ON SOME PROBLEMS RELATED TO HERMITE AND LAGUERRE EXPANSIONS

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A
Thesis submitted
to the Indian Statistical Institute in
partial fulfilment of the requirements for the award of
DOCTOR OF PHILOSOPHY.

BANGALORE
March 1996

To my parents

ACKNOWLEDGEMENTS

I take this opportunity to express my deep gratitude to my thesis supervisor Prof. S. Thangavelu for the constant encouragement he gave me throughout and also for the friendship and affection he has towards me. I thank him for bringing me to the exciting world of harmonic analysis and giving me a good training in hard analysis.

I wish to thank Prof. Alladi Sitaram for his inspiring lectures on Fourier analysis and harmonic analysis on Lie groups, which helped me to get a wider knowledge in the field. I also wish to thank professors V.S. Sunder, G. Misra, T.S.S.R.K. Rao and V. Pati for teaching me during the course work at I.S.I, and all the other members of the faculty for their encouragement and moral support. I also take this opportunity to thank my M.Sc teachers in Calicut university for their encouragement.

I am obliged to National Board for Higher Mathematics for the financial support and to the Indian Statistical Institute for providing me with excellent facilities for my research.

I wish to express my deep gratitude to Prof. Adimurthi for giving me an opportunity to visit the T.I.F.R. Bangalore centre, during July 1992 to August 1993. I wish to thank him for all the facilities given to me during my stay there.

It is pleasure to thank Dr. Ratikanta Panda, Dr. Nandakumaran, Dr. A. K. Vijayarajan and Ms. Rama Rawat for their encouragement and friendship. I thank my friend Mr. Mihir Das for his careful reading of the manuscript.

I would like to thank my parents, sister and brother-in-law for their constant encouragement and moral support without which this work would not have been possible.

CONTENTS

INTRODUCTION

1 An overview of the thesis

The first three chapters of this thesis are concerned with the spherical means associated to the Hermite and Laguerre expansions. The study of spherical means has a very long history. The classic work of F. John [9] deals with various applications of the spherical means to the theory of partial differential equations. They entered Fourier analysis with the celebrated theorem of E. Stein on spherical analogue of the Lebesgue differentiation theorem. Ever since they have appeared again and again in several areas of analysis like integral geometry, inversion of Fourier transforms and related areas.

For a locally integrable function f on $\mathbb{R}^n$ the spherical means are defined by

$$f * \mu_r(x) = \int\limits_{|y|=r} f(x-y)\, d\mu_r$$

where μ_r is the normalised surface measure on the sphere $|y| = r$. Associated to the spherical means is the spherical maximal function

$$Mf(x) = \sup_{r>0} |f * \mu_r(x)|.$$

In 1976, Stein [21] established that when $n \geq 3$ and $p > \frac{n}{n-1}$ the maximal operator M is bounded on $L^p(\mathbb{R}^n)$:

$$\|Mf\|_p \leq C\, \|f\|_p, \quad f \in L^p(\mathbb{R}^n).$$

From this follows the convergence of spherical means :

$$\lim_{r\to 0} \int_{|y|=r} f(x-y)\, d\mu_r = f(x) \quad a.e.$$

This is the spherical analogue of the Lebesgue differentiation theorem

$$\lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r} f(x-y)\, dy = f(x) \quad a.e.$$

where B_r is the ball of radius r centered at 0. The case $n = 2$ remained open for almost a decade and in 1987 Bourgain [4] showed that the above maximal inequality holds in that case as well.

The connection between the spherical means and the wave equation is explained by the following fact. When $n = 3$ the function

$$u(x,t) = ct\ (f * \mu_t(x))$$

with a suitable constant c satisfies the wave equation

$$\partial_t^2 u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t)$$

with initial conditions

$$u(x,0) = 0, \quad \partial_t u(x,0) = f(x).$$

When $n = 1$ we have the familiar solution

$$u(x,t) = \frac{1}{2}\int_{-t}^{t} f(x-s)ds$$

and the classical differentiation theorem, namely

$$\lim_{t\to 0} \frac{1}{2t}\int_{-t}^{t} f(x-s)ds = f(x) \ \ a.e.$$

can be generalised to yield

$$\lim_{t\to 0} \frac{u(x,t)}{t} = f(x) \ \ a.e.$$

whenever u is a solution of the wave equation with initial condition $f \in L^p_{loc}(\mathbb{R}^n)$, $p > \frac{2n}{n+1}$. Several authors have studied almost everywhere convergence of solutions to the L^p initial data for general hyperbolic differential equations, see for instance Sogge [21].

Various refinements of the regularity properties of the spherical means were obtained by several authors, see the works of Oberlin - Stein [15], Sjölin [19] and Peyriere - Sjölin [16].

Motivated by these works Colzani [6] studied spherical means on compact symmetric spaces using a scale of Sobolev spaces and proved a localisation theorem for spherical harmonic expansions. In a recent paper Pinsky [17] has used the regularity of spherical means to study pointwise inversion of Fourier integrals.

Our point of departure is the study of twisted spherical means on $\mathbb{C}^n$ which S.Thangavelu has initiated in [34]. Let μ_r stand for the normalised surface measure on the sphere $|w| = r$ in $\mathbb{C}^n$ and define the twisted spherical means

$$f \times \mu_r(z) = \int_{|w|=r} f(z-w)\, e^{\frac{i}{2} Im z.\bar{w}}\, d\mu_r.$$

It has been proved in [33] that $f \times \mu_r$ has the expansion

$$f \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r)\, f \times \varphi_k(z).$$

Here $\varphi_k(z)$ stands for the k th Laguerre function $L_k^{n-1}(\frac{1}{2}|z|^2)\, e^{-\frac{1}{4}|z|^2}$ and $f \times \varphi_k(z)$ denotes the twisted convolution $\int_{\mathbb{C}^n} f(z-w)\, \varphi_k(w)\, e^{\frac{i}{2} Im z \cdot \bar{w}}\, dw$ of f and φ_k. By measuring the regularity of these means in terms of certain Sobolev spaces a localisation theorem for special Hermite expansions was established. As the twisted spherical means are dominated by the ordinary spherical means on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ it follows that the maximal function

$$M_0 f(z) = \sup_{r>0} |f \times \mu_r(z)|$$

is bounded on $L^p(\mathbb{C}^n)$ for $p > \frac{2n}{2n-1}$.

We may extend this result in two possible directions. Using the local co-ordinates on the sphere $|w| = r$ in $\mathbb{C}^n$ it is easy to see that

$$\begin{aligned} f \times \mu_r(z) = \quad & C_n \int_0^{\pi} f\left((r^2 + |z|^2 + 2r|z|cos\theta)^{1/2}\right) \\ & \times \frac{J_{n-3/2}(\frac{1}{2} r|z|sin\theta)}{(\frac{1}{2} r|z|sin\theta)^{n-3/2}}\, sin^{2n-2}\theta\, d\theta. \end{aligned}$$

for a suitable constant C_n. This motivates us to define the Laguerre mean of order α of a function on $\mathbb{R}_+$ as

$$T_r{}^{\alpha} f(z) = \frac{2^{\alpha}\,\Gamma(\alpha+1)}{\sqrt{2\pi}} \int_0^{\pi} f\left((r^2+z^2+2\,r\,z\,cos\theta)^{1/2}\right) \times \frac{J_{\alpha-1/2}(\frac{1}{2}\,r\,z\,sin\theta)}{(\frac{1}{2}\,r\,z\,sin\theta)^{\alpha-1/2}}\,sin^{2\alpha}\theta\,d\theta.$$

Then $T_r{}^{\alpha}$ is a bounded self adjoint operator on $L^2(\mathbb{R}_+, x^{2\alpha+1}dx)$. For $\alpha > -1$ let $L_k^{\alpha}(t)$ be the k th Laguerre polynomial of type α. Let

$$\psi_k^{\alpha}(t) = \frac{\Gamma(k+1)\,\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}\,L_k^{\alpha}\left(\frac{1}{2}t^2\right)\,e^{-\frac{1}{4}t^2}.$$

We have the interesting (product) formula, see [36]

$$T_r{}^{\alpha}\,\psi_k^{\alpha}(z) = \psi_k^{\alpha}(r)\,\psi_k^{\alpha}(z),$$

for $\alpha > -\frac{1}{2}, r \geq 0,\; z \geq 0$.

The functions $\tilde{\psi}_k^{\alpha}(r) = \left(\frac{2^{-\alpha}\,\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} L_k^{\alpha}\left(\frac{1}{2}t^2\right)\,e^{-\frac{1}{4}t^2}$ form an orthonormal basis for $L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr\,)$. Therefore every function f in $L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr\,)$, has the Laguerre expansion

$$f(t) = \sum_{k=0}^{\infty} \langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}\,\tilde{\psi}_k^{\alpha}(t).$$

Here $\langle\,,\rangle_{\alpha}$ denotes the obvious innerproduct in $L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr\,)$,

In view of the above product formula it is easy to see that $T_r{}^{\alpha} f(z)$ has the series expansion

$$T_r{}^{\alpha} f(z) = \sum_{k=0}^{\infty} \langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}\,\psi_k^{\alpha}(r)\,\tilde{\psi}_k^{\alpha}(z),$$

for $r \geq 0, z \geq 0, \alpha > -1/2$.

When f is a radial funcion $f \times \varphi_k$ reduces to

$$f \times \varphi_k(z) = C_n \frac{k!(n-1)!}{(k+n-1)!}\left(\int_0^{\infty} f(s)\,\varphi_k(s)\,s^{2n-1}\,ds\right)\varphi_k(z)$$

where C_n is a constant. Therefore, $f \times \mu_r(z)$ takes the form

$$f \times \mu_r(z) = C_n \sum_{k=0}^{\infty} \left(\frac{k!\,(n-1)!}{(k+n-1)!} \right)^2 \int_0^{\infty} f(s)\, \varphi_k(s)\, s^{2n-1}\, ds\, \varphi_k(r)\, \varphi_k(z).$$

Thus for a radial function f on $\mathbb{C}^n$ the twisted spherical mean of f coincides with the Laguerre mean of order $n-1$, considered as a function on $\mathbb{R}_+$.

From the Laguerre expansion of f, using the orthogonality of ψ_k^α and the fact that $\sum_{k=0}^{N} L_k^\alpha = L_N^{\alpha+1}$ we see that the Nth partial sum for the Laguerre expansion can be written in the form

$$S_N f(t) = \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^{\infty} T_r^\alpha f(t)\, \varphi_N^{\alpha+1}(r)\, r^{2\alpha+1}\, dr,$$

where $\varphi_N^\alpha(r)$ denotes the Laguerre function $L_N^\alpha\left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2}$. For each $s \in \mathbb{R}$, $\alpha > -1$, let

$$|f|_{s,\alpha}^2 = \sum_{k=0}^{\infty} (2k+\alpha+1)^{2s} \langle f, \tilde{\psi}_k^\alpha \rangle_\alpha^2.$$

We define the Laguerre Sobolev space W_α^s associated with the Laguerre differential operator $Q_\alpha = -\left(\frac{d^2}{dr^2} + \frac{2\alpha+1}{r}\frac{d}{dr} - \frac{r^2}{4} \right)$ by

$$W_\alpha^s = \{ f \in L^2(\mathbb{R}_+, r^{2\alpha+1} dr) : |f|_{s,\alpha} < \infty \}.$$

By studying the regularity of $T_r^\alpha f(t)$ as a function of r using the above Sobolev space we prove a localisation theorem for Laguerre expansions.

Now let us consider the maximal function associated with this spherical mean. Define the maximal function T_*^α by

$$T_*^\alpha f(t) = \sup_{r>0} |T_r^\alpha f(t)|.$$

We will show that for $f \in L^p(\mathbb{R}_+, t^{2\alpha+1} dt), p > \frac{2\alpha+2}{2\alpha+1}$ the inequality

$$\int_0^{\infty} |T_*^\alpha f(t)|^p\, t^{2\alpha+1}\, dt \leq C \int_0^{\infty} |f(t)|^p\, t^{2\alpha+1} dt$$

holds. This is the analogue of Stein's theorem for the Laguerre means. When $\alpha = \frac{1}{2}, n = 3$ and f radial, the function

$$u(x,t) = c\, t\, T_t^{\frac{1}{2}} f(\sqrt{2}\, x)$$

solves the Darboux equation

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{t^2}{4}-\frac{3}{2}\right)u(x,t)=\frac{1}{2}(-\Delta+|x|^2-3)u(x,t)$$

with the initial conditions

$$u(x,0)=0, \qquad \partial_t u(x,0)=f(x).$$

From the maximal theorem for the Laguerre means we obtain

$$\lim_{t\to 0}\frac{u(x,t)}{t}=f(x)\ a.e.$$

for $f\in L^p(\mathbb{R}^3), p>\frac{3}{2}$. When $n=1$ and f even, the function $T_t^{-\frac{1}{2}}f(x)$ solves another Cauchy problem

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{t^2}{4}\right)v(x,t)=\left(-\frac{\partial^2}{\partial x^2}+\frac{x^2}{4}\right)v(x,t)$$

with initial conditions

$$v(x,0)=f(x), \qquad \partial_t v(x,0)=0.$$

However, we consider only the case $\alpha\geq 0$ for technical reasons.

The second extension we have in mind is the study of the maximal function associated to the Weyl transform of the measure μ_r. For an integrable function f on $\mathbb{C}^n$ the Weyl transform W associates an operator on $L^2(\mathbb{R}^n)$ defined by

$$W(f)=\int_{\mathbb{C}^n}f(z)\,\pi(z)\,dz.$$

Here $\pi(z)$ is the unitary operator on $L^2(\mathbb{R}^n)$ given by

$$\pi(z)\varphi(\xi)=e^{i(x.\xi+\frac{1}{2}x.y)}\varphi(\xi+y),$$

for $\varphi\in L^2(\mathbb{R}^n)$. In fact this is related to the unitary representation π_1 of the Heisenberg group. For details we refer to chapter 4. We can define the Weyl transform of the normalised surface measure μ_r on the sphere $|z|=r$ in $\mathbb{C}^n$ as

$$W(\mu_r)=\int_{|z|=r}\pi(z)\,d\mu_r(z).$$

It is a well known fact that

$$W(\mu_r)f(x) = \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!}\,\varphi_k(r)\,P_kf(x)$$

where P_kf are the projections of f onto the k th eigenspace of the Hermite operator $H = -\Delta + |x|^2$. Using a transference principle, originally due to Calderón, we will show that

$$\| \; sup_{r>0}|W(\mu_r)\,f\,(x)|\,\|_p \leq C\,\|f\|_p$$

holds for $f \in L^p(\mathbb{R}^n), p > \frac{2n}{2n-1}$. These operators $W(\mu_r)$ may be called spherical means for the Hermite expansions. As in the case of Laguerre means measuring the regularity of these operators in terms of Hermite - Sobolev spaces we can prove a localisation theorem for Hermite expansions.

More generally we can look at expansions of the form

$$S_t^\alpha f\,(x) = \sum_{k=0}^{\infty} \psi_k^\alpha(t)\,P_kf\,(x)$$

which may be called Hermite - Laguerre expansions. When $\alpha = \frac{1}{2}$ the functions $u(x,t) = t\,S_t^{\frac{1}{2}}f\,(x)$ solves the Darboux equation

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{t^2}{4} - \frac{3}{2}\right)u(x,t) = \left(-\Delta + |x|^2 - n\right)u(x,t)$$

with the initial conditions $u(x,0) = 0, \quad \partial_t u(x,0) = f(x)$. The solutions of the wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = -\left(-\Delta + |x|^2\right)u(x,t)$$

can be expressed in terms of $S_t^{\frac{1}{2}}f$ and hence we prove pointwise convergence of solutions of the wave equation to the initial condition f. The function $S_t^{-\frac{1}{2}}f$ also solves the Darboux equation with another initial condition.

When $t \to 0$ the functions $\psi_k^\alpha(t) \to 1$ and therefore it is reasonable to expect that $S_t^\alpha f \to f$ in the norm for f in $L^p(\mathbb{R}^n)$. The norm convergence of $S_t^\alpha f$ to f is equivalent to the uniform boundedness of S_t^α on $L^p(I\!R^n)$. For certain values of α we show that this is indeed the case. For other values of α we establish some $L^p - L^{p'}$ inequalities for these operators. As a consequence we obtain some interesting estimates for the Hermite projection operators.

In chapter 4 we study certain analogues of Besicovitch - Wiener theorem for the Fourier transform on the Heisenberg group. Consider the discrete measure μ given by

$$\mu = \sum_{j=0}^{\infty} c_j\, \delta_{a_j}$$

with $\sum_{j=0}^{\infty} c_j^2 < \infty$. The Fourier transform of such a measure is given by the almost periodic function

$$F(\xi) = \hat{\mu}(\xi) = \sum_{j=0}^{\infty} c_j\, e^{ia_j\cdot\xi}.$$

For such functions we have

$$lim_{r\to\infty} r^{-n} \int_{B_r(y)} |\hat{\mu}(\xi)|^2 d\xi = c \sum_{j=0}^{\infty} |c_j|^2$$

for any fixed y. The left hand side is the so called Bohr means of the almost periodic function F and it was first proved by H. Bohr for uniformly almost periodic functions and the above general form was proved by A. Besicovitch [3]. Wiener [37] considered finite measures of the form

$$\sum_{j=0}^{\infty} c_j\, \delta_{a_j} + \nu$$

where ν is absolutely continuous with respect to the Lebesgue measure. He proved that ν contributes nothing to the Bohr means of $\hat{\mu}$.

For the Fourier transform we also have the Plancherel theorem which can be written as

$$lim_{r\to\infty} \int_{B_r(y)} |\hat{f}(\xi)|^2\, d\xi = (2\pi)^n \int |f(x)|^2\, dx.$$

If we think of this as the Plancherel theorem for the n-dimensional measure fdx, then the Besicovitch - Wiener theorem can be considered as the 0 - dimensional analogue for the discrete measure μ. For the surface measure on $|x| = t$ there is a result due to Agmon - Hörmander which gives the $(n-1)$ - dimensional case of the above theme. Recently Strichartz [28] has made a far-reaching generalisation of the above theme by considering measures which are fractal in nature. In the general setup equalities have to be replaced by inequalities. Results of the type

$$limsup_{r\to\infty} r^{\alpha-n} \int_{B_r(y)} |(\widehat{fd\mu})(\xi)|^2 d\xi \leq C \int |f|^2 d\mu.$$

have been proved by Strichartz for a large class of α-dimensional measures.

Our point of departure is the Strichartz [27] spectral decomposition of an L^2 function on H^n in terms of eigenfunctions of the sublaplacian on H^n. This decomposition is written as,

$$f = (2\pi)^{-n-1} \sum_{k=0}^{\infty} f * e_k^\lambda(z,t)$$

where for each $\lambda > 0$, e_k^λ are the elementary spherical functions defined by

$$e_k^\lambda(z,t) = e^{-i\lambda t}\varphi_k^\lambda(z).$$

Then the Plancherel theorem reads

$$\|f\|_2^2 = 2\pi \sum_{k=0}^{\infty} \int_{\mathbb{C}^n} \int_{-\infty}^{\infty} |f * e_k^\lambda(z,0)|^2 \, d\lambda \, dz.$$

For a finite measure μ on H^n we can write down the spectral decomposition as

$$\mu = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \mu * e_k^\lambda(z,t)$$

where the above sum converges in the distribution sense. For discrete measures of the form $\mu = \sum c_j \, \delta(z_j, t_j)$ with $\{c_j\} \in l^1 \cap l^2$ we show that

$$lim_{N\to\infty} N^{-n} \sum_{k=0}^{N} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 dz = \frac{(2\pi)^n}{|\lambda|^n n!} \sum_{j=0}^{\infty} |c_j|^2.$$

This is analogous to the Besicovitch - Wiener theorem for the Fourier transform on the Euclidean space.

We also consider measures of the form $g\, d\mu_r$ where μ_r is the normalised surface measure on the sphere $S_r = \{(z,0) : |z| = r\} \subset H^n$ and $g \in L^2(I\!R)$. For this measure we prove

$$lim_{N\to\infty} N^{-\frac{1}{2}} \sum_{k=0}^{N} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu_r * e_k^\lambda(z,0)|^2 \, |\lambda|^{2n-\frac{1}{2}} \, d\lambda \, dz = \frac{\pi^{n-1}\, 2^{2n-\frac{1}{2}}\,(n-1)!}{r^{2n-1}} \|g\|_2^2.$$

This is analogous to the Agmon-Hörmander theorem. We can also replace the measure $g\,dt$ by the discrete measure $\sum_{j=0}^{\infty} c_j\, \delta(z_j, t_j)$ and get similar results.

