A CONTRIBUTION TO INVENTORY THEORY

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[redacted]

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To my parents

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A CONTRIBUTION TO INVENTORY THEORY

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PART I : FOUNDATION

Chapter 1

FRAMEWORK AND PROBLEMS

We have called this thesis simply a contribution to inventory theory. Much obviously remains to be stated. However, at this point we can only say that the contribution is essentially methodological in nature (or so, at least, it is intended to be) and consists in itself simply of a series of logical constructions devoted to this end. We will be able to be more concrete about the nature of our objectives after we have stated the background we have started from in writing the thesis. But even this has to wait for the moment till we are through with a few clarifications of the content.

1. The first point to be stated is that we concern ourselves exclusively with socalled inventory problems of a trader, i.e., a person actually meeting the demand for a product by first buying it from a source or origin and then selling it over —— these institutional specifications are just built into our framework from the very beginning.

Though our choice of the trader as reference point necessarily restricts the scope of the whole exercise, it is not entirely arbitrary but is rooted in the very nature of inventory problems as they are stated in the subject. The basic problems are stated simply as ''when to replenish the inventory and how much to order for replenishment'', it being asserted that ''essentially every decision which is made in controlling inventory in any organisation'' is in one way or another related to these two questions[1]. Our point now is simply that these questions arise almost definitionally in the context of trade, i.e., the trader's activity as just described[2]. Their 'application' to other contexts, e.g., manufacturing or production in general, appears to rest ultimately on thinking of the context itself by analogy from trade. The logical starting point of inventory theory as it exists is therefore simply the trader.

2. Having come this far with the subject as it exists (also called the standard or conventional theory), we have to state that the questions which we ourselves come to set up as inventory problems for the trader will be somewhat different from the two questions of Hadley and Whitin. This is rooted simply in our seeing the whole activity of 'trade' in a different framework as compared to the existing theory. The framework orginates in the ''background'' referred earlier. We will come to this in the due course. For the present, we point out that our precise approach to inventory problems (in a sense of the word presently clarified) will also be

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1/ G. Hadley and T.M. Whitin, Analysis of Inventory Systems, p. 1. This will be our standard reference for the existing theory in the thesis.

2/ However the activity as just described does not imply that the trader in fact sells from a stock or inventory of his goods, which is simply presumed in these questions. Hence we say 'almost definitional'. Obviously, the existence of stock or inventory is prior to all ''inventory problems'' and so basic to the subject. We will take up the questions in this light later on.

somewhat different from the existing theory, having its roots again in the framework referred. It appears best to start here from the approach and then work out the framework and problems, taking up the ''background'' inter alia of this.

3.1 Let us clarify the meaning of the word ''approach'' in the present context. Above, we have understood an inventory problem simply as a decision problem for the trader, concerned in some way or other with the stock of his goods or rather the maintainance of this stock[3/]. Nothing is stated in this about the decision criterion or objective function for resolving the decision. ''Approach'' here is understood simply as choice of the decision criterion. In the subject as it exists, the criterion is left simply as minimum cost as per a definition of cost deemed appropriate for the purpose. We will go beyond this and take the criterion to be maximum rate of profit (profit per unit of time per unit of capital) as per appropriate definitions of the two terms 'capital' and 'profit'.

3.2 Let us proceed to some elementary clarifications of the decision criterion or objective function. The trader is obviously interested in the profit earned from his business. But he has to put in a certain capital for the profit and so his decision criterion is logically defined

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3/ If the stock were not in some sense to be maintained, the question of its 'replenishment' would not have simply arisen.

in terms of the rate of profit, not just profit[4]. This would not have mattered if capital could be treated simply as given a priori for the problems concerned. This however is not the case with us. Capital is not only an ''unknown'' in the general structure of an inventory problem, it is in fact the fundamental decision variable through all these problems, and therefore also the fundamental binding factor of all of them. As this suggests, the logical starting point for our framework is defined by capital. Let us however be through with the points regarding the decision criterion before coming to this.

3.3 Let us return for a minute to the notion of cost in conventional inventory theory. The concept is invariably defined to include a 'cost' of capital reflecting its earning in alternative lines of investment (other than in the 'process' under reference), called the interest charge/or cost[5]. This is only an indirect or roundabout way of reckoning a point — that capital yields profit, — which we incorporate directly in our decision criterion[6]. As a corollary we simply cut out the ground from under the socalled ''interest cost'' in our framework, and do not simply meet it any more, except for purpose of comparison with the standard theory.

4/ Stated somewhat informally, our point here is that an absolute magnitude of profit means little to a trader unless he is told what capital he has to put in for it. This already implies the criterion of rate of profit for the comparison of alternative policies.

5/ See, e.g., Hadley and Whitin, p. 13.

6/ A straightforward formal way of recognising this is that since capital occurs in the denominator of rate of profit, a larger value of capital is always penalised in our decision criterion, making it superfluous to include a separate interest charge for this purpose. This also does away with an arbitrary parameter in the general cost function, viz., the rate of interest, which is a pure analytical gain.

3.4 A second point about the decision criterion is as follows. As will be seen, inventory problems by their nature focus attention upon uncertainties faced by the trader in carrying out his activity, implying that the profit made by him is also ultimately an uncertain magnitude. It is therefore meaningless to talk of the maximum profit or rate of profit. Our first step in this context will be to translate the ''uncertainties'' into the language of probability theory so that we can use its ''tools'' for our purpose. This defines an important part of the whole framework. Granting the step, we automatically convert profit into a random variable in the sense of probability theory so that we can speak of its expectation in this sense(probability theory). With this, we simply define the decision criterion to be the maximum expected rate of profit. This is taken to be understood as part of the appropriate definition of the term 'profit' for our purpose.

4.1 Let us now begin on the framework. Let us start back from the definition of a trader as a pure re-seller of goods. We can say that the 'boundary' of this process (re-selling) is set simply by the 'buying' on the one hand and 'selling' on the other hand of the trader, in the sense that the goods simply cross this boundary from the 'outside' to 'inside' and vice versa through the 'buying' and 'selling' respectively. More simply, buyings are flows into and sellings flows out of the process (or 'inflows' and 'outflows' respectively)[7/]. We. note at once that the buyings and sellings are also simultaneously money-flows in the opposite direction of the goods-flows, so that we have both a financial or monetary plane of the process as well as a purely physical plane underlying it. This is important to recognise from the beginning, for such terms as 'capital', 'cost', 'profit', i.e., the whole decision criterion, is defined only on the monetary plane. One important point of our framework is a complete integration of these two planes of the process, which is completely absent in the conventional framework which simply conceives the process on a pure physical plane and then merely transforms its variables into monetary magnitudes through an exogeneous process of costing or valuation, etc., — the financial plane as such is never encountered.

4.2 Now, once the 'inside' and 'outside' of our process are clearly distinguished, one can approach the process itself from inside or from outside, i.e., pass from inside to outside or vice versa (and thus build up a total picture). We will adopt the former approach for this takes us at once to the heart of the process.

7/ We must acknowledge our debt to N. Georgescu-Roegen for our understanding of the methodology of the very concept of ''process'', in particular the fundamental analytical significance of the notion of ''boundary'' for this purpose. The criterion of ''crossing of the boundary'' for defining the 'flows', i.e., inflows and outflows, are also from him. However he himself used the term 'flow' in a somewhat different sense. The concepts inflow and outflow defined here correspond to his concept of 'input' and 'output' defined in reference to a production process. See : N. Georgescu-Roegen, The Entropy Law and the Economic process, pp. 211-5 .

This 'heart', we can say, consists simply of a two-way connection between the 'buying' and the 'selling' (or sale and purchase) of the trader. The connection of 'selling' to 'buying' is purely definitional, for it is only the goods bought that are sold over by the trader, i.e., the connection is implicit in the very notion of re-sale. This is on the physical plane. The reverse connection occurs on the financial-plane and consists simply of the fact that the goods bought are paid for ultimately out of the sale-proceeds realised through the process[8/]. This is also the basic point of our integration of the goods and the money plane of the process.

4.3 Let us now see the above interconnection in time. This is where the stock comes in[9/], for once we have the trader as selling out of a stock, his purchase is also into the stock, and the whole connection of purchase to sale is via the stock. There cannot therefore be any straight connection between the quantities of sale and purchase through time. The same point also holds true on the financial plane leading to some significant new concepts.

8/ More generally, we can say that all expenses actually incurred by the trader for his 'process', not just the payment for the goods, are financed ultimately out of the sale proceeds. As will be seen later, the point is of some importance for the setting up of inventory problems. For the time we implicitly take the expenses or actual cost to be defined by the payment for goods bought.

9/ We are still simply presuming the existence of a stock of goods in the process of trade, the logical validity of which we look into later on.

The basic point is simply that the successive purchases must be financed out of sale proceeds already realised. We therefore visualise the trader as laying aside or setting apart part of the sale-proceeds as they come in for future purchase, and call it the ploughback from sale-proceeds (or just the 'ploughback' for short)[10/]. Till spent, the ploughback then simply accumulates into a 'fund' inside the process, and the purchase is actually financed from this fund which we call the purchase fund of the trader. Thus just as the sales are connected to purchase only via the stock of goods —— which we may in parallal call the sales stock[11/] —— so also the purchases get connected to sales via a stock of money, viz. the purchase fund.

4.4 We can now take up the notion of capital. Intuitively, capital is the sum of money setting up the process. We can view this ''setting up'' at its simplest as simply a purchase which itself converts the capital from its original 'money form' to a 'goods form', viz., the stock of goods created by the purchase. However, as just noted, the matter does

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10/ This can however remain as a purely notional division of sale-proceeds till either the purchase is actually made or the profit is actually taken out of the process. The significance of this point will become clear later.

11/ The term is appropriate for the stock is held precisely for the purpose of sale, and the sale is in fact from the stock. Exactly parallel points also apply to the term, purchase-fund.

not end there. As the goods are sold from the stock, a purchase-fund is simultaneously created inside the process by the ploughback from sale-proceeds, and so capital simply continues to exist in both a money form (purchase fund) and a goods or physical form (the sales stock). Viewed purely from inside, the process itself then appears purely as a process of mutual transformation of these two forms of capital, effected respectively by the buying and sellings.

4.5 We will continue with this 'inside view' of the process in the body of the thesis. At this point we just mention that our fundamental assumption on the process itself will be that capital is maintained intact through the process so that we can speak of it as a determinate magnitude (or constant) of the process. It is precisely because of this that we can meaningfully speak of it as a decision variable.

5.1 We are now in a position to state the background of our work. The genesis of the 'inside view' of the process of trade set out above, and so also the notion of capital in its two forms, goes back all the way to Adam Smith, the relevant passage being as follows :

> ''The goods of the merchant yield him no revenue or profit till he sells them for money, and the money yields him as little till it is again exchanged for goods. This capital is continually going from him in one shape, and returning to him in another, and it is only by means of such circulation, or successive exchanges, that it can yield him any profit''.12/

12/ Adam Smith, The Wealth of Nations, pp. 262-3, (Modern Library Edition).

However, we have ourselves come to know of this only in the second hand. Our direct source is a detailed analytical ''reconstruction'' of this passage by one of our colleagues in the Institute, P. Gajapathi, in the course of his research for the Ph.D. degree which regrettably is not yet finished[13/]. The whole conceptual framework set out above, including the notions of 'sales stock', 'purchase fund' and 'ploughback', is straight from this work. A more detailed statement of this framework is given in Chapter 3 below (Sections 1 and 2) from which we ourselves take off in our own direction of interest (inventory theory) which in turn requires some specific extensions of the framework itself. These are taken up in Section 3 of the chapter referred. This sets out the fundamental background to our work.

5.2 The work of another colleague of ours in the Institute, V. Narasimhan, also comes into the background[14/]. She too took off from the work of P. Gajapathi and then developed the matter as ''tool'' for her subject of price formation (of foodgrains, in which traders were a priori deemed to play a ''central role''). This focussed attention on the outside of the

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13/ P. Gajapathi, Wage, Capital, Value and Rate of Profit : Reconstruction of the Classical Framework (unpublished manuscript, Indian Statistical Institute, 1986), Chapter 3 (''Merchant's Capital and the Process of Trade — Smith's Basic Process-View of Capital'').

14/ V. Narasimhan, An Essay on the Formation and Dynamics of the Marketed Surplus and Price of Foodgrains (unpublished Ph.D. thesis, Indian Statistical Institute, 1986), Chapter 5 (''A Digression into the Process of Trade'').

process (e.g., demand and supply, competition, structure of the market, etc.) which is of obvious interest for our purpose, and her formulations in this regard again provide us with 'starting points' in the thesis.

6.1 This sets out the full background we have started from in writing the thesis. Our basic objective has been simply to carry on with the basic line of reasoning or mode of argument met in this background[15/] into the area of inventory theory, developing alongside the tools and concepts needed for this purpose. By the very nature of this effort, we have a fresh look at socalled inventory problems of the trader. These are eventually formulated as a fresh sequence of inventory models which we contribute to the literature. Let us just mention that the tools and concepts just referred can, we believe, serve the purpose of inventory theory in general without being necessarily tied down to these ''models''. They belong simply to the approach and framework taken.

6.2 Thus the thesis as a whole breaks up into two broad parts which we call simply the ''foundation'' and the ''models''. The former consists of the approach and framework (this chapter) as well as the detailed construction of the basic tools and concepts already suggested by the framework (Chapters 2,3,4) and the latter of the sequence of inventory models.

15/ It is difficult to write about this a priori. Let us just say that it consists of seeing things clearly in time and therefore necessarily beginning from some explicit stationarity postulate. Let us add that for our purpose it has not been necessary to step out of the stationarity postulates introduced (e.g., that the trader maintains intact his capital through his buyings and sellings).

6.3 This clarifies the basic intents and contents of the thesis. Let us just add one more clarification though this may be redundant. We grant completely the 'practical nature' of our subject. However, this does not free one of the necessity of examining the logical foundations of the subject. Our whole exercise takes place at this plane. There would be a long way to go from this plane to the plane of practical interest. This is explicitly admitted beforehand.

7.1 Let us now resume our development of the framework. The first question that comes up before we can move into our subject is simply the existence of stock (see footnote 2 above). We will however treat the question and the nature of inventory problems as an essentially single ''methodological unit'' of the subject. This is simply because any logical basis of the existence of stock already makes room for certain inventory problems (quite possibly precluding others), and so the two necessarily go together.

7.2 We have to begin from the outside of the process, i.e., the 'outside' faced by the trader in his buying and selling. Let us begin with the 'selling' where this has a clearcut significance. This is because any sale by the trader is but a ''meeting'' of some demand come from outside. Let us visualise this ''coming'' of demand as simply 'customers' coming to our trader and presenting their 'demands' before him to meet. We note in the passing that these particular demands are in their very nature random events in a pure a priori sense — nothing, even in theory, can be hypothesised about 'when' precisely a particular customer would come and what precisely (what quantity) he would demand. So any meaningful hypothesis about demand has to be about some notion of demand in the total or as a mass summing over the 'particular' demands. This is an important point of our total conceptual framework and we simply took the first opportunity of mentioning it. The point itself is to be developed later on.

7.3 Let us now look into the relation between sale and demand. We assume simply that the trader can sell only as and when particular demands are placed before him, and in these quantities[16]. Stated differently, he cannot postpone the meeting of a 'demand' to any point be chooses. If he cannot meet a demand right then and there, then the demand is simply lost to him — it goes to some other trader to be satisfied. This is the view of the market from which the demand comes. We grant that there can be cases where postponement of meeting a demand is possible. However, such cases we think are only exceptions to the 'general rule' stated above, exceptions in a sense only proving the rule. So, we keep it out of our framework. This is justified as logical starting point for the very notion of inventory problems.

16/ We can call this somewhat loosely the ''sales rule'' of the process — loosely because nothing is stated in it about the actual sales. We will fix the term in a more precise sense in the body of the thesis.

7.4 This again is a point of departure from the existing structure of inventory theory where both cases are simply allowed on par, called respectively the ''lost sales case'' and the ''backorders case''. This done, the analysis itself proceeds in the main through the second case on the ground that ''it is much easier to treat the backorders case than the lost sales case''[17]. No comment is necessary.

7.5 Let us now turn to the purchase side of the trader. Suppose we leave the outside as completely ''passive'' or ''neutral'' here, in the sense that the trader is left free to buy any quantity at any point of time he chooses — the 'outside' (or ''market'') simply supplies these quantities, and the matter ends there. Suppose also that there is no 'lag' between the purchase or purchase-decision (the socalled ''order'') and the actual flow of the goods into the process (i.e., the ''delivery'' or arrival of the goods)[18]. Under these circumstances, the trader can buy exactly as and when the 'demands' occur, and precisely in those quantities, and thus ''meet'' the whole demand through time without any carrying of 'stock'.

17/ See: Hadley and Whitin, p.9.

18/ This particular supposition (no delivery lag) is maintained all through the thesis. It constitutes an important difference between the structure of our inventory problems and those of the existing theory where the delivery lag is simply an axiom all through. Reasons for our abstraction from the delivery lag are given later.

7.6 This brings us to the <u>logical problem of existence of stock.</u> The problem consists simply of precluding or ruling out the above case on some logical ground. If we keep to all the suppositions made above, the 'ground' must lie <u>inside</u> the process in the sense of the trader himself <u>choosing</u> to operate on the basis of stock even though he can do without. Alternatively the 'ground' may be placed <u>outside</u> the process by dropping the assumption of a ''passive'' outside in the sense understood above (i.e., on the purchase side of the trader). We shall now proceed along the latter route picking up the former at a later point. This is simply to define a logical, or methodological, ordering of issues from our point of view.

8. Our outside basis of the existence of stock consists simply of this. We visualise our trader as buying his goods in a <u>periodic market</u> which by difinition meets at a <u>fixed, regular interval through time</u>, say every Monday and therefore at a ''weekly'' interval. Clearly, then, his whole sale through any such ''week'' must be made out of a <u>stock at the beginning</u>, more precisely the <u>stock immediately after his purchase at the ''beginning'' of the week</u> (i.e., Monday) <u>when the market meets.</u> To complete the picture, we may simply suppose that the trader remains ''closed'' to sale at the time of purchase (say Monday mornings) so that there is a complete disjunction between his sale and purchase points in time. Any sale then must be made out of a stock created by past purchases. Existence of stock is here simply synonymous with the non-vacuity of the process, i.e., the fact of any sale at all.

Having spelt out the case, let us be clear that we mean it as no more than something purely provisional in our framework, removable (i.e., substitutible by other assumptions) after it has served its purpose. The 'purpose' consists simply of defining a meaningful inventory problem for the trader in the simplest possible way. Let us now come to this.

9.1 To clarify the conceptual roots of the problem we have in mind, we start back from the mere fact of selling out of a stock. It is granted that the stock is also ''bought into'', i.e., replenished. But the fundamental conceptual point about stock is simply that it is carried over from the past and therefore appears as a given at any given moment. By this very fact, we now have an inside condition upon 'sale', viz., the stock at any given moment has to be sufficient to meet the demand that may be placed at that moment; otherwise the sale is simply lost or foregone. Foregoing 'sale' is foregoing the profit that would have been made upon the sale, which is certainly a ''loss'' to the trader, the very prospect of which he would like to avoid as far as possible[19].

19/ The point however needs be looked into at greater depth. Granting that profit is made only upon sale, it remains the point that the profit that is made upon the sale of any given time depends upon the selling price then obtaining. Should the trader expect a rise in the price in the future, the same profit motive may induce him not to sell now in order to be able to sell in the future at a higher price and consequently greater profit. ''Not to sell'' here means simply foregoing some sale that would have been made, i.e., turning down the 'demands' concerned, given a stock to sell from which is now 'saved' for sales in the 'future'. This is pure speculation, speculation by means of withdrawal of stock from current sale.

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In this, we already have the genesis of an inventory problem for the trader. To actually make out the problem however we must step out of the momentary view of stock, sale and demand just taken. We must view them over time retaining intact in some meaningful way the very notion of 'sales foregone'. This is precisely our whole motivation for the assumption of a periodic market for the trader to buy from.

9.2 Let us see how the objective is accomplished. Let us focus our attention upon a particular ''week'', i.e., the period from one particular ''meeting'' of the market to the next. The trader begins the period with a certain purchase which (the amount bought) just gets added to the stock brought over from the previous period giving him a total stock for the period, which we may also call simply the ''stock begun with''. (This expression is thus understood synonymously as the ''stock immediately after the purchase''.) He then simply waits for the customers to arrive with their demands in order to make his sales through the period or week.

Contd. Footnote 19

Having come this far, we now simply state that we do not simply bring in speculation in any form into our framework of trade process here. So the whole speculative motive of stock holding is also left out. Going down one step further, we simply assume given and unchanging prices in our framework all through, so that there is no basis for speculation. This is what justifies our considering 'sales foregone' to be necessarily a ''loss'' to the trader.

The points made in this footnote are all taken from the thesis of V. Narasimhan referred earlier.

Suppose now, he knew beforehand what exactly the total demand over the period would be. He could have then simply bought at the beginning the precise amount required to meet this whole demand (which is nothing but the total demand itself minus the stock carried over from the previous period) and thus avoid the whole prospect of 'sales foregone'. However, such knowledge is not very meaningfully defined in the face of the uncertainty of demand already suggested by the randomness of 'particular demands'. Pending a more rigorous formulation of the concept, we give a more or less intuitive statement of the problem defined through it.

9.3 Let us start from capital and grant that the larger the capital, the greater is the stock begun with in any particular ''week''. Obviously, to that extent, the prospect of sales foregone is also avoided. Going to the extreme, let us supose that the prospect is completely avoided by putting in a large enough amount of capital. What return this fetches in terms of the expected profit for the ''week'' is however left completely in the open. There is therefore a genuine problem of the optimum capital as per our criterion of maximum rate of profit.

This roughly outlines the first substantive problem and corresponding ''model'' that we set up in the thesis. For the time, we refer to the model simply as Model I. To be in the clear, we point out that the whole model is defined in terms of only two basic substantive referrants viz., (a) the periodic market; and (b) the uncertainty of demand.

10.1 Let us now return to our framework and take some further steps in its development, all of which come into a rigorous formulation of the above problem. However, our direct reference now on will be the framework itself and not the model which in turn comes in mainly as ''illustration'' of some general point.

Methodologically the basic step is this. Above, we have argued in terms of a particular 'week' culled out of the succession of weeks and given a status in itself. This is not our method in the thesis. We consider any particular week to be essentially the same as any other for the trader and argue in terms of the general conditions of the process. We can call this the fundamental stationarity postulate underlying our whole framework and methodology.

10.2 On the ''inside'', the postulate is already reflected in our condition that capital is maintained intact through the process. Having said this, let us just point out that we do not assume this condition a priori in the sense of imposing it upon the 'process' from outside. We derive it from more basic postulates on the inside working of the process. This can however be clarified only through a fuller discussion of the concept of capital, which we give in Chapter 3 of the thesis.

Now, the condition that capital is maintained intact through the process sets only a framework for possible timepaths of the sales stock within capital. The timepath itself has to be thought of on its own within the framework. This takes us further inside the complete framework of capital to be developed in the chapter just referred. For our purpose here we simply state an end-proposition of this development, viz. in our framework, we have a one-to-one correspondence between (a) the capital put into and maintained through the process and (b) the sales stock immediately after purchase.

10.3 Let us now point out the implications of this proposition. First, since capital in our framework is constant through time (this is the same as the condition of maintaining capital intact), so is the stock immediately after purchase. Stated in more operational terms, this means that every purchase simply restores the stock back to a predefined level (defined in reference to capital). This gives us a precise notion of maintenance of stock where we can say that the 'level' at which the stock is maintained is simply the level of stock immediately after a purchase. More simply, we can also call it the 'stock maintained' (or stock maintained through the successive purchases). Thus all the notions, ''stock maintained'', ''stock immediately after a purchase'' and ''stock begun with'' are simply equivalent in this framework, and we can pass from one to the other depending upon the context.

10.4 Second, we have already stated that we have capital as the fundamental decision variable of the trader in our framework. Given the one-to-one relation between capital and stock maintained, we can obviously equivalently treat the latter as the decision variable.

Let us take off a minute here to clarify the logical structure that goes with treating the ''stock maintained'' as decision variable in terms of our Model I. Suppose one did cull out a particular ''week'' from the succession of ''weeks'' and pose the problem in respect of it. By the very procedure, one is left on the one hand with the stock carried over from the previous week as a pure datum and on the other with the amount bought in the week as the only meaningful decision variable. However, we have already given up this framework. In our framework, we can directly think of the stock begun with, i.e., the stock maintained through purchases as decision variable, leaving in the open the precise status of its two components (amount bought and amount carried over from the past). We can now close this point by saying that in the problem considered, both the components appear as endogenous variables, in fact as random variables reflecting ultimately the randomness or uncertainty of the environment. No meaning can therefore be given to the amount bought as a decision variable.

Let us return for a minute to the framework of inventory theory as it exists. In it, the ''amount bought'' is simply postulated a priori to be a decision variable of the trader all through, i.e., in all inventory problems. This clearly sets apart the two frameworks afresh.

11.1 Our next step in the framework concerns the treatment of demand. Note that in the problem, the relevant magnitude of demand is already a 'total', viz. the demand over a ''week''. We have already said that any meaningful hypothesis about 'demand' (i.e., the demand faced by a trader) has to be about the 'total demand' summed over particular demands, and not the particular demands themselves. Let us now specify two basic hypotheses on the 'total demand' as part of our framework.

11.2 First, we have any particular week to be ''essentially the same'' as any other for our trader. So, the demand faced over any particular week is also essentially the same as that over any other and we can call this simply a ''week's demand''. (The precise significance of the qualification ''essentially'' in this context will be presently clarified.) Note that the expression ''week's demand'' really means here the demand over a week's length of time, for which particular ''week'' it is, i.e., its location in the real time axis, does not matter. Next, we note that the term ''week'' comes in here purely exogeneously, not from within the demand conditions underlying the demand actually faced by the trader. For logical clarity therefore it is best to give up the reference to 'week' and think of the 'total demand' defined in reference to any period of time in general. Whatever the period chosen, it would still remain the point that the demand depends only upon the length of the period, i.e., the ''time'' elapsing between the beginning and the end of the period, and not upon its location in time (represented by the beginning or end). This is the first hypothesis. It constitutes a general stationarity postulate on the demand, more precisely the demand conditions, faced by the trader. The difference between the two ('demand' and 'demand conditions') is simply that even when the demand conditions remain stationary or unchanged over time, the demand as a magnitude is not necessarily the same period after period. That is why we said above that the demand faced by the trader over a week is essentially the same as that over any other. Let us now come to the clarification of this, which also brings us to our second hypothesis concerning demand.

11.3 We simply assume that the demand conditions are such that the demand over any given length of time is a random variable in the sense of probability theory, i.e., having a well defined probability distribution[19]. (We call this the probability distribution of demand over time under stationary demand conditions.) So, clearly, the actual demand is not necessarily the same 'period after period'. But the whole conception is based on the assumption of unchanged demand conditions through time, which is already evident from the fact that 'demand' as a magnitude is conceived in reference to a pure length of time independently of historical time[20].

19/ The concept of uncertainty of demand is meant by us in this thesis as simply synonymous with this assumption.

20/ The whole treatment of ''time'' set out above represents our second point of acknowledgement to N. Georgescu-Roegen. We refer in particular to the distinction between what he wrote as ''Time, T'' and ''time, t''. To quote —

Contd.....

Our assumption that demand over any length of time is a random variable has its basis in the fact that 'particular demands' which it is constituted of are pure random events in themselves. However, between the two, i.e., the particular demands and the total demand, there lies the significant difference that we apply the tools of probability theory only to the latter, not the former. Since any such application constitutes the making of a 'hypothesis', this only takes us back to our initial proposition that any meaningful hypothesis on the 'demand' faced by a trader has to be on the 'total demand' and not the 'particular demands'.

Contd. footnote 20

> ''T represents Time, conceived as the stream of consciousness or, if you wish, as a continuous succession of ''moments'', but t represents the measure of an interval (T', T'') by a mechanical clock''. (op. cit., p.135)

In this language, the 'demand' we talk of is time-dependent but not Time-dependent. The distinction being made clear, we will also find it convenient at points to write in his manner (Time and time), but this becomes unnecessary when we write explicitly of time points and time lengths, where the small case ''t'' beginning is simply retained for both words, i.e., we do not write them as Time point and time length.

We also refer in this context to Roegen's insistence on beginning with the notion of ''stationary state'' for any rigorous programme of study of ''growth'', which actually constitutes a far-reaching defence of the same methodology across a wide range of phenomenal domains, (op. cit., pp.228-30). Our whole exercise in this thesis is ultimately an exercise in the methodology suggested by him and owes greatly to his writings. This includes in particular our attempt at setting out the probabilistic conception of demand on the basis of unchanged demand conditions.

11.4 This again constitutes an important point of difference from the whole approach of inventory theory as it exists.

Let us just quote from our standard reference. The point that the demand over a period is but the sum of the particular demands occuring in the period is stated by them(Hadley and Whitin) in the following words :

> ''Let us begin by noting that the number of units demanded in any time period will depend on the time between demands and on the number of units demanded when a demand occurs''21/

Note that a logical status is already given in this to the ''time'' elapsing between successive particular demands and their ''quantities''. From this Hadley and Whitin go on to write :

> ''In the real world, both the time elapsed between demands and the quantity demanded can be random variables''22/

We on the other hand do not simply grant any logical status to these ''variables'' and therefore the question of their being random variables in the sense of probability theory does not simply arise. They remain simply as raw data for the 'theory', not as its logical starting point which we locate only in the 'total'. Both 'stationarity' and 'randomness' (in the sense of probability theory) are therefore conceived by us directly in terms of 'total demand' as outlined above. The sequence of particular demands just drops out of the scene at this level.

21/ See :Hadley and Whitin , p.107.

22/ Hadley and Whitin ., p. 107.

11.5 Before leaving off the subject (demand), let us just explain what it means to ''abstract'' from the uncertainty or randomness of demand in our framework. The basic point is that this does not in any way dispute the fact that the underlying 'particular demands' are pure random events in the a priori sense. All that the ''abstraction'' means is that the randomness in some sense cancels out over time, leaving one with a determinate magnitude for the demand over a period. Like all abstractions, it is the 'purpose' which justifies or rejects it.

Let us make one more point in this context. The idea of cancellation of randomness (or indeterminacies) of particular events over time just referred is admittedly a very general one. When used to justify the abstraction from uncertainty (in our sense of the term), it is used in a ''strong'' sense. It can also be used in a ''weak'' sense as follows : As of any given ''time'' entering it, one still has demand as a random variable, but the ''randomness'' of it (some measure of uncertainty) decreases with time, so that one is left with a more and more determinate magnitude of ''demand'' the longer the ''time'' in reference to which it is defined in the first place[23]. This can be treated as an intrinsic property of what we called the ''probability distribution of demand over time''. In the body of the thesis, we will attempt a rigorous formulation of the

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23/ We should perhaps stress again that all this is based upon the assumption of unchanged demand conditions so that no meaningful notion of the socalled ''uncertainty of the future'' simply enters the conception.

property, which we will call by the name of the Second Law of the probability distribution of demand over time (Sec.4, Ch.2). Needless to say, this has to wait till the prior formulations are over.

12.1 We have one more element of our general framework to introduce. So far, we have implicitly taken the demand faced by a trader as given —— coming from outside and therefore given. However, no demand is really 'given' for a trader in any absolute sense. It comes to him from the market as a whole only via the competition from other traders. The elementary meaning of this competition is that if a demand (customer) cannot be satisfied by a trader, it simply goes elsewhere. This is already taken account of in the notion of sales foregone. However, the matter does not end here. Should a trader persistently fail to satisfy the demand that comes to him when others do not, the demand eventually ceases to come to him at all. Hence, over Time demand is not really a given for a trader even under stationary demand conditions in the market, but depends upon what proportion of it is satisfied by him at present, which depends upon the stock maintained[24].

We cannot take account of this ''dynamics of demand'' as we may call it, in the objective relations of our framework. But we can implicitly include it at the subjective level of the trader's decision making

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24/ This paragraph is based on ideas taken from the thesis of V. Narasimhan referred earlier.

in the following way. First, we retain the assumption of given demand and then make the whole ''dynamics'' spoken of an implicit element of this given demand in the sense of the trader's recognising it as basic to the continuation of this given demand into the future. To reflect this in his decision making we have to go beyond just the profits made or realised (as of any given capital) as his ''goal'' or ''objective'' for no special significance is then attached to the loss of customers or demand on account of demand not met. So, the demand not met or sales foregone, as we call it, enters as an additional argument in the objective function over and above what it already implies in terms of profit made. However, we do not go beyond the term 'profit' for this extension, for the whole argument remains within the general conception of the socalled profit motive. This leads then to a reformulation of the very notion of 'profit' in our decision criterion of maximum rate of profit. The precise reformulation will be given in chapter 4 of the thesis. It is simply part of the ''appropriate definition'' of the term that we spoke of at the beginning.

12.2 Viewed at the purely formal level, our reformulation of the profit concept is equivalent to taking account of the socalled ''stockout cost'' or more precisely the ''cost or penalty for lost sales'' of the conventional framework. However the conceptual basis of the two are different. In the conventional framework, the decision criterion is simply minimum cost and this by definition requires a cost or penalty to be attached to ''lost sales'' for otherwise the trader may simply lose all sales and end up with the absolute minimum of zero cost ! No such essentially tautologous argument is involved in our notion.

13.1 We have now completed the statement of our framework or rather the elements of our framework (for no attempt is made here to define a general inventory problem for the trader which a complete statement of the framework amounts to). This prepares us to resume the statement of specific problems within the framework that we take up in the thesis. However, instead of proceeding directly to this we will now have a digression and present a brief review of the basic framework of the existing theory as we understand this. This is simply to define the background to our further problems in the thesis.

13.2 The starting point of existing inventory theory is that the trader is free to choose the Time of his purchase. Let us follow up the logical consequences of this.

The first consequence is simply that the whole question of existence or holding of stock is now opened up. The basic answer to this question in the existing framework of inventory theory is found in the line of reasoning leading to the socalled ''lotsize formula'' or ''square root formula'' which represented the very beginning of inventory theory in the sense of formal (mathematical) analysis of inventory problems. Let us follow Arrow's valuable historical review of the subject on this[25]. Citing early references, he wrote :

> ''... they assumed that in addition to the price paid for the goods ordered, <u>there is a procurement cost to each order which is independent of the magnitude of the order</u>[26]. In that case, there is an incentive not to order continuously but to order a larger amount less often. Such a policy, however, implies the holding of inventory in the intervals between the orderings'' (*op. cit.*, p.5; underlining added).

One word before we proceed on with this. We mentioned earlier that a delivery lag is entered simply as an a priori axiom in the existing framework of inventory theory. The logical significance of the axiom is not explained thereby. To get to it, we shall initially abstract from it, and so identify the notions of 'ordering' and 'buying'. After we have gotten off the ground this way, we will be in a position to see what precise logical significance the introduction of delivery lag brings with it.

Let us now return to the passage from Arrow. He did not explicitly refer to <u>demand</u> in this passage and the subsequent analysis. But we can read the policy of ''ordering continuously'' simply as ordering (i.e.,

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25/ See K.J. Arrow, S. Karlin and H. Scarf, <u>Studies in the Mathematical Theory of Inventory and Production</u>, Chapter 1 (''Historical Background'') by K.J. Arrow.

26/ We shall call this cost (in a somewhat generalised sense) the <u>transaction cost</u> of the trader. The term will be explained later.

buying, as just explained) as and when the demand occurs, i.e., demand itself is implicitly taken to occur continuously in time[27/]. As we have already seen, no stock is then ever held in the process; besides, demand is met all through. (We call this the condition of ''full demand satisfaction''.)

13.3 As against this, we now have the policy of buying ''larger amount, less often'', necessarily implying ''the holding of stock'' between purchases. This establishes the existence of stock[27'/]. Let us now note that the condition of full demand satisfaction is maintained intact through any ''policy'' of the type mentioned provided only that a fresh purchase is made before the stock is sold out, i.e., fallen to zero. Going to the extreme, we now define a policy of buying only as and when the stock has fallen to zero, which obviously answers the question, when to buy, and in this sense defines a well-defined ''purchase rule'' for the trader as we may call it in parallel to the term ''sales rule'' introduced earlier.

27/ This is a convenient point to state one implicit assumption in our own treatment of demand, not mentioned earlier. Mathematically, the passage from 'discrete' to 'continuous' demand is accomplished via the postulate of a ''dense'' collection of discrete demands over any stretch of time. Conceptually, the postulate is implicit in our very starting point in the notion of demand as a mass as distinct from the 'particular demands'. Our whole treatment of the concept of 'demand' is therefore to be understood as of a dense sequence of particular demands as the underlying ''raw data'' of the concept.

27'/ It is obvious that the basis of existence of stock is in this case located inside the process. We will later include this within the inventory problems studied in the thesis.

Returning to Arrow, we find him building up his arguments to the square root formula precisely on the basis of this ''rule''. In his words:

> ''Once an amount x is ordered, the inventory is allowed to run down to zero before another order is delivered'' (op. cit., p. 6).

Equating 'order' to 'delivery', i.e., ignoring the delivery lag, this boils down to the purchase rule stated above. Though not stated explicitly by Arrow, there is a simple rationale behind this rule which we bring out in a few words.

13.4 Let us consider an arbitrary sequence of purchase where the amount, but not the Time, of each purchase is given a priori. It is then clear that the time between any two successive purchases is at a maximum subject to the condition of full demand satisfaction under the stated rule. So, the transaction cost per unit of time covered by the whole sequence is at a minimum under this rule under the stated condition. This is the rationale. Since nothing is stated here about the amounts bought, the ''rule'' becomes an integral part of the optimum policy of the trader regardless of the precise optimality criterion, and hence can be assumed a priori in the formal structure of relations of the 'model', as in Arrow.

13.5 We are now only one step away from the basic framework of the existing theory. It is clear that going by the element of transaction cost alone, we are led to a policy of an arbitrarily large amount of purchase in any single act of purchase implying an indefinite postponement of the next purchase. This tendency is now counter balanced by the interest charge upon the ''capital'' locked up in the holding of stock. The ''amount bought'' is then determined by balancing the two opposite forces. The square root formula itself is obtained by cost minimization subject to the two further conditions (a) that the same amount is bought in all successive purchases, and (b) that demand occurs at a given rate through time[28], implying that there is no uncertainty of demand.

13.6 This completes the statement of the basic framework of the existing theory in a strict sense. Pending clarification of this remark, we now point out a simple bridge or passage between our framework and the above framework defined by the ''purchase rule'' above.

This is simply that under this rule, the stock immediately after a purchase is the same as the amount bought and so if one is a constant through time, so is the other. With this, we can simply extend the string of our basic equivalences as regards the choice of the decision variable from 'capital' originally postulated to the 'amount bought' which is the decision variable in the conventional framework.

13.7 Let us now proceed on with our review of the conventional framework. First the delivery lag. Arrow himself wrote in continuation of the arguments presented above that

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28/ We will later give a rigorous conceptual underpinning to the term (Section 1, Chapter 2) which stands in clear contrast to its indiscriminate use in the general subject of economics.

''lags in delivery were irrelevant in the last section (i.e., arguments discussed above), where certainty is assumed; the time of ordering is simply made early enough to ensure delivery at the needed moment. (op. cit., p.7, foot note; underlining added.)

Clearly, ''needed moment'' here meant simply the moment when the stock has fallen to zero 29/. Thus the delivery lag is of no consequence — a pure ornament or encumbrance, as one likes — so long as the uncertainty of demand is abstracted from. Let us now look into the implications of the uncertainty.

The elementary implication is that there is now no way of ''ordering'' that the trader can devise such that the stock will have fallen exactly to zero at the time of delivery. If it has not yet fallen to zero, then it is simply carried over to the next ''cycle'' of operations 30/. This is of no consequence in itself. But if the stock has already fallen to zero before the delivery, then some amount of sale must have necessarily been foregone (or backordered) within the ''cycle'' concerned. This is the significance of the delivery lag in the conventional

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29/ This is assuming the prior holding of stock. Without this, the ''needed moment'' is simply the moment when a demand occurs, taking one back precisely to the case of zero stock all through. Thus the delivery lag by itself (without uncertainty) does not ensure the existence of stock. Again the point is noted by Arrow.

