $*$-homomorphisms from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+)) \cong L^\infty(\mathbb{R}) \otimes \mathcal{B}(L^2(\mathbb{P}))$ by setting $j_t(\phi)$ to be the multiplication by $\phi(.+B_t)$ in $L^2(\mathbb{R}) \otimes L^2(\mathbb{P})$. Denote by $\mathcal{A}^{(2)}$ the set of functions ϕ in $L^\infty(\mathbb{R})$ which have second order continuous derivatives such that ϕ' and ϕ'' belong to $L^\infty(\mathbb{R})$. It is then clear that for $\phi \in \mathcal{A}^{(2)}$, $j_t(\phi)$ satifies the stochastic differential equation

$$dj_t(\phi) = j_t(\phi')dB_t + \frac{1}{2}j_t(\phi'')dt;\ \ j_0(\phi) = \phi.$$

Translating this into the Fock space language and noting that dB_t corresponds to $a(dt) + a^\dagger(dt)$, we obtain the following E-H flow equation :

$$dJ_t = J_t \circ (a_\delta(dt) + a_\delta^\dagger(dt) + \mathcal{I}_\mathcal{L}(dt)),\ \ J_0 = id,$$

where J_t is defined by $J_t(\phi \otimes e(f))u = j_t(\phi)(ue(f))$ for $u \in h$, $f \in L^2(I\!R_+)$, and $\delta(\phi) = \phi'$, $\mathcal{L}(\phi) = \frac{1}{2}\phi''$ for $\phi \in \mathcal{A}^{(2)}$. This is clearly a Poisson-free E-H dilation of the semigroup generated by $\mathcal{L}$. We shall now show that it is possible to construct a sequence $T_t^{(n)}$ of uniformly continuous q.d.s. on $\mathcal{A}$ which approximates T_t in a suitable sense and $T_t^{(n)}$ admits an E-H dilation $j_t^{(n)}$ such that the Poisson or Λ-term in the flow equation for $j_t^{(n)}$ tends to zero as $n \to \infty$ in an appropriate sense.

Let $\mathcal{L}_n : \mathcal{A} \to \mathcal{A}$ be defined by $\mathcal{L}_n(\phi)(x) = \frac{n}{2}(\phi(x + \frac{1}{\sqrt{n}}) + \phi(x - \frac{1}{\sqrt{n}}) - 2\phi(x))$. Clearly, for $\phi \in \mathcal{A}^{(2)}$, $\mathcal{L}_n(\phi)$ converges to $\mathcal{L}(\phi)$ pointwise (since ϕ'' is continuous) and hence strongly (by the dominated convergence theorem). To obtain structure maps for constructing an E-H dilation of $T_t^{(n)} \equiv e^{t\mathcal{L}_n}$, let us consider the unitary operator $\mathcal{T}_n$ in $\mathcal{B}(h)$ given by $\mathcal{T}_n f(x) = f(x + \frac{1}{\sqrt{n}})$. Let $k_0 = \mathbb{C}^2$ and $R_n \in \mathcal{B}(h, h \otimes k_0)$ be the operator $R_n f = \sqrt{n/2}((\mathcal{T}_n - I)f \oplus (\mathcal{T}_n^* - I)f)$. Define a $*$-homomorphism π_n of $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(k_0)$ by setting $\pi_n(\phi) = \begin{pmatrix} \mathcal{T}_n \phi \mathcal{T}_n^* & 0 \\ 0 & \mathcal{T}_n^* \phi \mathcal{T}_n \end{pmatrix}$. Since it is easy to see that $\mathcal{T}_n \phi \mathcal{T}_n^*$ is multiplication by $\phi(. + \frac{1}{\sqrt{n}})$ and $\mathcal{T}_n^* \phi \mathcal{T}_n$ is multiplication by $\phi(. - \frac{1}{\sqrt{n}})$, it follows that $\pi_n \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0))$. Now, define a π_n-derivation δ_n by setting $\delta_n(\phi) = R_n \phi - \pi_n(\phi) R_n$. Clearly, $\delta_n(\phi)(x) = \sqrt{n/2}\{\phi(x + \frac{1}{\sqrt{n}}) - \phi(x)$, $\phi(x - \frac{1}{\sqrt{n}}) - \phi(x)\}$, which in particular shows that $\delta_n(\phi) \in \mathcal{A} \otimes k_0$. Then we verify that $\mathcal{L}_n(\phi) = R_n^* \pi_n(\phi) R_n - \frac{1}{2} R_n^* R_n \phi - \frac{1}{2} \phi R_n^* R_n$. Thus, having got the structure maps $(\mathcal{L}_n, \delta_n, \pi_n - id)$, we can construct an E-H flow $j_t^{(n)}$ for $T_t^{(n)}$. The Λ-coefficient in this flow equation is $\pi_n - id$. It is easy to see that for any continuous function ϕ in $\mathcal{A}$, $\pi_n(\phi) - \phi$ converges to 0 strongly. If ϕ is everywhere continuously differentiable with its derivative in $\mathcal{A}$, then this convergence will take place in the norm of $\mathcal{A}$. In fact, if we denote by $\mathcal{A}^\infty$ the set of all smooth functions having the derivatives of all order in $\mathcal{A}$, then for any fixed $\phi \in \mathcal{A}^\infty$, $\mathcal{L}_n(\phi)$, $\delta_n(\phi)$ and $\pi_n(\phi)$ converge respectively to $\mathcal{L}(\phi)$, $\tilde{\delta}(\phi)$ and $\tilde{\pi}(\phi)$ in norm, where $\tilde{\delta}(\phi) = \frac{1}{\sqrt{2}}(\phi', -\phi')$ and $\tilde{\pi}(\phi) = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$. For ϕ in $\mathcal{A}^{(2)}$, the above convergences take place strongly.

Let $(B_t^{(1)}, B_t^{(2)})_{t \geq 0}$ be a two-dimensional standard Brownian motion and $I\!P_2$ be the measure induced by this process. Then $L^2(I\!P_2) \cong \Gamma(L^2(I\!R_+ \otimes \mathbb{C}^2)) = \Gamma(L^2(I\!R_+, k_0))$. Define $\tilde{B}_t = \frac{1}{\sqrt{2}}(B_t^{(1)} - B_t^{(t)})$. Clearly, $\tilde{B}_t$ is a one-dimensional standard Brownian motion. If we set $\tilde{j}_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(I\!R_+, k_0))) \cong \mathcal{A} \otimes \mathcal{B}(L^2(I\!P_2))$ by $\tilde{j}_t(\phi) =$ multiplication by $\phi(. + \tilde{B}_t)$, then this time-indexed family of

$*$-homomorphisms satisfies the following flow equation :

$$d\tilde{j}_t(\phi) = \frac{1}{\sqrt{2}}\tilde{j}_t(\phi')(dB_t^{(1)} - dB_t^{(2)}) + \frac{1}{2}\tilde{j}_t(\phi'')dt, \;\; \tilde{j}_0(\phi) = \phi;$$

where $\phi \in \mathcal{A}^{(2)}$. In the Fock-space language, this becomes upon setting $\tilde{J}_t(\phi \otimes e(f))u = \tilde{j}_t(\phi)(ue(f))$ for $u \in h, f \in L^2(I\!R_+, k_0)$:

$$d\tilde{J}_t = \tilde{J}_t \circ (a_{\tilde{\delta}}(dt) + a^\dagger_{\tilde{\delta}}(dt) + \mathcal{I}_{\mathcal{L}}(dt)), \;\; \tilde{J}_0 = id.$$

Thus we see that although the approximating sequence $j_t^{(n)}$ satisfies an E-H equation with Poisson terms, in the limit the contributions of Poisson terms disappear, giving a Poisson-free flow equation for $\tilde{j}_t$.

4.5 Implementation of $E - H$ flow

Recall the notations of theorems 4.2.1 and 4.2.3. We have for $x \in \mathcal{A}$, $\pi(x) = \Sigma^*(x \otimes 1_{k_0})\Sigma \in \mathcal{A} \otimes \mathcal{B}(k_0)$, $\delta(x) = Rx - \pi(x)R \in \mathcal{A} \otimes k_0$ for a suitable Hilbert space k_0, where $R \in \mathcal{B}(h, h \otimes k_0)$ and Σ is a partial isometry in $h \otimes k_0$. Now let us consider the H-P type q.s.d.e. :

$$dV_t = V_t(a^\dagger_R(dt) + \Lambda_{\Sigma^* - I}(dt) - a_{\Sigma R}(dt) + (iH - \frac{1}{2}R^*R)dt), V_0 = I. \tag{4.8}$$

Then by proposition 3.1.2, there is a contraction valued unique solution V_t as a regular process on $h \otimes \Gamma$. The following theorem shows that every Evans-Hudson type flow J_t satisfying equation (3.9) is actually implemented by a process V_t satisfying equation (4.8).

Theorem 4.5.1 *The flow J_t satisfying the equation (3.9) is implemented by a partial isometry valued process V_t satisfying (4.8) , that is, $J_t(x \otimes e(f))u = V_t(x \otimes 1_\Gamma)V_t^*(ue(f))$. Furthermore, the projection-valued processes $P_t \equiv V_tV_t^*$ and $Q_t \equiv V_t^*V_t$ belong to $\mathcal{A} \otimes \mathcal{B}(\Gamma)$ and $\mathcal{A}' \otimes \mathcal{B}(\Gamma)$ respectively.*

We need a lemma for the proof of this theorem.

Lemma 4.5.2 *If $\mathcal{B}$ is a von Neumann algebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and p is a projection in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{B} \ni x \mapsto pxp$ is a $*$-homomorphism of $\mathcal{A}$, then $p \in \mathcal{B}'$.*

Proof of the lemma : Let q be any projection in $\mathcal{B}$. We have by the hypothesis that.

$$pqp = pq^n p = (pqp)^n \ \forall n \geq 1.$$

But

$$(pqp)^n = \underbrace{pqp \cdot pqp \ldots}_{\text{n times}} = (pq)^n p \xrightarrow{s} (p \wedge q)p = p \wedge q,$$

by von Neumann's Theorem, where $p \wedge q$ denotes the projection onto $\mathrm{Ran}(p) \bigcap \mathrm{Ran}(q)$. Thus we have, $(qp)^* qp = pqp = p \wedge q$, which implies that qp is a partial isometry with the initial space $\mathrm{Ran}(p \wedge q)$ and hence $qp.p \wedge q = qp$. But $qp.p \wedge q = p \wedge q$, and thus $qp = p \wedge q = (p \wedge q)^*$ (since $p \wedge q$ is a projection) $= pq$. This completes the proof because $\mathcal{B}$ is generated by its projections. □

Proof of the theorem : Setting $J_t'(x \otimes e(f))u = V_t(x \otimes 1_\Gamma)V_t^*(ue(f))$ for $u \in h, f \in \mathcal{C}$, and using equation (4.8) we verify easily that $J_0' = id$ and J_t' satisfies the same flow equation (3.9) as does J_t. By the uniqueness of the solution of the initial value problem (3.9) we conclude that $J_t = J_t'$. Now, as in theorem 3.2.6 , if we set $j_t(x)ue(f) = J_t(x \otimes e(f))u$, it follows that $j_t(x) = V_t(x \otimes 1_\Gamma)V_t^*$ and that $j_t(.)$ is a $*$-homomorphism of $\mathcal{A}$. Therefore, $V_t(xy \otimes 1_\Gamma)V_t^* = V_t(x \otimes 1_\Gamma)Q_t(y \otimes 1_\Gamma)V_t^*$ for $x, y \in \mathcal{A}$. In particular, $P_t = j_t(1) = V_t V_t^*$ is a projection, that is . V_t is a partial isometry valued regular process. It also follows from the same identity that $Q_t(xy \otimes 1_\Gamma)Q_t = Q_t(x \otimes 1_\Gamma)Q_t(y \otimes 1_\Gamma)Q_t$, that is, $x \otimes 1_\Gamma \mapsto Q_t(x \otimes 1_\Gamma)Q_t$ is a $*$-homomorphism of $\mathcal{A} \otimes 1_\Gamma$. Therefore $Q_t \in (\mathcal{A} \otimes 1_\Gamma)' = \mathcal{A}' \otimes \mathcal{B}(\Gamma)$, by the lemma 4.5.2. □

4.6 Weak Markov process associated with the $E - H$ type flow.

Here we consider the solution J_t of the equation (3.9) or the associated j_t and construct a weak Markov process (see [5]) with respect to the Fock filtration. The next theorem summarizes the results :

Theorem 4.6.1 *(i) Let j_t be as in theorem 4.2.3. Set $F_t \equiv j_t(1)\mathbb{E}_t$ where $\mathbb{E}_t$ is the conditional expectation operator given by $\mathbb{E}_t(ue(f)) = ue(f_t)$. then there exists a nonzero projection $j_\infty(1)$ such that the family of projections $\{j_t(1)\}$ and $\{F_t\}$ decreases and increases to $j_\infty(1)$ respectively.*

(ii) The triple $\{j_t, h \otimes \Gamma, F_t\}$ is a weak Markov process as defined in [5], that is. $\mathbb{E}_0^F j_0(x) = xF_0$, $j_t(x)F_t = F_t j_t(x) = F_t j_t(x) F_t$, $\mathbb{E}_s^F j_t(x) = j_s(T_{t-s}(x))F_s$ for $0 \leq s \leq t < \infty, x \in \mathcal{A}$, where $\mathbb{E}_s^F(X) = F_s X F_s$ for $X \in \mathcal{B}(h \otimes \Gamma)$.

(iii) If we set $k_t(x) = j_t(x)F_t = j_t(x)\mathbb{E}_t = \mathbb{E}_t j_t(x)$, then the triple $\{k_t, h \otimes \Gamma, F\}$ is a conservative weak Markov flow subordinate to $\{F_t\}$ (see [5]), that is, k_t satisfies the properties listed in (ii) above and also $k_t(1) = F_t$.

Proof : (i) In the notation of theorem 4.2.3, $j_t(1) = V_t V_t^*$ and therefore taking $W_t = V_t^*$ and using the relation (3.5) with T replaced by Σ^* we see that $\{j_t(1)\}$ is a decreasing family of projections. On the other hand, a simple computation shows that for $t > s > 0$,

$$\langle ve(g), (F_t - F_s)(ue(f)) \rangle =$$
$$\langle W_t ve(g), W_t ue(f) \rangle e^{-\int_t^\infty \langle g(\tau), f(\tau) \rangle d\tau} - \langle W_s ve(g), W_s ue(f) \rangle e^{-\int_s^\infty \langle g(\tau), f(\tau) \rangle d\tau} =$$
$$\int_s^t exp\{-\int_\tau^\infty \langle g(\tau'), f(\tau') \rangle d\tau'\} \langle W_\tau ve(g), \langle g(\tau), \Sigma^* \Sigma_{f(\tau)} \rangle W_\tau ue(f) \rangle d\tau,$$

from which it follows that $\{F_t\}$ is increasing family of projections. Since $\mathbb{E}_t$ increases to I on $h \otimes \Gamma$, and since $j_t(1)$ converges strongly to say $j_\infty(1)$, we have that F_t increases to $j_\infty(1)$. Therefore $j_\infty(1)$ cannot be the zero projection.

(ii) Let $u, v \in h$, $f, g \in k$, $x \in \mathcal{A}$. Since $j_s(1)j_t(1) = j_t(1)$ for $s \leq t$, we have,

$$\begin{aligned}
\langle F_s j_t(x) F_s(ve(g)), ue(f) \rangle &= \langle j_s(1) j_t(x) j_s(1) ve(g_s), ue(f_s) \rangle \\
&= \langle j_t(x) ve(g_s), ue(f_s) \rangle = \langle J_t(x \otimes e(g_s))v, ue(f_s) \rangle \\
&= \langle J_s(x \otimes e(g_s))v, ue(f_s) \rangle + \int_s^t \langle J_\tau(\mathcal{L}(x) e(g_s))v, ue(f_s) \rangle d\tau \\
&= \langle j_s(x) F_s(ve(g)), ue(f) \rangle + \int_s^t \langle F_s j_\tau(\mathcal{L}(x)) F_s(ve(g)), ue(f) \rangle d\tau,
\end{aligned}$$

because $f_s(\tau) = 0$, $g_s(\tau) = 0$ for $\tau > s$. Thus, if we denote by Ξ_t the map $\mathcal{A} \ni x \mapsto F_s j_t(x) F_s$ then $\frac{d\Xi}{dt} = \Xi_t \circ \mathcal{L}$, for $s \geq t$. On the other hand, denoting by Π_t the map given by, $\Pi_t(x) = j_s(T_{t-s}(x))F_s$, we can easily verify that $\frac{d\Pi}{dt} = \Pi_t \circ \mathcal{L}$. Since $\Xi_s(x) = \mathbb{E}_s j_s(x) = j_s(x) F_s = \Pi_s(x)$, the initial values of Π and Ξ agree. Thus by the standard uniqueness result of differential equations, we conclude that $\Pi_t \equiv \Xi_t$ for all $t \geq s$.

(iii) The proof of this part is obvious from the definitions. □

We have no result with regards to minimality of the above process in the sense of [5]. However, if we denote the closed linear span of $\{j_{t_1}(x_1)j_{t_2}(x_2)\dots j_{t_n}(x_n)ue(0)|$ $x_j \in \mathcal{A},\ t > t_1 \geq t_2 \geq \dots \geq t_n > 0\ \}$ by $\mathcal{K}'_t$ for $0 \leq t \leq \infty$, then it is an easy observation that $\mathcal{K}'_t$ is contained in the range of F_t for each $t < \infty$, and thus $\mathcal{K}'_\infty$ is contained in $j_\infty(1)(h \otimes \Gamma)$. We suspect that $\mathcal{K}'_\infty = j_\infty(1)(h \otimes \Gamma)$, which we have not been able to prove. If this turns out to be true, then the above provides a complete general theory of stochastic dilation for a uniformly continuous quantum dynamical semigroup on a von Neumann algebra. It should also be noted that the final weak Markov process (j_s, F_s) is actually living in $h \otimes \Gamma(k)$ and its filtration is subordinate to that in the Fock space.

