SOME CONTRIBUTIONS TO LINEAR COMPLEMENTARITY PROBLEM

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TO
MY BELOVED
LATE YOUNGER BROTHER

G. VIJAYASARATHY

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CONTENTS

GLOSSARY OF NOTATION

Sets

$\bar{n}$	the set $\{1, 2, \ldots, n\}$, n is any positive integer
n^*	all nonempty subsets of $\bar{n}$
$\lvert\alpha\rvert$	cardinality of the set α
$\alpha,\ \beta,\ \gamma$	denote the subsets of $\bar{n}$
$\bar{\alpha}$	complement of the set α relative to $\bar{n}$
$\alpha \setminus \beta$	the set $\alpha \cap \beta^c$
$\alpha \triangle \beta$	the set $(\alpha \cup \beta) \setminus (\alpha \cap \beta)$

Spaces

$\boldsymbol{R}^n$	real n-dimentional space
$\boldsymbol{R}^{m \times n}$	the space of $m \times n$ real matrices
$\boldsymbol{R}^n_+$	the nonnegative orthant of $\boldsymbol{R}^n$
$\boldsymbol{R}^n_{++}$	the positive orthant

Vectors

z^t	transpose of z
$x^t y$	the standard inner product between x and y
$x \geq y$	$x_i \geq y_i$ for all $i \in \bar{n}$
$x > y$	$x_i > y_i$ for all $i \in \bar{n}$
$supp(z)$	support of z, i.e., $\{i \in \bar{n} : z_i \neq 0\}$
probability vector	a nonnegative vector with sum of its coordinates equal to one

Matrices

$A = (a_{ij})$	a matrix with a_{ij}'s as its entries. We denote the entries of a matrix by the corresponding lower case letters, for example entries of B are denoted by b_{ij}
$A \leq B$	$a_{ij} \leq b_{ij}$ for all $i,\ j$

Matrices

$A < B$	$a_{ij} < b_{ij}$ for all i, j
$det\,A$	determinant of matrix A
I	the identity matrix
$A_{\alpha\beta}$	the submatrix of A obtained by dropping rows and columns of A corresponding $\bar{\alpha}$ and $\bar{\beta}$ respectively
$A_{\alpha.}$	stands for $A_{\alpha\bar{n}}$, $A \in \boldsymbol{R}^{m\times n}$
$A_{.\beta}$	stands for $A_{\bar{n}\beta}$, $A \in \boldsymbol{R}^{m\times n}$
$A_{i.}$	i^{th} row of A
$A_{.j}$	j^{th} column of A
diag $(a_1, \ldots, a_n)$	diagonal matrix with $a_{ii} = a_i$
$v(A)$	value of (matrix game) A
$\wp_\alpha(A)$	principal pivotal transoform of A with resepect to α
$SP(A)$	sign pattern of A

Sign Symbols

$\ominus$	nonpositive real
$\oplus$	nonnegative real

LCP Notations

(q, A)	LCP with data q and A
$F(q, A)$	set of all feasible solutions of (q, A)
$S(q, A)$	set of all solutions of (q, A)
$K(A)$	set of all q such that $S(q, A) \neq \phi$
$C_A(\alpha)$	complementary submatrix, $\alpha \subseteq \bar{n}$
$pos\,A$	cone generated by columns of A, $\{Ax : x \geq 0\}$

MATRIX CLASSES

Symbol	Definition
$\boldsymbol{C}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : x^tAx \geq 0 \ \forall x \in \boldsymbol{R}^n_+\}$
$\boldsymbol{C}_o^+$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C}_o : [x^tAx = 0, \ x \in \boldsymbol{R}^n_+] \Rightarrow (A + A^t)x = 0\}$
$\boldsymbol{E}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\ 0 \neq x \in \boldsymbol{R}^n_+ \ \exists\ k \in \bar{n} \ni x_k > 0 \text{ and } (Ax)_k > 0\}$
$\boldsymbol{E}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\ 0 \neq x \in \boldsymbol{R}^n_+ \ \exists\ k \in \bar{n} \ni x_k > 0 \text{ and } (Ax)_k \geq 0\}$
$\boldsymbol{E}_o^f$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in \bar{n}, det\, A_{\alpha\alpha} \neq 0 \Rightarrow \wp_\alpha(A) \in \boldsymbol{E}_o\}$
$\boldsymbol{N}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, det\, A_{\alpha\alpha} < 0\}$
$\boldsymbol{N}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, det\, A_{\alpha\alpha} \leq 0\}$
$\boldsymbol{P}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, det\, A_{\alpha\alpha} > 0\}$
$\boldsymbol{P}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, det\, A_{\alpha\alpha} \geq 0\}$
$\boldsymbol{Q}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : S(q, A) \neq \phi \ \forall\ q \in \boldsymbol{R}^n\}$
$\boldsymbol{Q}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall q \in \boldsymbol{R}^n, \ F(q, A) \neq \phi \Rightarrow S(q, A) \neq \phi\}$
$\bar{\boldsymbol{Q}}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, A_{\alpha\alpha} \in \boldsymbol{Q}\}$
$\bar{\boldsymbol{Q}}_0$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, A_{\alpha\alpha} \in \boldsymbol{Q}_o\}$
$\boldsymbol{R}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : (q, A)$ has a unique solution $\forall\ q \geq 0\}$
$\boldsymbol{R}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : (0, A)$ has a unique solution $\}$
$\boldsymbol{S}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : v(A) > 0\}$
$\bar{\boldsymbol{S}}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, v(A_{\alpha\alpha}) > 0\}$
$\boldsymbol{S}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : v(A) \geq 0\}$
$\bar{\boldsymbol{S}}_o$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall \alpha \in n^*, v(A_{\alpha\alpha}) \geq 0\}$
$\boldsymbol{U}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \lvert S(q, A)\rvert = 1 \ \forall\ q \in$ interior of $K(A)\}$.
$\boldsymbol{V}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : \forall\ \alpha \in \bar{n}, SP(A_{\alpha\alpha} z_\alpha) = (0, 0, \ldots, 0, \ominus)^t \Rightarrow z_\alpha \not> 0\}$
$\boldsymbol{Z}$	$\cup_n\{A \in \boldsymbol{R}^{n\times n} : a_{ij} \leq 0 \ \forall\ i \neq j\}$

ABSTRACT

This dissertation deals with a number of problems related to linear complementarity problem (LCP). Given a real square matrix A of order n and a real n-vector q, the LCP is to find a nonnegative n-vector z such that $Az + q \geq 0$ and $z^t(Az + q) = 0$. There is vast literature on LCP, evolved during the last four decades. LCP plays a crucial role in the study of mathematical programming from the view point of algorithms as well as applications. The inherent nature of the problem has led the researchers to introduce and study a variety of matrix classes in connection with LCP. Most of the work of this dissertation pertains to LCP within the class of semimonotone matrices ($\boldsymbol{E}_0$) introduced by Eaves. Besides, results on the matrix classes $\boldsymbol{Q}$ and $\boldsymbol{Q}_0$, which are of fundamental interest in the theory of LCP, are presented. The usefulness of these results is demonstrated through a number of applications. The gist of the dissertation is presented below in a chapter-wise summary.

Chapter 1 is introductory in nature and presents LCP and related material that is needed for the discussion in the subsequent chapters. In each of the chapters from 2 to 5, the first section introduces the background of the contents of that chapter while the second section presents the related results from the literature; the subsequent sections are devoted to our work pertaining to that chapter.

Al-Khayyal (1991) specified a condition (through polyhedral sets) for a matrix to belong to $\boldsymbol{P}_0$ and raised the question whether the same could be sufficient for membership in $\boldsymbol{Q}_0$. In Chapter 2, while answering his question in the affirmative, the author further relaxed the condition and obtained a new one which is sufficient for membership in $\boldsymbol{Q}_0$. It has also been established that, if $A \in \boldsymbol{R}^{n \times n}$ satisfies the relaxed condition and q is such that (q, A) has a feasible solution, then the solutions of (q, A) can be obtained by solving a suitable linear programming problem (LPP). Examples of matrices which satisfy the relaxed condition have been provided. The problem of solving LCP as LPP has been studied at length by several authors. It is known that when LPPs have solutions, they have solutions with bases. However, this is not so with LCP. A linear complementarity problem may have a solution but may not have a complementary basis. The author analyses this aspect in Section 2.4 and provides a sufficient condition under which the existence of complementary bases can be guaranteed.

In Chapter 3, results pertaining to the matrix classes $\boldsymbol{Q}$ and $\boldsymbol{Q}_0$ are presented. Though a number of subclasses of $\boldsymbol{Q}$ and $\boldsymbol{Q}_0$ have been identified, the

problem of testing whether a given matrix A belongs to $\boldsymbol{Q}$ or $\boldsymbol{Q}_0$, in general, remains complex. In this regard, we present some elementary propositions providing sufficient conditions under which the membership in $\boldsymbol{Q}$ can be asserted. The usefulness of these propositions is demonstrated through a number of applications. Section 3.4 is devoted to our results on $\boldsymbol{Q}_0$-matrices. Many of these results are analogies of known results on $\boldsymbol{Q}$-matrices. The main results are : characterization of nonnegative $\boldsymbol{Q}_0$-matrices, necessary and sufficient conditions, in some special cases, for a matrix to be completely $\boldsymbol{Q}_0$, sufficient conditions for principal submatrices of order $(n-1)$ of $n \times n$ to be in $\boldsymbol{Q}_0$, and necessary conditions on $\boldsymbol{Q}_0$-matrices in some special cases. As an application of characterization of nonnegative $\boldsymbol{Q}_0$-matrices, it is established that, if $A + A^t$ is a nonnegative $\boldsymbol{Q}_0$-matrix, both A and A^t are $\boldsymbol{Q}_0$-matrices.

Results in Chapter 4 are concerned with the class of semimonotone matrices introduced by Eaves (1971). In Section 4.3, we settle a conjecture initially posed by Pang (1979) and later modified by Jeter and Pye (1984) and Gowda (1990). The conjecture in its modified version states that every copositive $\boldsymbol{Q}$-matrix is an $\boldsymbol{R}_0$-matrix. A counter example is constructed to show that the conjecture is false. We derive conditions under which copositive (semimonotone) $\boldsymbol{Q}$-matrices belong to $\boldsymbol{R}_0$. It is well known that symmetric semimonotone $\boldsymbol{Q}$-matrices are completely $\boldsymbol{Q}$. We show, in Section 4.6, that symmetric semimonotone $\boldsymbol{Q}_0$-matrices are completely $\boldsymbol{Q}_0$. Another important result in this section is the extension of Pang's result on $\boldsymbol{E}_0 \cap \boldsymbol{Q}$-matrices which states that if A is in $\boldsymbol{E}_0 \cap \boldsymbol{Q}_0$, then every nontrivial solution of $(0, A)$ has at least two nonzero coordinates. It is established that if A is a $\boldsymbol{E}_0 \cap \boldsymbol{Q}_0$-matrix and if every row of A has a positive entry, then every nontrivial solution of $(0, A)$ has at least two nonzero coordinates.

