STUDY OF MODULI OF BUNDLES

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INTRODUCTION

Hitchin [Hi2] realized the importance of studying pairs (E, ϕ) where E is a vector bundle and ϕ is a homomorphism of E into $E \otimes L$ for a fixed line bundle L. When C is a smooth projective algebraic curve this has since been studied quite extensively. One can construct a covering of C in this situation and the given data can be completely recovered by this covering map and a line bundle on the covering curve (see Beauville, Narasimhan and Ramanan [BNR]). When L is the canonical bundle this procedure gives a completely integrable system on the cotangent bundle of the moduli of vector bundles.

The aim in the first chapter, which based on a joint work with S. Ramanan [BR], is firstly to extend some of these results to the case of arbitrary principal bundles, bundles with parabolic structures, etc., but more importantly, to set forth a systematic infinitesimal study of the moduli functor. Some of the results we prove are indeed valid for higher dimensional varieties.

The computations here are made in the following framework. We start with an algebraic group G and a vector bundle V on a smooth algebraic variety. We consider deformations of pairs (P, θ) consisting a principal G-bundle P and a section θ of $ad(P) \otimes V$ satisfying the condition $\theta \wedge \theta = 0$. We calculate the infinitesimal deformation as the hypercohomology of a complex natually associated to the pair (P, θ). This result is obtained as a corollary of an identification of infinitesimal deformations of a more general object. We then address the question of smoothness of the related functor and obtain a necessary condition for smoothness.

We then take $V = K$ and define a natural 1-form on the moduli space of Higgs bundles on curves; the exterior derivative of this form defines a symplectic structure on the moduli space. If we confine ourselves to the pairs where P is stable, the moduli space can be identified with the cotangent bundle of the moduli space of G-bundles, and the symplectic structure can be identified with the Hamiltonian structure. In the case when $G = SL(n)$, Hitchin considered the global analogue of the map which maps $\mathfrak{g}$ into $\mathbb{C}^{n-1}$ given by the coefficients of the characteristic polynomial. We look at the analogue of the Kostant map of $\mathfrak{g}$ into $\mathbb{C}^{l}$ for all semisimple groups and show that the fibres are Lagrangian at the smooth points of the fibre and also that the symplectic form vanishes on any smooth variety in the fibre over 0. As in Laumon [L], this leads to the existence of very stable bundles. we then extend these results to pairs with parabolic structures.

Now we come to the description of the second chapter.

M. Green and R. Lazarsfeld developed a very interesting deformation theory for cohomology of holomorphic vector bundles. Given a family of holomorphic vector bundles E_T, on a Kähler manifold X, parametrized by T, the infinitesimal deformation of cohomology is given by a map $T_t(T) \otimes H^i(X, E_t) \longrightarrow H^{i+1}(X, E_t)$. The basic theorem is that the deformation of cohomology is guided by the deformation of the bundle itself. In other words the above map is obtained using the infinitesimal deformation of bundles, $T_t(T) \longrightarrow H^1(X, End(E_t))$, and the cup product, $H^1(X, End(E_t)) \otimes H^i(X, E_t) \longrightarrow H^{i+1}(X, E_t)$. In fact, in [GL2], the set up for deformation of cohomology of general elliptic operators is established. But of course the above mentioned theorem is only for a special class of elliptic operator namely, the class of $\overline{\partial}$ operators.

Here we consider a slight variation of the class of operators analyzed in [GL2]. A Higgs bundle is a holomorphic bundle (X, E), along with a section of $\theta \in End(E) \otimes \Omega_X^1$ satisfying some integrability condition. A Higgs bundle comes naturally with a complex

$$D^{\cdot} \ : \ D^0 = E \longrightarrow D^1 = E \otimes \Omega_X^1 \longrightarrow D^2 = E \otimes \Omega_X^2 \longrightarrow \ ....$$

The infinitesimal deformation of a family of triplets (X, E, θ) parametrized by T is given by a map of the tangent space of T into the 1-st hypercohomology of a complex obtained using the sheaf of first order operators on E. We show that the infinitesimal deformation of the hypercohomology

$\mathbb{H}^i(D.)$, in the sense of [GL2], is given by cupping with the corresponding infinitesimal deformation class of the triplet (X, E, θ). When the $\theta = 0$ over the family and X is not deformed, then this coincides with the result of [GL2] stated at the beginning.

Next we consider the deformation of cohomology of a local system. There again the corresponding result holds namely, the infinitesimal deformation of the cohomology of a local system is given by cupping with the corresponding infinitesimal deformation of the local system itself.

There is a natural correspondence between the set of irreducible local systems and the set of stable Higgs bundles with vanishing Chern classes [S2]. Also there is a natural isomorphism $\mathbb{H}^i(D.)$ with the i-th cohomology of the corresponding local system [S2]. We prove that the deformations of $\mathbb{H}^i(D.)$ and the deformations of the cohomology of local system are compatible with the identification of cohomologies. We then give some applications; one of them being a new proof of the main theorem of [A].

Finally we prove some weaker versions of the results about symplectic structure on the moduli of Higgs bundles over a curve proved in the first chapter, in the case of higher dimensional varieties. More specifically, we show that the space of infinitesimal deformations of a Higgs bundle on a smooth projective variety admits a natural 1-form which is defined analogously as in the case of curves. When we restrict ourselves to Higgs bundles satisfying $c_2(End(E)) = 0$, the exterior derivative of the above 1-form is actually nondegenerate on the infinitesimal deformation space, and hence defines a symplectic structure on the smooth locus of the moduli of Higgs bundles. This 2-form, when restricted to a cotangent bundle of a moduli of stable bundles, coincides with the natural symplectic form. Finally we prove that the coordinates of the Hitchin map Poisson-commute.

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CHAPTER ONE : HITCHIN PAIRS

§1.1 INFINITESIMAL DEFORMATIONS.

Let X be a smooth complete algebraic variety over $\mathbb{C}$. In what follows we will always assume varieties to be defined over $\mathbb{C}$. Let G be an algebraic group, and $\mathfrak{g}$ its Lie algebra.

Let $P \xrightarrow{p} X$ be a principal G-bundle over X; in other words, G acts equivariantly on P and the action satisfies the following properties: there is some open cover $\{U_i\}_{i \in I}$ of X, such that the restriction $p^{-1}U_i$ is isomorphic to $U_i \times G$ as G-spaces with identity map on U_i. The bundle associated to P for the adjoint action of G on itself, is denoted by $Ad(P)$. Note that $Ad(P)$ is a group scheme over X, and it has a natural action on all bundles associated to P. Similarly the bundle associated to P for the adjoint action of G on its Lie algebra $\mathfrak{g}$ is denoted by $ad(P)$. Note that $ad(P)$ is the Lie algebra bundle corresponding to $Ad(P)$, and it has a natural map to infinitesimal automorphisms of any bundle associated to P.

N.J. Hitchin in [Hi1] introduced the concept of a *Higgs bundle* on a Riemann surface; it is a pair of the form (E, θ), where E is a holomorphic vector bundle on Riemann surface M and $\theta \in H^0(M, End(E) \otimes K_M)$. This was generalized for higher dimension by C. Simpson. For a holomorphic vector bundle E on a complex manifold M, the algebra structure on $End(E)$ and the exterior algebra structure on $\oplus \Omega^i_M$ combines to give an algebra structure on $End(E) \otimes (\oplus \Omega^i_M)$; this algebra structure is also denoted by $\wedge$. A Higgs bundle on M is a pair (E, θ), where E is a holomorphic bundle on M, and $\theta \in H^0(M, End(E) \otimes \Omega^1_M)$ with the property that $\theta \wedge \theta = 0$.

Let V be a vector bundle on X. The following is a generalization of Higgs bundles which, following the terminology of [BR] we will call a *Hitchin pair*.

Definition (1.1.1). A *Hitchin pair* is a pair of the form (P, θ), where P is a principal G bundle on X, and $\theta \in H^0(X, ad(P) \otimes V)$ satisfying the condition $\theta \wedge \theta = 0$ as an element of $H^0(X, ad(P) \otimes \overset{2}{\wedge} V)$.

Fix two finite dimensional representations ρ and ρ' of the group G, on the vector spaces W and W_1 respectively. Also fix two vector bundles V and V_1 on the variety X. Let

$$h : V \otimes W \longrightarrow V_1 \otimes W_1$$

be a *G-equivariant symmetric* bundle map of degree d; where by W_X we mean the trivial bundle $X \times W$, similarly for $W_{1,X}$. Let P be a principal G-bundle on X; ρP and $\rho_1 P$ are the vector bundles associated to P for the representations ρ and ρ' respectively. The above map h would induce

$$\bar{h} : \rho P \otimes V \longrightarrow \rho' P \otimes V_1,$$

a symmetric bundle map of degree d.

The subspace of W_1 spanned by the G-invariant vectors is denoted by W_1^G. Let

$$\beta \in H^0(X, W_{1,X}^G \otimes V_1)$$

Note that for any principal bundle P, since β being G-invariant, it gives an element of $H^0(X, \rho_1 P \otimes V_1)$, which will also be denoted by β.

If we denote the principal G-bundle $P \times Spec\mathbb{C}[\epsilon]$ over $X[\epsilon]$ by $P[\epsilon]$, then the bundle of automorphisms of $P[\epsilon]$ which induce identity over the closed point is just $ad(P)$. Note that $ad(P)$ is a bundle of Lie algebras. For a section s of $ad(P)$ the corresponding automorphism of $P[\epsilon]$ will be denoted by $1 + s\epsilon$. Then it is obvious that $s_1 + s_2$ corresponds to the composite of the automorphisms corresponding to s_1 and s_2. If $\bar{\rho}$ is a representation of G then $ad(P)$ acts on $\bar{\rho}P$. Moreover if $v + w\epsilon$ is a section of $(\bar{\rho}P \otimes V)[\epsilon]$, then we have

$$\bar{\rho}(1 + s\epsilon)(v + w\epsilon) = v + w\epsilon + \bar{\rho}(s)(v)\epsilon$$

This justifies our notation as well.

Primarily we are interested in an infinitesimal study of deformations of the pairs (P,θ), where P is a principal G-bundle and $\theta \in H^0(X,\rho P \otimes V)$, such that

$$\bar{h}(\theta) = \beta$$

Let (P,θ), with $\theta \in H^0(X,\rho P \otimes V)$, be a pair such that

$$\bar{h}(\theta) = \beta$$

To any (parameter) scheme $T = Spec(A)$ with A an Artinian local algebra, associate the set of all isomorphic classes of pairs $(\bar{P},\bar{\theta})$, on $T \times X$ such that $\bar{h}(\bar{\theta},....,\bar{\theta}) = \beta$, and a given isomorphism of the restriction to $\mathrm{m} \times X$, where m is the closed point of $SpecA$, with (P,θ). This defines a functor on the category of Artinian local $\mathbb{C}$-algebras with values in sets. We may call this functor the *formal deformation functor* of (P,θ), and denote it by $\mathcal{F}_{P,\theta}$, or simply by $\mathcal{F}$. The space of infinitesimal deformations of $\mathcal{F}$ is defined to be $\mathcal{F}(\mathbb{C}[\epsilon])$, with $\epsilon^2 = 0$.

Define $e(\theta)$ to be the map $adP \longrightarrow \rho P \otimes V$, given by $e(\theta)(s) = -\rho(s)(\theta)$.

Lemma (1.1.2). *The following is a complex of sheaves on X*

$$C^\cdot : C^0 = ad(P) \stackrel{d_0=e(\theta)}{\longrightarrow} C^1 = \rho P \otimes V \stackrel{d_1}{\longrightarrow} \rho' P \otimes V_1 \rightarrow 0,$$

where $d_1(s) = d\bar{h}(s,\theta,....,\theta)$.

Proof. This is obvious. Let $Ad(P)$ be the *gauge bundle* defined earlier. We mentioned that $Ad(P)$ acts on all the associated bundles, hence in particular on $\rho P \otimes V$ and $\rho' P \otimes V_1$. For $g \in \Gamma(U, Ad(P))$, where $U \subset X$ is an open set,

$$\bar{h}(g(\theta),....,g(\theta)) = g(\bar{h}((\theta),....,g(\theta))) = g(\beta),$$

since h is G-equivariant. But, β being G-invariant, we have, $\bar{h}(g(\theta),....,g(\theta)) = \beta$. The infinitesimal version of this equality is: $d_0 \circ d_1 = 0$. □

Theorem (1.1.3). *The space of infinitesimal deformations of $\mathcal{F}$ is canonically isomorphic to the first hypercohomology $\mathbb{H}^1(C^\cdot)$, where $C^\cdot$ is the complex (1.1.2).*

Proof. Let $\mathcal{U} = \{U_i = SpecA_i\}_{i\in I}$ be a finite covering of X by affine open sets. Denote the restriction $ad(P)|U_i$ by M_i, M_i is an A_i-module. Also denote the restriction on U_i, $\rho P|U_i$ by R_i, and the restriction on U_i, $\rho_1 P|U_i$ by S_i; R_i and S_i are also A_i-modules. On the intersection $U_{ij} := U_i \cup U_j = SpecA_{ij}$, denote $ad(P)|U_{ij} = M_{ij}$, $\rho P|U_{ij} = R_{ij}$ and $\rho_1 P|U_{ij} = S_{ij}$, M_{ij}, R_{ij} and S_{ij} are A_{ij}-modules. Similar notations are used for higher order intersections. We will use the same notation M_i to denote the space of global sections of the sheaf M_i on U_i; similarly for R_i, S_i, etc. Consider the following Cech resolution of $C^\cdot$:

1.1.4

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^0 & \xrightarrow{d_0} & C^1 & \xrightarrow{d_1} & C^2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma M_i & \xrightarrow{d_0} & \Sigma R_i \otimes \Gamma(U_i, V) & \xrightarrow{d_1} & \Sigma S_i \otimes \Gamma(U_i, V_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma M_{ij} & \xrightarrow{d_0} & \Sigma R_{ij} \otimes \Gamma(U_{ij}, V) & \xrightarrow{d_1} & \Sigma S_{ij} \otimes \Gamma(U_{ij}, V_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \cdot & & \cdot & & \cdot \\
 & & \cdot & & \cdot & & \cdot \\
 & & \cdot & & \cdot & & \cdot
\end{array}$$

This is an acyclic resolution of $C^\cdot$ and hence $\mathbb{H}^1(C^\cdot)$ can be computed as the quotient Z/B. Here Z consists of pairs (s_{ij}, t_i), where

$$s_{ij} \in \Gamma(U_{ij}, ad(P)) = M_{ij},$$

and

$$t_i \in \Gamma(U_i, \rho P \otimes V) = R_i \otimes \Gamma(U_i, V)$$

such that

1. $s_{ij} + s_{jk} = s_{ik}$ as elements of M_{ijk}
2. $t_i - t_j = d_0(s_{ij})$ as elements of $M_{ij} \otimes \Gamma(U_{ij}, V)$
3. $d_1(t_i) = 0$,

and B is the subspace of Z consisting of pairs $(s_i - s_j, d_0(s_i))$, with $s_i \in M_i$.

Starting with an element (s_{ij}, t_i) of Z, we will construct a principal G-bundle $\bar{P}$ on $Spec(\mathbb{C}[\epsilon]) \times X$ and $\bar{\theta} \in H^0(\rho(\bar{P}) \otimes V)$, satisfying $\bar{h}\bar{\theta} = \beta$, such that there are isomorphisms $\bar{P}|X \longrightarrow P$ and $\bar{\theta}|X = \theta$. Take the bundle $(p_2)^*(P|U_i)$ on $U_i[\epsilon] := Spec(\mathbb{C}\,[\epsilon]) \times U_i$ for every i, where p_2 is the projection $Spec(\mathbb{C}\,[\epsilon]) \times U_i \longrightarrow U_i$. This bundle will be denoted by $\bar{P}_i$. Now we may identify the restrictions of $\bar{P}_i$ and $\bar{P}_j$ to $U_{ij}[\epsilon]$ by means of the isomorphism $1 + s_{ij}\epsilon$ of $\bar{P}_{ij}$. Condition (1) above ensures the compatibility of these identifications, and hence we get a principal G-bundle $\bar{P}$ on $Spec(\mathbb{C}\,[\epsilon]) \times X$.

On $U_i[\epsilon]$ we have $\theta_i + t_i\epsilon \in \Gamma(U_i[\epsilon], \rho\bar{P}_i \otimes V)$. We claim that these sections of $\rho\bar{P}_i \otimes V$ on $U_i[\epsilon]$ patch together to give a global section $\theta + t\epsilon \in H^0(Spec\mathbb{C}\,[\epsilon] \times X, \rho\bar{P})$. Indeed we have to show that over U_{ij} the following identity holds.

$$\rho(1 + s_{ij}\epsilon)(\theta_i + t_i\epsilon) = \theta_j + t_j\epsilon.$$

But this follows from the cocycle condition (2). Now

$$\bar{h}((\theta_i + t_i\epsilon), ..., (\theta_i + t_i\epsilon)) = \bar{h}(\theta_i, ..., \theta_i) + d\bar{h}(t_i, \theta_i, ..., \theta_i)\epsilon.$$

So, condition (3) implies that $(\bar{P}, \theta + t\epsilon) \in \mathcal{F}(Spec(\mathbb{C}\,[\epsilon])$.

Thus we have associated to $((s_{ij}), (t_i))$ an infinitesimal deformation of (P, θ).

Suppose $((s_{ij}),(t_i)) \in B$, that is to say $s_{ij} = s_i - s_j, t_i = d_0(s_i)$. Then the identification $\bar{P}_i \longrightarrow \bar{P}_j$ on U_{ij} is given by the automorphism $1 + (s_i - s_j)\epsilon$. Hence if we consider the automorphism of $\bar{P}_i$ given by $1 + s_i\epsilon$, the following diagram commutes.

1.1.5

$$\begin{array}{ccc} \bar{P}_{ij} & \xrightarrow{1+s_i\epsilon} & \bar{P}_{ij} \\ \Big\downarrow{\scriptstyle 1+s_{ij}\epsilon} & & \Big\downarrow{\scriptstyle id} \\ \bar{P}_{ij} & \xrightarrow{1+s_j\epsilon} & \bar{P}_{ij} \end{array}$$

This shows that patching by $1 + s_{ij}\epsilon$ and patching by identity give isomorphic bundles. In other words, if $(s_{ij}, t_i) \in B$, then the construction leads to a bundle isomorphic to $p_2^*(P)$ globally. Moreover we have the identity $(1 + s_i\epsilon)(\theta_i + t_i\epsilon) = \theta_i + t_i\epsilon + s_i(\theta)\epsilon = \theta_i$ in view of the definition of coboundary, implying that $e(\theta_i)(s_i) = t_i$.