4.7 Dilation in the C^* algebraic set-up

Let us prove the existence of a canonical dilation for a uniformly continuous quantum dynamical semigroup on a *separable* unital C^* algebra. Assume in this section that $\mathcal{A}$ is a separable unital C^* algebra in $\mathcal{B}(h)$ and T_t is a uniformly continuous quantum dynamical semigroup acting on $\mathcal{A}$ with the bounded generator $\mathcal{L}$. Let us first introduce some useful notations. Recall the discussions of Hilbert C^* modules and related topics in the first chapter (section 3). Let k_0 be a separable Hilbert space. Fix an orthonormal basis $\{e_1, e_2, \dots\}$ of k_0. We denote the Hilbert modules $\mathcal{A} \otimes_{C^*} k_0$ and $\mathcal{A} \otimes_{C^*} \Gamma$ by F and G respectively, where $\Gamma = \Gamma(L^2(\mathbb{R}_+, k_0))$. The C^* algebras $\mathcal{A} \otimes \mathcal{B}_0(k_0)$ and $\mathcal{A} \otimes \mathcal{B}_0(\Gamma)$ will be denoted by $\mathcal{F}$ and $\mathcal{G}$ respectively. Note that $\mathcal{L}(F) = \mathcal{M}(\mathcal{F})$ and $\mathcal{L}(G) = \mathcal{M}(\mathcal{G})$. Recall the notation η_f for $\eta \in \mathcal{B}(h, h \otimes k_0)$ and $f \in k_0$. If $\eta \in \mathcal{A} \otimes_{\text{alg}} \mathcal{B}_0(k_0)$, say of the form $\eta = \sum_{i,j=1}^n x_{ij} \otimes |e_i\rangle\langle e_j|$, we note that $\eta_f \in \mathcal{A} \otimes_{\text{alg}} k_0$ given by

$$\eta_f = \sum_{ij=1}^{n} \langle e_j, f\rangle x_{ij} \otimes |e_i\rangle.$$

We observe that $\forall\ u \in h$, $\|\eta_f u\|^2 = \sum_i \|x_i^f u\|^2$, where $x_i^f = \sum_j \langle e_j, f\rangle x_{ij}$. On the other hand, taking $a = \frac{1 \otimes |f\rangle\langle f|}{\|f\|^2}$ if f is nonzero and $a = 0$ when $f = 0$, we see that $\|\eta a(u \otimes f)\|^2 = \|\frac{1}{\|f\|^2} \sum_{i,j} \langle e_j, f\rangle x_{ij} u \otimes |e_i\rangle\langle f, f\rangle\|^2 = \sum_i \|x_i^f u\|^2$. That is,

$\|\eta_f u\| = \|(\eta a)(u \otimes f)\| \leq \|\eta a\|\|u\|\|f\|$, and hence $\|\eta_f\| \leq \|\eta a\|\|f\|$. This shows that given an element $\eta \in \mathcal{L}(F) = \mathcal{M}(\mathcal{F})$ and $f \in k_0$, if we choose a net $\eta^{(\alpha)}$ from $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}_0(k_0)$ converging in the strict topology of the multiplier to η, then $\eta_f^{(\alpha)}$ will be norm-Cauchy. This shows that for all $\eta \in \mathcal{L}(F)$, η_f belongs to F. Furthermore, for a bounded linear map $\sigma : \mathcal{A} \to \mathcal{L}(F)$ and $f \in k_0$ we set the map $\sigma_f : \mathcal{A} \to F$ by $\sigma_f(x) = \sigma(x)_f$.

Let us now prove the existence theorem for a canonical E-H dilation.

Theorem 4.7.1 *Given a uniformly continuous, conservative quantum dynamical semigroup T_t on a separable unital C^* algebra $\mathcal{A}$ with generator $\mathcal{L}$, there exists a separable Hilbert space k_0 and a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(F) = \mathcal{M}(\mathcal{F})$, a π-derivation $\delta : \mathcal{A} \to F$ such that $\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y) = \delta(x)^*\delta(y)$.*

Furthermore, we can extend the maps $\mathcal{L}, \delta, \pi$ to the universal enveloping von Neumann algebra $\tilde{\mathcal{A}}$ of $\mathcal{A}$ such that the E-H type q.s.d.e. $dJ_t = J_t \circ (a_\delta(dt) + a_\delta^\dagger(dt) + \Lambda_{\pi - id}(dt) + \mathcal{I}_{\mathcal{L}}(dt))$ with the initial condition $J_0 \equiv id$ admits a unique solution as a map from $\tilde{\mathcal{A}} \otimes_s \Gamma$ to itself. The restricton of J_t on G takes value in G and there is a $*$-homomorphism $j_t : \mathcal{A} \to \mathcal{L}(G) = \mathcal{M}(\mathcal{G})$ satisfying $j_t(x)(ue(f)) = J_t(x \otimes e(f))u$ for all $x \in \mathcal{A}$, $u \in \tilde{h}$, $f \in L^2(\mathbb{R}_+, k_0)$, where $\tilde{h}$ denotes the universal enveloping GNS space of $\mathcal{A}$.

Proof : Let us imbed $\mathcal{A}$ in its universal enveloping GNS space $\tilde{h}$, where the weak closure of the image of $\mathcal{A}$ in $\tilde{h}$ is the universal enveloping von Neumann algebra $\tilde{\mathcal{A}}$ of $\mathcal{A}$. We identify $\mathcal{A}$ with its imbedding inside $\mathcal{B}(\tilde{h})$. By the remark 1.2.22, we obtain a Hilbert space $\mathcal{K}$, a $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, a ρ-derivation $\alpha : \mathcal{A} \to \mathcal{B}(\tilde{h}, \mathcal{K})$ such that $\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y) = \alpha(x)^*\alpha(y)$. Now consider the Hilbert $\mathcal{A}$-module E defined as the closure of the algebraic linear span of elements of the form $\alpha(x)y$, where $x, y \in \mathcal{A}$, with respect to the operator norm of $\mathcal{B}(\tilde{h}, \mathcal{K})$. $\mathcal{A}$ acts on E by the usual right multiplication, and the inner product of E is inherited from that of $\mathcal{B}(\tilde{h}, \mathcal{K})$, namely, $\langle L, M \rangle = L^*M$ for $L, M \in E$. We note that $\langle \alpha(x)y, \alpha(a)b \rangle = y^*\alpha(x)^*\alpha(a)b \in \mathcal{A}$, and thus E is indeed a Hilbert $\mathcal{A}$-module. We identify ρ with a left action $\hat{\rho}$ given by, $\hat{\rho}(x)(\alpha(y)) = \alpha(xy) - \alpha(x)y$ and extending it $\mathcal{A}$-linearly. Furthermore, since $\mathcal{A}$ is separable, E is countably generated as a Hilbert $\mathcal{A}$-module. To see this, one can choose any countable family $\{x_i\}$ of dense subset of $\mathcal{A}$ and note that E is the closed $\mathcal{A}$-linear span of $\{\alpha(x_i)\}$. By the theorem 1.3.4, we obtain a separable Hilbert space k_0, an isometric $\mathcal{A}$-linear map $t : E \to F$ which imbeds E as a complemented closed submodule of F. Clearly, by the theorem 1.3.5, $t\hat{\rho}(x)t^* \in \mathcal{L}(F)$. We set $\delta(x) = t(\alpha(x))$ and $\pi(x) = t\hat{\rho}(x)t^*$ to complete the first part of the proof.

For the second part, first note that π being a $*$-homomorphism it admits an extension as a $*$-homomorphism of $\tilde{\mathcal{A}}$ into $(\mathcal{L}(F))'' = \tilde{\mathcal{A}} \otimes_s \mathcal{B}(k_0)$. Also note that by the remark 1.2.22, we obtain R in the ultraweak closure of $\{\delta(x)y : x, y \in \mathcal{A}\}$ such that $\mathcal{L}(x) = R^*\pi(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H,x]$ and $\delta(x) = Rx - \pi(x)R$ for all $x \in \mathcal{A}$, and for some $H \in \tilde{\mathcal{A}}$. We extend $\mathcal{L}$ and δ by the same expressions as above with the extended π on $\tilde{\mathcal{A}}$. Since R is in particular in the ultraweak closure of F, which is same as $\tilde{\mathcal{A}} \otimes_s k_0$, we see that the extended maps $\mathcal{L}$ and δ map $\tilde{\mathcal{A}}$ into itself. Now, by the results of section 4.2, we obtain J_t and j_t as mentioned in the statement of the present theorem. It remains to show that $J_t(x \otimes e(f)) \in G$ for $x \in \mathcal{A}$, and $j_t(x) \in \mathcal{L}(G)$.

To this end, first note the following. If $\beta \in \mathcal{B}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}} \otimes_s k_0)$ such that $\beta(x) \in F$ for all $x \in \mathcal{A}$, then we claim that $a^\dagger_\beta(\Delta)(x \otimes e(f))$ belongs to G for any bounded subinterval Δ of $\mathbb{R}_+$ and $x \in \mathcal{A}$, $f \in L^2(\mathbb{R}_+, k_0)$. This follows from the definition of $a^\dagger_\beta(\Delta)(x \otimes f^{\otimes^n})$ which belongs to G. This enables us to prove that $a^\dagger_\delta(.)$ and $\Lambda_{\pi - id}(.)$ maps $x \otimes e(f)$ into G for $x \in \mathcal{A}$. That $a_\delta(.)$ and $\mathcal{I}_{\mathcal{L}}(.)$ also do so is quite clear. Now, recall the iterates $J_t^{(n)}$ constructed in the proof of theorem 3.2.6 of chapter 3 and by the estimates made in that proof, it is clear that $\|(J_t(x \otimes e(f)) - \sum_{m \leq n} J_t^{(m)}(x \otimes e(f)))u\| \leq \|u\|\|x\|\|e(f)\|\|E_t\| \sum_{m=n+1}^{\infty} C^{\frac{m}{2}}.(m!)^{-\frac{1}{4}}$, for come constant C, and thus, $\|J_t(x \otimes e(f)) - \sum_{m \leq n} J_t^{(m)}(x \otimes e(f))\| \to 0$. But by iterative construction, each $J_t^{(m)}(x \otimes e(f))$ belongs to G.

We now prove that $j_t(x) \in \mathcal{L}(G) = \mathcal{M}(\mathcal{G})$ for $x \in \mathcal{A}$. By definition of multiplier, we have to show that for all $A \in \mathcal{G}$, $j_t(x)A$ and $Aj_t(x)$ belong to $\mathcal{G}$. It is enough to verify that $j_t(x)A \in \mathcal{G}$ for all x, A, as $\mathcal{G}$ is $*$-closed and $j_t(x^*) = j_t(x)^*$. Since $\mathcal{G}$ is the operator-norm closure of elements which are finite linear combinations of the form $y \otimes |e(f)\rangle\langle e(g)|$ for $y \in \mathcal{A}$, $f, g \in L^2(\mathbb{R}_+, k_0)$, it suffices to show that for fixed $x \in \mathcal{A}$ and $t \geq 0$, $j_t(x)(y \otimes |e(f)\rangle\langle e(g)|)$ is in $\mathcal{G}$ for $y \in \mathcal{A}$, $f, g \in L^2(\mathbb{R}_+, k_0)$. Since $J_t(x \otimes e(f)) \in G = \mathcal{A} \otimes_{C^*} \Gamma$, we can choose a sequence L_n of the form $\sum_{i=1}^{k_n} z_i^{(n)} \otimes \rho_i^{(n)}$ for $z_i^{(n)} \in \mathcal{A}$, $\rho_i^{(n)} \in \Gamma$ such that L_n converges in the norm of G to $J_t(x \otimes e(f))$. Now, observe that for $u \in \tilde{h}$ and $\eta \in \Gamma$, $j_t(x)(y \otimes |e(f)\rangle\langle e(g)|)(u \otimes \eta) = \langle e(g), \eta\rangle J_t(x \otimes e(f))yu = \lim_{n \to \infty} \langle e(g), \eta\rangle L_n yu$. Choose an orthonormal basis $\{\gamma_l\}$

of Γ and take a vector $W \equiv \sum_l w_l \otimes \gamma_l$ of Γ. It is easy to see that

$$\|\{(j_t(x)(y \otimes |e(f)\rangle\langle e(g)|) - \sum_{i=1}^{k_n} z_i^{(n)} y \otimes |\rho_i^{(n)}\rangle\langle e(g)|)\}W\| \leq$$

$$\sum_l |\langle e(g), \gamma_l\rangle| \|(J_t(x \otimes e(f))y - L_n y)w_l\| \leq$$

$$\|J_t(x \otimes e(f)) - L_n\|\|y\|\|e(g)\|(\sum_l |w_l|^2)^{\frac{1}{2}} = \|J_t(x \otimes e(f)) - L_n\|\|y\|\|e(g)\|\|W\|;$$

and hence $j_t(x)(y \otimes |e(f)\rangle\langle e(g)|)$ is norm-limit of $\sum_{i=1}^{k_n} z_i^{(n)} y \otimes |\rho_i^{(n)}\rangle\langle e(g)| \in \mathcal{A}_{\text{alg}}\mathcal{B}_0(\Gamma)$. This completes the proof. □

Chapter 5

Dilation of completely positive flows

In quantum probability the basic notion of a stochastic process is a time-indexed family of *-homomorphisms between operator algebras $(j_t : \mathcal{A} \to \mathcal{B})$. When composed with a conditional expectation onto a subalgebra $\mathcal{C}$ the resulting family of maps $(k_t = \mathbb{E} \circ j_t)$ is no longer composed of *-homomorphisms, however each k_t completely positive and contractive. If $\mathcal{C}$ is of the form $\mathcal{C}_1 \otimes \mathcal{B}(\Gamma_0)$ for a Fock space Γ_0 carrying quantum noise, and if k is adapted to the filtration of the noise and is a cocycle with respect to the natural shift, then, under some regularity assumptions, k is necessarily governed by a quantum stochastic differential equation of the form $dk_t = k_t \circ \theta^{\alpha}_{\beta}\, d\Lambda^{\beta}_{\alpha}(t)$ ([37]).

Here we consider the converse problem, and realise every adapted, regular, completely positive, contractive flow $(k_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma_0))$ as a conditioned *-homomorphic flow: $k_t = \mathbb{E}_0 \circ j_t$. The special case when k is a deterministic CP contractive semigroup has been treated in the previous chapters.

To achieve the above stochastic dilation we combine the special form that the coefficient matrix θ has, by virtue of k being completely positive and contractive ([35],[36]), with the technique developed in [25] (utilised in chapter 4) for obtaining structure maps (that is QSDE coefficient matrices) from CP semigroups.

5.1 Introduction

In the previous chapter, we have proved that for a uniformly continuous, normal, contractive semigroup T on a von Neumann algebra $\mathcal{A}$ one can always obtain an Evans-Hudson dilation in the Fock space over a suitably chosen noise space. In this Fock space picture one may think of T itself as a flow satisfying the QSDE $dT_t = T_t \circ \theta_0^0 \, d\Lambda_0^0(t)$, since $d\Lambda_0^0(t) = dt$ and where θ_0^0 is the generator of T. That is, T is a flow with only deterministic (time) components and we construct its dilation j by adding suitable noise or random components which are averaged out by the vacuum expectation to give back T. This idea admits a natural generalisation if one replaces T by a more general flow $k = (k_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B})$, a family of completely positive (CP) and contractive maps, that satisfes a QSDE of Evans-Hudson type. An obvious way of constructing such a flow is to take a conditional expectation of the *-homomorphic flow j which averages out some but not all of the noise space. It is now natural to ask the converse question: given a CP contractive flow k with some noise space $_0$, can we enlarge the noise space by adding another Hilbert space, $_1$ say, and obtain a *-homomorphic flow j acting on $\otimes \Gamma(L^2(\mathbb{R}_+; {}_0 \oplus {}_1))$ so that $\mathbb{E}_1 \circ j_t = k_t$, where $\mathbb{E}_1$ denotes the conditional expectation that averages out the noise corresponding to $_1$? This is a generalisation of the dilation problem for a CP semigroup. In the present chapter we give an affirmative answer to the above question in the case when $\mathcal{A}$ is a von Neumann algebra and the CP contractive flow k is ultraweakly continuous, thus extending the results of [25]. The construction is based on the structure theorem for CP contractive flows ([36],[35]), combined with techniques introduced in [25] to obtain "structure maps". In Section 5.3 we present a suitable modification of the structure theorem for CP contractive flows and then construct the required dilation in Section 5.5 using the new form for θ.

The commutative (or classical) version of such CP flows appear often in stochastic filtering theory and in measure-valued or super-processes. We shall discuss potential applications of our theory in these two contexts at the end of the chapter. Similar questions have been posed by Belavkin [3], [4]. However the dilations obtained there are different for two reasons. Firstly the algebra $\mathcal{A}$ under consideration is the full algebra $\mathcal{B}()$, and secondly the CP flow is implemented through conjugation by the solution V, say, of a Hudson-Parthasarathy type QSDE. That is k has the form $k_t(a) = V_t^*(a \otimes 1)V_t$. In the present context, a similar implementation of the CP flow on a von Neumann algebra by inner perturbations of a *-homomorphic flow (in the spirit of [16]) will be treated elsewhere.

5.2 Some notations and terminologies

Fix here, for the rest of the chapter, a Hilbert space $\mathbf{h}$, the *initial space*, and $\mathcal{A} \subset \mathcal{B}(\mathbf{h})$, a unital C^*-algebra.

In [36], [37] the noise space $\mathbf{k}$ is always assumed to be separable since the quantum stochastic calculus with infinite degrees of freedom used there is the version developed by Mohari and Sinha ([42]). We believe that this restriction of separability can be removed by a suitable modification of the arguments in [36], [37] replacing the coordinatized calculus by our coordinate-free version. However, in the present chapter we do not want to take up the problem of extending the theory of [36], [37] and shall assume the separability of the noise space (but not of the initial space). An orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathbf{k}$ is chosen, and the noise differentials $\{d\Lambda_\beta^\alpha\}_{\alpha,\beta\geq 0}$ are defined in terms of these vectors. They fall into four distinct classes: $d\Lambda_0^0 = dt$, the time component; $d\Lambda_0^i = dA^i$, the ith annihilation component $(i \geq 1)$; $d\Lambda_j^0 = dA_j^\dagger$, the jth creation component $(j \geq 1)$; and $d\Lambda_j^i$, the (i,j)th conservation or gauge component $(i, j \geq 1)$. The generator of a quantum stochastic flow is then an infinite family $\theta = \{\theta_\beta^\alpha\} \subset \mathcal{B}(\mathcal{A})$. But as noted in [35], the Hilbert spaces $\mathbf{h} \otimes (\mathbb{C} \oplus \mathbf{k})$ and $\oplus_{\gamma\geq 0}\mathbf{h}$ can be identified by use of this basis (adding $e_0 = 1 \in \mathbb{C}$) to give a basis of $(\mathbb{C} \oplus \mathbf{k})$, and if θ is the generator of a CP contractive flow then there is a bounded map $\widetilde{\theta} : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k})$ which has components θ_β^α with respect to this basis.

This global boundedness property connects with the approach in the previous chapters, all of which is essentially contained in [25] where the calculus was reformulated in a coordinate free manner using Hilbert module techniques. The modules encountered by us in [25] are all of the form $\mathcal{A}'' \otimes \mathcal{B}(\mathbf{k}_1; \mathbf{k}_2)$, for any two Hilbert spaces $\mathbf{k}_1$ and $\mathbf{k}_2$. This can be characterised as the set $\{T \in \mathcal{B}(\mathbf{h} \otimes \mathbf{k}_1; \mathbf{h} \otimes \mathbf{k}_2) : (a' \otimes 1_2)T = T(a' \otimes 1_1) \forall a' \in \mathcal{A}'\}$, and coincides with the closure of $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}(\mathbf{k}_1; \mathbf{k}_2)$ in the weak, ultra-weak, strong and ultra-strong topologies. In the above (and throughout the chapter) we use the notation 1_i to denote the identity operator on the Hilbert space $\mathbf{k}_i$. Since $\mathcal{B}(\mathbb{C}; \mathbf{k})$ is naturally identified with $\mathbf{k}$, we write $\mathcal{A}'' \otimes \mathbf{k}$ for $\mathcal{A}'' \otimes \mathcal{B}(\mathbb{C}; \mathbf{k})$.