The results in Chapter 5 pertain to the class of fully semimonotone $(\boldsymbol{E}_0^f)$ matrices introduced by Cottle and Stone (1983). Stone (1981) proved that, within the class of $\boldsymbol{Q}_0$-matrices, the $\boldsymbol{U}$-matrices are $\boldsymbol{P}_0$-matrices, and conjectured that the same must be true for $\boldsymbol{E}_0^f$. It is established that this conjecture is true for matrices of order upto 4×4, and in a number of special cases (of any order). The special cases include $\boldsymbol{E}_0^f$-matrices which are either symmetric or nonnegative or copositive-plus or $\boldsymbol{Z}$-martrices or $\boldsymbol{E}$-matrices. In the sequel we introduce a subclass of $\boldsymbol{E}_0^f$, the class of fully copositive $(\boldsymbol{C}_0^f)$ matrices, and show that $\boldsymbol{C}_0^f \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$.

CHAPTER 1

INTRODUCTION

1.1. THE LINEAR COMPLEMENTARITY PROBLEM

This dissertation deals with and is centered around the linear complementarity problem (LCP). LCP is a combination of a linear and nonlinear system of inequalities and equations, and may be stated as follows:

Given a real square matrix A of order n and a real n-vector q, the linear complementarity *problem* with data A and q is to find a real n-vector z satisfying the following conditions:

$$Az + q \geq 0, \tag{1.1}$$

$$z \geq 0, \tag{1.2}$$

$$\text{and} \quad z^t(Az + q) = 0. \tag{1.3}$$

We shall denote the LCP with data A and q by (q, A). By a feasible solution to (q, A), we mean a real n-vector z which satisfies conditions (1.1) and (1.2). Further, if z also satisfies (1.3), then z is called a solution to (q, A).

LCP is treated as a part of optimization theory and equilibrium problems. Though the origin of the subject dates back to the year 1940, it picked up momentum only in the mid sixties. LCP has gained importance as it is the unification of several optimization and equilibrium problems; in particular, the linear and quadratic programming problems as well as the equilibrium problems (both physical and economic) can be formulated as LCPs. It was the algorithm of Lemke and Howson (1964), which was developed by them to solve the bimatrix games, that brought the LCP into limelight. Ever since, the subject has been a fertile field for researchers.

LCP has a wide range of applications encompassing fields such as control theory, economics, engineering, game theory and optimization (see Lemke and

Howson (1964), Cottle and Dantzig (1968), Dantzig and Manne (1974), Cohen (1975), and Balas (1981)). Despite four decades of extensive research satisfactory answers have not yet been found to the following two fundamental questions :

(i) for a given general matrix $A \in \boldsymbol{R}^{n \times n}$, what are the conditions under which (q, A) has a solution for every $q \in \boldsymbol{R}^n$?

(ii) for a given general matrix $A \in \boldsymbol{R}^{n \times n}$, what are the conditions under which (q, A) has a solution whenever it has a feasible solution ?

Basically these are the questions which have triggered the researchers to study the matrix classes extensively in connection with LCP. Thus the study of LCP has drifted, partially, to the study of a host of matrix classes in relation to linear complementarity problems. Most of the work in this dissertation also deals with the study of well known classes of matrices. A brief overview of this dissertation is given in the remaining part of this section.

In section 1.2 of this chapter we present a brief background of LCP. Some important preliminary results on LCP are outlined. Since we make use of some results from Linear Programming (LP) and Game Theory, a brief discussion on these topics with some important results is included in sections 1.3 and 1.4. The notion of sign pattern of matrices is introduced in section 1.5. Using sign pattern of matrices, several results (in Chapter 5) have been derived.

Aganagic and Cottle (1987) gave an impressive characterization of $\boldsymbol{Q}_o$-matrices within the class of $\boldsymbol{P}_o$-matrices and established that (q, A) can be processed by Lemke's algorithm, with a suitable apparatus to resolve degeneracy, when A is a $\boldsymbol{P}_o \cap \boldsymbol{Q}_o$-matrix (see glossary for definitions $\boldsymbol{P}_o$ and $\boldsymbol{Q}_o$). Al-Khayyal (1991) specified a sufficient condition, through polyhedral sets, for a matrix to be in $\boldsymbol{P}_o$ and conjectured that the same must be sufficient for membership in $\boldsymbol{Q}_o$. This problem is addressed in Chapter 2. While answering his question affirmatively we further relax his condition and obtain a new one which is sufficient for membership in $\boldsymbol{Q}_o$.

Further, it is established that if $A \in \boldsymbol{R}^{n \times n}$ satisfies the relaxed condition and $q \in \boldsymbol{R}^n$ is such that (q, A) has a feasible solution, then (q, A) can be solved as a linear programming problem (LPP). Thus, the condition specified by Al-Khayyal and our relaxed condition are both concerned with the relationship between LPP and LCP. This aspect has been studied at length by several authors. For details see Mangasarian (1976a, 1976b, 1978), Cottle and Pang (1978a ,1978b), Al-Khayyal (1989, 1991) . In section 2.3 of Chapter 2 we identify some classes of matrices and provide two examples which satisfy the relaxed condition.

It is well known that, when LPPs have optimal solutions, they have optimal solutions with optimal bases. However, this is not so with LCP. An LCP may have a solution but not necessarily with a complementary basis. We analyse this aspect in section 2.4 of Chapter 2.

Despite the vast literature on the subject, there are no efficient methods to test whether a given matrix belongs to $\boldsymbol{Q}$ or $\boldsymbol{Q}_0$. For 2×2 matrices, one can easily check for membership in $\boldsymbol{Q}$ or $\boldsymbol{Q}_0$ graphically. For higher dimensional matrices, in general, it is very difficult to check this. But, a number of sufficient conditions are available to assert membership in $\boldsymbol{Q}$ or $\boldsymbol{Q}_0$. A large number of subclasses of $\boldsymbol{Q}$ and $\boldsymbol{Q}_0$ have also been identified (see Cottle, Pang and Stone (1992) and Murty (1988)). In this regard, we present some useful elementary propositions in section 3.3 of Chapter 3 which provide sufficient conditions for membership in $\boldsymbol{Q}$. The usefullness of these propositions is demonstrated through a number of applications.

Several properties of $\boldsymbol{Q}_0$-matrices are presented in section 3.4. Murty (1972) gave a characterization of nonnegative $\boldsymbol{Q}$-matrices. We present a characterization of nonnegative $\boldsymbol{Q}_0$-matrices and establish several applications of this result. Jeter and Pye (1985) studied the connections of $\boldsymbol{Q}$-matrices with their principal submatrices. While extending these results to $\boldsymbol{Q}_0$-matrices, we establish that stronger conclusions are possible. In particular, we derive conditions, in terms of *principal pivotal transforms*, for a $n \times n$

Q_o-matrix to have all its principal submatrices of order $(n-1)$ in Q_o. As an interesting application of this result, we show, in Chapter 4, that a symmetric semimonotone matrix is Q_o if, and only if, it is *completely-Q_o*. The study of completely Q_o-matrices, in general, is a complex problem (see Cottle (1980), and Fredricksen, Watson and Murty (1986)).

Chapter 4 deals with results on semimonotone and copositive matrices. Semimonotone class is one of the large classes and contains P_o and copositive matrices among others (see Cottle, Pang and Stone (1992)). If A is such that $(0, A)$, the homogeneous LCP, has a unique solution, namely $z = 0$, then A is called an R_o-matrix. The homogeneous LCP plays a key role in the study of LCP (see Cottle, Pang and Stone (1992)). While Aganagic and Cottle (1979) proved that $P_o \cap Q$ is same as $P_o \cap R_o$, Pang (1979b) proved that semimonotone R_o-matrices are Q-matrices and conjectured that the converse must be true. Subsequently, Jeter and Pye (1989) produced a counter example to disprove this. Later Gowda (1990) showed that Pang's conjecture is true in the case of symmetric matrices. It is known that symmetric semimonotone matrices are copositive. Jeter and Pye as well as Gowda raised the question : Are copositive Q-matrices contained in R_o? As a counter example, we provide (in Chapter 4) a 4×4 copositive Q-matrix which is not R_o.

Also Chapter 4 contains some more results on copositive and semimonotone matrices. These results provide sufficient conditions under which Q-matrices; that are either copositive or semimonotone, belong to R_o. Primarily, these results are of theoretical interest and may not have any bearing on algorithmic aspects. At the end of the chapter, some examples are provided to dispel some thoughts on possible extensions of our results.

Cottle and Stone (1983) introduced the classes U and E_0^f-matrices and investigated their properties. They observed that $U \subseteq E_0^f$. Stone (1981) showed that $U \cap Q_o \subseteq P_o$ and conjectured that $E_0^f \cap Q_o \subseteq P_o$. In Chapter 5, we show that this conjecture is true for matrices of order upto 4×4 and partially resolve it for higher order matrices. This is done by showing that if $A \in E_0^f \cap Q_o$ and if every proper principal minor of A is nonnegative, then $A \in \boldsymbol{P}_o$. From this key result, we deduce that the conjecture is true for symmetric matrices. Another important corollary of the key result is that $\bar{Q}_0 \cap \boldsymbol{E}_o^f \subseteq \boldsymbol{P}_o$, where $\bar{Q}_0$ is the class of completely $\boldsymbol{Q}_o$-matrices. As a consequence of this corollary we identify large subclasses of $\boldsymbol{E}_o^f \cap \boldsymbol{Q}_o$ for which the conjecture is true. In the sequel, we introduce a subclass of $\boldsymbol{E}_o^f$-matrices, namely, the class of fully copositive matrices ($\boldsymbol{C}_o^f$) and derive some results for this class. In particular, we show that $\boldsymbol{C}_o^f \cap \boldsymbol{Q}_o \subseteq \boldsymbol{P}_o$.

As mentioned earlier, Aganagic and Cottle (1987) characterized $\boldsymbol{Q}_o$-matrices with nonnegative principal minors and established that Lemke's algorithm, with a suitable apparatus to resolve degeneracy, processes (q, A) whenever $A \in \boldsymbol{P}_o \cap \boldsymbol{Q}_o$. Appealing to this result, we deduce (section 5.4) that, for A in a number of subclasses of $\boldsymbol{E}_o^f \cap \boldsymbol{Q}_o$-matrices, (q, A) can be processed by Lemke's algorithm.

1.2. LCP AND MATRIX CLASSES

The linear complementarity problem with data A and q is defined in section 1.1. To facilitate our future discussions we give below another form in which LCP is often presented.

Given $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$, LCP (q, A) is to find w and z in $\boldsymbol{R}^n$ such that :

$$w - Az = q, \tag{1.4}$$

$$w \geq 0, \quad z \geq 0, \tag{1.5}$$

and

$$w^t z = 0. \tag{1.6}$$

For convenience, we call the variables (coordinates) of z as primary variables and those of w as secondary variables. Conditions (1.4) and (1.5) are called the feasibility conditions, whereas condition (1.6) is called the complementarity condition.

For any $q \in \boldsymbol{R}^n$ and $A \in \boldsymbol{R}^{n \times n}$, define the sets $F(q, A)$ and $S(q, A)$ by :

$$F(q, A) = \{ z \in \boldsymbol{R}_+^n \ : \ Az + q \geq 0 \}, \tag{1.7}$$

$$S(q, A) = \{ z \in F(q, A) : (Az + q)^t z = 0 \}. \tag{1.8}$$

We say that (q, A) is feasible if $F(q, A) \neq \phi$. Any element of $F(q, A)$ is called a feasible solution to (q, A). Say that (q, A) has a solution if $S(q, A) \neq \phi$. Any element of $S(q, A)$ is called a solution to (q, A). When we say ' let (w, z) be a solution to (q, A)', we mean $z \in S(q, A)$ and $w = Az + q$.