Thus if the pair $(s_{ij}, t_i) \in B$, then the associated pair $(\bar{P}, \bar{\theta})$ is isomorphic to the trivial pair $(p_2^*P, p_2^*\theta)$. Hence we have given a map from $I\!H^1(C^\cdot)$ into the space of infinitesimal deformations.

Now we want to construct a map from the set of infinitesimal deformations of the pair (P, θ) to $I\!H^1(C^\cdot)$. Let $\bar{P}$ be a G-bundle and $\bar{\theta} \in \Gamma(\rho(\bar{P}) \otimes V)$ such that $(\bar{P}|X, \bar{\theta}|X)$ is the given pair (P, θ). Using the fact that U_i is affine, it is easy to see that the G-bundle $\bar{P}_i := \bar{P}|U_i[\epsilon]$ is the pull back of a bundle on U_i. Clearly then $\bar{P}_i$ is the pull back of P_i.

So $\bar{P}$ is obtained by gluing $P_i[\epsilon] = p_2^*P|(Spec\mathbb{C}\ [\epsilon] \times U_i)$ and $P_j[\epsilon] = p_2^*P|(Spec\mathbb{C}\ [\epsilon] \times U_j)$, over $U_{ij}[\epsilon]$ using some automorphism of $P_{ij}[\epsilon]$; this automorphism is of the form $1 + s_{ij}\epsilon$, where $s_{ij} \in \Gamma(U_{ij}, ad(P))$. Clearly these s_{ij}'s satisfy the condition $s_{ij} + s_{jk} = s_{ik}$ on U_{ijk}.

It also follows that the homomorphism $\bar{\theta}$ is given by $\theta + t_i\epsilon$, where $t_i \in \Gamma(U_i, \rho P) \otimes V)$. As $\bar{\theta}$ is a global homomorphism, the t_i should satisfy the following compatibility condition on U_{ij}.

$$(1 + s_{ij}\epsilon)(\theta + t_i\epsilon) = (\theta + t_j\epsilon)$$

This implies that $d_0(s_{ij}) = (t_i - t_j)$. Moreover, $\bar{h}(\bar{\theta}, ..., \bar{\theta}) = \beta$, which would imply that $\bar{h}(t_i, \theta, ..., \theta) = 0$. Thus $((s_{ij}), (t_i))$ belongs to Z. This gives a map from the set of infinitesimal deformations to $\mathbb{H}^1(C^{\cdot})$.

We now want to show that the above map is the inverse of the map from $\mathbb{H}^1(C^{\cdot})$ to infinitesimal deformations constructed earlier. But this is quite obvious. The family on $(\bar{P}, \bar{\theta})$ constructed earlier from $(s_{ij}, t_i) \in Z$ comes with an isomorphism of $\bar{P}_i$ with $p_2^* P|U_i$ and an isomorphism of $\bar{\theta}|U_i[\epsilon]$ with $\theta + t_i\epsilon$. Hence the cocycle constructed for this family $(\bar{P}, \bar{\theta})$ along with the above isomorphisms is indeed (s_{ij}, t_i). Hence the composition is identity. It is clear that composing the other way what is got is also identity. □

Corollary (1.1.6). The infinitesimal deformations of a Hitchin pair (P, θ) is given by the first hypercohomology of the the following complex

$$ad(P) \xrightarrow{\wedge\theta} ad(P) \otimes V \xrightarrow{\wedge\theta} ad(P) \otimes \overset{2}{\wedge} V \longrightarrow 0$$

Let $f : G \xrightarrow{f} H$ be a homomorphism of algebraic groups, P is a principal G bundle on X. Using f, G acts on H by left translations, and the bundle associated to P for this action has a natural structure of a principal H-bundle, called the *extension of structure group*; this bundle will be denoted by $f(P)$. We will assume that are given homomorphisms (still denoted by f) of G-representation space ρ and ρ' into H-representation spaces ρ_1 and ρ_1' respectively, which are compatible with f. Also assume are given section $\theta \in H^0(X, \rho P \otimes V)$ and $\theta' \in H^0(X, \rho_1 f(P) \otimes V)$ which are compatible in the obvious sense. Then we have a morphism of the functor $\mathcal{F}_{(P,\theta)}$ into the functor $\mathcal{F}_{(f(P), f(\theta))}$. The corresponding infinitesimal map is given by the natural morphism of the complex $C^{\cdot}$ given by (P, θ) into that given by $(f(P), f(\theta))$.

Remark (1.1.7). Let X be a complete curve; and (P, θ) be a Hitchin pair (definition (1.1.1)). Then we have the obvious exact sequence

$$0 \longrightarrow (\rho P \otimes V)[1] \longrightarrow C^{\cdot} \longrightarrow ad(P) \longrightarrow 0$$

Here we write, in conformity with the notation of the theory of derived categories, $(E^\cdot)[1]$ to mean that there is a shift in the indices of the complex. Also any sheaf is considered as a complex with the only nontrivial entry at the 0-th degree.

Now this exact sequence of complexes yields the cohomology exact sequence

$$0 \longrightarrow \mathbb{H}^0(C^\cdot) \longrightarrow H^0(adP) \longrightarrow H^0(\rho P \otimes V) \longrightarrow \mathbb{H}^1(C^\cdot)$$
$$\longrightarrow H^1(adP) \longrightarrow H^1(\rho P \otimes V) \longrightarrow \mathbb{H}^2(C^\cdot) \longrightarrow 0.$$

It is easy to see that the above map $H^0(adP) \longrightarrow H^0(\rho P \otimes V)$ is the map $e(\phi)$. In particular, $\mathbb{H}^0(C^\cdot)$ is isomorphic to the space of sections of adP satisfying $\rho(s)\phi = 0$. Moreover the map $\mathbb{H}^1(C^\cdot) \longrightarrow H^1(adP)$ is clearly the differential of the 'forget' map from the functor $\mathcal{F}$ into the moduli functor of principal bundles.

Remark (1.1.8). When X is a complete curve, the moduli space of semistable Hitchin pairs (P, θ) has been constructed, to our knowledge, only for a few special cases, e.g $G = GL(n)$ or $SL(n)$, $\rho = ad$ and V is a line bundle. Whenever such a moduli space exists, the infinitesimal deformation space is the Zariski tangent space at (P, θ) to such a moduli space if the stable pair admits no nontrivial automorphism. In particular, this applies to stable pairs of vector bundles. Indeed, this is also true for the 'local' or 'formal' moduli space at the points (P, θ) where the pair (P, θ) does not admit any nontrivial automorphisms.

The most important example of Hitchin pair is of course when ρ is the adjoint representation in $\mathfrak{g}$ and $V = \Omega^1_X$, *i.e.* the Higgs bundle, since stable Higgs bundles can be identified with the irreducible representations of $\pi_1(X)$ in G. If G is a semisimple group of rank l, then there are l primitive adjoint invariant polynomials which generate all invariant polynomials. These are of degrees m_i, called *the exponents* of the Lie group. Putting all these invariant polynomials together we obtain a morphism of the moduli functor $\mathcal{F}$ into the affine space $\Sigma\Gamma(X, S^{m_i}\Omega^1_X)$. Hitchin proved the properness of this map when X is a complete curve. Simpson proved the analogous statement for higher dimension. Note that when X is a complete curve, the dimension of the moduli space of pairs (assuming that it exists) is $2(dimG)(g-1)$, while the dimension of the affine space mentioned above is $\Sigma(2m_i(g-1)+1-g) = (g-1)(\Sigma(2m_i-1))$. But then it is wellknown that $\Sigma(2m_i - 1) = dimG$. Hence its dimension is $(dimG)(g-1) = 1/2$(dim of the moduli space). We will come back to this point later.

§1.2 SMOOTHNESS OF THE FUNCTOR $\mathcal{F}$

Let (P, θ) be a Hitchin pair. The formal deformation functor $\mathcal{F}$ of (P, θ) may not always be pro-representable. We now investigate when it is and if so, when the representing complete local algebra is regular.

Theorem (1.2.1). *Let P is a principal G-bundle, V a vector bundle and ρ a linear representation of G. Let θ be a section of $\rho P \otimes V$ such that the group of automorphisms of P which fix the section θ is generated by $\Gamma(AutV)$ and $\Gamma(Z)$, where Z is the bundle associated to P with $ker\rho$ as fibre. Then $\mathcal{F}$ is pro-representable. Moreover if $\mathbb{H}^2(C^\cdot) = 0$ then the representing complete local algebra is regular.*

Proof. We will use the work of Schlessinger [Sch] in proving this. It is a straightforward matter to see that the functor $\mathcal{F}$ satisfies the conditions H1 and H2. By Theorem 2.1 the vector space of infinitesimal deformations of $\mathcal{F}$ is isomorphic to $\mathbb{H}^1(C^\cdot)$. Clearly this is finite dimensional, proving that $\mathcal{F}$ satisfies H3 as well. By Remark following ([Sch], Theorem 2.11), we only have to check that any automorphism of $(\bar{P}, \bar{\theta})$ on $SpecS \times X$ can be extended to $(\tilde{P}, \tilde{\theta})$ on $SpecT \times X$ whenever S is defined by an ideal I with $I\mathrm{m} = 0$ (where m is the maximal ideal. This would follow if we could show that $\Gamma(SpecS \times X, G(P, \theta))$ is generated by $\Gamma(AutV)$ and $\Gamma(Z)$. The latter statement has been assumed for $S = \mathbb{C}$, and can in general be shown to be true for all Artinian local algebras by induction on length. Finally, assume that $\mathbb{H}^2(C^\cdot) = 0$. Then we will show that the morphism $\mathcal{F}(\mathbb{C}[\epsilon]/(\epsilon^{n+1})) \to \mathcal{F}(\mathbb{C}[\epsilon]/(\epsilon^n))$ induced by the natural surjection $p : \mathbb{C}[\epsilon]/(\epsilon^{n+1}) \to \mathbb{C}[\epsilon]/(\epsilon^n)$, is surjective. Define $\bar{G}(n)$ to be the following complex of sheaves on $X \times Spec(\mathbb{C}[\epsilon]/(\epsilon^n)$,

$$\bar{G}(n) : G(n) \longrightarrow \rho P \otimes V \longrightarrow 0$$

where $G(n)$ is the gauge group of the trivial family P of principal G-bundles on $X \times Spec(\mathbb{C}[\epsilon]/(\epsilon^n))$ and the map is given by $g \longmapsto \rho(\theta)$. (By a trivial family we mean the pull back of P from X). We have the following exact sequence of complexes of sheaves,

$$0 \longrightarrow C^{\cdot} \otimes (\epsilon^n) \longrightarrow \bar{G}(n+1) \longrightarrow \bar{G}(n) \longrightarrow 0$$

This in turn induces the long exact sequence of cohomologies

$$\mathbb{H}^1(\bar{G}(n+1)) \longrightarrow \mathbb{H}^1(\bar{G}(n)) \longrightarrow \mathbb{H}^2(C^{\cdot}) \otimes (\epsilon^n)$$

We have here used the fact that $\epsilon^n.\epsilon^n = 0$ to deduce that $\mathbb{H}^2(C^{\cdot} \otimes (\epsilon^n)) = \mathbb{H}^2(C^{\cdot}) \otimes \epsilon^n$. Now by assumption $\mathbb{H}^2(C^{\cdot}) = 0$. So, $\mathbb{H}^1(\bar{G}(n+1)) \to \mathbb{H}^1(\bar{G}(n))$ is surjective. Hence $\mathcal{F}(\mathbb{C}[\epsilon]/(\epsilon^{n+1})) \to \mathcal{F}(\mathbb{C}[\epsilon]/(\epsilon^n))$ is surjective. If $\mathcal{F}$ is represented by a complete local ring A, then this shows that any homomorphism $A \to \mathbb{C}[X]/(X^r)$ can be lifted to a homomorphism $A \to \mathbb{C}[X]/(X^{r+1})$. This completes the proof in the light of the following

Lemma (1.2.2) *Let R be a local $\mathbb{C}$-algebra. If any algebra homomorphism of R into $\mathbb{C}[X]/(X^r)$ can be lifted to $\mathbb{C}[X]/(X^{r+1})$, for all $r \geq 1$, then R is regular.*

Proof. Let $(f_1, ..., f_N)$ be a minimal set of generators for the maximal ideal. It is enough to show that if θ is a homogeneous polynomial of degree r such that $\theta(f_1, ..., f_N) \in \mathrm{m}^{r+1}$, then θ is identically zero. In fact we will show that $\theta(a_1, ..., a_N) = 0$, for every $(a_1, ..., a_N) \in \mathbb{C}^N$. Consider the algebra homomorphism $l : R \longrightarrow \mathbb{C}[X]/(X^2)$, such that $l(f_i) = a_i X$. Any lift of l to $\mathbb{C}[X]/(X^{r+1})$ takes $\theta(f_1, ..., f_N)$ to zero by assumption. On the other hand $\theta(f_1, ..., f_N) \equiv \theta(a_1 X, ..., a_N X)$ modulo X^{r+1}. But, $\theta(a_1 X, ..., a_N X) = \theta(a_1, ..., a_N) X^r$ by homogeneity and hence $\theta(a_1, ..., a_N) = 0$.

§1.3 HIGGS BUNDLES ON A CURVE.

Let X be a smooth complete curve; the canonical bundle of X is denoted by K. Let G be a reductive algebraic group over $\mathbb{C}$. Since G is reductive, the Lie algebra $\mathfrak{g}$ of G admits a nondegenerate symmetric invariant bilinear form; choose and fix such a form B on $\mathfrak{g}$. A family of Higgs G-bunles on X parametrized by T is a pair $(\bar{P},\bar{\theta})$, where $\bar{P}$ is a bundle over $\bar{X} := X \times T$ and $\bar{\theta} \in H^0(X \times T, \Omega^1_{\bar{X}/T})$. Note that given a morphism $f : T \longrightarrow S$, and a family $\mathcal{F}_S := (\bar{P},\bar{\theta})$ of Higgs bundles on S, the pullback of the family $f^*\mathcal{F}_S := (f^*\bar{P}, f^*\bar{\theta})$ is a family of Higgs bundles parametrized by T. A n-form on the *moduli functor* (also called *moduli stack*) of Higgs G-bundles on X is a datum consisting of a n-form α_T on T for any family of Higgs bundles parmetrized by T, and an equality $f^*\alpha_S \cong \alpha_T$ for base change,

$$\begin{array}{ccc} \mathcal{F}_T & \longrightarrow & f^*\mathcal{F}_S \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S, \end{array}$$

which for every composition

$$\begin{array}{ccccc} \mathcal{F}_T & \longrightarrow & f^*\mathcal{F}_S & \longrightarrow & g^*\mathcal{F}_{S_1} \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{f} & S & \xrightarrow{g} & S_1. \end{array}$$

satisfies the cocycle condition

$$\begin{array}{ccc} f^*(g^*\alpha_{S_1}) & \rightarrow & f^*\alpha_S \\ \downarrow{\cong} & & \downarrow{\rho_f} \\ (gf)^*\alpha_{S_1} & \rightarrow & \alpha_T. \end{array}$$

Now we will construct a 1-form and a 2-form on the moduli functor of Higgs bundles which henceforth will be denoted by $\mathcal{M}$. What we call $\mathcal{M}$, in [L] is denoted by $T^*Fib_{X,n}$.

We have seen that the infinitesimal deformations of a Higgs bundle (P, θ) is given by the 1-st hypercohomolgy of the following complex $C^{\cdot}$

$$C^{\cdot} : C^0 = ad(P) \xrightarrow{\wedge\theta} C^1 = ad(P) \otimes K \to 0.$$

From the complex $C^{\cdot}$ there is a natural projection to the complex

$$D. : D^0 = ad(P) \longrightarrow 0.$$

This projection induces a map at the level of hypercohomology which is denoted by F, *i.e.* $F_i : \mathbb{H}^i(C^{\cdot}) \to \mathbb{H}^i(D.) = H^i(X, ad(P))$. The tangent space to $\mathcal{M}$ at (P, θ) is given by $\mathbb{H}^1(C^{\cdot})$. Using the form B on $\mathfrak{g}$ there is an isomorphism $ad(P) \longrightarrow ad(P)^*$. Hence the two vector spaces $H^0(X, ad(P) \otimes K)$ and $H^1(X, ad(P))$ are dual of each other by Serre duality. So

$$\alpha \longrightarrow < F_1(\alpha), \theta >$$

defines a 1-form on $\mathbb{H}^1(C^{\cdot})$, and it is easy to see that this actually defines a 1-form on $\mathcal{M}$, which is denoted by Φ.

Before we describe the 2-form on $\mathcal{M}$ we will first recall the duality theorem for hypercohomology of a complex of locally free sheaves. The duality theorem asserts that the i-th hypercohomology of a complex

$$0 \to A_0 \to A_1 \to \ldots \to A_r \to 0$$

is dual to the $(r+1-i)$-th hypercohomology of the complex $\check{A}$ given by $(\check{A})_k = A^*_{r-k} \otimes K$, the differentials being transposes of the differentials in $C^{\cdot}$ tensored with identity [ref]. We see that $\check{C}^{\cdot} = C^{\cdot}$ and hence we have dualities between $\mathbb{H}^1(C^{\cdot})$ and $\mathbb{H}^1(C^{\cdot})$ on the one hand and $\mathbb{H}^0(C^{\cdot})$ and $\mathbb{H}^2(C^{\cdot})$ on the other. The first one for example can be made a little more concrete as follows. Consider the following diagram of complexes

$$\begin{array}{ccccc} adP \otimes adP & \xrightarrow{\alpha} & I & \xrightarrow{\alpha} & J \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

In the top horizontal line, I denotes the bundle

$$(adP \otimes (adP)^* \otimes K) \oplus ((adP)^* \otimes K) \otimes adP)$$

, J denotes the bundle

$$((adP)^* \otimes K) \otimes ((adP)^* \otimes K)$$

and the map α is given by $\wedge\, \theta \otimes Id + Id \otimes \wedge\, \theta$. This yields the following pairing

1.3.1

$$I\!H^1(C^\cdot) \otimes I\!H^1(C^\cdot) \to I\!H^2(C^\cdot \otimes C^\cdot) \to I\!H^2(K[1]) = H^1(K) = \mathbb{C}.$$

This pairing is clearly a 2-form on $I\!H^1(C^\cdot)$. Hence it defines a 2-form on $\mathcal{M}$, which is denoted by Ω.

A symplectic structure on a manifold X is a closed nondegeerate 2-form on X. Any 2-form on X induces a smooth bundle homomorphism $T_X \to T_X^*$. A 2-form Θ is called nondegenerate if the above homomorphism corresponding to Θ is an isomorphism.

For any smooth manifold X, the total space of the cotangent bundle T_X^* admits a natural symplectic structure. Indeed if $p : T_X^* \longrightarrow X$ is the projection then for $v \in T_{x,\omega}T_X^*$, the correpondence $v \longmapsto < dp(v), \omega >$ defines a 1-form on T_X^*, which is denoted by θ. It is easy to see that the form $\Theta := d\theta$ is a symplectic structure on T_X^*.