30/ A cycle of operations is defined to be the set of operations (buyings and sellings) from one purchase to the next, including the former and excluding the latter.

framework, and it also brings us to complete the statement of the framework itself. There are quite a number of points to make in this context, and so let us proceed in order.

13.8 Let us start back from the question, when to order. We note first of all that what is meant by this question in the conventional framework is at what level of the stock to place an order, i.e., the question is implicitly framed in terms of the stock[31]. As just seen, (a) if there is no delivery lag, then the order is placed when the stock has fallen to zero — this is independent of whether there is uncertainty of demand or not; and (b) if there is no uncertainty of demand, then the order is so placed that the stock falls to zero when the delivery is made — this is independent of whether there is any delivery lag or not. Thus, the whole question of when to order comes of its own or becomes a genuine inventory problem only on the joint basis of delivery lag and uncertainty of demand, and the substance of the problem is then defined simply by the prospect of sales foregone[32]. As such, this whole framework becomes an alternative to our own framework of a 'periodic' market'.

31/ All the external factors are then automatically ruled out of the framework of the question. The limitation of this in the a priori sense is clear from the fact that no room is then left for such a thing as speculative purchase.

32/ We do not refer any more to the other alternative, viz., ''backorders'', simply becuase this takes us too far away from our own subject matter in the thesis. Our substantive view on the question has already been given (see pp. 13-4 above).

Let us now simply count the substantive factors or elements of the framework of the existing theory as met so far. We have : (a) the transaction cost; (b) the interest cost; (c) the delivery lag, (d) the uncertainty of demand and (e) the prospect of sales foregone (and the 'cost' assigned to it, viz., the socalled stockout cost). Out of these, a basic framework is already set up by (a) and (b) which answers the question of how much to order or buy at a time by the socalled ''square root formula'', the question of when to order or buy being a priori answered by the 'purchase rule' defined in reference to (a). Factors (c) - (e) are then superimposed upon this basic structure to define meaningful problem around the question of when (at what stock) to place the order. This explains our earlier statement that the basic framework in a strict sense is defined simply by (a) and (b).

14.1 This completes our review of the basic framework of the existing theory. In the process, we have also set out the essential logical structure of this framework, as we see it. On returning to our own framework, let us first explain our reasons for ignoring the delivery lag in this thesis.

Stated in a word, the reason is that we already make room for the essential problem that comes with the delivery lag in the existing framework by the concept of a 'periodic market'. Needless to say, this is a purely methodological and not empirical defense. Nothing in fact is argued in empirical terms. However, the methodological case in favour of our concept is a strong one. To convince one of this, we need only point out that our Model I is a primitive inventory model, not an extension of or superimposition upon some ''earlier'' model. All the extraneous elements coming through the ''earlier'' model are therefore simply left out, leading to a clear focus on the essential problem avoiding all unnecessary complications.

Before leaving this off, let us just mention that there is nothing in our framework as such which bars the entry of the ''delivery lag''. This is a question of 'purpose' just gone over.

14.2 Let us now resume the statement of our inventory problems. The first problem to be stated starting from the above background is simply a reexamination of the basic model of existing inventory theory, viz., the model leading to the square root formula[33/], from the stand point of our decision criterion. To do this, we have to first recast the model in our own framework of ''capital and profit'', as we may put it, from out of its original framing in terms of ''cost'' alone, i.e., we take it out of the latter framework and put it in the former. This defines Model III of the thesis[33'/].

33/ We also call this for short the ''standard model''. It is to be found in all text books or treatises on inventory theory, e.g., in Arrow, et. al. pp. 5-6; in Hadley and Whitin, pp.28-34 (where it is called the ''Simplest Lot Size Model'') as well as in certain text books on economic theory, e.g. W.J. Baumol, Economic Theory and Operations Analysis, pp.5-10 (discussed under the heading, ''A Simple Inventory Problem'').

33'/ We will come to Model II last of all in the present sequence.

To avoid possible confusion, we mention here some apparent differences in the scope and structure of the two models arising out of this ''recasting''. This is apart from the difference in the decision criterion which of course is a fundamental substantive and not ''apparent'' difference. First, as already explained, our decision criterion leaves no logical ground for the socalled interest cost of the standard model. Hence it is simply absent from our model. Second, the standard model formally includes a delivery lag although, as just seen, this is of no analytical significance in itself in the model. We will simply ignore it. Third, the standard model abstracts completely from the uncertainty of demand. We shall also begin with this case for its simplicity as well as comparability with the standard model. But after this, we go on to take account of the uncertainty of demand within the scope of the same model. This we can do because the uncertainty of demand does not bring in any new substantive issue in our model, thanks to our assumption of no delivery-lag. It of course changes the formal structure of the model. Finally, our starting from a rigorous framework of capital gives us a different 'formula' for capital as compared with the 'formula' used in the conventional framework. This does not however stand in the way of a meaningful comparison of the two models.

14.3 Let us take up clarification of the term transaction cost in this context, now that it is part of our Model III. What we mean by the term is simply a cost associated with repeated transactions (purchases) per se independently of the amount bought. This does not necessarily mean an actual cost in the sense of spending. It may represent simply the trader's own evaluation of the ''inconvenience'' felt in making the transactions, expressed in money. In the literature it is typically left simply as a component of the general term, ''procurement 'cost'' (or purchase cost). This does not serve the purpose of fixity of reference. Terms like ''fixed charge'' or ''fixed cost'' which are sometimes used appear too narrow in their connotation. Hence we have used the term, transaction cost, which, we think clearly points to the qualitative basis of the cost. This also leaves us free to use the term purchase cost (or cost of purchase) unambiguously to mean simply the cost of the goods bought, i.e., the amount paid.

15. The starting point of our next problem in the thesis is as follows. To say that the trader is left free to choose the Time of his purchase (which is really a property of the market be buys from) is not necessarily to give the 'choice' itself any basic status in the actual carrying out or organisation of the process. Analytically, the 'choice' simply means that the Time of the successive purchases is left to circumstances as per some definite ''rule'' devised for the purpose (e.g. the ''purchase rule'' as stated). In place of this, the trader may himself close the question by deciding a priori to carryout or organise his whole process on the basis of some fixed, or regular, or constant interval of purchase through

Time[34]. The Time of any particular purchase is then simply the Time of the last purchase advanced by this interval, and drops out of the scene of decision making. But the purchase interval itself (or more correctly, the length of this interval[35] becomes an object of choice in this set-up. This is the set up for our next problem formulated as Model IV of the thesis, called the Composite Model.

16. Let us make a short digression at this point. Once we have the purchase interval as trader-decision, we can also treat it as prior decision for some other decision. So, we can now see our first problem, equivalently the first model, as defined on this basis. This gets us rid of the assumption of a periodic market in the layout of the model, which in turn clarifies our earlier statement that the assumption has a purely provisional status in our scheme. The purchase interval itself is then left simply as a parameter in the model, opening up the road straight to the examination of the effects of variations in this parameter on relevent variables. This constitutes an important part of our analysis of the model.

34/ All the terms, fixed, regular and constant, are used here synonymously. The rule itself is understood simply as an a priori organisational principle followed by the trader.

35/ This is one term which we have to use freely in the thesis, meaning some times an 'interval' (or period) in the proper sense, some times only the length of the interval; some times a 'constant', some times a 'variable' through Time; some times a given datum, some times an object of choice and so on, depending upon the context.

17. Let us return to the composite model. The assumption of a fixed purchase interval serves two important methodological purposes in the model. First, it secures the existence of stock[36/], and secondly, it brings back the problem of sales foregone ruled out by construction in Model III. This establishes its connection with Model I. Next, the choice of the purchase interval is made meaningful by inclusion of the transaction cost. This establishes its connection with Model III. This explains the term, 'composite model'. The model can also be looked upon as a fundamental conceptual transformation of the 'model' implicit in the basic framework of existing inventory theory as already outlined. This is evident from the fact that the basic forces considered are the same uncertainty of demand and the various 'costs', but they appear under different substantive questions, viz. at what interval to purchase and what stock to maintain through the successive purchases, not when to buy and what amount.

36/ Needless to say, this is also an internal basis of the existence of stock, i.e., it lies inside the process of trade.

18.1 It remains to come to Model II of the thesis. Our object here has been simply to build an element of uncertainty of supply into the structure of an inventory problem, which appears to go largely by default in the literature. The reason for this is not far to seek, and that also sets the background to the way we introduce the element in our model. So, let us start from this.

Let us remember that the formal structure of an inventory problem in the subject as it exists is set by the two questions, when to buy (or order) and how much. The implicit assumption behind this is simply that the goods are always available in sufficient quantities in the 'market' or 'source' for the trader to buy what he wants. This simply rules out of hand any notion of uncertainty of supply in an operationally meaningful sense, for whatever uncertainty there may be in an a priori sense certainly does not touch the trader if he can buy what he wants (what quantity) when he wants. It follows that to be 'operational', the notion of uncertainty of supply must mean an exogenous restriction on the trader's purchase as well.

Let us now be absolutely clear that we take but a small step from this background in setting up our model. Proceeding in line with the background, we grant that on the whole the trader is able to buy what he wants, but not always — there are times when he has to go simply without any purchase at all. The idea is that there is some sudden disruption of the whole supply, or supply system, leaving the 'source' the trader buys from without any supply. Even if there is a limited supply, we may assume realistically that it is cornered by big traders who fall outside the scope of our analysis, thus taking us back to the point of ''zero purchase'' for our trader. We call this the availability problem and build our model around it.

18.2 Let us immediately point out a basic limitation of the model in handling the problem. Let us start from the trader going to the market and finding himself unable to make his purchase because the 'supply' there has suddenly disappeared. It would be logical to think that he (the trader) then simply stays on in the market till the goods become again available, implying in turn that the market meets continuously in Time. Interestingly, this takes one back to the existing framework of inventory theory in the sense that it can be interpreted as the case of a variable and uncertain ''delivery lag'', with, implicitly, a high probability of zero lag, for ''on the whole'' the trader is still seen to buy what he wants when he wants. Stated slightly differently, the whole 'event' which brings about the ''lag'' is by nature an accident, and hence has only a small probability of occurrence. It is however rather late in the day for us to go into all this(''delivery lag''). We simply go back to our original starting point of a periodic market and cast the whole problem in this rigid framework. The rigidity is simply that the trader having failed to make a purchase is left with no option other than waiting till the next meeting of the market for his purchase. This is the basic limitation of our formulation of the availability problem in Model II. Further details of the model are left to the chapter concerned.

19.1 Let us now say a few words in clarification of the titles given to the various models. We have titled Model I and Model II completely symmetrically as ''Optimum stock under uncertain demand and a given purchase interval'' and ''Optimum stock under uncertain supply and a given purchase interval''. The first point to be clarified is that the terms ''uncertain demand'' and ''uncertain supply'' in the two titles are not to be taken to connote their mutual exclusiveness in the two models. We do consider uncertainty of demand in Model II though we begin here with the case of certain demand.

Next, it is to be clarified that the expression ''a given purchase interval'' has rather different substantive connotations in the two models. Firstly, in Model I the expression connotes <u>either</u> that the trader buys from a periodic market <u>or</u> that he himself decides to buy at a fixed interval through Time which (the interval) is taken as given in the sense of a prior decision in the model. In Model II, only the former interpretation is really admissible, for there is nothing to explain why the trader sticks to his self-imposed rule of the second interpretation even in the case of failure to make a purchase. Secondly, we can speak of a ''given purchase interval'' in Model II only in a <u>notional</u> sense, for it is only the interval between the trader's successive <u>intended</u> purchases, and not the <u>actual</u> purchases, that can be given a priori when the availability problem is there. So, this sense of the term must be taken as implicitly understood in the context of Model II. No such restriction on the term is obviously placed in Model I.

Finally, it is to be mentioned that the term ''stock'' in the titles of the two models is to be taken for ''stock maintained'' in the sense of this term already explained. Even here there arise special problems of interpretation in the context of Model II. But we leave the matter to the chapter concerned.

19.2 Let us now come to the title given to Model III : ''Transaction cost and the optimum purchase and purchase interval''. The point to be clarified is simply that ''purchase interval'' here does not represent a decision variable (or optimising variable) separately from ''purchase'', i.e., the amount bought in a purchase. We have nevertheless put it in the title simply to make clear, and stress, the line that separates Model I and Model II on the one hand, and Model III on the other. Model IV is titled simply ''The composite model''. This is already explained. Let us just be clear that this is a ''composite'' of Model I and Model III — Model II does not come into it.

20. It is now time to bring this long introductory chapter to an end. Let us just say that the whole thesis remains ultimately as a conceptual effort trying to integrate inventory theory within the mainstream of economic theory. This is already evident from the statement of the ''background'' we start from before coming to inventory problems and from the way we set up the problems. In a nutshell, the problems taken together

are defined through the four concepts of capital, profit, demand and supply [37/]. These are all fundamental concepts of economic theory. This is where all the ideas of the thesis come from, and ultimately go back to. The thesis is thus a contribution to inventory theory seen through economic theory.

37/ Our basic points regarding the first three of these concepts were already set out as part of development of the ''framework'' and points regarding the fourth are now set out through the statement of the ''availability problem''.

Chapter 2

TOOLS AND CONCEPTS, I : DEMAND

Section 1 : The Concept of Rate of Demand

1. In Chapter 1 we summed up our stationarity postulate on demand conditions with the statement that demand in our sense —— demand in the total or as a mass aggregated over particular demands —— is time-dependent, not Time- dependent. Our object now is to construct certain fundamental tools of analysis out of a mathematical formulation of this postulate. In this section, we abstract from the uncertainty of demand, which we come to deal with from the next section onwards. As already explained, from the tool point of view, this is a passage from a deterministic to probabilistic treatment of demand. Let us also repeat that both the notions, stationarity and uncertainty (or randomness), are understood by us in reference to total demand, not the sequence of particular demands as such. Time already enters this concept, and that is the starting point of our formulation.

2.1 Mathematically, demand in our sense is the total of particular demands occuring within some interval of time, say (t, t'), where t and t' $(>t)$ are two arbitrary time points. Let $D(t, t')$ be this total demand. Let us now go over to an alternative representation of the interval (t, t'). What we do is simply separate out the time and Time in the interval (t, t'), the former being defined by the length of the interval, which we will denote by τ, and the latter being defined by the location of this length on the real time axis, which can be represented either by the " beginning" or by the " end" of the interval, i.e., by t or t'. Choosing the " beginning" for this purpose, we have the interval represented by the pair (τ, t). Stated in words, (τ, t) is the time-interval of length τ beginning at time-point t. So, the demand $D(t, t')$ is now re-written $D(\tau, t)$.

2.2 Now, as a pure thought experiment, we can obviously vary τ and t independently of one another in $D(\tau, t)$ and thus consider $D(\tau, t)$ as a function of two variables. Following this, we can say that the dependance of $D(\tau, t)$ on τ and t gives us respectively the " time-dependence" and " Time-dependence" of demand. With this, we can represent the postulate of stationary demand by the condition that $D(\tau, t)$ is independent of t or in other words, it is a function of only one variable, τ. For simplicity of notation, we use the same letter, D, to denote the function. Our condition is mathematically stated as

(1) $D(\tau, t) = D(\tau)$, all t.[1]

2.3 We now point out an extremely significant implication of the formulation of the notion of stationarity. On the face of it, it could appear that the mathematical form of $D(\tau)$ is left free in (1). This

[1] Equations are numbered consecutively in each chapter and then referred by that number within the same chapter and by a pair of numbers in other chapters, the first number referring to the chapter and the second number referring to the equation. Thus, e.g., the notation (2) refers to the equation number 2 of the same chapter and (3.2) to equation number 2 of chapter 3. When an equation is repeated within the same chapter, the original equation number is also repeated, but when it is repeated in a different chapter a fresh number is given to it. These conventions apply to the text excluding the appendices. In each appendix equations whether new or repeated are numbered afresh.

however is <u>not</u> so, $D(\tau)$ must in fact be <u>proportional</u> to τ, in other words, there must exist a positive constant, d, such that

$$(2) \qquad D(\tau) = d.\tau \,,$$

Proof of this claim is given in the appendix at the end of this chapter.

3.1 The constant d in (2) defines the concept, <u>rate of demand</u>. As such, this term is understood by us only in reference to stationary demand conditions, as something implicit in this very notion. This is very distinct from using the term to mean just a dimensional transformation of the quantity, total demand in a certain interval of time, without any further backing, which appears to be its conventional usage[2/]. This clarifies our distinctive use of the term and the conceptual underpinning we give to it, already hinted previously (see p. 33, fn. 28).

3.2 Lest there be any ambiguity about our position, let us clarify the essential point we are trying to make. Let us switch over from the abstract notion of a rate of demand to such seemingly concrete expressions as, e.g., a day's demand or daily demand. We can clearly say that stationarity of demand conditions is already implicit in these terms, for one does not mean by these terms the demand on a particular day. One means simply the demand over a day's length of time as per certain conditions of demand implicitly taken as given. But this itself is a valid

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[2/] A simple example serves to bring out the arbitrainess of such a definition. Consider a once-and-for-all occurance of a demand, say at time point t_0. We can define any number of intervals containing t_0 and give <u>any</u> value to the " rate of demand" as per this definition that we may like!

entity only if these conditions are in fact stationary, for an abstraction is already made from Time in the expression. Our concept of rate of demand simply gives a rigorous mathematical expression to these intuitive notions. However, we grant that in the expression, a day's demand or daily demand, the " day" may come in as a natural unit of measurement of time for the very measurement of demand. As per this scale of measurement, one can talk of so many days' demand, but not of so many hours' demand in the sense of something directly measured. In our formulation on the other hand a direct measurability of demand in terms of any length of time is simply presumed. This is a difference in the precise quantitative framing of the notion of stationarity. It does not free the notion of rate of demand of the need of any such framing. This is our point.

Section 2 : The Probability Distribution of Demand over Time[3]

1. Let us start off directly with $D(\tau)$, the demand over a certain " time", τ, as a random variable. Obviously, the probability distribution of this random variable is defined parametrically wrt τ[4]. Let

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[3] This title is an abbreviation of the fuller expression, " probability distribution of demand over Time under stationary demand conditions". This is simply taken as understood in all similar uses of the term, e.g., in the titles of the following sections of this chapter. Depending upon the context, we will shorten the expression still further to just the " probability distribution of demand".

[4] Throughout the thesis, we use the term, probability distribution of a random variable synonymously with the probability density function (pdf) of this random variable.

us denote this distribution by the function $f(u, \tau)$. Loosely speaking, $f(u, \tau)$ is the probability that a certain quantity, u, will be demanded over a certain length of time, τ.

2. Let us remember that the whole concept is defined on the assumption of stationary or unchanged demand conditions through Time. But given this assumption, the notions of " demand over a certain time" and " time taken for a certain demand to occur" are nothing but two sides of the same coin. Hence if one is a random variable so is the other, and their probability distribution are one and the same. So, if we now denote the time taken for u demand to occur by T(u), then we have $f(u, \tau)$ as the probability distribution of T(u) as well.

In sum, $f(u, \tau)$ represents the probability distribution of both $D(\tau)$ and T(u). In the first case, we read u as variable and τ as parameter and in the second case we read u as parameter and τ as variable in $f(u, \tau)$. The appropriate interpretation depends upon the context and has to be so understood. (This cannot be always explicitly specified in writing).

3. For our own purpose in the thesis, the main interest is focussed upon $D(\tau)$ as the random variable. In the rest of the chapter we prepare the grounds for this by discussing the variational properties of the probability distribution of this random variable, i.e., of the properties of $f(u, \tau)$ defined through shifts in the parameter, τ. Nothing is discussed here about the precise shape of $f(u, \tau)$ for a given τ, which we simply leave in the open to the stage of our actual use of the tool.

Though, as just stated, our own interest in the thesis focuses mainly on D(τ), we also on occasion make explicit use of the variable T(u). The relevant variational properties of the probability distribution of T(u) are mentioned below alongside our main line of analysis in the rest of the chapter.

Section 3 : First Law of the Probability Distribution of Demand over Time

1.1 Conceptually it is obvious that a change in the value of parameter, τ , effects a rightward or leftward shift of whole probability distribution of D(τ) depending on whether the change means a rise or fall in the value of τ . Within this, our First Law of the Probability Distribution of Demand stipulates simply that the expected value or mean of the pdf of D(τ) shifts proportionately with τ . This is understood simply as a conceptual extension of the form of D(τ) established in (2) for the case of deterministic demand to the case of probabilistic demand. Following this we also identify the factor of proportionality in this law with that in (2), i.e., the rate of demand. This effectively redefines the concept of rate of demand as simply the expected demand per unit of time. It needs hardly be stressed again that the whole conception is based upon a prior postulate of stationarity of demand conditions for otherwise the entity D(τ) is not defined at all.

1.2 It remains to spell out the conception which is really to establish a conceptual connection between the two cases of probabilistic and deterministic demand. First, we give a mathematical statement of our

First Law of the Probability Distribution of Demand. The statement is simply that there exists a positive constant, c, such that

(3) $E(D(\tau)) = c\tau$

where 'E' denotes the expectation operator[5]. This is the first step. The second step consists of the equation :

(4) $c = d$,

1.3 Let us now clarify the whole conception set out. The essence of this consists simply of thinking of the case of deterministic demand as the degenerate case of probabilistic demand where the " whole" probability of demand gets concentrated at a single point, i.e., mathematically, the domain of $f(u, \tau)$ shrinks to a single point. This point is then simply identified as the mean demand, $E(D(\tau))$, i.e., the shrinking of the whole pdf of $D(\tau)$ is taken to be defined about its mean. This is taken to be something purely definitional or logical in the sense that a shrinking of the pdf of a random variable is meaningfully conceivable only about its mean. Going back to step 1 of the argument, we must now equate $E(D(\tau))$ with $D(\tau)$ itself understood as a scalar or determinate magnitude. But we have already seen in section 1 that in this case $D(\tau)$ is in fact proportional to τ where the factor of proportionality was denoted d. So, we must now have $E(D(\tau))$ as also proportional to τ with the same factor of proportionality. This simulteneously establishes the two propositions stated in (3) and (4) above. Both the propositions are in

[5] We will also use the alternative notation $\bar{u}(\tau)$ for $E(D(\tau))$ in line with the fact that u denotes realization of the random variable $D(\tau)$.

this sense purely logical or definitional, and do not constitute any " additional" assumption on the probability distribution of demand.

2.1 Before ending, we point out two things. First, equations (3) and (4) have their automatic counterparts in the probability distribution of the " time taken for certain demand to occur", i.e., the variable $T(u)$. Stated jointly, the counterpart is simply :

(5) $$E(T(u)) = \frac{u}{d}.$$

Thus we have the interpretation of d as the reciprocal of the mean time taken for one unit of demand to occur.

2.2 Our second observation is in a sense purely terminological. First, we point out that it would be incorrect language in our set up to talk of an expected rate of demand. The rate of demand itself is understood as the expected value of the random variable $D(\tau)/\tau$. It is meaningless to talk of expectation of an expected value. Next, we point out the connection between our language and the intuitive content of the term, rate of demand, as denoting an average. The " average" is now rigorously understood in a double sense : (a) in the sense of a time average as connoted by the expression $D(\tau)/\tau$ and (b) in the sense of statistical average or mean which is simply the mean of the above " average" taken as a random variable.

Section 4 : Second Law of Probability Distribution of Demand over Time

1. The Second Law of Probability Distribution of Demand over Time was defined in Chapter 1 to mean that the uncertainty or randomness of demand

decreases with " time". Stated more explicitly, the demand over a longer period has a small uncertainty associated with it, given no change in the underlying conditions of demand. No precise definition of the uncertainty of demand being greater or smaller was however given in Chapter 1. Our first object now is to give a suitable definition of this. We will first give a definition of rise or fall in uncertainty of demand for a given value of the parameter, τ, in $D(\tau)$, and then go from this to changes in uncertainty defined through the variation of τ, which is ultimately what is relevant for our purpose.

2.1 Let us begin. In the previous section, we conceived deterministic demand as the degenerate case of probabilistic demand where the whole probability of demand is concentrated at a single point (the mean demand). Clearly we can think of this " concentration" as the limit or culmination of a process of increasing concentration of probability around the mean. Such a process means by definition a process of decreasing uncertainty. Since τ is fixed here, the decrease occurs out of exogeneous shifts in the probability distribution of $D(\tau)$. Next, we can understand an increase or decrease in the concentration of probability around the mean simply as an increase or decrease in the probability under some given neighbourhood of the mean, specified in advance. We can then say that there is an increase or decrease of uncertainty on account of the shift in probability distribution according as the probability under this neighbourhood has decreased or increased, i.e., the two are inversely related. Choice of the neighbourhood itself is based on a prior consideration which we must leave in the open. From a purely analytical point of view, this introduces an element of arbitrariness in the procedure, which is admitted. However, it will be seen later that this does not create any fundamental problem for our specific purpose.

Let us now proceed to step 2 of the analysis. The basic point to recognise here is that a change in the value of τ — shortening or lengthening the period — means by definition a change in the whole scale of demand. This is already reflected in the fact that the mean demand varies proportionately with τ (the First Law of Probability Distribution of Demand over Time). However, this is not the end of the matter. A larger mean will typically have a larger variability of Demand associated with it in absolute units. So, if we keep to a fixed neighbourhood of the mean in the definition of change in uncertainty, then we automatically have the uncertainty as an increasing function of τ. This is purely tautological and connotes nothing of the substance of the matter.

2.2 Let us now go to a more meaningful definition of change in uncertainty (rise or fall) brought about by the variation of τ. The concept of a " fixed" neighbourhood of the mean in the statement above is implicitly understood as a neighbourhood defined by a fixed absolute deviation of the mean allowed in the neighbourhood. We now simply replace this fixed absolute deviation from the mean by a fixed relative deviation to define the neighbourhood. To be in the clear, what we suggest is that if δ_0 was the fixed absolute deviation from mean defining the neighbourhood for a given value of $\tau = t_0$, then we first express δ_0

as a relative deviation from the mean, i.e., as $\delta_0 / \bar{u}(\tau_0) = \varepsilon$[5], say, where $\bar{u}(\tau_0)$ is the mean of the pdf, $f(u, \tau_0)$, and then define a neighbourhood of $\bar{u}(\tau_0)$ by this fixed relative deviation, ε, for all values of τ. This done, we say that there is an increasing or decreasing uncertainty of demand with τ as the probability under this neighbourhood decreases or increases with τ.

3. Before giving a more formal statement of the definition, we stop for a few general observations on it. First, one may call the change in uncertainty defined here the change in the <u>relative uncertainty</u> as compared to the change in the <u>absolute uncertainty</u> originally defined. We will however talk of the change in uncertainty defined here simply as change in uncertainty through the variation of τ. Next, we point out that since $\bar{u}(\tau)$ is proportional to τ (First Law of the Probability Distribution of Demand), the absolute deviation allowed from the mean in our neighbourhood now also increases proportionately with τ. Thus whether we say that we allow a fixed relative deviation from the mean or a variable absolute deviation varying proportionately with τ in our neighbourhood for defining change in uncertainty over time comes to the same thing. Finally, we mention that the neighbourhood must in any case be strictly contained within the domain of the probability distribution of $D(\tau)$ for all values of τ. This is simply taken as understood all through. Thus the parameter ε is not left completely arbitrary in the analytical sense but must satisfy the above restriction. We will follow up this point a little further at the end of this section.

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[5] The symbol 'ε' is to be read as 'epsilon' or 'belonging to' depending upon the precise context.

4. Let us now give a formal statement of our definition of the rise or fall in uncertainty of demand through the variation of τ. First we define a neighbourhood of $\bar{u}(\tau)$ as per the parameter, ε, which we denote by $N_\varepsilon(\bar{u}(\tau))$:

$$N_\varepsilon(\bar{u}(\tau)) = \left\{ u : (1-\varepsilon)\,\bar{u}(\tau) < u < (1+\varepsilon)\,\bar{u}(\tau) \right\}$$

Next, we define the probability integral, $I_\varepsilon(\tau)$:

$$(5') \qquad I_\varepsilon(\tau) = \int_{N_\varepsilon(\bar{u}(\tau))} f(u, \tau)\, du.$$

We then say that there is a rise or fall in the uncertainty of demand with τ as :

$$\frac{dI_\varepsilon(\tau)}{d\tau} \lesseqgtr 0 .$$

5. Given this definition, our Second Law of the Probability Distribution of Demand over Time is defined simply by the condition :

$$(6) \qquad \frac{d\,I_\varepsilon(\tau)}{d\tau} > 0, \quad \text{all } \tau .$$

This is for a fixed ε entering as parameter in the whole concept. However the definition can be immediately generalised by allowing ε to vary over a " meaningful range" and yet requiring (6) to be satisfied. This range is already circumscribed by the condition that $N_\varepsilon(\tau)$ must be strictly contained within the domain of the pdf of $D(\tau)$. This apart, there also remain the a priori consideration entering the choice of the neighbourhood mentioned at the beginning. It is however clear that once

ε is allowed to vary over its meaningful range we get rid of any arbitrariness in the statement of the " Law" concerned.

Section 5 : The Case of an Unchanged Structure of the Probability Distribution of Demand over Time

1. What we call the case of an " unchanged structure" of the probability distribution of demand over time is on the one hand a straight formal generalization from our First Law of the Probability Distribution of Demand and on the other a straight substantive denial of our Second Law. That is, starting from the First Law, we have (a) this case and (b) the case defined by the Second Law, as mutually exclusive alternatives[6]. Let us now simply follow out this preview of the matter.

2.1 According to our First Law of the Probability Distribution of Demand, the mean value, $\bar{u}(\tau)$, of the random variable, $D(\tau)$, varies proportionately with τ. Underlying this is the broader relation that the whole pdf of $D(\tau)$ undergoes a rightward or leftward shift with a rise or fall in the value of τ. Our generalisation from the First Law

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[6] It is to be remembered in this context that while the First Law is purely " logical" in nature, the Second Law is based on a priori intuitive considerations relating to " demand" in its specific sense in the thesis, i.e., the demand faced by the trader through Time under stationary conditions, which are also somewhat " provisional" in nature. There is thus a difference in the " level of conception" of the two Laws. The concept of an " unchanged structure" of the probability distribution of demand defined here belongs to the same general level as the First Law. This is what motivates our formulation of the concept in the first place. In the actual problems dealt with in the thesis, we make use of both cases under appropriate motivation.

is simply that this shift as a whole is a proportional shift. This is what defines the case of an unchanged structure of the probability distribution of demand over time in our sense of the term.

2.2 Let us formalise. The essential task is to define formally what we mean by a " proportional shift" of the whole pdf of $D(\tau)$ with τ. What we mean is this. Let us consider an arbitrary interval, (a, b), contained within the domain of the pdf, $f(u, \tau)$, for a given value of τ. By a proportional shift in the whole pdf of $D(\tau)$ with τ, we then mean that $f(u, \tau)$ satisfies the following condition :

$$(7) \qquad \int_a^b f(u, \tau)\, du = \int_{\lambda a}^{\lambda b} f(u, \lambda \tau)\, du, \text{ for all } \lambda > 0.$$

In other words, what we mean is that a proportionate blow up or contraction of the interval (a, b) in proportion with the variation of τ (represented here by the parameter, λ) leaves the probability under the interval invariant. This is true for all intervals contained within the domain of the pdf. Hence in particular the condition implies that the domain itself is proportionately blown up or contracted with τ.[7]

2.3 Let us now make some formal deductions from (7). First, we effect a transformation of variables in the RHS of (7) by defining the variable, v, by

$$v = u/\lambda,$$

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[7] Obviously, this condition becomes vacuous if the domain is $(0, \infty)$.

So,

$$v = \begin{cases} a, \text{ when } u = \lambda a \\ b, \quad " \quad u = \lambda b \end{cases}$$

and

$$du = \lambda dv,$$

With this substitution, we can now write (7) as :

$$\int_a^b f(u, \tau)\, du = \int_a^b f(\lambda v, \lambda \tau)\, \lambda dv,$$

But v is a pure dummy variable in the RHS integral. Hence we can write it back as u. This gives us the equation :

$$\int_a^b f(u, \tau)\, du = \int_a^b \lambda f(\lambda u, \lambda \tau)\, du$$

or,

$$(8) \qquad \int_a^b \left[f(u, \tau) - \lambda f(\lambda u, \lambda \tau) \right] du = 0.$$

This is true for all intervals, (a, b), contained within the domain of $f(u, \tau)$ (τ given). Granting that $f(u, \tau)$ is a continuous function of u over its whole domain for any given τ, this means that the function integrated in (8) must <u>identically vanish</u> over the whole domain, i.e.,

$$f(u, \tau) - \lambda f(\lambda u, \lambda \tau) = 0, \text{ for all } \lambda > 0$$

or,

$$(9) \qquad f(\lambda u, \lambda \tau) = \frac{1}{\lambda} f(u, \tau), \text{ for all } \lambda > 0.$$

2.4 This completes our formal deductions from (7). Conversely, it also can be shown that (7) follows from (9). Hence (7) and (9) are equivalent. A parallel equivalence can also be stated between (9) and the analogue of (7) in terms of τ as the variable of integration. With these, we now formally <u>define</u> the case of an unchanged structure of the probability distribution of demand over Time by (9).

3.1 That the case of an unchanged structure of the probability distribution of demand constitutes a denial of our Second Law of this distribution becomes obvious on taking the interval (a, b) to be a neighbourhood of the mean demand, $\bar{u}(\tau)$, e.g., $N_\varepsilon(\bar{u}(\tau))$ of the previous section. According to the Second Law, the probability under this neighbourhood should <u>increase</u> with τ, but according to (7), the probability is left invariant[8]. This can be stated compactly in terms of the function, $I_\varepsilon(\tau)$, defined in (5′) as :

$$(10) \qquad \frac{d\, I_\varepsilon(\tau)}{d\tau} \begin{cases} > 0, & \text{under the condition of the Second Law of Probability Distribution of Demand over Time.} \\ = 0, & \text{under the condition of unchanged structure of the probability distribution of demand over Time.} \end{cases}$$

3.2 Let us clarify this a little further. First, we point out that our whole concept of an unchanged structure of the probability distribution of demand is in essence a formalisation of what can be called the

[8] This invariance follows from the fact that by our First Law, the mean demand, $\bar{u}(\tau)$, shifts proportionately with τ and hence so does the neighbourhood around it defined by $(1 \pm \varepsilon)\,\bar{u}(\tau)$.

<u>pure scale effect</u> of variation of τ on this distribution. (9) itself defines, we may say, a pure " scaling " of the function, $f(u, \tau)$. The notion of " scale effect" of variation of τ was already met in our discussion of the Second Law where it was seen to <u>modify</u> our implicit preliminary statement of the Law in terms of absolute uncertainty and require us to state it in terms of relative uncertainty. The Law itself, we can now say, defines in turn a <u>modification of the pure scale effect</u>. Hence its denial of the concept of unchanged structure of probability distribution of demand.

4. Let us end the discussion by pointing out that the condition of unchanged structure of the probability distribution of demand over time makes this whole probability distribution " scale free" in the sense that the probability of a certain demand occuring over a certain time (equivalently, the probability of a certain time being taken for a certain demand to occur) depends only upon the <u>ratio</u> of the two and not upon their absolute magnitudes. So, if we transform our random variable into a ratio form, $D(\tau)/\tau$ and $T(u)/u$, we have their probability distributions as independent of τ and u respectively. In effect, we get rid of parameters in defining our probability distribution.

To prove these assertions, we make use of the general theorem that if X is a random variable with pdf $f(x)$ and c is a positive constant, then cX is a random variable with pdf $\frac{1}{c} f\left(\frac{y}{c}\right)$, where y denotes a realisation of cX. So, since $D(\tau)$ is a random variable with pdf $f(u, \tau)$ for given τ $D(\tau)/\tau$ is a random variable with pdf $\tau f(\tau x, \tau)$, where x denotes a realisation of $D(\tau)/\tau$. But by (9)

$$\tau f(\tau x, \tau) = f(x, 1).$$

This proves that the pdf of $D(\tau)/\tau$ is independent of τ. By similar reasoning it can be proved that the pdf of $T(u)/u$ is given by $f(1, y)$, where y denotes a realisation of $T(u)/u$, which is independent of u. This proves our assertions.

Appendix to Chapter 2

Proof of the Claim, $D(\tau) = d.\tau$

Let us begin by recapitulating the definition of $D(\tau)$. $D(\tau)$ denotes the demand over a period of length, τ, this being a well-defined notion by our stationarity postulate that the demand over a period depends only upon the length of the period and not upon its location in the real time axis. So, formally, $D(\tau)$ is defined by :

(i) $D(\tau) = D(\tau, t)$, all t

where $D(\tau, t)$ is the demand over the period of length τ beginning at time point t.

Let us also remember in this context the following point made in Chapter 1. $D(\tau, t)$ is by definition the total demand aggregated over the particular demands occuring within the time interval $(t, t+\tau)$. It is also assumed that over any such interval there is in fact a " dense" collection (or sequence) of particular demands. As already explained in Chapter 1, the assumption is implicit in the treatment of demand as a " continuous function of time" (see p. 31 , fn. 27). We now take the step of explicitly postulating this condition for our own function $D(\tau)$. That is, we assume allthrough that <u>$D(\tau)$ is a continuous function of τ</u>, where $\tau \varepsilon (0, \infty)$.

Let us now come to the claim to be proved. To prove it, it is obviously sufficient to show that for any two time-lengths, τ and τ' :

(ii) $\dfrac{D(\tau)}{\tau} = \dfrac{D(\tau')}{\tau'}$.

Let us first consider the case where one of the time-lengths, say τ', is a <u>multiple</u> of the other, i.e., τ; i.e., there exists a positive integer N such that

(iii) $\tau' = N\tau$,

Let us now consider the expression $D(\tau')$. This is the total demand occuring over any period of length τ'. Let us fix the period as (τ', t_0) where t_0 is an arbitrarily fixed time point. Let us now divide the period into N subperiods, each of length $\tau'/N = \tau$, beginning respectively at time points $t_0, t_1, \ldots\ldots, t_{N-1}$ where by definition

$$t_k = t_0 + k\tau \ , \quad k = 0, 1, \ldots\ldots, N-1.$$

By definition

$$\begin{aligned} D(\tau') = D(\tau', t_0) &= D(\tau, t_0) + D(\tau, t_1) + \ldots + D(\tau, t_{N-1}) \\ &= \underbrace{D(\tau) + D(\tau) + \ldots\ldots + D(\tau)}_{N \text{ terms}}, \text{ by (i)} \\ &= ND(\tau) \\ &= \frac{\tau'}{\tau} D(\tau), \quad \text{by (iii)}. \end{aligned}$$

So, we have the result :

$$\frac{D(\tau)}{\tau} = \frac{D(\tau')}{\tau'}$$

This proves (ii) for the case $\tau' = N\tau$.

It is more or less obvious that we can use the same type of proof for the case where τ and τ' are both <u>multiples of some common time-length</u>, say τ_0, i.e., there exist positive integers N and N$'$ such that

$$\text{(iv)} \quad \begin{cases} \tau = N\tau_0 , \\ \tau' = N'\tau_0 . \end{cases}$$

The proof is as follows. By steps similar to the above, we have

$$D(\tau) = N\, D(\tau_0),$$
$$D(\tau') = N' D(\tau_0).$$
$$\therefore \left.\begin{aligned} D(\tau) &= \frac{\tau}{\tau_0}\, D(\tau_0), \\ D(\tau') &= \frac{\tau'}{\tau_0}\, D(\tau_0) \end{aligned}\right\}, \text{ by (iv)}.$$
$$\therefore \frac{D(\tau)}{\tau} = \frac{D(\tau_0)}{\tau_0},$$
$$\frac{D(\tau')}{\tau'} = \frac{D(\tau_0)}{\tau_0}.$$
$$\text{(v)} \qquad \therefore \frac{D(\tau)}{\tau} = \frac{D(\tau')}{\tau'}, \text{ where (iv) holds.}$$

For all practical purposes, this may be taken to prove our claim, for by choosing τ_0 small enough, we can always approximate τ and τ' by relations as in (iv). Let us however give a rigorous proof of the claim for the general case. What remains to be proved is simply that (ii) continues to hold when *at least one* of τ and τ' is an irrational number[1], for in case they are both rational, then (iv) is automatically satisfied[2].