In this spirit we can also prove similar results for the Hermite expansion. For measures of the form $\mu = \sum c_j\, \delta_{a_j} + \nu$, where ν is absolutely continuous with respect to the Lebesgue measure on $I\!R^n$, Strichartz has established a Besicovich - Wiener type result for this expansion. We consider measures of the form $f\, d\nu_1$ where ν_1 is the normalised surface measure on the sphere $|x| = 1$ in $I\!R^n$, and f a square integrable function on the sphere $|x| = 1$. For this type of measures we prove

$$lim_{N\to\infty} N^{-\frac{1}{2}} \sum_{k=0}^{\infty} \| P_k\,(f\, d\nu)\|_2^2 = \frac{2}{\pi} \int_{|x|=1} |f\,|^2 d\nu_1.$$

Similar results can be proved for special Hermite expansions also. By this term we mean an expansion of the type

$$f = (2\pi)^{-n} \sum f \times \varphi_k$$

where f is a function on $\mathbb{C}^n$. These results are in a way particular cases of results for Heisenberg group when we consider functions on the Heisenberg group that are independent of t.

2 Some open problems

We would like to conclude this introduction with some open problems which warrant further investigation. These problems may perhaps require new ideas for their complete solution.

(1) Theorem 2.1.3 concerning the almost everywhere convergence to initial data of solutions of certain Darboux's equation deals only with radial functions. It is interesting to know if the theorem remains true in the general case. Likewise, theorem 2.2.4 about solutions of the wave equation associated to the Hermite operator is proved only in the lower dimensions, $n \leq 3$. It is worthwhile to see what happens in the higher dimensional situation.

(2) In chapter 3 we have proved a localisation theorem for Hermite expansions, see corollary 3.3.4 under the assumption that $f \in W_H^{\frac{1}{2}}(I\!R^n)$. It can be shown some regularity of that sort is needed for the localisation to hold. However the optimal condition which ensures the localisation is not known.

(3) Concerning the Hermite - Laguerre expansions we have shown in chapter 3 that the operators $S_t^\alpha f$ are uniformly bounded on $L^p(I\!R^n)$, $1 < p < \infty$ provided $[\alpha] > \frac{n}{2}$. It is natural to ask which is the smallest value of α for which the uniform boundedness and hence norm convergence holds for all $1 < p < \infty$. We conjecture that this critical index α is $\frac{n-1}{2}$. Also the problem about almost everywhere convergence of S_t^α needs further investigation.

(4) In the last chapter of the thesis we have extended some results of Strichartz to the Heisenberg group setup but only treated the discrete and the surface measure. It is worthwhile to treat, as in the case of Euclidean Fourier transform, more general mearsures of fractal nature and obtain analogous results.

3 References

This thesis is based on the following three papers :

[1] Ratnakumar P.K, A localisation theorem for Laguerre expansions, *Proc. Ind. Acad. of Sci.,* 105 (1995), 303 - 314.

[2] Ratnakumar P.K and S. Thangavelu, Analogues of Besicovitch-Wiener theorem for the Heisenberg group. *Journal of Fourier Analysis and Applications,* to appear.

[3] Ratnakumar P.K and S. Thangavelu, Spherical means, Wave equations and Hermite - Laguerre expansions, preprint.

For the background material concerning Hermite and Laguerre expansions we refer to the monograph

[4] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical notes, 42, Princeton Univ. Press, Princeton. (1993).

We would like to mention that we closely follow the notation and terminology of the above reference [4]. For any undefined term in the thesis we refer the reader to the same monograph.

Chapter 1

SPHERICAL MEANS AND MAXIMAL FUNCTIONS

In this chapter we discuss the boundedness properties of the maximal spherical means associated with the Hermite and Laguerre expansions. To study the boundedness of the maximal Laguerre means we embed the Laguerre means T_r^α into an analytic family of operators and then apply Stein's analytic interpolation theorem. For this we introduce a g function associated with the Laguerre expansion. The spherical mean for the Hermite expasion is defined to be the Weyl transform of the normalised surface measure μ_t on the sphere $|z| = t$ in $\mathbb{C}^n$. We prove that when $p > \frac{2n}{2n-1}$ the maximal operator $sup_{0<t<1}|W(\mu_t)f|$ is bounded on $L^p(\mathbb{R}^n)$. This is done by using a transference principle due to A.P. Calderón. We define an action of the reduced Heisenberg group on $\mathbb{R}^n \times \mathbb{T}$ in such a way that the twisted convolution operator with the measure μ_t on $L^p(\mathbb{C}^n)$ is transfered to the operator $W(\mu_t)$ defined on $L^p(\mathbb{R}^n)$.

1.1 A maximal theorem for Laguerre means

Recall the definition of the Laguerre means $T_r^\alpha f$ of a function f on $\mathbb{R}_+$ given in the introduction

$$T_r^\alpha f(x) = \frac{2^\alpha \Gamma(\alpha+1)}{\sqrt{2\pi}} \int_0^\pi f((r,x)_\theta) \frac{J_{\alpha-\frac{1}{2}}(\frac{1}{2} rx sin\theta)}{(\frac{1}{2} rx sin\theta)^{\alpha-1/2}} sin^{2\alpha}\theta \, d\theta,$$

for $\alpha > -\frac{1}{2}$, where $x \in I\!R_+$ and $(r,x)_\theta = (r^2+x^2+2r\,x\,cos\theta)^{1/2}$. The Laguerre polynomials $L_k^\alpha(x)$, of type $\alpha > -1$ are defined by

$$\frac{1}{k!}\,\frac{d^k}{dx^k}(e^{-x}x^{\,k+\alpha}) = L_k^\alpha(x)e^{-x}x^\alpha.$$

We have seen in the introduction that $T_r{}^\alpha f$ has the series expansion

$$T_r{}^\alpha f(z) = \sum_{k=0}^{\infty}(f,\tilde{\psi}_k^\alpha)_\alpha\ \psi_k^\alpha(r)\,\tilde{\psi}_k^\alpha(z).$$

Using techniques similar to those in [24] we show that the maximal function

$$T_*{}^\alpha f\,(x) = sup_{r>0}|T_r{}^\alpha f\,(x)|$$

is bounded on $L^p(I\!R_+)$, if $p > \frac{2(\alpha+1)}{2\alpha+1}$.

The generalised Euclidean translation $\tau_r{}^\alpha f$ of f is defined by

$$\tau_r{}^\alpha f\,(x) = \frac{2^{-\alpha}\,\pi^{-1/2}}{\Gamma(\alpha+1/2)}\int_0^{\pi} f((r,x)_\theta)\,sin^{2\alpha}\theta\,d\theta.$$

Clearly $|T_r{}^\alpha f\,(x)| \le C\,\tau_r{}^\alpha\,|f\,|(x)$. Thus it is enough to prove L^p boundedness for the maximal operator

$$mf\,(x) = sup_{r>0}|\tau_r{}^\alpha f\,(x)|$$

for f on $I\!R_+$. The Hankel transform of a function on $I\!R_+$ is defined by

$$\mathcal{H}f\,(x) = \int_0^{\infty} f\,(y)\ \frac{J_\alpha(xy)}{(xy)^\alpha}\,y^{2\alpha+1}\,dy$$

where, for $Re\,\nu > -1$ and $x \in I\!R_+$, J_ν is the Bessel function given by the series

$$J_\nu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\,(x/2)^{2k+\nu}}{\Gamma(k+1)\,\Gamma(k+\nu+1)}.$$

Since the operator $\tau_r{}^\alpha$ is self adjoint on $L^2(I\!R_+, y^{2\alpha+1}\,dy)$ and satisfies

$$\tau_r{}^\alpha\left(\frac{J_\alpha(zy)}{(zy)^\alpha}\right) = \frac{J_\alpha(zr)}{(zr)^\alpha}\ \frac{J_\alpha(zy)}{(zy)^\alpha}$$

(here the translation is taken only in the y variable), we see that

$$\mathcal{H}(\tau_r{}^{\alpha} f)(x) = \mathcal{H}f(x)\,\frac{J_\alpha(rx)}{(rx)^{\alpha}}.$$

The generalised Euclidean convolution of two functions f and g on $\mathbb{R}_+$ is defined to be

$$f * g(x) = \int_0^{\infty} f(y)\,\tau_x{}^{\alpha} g(y)\,y^{2\alpha+1}\,dy.$$

Then we have

$$\mathcal{H}(f * g) = \mathcal{H}(f)\,\mathcal{H}(g). \qquad (1.1.1)$$

For $Re\,\beta > 0$, let N^{β} denote the function on $\mathbb{R}_+$ given by

$$N^{\beta}(y) = (1-y^2)^{\beta-1}\chi_{(0,1)}(y)$$

and let $N_r^{\beta}(y) = N^{\beta}(y/r)\,r^{-(2\alpha+2)}$, the r - dilation of $N^{\beta}(y)$. Let $\tau_r^{\alpha,\beta}$ be the operator defined by

$$\tau_r^{\alpha,\beta} f(x) = f * N_r^{\beta}(x). \qquad (1.1.2)$$

For $Re\,\beta > 0,\ \alpha > -1$, we have the formula

$$\frac{J_{\alpha+\beta}(w)}{w^{\alpha+\beta}} = \frac{2^{1-\beta}}{\Gamma(\beta)}\int_0^1 \frac{J_\alpha(s\,w)}{(s\,w)^{\alpha}}(1-s^2)^{\beta-1}s^{2\alpha+1}\,ds. \qquad (1.1.3)$$

Replacing w by rw and applying the change of variable $rs = t$ the above equation becomes

$$\frac{J_{\alpha+\beta}(r\,w)}{(r\,w)^{\alpha+\beta}} = \frac{2^{-\beta}}{\Gamma(\beta)}\mathcal{H}(N_r^{\beta})(w). \qquad (1.1.4)$$

Thus in view of (1.1.1) and (1.1.4) we see that

$$\mathcal{H}(\tau_r^{\alpha,\beta} f)(w) = 2^{\beta}\,\Gamma(\beta)\,\mathcal{H}f(w)\,\frac{J_{\alpha+\beta}(r\,w)}{(r\,w)^{\alpha+\beta}}$$

which converges to $\mathcal{H}(\tau_r{}^{\alpha} f)(w)$ as $\beta \to 0+$. From (1.1.3) we get

$$\frac{J_{\alpha+\beta}(r\,w)}{(r\,w)^{\alpha+\beta}}\ \mathcal{H}(f)(w)$$
$$= \frac{2^{1+\mu-\beta}}{\Gamma(\beta-\mu)}\int_0^1 \frac{J_{\alpha+\mu}(srw)}{(srw)^{\alpha+\mu}}\mathcal{H}f(w)\,(1-s^2)^{\beta-\mu-1}\,s^{2\alpha+2\mu+1}\,ds$$

for $Re\,\beta > \mu$, and $\alpha + \mu > -1$. Then using (1.1.1), (1.1.4) in the above equation and taking inverse Hankel transform on both sides, we get

$$\tau_r^{\alpha,\beta} f(w) \qquad (1.1.5)$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\mu)\Gamma(\mu)} \int_0^1 \tau_{sr}^{\alpha,\mu} f(w)\,(1-s^2)^{\beta-\mu-1}\, s^{2\alpha+2\mu+1}\, ds.$$

Therefore, by Cauchy - Schwarz

$$|\tau_r^{\alpha,\beta} f(w)| \le \frac{\Gamma(\beta)}{\Gamma(\beta-\mu)\Gamma(\mu)} C_{\beta,\mu} \left(\frac{1}{r}\int_0^r |\tau_s^{\alpha,\mu} f(w)|^2\, ds\right)^{1/2}, \qquad (1.1.6)$$

where

$$C_{\beta,\mu} = \left(\int_0^1 (1-s^2)^{2\beta-2\mu-2}\, s^{4\alpha+4\mu+2}\, ds\right)^{1/2}$$

which is finite whenever $Re\,\beta > \mu + 1/2$ and $\mu > -\alpha - 3/4$. Now we prove the following theorem, closely following the ideas in [24].

Theorem 1.1.1 *Let* $\tau^{\beta} f(x) = sup_{\,r>0} |\tau_r^{\alpha,\beta} f(x)|$. *Then* τ^{β} *is a bounded operator on* $L^2(I\!R_+, r^{2\alpha+1} dr)$ *whenever* $Re\,\beta > -\alpha$.

Proof : In view of the inequality (1.1.6), it is enough to prove that the operator

$$m^* f(x) = sup_{r>0} \left(\frac{1}{r}\int_0^r |\tau_s^{\alpha,\mu} f(x)|^2 ds\right)^{1/2}$$

is bounded on $L^2(I\!R_+, r^{2\alpha+1} dr)$. Now by triangle inequality

$$\left(\frac{1}{r}\int_0^r |\tau_s^{\alpha,\mu} f(x)|^2 ds\right)^{1/2} \le \left(\frac{1}{r}\int_0^r |\tau_s^{\alpha,\mu} f(x) - f * \varphi_s(x)|^2 ds\right)^{1/2} + \left(\frac{1}{r}\int_0^r |f * \varphi_s(x)|^2 ds\right)^{1/2}$$

where $\varphi \in C_0^\infty(I\!R_+)$ to be chosen later, and $\varphi_s(x) = s^{-(2\alpha+2)}\varphi(x/s)$.

Introduce the g function,

$$g(f,x) = \left(\int_0^\infty |\tau_s^{\alpha,\mu} f(x) - f * \varphi_s(x)|^2 \frac{ds}{s}\right)^{1/2}.$$

Then we see that

$$sup_{r>0}\left(\frac{1}{r}\int_0^r |\tau_s^{\alpha,\mu}f(x)|^2 ds\right)^{1/2} \leq g(f,x)$$
$$+\,sup_{r>0}\left(\frac{1}{r}\int_0^r |f*\varphi_s(x)|^2\, ds\right)^{1/2}.$$

The second term of the above inequality can be dominated by some constant times $sup_{s>0}|f*\varphi_s(x)|$. Stempak has studied such maximal operators in connection with almost everywhere convergence of Laguerre expansions. He has proved that these maximal operators are bounded on $L^p(I\!R_+, r^{2\alpha+1}dr)$, for $p>1$. For a proof of this we refer to [25]. Thus it follows that the second term defines a bounded operator on $L^2(I\!R_+, r^{2\alpha+1}dr)$.

We now proceed to prove that the g function is a bounded operator on $L^2(I\!R_+, r^{2\alpha+1}dr)$. By taking Hankel transform this is equivalent to proving that

$$\int_0^\infty |\frac{J_{\alpha+\mu}(s)}{s^{\alpha+\mu}} - \mathcal{H}(\varphi)(s)|^2\, \frac{ds}{s} \leq M. \qquad (1.1.7)$$

We have $\frac{J_{\alpha+\mu}(s)}{s^{\alpha+\mu}} = \frac{1}{\Gamma(\alpha+\mu+1)}$ when $s=0$. By choosing the function $\varphi \in C_0^\infty(I\!R_+)$ such that $\mathcal{H}(\varphi)(0) = \frac{1}{\Gamma(\alpha+\mu+1)}$ we can make $\frac{1}{s}\{\frac{J_{\alpha+\mu}(s)}{s^{\alpha+\mu}} - \mathcal{H}\varphi(s)\}$ bounded near the origin. For large s we have $|\frac{J_{\alpha+\mu}(s)}{s^{\alpha+\mu}}| \leq Cs^{-\alpha-\mu-1/2}$ and therefore

$$\int_1^\infty |\frac{J_{\alpha+\mu}(s)}{s^{\alpha+\mu}}|^2 \frac{ds}{s} < \infty$$

if $\mu > -\alpha - \frac{1}{2}$. Also $\int_1^\infty |\mathcal{H}(\varphi)(s)|^2 \frac{ds}{s} \leq M$. Therefore, it follows that

$$\int_0^\infty |\frac{J_{\alpha+\mu}(s)}{s^{\alpha+\mu}} - \mathcal{H}(\varphi)(s)|^2\, \frac{ds}{s} \leq M.$$

This proves the theorem. □

Theorem 1.1.2 *τ^β is a bounded operator on L^∞ if $Re\,\beta > 0$, and bounded on $L^p, p>1$ if $Re\,\beta \geq 1$.*

Proof : The proof follows from the fact that

$$\tau_r^{\alpha,\beta} f(x) = f * N_r^{\beta}(x)$$

and $sup_{r>0}|f * N_r^{\beta}(y)|$ defines a bounded operator on L^∞ if $Re\,\beta > 0$, and on $L^p\ \forall\, p > 1$ if $Re\,\beta \geq 1$. We refer to [25] or [36] for a proof of this fact. □

Now we prove the L^p boundedness of the maximal operator m.

Theorem 1.1.3 *Let* $\alpha > 0$.

(*i*) *For* $1 < p \leq 2$, *we have* $\|mf\|_p \leq C_{p,\alpha}\|f\|_p$ *whenever* $p > \frac{2(\alpha+1)}{2\alpha+1}$.

(*ii*) *For* $2 < p \leq \infty$, $\|mf\|_p \leq C_{p,\alpha}\|f\|_p$.

Proof : We prove the above theorem by using Stein's analytic interpolation theorem, see [23]. Let $r(x)$ be a nonnegative measurable function on $I\!R_+$ and $\epsilon > 0$. Consider the family of operators

$$G_\epsilon^z f(x) = \tau_{r(x)}^{\alpha,(\epsilon-(1+\alpha))z+1} f(x).$$

Using (1.1.5) it is easy to see that the family G_ϵ^z defines an analytic family of operators. Also using (1.1.5) one can see that G_ϵ^z defines an admissible family of operators in the sense of Stein.

By theorem 1.1.2 G_ϵ^{iy} is a bounded operator on $L^{p_1}(I\!R_+, r^{2\alpha+1}dr)$ for $p_1 > 1$. Also by theorem 1.1.1 G_ϵ^{1+iy} is a bounded operator on $L^2(I\!R_+, r^{2\alpha+1}dr)$. Therefore by analytic interpolation theorem we see that the operator G_ϵ^t is bounded on $L^{p_t}(I\!R_+, r^{2\alpha+1}dr)$ for $\frac{1}{p_t} = \frac{1-t}{p_1} + \frac{t}{2} < 1 - \frac{t}{2}$, since $p_1 > 1$. When $\alpha > \epsilon$, choosing $t = \frac{1}{1+\alpha-\epsilon}$ we see that the operator $G_\epsilon^{\frac{1}{1+\alpha-\epsilon}} = \tau_{r(x)}^{\alpha,0} = \tau_{r(x)}^{\alpha}$ is bounded on $L^p(I\!R_+, r^{2\alpha+1}dr)$ whenever $p > \frac{2(\alpha+1-\epsilon)}{(2\alpha+1-2\epsilon)}$. Letting $\epsilon \to 0$ we see that

$$\|\tau_{r(x)}^{\alpha} f\|_p \leq C\,\|f\|_p$$

for $p > \frac{2(\alpha+1)}{2\alpha+1}$. Since the right hand side of the above equation is independent of r, we see that

$$\| sup_{r>0} |\tau^{\alpha}_{r(x)} f| \|_p \leq C \|f\|_p.$$

This proves the first part of the theorem. Since m is bounded on $L^2(\mathbb{R}_+, r^{2\alpha+1}dr)$ and $L^\infty(\mathbb{R}_+, r^{2\alpha+1}dr)$ the proof of the second part follows from the Marcinkiewicz interpolation theorem. □

1.2 A maximal theorem for the Weyl transform

In this section we consider the maximal function associated to the Weyl transform of the measures μ_r. Recall that $W(\mu_r)f$ was defined by

$$W(\mu_r)f(\xi) = \int_{|z|=r} e^{i(x.\xi+\frac{1}{2}x.y)} f(\xi + y)\, d\mu_r(z) \tag{1.2.1}$$

where $z = x + iy$ and $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Defining

$$Mf(x) = \sup_{r>0} |W(\mu_r)f(x)|$$

we are interested in estimates of the form

$$\|Mf\|_p \leq C \|f\|_p$$

for the maximal operator. The maximal function

$$M_0 f(z) = \sup_{r>0} |f \times \mu_r(z)|$$

is bounded on $L^p(\mathbb{C}^n)$ for $p > \frac{2n}{2n-1}$, as observed in the introduction. We use a transference principle first observed by Calderón [5] to transfer the maximal theorem for M_0 to a maximal theorem for M on $L^p(\mathbb{R}^n)$.