Complementary Cones

The concept of complementary cones first appeared in Samuelson, Thrall and Wesler (1958). Later Murty (1972) studied them in greater depth and obtained some remarkable results on the number and parity of solutions to linear complementarity problems.

Definition 1.2.1. Let $A \in R^{m \times n}$. The set $\{q \in R^m : q = Ax \text{ for some } x \in R^n_+\}$ is called the *convex cone* generated by columns of A and is denoted by *pos* A. The columns of A are called the generators of the cone.

Definition 1.2.2. Let $A \in R^{n \times n}$. For any $\alpha \subseteq \bar{n}$, the matrix B, defined by

$$B_{.j} = -A_{.j} \text{ if } j \in \alpha \text{ and } B_{.j} = I_{.j} \text{ if } j \in \bar{\alpha}$$

is called a complementary submatrix of $[\, I : -A \,]$ with respect to α and is denoted by $C_A(\alpha)$. If, in addition, $det\, A_{\alpha\alpha} \neq 0$, then B is called a *complementary basis*.

Definition 1.2.3. For any $\alpha \subseteq \bar{n}$, $pos\, C_A(\alpha)$, the cone generated by $C_A(\alpha)$, is called the *complementary cone* of $[\, I : -A \,]$ with respect to α.

Note that there are 2^n complementary cones (not necessarily distinct). Further, (q, A) has a solution if, and only if, $q \in pos\, C_A(\alpha)$ for some $\alpha \subseteq \bar{n}$. In addition, if $q \in pos\, C_A(\alpha)$ and $det\, C_A(\alpha) \neq 0$, then $z \in S(q, A)$ where $z_\alpha = -(A_{\alpha\alpha})^{-1} q_\alpha$ and $z_{\bar{\alpha}} = 0$.

Lemma 1.2.4. Let $A \in R^{n \times n}$ and $q \in R^n$. Then (q, A) has a solution if, and only if, there exists a subset α of $\bar{n}$ such that q belongs to the complementary cone with respect to α.

Proof. Suppose (q, A) has solution, say $z \in S(q, A)$. Let $\alpha = supp(z)$ and $w = Az + q$. Then by complementarity, $w_i = 0$ for $i \in \alpha$. It is clear that q belongs to the cone generated by $C_A(\alpha)$. Conversely, if q belongs to a complementary cone generated by $C_A(\alpha)$ for some $\alpha \subseteq \bar{n}$, then there exists a nonzero $x \in \boldsymbol{R}^n_+$ such that $q = Bx$, where $B = C_A(\alpha)$. Define $z \in \boldsymbol{R}^n_+$ by $z_i = x_i$ if $i \in \alpha$ and $z_i = 0$ if $i \in \bar{\alpha}$. It is easy to check that $z \in S(q, A)$.□

Definition 1.2.5. Let $A \in \boldsymbol{R}^{n \times n}$ and let $\alpha \subseteq \bar{n}$. The complementary cone with respect to α is said to be *nondegenerate* or *full* if $det\, A_{\alpha\alpha} \neq 0$. Otherwise it is called a degenerate complementary cone. Further, if $pos\, C_A(\alpha)$ is degenerate, then it is said to be strongly degenerate if there exists a nonzero $x \in \boldsymbol{R}^n_+$ such that $C_A(\alpha)x = 0$; otherwise it is called a weakly degenerate cone.

Remark 1.2.6. If a complementary cone is nondegenerate, then it has a nonempty interior. On the other hand, if it is degenerate, then it must be contained in a hyperplane.

Definition 1.2.7. Let $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$. The matrix A is said to be *nondegenerate* if $det\, A_{\alpha\alpha} \neq 0$ for all $\alpha \in n^*$. Any solution z of (q, A) is said to be *nondegenerate* if $z + Az + q > 0$. Otherwise it is called a degenerate solution. A vector q is said to be *nondegenerate* with respect to A if every solution of (q, A) is nondegenerate.

Consider the following examples.

Example 1.2.8. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

The complementary cones corresponding to these matrices are depicted in figures A through D on page 8.