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[1] This is understood as of given unit of measurement of " time".

[2] Suppose τ and τ' are both rational numbers. This means that there exist positive integers a, b, a' and b' such that $\tau = a/b$ and $\tau' = a'/b'$. It is now easily verified that (iv) is satisfied, by putting $\tau_0 = 1/(bb')$, $N = ab'$ and $N' = a'b$. Note that, by definition, $\tau_0 > 0$ and N and N' are positive integers.

The method we adopt for proving this is simply to prove the equivalent proposition that for any $\varepsilon > 0$,

$$\left| \frac{D(\tau)}{\tau} - \frac{D(\tau')}{\tau'} \right| < \varepsilon .$$

The proof is as follows.

Take any $\varepsilon > 0$.

Now, take a number $\varepsilon_1 > 0$ such that $\tau + \varepsilon_1$ is a rational number and

$$\text{(vi)} \quad \left| \frac{D(\tau)}{\tau} - \frac{D(\tau + \varepsilon_1)}{\tau + \varepsilon_1} \right| < \varepsilon/2 .$$

(Such an ε_1 exists simply because $D(\tau)$ is contineous in $\tau \varepsilon (0, \infty)$). Similarly, take a number $\varepsilon_2 > 0$ such that $\tau' + \varepsilon_2$ is a rational number and

$$\text{(vii)} \quad \left| \frac{D(\tau')}{\tau'} - \frac{D(\tau' + \varepsilon_2)}{\tau' + \varepsilon_2} \right| < \varepsilon/2 .$$

Now, since both $\tau + \varepsilon_1$ and $\tau' + \varepsilon_2$ are rational numbers, so, by the last footnote, (iv) holds for $\tau + \varepsilon_1$ and $\tau' + \varepsilon_2$ and hence by (v) we have :

$$\text{(viii)} \quad \frac{D(\tau + \varepsilon_1)}{\tau + \varepsilon_1} = \frac{D(\tau' + \varepsilon_2)}{\tau' + \varepsilon_2} .$$

Now,

$$\left| \frac{D(\tau)}{\tau} - \frac{D(\tau')}{\tau'} \right|$$

$$= \left| \frac{D(\tau)}{\tau} - \frac{D(\tau + \varepsilon_1)}{\tau + \varepsilon_1} + \frac{D(\tau' + \varepsilon_2)}{\tau' + \varepsilon_2} - \frac{D(\tau')}{\tau'} \right| , \text{ by (viii)}$$

$$\leq \left| \frac{D(\tau)}{\tau} - \frac{D(\tau + \varepsilon_1)}{\tau + \varepsilon_1} \right| + \left| \frac{D(\tau' + \varepsilon_2)}{\tau' + \varepsilon_2} - \frac{D(\tau')}{\tau'} \right|$$

$$= \left| \frac{D(\tau)}{\tau} - \frac{D(\tau + \varepsilon_1)}{\tau + \varepsilon_1} \right| + \left| \frac{D(\tau')}{\tau'} - \frac{D(\tau' + \varepsilon_2)}{\tau' + \varepsilon_2} \right|$$

$< \varepsilon/2 + \varepsilon/2,$ by (vi) and (vii)

$= \varepsilon$.

$$\therefore \left| \frac{D(\tau)}{\tau} - \frac{D(\tau')}{\tau'} \right| < \varepsilon .$$

This completes the proof of our claim.

Chapter 3

TOOLS AND CONCEPTS, II : CAPITAL

Section 1 : The Condition of Maintaining Capital Intact — the ''Ploughback Rule'' of the Process [1]

Our object here is to spell out the conditions of ''maintaining capital intact'' which we maintain all through as part of our basic framework of the process of trade. For this purpose, we simply go back to the internal transformation going on inside the trader's capital described by *Adam Smith in the passage* quoted in Chapter 1 and spell it out using terms established there.

1. Any purchase by the trader is by definition a drawal from his purchase fund and an addition to his sales stock by the amount drawn from the purchase fund divided by the price paid. Let us now define his capital as a magnitude to be the sum of the money in the purchase fund and goods in the sales stock valued at cost, i.e., at the price paid per unit of the goods bought[2]. It is then clear that there is no change in the capital as such through any purchase.

1/ This section and the next are based entirely on the work of P. Gajapathi referred in Chapter 1.

2/ It is assumed all through that this is a constant price in Time. There is then no ambiguity about the valuation of stock.

2. Let us now turn to the sales. We have to begin with a prior assumption. We assume that there is a well defined 'cost' behind any sale, which we further equate to the cost of purchase of the amount sold, i.e., the amount sold times the buying price of the trader. In other words, we assume that the whole spending in theprocess is given simply by the ''cost of purchase'' in the sense just defined[3].

Starting from any sale, we now require the trader to lay aside from his sale proceeds precisely the cost of purchase of the amount sold and put this into the purchase fund. We call this the ''ploughback rule'' of the process. It is clear that under this rule, there is again no change in the trader's capital as such through any sale — the amount sold is simply withdrawn from the sales stock and the cost-value of this amount is put back into the purchase fund.

3.1 It follows from the definition of capital and the ploughback rule set out above that there is no change in the trader's capital through the whole sequence of sales and purchases, i.e., through the process as a whole. This comes to the same as saying that capital is maintained intact through the process. Granted the definition of capital, we thus rigorously ''deduce'' this condition from a substantive ''rule'' on the process. A moment's reflection shows that the condition and the rule are in fact equivalent.

3/ This rules out the socalled transaction cost (except in the sense of a pure psychic cost) which in turn comes to define a clear point of extension of the ''framework'' now being developed. The matter is taken up in Section 3 below. Till then we continue with the assumption just introduced.

3.2 We can now give a compact mathematical representation of the definition of capital together with the condition of maintenance of capital just stated by the following <u>formula</u> :

(1) $K = qS_t + F_t$.

''t'' here stands for any arbitrary point of time and S_t and F_t for the respective amounts in the sales stock and the purchase fund at time point, t. q is the price paid by the trader per unit of goods bought. Since this is by assumption a constant through Time, there is no ''time subscript'' to it. K stands for <u>capital</u> as per the definition given. The condition of ''maintenance of capital'' is indicated by the absence of any time subscript to K, implying that this too is a constant through Time.

3.3 One point of some subtlety about the formula may be noted. The time point of any sale or purchase is by definition a point of discontinuity of S_t and F_t considered as ''functions'' of t. This does not however affect our formula. For any point of discontinuity of S_t and F_t, we may take them to be defined in <u>either</u> the ''left hand limit'' sense (S_t^- and F_t^-) <u>or</u> in the ''right hand limit'' sense (S_t^+ and F_t^+) in

(1), and (1) still holds. This is taken as understood throughout the discussion. In economic terms, the left hand limit gives us the values of S_t and F_t immediately before a sale or purchase and similarly the right hand limit gives us the values immediately after the event concerned. According to our substantive statements, the sum, $qS_t + F_t$, is left intact in each case i.e., the right hand and the left hand limit of K are the same at all purchase points and sale points.

4. Before ending, let us take up a few conceptual points. It was mentioned at the beginning that the ploughback may represent a purely notional division of sale proceeds inside the process until a fresh purchase is made or the profit is taken out. It follows from this that the purchase fund and hence capital itself may also remain as purely notional entities inside the process without an actual existence. This must be granted for what is observable from outside is only the purchase and from the fact of a purchase we may infer only that a sufficient fund must have existed inside the process to finance it at the Time concerned, no more. None of this however detracts from the significance of our formulations.

Section 2 : The ''Spending Rule'' and the Concept of Stock Maintained

1. ''Capital and ''stock'' each has its own special conceptual problems. Maintenance of capital can be understood in a point-to-point or continuous sense, as in our formula. The concept of maintenance of

stock cannot be so understood because of the simple fact that the stock goes up and down with every buying and selling and there is no perfect synchronisation of the sales and purchases over Time (if there were, the stock would not have simply existed !). The concept of maintenance of stock is therefore necessarily to be understood in a <u>freer</u> sense. On the other side, capital, as just seen, can remain ultimately as a <u>notional</u> entity inside the process. But the stock is necessarily a physical entity having an <u>actual</u> existence, though, needless to mention, it also exists inside the process and is therefore not observable from outside.

2.1 So much for the a priori conceptual clarifications of our subject. Let us now begin with the ''spending rule''. In the background, we posit a well-defined purchase fund existing inside the process at any moment for the trader to spend from to buy his goods. The ''rule'' is simply that <u>in any purchase, the trader spends his whole preexisting purchase fund</u>.

2.2 Let us now follow through the consequences of the rule. The immediate consequence is that the <u>purchase fund immediately after a purchase is zero</u>. We can put this in symbols as :

$$(2) \qquad F_t^+ = 0, \quad t \in T_P ,$$

where T_P denotes the set of successive purchase points. Substituting the value in (1) we then have

(3) $$K = qS_t^+ \ , \quad t \ \varepsilon \ T_P \ .$$

S_t^+ here ($t \ \varepsilon \ T_P$) is nothing but the stock immediately after a purchase. We can now connect up with our statement of the framework of trade process given in Chapter 1. (3) establishes at once (a) the one-to-one correspondence between capital and the stock immediately after purchase and (b) hence the ''constancy'' or ''sameness'' of the latter magnitude through the successive purchases, i.e., for all $t \ \varepsilon \ T_P$, which in turn gives (c) a precise and rigorous definition of the notion of maintenance of stock and (d) thence identifies the ''level'' at which the stock is maintained simply as the stock immediately after purchase or the stock ''restored'' by the purchase, as we may also put it (hence the fuller expression : stock maintained through successive purchases).

In view of the above, we can now simplify the notation in (3) and write it as :

(4) $$K = qS$$

where S denotes the stock immediately after a purchase, equivalently the ''level'' at which the stock is maintained through the successive purchases. The absence of any ''time-subscript'' to S already denotes its ''constancy'' through Time which in turn is implicit in the very notion of ''maintenance'' of stock.

2.3 A clear idea of mechanism behind (a) to (c) is implicit in our two rules, i.e., the ploughback rule and the spending rule. Let us now spell this out.

According to the spending rule, the purchase fund just before a purchase, i.e., F_t^- for any $t \varepsilon T_p$, is simply the total ploughback made over the Time from the last purchase to the purchase under reference. According to the ''ploughback rule'', this total ploughback is nothing but the cost of purchase of the amount sold over the Time referred and so, by the ''spending rule'' again, the amount bought in the purchase under reference is simply the amount sold between last purchase and this purchase. It follows from this that the purchase itself restores the stock back to its value immediately after the last purchase. Since this is true for any arbitrary purchase, the stock immediately after purchase is at the same level after each successive purchase.

Section 3 : Transaction Cost and Capital

1. The concept of transaction cost was explained in Chapter 1 where it was pointed out that this could be either an actual cost in the sense of money spent or a purely notional cost. In the latter case, our framework of capital is left intact, for there is then no question of financing the cost, which is the link between 'capital' and 'cost'.

2.1 Let us now suppose that the transaction cost represents an actual spending. (This is taken as understood all through the remaining discussion here.) The fundamental problem with this from our point of view is that the cost is incurred per 'act' of purchase, not per unit of the goods bought. So, given any sale, we do not know a priori what 'transaction cost' is incurred for it, and cannot define a ''ploughback'' from the sale-proceeds covering the ''full cost''. This was already hinted at the beginning of this chapter.

2.2 There may be pure ''accounting'' solutions to this problem of costing. But we opt for a more operational approach to our general problem. This consists in essence of giving up the attempt to define a ploughback in reference to each particular sale and substituting this by a ploughback defined in reference to the total sale between two successive purchases, equivalently in reference to the corresponding purchase interval as a whole. Note that this at once makes the purchase fund undefined at any point between two successive purchases. Hence capital as a whole also becomes undefined at these points. This does not however create any problem in the operational sense, for the purchase fund itself becomes operationally relevant only at the time of successive purchases. So, from a strictly operational standpoint, we can make do with a notion of capital which exists in its physical form (sales stock) all through but in its financial form (purchase fund) only at the time of successive purchases. This is the approach. Obviously, this amounts to a re-framing of the whole notion of capital.

3.1 Let us now return to the ploughback rule. Obviously, we must retain the essence of our earlier ploughback rule in this new framework. This essence was simply that the ploughback enabled the trader to finance precisely the replenishment of the stock sold from out of the sale-proceeds. Let us now see what this financing boils down to when there is a transaction cost to be financed and the ploughback is defined for a purchase interval as a whole. In a word, the transaction cost itself has to be first ploughed back; after this, the actual cost of the goods sold over the interval (= amount sold x buying price) has also to be ploughed back. The ploughback for the interval as a whole is simply the sum of these two costs which we can call the ''whole cost'' of the sale over the interval. This defines the new ploughback rule which is obviously well defined.

3.2 Does this solve our problem ? We now have a well defined purchase fund at each purchase point (just before the purchase); the sales stock is also well defined at these points. But what about the value of this stock ? Without any well defined value per unit of the goods (value in the sense of cost-value), we cannot simply value the stock in the sense intended.

Again, there may be pure accounting solutions to the problem, but we continue with a more operational approach. However, this time, the approach is defined through a substantive assumption on the process and thus restricts the scope of the analysis. The assumption is nothing but the ''purchase rule'' as stated in Chapter 1 in the context of the question 'when to buy', viz. a fresh purchase is made only when the stock is sold out. (See p. 31).

The immediate implication of the assumption for our purpose is that capital just before a purchase consists only of the purchase fund, which, we just saw, is well defined at this point. What is the magnitude of this fund ? We now have the whole stock immediately after a purchase as sold over the ensuing purchase interval (purchase rule) and the whole cost of this sale as ploughed back into the purchase fund (ploughback rule). These two propositions serve to give us a well defined magnitude of capital, in the process of deriving which we also prove afresh the constancy of capital through Time. Let us argue this out.

4.1 Let us remember that capital is now defined only at the successive purchase points, more precisely, just before each successive purchase. Hence to prove the constancy of capital through Time, it suffices to prove that capital just before any two successive purchases has the same value.

Let now t and t' (> t) represent any two successive purchase points. Since a purchase is made only when the stock has fallen to zero (purchase rule), we have :

(5) $$K_t^- = F_t^-$$

and

(6) $$K_{t'}^- = F_{t'}^- .$$

Now, $F^-_{t'}$ is given by the ''whole cost'' of the sale between t and t' (ploughback rule) and the sale in turn is given by the stock immediately after the last purchase, i.e., S^+_t (purchase rule). It then follows from the definition of ''whole cost'' that

$$(7) \qquad F^-_{t'} = A + qS^+_t$$

where A denote the <u>transaction cost per act of purchase.</u> By the spending rule of the process, S^+_t is simply what is bought by spending the full amount, F^-_t. So,

$$(8) \qquad S^+_t = \frac{F^-_t - A^{4/}}{q} .$$

So, we have :

$$K^-_{t'} = F^-_{t'} , \text{ by } (6)$$

$$= A + qS^+_t , \text{ by } (7)$$

$$= F^-_t , \text{ by } (8)$$

$$= K^-_t , \text{ by } (5) .$$

4/ Note that this equation implicitly restricts F^-_t to take values <u>larger than A.</u> This does not however create any problem for our purpose.

(9) $\therefore \qquad K^-_{t'} = K^-_t$.

This proves the constancy of capital through Time. So we can now write K^-_t simply as K. From (9), (6) and (7) it then follows that :

$$K = A + qS^+_t .$$

It follows that the stock immediately after a purchase is also a constant through Time, and so we can write it, i.e., S^+_t , simply as S. The magnitude of capital is then given by the equation :

(10) $$K = A + qS .$$

This is our new or 'revised' formula for capital in the presence of transaction cost.

4.2 Before ending, we note that (10) implicitly defines a rule of valuation of the stock immediately after purchase. So, even though we scrupulously avoided the problem of stock valuation in our approach, the arguments nevertheless give us precisely a rule for this in the end. It is however to be clearly pointed out that (a) the rule is defined only in respect of the stock immediately after purchase, not the stock at any point, and (b) even this is defined only on the basis of the assumption that the stock immediately after purchase is the same as the amount bought. So, whatis really valued is simply the amount bought. The problem of stock valuation is effectively evaded thanks to the absence of any stock carried over from one purchase interval to the next.

5.1 Before ending this section, we have to point out that our whole analysis here is based on a tacit assumption that the whole transaction cost per act of purchase, A, is actually spent precisely at the successive purchase points. Whether or not this is so however depends upon the precise nature of the organisational set up for purchase within the process. We will not complicate our scheme by trying to accomodate variations on this point. However, the problem motivates us to take a fresh look at the conceptual basis of our whole framework.

5.2 Let us start from the point that all spendings in a going process of trade must be *financed* ultimately out of its sale-proceeds. Given the basic nature of the process, we are then led to conceive the payment for the successive purchases as coming out of the sale-proceeds via a purchase fund created by ploughbacks from sale-proceeds. When the transaction cost is also incurred precisely at these points, we have a clear basis for extending the scheme to cover it. What if the hypothesis here is not true ? Suppose the actual spending on account of the transaction cost is spread out in Time in some general fashion. It appears quite permissible to suppose that the spending is in this case financed directly out of sale-proceeds, without being routed through the purchase fund. The transaction cost does not then enter into capital at all, taking us back to the earlier formulas for capital exactly in the case of a notional transaction cost.

5.3 We are thus led to a <u>general division of the trade process with a transaction cost into two cases, one where the transaction cost enters capital and one where it does not.</u> For the former, the formula for capital is taken to be given by (10) while for the latter, it is given by definition by (4). This is the ultimate upshot of the element of transaction cost in a trade-process for our purpose here. Let us just end by pointing out that the division just suggested may also be defended in terms of a purely accounting distinction in the sense of there being two distinct rules of accounting for the two categories of cost.

Chapter 4

TOOLS AND CONCEPS, III : PROFIT

1.1 This Chapter has the entirely limited objective of fixing an "appropriate definition" of profit in the decision criterion of a trader for his inventory problems in view of the substantive considerations in this regard stated at length in Chapter 1 (see pp. 27-9). Our task therefore boils down essentially to a formalisation of these ideas. Before beginning, we just mention that the form of the decision criterion is implicitly taken to be maximum rate of profit, our concern here being solely with the term 'profit' in this context. The criterion as such does not explicitly enter our discussion of this Chapter (see however footnote 1 below).

1.2 Let us now recapitulate the background. We are concerned with a trader operating under given conditions of the market which gives him a certain rate of demand. This sets the limit to his sales, subject to which he carries out his profit making. However, this is only <u>of the present</u> (or the <u>shortrun</u>). Whether or not demand continues to come to him at the same rate in the <u>future</u> depends upon what proportion of it is in fact met by the trader at present, given no change in the overall condition of the market. Though in itself this is an objective relation, we can enter it in our framework only at the subjective level of the trader's decision criterion. This is what brings us to the concept of profit. The important point is that we have to go beyond just the profit actually made for an adequate notion of profit in this context, for the profit made is only of the present or the shortrun and fails to bring out anything of the trader's concern for his own future (or the longrun) which is the substantive point raised.

2.1 Let us now argue from this point. The future or the longrun here is already present in the proportion of demand actually met in the present or the shortrun. The higher the proportion, the better the future prospects. We can also put this equivalently by saying that the smaller the demand that is not met at present, the better the future prospects. This translates back the argument into absolute magnitudes, which is analytically convenient for our purpose. Let us also introduce the expression " sales foregone" for " demand not met". We thus have the proposition that the smaller the sales foregone, the better the future prospects.

Starting back from profit made, we can now fix our appropriate definition of profit by assigning a negative weight to sales foregone and adding this to profit made. Profit is then expressed as a weighted sum of the two, with a weight, 1, to profit made, representing the present, so that the weight assigned to sales foregone, representing the future, is implicitly a relative weight reflecting the trader's own concern for his future vis-a-vis the present. The greater the value of this weight in absolute terms (neglecting the 'sign'), the greater is the extent of this concern shown.

2.2 Let us now take a purely formal step for this definition. This consists simply of going from the " sales foregone" talked of to the profit foregone upon the sales foregone (or just " profit foregone" for short). Since profit foregone and profit made are measured in the same unit, the weight upon the former becomes a pure dimensionless number which is analytically very neat. There is also the conceptual gain of remaining entirely within the term profit, with the implicit correspondence between profit made and present profit on the one hand and profit foregone and future profit on the other taken as understood. We can call this a reformulation of the very concept of profit.

2.3 Let us now put the matter in symbols. Let P_m and P_f stand respectively for profit made and profit foregone[1]. Thus profit as per our reformulation is defined by :

(1) $P = P_m - \lambda P_f, \; \lambda > 0$

where λ is the weight expressing the trader's own concern for the future vis-a-vis the present. It remains by definition as a subjective parameter in the definition of profit just given with the interpretation that a higher value of λ means a greater concern for the future vis-a-vis the present.

3.1 Can we go any further than this, in the sense of carrying the interpretation to numerical values of λ? Going purely by the form of (1), it may appear that the value $\lambda = 1$ --- equal weight to P_m and P_f --- can be given the interpretation of a cut off point between farsightedness and

[1] P_f and P_m are taken to be measured in reference to some unspecified interval of time as of the given state and can be expressed either in absolute monetary unit or in the unit of money per unit of time. This does not matter for the discussion here, but for future reference we mention that when we come to the rate of profit, we must understand 'profit' in its numerator in the latter form, i.e., as profit per unit of time.

myopia or the relative overvaluation and undervaluation of the future. However, we have far too many implicit steps in our argument to permit such interpretation, at least in any straight sense. The upshot is that we are left without any a priori benchmark of reference for judgements on the order of magnitude of λ. This is a matter left entirely to inferences from the operational consequences of different values of λ in the context of the substantive problem of reference. This limitation of the formulation is simply admitted.

3.2 Let us add one final observation in this respect. There is a tautological relation between P_m and P_f in any given situation, for the making of a profit is the same as not foregoing that profit. The motive for avoiding P_f from this standpoint is already included in the argument, P_m, in (1). The weight, λ, on P_f in P in (1) comes in as something <u>over and above</u> this definitional implication of variation of P_f for P_m. It is in this sense an " additional weight" upon P_f for its connotations regarding the future. This is a point to be borne in mind in interpreting specific values of λ.

PART II : MODELS

Chapter 5

MODEL I : OPTIMUM STOCK UNDER UNCERTAIN DEMAND AND A GIVEN PURCHASE INTERVAL

Section 1 : Structure of the Model

1.1 We now begin on the sequence of our inventory models in the thesis, already outlined in Chapter 1. As stated there, the first model or Model I, which is analysed through this chapter and the next, is based on the assumption that the trader makes his successive purchases at a fixed interval through Time, the interval being given from outside the model. For convenience of expression, we will refer to this interval as a " week".

1.2 As a result of this assumption, we have the trader's total sale in any week limited by the stock at the beginning of the week, understood as the stock immediately after the purchase made at this point, i.e., the stock carried over from the previous week plus the amount just bought. The sale referred is also limited by the total demand occurring in the week. The sale is thus subject to a " dual constraint", and between the two we have the whole problem of sales foregone which lies at the heart of the inventory problem of this model.

1.3 However, before proceeding to this, we have to state that in our model the stock at the beginning of each week (in the sense of the term just defined) is the same (i.e., of the same magnitude). It is called simply the " stock" and denoted by S. For any given value of S, there is a certain capital put in and maintained through the process, denoted K. S and K are related by the formula :

(1) $K = qS$

where q denotes the trader's buying price per unit of his goods, assumed constant through Time. All this comes out from our basic framework of capital —— equivalently, the " ploughback rule" and the " spending rule" of the trade process ——— set out in Chapter 3, Sections 1 and 2. For completeness, we mention that it is assumed in the background that the only cost incurred in this process is the cost of purchase of the goods bought and sold by the trader.

1.4 Let us now return to the idea of " dual constraint" upon the trader's sale in a week. Let us fix our attention upon a particular week and denote the total demand occurring in the week by D. The dual constraint is then given by the following two inequalities :

(a) $X \leq S$

(b) $X \leq D$

where X denotes the trader's sale in the week under reference.

We will assume that in any particular week, at least one of (a) and (b) is a binding constraint, and so we can write them compactly as :

(2) $X = \min \{ S, D \}$.

This fixes the total sale in any week.

Let us now note the simple point that D comes from outside the trade process while S is ultimately chosen by the trader. Hence the substantive assumption underlying (2) is that the trader goes on meeting all the 'demands' that come so long as he has a stock to do this. This fixes the complete " sales rule" of the process kept in the open earlier (p.13,fn).

1.5 It remains to state our assumptions regarding demand. We assume that demand is a random variable in the sense explained at length in Chapters 1 and 2 (see pp.23-7, pp.50-1). To recapitulate, this means that the total demand over any given interval of time is a random variable depending only upon the length of the interval and not upon its location in the time axis. This random variable is then denoted as $D(\tau)$ where τ stands for length of the interval under reference.

As a result of this assumption, we can go from (2) to a statement of the total sale of any week, which is given by

$$X = \min \left\{ S, D(\tau) \right\}$$

for τ = a week's length of time or the given purchase interval in the pure " time length" sense of the term[1]. Since however τ is simply a given datum in our model, we can simplify the notation by writing back D for $D(\tau)$. This takes us back to (2) as the equation for the sale in any week in general and not a given week.

2.1 Statement of the model is now completed by saying simply that either S or K — it does not matter which[2] — is so chosen that expected rate of profit is at its maximum where " profit" is understood as per our

[1] See footnote 35, p. 40.

[2] In the formal structure of the model, the decision variable is specified as S.

own reformulation of the concept set out in Chapter 4. We can also put this point in more behaviourial terms by saying that once the implications of alternative choices (or decisions) of S are set out before the trader, he would in fact choose the one yielding him the highest expected rate of profit. The value of S so chosen defines the " optimum stock" in the model.

2.2 Let us now proceed to define the terms that come in through the decision criterion just stated. First, the profit made in a week, denoted P_m, is given by :

(3) $$P_m = \pi X,$$

where π denotes the profit margin per unit of sale, i.e.,

$$\pi = p - q$$

and p is the trader's selling price per unit of his goods, assumed to be a constant through Time. Next, the sale foregone and profit foregone in a week, denoted respectively X_f and P_f, are defined by

$$X_f = D - X$$

$$P_f = \pi X_f$$

(4) $$= \pi (D - X)$$

Profit in a week, denoted P, is then defined by

$$P = P_m - \lambda P_f$$

$$= \pi X - \lambda \pi (D - X), \text{ by (3) and (4)}$$

(5) $\quad = (1 + \lambda)\,\pi X - \lambda\pi D$

where λ denotes the subjective parameter in profit as explained in Chapter 4.

2.3 Let us take off a minute here to link up with a previous observation. Let us fix our attention on the first term in the RHS of (5). λ clearly enters here as something added to unity which is the original weight upon $P_m = \pi X$ in P, this " additional weight" being simply a transfer from P_f to P_m. This bears out the previous observation that λ is really an " additional weight" upon P_f for its connotations regarding the future quite apart from what P_f already implies about P_m, which is purely of the present (see p. 87).

2.4 To return to the definition of terms in our model, the rate of profit, r, and expected rate of profit, ρ, are defined by :

(6) $\quad r = \dfrac{P/\tau}{K}$ [3]

and

(7) $\quad \rho = E(r)$

where E(.) denotes the expectation operator.

3. Clearly, after substitutions from (6), (5), (2) and (1) in (7), we can express ρ as :

[3] Note that division of P by τ converts the numerator of the expression to profit per unit of time which is the appropriate dimensionality of profit in the expression, rate of profit (see footnote 1, p. 86).

$$(8) \qquad \rho = E \left[\frac{(1+\lambda)\, \pi \min\{S, D\} - \lambda \pi D}{\tau q S} \right].$$

The whole structure of the model is summed up in this equation. In particular, it is seen that since λ, π, τ and D are all given from outside the model, ρ is obtained as a function of S, say $\rho(S)$. So, we can talk of $\rho(S)$ being maximized, i.e., maximized wrt S. This shows that the inventory problem in the model is indeed well defined in the *a priori* sense. Whether or not it is also well defined in the *a posteriori* sense, i.e., a solution to this problem exists or not, is of course left open. This brings us to the domain of analysis of the model which we begin in the next section and continue through to the next chapter. Just to draw the line, we mention that the subject reserved for the next chapter is the effect of variation of the purchase interval, τ, on the optimum stock. The effect of variation of all other "givens" or "data" of the model are covered within the course of this chapter. These givens, i.e., givens apart from the purchase interval, τ, consist of (i) the two prices, p and q, which we may also refer to as the whole cost-price data in the model; (ii) the subjective parameter in profit, λ; and (iii) the demand conditions as represented by the pdf of D, to be denoted by f(u)[4]. This gives another summing up of the structure of the model.

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[4] Note that τ by definition enters f(u) as parameter. In line with the notational scheme introduced earlier, this is not explicitly shown here.

Section 2 : The Optimum Stock

1.1 Before we take up our problem of optimum stock under the conditions of uncertain demand and a given purchase interval, which are the defining characteristics of the present model, let us just see it in abstraction from the whole uncertainty of demand. This, we may say, defines the " simplest inventory model" in our whole analytical scheme. As such, it is also the logical starting point of all the subsequent models. Our object here is simply to do the preliminary groundwork of working out the model and thus set the stage for the rest.

1.2 Formally, the simplest model is obtained from Model I by simply treating the demand in a " week", D, as a determinate magnitude or scalar, not a random variable. It is actually a demand repeating itself over week after week or the demand per week. The sale per week in this sense, X, is set by the minimum of D and the " stock", S, which is also actually the stock at the beginning of every week in succession.

Obviously, $X = S$ for $S \leq D$ and $X = D$ for $S > D$. Let us now start from the critical value of S dividing the two regions, i.e., the value $S = D$, and see what moving into either region implies in terms of the rate of profit, which is obviously a scalar now.

1.3 Now, capital remains proportional to S all through, and so it increases or decreases proportionately with S as S is increased above or decreased below D. Let us now turn to profit. If S is increased above D, then X just stays constant and hence so does profit (= profit made, for there is no sale foregone in this region). The rate of profit must therefore fall. If on the other hand, S is reduced below D, then, firstly, X falls by the same amount and hence profit made falls proportionately with S, but profit as per our definition falls still further, for there now emerges a 'sale foregone', D–S. It follows that the rate of profit must again fall. In other words, the rate of profit falls in any case, whether we move into the region, $S > D$, or into the region, $S < D$. Hence it is at a maximum when $S = D$, and only then. The optimum stock, which we may still denote by S^*, is therefore given by $S^* = D$.

The point to note is simply the critical significance of assigning a negative weight to 'profit foregone' in the definition of profit, in the above derivation of optimum policy. If this weight were not there, i.e., $\lambda = 0$, then profit would have simply coincided with profit made and the two would have fallen proportionately with S as S were reduced below D. Since capital too falls proportionately, the rate of profit would have been left invariant. There would then be nothing to discriminate between values of S over the whole range, $0 < S \leqq D$, and the optimum policy would have been left indeterminate. Starting back from this point, we can now see that it is $\lambda > 0$ which alone gives us a determinate or well defined policy in the simplest model. To repeat, if $\lambda = 0$, then S^* is left indeterminate over the whole range, $0 < S \leqq D$, and $\lambda > 0$ converts this indeterminate S^* to the determinate value $S^* = D$.

1.4 For completeness, we just set down the algebraic steps corresponding to the deductions just made. For this, we express the rate of profit as a function of S with λ explicitly as a parameter, denoted $r_\lambda(S)$. We can now argue straight from the definition of $r_\lambda(S)$ as follows.

$$r_\lambda(S) = \frac{\pi}{\tau q} \cdot \frac{(1+\lambda)\min\{S, D\} - \lambda D}{S}, \text{ by definition}$$

$$= \begin{cases} \frac{\pi}{\tau q} \cdot \frac{1}{S}[(1+\lambda) S - \lambda D], & S \leq D, \\ \frac{\pi}{\tau q} \cdot \frac{D}{S}, & S > D. \end{cases}$$

Let us simplify the notations by writing r^* for $\pi / \tau q$ and x for D/S so that $r_\lambda(S)$ is also implicitly converted into a function of x. We then have :

$$(9) \qquad r_\lambda(S) = \begin{cases} r^*[1 - \lambda(x-1)], & x \geq 1, \\ r^* x, & x < 1. \end{cases}$$

It is clear from (9) that (a) r^* is the maximum value of $r_\lambda(S)$ over all $S > 0$ for all $\lambda \geq 0$; and (b) this maximum is obtained at the <u>unique</u> value, $x = 1$ (i.e., $S = D$) iff $\lambda > 0$, $r_0(S)$ being simply the constant, r^*, for all x, $1 \leq x < \infty$ (i.e., $0 < S \leq D$).

2. We are now through with the simplest model. Returning to Model I, the optimum stock in it is obtained by maximizing the function $\rho(S)$ defined in the last section. Let us take up this maximisation.

We note, to begin with, that we can write $\rho(S)$ as :

$$\rho(S) = \frac{\pi}{\tau q} R(S),$$

where

$$(10) \qquad R(S) = \frac{1}{S} E[(1+\lambda)\min\{S, D\} - \lambda D].$$

Two things follow from this. First, the constant factorised out of $\rho(S)$ in defining $R(S)$ does not play any role in the maximization of ρ. This means in particular that the whole optimization in our model is independent of the whole cost-price data, (p, q). The optimization is in this sense defined in purely " physical terms". Note that we cannot make the same independence assertion wrt τ which is also an element of the constant factorised out. This is because τ implicitly enters $R(S)$ through the pdf of D, $f(u)$.

Secondly, we note that the 'constant' under reference is nothing but the maximum rate of profit of the simplest model, which we denoted by r^*. We also note that the random variable whose expectation is taken in (10) has by definition value less than or equal to S. Hence $R(S) \leq 1$, so that

$$(11) \qquad \rho(S) \leq r^* .$$

Thus the expected rate of profit in our model is bounded above by the maximum rate of profit of the simplest model.

3. Returning to our problem, the first step we have to take is to find out the expected value of $\min\{S, D\}$, i.e., the expected sale in a week or $E(X)$. The expectation is taken over the random variable, D, the pdf of which is already denoted as $f(u)$. Hence by definition

$$E(X) = E(\min\{S, D\})$$

$$(12) \qquad = \int_a^S uf(u)du + S\int_S^b f(u)du ,$$

where (a, b) denotes the domain of the pdf, f(u)[5]. The expression for E(X) just obtained simply expresses the fact that in our model the demand in a week, u, is fully satisfied upto S, i.e., the stock at the beginning of the week, and hence sale coincides with demand in this range, beyond which it just stays fixed at the level, S.

Let us now simplify the notation by writing (12) as :

$$(13)\qquad E(X) = \int^{S} u f(u)du + S \int_{S} f(u)du$$

i.e., we simply omit the limits, a and b, in the RHS integrals in (12). This is a general convention followed throughout the rest of the thesis. Stated explicitly, the convention is that the omitted limits of an integral stand for the limits or infimum and supremum of the pdf entering the integral concerned.

Let us now find out the effect of a marginal change in the value of S upon E(X). Firstly we assume that f(u) is continuous over [a, b], when b < ∞ or over [a, b) when b = ∞[6]. Then we have :

$$\frac{dE(X)}{dS} = Sf(S) - Sf(S) + \int_{S} f(u)du, \quad \text{by (13)}$$

$$(14)\qquad = \int_{S} f(u)du$$

4.1 We can now take up the maximization of ρ (S). Maximization of ρ (S) is obviously equivalent to maximization of R(S), i.e., the maximizing

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[5] Note again that a and b are functions of τ which is not explicitly shown here. Also for the record, we note in this context that the domain (a, b) of f(u) satisfy by definition the condition :

$$0 \leq a < b \leq \infty.$$

[6] The continuity of f(u) is taken as granted through our whole analysis.

value of the variable, S, is the same in both cases. So, we deal with the simpler expression, R(S). Differentiating the function and substituting from (14) and (13) we obtain :

$$R'(S) = \frac{1}{S^2}\left\{(1+\lambda)\left[S\int_S f(u)du - \int^S uf(u)du - S\int_S f(u)du\right] + \lambda\bar{u}\right\}$$

$$= \frac{1}{S^2}\left\{\lambda\bar{u} - (1+\lambda)\int^S uf(u)du\right\}$$

$$\gtreqless 0$$

as

$$(1+\lambda)\int^S uf(u)du \lesseqgtr \lambda\bar{u} .$$

Clearly, the optimum value of S, to be denoted S^*, is obtained by solving the equation :

$$(15) \qquad (1+\lambda)\int^{S^*} uf(u)du = \lambda\bar{u} .$$

(15) is the fundamental optimality condition of the model. We can also write it as :

$$(16) \qquad \int^{S^*} uf(u)du = \alpha\bar{u} ,$$

where

$$\alpha = \lambda/(1+\lambda)$$

and so

$$0 < \alpha < 1 .$$

4.2 Now, since f(u) is continuous over its domain, the LHS of (16) is obtained as a monotonically increasing and continuous function of S^* going from 0 to $\bar{u}$ as S^* is varied over the domain of $f(u)$. It follows that for any α, $0 < \alpha < 1$, there exists a unique solution of (16) belonging to the domain of $f(u)$. So, the optimum stock, S^*, is indeed meaningfully determined by the optimality condition (16).

4.3 Further, it is also clear that S^* itself goes from a to b as α is varied over (o, 1) — which is to say that λ is varied over $(0, \infty)$ — monotonically increasingly with α over this whole range of α, i.e., $0 < \alpha < 1$. There is in fact a clear functional form relating S^* to α. We shall note this a little later.

5.1 Now, we have already pointed out the lack of any clear a priori benchmark for judging "high" and "low" values of λ (see p. 87). Hence, from a practical standpoint, little is said about the actual optimum policy. Accepting the criticism, let us now just reverse the question and see if we can talk meaningfully of an a priori benchmark for S^* itself.

This is where the "stage-setting" by the simplest model becomes relevant. S^* in this model is simply equal to D which is the demand per week abstracted from all uncertainty. Reintroducing the uncertainty, we simply identify this D with the mean demand of a week, i.e., $\bar{u}$ 7/. This establishes $\bar{u}$ as a clear a priori benchmark of reference for S^* in general. But, given the probability distribution of demand, $f(u)$, $S^* = \bar{u}$ in our model only for a critical value of λ. Given the complete openness of the

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7/ See in this connection the discussion in Section 3 of Chapter 2.

order of magnitude of λ, it is of interest to see what order of magnitude the critical value of λ takes for the given $f(u)$. This is a matter of explicit numerical calculation of the optimal policy in our model. We take this up in an appendix at the end of the chapter (Appendix A), which also includes the sensitivity analysis of the optimal policy wrt λ over a suitable range defined there.

5.2 Pending the numerical calculations let us see what, if anything, can be concluded regarding the relation between S^* and $\bar{u}$ by pure analytical means. For this, it is convenient to transform slightly the statement of the optimality condition. Let us just write $\bar{u}$ in (15) explicitly in terms of $f(u)$. This gives us :

$$(1 + \lambda) \int^{S^*} uf(u)du = \lambda \int uf(u)du$$

$$= \lambda \int^{S^*} uf(u)du + \lambda \int_{S^*} uf(u)du$$

or

$$(17) \qquad \int^{S^*} uf(u)du = \lambda \int_{S^*} uf(u)du.$$

Let us put $\lambda = 1$ in (17). S^* must then be <u>greater</u> than the <u>median</u> of $f(u)$, say u_m. This follows simply from the definition of u_m, viz.

$$\int^{u_m} f(u)du = \int_{u_m} f(u)du.$$

A forteriori, the result stated holds for $\lambda > 1$ as well. Hence we have the general result :

$$S^* > u_m \quad \text{if} \quad \lambda \geq 1.$$

From this, we conclude that unless $f(u)$ is highly positively skewed,

(18) $S^* > \bar{u}$, if $\lambda \geq 1$.