The transference principle of Calderón deals with operators T on $L^p(\mathbb{R}^n)$ which are semi-local in the sense that there exists $R > 0$ such that whenever f is supported in a ball $|x| \leq r$, Tf will be supported in $|x| \leq R + r$. If $\mathbb{R}^n$ acts on a measure space X by measure-preserving transformations then T can be transferred to an operator T_0 acting on $L^p(X)$. Moreover, if T commutes with the above action then any $L^p(\mathbb{R}^n)$ inequality for T gives a similar inequality for T_0 on $L^p(X)$. An application of this principle for ergodic averages on spheres has been given in Jones [10].

In order to apply the transference principle we have to find a suitable group G acting on $\mathbb{R}^n$ such that under transference the operators $f \times \mu_r$ will go to $W(\mu_r)f$. This is achieved by considering the following action of the reduced Heisenberg group on $\mathbb{R}^n$.

Let $\mathbb{T}$ be the one dimensional torus identified with $[0, 2\pi)$ and let $G = \mathbb{C}^n \times \mathbb{T}$ be the reduced Heisenberg group with the group law given by

$$(z,t)(w,s) = \left(z + w,\ t + s + \frac{1}{2} Im\, z \cdot \overline{w}\right)$$

where $t + s + \frac{1}{2} Im\, z \cdot \overline{w}$ is taken mod 2π. This group defines an action on $\mathbb{R}^n \times \mathbb{T}$ in the following way : let $g = (z,t), z = x + iy$ and let $(\xi, s) \in \mathbb{R}^n \times \mathbb{T}$. Define

$$U(g)(\xi, s) = \left(\xi - y,\ s - t + \frac{1}{2} x \cdot y - x \cdot \xi\right).$$

It is easily verified that

$$U(g)U(g') = U(gg'), \quad U(0) = id$$

and $U(g)$ is measure preserving.

We use this action to transfer functions on $\mathbb{R}^n \times \mathbb{T}$ to functions on G. Suppose Φ is a function on $\mathbb{R}^n \times \mathbb{T}$ and $(\xi, s) \in \mathbb{R}^n \times \mathbb{T}$. We define

$$F_{(\xi,s)}(z,t) = \Phi(U(z,t)(\xi,s))$$

to be the transferred function on $G = \mathbb{C}^n \times \mathbb{T}$. If T is an operator acting on functions $F(z,t)$ we can define the transferred operator T_0 on functions $\Phi(\xi, s)$ by setting

$$T_0\Phi(\xi, s) = (TF_{(\xi,s)})(0).$$

Suppose now T is a convolution operator on G, say

$$TF(g) = K * F(g) = \int_G K(gh^{-1})\, F(h)\, dh.$$

If we further assume that $K(z,t) = k(z)\, e^{-it}$, $F(z,t) = f(z)\, e^{-it}$ then

$$\begin{aligned} TF(z,t) &= \int_G K\left(z-w, t-s-\frac{1}{2} Im z \cdot \overline{w}\right) f(w)\, e^{-is}\, dw\, ds \\ &= (2\pi)\, e^{-it}\, k \times f(z). \end{aligned}$$

Thus when K and F are of the special form above TF reduces to a twisted convolution operator. If Φ is of the form $\Phi(\xi, s) = \varphi(\xi)\, e^{is}$ then the transferred function becomes

$$F_{(\xi,s)}(z,t) = e^{is} e^{-it} f_\xi(z)$$

where the function $f_\xi(z)$ is given by

$$f_\xi(z) = \varphi(\xi - y) e^{-ix.\xi + \frac{i}{2} x.y}.$$

Therefore,

$$\begin{aligned} T_0\Phi(\xi, s) &= (TF_{(\xi,s)})(0) \\ &= 2\pi\, e^{is}\, k \times f_\xi(0) \end{aligned}$$

and we see that

$$\begin{aligned} k \times f_\xi(0) &= \int_{\mathbb{C}^n} k(-w)\, f_\xi(w)\, dw \\ &= \int_{\mathbb{C}^n} k(z)\, e^{i(x.\xi + \frac{1}{2} x.y)}\, \varphi(\xi + y)\, dx\, dy. \end{aligned}$$

If we recall the definition of the Weyl transform we get the relation

$$T_0\Phi(\xi, s) = (2\pi)\, e^{is}\, W(k)\, \varphi(\xi).$$

We now state and prove the following transference result. The proof is modelled after the proof of theorem 2.2 in Jones [10]. However, we give a complete proof for the sake of completeness.

Theorem 1.2.1 *Assume that k is a compactly supported distribution on $\mathbb{C}^n$ such that the operator $f \to k \times f$ is bounded on $L^p(\mathbb{C}^n)$. Then $W(k)$ is bounded on $L^p(\mathbb{R}^n)$.*

Proof : We first observe that for functions of the form $F(z,t) = e^{-it} f(z)$ the inequality

$$\|T\, F\|_p \leq C\, \|F\|_p$$

holds whenever $\|k \times f\|_p \leq C\, \|f\|_p$ holds.

Now consider the equations

$$\begin{aligned}\int_{\mathbb{R}^n} |W(k)\varphi(\xi)|^p d\xi &= \frac{1}{2\pi}\int_{\mathbb{R}^n}\int_0^{2\pi} |W(k)\varphi(\xi)e^{is}|^p\, ds\, d\xi \\ &= \frac{1}{(2\pi)^{p+1}}\int_{\mathbb{R}^n}\int_0^{2\pi} |T F_{(\xi,s)}(0)|^p\, ds\, d\xi \\ &= \frac{1}{(2\pi)^{p+1}}\int_{\mathbb{R}^n\times\mathbb{T}} |T F_a(0)|^p\, da\end{aligned}$$

where $a = (\xi, s)$. Since the measure $da = d\xi\, ds$ is invariant under the action of $U(g),\ g \in G$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |W(k)\varphi(\xi)|^p\, d\xi &= \frac{1}{(2\pi)^{p+1}}\int_{\mathbb{R}^n\times\mathbb{T}} |T F_{U(g)a}(0)|^p\, da \\ &= \frac{1}{(2\pi)^{p+2}}\frac{1}{\Omega_n R^{2n}}\int_{|z|\leq R}\int_{\mathbb{T}}\int_{\mathbb{R}^n\times\mathbb{T}} |T F_{U(g)a}(0)|^p\, da\, dg\end{aligned}$$

where $\Omega_n = \frac{\pi^n}{\Gamma(n+1)}$. Let us now calculate $T F_{U(g)a}(0)$. By definition

$$T F_{U(g)a}(0) = \int_G K(h^{-1})\, F_{U(g)a}(h)\, dh$$

$$\begin{aligned}&= \int_G K(h^{-1})\, \Phi(U(h)U(g)a)\, dh = \int_G K(h^{-1})\, \Phi(U(hg)a)\, dh \\ &= \int_G K(gh^{-1})\, \Phi(U(h)a)\, dh.\end{aligned}$$

In other words,

$$T F_{U(g)a}(0) = \int_G K(gh^{-1})\, F_a(h)\, dh.$$

Consequently,

$$\int_{\mathbf{R}^n} |W(k)\,\varphi(\xi)|^p\,d\xi$$
$$= C_n\,R^{-2n} \int_{B_R\times\Pi'} \int_{\mathbf{R}^n\times\Pi'} \left| \int_G K(gh^{-1}) F_a(h)\,dh \right|^p dg\,da$$

where B_R is the ball $|z| \leq R$.

If we assume that $k(w)$ is supported in $|w| \leq A$ then for $|z| \leq R$ what matters is the values of $F_a(h)$ for $|w| \leq R + A$, where $h = (w, s)$. Thus

$$\int_{B_R\times\Pi'} \int_{\mathbf{R}^n\times\Pi'} \left| \int_G K(gh^{-1})\,F_a(h)\,dh \right|^p dg\,da$$
$$= \int_{B_R\times\Pi'} \int_{\mathbf{R}^n\times\Pi'} \left| \int_G K(gh^{-1})\tilde{F}_a(h)dh \right|^p dg\,da$$

where $\tilde{F}_a(h) = F_a(h)$ for $|w| \leq R + A$ and 0 elsewhere. Since the operator T is bounded on functions of the form $F(z,t) = e^{-it} f(z)$ the above integral is dominated by

$$\int_{\mathbf{R}^n\times\Pi'} \left(\int_G |\tilde{F}_a(h)|^p\,dh \right) da \quad \leq C \int_{|w|\leq A+R} dw \int_{\mathbf{R}^n\times\Pi'} |F_a(h)|^p\,da$$
$$\leq C(A+R)^{2n} \int_{\mathbf{R}^n} |\varphi(\xi)|^p\,d\xi.$$

Finally, we have proved that

$$\int_{\mathbf{R}^n} |W(k)\varphi(\xi)|^p\,d\xi \leq C\left(\frac{A+R}{R}\right)^{2n} \int_{\mathbf{R}^n} |\varphi(\xi)|^p\,d\xi.$$

Letting $R \to \infty$ we conclude that

$$\int_{\mathbf{R}^n} |W(k)\varphi(\xi)|^p\,d\xi \leq C \int_{\mathbf{R}^n} |\varphi(\xi)|^p\,d\xi.$$

This completes the proof of the theorem. □

In the above proof we have only used the semi-local property of the operator T. Therefore, if we let

$$M_N f(x) = \sup_{0<r\leq N} |W(\mu_r) f(x)|$$

then M_N are semi-local and the above arguments will lead to

$$\|M_N f\|_p \leq C\,\|f\|_p\,, \qquad p > \frac{2n}{2n-1}$$

with C independent of N. This follows from the maximal theorem for $f \to f \times \mu_r$. Letting $N \to \infty$ we obtain the following result.

Theorem 1.2.2 *For $f \in L^p(\mathbb{R}^n), p > \frac{2n}{2n-1}$ we have*

$$\|Mf\|_p \leq C\,\|f\|_p.$$

Consequently, $W(\mu_r)f \to f$ a.e. as $r \to 0$.

We use this result in the following sections to study solutions of certain wave equations and also to study Hermite - Laguerre expansions. We conclude this section with the following remark. A general transference theorem for the Weyl transform has been proved by Mauceri [13]. In fact, he has shown that if k is any distribution on $\mathbb{C}^n$ such that $f \to k \times f$ is bounded on $L^p(\mathbb{C}^n)$, then $W(k)$ is bounded on $L^p(\mathbb{R}^n)$. Since we are interested in maximal functions associated to the Weyl transform we have to consider compactly supported kernels.

Chapter 2

SPHERICAL MEANS AND DIFFERENTIAL EQUATIONS

In this chapter we deal with some partial differential equations associated with Hermite and Laguerre differential operators. We consider some Cauchy problems with L^p initial data, namely Darboux equations and wave equations associated with the Hermite operator. We study the almost everywhere convergence of solutions to the initial data for these equations, using the maximal theorem for the spherical means established in the previous chapter. For the one dimensional wave equation for the Hermite operator we show that $\frac{u(x,t)}{t}$ converges almost everywhere to the initial data $f \in L^p \cap L^1$, for $p > 2$. This is done by expressing the solutions in terms of the spherical mean $W(\mu_t)f$. In higher dimensions we prove an almost everywhere convergence of some Riesz means of the solutions to the initial data. In the case of Darboux's equation we show that the solution converges almost everywhere to the initial value as t tend to 0 for $p > \frac{2n}{2n-1}$ for the L^P initial data provided $H^{n-\frac{1}{2}}f$ is also in $L^p(\mathbb{R}^n)$.

2.1 The equation of Darboux and the Laguerre means

If μ_t is the normalised surface measure on the sphere $|x| = t$ in $\mathbb{R}^n$ then it is well known that the spherical mean $f * \mu_t(x)$ satisfies the Darboux equation:

$$\left(\partial_t^2 + \frac{n-1}{t}\,\partial_t\right) u(x,t) = \Delta u(x,t).$$

Observe that the differential operator on the left hand side is the radial part of the Laplacian Δ on $\mathbb{R}^n$. In a similar way the twisted spherical means $f \times \mu_t(z)$ satisfies

$$\left(\partial_t^2 + \frac{2n-1}{t}\,\partial_t - \frac{t^2}{4}\right) u(z,t) = -L\,u(z,t)$$

where L is the operator

$$L = -\Delta_z + \frac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\,\frac{\partial}{\partial y_j} - y_j\,\frac{\partial}{\partial x_j}\right)$$

and the Weyl transform $W(\mu_t)f$ satisfies

$$\left(-\partial_t^2 - \frac{2n-1}{t}\partial_t + \frac{t^2}{4}\right) v(x,t) = H\,v(x,t)$$

where $H = -\Delta + |x|^2$ is the Hermite operator.

Recall that $W(\mu_t)f$ is given by

$$W(\mu_t)f(x) = \sum_{k=0}^{\infty}\frac{k!\,(n-1)!}{(k+n-1)!}\,\varphi_k(t)\,P_kf(x).$$

Generalising this, we consider

$$S_t^{\alpha} f(x) = \sum_{k=0}^{\infty}\psi_k^{\alpha}(t)\,P_kf(x)$$

where,

$$\psi_k^{\alpha}(t) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}\,L_k^{\alpha}\left(\frac{1}{2}t^2\right)\,e^{-\frac{1}{4}t^2}.$$

Here P_k is the Hermite projection operator given by

$$P_kf(x) = \sum_{|\beta|=k}\langle f,\Phi_\beta\rangle\,\Phi_\beta,$$

where Φ_β are the n - dimensional Hermite functions, (for definition and properties see chap 4,sec 3). The Laguerre functions ψ_k^{α} satisfy

$$\left(-\partial_t^2 - \frac{2\alpha+1}{t}\,\partial_t + \frac{1}{4}t^2\right)\psi_k^{\alpha} = (2\,k+\alpha+1)\,\psi_k^{\alpha}$$

and consequently, $v_\alpha(x,t) = S_t^\alpha f(x)$ satisfy

$$\left(-\partial_t^2 - \frac{2\alpha+1}{t}\partial_t + \frac{1}{4}t^2\right) v_\alpha = (H+\alpha+1-n)\, v_\alpha.$$

These equations may be called Darboux equations for the Hermite operator. In this section we are particularly interested in the cases $\alpha = \pm\frac{1}{2}$. Using the results of the previous sections we study solutions of these equations.

When $\alpha = -\frac{1}{2}$ the function $v_{-\frac{1}{2}}(x,t) = S_t^{-\frac{1}{2}} f(x)$ satisfies the equation

$$\left(-\partial_t^2 + \frac{1}{4}t^2 - \frac{1}{2}\right) v_{-\frac{1}{2}}(x,t) = (H-n)\, v_{-\frac{1}{2}}(x,t) \tag{2.1.1}$$

with the initial conditions

$$v_{-\frac{1}{2}}(x,0) = f(x), \qquad \partial_t v_{-\frac{1}{2}}(x,0) = 0. \tag{2.1.2}$$

The first condition is verified as $\psi_k^{-\frac{1}{2}}(0) = 1$ for all k. To verify the second condition observe that the function $\psi_k^{-\frac{1}{2}}(t)$ is expressible in terms of the Hermite polynomial $H_{2k}(t)$ as

$$\psi_k^{-\frac{1}{2}}(t) = \frac{(-1)^k 2^{-2k}\sqrt{\pi}}{\Gamma(k+\frac{1}{2})} H_{2k}\left(\frac{t}{\sqrt{2}}\right) e^{-\frac{1}{4}t^2}.$$

Now since the vector field $(-\partial_t + \frac{1}{2}t)$ takes H_{2k} into H_{2k+1} and since $H_{2k+1}(0) = 0$ the second condition is also verified.

We now prove the following maximal estimate concerning solutions of the above Darboux equation. We define

$$u^*(x) = \sup_{0<t\leq 1} |u(x,t)|$$

whenever u is a function on $\mathbb{R}^n \times \mathbb{R}^+$.

Theorem 2.1.1 *Let u be a tempered solution of* (2.1.1) - (2.1.2) *with $f \in L^p(\mathbb{R}^n)$, $p > \frac{2n}{2n-1}$. Then the following maximal estimate holds :*

$$\|u^*\|_p \leq C \|H^{n-\frac{1}{2}} f\|_p$$

where $H^{n-\frac{1}{2}}f$ is defined using spectral theorem.

Proof : We express $\psi_k^{-\frac{1}{2}}(t)$ in terms of $\psi_k^{n-1}(t)$ using the following formula, see [2]. For $-1 < \beta < \alpha$ one has

$$e^{-t} L_k^{\beta}(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_t^{\infty} (s-t)^{\alpha-\beta-1} e^{-s} L_k^{\alpha}(s)\, ds.$$

From this we get

$$e^{-\frac{t^2}{2}} L_k^{-\frac{1}{2}} \left(\frac{1}{2}t^2\right) = \frac{2^{-n+\frac{3}{2}}}{\Gamma(n-\frac{1}{2})} \int_t^{\infty} s(s^2-t^2)^{n-\frac{3}{2}} L_k^{n-1} \left(\frac{1}{2}s^2\right) e^{-\frac{s^2}{2}}\, ds.$$

Consequently,

$$\psi_k^{-\frac{1}{2}}(t) = C_n \frac{\Gamma(k+n)}{\Gamma(k+\frac{1}{2})} \int_t^{\infty} s\,(s^2-t^2)^{n-\frac{3}{2}} e^{-\frac{1}{4}(s^2-t^2)} \psi_k^{n-1}(s)\, ds$$

where C_n is a constant. If we let T_n to be the operator

$$T_n f\,(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k+n)}{\Gamma(k+\frac{1}{2})} P_k f\,(x)$$

then we have the relation

$$\sum_{k=0}^{\infty} \psi_k^{-\frac{1}{2}}(t)\, P_k f\,(x) = C_n \int_t^{\infty} s\,(s^2-t^2)^{n-\frac{3}{2}} e^{-\frac{1}{4}(s^2-t^2)}\, W(\mu_s)\, T_n f\,(x)\, ds.$$

Now any tempered solution of the equation under consideration is given by

$$u(x,t) = \sum_{k=0}^{\infty} \psi_k^{-\frac{1}{2}}(t)\, P_k f\,(x)$$

and so from the above relation we obtain

$$u^*(x) \le C \sup_{t>0} |W(\mu_t)\, T_n f\,(x)|.$$

From this and the maximal theorem for the Weyl transform $W(\mu_t)$ we get for $p > \frac{2n}{2n-1}$

$$\|u^*\|_p \le C\, \|T_n f\,\|_p.$$

Finally, it can be checked that the sequence

$$m(k) = \frac{\Gamma(k+n)}{\Gamma(k+\frac{1}{2})}\,(2\,k+n)^{-n+\frac{1}{2}}$$

verifies the conditions of the Marcinkiewicz multiplier theorem for the Hermite expansions (see chap 3, sec 4) and consequently

$$\|T_n f\|_p \leq C\,\|H^{n-\frac{1}{2}} f\|_p, \quad 1 < p < \infty.$$

This completes the proof of the theorem. □

Corollary 2.1.2 *If $f \in L^p(\mathbb{R}^n)$, $H^{n-\frac{1}{2}} f \in L^p(\mathbb{R}^n)$ with $p > \frac{2n}{2n-1}$ then $u(x,t) \to f(x)$ a.e. as $t \to 0$.*

Next we consider the other Cauchy problem

$$\left(-\partial_t^2 + \frac{t^2}{4} - \frac{3}{2}\right)\, v(x,t) = \frac{1}{2}\,(H-n)\,v(x,t) \tag{2.1.3}$$

with the initial conditions

$$v(x,0) = 0, \qquad \partial_t v(x,0) = f(x). \tag{2.1.4}$$

If v is a tempered solution of this problem then it is given by

$$v(x,t) = t\sum_{k=0}^{\infty} \psi_k^{\frac{1}{2}}(t) P_{2k} f(x). \tag{2.1.5}$$

To see this we only have to verify :

$$\left(-\partial_t^2 + \frac{t^2}{4} - \frac{3}{2}\right)\, (t\,\psi_k^{\frac{1}{2}}(t)) = 2\,k\,t\,\psi_k^{\frac{1}{2}}(t) \tag{2.1.6}$$

$$\lim_{t\to 0} t\,\psi_k^{\frac{1}{2}}(t) = 0 \tag{2.1.7}$$

$$\lim_{t\to 0} \left(-\partial_t + \frac{1}{2}t\right)\, t\,\psi_k^{\frac{1}{2}}(t) = -1. \tag{2.1.8}$$

That (2.1.6) and (2.1.7) are true follows from

$$t\,L_k^{\frac{1}{2}}\left(\frac{1}{2}t^2\right)\,e^{-\frac{1}{4}t^2}=\frac{(-1)^k\,2^{-2k}}{\sqrt{2}\,\Gamma(k+1)}\,H_{2k+1}\left(\frac{t}{\sqrt{2}}\right)\,e^{-\frac{t^2}{4}}.$$

Applying $(-\partial_t+\frac{1}{2}t)$ to $H_{2k+1}(\frac{t}{\sqrt{2}})\,e^{-\frac{t^2}{4}}$ we get

$$(-\partial_t+\frac{1}{2}t)\left(H_{2k+1}\left(\frac{t}{\sqrt{2}}\right)\,e^{-\frac{t^2}{4}}\right)=\frac{1}{\sqrt{2}}\,H_{2k+2}\left(\frac{t}{\sqrt{2}}\right)\,e^{-\frac{t^2}{4}}.$$

Using the explicit values for $H_{2k+2}(0)$ given in Szegö [29] we can check that (2.1.8) is true.