So with the global boundedness of θ guaranteed by Theorem 5.2 of [36] when k

is CP and contractive. we will write

$$\theta = \begin{pmatrix} \tau & \nu \\ \chi & \sigma - \iota \end{pmatrix} \tag{5.1}$$

where $\tau \in \mathcal{B}(\mathcal{A}), \nu : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathrm{k}; \mathbb{C}), \chi : \mathcal{A} \to \mathcal{A}'' \otimes \mathrm{k}$ and $\sigma, \iota : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathrm{k})$. Throughout the chapter ι will denote the map $\iota(a) = a \otimes 1_{\mathrm{k}}$ for the relevant k. and by $\widehat{\theta}$ we will denote the associated mapping matrix

$$\widehat{\theta} = \begin{pmatrix} \tau & \nu \\ \chi & \sigma \end{pmatrix}.$$

Let us introduce some more notations here. For a Hilbert space $\mathbf{k}$, we denote by $\mathcal{B}_{[a,b],\mathrm{k}}$ and $\mathcal{B}_{[a,b),\mathrm{k}}$ the spaces $\mathcal{B}(\Gamma(L^2([a,b],\mathrm{k})))$ and $\mathcal{B}(\Gamma(L^2([a,b),\mathrm{k})))$ respectively where $0 \leq a < b \leq \infty$. $\mathcal{B}_{[0,\infty),\mathrm{k}}$ is denoted simply by $\mathcal{B}_{\mathrm{k}}$. For any finite or infinite interval Δ, $1_{\Delta,\mathrm{k}}$ will denote the identity on $\Gamma_{\Delta,\mathrm{k}} \equiv \Gamma(L^2(\Delta,\mathrm{k}))$ and Ω_{k} will denote the Fock vacuum vector in $\Gamma(L^2(\mathbb{R}_+,\mathrm{k}))$. If $\mathcal{A}$ is a von Neumann algebra and is a Hilbert space with subspace k_1, the vacuum conditional expectation $\mathbb{E}_1 : \mathcal{A} \otimes \mathcal{B}_{\mathrm{k}} \to \mathcal{A} \otimes \mathcal{B}_{\mathrm{k}_1}$ is given by $\mathbb{E}_1[c] = E^* c E$ where E is the isometry $\mathrm{h} \otimes \Gamma_{\mathrm{k}_1} \ni \xi \to \xi \otimes \Omega_{\mathrm{k}_1^\perp} \in \mathrm{h} \otimes \Gamma_{\mathrm{k}}$. When $\mathrm{k}_1 = \{0\}$ it is denoted $\mathbb{E}$. Throughout the chapter, we shall use the Einsteine summation convention. The Greek indices α, β etc. will vary over $0, 1, 2, \ldots$, whereas the Roman indices i, j etc. will vary over $1, 2, \ldots$. Given a Hilbert space k, a *contraction process on $\mathcal{A}$ with noise dimension space* k is a weakly measurable family of contractions $k = (k_t)_{t \geq 0}$ satisfying $k_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}_{[0,t],\mathrm{k}} \otimes 1_{[t,\infty),\mathrm{k}} \subset \mathcal{A}'' \otimes \mathcal{B}_{\mathrm{k}}$. In other words k is adapted to the Fock filtration. When $\mathcal{A}$ is a von Neumann algebra k is called normal if the map $a \mapsto k_t(a)$ is ultraweakly continuous for each t.

A completely positive, contractive, normal process k on $\mathcal{A}$, with noise dimension space k, is a *stochastic cocycle* if

$$k_{s+t} = \widehat{k_s} \circ \sigma_s \circ k_t \ (s, t \geq 0)$$

where σ_s is the right shift $\mathcal{A} \otimes \mathcal{B}_{\mathrm{k}} \to \mathcal{A} \otimes \mathcal{B}_{[s,\infty),\mathrm{k}} \subset \mathcal{A} \otimes \mathcal{B}_{\mathrm{k}}$, and $\widehat{k_s}$ is the normal extension of the map $\mathcal{A} \otimes_{\mathrm{alg.}} \mathcal{B}_{[s,\infty),\mathrm{k}} \ni a \otimes b \mapsto k_s(a)(1_{[0,s],\mathrm{k}} \otimes b)$. For a stochastic cocycle k the family $(\mathbb{E} \circ k_t)$ is a one-parameter semigroup, called the *associated Markov semigroup*. A *regular cocycle* is one that is pointwise weakly continuous and whose Markov semigroup is uniformly continuous.

Theorem 5.2.1 *([37]) Let $\mathcal{A}$ be a von Neumann algebra, let $\mathbf{k}$ be a separable noise dimension space, and let k be a completely positive, contractive, normal process. Then the following are equivalent:*

(i) *k is a regular cocycle.*

(ii) *k weakly/strongly satisfies a QSDE of the form $dk = k \circ \theta^\alpha_\beta \, d\Lambda^\beta_\alpha, k_0(a) = a \otimes 1_{\Gamma_\mathbf{k}}$ for a bounded coefficient matrix $\theta = (\theta^\alpha_\beta) : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k})$.*

Let k and l be a pair of contraction processes on $\mathcal{A}$ with noise dimension spaces $\mathbf{k}_1$ and $\mathbf{k}$ respectively, where $\mathbf{k}_1$ is a subspace of $\mathbf{k}$. l is a *stochastic dilation* of k if

$$k_t = \mathbb{E}_1 \circ l_t \ (t \geq 0).$$

Thus in this terminology, if k is a stochastic cocycle with Markov semigroup P, then k is a stochastic dilation of P. When k and l are processes that satisfy QSDEs it is possible to determine if l is a stochastic dilation of k by inspecting their generators.

Lemma 5.2.2 *Let $\mathbf{k}_1, \mathbf{k}_2$ be Hilbert spaces, and let k and l be contractive processes on $\mathcal{A}$ with noise spaces $\mathbf{k}_1$ and $\mathbf{k}_1 \oplus \mathbf{k}_2$ respectively. Suppose that $\theta : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k}_1)$ and $\phi : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k}_1 \oplus \mathbf{k}_2)$ are bounded mapping matrices such that k and l weakly satisfy the QSDEs*

$$dk_t = k_t \circ \theta^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \ dl_t = l_t \circ \phi^\alpha_\beta \, d\Lambda^\beta_\alpha(t).$$

Then l is a stochastic dilation of k if and only if ϕ has the form

$$\phi = \begin{pmatrix} \theta & \eta \\ \lambda & \psi \end{pmatrix}, \tag{5.2}$$

where $\eta : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathbf{k}_2; \mathbb{C} \oplus \mathbf{k}_1), \lambda : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k}_1; \mathbf{k}_2)$ and $\psi : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathbf{k}_2)$ are bounded linear maps.

Proof : If l is a stochastic dilation of k then $\langle ue(f^1 \oplus 0), l_t(a)ve(g^1 \oplus 0)\rangle =$ $\langle ue(f^1), k_t(a)ve(g^1)\rangle$. It is then straightforward to show that ϕ has the form (5.2) by applying the first fundamental formula of quantum stochastic calculus to each side, differentiating the resulting expressions at $t = 0$, and varying f and g. In the other direction, if ϕ is of the form (5.2) then the process $\mathbb{E}_1 \circ l$ is also a weak solution to the QSDE satisfied by k. Thus by uniqueness of solutions ([36], Theorem 3.1) we have $k = \mathbb{E}_1 \circ l$. □

5.3 Structural form for a cocycle generator

In this section we refine the characterisation of CP cocycle generators found in [36] and [35] in such a way that when $\mathcal{A}$ is a von Neumann algebra, the constituents may be used to determine a *-homomorphic dilating cocycle. The refinement uses the technique introduced in [25] for dilating CP semigroups.

In the following we do not assume that the flow k is contractive, therefore θ need not be bounded, and so we must work with individual components θ^α_β of θ. Also, let $\widehat{\mathsf{h}}_{00}$ be the subspace of $\oplus_{\gamma\geq 0}\mathsf{h} = \mathbf{h}\otimes(\mathbb{C}\oplus \mathsf{k}_0)$ consisting of vectors with only finitely many non-zero components with respect to the chosen basis.

Theorem 5.3.1 *Let θ be a mapping matrix on $\mathcal{A}$ of the form* (5.1) *that weakly generates a flow k with separable noise space k_0.*
(a) The following are equivalent:
(i) k is completely positive.
(ii) θ is real and there is a quintuple $\mathcal{R} = (\mathbf{k}_1, \pi, h, d, \{s_j\})$ consisting of a Hilbert space k_1, a *-homomorphism $\pi : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathsf{k}_1)$ and operators $h = h^* \in \mathcal{A}''$, $d \in \mathcal{A}''\otimes \mathsf{k}_1$ and $s_j \in \mathcal{A}''\otimes\mathsf{k}_1$ $(j = 1, 2, \ldots, \dim(\mathsf{k}_0))$ such that

$$\begin{pmatrix} \tau(a) & \nu_j(a) \\ \chi^i(a) & \sigma^i_j(a) \end{pmatrix} = \begin{pmatrix} \mathcal{L}(a) + \frac{1}{2}\{t, a\} & \delta^\dagger(a)s_j + a(c^j)^* \\ s_i^*\delta(a) + c^i a & s_i^*\pi(a)s_j \end{pmatrix} \quad (5.1i)$$

$$\pi(1)d = d \text{ and } \pi(1)\mathrm{s_j} = \mathrm{s_j}\ \forall \mathrm{j} \quad (5.1ii)$$

$$\mathrm{Ran}(\pi(1)) = \overline{\mathcal{K}_0} \quad (5.1iii)$$

where $t = \tau(1), c^i = \chi^i(1), \delta = \delta_{d,\pi}, \mathcal{L} = \mathcal{L}_{d,\pi,h}$ and $\mathcal{K}_0 = \{\delta(a)u^0 + \pi(a_i)s_j u^j : a \in \mathcal{A}, (u^\alpha) \in \widehat{\mathsf{h}}_{00}\}$.

(b) If $\mathcal{R}_1$ and $\mathcal{R}_2$ are quintuples satisfying (5.1) *then there is a unique partial isometry $V : \mathbf{h}\otimes\mathbf{k}_1 \to \mathbf{h}\otimes\mathbf{k}_2$ satisfying*

$$V^*V = \pi_1(1), VV^* = \pi_2(1) \quad (5.2i)$$

$$\pi_2(a) = V\pi_1(a)V^*, \delta_2 = V\delta_1;\ s^2_j = Vs^1_j. \quad (5.2ii)$$

Proof : The implication (aii $\Rightarrow$ ai) is contained in [36], Theorem 4.1.

(ai $\Rightarrow$ aii): Suppose that k is completely positive. Then, by Theorem 4.1 of [36], θ is real and there is a quadruple $\mathcal{Q} = (\rho, \mathcal{H}, \gamma, \{D_i\})$ consisting of a representation

$(\rho, \mathcal{H})$ of $\mathcal{A}$, a ρ-derivation $\gamma : \mathcal{A} \to \mathcal{B}(\mathbf{h}; \mathcal{H})$ and a family of operators $\{D_i : i = 1, \dots, \dim(\mathbf{k}_1)\}$ in $\mathcal{B}(\mathbf{h}; \mathcal{H})$ such that

$$\begin{pmatrix} \partial\tau(a,b) & \nu_j(a) \\ \chi^i(a) & \sigma^i_j(a) \end{pmatrix} = \begin{pmatrix} \gamma(a)^*\gamma(b) & \gamma^\dagger(a)D_j + a\nu_j(1) \\ D_j^*\gamma(a) + \chi^i(1)a & D_i^*\rho(a)D_j \end{pmatrix} \quad 5.3i$$

$$\rho \text{ is unital} \quad 5.3ii$$

$$\mathcal{H} = \overline{\mathcal{H}_0} \quad 5.3iii$$

where $\mathcal{H}_0 = \mathrm{lin}\{\gamma(a)u^0 + \rho(a)D_iu^i : a \in \mathcal{A}, (u^\alpha) \in \widehat{\mathbf{h}}_{00}\}$. For each unitary $w \in \mathcal{A}'$ define bounded linear operators $\gamma^w : \mathcal{A} \to \mathcal{B}(\mathbf{h}; \mathcal{H})$ and $D_j^w \in \mathcal{B}(\mathbf{h}; \mathcal{H})$ by $\gamma^w(a) = \gamma(a)w$ and $D_j^w = D_jw$, and note the following relations:

$$\gamma^w(a)^*\gamma^w(b) = w^*\partial\tau(a,b)w = \partial\tau(a,b) = \gamma(a)^*\gamma(b);$$

$$(D_j^w)^*\gamma^w(a) = w^*D_i^*\gamma(a)w = w^*(\chi^i(a) - \chi^i(1)a)w = \chi^i(a) - \chi^i(1)a = D_i^*\gamma(a);$$

$$(D_i^w)^*\rho(a)D_j^w = w^*D_i^*\rho(a)D_jw = w^*\sigma^i_j(a)w = \sigma^i_j(a);$$

$$\mathcal{H}_0^w = \mathcal{H}_0.$$

In other words the quadruple $\mathcal{Q}^w = (\rho, \mathcal{H}, \gamma^w, \{D_i^w\})$ also satisfies (5.3). Hence, by the uniqueness part of Theorem 4.1 in [36], there is a unique unitary operator $\rho'(w)$ on $\mathcal{H}$ such that

$$\rho'(w)D_i = D_iw; \ \rho'(w)\gamma(a) = \gamma(a)w; \quad \rho'(w)\rho(a) = \rho(a)\rho'(w). \tag{5.4}$$

The resulting map ρ' is easily seen to be a unitary representation of the group of unitaries in $\mathcal{A}'$ by checking matrix elements against vectors from the dense subspace $\mathcal{H}_0$, and it follows that ρ' extends linearly to a normal, unital representation of $\mathcal{A}'$. Hence by 1.1.6, there is a Hilbert space $\mathbf{k}_1$ and an isometry $V : \mathcal{H} \to \mathbf{h}\otimes\mathbf{k}_1$ such that $\rho'(x') = V^*(x'\otimes 1_1)V$ and $p := VV^* \in (\mathcal{A}'\otimes 1_1)'$. Now define $\pi' : \mathcal{A}' \to \mathcal{B}(\mathbf{h}\otimes\mathbf{k}_1), \pi : \mathcal{A} \to \mathcal{B}(\mathbf{h}\otimes\mathbf{k}_1), \delta : \mathcal{A} \to \mathcal{B}(\mathbf{h}; \mathbf{h}\otimes\mathbf{k}_1)$, and $s_j \in \mathcal{B}(\mathbf{h}; \mathbf{h}\otimes\mathbf{k}_1), j \geq 0$ by

$$\pi'(x') = (x'\otimes 1_1)p, \ \pi(a) = V\rho(a)V^*, \ \delta(a) = V\gamma(a), \ s_j = VD_j. \tag{5.5}$$

Since $p \in \mathcal{A}'' \otimes \mathcal{B}(\mathbf{k}_1)$, algebraic manipulations applied to (5.4) reveal the following identities:

$$\pi(1) = p; \ p\delta(a) = \delta(a); \ ps_j = s_j,$$

$$\pi(a)(x'\otimes 1_1) = (x'\otimes 1_1)\pi(a) = \pi'(x')\pi(a),$$

$$\delta(a)x' = (x'\otimes 1_1)\delta(a),$$

$$s_jx' = (x'\otimes 1_1)s_j.$$

Thus $\pi(\mathcal{A}) \subset \mathcal{A}'' \otimes \mathcal{B}(\mathbf{k}_1), \delta(\mathcal{A}) \subset \mathcal{A}'' \otimes \mathbf{k}_1$ and $s_j \in \mathcal{A}'' \otimes \mathbf{k}_1$. Moreover

$$\operatorname{Ran}(\pi(1)) = \operatorname{Ran}(V) = \overline{V\mathcal{H}_0} = \overline{\mathcal{K}_0}.$$

By Theorem 2.1 of [8], there is an operator $d \in \overline{\operatorname{lin}}^{uw}\{\delta(a)b : a, b \in \mathcal{A}\} \subset \mathcal{A}'' \otimes \mathbf{k}_1$ such that $\delta = \delta_{d,\pi}$. Since $\delta(1) = 0, \pi(1)d = d$, and since $\partial\tau(a,b) = \delta_{d,\pi}(a)^*\delta_{d,\pi}(b)$ it follows from Lemma 1.2.24 that $\pi(\cdot) = \mathcal{L}_{d,\pi,h}(\cdot) + \frac{1}{2}\{\tau(1), \cdot\}$ for some $h = h^* \in \mathcal{A}''$. This completes the proof of part (a).

(b) Writing $\mathcal{K}^i$ for $\overline{\mathcal{K}_0^i} \subset \mathbf{h} \otimes \mathbf{k}_i, i = 1, 2$. Theorem 4.1 of [36] ensures the existence of a unique unitary operator $V_0 : \mathcal{K}^1 \to \mathcal{K}^2$ satisfying

$$V_0 s_j^1 = s_j^2;\; V_0\delta_1 = \delta_1;\; V_0\pi_1(a) = \pi_2(a)V_0.$$

Let V be the unique extension of V_0 to $\mathbf{h} \otimes \mathbf{k}_1$ that satisfies (5.2i). Then V satisfies (5.2ii), and is clearly the unique partial isometry satisfying (5.2). □

Remark 5.3.2 *If the initial space* $\mathbf{h}$ *is separable, then, by Proposition 4.9 of [36], the representation space* $\mathcal{H}$ *in the nonstructural quadruple* $\mathcal{Q}$ *may be assumed to be separable. Thus the von Neumann algebra* $\rho(\mathcal{A})'$ *is* σ*-finite, and so we may assume that the Hilbert space* $\mathbf{k}_1$ *is separable (see [14], p.62).*

Theorem 5.3.3 *Let* θ *be a mapping matrix of the form* (5.1) *that weakly generates a flow* k*. The following are equivalent:*

(i) k *is completely positive and contractive.*

(ii) θ *is real and bounded, and there is a quintuple* $\mathcal{S} = (\mathbf{k}_1, \pi, h, d, s)$ *consisting of a Hilbert space* $\mathbf{k}_1$*, a *-homomorphism* $\pi : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\mathbf{k}_1)$ *and operators* $h = h^* \in \mathcal{A}'', d \in \mathcal{A} \otimes \mathbf{k}_1$ *and a contraction* $s \in \mathcal{A}'' \otimes \mathcal{B}(\mathbf{k}_0; \mathbf{k}_1)$ *such that*

$$\widehat{\theta}(a) = \begin{pmatrix} \mathcal{L}(a) + \frac{1}{2}\{t, a\} & \delta^\dagger(a)s + ac^* \\ s^*\delta(a) + ca & s^*\pi(a)s \end{pmatrix} \quad 5.6i$$

$$\pi(1)d = d \text{ and } \pi(1)s = s \quad 5.6ii$$

$$\operatorname{Ran}(\pi(1)) = \overline{\mathcal{K}_0} \quad 5.6iii$$

$$\begin{pmatrix} -t & -c^* \\ -c & 1 - s^*s \end{pmatrix} \geq 0 \quad 5.6iv$$

where $t = \tau(1), c = \chi(1), \delta = \delta_{d,\pi}, \mathcal{L} = \mathcal{L}_{d,\pi,h}$ *and* $\mathcal{K}_0 = \{\delta(a)u^0 + \pi(a)su_1 : a \in \mathcal{A}, (u^0, u_1) \in \widehat{\mathbf{h}}_{00}\}$.