This, it appears, is all that can be concluded regarding the relation between S^* and $\bar{u}$ on pure analytical grounds. The rest is left to numerical calculations. However, we note that since S^* is an increasing function of λ, we also have the implicit result here that $\lambda = 1$ is an upper bound to the " critical value" of λ defined earlier, at least for the case of $f(u)$ being " not highly positively skewed". How far off this upper bound is from the actual critical value of λ is however a matter of numerical calculations.

6.1 Let us now continue on. It is obviously possible to obtain the order of magnitude of λ from the order of magnitude of $S^*/\bar{u}$ deemed in some sense " plausible" in the given environment. However no precise interpretation to the value of λ in a policy-sense is defined thereby. We cannot therefore see this as a meaningful procedure for a posteriori judgements on the order of magnitude of λ, which we spoke of earlier (p. 87). Let us now suggest one such procedure. This is going outside the strict analysis of the model. The matter is taken up for its general significance in our whole conceptual framework in the thesis.

6.2 The procedure is as follows. We take it for granted that our trader is a priori reconciled to the prospect of foregoing some sale in a week on account of his running out of stock. The question is what margin of such loss is allowed by him. This cannot be set in advance either in quantity or as proportion of sale for sale itself is uncertain. It can however be quite meaningfully set in advance in probability, in the sense, simply, of a targeted value for the probability of "full demand satisfaction" in a week. If we now grant the judgement of "plausible" values of this target in a definite environment, we indeed have the basis for a posteriori judgements on the order of magnitude of λ in that environment in a clear policy-sense. This is the procedure. It amounts simply to interpreting the "target" under reference in terms of profit calculations as per the whole scheme set out in the chapter on profit (i.e. Chapter 4).

6.3 Let us now come to the formal procedure. Given the stock S, the probability of full demand satisfaction is simply F(S), where F denotes the distribution function of the random variable D. So, if β^*, is the targeted value of the probability of full demand satisfaction, we then can at once find the stock implied by it and interpret it as the "optimum stock" in our sense. This is the first step. Mathematically, it amounts simply to the equation :

$$(19) \qquad \beta^* = F(S^*).$$

Let us now specify the form of the mathematical relation between S^* and λ left in the open earlier.

To arrive at this form, we write our optimality condition, (16), in the following form :

$$\frac{1}{\bar{u}} \int^{S^*} uf(u)du = \alpha .$$

The LHS of this equation is *nothing* but the <u>first-moment distribution function of the random variable,</u> D, $F_1(u)$[8], evaluated at $u = S^*$. Noting that α is nothing but a transformation of λ, the mathematical form of the relation between S^* and λ is implicitly specified by :

(20) $F_1(S^*) = \alpha$.

This is the second step. The procedure then is simply to infer the value of λ from the value of β^* deemed " plausible" in some sense by solving the following equation :

(21) $\bar{\alpha} = F_1(F^{-1}(\beta^*))$

which follows from (20) and (19). This obviously gives us a well defined computational programme. We take it up in the appendix already referred.

6.4 Let us again see what can be argued regarding the relation between α and β^* on pure analytical grounds. We note that for any positively valued random variable with pdf $f(u)$, F and F_1 satisfy the relation :

(22) $F(u) > F_1(u)$,

for all u in the domain of the pdf $f(u)$, where F and F_1 denote respectively the distribution function and the first moment distribution

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[8] The first-moment distribution function of any positively valued random variable X having pdf $f(u)$ with domain as (a, b) such that $0 \leq a < b \leq \infty$ and $E(X) = \bar{x}$, say, exists, is defined as :

$$F_1(x) = \begin{cases} 0 & , \text{ for all } x \text{ such that } -\infty < x \leq a , \\ \frac{1}{\bar{x}} \int^x uf(u)du & , \text{ " " } x \text{ " " } a < x < b , \\ 1 & , \text{ " " } x \text{ " " } b \leq x < \infty . \end{cases}$$

function of the random variable concerned[9]. Hence

(23) $\beta^* > \alpha$.

This is the analytical result concerning the relation between β^* and α. Stated in substantive terms, the result is that for any λ, the optimum stock is so chosen that the probability of full demand satisfaction is greater than the ratio, $\lambda/(1 + \lambda)$. Obviously, given the value of β^*, λ itself is bounded above by this relation, i.e., (23), e.g., if $\beta = 0.5$, $\lambda < 1$. How " far off" this upper bound on λ is from the actual value of λ for the given β^* is however again a matter that has to be left to explicit numerical calculations. This concludes our analysis of the optimal policy in our model vis-a-vis the subjective parameter in profit, λ.

Section 3 : Effect of Change in Demand Conditions on the Optimum Stock --- the General Case

1. Change in demand conditions in our model is represented by a perturbation of the pdf, f(u). Let us denote the perturbation by $\Delta f(u)$ and the change in the value of the optimum stock resulting from the perturbation by ΔS^*. Our object here is to see what can be said about the sign of ΔS^* just from this. This is a purely formal problem devoid of any substantive significance as the whole notion of " change in demand conditions" is left undefined, i.e., nothing is said about the nature of the change in an a priori sense. We shall come to the substantive problems

[9] This is a well-known result of Income Distribution Theory, more familiarly stated as " the Lorenz Curve, except for degenerate distributions, lies below the diagonal line". A proof of this result however is given in Appendix B below for the sake of completeness.

in this sense, where the change in the demand conditions is properly defined, in the following sections. The whole of this section is simply a formal preparation for these substantive problems.

2. Let us now begin. We begin with the purely formal meaning of saying that $\Delta f(u)$ defines a perturbation of the pdf, $f(u)$. Obviously, this means that the pdf before and after the perturbation are respectively $f(u)$ and $(f(u) + \Delta f(u))$. Note that the two pdf's (i.e., probability density functions) are not necessarily defined over the same domain. Let the respective domains be (a, b) and (a', b'). The domain of the function, $\Delta f(u)$ is then $(\min\{a, a'\}, \max\{b, b'\})$.

Following our general notational convention, we will omit the limits of the integral over $f(u)$, $f(u) + \Delta f(u)$ and $\Delta f(u)$ whenever they coincide with the limits of their respective domains specified above. The differences in these limits are of course to be clearly kept in mind.

Next, we point out that by definition

(24) $\int f(u)du = \int (f(u) + \Delta f(u))du = 1.$

Hence

(25) $\int \Delta f(u)du = 0.$

Let us now write down the optimum conditions determining S^* and $S^* + \Delta S^*$:

(16) $\int^{S^*} uf(u)du = \alpha\, \bar{u},$

(26) $\int^{S^*+\Delta S^*} u(f(u) + \Delta f(u))du = \alpha\, (\bar{u} + \Delta\bar{u}),$

where

$$\Delta\overline{u} = \int u\Delta f(u)du.$$

3.1 Let us now argue. Obviously,

$$\Delta S^{*} \gtreqless 0$$

$$\iff S^{*} + \Delta S^{*} \gtreqless S^{*}$$ [10]

$$\iff \int^{S^{*}+\Delta S^{*}} u(f(u) + \Delta f(u))du \gtreqless \int^{S^{*}} u(f(u) + \Delta f(u))du,$$ since $(f(u) + \Delta f(u))$ is a positively valued function over its whole domain.

$$\iff \alpha\,(\overline{u} + \Delta\overline{u}\,) \gtreqless \int^{S^{*}} u(f(u) + \Delta f(u))du,$$ by (26).

Let us now consider the RHS integral in the above expression. Its lower limit is a' where $a' \gtreqless a$. Consider first the case $a' \leq a$. Clearly, we can write

$$(27)\quad \int^{S^{*}} u(f(u) + \Delta f(u))du = \int^{S^{*}} u\, f(u)du + \int^{S^{*}} u\Delta f(u)du\,,$$

for (a) the lower limit of the second integral on the RHS of this equation is in this case a', i.e., the same as the lower limit of the LHS integral; and (b) we can consider the first integral on the RHS to be defined with a' as lower limit, for by definition :

$$f(u) = 0,\quad a' < u < a.$$

[10] The symbol "$\iff$" is to be understood in this and similar sequence of statements in a "case by case sense", i.e., the first case distinguished in the prior statement ($\Delta S^{*} > 0$) holds if and only if the first case distinguished in the posterior statement ($S^{*} + \Delta S^{*} > S^{*}$) holds, similarly for the other cases.

We now claim that (27) is true for the other case as well, i.e., for $a' > a$. The proof is as follows, $a' > a$ means by definition

$$\Delta f(u) = -f(u), \quad a < u < a',$$

Hence

$$(28) \quad \int^{a'} u \Delta f(u)du = - \int^{a'} uf(u)du.$$

If we now break up the LHS integral of (27) into its two parts and add the two integrals in (28) to their respective commensurate parts in terms of the functions integrated in this break up, then we simply get back to the RHS of (27). Hence the two sides of (27) are equal. This proves the claim made.

Let us now proceed on. Using (27), we can write,

$$\Delta S^* \gtreqless 0$$

$$\iff \alpha\,(\bar{u} + \Delta\,\bar{u}) \gtreqless \int^{S^*} uf(u)du + \int^{S^*} u\Delta f(u)du$$

$$\iff \alpha\,(\bar{u} + \Delta\,\bar{u}) \gtreqless \alpha\,\bar{u} + \int^{S^*} \Delta f(u)du, \qquad \text{by (16)}$$

$$\iff \alpha\,\Delta\,\bar{u} \gtreqless \int^{S^*} \Delta f(u)du$$

$$\iff \int^{S^*} u\Delta f(u)du \lesseqgtr \alpha\Delta\,\bar{u}$$

Hence we arrive at the conclusion :

$$(29) \quad \Delta S^* \gtreqless 0 \quad \text{as} \quad \int^{S^*} u\Delta f(u)du \lesseqgtr \alpha\Delta\bar{u}$$

This gives us a complete criterion for finding out the sign of ΔS^* in terms of the location of S^* on the u-axis. The criterion is simply whether the integral of the perturbation function, $\Delta f(u)$, weighted by u, upto S^* is less than or greater than α times the shift in the mean demand associated with the perturbation, the cases implying respectively an increase and decrease in the value of the optimum stock with the perturbation. This is the basic result of this section.

3.2 Let us be a little clearer about the nature of this result. It is convenient for this to introduce the notion of a " critical value" of S^* defined as follows. First, we define the function $\varphi(u)$ by

(30) $\varphi(u) = \int^{u} v\Delta f(v)dv.$

The critical value of S^* is then simply the solution (if it exists) of the equation :

$$\varphi(u) = \alpha \Delta \bar{u} .$$

Let us denote the critical value of S^* if it exists by Z_α . We can then restate our basic result as saying that ΔS^* changes its sign as S^* crosses Z_α . Further, ΔS^* changes sign from negative to positive if $\varphi(u)$ is decreasing in the neighbourhood of Z_α and from positive to negative if $\varphi(u)$ is increasing in the neighbourhood of Z_α.

We now note that even granting the existence of Z_α , there is no guarantee that any crossing of Z_α by S^* is really defined. For one thing S^* belongs by definition to the interval (a, b) but Z_α may be located outside this interval. Secondly, granting that Z_α also exists in this interval, to vary S^* in the present context is simply to vary the parameter α . But Z_α is then also varied and hence even though S^* is varied over the whole interval (a, b) as α is varied over (0, 1), a "crossing" may fail to occur.

We are thus led to two successive questions so far as the sign of ΔS^* is concerned : (a) the existence of Z_α ; and (b) the possibility of S^* crossing Z_α with the variation of α. This is where the task of this section ends. As just stated, answers to both questions depend ultimately upon the precise nature of the perturbation considered. This is where the whole substantive interpretation or a priori meaning of the "change in demand conditions" begun with comes in. This takes us to the next section.

Section 4 : The Case of a Rise or Fall of Demand

1. In case of no uncertainty, 'demand', D, is a scalar and rise or fall in demand has a clear-cut meaning — D rises or falls. In the case of uncertainty, with the demand conditions as a whole represented by a probability distribution, $f(u)$, it may appear off hand that we need merely replace the scalar D of the certainty case by the <u>mean demand</u>, $\bar{u}$, and say that 'demand' has risen or fallen as $\Delta\bar{u} \gtrless 0$. However, this is really completely arbitrary for the simple reason that unless specifically restricted, any arbitrary perturbation of $f(u)$ will generate a positive or negative $\Delta\bar{u}$. The proper definition is that there is a rise in demand if the <u>probability of "large demand" rises</u> and similarly there is a fall in demand if the <u>probability of "small demand" rises</u>. Since total probability remains unchanged, there is a corresponding fall in the probability of small demand in one case and large demand in the other. Stated differently, the rise and fall in demand are defined respectively by a <u>rightward</u>

and leftward shift of the pdf, f(u), as a whole. By definition, the mean demand, $\bar{u}$, rises in one case and falls in the other. But this change in $\bar{u}$ is only " associated" with the underlying change in demand conditions as specified, it does not " define" the rise and fall in demand. However, if the variation of $\bar{u}$ is understood in a prior frame defined by the shift of the pdf as described then we can represent the cases of rise and fall in demand by $\Delta \bar{u} > 0$ and $\Delta \bar{u} < 0$.

2. Let us now settle down to our problem — the effect of a rise or fall in demand on the value of optimum stock in our model[11]. Nothing would perhaps be considered more " intuitively obvious" than this, that a rise or fall in demand must bring about a corresponding rise or fall in the value of the optimum stock. Stated slightly differently, the proposition is that the optimum stock is an increasing function of mean demand, given the prior framing of the variation of mean demand just defined.

Broadly speaking, our object here is simply a rigorous examination of this proposition in terms of our model. The general analysis of the previous section already throws some doubt on the proposition, for according to it, the direction of change in the optimum stock with a change in demand conditions depends entirely upon the original location of the optimum stock on the u-axis relative to a critical value, Z_{α}, i.e., it depends upon whether $S^* < Z_{\alpha}$ or $S^* > Z_{\alpha}$. The existence of Z_{α} for the general case is however left in the open. So, the first question is whether Z_{α} exists in the present case. Let us take this up.

[11] We will consider only the case of rise in demand. This is no limitation of the scope as the other case can be dealt with symmetrically.

3.1 First, we formally define the class of perturbations of the pdf of D that we are concerned with. By the definition of " rise in demand", the perturbation Δf is now defined through a parameter, u_0, separating the region of " low" or " small" values of demand, $u < u_0$ and the region of " high" or " large" values of demand, $u > u_0$. Δf is then required to satisfy the condition :

(31) $\Delta f \lesseqgtr 0$ as $u \lesseqgtr u_0$ [12].

(31) defines the class of perturbations we are concerned with.

3.2 Note that it follows at once from (25) and (31) that

(32) $\Delta \bar{u} = \int u \Delta f(u) du > 0.$

Thus, we have the rise in mean demand following as a corollary to the concept of " rise in demand" defined above.

We also note that there is now a rightward shift of the domain of $f(u)$, if there is a shift of the domain at all. Hence

(33) $a' \geq a, \; b' \geq b.$

It follows that the domain of the perturbation $\Delta f(u)$ is now defined by (a, b').

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[12] This condition can be weakened but it unnecessarily complicates the analysis, and so we restrict ourselves to the condition as stated. It is also to be stated that on a priori grounds, u_0 should be in the " vicinity" of $\bar{u}$. However, this does not explicitly enter our analysis.

4.1 Let us now return to the function $\varphi(u)$ defined in the last section. Since $\varphi(u)$ is defined over the same domain as of $\Delta f(u)$, its domain is now (a, b'). Further, by the definition of $\varphi(u)$, we have :

(34) $\lim_{u \uparrow b'} \varphi(u) = \Delta \bar{u} > 0.$

Now, since the function $\Delta f(u)$ changes its sign from negative to positive at u_0, so does the function, $v \Delta f(v)$, which is integrated over in $\varphi(u)$. It follows that $\varphi(u)$ is monotonically decreasing upto $u = u_0$ and monotonically increasing thereafter $(u > u_0)$. Further,

(35) $\varphi(u_0) < 0.$

It follows at once from (34) and (35) that there exists an $x \varepsilon (u_0, b')$ such that

(36) $\varphi(x) = 0$.[13]

Further, x is unique since $\varphi(u)$ is monotonic over the range, $u_0 < u < b'$, where x is already located. Also, $\varphi(u)$ is positive over the range $x < u < b'$.

From (34) and (36) and the monotonicity of $\varphi(u)$ over the range, $u_0 < u < b'$ (and hence over $x < u < b'$) we now conclude that the equation

$$\varphi(u) = \alpha \Delta \bar{u}$$

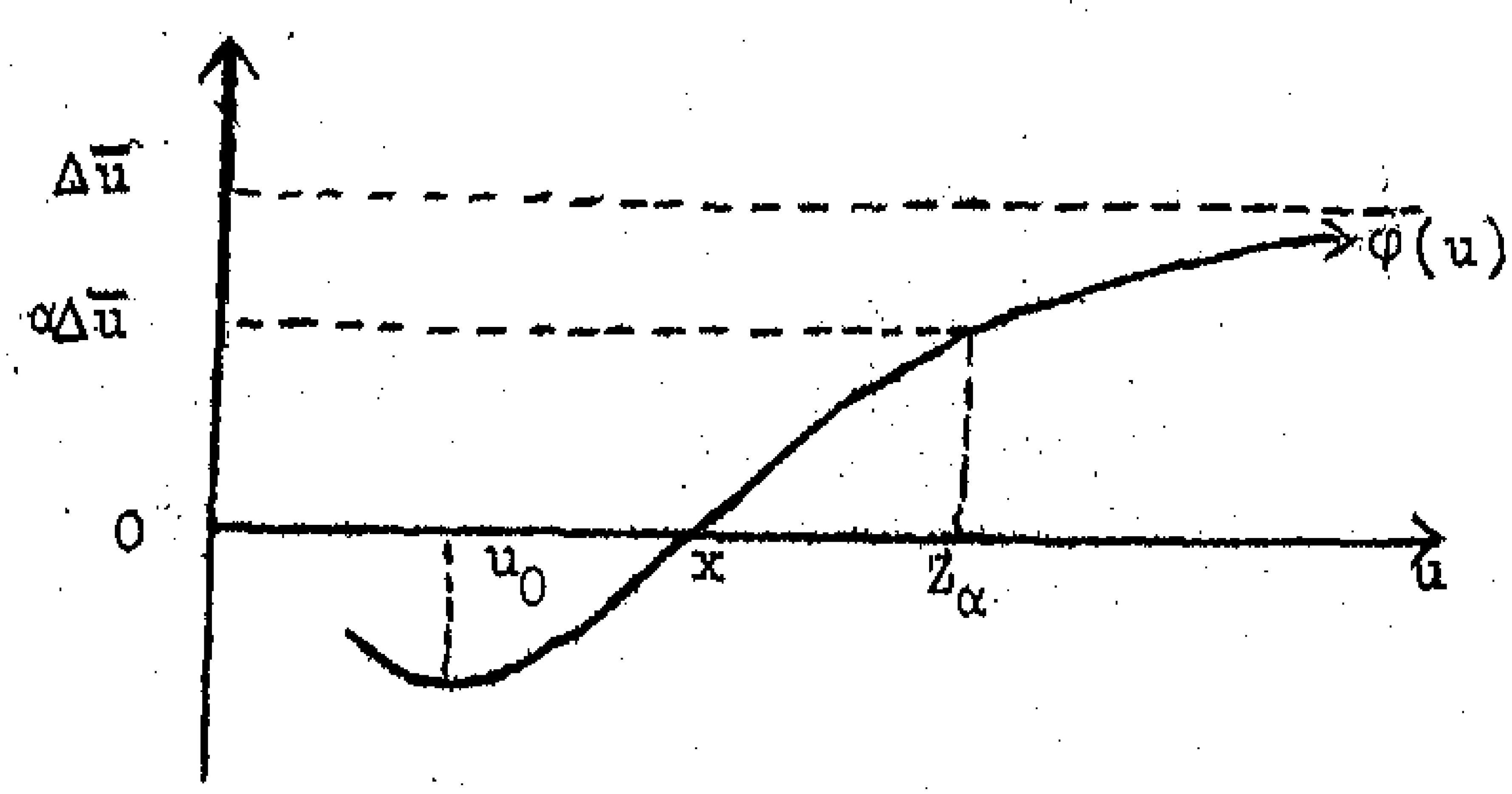
has a unique solution for any $\alpha \varepsilon (0, 1)$. This answers the question raised. Infact, we now have a unique $Z_\alpha \varepsilon (x, b')$ for any $\alpha \varepsilon (0, 1)$.

[13] In general, $x \lesseqgtr b$. But if $b = b'$, then obviously $x < b$.

So, ΔS^* has <u>one sign</u> for $S^* < Z_\alpha$ and the <u>opposite sign</u> for $S^* > Z_\alpha$. Since $\varphi(u)$ is infact an increasing function in the neighbourhood of Z_α, the sign is <u>positive for $S^* < Z_\alpha$</u> and <u>negative for $S^* > Z_\alpha$</u>. So, we have the general result

(37) $\Delta S^* \gtreqless 0$ as $S^* \lesseqgtr Z_\alpha$.

The determination of Z_α is graphically depicted in Figure 1 below.



<u>Figure 1</u>

4.2 Thus we do have the counterintuitive possibility that the optimum stock may indeed fall with a rise in demand, the necessary and sufficient condition for this being that the optimum stock lies above a critical value.

Let us see why. First we note that Z_α is a priori a " very high value" of demand, for

(38) $Z_\alpha > x > u_0$

and u_0 **is** the original cut off point between " low" and " high" values of demand **started** out with. So, $S^* > Z_\alpha$ means that the optimum stock begun with **is** already on the very high side.

Let us now start back from the optimality condition (16), i.e.

$$\int^{S^*} uf(u)du = \alpha \bar{u} .$$

Under our perturbation, the RHS increases by $\alpha\Delta\bar{u} < \Delta\bar{u}$ while the LHS changes by

$$\int^{S^*} u \Delta f(u)du .$$

Remembering that $\Delta \bar{u}$ is nothing but the integral over the whole domain (i.e., S^* replaced by the upper limit of the domain), a high value of S^* means that a significant part of $\Delta \bar{u}$ is already accounted for under our integral. If this part be greater than $\alpha \Delta \bar{u}$ —— which a priori is certainly " possible" for α is a given fraction —— then there must be a <u>fall</u> in the optimum stock to maintain the optimum condition under the perturbation. In a nutshell, what this means is that there has already occurred sufficient rise in the probability of values of demand <u>under S^*</u> to account for α-fraction of the rise in the <u>mean demand</u>.

5.1 Let us now proceed on. So far, we have merely established the possibility of a fall in the value of optimum stock with a rise in demand. We will not directly pursue this counter intuitive possibility any further. Instead, we follow the complementary line of analysis where the question becomes, <u>under what conditions the intuitive proposition holds</u>. More **precisely**, we look for such conditions entirely in the <u>nature of the</u>

perturbation, Δf. Thus our question stated in full is : Under what further restrictions on the perturbation, Δf, defining the " rise in demand", does one have a rise in the optimum stock with a rise in demand?

Two alternative restrictions serving this end are stated below as H1 (hypothesis 1) and H2 (hypothesis 2). Before stating them, we make a simple observation on our problem.

5.2 We already know that

$$\Delta S^* \gtrless 0 \quad \text{as } S^* \lesseqgtr Z_\alpha$$

Further,

$$Z_\alpha > x \qquad \text{(see (38))}.$$

It follows that $\Delta S^* \leq 0$ is possible only for $S^* > x$. Hence our search boils down to finding suitable restrictions on $\Delta f(u)$ such that

(39) $\Delta S^* > 0$, for $S^* > x$.

This is the observation.

5.3 Our two alternative hypotheses for ensuring (39) are :

H1 : $\Delta f(u)/f(u) < \Delta\bar{u}/\bar{u}$, for all $u \in (x, b)$[14]

H2 (a) : $b = b'$[15]

(b) : $\Delta f(u)/f(u)$ is increasing over (a, b).

[14] If $x \geq b$, then for all S^*, we have : $S^* < b \leq x$; hence $S^* < x < Z_\alpha$ and hence $S^* < Z_\alpha$ and hence $\Delta S^* > 0$. So, in this hypothesis we consider only the case $x < b$.

[15] In this case (i.e., when $b = b'$) we have : $x < b' = b$ and hence $x < b$.

Both the hypotheses obviously have a clear intuitive content. H1 states that the relative rise in the probability of demand for values of demand in the range (x, b) is less than the relative rise in the mean demand arising out of our perturbation $\Delta f(u)$. H2 requires firstly the upper limit of the domain of f(u) to be unchanged under the perturbation $\Delta f(u)$ and secondly the relative change in the probability of demand under the perturbation to be an increasing function of u over the whole domain of the original pdf, (a, b).

6. Let us now prove that H1 implies (39). The proof is as follows. We are interested only in values of S^* such that

$$S^* > x,$$

Given this, we can obviously write

$$\int^{S^*} u\,\Delta f(u)du = \int^{x} u\,\Delta f(u)du + \int_{x}^{S^*} u\,\Delta f(u)du.$$

But by the definition of x, the first integral on the RHS in zero. Hence

$$\begin{aligned}\int^{S^*} u\,\Delta f(u)du &= \int_{x}^{S^*} u\,\Delta f(u)du \\ &= \int_{x}^{S^*} u\,\frac{\Delta f(u)}{f(u)}\,f(u)du \\ &< \int_{x}^{S^*} u\,\frac{\Delta\bar{u}}{\bar{u}}\,f(u)du, \text{ by H1} \\ &= \frac{\Delta\bar{u}}{\bar{u}}\int_{x}^{S^*} uf(u)du\end{aligned}$$

$$< \frac{\Delta\bar{u}}{\bar{u}} \int^{S^*} uf(u)du$$

$$= \frac{\Delta\bar{u}}{\bar{u}} \alpha\bar{u}, \text{ by the optimality condition (16).}$$

$$= \alpha \Delta\bar{u} .$$

From the general criterion for the sign of ΔS^* given in (29), we can now conclude :

$\Delta S^* > 0.$

This completes the proof.

7. Let us now prove that H2 implies (39). Let us remember that S^* is a monotonically increasing function of α going from a to b as α goes from 0 to 1. We now define the value of α, say α_x, at which $S^* = x$. Obviously α_x is well defined and $0 < \alpha_x < 1$. For the rest of the proof, we shall consider S^* explicitly as function of α, though for notational simplicitly we will not write so.

Let us now define the function

(40) $V(\alpha) = \varphi(S^*)/\alpha\,\bar{u}$, for all α, $\alpha_x < \alpha < 1$.

Now, as α is varied over the interval $(\alpha_x, 1)$, S^* varies over the interval (x, b) and so, as already concluded that $\varphi(u)$ is positive over the range $x < u < b'$ (= b, in the present case), for all α, $\alpha_x < \alpha < 1$, $\varphi(S^*) > 0$. Hence

$V(\alpha) > 0$, for all α, $\alpha_x < \alpha < 1$.

Hence by logarithmic differentiation of $V(\alpha)$, for all α, $\alpha_x < \alpha < 1$,

(41) $$\frac{V'(\alpha)}{V(\alpha)} = \frac{\varphi'(S^*)}{\varphi(S^*)} - \frac{\bar{u}}{\alpha\bar{u}} .$$

Now, by definition of $\varphi(u)$,

$$\varphi(S^*) = \int^{S^*} u\Delta f(u)du,$$

Differentiating both sides of this equation wrt α, we get :

(42) $$\varphi'(S^*) = S^* \Delta f(S^*) \frac{dS^*}{d\alpha} .$$

Now, by the optimality condition (16),

(16) $$\int^{S^*} uf(u)du = \alpha \bar{u} .$$

Differentiating both sides of (16) wrt α, we get :

$$S^* f(S^*) \frac{dS^*}{d\alpha} = \bar{u} .$$

$$\therefore S^* \frac{dS^*}{d\alpha} = \frac{\bar{u}}{f(S^*)} .$$

Substituting this value in (42), we get :

$$\varphi'(S^*) = \bar{u} \Delta f(S^*)/f(S^*).$$

Again, substituting this value in (41), we get :

$$\frac{V'(\alpha)}{V(\alpha)} = \frac{\bar{u}\Delta f(S^*)/f(S^*)}{\varphi(S^*)} - \frac{\bar{u}}{\alpha\bar{u}}$$

$$= \frac{\bar{u}}{\varphi(S^*)} \left\{ \frac{\Delta f(S^*)}{f(S^*)} - \frac{\varphi(S^*)}{\alpha\bar{u}} \right\}$$

(43) $$= \frac{\bar{u}}{\varphi(S^*)} \left\{ \frac{\Delta f(S^*)}{f(S^*)} - V(\alpha) \right\},$$ by the definition of $V(\alpha)$.

Now,

$$V(\alpha) = \varphi(S^*)/\alpha\bar{u}$$

$$= \int^{S^*} u\Delta f(u)du/\alpha\bar{u},$$ by the definition of $\varphi(u)$.

$$= \int^{S^*} u \frac{\Delta f(u)}{f(u)} f(u)du/\alpha\bar{u}$$

$$< \frac{\Delta f(S^*)}{f(S^*)} \int^{S^*} uf(u)du/\alpha\bar{u},$$ by H2(b).

$$= \frac{\Delta f(S^*)}{f(S^*)},$$ by (16).

∴ from (43) we get :

$$\frac{V'(\alpha)}{V(\alpha)} > 0, \text{ for all } \alpha,\ \alpha_x < \alpha < 1.$$

But as already concluded, $V(\alpha) > 0$, for all α, $\alpha_x < \alpha < 1$.

$$\therefore\ V'(\alpha) > 0, \quad \text{for all } \alpha,\ \alpha_x < \alpha < 1.$$

Thus $V(\alpha)$ is an increasing function of α over its whole domain. So, for all α, $\alpha_x < \alpha < 1$, we have :

$$V(\alpha) < \lim_{\alpha \uparrow 1} V(\alpha)$$

$$= \lim_{\alpha \uparrow 1} (\varphi(S^*)/\alpha\bar{u}),$$ by the definition of $V(\alpha)$,

$$= (\lim_{\alpha \uparrow 1} \varphi(S^*))/\alpha\bar{u}$$

$$= \left(\lim_{\alpha \uparrow 1} \int^{S^*} u\Delta f(u)du\right)/\alpha\bar{u}, \quad \text{by the definition of } \varphi(u)$$

$$= \int u\Delta f(u)du/\alpha\bar{u}, \quad \text{by H2(a)}$$

$$= \Delta\bar{u} / \bar{u} .$$

Hence from the definition of $V(\alpha)$, it follows that for all α, $\alpha_x < \alpha < 1$,:

$$\varphi(S^*)/\alpha\bar{u} < \Delta\bar{u}/\bar{u}$$

$$\iff \varphi(S^*) < \alpha\Delta\bar{u}$$

$$\iff \int^{S^*} u\Delta f(u)du < \alpha\Delta\bar{u} \quad \text{by the definition of } \varphi(u).$$

So, we have :

$$\int^{S^*} u\Delta f(u)du < \alpha\Delta\bar{u} \quad \text{for all } \alpha,\ \alpha_x < \alpha < 1.$$

But it has already been concluded that as α is varied over the interval $(\alpha_x, 1)$, S^* varies over the interval (x, b). So, we have :

$$\int^{S^*} u\Delta f(u)du < \alpha\Delta\bar{u}, \quad \text{for all } S^*, \quad x \leq S^* < b.$$

Again from the general criterion for the sign of ΔS^* given in (29), we can now conclude :

$$\Delta S^* \geq 0, \quad \text{for all } S^*, \quad x < S^* < b.$$

This completes the proof.

8. We have now concluded the successive tasks we had set before ourselves in this section. Let us end by putting them in perspective. We started out with the simple intuitive expectation that the optimum

stock must increase when there is a rise in demand. This is what set the whole perspective of our analysis. First, we showed that going purely by the definition of " rise in demand", one can conclude only that the optimum stock rises or falls with the rise in demand as it lies below or above a critical value. However, it could not be said whether there was meaningful crossing of this critical value by the optimum stock or not. The analysis in this sense remains open ended. However a possibility was certainly indicated that the intuitive proposition is not true. Next, we derived two alternative <u>sufficient</u> conditions for the validity of the intuitive proposition in terms of the precise nature of the perturbation of the probability distribution of demand defining the " rise in demand". However, we have only interpreted these conditions in a priori terms, we do not defend them as " plausible" in any a priori sense. The openendedness or inconclusiveness of the analysis therefore simply remains in the substantive sense. The main thrust of the analysis therefore lies simply in throwing doubt on the intuitive expectation begun with.

<u>Section 5</u> : <u>The Case of Rise or Fall in the Uncertainty of Demand</u>

1. The basic formalisation of the idea of a rise or fall in the uncertainty of demand faced by a trader has already been suggested in Chapter 2 (Sections 4 and 5), viz. they mean respectively a <u>lesser</u> or a <u>greater concentration of the total probability under some given neighbourhood of mean demand</u>. We will use a slightly more general concept here : We simply replace the term " neighbourhood" in this definition by an " interval". So, a rise or fall in the uncertainty of demand is taken to

mean a lower or greater concentration of the total probability under a given interval containing the mean demand. At the same time, we will take the mean demand itself to remain unchanged under the change in demand conditions. This isolates the change in uncertainty from any change in the 'level of demand' that may go with the change in the demand condition. We may say that this defines a pure change in the uncertainty of demand. Our general object here is to look into the effects of such a change on the optimum stock in our model.

2. Let us now state the concept of change in uncertainty of demand in terms of a perturbation of the probability distribution of demand, $\Delta f(u)$. The relevent perturbation is defined ~~through two parameters~~, u_1 and u_2, $u_2 > u_1$ say, where (a) the interval (u_1, u_2) is contained within the domain of $f(u)$, (a, b), and (b) the mean demand, $\overline{u}$, belongs to (u_1, u_2). That is,

$$(44) \qquad a < u_1 < \overline{u} < u_2 < b$$

We then say that there is a rise in the uncertainty of demand if

$$(45) \qquad \Delta f(u) \begin{cases} > 0, & a < u < u_1 \\ < 0, & u_1 < u < u_2 \\ > 0, & u_2 < u < b \end{cases}$$

and

$$(46) \qquad \Delta\overline{u} = \int u \Delta f(u) du = 0.$$

(44) - (46) define the class of perturbations we are concerned with[16].

[16] The case of fall in the uncertainty of demand is defined analogously. We will restrict our analysis only to the case of rise in uncertainty. This does not imply any restriction of the scope as the other case can be dealt with symmetrically.

Note that if the domain of the new pdf $(f(u) + \Delta f(u))$ is (a', b') then we now have by definition

(47) $a' \leq a$ and $b' \geq b$.

This simply expresses the point that a rise in uncertainty means a greater dispersion or range of values of the random variable. We also note that the domain of perturbation $\Delta f(u)$ is now given by (a', b').

3.1 Let us now settle down to the analysis. First, substituting (46) in our general criterion for the sign of ΔS^* given in (29) of Section 3 of this Chapter, we obtain the result :

(48) $\Delta S^* \gtreqless 0$ as $\int^{S^*} u \Delta f(u) du \lesseqgtr 0$.

This in turn implies that the " critical value" of S^*, Z_α, is now defined by :

$$\varphi(Z_\alpha) = 0$$

Note that this makes Z_α <u>independent of α</u>. So we now write it simply as Z. That is, Z is defined by :

$$\varphi(Z) = 0.$$

Let us now take up the question of existence of Z. Let us explicitly recall the function $\varphi(u)$:

$$\varphi(u) = \int^{u} v \Delta f(v) dv.$$

It follows at once from the sign restriction on $\Delta f(u)$ over various intervals given in (45) that (a) $\varphi(u_1) > 0$ and (b) $\varphi(u)$ is decreasing over the interval (u_1, u_2). It also follows from (46) that

$$\varphi(u_2) = \int^{u_2} v\Delta f(v)dv$$

$$= \int v\Delta f(v)dv - \int_{u_2} v\Delta f(v)dv$$

$$= - \int_{u_2} v\Delta f(v)dv$$

(c) < 0, by (45)

From (a) - (c) we now conclude that Z indeed exists in the present case. Further, Z is unique and located in the interval (u_1, u_2). Finally $\varphi(u)$ is decreasing in the neighbourhood of Z. The determination of Z is graphically depicted in Figure 2 below. Figures 1 and 2 also sum up between themselves the basic qualitative properties of the function $\varphi(u)$ for the respective case of " rise in demand" and " rise in the uncertainty of demand".

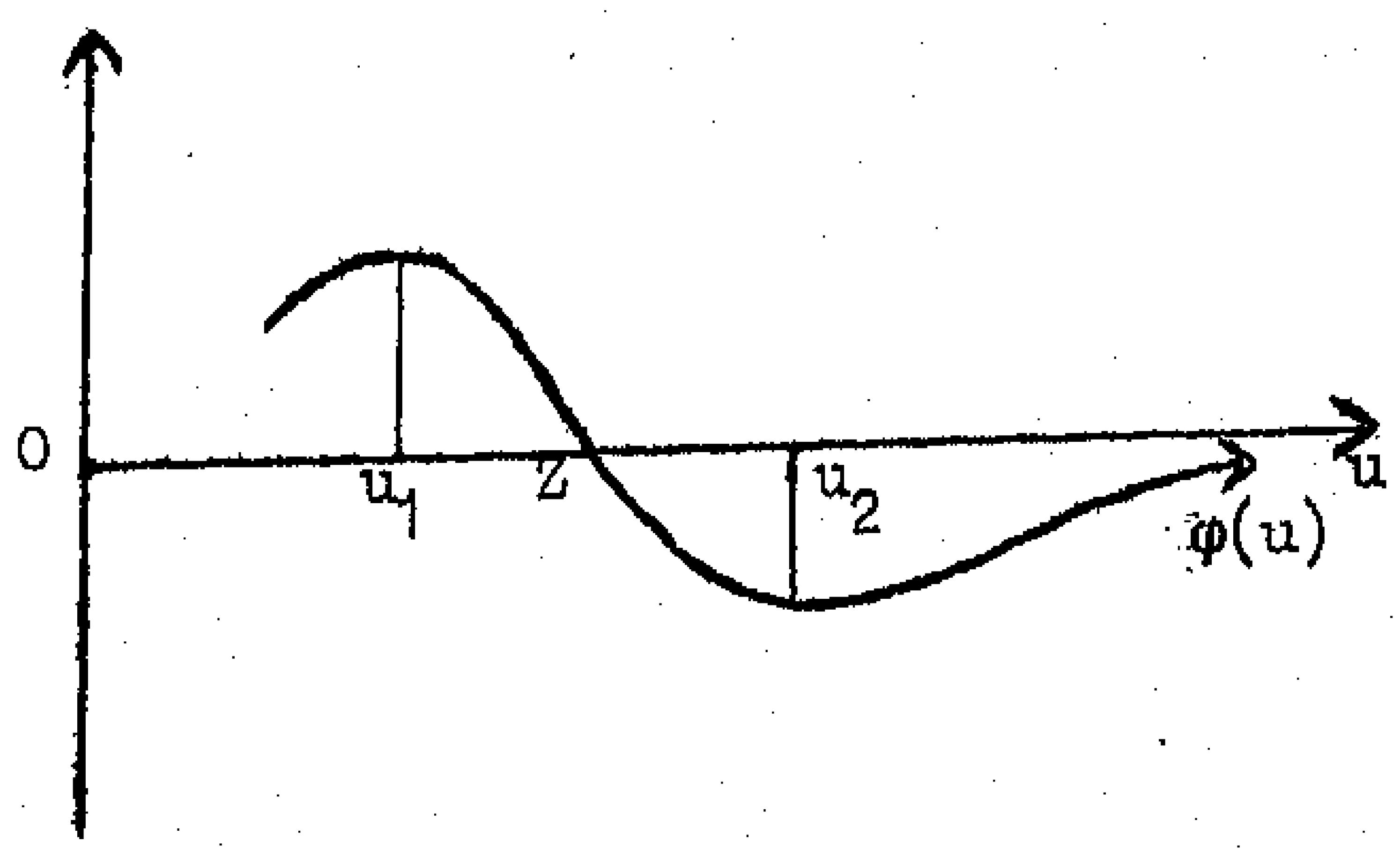


Figure 2

3.2 Two important conclusions follow from these points. First, from the decreasingness of $\varphi(u)$ in the neighbourhood of Z, we conclude that

(49) $\Delta S^* \gtreqless 0$ as $S^* \gtreqless Z$.