Next we want to prove that on $\mathcal{M}$ the 2-form Ω defines a symplectic structure. But before that we need the following result

Theorem (1.3.2). *On $\mathcal{M}$ the 2-form Ω coincides with $d\Phi$.*

Proof. We need some general facts about Higgs bundles. Let (P, θ) be any Higgs bundle. Define $\Omega^{p,q}(ad(P)) := C^\infty(X, ad(P))$, *i.e.* the space of all smooth (p,q)-forms on X with values in $ad(P)$. Let $\overline{\partial}_P$ be the dolbeault operator which defines the holomorphic structure on $ad(P)$. The complex $C^\cdot$, defined in (1.1.2), admits the following Dolbeault resolution

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0 & \xrightarrow{\theta} & C^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{0,0}(ad(P)) & \xrightarrow{\theta} & \Omega^{1,0}(ad(P)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\partial}_P} & & \downarrow{\scriptstyle \overline{\partial}_P} & & \\
0 & \longrightarrow & \Omega^{0,1}(ad(P)) & \xrightarrow{\theta} & \Omega^{1,1}(ad(P)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

This being a fine resolution, the i-th hypercohomology $\mathbb{H}^i(C.)$ can be computed as the cohomology of the following complex

$$\sum_{j=0}^{i-1} \Omega^{j,i-j-1}(ad(P)) \xrightarrow{\overline{\partial}_P+\theta} \sum_{j=0}^{i} \Omega^{j,i-j}(ad(P)) \xrightarrow{\overline{\partial}_P+\theta} \sum_{j=0}^{i+1} \Omega^{j,i-j+1}(ad(P)).$$

Let R be the ring $\mathbb{C}[\epsilon_1, \epsilon_2]/({\epsilon_1}^3, {\epsilon_1}^2\epsilon_2, \epsilon_1{\epsilon_2}^2, {\epsilon_2}^3)$. Let $(\bar{P}, \bar{\theta})$ be a family of Higgs bundles on X parametrized by $Spec\, R$, in other words $\bar{P}$ is a holomorphic vector bundle on $\bar{X} := X \times Spec\, R$ and $\bar{\theta} \in H^0(X \times Spec\, R, ad(\bar{P}) \otimes \Omega^1_{\bar{X}/Spec R})$. Let the restriction of this family to the closed point $c := Spec\,(R/\mathrm{m})$, where m is the maximal ideal, be (P, θ). Θ and Φ will give a 1-form and a 2-form on $Spec\, R$ respectively; these forms will also be denoted by Φ and Ω respectively.

The family of bundles $\bar{P}$ parametrized by $Spec\, R$ is C^∞ trivial, in other words there is an C^∞ isomorphism $f : \bar{P} \longrightarrow p^*P$, where $p : \bar{X} \longrightarrow X$ is the natural projection. Using this isomorphism f, the dolbeault operator that defines the holomorphic structure on $ad(\bar{P})$ can be expressed in the following form

$$\overline{\partial}_P + A_1\epsilon_1 + A_2\epsilon_2 + B_1\epsilon_1^2 + B_2\epsilon_2^2 + C\epsilon_1\epsilon_2,$$

where A_i, B_i and C are smooth sections of $ad(P) \otimes \Omega^{0,1}_X$.

Using f, $\tilde{\theta}$ can be expressed in the following form

$$\theta + \alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \beta_1\epsilon_1^2 + \beta_2\epsilon_2^2 + \gamma\epsilon_1\epsilon_2,$$

where $\alpha_i, \beta_i, \gamma \in \Omega^{1,0}(ad(P))$. For $\phi \in \Omega^{0,1}(ad(P))$ and $\psi \in \Omega^{1,0}(ad(P))$, define $(\phi, \psi) := \int_X B(\phi \wedge \psi)$, where the (1,1)-form $B(\phi \wedge \psi)$ is defined using the G invariant bilinear form B on $\mathfrak{g}$. Recall that if $\overline{\partial}_V + h\epsilon$ is the holomorphic structure of the adjoint bundle of a family of G bundles over $\mathbb{C}[\epsilon]/\epsilon^2$ then, h represents the element of $H^1(X, ad(V))$ which corresponds to this infinitesimal deformation. The 1-form Φ on $Spec\, R$ is

$$(A_1, \tilde{\theta})d\epsilon_1 + (A_2, \tilde{\theta})d\epsilon_2 + (B_1, \tilde{\theta})d(\epsilon_1^2) + (B_2, \tilde{\theta})d(\epsilon_2^2) + (C, \tilde{\theta})d(\epsilon_1\epsilon_2).$$

Taking exterior derivation we get

$$\begin{aligned} d\Phi = \; & (A_1, \alpha_2 + 2\beta_2\epsilon_2 + \gamma\epsilon_1)d\epsilon_2 \wedge d\epsilon_1 + (A_2, \alpha_1 + 2\beta_1\epsilon_1 + \gamma\epsilon_2)d\epsilon_1 \wedge d\epsilon_2 \\ & + (B_1, \alpha_2 + 2\beta_2\epsilon_2)2\epsilon_1 d\epsilon_2 \wedge d\epsilon_1 + (B_2, \alpha_1 + 2\beta_1\epsilon_1)2\epsilon_2 d\epsilon_1 \wedge d\epsilon_2 \\ & + (C, \alpha_2 + \gamma\epsilon_1)\epsilon_2 d\epsilon_2 \wedge d\epsilon_1 + (C, \alpha_1 + \gamma\epsilon_2)\epsilon_1 d\epsilon_1 \wedge d\epsilon_2. \end{aligned}$$

Hence $d\Phi(c) = [-(A_1, \alpha_2) + (A_2, \alpha_1)]d\epsilon_1 \wedge d\epsilon_2$. From the definition of Ω it is easy to check that this is same as $\Omega(c)$. Evaluation of the exterior derivative of a differential form at a point depends only on the restriction of the form to the second order neighborhood of the point in question. This completes the proof. □

So the 2-form Ω on $\mathcal{M}$ is exact. To prove that it defines a symplectic structure all we have show is nondegeracy of Ω. But follows from the fact that the complex $C^{\cdot}$ is dual to itself. Hence we have proved

Corollary (1.3.3). *The 2-form Ω on $\mathcal{M}$ defines a symplectic structure.*

Definition (1.3.4). A principal bundle P on X is said to be *stable* if for any holomorphic reduction P_H for a maximal parabolic subgroup H, given by $f_H : X \to P/H$ the following condition holds:

$$degree(f_H^* P(\chi)) \quad < 0$$

where χ is a dominant character with respect to a Borel subgroup contained in H, $P(\chi)$ is the associated line bundle on P/H.

A Ramanathan [R] constructed the moduli of stable principal G-bundles on a Riemann surface as quasi-projective variety. Let $\mathcal{F}_G$ be the moduli of stable G-bundles on X. The cotangent space at P, $T^*_P\mathcal{F}_G = H^0(X, ad(P) \otimes K)$. It is easy to see that the construction of the forms Φ and Ω on $\mathcal{M}$ can be repeated to on $T^*\mathcal{F}_G$. The forms on $T^*\mathcal{F}_G$, thus obtained are also denoted by Φ and Ω respectively. Also Thm.(1.3.2) implies that on $T^*\mathcal{F}_G$, $d\Phi = \Omega$. We saw earlier that the total space of a cotangent bundle admits a natural 1-form. The natural 1-form on $T^*\mathcal{F}_G$ turns out to be Φ itself. Indeed it is quite obvious once the definition of the natural 1-form on $T^*\mathcal{F}_G$ and the definition of Φ are compared. We also noted earlier that the exterior derivative of the natural 1-form on the total space of a cotangent bundle gives the canonical symplectic structure. Hence we have the following

Proposition (1.3.5). *The 2-form Ω on $T^*\mathcal{F}_G$ is the canonical symplectic form.*

§1.4 HITCHIN MAP AND VERY STABLE BUNDLES.

A homogeneous polynomial f on the Lie-algebra $\mathfrak{g}$, *i.e.* an element of $S^i\mathfrak{g}^*$, is called *G-invariant* if it is invariant under the adjoint action of G on $\mathfrak{g}$. Let $\mathcal{I}$ be the space of all G-invariant polynomials $\mathfrak{g}$. Let $f \in \mathcal{I}$ be a homogeneous G-invariant polynomial of degree d. If P is a principal G-bundle on X, and $\theta \in H^0(X, ad(P) \otimes K)$, then the evaluation of f on θ gives an element $f_\theta \in H^0(X, K^d)$. The correspondence $f \times \theta \longmapsto f_\theta$ gives a map

$$\mathcal{H}: \ \mathcal{I} \times T^*\mathcal{F}_G \longrightarrow \oplus_i H^0(X, K^i),$$

which is called the *Hitchin map.*

The Lie algebra $\mathfrak{g}$ of G has a direct sum decomposition of ideals

$$\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_s$$

where $\mathfrak{g}_c$ is the centre of $\mathfrak{g}$ and $\mathfrak{g}_s := [\mathfrak{g}, \mathfrak{g}]$, a semisimple Lie algebra. The nilpotent cone in $\mathfrak{g}_s$ will be denoted by $\mathfrak{n}$. Note that since $\mathfrak{n}$ is $IntG$-invariant it makes sense to talk about a section of $adP \otimes K$ to be in $\mathfrak{n}$. Such sections will be referred to as *nilpotent sections.*

Definition (1.4.1). A principal G-bundle P is said to be *very stable* if $H^0(X, adP \otimes K)$ does not contain any nonzero nilpotent section.

If Q is any reduction of P to a structure group H which is maximal parabolic, then the isotropy representation of H has as its determinant a dominant character. Hence if P is not stable then there is a reduction of structure group to some H such that the degree of the vector bundle associated to the isotropy representation of H is negative. But the nilpotent radical $\mathfrak{h}'$ of $\mathfrak{h}$, the Lie algebra of H is an H-module which is dual to the isotropy representation. Hence the associated bundle adQ' with $\mathfrak{h}'$ as fibre has positive degree, and consequently the Riemann-Roch theorem implies that $\Gamma(adQ' \otimes K)$ is nonzero. This means that P is not very stable. In other words, we have

Theorem (1.4.2). *Very stable bundles are stable.*

In this section we wish to prove that very stable bundles form a nonempty Zariski open set in $\mathcal{F}_G$. In the light of the following lemma, it is enough to show the existence of one very stable bundle.

Lemma (1.4.3) *The set of very stable bundles contitute a Zariski open (possibly empty) set of $\mathcal{F}$*

Proof. For a stable bundle P, let $ad(P_s)$ be the subbundle of adP defined by the Lie subalgebra $\mathfrak{g}_s$ of $\mathfrak{g}$. $H^0(X, ad(P_s)) = 0$. So considering $H^0(X, ad(P_s) \otimes K)$ as the fiber over P, we get a vector bundle on $\mathcal{F}_G$ and let $p : \mathcal{P} \to \mathcal{F}_G$ be the associated projective bundle (the space of 1-dimensional subspaces). Let $N_i \subset \mathcal{P}$ be the subset of consisting of all those sections s such that $ad(s)^i = 0$. Clearly N_i is a Zariski closed subspace of $\mathcal{P}$; and that $N_i's$ stabilise after some finite step, *i.e.*

$$N_j = N_{j+1} = Nj + 2 \ldots$$

Since the morphism p is proper, $p(N_i)$ is a Zariski closed subset of $\mathcal{F}_G$. But the the compliment of $\cup_i p(N_i)$ in $\mathcal{F}_G$ is precisely the set of very stable bundles. This completes the proof. □

Indeed let us denote by $\tilde{N}$ the subvariety of $T^*\mathcal{F}_G$ consisting of elements of the form (P, θ) where θ nilpotent. Then the set of very stable bundles in $\mathcal{F}_G$ is given by $\mathcal{F}_G$ - $p(\tilde{N})$, where $p : T^*\mathcal{F}_G \to \mathcal{F}_G$ is the projection from the cotangent bundle. Clearly, $p(\tilde{N})$ is closed in $\mathcal{F}_G$.

Theorem (1.4.4). *The symplectic form Ω on $T^*\mathcal{F}_G$ vanishes when restricted to any smooth variety contained in $\tilde{N}$.*

Proof. Take a point of $\tilde{N}$, namely a principal G-bundle and a nilpotent section θ of $ad(P) \otimes K$. Then at the generic point of X we get a nilpotent element of $\mathfrak{g}$. This gives rise to a canonically defined parabolic subalgebra of $\mathfrak{g}$ whose nilpotent radical contains it. (Indeed this parabolic algebra exists over the algebraic closure of the function field of X and, in view of its canonical description, one sees that it descends under the Galois action to the function field of X). Clearly then there exists a subbundle of $ad(P)$ such that the generic fibre is the above mentioned parabolic subalgebra. Now since the parabolic subalgebra has the corresponding parabolic group H as its normaliser in G, it follows that there is a reduction of P to an H-bundle Q and a section ψ of $ad(Q) \otimes K$ such that θ comes from ψ. Denoting by $(adQ)'$ the bundle associated to Q with the nilpotent radical $\mathfrak{h}'$ of $\mathfrak{h}$ as fibre, we see that the pair (Q, ψ), with ψ considered as a section of $(adQ)' \otimes K$ has as infinitesimal deformation space, the hypercohomology of the complex (Theorem 1.1.3)

$$D^\cdot : \ (adQ)' \xrightarrow{ad\psi} (adQ)' \otimes K \longrightarrow 0$$

Now given a deformation of (P, θ), parametrised by an integral algebraic scheme with all θ nilpotent, it is easy to see that we may find a nonempty open set U in the parameter scheme and reduce the structure group to a parabolic group H on the whole of $U \times X$, such that θ lies in $(adQ)' \otimes K$ at all points of U, where $(adQ)'$ denotes the nilpotent radical bundle as above. This means that in order to compute the restriction of the symplectic form on this family, we have only to compare $\mathbb{H}^1(D^\cdot)$ with $\mathbb{H}^1(C^\cdot)$ and compute the restriction of the form. This amounts, in view of the definition of the form Ω, to computing the bilinear form on $\mathbb{H}^1(D^\cdot)$ given by the pairing $(adQ)' \otimes (adQ)' \otimes K \to K$. But the Killing form itself vanishes on $(adQ)'$ thus proving the following

Proposition (1.4.5). *If (P, θ) is a family of pairs in $\tilde{N}$ parametrised by an integral scheme T then the pullback of the symplectic form on N to T is zero.*

Clearly the above proposition implies Theorem (1.4.4). □

Remark (1.4.6). When G is semisimple, the fibre of the Hitchin map from the moduli $\mathcal{M}$ of Higgs bundles over 0 is not reduced. For example the variety $\mathcal{F}_G$ itself is imbedded in $\mathcal{M}$ by mapping a bundle P into $(P, 0)$. Clearly $\mathcal{F}_G$ is contained in the Hitchin fibre over 0 and the differential of the Hitchin map is 0 all along its points. But over the open set of very stable points in M, this is precisely the set theoretic inverse image $h^{-1}(0)$. This shows that the subvariety $h^{-1}(0)$ contains a nonreduced component whose reduced variety is M.

Now we will deduce from Theorem (1.4.4) a general statement asserting the existence of very stable bundles for any structure group G.

Corollary (1.4.7). *The set of very stable G-bundles form a nonempty Zariski open set of $\mathcal{F}_G$.*

Proof. From Theorem (1.4.4) it follows that $dim\tilde{N} \leq 1/2 dim T^*\mathcal{F}_G = dim\mathcal{F}_G$, where $\tilde{N}$ is the family of pairs (P, θ) with P being a stable principal bundle and θ a nilpotent section of $ad(P) \otimes K$. But on $\tilde{N}$ there is a free action of the group $\mathbb{C}^* = \mathbb{C} - 0$ namely $(P, \theta) \to (P, \lambda\theta)$, for any $\lambda \in \mathbb{C}^*$, and this action commutes with the projection to $\mathcal{F}_G$. So $dim\, p(\tilde{N}) \leq dim\mathcal{F}_G - 1$. This proves the existence of very stable principal G-bundles.

Proposition (1.4.8). *The infinitesimal deformation space of a pair (P, θ) so that the deformation is into pairs (P', θ') where θ' is in the same orbit as that of θ at the generic point of X, under the adjoint action of G, is given by the first hypercohomology of the complex $(D^\cdot)$ where $D^\cdot$ is given below.*

1.4.9

$$ad(P) \xrightarrow{ad(\theta)} F \longrightarrow 0$$

Here F denotes the subbundle of $ad(P) \otimes K$ generated by the image of $ad(\theta)$.

Proof. Note that the long exact sequence of cohomologies for the inclusion of $D^\cdot$ in $C^\cdot$ givces an injection of $\mathbb{H}^1(D^\cdot)$ into $\mathbb{H}^1(C^\cdot)$. The construction of the map from $\mathbb{H}^1(C^\cdot)$ into the space of infinitesimal deformations in Theorem.(1.1.3) would immediately imply that the image of $\mathbb{H}^1(D^\cdot)$ is contained in the infinitesimal deformations of the type described in the proposition. Conversely, given an infinitesimal deformation it is easy to see that the corresponding element in $\mathbb{H}^1(C^\cdot)$ as described in Theorem 2.3 belongs to $H^1(D^\cdot)$. □

Theorem (1.4.10). *If a pair (P, θ) is a smooth point of the fibre of the Hitchin map $\mathcal{H}$, then the symplectic form Ω restricts to the zero form on the tangent space of the fibre.*

Proof. The assumption is equivalent to saying that the Hitchin map has surjective differential. We will now show that this implies that for any fixed point $x \in X$, the element of $\mathfrak{g}$ given by evaluating θ at x is a regular element, that is to say, has centraliser of dimension $l = rank(G)$. Consider triples (P, θ, α) where (P, θ) are as before and α is an isomorphism of P_x with a fixed G-set on which G acts simply transitively. Then these triples form a G-principal bundle over $\mathcal{M}$. In any case, the map $(P, \theta, \alpha) \to h(P, \theta)$ has surjective differential. Since evaluation maps $\Gamma(K^i) \to (K^i)_x$ are all surjective for $i \geq 1$, it follows that the composite of the differential of the Hitchin map with evaluation at x is also surjective. Now the Hitchin map followed by composing the evaluation map at x is obtained by evaluating θ at x

and identifying adP_x with $\mathfrak{g}$ using α, and then applying the Kostant map $\mathfrak{g} \to \mathbb{C}^{\,l}$ given by invariant polynomials. Infinitesimalising this commutative diagram, we conclude that the Kostant map has surjective differential at the point $(\theta)_x$. It has in fact been shown in ([ref], Theorem 0.1) that this is equivalent to θ being regular.