Proof : By [36], Proposition 5.1 and Theorem 5.2, (ii) implies (i). Conversely if (i) holds then θ is bounded and satisfies $\theta(1) \leq 0$, so that σ is completely positive and contractive. Letting $(\mathrm{k}_1, \pi, h, d, \{s_i\})$ be the quintuple of Theorem 5.3.1, define an operator $s : \mathrm{h} \otimes \mathrm{k}_0 \to \mathrm{h} \otimes \mathrm{k}_1$ with dense domain h_{00} by $s\mathrm{v} = \mathrm{s}_i \mathrm{v}^i$. Since $\pi(1)s_j = s_j$ for each j, and

$$\|s\mathrm{v}\|^2 = \langle \mathrm{s}_i \mathrm{v}^i, \mathrm{s}_j \mathrm{v}^j \rangle = \langle \mathrm{v}^i, \mathrm{s}_i^* \pi(1) \mathrm{s}_j \mathrm{v}^j \rangle = \langle \mathrm{v}^i, \sigma_j^i(1) \mathrm{v}^j \rangle = \langle \mathrm{v}, \sigma(1) \mathrm{v} \rangle \leq \|\mathrm{v}\|^2,$$

it follows that s is a contraction. The remaining properties now follow easily from Theorem 5.3.1. □

5.4 Implications of contractivity

In this section we extract some consequences of the operator inequality (5.6iv) which are needed for defining both methods of dilation and also their comparison in the next section. We summarise these consequences in the form of three lemmas.

Lemma 5.4.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $X_{ij} \in \mathcal{B}(\mathcal{H}_j; \mathcal{H}_i)$ for $i, j = 1, 2$, and suppose that the element $X \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with the X_{ij} as components is positive, that is*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \geq 0.$$

Then we have the following:

(i) *X_{11} and X_{22} are positive and $X_{12} = X_{21}^*$.*

(ii) *There is a unique operator $K \in \mathcal{B}(\mathcal{H}_2; \mathcal{H}_1)$ such that $KX_{22}^{1/2} = X_{12}$ and $\mathrm{Ker}(K) \supset \mathrm{Ker}(X_{22}^{1/2})$. Moreover this K satisfies $KK^* \leq X_{11}$.*

Proof : (i) is straightforward.

(ii) First note that since $X_{22}^{1/2}$ is self-adjoint we have the decomposition $\mathcal{H}_2 = \overline{\mathrm{Ran}(X_{22}^{1/2})} \oplus \mathrm{Ker}(X_{22}^{1/2})$. A standard argument, as in the proof of the Cauchy-Schwarz inequality, gives

$$|\langle \xi, X_{12}\eta \rangle|^2 \leq \langle \xi, X_{11}\xi \rangle \|X_{22}^{1/2}\eta\|^2 \; \forall \xi \in \mathcal{H}_1, \eta \in \mathcal{H}_2.$$

In particular $\mathrm{Ker}(X_{22}^{1/2}) \subset \mathrm{Ker}(X_{12})$, so there is a well-defined linear map $K_0 : \mathrm{Ran}(X_{22}^{1/2}) \to \mathcal{H}_1$ such that $K_0(X_{22}^{1/2}\eta) = X_{12}\eta$, and also note that K_0 is bounded.

Let $\overline{K_0}$ denote the unique continuous extension to $\overline{\mathrm{Ran}(X_{22}^{1/2})}$, then if we extend $\overline{K_0}$ to all of $\mathcal{H}_2$ by setting to be zero on $\mathrm{Ker}(X_{22}^{1/2})$ we obtain the required K, which is obviously unique.

For each positive integer n let ϕ_n be the function on $[0,\infty)$ defined by

$$\phi_n(\lambda) = \{ \begin{array}{ll} \lambda^{-1/2}, & \lambda \geq 1/n, \\ 0, & \text{otherwise}. \end{array}$$

Let $B_n = \phi_n(X_{22})$, and observe that $(X_{22}^{1/2}B_n) = (B_n X_{22}^{1/2})$ converges strongly to the identity. In particular $K^* = \mathrm{s}-\lim_n B_n X_{21}$. So now fix $\xi \in \mathcal{H}_1$ and set $\eta_n = -B_n^2 X_{21}\xi$. Then

$$\begin{aligned} 0 \leq & \left\langle \begin{pmatrix} \xi \\ \eta_n \end{pmatrix}, X \begin{pmatrix} \xi \\ \eta_n \end{pmatrix} \right\rangle \\ & = \langle \xi, X_{11}\xi\rangle - 2\mathrm{Re}\langle \xi, X_{12}B_n^2X_{21}\xi\rangle + \|X_{22}^{1/2}B_n^2X_{21}\xi\|^2 \\ & \longrightarrow \langle \xi, X_{11}\xi\rangle - \|K^*\xi\|^2, \end{aligned}$$

from which the inequality follows. □

Lemma 5.4.2 *Let $\mathcal{H}_1, \mathcal{H}_2, X_{ij}$ and K be as in* Lemma 5.4.1. *Let $\mathcal{H}_3$ be another Hilbert space and suppose that $X_{22} = 1 - D^*D$ for some contraction $D \in \mathcal{B}(\mathcal{H}_2;\mathcal{H}_3)$. Then $L \in \mathcal{B}(\mathcal{H}_1;\mathcal{H}_3)$ defined by $L = DK^*$ satisfies*

$$(1-DD^*)^{1/2}L = DX_{21};\ \mathrm{Ker}(L^*) \supset \mathrm{Ker}((1-DD^*)^{1/2});\ L^*L \leq X_{11} - X_{21}^*X_{21}.$$

Furthermore, if the Hilbert spaces $\mathcal{H}_i$ are of the form $\mathcal{H}_i = \mathbf{h} \otimes \mathcal{K}_i, i = 1,2,3,$ for some Hilbert spaces $\mathcal{H}, \mathcal{K}_i$, and if $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra such that $X \in \mathcal{C} \otimes \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ and $D \in \mathcal{C} \otimes \mathcal{B}(\mathcal{K}_2;\mathcal{K}_3)$, then

$$K \in \mathcal{C} \otimes \mathcal{B}(\mathcal{K}_2;\mathcal{K}_1) \text{ and } \mathrm{L} \in \mathcal{C} \otimes \mathcal{B}(\mathcal{K}_1;\mathcal{K}_3).$$

Proof : Since $(1-DD^*)^{1/2}D = D(1-D^*D)^{1/2}$ we have

$$(1-DD^*)^{1/2}L = D(1-D^*D)^{1/2}K^* = DX_{22}^{1/2}K^* = DX_{21}.$$

Also, if $\eta \in \mathrm{Ker}((1-DD^*)^{1/2})$ then $D^*\eta \in \mathrm{Ker}((1-D^*D)^{1/2}) = \mathrm{Ker}(X_{22}^{1/2})$, so $L^*\eta = 0$, and the inequality for KK^* in Lemma 5.4.1 gives

$$L^*L = KK^* - KX_{22}K^* = KK^* - X_{21}^*X_{21} \leq X_{11} - X_{21}^*X_{21}.$$

For the second part note that $X \in \mathcal{C} \otimes \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ implies $B_n \in \mathcal{C} \otimes \mathcal{B}(\mathrm{k}_2)$ and so

$$K^*(c' \otimes 1_{\mathcal{K}_1}) = \mathrm{s} - \lim_n \mathrm{B_n X_{21}}(\mathrm{c}' \otimes 1_{\mathcal{K}_1}) = \mathrm{s} - \lim(\mathrm{c}' \otimes 1_{\mathcal{K}_2})\mathrm{B_n X_{21}} = (\mathrm{c}' \otimes 1_{\mathcal{K}_2})\mathrm{K}^*$$

for all $c' \in \mathcal{C}'$. The result for L now follows from its definition and the hypothesis on D. □

The final lemma summarises some easily proved facts about linear equations in operators.

Lemma 5.4.3 *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and suppose $A \in \mathcal{B}(\mathcal{H}_2), B, X_0 \in \mathcal{B}(\mathcal{H}_1; \mathcal{H}_2)$ are such that $AX_0 = B$. Then any solution of the equation $AX = B$ can be written as $X = X_0 + X_1$ where $AX_1 = 0$. Moreover if $X_0^*|_{\mathrm{Ker(A)}} = 0$ then $X_0^* X_1 = 0$.*

5.5 Dilation

In this section we give the main result of the chapter, namely that any regular stochastic cocycle on a von Neumann algebra has a *-homomorphic stochastic dilation. In fact we give two methods, and discuss the relationship between them after the main result.

Theorem 5.5.1 *Every regular stochastic cocycle on a von Neumann algebra with separable noise dimension space k_0 admits a normal *-homomorphic stochastic dilation.*

Proof : Let k be a regular stochastic cocycle on $\mathcal{A}$ with generator θ and let $(\mathbf{k}_1, \pi, h, d, s)$ be the data for a standard structural form for θ from Theorem 5.3.3. Thus,

$$\widehat{\theta}(a) = \begin{pmatrix} \tau(a) & \delta^\dagger(a)s + ac^* \\ s^*\delta(a) + ca & s^*\pi(a)s \end{pmatrix} \text{ and } \begin{pmatrix} -t & -c^* \\ -c & (1 - s^*s) \end{pmatrix} \geq 0, \tag{5.7}$$

where $t = \tau(1) \leq 0$ as before.

We give two classes of dilation for k.

Direct approach: First note that $-t \geq c^*c$ by Lemma 5.4.2, and let $g \in \mathcal{A} \otimes \mathrm{k}_1$ satisfying the constraints

$$sc + rg = 0 \text{ and } \mathrm{g^*g} \leq -\mathrm{t} - \mathrm{c^*c},$$

where $r = (1 - ss^*)^{1/2} \in \mathcal{A} \otimes \mathcal{B}(\mathbf{k}_1)$. A particular example of such an operator is given by taking g_0 to be the operator L from Lemma 5.4.2.

Define $\gamma : \mathcal{A} \to \mathcal{A} \otimes (\mathbf{k}_0 \oplus \mathbf{k}_1 \oplus \mathbb{C})$ and $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbf{k}_0 \oplus \mathbf{k}_1 \oplus \mathbb{C})$ by

$$\gamma(a) = \begin{pmatrix} s^*\delta(a) + ca \\ r\delta(a) + ga \\ ea \end{pmatrix} \text{ and } \rho(a) = \begin{pmatrix} s^*\pi(a)s & s^*\pi(a)r & 0 \\ r\pi(a)s & r\pi(a)r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where $e \in \mathcal{A}$ satisfies $e^*e = -t - c^*c - g^*g$. It is easily verified that ρ is a *-homomorphism and γ is a ρ-derivation satisfying

$$\gamma(a)^*\gamma(b) = \tau(a^*b) - a^*\tau(b) - \tau(a^*)b = \partial\tau(a,b) - a^*\tau(1)b. \tag{5.8}$$

So if we define $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k}_0 \oplus \mathbf{k}_1 \oplus \mathbb{C})$ by

$$\phi = \begin{pmatrix} \tau & \gamma^\dagger \\ \gamma & \rho - \iota \end{pmatrix} \tag{5.9}$$

then ϕ is clearly ultraweakly continuous, and assumes the correct algebraic form for it to be the generator of a *-homomorphic flow ([36], Theorem 6.5). Since the top left corner component is θ by construction, the solution j to the QSDE is a dilation of k by Lemma 5.2.2. If $\mathbf{k}_1$ is separable, which is the case if $\mathbf{h}$ is separable, then j may be constructed using the methods detailed in [36], otherwise the module based calculus of chapter 3 must be employed.

Approach by reduction: This time let $f \in \mathcal{A} \otimes \mathcal{B}(\mathbf{k}_0)$ and $l \in \mathcal{A} \otimes \mathbf{k}_0$ be a solution to the simultaneous (in)equalities $f^*f \leq 1 - s^*s$, $f^*l = c$ and $l^*l \leq -t$. An example of such a pair is given by taking $f_0 = (1 - s^*s)^{1/2}$ and letting l_0 be the operator $-K^*$ from Lemma 5.4.1. Then we have

$$\widetilde{\theta}(a) = \begin{pmatrix} \tau(a) & \widetilde{\delta}^\dagger(a)\widetilde{s} \\ \widetilde{s}^*\widetilde{\delta}(a) & \widetilde{s}^*\widetilde{\pi}(a)\widetilde{s} \end{pmatrix} \tag{5.10}$$

where $\widetilde{\pi} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbf{k}_1 \oplus \mathbf{k}_0)$, $\widetilde{s} \in \mathcal{A} \otimes \mathcal{B}(\mathbf{k}_0; \mathbf{k}_1 \oplus \mathbf{k}_0)$ and $\widetilde{\delta} : \mathcal{A} \to \mathcal{A} \otimes (\mathbf{k}_1 \oplus \mathbf{k}_0)$ are, respectively, a *-homomorphism, a contraction and a $\widetilde{\pi}$-derivation given by

$$\widetilde{\pi}(a) = \begin{pmatrix} \pi(a) & 0 \\ 0 & 0 \end{pmatrix}, \ \widetilde{s} = \begin{pmatrix} s \\ f \end{pmatrix}, \ \widetilde{\delta}(a) = \begin{pmatrix} \delta(a) \\ la \end{pmatrix}.$$

Now define $\gamma : \mathcal{A} \to \mathcal{A} \otimes (\mathbf{k}_0 \oplus (\mathbf{k}_1 \oplus \mathbf{k}_0) \oplus \mathbb{C})$ and $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbf{k}_0 \oplus (\mathbf{k}_1 \oplus \mathbf{k}_0) \oplus \mathbb{C})$ by

$$\gamma(a) = \begin{pmatrix} \widetilde{s}^*\widetilde{\delta}(a) \\ \widetilde{r}\,\widetilde{\delta}(a) \\ na \end{pmatrix}, \ \rho(a) = \begin{pmatrix} \widetilde{s}^*\widetilde{\pi}(a)\widetilde{s} & \widetilde{s}^*\widetilde{\pi}(a)\widetilde{r} & 0 \\ \widetilde{r}\,\widetilde{\pi}(a)\widetilde{s} & \widetilde{r}\,\widetilde{\pi}(a)\widetilde{r} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\widetilde{r} = (1 - \widetilde{s}\,\widetilde{s}^*)^{1/2} \in \mathcal{A} \otimes \mathcal{B}(\mathbf{k}_1 \oplus \mathbf{k}_0)$ and $n \in \mathcal{A}$ satisfies $n^*n = (-t - l^*l)^{1/2} \in \mathcal{A}$. As before ρ is a *-homomorphism, γ a ρ-derivation satisfying (5.8), and so constructing $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k}_0 \oplus (\mathbf{k}_1 \oplus \mathbf{k}_2) \oplus \mathbb{C})$ as in (5.9) gives the generator of a *-homomorphic flow that will be a dilation of k by Lemma 5.2.2. □

In the approach by reduction the generator θ of the cocycle k was rewritten so that the term $c = \chi(1)$ was absorbed into the π-derivation to give a $\widetilde{\pi}$-derivation $\widetilde{\delta}$, which will not satisfy $\widetilde{\delta}(1) = 0$ in general. However if $c = 0$ then $\widehat{\theta}$ is already in the form (5.10) and so we can skip this part of the construction (and must choose n so that $n^*n = -t$ for the dilation). This has the effect of reducing the noise space of the dilating process from $\mathbf{k}_0 \oplus (\mathbf{k}_1 \oplus \mathbf{k}_0) \oplus \mathbb{C}$ to $\mathbf{k}_0 \oplus \mathbf{k}_1 \oplus \mathbb{C}$. Similarly if e in the direct approach, or n in the approach by reduction is zero, then the final copy of $\mathbb{C}$ may be omitted.

Note that if k is unital, then $\theta(1) = 0$ ([36], Proposition 5.1), so c and t are zero, and s is an isometry. The the only possible solution to the constraints for g, f and l is to set them all to be zero, and also it follows that $e = n = 0$. Analysing the two methods then shows that the two dilations actually coincide in this case.

When k is non-unital however the relationship between the methods of dilation is less clear. Let $\phi^{g,e}$ be the generator of the dilation obtained by the direct route for a given pair (g, e). Then g and e appear as components of the matrix $\phi^{g,e}(1)$ and so the dilation for different values of the pair (g, e) will be distinct ([36], Theorem 3.1). For the approach by reduction, although n is readily deduced from $\phi^{f,l,n}(1)$ since it appears as a component, there is no obvious way to obtain f and l. However if we fix $f = (1 - s^*s)^{1/2}$ then the operators l that satisfy the relevant (in)equalities are in bijective correspondence with the g, as shown below.

Proposition 5.5.2 *Let $\theta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbb{C} \oplus \mathbf{k}_0)$ have the form (5.7), and set $f = (1 - s^*s)^{1/2}$. Then the map*

$$B : \mathcal{A} \otimes \mathbf{k}_0 \ni l \mapsto -sl \in \mathcal{A} \otimes \mathbf{k}_1$$

gives a bijection of $\mathcal{F} = \{l : fl = c, l^*l \leq -t\}$ *onto* $\mathcal{G} = \{g : sc + rg = 0, g^*g \leq -t - c^*c\}$.

Proof : Let $l \in \mathcal{F}$ and put $g = -sl$. Then $rg = -rsl = -sfl = -sc$. Moreover. $g^*g = l^*s^*sl = l^*l - l^*f^2l = l^*l - c^*c \leq -t - c^*c$, and so $g \in \mathcal{G}$. If $-sl = -sl'$ then

$$0 = s^*s(l - l') = (1 - f^2)(l - l'),$$

and since $f^2l = f^2l'$. B is injective.

It remains to show that $B(\mathcal{F}) = \mathcal{G}$. Let g_0 and l_0 be the particular solutions to their respective equations discussed in the proof of Theorem 5.5.1, and let $g \in \mathcal{G}$. Write $g = g_0 + g_1 = g_1 - sl_0$ as in Lemma 5.4.3. Let $l = l_0 - s^*g_1$, then $-sl = g_0 + ss^*g_1 = g_0 + (1 - r^2)g_1 = g$, as $rg_1 = 0$.