Secondly, from the fact that Z lies in the interval (u_1, u_2), and is independent of α, it follows that as α is varied, S^* must indeed cross its " critical value" Z. This is so because S^* itself increases monotonically from a to b as α is increased from 0 to 1. Hence there is a value of α, say α_Z, at which $S^* = Z$, $0 < \alpha_Z < 1$. With this, we can transform (49) into an equivalent statement in terms of the actual value of α, viz.

(50) $\Delta S^* \gtreqless 0$ as $\alpha \gtreqless \alpha_Z$

The general conclusion reached is thus that the direction of change in the value of the optimum stock with a rise in the uncertainty of demand is a priori <u>ambiguous</u>. The optimum stock increases for high values of α ($\alpha > \alpha_Z$), decreases for low values of α ($\alpha < \alpha_Z$). Let us now see this result in perspective.

4.1 One general perspective is defined by the analysis of the effect of rise or fall in demand on the optimum stock discussed in the previous section. On the face of it, it may appear that the two cases run parallel. This however is so only in an absolute a <u>priori</u> sense, viz, we cannot a priori predict the sign of ΔS^* in either case (rise or fall in demand and rise or fall in uncertainty of demand), not in the <u>analytical sense</u>. Our result here is <u>analytically complete</u>, for given the value of α, we can indeed predict the sign of ΔS^* from (50). No such prediction was possible in the last section under the general notion of " rise or fall in demand".

4.2 A second general perspective in which to see our result is the intuitive expectation regarding the effect of change in the uncertainty of demand faced by a trader upon his stock holding. Nothing has been said on this so far. Though we cannot put it on par with the intuitive expectation in the problem of the last section, there is, we believe, a general feeling that the trader would hold a larger stock in a more uncertain situation, the idea at bottom being that the stock is a " hedge against uncertainty". The " situation " here may not be quite compatible with the situation portrayed in our model. This granted, it is nevertheless to be pointed out that within our model, one gets a conclusive refutation of the expectation — both cases of a rise or fall in the optimum stock with a rise in the uncertainty of demand are equally well defined in a general sense. By way of elucidation, we just say the following.

Let us first pass from the optimum stock in our model to the implicit target regarding the probability of " full demand satisfaction" defined by it (see p. 103 above). Now, a greater uncertainty of demand certainly means a larger probability of " high" values of demand. Given no change in the magnitude of stock, this means in turn a fall in the probability of full demand satisfaction. The stock therefore has to be increased to maintain the same probability of full demand satisfaction as target. This is true. But the same change in uncertainty also means a larger probability of " small" values of demand, and these are automatically satisfied within the stock begun with. Given the target, the stock required therefore rises on one count and falls on another count on account of the rise in uncertainty of demand. Going purely by this, it is therefore not possible to argue conclusively which way the stock should change. However, if the target is sufficiently high, then the first effect predominates and the stock increases. This follows from (50) on noting that the " target" under reference and α are positively related.

Appendices to Chapter 5

Appendix A : Numerical Results

To obtain numerical results in our model, we must fix an explicit numerical form of the pdf, $f(u)$. We will assume that our random variable D follows the Lognormal Distribution $\Lambda(\mu, \sigma^2)$ with numerical values of the two parameters, μ and σ, to be specified as necessary. Properties of the distribution used are all taken from J. Aitchison and J.A.C. Brown, The Lognormal Distribution, pp. 7-12, without any further reference.

Let us begin by explicitly solving for S^* under this assumption. From the optimum condition as stated in (16) we obtain :

$$\begin{aligned}
\alpha &= F_1(S^*) \\
&= \Lambda_1 (S^* \mid \mu, \sigma^2) \\
&= \Lambda (S^* \mid \mu + \sigma^2, \sigma^2) \\
&= N (\log S^* \mid \mu + \sigma^2, \sigma^2) \\
&= \Phi \left[\frac{1}{\sigma} (\log S^* - \mu - \sigma^2) \right] \\
&= \Phi \left[\frac{1}{\sigma} (\log S^* - \mu) - \sigma \right]
\end{aligned}$$

Solving this equation for S^* we get :

(i) $$S^* = \exp \left[\mu + \sigma (\sigma + \Phi^{-1}(\alpha)) \right].$$

Let us now find out $S^*/\bar{u}$ and $\beta^* = F(S^*)$ which are the actual variables of interest to us.

$\bar{u}$ is given by :

$$\bar{u} = \exp \left[\mu + \frac{1}{2} \sigma^2 \right].$$

(ii) $\therefore\ S^*/\bar{u} = \exp[\sigma(\frac{1}{2}\sigma + \Phi^{-1}(\alpha))]$.

$$\begin{aligned} \beta^* &= F(S^*) \\ &= \Lambda(S^* \mid \mu, \sigma^2) \\ &= N(\log S^* \mid \mu, \sigma^2) \\ &= \Phi[\tfrac{1}{\sigma}(\log S^* - \mu)] \end{aligned}$$

(iii) $= \Phi[\sigma + \Phi^{-1}(\alpha)]$, by (i).

Solving for α in terms of β^* from (iii), we get :

(iv) $\alpha = \Phi[\Phi^{-1}(\beta^*) - \sigma]$.

Let us now note the important point that <u>μ does not simply enter the expressions of $S^*/\bar{u}$ and β^*</u>. Hence it is unnecessary for our purpose to specify any numerical value of μ. Our numerical results would hold for any value of μ, and so the whole 'model' is summed up for our purpose by only two parameters, α and σ or rather λ and σ, for α is given by :

(v) $\alpha = \lambda/(1 + \lambda), \quad 0 < \lambda < 1,$

Let us take a minute off to note the substantive implication of the property of the model now that $S^*/\bar{u}$ is independent of μ. A change in the value of μ merely expresses a rise or fall in the total demand as it means by definition a rightward or leftward shift of the whole pdf of D. So, we have the result now that the rise or fall in demand brought about by the change in μ, brings about a corresponding rise or fall in the value of the optimum stock, S^*, which is in fact <u>proportional</u> to the change in the

<u>mean demand</u>, $\overline{u}$, associated with this change in demand. However, this is not as complete a result as it may appear, for a rightward or leftward shift of the pdf of D can also come about through the joint variation of μ and σ and nothing is stated here about the response of S^* to the rise or fall in total demand so brought about. Nevertheless, though partial, the result is obviously very strong when seen in the background of the analysis of section 4 of this chapter.

Let us now begin on the calculation of the <u>critical value of λ</u> which we defined by the condition :

(vi) $S^*/\overline{u} = 1.$

Obviously, the critical value of λ depends upon σ, and so our task boils down to finding the locus of (λ, σ) satisfying (vi).

Now,

$$S^*/\overline{u} = 1 \iff \exp[\sigma(\tfrac{1}{2}\sigma + \Phi^{-1}(\alpha))] = 1, \quad \text{by (ii)},$$

$$\iff \sigma(\tfrac{1}{2}\sigma + \Phi^{-1}(\alpha)) = 0$$

$$\iff \tfrac{1}{2}\sigma + \Phi^{-1}(\alpha) = 0$$

$$\iff \sigma + 2\,\Phi^{-1}(\alpha) = 0 .$$

Hence our locus is defined by the relation :

(vii) $\sigma + 2\,\Phi^{-1}(\alpha) = 0,$

which we can also write as :

(viii) $\alpha - \Phi(-\tfrac{1}{2}\sigma) = 0.$

Now, $\Phi(x)$ is a monotonically increasing function going from 0 to 1 as x goes from $-\infty$ to ∞. It follows from this and (vii) or (viii) that along our locus, as σ increases, α decreases (and hence so does λ), σ being defined on the whole domain $(0, \infty)$ and α on the domain $(0, \frac{1}{2})$ (and hence λ on the domain (0, 1)). In particular, σ goes to zero as α approaches the value $\frac{1}{2}$ from below which is to say that λ approaches the value, 1, from below.

Two properties regarding the critical values of λ is established by this. One, every value between 0 and 1 is a critical value of λ for a unique value of $\sigma > 0$; and two, $\lambda = 1$ is an upper bound to the critical values of λ. It is interesting to observe that this upper bound was established in Section 2 of this chapter under the condition that the pdf of D is " not highly positively skewed". The Lognormal Distribution however is positively skewed and for the purpose here, can be made as highly skewed as one likes.

In view of the results just established, we have calculated points on our locus by first fixing λ at an equal interval of 0.1 within the stated interval, (0, 1), and then finding the values of σ on our locus. The results are given in Table 1 below.

Table 1

Locus of (λ, σ) for $S^*/\bar{u} = 1$

λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
σ	2.68	1.91	1.48	1.11	0.88	0.61	0.46	0.30	0.15

Let us now turn to the variability of $S^*/\bar{u}$ wrt (λ, σ). Note that σ can be considered to be a measure of the degree of uncertainty as it is simply a transformation of the coefficient of variation[1]. So, different values of σ represent different degrees of uncertainty of demand faced by the trader in our model. This is what sums up the whole environment or demand conditions so far as the optimum policy in both forms, $S^*/\bar{u}$ and β^*, are concerned, for μ does not enter these expressions. We choose 3 alternative values of σ : 0.5, 1.0, 2.0, as sufficient variation of the " environment" for our purpose. The critical values of λ for these three values of σ are located approximately as 0.7, 0.4 and 0.2 respectively from Table 1. Clearly, if we fix $\sigma = 0.5$ and let λ vary over a range crossing 0.7, then we also observe how precisely $S^*/\bar{u}$ crosses the value, 1. Taking the range of λ as : 0.25 - 1.25, we have calculated this variation of $S^*/\bar{u}$ and then simply repeated the same exercise for the other two values of σ chosen. The results are shown in Table 2 below.

Table 2

Sensitivity of $S^*/\bar{u}$ wrt λ for Given Values of σ

λ \ σ	0.5	1.0	2.0
0.25	0.74	0.71	1.37
0.50	0.90	1.06	3.06
0.75	1.03	1.38	5.19
1.00	1.13	1.65	7.34
1.25	1.22	1.91	9.99

[1] Because, the coefficient of variation, η, for the Lognormal Distribution is given by : $\eta^2 = e^{\sigma^2} - 1$.

It is seen that $S^*/\bar{u}$ crosses the value, 1, only relatively slowly with λ for $\sigma = 0.5$, for λ increases 5 times over the depicted range but $S^*/\bar{u}$ increases less than 2 times. The overall sensitivity of $S^*/\bar{u}$ wrt λ is somewhat greater than this for $\sigma = 1.0$ but not too great. For $\sigma = 2.0$ however we have a much greater sensitivity. The actual values of $S^*/\bar{u}$ in this case are also very high for, say, $\lambda \geq 0.75$. In all, we can say that so long as the degree of uncertainty is not too high, $S^*/\bar{u}$ shows only a moderate variation with λ over the chosen range.

The broad pattern of variation of $S^*/\bar{u}$ with σ, i.e., the degree of uncertainty, that emerges from Table 2 is that $S^*/\bar{u}$ increases significantly with σ, particularly for high values of λ in the given range. Thus while σ increases 4 times over its range in Table 2, $S^*/\bar{u}$ increases more than 6 times for $\lambda = 1.0$ and more than 8 times for $\lambda = 1.25$. This is again a strong result when seen in the background of the analysis in Section 5 of this chapter. However, even within our limited range, we confirm that $S^*/\bar{u}$ is not positively related with σ all through (it falls from 0.74 to 0.71 as σ is increased from 0.5 to 1.0 for $\lambda = 0.25$).

Let us now come to Table 3. The underlying idea here is to pass from " plausible " values of β^* interpreted as the probability of full demand satisfaction targeted by the trader to the implied values of λ, i.e., the subjective parameter in our definition of 'profit'. So we start by fixing " plausible" values of β^*. We have chosen 4 different values of β^* : 0.5, 0.8, 0.9, 0.95. Obviously, the plausibility has to be judged relative to the " environment", i.e., the degree of uncertainty. This is left to implicit interpretations. As for the degree of uncertainty, we have kept to the same values of σ considered in Table 2, i.e., $\sigma = 0.5,\ 1.0,\ 2.0$. The implied values of λ for the given values of β^* and σ are shown in columns 1-3 of Table 3 given below. For completeness we have also given in parenthesis the value of $S^*/\bar{u}$ obtained for each combination of (σ, β^*). Column 4 of the table gives the upper bound over the implied value of λ as per relation (23) of Section 2 of this chapter, denoted in Table 3 by U.

Table 3

Values of λ implied by β^* for Given Values of σ

β^* \ σ	0.5	1.0	2.0	U
0.50	0.45 (0.88)	0.19 (0.61)	0.02 (0.14)	1
0.80	1.73 (1.34)	0.78 (1.41)	0.14 (0.73)	4
0.90	3.61 (1.68)	1.57 (2.19)	0.31 (1.76)	9
0.95	6.93 (2.01)	2.85 (3.14)	0.57 (3.63)	19

Note 1 : Figures in paranthesis give associated values of $S^*/\bar{u}$.

Note 2 : U (column 4) stands for the upper bound on λ calculated from relation (23) of Section 2 of this chapter.

To pick off hand, it is seen that for a moderate degree of uncertainty ($\sigma = 1.0$), the target $\beta^* = 0.5$ implies a value of λ as low as 0.19 and gives a value of 0.61 for $S^*/\bar{u}$. If for the same σ, β^* is raised to 0.8 then the implied value of λ and $S^*/\bar{u}$ are raised respectively to 0.78 and 1.41. If on the other hand, the degree of uncertainty is as high as $\sigma = 2.0$, then even as high a value of β^* as 0.95 implies a value of only 0.57 for λ though $S^*/\bar{u}$ is of course quite high (3.63). Finally, we point out that the value of U in each case is far above the value of λ actually implied by β^* (Column 4 of Table 3 versus any other column).

Appendix B : Proof of the Result, $F(u) > F_1(u)$

Let Y be a random variable with pdf $f(u)$ having domain as (a, b), where $0 \leq a < b \leq \infty$. Let $F(u)$ and $F_1(u)$ denote the distribution function and the first moment distribution function respectively of the random variable Y. Then we have to show that $F(u) > F_1(u)$, for all u such that $a < u < b$.

Take any $u = u_0$, say, such that $a < u_0 < b$. To show :

$F(u_0) > F_1(u_0)$.

Firstly denote $E(Y) = \bar{y}$.

(i) $\therefore\ \bar{y} = \int uf(u)du$, by definition of $E(Y)$

$$= \int^{u_0} uf(u)du + \int_{u_0} uf(u)du, \text{ since } a < u_0 < b$$

$$= F(u_0) \int^{u_0} u \frac{f(u)}{F(u_0)}\, du + (1 - F(u_0)) \int_{u_0} u \frac{f(u)}{1 - F(u_0)}\, du$$

or,

(ii) $\bar{y} = F(u_0)\, E_1(Y) + (1 - F(u_0))\, E_2(Y)$,

where

$$E_1(Y) = \int^{u_0} u \frac{f(u)}{F(u_0)}\, du$$

$$E_2(Y) = \int_{u_0} u \frac{f(u)}{1 - F(u_0)}\, du.$$

$\therefore$ $E_1(Y) =$ the expectation of the random variable Y truncated from above at $u = u_0$, and

$E_2(Y) =$ the expectation of the random variable Y truncated from below at $u = u_0$.

$\therefore$ $E_1(Y) < u_0 < E_2(Y)$. $\therefore$ $E_1(Y) < E_2(Y)$.

But by (ii), evidently, $\bar{y}$ is a weighted average of $E_1(Y)$ and $E_2(Y)$.

$$\therefore \bar{y} < E_2(Y)$$

$$= \int_{u_0} u \frac{f(u)}{1 - F(u_0)}\, du, \text{ by definition of } E_2(Y),$$

$$= \frac{1}{1 - F(u_0)} \int_{u_0} uf(u)\, du.$$

$$\therefore (1 - F(u_0))\,\bar{y} < \int_{u_0} uf(u)\, du$$

$$= \int uf(u)du - \int^{u_0} uf(u)\, du$$

$$= \bar{y} - \int^{u_0} uf(u)du, \text{ by (i)}.$$

(iii) $\therefore$ $(1 - F(u_0)\,\bar{y} < \bar{y} - \int^{u_0} uf(u)du$.

But $0 \leq a < b \leq \infty$. $\therefore$ $E(Y) > 0$ or, $\bar{y} > 0$.

So, we can divide both sides of (iii) by $\bar{y}$. Doing this, we get :

$$1 - F(u_0) < 1 - \frac{1}{\bar{y}} \int^{u_0} uf(u)du,$$

$$\therefore \frac{1}{\bar{y}} \int^{u_0} uf(u)du < F(u_0)$$

or, $F_1(u_0) < F(u_0)$, since by definition of first moment distribution function,

$$F_1(u_0) = \frac{1}{\bar{y}} \int^{u_0} uf(u)du$$

or, $F(u_0) > F_1(u_0)$.

Hence the result.

Chapter 6

MODEL I CONTINUED : EFFECT OF VARIATIONS IN PURCHASE INTERVAL ON THE OPTIMUM STOCK

Section 1 : Setting Out the Problem

1. In the inventory model we have been discussing so far, it is assumed that the trader makes his successive purchases at a fixed interval given from outside. On this basis and other assumptions of the model, we have derived the optimum stock of the trader. Obviously the optimum stock is implicitly a function of the purchase interval, i.e., the constant time-length between the successive purchases. So we have

$$(1) \qquad S^* = S^*(\tau),$$

where S^* and τ denote respectively the optimum stock and the purchase interval. Our whole object in this chapter is simply to look into the properties of the function.

Let us see how τ entered the structure of the model. It entered in two places : first, in the definition of rate of profit as profit per unit of time (P/τ) per unit of capital (K) ; secondly in the definition of the probability distribution of demand over the purchase interval, i.e., in the pdf , $f(u, \tau)$. Let us now point out that in the first role, τ does not actually enter the decision making or optimisation in the model. To explain, in this role, it was just an element of the constant,

$\pi / q\tau$, which could be simply factorised out of the rate of profit prior to the optimisation. The constant as a whole did not play any role in the determination of S^*. It follows that for our purpose here the whole significance of τ lies in its second role. Stated somewhat differently, we can say that the whole effect of variation in τ on S^* — which it is our object to analyse in this chapter — is transmitted through its effects on the pdf , $f(u, \tau)$.

2. There are two specific backgrounds for this purpose. One background is given by the variational properties of the probability distribution of demand over ''time'' discussed in Chapter 2 (Sections 3-5). The other background is given by the effects of changes in demand conditions — i.e., of shifts in $f(u, \tau)$ as a whole — on the optimum stock in our model discussed in Sections 3-5 of the previous chapter. Obviously, it is quite a ''weighty'' background in the total that we now proceed from.

Let us recapitulate the first background. There were two stages in it. The first stage consisted of what we called the ''first law of probability distribution of demand over time'', which we saw as implicit in the very notion of stationary demand conditions on which the whole concept of probability distribution of demand is based. In the second stage, we distinguished between two alternative conceptions stated respectively as the ''second law of probability distribution of demand over time'' and the ''case of unchanged structure of the probability distribution of demand over time''. Let us now remember that the latter alternative was a pure formal generalisation of the first law. In fact, the first law can be deduced from it. So, we can say that the background as a whole consists of two alternative conceptions of the probability distribution of demand over time, one defined by the condition of unchanged structure of the distribution and the other by the first and the second law of the distribution taken together.

Obviously, we have to take up the problem of this chapter separately for these two cases. Formally, we have two different problems, not one. We will now immediately take up and dispose of the problem in the first case, i.e., under the condition of unchanged structure of the probability distribution of demand over time. The whole remainder of the chapter is then devoted to the other case, defined by our two ''laws'' taken together.

3. Let us first write down the optimality condition of the model, by which $S^*(\tau)$ is determined. Written out in full, the condition is

$$(2) \qquad \int^{S^*(\tau)} u\, f(u, \tau)\, du = \alpha\, \bar{u}(\tau),$$

where $\bar{u}(\tau)$ is the mean of the pdf, $f(u, \tau)$ and α is a transformation of the subjective parameter in the definition of profit, λ, defined by

$$\alpha = \lambda / 1 + \lambda ,$$

and hence

$$0 < \alpha < 1 \quad \text{as } \lambda > 0.$$

Now, the concept of unchanged structure of the probability distribution of demand over time was defined in Chapter 2 by the condition (see (2.9)) :

$$f(\lambda u, \lambda\tau) = \frac{1}{\lambda} f(u, \tau), \text{ all } \lambda > 0.$$

Using this property, we can write (2) as :

$$(3) \qquad \int^{S^*(\tau)} \frac{u}{\tau} f\left(\frac{u}{\tau}, 1\right) du = \alpha \, \bar{u}(\tau).$$

Let us now effect a transformation of variable in (3) defined by

$$v = u/\tau$$

so that

$$dv = \frac{1}{\tau} du$$

and

$$v = S^*(\tau)/\tau \quad \text{when} \quad u = S^*(\tau).$$

Let us also remember that the lower limit of the integral in (3) is the lower limit of the domain of $f(u, \tau)$, say $a(\tau)$. It again can be deduced from the condition of unchanged structure of the pdf, $f(u, \tau)$, that $a(\tau)$ is in fact proportional to τ, say

$$a(\tau) = a.\tau.$$

Thus

$$v = a \quad \text{when} \quad u = a(\tau) = a.\tau.$$

With this we can write (3) as

$$\int^{S^*(\tau)/\tau} v\, g(v)\, \tau \, dv = \alpha\, \bar{u}(\tau)$$

or

(4) $$\tau \int^{S^*(\tau)/\tau} v\, g(v)\, dv = \alpha\, \bar{u}(\tau),$$

where

$$g(v) = f(v, 1)$$

and it is understood that the lower limit of the integral in (4) and the integral immediatelyprevious to (4) is a.

Now, as already stated, the condition of unchanged structure of the probability distribution of demand over time implies our first law of the probability distribution of demand over time which in turn states that $\bar{u}(\tau)$ is proportional to τ, the factor of proportionality defining the so-called ''rate of demand'' which we denote by d. Hence

$$\bar{u}(\tau) = d. \tau.$$

Substituting this relation in (4) we get

$$\tau \int^{S^*(\tau)/\tau} v\, g(v)\, dv = \alpha\, d.\tau.$$

Cancelling the common factor, τ, on both sides, this gives us

$$\int^{S^*(\tau)/\tau} v\, g(v)\, dv = \alpha\, d.$$

Obviously we can say that it is $S^*(\tau)/\tau$ which is determined by this equation. Since τ does not enter anywhere else in the equation, $S^*(\tau)/\tau$ is in fact determined independently of τ. This is saying the same as that $S^*(\tau)$ varies proportionately with τ. This completely solves our problem for the case under reference.

4. The other case is not so easily disposed off. In fact, it appears analytically quite intractable, for reasons that are already clear from the second background to our problem in this chapter. Let us go over this a little systematically.

Let us start from the first background. Stated in conceptual terms, our first law of the probability distribution of demand over time is that the mean demand varies proportionately with ''time'', i.e., the length of the interval over which the demand is defined in the first place. So, a longer ''time'' in this sense means, firstly, a greater mean demand and, secondly, a proportionately greater mean demand. In the background, the shift in the mean demand is seen to come from a rightward shift of the pdf of $D(\tau)$ as a whole.

Let us now move to the second background. In Section 4 of the previous chapter we examined the effect of a general rightward shift of the pdf of $D(\tau)$ on the optimum stock, $S^*(\tau)$, and saw that no firm conclusion could be obtained by analytical means as regards even the ''sign'' of this effect, i.e., as regards whether the optimum stock increased or decreased with the ''rise in demand'' as portrayed by the shift of the pdf of $D(\tau)$. True, τ was implicitly held constant in this discussion. But this is of no analytical significance in itself — the ''inconclusiveness'' of the analytical exercise of Section 4 would remain whether the ''rise in demand'' came from a variation of τ or anything else. Thus, at a general level, we can expect to make little headway with our problem here without simplifying it down in some way.

This is not the end of the matter. Let us return once again to the first background. Stated in conceptual terms, our second law of the probability distribution of demand over time is that the uncertainty of demand decreases with ''time'' 1/. This had an intuitive basis stated in Chapter 1 (see pp. 26-7) which we shall return to later on to clarify our whole motivation for the problem of this chapter. Pending this, let us now turn to the second background and see what we could say about the effect of rise or fall in the uncertainty of demand 2/ on the optimum stock in our model (Section 5, last chapter). Unlike in Section 4, the exercise in this regard was ''conclusive'' in an analytical sense. However, the conclusion was simply that the optimum stock either increased or decreased with a rise in uncertainty depending upon whether α was greater or smaller than a critical value defined in reference to the shift in the probability distribution of demand. At a general level, we are left with the same ambiguity about the direction of change of the optimum stock as in the case of a rise or fall in demand. It follows that we can hope to say little meaningful at a general level about the change in the optimum stock following from a lengthening (or shortening) of the purchase interval

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1/ Change in uncertainty is taken all through this chapter to mean change in relative uncertainty, i.e., the change in uncertainty relative to the change in the mean demand that occurs with the variation of τ.

2/ It is true that this rise or fall in uncertainty was conceived in absolute terms. However, since the definition included a condition that there was no change in the mean demand, it was equivalently a rise or fall in relative uncertainty as well. This ensures the consistency of our terms in this chapter.

when the lengthening produces simultaneously a <u>rise</u> in demand, or the <u>level</u> of demand (first law), <u>and</u> a <u>fall</u> in the uncertainty of demand (second law). This is the ''analytical intractability'' of the problem, and it also motivates <u>simplifying</u> the problem which we will come to in a minute.

5. First, a few words on the motivation of the whole direction now being followed. This goes back to the very beginning notions in our general problem area. We start from the following intuitive proposition. If we ask a trader what 'demand' he faces over, say, (a) <u>an hour's length of time</u>, and (b) <u>a day's length of time</u> (i.e., the working length of a 'day', say ten hours), then his answer to (b) will certainly be <u>ten times</u> the answer to (a). This is our first law of the probability distribution of demand over time. Note that the postulate of stationary demand conditions is already implicit in the very nature, or framing, of our question, and this is already <u>implicitly granted by the trader</u> — otherwise he could not, or would not, have simply given <u>any</u> answer to either question. He would have considered them ''meaningless''.

Let us pass on. Our beginning proposition is not yet completed. It is completed by the point that while saying that the demand he faces over a day's length of time is ten times the demand he faces over an hour's length of time, the trader will also be <u>more confident about the second number than about the first</u>. This is the proposition as a whole. The trader's ''greater confidence'' about his answer to (b) than to (a) must have a basis in the real demand conditions faced by him, and this, we argue, can only be a greater uncertainty of the demand over an hour's length of time as compared to the uncertainty over a day's length of time. This is our second law of probability distribution of demand over time.

The case we are proceeding with is thus conceived within our context of reference and not from outside. It is therefore of natural interest to follow through its implications for the ''inventory decisions'' of the trader. True, its direct point of reference would be the decision on purchase interval which we take as given here 3/. Nevertheless, our exercise here can be seen as ''groundwork'' for the decision. This is all that we claim about the significance of our exercise from the standpoint of decision making. We will return to this point at the end of our whole exercise.

6. Let us now come to the simplification under which we carry out our exercise. This is indeed a drastic simplification. We simply assume the simplest form of the probability distribution of demand, viz., the Rectangular Distribution.

Let us just set this down and complete the necessary groundwork for our exercise. Let $(a(\tau), b(\tau))$ be the domain of the pdf of $D(\tau)$, i.e., $f(u, \tau)$. By the definition of the Rectangular Distribution

$$(5) \qquad f(u, \tau) = \frac{1}{b(\tau) - a(\tau)}, \quad a(\tau) \leqslant u < b(\tau).$$

3/ This decision is taken up as an integral part of Model IV of the thesis. Unfortunately, the complexity of the model does not allow us to take account of the second law there.

The mean demand, $\bar{u}(\tau)$ is now given by

$$(6) \qquad \bar{u}(\tau) = \tfrac{1}{2}[a(\tau) + b(\tau)]$$

and so, by our first law of the probability distribution of demand, we have

$$(7) \qquad \bar{u}(\tau) = \tfrac{1}{2}[a(\tau) + b(\tau)] = d. \tau.$$

Let us now come to the second law. In view of the simple structure of our probability distribution of demand now, we can specify this in a very simple form. However one defines ''uncertainty'' in an absolute sense, a ''greater'' (absolute) uncertainty in the present case must mean a higher value of the <u>range</u> of the distribution, $b(\tau) - a(\tau)$ and vice versa. So, we have the range itself as general measure of (absolute) uncertainty. Dividing the range by the mean demand, we then have a general measure of uncertainty in the relative sense, which is relevant for our purpose. Let us denote the measure by $z(\tau)$:

$$(8) \qquad z(\tau) = \frac{b(\tau) - a(\tau)}{\bar{u}(\tau)} .$$

Clearly, the uncertainty of demand increases or decreases with τ as $z(\tau)$ is a decreasing or increasing function of τ. So our second law of the probability distribution of demand is now defined by the condition :

$$(9) \qquad \frac{d\,z(\tau)}{d\,\tau} < 0.$$

We will also add to this the asymptotic property that $z(\tau)$ tends to vanish as $\tau \to \infty$ or,

$$(10) \qquad \lim_{\tau \to \infty} z(\tau) = 0.$$

Henceforth we will understand the second law to be defined by (9) and (10) together.

This completes the groundwork. We are now set for the analysis.

Section 2 : Decomposition of the Total Effect into a Mean Effect and an Uncertainty Effect

1. Our first task now is to work out the optimum stock, $S^*(\tau)$, in terms of the Rectangular Distribution with parameters $a(\tau)$ and $b(\tau)$ as introduced already in Section 1 of this chapter. For notational simplicity, unless specifically necessary for the argument, we will drop the argument τ in these and related variables. We will also write c for the range of the distribution, i.e.,

(11) $c = b - a.$

Then, we have from (7) that :

(12) $\bar{u} = a + c/2$

and from (8) that :

(13) $z = c/\bar{u}.$

Now, under the Rectangular Distribution,

$$\int^{S^*} uf(u)du = \frac{1}{c} \int^{S^*} udu$$

$$= \frac{1}{2c} (S^{*2} - a^2).$$

∴ by the optimality condition (5.16), we have :

$$\frac{1}{2c} (S^{*2} - a^2) = \alpha \bar{u}$$

$$= \alpha (a + c/2), \text{ by (12)}.$$

Solving explicitly for S^*, we obtain :

$$(14) \qquad S^* = \sqrt{a^2 + 2\alpha ac + \alpha c^2}\ .$$

2. Now, the substance of our problem is defined by our two laws of probability distribution of demand, and these in turn are defined directly in turns of the <u>mean demand</u>, $\bar{u}$, and the measure of uncertainty of demand, z. Let us therefore express S^* in terms of $\bar{u}$ and z in place of a and c. The steps are as follows.

It is convenient to work with S^{*2} rather than S^*. From (14) we have :

$$\begin{aligned} S^{*2} &= a^2 + 2\alpha ac + \alpha c^2 \\ &= (a^2 + 2\alpha ac + \alpha^2 c^2) + \alpha c^2 - \alpha^2 c^2 \\ &= (a + \alpha c)^2 + \alpha(1 - \alpha)c^2 \\ &= \left[(a + c/2) + (\alpha - \tfrac{1}{2})c\right]^2 + \alpha(1 - \alpha)\, c^2. \end{aligned}$$

We can now express S^{*2} in terms of $\bar{u}$ and z by (12) and (13). This gives us :

$$\begin{aligned} S^{*2} &= \left[\bar{u} + (\alpha - \tfrac{1}{2})\, \bar{u}\, z\right]^2 + \alpha(1 - \alpha)\,(\bar{u}\, z)^2 \\ &= \bar{u}^2 \left[1 + (\alpha - \tfrac{1}{2})\, z\right]^2 + \alpha(1 - \alpha)\, \bar{u}^2 z^2 \\ &= \bar{u}^2 \left\{ \left[1 + (\alpha - \tfrac{1}{2}) z\right]^2 + \alpha(1 - \alpha)\, z^2 \right\} . \end{aligned}$$

After simplification, this reduces to :

$$(15) \qquad S^{*2} = \bar{u}^2 \left\{ \frac{1}{4} z^2 + (2\alpha - 1)\, z + 1 \right\} .$$

Let us now define the function, V(z) by :

$$(15') \quad V(z) = \frac{1}{4} z^2 + (2\alpha - 1) z + 1.$$

Solving back for S^* from (15), we can then write :

$$(16) \quad S^* = \bar{u} \sqrt{V(z)}.$$

3. The optimum stock is thus expressed as product of $\bar{u}$ and a function of z, $\sqrt{V(z)}$. This is an important result. It shows firstly that for any given z, S^* is simply proportional to $\bar{u}$. Now, in our problem, both $\bar{u}$ and z are functions of τ and so they both vary together. But there is a clear conceptual separation in this, for the behaviour of the two functions, $\bar{u}(\tau)$ and $z(\tau)$, are governed respectively by the first and the second law of probability distribution of demand. It follows that if we keep the second law in abeyance, then $z(\tau)$ is constant. By (16), S^* then varies proportionately with $\bar{u}$, and $\bar{u}$ in turn varies proportionately with τ by the first law of the probability distribution of demand. Consequently, S^* varies proportionately with τ[4/]. With this, the second law can be seen as modification of what we may call a " law of proportionate variation of S^* with τ". We shall study the nature of this modification in the next section.

Let us end by noting a simple decomposition of the total effect of variation in τ on S^* which follows at once from (16). By differentiating both sides of this equation, we obtain :

[4/] This is exactly how S^* varies with τ in the case of unchanged structure of the probability distribution of demand. This is not surprising as the concept of " unchanged structure" implies that $z(\tau)$ is constant.

$$(17) \qquad \frac{dS^*}{d\tau} = \sqrt{V(z)}\,\frac{d\bar{u}}{d\tau} + \frac{\bar{u}V'(z)}{2\sqrt{V(z)}} \cdot \frac{dz}{d\tau} \,.$$

Now, we have just seen that behaviour of $\bar{u}(\tau)$ and $z(\tau)$ are governed respectively by the first and the second law of the probability distribution of demand. It follows that the two terms on the RHS of (17) give us the effect of variation of τ on S^* as governed respectively by these two laws. We can also call them equivalently the "mean effect" and the "uncertainty effect" of the variation in τ on S^*. (17) then expresses the "total effect" of the variation in τ on S^* as the sum of these two effects. This is the decomposition.

Section 3 : Behaviour of the Optimum Stock Relative to Mean Demand

1.1 In the last section we saw that if we abstract from our second law of probability distribution of demand, then we have the optimum stock of our trader as varying proportionately with his purchase interval, given the simplification of a Rectangular Probability Distribution of demand. Our object here is to see how this relation is modified once the second law is re-introduced into the picture. Before beginning, we note that the mean demand faced by the trader over the purchase interval in any case varies proportionately with the interval itself (first law). As a result, we can restate our question as simply examining the behaviour of the optimum stock relative to the mean demand with variations in the purchase interval, i.e., the behaviour of the ratio, $S^*(\tau)/\bar{u}(\tau)$. This is the question we directly address ourselves to now on.

1.2 From a general practical as well as theoretical standpoint, it is the variation of τ within a range of relatively " small" values that is of interest. However, we will come to this only after establishing a " standard" for the value of $S^*(\tau)/\bar{u}(\tau)$ which we obtain by going to the other end i.e., τ large. Let us begin on this.

2.1 In Section 1 of this chapter we defined the second law of the probability distribution of demand under the present set up (Rectangular Probability Distribution of demand) by the condition that the uncertainty of demand, as measured by the variable $z(\tau)$, decreases with " time" and in the limit, vanishes, i.e., $z(\tau)$ is a decreasing function going to zero as $\tau \longrightarrow \infty$. For our purpose now, it is the limiting property of $z(\tau)$ that is relevant.

Let us write down the expression of $S^*(\tau)/\bar{u}(\tau)$:

(18) $$S^*(\tau)/\bar{u}(\tau) = \sqrt{V(z(\tau))} ,$$

where

(19) $$V(z(\tau)) = \frac{1}{4} z^2(\tau) + (2\alpha - 1)\, z(\tau) + 1 .$$

It follows at once from (19) that if $z(\tau) \longrightarrow 0$ as $\tau \longrightarrow \infty$, then $V(z(\tau)) \longrightarrow 1$ as $\tau \longrightarrow \infty$ and hence from (18) we obtain :

(20) $$\lim_{\tau \longrightarrow \infty} (S^*(\tau)/\bar{u}(\tau)) = 1 .$$

This is the " standard" referred earlier.

The result just obtained is only to be expected. Under our hypothesis, the uncertainty of demand vanishes in the limit, i.e., the demand becomes " certain". This simply takes us back to the " simplest inventory model" introduced and discussed in the previous chapter (see pp. 94-6) where we saw that the optimum stock is simply equal to the demand over a purchase interval. Viewed through the notion of probability distribution of demand, this " demand" is simply the mean demand, and (18) asserts precisely that, the optimum stock converges to the mean demand as $\tau \longrightarrow \infty$.

2.2 Let us proceed one step further with the behaviour of $S^*(\tau)/\bar{u}(\tau)$ for τ large. Given that $z(\tau)$ converges to zero as $\tau \longrightarrow \infty$, we argue that $z^2(\tau)$ becomes " negligible" for τ large. It then follows from (19) that, for τ large :

$$(21)\begin{cases} V(z(\tau)) > 1, & \text{if } \alpha \geq \frac{1}{2}, \text{ and} \\ V(z(\tau)) < 1, & \text{if } \alpha < \frac{1}{2}. \end{cases}$$

From (18), (20) and (21) we then conclude that $S^*(\tau)/\bar{u}(\tau)$ approaches 1 from above or below as $\tau \longrightarrow \infty$ according as $\alpha \geq \frac{1}{2}$ or $\alpha < \frac{1}{2}$.

3.1 We are now ready to go into the behaviour of $S^*(\tau)/\bar{u}(\tau)$ for small values of τ. Since we have already obtained the behaviour of this variable for large values of τ, this actually becomes part of the study of the global behaviour of $S^*(\tau)/\bar{u}(\tau)$. So we begin on this.

First we assume that the limit of $z(\tau)$ as $\tau \longrightarrow \infty$ exists, and denote it by z_0. Stated in mathematical terms, $\lim_{\tau \to 0} z(\tau) = z_0$.

Note that $z(\tau)$ is not defined at $\tau = 0$. Since $z(\tau)$ decreases for all $\tau > 0$, z_0 is the supremum of the function $z(\tau)$ for $\tau \in (0,\infty)$. So, as τ increases from 0 to ∞, $z(\tau)$ decreases from z_0 to 0. z_0 now enters our analysis as a basic parameter. It will be seen as we go on, that the global behaviour of $S^*(\tau)/\bar{u}(\tau)$ is quite different in different regions of the plane defined by the two parameters, α and z_0.

3.2 Let us now consider the value of $V(z)$ at $z = z_0$, to be denoted by V_0, i.e., $V_0 = V(z_0)$. From (19) we obtain :

$$(22) \qquad V_0 = \frac{1}{4} z_0^2 + (2\alpha - 1) z_0 + 1.$$

Clearly,

$$V_0 > 1 \text{ if } \alpha \geq \frac{1}{2}.$$

It is thus seen that our ratio, $S^*(\tau)/\bar{u}(\tau)$, lies above 1 for both large and small values of τ if $\alpha \geq \frac{1}{2}$.