Note that in the case of A there are 4 complementary cones, all of them distinct

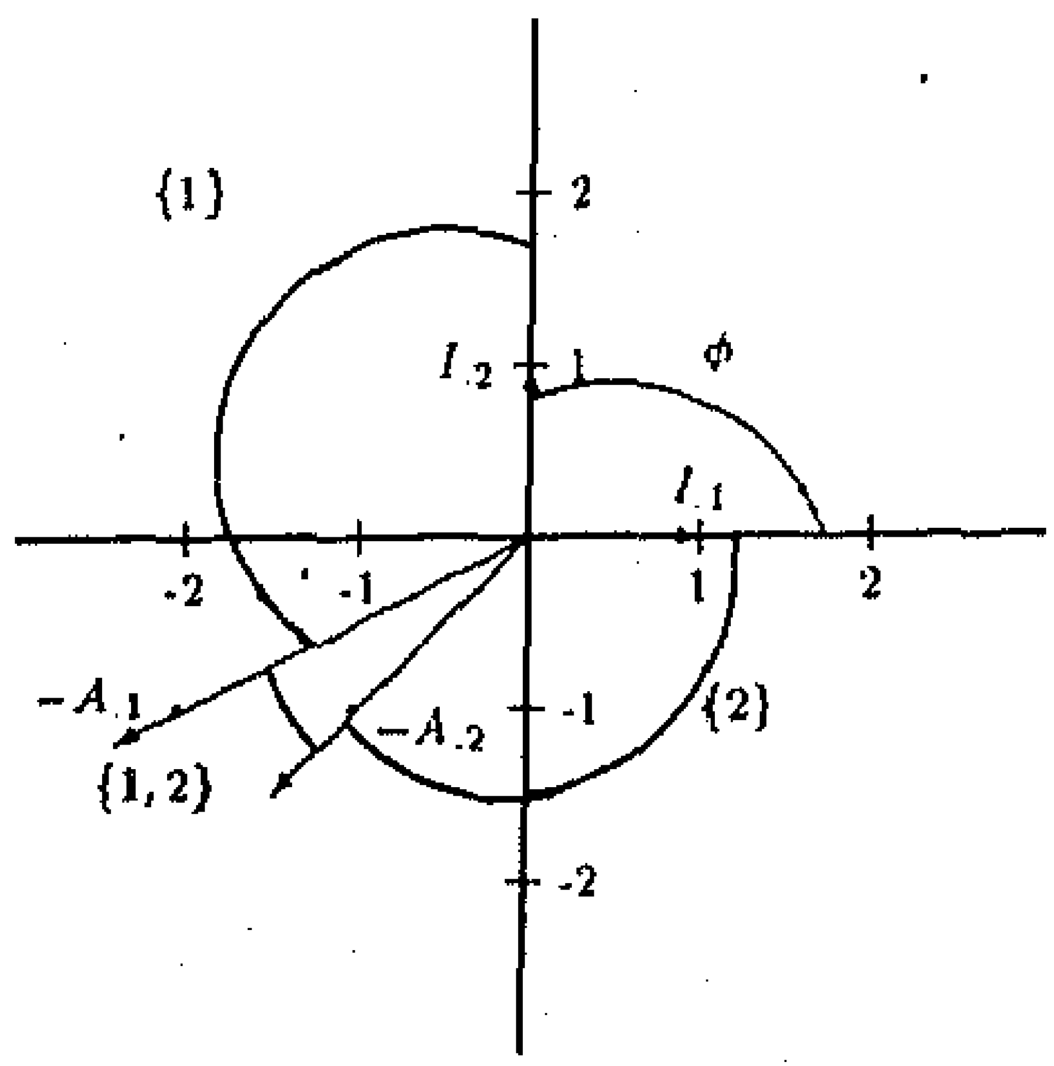

Figure A

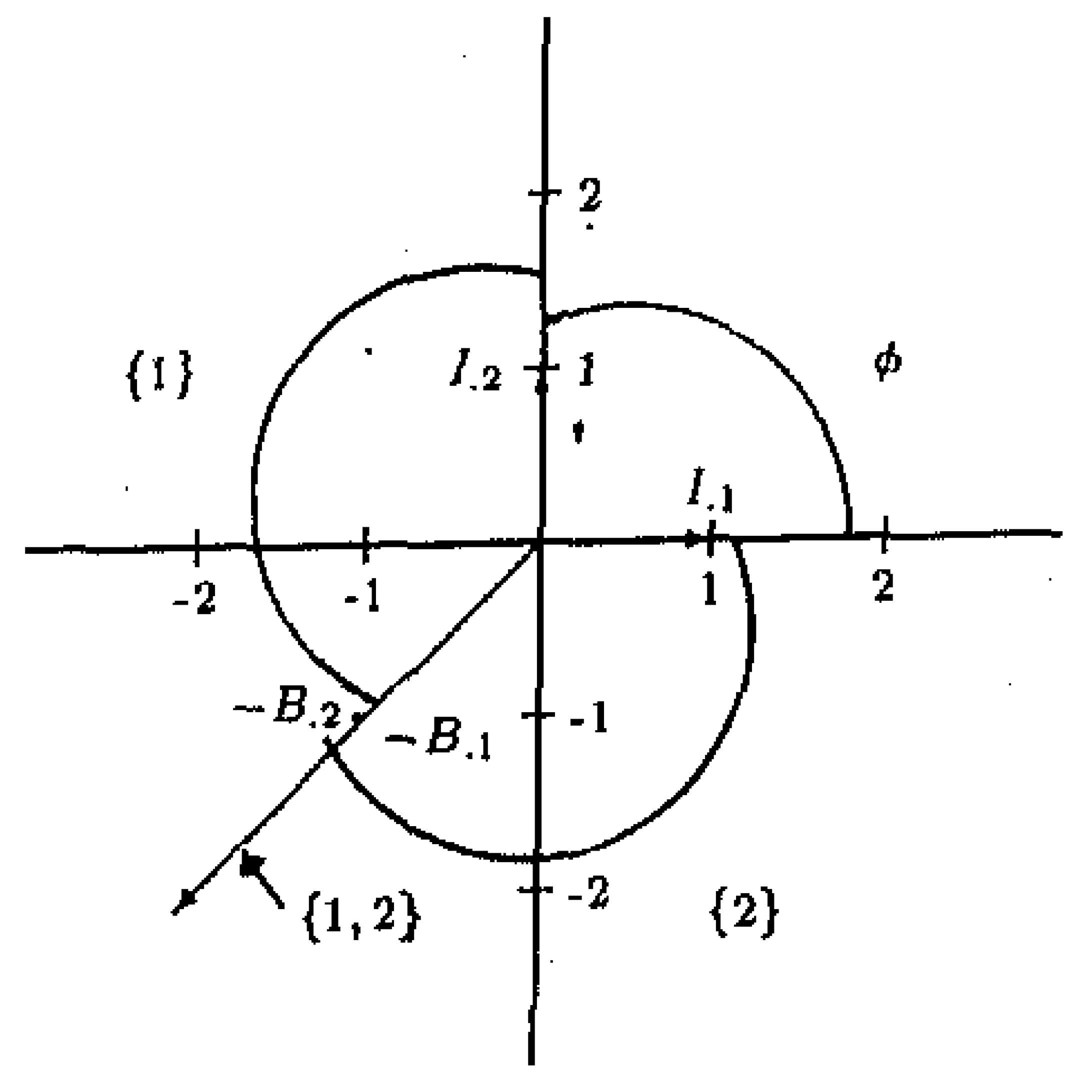

Figure B

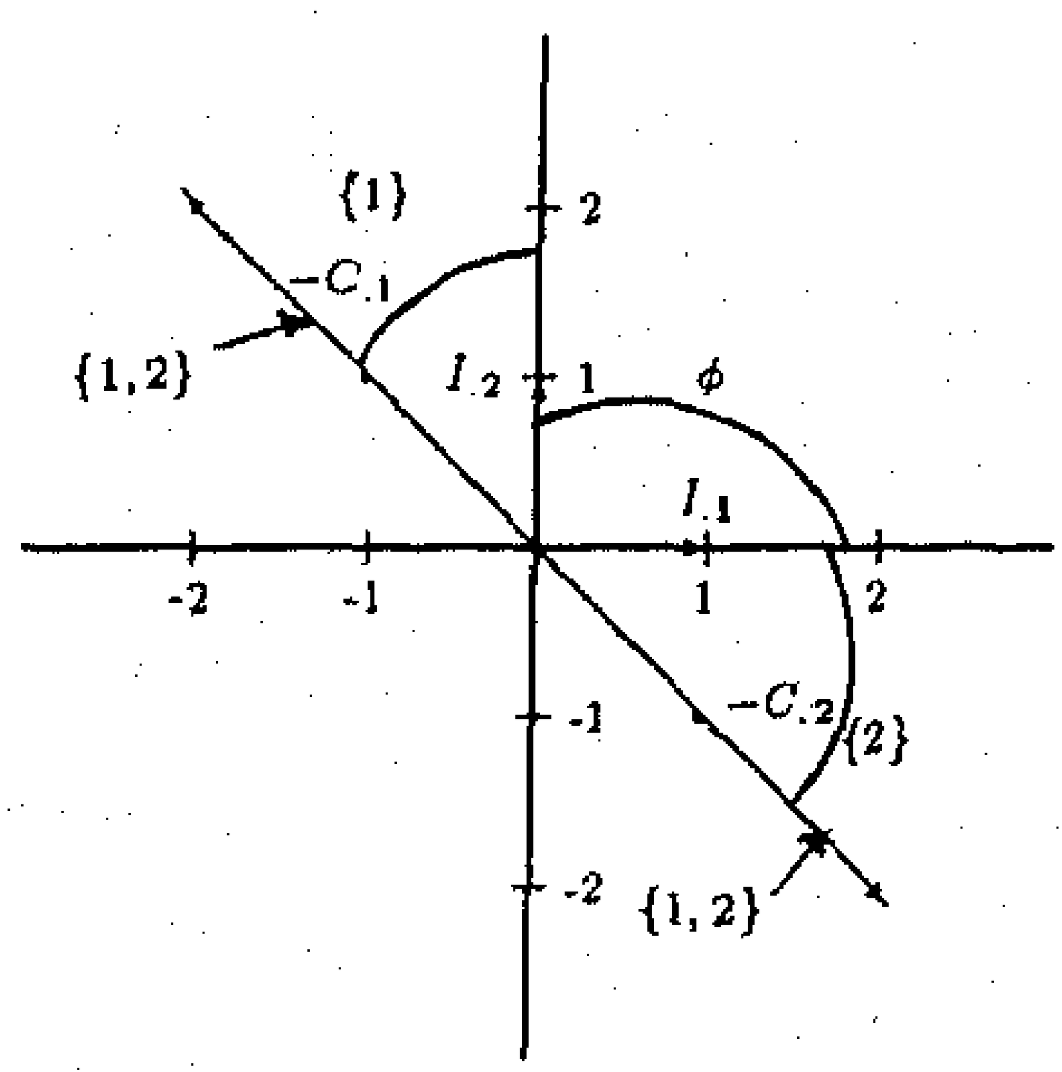

Figure C

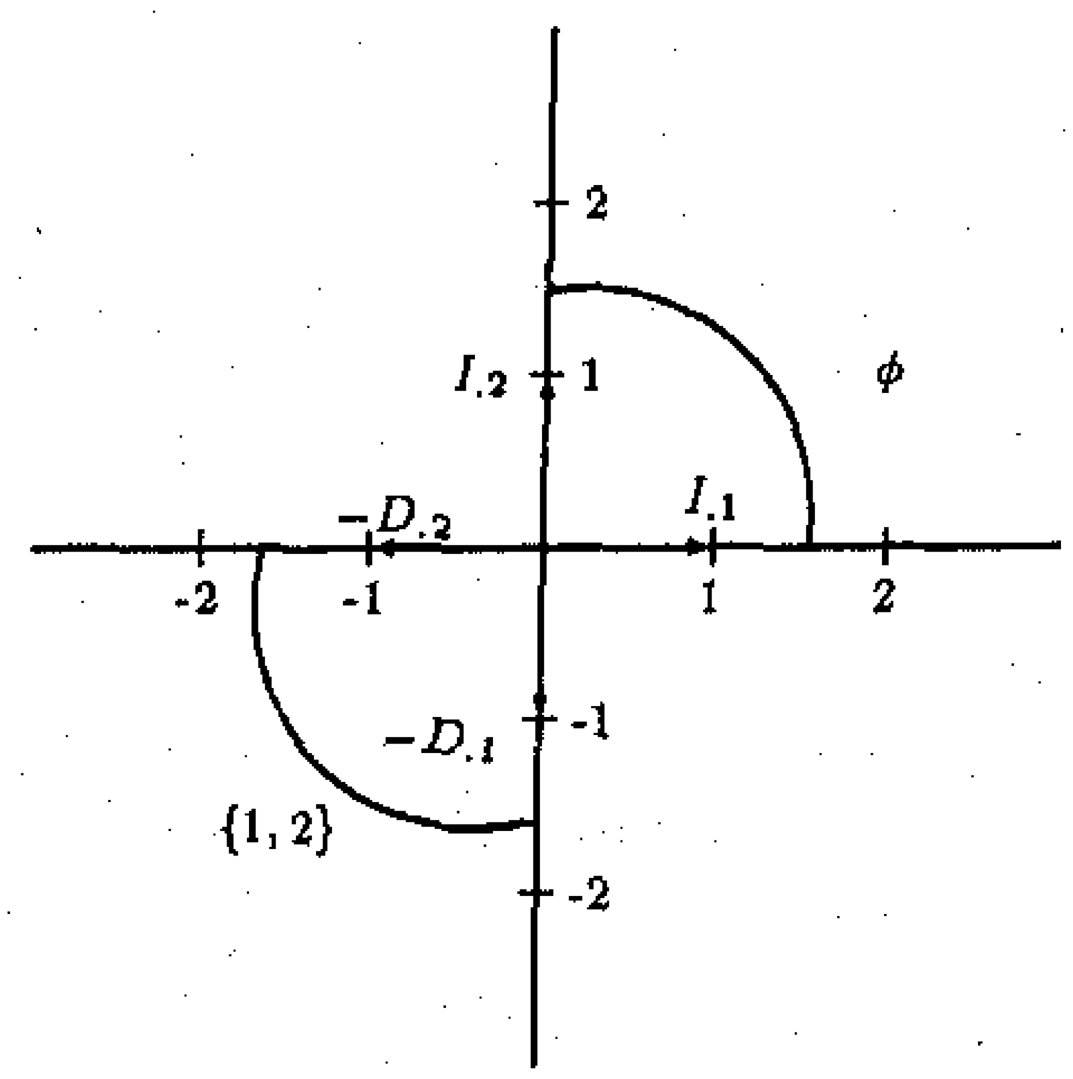

Figure D

Complementary Cones

and nondegenerate. In this case, (q, A) has a unique solution for every $q \in R^2$. In the case of B, there are four complementary cones, all of them distinct but one of them, namely $C_B(\{1,2\})$, is weakly degenerate. In this case, (q, B) has a solution for every $q \in R^2$ but not necessarily unique. In the case of C, there are three nondegenerate and one degenerate complementary cones. Notice that there are q's for which (q, C) has no solution but such q's have no feasible solutions either. Observe that $pos\, C_C(\{1,2\})$ is a strongly degenerate cone. Lastly, in the case of D, we have only two nondegenerate complementary cones and two degenerate complementary cones. In this case, note that there are q's for which (q, D) has feasible solutions but no (complementary) solutions. Further, note that A is a nondegenerate matrix. Also any $z \in S(q, A)$, where q is in the interior of any of the complementary cones, is a nondegenerate solution of (q, A). This last statement is true for the other three matrices as well.

For any $A \in \boldsymbol{R}^{n\times n}$, we use the notation $K(A)$ for the union of all complementary cones corresponding to A. Thus, in the Example 1.2.8,

$$K(A) = K(B) = R^2, \;\; K(D) = R^2_+ \cup R^2_-,$$

where $R^2_- = \{y : y = -x, \; x \in R^2_+\}$, and $K(C) = \{x \in \boldsymbol{R}^2 : x_1 + x_2 \geq 0\}$.

Principal Pivotal Transforms

Definition 1.2.9. Let $A \in \boldsymbol{R}^{n\times n}$ and $\alpha \in n^*$. Assume that $det\, A_{\alpha\alpha} \neq 0$. The *principal pivotal transform* (PPT) of A with respect to α is defined as the $n \times n$ matrix given by :

$$M = \begin{bmatrix} (A_{\alpha\alpha})^{-1} & -(A_{\alpha\alpha})^{-1} A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1} & (\,A/A_{\alpha\alpha}\,) \end{bmatrix}$$

where $(\,A/A_{\alpha\alpha}\,) = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$ is called the Schur complement of A with respect to α (or $A_{\alpha\alpha}$). We denote the PPT of A with respect to α by $\wp_\alpha(A)$.

Remark 1.2.10. Note that PPT is defined only for those α for which $det\, A_{\alpha\alpha} \neq 0$. We call such PPTs as legitimate PPTs. Whenever we refer to a

PPT we implicitly mean that it exists. When there is no ambiguity, we simply say PPT instead of legitimate PPT.

Remark 1.2.11. For $\alpha = \phi$, we define, by convention, the PPT as the matrix A itself. And if $\alpha = \bar{n}$, then $\wp_\alpha(A)$, by definition, is A^{-1} provided it exists. Further, for any $\alpha \subseteq \bar{n}$, $B = \wp_\alpha(A)$ if, and only if, $A = \wp_\alpha(B)$.

The principal pivotal transforms were introduced by Tucker (1963). The PPTs play an important role in the study of LCP. For details on the subject, see Cottle (1968), Cottle and Dantzig (1968), Parsons (1970), and Murty (1988).

Definition 1.2.12. Let $A \in \boldsymbol{R}^{n \times n}$. If P is any permutation matrix in $\boldsymbol{R}^{n \times n}$, then PAP^t is called a *principal rearrangement* of A.