Since $fs^*g_1 = s^*rg_1 = 0$, we have $fl = fl_0 - fs^*g_1 = c$ and $\mathrm{Ran}(s^*g_1) \subset \mathrm{Ker}(f) \subset \mathrm{Ker}(l_0^*)$. So finally we have

$$\begin{aligned} & l^*l \\ = \;& l_0^*l_0 + g_1^*ss^*g_1 = l_0^*l_0 + g_1^*g_1 - g_1^*r^2g_1 \\ = \;& l_0^*l_0 + g_1^*g_1 = l_0^*l_0 + g^*g - g_0^*g_0 \\ = \;& l_0^*l_0 - l_0^*s^*sl_0 + g^*g = l_0^*f^2l_0 + g^*g \\ = \;& c^*c + g^*g \leq -t. \end{aligned}$$

This shows that $l \in \mathcal{F}$ and completes the proof. □

Suppose now that k is a CP contractive cocycle whose generator θ has the form

$$\theta = \begin{pmatrix} \tau & \chi^\dagger \\ \chi & 0 \end{pmatrix},$$

that is there are no Poisson terms in the QSDE satisfied by k. Let $\mathcal{S} = (\mathbf{k}_1, \pi, h, d, s)$ be a quintuple satisfying the conditions of (5.6). So in particular $s^*\pi(a)s = a \otimes 1$. and since $\pi(1)s = s$, we have that s is an isometry. Note also that the inequality (5.6iv) implies that $c = \chi(1) = 0$. We now consider two cases:

$ss^* = 1$: In this case $s^*\delta_{d,\pi} = \delta_{s^*d,\iota}$ and $\mathcal{L}_{d,\pi,h} = \mathcal{L}_{s^*d,\iota,h}$. So note that by the identity (1.1)

$$\delta_{s^*d,\iota}(a)^*\delta_{s^*d,\iota}(b) = \partial\mathcal{L}_{s^*d,\iota,h}(a, b) = \partial\tau(a, b).$$

It follows from (5.8) that k is *-homomorphic if and only if $\tau(1) = 0$. So if $\tau(1) = 0$ then there is no need to dilate k; otherwise note that $(1 - ss^*)^{1/2} = r = 0$. Thus the generators of the two types of dilation are

$$\widehat{\phi}^{g,e}(a) = \begin{pmatrix} \tau(a) & \delta^\dagger(a) & ag^* & ae^* \\ \delta(a) & \iota(a) & 0 & 0 \\ ga & 0 & 0 & 0 \\ ea & 0 & 0 & 0 \end{pmatrix}, \widehat{\phi}^{0,l,n}(a) = \begin{pmatrix} \tau(a) & \delta^\dagger(a) & 0 & an^* \\ \delta(a) & \iota(a) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ na & 0 & 0 & 0 \end{pmatrix}.$$

Note that at least one of g or e must be non-zero, and that $n \neq 0$. So even after removing noise dimensions that are not required for the dilation we see that the QSDEs for $j^{g,e}$ and $j^{0,l,n}$ will contain Poisson terms.

$ss^* \neq 1$: In this case it is not so easy to see if θ is the generator of a *-homomorphic cocycle. However if there is a need to dilate, that is if k is not already *-homomorphic, then note that since $r = (1 - ss^*)^{1/2} = 1 - ss^*$ is a non-trivial projection we have $r\pi(a)r \neq a \otimes 1$. A similar analysis to the above shows once again that Poisson terms will appear in the dilation.

5.6 Unitary dilation

In this section we consider contraction cocycles on a Hilbert space, and their stochastic dilation to unitary cocycles.

A *stochastic contraction cocycle* on the Hilbert space $\mathbf{h}$ is a contractive operator process $(X_t)_{t\geq 0}$ on $\mathbf{h}$ adapted to the Fock filtration and satisfying

$$X_{s+t} = X_s \sigma_s(X_t)$$

in which the right shift σ_s acts on $\mathcal{B}(\mathbf{h} \otimes \Gamma_{\mathbf{k}})$ by

$$\sigma_s(X) = 1_s \otimes S_s X S_s^*,$$

where S_s is the right shift $\mathbf{h}\otimes\Gamma_{\mathbf{k}} \to \mathbf{h}\otimes\Omega_{L^2([0,s),\mathbf{k})}\otimes\Gamma_{[s,\infty),\mathbf{k}}$, 1_s is the identity operator on $\mathbf{h}\otimes\Gamma_{[0,s),\mathbf{k}}$, and some natural identifications of tensor products are invoked ([37]). The Markov semigroup of X is defined by $P_t = \mathbb{E}[X_t] = E^* X_t E$ $(t \geq 0)$. The process X is called *Markov regular* if P is norm continuous $\mathbb{R}_+ \mapsto \mathcal{B}(\mathbf{h})$. The counterpart to Theorem 5.2.1 is

Theorem 5.6.1 (**[37]**) *Let X be a contraction process on the Hilbert space* h. *with separable noise dimension space* k_0. *Then the following are equivalent:*
(i) X is a Markov regular stochastic cocycle on h.
(ii) X weakly satisfies a QSDE of the form

$$dX_t = X_t F^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \; X_0 = 1, \tag{5.11}$$

for a matrix $F = [F^\alpha_\beta]$ of bounded operators on h.
In this case X satisfies the equation strongly, and F defines an element of $\mathcal{B}(\mathrm{h}\otimes\widehat{k_0})$.

The Propositions 7.5 and 7.6 of [36] may be stated as follows.

Theorem 5.6.2 *Let X be a process on the Hilbert space* **h** *with separable noise dimension space* k_0, *weakly satisfying a QSDE of the form 5.11. Then the following equivalences hold*
(ai) X is a contractive process.
(aii) F is bounded with block matrix form

$$F = \begin{pmatrix} iH - \frac{1}{2}(M^*M + B^2) & BV(1 - W^*W)^{1/2} - M^*W \\ M & W - 1 \end{pmatrix}$$

where $H = H^*, B \geq 0, \|V\|, \|W\| \leq 1$.
(aiii) F is bounded with block matrix form

$$F = \begin{pmatrix} iH - \frac{1}{2}(LL^* + C^2) & -L \\ WL^* - (1 - WW^*)^{1/2}V'C & W - 1 \end{pmatrix}$$

where $H = H^*, C \geq 0, \|V'\|, \|W\| \leq 1$.
(bi) X is isometric.
(bii) F is bounded and has block matrix form

$$F = \begin{pmatrix} iH - \frac{1}{2}M^*M & -M^*W \\ M & W - 1 \end{pmatrix}$$

where $H = H^*, W^*W = 1$.
(ci) X is coisometric.
(cii) F is bounded with block matrix form

$$F = \begin{pmatrix} iH - \frac{1}{2}LL^* & -L \\ WL^* & W - 1 \end{pmatrix}$$

where $H = H^*, WW^* = 1$.

The representation (aii) is unique provided that V *satisfies* $\mathrm{Ker}(V) \supset \mathrm{Ker}(1 - W^*W)$ *and* $\mathrm{Ran}(V) \subset \overline{\mathrm{Ran}}(B)$, *which may be easily arranged. Uniqueness of the representation (aiii) may be similarly arranged.*

Remark 5.6.3 *If* X *weakly satisfies the QSDE 5.11 and the Poisson terms of the coefficient matrix* F *are* $(W-1)$ *where* W *is unitary, then* X *is isometric if and only if* X *is coisometric.*

Theorem 5.6.4 *Every Markov regular stochastic contraction cocycle on the Hilbert space* h, *with separable noise dimension space* $\mathbf{k}_0$, *admits a unitary stochastic dilation.*

Proof : Let X be a regular contractive cocycle. By Theorems 5.6.1 and 5.6.2 it satisfies the QSDE 5.11 in which the coefficient matrix matrix has the form

$$F = \begin{pmatrix} iH - \frac{1}{2}(M^*M + B^2) & -L \\ M & W-1 \end{pmatrix}$$

where $H = H^*, B \geq 0, \|W\| \leq 1$ and $L = -BV(1-WW^*)^{1/2} + M^*W$ for some V such that $\|V\| \leq 1$.

Let $G \in \mathcal{B}(\mathbf{h} \otimes (\mathbb{C} \oplus \mathbf{k}_0 \oplus \mathbf{k}_0 \oplus \mathbb{C}))$ be the operator with block matrix form

$$G = \begin{pmatrix} iH - \frac{1}{2}(M^*M + B^2) & -L & -L_1 & -M_2^*U_2 \\ M & W-1 & (1-WW^*)^{1/2} & 0 \\ -V^*B & (1-W^*W)^{1/2} & -W^*-1 & 0 \\ M_2 & 0 & 0 & U_2 - 1 \end{pmatrix}$$

in which $L_1 = BVW^* + M^*(1-WW^*)^{1/2}, U_2 \in \mathcal{B}(\mathbf{h})$ is unitary, and $M_2 \in \mathcal{B}(\mathbf{h})$ satisfies $M_2^*M_2 = B(1-V^*V)B$.

Viewing G as an operator in $\mathcal{B}(\mathbf{h} \otimes \widehat{\mathbf{k}})$, where $\mathbf{k} = \mathbf{k}_0 \oplus \mathbf{k}_0 \oplus \mathbb{C}$, and bearing in mind the remark after Theorem 5.6.2, it is easily checked that G is the generator of a unitary cocycle Y. It follows, by the operator process analogue of Lemma 5.2.2, that Y is a stochastic dilation of X. □

Remark 5.6.5 *(i) In some cases stochastic dilation may be realised more efficiently. Thus if W is unitary, or B is zero, then the third (respectively fourth) row and column is redundant.*

(ii) If $\mathbf{k}_0 = \{0\}$ then, with $M = iB, W = I$ and $U_2 = I$, so that

$$\begin{pmatrix} iH - \frac{1}{2}B^2 & -iB \\ iB & 0 \end{pmatrix},$$

the dilation is effected by the solution X of the QSDE

$$dX_t = \left(iBdQ_t + (iH - \frac{1}{2}B^2)dt\right)$$

where $Q_t = (A_t + A_t^*)$. Since the process Q is a classical Brownian motion, the dilation is "essentially commutative" in the terminology of [32]. When the operators B and H commute it is given explicitly by

$$X_t = e^{i(B\otimes Q_t + tH\otimes I)}.$$

(iii) Throughout this section we have in fact been dealing with left cocycles, in the terminology of [37]. A right stochastic contractive cocycle is a contractive operator process Z that satisfies $Z_{s+t} = \sigma_s(Z_t)Z_s$, and such cocycles that are Markov regular necessarily satisfy the right H-P equation, that is $dZ_t = G^\alpha_\beta Z_t d\Lambda^\beta_\alpha(t)$ for some operator matrix G. The dilation theorem for such processes is an immediate consequence of the time reversal techniques developed in [37].

5.7 Applications to classical probability

One of the most obvious examples in classical probability to which the theory of CP-valued quantum processes can be applied is that of measure valued processes. Consider for example a compact Hausdorff set X and let $\mathcal{P}(X)$ denote the set of all regular Borel probability measures on X. Let $\{\mu_t^x\}_{t\geq 0, x\in X}$ be a family of maps defined on some measure space $(\Omega, \mathbf{F}, \nu)$ such that for each $x \in X, \mu_t^x : \Omega \to \mathcal{P}(X)$ is a measurable map, where we view $\mathcal{P}(X)$ as a metric space in the topology of weak convergence and equip it with the Borel structure arising from this topology. Assume furthermore that for each $f \in C(X)$, the map $X \ni x \mapsto \langle f, \mu_t^x\rangle \equiv \int f\, d\mu_t^x$ is continuous for almost all $\omega \in \Omega$. Thus $\{\mu_t^x\}$ may be identified with the family of $\mathbb{C}$-valued processes $\langle f, \mu_t^x\rangle$. Let us now assume that there is a family $\{N_t^i\}_{i=1}^m$ of independent Poisson processes such that N^i with intensity parameters $\lambda_i > 0$. and that the following stochastic differential equation is satisfied:

$$d\langle f, \mu_t^x\rangle = \langle \alpha_0(f), \nu_t^x\rangle\, dt + \sum_{i=1}^{m} \langle \alpha_i(f), \mu_t^x\rangle\, dN_t^i,\ \mu_0^x = \delta_x, \tag{5.12}$$

where $\{\alpha_i\}_{i=0}^m$ are bounded linear maps on $C(X)$, with $\alpha_i(\overline{f}) = \overline{\alpha_i(f)}$. We call $\{\mu_t^x\}$ a *measure-valued flow* in such a case.

Now let $\mathbf{h}$ be the direct sum of the Hilbert spaces arising from the GNS construction associated with the irreducible representations of $C(X)$, and let $\mathcal{A} \subset \mathcal{B}(\mathbf{h})$ be the enveloping von Neumann algebra of $C(X)$. Clearly the α_i can be extended to normal, bounded linear maps on $\mathcal{A}$, which we also denote by α_i. It is well known that the Fock space $\Gamma(L^2(\mathbb{R}_+; \mathbb{C}^m))$ is canonically isomorphic to $L^2(\mathbb{P}) \equiv L^2(\Omega, \mathbf{F}, \mathbf{P})$ where $(\Omega, \mathbf{F}, \mathbf{P})$ is the measure space induced by the processes $\{N_t^i\}_{i=1}^m$ ([38], p.71). Define maps $k_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+; \mathbb{C}^d))) \cong \mathcal{A} \otimes \mathcal{B}(L^2(\mathbb{P}))$ as follows: first define $k_t : C(X) \to C(X) \otimes \mathcal{B}(L^2(\mathbb{P})$ by setting $k_t(f)(x)$ to be multiplication in $L^2(\mathbb{P})$ by $\langle f, \mu_t^x\rangle$, and then extend to $\mathcal{A}$ by continuity. It is clear that k satisfies the equation

$$dk_t(a) = k_t(\alpha_0(a))\, dt + \sum_{i=1}^{m} k_t(\alpha_i(a))\, dN_t^i,\ \forall a \in \mathcal{A}.$$

However N_t^i is realised in $\mathcal{B}(L^2(\mathbb{P})) \cong \mathcal{B}(\Gamma(L^2(\mathbb{R}_+; \mathbb{C}^m)))$ as the operator $\Lambda_i^i(t) + \sqrt{\lambda_i}(A_i(t) + A_i^\dagger(t)) + \lambda_i t$ ([38], p.74), and thus k satisfies the QSDE: $dk = k \circ \theta_\beta^\alpha\, d\Lambda_\alpha^\beta$ where we set

$$\theta_0^0 = \alpha_0 + \sum_{i=1}^{m} \lambda_i \alpha_i,\ \theta_0^i = \theta_i^0 = \sqrt{\lambda_i}\alpha_i, \text{ and } \theta_j^i = \delta_j^i \alpha_i. \tag{5.13}$$

By construction k is a CP contractive flow, and so by the general theory above we can construct a *-homomorphic dilation j of k. Furthermore, since $\mathcal{A}$ is abelian, the family $\{j_t(a) : t \geq 0, a \in \mathcal{A}\}$ will be abelian (see [45], Theorem 28.8). On the other hand, given any *-homomorphism π from $C(X)$ onto a commutative C^*-algebra $\mathcal{C}$, by Gelfand theory there exists a map η from the Gelfand spectrum of $\mathcal{C}$ to X such that π is unitarily equivalent to the *-homomorphism $f \mapsto f \circ \eta$. In view of this we obtain a time indexed family η_t of maps from some abstract space to X such that for each t, j_t is unitarily equivalent to the *-homomorphism $f \mapsto f \circ \eta_t$. However we can as yet say very little about $\{\eta_t\}$ as a process. Roughly speaking the dilation of k may be viewed as a lifting of a measure-valued process μ^x to a process taking values in the underlying state space X.

As a final application, another source of CP flows arising in classical probability come from filtering theory. One considers the signal process Z_t on a space $(\Omega, \mathbf{F}, \mathbf{P})$ which is a Markov process with (possibly unbounded) generator L, and also an observation process Y_t which satisfies an equation of the form $dY_t = h_t(Z_t)\,dt + dW_t$ for some Wiener process W independent of Z and some h that is sufficiently well-behaved. Assume for simplicity that Z and Y are $\mathbb{R}^m$-valued processes. Then Bayes' formula gives, for any $\mathbf{F}_t^Z$-measurable and integrable random variable g,

$$\mathbb{E}[g|\mathbf{F}_t^Y] = \sigma_t'(g, Y)/\sigma_t'(1, Y)$$

where

$$\sigma_t'(g, Y(\omega)) = \int g(\omega') \exp\left\{\sum_{i=1}^m \int_0^t h_s^i(X_s(\omega'))\,dY_s^i(\omega') - \frac{1}{2}\int_0^t |h_s(X_s(\omega'))|^2\,ds\right\} d\mathbb{P}(\omega'),$$

and h^j and Y^j denotes the jth component of h and Y respectively. The map $\widehat{\sigma}_t$ is then defined by $\widehat{\sigma}_t(f, Y) = \sigma_t'(f(X_t), Y)$ for all f such that the right hand side makes sense. Furthermore $\widehat{\sigma}_t$ satisfies the well known Zakai equation (under certain assumptions) which is of the form

$$d\widehat{\sigma}_t(f, Y) = \widehat{\sigma}_t(L_t f, Y)\,dt + \sum_{j=1}^m \widehat{\sigma}_t(h_t^j f, Y)\,dY_t^j.$$

This Zakai equation has a formal similarity with the flow equation considered by us, although L and h^j are in general unbounded. We refer the reader to [29] for a comprehensive account of the Zakai equation and related topics. We remark that $f \mapsto \widehat{\sigma}_t(f, Y)$ is indeed a CP map on the underlying function algebra, but in general it will not be a *-homomorphism.

Chapter 6

Construction of minimal semigroups

Given a formal unbounded generator, the minimal quantum dynamical semigroup on a von Neumann algebra is constructed. A set of equivalent necessary and sufficient conditions for the conservativity of the minimal semigroup is given and in the case when it is not conservative. a distinguished family of conservative perturbations of the semigroup is studied. Finally, some of these results are applied to the classical Markov semigroup with arbitrary state space.

The structure of generators of uniformly continuous quantum dynamical semigroups (i. e. semigroups of normal completely positive linear maps) on a von Neumann algebra was completely characterized by Christensen and Evans [8] and Lindblad [34]. However, no such structure theorem is available when the semigroup is not uniformly continuous but only continuity with respect to some other suitable weaker topology is given. There have been several attempts to construct a "minimal" semigroup starting from a given unbounded operator. Such endeavours have proved to be successful when the underlying von Neumann algebra is either a commutative one (in particular classical Markov processes with countable state space, studied by Feller [20] and Kato [31]) or the full algebra $\mathcal{B}(\mathcal{H})$ for some complex separable Hilbert space $\mathcal{H}$ (Davies [12], Mohari and Sinha [43], Chebotarev [9]). Our aim here is to construct a minimal semigroup on an *arbitrary von Neumann algebra*, starting with suitable assumptions on the formal generator; then study the conservativity property and apply the construction to a large class ofclassical Markov semigroups.

as well as to the irrational rotation algebra $\mathcal{A}_\theta$. The previously considered cases of commutative function algebras and $\mathcal{B}(\mathcal{H})$ are special cases of this construction.

6.1 Construction of minimal semigroup

Suppose that h and $\mathcal{K}$ are two Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(h)$ is a von Neumann algebra, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a normal, unital, *-representation; $(P_t)_{t\geq 0}$ is a C_0 -contraction semigroup on h; and $R : h \to \mathcal{K}$ is a closed, densely defined, linear (possibly unbounded) map. Formally we introduce a map $\mathcal{L}$ by, $\mathcal{L}(x) = R^*\pi(x)R + xG + G^*x$, $x \in \mathcal{A}$; where G is the generator of $(P_t)_{t\geq 0}$. Let us make the following assumptions on G and R :

(A1) $G\ \eta\ \mathcal{A}$ and $R^*\pi(x)R\ \eta\ \mathcal{A}, \forall\, x \in \mathcal{A}$; where η denotes affiliation to a von Neumann algebra (see [50], [14] for the definition of affiliation and relevant discussion).

(A2) $\mathcal{D}(G) \subseteq \mathcal{D}(R)$ and $\forall\, u, v \in \mathcal{D}(G)$, $< Ru, Rv > + < u, Gv > + < Gu, v > = 0$, where $\mathcal{D}(E)$ denotes the domain of a linear map E.