Theorem 2.1.3 *Let* $n=3$ *and* $f\in L^p(R^n)$ *be radial. If* v *is the solution of* (2.1.3) - (2.1.4) *given by* (2.1.5) *then*

$$\lim_{t\to 0}\frac{v(x,t)}{t}=f(x)\quad a.e.$$

provided $p>\frac{3}{2}$.

Proof : We will show that when f is radial and $p>\frac{3}{2}$ we have the maximal estimate

$$\|\sup_{t>0}|\frac{v(x,t)}{t}|\|_p\leq C\|f\|_p.$$

To prove this we observe that when f is radial $P_{2k+1}f=0$ and

$$P_{2k}f(x)=\frac{2\,\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}\left(\int_0^\infty f(s)\,L_k^{\frac{1}{2}}(s^2)\,e^{-\frac{1}{2}s^2}s^2\,ds\right)L_k^{\frac{1}{2}}(|x|^2)\,e^{-\frac{1}{2}|x|^2}.$$

For a proof of this we refer to theorem 3.4.1 of [36]. Thus $v(x,t)$ is expressed in terms of Laguerre means of order $\frac{1}{2}$:

$$v(x,t)=C\,t\,T_t^{\frac{1}{2}}f(\sqrt{2}|x|).$$

By appealing to the maximal theorem for Laguerre means we complete the proof. □

It would be interesting to see if the above theorem remains true for all functions, not necessarily radial. We conclude this section with the following result showing a finite propagation speed in the three dimensional case.

Theorem 2.1.4 *Let $n = 3$ and let v be the solution given by* (2.1.5). *If f is radial and supported in $|x| \leq r$, then $v(x,t)$ is supported in $|x| \leq \frac{1}{\sqrt{2}}(r+t)$.*

Proof : In the proof of the previous theorem we observed that when f is a radial function the solution $v(x,t)$ given by (2.1.5) can be written in terms of the Laguerre means :

$$v(x,t) = C\,t\,T_t^{\frac{1}{2}} f(\sqrt{2}|x|).$$

From the integral representation of the Laguerre means it follows that when f is supported $|x| \leq r$, $T_t^{\frac{1}{2}} f(x)$ is supported in $|x| \leq r+t$ (see lemma 3.2.2). This proves the theorem.□

2.2 Wave equation for the Hermite operator

In this section we study the pointwise convergence of solutions of the wave equation

$$\partial_t^2 u(x,t) = -(-\Delta + |x|^2)u(x,t)$$

to the initial data. We use the maximal theorem for the Weyl transform of μ_t which we established in chapter 1 in order to prove almost everywhere convergence of u. Actually certain Riesz means of solutions u can be expressed in terms of $W(\mu_t)$ and this enables us to study the pointwise convergence.

First we consider the one dimensional case. In this case it is convenient to consider the operator $(H + \frac{1}{2})$ rather than H. So, we look at solutions of the Cauchy problem

$$\partial_t^2 u(x,t) = -\left(H + \frac{1}{2}\right) u(x,t)$$

with initial conditions

$$u(x,0) = 0, \qquad \partial_t u(x,0) = f(x).$$

Formally, the solution of this problem is given by

$$u(x,t) = \left(H + \frac{1}{2}\right)^{-\frac{1}{2}} sin\left(t\left(H + \frac{1}{2}\right)^{\frac{1}{2}}\right) f(x)$$

where $(H+\frac{1}{2})^{-\frac{1}{2}}$ and $\sin\left(t(H+\frac{1}{2})^{\frac{1}{2}}\right)$ are defined using spectral theorem. Thus the solution $u(x,t)$ has the expansion

$$u(x,t) = 2^{-\frac{1}{2}} \sum_{k=0}^{\infty} (4k+3)^{-\frac{1}{2}} \sin\left(\left(2k+\frac{3}{2}\right)^{1/2} t\right) P_k f(x).$$

We establish the following theorem concerning the almost everywhere convergence of $u(x,t)$.

Theorem 2.2.1 *Let u be the solution of the above Cauchy problem. Then*

$$\lim_{t\to 0} \frac{u(x,t)}{t} = f(x)$$

for a.e. $x \in I\!R$ *provided* $f \in L^p \cap L^1(I\!R), p > 2$.

In the proof of this theorem (as well as in the proofs of other results in this section) a crucial role is played by the Hilb type asymptotic formulas for the Laguerre functions (see Szegö [29]):

$$L_k^{\alpha}(t)\, e^{-t}\, t^{\alpha/2} = K^{-\frac{\alpha}{2}} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} J_{\alpha}(2\sqrt{Kt}) + O(k^{\frac{\alpha}{2}-\frac{3}{4}}).$$

Here $K = k + \frac{\alpha+1}{2}$ and the error term can be replaced by

$$t^{\frac{5}{4}}\, O(k^{\frac{\alpha}{2}-\frac{3}{4}}), \quad k^{-1} \le t \le 1.$$

Taking $\alpha = \frac{1}{2}$ in the above formula we can write

$$\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} L_k^{\frac{1}{2}}\left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2} = (\sqrt{K}\,t)^{-\frac{1}{2}} J_{\frac{1}{2}}(\sqrt{K}t) - m_k(t)$$

where $K = 2k + \frac{3}{2}$ and the error term $m_k(t)$ satisfies

$$|m_k(t)| \le C\, k^{-\frac{3}{4}}, \quad 0 \le t \le 1$$

uniformly in t. Since

$$\frac{\sin s}{s} = s^{-\frac{1}{2}} J_{\frac{1}{2}}(s)$$

the solution u is expressed as

$$\frac{u(x,t)}{t} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} L_k^{\frac{1}{2}}\left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2} P_k f(x) + \sum_{k=0}^{\infty} m_k(t) P_k f(x).$$

Let us write this as

$$\frac{u(x,t)}{t} = S_t^{\frac{1}{2}} f(x) + m_t f(x).$$

We first prove the following equiconvergence result.

Proposition 2.2.2 *For f in $L^1(\mathbb{R})$ we have*

$$\lim_{t\to 0}\left(\frac{u(x,t)}{t} - S_t^{\frac{1}{2}} f(x)\right) = 0.$$

Proof : For $f = \sum_{k=0}^{N} P_k f$ it is clear that both $\frac{u(x,t)}{t}$ and $S_t^{\frac{1}{2}} f(x)$ converge to $f(x)$ pointwise as $t \to 0$. Therefore, it is enough to prove the inequality

$$\sup_{t>0} |m_t f(x)| \le C \int |f(x)|\, dx.$$

The operator m_t is given by the kernel

$$m_t(x,y) = \sum_{k=0}^{\infty} m_k(t)\, h_k(x)\, h_k(y)$$

where h_k are normalised Hermite functions on $\mathbb{R}$. Using the estimates for $m_k(t)$ and applying Cauchy - Schwarz we get

$$(m_t(x,y))^2 \le C\, B(x)\, B(y),$$

where we have written

$$B(x) = \sum_{k=0}^{\infty} (2k+1)^{-\frac{3}{4}} (h_k(x))^2.$$

Now we use the generating function

$$\sum_{k=0}^{\infty} r^k\, h_k(x)\, h_k(y) = \pi^{-\frac{1}{2}}(1-r^2)^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(x^2+y^2) + \frac{2r}{1-r^2}xy}$$

to estimate $B(x)$. First we have

$$\sum_{k=0}^{\infty} e^{-(2k+1)t} (h_k(x))^2 = \pi^{-\frac{1}{2}} (sinh2t)^{-\frac{1}{2}} e^{-x^2.tanh2t}$$

and since

$$(2k+1)^{-\frac{3}{4}} = \frac{1}{\Gamma(\frac{3}{4})} \int_0^{\infty} e^{-(2k+1)t} t^{-\frac{1}{4}} dt$$

we have the expression

$$B(x) = \frac{1}{\Gamma(\frac{3}{4})} \pi^{-\frac{1}{2}} \int_0^{\infty} t^{-\frac{1}{4}} (\sinh 2t)^{-\frac{1}{2}} e^{-x^2 tanh\, 2t} dt.$$

From this we obtain the estimates $B(x) \leq C$ and for $x^2 \geq 1$

$$B(x) \leq C \int_0^{\infty} t^{\frac{1}{4}-1} e^{-4tx^2} dt = C\, x^{-\frac{1}{2}}.$$

In getting these estimates we have used the facts that $\sinh t \sim t, \tanh t \sim t$ as $t \to 0$ and $tanh\, t = O(1)$ as $t \to \infty$. Therefore, we have the inequality

$$|m_t f(x)| \leq C(1+x^2)^{-\frac{1}{8}} \int |f(y)| (1+y^2)^{-\frac{1}{8}} dy$$

and this proves the proposition. □

Next we prove the following result concerning the maximal function

$$S_*^{\frac{1}{2}} f(x) = \sup_{0<t\leq 1} |S_t^{\frac{1}{2}} f(x)|.$$

Proposition 2.2.3 *Let $f \in L^p(\mathbb{R}), p > 2$. Then*

$$\|S_*^{\frac{1}{2}} f\|_p \leq C \|f\|_p$$

and consequently $S_t^{\frac{1}{2}} f(x) \to f(x)$ a.e. as $t \to 0$.

Proof : The idea of the proof is to express $S_t^{\frac{1}{2}}f$ in terms of $S_t^0 f$. To this end we make use of the formula

$$\frac{\Gamma(k+1)\Gamma(\mu+\nu+1)}{\Gamma(k+\mu+\nu+1)} L_k^{\mu+\nu}(t) = \frac{\Gamma(k+1)\Gamma(\mu+\nu+1)}{\Gamma(\nu)\Gamma(k+\mu+1)} \times \int_0^1 s^{\mu}(1-s)^{\nu-1} L_k^{\mu}(st)\, ds \qquad (2.2.1)$$

which is valid for $\mu > -1, \nu > 0$. Taking $\mu = 0$ and $\nu = \frac{1}{2}$ we get

$$\frac{\Gamma(k+1)\Gamma(\frac{1}{2}+1)}{\Gamma(k+\frac{1}{2}+1)} L_k^{\frac{1}{2}}\left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2}$$

$$= \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} \int_0^1 (1-s)^{-\frac{1}{2}} e^{-\frac{t^2}{4}(1-s)} L_k^0\left(\frac{1}{2}t^2 s\right) e^{-\frac{1}{4}t^2 s}\, ds.$$

From this formula it is clear that

$$S_t^{\frac{1}{2}}f(x) = \frac{1}{2}\int_0^1 (1-s)^{-\frac{1}{2}} e^{-\frac{t^2}{4}(1-s)} S_{st^2}^0 f(x)\, ds$$

which immediately gives

$$|S_*^{\frac{1}{2}}f(x)| \le C \sup_{0<t\le 1} |S_t^0 f(x)|.$$

Since $S_t^0 = W(\mu_t)f$, the proposition follows from the theorem 1.2.2. □

Combining propositions 2.2.2 and 2.2.3 we immediately obtain theorem 2.2.1.

Next we consider the following Cauchy problem for the n-dimensional Hermite operator $H = -\Delta + |x|^2$:

$$\partial_t^2 v(x,t) = -Hv(x,t),$$

$$v(x,0) = f(x), \quad \partial_t v(x,0) = 0.$$

Formally, the solution is given by

$$v(x,t) = (\cos\left(tH^{\frac{1}{2}}\right))f(x).$$

In this case we are unable to prove the pointwise convergence of the solution to the initial data. This situation is similar to the case of solutions of the Schrödinger equation on $\mathbb{R}^n$. Motivated by the works [20] and [31] we now consider what may be called the Riesz means of the solution v. These are defined by

$$v^\alpha(x,t) = \int_0^t \left(1 - \frac{s^2}{t^2}\right)^{\alpha - \frac{1}{2}} v(x,s)ds.$$

In small dimensions we are able to prove the following theorem about pointwise convergence of the Riesz means.

Theorem 2.2.4 *Let $n = 1, 2$ or 3. Then we have*

$$\lim_{t \to 0} \frac{v^{n-1}(x,t)}{t} = f(x) \quad a.e.\ x \in \mathbb{R}^n$$

provided $f \in L^p \cap L^1(\mathbb{R}^n)$, $p > \frac{2n}{2n-1}$.

The proof of this theorem is similar to that of the previous theorem. We express $v^{n-1}(x,t)$ in terms of $J_{n-1}(tH^{\frac{1}{2}})f$ which is then compared with $W(\mu_t)f$. To start with we have the formula for the Bessel function

$$z^{-\alpha} J_\alpha(z) = \frac{2^{-\alpha}}{\Gamma(\alpha + \frac{1}{2})\,\Gamma(\frac{1}{2})} \int_{-1}^1 (1 - s^2)^{\alpha - \frac{1}{2}} e^{isz}\, ds.$$

This gives us the relation

$$(tH^{\frac{1}{2}})^{-n+1} J_{n-1}(tH^{\frac{1}{2}}) = \frac{2^{2-n}}{\Gamma(n - \frac{1}{2})\,\Gamma(\frac{1}{2})} \int_0^1 (1 - s^2)^{n - \frac{3}{2}} (\cos tsH^{\frac{1}{2}})\, ds.$$

In other words

$$\frac{v^{n-1}(x,t)}{t} = \frac{\Gamma(n - \frac{1}{2})\,\Gamma(\frac{1}{2})}{2^{2-n}} J_{n-1}(tH^{\frac{1}{2}})\,(tH^{\frac{1}{2}})^{-n+1} f(x).$$

The Bessel function can be written as the sum of the Laguerre function and an error term using Hilb's asymptotic formula, viz.

$$\frac{2^{n-1} J_{n-1}(t\sqrt{2k+n})}{(t\sqrt{2k+n})^{n-1}} = \frac{\Gamma(k+1)}{\Gamma(k+n)} e^{-\frac{t^2}{4}} L_k^{n-1}\left(\frac{1}{2}t^2\right) + m_k^{n-1}(t).$$

The error term can be seen to satisfy the uniform bound

$$|m_k^{n-1}(t)| \leq C\,(2k+n)^{-a(n)}, \quad 0 \leq t \leq 1$$

with $a(1) = \frac{3}{4}$, $a(2) = \frac{7}{4}$ and $a(n) = 2$ for $n \geq 3$. As before we first prove the equi-convergence result.

Proposition 2.2.5 *Let $n = 1, 2$ or 3 and $f \in L^1(I\!R^n)$. Then*

$$\lim_{t \to 0} \left(\frac{v^{n-1}(x,t)}{t} - S_t^{n-1} f(x) \right) = 0.$$

Proof : Again it is enough to prove the estimate

$$\sup_{0<t\leq 1} |m_t^{n-1} f(x)| \leq C \int_{I\!R^n} |f(x)| dx$$

where

$$m_t^{n-1} f(x) = \sum_{k=0}^{\infty} m_k^{n-1}(t)\, P_k f(x).$$

The proof of the above estimate is similar to the one-dimensional case. If $\Phi_k(x,y)$ stands for the kernel of P_k, then the kernel of m_t^{n-1} is estimated by

$$|m_t^{n-1}(x,y)| \leq C \sum_{k=0}^{\infty} (2k+n)^{-a(n)}\, |\Phi_k(x,y)|$$

which by Cauchy-Schwarz is dominated by $(B(x)B(y))^{\frac{1}{2}}$, where

$$B(x) = \sum_{k=0}^{\infty} (2k+n)^{-a(n)}\, \Phi_k(x,x).$$

As before, the generating function identity for the n-dimensional Hermite functions shows that

$$B(x) = C_n \int_0^\infty t^{a(n)-1} (\sinh 2t)^{-\frac{n}{2}} e^{-(\tanh 2t)|x|^2} dt.$$

Thus we get the estimate

$$|m_t^{n-1}(x,y)| \leq C (1+|x|)^{-b(n)} (1+|y|)^{-b(n)}$$

where $b(1) = \frac{1}{4}$, $b(2) = \frac{3}{4}$ and $b(3) = \frac{1}{2}$. We note that the above integral defining $B(x)$ does not converge if $n > 3$. This completes the proof of the proposition. □

To complete the proof of theorem 2.2.4 we observe that $S_t^{n-1} f = W(\mu_t) f$ and consequently $S_*^{n-1} f(x)$ is bounded on $L^p(\mathbb{R}^n), p > \frac{2n}{n-1}$. Therefore, $S_t^{n-1} f(x)$ and hence $\frac{v^{n-1}(x,t)}{t}$ converge to $f(x)$ a.e. as $t \to 0$. Hence the theorem. □

Chapter 3

SOME CONVERGENCE RESULTS FOR HERMITE AND LAGUERRE EXPANSIONS

In this chapter we prove localisation theorems for Hermite and Laguerre expansions. This is done using the spherical means associated with these expansions. In this connection we study the regularity properties of these spherical means in terms of Laguerre Sobolev spaces. We also study the convergence of Hermite - Laguerre expansions $S_t^\alpha f$ encountered in the previous section. We show that for large α, $S_t^\alpha f$ converges to f in L^p norm as t tends to 0 for $1 < p < \infty$. The norm convergence of $S_t^\alpha f$ is equivalent to the uniform boundedness of S_t^α on $L^p(\mathbb{R}^n)$. For α large this is a consequence of the Marcinkiewicz multiplier theorem for Hermite expansion. For other values of α we establish some $L^p - L^{p'}$ inequalities for these operators.

3.1 The Sobolev spaces $W_\alpha^s(\mathbb{R}_+)$ and $W_H^s(\mathbb{R}^n)$

In this section we discuss some Sobolev spaces associated with Hermite and Laguerre differential operators. The usual Sobolev space $H^s(\mathbb{R}^n)$ is defined to be

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : (-\Delta + 1)^s f \in L^2(\mathbb{R}^n)\}$$

using the operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. Since we are interested in studying the regularity of $T_r^\alpha f(x)$ as a function of r it is convenient to define a Sobolev space in terms of Laguerre differential operator

$$Q_\alpha = -\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx} - \frac{x^2}{4}\right).$$

Then the normalised functions

$$\tilde{\psi}_k^\alpha(x) = \left(\frac{2^{-\alpha}\,\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{\frac{1}{2}} L_k^\alpha\left(\frac{1}{2}x^2\right) e^{-\frac{x^2}{4}}$$

are eigenfunctions of Q_α with eigenvalue $2k+\alpha+1$, which form an orthonormal basis for $L^2(\mathbb{R}_+, x^{2\alpha+1}\,dx)$. We denote the norm in $L^2(\mathbb{R}_+, x^{2\alpha+1}\,dx)$ by $\|\cdot\|_\alpha$.

For $\alpha > -1$ we define

$$W_\alpha^s(\mathbb{R}_+) = \{f \in L^2(\mathbb{R}_+, r^{2\alpha+1}dr) : Q_\alpha^s f \in L^2(\mathbb{R}_+, r^{2\alpha+1}dr)\}$$

where Q_α^s is defined using spectral theorem. In other words

$$f = \sum_{k=0}^{\infty} (f, \tilde{\psi}_k^\alpha)_\alpha\, \tilde{\psi}_k^\alpha$$

belongs to $W_\alpha^s(\mathbb{R}_+)$ if and only if

$$\left(\sum_{k=0}^{\infty} (2k+\alpha+1)^{2s}\, |\langle f, \tilde{\psi}_k^\alpha\rangle_\alpha|^2\right)^{1/2} < \infty.$$

Here $\langle\, ,\rangle_\alpha$ denotes the obvious inner product in $L^2(\mathbb{R}_+, r^{2\alpha+1}dr)$. The last expression defines the norm denoted by $|\cdot|_s$. in $W_\alpha^s(\mathbb{R}_+)$,

Similarly one can also define the Hermite Sobolev space W_H^s, using the Hermite operator $H = -\Delta + |x|^2$. Thus $f = \sum_{k=0}^{\infty} P_k(f) \in W_H^s(\mathbb{R}^n)$ if and only if

$$\sum_{k=0}^{\infty} (2k+n)^s P_k(f) \in L^2(\mathbb{R}^n).$$

We use the same symbol $|\cdot|_s$ for the norm in W_H^s also.