Lemma 1.2.13. Given any $\alpha \subseteq \bar{n}$, there exists a permutation matrix P such that

$$PAP^t = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

Proof. Suppose $\alpha = \{k_1, k_2, \ldots, k_m\}$ and $\bar{\alpha} = \{k_{m+1}, k_{m+2}, \ldots, k_n\}$. Define $P \in \boldsymbol{R}^{n \times n}$ by :

$$\text{for } i \in \bar{n}, \quad p_{ij} = 1 \text{ if } j = k_i \text{ and } p_{ij} = 0 \text{ otherwise}.$$

With this P note that

$$PAP^t = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}. \qquad \square$$

Remark 1.2.14. Observe that in the above lemma, though α is a subset of $\bar{n}$, the order in which α is written is important. For example, for $n = 2$, take $\alpha = \{1, 2\}$. In this case

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $PAP^t = A$. But if we write $\alpha = \{2, 1\}$, then

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad PAP^t = \begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix} (\neq A).$$

Note that LCP with data $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$ is to find $w, z \in \boldsymbol{R}^n$ such that

$$[\, I \; : \; -A \,] \begin{bmatrix} w \\ z \end{bmatrix} = q, \tag{1.9}$$

$$w \geq 0, \;\; z \geq 0, \tag{1.10}$$

and $$w^t z = 0. \tag{1.11}$$

In the light of Lemma 1.2.13, for any given $\alpha \subseteq \bar{n}$, we can rewrite the above LCP in terms of

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

To see this, first note that if P is any permutation matrix, then $PP^t = P^tP = I$. So, from (1.9) we have :

$$[\, P^tP \; : \; -P^tPAP^tP \,] \begin{bmatrix} w \\ z \end{bmatrix} = q,$$

$$\text{or } [\, I \; : \; -PAP^t \,] \begin{bmatrix} Pw \\ Pz \end{bmatrix} = Pq. \tag{1.12}$$

Note that (1.12) is equivalent to

$$I \begin{bmatrix} w_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} - \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} q_\alpha \\ q_{\bar{\alpha}} \end{bmatrix}. \tag{1.13}$$

We shall now present some important results connecting PPTs and LCP. These results may be found in references mentioned immediately after Remark 1.2.11.

Lemma 1.2.15. Let $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$. Suppose $\alpha \in n^*$ is such that $\wp_\alpha(A)$ exists. Then $S(q, A) \neq \phi$ if, and only if, $S(p, B) \neq \phi$, where $B = \wp_\alpha(A)$, $p_\alpha = -(A_{\alpha\alpha})^{-1} q_\alpha$ and $p_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1} q_\alpha$.
Further, $|S(q, A)| = |S(p, B)|$.

Proof. Suppose $S(q, A) \neq \phi$. Let (w, z) be a solution of (q, A). Then we have :

$$\begin{bmatrix} w_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} - \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} q_\alpha \\ q_{\bar{\alpha}} \end{bmatrix}, w \geq 0, \;\; z \geq 0, \;\; \text{and } w^t z = 0.$$

Premultiplying the equation by $(C_A(\alpha))^{-1}$ and rewriting it, we get

$$\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} - B \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} p_\alpha \\ p_{\bar{\alpha}} \end{bmatrix}.$$

Also

$$\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} \geq 0, \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} \geq 0, \text{ and } \begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix}^t \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} = w^t z = 0.$$

In other words,

$$\begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} \in S(p, B).$$

Proof of the converse is exactly in the reverse direction. Hence there is a one to one correspondence between solutions of (q, A) and solutions of (p, B). Consequently, $|S(q, A)| = |S(p, B)|$. □

Remark 1.2.16. Note $q \in pos\, C_A(\alpha)$ if, and only if, $p \in pos\, C_B(\alpha)$ and p in the above lemma is called the principal pivotal transform of q with respect to α (and A). Note that, if there exists a principal pivotal transform p of q such that $p \geq 0$ then it is easy to get a solution to (q, A) using this PPT.

Remark 1.2.17. From the Lemma 1.2.15, it is clear that complementary cones corresponding to A and $\wp_\alpha(A)$ have one to one correspondence through the nonsingular linear transformation $q \to (C_A(\alpha))^{-1}q$. Further, q is in the interior of $pos\, C_A(\alpha)$ if, and only if, p is in the interior of $pos\, C_B(\alpha)$.

We close our discussion on PPTs with the following results concerning determinants of PPTs. The proofs of these results may be found in Cottle, Pang and Stone (1992).

Theorem 1.2.18. Suppose $A \in R^{nxn}$ and $A_{\alpha\alpha}$ is a nonsingular principal submatrix of A. Then the determinant of Schur complement ($A/A_{\alpha\alpha}$) of A with respect to α is given by :

$$det(\ A/A_{\alpha\alpha}\ ) = \frac{det\, A}{det\, A_{\alpha\alpha}}.$$

If, in addition, $F = (A/A_{\alpha\alpha})$ is nonsingular, then :

$$A^{-1} = \begin{bmatrix} D & -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}F^{-1} \\ -F^{-1}A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1} & F^{-1} \end{bmatrix}, \tag{1.14}$$

where $D = (A_{\alpha\alpha})^{-1} + (A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}F^{-1}A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$.

Theorem 1.2.19. Suppose $A \in \boldsymbol{R}^{n \times n}$ and $\alpha \subseteq \bar{n}$ is such that $det\, A_{\alpha\alpha} \neq 0$. If $M = \wp_\alpha(A)$ and β is any subset of $\bar{n}$, then

$$det\, M_{\beta\beta} = \frac{det\, A_{\gamma\gamma}}{det\, A_{\alpha\alpha}}, \tag{1.15}$$

where $\gamma = \alpha \triangle \beta$.

LCP and Matrix Classes

As mentioned earlier, much of the research in LCP has been devoted to the study of matrix classes and there are several reasons for this. Many of the matrix classes arose in a natural way from practical applications. For example, the positive semidefinite matrices (see glossary for definitions of various matrix classes) are commonly found in several applications. The class of adequate matrices, introduced by Ingleton (1966), arose from a study of dynamical systems subject to smooth unilateral conditions. The class of $\boldsymbol{Z}$-matrices arises in optimal stopping and capital stock problems (see Cohen (1975), and Dantzig and Manne (1974)).

Another important reason for the study of matrix classes in LCP is that they offer certain nice features from the view point of algorithms as well as analytical properties. For example, the question - what are the matrices A for which (q, A) can be processed by a particular algorithm for any q? - is of primary importance (from the algorithmic view point). This question has been studied extensively by several authors (see Lemke and Howson (1964), Lemke (1965, 1978), Cottle and Dantzig (1968), Murty (1974), and Watson (1974)). The other types of questions which have received great attention in the literature are : 'what are the matrices A for which (q, A) has a solution for every q (or whenever (q, A) has a feasible solution) ?'; 'what are the matrices A for which (q, A) has a unique solution for every q ?' etc.

The type of questions mentioned above lead to matrix classes such as positive definite, positive semidefinite, $\boldsymbol{Q}$, $\boldsymbol{Q}_0$, $\boldsymbol{P}$, $\boldsymbol{P}_0$, $\boldsymbol{E}_0$, $\boldsymbol{R}_0$ etc. ,We shall discuss these classes in more detail in the subsequent chapters as and when they become relevant. Many results are concerned with characterization of the matrix classes in connection with LCP. Basically, there are two types of characterizations -constructive and analytical. Usually, the constructive characterizations (if efficient) are useful in verifying whether a given matrix belongs to a particular class or not, while the analytical characterizations are important from the theoretical view point. For example, ' $A \in \boldsymbol{P}$ if, and only if, $\{x \in \boldsymbol{R}^n : x_i(Ax)_i \leq 0 \ \forall\, i\} = \{0\}$ ' is an analytical characterization of $\boldsymbol{P}$ and has been used to derive several results; whereas ' $A \in \boldsymbol{P}$ if, and only if, $det\, A_{\alpha\alpha} > 0 \ \forall\, \alpha \in n^*$ ' is a constructive characterization (though *inefficient*) of $\boldsymbol{P}$ and is useful (when n is small) in verifying whether $A \in \boldsymbol{P}$ or not. For more details on the subject LCP, the reader may refer to the excellent books by Cottle, Pang and Stone (1992) and Murty (1988).

1.3. LINEAR PROGRAMMING

In this section we briefly introduce the Linear Programming Problem and present some results that are used in the subsequent chapters.

Statement of LPP

Let $A \in \boldsymbol{R}^{m \times n}, c \in \boldsymbol{R}^n$ and $b \in R^m$. Consider the Linear Programming Problem (LPP), in the following form : find $z \in \boldsymbol{R}^n$ to

$$\text{minimize} \quad c^t z \tag{1.16}$$

$$\text{subject to} \quad Az = b, \tag{1.17}$$

$$\text{and} \quad z \geq 0. \tag{1.18}$$

Consider another LPP given by :

$$\text{maximize} \quad b^t y \tag{1.19}$$

$$\text{subject to} \quad A^t y \leq c, \tag{1.20}$$

$$\text{and} \quad y \in R^m. \tag{1.21}$$

If one of the above two problems is called the primal problem, then the other LPP is called the dual of the primal LP. In this section we shall call the first problem, specified in (1.16) to (1.18), as the primal.

Definition 1.3.1. A vector $z \in \boldsymbol{R}^n$ is called a feasible solution to the primal if it satisfies the constraints (1.17) and (1.18). If, in addition, the columns of A corresponding to positive z_i's are linearly independent, then z is called a basic feasible solution (BFS). Further, the set $S = \{z \in \boldsymbol{R}^n : \ Az = b, \ \ z \geq 0\}$ is called the feasible region of the LPP and z is called an optimal solution of the LPP if $z \in S$ and $c^t z \leq c^t x$ for all $x \in S$.

Definition 1.3.2. If $z \in S$ satisfies

$$[x, y \in S, \ 0 < \lambda < 1 \text{ and } z = \lambda x + (1 - \lambda) y] \Rightarrow x = y = z,$$

then z is called an *extreme point* of S.

Theorem 1.3.3. A vector $z \in S$ is an extreme point of S if, and only if, it is a basic feasible solution of the LPP. □

Theorem 1.3.4. If (the Primal) LPP has an optimal solution, then it has optimal basic feasible solution. □

Theorem 1.3.5. If the primal and the dual LPP's have feasible solutions, then they both have optimal solutions and their optimum objective values are equal. □

Theorem 1.3.6. Suppose $\bar{x}$ and $\bar{y}$ are feasible solutions to the primal and the dual LPPs, specified in (1.16) to (1.21), respectively. Then they are optimal to their *respective* problems if, and only if,

$$\text{for each } i \in \bar{n}, \qquad \bar{x}_i > 0 \Rightarrow (A^t\bar{y})_i = c_i. \ \square$$

This theorem is known as the complementary slackness theorem. Proofs of these results may be found in Murty (1976).

In the rest of this section, we shall assume that

$$A \in \boldsymbol{R}^{m \times n}, \ \ b \in R^m \text{ and } S = \{z \in \boldsymbol{R}^n \ : \ Az \leq b\}.$$

Definition 1.3.7. A nonzero vector $d \in \boldsymbol{R}^n$ is called a *direction* of S if $x + \lambda d \in S \ \forall \ x \in S$ and for all real numbers $\lambda \geq 0$. Two directions d and $\bar{d}$ are said to be distinct if $d \neq \lambda\bar{d}$ for any positive real number λ. A direction d is said to be an extreme direction if it cannot be written as a positive linear combination of two distinct directions, that is, if $d = \lambda_1\bar{d} + \lambda_2 d^*$, where $\bar{d}$ and d^* are any two directions of S and λ_1 and λ_2 are any two real numbers, then at least one of λ_1 and λ_2 is nonpositive.