Finally, the infinitesimal deformations of the pair (P, θ) into nilpotent pairs are given by elements of the complex $D^\cdot$ by (1.4.8) above. But in this case the map $ad(P) \xrightarrow{ad(\theta)} F$ is surjective since the map $ad(\theta)$ is of constant rank at all points of X. Thus $D^\cdot$ is quasi-isomorphic to the single member subcomplex Z_θ where Z_θ is the centraliser of θ. Now it is obvious that the symplectic form restricts to the zero form even at the complex level since there is no nonzero term at degree 1 in the complex. □

Remark. Since the fibre of the Hitchin map over 0 consists of a pure m-dimensional variety (m = dimension of $\mathcal{F}_G$) as we have seen in Proposition (1.4.5), the generic fibre has also dimension m. It is on the other hand certainly reduced, and hence has smooth points. Thus the assumption in Thmeorem (1.4.10) is satisfied on a nonempty Zariski open set. In particular, the Hitchin map gives an algebraically completely integrable system.

§1.5 PARABOLIC BUNDLES ON A CURVE.

Let G be a *connected reductive algebraic group* over $\mathbb{C}$, and $\mathfrak{g}$ its Lie algebra. Let Z be the center of G, and X a compact Riemann surface. $\mathbb{C}$.

$$I := \quad \{p_1, ..., p_n\} \subset X$$

is a finite set whose elements will be referred as *parabolic points*. For a holomorphic principal G-bunlde P on X, the associated bundle for the adjoint representation of G in $\mathfrak{g}$ will be denoted by $ad(P)$. Define K' to be the set of all those elements of $\mathfrak{g}$ which are contained in the Lie algebra of *some* maximal compact subgroup of G. The eigenvalues of the adjoint action on $\mathfrak{g}$ of an element $\gamma \in K'$ are imaginary. Hence γ will give a flag structure on $\mathfrak{g}$, indexed by real numbers, in the following way: for $r \in \mathcal{R}$, take the subspace γ^r of $\mathfrak{g}$ generated by the eigenspaces of $ad(\gamma)$ corresponding to all the eigenvalues λ satisfying $\frac{\lambda}{2\pi i} \geq r$. Let $P_\gamma \subset \mathfrak{g}$ be the parabolic subalgebra associated with γ; it consists of all those elements of $\mathfrak{g}$ whose adjoint action preserves the flag structure; equivalently the subalgebra spanned by the eigenspaces corresponding to all those eigenvalues of $ad(\gamma)$ which are of the type $2\pi ir$ where $r \geq 0$. $\gamma' \in \mathfrak{g}$ is said to be equivalent to γ if $\gamma - \gamma'$ belongs to the nilpotent part of P_γ. This relation is in fact an equivalence relation; the set of equivalence classes is denoted by $\bar{K}$. Note that both K' and the equivalence relation on it being G invariant, $\bar{K}$ makes sense on $ad(P)$. The equivalence classes in $ad(P)_x$ for $x \in X$ will be denoted by $\bar{K}_{P,x}$.

Definition (1.5.1). A *parabolic structure* on a principal G-bundle $P \longrightarrow X$ at a parabolic point x is a choice of an element $l_x \in \bar{K}_{P,x}$. A parabolic bundle is a bundle along with a parabolic structure at each parabolic point.

Definition (1.5.2). A parabolic G- bundle P is said to be *stable* if for a maximal parabolic subgroup H and any holomorphic reduction P_H, given by $f_H : X \to P/H$ the following condition holds:

$$degree(f_H^* P(\chi)) + \sum_{x \in I} \bar{\chi}(\bar{l}_x) \quad < 0,$$

where χ is a dominant character with respect to a Borel subgroup contained in H, $P(\chi)$ is the associated line bundle on P/H, $\bar{\chi} : \mathcal{H} \longrightarrow \mathbb{C}$ is the Lie algebra homomorphism induced by χ and $\bar{l}_x \in ad(P_H)$ is a conjugate of l_x.

U. Bhosle and A. Ramanathan in [BhR] constructed the moduli of stable paraboilic bundles of a fixed topological type and with fixed parabolic structures. The moduli space is a smooth quasi-projective variety of $\mathbb{C}$ of dimension

$$dim\, Z \ + \ (g-1) dim\, G \ + \ \sum_{x \in I} dim\, G/P_{l_x}.$$

For a parabolic G-bundle P, define $ad^1(P)$ to be the subsheaf of $ad(P)$ whose adjoint action on $ad(P)$ preserves the flag structure at the parabolic points. In other words the image of the bundle homomorphism

$$ad^1(P) \longrightarrow ad(P)$$

at the parabolic points is precisely the parabolic subalgebra corresponding to the parabolic structure. Also define $ad^0(P)$ to be the subsheaf of $ad^1(P)$ whose adjiont action is nilpotent with respect to the flag, in other words the image of the bundle homomorphism

$$ad^0(P) \longrightarrow ad(P)$$

at the parabolic points is precisely the nilpotent part of the parabolic subalgebra corresponding to the parabolic structure.

Let $\mathcal{M}_{\mathcal{P}}$ be a moduli of stable paraboilic G-bundles.

Lemma (1.5.3). *For $P \in \mathcal{M}_{\mathcal{P}}$, the tangent space $T_P\mathcal{M}_{\mathcal{P}}$ is naturally isomorphic to $H^1(X, ad^1(P))$.*

Proof. The proof is similiar in spirit to (1.1.3). As in (1.1.3), let $\mathcal{U} = \{U_i = SpecA_i\}_{i \in I}$ be a finite affine covering of X, such that the parabolic point $p_j \in U_j$. Given any 1-cocycle s_{ij} for the Cech resolution of $ad^1(P)$ as in (1.1.4), if we repeat the construction of an infinitesimal deformation of principal bundle P as in (1.1.3), we actually get a infinitesimal deformation of parabolic bundle P. This is because the adjoint action of any element in P_γ for a $\gamma \in \bar{K}$ preserves the set of elements of $\mathfrak{g}$ constituting the class γ.

On the other hand given a infinitesimal deformation of the parabolic bundle P, if we repeat the construction of 1-cocycle for the Cech resolution, then since the parabolic structure on any parabolic point is fixed, the cocycle must come from $ad^1(P)$. Indeed the Lie-algebra of P_γ for a $\gamma \in \bar{K}$ is precisely the subalgebra of $\mathfrak{g}$ which preserves the set of elements of $\mathfrak{g}$ constituting the class γ. □

Recall that there is a nodegenerate invariant form B on $\mathfrak{g}$. Now is easy to see that for any $\gamma \in \bar{K}$, the annihilator of the Lie-algebra of P_γ is the nilpotent part of P_γ. For any parabolic bundle P, let $\mathfrak{p}_{I_x}$ be the parabolic subalgebra of $ad(P)_x$ given by the parabolic structure l_x at a parabolic point $x \in I$. There is the following natural exact sequence of sheaves on X

$$0 \longrightarrow ad(P) \otimes \mathcal{O}(-I) \longrightarrow ad^1(P) \longrightarrow \oplus_{x \in I} \mathfrak{p}_{l_x} \longrightarrow 0$$

So taking the dual of the above sequence we get

$$0 \longrightarrow ad^1(P)^* \longrightarrow ad(P) \otimes \mathcal{O}(I) \longrightarrow \oplus_{x \in I}\, ad(P)_x/\mathfrak{p}_{l_x} \longrightarrow 0$$

Now Serre duality implies that the dual of $H^1(X, ad^1(P))$ is $H^0(X, ad^0(P) \otimes K \otimes \mathcal{O}(I))$. Hence we have

Lemma (1.5.4). *For $P \in \mathcal{M}_{\mathcal{P}}$, the cotangent space $T^*_P\mathcal{M}_{\mathcal{P}}$ is naturally isomorphic to $H^0(X, ad^0(P) \otimes K \otimes \mathcal{O}(I))$.*

Another way to prove the above lemma is as follows. The form B on $\mathfrak{g}$ induces a nondegenerate pairing $ad(P) \otimes ad(P) \to \mathbb{C}$. Now, since for any $\gamma \in \bar{K}$, the annihilator of $\mathfrak{p}_\gamma$ is the nilpotent part of it $\mathfrak{p}_\gamma$, clearly we have a nondegenerate pairing

$$ad^1(P) \otimes ad^0(P) \otimes \mathcal{O}(I) \longrightarrow \mathbb{C}$$

Hence the two vector spaces $H^0(X, ad^0(P) \otimes K \otimes \mathcal{O}(I))$ and $H^1(X, ad^1(P))$ are duals of each other by Serre duality.

Now we will consider deformations of pairs of the form (P, θ), where P is a parabolic G-bundle of with a given parabolic structure, and

$$\theta \in H^0(X, ad^0(P) \otimes K \otimes \mathcal{O}(I))$$

First note that for such a pair (P, θ), the adjoint action $ad(\theta)$ on $ad(P)$ maps $ad^1(P)$ into $ad^0(P) \otimes K \otimes \mathcal{O}(I)$. Consider the following complex $C^{\cdot}$ of sheaves on X

(1.5.5)

$$C^{\cdot} : C^0 = ad^1(P) \xrightarrow{\wedge\theta} C^1 = ad(P) \otimes K \to 0$$

Lemma (1.5.6). *The space of infinitesimal deformations of the pair* (P, θ) *is canonically parametrized by the 1-st hypercohomology* $\mathbb{H}^1(C^\cdot)$.

Proof. Again the proof of Theorem (1.1.3) can be adapted for this situation, and we omit the details. □

Now assume that $P \in \mathcal{M}_{\mathcal{P}}$, and $(P, \theta) \in T^*_P\mathcal{M}_{\mathcal{P}}$. From the complex $C^\cdot$ defined in (1.5.5), there is a natural projection to the complex

$$D^\cdot : \ D^0 = ad^1(P) \longrightarrow 0.$$

The induced map of of first hypercohomologies is denoted by F. The tangent space to $T^*\mathcal{M}_{\mathcal{P}}$ at (P, θ) is given by $\mathbb{H}^1(C^\cdot)$. We saw that

$$H^0(X, ad^0(P) \otimes K \otimes \mathcal{O}(I)) = H^1(X, ad^1(P))^*,$$

and hence

$$\alpha \longrightarrow < F(\alpha), \theta >$$

defines a 1-form on $\mathbb{H}^1(C^\cdot)$. It is easy to see that this actually defines a 1-form on $T^*\mathcal{M}_{\mathcal{P}}$; this form is denoted by Φ. The total space $T^*\mathcal{M}_{\mathcal{P}}$, being a total space of a cotangent bundle has a natural 1-form, comparing definotions, it is easy to see that the form Φ is indeed the canonical 1-form.

Now we will define a 2-form on $T^*\mathcal{M}_{\mathcal{P}}$. In §1.3 we defined the dual of a complex. Clearly, the complex $C^\cdot$ defined in (1.5.5), is dual of itself. Hence there is a natural map from the complex $C^\cdot \otimes C^\cdot$ into the complex

$$0 \longrightarrow K$$

So, as in (1.3.1), we have the following maps of cohomologies

$$\mathbb{H}^1(C^\cdot) \otimes \mathbb{H}^1(C^\cdot) \to \mathbb{H}^2(C^\cdot \otimes C^\cdot) \to \mathbb{H}^2(K[1]) = H^1(K) = \mathbb{C}$$

This defines a 2-form on $\mathbb{H}^1(C^\cdot)$. Now $\mathbb{H}^1(C^\cdot)$ being the tangent space of $T^*\mathcal{M}_{\mathcal{P}}$, the above pairing defines a 2-form on $T^*\mathcal{M}_{\mathcal{P}}$. This 2-form is denoted by Ω. Now we are in a position to state the parabolic analogue of Thm.(1.3.2).

Theorem (1.5.7). *On the total space $T^*\mathcal{M}_{\mathcal{P}}$ the 2-form Ω coincides with $d\Phi$.*

Proof. There is the following Dolbeault resolution of the complex $C^\cdot$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0 & \xrightarrow{\theta} & C^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{0,0}(ad^1(P)) & \xrightarrow{\theta} & \Omega^{1,0}(ad^0(P)\otimes\mathcal{O}(I)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial_P} & & \downarrow{\scriptstyle \bar\partial_P} & & \\
0 & \longrightarrow & \Omega^{0,1}(ad^1(P)) & \xrightarrow{\theta} & \Omega^{1,1}(ad^0(P)\otimes\mathcal{O}(I)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

This is a fine resolution, and hence the i-th hypercohomology $\mathbb{H}^i(C.)$ can be computed as the cohomology of the diagonal complex.

Let R be the ring $\mathbb{C}[\epsilon_1,\epsilon_2]/(\epsilon_1{}^3,\epsilon_1{}^2\epsilon_2,\epsilon_1\epsilon_2{}^2,\epsilon_2{}^3)$. Let $\bar{P}$ be a family of parabolic bundles parametrized by $Spec\,R$. Let $\bar\theta \in H^0(X\times Spec\,R, ad(\bar{P})\otimes \Omega^1_{\bar{X}/Spec R}\otimes\mathcal{O}(\bar{I}))$, $\bar{I}$ is the divisor $I\times Spec\,R$. Let the restriction of this pair $(\bar{P},\bar\theta)$ to the closed point be (P,θ). Now the proof of Theorem (1.3.2) can be adapted to this situation and we omit the details. □

Recall that the exterior derivative of the canonical 1-form on the total space of a cotangent bundle is the canonical symplectic form. Hence we have the following corollary of the above Theorem.

Corollary (1.5.8). *The 2-form Ω on $T^*\mathcal{M}_{\mathcal{P}}$ is the canonical symplectic form.*

Now we will define a parabolic analogue Hitchin map. Note that all G invariant polynomials on $\mathfrak{g}$ vanish on $\mathfrak{n}$, the nilpotent cone of the semisimple part of $\mathfrak{g}$. Hence if f is a homogeneous invariant polynomial of degree d, and $\theta \in H^0(X, ad^0(P) \otimes K \otimes \mathcal{O}(I))$, then the evaluation f_θ of f on θ is a section of $K^d\mathcal{O}((d-1)I)$. Recall that the space of invariant polynomials on $\mathfrak{g}$ were denoted by $\mathcal{I}$. Thus we have the follwing map

$$\mathcal{H}_{\mathcal{P}} : \ \mathcal{I} \times T^*\mathcal{M}_{\mathcal{P}} \longrightarrow \bigoplus_{i \geq 0} H^0(X, K^i \otimes \mathcal{O}((i-1)I))$$

This map will be called the *Hitchin map*.

We will recall the definition of Poisson bracket. Let Y be a symplectic manifold with a Ω being the symplectic form. For a smooth function f on Y, let $\bar{df}$ be the smooth vector field on Y defined by the following property: for any tangent vector $v \in T_yY$,

$$\Omega(v, \bar{df}) = v(f)$$

The Poisson bracket of two smooth functions f_1 and f_2 are defined as follows

$$\{f_1 , f_2\} := df_1(\bar{df}_2)$$

f_1 and f_2 are said to Poisson-commute if $\{f_1, f_2\} = 0$.

Now we will prove that the Hitchin map $\mathcal{H}_{\mathcal{P}}$ Poisson-commutes, *i.e.* if f_1 and f_2 are two linear functional on $\oplus_i H^0(X, K \otimes \mathcal{O}((i-1)I))$, then $f_1 \circ \mathcal{H}_{\mathcal{P}}$ and $f_2 \circ \mathcal{H}_{\mathcal{P}}$ Poisson-commute.

Theorem (1.5.9). *The function $\mathcal{H}_{\mathcal{P}}$ Poisson-commutes with respect to the canonical symplectic structure on $T^*\mathcal{M}_{\mathcal{P}}$.*

Proof. Let $P \in \mathcal{M}_{\mathcal{P}}$, and $\theta \in T^*_P\mathcal{M}_{\mathcal{P}}$. Note that to check Poisson-commutativity of $\mathcal{H}_{\mathcal{P}}$ at θ, only involves 1-st order neighborhood of θ in $T^*_P\mathcal{M}_{\mathcal{P}}$.

Let $(\alpha_i, \beta_i) \in \Omega^{0,1}(ad^1(P)) \oplus \Omega^{1,0}(ad^0(P) \otimes \mathcal{O}(I))$, $i = \{1, ...d\}$ be representatives of a basis of $\mathbb{H}^1(C.)$ in the Dolbeault resolution of $C.$ given in (1.5.7). After a rearrangement of indices, let $\{\alpha_i\}_{i=1}^{k}$ be a maximal linearly independent subset of the set of sections $\{\alpha_i\}_{i=1}^{d}$, and similarly $\{\beta_i\}_{i=1}^{l}$ is a maximal linearly independent subset of $\{\beta_i\}_{i=1}^{d}$.

For $\phi \in \Omega^{0,1}(ad^1(P))$ and $\psi \in \Omega^{1,0}(ad^0(P) \otimes \mathcal{O}(I))$, define

$$(\phi, \psi) := \int_X B(\phi \wedge \psi)$$

, where the (1,1)-form $B(\phi \wedge \psi)$ on X is defined using the nondegenerate form B on $\mathfrak{g}$.

It is easy to see that for any nonzero $s \in \Omega^{0,1}(ad^1(P))$, there is a $t \in \Omega^{1,0}(ad^0(P) \otimes \mathcal{O}(I))$ such that $(s,t) \neq 0$. Conversely, for any $0 \neq t \in \Omega^{1,0}(ad^1(P))$, there is a $s \in \Omega^{0,1}(ad^0(P) \otimes \mathcal{O}(I))$ such that $(s,t) \neq 0$. Using this it can be proved that the collection $\left\{\{\alpha\}_{i=1}^{k}, \{\beta\}_{i=1}^{l}\right\}$ can be increased to a collection $\left\{\{\alpha\}_{i=1}^{m}, \{\beta\}_{i=1}^{m}\right\}$, such that the $m \times m$ matrix $\left((\alpha_i, \beta_j)\right)_{i,j}$ is nonsingular. First extend the $k \times l$ matrix $\left((\alpha_i, \beta_j)\right)_{i,j}$ to some invertible $m \times m$ matrix A. Then the equation $\left((\alpha_i, \beta_j)\right)_{i,j} = A$ can be solved (in fact space of solutions will be infinite dimensional). Let V be the formal vector space over $\mathbb{C}$ generated by the set $\{\alpha\}_{i=1}^{m} \cup \{\beta\}_{i=1}^{m}$. We will call the vectors corresponding to α_i and β_i by e_i and f_i respectively. Let ω be the symplectic form on V defined by the following conditions: $\omega(\alpha_i, \alpha_j) = 0 = \omega(\beta_i, \beta_j)$ and $\omega(\alpha_i, \beta_j) = (a_i, \beta_j)$.