Remark 6.1.1 *(i) Note that for a uniformly continuous quantum dynamical semigroup, its generator is given by Christensen-Evans form* [8] *:*
$\mathcal{L}(x) = R^*\pi(x)R + xG + G^*x$, where $R \in \mathcal{B}(h, \mathcal{K})$ such that $R^*\pi(x)R$ and G are in $\mathcal{A}$ for $x \in \mathcal{A}$. Thus the assumption (A1) *is a naturally possible generalization of the* [8] *properties to the case where $\mathcal{L}$ is unbounded. Note also that* (A2) *corresponds to formal statement that $\mathcal{L}(1) = 0$. It is also clear that when $\mathcal{A} = \mathcal{B}(h)$,* (A1) *is trivially satisfied.*
(ii) However, it is to be noted that the assumption (A1) *does not cover the case of the heat semigroup.*
(iii) Note that (A2) *is equivalent to the following :*

$$(A2') : (1 - G^*)^{-1}R^*R(1 - G)^{-1} + (1 - G^*)^{-1}G(1 - G)^{-1}+$$
$$(1 - G^*)^{-1}G^*(1 - G)^{-1} = 0.$$

Let us now consider $\pi(\mathcal{A})$ as a von Neumann algebra in $\mathcal{B}(\mathcal{K})$ and define a map π_* from $\pi(\mathcal{A}_h)_*$ to $\mathcal{A}_{h,*}$ by, $(\pi_*\psi)(a) = \psi(\pi(a))$, $a \in \mathcal{A}_h$, and $\psi \in \Omega_{\pi(\mathcal{A}_h)} \equiv (\pi(\mathcal{A}_h))_*$. Modulo canonical identification, π_* can be viewed as a map from $\mathcal{B}_1^{s.a.}(\mathcal{K})/(\pi(\mathcal{A}_h))^\perp$ to $\mathcal{B}_1^{s.a.}(h)/\mathcal{A}_h^\perp$; and we won't notationally distinguish between these two views. It is clear that $tr((\pi_*[\eta]_h)x) = tr([\eta]_h\pi(x))\ \forall\, x \in \mathcal{A}_h, \eta \in \mathcal{B}_1^{s.a.}(\mathcal{K})$.

Lemma 6.1.2 $||\pi_*|| \leq 1$, π_* *is positive. and the dual of* π_* *is the restriction of* π *to* $\mathcal{A}_h$.

Proof : For $\psi \in \Omega_{\pi(\mathcal{A}_h)}$, we have $||\pi_*\psi|| = sup_{a\in\mathcal{A}_h,||a||\leq 1}|(\pi_*\psi)(a)| = sup_{a\in\mathcal{A}_h,||a||\leq 1}|\psi(\pi(a))| \leq sup_{b\in\pi(\mathcal{A}_h),||b||\leq 1}|\psi(b)|$ (since $||\pi(c)|| \leq ||c|| \; \forall \; c$, for any $*$-representation π) $= ||\psi||$.

The map π_* is positive because for any positive functional $\psi \in \Omega_{\pi(\mathcal{A}_h)}$, $(\pi_*\psi)(a^*a)$ $= \psi(\pi(a)^*\pi(a)) \geq 0 \; \forall \; a \in \mathcal{A}$. The assertion about its dual is straightforward. □

Before proceeding further, we note a standard fact about operators affiliated to a von Neumann algebra , the proof of which is easy and is omitted.

Lemma 6.1.3 *A possibly unbounded, closed, densely defined operator* B *acting on a Hilbert space* h *is affiliated to a von Neumann algebra* $\mathcal{B}$ *if and only if for every* a' $\in \mathcal{B}'$ *and* $u \in \mathcal{D}(B)$, *we have that* $a'u \in \mathcal{D}(B)$ *and* $Ba'u = a'Bu$, *where* $\mathcal{B}'$ *denotes the commutant of* $\mathcal{B}$. *Moreover, if* B *and* C *are two closed, densely defined operators affiliated to* $\mathcal{B}$, *such that* BC *is also a closed, densely defined operator, then* BC *is affiliated to* $\mathcal{B}$.

Now, let us define $S_t : \mathcal{B}_1^{s.a.}(h) \to \mathcal{B}_1^{s.a.}(h)$ by, $S_t(\rho) = P_t\rho P_t^*$ ($t \geq 0$). It is immediate that $(S_t)_{t\geq 0}$ is a positive, C_0-contraction semigroup.

Lemma 6.1.4 $(\lambda - G)^{-1} \in \mathcal{A} \; \forall$ *positive* λ. *Moreover,* $\forall \; x \in \mathcal{A}_h$, $P_t^*xP_t \in \mathcal{A}_h \; \forall$ $t \geq 0$ *and* $(1 - G^*)^{-1}R^*\pi(x)R(1 - G)^{-1} \in \mathcal{A}_h$.

Proof: We have for $\lambda > 0$, $a' \in \mathcal{A}'$ and $u \in h$, $(\lambda - G)a'(\lambda - G)^{-1}u = a'(\lambda - G)(\lambda - G)^{-1}u$ (by A1 and 6.1.3)

$\Rightarrow (\lambda - G)a'(\lambda - G)^{-1} = a'$

$\Rightarrow a'(\lambda - G)^{-1} = (\lambda - G)^{-1}a^{-1}$, for all $a' \in \mathcal{A}'$

$\Rightarrow (\lambda - G)^{-1} \in \mathcal{A}'' = \mathcal{A}$, because $(\lambda - G)^{-1}$ is bounded. This also gives, $P_t = s - lim_{n\to\infty}(n/t(n/t - G)^{-1})^n \in \mathcal{A}$. Thus, for any x in $\mathcal{A}_h$, $P_t^*xP_t$ is a self-adjoint element in $\mathcal{A}$, i. e. belongs to $\mathcal{A}_h$.

Now, let $x \in \mathcal{A}_h$. Let us consider the polar decomposition of R as, $R = U|R|$, where $|R|$ is a positive operator affiliated to $\mathcal{A}$ (because $|R| = (R^*R)^{1/2}$, R^*R $= R^*\pi(1)R \; \eta \; \mathcal{A}$) and U is a partial isometry with the closure of range of $|R|$ as the initial space and the closure of range of R as the final space. It is easy to note that $|R|(1 - G)^{-1}$ is bounded (as $Ran((1 - G)^{-1}) \subseteq \mathcal{D}(R) = \mathcal{D}(|R|)$)

and by 6.1.3 (second part), it is affiliated to $\mathcal{A}$, hence $\in \mathcal{A}$. For any bounded continuous function f from $[0,\infty)$ to $\mathbb{R}$, $f(|R|) \in \mathcal{A}$, as $|R| \; \eta \; \mathcal{A}$. In particular, $n(1+n|R|)^{-1} \in \mathcal{A}$ for any positive integer n. So, $n^2(1+n|R|)^{-1}R^*\pi(x)R(1+n|R|)^{-1} = (n(1+n|R|)^{-1}|R|)U^*\pi(x)U(n|R|(1+n|R|)^{-1}) \in \mathcal{A}$ (since it is clearly bounded and is affiliated to $\mathcal{A}$ by 6.1.3). But $n|R|(1+n|R|)^{-1} \to 1$ strongly as $n \to \infty$: hence $U^*\pi(x)U = s - lim_{n\to\infty} n^2(1+n|R|)^{-1}R^*\pi(x)R(1+n|R|)^{-1} \in \mathcal{A}$. Thus, $(1-G^*)^{-1}R^*\pi(x)R(1-G)^{-1} = (|R|(1-G)^{-1})^*(U^*\pi(x)U)(|R|(1-G)^{-1}) \in \mathcal{A}$, (as $|R|(1-G)^{-1} \in \mathcal{A}$) and being also self-adjoint, it belongs to $\mathcal{A}_h$.

□

Lemma 6.1.5 *For $t \geq 0$, S_t induces a linear map $\widetilde{S_t} : \mathcal{A}_{h,*} \to \mathcal{A}_{h,*}$, and $(\widetilde{S_t})_{t\geq 0}$ is a positive, C_0-contraction semigroup on $\mathcal{A}_{h,*}$ (positivity means that $\widetilde{S_t}([\rho]_h)$ is a positive element whenever $[\rho]_h$ is so in $\mathcal{A}_{h,*}$).*

Proof: If we take $\rho_1, \rho_2 \in \mathcal{B}_1^{s.a.}(h)$ such that $\rho_1 \sim \rho_2$, then for any $x \in \mathcal{A}_h$ and $t \geq 0$, $tr(S_t(\rho_1)x) = tr(P_t\rho_1P_t^*x) = tr(\rho_1P_t^*xP_t) = tr(\rho_2P_t^*xP_t)$
(by 6.1.4) $= tr(S_t(\rho_2)x)$. Thus, $S_t(\rho_1) \sim S_t(\rho_2)$, which proves that S_t induces a map $\widetilde{S_t}$. Semigroup property and positivity of $(\widetilde{S_t})_{t\geq 0}$ are immediate, whereas strong continuity follows from the fact that $(S_t)_{t\geq 0}$ is strongly continuous and $||\widetilde{S_t}([\rho]_h) - [\rho]_h|| \leq ||S_t(\rho) - \rho||_1 \; \forall \rho \in \mathcal{B}_1^{s.a.}(h), t \geq 0$. □

Let us denote the generator of $(S_t)_{t\geq 0}$ by Z. Since each S_t induces an operator on $\mathcal{A}_{h,*}$, Z will do so. It is easy to see that the generator of $(\widetilde{S_t})_{t\geq 0}$ will be a closed extension of the map induced by Z. We, by slight abuse of notation, denote by $\tilde{Z}$ the generator of $(\widetilde{S_t})_{t\geq 0}$. Let us define $\varphi : \mathcal{B}_1^{s.a.}(h) \to \mathcal{B}_1^{s.a.}(h)$ by, $\varphi(\rho) = (1-G)^{-1}\rho(1-G^*)^{-1}$. Since $(1-G)^{-1}$ and $(1-G^*)^{-1}$ belong to $\mathcal{A}$, φ will induce a map $\tilde{\varphi}$ from $\mathcal{A}_{h,*}$ to $\mathcal{A}_{h,*}$, which can be proven in a way similar to the proof of 6.1.5. Let us denote by $\mathcal{D}$ and $\tilde{\mathcal{D}}$ the ranges of φ and $\tilde{\varphi}$ respectively.

Lemma 6.1.6 *$\mathcal{D}$ and $\tilde{\mathcal{D}}$ are dense in $\mathcal{B}_1^{s.a.}(h)$ and $\mathcal{A}_{h,*}$ respectively. Moreover, they are cores for Z and $\tilde{Z}$ respectively.*

Proof: Since $\mathcal{D}(G)$ is dense in h, the real linear span of rank-one operators of the form $|u><u|$, $u \in \mathcal{D}(G)$ is dense in $\mathcal{B}_1^{s.a.}(h)$. But for $u \in \mathcal{D}(G) = Ran((1-G)^{-1})$. $|u><u|$ is clearly in $\mathcal{D}$, which proves the density of $\mathcal{D}$ in $\mathcal{B}_1^{s.a.}(h)$. Core property of $\mathcal{D}$ follows because each S_t leaves $\mathcal{D}$ invariant. The assertions about $\tilde{\mathcal{D}}$ follow similarly, only thing to note is that $||[\rho]_h - [\sigma]_h|| \leq ||\rho - \sigma||_1 \; \forall \rho, \sigma \in \mathcal{B}_1^{s.a.}(h)$. □

Lemma 6.1.7 *Given $[\rho]_h \in \tilde{\mathcal{D}}$ and $\epsilon > 0$, we can get positive elements $[\rho_1]_h$ and $[\rho_2]_h$ in $\tilde{\mathcal{D}}$ such that $[\rho]_h = [\rho_1]_h - [\rho_2]_h$ and $||[\rho_1]_h|| + ||[\rho_2]_h|| \leq ||[\rho]_h|| + \epsilon$.*

Proof : Let us consider a general $[\rho]_h \in \tilde{\mathcal{D}}$ such that $\rho = \varphi(\sigma)$, $\sigma \in \mathcal{B}_1^{s.a.}(h)$. Given $\epsilon > 0$, we can choose sufficiently large n such that

$||[\sigma_n]_h - [\rho]_h|| \leq \epsilon$, where $\sigma_n = (1 - G/n)(1 - G)^{-1}\sigma((1 - G/n)(1 - G)^{-1})^*$
$= (1 - G/n)\varphi(\sigma)(1 - G^*/n) = (1 - G/n)\rho(1 - G^*/n)$. By the non- commutative Hahn decomposition (Theorem 4.2 (ii) of page 140 of [51]), we can get two positive elements σ_n^+, σ_n^- (not necessarily the positive and negative parts of σ_n) such that $||[\sigma_n]_h|| = ||[\sigma_n^+]_h|| + ||[\sigma_n^-]_h||$ and $[\sigma_n]_h = [\sigma_n^+]_h - [\sigma_n^-]_h$. Take $\rho_1 = (1 - G/n)^{-1}\sigma_n^+(1 - G^*/n)^{-1}$, $\rho_2 =$
$(1 - G/n)^{-1}\sigma_n^-(1 - G^*/n)^{-1}$: and observe that $||[\rho_1]_h|| + ||[\rho_2]_h||$
$= ||(1 - G/n)^{-1}\sigma_n^+(1 - G^*/n)^{-1}||_1 + ||(1 - G/n)^{-1}\sigma_n^-(1 - G^*/n)^{-1}||_1$ (as ρ_1, ρ_2 are positive)
$\leq ||\sigma_n^+||_1 + ||\sigma_n^-||_1$
$= ||[\sigma_n^+]_h|| + ||[\sigma_n^-]_h||$
$= ||[\sigma_n]_h|| \leq ||[\rho]_h|| + \epsilon$.

Now, $[\rho_1]_h - [\rho_2]_h = [(1 - G/n)^{-1}(\sigma_n^+ - \sigma_n^-)(1 - G^*/n)^{-1}]_h =$
$[(1 - G/n)^{-1}\sigma_n(1 - G^*/n)^{-1}]_h = [\rho]_h$, because $(\sigma_n^+ - \sigma_n^-) \sim \sigma_n$ implies $(1 - G/n)^{-1}(\sigma_n^+ - \sigma_n^-)(1 - G^*/n)^{-1} \sim (1 - G/n)^{-1}\sigma_n(1 - G^*/n)^{-1}$.

□

Let us now define $J : \mathcal{D} \to \mathcal{A}_{h,*}$ by,
$J(\varphi(\rho)) = \pi_*([R(1 - G)^{-1}\rho(1 - G^*)^{-1}R^*]_h)$, $\rho \in \mathcal{B}_1^{s.a.}(h)$; where $(1 - G^*)^{-1}R^*$ is to be interpreted as the bounded operator $(R(1 - G)^{-1})^*$. Next two lemmas give some useful properties of the map $\tilde{J}$ induced by J.

Lemma 6.1.8 *The map $\tilde{J} : \tilde{\mathcal{D}} \to \mathcal{A}_{h,*}$ given by, $\tilde{J}(\tilde{\varphi}([\sigma]_h)) = J(\varphi(\sigma))$ for $\sigma \in \mathcal{B}_1^{s.a.}(h)$ is well-defined and linear.*

Proof : It is enough to prove that whenever $\varphi(\sigma) \sim 0$, we must have $J(\sigma) = [0]_h$. Given $\varphi(\sigma) \sim 0$, we first show that $\sigma \sim 0$. Let us denote by G_n the operator $(1 - G/n)^{-1}$, $n = 1, 2, ...$ Clearly, each G_n is in $\mathcal{A}$, and $G_n \to 1$ strongly. So, $(1 - GG_n)(1 - G)^{-1} \to 1$ strongly. This implies that $(1 - GG_n)(1 - G)^{-1}\sigma(1 - G^*)^{-1}(1 - GG_n)^*x$ converges to σx in trace-norm as n tends to ∞, for any $x \in \mathcal{A}_h$, and hence $tr(\sigma x)$

$= lim_{n\to\infty} tr((1-GG_n)(1-G)^{-1}\sigma(1-G^*)^{-1}(1-GG_n)^*x)$
$= lim_{n\to\infty} tr(\varphi(\sigma).(1-GG_n)^*x(1-GG_n)) = 0 \ \forall\ x \in \mathcal{A}_h$; where in the last step we have noted that $(1-GG_n) \in \mathcal{A}$, which implies $(1-GG_n)^*x(1-GG_n) \in \mathcal{A}_h$. Hence, $\sigma \sim 0$.

Now, $\forall\ x \in \mathcal{A}_h$, $tr(J(\sigma x)) = tr(R(1-G)^{-1}\sigma(1-G^*)^{-1}R^*\pi(x)) =$ $tr(\sigma(1-G^*)^{-1}R^*\pi(x)R(1-G)^{-1}) = 0$, because $(1-G^*)^{-1}R^*\pi(x)R(1-G)^{-1} \in \mathcal{A}_h$. This completes the proof. □

Lemma 6.1.9 *$\tilde{J}$ is positive, and $tr(\tilde{J}([\rho]_h) + \tilde{Z}([\rho]_h)) = 0 \ \forall\ [\rho]_h \in \tilde{\mathcal{D}}$.*

proof : Positivity of $\tilde{J}$ follows from the positivity of π_*. An easy computation shows that for $\rho \in \mathcal{D}$, $Z(\rho) = G\rho + \rho G^*$; hence for $[\rho]_h \in \tilde{\mathcal{D}}$, $tr(\tilde{Z}([\rho]_h)) = tr(Z(\rho)) =$ $tr(G\rho + \rho G^*)$. Now for any $\sigma \in \mathcal{B}_1^{s.a.}(h)$, we have $tr(\tilde{J}(\tilde{\varphi}([\sigma]_h))) = tr(J(\varphi(\sigma))) =$ $tr(R(1-G)^{-1}\sigma(1-G^*)^{-1}R^*\pi(1))$
$= tr(\sigma(1-G^*)^{-1}R^*R(1-G)^{-1})$
$= -tr(\sigma(1-G^*)^{-1}G(1-G)^{-1} + \sigma(1-G^*)^{-1}G^*(1-G)^{-1})$ (by A2$'$)
$= -tr(G\varphi(\sigma) + \varphi(\sigma)G^*) = -tr(\tilde{Z}(\tilde{\varphi}([\sigma]_h)))$, which completes the proof. □

For $\lambda > 0$, $(\lambda - Z)^{-1}$ can be expressed as $\int_0^\infty e^{-\lambda t}S_t dt$, hence $(\lambda - Z)^{-1}$ leaves $\mathcal{D}$ invariant and is positive. Similar statements are valid about $(\lambda - \tilde{Z})^{-1}$.

Let us define $\tilde{B}(\lambda) : \tilde{\mathcal{D}} \to \mathcal{A}_{h,*}$ by, $\tilde{B}(\lambda) = \tilde{J}(\lambda - \tilde{Z})^{-1}$.