Theorem 1.3.8. S has finitely many extreme points and extreme directions. Further, if $x^1, x^2, \ldots, x^k$ are all the extreme points of S and $d^1, d^2, \ldots, d^l$ are all the extreme directions of S, then for every $x \in S$, there exist $\lambda_1, \lambda_2, \ldots, \lambda_k$, all in [0,1], with $\sum_{i=1}^k \lambda_i = 1$, and nonnegative reals $\mu_1, \mu_2, \ldots, \mu_l$ such that $x = \sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^l \mu_j d^j$.
For details of proof see Murty (1976).

This section ends with the following theorem.

Theorem 1.3.9. Assume that rank $(A) = n$. Then we have :

(a) if $\bar{x}$ is an extreme point of S, then at least n of the m constraints of S must be binding, i.e., $|\{i : (A\bar{x})_i = b_i\}| \geq n$,

(b) if $c \in \boldsymbol{R}^n$ is such that the LPP

$$\text{minimize} \quad c^t x \quad \text{subject to} \quad x \in S \tag{1.22}$$

has an optimal solution, then there exists an optimal solution $\bar{x}$ which is an extreme point of S.

Proof. (a) Let $\alpha = \{i : (A\bar{x})_i = b_i\}$, and $k = |\alpha|$. Suppose $k < n$. The columns of the $k \times n$ matrix $A_{\alpha.}$ must be linearly dependent. So there exists a $d \in \boldsymbol{R}^n$ such that $d \neq 0$ and $A_{\alpha.}d = 0$.
Let $\bar{\alpha} = \{i \in \bar{m} : i \notin \alpha\}$. From the definition of α, it is clear that $(A\bar{x})_{\bar{\alpha}} < b_{\bar{\alpha}}$. We can choose a real number $\mu \neq 0$ such that

$$(A(\bar{x} \pm \mu d))_{\bar{\alpha}} \leq b_{\bar{\alpha}}.$$

Letting $y = \mu d$, we observe that $\bar{x} + y$ and $\bar{x} - y$ are in S. Since $y \neq 0$, $\bar{x} + y \neq \bar{x} - y$. But this contradicts the hypothesis that $\bar{x}$ is an extreme point of S as $\bar{x} = \frac{1}{2}(\bar{x} + y) + \frac{1}{2}(\bar{x} - y)$. It follows that $k \geq n$.

(b) Suppose $\bar{x}$ is an optimal solution to the LPP given by (1.22). If $\bar{x}$ is an *extreme* point of S, we are done. Suppose not. Let $x^1, x^2, \ldots, x^r$ be all the extreme points of S and let $d^1, d^2, \ldots, d^s$ be all the extreme directions of S. If any of the x^is is optimal, we are through. So assume $c^t x^i > c^t \bar{x}$ for $i = 1, 2, \ldots, r$. From Theorem 1.3.8, there exist $\lambda_1, \lambda_2, \ldots, \lambda_r$, all in [0,1], with $\sum_{i=1}^r \lambda_i = 1$, and nonnegative reals $\mu_1, \mu_2, \ldots, \mu_s$ such that

$$\bar{x} = \sum_{i=1}^r \lambda_i x^i + \sum_{j=1}^s \mu_j d^j.$$

Since the problem has an optimal solution, $c^t d^j \geq 0$ for $j = 1, 2, \ldots, s$ (otherwise $c^t(\bar{x} + \lambda d^j) \to -\infty$ as $\lambda \to \infty$). Then,

$$c^t \bar{x} = c^t\left(\sum_{i=1}^r \lambda_i x^i + \sum_{j=1}^s \mu_j d^j\right) = \sum_{i=1}^r \lambda_i c^t x^i + \sum_{j=1}^s \mu_j c^t d^j > c^t \bar{x}$$

as $c^t x^i > c^t \bar{x}$, $\mu_j \geq 0$ and $c^t d^j \geq 0$. It follows that the LPP has an extreme point optimal solution. □

1.4. GAME THEORY

A *two-person-zero-sum game* consists of two players, designated as player I and player II, each having a finite set of strategies. Let $S = \{s_1, s_2, \ldots, s_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$ be the sets of strategies for players I and II respectively. The game is played as follows. Each player chooses a strategy from his set of strategies. If (s_j, t_i) is the pair of strategies chosen, then player II pays player I \$ $a_{ij}(a_{ij} < 0$ means player II receives \$ $-a_{ij}$). The $m \times n$ matrix $A = (a_{ij})$ is called the (player I's) payoff matrix. The elements of S and T are called *pure strategies*. If there exist $i_0 \in \bar{m}$ and $j_0 \in \bar{n}$ such that $a_{i_0 j} \leq a_{i_0 j_0} \leq a_{i j_0}$ $\forall\, i \in \bar{m}$ and $\forall j \in \bar{n}$, then the game is said to have a solution in pure strategies with s_{j_0} and t_{i_0} as optimal strategies for player I and player II respectively. In this case $a_{i_0 j_0}$ is called the *value of the game* and is denoted by $v(A)$. Often the games do not have solutions in pure strategies. This forces the players to choose their strategies with some probabilities.

Any probability vectors p on S and q on T are called *mixed strategies* for the respective players. The real number $q^t Ap$ is called the expected payoff with respect to (p, q). If there exist mixed strategies $\bar{p}$ and $\bar{q}$, for I and II respectively, such that

$$\bar{q}^t Ap \leq \bar{q}^t A\bar{p} \leq q^t A\bar{p}$$

for all mixed strategies p and q for I and II respectively, then the game is said to have a solution in mixed strategies. In this case, $\bar{p}$ and $\bar{q}$ are called the optimal mixed strategies for I and II respectively, and $\bar{q}^t A\bar{p}$ is called the *value of the game* and is denoted by $v(A)$.

The games described above are also called matrix games. We shall now present a few important theorems on matrix games. For proofs and other details see Kapalanasky (1945) and Owen (1982).

Theorem 1.4.1. (Von Neumann) Every matrix game has a solution in mixed strategies. □

Definition 1.4.2. A mixed strategy x is said to be *completely mixed* if $x > 0$.

Theorem 1.4.3. If the value of a game A is positive, then player I has a completely mixed strategy (not necessarily optimal), $p > 0$ such that $Ap > 0$. Proof of this theorem is easy. □

Theorem 1.4.4. (Kapalanasky) If player II has a completely mixed optimal strategy q, then $Ap = ve$ for every optimal mixed strategy p of player I. Here $v = v(A)$ and $c = (1, 1, \ldots, 1)^t$.□

Theorem 1.4.5. Let $A \in R^{n \times n}$ and let M be a PPT of A with respect to some $\alpha \in n^*$. Consider the games with payoff matrices A and M. Then $v(A) > 0$ if, and only if, $v(M) > 0$.

Proof. Suffices to show one way. Suppose $v(A) > 0$. Get a probability vector x such that $x > 0$ and $Ax > 0$. Let $y = Ax$. We may assume without loss of generality, that

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

It can be seen from Lemma 1.2.15, that

$$\begin{bmatrix} x_\alpha \\ y_{\bar{\alpha}} \end{bmatrix} - M \begin{bmatrix} y_\alpha \\ x_{\bar{\alpha}} \end{bmatrix} = 0.$$

Since

$$\begin{bmatrix} x_\alpha \\ y_{\bar{\alpha}} \end{bmatrix} > 0 \text{ and } \begin{bmatrix} y_\alpha \\ x_{\bar{\alpha}} \end{bmatrix} > 0,$$

it follows that $v(M) > 0$. □

Theorem 1.4.6. The value of a game A is positive (nonnegative) if, and only if, there exists a $0 \neq x \geq 0$ such that $Ax > 0$ ($Ax \geq 0$). Similarly, the value of A is negative (nonpositive) if, and only if, there exists a $0 \neq y \geq 0$ such that $A^t y < 0$ ($A^t y \leq 0$). □

Definition 1.4.7. Let $A \in R^{n \times n}$. The game A is said to be completely mixed if every optimal strategy (for either player) is completely mixed.

Theorem 1.4.8. (Kapalanasky) A matrix game A with value zero is completely mixed if, and only if,

(a) A is a square matrix with $\text{rank}(A) = n - 1$, where n is the order of the matrix, and

(b) all the cofactors A_{ij} of A are all different from 0 and have the same sign. □

1.5. COMPUTATIONS WITH SIGN PATTERNS

In chapters 3, 4 and 5, proofs of several results are based on sign structures of matrices. We devote this section to notations and the nature of computations involving sign structures. The following notations will be used throughout this dissertation.

Sign Symbols

'$-$' stands for negative real numbers,

'$\ominus$' stands for nonpositive real numbers,

'$\oplus$' stands for nonnegative real numbers,

'$+$' stands for positive real numbers,

Definition 1.5.1. Let $A \in R^{m \times n}$. A sign pattern of the matrix A is defined as an $m \times n$ matrix whose entries are either the corresponding entries of A or their possible sign symbols. A sign pattern matrix of A is denoted by $SP(A)$.

It may be noted that $SP(A)$ is not unique. The following examples help in understanding the definition.

Example 1.5.2. Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 0 & -1 \\ 4 & -2 & 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ \oplus & \star & \star \\ \star & - & + \end{bmatrix}, \quad \begin{bmatrix} & 1 & \oplus \\ + & 0 & - \\ \oplus & \ominus & 1 \end{bmatrix}$$

are all sign patterns of A. Here a '$\star$' in a particular position stands for the corresponding entry. Note that

$$\begin{bmatrix} + & \star & 0 \\ 3 & \star & 0 \\ - & \star & 1 \end{bmatrix}$$

is not a sign pattern of A as '+' in (1,1) position of the above matrix is not a possible sign of a_{11}. The following examples illustrate the nature of computations with sign patterns.

Example 1.5.3. Let $A \in \boldsymbol{R}^{4\times 4}$ be such that

$$SP(A) = \begin{bmatrix} \oplus & - & \oplus & \star \\ \oplus & 0 & \oplus & - \\ + & - & 0 & + \\ - & 0 & + & 0 \end{bmatrix}.$$

Let $\alpha = \{3,4\}$. From $SP(A_{\alpha\alpha})$, it is clear that $det\, A_{\alpha\alpha} < 0$. Let $B = \wp_\alpha(A)$. We shall compute $SP(B)$. From $SP(A_{\alpha\alpha})$ it is easy to see

$$SP((A_{\alpha\alpha})^{-1}) = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}.$$

Since $B_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$,

$$\begin{aligned} SP(B_{\alpha\bar{\alpha}}) &= -SP((A_{\alpha\alpha})^{-1})SP(A_{\alpha\bar{\alpha}}) \\ &= -\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}\begin{bmatrix} + & - \\ - & 0 \end{bmatrix} \\ &= \begin{bmatrix} + & 0 \\ - & + \end{bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} SP(B_{\bar{\alpha}\alpha}) &= SP(A_{\bar{\alpha}\alpha})SP((A_{\alpha\alpha})^{-1}) \\ &= \begin{bmatrix} \oplus & \star \\ \oplus & - \end{bmatrix} \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \\ &= \begin{bmatrix} \star & \oplus \\ - & \oplus \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} SP(B_{\bar{\alpha}\bar{\alpha}}) &= SP(A_{\bar{\alpha}\bar{\alpha}}) - SP(A_{\bar{\alpha}\alpha})SP((A_{\alpha\alpha})^{-1}A'_{\alpha\bar{\alpha}}) \\ &= \begin{bmatrix} \oplus & - \\ \oplus & 0 \end{bmatrix} - \begin{bmatrix} \oplus & \star \\ \oplus & - \end{bmatrix} \begin{bmatrix} - & 0 \\ + & - \end{bmatrix} \\ &= \begin{bmatrix} \oplus & - \\ \oplus & 0 \end{bmatrix} - \begin{bmatrix} \star & \star \\ - & + \end{bmatrix} \\ &= \begin{bmatrix} \star & \star \\ + & - \end{bmatrix}. \end{aligned}$$

Thus,

$$SP(B) = \begin{bmatrix} \star & \star & \star & \oplus \\ + & - & - & \oplus \\ + & 0 & 0 & + \\ - & + & + & 0 \end{bmatrix}.$$

Remark 1.5.4. In the above computations we used the '=' sign to equate the sign patterns. Clearly this is not in the strict mathematical sense as we have not defined the equality of two sign patterns formally. However, we hope that the reader understands what is intended to be conveyed through these computations.