Let R be the ring $\mathbb{C}[\epsilon_1, ..., \epsilon_d]/I$, where I is the ideal generated by $\{\epsilon_i \epsilon_j\}$, $1 \leq i, j \leq d$. For the bundle P, let $\overline{\partial}_P$ be the dolbeault operator which defines the holomorphic structure on $ad(P)$. Then the operator $\overline{\partial}_P + \sum_{i=1}^{d} \epsilon_i \alpha_i$ defines a holomorphic structure on the C^∞ bundle $p_1^* P$ on $X \times Spec\, R$, this bundle is denoted by $\bar{P}$. $\bar{\theta} := \theta + \sum_{i=1}^{d} \epsilon_i \beta_i$ defines a section of $T^*\mathcal{M}_P$ over $\bar{P}$. So the infinitesimal deformation in the direction ϵ_i of this family is represented by (α_i, β_i).

Let R' be the ring $\mathbb{C}[e_1, ..., e_m, f_1, ..., f_m]/\mathfrak{m}^2$, $\mathfrak{m}$ being the maximal ideal of $\mathbb{C}[e_1, ..., e_m, f_1, ..., f_m]$. There is a natural inclusion $h : Spec\, R \longrightarrow Spec\, R'$ (recall that we enlarged a basis of $\{\alpha_i\}_{i=1}^{d}$ and a basis of $\{\beta_i\}_{i=1}^{d}$), defined by the condition $\epsilon_i \longmapsto e_i + f_i$. Let $c \in Spec\, R$ be the closed point. The symplectic form ω on V so defined that $h^*\omega(c)$ coincides with the symplectic form on $T_c Spec\, R$ defined by (1.3.1).

There is a function $H' : Spec\, R' \longrightarrow \oplus_{i=1}^{r} C^{\infty}(X, K^i \otimes \mathcal{O}((i-1)I))$ which is defined in the same way the Hitchin map is defined. Clearly the restriction of $\mathcal{H}_{\mathcal{P}}$ to $Spec\, R$ is h^*H'. So in order to show that $\mathcal{H}_{\mathcal{P}}$ Poisson-commutes at (P, θ), it is enough to show that H' Poisson-commutes at c', the closed point of $Spec\, R'$. But the function H' depends only on the f_i's, and the subspace of V spanned by $\{f_i\}_{i=i}^{m}$ is isotropic. *i.e.* the restriction of ω vanishes on this subspace. Hence H' Poisson-commutes at c'. This completes the proof. □

Now we introduce the parabolic analogue of very stability

Definition (1.5.10). A parabolic G-bundle P is said to be *very stable* if $H^0(X, ad^0(P) \otimes K \otimes \mathcal{O}(I))$ does not contain any nonzero nilpotent section.

Imitating the argument in (1.4.2) we get that any very stable parabolic bundle is actually stable parabolic. Also repeating the argument in §1.4 it can be easilly proved that the components of the preimage of 0 under $\mathcal{H}_{\mathcal{P}}$ are Lagrangian, *i.e.* the restriction of the symplectic form vanishes. For the same reason as in (1.4.7), the set of very stable parabolic bundles is a nonempty Zariski open set in the moduli of stable parabolic bundles. Moreover the fact that the fiber of $\mathcal{H}_{\mathcal{P}}$ over 0 is Lagrangian combines with the Poisson-commutativity of $\mathcal{H}_{\mathcal{P}}$ to imply that $\mathcal{H}_{\mathcal{P}}$ actually gives a completely integrable structure.

CHAPTER TWO : DEFORMATION OF COHOMOLOGY

§ 2.1 DEFORMATIONS OF A HIGGS BUNDLE.

Let X be a compact complex manifold of dimension n.

Definition (2.1.1). A *Higgs bundle* on X is a pair (E,θ), where E is a holomorphic vector bundle on X and $\theta \in H^0(X, End(E) \otimes \Omega^1_X)$, satisfying the condition that $\theta \wedge \theta = 0$ as a section of $H^0(X, End(E) \otimes \Omega^2_X)$.

Note that the Lie algebra structure on $End(E)$ and the exterior algebra structure on $\oplus \Omega^i_X$ induces an algebra structure on $\oplus(End(E) \otimes \Omega^i_X)$. As we mentioned in the previous chapter, Hitchin introduced the concept of Higgs bundles on curves [Hi1]; in higher dimensions it was defined in [S1].

Let E be a holomorphic vector bundle of rank r on X. $\mathcal{D}^i(E)$ denotes the sheaf of holomorphic differential operators of degree $\leq i$ on sections of E taking into itself. The following sequence of sheaves on X

$$0 \longrightarrow End(E) = \mathcal{D}^0(E) \longrightarrow \mathcal{D}^1(E) \xrightarrow{\sigma} T_X \otimes End(E) \longrightarrow 0$$

is exact, where σ denotes the symbol map. Define the *Atiyah algebra* of E to be

$$\mathcal{A}(E) := \{D \in \mathcal{D}^1(E) \; : \; \sigma(D) = id_E \otimes T_X\}.$$

$\mathcal{D}^1(E)$ is equipped with a natural Lie algebra (taking commutator) structure and $\mathcal{A}(E)$ is a sub-algebra.

There is a natural action of $\mathcal{A}(E)$ on $End(E) \otimes \Omega^i_X$ which is described below. First an operator $D \in \Gamma(U, \mathcal{A}(E))$ on $E|_U$, where $U \subset X$ is an open set, induces an operator on $End(E)|_U$. This operator, which will be again denoted by D, can be defined as follows: for $\alpha \in \Gamma(U, End(E))$ and $s \in \Gamma(U, E)$

$$D(\alpha)(s) = D(\alpha(s)) - \alpha D(s).$$

Now let $\beta \in \Gamma(U, End(E) \otimes \Omega^i_X)$ be $\sum_k \alpha_k \otimes \omega_k$, where $\alpha_k \in \Gamma(U, End(E))$ and $\omega_k \in \Gamma(U, \Omega^i_X)$. For $D \in \Gamma(U, \mathcal{A})$, define the operator on $End(E) \otimes \Omega^i_X$, which will be again denoted by D, using the following relation:

(2.1.2)

$$D(\beta) = \sum_k \left[D(\alpha_k) \otimes \omega_k + \alpha_k \otimes L\sigma(D)\omega_k\right],$$

where L is the Lie derivative. This operator D is well defined *i.e.* does not depend upon the decomposition of β.

For a Higgs bundle (E, θ) the map $D \longmapsto D(\theta)$ defines a map $d_\theta : \mathcal{A} \longrightarrow End(E) \otimes \Omega^1_X$.

Let m_θ be the operator on $End(E) \otimes \Omega^i_X$ given by the right multiplication by θ. For $\alpha \in End(E) \otimes \Omega^1_X$ $m_\theta(\alpha) = \alpha \wedge \theta + \theta \wedge \alpha$ where $\wedge$ is defined using the composition of operators in $End(E)$ and the exterior algebra structure on $\oplus \Omega^i_X$. $\theta \wedge \theta = 0$ implies that $m_\theta d_\theta = 0$. So the following sequence of sheaves on X is a complex.

(2.1.3)

$$C^\cdot : C^0 = \mathcal{A}(E) \xrightarrow{d_\theta} C^1 = End(E) \otimes \Omega^1_X \xrightarrow{m_\theta} C^2 = End(E) \otimes \Omega^2_X \longrightarrow 0.$$

Proposition (2.1.4). *There is a natural map from the space of infinitesimal deformations of the triplet (X, E, θ) to the first hypercohomology $\mathbb{H}^1(X, C^\cdot)$.*

Proof. Let $\bar{X} \xrightarrow{p} T$ be a holomorphic fibration over some open ball $T \subset \mathbb{C}^k$ containing $\{0\}$ such that $p^{-1}(0) = X$. $(\bar{E}, \bar{\theta})$ is a family of Higgs bundles on $\bar{X}$ *i.e.* $\bar{\theta} \in H^0(\bar{X}, \Omega^1_{\bar{X}/T})$ and $\bar{\theta} \wedge \bar{\theta} = 0$, such that the restriction $(\bar{E}|_{p^{-1}(0)}, \bar{\theta}|_{p^{-1}(0)})$ is (E, θ). For a suitable finite Stein covering $\{U_\alpha\}$ of $\bar{X}$ choose co-ordinates of the form $(z_1, ..., z_n, t_1, ..., t_k)$ on each U_α, where $(t_1, ..., t_k)$ are the co-ordinates on T; and $\frac{\partial}{\partial t_{\alpha,i}}$, $i \in \{1, ..., k\}$ are vector fields on U_α. Once trivializations of $\bar{E}|_{U_\alpha}$ are chosen, $\frac{\partial}{\partial t_{\alpha,i}}$ can be interpreted as first order operators on $\bar{E}|_{U_\alpha}$. The symbol of this operator is $\frac{\partial}{\partial t_{\alpha,i}} \otimes id_{\bar{E}|_{U_\alpha}}$. The vector field $\frac{\partial}{\partial t_{\alpha,i}} - \frac{\partial}{\partial t_{\beta,i}}$ projects to zero on T. Hence the operators

$$D_{\alpha,\beta,i} := \frac{\partial}{\partial t_{\alpha,i}}(0) - \frac{\partial}{\partial t_{\beta,i}}(0)$$

defined on $E|_{U_\alpha \cap U_\beta}$ gives a 1-cocycle of $\mathcal{A}$ on X.

Using the trivialization of $\bar{E}$ on U_α one can define

$$s_{\alpha,i} := d_\theta(\frac{\partial}{\partial t_{\alpha,i}})$$

as an element of $\Gamma(U_\alpha, End(\bar{E}|_{U_\alpha}) \otimes \Omega^1_{\bar{X}/T})$. Clearly $(D_{\alpha,\beta,i}, s_{\alpha,i})$ gives a 1-cocycle of the Cech resolution of $C^\cdot$ for the open cover $\{U_\alpha \cap X\}$ of X.

Though this construction of co-cycle involved choices of co-ordinates on $\bar{X}$ and *trivializations of* $\bar{E}|_{U_\alpha}$, *the association of the cohomology class given* by $(D_{\alpha,\beta,i}, s_{\alpha,i})$ to the tangent vector $\frac{\partial}{\partial t_i} \in T_0(T)$ does not depend upon *these choices. This completes the proof.* □

In the special case where X is a smooth algebraic variety over $\mathbb{C}$, the infinitesimal deformation functor in the sense of [Sch] can be defined in the following way. To any (parameter) scheme $\mathcal{A} = Spec(A)$ with A an Artinian local algebra, associate the set of all isomorphic classes triplets $(\bar{X}, \bar{E}, \bar{\theta})$ on $\mathcal{A}$, where $\bar{X}$ is a flat A scheme and $\bar{E}$ is a vector bundle on $\bar{X}$ and $\bar{\theta} \in H^0(\bar{X}, \Omega^1_{\bar{X}/A})$, *along with an isomorphism of the restriction to the* closed point $\mathfrak{m}$ of $SpecA$ with (X, E, θ). Two families $(\bar{X}, \bar{E}, \bar{\theta})$ and $(\bar{X}', \bar{E}', \bar{\theta}')$ *on* A *are said to be equivalent if there is an* A-isomorphism $\bar{X} \xrightarrow{\cong} \bar{X}'$ and a compatible bundle isomorphism $\bar{E} \xrightarrow{\cong} \bar{E}'$ such that $\bar{\theta}$ is mapped to $\bar{\theta}'$ and the restriction of these morphisms to $\mathfrak{m}$ coincides with the identity map on (X, E, θ) after the identification. This defines a functor on the category of Artinian local $\mathbb{C}$ -algebras with values in sets. We call this functor the *formal deformation functor* of (X, E, θ), and denote it by $\mathcal{F}$. Let $C^\cdot$ be the complex defined in (2.1.3). The following theorem is similar in spirit to Theorem (1.1.3).

Theorem (2.1.5). *The space of infinitesimal deformations of the triplet* (X, E, θ) *that is to say, the set* $\mathcal{F}(\mathbb{C}[\epsilon])$ *with* $\epsilon^2 = 0$ *is canonically isomorphic to* $\mathbb{H}^1(C^\cdot)$.

Proof. Let $\mathcal{U} = \{U_i = SpecA_i\}$ be a finite covering of X by affine open sets. Let $U_{ij} := U_i \cap U_j = SpecA_{ij}$, and similarly $U_{i_1} \cap U_{i_2} \cap \dots U_{i_m}$ is denoted by $U_{i_1 \dots i_m}$. Also define $U_i[\epsilon] := SpecA_i \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$ and $U_{i_1 \dots i_m}[\epsilon] := SpecA_{i_1 \dots i_m} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$. We consider the following Cech resolution of $C^{\cdot}$:

2.1.6

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^0 & \xrightarrow{d_\theta} & C^1 & \xrightarrow{m_\theta} & C^2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma \tilde{R}_i & \longrightarrow & \Sigma \tilde{M}_i \otimes \Gamma(U_i, \Omega^1) & \longrightarrow & \Sigma \tilde{M}_i \otimes \Gamma(U_i, \Omega^2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma \tilde{R}_{ij} & \longrightarrow & \Sigma \tilde{M}_{ij} \otimes \Gamma(U_{ij}, \Omega^1) & \longrightarrow & \Sigma \tilde{M}_{ij} \otimes \Gamma(U_{ij}, \Omega^2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

where $End(E)|_{U_i} = \tilde{M}_i$ and $\mathcal{A}(E)|_{U_i} = \tilde{R}_i$ are A_i-modules and $End(E)|_{U_{ij}} = \tilde{M}_{ij}$ and $\mathcal{A}(E)|_{U_{ij}} = \tilde{R}_{ij}$ are A_{ij} modules etc. This being an acyclic resolution of $C^{\cdot}$ the hypercohomology $H^1(C^{\cdot})$ can be computed as Z/B. Here Z consists of pairs (s_{ij}, t_i), where $s_{ij} \in \Gamma(U_{ij}, \mathcal{A}(E)) = R_{ij}$, and $t_i \in \Gamma(U_i, End(E) \otimes \Omega^1) = M_i \otimes \Gamma(U_i, \Omega^1)$ such that

1. $s_{ij} + s_{jk} = s_{ik}$ as elements of R_{ijk}
2. $t_j - t_i = d_\theta(s_{ij})$ as elements of $M_{ij} \otimes \Gamma(U_{ij}, \Omega^1)$
3. $m_\theta(t_i) = 0$.

And B is the subspace of Z consisting of pairs $(s_j - s_i, d_\theta(s_i))$, with $s_i \in R_i$.

Let (s_{ij}, t_i) be an element of Z. The condition (1) implies that s_{ij} is a 1-cocycle of $\mathcal{A}(E)$. Hence using the symbol map $\mathcal{A}(E) \xrightarrow{\sigma} T_X$ we get a 1-cocycle σs_{ij} of T_X. The vector space $H^1(X, T_X)$ being the space of infinitesimal deformations of X, there is a flat $\mathbb{C}[\epsilon]$ scheme X_ϵ along with an isomorphism of $X_\epsilon \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}$ with X. In fact X_ϵ is constructed by by gluing pairs U_i and U_j along U_{ij} using the automorphism $1 + \epsilon\sigma s_{ij}$ of U_{ij}.

Consider the bundle $(p_1)^*(E|U_i)$ on $U_i[\epsilon]$ for every i, where p_1 is the projection $U_i \times Spec(\mathbb{C}[\epsilon]) \longrightarrow U_i$. This bundle will be denoted by $\bar{E}_i$. The operator $1 + \epsilon s_{ij}$ on $E|U_{ij}$ is compatible with the operator $1 + \epsilon\sigma s_{ij}$ on U_{ij} in the sense that

$$(1 + \epsilon s_{ij})(fs) = (1 + \epsilon\sigma s_{ij})f(1 + \epsilon s_{ij})s.$$

Now we may identify the restrictions of $\bar{E}_i$ and $\bar{E}_j$ to U_{ij} by means of the isomorphism $1 + \epsilon s_{ij}$ of $\bar{E}_{ij}$. The above compatibility of operators ensures the compatibility of these identifications and hence we get a bundle E_ϵ on $Spec(\mathbb{C}[\epsilon]) \times X$.

On $Spec(U_i \times \mathbb{C}[\epsilon])$ we have $\theta_i + t_i\epsilon \in \Gamma(U_i[\epsilon], End(E) \otimes \Omega^1)$. We claim that these sections of $End(E)|U_i \otimes \Omega^1$ on $U_i[\epsilon]$ patch together to give a global section $\theta_\epsilon \in H^0(X_\epsilon, End(E_\epsilon) \otimes \Omega^1_{X_\epsilon/\mathbb{C}[\epsilon]})$. Indeed we have to show that over U_{ij} the following identity holds.

$$(1 + \epsilon s_{ij})(\theta_i + t_i\epsilon) = \theta_j + t_j\epsilon$$

But this follows from cocycle condition (2).

The condition (3) implies that $\theta_\epsilon \wedge \theta_\epsilon = 0$.

Thus we have associated to the 1-cocycle $((s_{ij}), (t_i))$ an infinitesimal deformation $(X_\epsilon, E_\epsilon, \theta_\epsilon)$ of (X, E, θ).

Now suppose that $((s_{ij}), (t_i)) \in B$, that is to say $s_{ij} = s_j - s_i, t_i = d_\theta(s_i)$. Then the commutativity of the following diagram

$$\begin{array}{ccc} U_{ij}[\epsilon] & \xrightarrow{1+\epsilon\sigma s_i} & U_{ij}[\epsilon] \\ \downarrow{\scriptstyle 1+\epsilon\sigma s_{ij}} & & \downarrow{\scriptstyle id} \\ U_{ij}[\epsilon] & \xrightarrow{1+\epsilon\sigma s_j} & U_{ij}[\epsilon] \end{array}$$

implies that the family X_ϵ as above is trivial. Similarly the commutativity of the diagram

$$\begin{array}{ccc} \bar{E}_{ij} & \xrightarrow{1+\epsilon s_i} & \bar{E}_{ij} \\ \big\downarrow{\scriptstyle 1+\epsilon s_{ij}} & & \big\downarrow{\scriptstyle id} \\ \bar{E}_{ij} & \xrightarrow{1+\epsilon s_j} & \bar{E}_{ij} \end{array}$$

implies that the E_ϵ is a pull back of E on X. Moreover the identity $(1+\epsilon s_i)(\theta+\epsilon t_i) = \theta + \epsilon t_i + \epsilon s_i\theta = \theta$ implies that θ_ϵ is pullback of θ. Hence the family $(X_\epsilon, E_\epsilon, \theta_\epsilon)$ associated to $(s_j - s_i, d_\theta s_i)$ is trivial. So we have a map from $\mathbb{H}^1(C^\cdot)$ to $\mathcal{F}(\mathbb{C}[\epsilon])$.

To construct the inverse map, let $(X[\epsilon], E[\epsilon], \theta[\epsilon]) \in \mathcal{F}(\mathbb{C}[\epsilon])$. $X[\epsilon]$ is obtained by gluing $U_i[\epsilon]$ and $U_j[\epsilon]$ along $U_{ij}[\epsilon]$ by some suitable isomorphism of $U_{ij}[\epsilon]$.