Lemma 6.1.10 *$\tilde{B}(\lambda)$ extends to a positive linear contractive map from $\mathcal{A}_{h,*}$ to $\mathcal{A}_{h,*}$, which we denote by the same notation.*

proof: For positive $[\rho]_h \in \tilde{\mathcal{D}}$, with $\rho \in \mathcal{D}$, we have $\|\tilde{B}(\lambda)([\rho]_h)\| = tr(\tilde{B}(\lambda)([\rho]_h))$ $= tr(\tilde{J}(\lambda - \tilde{Z})^{-1}([\rho]_h)) = -tr(\tilde{Z}(\lambda - \tilde{Z})^{-1}([\rho]_h)) = tr(\rho) - \lambda tr((\lambda - \tilde{Z})^{-1}([\rho]_h)) \leq$ $tr(\rho) = \|[\rho]_h\|$. For an arbitrary $[\rho]_h$ in $\tilde{\mathcal{D}}$ and any positive number ϵ, we choose two positive elements $[\rho_1]_h$ and $[\rho_2]_h$ satisfying the conclusions of 6.1.7. Therefore, $\|\tilde{B}(\lambda)([\rho]_h)\| \leq \|\tilde{B}(\lambda)([\rho_1]_h)\| + \|\tilde{B}(\lambda)([\rho_2]_h)\| \leq \|[\rho_1]_h\| + \|[\rho_2]_h\| \leq \|[\rho]_h\| + \epsilon$. This proves that $\|\tilde{B}(\lambda)([\rho]_h)\| \leq \|[\rho]_h\| \ \forall\ [\rho]_h \in \tilde{\mathcal{D}}$, and we complete the proof by the density of $\tilde{\mathcal{D}}$ in $\mathcal{A}_{h,*}$. □

Lemma 6.1.11 *$\tilde{J}$ extends to $\tilde{J}'$ on $\mathcal{D}(\tilde{Z})$ such that $tr(\tilde{Z}([\rho]_h) + \tilde{J}'([\rho]_h)) = 0 \ \forall\ [\rho]_h \in \mathcal{D}(\tilde{Z})$.*

Proof : It is enough to take $\widetilde{J}'([\rho]_h) = \widetilde{B}(1)(1-\widetilde{Z})([\rho]_h)$, for $[\rho]_h \in \mathcal{D}(\widetilde{Z})$, and to note that $\widetilde{\mathcal{D}}$ is a core for $\widetilde{Z}$. □

By a slight abuse of notation, we shall continue to denote $\widetilde{J}'$ by $\widetilde{J}$.

Theorem 6.1.12 *For* $0 \leq r < 1$, *define* $\widetilde{G_r} = \widetilde{Z} + r\widetilde{J}$. *Then* $\widetilde{G_r}$ *generates a positive,* C_0 *contraction semigroup, say* $(\widetilde{T_t}^{(r)})_{t\geq 0}$, *on* $\mathcal{A}_{h,*}$; *and* $(\lambda - \widetilde{G_r})^{-1} = (\lambda - \widetilde{Z})^{-1}\sum_{n=0}^{\infty} r^n \widetilde{B}(\lambda)^n$ *for* $\lambda > 0$.

Proof : Fix $\lambda > 0$, $r \in [0,1)$. We have, $(\lambda - \widetilde{G_r}) = (1 - r\widetilde{J}(\lambda - \widetilde{Z})^{-1})(\lambda - \widetilde{Z})$ $= (1 - r\widetilde{B}(\lambda))(\lambda - \widetilde{Z})$. Since $\|r\widetilde{B}(\lambda)\| < 1$, we can use the Neumann series to get $(1 - r\widetilde{B}(\lambda))^{-1} = \sum_{n=0}^{\infty} r^n \widetilde{B}(\lambda)^n$, which shows in particular that $(\lambda - \widetilde{G_r})^{-1} = (\lambda - \widetilde{Z})^{-1}\sum_{n=0}^{\infty} r^n \widetilde{B}(\lambda)^n$ exists as a bounded operator. Now, for any positive $[\rho]_h \in \mathcal{A}_{h,*}$, let us denote by $[\sigma]_h$ the element $(\lambda - \widetilde{G_r})^{-1}([\rho]_h)$. Clearly, $(\lambda - \widetilde{G_r})^{-1}$ is positive since $(\lambda - \widetilde{Z})^{-1}$ and $\widetilde{B}(\lambda)$ are so; and hence $[\sigma]_h$ is positive. Thus, $\|[\rho]_h\| = tr([\rho]_h)$

$= \lambda tr([\sigma]_h) + (1-r)tr(\widetilde{J}([\sigma]_h))$ (as $tr((\widetilde{J} + \widetilde{Z})([\sigma]_h)) = 0$)

$\geq \lambda tr([\sigma]_h)$

$= \lambda \|[\sigma]_h\|$.

So, $\|(\lambda - \widetilde{G_r})^{-1}([\rho]_h)\| \leq \|[\rho]_h\|/\lambda$ for all positive $[\rho]_h$ in $\mathcal{A}_{h,*}$ and hence for all $[\rho]_h$ in $\mathcal{A}_{h,*}$, since we can decompose any $[\rho]_h$ as $[\rho]_h = [\rho_1]_h - [\rho_2]_h$ with $\|[\rho]_h\| = \|[\rho_1]_h\| + \|[\rho_2]_h\|$ where $[\rho_1]_h$ and $[\rho_2]_h$ are positive (Theorem 4.2 (ii) of page 140 of [51]). We complete the proof of the theorem by appealing to the Hille- Yosida Theorem. and also noting that positivity of $(\lambda - \widetilde{G_r})^{-1}$ implies positivity of $\widetilde{T_t}^{(r)}$.
□

Theorem 6.1.13 *As* $r \uparrow 1$, $\widetilde{T_t}^{(r)} \uparrow \widetilde{T_t}^{(min)}$, *where* $(\widetilde{T_t}^{(min)})_{t\geq 0}$ *is a positive,* C_0 *contraction semigroup on* $\mathcal{A}_{h,*}$; *and the above convergence is strong and uniform in* t *over compact subsets of* $[0,\infty)$.

Proof : We see that for positive element $[\rho]_h \in \mathcal{A}_{h,*}$ $\widetilde{T_t}^{(r)}([\rho]_h) \geq \widetilde{T_t}^{(s)}([\rho]_h) \geq 0$ for r, s such that $1 > r \geq s \geq 0$. This follows from the series expansion of $(\lambda - \widetilde{G_r})^{-1}$ as in the preceeding theorem and from the fact that $\widetilde{T_t}^{(r)} = s - lim_{n\to\infty}(n/t)^n(n/t - \widetilde{G_r})^{-n}$. Moreover, $tr(\widetilde{T_t}^{(r)}([\rho]_h)) \leq \|x\|\|[\rho]_h\|$, since $\|\widetilde{T_t}^{(r)}\| \leq 1$. So $(tr(\widetilde{T_t}^{(r)}([\rho]_h)))_{r\in(0,1]}$ is an increasing bounded net of positive numbers and hence it converges to some finite positive limit as $r \uparrow 1$. For $r \geq s$ we have $\|\widetilde{T_t}^{(r)}([\rho]_h) - \widetilde{T_t}^{(s)}([\rho]_h)\| = tr(\widetilde{T_t}^{(r)}([\rho]_h) -$

$\widetilde{T}_t^{(s)}([\rho]_h)) \to 0$ as $r, s \to 1$ and hence $\widetilde{T}_t^{(r)}([\rho]_h)$ converges in the norm of $\mathcal{A}_{h,*}$ to say $\widetilde{T}_t^{(min)}([\rho]_h)$ as $r \uparrow 1$. We extend $\widetilde{T}_t^{(min)}$ to the whole of $\mathcal{A}_{h,*}$ by linearity using Theorem 4.2 (ii) of page 140 of [51] ; and observe that this extension will be a positive, contractive, linear map from $\mathcal{A}_{h,*}$ to itself, since each $\widetilde{T}_t^{(r)}$ is so. Semigroup property of $(\widetilde{T}_t^{(min)})_{t\geq 0}$ is obvious.

We now show that the convergence is uniform over compacts in t, which will also prove strong continuity of $(\widetilde{T}_t^{(min)})_{t\geq 0}$. Suppose that this is not true. Then, there exist positive $[\rho]_h \in \mathcal{A}_{h,*}$, positive number ϵ_0, and sequences $(r_n)_{n=1,2,\dots}$, $(t_n)_{n=1,2,\dots}$ such that $0 \leq r_n \uparrow 1$, $0 \leq t_n \to t_0$ for some $t_0 \geq 0$; and $||\widetilde{T_{t_n}}^{(r_n)}([\rho]_h) - \widetilde{T_{t_n}}^{(min)}([\rho]_h)|| \geq \epsilon_0$ $\forall$ positive integer n. Since $\widetilde{T_{t_n}}^{(r_n)}([\rho]_h)$ and $(-\widetilde{T_{t_n}}^{(r_n)}([\rho]_h) + \widetilde{T_{t_n}}^{(min)}([\rho]_h))$ are positive, we have by 1.1.10 that $||\widetilde{T_{t_n}}^{(min)}([\rho]_h)|| = ||\widetilde{T_{t_n}}^{(r_n)}([\rho]_h)|| + ||\widetilde{T_{t_n}}^{(min)}([\rho]_h) - \widetilde{T_{t_n}}^{(r_n)}([\rho]_h)||$, which implies, $||\widetilde{T_{t_n}}^{(min)}([\rho]_h)|| - ||\widetilde{T_{t_n}}^{(r_n)}([\rho]_h)|| \geq \epsilon_0$. So, $\forall\, m \leq n$, $||\widetilde{T_{t_n}}^{(r_m)}([\rho]_h)|| \leq ||\widetilde{T_{t_n}}^{(r_n)}([\rho]_h)|| \leq ||\widetilde{T_{t_n}}^{(min)}([\rho]_h)|| - \epsilon_0$. Keeping m fixed and letting n tend to ∞, we obtain that $||\widetilde{T_{t_0}}^{(r_m)}([\rho]_h)|| \leq ||\widetilde{T_{t_0}}^{(min)}([\rho]_h)|| - \epsilon_0$; and then by letting m tend to ∞, $||\widetilde{T_{t_0}}^{(min)}([\rho]_h)|| \leq ||\widetilde{T_{t_0}}^{(min)}([\rho]_h)|| - \epsilon_0$, a contradiction. □

Theorem 6.1.14 *The generator of $(\widetilde{T}_t^{(min)})_{t\geq 0}$, say $\widetilde{A}$, is an extension of $\widetilde{Z} + \widetilde{J}$; and we have the following minimality property :*

Whenever $(\widetilde{T}_t')_{t\geq 0}$ is a positive, C_0 contraction semigroup on $\mathcal{A}_{h,*}$ whose generator (say $\widetilde{A'}$) extends $\widetilde{Z} + \widetilde{J}$, we must have $\widetilde{T}_t' \geq \widetilde{T}_t^{(min)}$ $\forall\, t \geq 0$.

Proof : The first part of the theorem follows from the fact that $\widetilde{G_r}([\rho]_h) \to (\widetilde{Z} + \widetilde{J})([\rho]_h)$ as $r \uparrow 1$, $\forall\, [\rho]_h \in \mathcal{D}(\widetilde{Z})$. For minimality, it is required to observe that for $\lambda > 0$, $(\lambda - \widetilde{A}')^{-1} - (\lambda - \widetilde{G_r})^{-1} = (\lambda - \widetilde{A}')^{-1}(\widetilde{A}' - \widetilde{G_r})(\lambda - \widetilde{G_r})^{-1} \geq 0$, since the restriction of $\widetilde{A}'$ to the range of $(\lambda - \widetilde{G_r})^{-1}$, i. e. to $\mathcal{D}(\widetilde{G_r})$, is the same as $\widetilde{Z} + \widetilde{J} \geq \widetilde{G_r}$, and $(\lambda - \widetilde{A}')^{-1}$, $(\lambda - \widetilde{G_r})^{-1}$ are positive. We complete the proof by noting that $\widetilde{T}_t' = s - lim_{n\to\infty}(n/t)^n(n/t - \widetilde{A}')^{-n} \geq s - lim_{n\to\infty}(n/t)^n(n/t - \widetilde{G_r})^{-n} = \widetilde{T}_t^{(r)}$ $\forall$ r, t. □

For a linear map V from $\mathcal{A}_{h,*}$ to $\mathcal{A}_{h,*}$, we consider its canonical extension $\overline{V}$ from $\mathcal{A}_*$ to $\mathcal{A}_*$ defined as, $\overline{V}([\rho]) = V([Re(\rho)]_h) + i.V([Im(\rho)]_h)$. We will say that V is completely positive if the dual of $\overline{V}$, say $\overline{V}^*$, is completely positive as a map from $\mathcal{A}$ to $\mathcal{A}$, i. e. for any $x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n$ in $\mathcal{A}$, $\sum_{i,j=1}^n x_i^*\overline{V}^*(y_i^*y_j)x_j$ is positive. This is again equivalent to,

$\sum_{i,j=1}^n tr(x_i^*\overline{V}^*(y_i^*y_j)x_j\rho) \geq 0$ for all positive $\rho \in \mathcal{B}_1^{s.a.}(h)$.

$\sum_{i,j=1}^{n} tr(y_i^* y_j \overline{V}([x_j \rho x_i^*])) \geq 0$ for all positive $\rho \in \mathcal{B}_1^{s.a.}(h)$.
$\sum_{i,j=1}^{n} tr(y_j \overline{V}([x_j \rho x_i^*]) y_i^*) \geq 0$ for all positive $\rho \in \mathcal{B}_1^{s.a.}(h)$.
It is easy to note that complete positivity is preserved under taking finite sum, composition and strong limit of operators. Now we prove in the following theorem that the minimal semigroup constructed in the present section is not only positive but also completely positive.

Theorem 6.1.15 *$\widetilde{T}_t^{(min)}$ is completely positive for all $t \geq 0$.*

Proof : It is enough to prove that for any $r \in [0,1)$ and $t \geq 0$, $\widetilde{T}_t^{(r)}$ is completely positive. For this, it suffices to verify that $(\lambda - \widetilde{G_r})^{-1}$ is completely positive, because $\widetilde{T}_t^{(r)}$ is strong limit of $(n/t)^n(n/t - \widetilde{G_r})^{-n}$ as $n \to \infty$. But, $(\lambda - \widetilde{G_r})^{-1} = (\lambda - \widetilde{Z})^{-1}\sum_{j=0}^{\infty} r^j \widetilde{B}(\lambda)^j$, and so it is enough to check that $(\lambda - \widetilde{Z})^{-1}$ and $\widetilde{B}(\lambda)$ are completely positive.

Now, for $x_1, \ldots, x_n; y_1, \ldots, y_n \in \mathcal{A}$ and positive $\rho \in \mathcal{B}_1^{s.a.}(h)$, we have that
$\sum_{i,j=1}^{n} tr(y_j \overline{(\lambda - \widetilde{Z})^{-1}}([x_j \rho x_i^*]) y_i^*)$
$= \int_0^\infty e^{-\lambda t} \sum_{i,j=1}^{n} tr(y_j P_t x_j \rho x_i^* P_t^* y_i^*) dt$
$= \int_0^\infty e^{-\lambda t} tr(\eta_t \eta_t^*) dt$ (where $\eta_t = \sum_{i=1}^{n} y_i P_t x_i \rho^{1/2}$.)
≥ 0.

Finally, we want to show that $\sum_{i,j=1}^{n} tr(y_j \overline{\widetilde{B}(\lambda)}([x_j \rho x_i^*]) y_i^*) \geq 0$. By density of $\mathcal{D}(G)$ in $\mathcal{H}$, we choose sequences $\rho_i^{(k)} \in \mathcal{B}_2(h)$, $i = 1, 2, \ldots, n$ such that

$(1 - G)^{-1}\rho_i^{(k)} \to x_i \rho^{1/2}$ in Hilbert-Schmidt norm as $k \to \infty$, for each i. Recalling the definition of $\widetilde{B}(\lambda)$ and proceeding as in the previous paragraph, we get
$\sum_{i,j=1}^{n} tr(y_j \overline{\widetilde{B}(\lambda)}([(1 - G)^{-1}\rho_j^{(k)} \rho_i^{(k)*}(1 - G^*)^{-1}]) y_i^*)$
$= \sum_{i,j=1}^{n} tr(\overline{\widetilde{B}(\lambda)}([(1 - G)^{-1}\rho_j^{(k)} \rho_i^{(k)*}(1 - G^*)^{-1}]) y_i^* y_j)$
$= \int_0^\infty e^{-\lambda t} \sum_{i,j=1}^{n} tr(R(1 - G)^{-1} P_t \rho_j^{(k)} \rho_i^{(k)*} P_t^* (R(1 - G)^{-1})^* \pi(y_i^* y_j)) dt$
$= \int_0^\infty e^{-\lambda t} \sum_{i,j=1}^{n} tr(\pi(y_j) R(1 - G)^{-1} P_t \rho_j^{(k)} \rho_i^{(k)*} P_t^* (R(1 - G)^{-1})^* \pi(y_i)^*) dt$
≥ 0, clearly.
This completes the proof. □

6.2 Conservativity of the minimal semigroup and its perturbation

We say that $(\widetilde{T}_t^{(min)})_{t\geq 0}$ is conservative if $\widetilde{T}_t^{(min)^*}(1) = 1 \ \forall\ t \geq 0$, where $\widetilde{T}_t^{(min)^*}$ denotes the dual of $\widetilde{T}_t^{(min)}$. The above is equivalent to $tr(\widetilde{T}_t^{(min)}([\rho]_h)) = tr([\rho]_h)\ \forall\ [\rho]_h \in \mathcal{A}_{h,*}$. We now want to describe some useful criteria for conservativity.

For $\lambda > 0$, let $\beta_\lambda \equiv \{\ x \in \mathcal{A}_h, x \geq 0 :< Ru, \pi(x)Rv > + < u, xGv > + < Gu, xv > = \lambda < u, xv > \forall\ u, v \in \mathcal{D}(G)\ \}$.

Also, for $\lambda > 0$ and for $x \in \mathcal{A}_h$, we define a bilinear form $E^{\lambda,x}(u,v) \equiv \int_0^\infty e^{-\lambda t} < u, P_t^* R^* \pi(x) R P_t v > dt$. To see that it is well defined, we first note that by the assumption A2 we have, $||Ru||^2 = -2Re < Gu, u >$ and $||Rv||^2 = -2Re < Gv, v >$. Thus, $| < RP_t u, \pi(x) RP_t v > | \leq ||\pi(x)|| ||RP_t u||.||RP_t v|| \leq 2||x||.|Re < GP_t u, P_t u > |^{1/2} |Re < GP_t v, P_t v|^{1/2}$. Hence, $|E^{\lambda,x}(u,v)|$

$\leq 2||x|| \int_0^\infty e^{-\lambda t/2} |Re < GP_t u, P_t u > |^{1/2}.e^{-\lambda t/2} |Re < GP_t v, P_t v > |^{1/2} dt$

$\leq 2||x||(-\int_0^\infty e^{-\lambda t} Re < GP_t u, P_t u > dt)^{1/2}$

$(-\int_0^\infty e^{-\lambda t} Re < GP_t v, P_t v > dt)^{1/2}$

$= ||x||.(-\int_0^\infty e^{-\lambda t} d/dt ||P_t u||^2 dt)^{1/2} (-\int_0^\infty e^{-\lambda t} d/dt ||P_t v||^2 dt)^{1/2}$

$\leq ||x||.||u||.||v||$.