Example 1.5.5. Let

$$SP(A) = \begin{bmatrix} \oplus & 0 & - & - \\ \star & 0 & - & 0 \\ + & 0 & 0 & + \\ + & + & 0 & 0 \end{bmatrix}.$$

Further, assume that the all the diagonal entries of every legitimate PPT of A are nonnegative. Let $\alpha = \{2, 3, 4\}$. Clearly $det\, A_{\alpha\alpha} < 0$. Let $B = \wp_\alpha(A)$.

Using the adjoint formula we can check that

$$SP((A_{\alpha\alpha})^{-1}) = \begin{bmatrix} 0 & 0 & + \\ - & 0 & 0 \\ 0 & + & 0 \end{bmatrix}.$$

Note that

$$\begin{aligned} SP(B_{\alpha\bar{\alpha}}) &= -SP((A_{\alpha\alpha})^{-1})SP(A_{\alpha\bar{\alpha}}) \\ &= -\begin{bmatrix} 0 & 0 & + \\ - & 0 & 0 \\ 0 & + & 0 \end{bmatrix} \begin{bmatrix} \star \\ + \\ + \end{bmatrix} \\ &= \begin{bmatrix} - \\ \star \\ - \end{bmatrix}. \end{aligned}$$

Similarly $SP(B_{\bar{\alpha}\alpha}) = SP(A_{\bar{\alpha}\alpha})SP((A_{\alpha\alpha})^{-1}) = [+ \; - \; 0]$. By hypothesis, $b_{11} \geq 0$. Thus

$$SP(B) = \begin{bmatrix} \oplus & + & - & \oplus \\ - & 0 & 0 & + \\ \star & - & 0 & 0 \\ - & 0 & + & 0 \end{bmatrix}$$

CHAPTER 2

LCP AND LINEAR PROGRAMMING

2.1. INTRODUCTION

The main purpose of this chapter is to provide an answer to a question raised by Al-Khayyal (1991) in his paper on ' Necessary and sufficient conditions for the existence of complementary solutions and characterizations of the matrix classes $\boldsymbol{Q}$ and $\boldsymbol{Q}_0$.' He proved that a matrix $A \in \boldsymbol{R}^{n \times n}$ belongs to $\boldsymbol{P}_0$ under one condition on A and questioned whether that condition is also sufficient to conclude that A belongs to $\boldsymbol{Q}_0$. While answering this question in the affirmative, we established that solutions of (q, A) can be obtained by solving a suitable linear programming problem, provided A satisfies the condition and (q, A) has a feasible solution (see Murthy (1993)). Further, the condition is relaxed to obtain a sufficient condition for membership in $\boldsymbol{Q}_0$. Two classes of matrices which satisfy the relaxed condition are identified.

It is known that, unlike LPP, instances of linear complementarity problems may have complementary solutions without having complementary bases. We study this aspect in section 2.4 and provide a sufficient condition under which the existence of complementary bases can be assured. These results have some interesting applications as will be seen in the subsequent chapters.

It is well established that linear programming problems can be formulated as linear complementarity problems. It is natural to ask whether the converse is true. In other words, one is interested to know for what class of problems, solutions of LCP can be obtained by solving suitable linear programming problems. This problem has been studied extensively by several authors.

Mangasarian (1976a, 1976b) proposed the idea of solving LCPs as LPPs and obtained a number of conditions on the matrices under which solutions of LCPs can be obtained by solving suitable LPPs. Later he introduced the *slack* *linear complementarity problem* through which he enlarged the class of linear complementarity problems considered earlier by him (see Mangasarian (1978)).

In a classical approach to the above problem, Cottle and Pang (1978a) studied the problem through the concept of least elements of polyhedral sets. They introduced the class of hidden Z-matrices (the name appeared in Pang (1978)) and showed that many of the problems considered by Mangasarian fall out as special cases of their results. They also cite their computational experience on solving LCPs as LPPs. They further enriched their results in Cottle and Pang (1978b). For additional details on the subject see Pang (1977, 1979), Al-Khayyal (1989, 1991), and Mangasarian (1979a, 1979b).

In a similar attempt, Al-Khayyal (1991) studied the existence of solutions to LCPs through polyhedral sets (resulting from transposes of matrices in question) and obtained necessary and sufficient conditions for the same. Extending his results, he obtains sufficient conditions for membership in $\boldsymbol{Q}$, $\boldsymbol{P}$ and $\boldsymbol{P}_0$-matrix classes. He raised the question whether his sufficient condition for $\boldsymbol{P}_0$ is also sufficient for membership in $\boldsymbol{Q}_0$. As mentioned earlier, our main objective of the chapter is to answer this question.

Aganagic and Cottle (1987) gave a constructive characterization of $\boldsymbol{P}_0 \cap \boldsymbol{Q}_0$-matrices and showed that if $A \in \boldsymbol{P}_0 \cap \boldsymbol{Q}_0$, then Lemke's algorithm, with a suitable apparatus to resolve degeneracy, processes (q, A) for any $q \in \boldsymbol{R}^n$. An algorithm is said to process (q, A) if it either finds a solution or exhibits that the problem has no solution. Since the answer to Al-Khayyal's question is in the affirmative, his sufficient condition for $\boldsymbol{P}_0$ is also sufficient for membership in $\boldsymbol{P}_0 \cap \boldsymbol{Q}_0$.

2.2. PRELIMINARY RESULTS

Consider the LCP (q, A), where $A \in \boldsymbol{R}^{n\times n}$ and $q \in \boldsymbol{R}^n$. We shall present some important definitions and relevant results from the literature.

Let $A \in \boldsymbol{R}^{n\times n}$.

Definition 2.2.1. A is called a $\boldsymbol{Z}$-matrix if $a_{ij} \leq 0 \;\; \forall i, j, \;\; i \neq j$. The class of $\boldsymbol{Z}$-matrices is denoted by $\boldsymbol{Z}$.

Definition 2.2.2. A is said to be a $\boldsymbol{P}$-matrix ($\boldsymbol{P}_0$-matrix) if all principal minors of A are positive (nonnegative).

Definition 2.2.3. A is called a Minkowski matrix if it is in $\boldsymbol{P} \cap \boldsymbol{Z}$. The class of all Minkowski matrices is denoted by $\boldsymbol{K}$.

Theorem 2.2.4. (Mangasarian, 1976a). Suppose (q, A) has a feasible solution. Assume that there exist $X, Y \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{Z}$ and $r, s \in \boldsymbol{R}^n_+$ such that $AX = Y$ and $r^tX + s^tY > 0$. Then every optimal solution of the LPP

$$\text{minimize} \quad p^t z \quad \text{subject to} \quad Az + q \geq 0, \; z \geq 0$$

is a solution of (q, A), where $p = r + A^t s$. □

Definition 2.2.5. Let S be a nonempty subset of $\boldsymbol{R}^n$. An element $\bar{z} \in S$ is called a *least element* of S if $\bar{z} \leq z \; \forall \; z \in S$.

Definition 2.2.6. A matrix $A \in \boldsymbol{R}^{n\times n}$ is called a hidden $\boldsymbol{Z}$-matrix if there exist $X, Y \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{Z}$, and $r, s \in \boldsymbol{R}^n_+$ such that $AX = Y$ and $X^t r + Y^t s > 0$.

Remark 2.2.7. Note that, if S has a least element, then it must be unique.

Theorem 2.2.8. (Cottle and Pang, 1978a). Let $A \in \boldsymbol{R}^{n\times n}$ be a hidden $\boldsymbol{Z}$-matrix and let X and Y be as in the Definition 2.2.6. Suppose (q, A) has a feasible solution. Then the polyhedral set $S = \{x \in \boldsymbol{R}^n : Yx + q \geq 0, \; Xx \geq 0\}$ has a least element $\bar{x}$. Further, $\bar{z} = X\bar{x}$ solves (q, A) and can be obtained by solving the LPP

$$\text{minimize} \quad c^t z \quad \text{subject to} \quad Az + q \geq 0, \; z \geq 0, \tag{2.1}$$

where $c = X^{-1}r$ for any $r \in R^n_{++}$. □

Definition 2.2.9. Let $A \in R^{n \times n}$. A is said to be a Q-matrix if (q, A) has a solution for every $q \in R^n$. A is said to be Q_o-matrix if (q, A) has a solution for every q satisfying $F(q, A) \neq \phi$.

The following theorem characterizes the existence of solution to linear complementarity problems.

Theorem 2.2.10.(Mangasarian, 1979b). Let $A \in R^{n \times n}$ and $q \in R^n$. Then (q, A) has a solution if, and only if,the LPP

$$\text{minimize} \quad (r^t + s^t A)z$$

$$\text{subject to} \quad Az + q \geq 0, \quad z \geq 0$$

has an optimal solution for some $r, s \in R^n_+$ such that an associated dual optimal variable u satisfies

$$(I - A)^t u + r + A^t s > 0.$$

Further, each optimal solution of the above LPP is also a solution of (q, A). □

Theorem 2.2.11.(Al-Khayyal, 1991). Let $A \in R^{n \times n}$ and $q \in R^n$. Then (q, A) has a solution if, and only if,there exists a $\bar{y} \in R^n_+$ such that $\bar{y} \leq e$, and the LP

$$\text{minimize} \quad q^t v \text{ subject to } A^t v \leq e - \bar{y}, \; v \geq -\bar{y} \tag{2.2}$$

has an optimal solution $\bar{v}$ satisfying $(A - I)^t \bar{v} < e$. Here $e = (1, 1, \ldots, 1)^t \in R^n$.

Using this theorem Al-Khayyal derived sufficient conditions for A to be in Q, P and P_o. These conditions are finite (constructive) conditions but inefficient.

Consider $A \in R^{n \times n}$. Let $e = (1, 1, \ldots, 1)^t \in R^n$. Define the sets V_θ, for each θ, $0 \leq \theta \leq 1$, and $\hat{V}$ by

$$V_\theta = \{v \in R^n : \; A^t v \leq e, \quad v \geq -\theta e\}, \tag{2.3}$$

$$\text{and} \quad \hat{V} = \{v \in \boldsymbol{R}^n : (A - I)^t v < e\}. \tag{2.4}$$

Denote the closure of the set $\hat{V}$ by $\mathrm{cl}(\hat{V})$. Obviously, V_θ, $\theta \in [0, 1]$, and $\mathrm{cl}(\hat{V})$ are all polyhedral sets containing the zero vector. We use the notation $\mathrm{Ext}(V_\theta)$ for the set of all extreme points of V_θ.