We now prove an important property of the Laguerre Sobolev spaces. The following proposition says that multiplication by a suitable function transfers functions from one scale of Sobolev spaces to the other. This fact is crucial in the proof of localisation theorem for both Hermite and Laguerre expansions.

Proposition 3.1.1 *Let $\alpha > -1$ and let φ be a smooth function on $\mathbb{R}_+$ which satisfies the following conditions:*
(i) $\varphi \equiv 0$ near the origin in $\mathbb{R}_+$
(ii) $|\left(\frac{d}{dr}\right)^j \varphi(r)| = O\left(\frac{1}{r^{2+j}}\right)$ as $r \to \infty$ for $j = 0, 1, 2, 3 \dots 2m$.
Then the operator $M_\varphi : W_\alpha^{s}(\mathbb{R}_+) \to W_{\alpha+1}^{s}(\mathbb{R}_+)$ defined by $M_\varphi\, f = \varphi \cdot f$ is a bounded operator for all $s \leq m$.

The proof of this proposition needs the following lemmas. Before stating the first lemma we introduce, for each nonnegative integer k, the class C_k, consisting of all smooth functions on $\mathbb{R}_+$, vanishing near 0, also satisfying the decay condition, $|\left(\frac{d}{dr}\right)^j \varphi(r)| = O\left(\frac{1}{r^{2+k+j}}\right)$ as $r \to \infty$. The class C_k satisfies the following properties: (i) $C_{k+1} \subset C_k$. (ii) If $\varphi \in C_k$, $\frac{1}{r}\varphi \in C_{k+1}$, and $r\varphi \in C_{k-1}$, for $k > 1$. (iii) If $\varphi \in C_k$, $\varphi^{(j)} \in C_{k+j}$.

Lemma 3.1.2 *Under the above assumptions on m, φ and α we have* $Q_{\alpha+1}^m\, M_\varphi\, Q_\alpha^{-m} = \sum_{t+k \leq m} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q_\alpha^{t-m}$ *with* $\varphi_{k,t} \in C_k$.

Proof : We claim that $Q_{\alpha+1}^m\, M_\varphi$ can be written as a linear combination of the form

$$Q_{\alpha+1}^m\, M_\varphi = \sum_{t+k \leq m} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q_\alpha^t \qquad (3.1.1)$$

with $\varphi_{k,t} \in C_k$. First we note the following relations

$$\begin{aligned} Q_\alpha\, M_\varphi &= M_\varphi\, Q_\alpha - 2\, M_{\varphi'} \frac{d}{dr} - M_{(\varphi'' + \frac{2\alpha+1}{r}\varphi')} \\ Q_{\alpha+1} &= Q_\alpha - \frac{2}{r}\frac{d}{dr} \end{aligned} \qquad (3.1.2)$$

Using this relation in the above we get

$$Q_{\alpha+1}\, M_\varphi = M_\varphi\, Q_\alpha - 2\, M_{(\varphi'+\frac{\varphi}{r})}\, \frac{d}{dr} - M_{(\varphi''+\frac{2\alpha+1}{r}\varphi')}.$$

We also use the relation,

$$Q_\alpha \left(\frac{d}{dr}\right)^k = \left(\frac{d}{dr}\right)^k Q_\alpha + \sum_{j=0}^{k-1} b_j \left(\frac{1}{r}\right)^j \left(\frac{d}{dr}\right)^{k-j} + c_1\, r \left(\frac{d}{dr}\right)^{k-1} + c_2 \left(\frac{d}{dr}\right)^{k-2} \qquad (3.1.3)$$

where b_j, c_1, c_2, are constants. This can be easily proved by induction on k. We prove (3.1.1) by induction on m. (3.1.1) is clear for $m = 1$. Assume (3.1.1) for $m = j$. Now,

$$\begin{aligned}
Q^{j+1}_{\alpha+1}\, M_\varphi &= \left(Q_\alpha - \frac{2}{r}\frac{d}{dr}\right) (Q^j_{\alpha+1}\, M_\varphi) \\
&= \left(Q_\alpha - \frac{2}{r}\frac{d}{dr}\right) \left(\sum_{t+k\le j} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q^t_\alpha\right) \\
&= \sum_{t+k\le j} Q_\alpha \left(M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q^t_\alpha\right) - 2 \sum_{t+k\le j} \frac{1}{r}\frac{d}{dr} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q^t_\alpha \\
&= \sum_{t+k\le j} \left(M_{\varphi_{k,t}}\, Q_\alpha - 2\, M_{\varphi'_{k,t}} \frac{d}{dr} - M_{(\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t})}\right) \left(\frac{d}{dr}\right)^k Q^t_\alpha \\
&\quad -\frac{2}{r} \sum_{t+k\le j} \frac{d}{dr} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q^t_\alpha. \\
&= \sum_{t+k\le j} M_{\varphi_{k,t}}\, Q_\alpha \left(\frac{d}{dr}\right)^k Q^t_\alpha - 2 \sum_{t+k\le j} M_{\varphi'_{k,t}} \left(\frac{d}{dr}\right)^{k+1} Q^t_\alpha \\
&\quad - \sum_{t+k\le j} M_{(\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t})} \left(\frac{d}{dr}\right)^k Q^t_\alpha - \frac{2}{r} \sum_{t+k\le j} M_{\varphi'_{k,t}} \left(\frac{d}{dr}\right)^k Q^t_\alpha \\
&\quad -\frac{2}{r} \sum_{t+k\le j} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k+1} Q^t_\alpha.
\end{aligned} \qquad (3.1.4)$$

In the above computation we have used (3.1.2). In view of (3.1.3), the first term of the above is

$$= \sum_{t+k\le j} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q^{t+1}_\alpha + \sum_{i=0}^{k-1} b_i \left(\frac{1}{r}\right)^i \sum_{t+k\le j} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k-i} Q^t_\alpha$$

$$+\sum_{t+k\leq j} c_1\, r\, M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k-1} Q_\alpha^t + \sum_{t+k\leq j} c_2\, M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k-2} Q_\alpha^t$$

$$= \sum_{t+k\leq j+1} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k} Q_\alpha^t + \sum_{i=0}^{k-1} \sum_{t+k\leq j} b_i \left(\frac{1}{r}\right)^i M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k-i} Q_\alpha^t$$

$$+\sum_{t+k\leq j} c_1\, r\, M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k-1} Q_\alpha^t + \sum_{t+k\leq j} c_2\, M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^{k-2} Q_\alpha^t \qquad (3.1.5)$$

Now by induction hypothesis we have $\varphi_{k,t} \in C_k$. Note that in the second term of the above the coefficient of $\left(\frac{d}{dr}\right)^{k-i} Q_\alpha^t$ is $(1/r)^i \varphi_{k,t}$. We have $(1/r)^i \varphi_{k,t} \in C_{k+i} \subset C_k \subset C_{k-i}$ for $i \geq 0$ and also $r\varphi_{k,t} \in C_{k-1}$. Hence the first term in (3.1.4) is of the required form. The second term of (3.1.4) can be written as $-2\sum_{t+k\leq j+1} M_{\varphi'_{k-1,t}} \left(\frac{d}{dr}\right)^k Q_\alpha^t$, and $\varphi_{k,t} \in C_k$ by induction hypothesis. Therefore $\varphi'_{k-1,t} \in C_k$ in view of (iii). Hence the second term of (3.1.4) is also of the required form. In the third term the coefficient of $\left(\frac{d}{dr}\right)^k Q_\alpha^t$ is $M_{(\varphi''_{k,t} + \frac{2\alpha+1}{r}\varphi'_{k,t})}$ and $\varphi''_{k,t} + \frac{2\alpha+1}{r}\varphi'_{k,t} \in C_{k+2} \subset C_k$ by induction hypothesis and in view of (i),(ii) and (iii). Similarly $\frac{1}{r}\varphi'_{k,t}$ occuring in the fourth term belongs to $C_{k+2} \subset C_k$. Also $\frac{1}{r}\varphi_{k,t}$ occuring in the fifth term $\in C_{k+1} \subset C_k$. Therefore (3.1.1) holds for $m = j+1$ also. Thus we have $T^m f = Q_{\alpha+1}^m M_\varphi Q_\alpha^{-m} f = \sum_{t+k\leq m} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q_\alpha^{t-m} f$, which proves the first lemma. □

Lemma 3.1.3 $\left(\frac{d}{dr}\right)^i Q_\alpha^t : L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr) \to L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr)$ *is a bounded operator whenever i is a non negative integer and $i+t \leq 0$.*

Proof : We prove that $\frac{d}{dr}\, Q_\alpha{}^t$ is a bounded operator on $L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr)$ for $1+t \leq 0$. We first note that

$$\frac{d}{dr}\tilde{\psi}_k^\alpha(r) = -\frac{r}{\sqrt{2}}\left(k^{\frac{1}{2}}\,\tilde{\psi}_{k-1}^{\alpha+1}(r) + (k+\alpha+1)^{1/2}\,\tilde{\psi}_k^{\alpha+1}(r)\right). \qquad (3.1.6)$$

This can be seen as follows: Using the relations $\frac{d}{dr}L_k^\alpha(r) = -L_{k-1}^{\alpha+1}(r)$ and $L_k^{\alpha+1} - L_{k-1}^{\alpha+1} = L_k^\alpha$, we calculate

$$\frac{d}{dr}\left(L_k^\alpha\left(\frac{1}{2}r^2\right)e^{-\frac{r^2}{4}}\right) = -r\left(L_{k-1}^{\alpha+1}\left(\frac{1}{2}r^2\right) + \frac{1}{2}L_k^\alpha\left(\frac{1}{2}r^2\right)\right)e^{-\frac{r^2}{4}}$$

$$= -\frac{r}{2}\left(L_{k-1}^{\alpha+1}\left(\frac{1}{2}r^2\right) + L_{k-1}^{\alpha+1}\left(\frac{1}{2}r^2\right) + L_k^{\alpha}\left(\frac{1}{2}r^2\right)\right) e^{-\frac{r^2}{4}}$$
$$= -\frac{r}{2}\left(L_{k-1}^{\alpha+1}\left(\frac{1}{2}r^2\right) + L_k^{\alpha+1}\left(\frac{1}{2}r^2\right)\right) e^{-\frac{r^2}{4}}.$$

Now (3.1.6) follows from the definition of $\tilde{\psi}_k^{\alpha}$. Let $f \in L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr)$. By definition

$$\frac{d}{dr}Q_{\alpha}^{t} f(r) = \sum_{k=0}^{\infty}(2k+\alpha+1)^t \langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha} \frac{d}{dr}\tilde{\psi}_k^{\alpha}(r), \tag{3.1.7}$$

and using (3.1.6) we get

$$\begin{aligned}\frac{d}{dr}Q_{\alpha}^{t} f(r) &= -\frac{r}{\sqrt{2}}\sum_{k=1}^{\infty}(2k+\alpha+1)^t k^{\frac{1}{2}} \langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}\, \tilde{\psi}_{k-1}^{\alpha+1}(r)\\ &\quad -\frac{r}{\sqrt{2}}\sum_{k=0}^{\infty}(2k+\alpha+1)^t (k+\alpha+1)^{1/2} \langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}\, \tilde{\psi}_k^{\alpha+1}(r)\\ &= -\frac{r}{\sqrt{2}}\left(Tf(r) + Sf(r)\right) \end{aligned}\tag{3.1.8}$$

where

$$Tf(r) = \sum_{k=1}^{\infty}(2k+\alpha+1)^t k^{\frac{1}{2}} \langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}\, \tilde{\psi}_{k-1}^{\alpha+1}(r) \tag{3.1.9}$$

and

$$Sf(r) = \sum_{k=0}^{\infty}(2k+\alpha+1)^t (k+\alpha+1)^{1/2} \langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}\, \tilde{\psi}_k^{\alpha+1}. \tag{3.1.10}$$

Therefore,

$$\|\frac{d}{dr}Q_{\alpha}^{t} f(r)\|_{\alpha} \le \frac{1}{\sqrt{2}}\left(\|r\,Tf(r)\|_{\alpha} + \|r\,Sf(r)\|_{\alpha}\right). \tag{3.1.11}$$

Now using the expansion (3.1.9) we calculate,

$$\begin{aligned}\|r\,Tf(r)\|_{\alpha}^2 &= \int_0^{\infty} r^2 |Tf(r)|^2 r^{2\alpha+1}\,dr\\ &= \int_0^{\infty} |Tf(r)|^2 r^{2\alpha+3}\,dr\\ &= \sum_{k=1}^{\infty}(2k+\alpha+1)^{2t} k\, |\langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}|^2\\ &\le \sum_{k=1}^{\infty}(2k+\alpha+1)^{2t+1} |\langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}|^2\\ &\le \sum_{k=1}^{\infty} |\langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}|^2 = \|f\|_{\alpha}^2 \end{aligned}\tag{3.1.12}$$

since $1+t \leq 0$. Similarly one can see that

$$\|rSf(r)\|_\alpha^2 \leq \|f\|_\alpha^2. \tag{3.1.13}$$

Using (3.1.12) and (3.1.13) in (3.1.11) we see that $\|\frac{d}{dr}Q_\alpha^t f\|_\alpha \leq \sqrt{2}\|f\|_\alpha$ for $1+t \leq 0$. Similarly one can show that $\|\left(\frac{d}{dr}\right)^j Q_\alpha^t f\|_\alpha \leq c\|f\|_\alpha$ for some constant c, whenever $j+t \leq 0$, which proves the second lemma. □

Proof of proposition 3.1.1 : We have by definition $W_\alpha^s = Q_\alpha^{-s}(L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr))$. Therefore it is enough to prove that

$$Q_{\alpha+1}^s M_\varphi Q_\alpha^{-s} : L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr) \to L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr) \tag{3.1.14}$$

is a bounded operator. Put

$$T^t f = Q_{\alpha+1}^t M_\varphi Q_\alpha^{-t} f. \tag{3.1.15}$$

Then clearly,

$$\begin{aligned} \|T^0 f\|_{\alpha+1} &= \|\varphi f\|_{\alpha+1} \\ &\leq c_0\|f\|_\alpha, \end{aligned} \tag{3.1.16}$$

for some constant c_0 independent of f. We will also prove that, for any positive integer m

$$\|T^m f\|_{\alpha+1} \leq c_1\|f\|_\alpha, \tag{3.1.17}$$

for some constant c_1 independent of f.

Assuming (3.1.17) for a moment choose $f_1 \in L^2(\mathbb{R}_+, r^{2\alpha+1}\,dr)$ and $g_1 \in L^2(\mathbb{R}_+, r^{2\alpha+3}\,dr)$ to be finite linear combinations of $\tilde{\psi}_k^\alpha$'s and $\tilde{\psi}_k^{\alpha+1}$'s, respectively. Consider the function h which is holomorphic in the region $0 < Re(z) < m$ and continuous in $0 \leq Re(z) \leq m$, defined by:

$$h(z) = \langle T^z f_1\,,\, g_1\rangle_{\alpha+1} = \langle Q_{\alpha+1}^z M_\varphi Q_\alpha^{-z} f_1\,,\, g_1\rangle_{\alpha+1} \tag{3.1.18}$$

Then by (3.1.16) we have,

$$\begin{aligned} |h(iy)| &= |\langle Q_{\alpha+1}^{iy}\, M_\varphi\, Q_\alpha^{-iy} f_1\,,\, g_1\rangle_{\alpha+1}| \\ &= |\langle \varphi(r)\tilde{f}_1\,,\, \tilde{g}_1\rangle_{\alpha+1}| \end{aligned}$$

where $\tilde{f}_1 = Q_\alpha^{-iy} f_1$, and $\tilde{g}_1 = Q_{\alpha+1}^{-iy} g_1$. Therefore,

$$\begin{aligned} |h(iy)| &\le \|T^0 \tilde{f}_1\|_{\alpha+1}\, \|\tilde{g}_1\|_{\alpha+1} \\ &\le c_0\, \|\tilde{f}_1\|_\alpha\, \|\tilde{g}_1\|_{\alpha+1}, \end{aligned}$$

and since both Q_α^{-iy} and $Q_{\alpha+1}^{-iy}$ are unitary operators, we get

$$|h(iy)| \le c_0\, \|f_1\|_\alpha\, \|g_1\|_{\alpha+1}.$$

Similarly by using (3.1.17) we get

$$\begin{aligned} |h(m+iy)| &= |\langle Q_{\alpha+1}^{m+iy}\, M_\varphi\, Q_\alpha^{-m-iy} f_1\,,\, g_1\rangle_{\alpha+1}| \\ &= |\langle Q_{\alpha+1}^{m}\, M_\varphi\, Q_\alpha^{-m} \tilde{f}_1\,,\, \tilde{g}_1\rangle_{\alpha+1}| \\ &\le \|T^m \tilde{f}_1\|_{\alpha+1}\, \|\tilde{g}_1\|_{\alpha+1} \\ &\le c_1\, \|f_1\|_\alpha\, \|g_1\|_{\alpha+1}. \end{aligned}$$

Thus we have

$$|h(iy)| \le c_0\, \|f_1\|_\alpha\, \|g_1\|_{\alpha+1}$$

$$|h(m+iy)| \le c_1\, \|f_1\|_\alpha\, \|g_1\|_{\alpha+1}.$$

Since h is a bounded function we have by three lines theorem

$$|\, h(t+iy)\, | \le c_0^{1-t/m}\, c_1^{t/m}\, \|f_1\|_\alpha\, \|g_1\|_{\alpha+1},$$

for $0 < t < m$. In particular,

$$|h(t)| \le c_0^{1-t/m}\, c_1^{t/m}\, \|f_1\|_\alpha\, \|g_1\|_{\alpha+1},$$

that is,

$$|\langle T^{\,t} f_1 \,,\, g_1\rangle| \le c_0^{1-t/m}\, c_1^{t/m}\, \|f_1\|_\alpha \|g_1\|_{\alpha+1}.$$

Now taking supremum over all such $g_1 \in L^2(I\!R_+, r^{\,2\alpha+3}\,dr)$ with $\|g_1\|_{\alpha+1} \le 1$ we get $\|T^{\,t} f_1\|_{\alpha+1} \le c_0^{1-t/m}\, c_1^{t/m}\, \|f_1\|_\alpha$. Therefore $T^{\,t}$ is a bounded operator on a dense subset of $L^2(I\!R_+, r^{\,2\alpha+1}\,dr)$. Therefore it has a norm preserving extension to $L^2(I\!R_+, r^{\,2\alpha+1})$. Thus we have

$$\|T^{\,t} f\,\|_{\alpha+1} \le c_t\, \|f\,\|_\alpha$$

$\forall\, f \in L^2(I\!R_+, r^{\,2\alpha+1}\,dr)$, for $0 < t < m$, which proves (3.1.14).