The exact sequence of sheaves on $U_i[\epsilon]$

$$0 \longrightarrow End(\mathbb{C}^{\,r}) \longrightarrow Aut(\mathbb{C}[\epsilon]^r) \longrightarrow Aut(\mathbb{C}^{\,r}) \longrightarrow 0$$

gives the following exact sequence of cohomologies

$$H^1(U_i[\epsilon], End(\mathbb{C}^{\,r})) \longrightarrow H^1(U_i[\epsilon], Aut(\mathbb{C}[\epsilon]^r)) \longrightarrow H^1(U_i[\epsilon], Aut(\mathbb{C}^{\,r})) \longrightarrow H^2(U_i[\epsilon], End(\mathbb{C}^{\,r})).$$

But $H^1(U_i[\epsilon], End(\mathbb{C}^{\,r})) = 0 = H^2(U_i[\epsilon], End(\mathbb{C}^{\,r}))$ and hence

$$H^1(U_i[\epsilon], Aut(\mathbb{C}[\epsilon]^r)) = \quad H^1(U_i[\epsilon], Aut(\mathbb{C}^{\,r})).$$

which implies that any bundle on $U_i[\epsilon]$ is a pullback of a bundle on U_i.

Now it is easy to see that for $(X[\epsilon], E[\epsilon], \theta[\epsilon])$ imitating (2.1.4) a corresponding element $\gamma \in \mathbb{H}^1(C^\cdot)$ can be constructed. Also it is easy to see that the map $\mathcal{F}(\mathbb{C}[\epsilon]) \longrightarrow \mathbb{H}^1(C^\cdot)$ given by $(X[\epsilon], E[\epsilon], \theta[\epsilon]) \longmapsto \gamma$ is the inverse of the map constructed earlier. □

Remark (2.1.7). The space of infinitesimal deformations of the Higgs bundle (E, θ) with fixed base is parametrized by the 1-st hypercohomolgy of the following complex $C'^{\cdot}$

$$C'^{\cdot} : End(E) \xrightarrow{m_\theta} End(E) \otimes \Omega^1_X \xrightarrow{m_\theta} End(E) \otimes \Omega^2_X \longrightarrow 0.$$

The natural inclusion of $C'^{\cdot}$ in $C^{\cdot}$ induces a map of cohomologies which connects the infinitesimal deformations of (E, θ) with that of (X, E, θ).

(2.1.8) Remark. Let $\mathcal{F}_1$ denotes the formal deformation functor of a Higgs bundle (E, θ) on a fixed smooth complete varity X; so the difference with $\mathcal{F}$ is that the base X does not deform. Then from Theorem 1.2.1 it follows that if the subspace of $H^0(X, End(E))$ consisting of endomorphisms which commute with θ is just the scalar multiplications then $\mathcal{F}_1$ is pro-representable in the sense on [Sch]. This happens for example when (E, θ) is stable (see Section 2.4 for definition of stability). Moreover if $\mathbb{H}^2(C'^{\cdot}) = 0$ ($C'^{\cdot}$ is defined in (2.1.7)), the representing complete local algebra is regular (Theorem 1.2.1).

§ 2.2 DEFORMATIONS OF A COHOMOLOGY.

Let

(2.2.1)

$$\begin{array}{c}(X_T, E_T, \theta_T)\\ \big\downarrow p\\ T\end{array}$$

be a family of Higgs bundles parametrized by a complex manifold T. For $t \in T$ the Higgs bundle on $X_t := p^{-1}t$ is denoted by (E_t, θ_t). The condition that $\theta_t \wedge \theta_t = 0$ ensures that the following sequence $D^{\cdot}$ of bundles on X_t is a complex.

(2.2.2)

$$D^{\cdot} \ : \ D^0 = E_t \xrightarrow{\theta_t} D^1 = E_t \otimes \Omega^1_{X_t} \xrightarrow{\theta_t} \ldots. \xrightarrow{\theta_t} D^n = E_t \otimes \Omega^n_{X_t} \longrightarrow 0.$$

Let $\mathcal{O}_t$ be the germs of holomorphic functions at t and m the unique maximal ideal in $\mathcal{O}_t$. The following exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{m}D^{\cdot}/\mathrm{m}^2D^{\cdot} & \longrightarrow & D^{\cdot}/\mathrm{m}^2D^{\cdot} & \longrightarrow & D^{\cdot}/\mathrm{m}D^{\cdot} & \longrightarrow & 0\\ & & \big\downarrow \| & & & & \big\downarrow \| & & \\ & & D^{\cdot} \otimes T_t^* & & & & D^{\cdot} & & \end{array}$$

induces a map D. **(2.2.3)**

$$\delta: \quad T_t(T) \otimes \mathbb{H}^i(D^\cdot) \longrightarrow \quad \mathbb{H}^{i+1}(D^\cdot).$$

So for $v \in T_t(T)$ we have a complex $(\mathbb{H}^*(D^\cdot), \delta(v,-))$ as follows

(2.2.4)

$$0 \longrightarrow \mathbb{H}^0(D^\cdot) \xrightarrow{\delta(v,-)} \mathbb{H}^1(D^\cdot) \xrightarrow{\delta(v,-)} \mathbb{H}^2(D^\cdot) \xrightarrow{\delta(v,-)} \ldots \xrightarrow{\delta(v,-)} H^n(D^\cdot) \longrightarrow 0.$$

Denote by $C^{\cdot t}$ the complex on X_t defined as in (2.1.3). The algebra of operators $\mathcal{A}(E)$ acts naturally on $E \otimes \Omega^i_X$. Indeed for $D \in \Gamma(U, \mathcal{A}(E))$ and $s \in \Gamma(U, E \otimes \Omega^i_X)$ which is of the form $s = \sum_k s_k \otimes \alpha_k$, where $s_k \in \Gamma(U, E)$ and $\alpha_k \in \Gamma(U, \Omega^i_X)$, $D(s) = \sum_k D(s_k) \otimes \alpha_k + s_k \otimes L_{(\sigma D)}\alpha_k$. This action and the natural action of $\oplus_i End(E) \otimes \Omega^i_X$ on $\oplus_i E \otimes \Omega^i_X$ yields a map of complexes

$$f: \ C^{\cdot t} \otimes D^\cdot \longrightarrow D^\cdot.$$

f induces the following map of cohomologies

(2.2.5)

$$f_*: \ \mathbb{H}^1(C^{\cdot t}) \otimes \mathbb{H}^i(D^\cdot) \longrightarrow \mathbb{H}^{i+1}(D^\cdot).$$

Let $d: T_t(T) \longrightarrow \mathbb{H}^1(C^{\cdot t})$ be the infinitesimal deformation defined by (2.4).

Theorem (2.2.6). *The composition of maps $f_* \circ (d \otimes id)$, as a map from $T_t(T) \otimes \mathbb{H}^i(D^\cdot)$ to $\mathbb{H}^{i+1}(D^\cdot)$ coincides with δ.*

Proof. It is enough to prove when $dim\ T = 1$. Let ζ be a co-ordinate on an open subset U of T containing t with t orresponding to $\zeta = 0$.

In order to prove (2.2.6) we will use a Dolbeault resolution of $D^\cdot$. For notational simplicity The space of C^∞ sections of the bundle $E_U \otimes \Omega^{i,j}_{X_U/U}$ on X_U is denoted by $\Omega^{i,j}_R(E_U)$. Similarly, $\Omega^{i,j}_R(E_t)$ denotes the space of C^∞ sections of $E_t \otimes \Omega^{i,j}_{X_t}$ on X_t. Let $\overline{\partial}_E$ be the dolbeault operator which defines the holomorphic structure on E_T; it acts on $\sum_{i,j} \Omega^{i,j}_R(E_U)$. The following is a resolution of $D^\cdot$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^0 & \xrightarrow{\theta_t} & D^1 & \cdots & D^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_R^{0,0}(E_t) & \xrightarrow{\theta_t} & \Omega_R^{1,0}(E_t) & \cdots & \Omega_R^{n,0}(E_t) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial_E} & & \downarrow{\scriptstyle \bar\partial_E} & & \downarrow{\scriptstyle \bar\partial_E} & & \\
0 & \longrightarrow & \Omega_R^{0,1}(E_t) & \xrightarrow{\theta_t} & \Omega_R^{1,1}(E_t) & \cdots & \Omega_R^{n,1}(E_t) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial_E} & & \downarrow{\scriptstyle \bar\partial_E} & & \downarrow{\scriptstyle \bar\partial_E} & & \\
0 & \longrightarrow & \Omega_R^{0,2}(E_t) & \xrightarrow{\theta_t} & \Omega_R^{1,2}(E_t) & \cdots & \Omega_R^{n,2}(E_t) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial_E} & & \downarrow{\scriptstyle \bar\partial_E} & & \downarrow{\scriptstyle \bar\partial_E} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

This being a fine resolution, the i-th hypercohomology $\mathbb{H}^i(D^\cdot)$ can be computed as the cohomology of the following complex

$$\sum_{j=0}^{i-1}\Omega_R^{j,i-j-1}(E_t)\xrightarrow{\bar\partial_E+\theta_t}\sum_{j=0}^{i}\Omega_R^{j,i-j}(E_t)\xrightarrow{\bar\partial_E+\theta_t}\sum_{j=0}^{i+1}\Omega_R^{j,i-j+1}(E_t).$$

The family $X_U\xrightarrow{p}U$ is trivial as a C^∞ family *i.e.* there is a diffeomorphism of families $\lambda : X_U\longrightarrow X_t\times U$. Moreover, the family of bundles $E_U\longrightarrow X_U$ is also trivial in the sense that there is a C^∞ isomorphism of bundles $\rho : E_U\longrightarrow\lambda^*E_t$. The vector field on X_U gotten by pulling back the vector field $\frac{\partial}{\partial\zeta}$ on $X_t\times U$ by λ is denoted by n. The 1-st order operator $\frac{\partial}{\partial\zeta}$ on C^∞ sections of $\lambda^*(E_t)$ pulls back as an C^∞ section of $\mathcal{A}(E_U)$. This pulled back operator which is denoted by $D(\mathrm{n})$ has a natural action on sections of $\sum_{i,j}\Omega_R^{i,j}(E_U)$ as described in (2.1.2).

Let $\{U_\alpha\}$ be a finite cover of $p^{-1}(U)$ and $(z_{\alpha,1},\dots,z_{\alpha,n},\zeta)$ be a co-ordinate chart on U_α as in (2.4). Using the co-ordinate chart we have a vector field

$\frac{\partial}{\partial\zeta}$ on U_α which is denoted by $\frac{\partial}{\partial\zeta_\alpha}$. The bundle $E_U|U_\alpha$ on U_α is isomorphic to the pullback of $E_{t,\alpha} := E_t|(X_t \cap U_\alpha)$ on $X_t \cap U_\alpha$ by the projection $f_\alpha : U_\alpha \longrightarrow X_t \cap U_\alpha$ defined by $(z_{\alpha,1}, ..., z_{\alpha,n}, \zeta) \longmapsto (z_{\alpha,1}, ..., z_{\alpha,n})$. Fix such an isomorphism $h_\alpha : E_U|U\alpha \longrightarrow f_\alpha^* E_{t,\alpha}$. Then using pull back $\frac{\partial}{\partial\zeta_\alpha}$ gives an 1-st order operator $D_\alpha \in \Gamma(U_\alpha, \mathcal{A}(E|_{U_\alpha}))$.

$D_\alpha(0) - D(\mathrm{n})$ is a smooth section of $\mathcal{A}(E_{t,\alpha})$ on $X_t \cap U_\alpha$ and moreover on $U_{\alpha,\ \beta}$ the difference being holomorphic

$$D_\zeta := \overline{\partial}_E(D_\alpha(0) - D(\mathrm{n}))$$

is globally defined section of $\Omega^{0,1}(X_t, \mathcal{A}(E_t))$, where $\overline{\partial}_E$ is the dolbeault operator which defines the holo. structure on $\mathcal{A}(E_T)$.

Define

$$\mu_\zeta := D(\mathrm{n})(\theta_U)|_{X_t}$$

as a smooth section of $End(E_t) \otimes \Omega^{1,0}_{X_t}$.

The pair (D_ζ, μ_ζ) is a 1-cocycle for the dolbeault resolution of $C^{\cdot t}$ and represents the element in $\mathbb{H}^1(C^{\cdot t})$ which corresponds to $\frac{\partial}{\partial\zeta}$ by (2.4).

Let $\Phi \in \sum_{j=0}^{i} \Omega_R^{j,i-j}(E_t)$ be in the kernel of $\overline{\partial}_E + \theta_t$, $\bar{\Phi}$ denotes the element in $\mathbb{H}^i(D^\cdot)$ represented by Φ. $p_1 : X_t \times U \longrightarrow X_t$ is the projection and $p_1^*(\Phi)$ is the pullback of Φ as a section of $\oplus_j p_1^*(E_t \otimes \Omega_{X_t}^{j,i-j})$. Using λ and ρ, $p_1^*(\Phi)$ gives a smooth section $\widetilde{p_1^*\Phi}$ of $\oplus_j(E_U \otimes \Omega^{j,i-j}_{X_U/U})$.

In the above notation

$$\delta(\frac{\partial}{\partial\zeta} \otimes \bar{\Phi}) = D(\mathrm{n})((\overline{\partial}_E + \theta_U)(\widetilde{p_1^*\Phi}))|_{X_t},$$

as elements of $\mathbb{H}^{i+1}(D^\cdot)$. Note that $D(\mathrm{n})((\overline{\partial}_E + \theta_U)(\widetilde{p_1^*\Phi}))|_{X_t}$ on X_t is $\overline{\partial}_E + \theta_t$ closed. One way to see this is as follows: using λ and ρ we have $\overline{\partial}_{E_{t'}} = \overline{\partial}_{E_t} + A(t')$ where $A(t') : \sum_{j=0}^{i} \Omega_R^{j,i-j}(E_t) \longrightarrow \sum_{j=0}^{i+1} \Omega_R^{j,i-j+1}(E_t)$ is a 0-th order operator *i.e.* an endomorphism of bundles. That $D(\mathrm{n})((\overline{\partial}_E + \theta_U)(\widetilde{p_1^*\Phi}))|_{X_t}$ is $\overline{\partial}_E + \theta_t$ closed follows from the fact that θ is a holomorphic section and on X_t

$$\overline{\partial}_{E_t} D(\mathrm{n})(s) = D(\mathrm{n})(\overline{\partial}_E(s)) - \frac{\partial}{\partial\zeta} A(t')(s)\ .$$

On U_α we have

$$D(\mathrm{n})((\overline{\partial}_E+\theta_U)(\widetilde{p_1^*\Phi})) = D_\alpha(\overline{\partial}_E(\widetilde{p_1^*\Phi})) - (D_\alpha - D(\mathrm{n}))(\overline{\partial}_E(\widetilde{p_1^*\Phi})) + D(\mathrm{n})(\theta_U\widetilde{p_1^*\Phi})$$

$$= D_\alpha(\overline{\partial}_E(\widetilde{p_1^*\Phi})) - \overline{\partial}_E((D_\alpha - D(\mathrm{n}))(\widetilde{p_1^*\Phi})) + \overline{\partial}_E(D_\alpha - D(\mathrm{n}))\widetilde{p_1^*\Phi} + D(\mathrm{n})(\theta_U\widetilde{p_1^*\Phi}).$$

Since D_α is a holomorphic operator and by construction $D(\mathrm{n})(\widetilde{p_1^*\Phi}) = 0$, applying Leibniz rule the above equality implies that on X_t

$$D(\mathrm{n})((\overline{\partial}_E + \theta_U)(\widetilde{p_1^*\Phi}))|_{X_t} = D_\zeta(\Phi) + \mu_\zeta(\Phi).$$

But the right hand side of the above equality is $f_* \circ (d \otimes id)(\frac{\partial}{\partial\zeta} \otimes \bar{\Phi})$. This completes the proof. □

For a Higgs bundle (E,θ) on X the following complex of differential operators

(2.2.7)

$$0 \longrightarrow \Omega^0(E) \xrightarrow{\overline{\partial}_E+\theta} \oplus_k \Omega^{k,1-k}(E) \xrightarrow{\overline{\partial}_E+\theta} \ldots \xrightarrow{\overline{\partial}_E+\theta} \oplus_k \Omega^{k,n-k}(E) \longrightarrow 0$$

being elliptic (its symbol being same as that of $\overline{\partial}_E$) the Theorem 1.7 of [GL2] applies and in this special case ((1.7),(1.8),(1.9), [GL2]) gives the following

Proposition (2.2.8). *(i) If for some $t \in T$ in (2.2.1), for all nonzero $v \in T_t(T)$ the condition $\dim H^j(\mathbb{H}^*(D^\cdot), \delta(v,-)) \leq m$ (defined in (2.2.4)) is satisfied then for all $t' \in U - t$, where U is a neighborhood of t in T, $\dim \mathbb{H}^j(D^\cdot_{t'}) \leq m$.*

(ii) If for all nonzero $v \in T_t(T)$, $H^j(\mathbb{H}^*(D^\cdot), \delta(v,-)) = 0$ and $t \in S^j(\overline{\partial}_E + \theta) := \{s \in T | \ \mathbb{H}^j(D^\cdot_s) \neq 0\}$, then t is an isolated point of $S^j(\overline{\partial}_E + \theta)$.

(iii) If for some $v \in T_t(T)$, $H^j(\mathbb{H}^*(D^\cdot), \delta(v,-)) = 0$ then t is not an interior point of $S^j(\overline{\partial}_E + \theta)$.

§ 2.3 DEFORMATIONS OF COHOMOLOGY OF A LOCAL SYSTEM.

Let V is a C^∞ vector bundle of rank r on X and ∇ be a flat connection on V. Note that V has a holomorphic structure induced by ∇. The corresponding *local system, i.e.* the locally constant sheaf of flat sections, is denoted by V^∇. There is a 1-1 correspondence between the set of all flat connections on V and equivalence classes representations of $\pi_1(X)$ in $Gl(r,\mathbb{C})$.

The space of infinitesimal deformations of flat connections on V is given by $H^1(End(V^\nabla))$. So the infinitesimal deformations of the pair (X,∇) is given by $H^1(X, T_X \oplus End(V^\nabla))$.