Let us now focus attention for a minute on the function $V(z)$ defined in (15′) (i.e., we consider z here as independent variable, not a function of τ). By differentiating this function we obtain :

$$(23) \qquad V'(z) = \frac{1}{2} z + (2\alpha - 1).$$

Clearly, if $\alpha \geq \frac{1}{2}$, then $V'(z) > 0$ at all $z > 0$. Since z is a decreasing function of τ for all $\tau > 0$, we conclude that if $\alpha \geq \frac{1}{2}$ then V must decrease continuously with τ for all $\tau > 0$, and hence so must our ratio, $S^*/\bar{u}$, for it is simply the square root of V. Given the point already established that $S^*/\bar{u}$ converges to 1 as $\tau \longrightarrow \infty$, it follows that $S^*/\bar{u}$ monotonically decreases to 1 if $\alpha \geq \frac{1}{2}$. So, in particular, it stays above the "standard", $S^*/\bar{u} = 1$, all through.

3.3 We have now obtained the global behaviour of $S^*(\tau)/\bar{u}(\tau)$ for the case $\alpha \geq \frac{1}{2}$. For $\alpha < \frac{1}{2}$, it is necessary to proceed by distinguishing sub-cases defined jointly by the value of the two parameters, α and z_0. These cases will be presently defined. To put things on a systematic basis, we will simply conceive all the cases (including $\alpha \geq \frac{1}{2}$) in terms of these two parameters and denote them as Cases A, B, C and D. Case A is defined simply by :

Case A : $\alpha \geq \frac{1}{2}$.

This case has already been completely analysed.

3.4 Let us now define Case B. We note that $V_0 > 1$ is possible even under $\alpha < \frac{1}{2}$. Case B is defined simply by the joint condition (a) $\alpha < \frac{1}{2}$, and (b) $V_0 > 1$. Referring back to (22) the explicit definition of the case in terms of (α, z_0) is obtained as below :

Case B : $\alpha < \frac{1}{2}$ and $4(1 - 2\alpha) < z_0$.

Let us now return for a minute to the behaviour of $V(z)$ established in (23). It follows at once from this equation that

(24) $V'(z) \lesseqgtr 0$ as $z \lesseqgtr 2(1 - 2\alpha) = z^*$, say.

Thus given $\alpha < \frac{1}{2}$, we have $V(z)$ initially decline with z upto the critical value, $z = z^* = 2(1 - 2\alpha)$, and increase thereafter. So, V reaches its minimum, V^* say, at $z = z^*$. By direct substitutions, the value of V^* is found to be :

(25) $$V^* = 4\alpha(1 - \alpha) < 1.$$

Let us now return to the mainline of our discussion. From the definition of Case B it is seen at once that in this case

$$z_0 > 2z^* .$$

Since $z(\tau)$ declines monotonically beginning from the value z_0 and ending at the value 0, it must cross the critical value, z^*, on the way from 0 to ∞. It follows that $V(z(\tau))$ in this case <u>initially declines with τ until it reaches its minimum value V^* when $z(\tau) = z^*$ after which it increases continuously with τ</u>. But we also know from our parameter configuration that $V_0 > 1$, while $V^* < 1$, and in any case that $V \longrightarrow 1$ as $\tau \longrightarrow \infty$. It follows that starting from $V_0 > 1$, V in fact <u>crosses</u> the value, $V = 1$, and then approaches it <u>from below</u> after crossing its minimum value V^* located in the region $V < 1$.

3.5 Let us now define <u>Case C</u>. In the behaviour of $S^*(\tau)/\bar{u}(\tau)$ in case B just described, two properties of the case clearly stand out, viz., (a) $z_0 > z^*$, and (b) $V_0 > 1$. We now define Case C by simply retaining (a) and giving up (b). The explicit definition in terms of (α, z_0) is as follows :

<u>Case C</u> : $2(1 - 2\alpha) < z_0 \leq 4(1 - 2\alpha)$.

In this case again, V initially declines with τ until it reaches the value V^* after which it continuously increases, approaching the value 1 as $\tau \rightarrow \infty$. This is very similar to the behaviour of V in Case B. The only difference is that while in Case B, V (and hence $S^*/\bar{u}$) lies initially above the value $V = 1$, and then crosses it, now V lies below this value all through.

3.6 We now come to Case D which is simply residually defined. Stated in substantive terms, the case is defined by the sole condition, $z_0 \leq z^*$, i.e.,

Case D : $z_0 \leq 2(1 - 2\alpha)$.

Since we start here from a z_0 equal to or below the critical value, z^*, as τ is increased and therefore $z(\tau)$ is decreased, $V(z(\tau))$ simply increases all through and hence so does $S^*(\tau)/\bar{u}(\tau)$. This is the polar opposite of Case A. In Case A, $S^*(\tau)/\bar{u}(\tau)$ *monotonically decreases* to its limiting value, 1; in Case D, it monotonically increases to the same limiting value.

3.7 We have now obtained the global bahaviour of our ratio, $S^*(\tau)/\bar{u}(\tau)$, over all parameter configurations (α, z_0). In the process, we have distinguished alternative ranges of parameter configurations for which the behaviour of $S^*(\tau)/\bar{u}(\tau)$ is different in some respect or other within the common point that $S^*(\tau)/\bar{u}(\tau) \longrightarrow 1$ as $\tau \longrightarrow \infty$ in all cases. These different cases are graphically depicted as corresponding regions in the (α, z_0) plane in Figure 1 below.

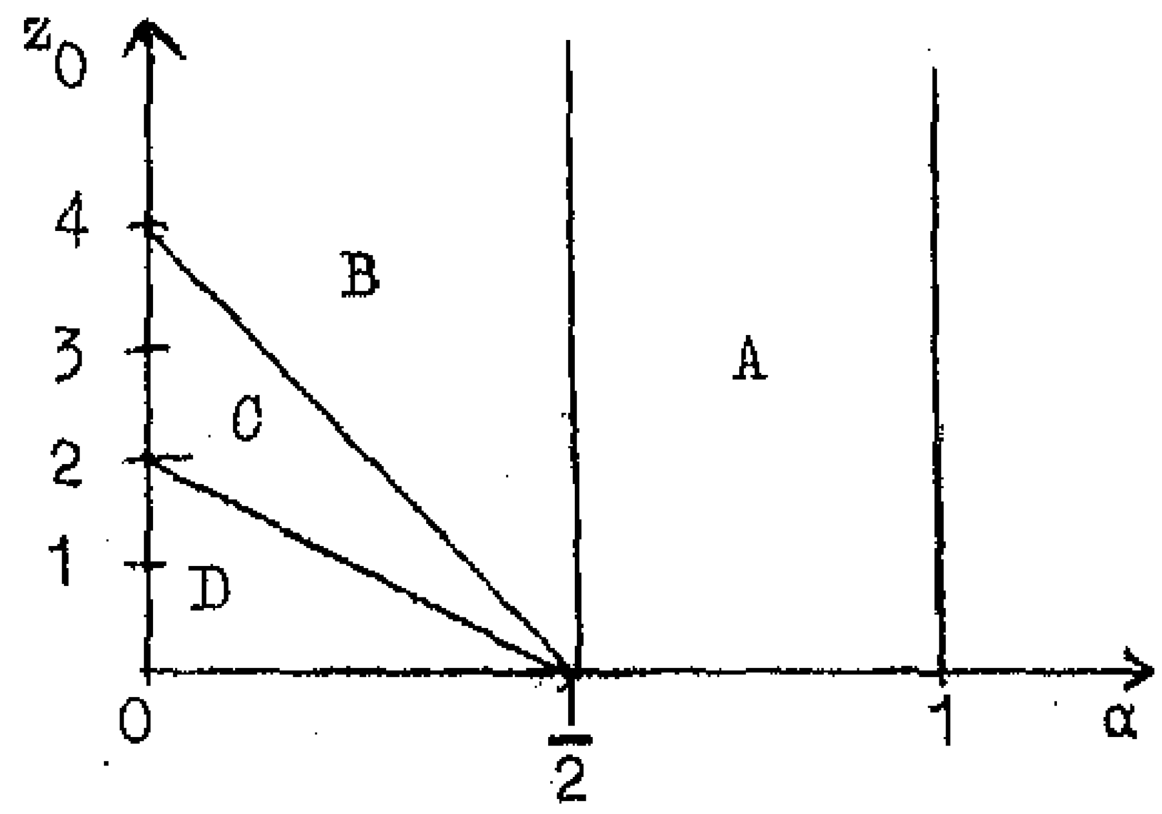

Figure 1

4. We are now ready to pass on to the behaviour of $S^*(\tau)/\bar{u}(\tau)$ for small values of τ, which, as we said earlier, is the real point of interest. Here the striking point is that over a wide range of parameter configurations, the ratio initially falls. This covers various possibilities, e.g. (a) the ratio falling all through (Case A) or falling initially and then rising (Cases B, C), and (b) the ratio lying initially above the "standard", 1 (Cases A, B) or below it (Case C). As for the range of parameter configurations, it is only for very small values of both parameters, α and z_0 (Case D) that we have the exception to our behaviour. In view of the actual numerical magnitudes involved, we may just argue to leave the case out of the present discussion[5/]. The behaviour of $S^*(\tau)/\bar{u}(\tau)$ for Cases A-C is graphically depicted in Figure 2 below.

[5/] See in this context the discussion on the order of magnitude of λ in Appendix A to Chapter 5.

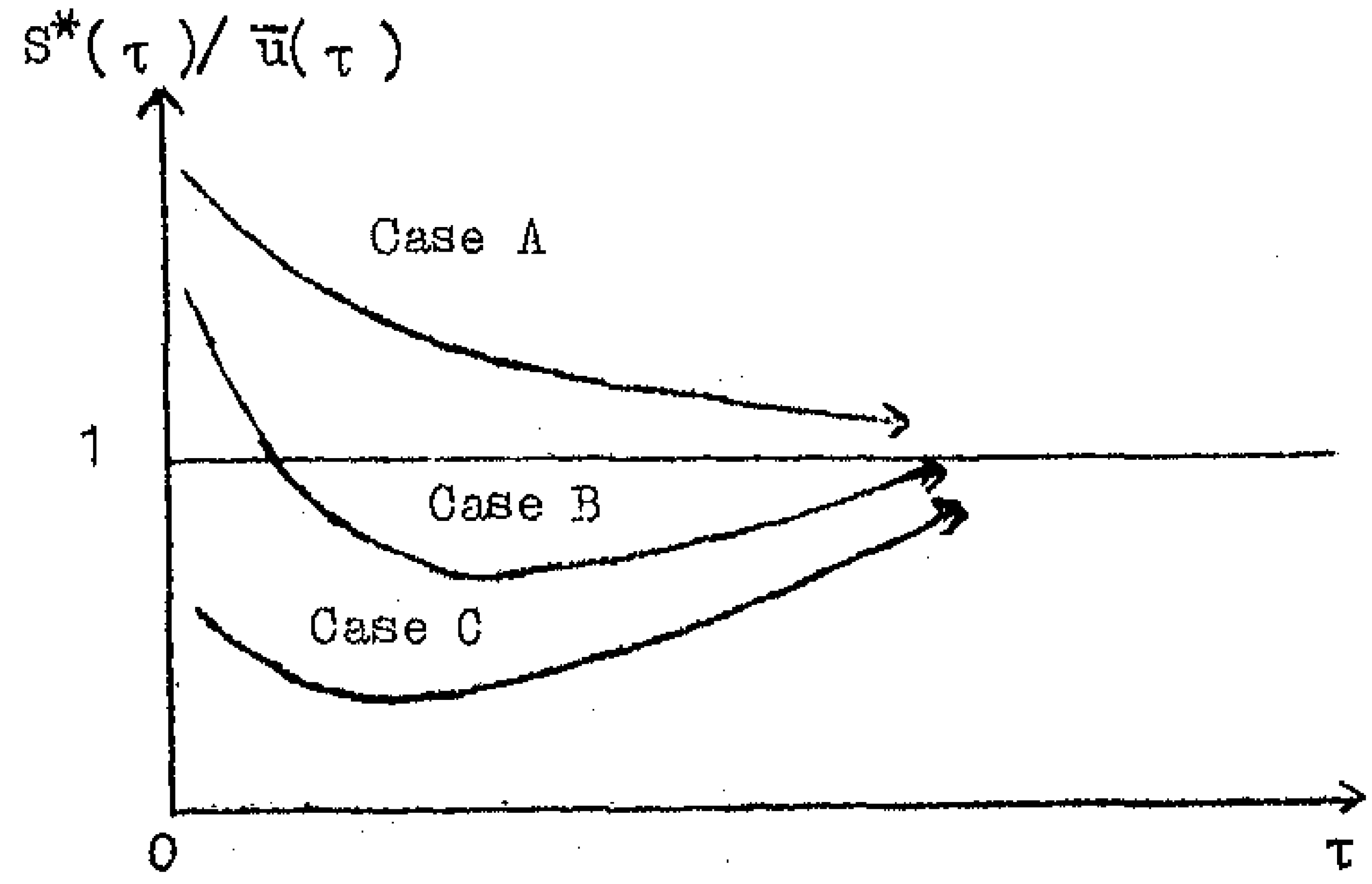

Figure 2

5. Let us now come to the significance of our result that the ratio, $S^*(\tau)/\bar{u}(\tau)$, initially falls with τ. We take that the purchase interval, τ, enters the model as " prior decision", so that what we have been discussing is simply the effect of the trader himself varying this prior decision upon his own " posterior decision" i.e., the optimum stock of the model, $S^*(\tau)$, in the form stated, $S^*(\tau)/\bar{u}(\tau)$. Since $\bar{u}(\tau)$ in turn is simply proportional to τ, this comes to the same as his studying the effects of varying τ upon the ratio, $S^*(\tau)/\tau$. This is precisely the 'form' we had started from. The question is what " feedback" does this study have on the prior decision itself, i.e., the choice of τ.

Clearly it is only at a general qualitative level that we can discuss this feedback — the actual choice or decision of τ can be discussed only in terms of a complete inventory model, which we do *not go into*. Let us now remember a very substantive point. The whole of inventory theory as it exists begins with a built in force favouring an arbitrarily small purchase interval, viz., the cost of capital per unit time. It is therefore only by introducing an opposite force, viz., the transaction cost, that it gets to erect a meaningful framework for inventory problems in the first place. Nothing of this is simply brought into play here.

Let us now put our result against this background. It is seen that if the trader lengthens his purchase interval starting from a very small value, then he reduces the optimum stock relative to the purchase interval, which we may also put as the stock required per unit time. It is then possible that his cost of capital per unit time is actually reduced and not increased by the lengthening of the purchase interval. This is the basic significance of our result. It suggests very broadly the possibility of quite another force (other than transaction cost) working against the choice of a very small purchase interval by the trader.

To clarify the nature of this force, we need only remember that the whole result under reference is the work of what we called the uncertainty effect of varying the purchase interval, as distinct from its mean effect. When the purchase interval is very small, the uncertainty of demand faced is very large and this induces the trader to hold a large stock relative to the mean demand. As the purchase interval is lengthened, the uncertainty is reduced, and this calls for a smaller stock relative to the mean demand, which comes to the same as a smaller stock holding per unit time, for the mean demand is simply proportional to purchase interval. So we can say that the 'force' against the choice of a very small purchase interval that is indicated here is simply the avoidance of the very high uncertainty of demand that goes with a very small purchase interval.

Chapter 7

MODEL II : OPTIMUM STOCK UNDER UNCERTAIN SUPPLY AND A GIVEN PURCHASE INTERVAL

Section 1 : Availability Problem and the Process of Trade

1. In Chapter 1 we broadly specified the nature of uncertainty of supply or the ''availability problem'' as we called it around which the model of this chapter is built up. The specifications included in particular the assumption that the trader buys from a periodic market. This already puts the model in a common framework of a given purchase interval (in the extended sense of the term explained in Chapter 1; see p. 44) along with Model I. The rest of the model remains to be built up. This is essentially a matter of looking at the availability problem through the process of trade as set out in Sections 1 and 2 of Chapter 3 and seeing what it implies about the structure of the process itself. This is what we set out to do in this section.

2.1 Let us briefly recapitulate the availability problem. The problem in a nutshell is that the trader may on occasion find a ''zero availability'' of the goods in the market, in which case he obviously has to go without making a purchase. In all other cases the availability is sufficient for his purpose, i.e., he can make his intended purchase. So, effectively only two alternative states of the market are defined, which we may denote by $A = 0$ and $A > 0$, where A denotes the total availability in the market. If $A = 0$, the actual purchase is zero; otherwise it is simply equal to the intended purchase. There are definite probabilities associated with the two states, but we will come to this later.

2.2 Let us now link up with the process of trade as described in Chapter 3. First there is a conceptual problem. In the process as described in Section 2 of the chapter referred, every purchase by the trader restored his sales stock back to a certain level defined in relation to the capital put into and maintained through the process. This level of the stock was then called the 'stock maintained', which by definition was the same as the stock immediately after a purchase. For reference, we put down the relation between capital, K, and the stock maintained, S, viz.

(1) $K = q S$,

where q denotes the trader's buying price per unit of the goods.

Let us now go back to the point that every purchase by the trader restores his stock back to the level S. It follows that if a purchase cannot be made, then the stock is not restored — in that sense, it is also not maintained. This is the conceptual point [1]. The point to remember however is that the whole notion of restoration or maintenance of stock was

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[1] This was just alluded to in Chapter 1. See p. 45.

earlier defined in relation to the purchases actually made or actual purchase, not the ''intended'' purchase in any sense distinct from the ''actual'' purchase. The distinction itself is made only in the present set-up. If we now restrict ourselves only to the purchases actually made, then we will still find the stock to be restored or maintained at the level S (for the same K and q obviously) in the present set-up. This follows simply from the analysis of Section 2 of Chapter 3 [2]. However, taken out of this restriction, i.e., at a general level, the notion now is properly understood as the stock sought to be maintained, and we will henceforth call it by this name. So, S now denotes the stock sought to be maintained or more fully the 'level' at which the stock is sought to be maintained.

2.3 Let us now bring in the condition of periodic market, which, for the convenience of language, we will call a ''weekly market''. The point about the maintenance of stock just stated can then also be expressed as follows : The stock is in fact maintained through the successive purchases actually made, but it is not necessarily maintained through the successive weeks. This is the distinguishing characteristic of the present model, setting it apart from Model I.

2/ As a corollary, the ''intended'' purchase in the present set-up is now closed by the relation that the intended purchase is simply to restore the stock back to the level, S, i.e., stated more fully, to buy back the whole amount sold since the last actual purchase.

Let us now define the <u>stock at the beginning of a ''week''</u> as the stock inclusive of the amount bought at this point, if any, and denote it by Q. In Model I, Q was always equal to S —— 'always', i.e., at the beginning of each week. In the present model, Q is equal to S if and only if a purchase is actually made at the beginning of the 'week' concerned. Otherwise Q falls short of S —— this is the whole meaning of the stock being ''not restored''. So, we have the general inequality :

(2) $Q \leq S.$

This is true for all 'weeks'. It is however to be noted that while S is by definition a constant through Time, Q is not so. To come to the behaviour of Q through Time, it is first necessary to look into the <u>sales side</u> of the process. Let us take our time on this, for it takes us straight to the heart of the inventory problem that arises with the availability problem.

3.1 By the fundamental ''sales rule'' of the trade-process, the total sale in a week is set by the <u>minimum</u> of the stock at the beginning of the week, i.e., Q, and the total demand occurring in the week. So, for any given week we have the relation :

(3) $X = \min \{Q, D\}$,

where X and D denote respectively the 'sale' and the 'demand' for the week under reference.

3.2 Let us now recall the problem of <u>sales foregone</u> which lay at the very heart of the whole inventory problem in Model I. The problem is simply the trader's inability to meet some 'demand' — and hence to make the profit upon the sale upon the demand — on account of his running out of 'stock'. Given the 'weekly' frame, this means simply that Q falls short of D, and hence so does X — 'sales foregone' is simply the gap, (D - X). In Model I, this meant simply S falling short of D, for Q was then identical with S. In the present set-up there arises the <u>independent possibility</u> of sales foregone, not because S is smaller than D, but because <u>Q is smaller than S</u>. This is the heart of the inventory problem in the present set-up. Going back to the starting point of <u>capital</u>, we can say that for the <u>same</u> capital the sales that would have been made in Model I may now have to be 'foregone' because though the capital is there, there are no goods in the market to be bought with the 'capital'. But no new means or instrument is defined to tackle the problem. Capital, or equivalently, the stock maintained, or sought to be maintained, remains the <u>one</u> decision variable of the trader, and <u>in this sense</u>, the fundamental structure of the problem is not altered. We can say that the availability problem simply ''feeds into'' the problem of sales foregone 3/ and generalises the inventory problem that arises out of it from within.

3/ This is very generally true, for the problem of sales foregone arises only out of the trader running out of stock, and if the goods are then not available for stock-replacement, even temporarily, <u>more</u> sales must be foregone.

3.3 Let us take off a minute here to outline the programme. It is clear from the nature of the problem just outlined that the trader would have a genuine problem of deciding about the capital on the basis of which to carry out the process even in abstraction from the uncertainty of demand. So, this is where we begin from (Section 2), and then go on to bring in the uncertainty of demand (Section 3). Thus we have two successive models in this chapter. The first is a generalisation of what we called the ''simplest inventory model'' (see pp. 94-6) and the second is a generalisation of Model I as well as of the first model just referred.

4. Let us now take up the behaviour of Q over Time. For an algebraic statement, we arbitrarily fix some particular week as our reference point and denote it by the subscript '0' and the preceding weeks by subscripts -1, -2, ... Using this notation, we already know that $Q_0 = S$ if $A_0 > 0$. If on the other hand $A_0 = 0$, then Q_0 is simply the stock brought over from the last week, which in turn is given by the stock at the beginning of the last week, i.e., Q_{-1}, minus the sale of the last week, i.e., X_{-1}. We thus end up with a general recursive relation in Q defined by :

$$(4) \qquad Q_0 = \begin{cases} S, & \text{if } A_0 > 0 \\ Q_{-1} - X_{-1}, & \text{if } A_0 = 0. \end{cases}$$

Note that by (2) and (3) we already have :

$S \geqq Q \geqq X$.

This is true for all 'weeks'. Hence from (4) :

(5) $0 \leqq Q_0 \leqq S$.

5.1 Let us now turn to the <u>probability of occurrence of the two states of the market</u>, $A = 0$ and $A > 0$ for any week, which we may also write as $A = 0$ and $A \neq 0$, since A is intrinsically non-negative. The a priori assumption is of course that $A = 0$ has only a ''small probability of occurrence'', say ε. This is to be made precise, for it is ultimately a <u>sequence of availabilities</u> $\{A_k\}$ that we are concerned with where k indexes the successive market meetings.

It would be possible to treat the A_k's as independently and identically distributed random variables so that the probability of '$A_k = 0$' is ε for each k independently of the past values of A. The probability of two successive occurrences of zero availability would then be ε^2 and so on.

5.2 We will however choose a <u>simpler alternative</u>. We will simply equate the probability of two successive occurrences of zero availability to zero, i.e., we assume that if $A_k = 0$ then $A_{k+1} \neq 0$. In other words, $A_k = 0$ is possible only if $A_{k-1} \neq 0$. With this we must now interpret the a priori probability of $A = 0$, i.e., ε, as the conditional probability of $A_k = 0$, given $A_{k-1} \neq 0$. Thus, we now give up the assumption that A_k's are independently distributed. The assumption of identical distribution is retained.

Let us now find out the unconditional probabilities of $A = 0$ and $A \neq 0$. We start with the two axioms already suggested :

Axiom I : $P(A_k \neq 0 | A_{k-1} = 0) = 1$.

Axiom II : $P(A_k = 0 | A_{k-1} \neq 0) = \varepsilon$.

Now,

$$P(A_k \neq 0) = P(A_k \neq 0, A_{k-1} \neq 0) + P(A_k \neq 0, A_{k-1} = 0)$$

$$= P(A_k \neq 0 | A_{k-1} \neq 0). P(A_{k-1} \neq 0)$$

$$+ P(A_k \neq 0 | A_{k-1} = 0). P(A_{k-1} = 0)$$

$$= [1 - P(A_k = 0 | A_{k-1} \neq 0)]. P(A_{k-1} \neq 0)$$

$$+ P(A_k \neq 0 | A_{k-1} = 0). P(A_{k-1} = 0)$$

$$= (1 - \varepsilon) P(A_{k-1} \neq 0) + 1.P(A_{k-1} = 0),$$

by Axioms I and II

$$= (1 - \varepsilon) P(A_{k-1} \neq 0) + 1 - P(A_{k-1} \neq 0).$$

If we now denote $P(A_k \neq 0)$ by x_k then the above equation gives us the following first order difference equation in x_k :

$$x_k = (1 - \varepsilon)\ x_{k-1} + 1 - x_{k-1}$$

$$(6) \qquad = 1 - \varepsilon\, x_{k-1} .$$

Since A_k's are identically distributed, we must have $x_k = x_{k-1} = x$, say. From (6), we then see at once that :

$$(7) \qquad x = \frac{1}{1 + \varepsilon} = 1 - \beta \quad , \text{ say.}$$

Hence, stated in full, our unconditional probabilities are given by :

$$(8) \qquad \begin{cases} P(A \neq 0) = \dfrac{1}{1 + \varepsilon} = 1 - \beta , \\[2ex] P(A = 0) = \dfrac{\varepsilon}{1 + \varepsilon} = \beta . \end{cases}$$

Let us now return to the recursive relation in Q_0 defined in (4). Obviously, (7) turns this relation into a <u>complete probability distribution of Q_0 defined in terms of S and X_{-1}</u>, viz. :

$$(9) \qquad Q_0 = \begin{cases} S, \text{ with probability } 1 - \beta, \\ S - X_{-1}, \text{ with probability } \beta. \end{cases}$$

This is the simplicity achieved by Axioms I and II. If A_k's were treated as independently and identically distributed, then we would have a probability distribution of Q_0 defined in terms of S and the whole sequence of past sales, X_{-1}, X_{-2}, ... This sequence is now truncated to just one value, X_{-1}.

6. We have now completed the groundwork necessary for the inventory models of this chapter. Before taking up the models, we take up one point about the structure of our trade process which, though not strictly necessary for the model building, throws some light on the general conceptual framework underlying it.

Let us refer back to the sales equation (3), which is in fact simply brought over from Model I. As explained there, the equation expresses a dual constraint on the trader's sale in a week, an ''inside constraint'' defined by his stock at the beginning of the week and an ''outside constraint'' defined by the total demand occurring in the week. We now point out that in an exactly parallel fashion, there is a dual constraint on the trader's purchase in a week where the inside constraint is defined by his purchase fund existing at the beginning of the week and the outside constraint by the total availability in the market bought from at this point. Denoting the amount bought and the purchase fund respectively by Z and F, we therefore have a parallel ''purchase equation'' defined by

$$(10) \qquad Z = \min \left(\frac{1}{q} F, A \right).$$

It would be possible to work out the recursive relation in Q defined in (4) starting from this equation and arguing explicitly through the ploughback rule and the spending rule of the process. To avoid undue repetition, we do not go over this. Let us just go back to (10) and point out the following. A dual constraint on the trader's purchase as expressed in (10) is <u>implicit</u> in our general framework of the trade process and hence of inventory problems of the trader in general from the very beginning. However, this is purely formal. Unlike the demand constraint on the trader's sale, which is purely definitional, no meaningful supply constraint on his purchase appears definable except under special conditions defined essentially through some substantive notion of supply uncertainty. In spite of the striking <u>formal</u> resemblance between (3) and (10), the conceptual parallelism thus does not go very far.

<u>Section 2</u> : <u>The Optimum Stock when Demand is Certain</u>

1. We now come to the basic object of this chapter, to build an inventory model around the " availability problem" and analyse it. As already stated, the model in this section simply extends our " simplest inventory model" introduced in the course of our analysis of Model I by introducing the availability problem in it. This already implicitly defines the model and we can proceed just on this basis.

Let us begin with a bird's- eye view of the extension of the simplest model defined by the present model. In both models we consider a trader who (a) gets his supply from a periodic market meeting at the beginning of each " week" [4], and (b) faces a given demand repeating itself week after week in the amount, D. But whereas in the simplest model there is an assured (and enough) supply each week, in the present model, the supply may at times be zero, being otherwise sufficient for the trader's requirements. There are certain " probabilities" attached to these two states of the market (zero supply and positive supply), but we shall pick them up on the way. Let us for the time see how much can be argued about the " working" of the present model just on this basis, having in mind the working of the simplest model set out on pp. 94 - 6 .

2.1 We already know that in the simplest model, the " best" policy of the trader is to put in a capital just sufficient to buy the quantity, D, i.e., put in the capital, K = qD, which in turn enables him to buy back precisely this amount at the beginning of each week, having successively sold off the whole amount in the preceding week. This he actually does week after week under the "best" policy, thereby meeting the whole demand over time, and just that, i.e., he does not have any stock left over at the end of any week after meeting the whole demand over that week.

2.2 Let us now come to the present model. Here, there is a clear chance that the trader may not be able to buy anything some week. Let us consider such a week, and call it the n-th week, or week (n). If he had put in the same capital, K = qD, as in his best course in the simplest

[4] Strictly speaking, the simplest model is defined independently of the assumption of a periodic market. It is made here for comparability.

model, then he would have simply no stock at the beginning of this week -- neither has he brought over any stock from the preceding week nor is he able to buy anything. He therefore has to forego the whole sale of the week, i.e., a sale of the amount, D.

2.3 Clearly, had he brought forward a stock in this amount - D - from the preceding week, i.e., week (n - 1), he would not have foregone any sale. Let us now follow out the logical implications of this case, i.e., his bringing forward a stock in amount D from week (n - 1) to week (n).

To say that a positive stock is carried forward from any week to the next is already to imply that no sale is foregone in the first week. So, the amount sold in week (n - 1) in our case is D, and since a stock, D, is carried over from this week to the next, the stock at its beginning must have been 2D. That is, the stock at the beginning of week (n - 1) in our case is 2D.

Let us now turn to week (n + 1). This brings us to the first statement of " probabilities" in the present model. We know that " zero supply" cannot occur in two consecutive weeks. Since in week (n) there is zero supply, in week (n + 1) there is positive supply. Hence a purchase is actually made in week (n + 1) and the amount bought is given by the purchase fund brought over from week (n) to week (n + 1) divided by q. What is the magnitude of this purchase fund? It consists, firstly, of the ploughback of week (n), the amount of which is qD; and secondly, of the ploughback of week (n - 1), for nothing could be spent in week (n), the amount of this too being again qD. Anything else? The answer is 'no', for by the same condition of no two consecutive weeks of zero supply, a purchase must have been made in week (n − 1) and that exhausted the pre-existing purchase fund. Hence, the purchase fund at the beginning of week (n + 1) is precisely 2qD and the amount bought is 2qD ÷ q = 2D. The stock at the beginning of this week is thus again 2D, which through the week again gets divided into one part sold over the week and another carried forward to the next week, i.e., week (n + 2).

2.4 Viewing things over, it is clear that under the case under reference, (a) the capital put in is K = 2qD; and (b) demand is fully satisfied every week. These are the relevant logical properties of this case for our purpose, and we can now go on to a comparison of the case begun with and the present case in terms of the expected rates of profit they yield. Before beginning, let us just restate this as a comparison of two alternative decisions of the trader, K = qD (alternative 1) and K = 2qD (alternative 2).

3.1 Obviously, the capital put in is doubled as we go from alternative 1 to alternative 2. What happens to profit, the expected profit to be precise? Let us start with profit made. Under alternative 2, D amount is sold every week and hence πD amount of profit is made every week, where π denotes the profit margin per unit of sale. Under alternative 1, D amount is sold in a week if the supply in that week is positive, nothing is sold if the supply is zero. The respective probabilities of these two states —— positive supply and zero supply — are $(1 - \beta)$ and β. Hence the expected sale in a week is $(1 - \beta) D + \beta . 0 = (1 - \beta) D$, and the expected profit made is $(1 - \beta)\,\pi D$. It follows that as we go from alternative 1 to alternative 2, the expected profit made increases by the amount, $\beta\pi D$. The proportionate increase is therefore $\beta\pi D/(1 - \beta)\,\pi D = \beta/(1 - \beta) = \varepsilon$, i.e., the conditional probability of zero supply in a week, given that the supply in the previous week was positive.

3.2 Let us now for a moment abstract from the negative contribution of profit foregone in 'profit'. We then see that as we go from alternative 1 to alternative 2, the expected profit increases by 100ε%, while 'capital' increases by 100%, and hence expected rate of profit certainly falls. The best policy for the trader (within the two alternatives under consideration) would then be the same as in the simplest model, i.e., $K = qD$. This is independent of the value of ε.

3.3 The negative contribution of 'profit foregone' certainly tilts the scale somewhat in favour of alternative 2, for there is no profit foregone under it. The expected profit foregone under alternative 1 is simply $\beta\pi D$. Weighting this by the factor λ measuring the "loss" implied by a unit of profit foregone in terms of profit made, and adding this to profit made, we find the expected profit under alternative 1 to be $(1 - \beta)\,\pi D - \lambda.\beta\,\pi D = [1 - (1 + \lambda)\,\beta]\,\pi D$. The expected profit under alternative 2 of course remains πD. The relative increase in expected profit as we go from alternative 1 to alternative 2 is now given by the ratio, $(1 + \lambda)\beta/[1 - (1 + \lambda)\,\beta]$, granting that the denominator of this ratio is positive. The relative increase in capital remains 100%.

We are now ready for the comparison of the expected rates of profit under the two alternatives, which we now denote by ρ_1 and ρ_2 respectively. We then conclude that :

$$\rho_1 \gtrless \rho_2 \quad \text{as} \quad \frac{(1+\lambda).\beta}{[1-(1+\lambda).\beta]} \lesseqgtr 1.$$

On simplification this yields :

$$(11) \quad \rho_1 \gtrless \rho_2 \quad \text{as } (1+\lambda) / \beta \lesseqgtr 0.5.$$

4.1 This is the basic result of the model. Let us now discuss the significance of this result.

Let us for a minute go back to the simplest model. There, we proved the optimality of K = qD by showing that the rate of profit was (a) a decreasing function of K for K > qD implying that the optimum capital was bounded above by K = qD; and (b) an increasing function of K for K < qD implying that the capital was bounded below by K = qD. By exactly the same arguments proving (a) and (b), it can be seen that in the present model, the expected rate of profit is (a) a decreasing function of K for K > 2qD and (b) an increasing function of K for K < qD. This implies that the optimum capital is respectively bounded above by K = 2qD and bounded below by K = qD. So, the optimum capital is located in the interval $[K_1, K_2] = [qD, 2qD]$. By the linear structure of the whole model, the search for the optimum capital can be confined to just the two extremities, $K = K_1$ and $K = K_2$. (11) therefore defines the criterion for optimum capital in the model as a whole. The optimum capital, to repeat, is either K_1 or K_2 depending upon whether the number $(1 + \lambda)\beta$ is less or greater than 0.5.

4.2 Let us now remember that ε and hence β is only a "small probability", say of the order of 0.05. It follows that unless λ is of the order of 9 or above, we will always have $(1 + \lambda)\beta < 0.5$. This order of magnitude of λ can a priori be ruled out of hand[5]. Let us explicitly postulate this as an a priori relation of the model :

(12) $(1 + \lambda)\beta < 0.5$[6].

We can now conclude that the optimum capital in the model is $K = qD$, the same as in the simplest model. Stated in symbols :

(13) $K^* = qD$,

where K^* denotes the optimum capital.

4.3 Our whole extension of the simplest model thus turns out to be simply an extension of the validity of its solution to the case of uncertain supply as defined by our "availability problem", under the

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[5] See in this connection the discussion on the "order of magnitude" of λ in Appendix A to Chapter 5.

[6] It may be noted that by definition, $\beta < 0.5$, for $\beta = \varepsilon/(1 + \varepsilon)$ and ε must belong to (0, 1).

parameter configuration defined by (12). At the same time, the point is clearly made that had the inequality in (12) been reversed, the optimum capital would have been different, in fact, $2K^*$. We also call attention to the point that it is not just the probability of zero supply, but this probability viewed through the prospective " loss" of sales foregone that ultimately determines the size of optimal capital.

5.1 We have now given a complete analysis of our model by pure logical arguments. It would be possible to close the section at this point. However, we will briefly go over the analysis again on a formal mathematical plane. This serves the purpose, firstly, of verification of results stated; and, secondly, of providing a convenient stepping stone for the analysis of the model of the next section.

5.2 Let us now take off from the analysis of the previous section, using freely the notation established there. We begin from the probability distribution of stock at the beginning of a week defined in (9) reproduced below :

$$(9) \qquad Q_0 = \begin{cases} S & \text{, with probability } 1 - \beta, \\ S - X_{-1} & \text{, with probability } \beta . \end{cases}$$

Using the sales equation (3), this gives us :

$$(13) \qquad Q_0 = \begin{cases} S & \text{, with probability } 1 - \beta, \\ S - \min\{Q_{-1}, D_{-1}\} & \text{, with probability } \beta . \end{cases}$$

Now, (a) $Q_0 = S - \min\{Q_{-1}, D_{-1}\}$ means $A_0 = 0$ and hence by Axiom I, $A_{-1} \neq 0$. By (4), this implies $Q_{-1} = S$. (b) In the present model, $D_{-1} = D$. Hence the whole 'recursiveness' in (13') disappears and we can write it simply as :

$$(14) \qquad Q = \begin{cases} S & \text{, with probability } 1 - \beta \text{ ,} \\ S - \min\{S, D\} \text{ ,} & \text{with probability } \beta \text{ .} \end{cases}$$

Let us now turn to the sale equation :

$$(3) \qquad X = \min\{Q, D\} \text{ .}$$

We now find out the expected value of X as a function of S from (14) and (3). It is at once verified by direct substitution that the expected value is given by :

$$(15) \qquad E(X(S)) = \begin{cases} (1-\beta)S & , \quad S < D \text{ ,} \\ (1-\beta)D + \beta(S - D), & D \leq S \leq 2D \text{ ,} \\ D & , \quad S > 2D. \end{cases}$$

5.3 Let us now go on to <u>profit</u>. We can start off from the general equation of profit in (5.5) reproduced below :

$$P = (1 + \lambda)\pi X - \lambda\pi D \text{ ,}$$

where P denotes profit in a week, π is the profit margin per unit of sale and λ is the subjective parameter weighting the " loss" of one unit of profit foregone in terms of profit made.

Obviously, P is also a function of S, say P(S). Taking expectation we find :

$$E(P(S)) = (1 + \lambda)\, \pi\, E(X(S)) - \lambda\, \pi D$$

$$(16) \qquad = \begin{cases} (1 + \lambda)\, \pi\, (1 - \beta)\, S - \lambda \pi D, & S < D, \\ (1 + \lambda)\, \pi\, [(1 - \beta)\, D + \beta (S - D)] - \lambda\, \pi D, & D \leqq S \leqq 2D, \\ (1 + \lambda)\, \pi D - \lambda \pi\, D, & S > 2D, \quad \text{from (15).} \end{cases}$$

The expected rate of profit, ρ, is defined by

$$\rho = \frac{E(P(S)/\tau)}{K}$$

$$(17) \qquad = \frac{E(P(S))}{\tau\, qS}, \qquad \text{from (1)},$$

where τ denotes a week's length of time. Clearly, ρ is a function of S, say $\rho(S)$.

It is convenient to define the variable, x, by

$$(18) \qquad x = D/S$$

and consider ρ as a function of x, say $\rho(x)$.