To prove (3.1.17) we proceed as follows. By Lemma 3.1.2 we see that $T^{\,m} f$ is of the form $\sum_{t+k\le m} M_{\varphi_{k,t}} \left(\frac{d}{dr}\right)^k Q_\alpha^{t-m} f$. Also by Lemma 3.1.3 $\left(\frac{d}{dr}\right)^k Q_\alpha^{t-m}$ defines a bounded operator on $L^2(I\!R_+, r^{\,2\alpha+1}\,dr)$, whenever $k+t-m \le 0$. Since $\varphi_{k,t}$ satisfies the conditions (i) and (ii) of the proposition 3.1.1 for $j = 0$, it follows that $M_{\varphi_{k,t}}$ maps $L^2(I\!R_+, r^{\,2\alpha+1}\,dr) \to L^2(I\!R_+, r^{\,2\alpha+3}\,dr)$ boundedly. Thus we see that $\|T^{\,m} f\|_{\alpha+1} \le c_1 \|f\,\|_\alpha$. This completes the proof of the proposition. □

3.2 A localisation theorem for Laguerre expansions

In this section we prove a localisation theorem for Laguerre expansions using Laguerre means. There are several types of Laguerre expansions on $I\!R_+ = (0, \infty)$ studied in the literature. The first one concerned with the Laguerre polynomials $L_k^\alpha(x)$ which form an orthonormal basis for $L^2(I\!R_+, e^{-x}\, x^\alpha\, dr)$. Thus every $f \in L^2(I\!R_+, e^{-x}\, x^\alpha\, dr)$ has the Laguerre expansion

$$f = \sum_{k=0}^{\infty} c_k\, a_k\, L_k^\alpha(x)$$

with $a_k = \int_0^\infty f(x)\, L_k^\alpha(x)\, e^{-x}\, x^\alpha\, dx$ and $c_k^{-1} = \int_0^\infty (L_k^\alpha(x))^2\, e^{-x}\, x^\alpha\, dx$. Considering the functions $\left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} L_k^\alpha(x)\, e^{-\frac{x}{2}}\, x^{\frac{\alpha}{2}}$ as an orthonormal system in $L^2(I\!R_+, dt)$ we get another type of Laguerre expansions. A third type of Laguerre expansion is obtained by considering the functions $\varphi_k^\alpha(x) = L_k^\alpha\left(\frac{1}{2}x^2\right) e^{-\frac{x^2}{4}}$ as an orthogonal system in $L^2(I\!R_+, x^{2\alpha+1}\,dx)$. Several authors have studied norm convergence and almost everywhere convergence of Riesz means of such expansions. Some references are Askey - Wainger [3], Muckenhoupt [14], Görlich-Markett [8], Markett [12], Sternpak [25], [26], Thangavelu [32]. For various results concerning Hermite and Laguerre expansions we refer to the monograph [36].

Here we will be dealing with the Laguerre functions $\tilde{\psi}_k^\alpha(x) = \left(\frac{2^{-\alpha}\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{\frac{1}{2}} \varphi_k^\alpha(x)$, which form an orthonormal basis for $L^2(I\!R_+, x^{2\alpha+1}\,dx)$. Thus every $f \in L^2(I\!R_+, x^{2\alpha+1}\,dx)$ has the expansion

$$f = \sum_{k=0}^{\infty} \langle f, \tilde{\psi}_k^\alpha \rangle_\alpha \tilde{\psi}_k^\alpha$$

Reacall that the Laguerre mean $T_r^\alpha f(z)$ of a function f on $I\!R_+$ is defined in the introduction by

$$T_r^\alpha f(z) = \frac{2^\alpha\,\Gamma(\alpha+1)}{\sqrt{2\pi}} \int_0^\pi f\left((r^2+z^2+2\,r\,z\,cos\theta)^{1/2}\right) \times \frac{J_{\alpha-1/2}(\frac{1}{2}\,r\,z\,sin\theta)}{(\frac{1}{2}\,r\,z\,sin\theta)^{\alpha-1/2}}\, sin^{2\alpha}\theta\, d\theta. \qquad (3.2.1)$$

We have also seen that the Laguerre means have the series expansion

$$T_r^\alpha f(z) = \sum_0^\infty \langle f, \tilde{\psi}_k^\alpha \rangle_\alpha\, \psi_k^\alpha(r)\, \tilde{\psi}_k^\alpha(z), \qquad (3.2.2)$$

for $r \geq 0, z \geq 0, \alpha > -1/2$. We start by proving the following regularity result for the Laguerre means in terms of the Sobolev space $W_\alpha^s(I\!R_+)$.

Lemma 3.2.1 *Let $f \in W_\alpha^s$ then the following hold :*
(i) For $z \neq 0$, $T_r^\alpha f(z) \in W_\alpha^{s+\frac{\alpha}{2}+\frac{1}{4}}(I\!R_+)$ as a function of r.
(ii) $T_r^\alpha f(0) \in W_\alpha^s$. if and only if $f \in W_\alpha^s(I\!R_+)$.

Proof : The proof needs the following asymptotic estimates. See[11].

$$\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \approx k^{-\alpha} \qquad (3.2.3)$$

$$\tilde{\psi}_k^{\alpha}(z) \approx \pi^{-\frac{1}{2}} k^{-1/4} 2^{\frac{1}{4}} |z|^{-\alpha-\frac{1}{2}} \cos\left(\sqrt{2k}\, z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right), \quad z \neq 0 \qquad (3.2.4)$$

$$\tilde{\psi}_k^{\alpha}(0) \approx \frac{2^{-\frac{\alpha}{2}}}{\Gamma(\alpha+1)} k^{\alpha/2} \text{ as } k \to \infty. \qquad (3.2.5)$$

From the above series expansion (3.2.2) for the Laguerre means we see that

$$\begin{aligned}\int_0^{\infty} |T_r^{\alpha} f(z)|^2 r^{2\alpha+1}\, dr &= 2^{\alpha} (\Gamma(\alpha+1))^2 \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} |\langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}|^2 |\tilde{\psi}_k^{\alpha}(z)|^2 \\ &\leq C(z) \sum_{k=0}^{\infty} (1+k)^{-\alpha} (1+k)^{-\frac{1}{2}} |\langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}|^2 \qquad (3.2.6)\end{aligned}$$

for $z \neq 0$, in view of (3.2.3) and (3.2.4). Also in view of 3.2.3 and 3.2.5 we can see that

$$\int_0^{\infty} |T_r^{\alpha} f(0)|^2 r^{2\alpha+1}\, dr \approx \sum_{k=0}^{\infty} |\langle f, \tilde{\psi}_k^{\alpha}\rangle_{\alpha}|^2. \qquad (3.2.7)$$

Now recalling the definition of the Sobolev space $W_{\alpha}^{s}(\mathbb{R}_+)$, from (3.2.6) it follows that $T_r^{\alpha} f(z) \in W_{\alpha}^{s+\frac{\alpha}{2}+\frac{1}{4}}(\mathbb{R}_+)$, as a function of r whenever $f \in W_{\alpha}^{s}(\mathbb{R}_+)$. Similarly from (3.2.7) it follows that $f \in W_{\alpha}^{s}(\mathbb{R}_+)$ if and only if $T_r^{\alpha} f(0) \in W_{\alpha}^{s}(\mathbb{R}_+)$. □

Now we prove a very useful property of Laguerre means.

Lemma 3.2.2 *(i) If f is supported in $0 \leq z \leq b$, then $T_r^{\alpha} f(z)$ as a function of r is supported in $0 \leq r \leq b+z$.*
(ii) If f vanishes in a neighbourhood of z then $T_r^{\alpha} f(z)$ as a function of r vanishes in a neighbourhood of origin in $\mathbb{R}_+$.

Proof : If f is supported in $z \leq b$ then the integral (3.2.1) vanishes unless $(r^2 + z^2 + 2rz\cos\theta)^{1/2} \leq b$. This implies $(r-z)^2 \leq b^2$. Therefore the integral (3.2.1) vanishes unless $|r-z| \leq b$ or $r \leq b+z$ which proves (i). To prove (ii) notice that if f vanishes in a neighbourhood $\{|y-z| < a\}, a > 0$ of z, the integral (3.2.1) is zero if $|(r^2+z^2+2rz\cos\theta)^{1/2} - z| \leq a$.

Since z is fixed this says that the above inequality holds for r in a neighbourhood of 0. Now consider the continuous function

$$g(r) = |(r^2 + z^2 + 2\,r\,z\,cos\theta)^{1/2} - z| - a,$$

defined on $I\!R_+$. We have $g(0) = -a < 0$. Therefore $g < 0$ in a neighbourhood of 0 as well. This means that for r in some neighbourhood of 0 we have $|(r^2+z^2+2r\,z\,cos\theta)^{1/2} - z| < a$. Thus $T_r^{\alpha} f(z) \equiv 0$ in that neighbourhood. □

Now we are in a position to prove a localisation theorem for Laguerre expansions. From the the series expansion (3.2.2) using the orthogonality of $\tilde{\psi}_k^{\alpha}$ we see that

$$\int_0^{\infty} T_r^{\alpha} f(z)\,\varphi_k^{\alpha}(r)\,r^{2\alpha+1}\,dr = 2^{\alpha}\,\Gamma(\alpha+1)\,(f, \tilde{\psi}_k^{\alpha})_{\alpha}\,\tilde{\psi}_k^{\alpha}(z).$$

In view of the relation $\sum_0^N L_k^{\alpha}(x) = L_N^{\alpha+1}$, we see that

$$\begin{aligned} S_N^{\alpha} f(z) &= \sum_{k=0}^{N} (f, \tilde{\psi}_k^{\alpha})_{\alpha}\,\tilde{\psi}_k^{\alpha}(z) \\ &= \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^{\infty} T_r^{\alpha} f(z) \sum_{k=0}^{N} \varphi_k^{\alpha}(r)\,r^{2\alpha+1}\,dr \\ &= \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^{\infty} T_r^{\alpha} f(z)\,\varphi_N^{\alpha+1}(r)\,r^{2\alpha+1}\,dr. \qquad (3.2.8) \end{aligned}$$

Using the above representation for the partial sums it is easy to prove the following theorem:

Theorem 3.2.3 *Let $\alpha > -\frac{1}{2}$ and let $f \in L^2(I\!R_+, x^{2\alpha+1}\,dx)$ be a function vanishing in a neighbourhood B_z of a point $z \in I\!R_+$. If $w \in B_z$ is such that $T_r^{\alpha} f(w) \in W_{\alpha}^{\frac{\alpha+1}{2}}(I\!R_+)$, as a function of r, then $S_N f(w) \to 0$ as $N \to \infty$.*

Proof : The proof uses the following fact: if $g \in L^2(I\!R_+, r^{2\alpha+1}\,dr)$, then the Fourier-Laguerre coefficients $(g, \tilde{\psi}_k^{\alpha})_{\alpha} \to 0$ as $k \to \infty$. Recalling the definition of $\tilde{\psi}_k^{\alpha}$ this means that

$$\int_0^{\infty} g(r)\,\varphi_k^{\alpha}(r)\,r^{2\alpha+1}\,dr = o(k^{\frac{\alpha}{2}})$$

as $k \to \infty$. Also if $g \in W_\alpha^s(I\!R_+)$ then,

$$\int_0^\infty g(r)\,\varphi_k^\alpha(r)\,r^{2\alpha+1}\,dr = o(k^{-s+\frac{\alpha}{2}}) \qquad (3.2.9)$$

as $k \to \infty$. From (3.2.8) we get

$$S_N^\alpha f(z) \;=\; \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\infty \frac{T_r{}^\alpha f(z)}{r^2}\,\varphi_N^{\alpha+1}(r)\,r^{2\alpha+3}\,dr.$$

Let $\tilde{h}$ be a smooth function on $I\!R_+$ such that $\tilde{h}(r) \equiv 1$ on the support of $T_r{}^\alpha f(z)$ and $\tilde{h}(r) \equiv 0$ in a neighbourhood of the origin in $I\!R_+$. Put $h(r) = \frac{\tilde{h}(r)}{r^2}$. Then the above equation can be written as

$$S_N^\alpha f(z) \;=\; \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\infty h(r)\;T_r{}^\alpha f(z)\;\varphi_N^{\alpha+1}(r)\,r^{2\alpha+3}\,dr$$

Now if $T_r{}^\alpha f(z) \in W_\alpha^{\frac{\alpha+1}{2}}(I\!R_+)$, we have by proposition 3.1.1 $h(r)\,T_r{}^\alpha f(z) \in W_{\alpha+1}^{\frac{\alpha+1}{2}}(I\!R_+)$. Therefore by (3.2.9),

$$S_N^\alpha f(z) = o(N^{\,(-\frac{\alpha+1}{2}+\frac{\alpha+1}{2})}) = o(1),$$

as $N \to \infty$ which proves the theorem. □

In view of Lemma 3.2.1, if $f \in W_\alpha^{1/4}(I\!R_+)$, then $T_r{}^\alpha f(z) \in W_\alpha^{\frac{\alpha+1}{2}}(I\!R_+)$, for $z \neq 0$. Thus we have the following corollary to the above theorem.

Corollary 3.2.4 *If $f \in W_\alpha^{1/4}(I\!R_+)$ then the conclusion of Theorem 3.2.3 holds at points $z \neq 0$.*

3.3 A localisation theorem for Hermite expansions

In this section we prove a localisation theorem for Hermite expansions. The techniques involved are essentially the same as in the previous section. The fact is that the spherical mean involved in this case is the Weyl transform of the normalised surface measure μ_r on the sphere in $\mathbb{C}^n$. We start by recalling the definition of the Hermite functions.

The Hermite polynomials $H_k(x)$ are defined on the real line by

$$(-1)^k \frac{d^k}{dx^k} e^{-x^2} = H_k(x)\, e^{-x^2}.$$

Then the functions $\tilde{h}_k(x) = H_k(x)\, e^{-\frac{x^2}{2}}$ are called the Hermite functions. The normalised functions $h_k(x) = (\frac{1}{2^k k! \sqrt{\pi}})^{1/2} \tilde{h}_k(x)$, $k = 0, 1, 2, \ldots$ form an orthonormal basis for $L^2(I\!R)$ and they are also the eigenfunctions of the differential operator $H = -\frac{d^2}{dx^2} + x^2$, called the Hermite operator, with eigenvalue $2k+1$. For each multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we define the Hermite functions on $I\!R^n$ by $\Phi_\alpha = \prod_{k=1}^{n} h_{\alpha_k}(x_k)$, $x \in I\!R^n$. Then the functions Φ_α are the normalised Hermite functions, and they are eigenfunctions of the $n-$ dimensional Hermite operator $-\Delta + |x|^2$ with eigenvalue $2|\alpha| + n$. The collection $\{\Phi_\alpha\}_\alpha$ forms an orthonormal basis for $L^2(I\!R^n)$. Thus for $f \in L^2(I\!R^n)$ we have the Hermite expansion

$$f(x) = \sum_{k=0}^{\infty} P_k f(x) \tag{3.3.1}$$

where P_k is the Hermite projection operator

$$P_k f(x) = \sum_{|\alpha|=k} \langle f, \Phi_\alpha \rangle\, \Phi_\alpha(x).$$

We have seen in the introduction that the Weyl transform of the measure μ_r has the expansion

$$W(\mu_r) = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r)\, P_k \tag{3.3.2}$$

where $\varphi_k(r) = L_k^{n-1}(\frac{r^2}{2})\, e^{-\frac{r^2}{4}}$ denotes the Laguerre functions of order $n-1$.

We start by studying the regularity properties of the operator $W(\mu_r)$ in terms of the above Sobolev spaces. We prove the following:

Proposition 3.3.1 *If $f \in W_H^s(I\!R^n)$, then $W(\mu_r)f(\xi) \in W_{n-1}^{s+\frac{n-1}{2}}(I\!R_+)$, for almost every $\xi \in I\!R^n$.*

Proof : From (3.3.2) we get

$$Q_{n-1}^{s+\frac{n-1}{2}} W(\mu_r)f(x) = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} (2k+n)^{s+\frac{n-1}{2}} \varphi_k(r) \; P_k f(x).$$

Integrating from 0 to ∞ with respect to the measure $r^{2n-1}dr$ and using the orthogonality of $(\frac{k!(n-1)!}{(k+n-1)!})^{1/2}\varphi_k(r)$ we get

$$|W(\mu_r)f(x)|^2_{s+\frac{n-1}{2}} = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} (2k+n)^{2s+n-1} |P_k f(x)|^2.$$

Now integrating both sides over $\mathbb{R}^n$, using $\frac{k!(n-1)!}{(k+n-1)!} \leq C(2k+n)^{-n+1}$, and the fact that P_k are orthogonal projections we get

$$\begin{aligned} \int_{\mathbb{R}^n} |W(\mu_r)f(x)|^2_{s+\frac{n-1}{2}} dx &\leq \sum_{k=0}^{\infty} (2k+n)^{2s} \|P_k f(x)\|_2^2 \\ &\leq C \|H^s f\|_2^2 = C |f|_s^2. \end{aligned}$$

Thus it follows that for almost every x in $\mathbb{R}^n$ $|W(\mu_r)f(x)|_{s+\frac{n-1}{2}} < \infty$ whenever $|f|_s < \infty$, which proves the proposition. □

The next lemma is analogous to lemma (3.2.2) in the case of Laguerre expansions.

Lemma 3.3.2 *If $f \equiv 0$ near a point $\xi_0 \in \mathbb{R}^n$, then $W(\mu_r)f(\xi_0)$ as a function of r vanishes in a neighbourhood of 0 in $\mathbb{R}_+$.*

Proof : Recall the integral representation for $W(\mu_r)f(\xi)$ given in the introduction.

$$W(\mu_r)f(\xi) = \int_{x^2+y^2=r^2} e^{i(x.\xi+\frac{1}{2}x.y)} f(\xi+y)\, d\mu_r(x,y). \qquad (3.3.3)$$

From the above integral we notice that if $f \equiv 0$ near a point $\xi_0 \in \mathbb{R}^n$, then $f(\xi_0+y)$ is also $\equiv 0$ for small $|y|$. Thus it follows that for small r

$$\int_{x^2+y^2=r^2} e^{i(x.\xi+\frac{1}{2}x.y)} f(\xi+y)\, d\mu_r(x,y) \equiv 0.$$

This proves the lemma. □

Now we are in a position to prove a localisation theorem for the Hermite expansions.

Theorem 3.3.3 *Let $f \in L^2(\mathbb{R}^n)$ be a function vanishing in a neighbourhood B_x of a point $x \in \mathbb{R}^n$. If $y \in B_x$ is such that $W(\mu_r)f(y) \in W_{n-1}^{n/2}(\mathbb{R}_+)$, as a function of r, then $S_N f(y) \to 0$ as $N \to \infty$.*

Proof : The proof is similar to the proof of theorem 3.2.3. From (3.3.2) using the orthogonality of φ_k we see that

$$P_k f(\xi) = \int_0^\infty W(\mu_r) f(\xi)\, \varphi_k(r)\, r^{2n-1}\, dr.$$

Therefore using the relation $\sum_{k=0}^N L_k^{n-1}(t) = L_N^n(t)$ we can write the partial sum of the Hermite expansion in the form

$$S_N f(\xi) = \sum_{k=0}^N P_k f(\xi) = \int_0^\infty W(\mu_r)\, f(\xi)\, \varphi_N^n(r)\, r^{2n-1}\, dr.$$

By lemma 3.3.2 we have $W(\mu_r)f(y)$ vanishes (as a function of r) in a neighbourhood of 0 in $\mathbb{R}_+$ for $y \in B_x$. Let $\tilde{h}(r)$ be a smooth function on $\mathbb{R}_+$, vanishing near zero and $\equiv 1$ on the support of $W(\mu_r)f(y)$. Letting $h(r) = \frac{1}{r^2}\tilde{h}(r)$, we get

$$S_N f(y) = \int_0^\infty h(r)\, W(\mu_r)\, f(y)\, \varphi_N^n(r)\, r^{2n+1}\, dr. \qquad (3.3.4)$$

Since $W(\mu_r)f(y) \in W_{n-1}^{n/2}(\mathbb{R}_+)$, in view of the proposition 3.1.1 we have $h(r)W(\mu_r)f(y) \in W_n^{n/2}(\mathbb{R}_+)$. Thus as in the case of localisation theorem for Laguerre expansions we see that right hand side of (3.3.4) is $= o(1)$ as $N \to \infty$, which proves the theorem. □

In view of the proposition 3.3.1 we have the following corollary to the above theorem.

Corollary 3.3.4 *If $f \in W_H^{1/2}(\mathbb{R}^n)$, then the conclusions of the above theorem holds.*

3.4 Convergence of Hermite - Laguerre expansions

In this section we are concerned with the pointwise and norm convergence of $S_t^\alpha f$ to f as $t \to 0$. Recall that $S_t^\alpha f$ is given by the expansion

$$S_t^\alpha f(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(\frac{1}{2}t^2)e^{-\frac{1}{4}t^2} P_k f(x).$$

This expansion may be called the Hermite-Laguerre expansion of f and the convergence of $S_t^\alpha f$ to f may be considered a new summability method for the Hermite expansion. As $S_t^{n-1} f = W(\mu_t) f$ the following theorem is a restatement of the maximal theorem in chapter 1.

Theorem 3.4.1 *Let $f \in L^p(I\!R^n)$, $p > \frac{2n}{2n-1}$. Then $S_t^{n-1} f(x) \to f(x)$ a.e. and also in the norm.*

We are interested in knowing similar properties of S_t^α with $0 \leq \alpha < n-1$. For α big enough we can obtain norm convergence as can be seen from the following result.