Let

(2.3.1)

$$\begin{array}{c}(X_T,\nabla^T)\\ \big\downarrow p\\ T\end{array}$$

be a family of local systems on a C^∞ vector bundle V. For $t \in T$ let V^{∇^t} be the local system on X_t. Imitating the derivation of (2.2.3) we have

(2.3.2)

$$\gamma: \quad T_t(T) \otimes H^i(V^{\nabla^t}) \longrightarrow \quad I\!H^{i+1}(V^{\nabla^t}).$$

Let $d : T_t(T) \longrightarrow H^1(X_t, End(V^{\nabla^t}))$ be the infinitesimal deformation map of the local system. The natural action of $End(V^{\nabla^t})$ on V^{∇^t} defines the homomorphism

(2.3.3)

$$g: \; H^1(X_t, End(V^{\nabla^t})) \otimes H^i(V^{\nabla^t}) \longrightarrow H^{i+1}(V^{\nabla^t}).$$

Theorem (2.3.4). *The composition of maps $g \circ (d \otimes id)$ coincides with γ.*

Proof. The proof is similar (actually simpler) to (2.26). As in (2.26) assume that $dim\, T = 1$ and ζ a co-ordinate on an open subset U of T containing t with t corresponding to $\zeta = 0$. Fix a deffeomorphism $\lambda : X_U \longrightarrow X_t \times U$.

For $t' \in U$, $A^i_{t'}(V)$ denotes the space of i-forms on $X_{t'}$ with values in V *i.e.* C^∞ sections of $V \otimes \overset{i}{\wedge} T^*_{\mathbb{C}}(X_{t'})$. Let n be the vector field on $X_t \times U$ given by $\frac{\partial}{\partial \zeta}$.

The following de Rham resolution of V^{∇^t} is a fine resolution.

$$0 \longrightarrow V^{\nabla^t} \longrightarrow A^0_t(V) \xrightarrow{\nabla^t} A^1_t(V) \xrightarrow{\nabla^t} A^2_t(V) \xrightarrow{\nabla^t} \dots \xrightarrow{\nabla^t} A^n_t \longrightarrow 0.$$

The cohomology class $d(\frac{\partial}{\partial \zeta})$ is represented by the cocycle $\frac{\partial}{\partial \zeta}\nabla^\zeta(0)$ for the de Rahm resolution of $End(V^{\nabla^t})$.

Let $\Phi \in A^i_t(V)$ be in the kernel of ∇^t. Using the projection $p_1 : X_t \times U \longrightarrow X_t$, Φ gives a section $p_1^*\Phi$ of $V \otimes p_1^*(\Omega^i_{X_t})$ on $X_t \times U$. Now

$$\gamma(\frac{\partial}{\partial \zeta} \otimes \Phi) = \mathrm{n}(\nabla^\zeta p_1^*\Phi)(0).$$

Since $\mathrm{n}(p_1^*\Phi) = 0$, we have $\mathrm{n}(\nabla^\zeta p_1^*\Phi)(0) = \frac{\partial}{\partial \zeta}(\nabla^\zeta)(0)\Phi$. This completes the proof. □

Since the connection operator is elliptic ((1.7),(1.8),(1.9), [GL2]) gives the following

Proposition (2.3.5). *(i) If for some $t \in T$ in (4.1), for all nonzero $v \in T_t(T)$ the condition $\dim H^j(H^*(V^{\nabla^t}), \gamma(v, -)) \le m$ (defined in (2.24)) is satisfied then for all t' in a punctured neighborhood of t in T, $\dim H^j(\nabla^{t'}) \le m$.*

(ii) If for all nonzero $v \in T_t(T)$, $H^j(H^*(V^{\nabla^t}), \gamma(v, -)) = 0$ and $t \in S^j(\nabla) := \{s \in T | \ H^j(\nabla^s) \neq 0\}$, then t is an isolated point of $S^j(\nabla)$.

(iii) If for some $v \in T_t(T)$, $H^j(H^*(V^{\nabla^t}), \gamma(v, -)) = 0$ then t is not an interior point of $S^j(\nabla)$.

§ 2.4 CORRESPONDENCE OF DEFORMATIONS.

Let X be a compact Kähler manifold of dimension n with ω being the Kähler form.

The Higgs bundle (E, θ) on X is called *stable* if for any proper subsheaf $0 \neq F \subset E$ with the quotient being torsion free, such that F is θ invariant *i.e.* $\theta(F) \subset F \otimes \Omega^1_X$, the following condition holds

$$deg(F)/rank(F) < \quad deg(E)/rank(E).$$

The degree is defined by $deg(E) := (c_1(E) \cup \omega^{n-1}) \cap [X]$.

Hitchin for curves [Hi1] and Simpson for compact Kähler manifolds [S1] proved that a stable Higgs bundle admits a unique (upto scaler) Hermitian-Einstein metric *i.e.* if the metric is denoted by K then,

$$\Lambda(D + \theta + \theta^*)^2 = \lambda id_E,$$

where $\lambda \in \mathbb{C}$, D is the holomorphic hermitian connection on E for K, θ^* and Λ are the adjoints of θ and the wedge product by ω respectively. If all the Chern classes of E vanish then $\lambda = 0$; in fact vanishing of first and second Chern classes is enough [S2]. In such a situation the connection $(D + \theta + \theta^*)^2$ is flat. In other words E is associated to a representation of $\pi_1(X)$ in $Gl(k, \mathbb{C})$. It follows from [S1],[C] that the connection $(D+\theta+\theta^*)^2$ irreducible.

Conversely, if ∇ is a irreducible flat connection on a vector bundle $V \longrightarrow X$ then it follows from [D] when $n = 1$ and from [C] in general that (V, ∇) admits a unique (upto scaler) *harmonic metric*, which means the following: For a hermitian metric K on V define D^c_K by $(D^c_K)^* = -i[\Lambda, \nabla]$, also define $G_K := (\nabla D^c_K + D^c_K \nabla)/4$. The metric K is called harmonic if

$$\Lambda G_K = 0.$$

A result due to Deligne ((2.1), [S2]) says that $\Lambda G_K = 0$ implies $G_K = 0$, this in turn implies that $(\nabla + D^c_K)/2$ induces a stable Higgs structure on the C^∞-bundle V. Note that the existence of flat connection implies that all the Chern classes of V vanish. The details of the above correspondence of stable Higgs bundles and irreducible flat connections can be found for example in [S2]. Irreducible unitary representations correspond to stable bundles *i.e.* $\theta = 0$.

For a stable Higgs bundle (E, θ) with $c_1(E) = 0 = c_2(E)$, let $D^\cdot$ be the complex of sheaves defined as in (2.22). If E^∇ be the corresponding local system then there is a natural isomorphism between $\mathbb{H}^i(D^\cdot)$ and $H^i(E^\nabla)$ ((2.22), [S2]) which is described briefly below. The kernel of the operator

$$\Delta_H \;:=\; (\overline{\partial}_E + \theta)^*(\overline{\partial}_E + \theta) \;+\; (\overline{\partial}_E + \theta)(\overline{\partial}_E + \theta)^*$$

on $A^i(E)$ (C^∞ sections of $E \otimes \overset{i}{\wedge} T^*_{\mathbb{C}}(X)$), where the adjoint is taken with respect to the Hermitian-Einstein metric and the Kähler metric, is canonically isomorphic to $\mathbb{H}^i(D^\cdot)$. Similarly, $H^i(E^\nabla)$ is given by the kernel of $\Delta := \nabla^*\nabla \;+\; \nabla\nabla^*$ on $A^i(E)$. Using analogues of Kähler identities (Sec.2, [S2]) it can be proved that $2\Delta_H = \Delta$. This relation among Laplacians induces an isomorphism $F : H^i(E^\nabla) \longrightarrow \mathbb{H}^i(D^\cdot)$. This isomorphism is not holomorphic over families.

Let $h : H^1(End(E^\nabla)) \longrightarrow \mathbb{H}^1(C^\cdot)$ be the composition of $\mathbb{H}^1(C'^\cdot) \longrightarrow \mathbb{H}^1(C^\cdot)$ (defined in (2.7)) with $H^1(End(E^\nabla)) \longrightarrow \mathbb{H}^1(C'^\cdot)$, the infinitesimal version of the correspondence between Higgs bundles and local systems. Recall the maps f_* and g in (2.26) and (4.3) respectively.

A natural question that arises is: how the deformations of $\mathbb{H}^i(D^\cdot)$ are related to the deformations of $H^i(E^\nabla)$. Unfortunately, the following diagram

$$\begin{array}{ccc} H^1(End(E^\nabla)) \otimes H^i(E^\nabla) & \xrightarrow{g} & H^{i+1}(E^\nabla) \\ \Big\downarrow{\scriptstyle h\otimes F} & & \Big\downarrow{\scriptstyle F} \\ \mathbb{H}^1(C^\cdot) \otimes \mathbb{H}^i(D^\cdot) & \xrightarrow{f_*} & \mathbb{H}^{i+1}(D^\cdot) \end{array}$$

does not commute. Indeed an infinitesimal deformation of a line bundle is given by $\bar{\psi}$ where $\psi \in H^0(X, \Omega^1_X)$ where as the corresponding infinitesimal deformation of representation in $U(1)$ is given by $\psi + \bar{\psi}$. But this is the general picture which we proceed to elaborate.

Let $(\alpha,\beta) \in A^{0,1}(End(E)) \oplus A^{1,0}(End(E))$ be a harmonic form representing an element $\gamma \mathbb{H}^1(C'^{\cdot})$. The infinitesimal deformation of Hermitian-Einstein metric corresponding to γ is given by H, a symmetric element of $End(E)$. Let $(\alpha',\beta') \in A^{1,0}(End(E)) \oplus A^{0,1}(End(E))$ be the infinitesimal deformations of $\partial_E + \theta^*$, ∂_E is the (1,0) component of the unitary connection. Differentiating the identity

$$(\partial_E x, y) + (x, \overline{\partial}_E y) = \partial(x,y),$$

where $x, y \in A^0(E)$, we get

$$(H\partial_E x, y) + (\alpha' x, y) + (Hx, \overline{\partial}_E y) + (x, \alpha y) = \partial(Hx, y).$$

After some cancellations we have

$$\alpha' \;=\; \alpha^* \;+\; \partial(H).$$

Differentiating the identity $(\theta^* x, y) = (x, \theta y)$ we get in the same way

$$\beta' \;=\; \beta^* \;+ [\theta^*, H].$$

Let $\Phi \in A^i(E)$ be a harmonic form. Then

$$((\partial H + [\theta^*, H])\Phi, x) \;=\; (\partial(H\Phi), x) + (H\theta^*\Phi, x) + ([\theta^*, H]\Phi, x)$$

$$= -(\theta^* H\Phi, x) + (H\theta^*\Phi, x) + ([\theta^*, H]\Phi, x) = 0.$$

Define $\bar{h} \;:\; H^1(End(E^\nabla)) \longrightarrow \mathbb{H}^1(C^\cdot)$ by $\xi \longmapsto h(\xi) + h(\xi)^*$, where $h(\xi)$ is the harmonic representative and the adjoint is with respect to the Hermitian-Einstein metric. Note that $h(\xi)^*$ is also a 1-cocycle for the complex $C^\cdot$. So combining everything we have

Theorem (2.4.1). *The following diagram commutes*

$$\begin{array}{ccc} H^1(End(E^\nabla)) \otimes H^i(E^\nabla) & \xrightarrow{\;g\;} & H^{i+1}(E^\nabla) \\ \Big\downarrow{\scriptstyle \bar{h}\otimes F} & & \Big\downarrow{\scriptstyle F} \\ \mathbb{H}^1(C^\cdot) \otimes \mathbb{H}^i(D^\cdot) & \xrightarrow{\;f_*\;} & \mathbb{H}^{i+1}(D^\cdot). \end{array}$$

In view of (2.2.6) and (2.3.4) the above theorem connects deformations of $\mathbb{H}^i(D^\cdot)$ with the deformations of $H^i(E^\nabla)$ on any family, even if there is a deformation of the Kähler structure. That is because a deformation of Kähler structure keeping everything else fixed has no effect on either of the cohomologies.

§ 2.5. SOME APPLICATIONS OF COHOMOLOGY DEFORMATIONS.

Let H be the space of all harmonic 1-forms on the Kähler manifold X. For a Higgs bundle (E, θ) the operator $\overline{\partial}_E + \theta$ will be called the *Higgs operator*, this operator on the C^∞ bundle E determines the Higgs structure. It is easy to see that for $\psi \in H$, the operator $\overline{\partial}_E + \theta + \psi \otimes id$ determines a new Higgs structure. This way, for fixed (E, θ), we have a family of Higgs bundles parametrized by H. Similarly, for flat connection (V, ∇) and $\psi \in H$, the operator $\nabla + \psi$ gives a flat connection. The $dim\, \mathbb{H}^i(D^\cdot)$ and $dim\, \mathbb{H}^i(V^\nabla)$ are constant on some Zariski open sets of H; the complex $D^\cdot$ is defined as in (2.2.2). Now ((2.2.8),i) and ((2.3.5),i) gives the following

Proposition (2.5.1). *(i) For a general Higgs bundle (E', θ') in the family on H as above, the restriction of f_*, $H \otimes \mathbb{H}^i(D^\cdot) \xrightarrow{f_*} \mathbb{H}^{i+1}(D^\cdot)$ is zero.*
(ii) For a general (E, ∇) in H the restriction of g, $H \otimes H^i(E^\nabla) \xrightarrow{g} H^{i+1}(E^\nabla)$ is zero.

The above proposition in turn implies the following

Theorem (2.5.2). *(i) For a general (E, ∇) in any component of M_F, the restriction of g, $H \otimes H^i(E^\nabla) \xrightarrow{g} H^{i+1}(E^\nabla)$ is zero.*
(ii) For a general stable Higgs bundle (E', θ') in any component of M_H the restriction of f_*, $H \otimes \mathbb{H}^i(D^\cdot) \xrightarrow{f_*} \mathbb{H}^{i+1}(D^\cdot)$ is zero.

For rank one local systems or Higgs bundles (2.5.2) holds without the assumption that X is projective; the moduli spaces M_F and M_H exists for any Kähler manifold. When rank is one and $\theta = 0$, ((6.2),ii) is Thm.(2.7) [GL2].

In the rest of this section we restrict our attention to rank one local systems, and Higgs line bundles having zero Chern class. Note that the moduli of such Higgs line bundles is the product $Pic^0(X) \times H^0(X, \Omega^1_X)$.

As before, M_F is the moduli of rank one local systems, and M_H is the moduli of topologically trivial Higgs line bundles. H denotes the space of all harmonic 1-forms on the Kähler manifold X.

Define

$$F^i_m := \{\nabla | \ \nabla \in M_F, dim\, H^i(X, \nabla) \geq m\}.$$

Similarly for Higgs line bundles define

$$S^i_m := \{(L, \theta) | \ (L, \theta) \in M_H, dim\, H^i(X, D^\cdot) \geq m\}.$$

Note that the natural deffeomorphism between M_F and M_H takes F^i_m to S^i_m.

The operation of taking tensor product equips M_F with a structure of an abelian group. Note that the group structure on $Pic^0(x)$ induces a group structure on M_H. It is easy to check that the correspondence between M_F and M_H is an isomorphism of groups. The identity element *i.e.* the constant sheaf $\mathbb{C}$ will be denoted by ∇^0. The map $\xi \longmapsto \nabla^0 + \xi$ defines a family of local systems on H. The induced map $\rho : H \longrightarrow M_F$ is a surjective group homomorphism.

Since ρ is a local deffeomorphism, for any $\nabla \in M_F$, $\nabla' \longmapsto (\rho)^{-1}(\nabla' - \nabla)$ is a deffeomorphism from some neighborhood U of ∇ to a neighborhood of 0 in H. This map is denoted by ζ, note that $\zeta(\nabla) = 0$. The map $H^i(X, \nabla') \longrightarrow H^{i+1}(X, \nabla')$, given by $\alpha \longmapsto \alpha \cup \zeta(\nabla')$ is denoted by $\cup\zeta$. Let $H' \subset H$ be a linear subspace. Define $U' := U \cap \zeta^{-1}H'$. We have the following complex D^∇ of sheaves on U'

(2.5.3)

$$\ldots \xrightarrow{\cup\zeta} H^{i-1}(X, \nabla) \otimes \mathcal{O}(U') \xrightarrow{\cup\zeta} H^i(X, \nabla) \otimes \mathcal{O}(U') \xrightarrow{\cup\zeta} H^{i+1}(X, \nabla) \otimes \mathcal{O}(U') \xrightarrow{\cup\zeta} \ldots$$

Let $\bar{\nabla} \longrightarrow X \times U'$ be a tautological family of local systems parametrized by U'. The sheaf on U', given by the i-th cohomology of D^{∇} is denoted by $\mathcal{H}^i(D^{\nabla})$.

Theorem (2.5.4). *The stalk of the i-th direct image* $(\mathcal{R}^i_{p_2*}(\bar{\nabla}))_{\nabla}$ *is isomorphic to the stalk* $(\mathcal{H}^i(D^{\nabla}))_{\nabla}$.

This theorem is similar to Thm.2.22 of [GL3]. The proof of (2.22 [GL3]) can be adapted almost without any change to prove (6.3), and we omit it. The operator $\overline{\partial}_{L_y}$ used in the proof of (2.22 [GL3]) (for example in (2.4 [GL3])) is to be replaced by ∇.

If H' is a one dimensional subspace of H generated by ψ, then from (6.4) and ((2.23), [GL3]) we get that there is a punctured neighborhood $V \subset M_F$ of 0, such that for any $\nabla' \in V$

$$H^i(\nabla') = H^i(D^{\nabla} \otimes \mathcal{O}_V/\mathrm{m}_{\nabla'}),$$

where $\mathrm{m}_{\nabla'}$ is the maximal ideal at ∇'. Note that the complex $D^{\nabla} \otimes \mathcal{O}_V/\mathrm{m}_{\nabla'}$ is isomorphic to the following complex which is supported at ∇' and is denoted by $D_{\nabla'}$

(2.5.5)

$$\ldots \xrightarrow{\cup\psi} H^{i-1}(X,\nabla) \xrightarrow{\cup\psi} H^i(X,\nabla) \xrightarrow{\cup\psi} H^{i+1}(X,\nabla) \xrightarrow{\cup\psi} \ldots$$

Let A be a irreducible component of F^i_m and is of dimension at least one. On some Zariski open subset A^0 of A, the dimension of the i-th cohomology of local system remains constant. Take any smooth point $\nabla \in A^0$ and any $\psi \in H$ such that $D(\zeta^{-1})(\psi) \in T_{\nabla}(A^0)$. In ((2.3.5) take T to be a curve in A^0 passing through ∇ and tangential along $D(\zeta^{-1})(\psi) \in T_{\nabla}(A^0)$. Then it follows from ((4.5),i) that $\psi \cup H^i(\nabla) = 0$. So for the complex $D_{\nabla'}$ defined in (6.5) *dim* $H^i(D_{\nabla'} \geq m$. Hence an open subset of $\zeta^{-1}(H')$, where H' is the line generated by ψ, is contained in F^i_m. This implies that A is a translation of a Lie subgroup of M_F.

We noted earlier that the correspondence between M_H and M_F is an isomorphism of groups. So we have the following following Theorem of [A]

Theorem (2.5.6). *(i) Any irreducible component of the subvariety F^i_m, of M_F, is a translation of some complex subgroup of M_F.*

(ii) Any irreducible component of S^i_m is a translation of some complex subgroup of M_H.