From (16) and (17) it is then seen :

$$(19) \qquad \rho(x) = \frac{\pi}{\tau q} \begin{cases} (1+\lambda)(1-\beta) - \lambda x, & 1 < x, \\ (1+\lambda)\left[(1-\beta)x + \beta(1-x)\right] - \lambda x, & \frac{1}{2} \leq x \leq 1, \\ x, & x < \frac{1}{2}. \end{cases}$$

5.4 This brings us to the concluding point of the analysis. It is now seen from (19) that $\rho(x)$ is a decreasing function for $x > 1$ and an increasing function for $x < \frac{1}{2}$. It follows that the maximum value of $\rho(x)$, if it exists, has to obtain at some value of x in the interval, $\frac{1}{2} \leq x \leq 1$. But $\rho(x)$ is a linear function over this interval. Hence the maximum exists and is obtained either at $x = 1$ or at $x = \frac{1}{2}$. By direct substitution, it is found :

$$\rho(1) = \frac{\pi}{\tau q}\left[(1+\lambda)(1-\beta) - \lambda\right] = \frac{\pi}{\tau q}\left[1 - (1+\lambda)\beta\right].$$

$$\rho(\tfrac{1}{2}) = \frac{\pi}{\tau q}\left[(1+\lambda)\cdot\frac{1}{2} - \frac{1}{2}\lambda\right] = \frac{\pi}{2\tau q}.$$

Hence after simplification :

$$(20) \qquad \rho(1) \gtrless \rho(\tfrac{1}{2}) \quad \text{as } (1+\lambda)\beta \lessgtr 0.5.$$

This is the same as (11). Hence in particular, it follows that under (12), the value of x maximising $\rho(x)$, say x^*, is given by :

$$x^* = 1.$$

Transforming back to S, and thence to K, this gives us :

$$S^* = D, \qquad \text{from (18)},$$

$$K^* = qS^* = qD, \qquad \text{from (1)},$$

where S^* and K^* are the values of S and K maximising ρ.

5.5 This concludes the verification of the results of our analysis of the model given earlier. Let us end by simply noting that S^* is the optimum stock in the model, 'stock' in the sense of the stock sought to be maintained. The special sense of the term is already clarified in Section 1. This ties up with the title of the section and the chapter as a whole.

Section 3 : The Optimum Stock when Demand Is Uncertain

1. As already stated, the inventary model of this section is an extension of Model I, discussed in the previous two chapters, by the introduction of our " availability problem" in it. The model can also be considered to be an extension of the model discussed in the previous section by introduction of the " uncertainty" of demand.

Let us start from this latter end. In the model of the previous section, D, the demand in a week, was a given quantity, repeating itself over week after week. In the present model, D is a random variable, meaning that the demand in each week is a random variable having the same pdf which we will denote by $f(u)$. In technical terms, if ...., D_{-1}, D_0, D_1, ... are the demands in ...., week − 1, week 0, week 1, ..., then the D_i's are iid. We will assume for simplicity that $f(u)$ has domain $(0, \infty)$. This is a major simplification of the structure of the model.

2. Let us now start from the probability distribution of stock at the beginning of a week given in (13′). As observed in the last section (see point (a) on p. 183), we can write this distribution in general as :

$$Q_0 = \begin{cases} S & \text{, with probability } 1 - \beta, \\ S - \min \{ S, D_{-1} \} & \text{, with probability } \beta . \end{cases}$$

The sale in our reference week (week (0)), X_0, according to the sales equation (3), is given by :

$$X_0 = \min \{ Q_0, D_0 \} .$$

3.1 We will now find out the expected value of X_0. This is a major step in the model-building. Obviously, the expectation is to be taken on the three distinct random variables, Q_0, D_0 and D_{-1}, the probability distribution of each of which is already specified. It will be convenient to denote the realised values of D_0 by u and the realised values of D_{-1} by v.

3.2 Let us first find out the <u>conditional expectation</u> of X_0, given D_0 and D_{-1}. This is given by :

$$E(X_0 | D_0, D_{-1}) = E\left(\min\left\{Q_0, D_0\right\} \middle| D_0, D_{-1}\right)$$

$$= (1-\beta)\min\left\{S, D_0\right\} + \beta \min\left\{S - \min\left\{S, D_{-1}\right\}, D_0\right\} .$$

This is a weighted average of two terms and hence the expected value of X_0 is the same weighted average of the expected value of these two terms. Let us denote these two expected values by E_1 and E_2 respectively. We thus have :

$$E(X_0) = (1-\beta) E_1 + \beta E_2 .$$

$$E_1 = E(\min\left\{S, D_0\right\}).$$

$$E_2 = E(\min\left\{S - \min\left\{S, D_{-1}\right\}, D_0\right\}).$$

It is clear that the first term under reference is greater than or equal to the second term, and hence $E_1 \geq E_2$[7]. Since $E(X_0)$ is a weighted average of E_0 and E_1, we have the relation :

$$E_1 \geq E(X_0) \geq E_2.$$

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[7] E_1 is the expected sale in week '0', given $A_0 > 0$ and hence $Q_0 = S$, and E_2 is the expected sale in week '0' given $A_0 = 0$ and hence $Q_0 \leq S$ but $Q_{-1} = S$.

3.3 Let us now find out E_1 and E_2.

(21) $$E_1 = \int^{S} uf(u)du + S \int_{S} f(u)du$$ [8].

As for E_2, we note first that

$$\min \left\{ S - \min \left\{ S, D_{-1} \right\}, D_0 \right\} = \begin{cases} D_0 & , \text{ if } S \geqq D_0 + D_{-1}, \\ S - D_{-1} & , \text{ if } D_0 + D_{-1} > S \geqq D_{-1}, \\ 0 & , \text{ if } D_{-1} > S. \end{cases}$$

From this we find :

(22) $$E_2 = \int^{S} \left[\int^{S-v} uf(u)\, du \right] f(u)dv + \int^{S} (S-v) \left[\int_{S-v} f(u)du \right] f(v)dv.$$

3.4 Now, both E_1 and E_2 are functions of S. Hence, so is $E(X_0)$. We will denote this function by V(S). That is,

$$V(S) = E(X_0)$$

(23) $$= (1 - \beta) \left[\int^{S} uf(u)du + S \int_{S} f(u)du \right] + \beta \left[\int^{S} \left[\int^{S-v} uf(u)du \right] f(v)dv + \int^{S} (S - v) \left[\int_{S-v} f(u)du \right] f(v)\, dv \right].$$

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[8] This is the same as the expected sale per week in Model I (see p. 98).

Note that we can write $E(X_0)$ simply as $E(X)$, i.e., the expected sale over <u>any</u> week, as $V(S)$ is independent of the particular week of reference.

By differentiation and simplification of terms, it is found that

$$(24) \qquad V'(S) = (1-\beta)\int_S f(u)du + \beta\int^S \left[\int_{S-v} f(u)du\right] f(v)dv > 0.$$

Differentiating once more, we obtain :

$$V''(S) = -(1-\beta)f(S) + \beta\left\{f(S) - \int^S f(S-v)f(v)dv\right\}$$

$$(25) \qquad = -(1-2\beta)f(S) - \beta\int^S f(S-v)f(v)dv < 0^{9/}$$

We thus have the important result that the <u>expected sale in a week is a strictly increasing function of the stock sought to be maintained, S, with a decreasing rate</u>.

4.1 We are now ready for the optimisation in our model. First, we have the general definition of profit in a week, P, given by (5.5) reproduced below :

$$P = (1+\lambda)\pi X - \lambda\pi D.$$

So,

$$E(P) = (1+\lambda)\pi E(X) - \lambda\pi E(D)$$

$$(26) \qquad = (1+\lambda)\pi V(S) - \lambda\pi\bar{u}$$

9/ This follows from the fact that β lies by definition in the interval (0, 0.5) (see footnote 6, on p.181).

where

$$\bar{u} = E(D) = \int u f(u) du.$$

The expected rate of profit, ρ , is now defined by

$$\rho = \frac{E(P/\tau)}{K}$$

$$= \frac{1}{\tau q S} \left\{ (1 + \lambda) \pi V(S) - \lambda \pi \bar{u} \right\}, \quad \text{from (26) and (1).}$$

(27) $$= \frac{(1 - \lambda)\pi}{\tau q} \cdot \frac{V(S) - \alpha \bar{u}}{S},$$

where

$$\alpha = \lambda/(1 + \lambda).$$

4.2 Obviously, ρ is a function of S, say $\rho(S)$. The optimisation in our model consists of " choosing" S such that $\rho(S)$ is at a maximum. This defines the <u>optimum stock</u> in the model, to be denoted by S^*. Now, mathematically, maximising $\rho(S)$ is equivalent to maximising the function R(S) defined by :

$$R(S) = \frac{1}{S} \left[V(S) - \alpha \bar{u} \right].$$

So, S^* can also be defined as the value of S at which R(S) is at a maximum.

By differentiating V(S) we obtain :

$$R'(S) = \frac{1}{S^2} \left\{ S V'(S) - \left[V(S) - \alpha \bar{u} \right] \right\}$$

$$\gtreqless 0$$

as

$$\alpha \bar{u} \gtrless V(S) - S\, V'(S) = H(S), \text{ say,}$$

where

(28) $H(S) = V(S) - S\, V'(S)$.

It is seen at once that H(S) is an increasing function of S, for

$$H'(S) = V'(S) - V'(S) - S\, V''(S)$$

(29) $$= -S\, V''(S) > 0 \text{, from (25).}$$

From this we conclude that S^*, if it exists, is obtained by solving the equation :

(30) $H(S^*) = \alpha \bar{u}$.

Further, granted the existence, S^* is uniquely determined from (30).

4.3 Let us now turn to the question of existence of a solution of (30). Since H(S) is an increasing function of S over $(0, \infty)$, the existence is proved if and only if it can be shown that :

(30′) $$\lim_{S \to 0} H(S) < \alpha \bar{u} < \lim_{S \to \infty} H(S).$$

To obtain these limits let us first write down the expression of H(S) in terms of the original parameters. Substituting for the values of V and

V' from (23) and (24) in (28) and simplifying the resulting expression, we obtain :

$$(31) \qquad H(S) = (1-\beta)\int^{S} uf(u)du + \beta\int^{S}\left[\int^{S-v} uf(u)du\right]f(v)dv - \beta\int^{S}\left[\int_{S-v} f(u)du\right]vf(v)dv.$$

It follows from (31) that :

$$\lim_{S \to 0} H(S) = 0.$$

This proves the first inequality in (30′). Let us now turn to the second inequality. There are three definite integrals on the RHS of (31), each a function of S. As $S \longrightarrow \infty$, the first integral goes unambiguously to $\bar{u}$, but about <u>each</u> of the second and third integrals, it can only be said that in the limit it lies in the interval $[0, \bar{u}]$. Noting the way the three integrals are combined in the RHS of (31), we can then conclude only that :

$$(1-\beta)\bar{u} + \beta\bar{u} \geq \lim_{S \to \infty} H(S) \geq (1-\beta)\bar{u} - \beta\bar{u}$$

or

$$(32) \qquad \bar{u} \geq \lim_{S \to \infty} H(S) \geq (1-2\beta)\bar{u} .$$

Since α lies in the interval (0, 1), it is clear that the second inequality in (30′) is automatically satisfied if the limiting value of H(S) in the above expression lies at its upper bound. Since this cannot be proved, we can only give a <u>sufficient condition</u> for the existence of S^*, viz., the lower bound of the limiting value of H(S) in (32) is <u>greater</u> than $\alpha \bar{u}$. The condition obviously is :

$$1 - 2\beta > \alpha .$$

On noting that $\alpha = \lambda/(1 + \lambda)$ and simplifying, the condition reduces to the parameter configuration we already imposed on our model in the previous section, viz. :

(12) $(1 + \lambda)\beta < 0.5$.

Thus reiterating this condition, we do indeed have the existence of an optimum stock in the present model.

5. It is however to be noted that analytically (12) plays a very different role in the models of these two sections. In the previous model, the existence of an optimum stock was guaranteed a priori, and (12) came in to ensure only a particular value of the optimum stock as distinct from another which necessarily obtained if (12) was violated. The condition in this sense played an <u>active role</u> in the determination of the magnitude of the optimum stock. In the present model, it comes in a logically prior role of guaranteeing the existence of an optimum stock. But since it appears only as a sufficient condition, it may be said to play a <u>passive role</u> in the background. Whether or not (12) is really required for the existence of S^* is left in the open.

6.1 Let us now turn to a more interesting matter from the standpoint of the 'lineage' of the present model. Let us first substitute for $H(S^*)$ from (31) in (30) after grouping the terms in (31). This gives us:

$$(33) \qquad (1-\beta)\int^{S^*} uf(u)du + \beta\int^{S^*}\Big[\int^{S^*-v} uf(u)du - v\int_{S^*-v} f(u)du\Big]f(v)dv = \alpha\,\bar{u}.$$

Obviously this is the optimising condition of the present model stated in full. We now note firstly that if $\beta = 0$, then (33) simply reduces to the optimising condition of Model I, i.e., (5.16). Comparison of (33) and (5.16) clearly reveals the essential continuity that runs from Model I to the present model.

6.2 Let us now turn to a <u>comparison of the value of the optimum stock in the two models</u>. Note that the LHS of (33), which is nothing but $H(S^*)$, is a <u>weighted average</u> of two terms, with weights $(1-\beta)$ and β. So, $H(S)$ lies between these two terms. Let us now see which is the <u>dominant term</u>. Let us denote the first term by $g(S^*)$. So,

$$g(S\ ) = \int^{S^*} uf(u)du.$$

Let us now turn to the second term. We can write it as :

$$\int^{S^*} h(v;\, S^*)\, f(v)dv,$$

where

$$h(v; S^*) = \int^{S^*-v} uf(u)du - v\int_{S^*-v} f(u)du$$

$$= g(S^* - v) - v\int_{S^*-v} f(u)du$$

$$< g(S^* - v)$$

$$< g(S^*).$$

$$\therefore \quad \int^{S^*} h(v;\ S^*)\ f(v)dv < \int^{S^*} g(S^*)\ f(v)dv$$

$$= g(S^*) \int^{S^*} f(v)dv$$

$$< g(S^*)\ .$$

This shows that the first term is the dominant term. From this we conclude that :

$$H(S^*) < \int^{S^*} uf(u)du$$

and hence

(34) $\int^{S^*} uf(u)du > \alpha\bar{u}$.

Remembering the optimising condition of Model I, i.e., (5.16), we now conclude that :

(35) $S^*_{II} > S^*_{I}$,

where S_I^* and S_{II}^* denote the optimum stock in Model I and the present model, i.e., Model II of the thesis, respectively.

6.3 It is thus seen that the net effect of the availability problem is to increase the optimum value of S (and hence also of capital). This is an important result. Since the availability problem is synonymous with the uncertainty of supply, it is shown that the optimum stock (or capital) increases as a result of this uncertainty.

There is a straight continuation of this result within the model, in that S^* is in fact an increasing function of β which is the a priori measure of the degree of supply uncertainty in the model. In other words, we have an increase in the optimum value of stock not only as we go from $\beta = 0$ (Model I) to $\beta > 0$ (Model II) but also as we increase the value of $\beta > 0$ within Model II. The result follows from two properties of H(S), viz. (a) H(S) is a weighted average of two terms, with weight $(1 - \beta)$ to the larger term, and β to the smaller term, and so a higher value of β automatically means a lower value of H(S); and (b) H(S) is an increasing function of S, so that if $H(S^*)$ is to remain equal to $\alpha \bar{u}$ (the optimising condition) through the variation of β, then S^* must increase with β.

6.4 We can now see the above result in a wider perspective. Let us start back from the intuitive feeling, noted earlier, that the magnitude of stock should be an increasing function of the degree of uncertainty as stock-holding is a "hedge" against uncertainty. This feeling is now confirmed in so far as the "uncertainty" under reference is the uncertainty of supply. This stands in strong contrast to the essentially ambiguous relation between the magnitude of optimum stock and the uncertainty of demand that obtains in Model I (see Section 5, Chapter 5).

7. Let us conclude by saying that it is really in the model of this section that the whole availability problem comes into its proper focus. It is in this sense the proper model of the availability problem. The background started from is Model I, where the uncertainty of demand is already there. The whole uncertainty of supply thus becomes a superimposition upon this more fundamental uncertainty. This, we believe, is the proper ordering of issues from the standpoint of inventory theory in general. The precise significance of this " supperimposition" is now analysed at length in terms of our two models.

Chapter 8

MODEL III : TRANSACTION COST AND THE OPTIMUM PURCHASE AND PURCHASE INTERVAL

Section 1 : The Case Where Transaction Cost Does Not Enter Capital

In Chapter 1, we left the model of this chapter simply as a ''recasting'' of the basic model of inventory theory in our own framework of ''capital and profit'' and pointed out certain apparent differences in the scope and structure of the two models arising out of this ''recasting'' or transformation. Let us now freshly go over the structure of our model on its own and solve it (this section). This prepares the ground for our re-examination of the ''standard model'' which we come to in the next section. Before beginning, let us just mention that the ''model'' of this section is a special case of the general model of this chapter. Its special assumptions will be made clear in the due course.

1.1 The starting point of the model is that the Time of purchase is left open to the trader and is then immediately taken to be closed by the ''purchase rule'' as stated in Chapter 1, viz. a purchase is made exactly as the stock gets sold out. Let us now get off the ground by noting the logical implications of this rule. The immediate implication is that the stock immediately after the purchase is equal to the amount bought. Using notations established in Chapter 3, we can put these two properties — the purchase rule and its corrollary just noted in the following terms :

$$\left.\begin{array}{ll}(1) & S_t^- = 0 \\ (2) & S_t^+ = Z_t\end{array}\right\} \quad \forall \quad t \in T_P .$$

1.2 Let us now proceed on. By the ''spending rule'' of the process, Z_t is obtained by dividing the trader's preexisting ''purchase fund'' at t by the price paid per unit of the goods; and by the ''ploughback rule'' of the process, taken together with this rule, the ''purchase fund'' referred is equal to the cost of purchase of the amount sold between the last purchase and this purchase. So, in the end, Z_t is simply equal to this amount. Let us denote the Time of the last purchase in reference to t by t_1 and the amount sold between t_1 and t by $X(t_1, t)$. We can then write our equation as :

$$(3) \quad Z_t = X(t_1, t).$$

(3) implies at once that the stock immediately after the present purchase (at t) is the same (in magnitude) as the stock immediately after the last purchase (at t_1). Stated in symbols :

$$(4) \quad S_t^+ = S_{t_1}^+ .$$

This brings us back to the condition of the stock being ''maintained'' through the process in our sense of the term.

1.3 Let us now simplify the notations in the same manner of Chapter 3. Stated in a word, we can now drop the ''time variable'', t and t_1, in S_t^-, S_t^+, Z_t, $X(t_1, t)$, i.e., write them as S^-, S^+, Z and X, meaning, respectively, the stock immediately <u>before</u> and <u>after</u> a purchase, the purchase itself (amount bought) and the amount sold between two successive purchases, these being all <u>independent</u> of which particular purchase is referred. (This is what justifies the new notation.) This done, we simplify the notation further by writing S for S^+, i.e., S is now understood simply as the stock immediately after a purchase.

1.4 Let us now resume the statement of our model. We note that by its ''purchase rule'' our model satisfies the property of ''full demand satisfaction'', i.e., the 'sale' in it is equal to 'demand'.

Let us now recapitulate our treatment of demand set out in Chapter 2. First, we now abstract from the uncertainty or randomness of demand. This is the first special assumption of the model of this section. Using notations established in Section 1 of Chapter 2, we can then express the <u>demand condition</u> in our model by the following string of relations :

$$D(t_0, t_1) = D(\tau)$$

(5) $$= d.\tau ,$$

where t_0 and t_1 $(> t_0)$ are two arbitrary time points, τ is the "time" between these two time points, i.e., the length of the interval (t_0, t_1), and d is the rate of demand per unit of time. In our model, d is a 'given' from outside.

Let us now take t_0 and t_1 to be elements of T_P, so that τ is the purchase interval. The condition of "full demand satisfaction" is then written as :

(6) $X = d.\tau$

It follows from this equation that τ is a constant through Time. Obviously, this is a property deduced from within the model, not imposed from outside.

2.1 So far, we have specified only the physical relations of the model. Let us now turn to the value relations, i.e., relations between variables expressed in money. The first point to be stated is that we now have transaction cost as defined in Chapter 1 (see pp. 38-9) as an element of the model. Our second special assumption for the model of this section is then simply that transaction cost does not enter capital[1]. Hence capital, K, in this model is defined simply by (3.4), i.e.,

(7) $K = q S$,

1/ The meaning of this is already discussed in Chapter 3, Section 3; see pp. 82-3.

where q is the trader's buying price per unit of his goods. Like d, q is also a given constant in our model.

2.2 Let us now look back. Since q and d are given for our purpose, we can start back from (7) where K is expressed in terms of S and express it successively in terms of Z, X and τ using respectively (2), (3) and (6). Let us write this down.

$$\begin{aligned} K &= qS\ , \quad \text{from (7)} \\ &= qZ\ , \quad \text{from (2)} \\ &= qX\ , \quad \text{from (3)} \\ &= qd\tau\ , \quad \text{from (6).} \end{aligned}$$

This establishes a one-to-one correspondence between capital (K), stock maintained or stock immediately after a purchase (S), amount bought in a purchase (Z) and the purchase interval (τ) in the model. Hence we can interchangeably use any one of these as 'decision variable', given that the whole model is a decision model[2].

2/ Note that the amount sold over a purchase interval cannot by definition be a decision variable of the trader. Keeping this is mind we omitted it in the statement of the one-to-one correspondence between our variables given above although it is part of the same logical chain.

3.1 Let us now turn to profit. The first point to be stated is that by property of ''full demand satisfaction'', the model does not have any 'profit foregone'. Hence 'profit' in our sense of the term established in Chapter 4 is now the same as profit made , and the two terms can be used interchangeably.

Now, the profit made over a purchase interval is given by :

$$(8) \quad P = \pi X - A,$$

where A represents the transaction cost associated with an act of purchase, and π denotes the profit margin per unit of sale, i.e.,

$$\pi = p - q,$$

where p is the price received per unit of goods sold or the selling price of the trader. Obviously, profit per unit of time is given simply by P/τ . From this we define the rate of profit, r, to be

$$(9) \quad r = \frac{P/\tau}{K}.$$

3.2 Let us now again look back. By simple substitution, we can express P, τ and K in terms of Z, viz.

$$(10) \quad P = \pi Z - A, \quad \text{from (8) and (3)},$$

$$(11) \quad \tau = Z/d, \quad \text{from (6) and (3)},$$

$$(12) \quad K = q Z, \quad \text{from (7) and (2)}.$$

Substituting these values in (9), we now express r as a function of Z :

$$r = r(Z) = \frac{\pi Z - A}{(Z/d)(q Z)}$$

$$= \frac{d}{q} \cdot \frac{\pi Z - A}{Z^2} \quad (13)$$

4. The statement of the model in purely formal terms is now completed by posing the ''maximization'' of r w.r.t Z as its <u>defining problem</u>. Substantively, this represents a <u>decision problem</u> for the trader with ''maximum rate of profit'' as the decision criterion and ''amount bought'' (in a purchase) as the decision variable under conditions completely summed up in the function $r(Z)$. Obviously, the problem could have alternatively been posed by specifying any one of K, S and τ as the optimizing or decision variable.

5. Before taking up the problem, we note a simple <u>implicit restriction on the range of choice of Z</u> in the model. From (10), we have

$$P > 0$$

if and only if

$$Z > A/\pi .$$

The restriction therefore is simply :

(14) $Z > A/\pi = Z_0$, say.

The interpretation is straightforward. The transaction cost, A, represents by definition a fixed cost incurred in each purchase interval regardless of the amount bought, i.e., Z. Clearly, this much cost has to be first recouped out of the net sale proceeds, i.e., the sale proceeds minus the cost value of the goods sold, before a profit is earned. The net sale proceeds per unit of sale is nothing but the profit margin, π. Hence at least A/π amount must be bought and sold over each purchase interval before any profit is made in that purchase interval. This is precisely what is stated in (14).

The above condition can be restated equivalently in terms of the purchase interval, τ. This gives us the restriction :

(15) $\tau > A/\pi d = \tau_0$, say.

We can call τ_0 the recoupment period of transaction cost within a purchase interval and the remainder, $\tau - \tau_0$, the profit period. It is as if the transaction cost is first recouped over a ''time'', τ_0, from any purchase, and the profit of the interval as a whole is then made over the remaining ''time'', $\tau - \tau_0$.

6.1 *Let us now take up the maximization.* By differentiation of $r(Z)$ we have

$$r'(Z) = \frac{d}{q\,Z^4} \{ \pi Z^2 - 2Z(\pi Z - A) \}$$

$$= \frac{d}{q\,Z^3} (2A - \pi Z)$$

$$\gtreqless 0$$

as

$$Z \lesseqgtr \frac{2A}{\pi} .$$

Clearly, the <u>optimum value</u> of Z, i.e., its value yielding the maximum rate of profit, is given by :

$$(16) \quad Z^* = \frac{2A}{\pi} .$$

Nothing could be simpler ! The optimum purchase is shown to be simply <u>twice</u> the critical minimum purchase for positive profit. It is proportional to the transaction cost and inversely proportional to the profit margin, hence related positively to the buying price and negatively to the selling price. We will spell out these properties in economic terms as part of the comparison of our model with the basic model of inventory theory, given in the next section.

6.2 For future reference we just note down the values of the optimum purchase interval, τ^*, and the maximum rate of profit, r^*. These are given by :

$$(17) \quad \tau^* = \frac{2A}{\pi d} = 2Z_0 ,$$

$$(18) \quad r^* = \frac{\pi^2 d}{4q\ A} .$$

Obviously, (17) is an alternative statement of the optimizing condition of the model in the sense that we could solve the model with τ instead of Z as the optimizing variable and derive this equation straight by optimization. It gives us the interesting decomposition that the optimum purchase interval is divided in two equal halves one (the first) recouping the transaction and the other yielding the profit of the interval.

Section 2 : Relation with the Standard Model

1.1 At the beginning, we stated our basic problem of this chapter as reexamination of the basic model of inventory theory (also called the ''standard model'') from the standpoint of our decision criterion of rate of profit. This is what motivated our model of the last section.

We now come to the standard model and attempt a comprehensive clarification of the relation between the two models, covering under this both the comparison of the properties of the two models taken simply as 'given' as well as the methodological relation per se in the sense of ''going'' or ''passing'' from one to the other.

1.2 The basic a priori difference between the two models is simply one of their decision criterion, viz. maximum rate of profit versus minimum cost. Associated with this is the difference that 'cost' in the standard model includes an interest charge upon capital (at a given rate of interest) which is absent from our model. As explained at the very beginning, the 'force' represented by the interest charge in the standard model is already present in our model by the very nature of its decision criterion, and so this 'difference' is in some sense a difference in the representation of the same force. It is this point that makes the whole question of ''going'' or ''passing'' from one model to the other meaningful in an a priori sense. Finally, as again mentioned at the beginning, the precise 'formula' for capital is not the same in the two models. We will however initially ignore this difference, i.e., we initially consider the standard model as of our 'formula' for capital. We may call this the ''modified'' standard model.

For reasons to become clear as we go on, almost all the points we make about our model vis-a-vis the modified standard model simply carry over to points about our model vis-a-vis the standard model. Quite apart from this, there is an a priori justification for considering the modified standard model, viz. our 'formula' for capital is prior to all inventory problems and the way they are sought to be resolved, i.e., whether by maximization of rate of profit or minimization of cost (or anything else for that matter). As will be seen later, this sort of 'priority' does not obtain for the conventional formula for capital (i.e., the formula of the conventional framework).

2.1 Let us now settle down to the work. The precise decision criterion or objective function of the standard model is the minimization of cost <u>per unit of time</u>. This 'cost' in turn is defined by :

$$(19) \quad C = \frac{1}{\tau} \; [qZ + A] \; + iK$$

where i is the <u>rate of interest</u>. Note that K is here taken to be given by our 'formula' (3.4). Hence C is really the cost per unit of time of the ''modified'' standard model.

2.2 Let us now express τ and K in (19) in terms of Z as per (11) and (12). This gives us :

$$(20) \quad C = C\,(Z) = qd + \frac{d.A}{Z} + iqZ.$$

We write C (Z), for q, d, A and i are all 'given' here.

By differentiation of $C(Z)$, we have :

$$C'(Z) = -\frac{d.A}{Z^2} + iq$$

$$\lesseqgtr 0$$

as

$$Z \lesseqgtr \sqrt{\frac{d\,A}{iq}}\,.$$

Obviously, then ;

$$(21') \quad Z^{*}_{mod} = \sqrt{\frac{d\,A}{iq}} \quad ,$$

where by Z^{*}_{mod} we mean the optimum solution (optimum purchase) in the modified standard model, i.e., the value of Z that minimizes $C(Z)$.

2.3 We now point out that save for a factor of proportionality, (21) is the same as the ''square root formula'' of the inventory theory, i.e., the solution of the standard model, as we call it. It is for this reason that almost all points that we make about our model vis-a-vis the modified standard model also apply to our model vis-a-vis the standard model. In presenting these points, we will therefore talk simply of the ''standard model'' for that is the ultimate point of our interest.

3. For ready reference, let us first write down the optimum value of Z in our model, i.e., under the criterion of maximum rate of profit. This is nothing but (16) :

$$(16) \quad Z^{*} = \frac{2A}{\pi} = \frac{2A}{p - q} .$$

Obviously, Z^{*}_{mod} and Z^{*} are defined over different sets of variables ; even for the common variables appearing in both, the precise relations are not the same. Let us now sort out the differences in a logical manner.

4. First, on the one hand, profit does not appear at all in the standard model, and hence the selling price, p, which is necessary in the general structure of relations concerned only for defining profit, also does not appear in it; on the other hand, the rate of interest, i, does not appear at all in our model. Correspondingly, p but not i appears in the arguments of Z^{*} while i but not p appears in the arguments of Z^{*}_{mod}. There is nothing more to explain about this.

5. Let us now come to the common factors, viz. transaction cost, A, and the buying price, q. The common essence of both models is simply the dependence of the optimum purchase upon transaction cost. Though the quantitative dependence is not the same in both cases, the qualitative features are identical : in both cases, the optimum purchase increases

with A, going to zero as A goes to zero. In our case, the increase is proportional; in the standard model, the increase is less than proportional (proportional to the square root of A).

6.1 The dependence of the optimum purchase on the price paid per unit of the goods bought, i.e., the buying price, is however the opposite of one another in the two cases ! This is obviously a fundamental difference in the two structures. The explanation appears to be as follows.

6.2 At the very beginning, we explained the logic of the determination of optimum purchase in the standard model as a ''balancing'' of two opposite forces of the transaction cost on the one hand and the interest charge on the other[3/]. So, it is the relative magnitudes of these two 'costs' that ultimately matter in the solution. The buying price, q, comes into this simply as determinant of capital upon which the ''interest'' is charged. So, the proper 'cost measure' in the interest charge is not just the rate of interest, i, but the product, iq. This is exactly how it appears in (21). Hence the relative significance of the transaction cost in the total cost as relevant for the purpose here is given by A/(iq). The optimum purchase is then an increasing function of this 'relative measure' of transaction cost and hence a decreasing function of the buying price q.

3/ See Chapter 1, pp. 32-3.

6.3 Nothing beyond cost is encountered in the standard model. In our model, on the other hand, we go up to profit, and in doing this, we have to talk of the recoupment of cost out of sale proceeds before we can talk of profit. This is already made transparent in the fact that the optimum purchase in our case is simply a ''blow up'' of the minimum purchase necessary for recouping the ''fixed'' transaction cost of a purchase interval. Since this recoupment is simply out of the net sale proceeds, which depends only upon the profit margin per unit of sale, for the rate of sale per unit of time is already given by the rate of demand, the higher the profit margin per unit of sale, the faster the recoupment of the fixed cost, A, and hence the smaller both the purchase interval and the purchase itself, i.e., τ^* and Z^*. But a higher buying price is cet par a smaller profit margin. The optimum purchase therefore varies positively with the buying price. To repeat, the higher the buying price, the longer it takes to recoup the fixed cost, A, out of the net sale proceeds, and hence the longer the purchase interval and greater the amount of purchase.

7.1 There remains one more ''difference'' between the solutions of the two models to talk of : the rate of demand, d, appears in the a priori structure of both models in exactly the same way but then it simply drops out of the optimum purchase in our model but reappears as determinant of the optimum purchase in the standard model. Obviously, this is another fundamental difference between the two structures.

7.2 Let us first explain our result. We go back to the fundamental starting point of the model (both our model and the standard model), viz. that the trader is free to choose the Time of his purchase. It follows at once from this that the stock held by him at any point cannot define a physical constraint upon his sales, for he can simply replace the stock (if finished) and sell from it. Viewed more comprehensively in Time, we can say that the stock maintained cannot be a physical constraint upon the rate of sale, for a higher rate can always be achieved within a given stock by simply ''turning it over'' more quickly, i.e., by selling off the same quantity in a shorter time and hence at the same time buying more frequently so as to maintain the stock through time. The question arises, what prevents an arbitrary high frequency of replenishment or, what comes to the same thing, an arbitrarily small purchase interval. The answer, within the model under reference, is provided by the transaction cost. But given the rate of sale, the frequency of stock replenishment and the magnitude of stock maintained are not two independent notions but simply transforms of one another within the same model. Starting from this, what the solution of the model brings out is a neat separation of the determinants of the rate of sale on the one hand and the stock maintained on the other, the former depending upon the demand conditions and the latter upon the cost conditions, represented respectively by the parameters d, and A and π. There is no 'interaction' between the two. This is the interpretation or explanation of our result.

7.3 It is to be stated that the interpretation owes much to the analysis of V. Narasimhan in her thesis referred at the beginning. To connect up, we mention that she explicitly treated the 'turnover period of stock (= ratio of stock maintained to the rate of sale) as a ''competetive norm'' and went on to write

> ''... the whole conception is here set out under the tacit assumption of a given state of the market. If there is a change in the state itself, there may be a change in the value of θ 4/ as well. One sees this very clearly by noting that when demand is high (in the market) all traders may be able to turn over their stocks faster — θ is low — and conversely, when demand is slack, θ is relatively large''.

Our own analysis, we can say, gives a rigorous justification of the possibility mentioned here as property of optimising behaviour in our sense.

7.4 Lest there be any confusion, we point out that no inferences can be made from the discussions above about the stock of big versus small traders. True, a big trader faces a higher rate of demand than a small trader. There is however no reason to believe that the transaction cost remains the same for a big and a small trader. Hence nothing can be inferred about the magnitude of stock of big and small traders from our analysis.

7.5 We have now explained why the optimum purchase and stock in our model are independent of the rate of demand or sale. There is nothing to discuss here about why the optimum purchase and stock of the standard model

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4/ θ denoted the turnover period of stock in her thesis. Given the structure of our model it boils down to the purchase interval, τ.

depend upon the rate of sale. This is a purely formal consequence of its very approach of ''cost minimization'', implicitly, for achieving a given rate of sale.

However, we note in the passing that much importance is attached in inventory theory to the precise quantitative dependence of the optimum stock and purchase upon the rate of sale established through its square root formula. Thus, we find Arrow in his historical review of the subject to conclude his discussion of the formula with the observation :

> ''The model of transfer costs[5/] actually leads to inclusions different from those in the vaguer models sketched above. In particular, the amount of inventory held is, on the average, proportional to the square root of annual sales, rather than proportional to this''. (op. cit. p.6)

This whole question of the precise manner of dependence of the ''inventory'' upon ''annual sales'' appears to be quite beside the point when looked at from the standpoint of our model.

8.1 This completes the strict comparison of properties of the two models. Let us now pass on to the question of the connection between the two whereby we can ''go'' or ''pass'' from one to the other. As is to be expected, such a connection is in fact established by the general proposition or theorem that maximization of rate of profit and maximization of

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5/ What is referred here is simply the ''standard model''.

profit or minimization of cost come to the same thing provided the 'cost' in the latter case includes an interest charge upon capital calculated at a value of the rate of interest set equal to the maximum rate of profit. For completeness of the argument, we append a proof of this general proposition or theorem at the end of this chapter.

8.2 Let us now verify this result in the present context. Suppose we set the value of 'i' in the standard model equal to r^* defined in (18). It is then seen at once that

$$Z^*_{mod} = \sqrt{\frac{dA}{iq}}$$

$$= \sqrt{\frac{dA}{r^* q}}$$

$$= \sqrt{\frac{dA \; 4qA}{q \; \pi^2 d}}$$

$$= \sqrt{\frac{4 \; A^2}{\pi^2}}$$

$$= \frac{2A}{\pi}$$

(22) $$= Z^* \; !$$

A moment's reflection shows that the converse of the above proposition is also true. So, the complete result is :

$$(23) \quad Z^*_{mod} = Z^* \quad \text{iff} \quad i = r^*.$$

This is the connection.

Having established the connection, let us point out that there remains an essential asymmetry within it. The point is that having obtained r^* from our model, one can set up the standard model with $i = r^*$ and obtain the <u>same</u> ''solution'' for it. In this sense, we can pass from our model to the standard model. But obviously one cannot take the reverse step, for there is no finding of r^* from within the standard model.

8.3 The above is purely methodological. Let us now come down to the <u>analytical</u> plane. First, we observe that Z^*_{mod} is a <u>decreasing function</u> of the rate of interest, i. This generalies the relation stated in (23) to :

$$(24) \quad Z^*_{mod} \gtreqless Z^* \quad \text{as} \quad i \lesseqgtr r^*.$$

On a priori grounds which we need not discuss here, one <u>may</u> further argue that the rate of interest must be <u>smaller</u> than the maximum rate of profit, i.e.,

$$i < r^*.$$

Obviously, one <u>then</u> has the result

(23) $Z^{*}_{mod} > Z^{*}$

9.1 We have now completed our main programme of the section. It remains to leave behind the "modified" standard model considered so far and come to the standard model itself, i.e., to the formula for capital used in the conventional framework of inventory theory.

9.2 The whole significance of capital in the conventional framework is that it defines the <u>base</u> of its "interest charge". Let us see how the interest is taken to be charged. The source is laid at the "holding" of stock or inventory, in the sense that for any given stock or inventory, the interest charge is taken to be proportional to the "time" for which it is held. Starting from this, the interest cost for a purchase interval (or equivalently a "cycle" of operations) as a whole is obtained by integration as proportional to the 'average level' of the stock over the cycle in the sense of a weighted average, the weights being the "time" (or duration) for which different levels or magnitudes of stock encountered are held over the cycle. Assuming the stock to decline at a uniform rate from a value, S, at the "beginning" of the cycle to the value, 0, at the end, the 'average' is then equated to $\frac{1}{2}S$. Finally, this 'physical base' is expressed in <u>money</u> by multiplying it by the buying price, q. This defines the <u>capital</u> of the conventional framework.

To avoid confusion of terms (symbols), let us denote 'capital' as conceived above as K_{st}. It is then given by

(26)
$$K_{st} = q\, S_{av}$$
$$= \tfrac{1}{2} qS ,$$

where S_{av} denotes the stock held ''on the average'' over a cycle/purchase interval. This is the 'formula' of capital in the conventional framework. We note in the passing that capital as just defined is often referred as capital ''locked up'' or ''tied up''. This is not understood by us, for capital is locked up or tied up in the process not only as sales stock but also as purchase fund. The latter is simply ignored here and hence so is the whole inside view of the trade process that goes with it.

9.3 Let us now turn to the formal analysis, which can be extremely brief. Obviously, in the standard model, capital is reckoned only at <u>half</u> of its value in the modified standard model, and hence so is the interest charge upon capital. This restraining force upon stock holding is therefore very significantly released or relaxed leading to a much larger stock, purchase and purchase interval. By the formal nature of the ''optimum condition'' (21), the optimum purchase is not quite doubled, it is raised by a factor of $\sqrt{2}$. Thus we have :

$$(27) \qquad Z^*_{st} = \sqrt{2} \; Z^*_{mod} = \sqrt{\frac{2\,dA}{iq}} \;,$$

where Z^*_{st} is the optimum purchase of the standard model. This is the ''square root formula'' of inventory theory.