This implies that the bilinear form is well defined and also it is bounded by $||x||.||u||.||v||$, which, by density of $\mathcal{D}(G)$ in h, gives that there exists a bounded operator $Q_\lambda(x)$ such that $< u, Q_\lambda(x) v > = E^{\lambda,x}(u,v)$ for all $u, v \in \mathcal{D}(G)$. By construction, $Q_\lambda(x)$ is self-adjoint for $x \in \mathcal{A}_h$ and it is positive if x is positive. Moreover, $||Q_\lambda(x)|| \leq ||x||$. We also claim that $Q_\lambda(x) \in \mathcal{A}_h$. For any $a' \in \mathcal{A}'$ and u, $v \in \mathcal{D}(G)$, say $u = (1-G)^{-1}u'$, $v = (1-G)^{-1}v'$, we have that,

$< u, Q_\lambda(x) a' v >$

$= \int_0^\infty e^{-\lambda t} < RP_t(1-G)^{-1}u', \pi(x) RP_t a'(1-G)^{-1}v' > dt$

$= \int_0^\infty e^{-\lambda t} < u', (1-G^*)^{-1} P_t^* R^* \pi(x) RP_t (1-G)^{-1} a' v' > dt$ (since $(1-G)^{-1}$ and a' commute by 6.1.4)

$= \int_0^\infty e^{-\lambda t} < u', a'(1-G^*)^{-1} P_t^* R^* \pi(x) RP_t(1-G)^{-1} v' > dt$ (as $(1-G^*)^{-1} P_t^* R^* \pi(x) RP_t (1-G)^{-1} = P_t^*(1-G^*)^{-1} R^* \pi(x) R(1-G)^{-1} P_t \in \mathcal{A}_h$ by 6.1.4)

$= \int_0^\infty e^{-\lambda t} < (a')^*(1-G)^{-1}u', P_t^* R^* \pi(x) RP_t (1-G)^{-1} v' > dt$

$= < (a')^* u, Q_\lambda(x) v >$

$= < u, a' Q_\lambda(x) v >$

By density of $\mathcal{D}(G)$ in h, we conclude that $Q_\lambda(x)$ commutes with a', and since $Q_\lambda(x)$ is also self-adjoint, so $Q_\lambda(x) \in \mathcal{A}_h$. Thus, $Q_\lambda : \mathcal{A}_h \to \mathcal{A}_h$ is a positive linear contraction.

Now we state the main result on conservativity.

Theorem 6.2.1 *The followings are equivalent : (i)* $(\widetilde{T_t}^{(min)})_{t\geq 0}$ *is conservative.*

(ii) $||\widetilde{T_t}^{(min)}([\rho]_h)|| = ||[\rho]_h||$ *for any positive* $[\rho]_h \in \mathcal{A}_{h,*}$.

(iii) For some (and hence for all) $\lambda > 0$, $(\widetilde{B}(\lambda))^n([\rho]_h) \to 0$ *as* $n \to \infty$, $\forall\ [\rho]_h \in \mathcal{A}_{h,*}$.

(iv) For some (and hence for all) $\lambda > 0$, $(\lambda - \widetilde{A})\widetilde{\mathcal{D}}$ *is dense in* $\mathcal{A}_{h,*}$, *where* $\widetilde{A}$ *is the generator of* $(\widetilde{T_t}^{(min)})_{t\geq 0}$.

(v) For some (and hence for all) $\lambda > 0$, $(\widetilde{A})^*x = \lambda x$ *has no nonzero solution in* $\mathcal{A}_h$, *where* $(\widetilde{A})^*$ *is the map dual to* $\widetilde{A}$.

(vi) β_λ *contains the only element* 0 *for some (and hence for all)* $\lambda > 0$.

(vii) $Q_\lambda(x) = x$ *has the only solution* $x = 0$ *for some (and hence for all)* $\lambda > 0$.

(viii) $\overline{X}_\lambda = 0$ *for some (and hence for all)* $\lambda > 0$, *where* $\overline{X}_\lambda = s - lim_{n\to\infty} Q_\lambda^n(1)$, *which exists and equals the maximal element of* $\beta_\lambda \bigcap \{0 \leq x \leq 1\}$.

The proof is omitted, since it is very similar to that for the case when $\mathcal{A}$ is $\mathcal{B}(h)$ ([43],[7], [47]).

It is easy to see that if the minimal semigroup is conservative then it is the only positive contractive strongly continuous semigroup with the prescribed generator. This fact follows in an exactly similar way as in [7] for the case of $\mathcal{B}(h)$. Moreover, when the minimal semigroup is not conservative, we can construct a family of "perturbed" semigroups each of whose generator extends $\widetilde{Z} + \widetilde{J}$. For a fixed real number m, we construct a semigroup $(\widetilde{T_t}^{(m)})_{t\geq 0}$. Fix a positive $[\omega]_h \in \mathcal{A}_{h,*}$ of norm 1, and define for $\lambda > 0$, $\widetilde{R_\lambda}^{(m)} \equiv (\lambda - \widetilde{A})^{-1}(I + H_\lambda)$, where I denotes the identity map on $\mathcal{A}_{h,*}$, and H_λ is defined as $H_\lambda([\rho]_h) = (m + 1 - \alpha_\lambda)^{-1} tr([\rho]_h \overline{X}_\lambda)[\omega]_h$; $\alpha_\lambda = tr([\omega]_h \overline{X}_\lambda)$. We denote by $\widetilde{R_\lambda}$ the resolvent of $\widetilde{A}$, i.e. $(\lambda - \widetilde{A})^{-1}$, and let $\widetilde{P_\lambda}$ be its dual. We have the following result.

Theorem 6.2.2 *(i)* $\overline{X}_\lambda = 1 - \lambda\widetilde{P_\lambda}(1)$, *and thus, the minimal semigroup* $(\widetilde{T_t}^{(min)})_{t\geq 0}$ *is conservative if and only if* $\widetilde{P_\lambda}(1) = \lambda^{-1}$.

(ii) For $\mu > 0$, $\mu \neq \lambda$, *we have,* $\widetilde{P_\lambda}(\overline{X}_\mu) = (\mu - \lambda)^{-1}(\overline{X}_\lambda - \overline{X}_\mu)$

(iii) $\widetilde{R_\lambda}^{(m)} - \widetilde{R_\mu}^{(m)} = (\mu - \lambda)\widetilde{R_\lambda}^{(m)}\widetilde{R_\mu}^{(m)}$.

(iv) $\widetilde{R_\lambda}^{(m)}$ *is one-to-one and it is the resolvent of a closed operator* $\tilde{A}^{(m)}$.
(v) For $m \geq 0$, $\|\widetilde{R_\lambda}^{(m)}\| \leq \lambda^{-1}$, *and hence* $\tilde{A}^{(m)}$ *is the generator of a positive, contractive,* C_0 *semigroup, which is conservative if and only if*
$m = 0$.
(vi) For $m \geq 0$, $\mathcal{D}(\tilde{A}^{(m)}) = \mathcal{D}(\tilde{A})$, *and for* $[\rho]_h \in \mathcal{D}(\tilde{A})$, $\tilde{A}^{(m)}([\rho]_h) = \tilde{A}([\rho]_h) - (m+1)^{-1}tr(\tilde{A}([\rho]_h))[\omega]_h$. *Moreover,* $\tilde{A}^{(m)}$ *and* $\tilde{A}$ *agree on* $\tilde{\mathcal{D}}$.

The proof is essentially identical to that of Theorem 3.1 of [47].

6.3 Application

To illustrate the above theory with a concrete example, let us consider a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$ and denote by $L^p(\mu)$ and $L^p(\mu)_{I\!R}$ the space of all measurable complex-valued (real-valued, respectively) functions on Ω with finite L^p-norm, for $p \geq 1$. We denote by h the Hilbert space $L^2(\mu)$ and by $\mathcal{A}$ the abelian von Neumann algebra $L^\infty(\mu)$, to be identified as multiplication operators on h. It is well-known that $\mathcal{A}_{h,*} \cong L^1(\mu)_{I\!R}$. Our aim is to construct the minimal semigroup when the generator is given by,

$(\mathcal{L}(\varphi)f)(\omega)$
$= (\int a(\omega, z)(\varphi(z) - \varphi(\omega))\mu(dz))f(\omega),$

whenever the right hand side exists ($\varphi \in \mathcal{A}$ and $f \in h$). Its formal predual is given by, $(A\psi)(x) = \int_\Omega a(y,x)\psi(y)\mu(dy) - (\int_\Omega a(x,y)\mu(dy))\psi(x)$, whenever the right hand side makes sense for $\psi \in L^1(\mu)$. We assume that $a : \Omega\times\Omega \to [0,\infty)$ is measurable and $\int a(x,y)\mu(dy)$ is finite for almost all x. This is the obvious generalization of classical semigroups studied by Feller [20] and Kato [31] where μ was chosen to be the counting measure on the set of positive integers.

Let us consider the Hilbert space $h \otimes k$, where $h = L^2(\mu(d\omega))$, $k = L^2(\mu(dz))$ and $\mathcal{K} = h \otimes k \cong L^2(\mu(d\omega) \otimes \mu(dz))$, where $\mu \otimes \mu$ denotes the measure-theoretic product of two copies of μ. Let $< .,. >_h$ and $< .,. >_{\mathcal{K}}$ denote the inner products in $\mathcal{H}$ and $\mathcal{K}$ respectively, whereas $\| \|_h$ and $\| \|_{\mathcal{K}}$ be respective norms. Define $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ by, $(\pi(\varphi)F)(\omega, z) = \varphi(z)F(\omega, z)$; $z, w \in \Omega$, $F \in \mathcal{K}$. It is easy to verify that π is a normal, *-representation. Note here that the representation chosen is *not* $(\hat{\pi}(\phi)F)(\omega, z) = \phi(\omega)F(\omega, z)$. This change in the choice of representation has deeper implication at the level of stochastic dilation as was observed in [44], (see also [41]) where it was shown that this necessitates introduction of a unitary operator-valued number process. Now, define $R : h \to \mathcal{K}$ as follows :

Let $\mathcal{D}(R)$ be the set of all f in h such that $\int\int a(\omega,z)|f(\omega)|^2\mu(d\omega)\mu(dz)$ is finite: and $(Rf)(\omega,z) = \sqrt{a(\omega,z)}f(\omega)$ for $f \in \mathcal{D}(R)$.

Theorem 6.3.1 *The operator* R *is densely defined and closed; and* $\mathcal{D}(R^*)$ *contains the set* $\mathcal{D}$ *of all* $G \in \mathcal{K}$ *such that* $\int \sqrt{a(.,z)}G(.,z)\mu(dz)$ *is in* $L^2(\mu)$. *For* $G \in \mathcal{D}$, $(R^*G)(\omega) = \int \sqrt{a(\omega,z)}G(\omega,z)\mu(dz)$; *and the set* $\hat{\mathcal{D}} \equiv \{f \in \mathcal{D}(R) : Rf \in \mathcal{D}\}$ *is a core for* $R^*\pi(\varphi)R$ *for any positive* $\varphi \in \mathcal{A}$. *Moreover, if we put an additional restriction, namely* $sup_{\omega\in\Delta}\int_\Delta a(\omega,z)\mu(dz) < \infty$ *for all measurable set* Δ *having finite* μ*-measure, then* $\mathcal{D}(R^*)$ *and* $\mathcal{D}$ *coincide.*

Proof : Let us denote by $g(\omega)$ the function $\int a(\omega,z)\mu(dz)$. Clearly, as $\mu\{\omega : g(\omega) = \infty\} = 0$, the linear span of elements of the form $f\chi_{\{\omega:g(\omega)\leq n\}}$, where n is any positive number and $f \in L^2(\mu)$ is dense in $L^2(\mu)$, where χ_B denotes indicator of B. But $f\chi_{\{\omega:g(\omega)\leq n\}} \in \mathcal{D}(R)$ for $f \in L^2(\mu)$, $n \geq 1$; which proves that R is densely defined. To see that R is closed, suppose that a sequence f_n in $\mathcal{D}(R)$ converges (in $\|\ \|_h$) to $f \in L^2(\mu)$ and Rf_n converges in $\|\ \|_{\mathcal{K}}$ to $\psi \in \mathcal{K}$. Then, we can choose a subsequence $(n_k)_{k=1,2,...}$ such that $f_{n_k} \to f$ a.e. (μ) and $Rf_{n_k} \to \psi$ a.e. $(\mu \otimes \mu)$. But, $\forall\ z \in \Omega$, $(Rf_{n_k})(\omega,z) = \sqrt{a(\omega,z)}f_{n_k}(\omega) \to \sqrt{a(\omega,z)}f(\omega)$ for almost all ω; and hence $\psi(\omega,z) = \sqrt{a(\omega,z)}f(\omega)$ a.e. $(\mu \otimes \mu)$. Since $\psi \in L^2(\mu \otimes \mu)$, it is clear that $f \in \mathcal{D}(R)$, which proves closedness of R.

Now we want to show first that $\mathcal{D} \subseteq \mathcal{D}(R^*)$. Suppose $G \in \mathcal{D}$. It is easy to see that if $\int \sqrt{a(\omega,z)}G(\omega,z)\mu(dz)$ is denoted by $h(\omega)$, then $h \in L^2(\mu)$ by hypothesis and $\forall\ f \in \mathcal{D}(R)$,

$< f,h >_h = \int\int \bar{f}(\omega)\sqrt{a(\omega,z)}G(\omega,z)\mu(dz)\mu(d\omega)$

$= \int\int \bar{f}(\omega)\sqrt{a(\omega,z)}G(\omega,z)\mu(d\omega)\mu(dz)$

$= < Rf,G >_{\mathcal{K}}$, where the interchange of the order of integration is justified because $\int\int \sqrt{a(\omega,z)}|\bar{f}(\omega)G(\omega,z)|\mu(dz)\mu(d\omega) \leq \|Rf\|_{\mathcal{K}}\|G\|_{\mathcal{K}} < \infty$, by Cauchy-Schwarz inequality. This proves $G \in \mathcal{D}(R^*)$ and $(R^*G)(\omega) = \int \sqrt{a(\omega,z)}G(\omega,z)\mu(dz)$.

To show that $\hat{\mathcal{D}}$ is a core for $R^*\pi(\varphi)R$ for positive $\varphi \in \mathcal{A}$, we consider the semigroup $(L_t)_{t\geq 0}$ where $L_t = e^{-tR^*\pi(\varphi)R}$. It is enough to show that for each t, $L_t\hat{\mathcal{D}} \subseteq \hat{\mathcal{D}}$, which is easy to verify.

Now, assume also that $sup_{\omega\in\Delta}\int_\Delta a(\omega,z)\mu(dz) < \infty$ for all Δ with finite μ-measure. Let $G \in \mathcal{D}(R^*)$. Then, $\exists\ \varphi \in h$ such that $< Rf,G >_{\mathcal{K}} = < f,\varphi >_h$

$\forall f \in \mathcal{D}(R)$. But $< Rf, G >_{\mathcal{K}} = \int \bar{f}(\omega)p(\omega)\mu(d\omega)$, as we have already computed (where $p(\omega) = \int \sqrt{a(\omega,z)}G(\omega,z)\mu(dz)$). Thus, $\int \bar{f}(\omega)(p(\omega) - \varphi(\omega))\mu(d\omega) = 0$, $\forall f \in \mathcal{D}(R)$.

Partitioning Ω into disjoint sets of finite μ-measure, say $\{\Omega_n\}_{(n=1,2,\ldots)}$ and choosing f to be $\chi_{\Omega_n \cap \{\omega:\varphi(\omega)-p(\omega)\geq\epsilon\}}$ for $\epsilon > 0$, we can deduce from the above identity that $\mu(\{\omega : \varphi(\omega) - p(\omega) \geq \epsilon; \omega \in \Omega_n\}) = 0$. Note that this argument requires $\chi_{\Omega_n \cap \{\omega:\varphi(\omega)-p(\omega)\geq\epsilon\}}$ to belong to $\mathcal{D}(R)$, which is a consequence of the assumption that $sup_{\omega\in\Delta}\int_\Delta a(\omega,z)\mu(dz) < \infty$ for all Δ with $\mu(\Delta) < \infty$. Similarly, one obtains that $\mu(\{\omega : \varphi(\omega) - p(\omega) \leq -\epsilon; \omega \in \Omega_n\}) = 0$ for all n and positive ϵ. Thus, $\varphi = p$ a.e. (μ), and hence $p \in L^2(\mu)$, proving $G \in \mathcal{D}$. This completes the proof. □

In order to incorporate the present example into the framework of the general theory developed in section 2, one only has to identify the generator G with $-1/2R^*R$, which is a multiplication operator by the measurable function $-1/2\int a(.,z)\mu(dz)$ and hence is affiliated to $\mathcal{A}$. Similarly, $R^*\pi(\varphi)R$ can be seen to be a multiplication operator by the measurable function $\int a(.,z)\varphi(z)\mu(dz)$ and hence is affiliated to $\mathcal{A}$. Thus, for positive $\varphi \in L^\infty(\mu)$ and $f \in \hat{\mathcal{D}}$, it is easy to verify that

$$((R^*\pi(\varphi)R - (1/2)R^*R\varphi - (1/2)\varphi R^*R)f)(\omega)$$
$$= ((R^*\pi(\varphi)R - \varphi R^*R)f)(\omega)$$
$$= (\int a(\omega,z)(\varphi(z) - \varphi(\omega))\mu(dz))f(\omega)$$
$$= (\mathcal{L}(\varphi)f)(\omega).$$

If we make the assumption that $sup_{\omega\in\Delta}\int_\Delta a(\omega,z)\mu(dz) < \infty$ for all Δ with finite μ-measure, then the above identity holds for all f in the domain of R^*R, which is the same as the domain of $R^*\pi(\varphi)R - \varphi R^*R$ for any $\varphi \in \mathcal{A}$ in this case.

Example 2

Consider another example where our theory works on a von Neumann algebra which is a type II_1 factor. (so it is neither commutative nor of the form $\mathcal{B}(\mathcal{H})$.)

We fix some irrational number θ and consider $h = L^2(\mathbb{R})$ and the C^* algebra $\mathcal{A}_\theta$ generated by the unitaries U and V where $(Uf)(s) = f(s+1)$, $(Vf)(s) = e^{2\pi is\theta}f(s)$. In this case U and V obey the commutation relation $UV = e^{2\pi i\theta}VU$. It is known [11] that the double commutant $\mathcal{A}_\theta''$ is a type II_1 factor in $\mathcal{B}(h)$. Let us consider a canonical derivation δ with the domain $\mathcal{D}$ consisting of all polynomials in U and V, and given by, $\delta(U) = U$, $\delta(V) = 0$. It is easy to see that $\delta(X) = [R, X]$, for $X \in \mathcal{D}$, where $(Rf)(s) = -sf(s)$ for all $f \in L^2(\mathbb{R})$ such that $sf(s)$ is also in $L^2(\mathbb{R})$. We note that R is affiliated to $\mathcal{A}_\theta''$, which follows beccause for each integer n, $\mathcal{A}_\theta$

contains multiplication by $e^{2\pi isn\theta}$ (which is nothing but V^n) and since θ is irrational, multiplication by e^{ias} belongs to $\mathcal{A}_\theta''$ for any $\alpha \in I\!R$. Thus, $R^*\pi(x)R = R^*xR\eta\mathcal{A}_\theta''$. We now take the formal expression $\mathcal{L}(X) = [R, [R, X]] = R^*XR - \frac{1}{2}R^*RX - \frac{1}{2}XR^*R$, for $X \in \mathcal{D}$, where $R^* = R$. It is easy to see that $\mathcal{L}(U^mV^n) = -m^2U^mV^n$.

Remark 6.3.2 *The probelm of stochastic dilation of the minimal semigroup for the case when h is separable can be dealt with exactly identically as was done in the case of $\mathcal{B}(h)$ by [40] and [19].*

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