§ 2.6 CONSTRUCTION OF SYMPLECTIC STRUCTURE.

Let X be a smooth projective variety over $\mathbb{C}$, and $L \in H^1(X, \Omega^1_X)$ be an ample class on X. Actually most of the things we do holds for compact Kähler manifolds.

In (2.1.7) we identified the space of infinitesimal deformations of a Higgs bundle (E, θ) on X. Since we will no longer consider deformations of X, the complex in (2.1.7) will be denoted by $C^\cdot$.

There is a natural 1-form, and also a natural 2-form on $\mathbb{H}^1(C^\cdot)$ which we will now describe. The projection from $C^\cdot$ to the complex $End(E) \xrightarrow{p} 0$ induces a homomorphism $\mathbb{H}^1(C^\cdot) \xrightarrow{p} H^1(X, End(E))$. Using the algebra structure on $End(E)$ and the trace map $tr : End(E) \longrightarrow \mathbb{C}$, we have

$$f : H^1(X, End(E)) \otimes H^0(X, End(E) \otimes \Omega^1_X) \longrightarrow H^1(X, \Omega^1_X).$$

The composition

(2.6.1)

$$\mathbb{H}^1(C^\cdot) \xrightarrow{p} H^1(X, End(E)) \xrightarrow{f(-,\theta)} H^1(X, \Omega^1_X) \xrightarrow{\cup L^{n-1}} H^n(X, \Omega^n_X) = \mathbb{C}.$$

defines a 1-form on $\mathbb{H}^1(C^\cdot)$, which is denoted by Θ.

Let $\Omega^1[1]$ be the shifted complex $0 \longrightarrow \Omega^1_X$, also denote the complex $0 \longrightarrow End(E) \otimes End(E) \otimes \Omega^1_X \oplus End(E) \otimes \Omega^1_X \otimes End(E)$ by $D^\cdot$. The following composition of morphisms of complexes is denoted by h.

$$C^\cdot \otimes C^\cdot \longrightarrow D^\cdot \longrightarrow \Omega^1[1].$$

So we have

(2.6.2)

$$\mathbb{H}^1(C^\cdot)\otimes\mathbb{H}^1(C^\cdot) \longrightarrow \mathbb{H}^2(C^\cdot\otimes C^\cdot) \xrightarrow{h} \mathbb{H}^2(\Omega^1[1])$$
$$= H^1(X,\Omega^1_X) \xrightarrow{\cup L^{n-1}} H^n(X,\Omega^n_X) = \mathbb{C}.$$

The above composition of maps defines a 2-form on $\mathbb{H}^1(C^\cdot)$, which is denoted by $\bar{P}$. Since the space of infinitesimal deformations of (E,θ) is parametrized by $\mathbb{H}^1(C^\cdot)$, on any family of Higgs bundles parametrized by T, Θ and $\bar{P}$ defines a 1-form and a 2-form on T respectively, which will again denoted by Θ and $\bar{P}$ respectively. Our next aim is to show that in certain sense $d\Theta = \bar{P}$.

Let R be the ring $\mathbb{C}[\epsilon_1,\epsilon_2]/({\epsilon_1}^3,{\epsilon_1}^2\epsilon_2,\epsilon_1{\epsilon_2}^2,{\epsilon_2}^3)$. Let $(\tilde{E},\tilde{\theta})$ be a family of Higgs bundles on X parametrized by $Spec\,R$, in other words $\tilde{E}$ is a holomorphic vector bundle on $\tilde{X} := X\times Spec\,R$ and $\tilde{\theta}\in H^0(X\times Spec\,R, End(\tilde{E})\otimes \Omega^1_{\tilde{X}/Spec R}$, with $\tilde{\theta}\wedge\tilde{\theta}=0$. Assume that the restriction of this family to the closed point $c := Spec(R/\mathrm{m})$, where m is the maximal ideal, is (E,θ). Θ and $\bar{P}$ will give a 1-form and a 2-form on $Spec\,R$ respectively; these forms will also be denoted by Θ and $\bar{P}$ respectively.

Proposition (2.6.3). *On the closed point $c\in Spec\,R$, the evaluation of 2-form $d\Theta(c)$ coincides with $\bar{P}(c)$.*

Proof. We need some general facts about Higgs bundles. Let (E,θ) be any Higgs bundle. Define $\Omega^{p,q}(End(E)) := C^\infty(X, End(E))$, *i.e.* the space of all smooth (p,q)-forms with values in $End(E)$. Let $\overline{\partial}_E$ be the dolbeault operator which defines the holomorphic structure on $End(E)$. Using the Dolbeault resolution of the complex $C^\cdot$ as in (3.6), the i-th hypercohomology $\mathbb{H}^i(C^\cdot)$ can be computed as the cohomology of the following complex

$$\sum_{j=0}^{i-1}\Omega^{j,i-j-1}(End(E)) \xrightarrow{\overline{\partial}_E+\theta} \sum_{j=0}^{i}\Omega^{j,i-j}(End(E)) \xrightarrow{\overline{\partial}_E+\theta} \sum_{j=0}^{i+1}\Omega^{j,i-j+1}(End(E)).$$

The given family of bundles $\tilde{E}$ parametrized by $Spec\,R$ is C^∞ trivial, in other words there is an C^∞ isomorphism $f:\,\tilde{E}\longrightarrow p^*E$, where $p:\tilde{X}\longrightarrow X$

is the natural projection. Using this isomorphism, the dolbeault operator that defines the holomorphic structure on $End(\bar{E})$ can be expressed in the following form

$$\overline{\partial}_E + A_1\epsilon_1 + A_2\epsilon_2 + B_1\epsilon_1^2 + B_2\epsilon_2^2 + C\epsilon_1\epsilon_2,$$

where A_i, B_i and C are smooth sections of $End(E) \otimes \Omega_X^{0,1}$.

Using f, $\bar{\theta}$ is of the form

$$\theta + \alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \beta_1\epsilon_1^2 + \beta_2\epsilon_2^2 + \gamma\epsilon_1\epsilon_2,$$

where $\alpha_i, \beta_i, \gamma \in \Omega^{1,0}(End(E))$. For $\phi \in \Omega^{0,1}(End(E))$ and $\psi \in \Omega^{1,0}(End(E))$, define $(\phi, \psi) := \int_X tr(\phi \wedge \psi)L^{n-1}$, where the (1,1)-form $tr(\phi \wedge \psi)$ is defined using the trace map. Recall that if $\overline{\partial}_V + h\epsilon$ is the holomorphic structure on a family over $\mathbb{C}[\epsilon]/\epsilon^2$ then, h represents the element of $H^1(X, V)$ which corresponds to this infinitesimal deformation. The 1-form Θ on $Spec\, R$ is

$$(A_1, \bar{\theta})d\epsilon_1 + (A_2, \bar{\theta})d\epsilon_2 + (B_1, \bar{\theta})d(\epsilon_1^2) + (B_2, \bar{\theta})d(\epsilon_2^2) + (C, \bar{\theta})d(\epsilon_1\epsilon_2).$$

Taking exterior derivation we get

$$\begin{aligned} d\Theta = {} & (A_1, \alpha_2 + 2\beta_2\epsilon_2 + \gamma\epsilon_1)d\epsilon_2 \wedge d\epsilon_1 + (A_2, \alpha_1 + 2\beta_1\epsilon_1 + \gamma\epsilon_2)d\epsilon_1 \wedge d\epsilon_2 \\ & + (B_1, \alpha_2 + 2\beta_2\epsilon_2)2\epsilon_1 d\epsilon_2 \wedge d\epsilon_1 + (B_2, \alpha_1 + 2\beta_1\epsilon_1)2\epsilon_2 d\epsilon_1 \wedge d\epsilon_2 \\ & + (C, \alpha_2 + \gamma\epsilon_1)\epsilon_2 d\epsilon_2 \wedge d\epsilon_1 + (C, \alpha_1 + \gamma\epsilon_2)\epsilon_1 d\epsilon_1 \wedge d\epsilon_2. \end{aligned}$$

Hence $d\Theta(c) = [-(A_1, \alpha_2) + (A_2, \alpha_1)]d\epsilon_1 \wedge d\epsilon_2$. From the definition (1.4) it is easy to check that this is same as $\bar{P}(c)$. This completes the proof. □

Let M_H be a moduli of stable Higgs bundles of rank r. There are two canonically defined forms Θ and $\bar{P}$ on M_H, and from (2.6.3), $d\Theta = \bar{P}$.

We put the following condition on M_H: for some (hence any) $(E, \theta) \in M_H$, the 2-nd Chern class $c_2(End(E)) = 0$. Henceforth we will always assume this condition. Note that when X is a Riemann surface this condition is automatically satisfied.

Let $(E,\theta) \in M_H$. (E,θ) being stable admits a Hermitian-Einstein metric [Hi1], [S1]. It is easy to check that the induced metric on the Higgs pair $(End(E), ad(\theta))$ is also Kähler-Einstein. Hence $(End(E), ad(\theta))$ is polystable [Hi1], [S2], *i.e.* direct sum of stable Higgs bundles of same slope. Since $c_1(End(E) = 0$ and by assumption $c_2(End(E) = 0$, the Kähler-Einstein metric is harmonic (p.19 Thm.1, [S2]).

The *Dolbeault complex* for $(End(E), ad(\theta))$, denoted by $D^\cdot$ is defined to be the following complex

$$End(E) \xrightarrow{ad(\theta)} End(E) \otimes \Omega_X^1 \xrightarrow{ad(\theta)} End(E) \otimes \Omega_X^2 \xrightarrow{ad(\theta)} \ldots \xrightarrow{ad(\theta)} End(E) \otimes \Omega_X^n \longrightarrow 0.$$

Note that there is a natural projection $h : D^\cdot \longrightarrow C^\cdot$ which induces isomorphism on $\mathbb{H}^1$; since both $\mathbb{H}^1$ and $\mathbb{H}^2$ of $ker(h)$ vanishes.

We saw that $(End(E), ad(\theta))$ is a harmonic bundle, and hence Lemma 2.6 of [S2], which is a generalization of hard Lefschetz theorem applies, and hence the map $L^{n-1} : \mathbb{H}^1(D^\cdot) \longrightarrow \mathbb{H}^{2n-1}(D^\cdot)$, given by cupping with L^{n-1}, is an isomorphism. Since $End(E)^*$ is isomorphic to $End(E)$, the Lemma 2.5 of [S2] says that the natural pairing $\mathbb{H}^1(D^\cdot) \otimes \mathbb{H}^{n-1}(D^\cdot) \longrightarrow \mathbb{C}$ is perfect. So combining this pairing with the isomorphism given by cupping with L^{n-1}, we have a perfect pairing $\mathbb{H}^1(D^\cdot) \otimes \mathbb{H}^1(D^\cdot) \longrightarrow \mathbb{C}$. Using the isomorphism between $\mathbb{H}^1(C^\cdot)$ and $\mathbb{H}^1(C^\cdot)$ the 2-form induced in $\mathbb{H}^1(C^\cdot)$ coincides with $\bar{P}$ defined by (2.6.2). Hence we have the following

Theorem (2.6.4). *The 2-form $\bar{P}$ on $\mathbb{H}^1(C^\cdot)$ is nondegenerate.*

Theorem 1.2.1 implies that the Zariski open subset of M_H consisting of all Higgs bundles for which $\mathbb{H}^2(C^\cdot) = 0$, is smooth (see (2.1.8)).

Recall that a symplectic structure on a manifold is a closed nondegenerate 2-form. From (2.6.3), the 2-form $\bar{P}$ is exact. So we have the following corollary of (2.6.4)

Corollary (2.6.5). *$\bar{P}$ defines a symplectic structure on M'_H.*

Let $M := \{(E,\theta) \in M_H | \ \theta = 0\}$. In other words M is a moduli of stable bundles on X. For such a stable bundle E, the nodegenerate form on

$T(M_H)|_M$ gives an isomorphism between T_E^*M and $H^0(X, End(E) \otimes \Omega_X^1)$. If E is stable then for any $\theta \in H^0(X, End(E) \otimes \Omega_X^1)$, the pair (E, θ) is stable. Also stability is an open condition, hence there is an open map $T^*M \xrightarrow{i} M_H$. Total space of any cotangent bundle admits a natural 1-form, and it is easy to see that this form on T^*M is $i^*\Theta$. Also, on the total space of a cotangent bundle, exterior derivation of the canonical 1-form defines a symplectic structure. Hence we have the following corollary of Prop.2.6.3.

Corollary (2.6.6). *The canonical symplectic structure on T^*M is given by $\bar{P}$.*

§2.7 POISSON-COMMUTATIVITY OF HITCHIN MAP.

As before, M_H denotes a moduli of stable Higgs bundle of rank r. There is a natural map

(2.7.1)

$$H: \; M_H \longrightarrow \bigoplus_{i=1}^{r} H^0(X, S^i\Omega_X^1),$$

which is defined as follows: let $p_i : A \longmapsto tr(A^i)$, be the invariant polynomial of degree i on $M(r, \mathbb{C})$. For $(E, \theta) \in M_H$, the correspondence $(E, \theta) \longmapsto p_i(\theta)$, gives a map $M_H \longrightarrow H^0(X, S^i\Omega_X^1)$, where S^i denotes the i-th symmetric product. The above map H, known as *Hitchin map*, was introduced in [Hi1], the higher dimensional analogue appeared in [S3].

Recall that on a symplectic manifold (Y, ω), two complex functions h_1 and h_2 are said to Poisson-commute if $\omega(\bar{df_1}, \bar{df_2}) = 0$, where $\bar{df_i}$, $i = 1, 2$ is the vector field on Y corresponding to the 1-form df_i. When X is a curve, Hitchin proved that the function H Poisson-commutes, *i.e.* if f_1 and f_2 are two linear functional on $\oplus_{i=1}^r H^0(X, S^i\Omega_X^1)$, then $f_1 \circ H$ and $f_2 \circ H$ Poisson-commute. The following theorem is generalization of the above fact to higher dimension.

Theorem (2.7.2). *The function H on M'_H Poisson-commutes.*

Proof. Let $(E,\theta) \in M_H$. Note that to check Poisson-commutativity of H at (E,θ) only involves 1-st order neighborhood of (E,θ).

Let $(\alpha_i,\beta_i) \in \Omega^{0,1}(End(E)) \oplus \Omega^{1,0}(End(E))$, $i = \{1,....d\}$ be representatives of a basis of $\mathbb{H}^1(C^\cdot)$ in the Dolbeault resolution of $C^\cdot$ given in (1.5). After a rearrangement of indices, let $\{\alpha_i\}_{i=1}^k$ be a maximal linearly independent subset of the set of sections $\{\alpha_i\}_{i=1}^d$, and similarly $\{\beta_i\}_{i=1}^l$ is a maximal linearly independent subset of $\{\beta_i\}_{i=1}^d$.

It is easy to see that for any nonzero $s \in \Omega^{0,1}(End(E))$, there is a $t \in \Omega^{1,0}(End(E))$ such that $(s,t) \neq 0$ (recall the notation introduced in the proof of (1.5)). Conversely, for any $0 \neq t \in \Omega^{1,0}(End(E))$, there is a $s \in \Omega^{0,1}(End(E))$ such that $(s,t) \neq 0$. Using this it can be proved that the collection $\{\{\alpha\}_{i=1}^k, \{\beta\}_{i=1}^l\}$ can be increased to a collection $\{\{\alpha\}_{i=1}^m, \{\beta\}_{i=1}^m\}$, such that the $m \times m$ matrix $((\alpha_i,\beta_j))_{i,j}$ is nonsingular. First extend the $k \times l$ matrix $((\alpha_i,\beta_j))_{i,j}$ to some invertible $m \times m$ matrix A. Then the equation $((\alpha_i,\beta_j))_{i,j} = A$ can be solved (in fact space of solutions will be infinite dimensional). Let V be the formal vector space over $\mathbb{C}$ generated by the set $\{\alpha\}_{i=1}^m \cup \{\beta\}_{i=1}^m$. We will call the vectors corresponding to α_i and β_i by e_i and f_i respectively. Let ω be the symplectic form on V defined by the following conditions: $\omega(\alpha_i,\alpha_j) = 0 = \omega(\beta_i,\beta_j)$ and $\omega(\alpha_i,\beta_j) = (a_i,\beta_j)$.

Let R be the ring $\mathbb{C}[\epsilon_1,....,\epsilon_d]/I$, where I is the ideal generated by $\{\epsilon_i\epsilon_j\}$, $1 \leq i,j \leq d$. For the Higgs bundle (E,θ), let $\overline{\partial}_E$ be the dolbeault operator which defines the holomorphic structure on $End(E)$. Then the operator $\overline{\partial}_E + \sum_{i=1}^d \epsilon_i\alpha_i$ defines a holomorphic structure on the C^∞ bundle p_1^*E on $X \times Spec\, R$, this bundle is denoted by $\bar{E}$. $\bar{\theta} := \theta + \sum_{i=1}^d \epsilon_i\beta_i$ defines a Higgs structure on $\bar{E}$. So the infinitesimal deformation in the direction ϵ_i of this family is represented by (α_i,β_i).

Let R' be the ring $\mathbb{C}[e_1,...,e_m,f_1,...,f_m]/\mathrm{m}^2$, m being the maximal ideal of $\mathbb{C}[e_1,...,e_m,f_1,...,f_m]$. There is a natural inclusion $h : Spec\, R \longrightarrow Spec\, R'$ (recall that we enlarged a basis of $\{\alpha_i\}_{i=1}^d$ and a basis of $\{\beta_i\}_{i=1}^d$), defined by the condition $\epsilon_i \longmapsto e_i + f_i$. Let $c \in Spec\, R$ be the closed point. The symplectic form ω on V so defined that $h^*\omega(c)$ coincides with the symplectic form on $T_c Spec\, R$ defined by (1.4).

There is a function $H' : Spec\, R' \longrightarrow \oplus_{i=1}^{r} C^{\infty}(X, S^i\Omega_X^1)$ defined as in (2.7.1). Clearly the restriction of H to $Spec\, R$ is h^*H'. So in order to show that H Poisson-commutes at (E, θ), it is enough to show that H' Poisson-commutes at c', the closed point of $Spec\, R'$. But the function H' depends only on the f_i's, and the subspace of V spanned by $\{f_i\}_{i=i}^{m}$ is isotropic. *i.e.* the restriction of ω vanishes on this subspace. Hence H' Poisson-commutes at c'. This completes the proof. □

Remark (2.7.3) If X is a curve of genus g, then $dim\, M_H = 2n^2(g-1)+2$, and the fiber of H over a general point is a jacobian of a spectral curve. The genus of a spectral curve is $n^2(g-1)+1$, hence dimension count implies that H is maximal rank, in other words H defines a completely integrable structure on M_H. But for higher dimension such explicit numbers are not available.

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