9.3 It follows that if one grants the condition

$$i < r^*,$$

then one has a <u>complete ordering</u> of Z^*, Z^*_{st} and Z^*_{mod}, viz.

$$(28) \qquad Z^*_{st} > Z^*_{mod} > Z^* .$$

9.4 Let us now wind up. Going back to our comparison of the properties of our model of the last section and the ''modified'' standard model, one sees that all the points made in this connection in fact stand when one compares our model with the standard model itself. The 'formula' for capital as such does not matter in this. But obviously the ''passage'' established between our model and the modified standard model is now snapped, for once capital is reckoned differently in the calculation of the interest charge (standard model) and the rate of profit (our model), the whole logic of imputing an interest cost at the maximum value of rate of profit falls to the ground. Of course, as a formal proposition the halfing/ of capital in the standard model can be exactly counter balanced by a doubling of the rate of interest, and so one can now replace (24) with the statement :

$$(29) \qquad Z^*_{st} \gtreqless Z^* \text{ as } i \lesseqgtr 2r^*.$$

But this has no significance beyond pure formalism.

Section 3 : The Case Where Transaction Cost Enters Capital

1. Where transaction cost enters capital, it doubly enters the rate of profit, so to say —— it enters once through profit, i.e., in the numerator of the rate of profit, and it enters again through capital in the denominator of this ratio. In either case, it lowers the value of rate of profit, for profit is decreased by the transaction cost and capital is increased or enhanced by it (see (3.10)). We have already studied the relation between the optimum purchase and transaction cost through the first relation, and seen the former to be an increasing function of the latter, in fact proportional to it. This simply expresses the force of avoiding the transaction cost in the model through the criterion of rate of profit. It follows from this that when second relation is superimposed upon this structure, the force of avoiding the transaction cost is strengthened, implying a higher value of the optimum purchase and purchase interval for the same value of all paramters, in other words, an upward shift of the function relating the optimum purchase to the transaction cost.

We will now show that these intuitive anticipations of the effect of transaction cost entering capital are indeed true. In the process, it will be seen that the optimum purchase still continues to be proportional to transaction cost, so that it is simply the factor of proportionality that is increased by the modification of our model under reference.

2.1 Let us now begin. The formula for capital in case transaction cost enters capital is given by (3.10). To avoid possible confusion of terms, let us denote this capital by K_{rev} (the suffix ''rev'' stands for ''revised''). We then have :

$$(30) \qquad K_{rev} = qS + A.$$

This is the only difference between the structure of our model of section 1 and the present model, i.e., we replace (7) of the previous model by (30) to define the present model. No difference from the essential logical structure of the previous model is connoted by this, for we still have a one-to-one correspondence between K, S, Z and τ and hence their interchangeability as decision variables.

2.2 Let us now go on to analyse the present model from our point of view (maximum rate of profit). Let r_{rev} stand for the rate of profit in the present model. By definition

$$r_{rev} = \frac{P/\tau}{K_{rev}}$$

$$= \frac{d.\,(\pi Z - A)}{Z.\,(qZ + A)}, \quad \text{from (10), (11), (30) and (2).}$$

$$= r_{rev}(Z),$$

where

$$r_{rev}(Z) = \frac{d.\,(\pi Z - A)}{Z.\,(qZ + A)}.$$

We will however find it convenient to write the function $r_{rev}(Z)$ as :

$$r_{rev}(Z) = \frac{d.\,(\pi - A/Z)}{qZ + A}.$$

By differentiation of $r_{rev}(Z)$, we now have :

$$r'_{rev}(Z) = \frac{d}{(qZ + A)^2}\left\{ (qZ + A)\frac{A}{Z^2} - (\pi - \frac{A}{Z})\,q \right\}$$

(31) $$= \frac{dZ^2}{(qZ + A)^2}\,(A^2 + 2qAZ - \pi qZ^2)$$

Let us now consider the quadratic equation :

$$\pi qZ^2 - 2qAZ - A^2 = 0 .$$

Its solutions are given by

$$Z = \frac{A}{\pi} (1 \pm \sqrt{p/q}) .$$

Clearly, we have two real roots of the equation, one positive and one negative, for $p > q$. Since Z must by definition be positive, it is only the positive root that is relevent for us. Let us denote it by Z^*_{rev}, i.e.,

$$(32) \qquad Z^*_{rev} = \frac{A}{\pi} (1 + \sqrt{p/q}) .$$

It is clear from (31) that

$$r'_{rev}(Z) \gtreqless 0$$

as

$$Z \lesseqgtr Z^*_{rev} .$$

Hence Z^*_{rev} indeed defines the <u>optimum purchase</u> in the present model.

3.1 (32) at once bears out the claims made at the beginning of this section. First, comparing (32) with (16) we have :

$$Z^*_{rev} - Z^* = \frac{A}{\pi} (\sqrt{p/q} - 1)$$

$$> 0, \text{ since } p > q.$$

Thus the optimum value of purchase in the present case is indeed greater than that in the previous case. (Hence so also is the optimum value of the purchase interval.) Next, the optimum purchase is again proportional to the transaction cost, A, but by a higher factor of proportionality than before.

3.2 We also note in this context that the values of the critical minimum purchase, Z_0, and the recoupment period of the transaction cost in a purchase interval, τ_0, are left unaffected by the present revision of the model. Hence the larger value of the optimum purchase interval in it is simply equivalent to a lengthening of the profit period in a purchase interval. By the same token, the proportion of the optimum purchase interval accounted for by the profit period is now turned into a variable depending on p and q.

3.3 One significant qualitative difference in the properties of the optimum solutions of the model in section 1 and this section is to be noted here. Previously, the optimum purchase increased monotonically with the buying price, q. This relation is now altered to give us the optimum purchase as <u>initially decreasing in q upto $q = p/4$ and then increasing</u>. The result can be verified by differentiating Z^*_{rev} in (32) wrt q.

4. Before ending, let us return for a minute to the standard model. The optimum purchase in it is obtained by minimising the cost per unit

of time of operating the process, inclusive of an interest charge upon capital at a given rate of interest, i. Our revised formula for capital obviously gives a different value of this base of the interest charge as compared to the values previously used. Does this change the value of the optimum purchase in the standard model ? The answer **is** no. This is simply because the only change that occurs in the cost function of the standard model by the inclusion of transaction cost in capital is the appearence of a new term, iA, which is a pure constant and hence leaves the optimum condition unaffected. Hence we can conclude by saying that whether transaction cost enters or does not enter capital makes some difference to the optimum purchase in our approach but not in the conventional approach.

Section 4 : The Case Where Demand Is Uncertain

1. Our object here is to replace the assumption of certain or deterministic demand assumed so far by that of an uncertain or probabilistic demand with a known probability distribution and see what difference this makes to our analysis of the problem of this chapter, keeping to all other assumptions introduced in it. Formally, this defines a ''probabilistic model'' for the problem of this chapter in place of the ''deterministic models'' considered so far. For simplicity we will consider only the case where transaction cost does not enter capital. But it will be obvious from the analysis given that all our conclusions in fact stand in respect of the other case also (i.e., for the case where transaction cost enters capital).

2.1 Let us now settle down to the analysis. The basic a priori difference between the structure of the deterministic and the probabilistic model is this. In the deterministic case or model, we have a one-to-one correspondence between capital (K), stock maintained (S), amount bought (Z) and purchase interval (τ) and hence can use them interchangably as decision variable in the model. In the probabilistic case on the other hand, this equivalence holds only in regard of K, S and Z. The purchase interval is now turned into a random variable, for given a purchase in amount Z, the time that elapses before the next purchase is simply the time taken to sell the constant Z (purchase rule) which in turn is the same as the time taken for Z demand to occur (sales rule) which is nothing but the random variable T (u) defined in Section 2 of Chapter 2 for u = Z, in case demand is of an uncertain or probabilistic nature. So, we cannot any longer talk of the purchase interval as decision variable. By the same token, the notion of an optimum purchase interval also drops out of the scene.

2.2 Let us now see what difference this makes for the actual analysis or working out of the model. Once we have the purchase interval as a random variable, we also have the profit per unit of time as a random variable, for this is given by dividing the profit of a purchase interval by the interval itself, i.e., the length of this interval to be precise. Obviously, this turns the rate of profit as well into a random variable and consequently the decision criterion becomes maximum expected rate of profit.

Let us now formalise these relations. Let $r(Z)$ denote the rate of profit considered as a function of the amount bought, Z. This is obtained simply by replacing τ in the definition of rate of profit (see (9)) by $T(Z)$. This gives us, on substitution from (10) and (12) :

$$(33) \qquad r(Z) = \frac{(\pi Z - A)/T(Z)}{qZ} .$$

Denoting the expected rate of profit by $\rho(Z)$ we then have by definition

$$\rho(Z) = E\,(r(Z))$$

$$= \frac{\pi Z - A}{qZ}\, E\left(\frac{1}{T(Z)}\right)$$

$$(34) \qquad = \frac{\pi Z - A}{qZ}\, \varphi(Z), \quad \text{say}$$

where

$$\varphi(Z) = E\left(\frac{1}{T(Z)}\right).$$

2.3 Let us now go back to the explicit equation for $r(Z)$ in the model of Section 1 of this chapter, given in (13). It is seen that $\rho(Z)$ is obtained from this equation simply by replacing the term, d/Z, by $\varphi(Z)$. Let us also remember that the optimum solution of this model, i.e., the value of Z maximizing $r(Z)$ in (13), is <u>independent</u> of the constant, d. It follows that <u>if</u> $\varphi(Z)$ is <u>proportional</u> to $1/Z$ then the optimum solution of the present model coincides with that of the original model or Z^*. <u>Certainty or uncertainty of demand would then have simply no effect on the optimum purchase.</u>

We now point out that the hypothesis of proportionality just mentioned is indeed true provided that the <u>probability distribution of demand has unchanged structure over time</u> in the sense of the term defined in Chapter 2. This follows from the fact that under this hypothesis the probability distribution of the random variable $T(Z)/Z$ is independent of Z (see p. 64). Hence the probability distribution of the random variable, $Z/T(Z)$, is also independent of Z. Hence, the expected value of this random variable, which is nothing but $Z\varphi(Z)$ is a constant (independent of Z), say C. It follows that

$$(35) \qquad \varphi(Z) = C/Z .$$

This proves the assertion made. We conclude that if the probability distribution of demand has unchanged structure over time, then the optimum purchase in our present probabilistic model is one and the same as the optimum purchase in the original deterministic model of Section 1 of this chapter.

Let us put this conclusion in symbols by denoting the optimum purchase in the present probabilistic model by Z^*_{prob}. The conclusion reached is then :

$$(36) \qquad Z^*_{prob} = Z^*,$$

if the probability distribution of demand has unchanged structure over time.

3. Let us now step outside the case considered. Our object is simply to see how (36) is modified when we do this, i.e., in essence, when we give up the simplification defined by (35) in respect of the substantive problem of optimum purchase. We will however not go into the full mathematical complications arising out of this. We will simply <u>assume</u> here that $\rho(Z)$ is maximized at a well defined value of Z, so that Z^*_{prob} remains well defined in mathematical sense. Obviously Z^*_{prob} is then obtained simply by setting the derivative of $\rho(Z)$ equal to zero. After simplification this equation boils down to :

$$(37) \qquad (\pi Z - A)\, Z\varphi'(Z) + A\,\varphi(Z) = 0.$$

This is the <u>optimum condition</u> of the present model and Z^*_{prob} is the solution of this equation.

Let us now note that on a priori grounds, $\varphi(Z)$ must be a decreasing function, for the larger the value of Z considered, the larger, on the average, must be the time taken for this demand (Z) to occur, and hence the shorter, on the average, the reciprocal of such ''time'', which (the last ''average'') is nothing but $\varphi(Z)$. The ''order'' of this decrease is given by the elasticity of the function, $-Z\varphi'(Z)/\varphi(Z)$. Let us denote this by ε, i.e., we write

$$\varepsilon = -\frac{Z\varphi'(Z)}{\varphi(Z)} \ (> 0) \ \underline{6/}$$

A clear connection with the case of unchanged structure of the probability distribution of demand over time is then defined by the fact that under this case, $\varepsilon = 1$ at all values of Z.

Since $\varphi'(Z) < 0$ and $Z > 0$, we can divide both sides of (37) by $Z\varphi'(Z)$. Doing this and arranging terms, we have the optimum condition stated as :

$$Z = \left(1 - \frac{\varphi(Z)}{Z\varphi'(Z)}\right)\frac{A}{\pi}.$$

Since Z^*_{prob} is nothing but the solution of this equation, we can write, using the notation, ε , just introduced, the following explicit expression for Z^*_{prob} :

6/ Obviously, ε is a function of Z. But this is not explicitly shown to keep our notations clean.

(38) $$Z^*_{prob} = \left(1 + \frac{1}{\varepsilon}\right)\frac{A}{\pi}$$ 7/

Let us now remember the expression for Z^* :

(16) $$Z^* = \frac{2A}{\pi}.$$

It follows from (38) and (16) that :

(39) $$Z^*_{prob} \gtreqless Z^* \quad \text{as} \quad \varepsilon \lesseqgtr 1.$$

Obviously this gives a complete answer, in qualitative terms, to our question, how (36) is modified when we give up the assumption that the probability distribution of demand has unchanged structure over time. An answer in quantitative terms is also obtained at once from (38), viz.

(40) $$Z^*_{prob} \approx Z^* \quad \text{if } \varepsilon \approx 1.$$

Let us conclude by just reiterating what we said at the beginning, viz. the introduction of uncertainty of demand does not bring with itself any new substantive issue into the main problem of this chapter, though the formal set up is altered (p. 38 , Chapter 1). This is self-evident from the analysis just given.

7/ In this equation, ε is evaluated at Z^*_{prob}.

Appendix to Chapter 8

Proof of the Equivalence of Maximization of Rate of Profit and Minimization of Cost Where Cost Includes an Interest Charge upon Capital Calculated at a Rate of Interest Set Equal to the Maximum Value of the Rate of Profit.

For simplicity we suppose that there is only one variable, denoted α, wrt which we talk of the maximization of rate of profit and minimization of cost. The rate of profit is defined by the function, $r(\alpha)$, where

$$(1) \qquad r(\alpha) = \frac{P(\alpha)}{K(\alpha)},$$

where $P(\alpha)$ and $K(\alpha)$ stand respectively for profit per unit of time and capital, each defined to be a function of α. $P(\alpha)$ is then given by;

$$(2) \qquad P(\alpha) = R - C(\alpha),$$

where R stands for revenue or sale-proceeds per unit of time which is treated as independent of α, and $C(\alpha)$ for cost per unit of time, defined as a function of α. $C(\alpha)$ does not include any interest charge upon capital.

Let us now define a new cost function, $C_n(\alpha)$, inclusive of an interest charge upon capital calculated at a given rate of interest, i. So, by definition

$$(3) \qquad C_n(\alpha) = C(\alpha) + iK(\alpha).$$

Let us now suppose that α^* maximizes the function $r(\alpha)$ over some open interval, say $0 < \alpha < \infty$. Assuming differentiability of $C(\alpha)$ and $K(\alpha)$ it then follows that

$$r'(\alpha^*) = 0 .$$

From (1) it is then seen that :

$$\text{(4)} \qquad \frac{P'(\alpha^*)}{K'(\alpha^*)} = \frac{P(\alpha^*)}{K(\alpha^*)} = r^*, \text{ say.}$$

Obviously, r^* is the maximum value of $r(\alpha)$. But from (2),

$$P'(\alpha) = -C'(\alpha), \text{ all } \alpha .$$

Hence, (4) can be written as :

$$\text{(5)} \qquad \frac{C'(\alpha^*)}{K(\alpha^*)} = -r^*$$

Let us now suppose that $\alpha(i)$ minimizes the function, $C_n(\alpha)$ over the same interval. It follows that

$$C_n'(\alpha(i)) = 0$$

and hence from (3) :

$$C'(\alpha(i)) + iK'(\alpha(i)) = 0.$$

Hence

$$(6) \qquad \frac{C'(\alpha(i))}{K'(\alpha(i))} = -i.$$

Comparing (5) and (6), it is now clear that :

$$(7) \qquad \alpha(i) = \alpha^{*} \quad \text{iff} \quad i = r^{*}.$$

This proves our result.

Chapter 9

MODEL IV : THE COMPOSITE MODEL

Section 1 : Genesis and Structure of the Model

The genesis of the inventory model of this chapter was set out at length in Chapter 1. Let us just go over this in purely formal terms.

1.1 Viewed in purely formal terms, the model is a straight extension of Model I. Model I was based on the twin assumption that (a) the trader made his successive purchases at a fixed interval through Time; and (b) this interval, or rather the length of this interval, τ, was given from outside the model. We now retain (a) but treat τ as explicit decision variable of the trader, to be determined from within the model. Model I as such however did not include any force to provide the substantive basis for this decision or determination. This force is now brought in from Model III and consists simply of an element of transaction cost in the trade process. This establishes the broad lineage of the model from Model III.

1.2 However, the precise lineage is not so straightforward as the lineage from Model I. This is simply because assumption (a) displaces the " purchase rule" on which the whole of Model III was based. The structure of the two models is therefore fundamentally different.

Let us pursue the point a little further. Model III itself was analysed on two distinct planes, (i) deterministic demand (Sections 1 and 3 of Chapter 8) and (ii) probabilistic demand (Section 4 of Chapter 8). In the first case, one simply comes back to (a), though not as assumption, but as deduction from within the model. Clearly however, it could have been introduced as assumption or prior restriction, without affecting the solution. So, we are free to take the deterministic version of Model III on this basis, i.e., with assumption (a) put in. The present model can then be seen as generalization of Model III in its deterministic version in this form in the sense of introducing uncertainty of demand in it, maintaining the form.

This is one way of looking at the lineage of the present model from Model III. But we ourselves took the step from certain to uncertain demand within the model. Viewed from the latter end, i.e., the probabilistic version of the model, the present model is not a generalization but restriction for it is defined simply by restricting the purchase points in Time which were left free in Model III, to be equally spaced in Time (which just restates assumption (a) above). Formally, this is just imposing an additional constraint upon the model. In sum, the present model can be looked upon either as generalization or as restriction of Model III depending upon the interpretation, or rather, the precise point of reference in Model III itself.

This completes the statement of the " lineage" of the model from Model I and Model III. Let us now set up the model on its own.

2.1 First, we make two simplifying assumptions for the model, which we take to be justified purely by the analytical complexity of the model itself. First, we assume that the transaction cost does not enter capital[1].

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[1] So, unless otherwise stated, our reference to Model III below is to be taken as reference to the model of Section 1 of Chapter 8. This is taken as understood all through.

Second, we assume that <u>the probability distribution of demand has unchanged structure over Time</u>.

2.2 As a result of the first assumption, <u>capital</u> in the model is defined by our original formula for it :

(1) $K = qS$

where K denotes the capital put in and maintained through the process, S the level at which the stock is maintained through the successive purchases, equivalently the stock immediately after purchase (henceforth simply the " stock") and q denotes the trader's buying price per unit of the goods, assumed constant through Time.

2.3 Let us now turn to the second assumption. We start from our basic random variable, $D(\tau)$, denoting the <u>demand over an interval of time of length</u> τ. Let $f(u, \tau)$ be the pdf of $D(\tau)$. Our assumption then is (see (2.9)) :

$$f(\lambda u, \lambda \tau) = \frac{1}{\lambda} f(u, \tau), \quad \text{for all } \lambda > 0.$$

As a result of this assumption, we can write $f(u, \tau)$ itself as follows :

$$f(u, \tau) = \frac{1}{\tau} f\left(\frac{u}{\tau}, 1\right)$$

(2) $$= \left(\frac{1}{\tau}\right) g(v), \text{ say},$$

where

$$v = u/\tau,$$

$$g(v) = f(v, 1).$$

$g(v)$ is simply a transformation of the original pdf, $f(u, \tau)$, into a new pdf — $f(u, \tau)$ was the pdf of $D(\tau)$, $g(v)$ is the pdf of $D(\tau)/\tau$, and what the transformation shows is simply that the latter is independent of τ. This implication of the condition of unchanged structure of probability distribution of demand over Time was already pointed out in Chapter 2 and is now simply expressed in terms of pdf.

Let us now remember that independently of this condition, the expected value of $D(\tau)$ is proportional to τ (our First Law of the Probability Distribution of Demand) defining the rate of demand, d, as this factor of proportionality. Obviously, d is now identified as the expected value of $g(v)$. Stated formally,

(2′) $d = E(D(\tau)/\tau)$

(2″) $= \int v g(v)\, dv$ 2/

3.1 We are now ready to resume our setting up of the model. We must begin with the statement of the total sale over a purchase interval, which is simply that the sale is set by the minimum of the stock at the beginning of the interval, i.e., S, and the demand over the interval, i.e., $D(\tau)$, for τ = the "time" between two successive purchases or the purchase interval in its pure " time length" sense. Denoting this sale by X, we then have :

(3) $X = \min \{ S, D(\tau) \}$.

2/ This means by definition that the integral is taken over the whole domain of the pdf g. But as per the convention set out on p. 98, the two limits of this integral are not shown explicitly.

Let us transform this into a statement of sale per unit of time, denoted x, x is defined by

$$x = X/\tau = \min \left\{ S/\tau \ , \ D(\tau)/\tau \right\}, \quad \text{by (3)}$$
$$= \min \left\{ s, \ D(\tau)/\tau \right\},$$

where

(3′) $s = S/\tau$.

Let us now derive the expression for the expected sale per unit of time, $E(x)$.

(4) $E(x) = \int^{s} v g(v) dv + s \int_{s} g(v) dv$, since $g(v)$ is the pdf of $D(\tau)/\tau$.

3.2 We now turn to profit. We have the definitions :

(5) $P = P_m - \lambda P_f$

(6) $P_m = \pi X - A$

(7) $P_f = \pi X_f$

(8) $X_f = D(\tau) - X$,

where P, P_m, P_f and X_f denote respectively the profit, profit made, profit foregone and sale foregone over a purchase interval. λ denotes the subjective parameter in profit, π denotes the profit margin per unit of sale (= $p - q$, where p is the trader's selling price per unit of the goods, assumed constant through Time) and A denotes the transaction cost per act of purchase.

After the necessary substitutions from (6) — (8) in (5) we have :

$$P = \pi X - A - \lambda \pi (D(\tau) - X)$$

$$= (1 + \lambda)\pi X - \lambda \pi D(\tau) - A.$$

From this, we find out first the <u>profit per unit of time</u> and then the <u>expected profit per unit</u> of time. These are given by :

$$P/\tau = (1 + \lambda) \pi x - \lambda \pi D(\tau)/\tau - A/\tau ,$$

$$E(P/\tau) = (1 + \lambda) \pi E(x) - \lambda \pi E(D(\tau)/\tau) - A/\tau$$

(9) $$= (1 + \lambda) \pi \left[\int^{s} vg(v)dv + s \int_{s} g(v)dv \right] - \lambda \pi d - A/\tau ,$$

by (4) and (2′).

3.3 The expected <u>rate</u> of profit, ρ, is now defined by :

$$\rho = \frac{E(P/\tau)}{K}$$

(10) $$= \frac{1}{qS} \left\{ (1 + \lambda) \pi \left[\int^{s} vg(v)dv + s \int_{s} g(v)dv \right] - \lambda \pi d - A/\tau \right\},$$

by (1) and (9).

Let us remember that s is nothing but a shorthand for S/τ . The variables in (10) are therefore just S and τ — all the rest, q, λ, π, g(v), d, A, are simply given data in the model. So, we have ρ as a function of S and τ , say $\rho(S, \tau)$. The statement of the model is now completed by the condition that S and τ be so chosen that $\rho(S, \tau)$ is at its maximum. The value of S and τ so chosen define the <u>optimum stock</u> and the <u>optimum purchase interval</u> of the model, denoted S* and τ^*.

4. We will take up the derivation of S^* and τ^* in the next section. Let us end this section once again with the background or genesis of the model and noting the obvious or natural analytical questions that arise in this context, which in fact set the main direction of our enquiry.

Though not explicitly stated so far, it is clear that the problem of optimum stock here is a straight extension of the problem of Model I and similarly the problem of optimum purchase interval is an extension of the problem of Model III in its deterministic version. It is thus of obvious interest to see how the extensions work out in terms of the solution of these two problems. Analytically, this is a matter of comparison of the respective solutions and interpretation of the differences. This sets the main direction.

Section 2 : Analysis of the Problems Posed

1. Let us begin by writing down and simplifying the expression of $\rho(S, \tau)$ for the purpose of finding the values of S^* and τ^*. By definition

$$\rho(S, \tau) = \frac{1}{qS}\left\{ (1+\lambda)\pi \left[\int^{s} vg(v)dv + s\int_{s} g(v)dv \right] - \lambda\pi\bar{d} - A/\tau \right\}, \quad s = S/\tau \text{ [3]}$$

$$= \frac{(1+\lambda)\pi}{qS}\left\{ \int^{s} vg(v)dv + s\int_{s} g(v)dv - \alpha\bar{d} - B/\tau \right\}$$

$$(11) \qquad = \frac{(1+\lambda)\pi}{q} R(S, \tau),$$

where

$$(12) \begin{cases} \alpha = \lambda/(1+\lambda), \\ B = A/\pi(1+\lambda), \\ R(S, \tau) = \frac{1}{S}\left\{ \int^{s} vg(v)dv + s\int_{s} g(v)dv - \alpha\bar{d} - B/\tau \right\}, \quad s = S/\tau . \end{cases}$$

Obviously, values of S^* and τ^* are found equivalently by maximizing $\rho(S, \tau)$ or $R(S, \tau)$. Let us write down the first order conditions of maximization of $R(S, \tau)$ [4]:

$$\frac{\partial R}{\partial S} = 0,$$

$$\frac{\partial R}{\partial \tau} = 0.$$

---

[3] For clearer notation, we freely use the variable, $s = S/\tau$ in writing our functions of S and τ, carrying the definition of s as a side relation to it.

[4] Throughout the discussion we take for granted that the second order conditions of maximization are satisfied.

These are the basic optimality conditions of the model.

Let us now find out the expressions of the two derivatives $\partial R/\partial S$ and $\partial R/\partial \tau$.

$$\frac{\partial R}{\partial S} = \frac{1}{S^2}\left\{ S\left[sg(s)\frac{1}{\tau} + \frac{1}{\tau}\int_s g(v)dv - sg(s)\frac{1}{\tau}\right] - \left[\int^s vg(v)dv + s\int_s g(v)dv - \alpha d - B/\tau\right]\right\},$$

since $\partial s/\partial S = 1/\tau$,

$$= \frac{1}{S^2}\left(\alpha d + B/\tau - \int^s vg(v)dv \right).$$

$$\frac{\partial R}{\partial \tau} = \frac{1}{S}\left[sg(s)(-S/\tau^2) + \int_s g(v)dv\,(-S/\tau^2) - sg(s)(-S/\tau^2) + B/\tau^2\right],$$ since $\partial s/\partial \tau = -S/\tau^2$,

$$= \frac{1}{S\tau^2}\left(B - S\int_s g(v)dv\right)$$

$$= \frac{1}{S\tau^2}\left(B - s\tau\int_s g(v)dv\right).$$

So, our optimality conditions are :

(13) $\int^s vg(v)dv = \alpha d + B/\tau$,

(14) $s\tau\int_s g(v)dv = B$.

Note that the optimality conditions here are expressed in terms of s and τ though the optimizing variables are S and τ. This does not create any problem simply by the definition of s (see $(3')$).

We have already denoted the optimum values of S and τ by S^* and τ^*. Let us now define : $s^* = S^*/\tau^*$. The full scheme of determination of S^* and τ^* are then obviously given by :

$$(15) \quad \int^{s^*} vg(v)dv = \alpha d + B/\tau^* ,$$

$$(16) \quad s^* \tau^* \int_{s^*} g(v)\, dv = B ,$$

$$(17) \quad S^* = s^* \tau^* .$$

Obviously, we can refer to (15) and (16) as the optimality conditions of our model.

2. Let us relegate the question of existence of a meaningful solution of (13) and (14) to the end of the chapter and settle down straight on the mainline of enquiry set at the end of the previous section, taking for granted that (13) and (14) have meaningful solution.

It is first necessary to clarify the substantive programme of work involved. Let us start from the two decision variables of the present model, S and τ. In Model I, S was the only decision variable, τ being simply taken as given. In Model III too there was only one decision variable, but this could be either S or τ (or some other transformation of these variables)[5/] by the very structure of the model — neither was taken as given. The upshot, for our present purpose, is that when we see the optimization in our model in the background of Model I, only (15), i.e., the optimality condition wrt S, comes into discussion, but when the

[5/] In our analysis of Model III, we used Z, the amount bought per purchase, as the explicit decision variable. This does not matter for the present purpose.

background is Model III, both the optimality conditions (15) and (16) come into discussion. All this however is purely procedural. Substantively, our interest focuses on the optimum stock when we see the present model in the background of Model I and on the optimum purchase interval when we see it in the background of Model III —— these are where the basic substantive forces on the respective determinations are introduced and worked out, being merely " compounded" in the present model.

3. Let us now begin. If we focus purely on the determination of S^*, then we find that the only substantive differences between Model I and the present model is that the former did not have any transaction cost in it which is now introduced. It follows that if we were to set the transaction cost, A, in the present model as equal to zero, we should simply get back the optimum stock of Model I. This is verified at once from (15), for if A = 0, then B = 0 and (15) then indeed boils down to the optimum condition of Model I, i.e. (5.16)[6]. Thus, formally, (15) simply generalizes the optimality condition of Model I by inclusion of a transaction cost.

We now note that the LHS integral of (15) is an increasing function of s^* and hence of S^*. It follows in a self explanatory notation that

(18) $S^*_{IV} > S^*_{I}$ [7].

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[6] To work this out, one has simply to transform back v and g(v) to u and f(u) and remember that $\overline{u}$ in (5.16) is the same as $d\tau$.

[7] Similar notation is adopted all through the following discussion when we explicitly compare optimum values across models.

To interpret this result, we just remember that when we did earlier bring the transaction cost into our discussion --- this was of course in Model III -- we did get the optimum stock as increasing function of the transaction cost (see (8.16)). Taking this to be a general result, it follows that the optimum stock must be larger when a transaction cost is present than when it is not. This is exactly what (18) says.

Another point may be noted in this context. One fundamental characteristic of S_I^* was that it was independent of the whole cost-price data of the model. S_{IV}^* however is dependent not only upon A but also upon (p, q). It infact depends only upon the ratio, A/π, so far as the cost-price data is concerned (see the definition of B in (12)). In other words, the transaction cost enters the optimum stock only relative to the profit margin. This is again in course with the analysis of Model III.

4.1 Let us now come to the optimization in the present model vis-a-vis Model III. There is infact something prior to the optimization that we have to first talk of. The very existence of transaction cost in an inventory model brings with itself a restriction on the values of its decision variables, which is purely in the nature of a viability condition (hence prior to optimality). This was expressed in Model III by a pair of lower bounds on S and τ denoted Z_o and τ_0 (see (8.14) and (8.15))[8].

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[8] We are implicitly passing here from Z to S, which is justified in Model III.

Let us therefore first see how these restrictions work out in the present model[9].

4.2 Let us start from (15). According to it :

$$\alpha d + B/\tau^* = \int^{s^*} vg(v)\, dv$$

$$\leq \int vg(v)dv$$

$$= d \qquad , \text{ by } (2') \text{ and } (2'').$$

Hence $B/\tau^* \leq (1 - \alpha)\, d$

or

$$\tau^* \geq \frac{B}{(1-\alpha)d}$$

$$= \frac{A}{\pi d} \quad , \text{ by } (12)$$

(19) $$= \tau_0 \quad , \text{ see } (8.15).$$

Thus we have exactly the same lower bound on τ^* as in Model III. Having said this, we must however note a *difference of interpretation* of the term, d, *in* (8.15) *and* (19). In (8.15) d stands for the ratio, $D(\tau)/\tau$, understood as a scalar; in (19) on the other hand, d stands only for the expected value of the same ratio understood as a random variable. So, what (19) really shows is that this difference in the interpretation of d, i.e., the rate of demand, does not matter so far as the viability condition on the purchase interval is concerned.

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[9] It is to be clearly stated that we limit ourselves here to purely formal deductions from (15) - (17) and interpreting the results in the light of this question without going into the a priori notion of "viability" in respect of the present model.

4.3 Let us now turn to the parallel relation in terms of S*. First, we must remember that the viability condition in Model III was directly on Z (the amount bought per purchase), not S. We speak of it as equivalently on S only because of the one-to-one correspondence, in that model, between Z and S. This correspondence is now no longer defined. In fact we no longer have Z as explicit variable in our model. It is therefore not possible to argue mechanically from the viability condition on Z in Model III to the present model. The whole argument has to be in terms of a broad conceptual connection between the notion of amount bought and the stock immediately after purchase.

Let us now turn to formal deductions. We now begin from (16). We can obviously write this equation as

$$S^* \int_{S^*} g(v)dv = B$$

or,

$$\begin{aligned} B/S^* &= \int_{S^*} g(v)dv \\ &\leq \int g(v)dv \\ &= 1 . \end{aligned}$$

Hence

$$\begin{aligned} S^* &\geq B \\ &= A/\pi\,(1 + \lambda), \quad \text{by (12)} \end{aligned}$$

$$(20) \qquad = Z_0 \,/\, (1 + \lambda) \quad \text{(see (8.14))}.$$

This is what we can now <u>interpret</u> as viability condition on S^* in our model defined through the transaction cost. The formal similarity with (8.14) is striking. As for the interpretation, we note that whereas in Model III, there was by definition no carry over of stock from one purchase interval to the next, in the present model, the prospect, or probability, of this always remains by the very fact of uncertainty of demand and a fixed purchase interval in Time. But in any purchase interval, a cost is incurred only for the amount bought, not for the stock brought over from the past. Since the stock immediately after a purchase, i.e., S, is by definition the sum of these two, this reduces the pressure on S which obtains on grounds of covering the fixed cost of a purchase interval, i.e., the transaction cost. Hence the lower bound on S defined through this is now reduced.

5.1 Let us now turn to the <u>lineage of the optimum purchase interval in the present model from that in Model III</u>, i.e., of τ^*_{IV} from τ^*_{III} (see notation introduced in footnote 7 of this Chapter).

As mentioned at the beginning, it is now necessary to work through both the optimality conditions of the model, (15) and (16). By simply substituting for B in (15) from (16), we get rid of both B and τ^* at the same stroke, giving us the equation :

$$(21) \qquad \int^{s^*} vg(v)dv = \alpha d + s^* \int_{s^*} g(v)dv.$$

5.2 Let us make a short digression at this point. In the scheme of derivations defined by (15 - (17), s^* and τ^* are taken to be jointly determined from (15) and (16) leaving S^* to be determined from (17). It is now seen that s^* is determined prior to the other variables by the equation just derived, i.e., (21). Remembering that $S^* = s^*\tau^*$, the values of S^* and τ^* can then be found directly from either (15) or (16). This is the proper scheme of determination implicit in the very structure of the optimality conditions of the model. It follows as a corollary that the question of a meaningful solution of the optimality conditions is fundamentally a question of meaningful solution of (21), i.e., a solution belonging to the domain of the pdf, g. As already stated, we take up the question at the end of the chapter.

5.3 Let us now return to our original question. First, we rewrite (21) in a more convenient form for us as follows. From (2′) and (2″), we have :

$$d = \int vg(v)dv$$

$$= \int^{s^*} vg(v)dv + \int_{s^*} vg(v)dv$$

and so

$$\int^{s^*} vg(v)dv = d - \int_{s^*} vg(v)dv.$$

Substituting this expression on the LHS of (21), we obtain :

$$d - \int_{s^*} vg(v)dv = \alpha d + s^* \int_{s^*} g(v)dv$$

or

$$\int_{s^*} vg(v)dv + s^* \int_{s^*} g(v)dv = (1 - \alpha)d. \tag{22}$$

This is the form wanted.

Let us now solve for τ^* from (16). This gives us :

$$\tau^* = \frac{B}{s^* \int_{s^*} g(v)dv}$$

$$= \frac{(1-\alpha)\,A}{\pi\, s^* \int_{s^*} g(v)dv}\,, \qquad \text{from (12) on noting that } 1-\alpha = 1/(1+\lambda),$$

$$= \frac{A}{\pi d} \cdot \frac{(1-\alpha)\,d}{s^* \int_{s^*} g(v)dv}$$

$$= \frac{A}{\pi d} \cdot \frac{\int_{s^*} vg(v)dv + s^* \int_{s^*} g(v)dv}{s^* \int_{s^*} g(v)dv}\,, \qquad \text{by (22)}$$

$$(23) \qquad = \frac{A}{\pi d} \left[1 + \frac{\int_{s^*} vg(v)dv}{s^* \int_{s^*} g(v)dv} \right].$$

Let us make a stop at this point. Earlier, we showed that the optimality condition wrt S in our model is a generalization of the optimality condition of Model I reflecting entirely the nature of the substantive generalization of the model itself. Let us now pose the parallel question in terms of τ and Model III. Having in mind the prior determination of s^* in the present model, we can take (23) as the generalized optimality condition wrt τ in our model as it is derived entirely from the original optimality condition wrt τ, i.e., (16), and the condition determining s^*, i.e., (22).

Now, the basic substantive difference between Model III and the present model is that demand is treated deterministically in the former and probabilistically in the latter. We can think of this as generalization precisely because we can think of the case of deterministic demand as degenerate case of probabilistic demand where the whole probability of demand gets concentrated at the single point of mean demand. We can also put this as the domain of the pdf, g, shrinking to the single point, d. If we now put in this restriction in (23), we find that both the integrals appearing in its RHS take on the value, d.[10] Hence they simply cancel one another giving us the result :

$$\tau^* = \frac{A}{\pi d} (1 + 1)$$

$$= \frac{2A}{\pi d} .$$

This is exactly the optimality condition wrt τ in Model III (see (8.17)). The question posed is thus answered. Let us now proceed on, leaving behind the case of certain demand. For any s* belonging to the interior of the domain of g, we obviously have the inequality :

$$s^* \int_{s^*} g(v)dv < \int_{s^*} vg(v)dv.$$

Substituting this in (23), we obtain at once the relation :

(24) $\tau^* > \frac{2A}{\pi d}$.

The conclusion reached is then simply :

(25) $\tau^*_{IV} > \tau^*_{III}$.

This runs exactly parallel to (18).

[10] This is granting that s* must in any case "belong" to the interior of the domain of the pdf g, which is infact proved at the end of the chapter.

5.4 Taking the two results, i.e., (18) and (25), together, we can say that the net effect of fusing our original models for the optimum stock and the optimum purchase interval, i.e., Model I and Model III respectively, into a single integrated model is simply to increase the value of both variables. This is, so to say, the " composite result" of the " composite model".

6. Let us now take up the question of existence of meaningful solution of our optimality conditions. We have already seen that the fundamental question is whether (21), or equivalently (22), has a solution s^*, belonging to the domain of g, say (a, b), where by definition :

$$0 \leq a < b \leq \infty .$$

Obviously, we can write (22) as :

$$G(s^*) = (1 - \alpha)d , \tag{26}$$

where

$$G(s^*) = \int_{s^*} vg(v)dv + s^* \int_{s^*} g(v)dv, \text{ for all } s^* \in (a, b). \tag{27}$$

It is assumed that g is continuous over its domain, (a, b), and hence so is $G(s^*)$ over the same domain. Now,

$$\lim_{s^* \to a} G(s^*) = \int vg(v)dv + a \int g(v)dv$$

$$= d + a$$

$$\geq d$$

$$> (1 - \alpha)\, d \text{ for all } \alpha \text{ such that } 0 < \alpha < 1. \tag{28}$$

Now since

$$s^* \int_{s^*} g(v)dv \; < \int_{s^*} vg(v)dv \quad \text{for all } s^* \in (a, b),$$

(27) gives us :

$$G(s^*) \; < \int_{s^*} vg(v)dv + \int_{s^*} vg(v)dv$$

$$= 2 \int_{s^*} vg(v)dv$$

The RHS of this equation goes to zero as $s^* \longrightarrow b$.

Hence :

(29) $$\lim_{s^* \to b} G(s^*) = 0.$$

By the continuity of $G(s^*)$ over (a, b) it follows at once from (28) and (29) that (22) has a solution, $s^* \in (a, b)$, for all values of the parameter, α, $0 < \alpha < 1$. The existence of $S^* > 0$ and $\tau^* > 0$ now follows at once from (16) and (17). It is thus shown that the optimality conditions of the model do have a meaningful solution.

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