Theorem 3.4.2 *For $[\alpha] > \frac{n}{2}$, $S_t^\alpha f$ converges to f in the norm, as $t \to 0$ for $f \in L^p(I\!R^n), 1 < p < \infty$.*

We will prove the above theorem by appealing to the Marcinkiewicz multiplier theorem for Hermite expansions. If $\{m(k)\}$ is a bounded sequence, then the operator T_m defined by

$$T_m f = \sum_{k=0}^{\infty} m(k) P_k f$$

is clearly bounded on $L^2(I\!R^n)$, but need not be bounded on $L^p(I\!R^n), p \neq 2$ unless some more conditions are imposed on the sequence. A sufficient condition on $m(k)$ is given by the Marcinkiewicz multiplier theorem proved in [36]:

Theorem 3.4.3 *Let Δ be the finite difference operator defined by $\Delta m(k) = m(k+1) - m(k)$. Assume that $J > \frac{n}{2}$ and for $j = 0, 1, 2, \cdots J$ the iterated finite differences $\Delta^j m(k)$ satisfy*

$$|\Delta^j m(k)| \leq C_j \, k^{-j}$$

where C_j are independent of k. Then the operator T_m is bounded on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$.

In view of this multiplier theorem we need only to verify that the sequence

$$m(k) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha \left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2}$$

satisfy the above estimates uniformly in t for $[\alpha] > \frac{n}{2}$. Using properties of the gamma function one can easily verify that the sequence

$$m_1(k) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}$$

satisfies the required estimates. Therefore, we have to consider the Laguerre polynomials alone. Now, the Laguerre polynomials verify the relation

$$L_{k+1}^\alpha(t) - L_k^\alpha(t) = L_k^{\alpha-1}(t)$$

and they also satisfy the uniform estimate, for $\alpha > -\frac{1}{3}$,

$$\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} \, |L_k^\alpha(t) e^{-\frac{1}{2}t}| \leq C$$

for all $t \in \mathbb{R}_+, k = 0, 1, 2, \cdots$. Therefore, if we use these two properties of the Laguerre functions, it is not difficult to show that the sequence $\{\psi_k^\alpha(t)\}$ verifies the conditions of the Marcinkiewicz multiplier theorem uniformly in t as long as $[\alpha] > \frac{n}{2}$. This completes the proof of theorem 3.4.2. □

Now it is natural to ask the following question : what is the smallest value of α such that S_t^α will be uniformly bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$? We conjecture that this critical value of α is $(\frac{n-1}{2})$. We do not consider this problem here, instead we look at the worst case, namely $\alpha = 0$, and prove some uniform $L^p - L^{p'}$ estimates for the operators S_t^α. When $\alpha = 0$ let us suppress α and write $L_k(t) = L_k^0(t)$. We consider the operator

$$S_t f = \sum_{k=0}^{\infty} L_k\left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2} P_k f.$$

For this operator we prove the following result.

Theorem 3.4.4 *For $f, g \in L^p(\mathbb{R}^n)$, $\frac{2n}{n+1} < p \leq 2$ we have the estimate*

$$\int_0^\infty |\langle S_t f, g\rangle|^2 t dt \leq C \|f\|_p^2 \|g\|_p^2.$$

As a corollary of this theorem we obtain the following interesting estimate regarding the Hermite projection operators.

Corollary 3.4.5 *For $f \in L^p(\mathbb{R}^n)$, $\frac{2n}{n+1} < p \leq 2$ we have*

$$\sum_{k=0}^{\infty} \|P_k f\|_2^4 \leq C \|f\|_p^4.$$

Proof : By taking $g = f$ in the theorem we have

$$\int_0^\infty |\langle S_t f, f\rangle|^2 t\, dt \leq C \|f\|_p^4.$$

But since P_k are projections

$$\langle S_t f, f\rangle = \sum_{k=0}^{\infty} L_k\left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2} \|P_k f\|_2^2.$$

The proof of the corollary is completed by noting that the family $\{L_k(\frac{1}{2}t^2)e^{-\frac{1}{4}t^2}\}$ is an orthonormal basis for $L^2(\mathbb{R}_+, t dt)$. □

Estimates of the form

$$\|P_k f\|_2 \leq C k^{\gamma(p)} \|f\|_p$$

are called $L^p - L^2$ restriction theorems for the Hermite projections and they play a crucial role in the study of Bochner-Riesz means for the Hermite expansions (see [36]). It is conjectured that

$$\|P_k f\|_2 \leq C\, k^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_p$$

for $1 \leq p < \frac{2n}{n+1}$. This conjecture has been verified only for the radial functions. The result of the corollary supports this conjecture.

We now turn to the proof of the theorem. We require the following two propositions. First we study mapping properties of the operator

$$K_t f(x) = \sum_{k=0}^{\infty} \left(\frac{2it+1}{2it-1}\right)^k P_k f(x)$$

for t real. Using the generating function identity (also known as Mehler's formula)

$$\sum_{\alpha} \Phi_\alpha(x)\, \Phi_\alpha(y)\, t^{|\alpha|} = \pi^{-\frac{n}{2}} (1-t^2)^{-\frac{n}{2}}\, e^{-\frac{1}{2}\frac{1+t^2}{1-t^2}(|x|^2+|y|^2)+\frac{2t}{1-t^2}x.y} \qquad (3.4.1)$$

for $0 < t < 1$ we can calculate the kernel of this operator. We can write

$$K_t f(x) = \int_{R^n} K_t(x,y) f(y)\, dy$$

and the kernel is given by

$$K_t(x,y) = c_n (1-2it)^n\, t^{-\frac{n}{2}}\, e^{iB(t,x,y)}$$

where B is real valued. We refer to [31] for this easy calculation. We are now ready to state and prove the following result.

Proposition 3.4.6 *For $1 \leq p \leq 2$ and $t > 0$*

$$\|K_t f\|_{p'} \leq C \left(\frac{t}{1+t^2}\right)^{-n(\frac{1}{p}-\frac{1}{2})} \|f\|_p.$$

Proof : By Riesz - Thorin convexity theorem it is enough to show that

$$\|K_t f\|_\infty \leq C\left(\frac{t}{1+t^2}\right)^{-\frac{n}{2}} \|f\|_1,$$

$$\|K_t f\|_2 \leq C \|f\|_2.$$

The first inequality follows from the explicit formula for the kernel and the second one follows from Plancherel theorem for the Hermite series since $\left(\frac{2it+1}{2it-1}\right)$ is of absolute value 1 for real t. □

In the next proposition we express the Laguerre functions $L_k(t)\, e^{-t/2}$ as the Fourier transform of certain functions.

Proposition 3.4.7 *For $t \geq 0$ we have*

$$L_k(t)e^{-\frac{1}{2}t} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-its}\frac{1}{2is-1}\left(\frac{2is+1}{2is-1}\right)^k ds.$$

Proof : The Laguerre functions are given by the generating function identity

$$\sum_{k=0}^{\infty} r^k L_k(t)e^{-\frac{1}{2}t} = (1-r)^{-1}e^{-\frac{1}{2}\frac{1+r}{1-r}t}$$

for $|r| < 1$. Therefore, it is enough to show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-its}\frac{1}{2is-1}\left\{\sum_{k=0}^{\infty} r^k \left(\frac{2is+1}{2is-1}\right)^k\right\} ds = (1-r)^{-1}e^{-\frac{1}{2}\frac{1+r}{1-r}t}$$

for $t \geq 0$. The geometric series on the left hand side can be summed and the integral can be evaluated using residue theorem. The value of the integral turns out to be zero for $t < 0$ and for $t \geq 0$ it is just the right hand side of the above equation. We leave the simple calculations to the interested reader. □

We now embark on the proof of theorem 3.4.4. Since

$$\langle S_t f, g\rangle = \sum_{k=0}^{\infty} L_k\left(\frac{2}{2}t^2\right) e^{-\frac{1}{4}t^2}\langle P_k f, g\rangle$$

and we are interested in the L^2 norm

$$\int_0^\infty |\langle S_t f, g\rangle|^2 \, t\, dt = \int_0^\infty |\sum_{k=0}^\infty L_k(t)\, e^{-\frac{1}{2}t} \langle P_k f, g\rangle|^2 \, dt.$$

In view of Plancherel theorem for the Fourier transform on $\mathbb{R}$ and proposition 3.4.7 we only need to show that

$$\int_{-\infty}^\infty \frac{1}{1+t^2} \, |\langle K_t f, g\rangle|^2 dt \leq C \|f\|_p^2 \, \|g\|_p^2.$$

By the result of propositon 3.4.6 we know that

$$|\langle K_t f, g\rangle| \leq C \left(\frac{t}{1+t^2}\right)^{-n(\frac{1}{p}-\frac{1}{2})} \|f\|_p \, \|g\|_p$$

and hence the above integral is dominated by

$$C^2 \|f\|_p^2 \, \|g\|_p^2 \int_{-\infty}^\infty \frac{1}{1+t^2} \left(\frac{t}{1+t^2}\right)^{-2n(\frac{1}{p}-\frac{1}{2})} dt.$$

This last integral is finite provided $\frac{2n}{n+1} < p \leq 2$. Hence the theorem. □

Theorem 3.4.8 *Let $\alpha \geq 0, s > \frac{1}{2}$ and $\frac{2n}{n+1} < p \leq 2$. Then we have the uniform estimates*

$$\|S_t^\alpha f\|_{p'} \leq C \, \|H^s f\|_p, \quad f \in L^p(\mathbb{R}^n).$$

Proof : Since for $\alpha \geq 0$, S_t^α can be expressed in terms of S_t, it is enough to prove a uniform estimate for the functions $S_t f$. To obtain this we need to get an estimate on $\sup_{t>0} |\langle S_t f, g\rangle|$ and this is achieved by the following trick. If a function $F(t)$ on $[0, \infty)$ has the expansion

$$F(t) = \sum_{k=0}^\infty a_k \, L_k(t)\, e^{-\frac{1}{2}t}$$

then

$$|F(t)|^2 \leq \left(\sum_{k=0}^\infty (2k+n)^{2s} \, |a_k|^2\right) \left(\sum_{k=0}^\infty (2k+n)^{-2s} \, |L_k(t)|^2 \, e^{-t}\right).$$

As the Laguerre functions $L_k(t)e^{-t/2}$ are uniformly bounded for $t \geq 0, k = 0, 1, 2, \cdots$ the second sum is finite provided $s > \frac{1}{2}$. Thus if $s > \frac{1}{2}$ we have

$$\|F\|_\infty^2 \leq C \sum_{k=0}^{\infty} (2k+n)^{2s} |a_k|^2.$$

Applying this trick to $\langle S_t f, g \rangle$ we have

$$\begin{aligned} \sup_{t>0} |\langle S_t f, g\rangle|^2 &\leq C \sum_{k=0}^{\infty} (2k+n)^{2s} |\langle P_k f, g\rangle|^2 \\ &= C \int_0^\infty \left| \sum_{k=0}^{\infty} L_k(t)\, e^{-\frac{1}{2}t} \langle P_k(H^s f), g\rangle \right|^2 dt \end{aligned}$$

where H^s is the fractional power of the Hermite operator defined by

$$H^s f = \sum_{k=0}^{\infty} (2k+n)^s P_k f.$$

Appealing to theorem 3.4.4 we see that

$$\sup_{t>0} |\langle S_t f, g\rangle| \leq C \|H^s f\|_p \|g\|_p$$

for $\frac{2n}{n+1} < p \leq 2, s > \frac{1}{2}$ which proves the theorem. □

The above theorem shows that the operators S_t^α are uniformly bounded from the Hermite Sobolev space $W_H^{s,p}(\mathbb{R}^n)$ into $L^{p'}(\mathbb{R}^n)$,. By allowing larger values of s we can show that S_t^α are bounded from $W_H^{s,p}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

Theorem 3.4.9 *Let $\alpha \geq 0, s > \frac{n+1}{2}$ and $1 \leq p \leq \infty$. Then holds the uniform estimate*

$$\|S_t^\alpha f\|_p \leq C \|H^s f\|_p, \qquad f \in L^p(\mathbb{R}^n).$$

Proof : We follow a similar line of reasoning as in the previous theorem. The proof is reduced to showing that for $s > \frac{n}{2}$

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2} |\langle K_t H^{-s} f, g\rangle|^2 \, dt \leq C \|f\|_p^2 \|g\|_{p'}^2.$$

This is guaranteed if we show that

$$\sup_{t>0} \|K_t H^{-s} f\|_p \leq C \|f\|_p, \; s > \frac{n}{2}.$$

The kernel of the operator $K_t H^{-s}$ can be explicitly calculated - this has been done in [31]. We refer to page 15 of that paper. From the calculations there it follows that the operator is uniformly bounded on $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$ provided $s > \frac{n}{2}$. This completes the proof of the theorem. □

Chapter 4

ANALOGUES OF BESICOVITCH - WIENER THEOREM FOR THE HEISENBERG GROUP

In this chapter we prove analogues of Besicovitch - Wiener theorem for the Heisenberg group H^n. We consider discrete measures of the form $\mu = \sum_{j=0}^{\infty} c_j\, \delta(a_j)$ with $\{c_j\} \in l^1 \cap l^2$, and consider spectral decomposition of μ in terms of the eigenfuctions of the sublaplacian on H^n. For this expansion we prove an analogue of Besicovitch - Wiener theorem. We also consider measures of the form $gd\mu_t$, where g is in $L^2(\mathbb{R})$ and μ_t is the normalised surface measure on the sphere $|z| = t \subset \mathbb{C}^n \subset H^n$ and prove Agmon - Hörmander type theorem for this expansion. We can also replace g by the measure $\sum c_j\, \delta_{a_j}$ with $\{c_j\} \in l^2$. We also consider the case of Hermite expansions on $\mathbb{R}^n$. We prove analogous results for measures of the form $f\, d\nu_1$ where ν_1 is the normalised surface measure on the sphere $|x| = 1$ in $\mathbb{R}^n$ and f is a square integrable function on this sphere. This is done by using a Hecke - Bochner type theorem for Hermite projection operators.

4.1 Fourier transform on the Heisenberg group

The $(2n+1)$ dimensional Heisenbeg group H^n is the set $\mathbb{C}^n \times \mathbb{R}$ with the operation

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2} Im z.\bar{w}\right)$$

where $z, w \in \mathbb{C}^n, t, s \in \mathbb{R}$. The Fourier transform on the Heisenberg group is defined using the infinite dimensional Schrödinger representations π_λ indexed by nonzero reals λ. These are all realised on $L^2(\mathbb{R}^n)$ and are given by $\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x.\xi+\frac{1}{2}x.y)} \varphi(\xi+y)$, for φ in $L^2(\mathbb{R}^n)$. Consequently the Fourier transform of a function f on H^n is the operator valued function

$$\hat{f}(\lambda) = \int f(z,t)\, \pi_\lambda(z,t)\, dz\, dt.$$

The Plancherel formula then reads

$$\|f\|_2^2 = (2\pi)^{-n-1} \int \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda.$$

where $\|T\|_{HS}$ is the Hilbert - Schmidt norm of the operator T.

Our point of departure is the following spectral decomposition of functions on H^n in terms of eigenfunctions of the sublaplacian on H^n. For $f \in H^n$ this expansion reads

$$f(z,t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} f * e_k^\lambda(z,t).$$

And the Plancherel formula for the above expansion is

$$\|f\|_2^2 = 2\pi \sum_{k=0}^{\infty} \int_{\mathbb{C}^n} \int_{-\infty}^{\infty} |f * e_k^\lambda(z,0)|^2\, d\lambda\, dz.$$

Here e_k^λ are the elementary spherical functions defined by

$$e_k^\lambda(z,t) = e^{-i\lambda t}\, \varphi_k^\lambda(z)$$

where $\varphi_k^\lambda(z) = L_k^{n-1}(\frac{1}{2}|\lambda||z|^2)\, e^{-\frac{1}{4}|\lambda||z|^2}$, L_k^{n-1} being the Laguerre polynomials of type $(n-1)$. For various properties of Fourier transform and spherical functions we refer to [33]. The book of Folland [7] gives a nice introduction to the Heisenberg group.

We now state and prove an analogue of the Besicovitch - Wiener theorem for the measure

$$\mu = \sum_{j=0}^{\infty} c_j\, \delta(z_j, t_j)$$

where $\delta(z_j, t_j)$ is the Dirac measure at the point (z_j, t_j). We need various properties of the Laguerre functions $\varphi_k(z)$. They satisfy the orthogonality relation $\varphi_k \times \varphi_j(z) = (2\pi)^n \varphi_k(z)\,\delta_{k,j}$ where $\delta_{k,j}$ is the Kronecker delta. Here $\varphi_k \times \varphi_j(z)$ denotes the twisted convolution $\int_{\mathbb{C}^n} \varphi_k(z-w)\,\varphi_j(w)\,e^{\frac{i}{2}Im z.\bar{w}}\,dw$ of φ_k and φ_j. They also satisfy the product formula

$$\int_{|w|=r} \varphi_k(z-w)\,e^{\frac{i}{2}Im z.\bar{w}}\,d\mu_r = \frac{k!(n-1)!}{(k+n-1)!}\,\varphi_k(r)\,\varphi_k(z)$$

where $\varphi_k(r)$ stands for $\varphi_k(w)$ with $|w| = r$, and μ_r is the normalised surface measure on the sphere $|w| = r$. We also need the following generating function identity

$$\sum_{k=0}^{\infty} \frac{k!}{(k+n-1)!}\,\varphi_k(r)\,\varphi_k(s)\,t^{2k} = (1-t^2)^{-1}\,e^{-\frac{1+t^2}{1-t^2}\frac{r^2+s^2}{4}}\,\frac{J_{n-1}(\frac{irst}{1-t^2})}{(\frac{irst}{2})^{n-1}}.$$

For these formulas we refer to [36]. We frequently use the following Tauberian theorem.

Theorem 4.1.1 *Let $\{\alpha_j\}$, $\{\lambda_j\}$ be sequences of real numbers such that $\alpha_j \geq 0$, $\lambda_j = j^{1/n} + o(j^{1/n})$. Then the following are equivalent :*

(1) $\sum_{j=1}^{\infty} e^{-\epsilon\lambda_j}\,\alpha_j \approx c_0\,\epsilon^{-(n-l)}$ as $\epsilon \downarrow 0$

(2) $\sum_{j=1}^{N} \alpha_j \approx c_0\,(\Gamma(n-l+1))^{-1}\,N^{\,1-l/n}$.

A proof of this theorem can be seen in [30].

Theorem 4.1.2 *Let $\mu = \sum_{j=0}^{\infty} c_j\,\delta(z_j, t_j)$, where the sequence (c_j) belongs to $l^1 \bigcap l^2$ and the z_j's are distinct. Then*

$$\lim_{N\to\infty} N^{-n} \sum_{k=0}^{N} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2\,dz = \frac{(2\pi)^n}{|\lambda|^n\,n!} \sum_{j=0}^{\infty} |c_j|^2.$$

Proof : We have $e_k^\lambda(z,t) = e^{-i\lambda t}\,\varphi_k(\sqrt{|\lambda|}z)$. Therefore,

$$\mu * e_k^\lambda(z,0) \;=\; \int_{H^n} e_k^\lambda\left(z-w, -s-\frac{1}{2}(Im z.\bar{w})\right)\,d\mu(w,s)$$

$$= \sum_{j=0}^{\infty} c_j \, e_k^{\lambda}\left(z - z_j, -t_j - \frac{1}{2}(Imz.\bar{z_j})\right)$$
$$= \sum_{j=0}^{\infty} c_j \, e^{-i\lambda(t_j + \frac{1}{2}(Imz.\bar{z_j}))} \, \varphi_k\left(\sqrt{|\lambda|}(z - z_j)\right).$$

Therefore,

$$\int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 \, dz$$
$$= \int_{\mathbb{C}^n} \sum_{j,p} c_j \, \bar{c_p} \, e^{-i\lambda(t_j - t_p + \frac{1}{2} Imz.\overline{z_j - z_p})} \varphi_k\left(\sqrt{|\lambda|}(z - z_j)\right) \, \varphi_k\left(\sqrt{|\lambda|}(z - z_p)\right) \, dz$$
$$= \sum_{j,p} c_j \, \bar{c_p} \, e^{-i\lambda(t_j - t_p)} \int_{\mathbb{C}^n} e^{i\frac{\lambda}{2} Im((z_j - z_p).\bar{z})} \, \varphi_k\left(\sqrt{|\lambda|}(z - z_j)\right) \varphi_k\left(\sqrt{|\lambda|}(z - z_p)\right) dz.$$

The interchange of the order of integration and the sum is justified by Fubini's theorem since (c_j) is in l^1 and

$$\int_{\mathbb{C}^n} |\varphi_k\left(\sqrt{|\lambda|}(z - z_j)\right)| \, |\varphi_k\left(\sqrt{|\lambda|}(z - z_p)\right)| \, dz \le |\lambda|^{-n} \int_{\mathbb{C}^n} |\varphi_k(z)|^2 \, dz.$$

A simple calculation shows that

$$\int_{\mathbb{C}^n} e^{i\frac{\lambda}{2} Im((z_j - z_p).\bar{z})} \, \varphi_k\left(\sqrt{|\lambda|}(z - z_j)\right) \, \varphi_k\left(\sqrt{|\lambda|}(z - z_p)\right) dz.$$
$$= |\lambda|^{-n} \, e^{i\frac{\lambda}{2} Imz_j.\bar{z_p}} \, \varphi_k \times \varphi_k\left(\sqrt{|\lambda|}(z_j - z_p)\right)$$
$$= (2\pi)^n \, |\lambda|^{-n} \, e^{i\frac{\lambda}{2} Imz_j.\bar{z_p}} \, \varphi_k\left(\sqrt{|\lambda|}(z_j - z_p)\right).$$

Thus we have

$$\int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 \, dz$$
$$= (2\pi)^n \, |\lambda|^{-n} \sum_{j,p} c_j \, \bar{c_p} \, e^{-i\lambda(t_j - t_p)} \, e^{i\frac{\lambda}{2} Imz_j.\bar{z_p}} \, \varphi_k\left(\sqrt{|\lambda|}(z_j - z_p)\right).$$

Now we consider the sum

$$\sum_{k=0}^{\infty} t^k \int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 dz.$$

As the functions $\varphi_k(z)$ are uniformly bounded (in fact, $|\varphi_k(z)| \le ck^{n-1}$), we can first sum with respect to k to get

$$\sum_{k=0}^{\infty} t^k \int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2 \, dz = (1-t)^{-n} \, (2\pi)^n \, |\lambda|^{-n}$$
$$\sum_{j,p} c_j \, \bar{c_p} \, e^{-i\lambda(t_j - t_p)} \, e^{i\frac{\lambda}{2} Imz_j.\bar{z_p}} \, e^{-\frac{1}{2}\frac{1+t}{1-t}|\lambda| \, |z_j - z_p|^2}.$$

In deriving the above we have used the generating function identity,(see Szegö [29]):

$$\sum_{k=0}^{\infty} t^k \varphi_k(z) = (1-t)^{-n} e^{-\frac{1}{2}\frac{1+t}{1-t}|z|^2}.$$

Since $|z_j - z_p| \neq 0$ for $j \neq p$ we see that

$$lim_{t\to 1-} e^{-\frac{1}{2}\frac{1+t}{1-t}|\lambda||z_j-z_p|^2} = 0$$

for $j \neq p$. Therefore,

$$lim_{t\to 1-}(1-t)^n \sum_{k=0}^{\infty} t^k \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \, dz = (2\pi)^n \, |\lambda|^{-n} \sum_{j=0}^{\infty} |c_j|^2.$$

By appealing to the Tauberian theorem we get the result. □

We next consider the surface measure on the sphere $S_r = \{(z,0) : |z| = r\} \subset H^n$. Let μ_r be the normalised surface measure on the sphere S_r.

Theorem 4.1.3 *Let $g \in L^2(I\!R)$ and let μ be the product of μ_r and $g\,dt$ on H^n. Then*

$$lim_{N\to\infty} N^{-\frac{1}{2}} \sum_{k=0}^{N} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \, |\lambda|^{2n-\frac{1}{2}} \, dz \, d\lambda = \frac{\pi^{n-1} 2^{2n-\frac{1}{2}} (n-1)!}{r^{2n-1}} \|g\|_2^2.$$

Proof : We have

$$\begin{aligned}
\mu * e_k^\lambda(z,0) &= \int_{H^n} e_k^\lambda(z-w, -s - \frac{1}{2} Im\, z\cdot\bar{w}) \, d\mu(w,s) \\
&= \int_{\mathbb{C}^n \times R} \varphi_k\left(\sqrt{|\lambda|}(z-w)\right) \, e^{-i\lambda(-s-\frac{1}{2}Im\, z\cdot\bar{w})} \, g(s) \, d\mu_r(w) \, ds \\
&= \int_{|w|=r} \varphi_k\left(\sqrt{|\lambda|}(z-w)\right) \, e^{i\lambda\frac{1}{2}Im\, z\cdot\bar{w}} \, d\mu_r(w) \int_R g(s) \, e^{i\lambda s} \, ds \\
&= \frac{k!\,(n-1)!}{(k+n-1)!} \varphi_k\left(\sqrt{|\lambda|} r\right) \, \varphi_k\left(\sqrt{|\lambda|}|z|\right) \, \hat{g}(\lambda).
\end{aligned}$$

Thus

$$\int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \, dz = |\lambda|^{-n} \, (2\pi)^n \frac{k!(n-1)!}{(k+n-1)!} \, |\varphi_k\left(\sqrt{|\lambda|} r\right)|^2 \, |\hat{g}(\lambda)|^2.$$

As in the previous theorem, we consider

$$\sum_{k=0}^{\infty} t^{2k} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \, |\lambda|^{2n-\frac{1}{2}} \, dz \, d\lambda$$
$$= \sum_{k=0}^{\infty} t^{2k} \frac{k!}{(k+n-1)!} (2\pi)^n \Gamma(n) \int_{-\infty}^{\infty} |\varphi_k\left(\sqrt{|\lambda|}r\right)|^2 \, |\hat{g}(\lambda)|^2 \, |\lambda|^{n-\frac{1}{2}} \, d\lambda.$$

First we claim that the sum can be taken inside the integral. To see this it is enough to check that the integral

$$\int \left\{ \sum_{k=0}^{\infty} t^{2k} \frac{k!}{(k+n-1)!} (2\pi)^n \Gamma(n) \, |\varphi_k\left(\sqrt{|\lambda|}r\right)|^2 \right\} |\hat{g}(\lambda)|^2 \, |\lambda|^{n-\frac{1}{2}} \, d\lambda$$

is finite. The generating function identity for the Laguerre functions gives

$$\sum_{k=0}^{\infty} t^{2k} \frac{k!}{(k+n-1)!} |\varphi_k\left(\sqrt{|\lambda|}r\right)|^2$$
$$= (1-t^2)^{-n} \left\{ i \frac{|\lambda| r^2 t}{2(1-t^2)} \right\}^{-n+1} J_{n-1}\left(i \frac{|\lambda| r^2 t}{1-t^2} \right) e^{-\frac{|\lambda| r^2}{2} \frac{1+t^2}{1-t^2}}.$$

The Bessel function $J_{n-1}(iz)$ has the estimate

$$|J_{n-1}(iz)| \le c \, z^{-\frac{1}{2}} e^{z} \qquad z \ge 1.$$

In view of this the above sum is bounded by

$$c_{r,t} \, |\lambda|^{-n+\frac{1}{2}} \, e^{-\frac{1}{2} \frac{1-t}{1+t} |\lambda| r^2}$$

and hence the integral under consideration is finite. Thus we have shown that

$$\sum_{k=0}^{\infty} t^{2k} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \, |\lambda|^{2n-\frac{1}{2}} \, dz \, d\lambda = \Gamma(n) (2\pi)^n (1-t^2)^{-n}$$
$$\int_{-\infty}^{\infty} |\hat{g}(\lambda)|^2 \left\{ i \frac{|\lambda| r^2 t}{2(1-t^2)} \right\}^{-n+1} J_{n-1}\left(i \frac{|\lambda| r^2 t}{1-t^2} \right) e^{-\frac{1}{2} \frac{1+t^2}{1-t^2} |\lambda| r^2} \, |\lambda|^{n-\frac{1}{2}} \, d\lambda.$$

Consider the function

$$(1-t^2)^{-n+\frac{1}{2}} \left\{ i \frac{|\lambda| r^2 t}{2(1-t^2)} \right\}^{-n+1} J_{n-1}\left(i \frac{|\lambda| r^2 t}{1-t^2} \right) e^{-\frac{1}{2} \frac{1+t^2}{1-t^2} |\lambda| r^2}.$$

Using the integral representation for the Bessel function this is equal to

$$(1-t^2)^{-n+1/2}\, e^{-\frac{|\lambda| r^2}{2}\frac{1+t^2}{1-t^2}}\, \frac{1}{\Gamma(1/2)\,\Gamma(n-1/2)} \int_{-1}^{1} (1-s^2)^{n-3/2}\, e^{-\frac{|\lambda| r^2 ts}{1-t^2}}\, ds$$
$$= \frac{(1-t^2)^{-n+1/2}}{\Gamma(1/2)\Gamma(n-1/2)} \int_{-1}^{1} (1-s^2)^{n-3/2}\, e^{-\frac{|\lambda| r^2}{2}\frac{1-t}{1+t}}\, e^{-\frac{|\lambda| r^2}{2}\frac{2t}{1-t^2}(1-s)}\, ds.$$

A simple computation shows that this is equal to

$$\frac{e^{-\frac{|\lambda| r^2}{2}\frac{1-t}{1+t}}}{\Gamma(1/2)\Gamma(n-1/2)} \int_0^{2/1-t^2} (y[2-y(1-t^2)])^{n-3/2}\, e^{-|\lambda| r^2 ty}\, dy.$$

As $t \to 1-$, this converges to

$$\frac{1}{\Gamma(1/2)\Gamma(n-1/2)} \int_0^{\infty} (2y)^{n-3/2}\, e^{-|\lambda| r^2 y}\, dy = 2^{n-3/2}\, \pi^{-1/2}\, (|\lambda| r^2)^{-n+\frac{1}{2}}.$$

Therefore we have proved

$$lim_{t\to 1-}(1-t^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} t^{2k} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2\, |\lambda|^{2n-\frac{1}{2}}\, dz\, d\lambda$$
$$= \Gamma(n)\, 2^{2n-\frac{3}{2}}\, \pi^{n-\frac{1}{2}}\, r^{-2n+1} \int_{-\infty}^{\infty} |\hat{g}(\lambda)|^2\, d\lambda$$

By appealing to Tauberian theorem we complete the proof. □

In the above theorem we can replace $g(t)\, dt$ by a discrete measure $\sum c_j\, \delta_{a_j}$, where a_j's are distinct. Then as a corollary we obtain the following result.

Corollary 4.1.4 *Let μ be the product of μ_r and the discrete measure $\sum_j c_j\, \delta_{a_j}$, where a_j's are distinct. Then*

$$lim_{M\to\infty} lim_{N\to\infty} \frac{N^{-\frac{1}{2}}}{M} \int_{-M}^{M} \sum_{k=0}^{N} \int_{\mathbb{C}^n} |\mu * e_k^{\lambda}(z,0)|^2\, |\lambda|^{2n-1/2}\, dz\, d\lambda$$
$$= \frac{\pi^{n-1}(n-1)!\, 2^{2n-1/2}}{r^{2n-1}} \sum_{j=0}^{\infty} |c_j|^2.$$

Proof : In this case

$$\mu * e_k^{\lambda}(z,0) = \int_{|w|=r} \varphi_k\left(\sqrt{|\lambda|}(z-w)\right)\, e^{i\frac{\lambda}{2} Im z.\bar{w}}\, d\mu_r(w)\, F(\lambda),$$

where

$$F(\lambda) = \sum_j c_j \, e^{i\lambda a_j}.$$

Therefore,

$$\mu * e_k^\lambda(z,0) = \frac{k!(n-1)!}{(k+n-1)!} \varphi_k\left(\sqrt{|\lambda|}r\right) \varphi_k\left(\sqrt{|\lambda|}z\right) \, F(\lambda).$$

Hence

$$\int_{-M}^{M}\int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \, |\lambda|^{2n-1/2} \, dz \, d\lambda$$
$$= (2\pi)^n \frac{k!(n-1)!}{(k+n-1)!} \int_{-M}^{M} \left\{\varphi_k\left(\sqrt{|\lambda|}r\right)\right\}^2 |F(\lambda)|^2 \, |\lambda|^{n-\frac{1}{2}} d\lambda.$$

As in the theorem we can show that

$$lim_{N\to\infty} N^{-\frac{1}{2}} \sum_{k=0}^{N} \int_{-M}^{M}\int_{\mathbb{C}^n} |\mu * e_k^\lambda(z,0)|^2 \; |\lambda|^{2n-1/2} \, dz \, d\lambda$$
$$= \frac{\pi^{n-1}(n-1)! \, 2^{2n-\frac{1}{2}}}{r^{2n-1}} \int_{-M}^{M} |F(\lambda)|^2 \, d\lambda.$$

By Wiener's theorem

$$lim_{M\to\infty} \frac{1}{M} \int_{-M}^{M} |F(\lambda)|^2 d\lambda = \sum_{j=0}^{\infty} |c_j|^2.$$

This completes the proof of the corollary. □

4.2 The case of Hermite expansions

Consider the normalised Hermite functions $\Phi_\alpha(x)$ on $\mathbb{R}^n$. These are indexed by multi indices $\alpha \in \mathbb{N}^n$ and are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$. In fact $H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha$ where $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $\{\Phi_\alpha\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. The Plancherel theorem for the Hermite expansion reads

$$\int |f|^2 dx = \sum_\alpha |\hat{f}(\alpha)|^2.$$

where $\hat{f}(\alpha) = \int f(x)\Phi_\alpha(x)dx$. The Hermite functions Φ_α satisfy the Mehler's formula

$$\sum_\alpha \Phi_\alpha(x)\,\Phi_\alpha(y)\,t^{|\alpha|} = \pi^{-\frac{n}{2}}\,(1-t^2)^{-\frac{n}{2}}\,e^{-\frac{1}{2}\frac{1+t^2}{1-t^2}(|x|^2+|y|^2)+\frac{2t}{1-t^2}x.y}$$

for $0 < t < 1$. We now prove an analogue of Wiener's theorem for Hermite expansions. The following result is due to Strichartz (see[28]).

Theorem 4.2.1 *Let $\mu = \sum c_j\,\delta(a_j) + \nu$ where ν is absolutely continuous and a_j's are distinct. Then*

$$lim_{N\to\infty} N^{-\frac{n}{2}} \sum_{|\alpha|\le N} |\hat{\mu}(\alpha)|^2 = \frac{(2\pi)^{-n/2}}{\Gamma(n/2+1)} \sum |c_j|^2$$

where $\hat{\mu}(\alpha) = \int \Phi_\alpha(x)d\mu$.

Proof : From Mehler's formula we see that

$$\begin{aligned}(1-t^2)^{n/2}\sum_\alpha |\hat{\mu}(\alpha)|^2\,t^{|\alpha|} &= \int\int (1-t^2)^{n/2}\sum_\alpha t^{|\alpha|}\,\Phi_\alpha(x)\,\Phi_\alpha(y)\,d\mu(x)\,\overline{d\mu(y)} \\ &= \pi^{-n/2}\int\int G_t(x,y)\,d\mu(x)\,d\mu(y).\end{aligned}$$

Where

$$\begin{aligned}G_t(x,y) &= exp\left(-\frac{1}{2}\frac{1+t^2}{1-t^2}(|x|^2+|y|^2) + \frac{2t}{1-t^2}(|x|^2-|y|^2)\right) \\ &= exp\left(-\frac{t}{1-t^2}|x-y|^2 - \frac{1}{2}\left(\frac{1-t}{1+t}\right)(|x|^2+|y|^2)\right).\end{aligned}$$

Notice that $G_t(x,y)$ is uniformly bounded by 1 and

$$lim_{t\to 1^-} G_t(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}.$$

Therefore, we see that

$$\begin{aligned}lim_{t\to 1^-}(1-t^2)^{n/2}\sum_\alpha |\hat{\mu}(\alpha)|^2\,t^{|\alpha|} &= \pi^{-n/2}\int\int \chi_{(x=y)}\,d\mu(x)\,\overline{d\mu(y)} \\ &= \pi^{-n/2}\int \mu(y)\,\overline{d\mu(y)} \\ &= \pi^{-n/2}\sum_{j=0}^{\infty}|c_j|^2.\end{aligned}$$

Setting $\epsilon = 1 - t$ we can rewrite this as

$$\begin{aligned} lim_{\epsilon \to 0+} \epsilon^{n/2} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} |\hat{\mu}(\alpha)|^2 e^{-\epsilon|\alpha|} &= lim_{\epsilon \to 0+} \epsilon^{n/2} \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} |\hat{\mu}(\alpha)|^2 \right) e^{-\epsilon k} \\ &= (2\pi)^{-n/2} \sum_{j=0}^{\infty} |c_j|^2. \end{aligned}$$

Now by using the Tauberian theorem we get the result. □

We now consider the case of the surface measure ν_r on the sphere $|x| = r$. More generally we consider measures of the form $f d\nu_r$, where f is a square integrable function on the sphere $|x| = r$. To treat such measures we need the following Hecke - Bochner type identity for the Hermite projection operators. Let $P_k(f)$ stand for the projection of f onto the k - th eigenspace spanned by $\Phi_\alpha(x), |\alpha| = k$, that is

$$P_k f(x) = \sum_{|\alpha|=k} \hat{f}(\alpha) \Phi_\alpha(x).$$

Let L_k^δ be the Laguerre polynomial of the type δ and define

$$R_k^\delta(f) = \frac{2\,\Gamma(k+1)}{\Gamma(k+\delta+1)} \int_0^\infty f(r)\, L_k^\delta(r^2)\, e^{-\frac{r^2}{2}}\, r^{2\delta+1}\, dr.$$

The following proposition has been proved in [35].

Proposition 4.2.2 *Assume that $f(x) = f_0(|x|)\, p(x)$ where $p(x)$ is a solid harmonic of degree m. Then*

$$P_{2k+m} f(x) = F_k(|x|)\, p(x)$$

where $F_k(r) = R_k^\delta(f)\, L_k^\delta(r^2)\, e^{-\frac{1}{2}r^2}$ with $\delta = \frac{n}{2} + m - 1$. For other values of k, $P_k(f) = 0$.

For a measure $d\mu$ on $I\!R^n$, let $P_k(d\mu)$ be defined by $P_k(d\mu)(x) = \sum_{|\alpha|=k} \left(\int \Phi_\alpha(y)\, d\mu(y) \right) \Phi_\alpha(x)$.

Theorem 4.2.3 *Let ν be the normalised surface measure on S^{n-1} and let $f \in L^2(S^{n-1}, d\nu)$. Then $lim_{N \to \infty} N^{-\frac{1}{2}} \sum_{k=0}^{N} \|P_k(f d\nu)\|_2^2 = \frac{2}{\pi} \int_{S^{n-1}} |f|^2 d\nu$.*

Proof : Expand f in terms of spherical harmonics $f = \sum c_m Y_m$ where Y_m is a spherical harmonic of degree m. In view of the proposition it is easy to see that

$$P_{2k+m}(Y_m d\nu) = \frac{2\,\Gamma(k+1)}{\Gamma(k+\delta+1)} L_k^\delta(1)\, e^{-\frac{1}{2}}\, L_k^\delta(|x|^2)\, e^{-\frac{1}{2}|x|^2}\, Y_m.$$

As various Y_m's are orthogonal to each other it is enough to prove the theorem when $f = Y_m$.

$$\begin{aligned}\sum_{k=0}^{\infty} t^k\, \|P_{2k+m}(Y_m d\nu)\|_2^2 &= 2\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} (L_k^\delta(1)\, e^{-\frac{1}{2}})^2\, t^k \\ &= 2\,(1-t)^{-1}\, e^{-\frac{1+t}{1-t}}\, \frac{J_\delta(\frac{2i\sqrt{t}}{1-t})}{(i\,\sqrt{t})^\delta}.\end{aligned}$$

Now we can proceed as in theorem 4.1.3 to conclude the proof. □

Similar results can also be proved in the case of special Hermite expansions. By the term we mean an expansion of the type

$$f = (2\pi)^{-n} \sum f \times \varphi_k$$

where f is a function on $\mathbb{C}^n$. These are in a way particular cases of results in the previous section when we consider functions on the Heisenberg group that are independent of t.

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