Theorem 2.2.12. (Al-Khayyal, 1991). Let $A \in \boldsymbol{R}^{n\times n}$. The set V_1, defined above ($\theta = 1$), is bounded if, and only if, the value of (the matrix game) A is positive. Further, if $V_1 \subseteq \hat{V}$, then A is a $\boldsymbol{Q}$-matrix. □

Theorem 2.2.13. (Al-Khayyal, 1991). Let $A \in \boldsymbol{R}^{n\times n}$. The set V_1 is a subset of $\hat{V}$ (subset of $\mathrm{cl}(\hat{V})$) if, and only if, each extreme point $\bar{v}$ of V_1 satisfies the following condition :
for each $i \in \bar{n}$,

$$\begin{aligned} &\text{either} \quad \{(A^t\bar{v})_i = 1 \quad \text{and} \quad \bar{v}_i > 0, (\bar{v}_i \geq 0,)\} \\ &\text{or} \quad \{(A^t\bar{v})_i < 0 \ ((A^t\bar{v})_i \leq 0) \quad \text{and} \quad \bar{v}_i = -1\}. \ \square \end{aligned} \tag{2.5}$$

Theorem 2.2.14. (Al-Khayyal, 1991). Let $A \in \boldsymbol{R}^{n\times n}$. If $V_1 \subseteq \hat{V}$, then $A \in \boldsymbol{P}$. If $V_1 \subseteq \mathrm{cl}(\hat{V})$, then $A \in \boldsymbol{P}_0$. □

Let $A \in \boldsymbol{R}^{n\times n}$ and let $\alpha \in n^*$. Consider the system :

$$A_{\alpha\alpha} z_\alpha \leq 0, \quad A_{\bar{\alpha}\alpha} z_\alpha \geq 0, \quad z_\alpha > 0. \tag{2.6}$$

Definition 2.2.15. $A \in \boldsymbol{R}^{n\times n}$ is said to have *property* (T), if for every $\alpha \in n^*$ the existence of solution to the system (2.6) implies there exists a nonzero vector $y_{\alpha_0} \geq 0$ such that

$$y^t_{\alpha_0} A_{\alpha_0\alpha} = 0 \quad \text{and} \quad y^t_{\alpha_0} A_{\alpha_0\bar{\alpha}} \leq 0$$

where $\alpha_0 = \{i \in \alpha : A_{i\alpha} z_\alpha = 0\}$.

Theorem 2.2.16. (Aganagic and Cottle, 1987). Let $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{P}_0$. Then $A \in \boldsymbol{Q}_0$ if, and only if, A and each of its principal pivotal transforms has property (T). Moreover, when $A \in \boldsymbol{P}_o \cap \boldsymbol{Q}_o$, for any $q \in \boldsymbol{R}^n$ Lemke's algorithm will either find a solution to (q, A) or else *exhibit that* (q, A) has no feasible solution. □

2.3. SUFFICIENT CONDITIONS FOR Q_o AND $P_o \cap Q_o$

Al-Khayyal (1991) raised the following question in his concluding remarks : Does $V_1 \subseteq \text{cl}(\hat{V})$ imply that A is a $\boldsymbol{Q}_o$-matrix ? As mentioned earlier, the answer is in the affirmative. In fact, we will show that $V_1 \subseteq \text{cl}(\hat{V})$ indeed implies that (q, A) can be solved as an LPP.

Theorem 2.3.1. Let $A \in \boldsymbol{R}^{n \times n}$. If there exists a $\theta \in (0, 1]$ such that $V_\theta \subseteq \text{cl}(\hat{V})$, *then every extreme point* v *of* V_θ *satisfies the following condition :*
for each $i \in \bar{n}$,

$$\text{either} \quad \{(A^t v)_i = 1 \text{ and } v_i \geq 0\} \quad \text{or} \quad \{(A^t v)_i < 1 \text{ and } v_i = -\theta\}. \qquad (2.7)$$

Proof. Let v be an extreme point of V_θ. Suppose for some $i \in \bar{n}$, $(A^t v)_i = 1$ and $v_i = -\theta$. Then $[(A - I)^t v]_i = 1 + \theta > 1$. This implies that v does not belong to $\text{cl}(\hat{V})$, which contradicts the hypothesis that $V_\theta \subseteq \text{cl}(\hat{V})$. Thus, out of $2n$ inequalities defining V_θ, at most n of them can hold as equalities. Since v is an extreme point of V_θ, and rank $([\ I : -A\ ]) = n$, by Theorem 1.3.9, it follows that at least n constraints must be binding. Hence exactly n of the $2n$ constraints must be binding. Therefore, if $(A^t v)_i < 1$, then the corresponding $v_i \geq -\theta$ must hold as an equality. That is, $v_i = -\theta$. On the other hand, if $(A^t v)_i = 1$, then, as $V_\theta \subseteq \text{cl}(\hat{V})$, $(A^t v)_i - v_i \leq 1$, and hence $v_i \geq 0$. □

Theorem 2.3.2. Suppose $A \in \boldsymbol{R}^{n \times n}$. Assume that for some $\theta, 0 < \theta \leq 1$, $V_\theta \subseteq \text{cl}(\hat{V})$. Then A is $\boldsymbol{Q}_o$-matrix.

Proof. Consider the following linear program (2.8) and its dual (2.9) :

$$\text{minimize} \quad q^t v \quad \text{subject to} \quad A^t v \leq e \quad \text{and} \quad v \geq -\theta e, \quad (2.8)$$

$$\text{minimize} \quad e^t(z + \theta w) \quad \text{subject to} \quad w - Az = q, \quad \text{and} \quad z \geq 0,\ w \geq 0, \quad (2.9)$$

where $e = (1, 1, \ldots, 1)^t \in \boldsymbol{R}^n$.

Suppose $q \in \boldsymbol{R}^n$ is such that $F(q, A) \neq \phi$, i.e., (q, A) has a feasible solution. Since $v = 0$ is a feasible solution of (2.8), both primal (2.8) and the dual (2.9) have feasible solutions and hence have optimal solutions. Let $(\bar{w}, \bar{z})$ be any optimal solution for the dual. From Theorem 1.3.9, we can choose an optimal solution $\bar{v}$ for the primal which is an extreme point of V_θ. Then $\bar{v}$ and $(\bar{w}, \bar{z})$ satisfy the complementary slackness conditions given by :

$$\bar{z}_i(1 - (A^t\bar{v})_i) = 0 \;\; \forall \;\; i \in \bar{n}, \tag{2.10}$$

$$\text{and} \quad \bar{w}_i(\theta + \bar{v}_i) = 0 \;\; \forall \;\; i \in \bar{n}. \tag{2.11}$$

Since $\bar{v}$ is an extreme point of V_θ, and $V_\theta \subseteq \text{cl}(\hat{V})$, it must satisfy (2.7). Our claim is that $(\bar{w}, \bar{z})$ is a solution of (q, A). Suffices to check complementarity condition $\bar{w}^t\bar{z} = 0$. If for some index i, $(A^t\bar{v})_i = 1$, then $\bar{v}_i \geq 0$ (because of (2.7)) and hence $\theta + \bar{v}_i \geq \theta$. Then, from (2.11), we have $\bar{w}_i = 0$.

On the other hand, if i is such that $(A^t\bar{v})_i \neq 1$, then from (2.7), we must have $(A^t\bar{v})_i < 1$ and $\bar{v}_i = -\theta$. This implies $1 - (A^t\bar{v})_i > 0$. From (2.10), then, we must have $\bar{z}_i = 0$. Thus, for each $i \in \bar{n}$, either $\bar{w}_i = 0$ or $\bar{z}_i = 0$. Therefore, $\bar{w}^t\bar{z} = 0$ and $(\bar{w}, \bar{z})$ is a solution of (q, A). It follows that $A \in \boldsymbol{Q}_o$. □

Remark 2.3.3. Note that the sets V_θ, $\theta \in [0, 1]$, are monotonically increasing in θ. That is, if $\theta \leq \bar{\theta}$, then $V_\theta \subseteq V_{\bar{\theta}}$. Taking $\theta = 1$, we answer Al-Khayyal's question in the following corollary.

Corollary 2.3.4. If $V_1 \subseteq \text{cl}(\hat{V})$, then $A \in \boldsymbol{Q}_o$.

Proof. Follows from Theorem 2.3.2 and Remark 2.3.3. □

Corollary 2.3.5. Suppose (q, A) has a feasible solution. Suppose that there exists an optimal solution $\bar{v}$ of the primal (2.8) satisfying (2.7) for some $\theta \in (0, 1]$. Then every optimal solution of the dual (2.9) is a solution to (q, A).

Proof. This follows from Theorem 2.3.2 and the fact that every pair of optimal solutions for the primal and the dual satisfies the complementary slackness conditions (2.10) and (2.11). □

Corollary 2.3.6. If $V_1 \subseteq \mathrm{cl}(\hat{V})$, then $A \in \boldsymbol{P}_o \cap \boldsymbol{Q}_o$.

Proof. This follows from Theorem 2.2.14 and Corollary 2.3.4. □

Consider the following relaxed condition : for some $\theta \in (0, 1]$,

$$(A^t v)_i = 1 \Rightarrow v_i > -\theta \ \forall\, i \in \bar{n}. \tag{2.12}$$

Remark 2.3.7. Note that, if any extreme point v of V_θ satisfies (2.7), then it also satisfies (2.12). And if $V_\theta \subseteq \mathrm{cl}(\hat{V})$, then every extreme point of V_θ satisfies both (2.7) and (2.12).

Theorem 2.3.8. If there exists a $\theta \in (0, 1]$ such that every extreme point v of V_θ satisfies (2.12), then $A \in \boldsymbol{Q}_o$. Further, every solution of the LPP given by (2.9) is a solution of (q, A).
Proof of this theorem is exactly similar to that of Theorem 2.3.2. □

For any matrix A, consider the conditions :

$$a_{ii} < 0 \ \ \forall \ \ i \in \bar{n}, a_{ij} \geq 0 \ \forall\, i, j, \ i \neq j, \ \text{ and } \sum_{i=1}^{n} a_{ij} < -1 \ \forall j \in \bar{n}, \tag{2.13}$$

$$a_{ii} > 0 \ \ \forall \ \ i \in \bar{n}, a_{ij} \leq 0 \ \forall\, i, j, \ i \neq j, \ \text{ and } \sum_{i=1}^{n} a_{ij} > -1 \ \forall j \in \bar{n}, \tag{2.14}$$

where $A = (a_{ij}) \in \boldsymbol{R}^{n \times n}$.

Theorem 2.3.9. Suppose $A \in \boldsymbol{R}^{n \times n}$ satisfies either (2.13) or (2.14). Then A is a $\boldsymbol{Q}_o$-matrix and every optimal solution of (2.9) is a solution of (q, A).

Proof. Suppose A satisfies (2.13). We will show that every element v of $V_1 = \{x \in \boldsymbol{R}^n : A^t x \leq e, \ x \geq -e\}$ satisfies (2.12). Fix $v \in V_1$. Suppose there exists an index i such that $(A^t v)_i = 1$ and $v_i = 1$. Then

$$-a_{ii} + \sum_{j=1, j \neq i}^{n} a_{ji} v_j = 1.$$

Since $v_j \geq -1 \ \forall \ j$ and $a_{ji} \geq 0 \ \forall \ j \neq i$, we have

$$\sum_{j=1, j\neq i}^{n} a_{ji} v_j \geq - \sum_{j=1, j\neq i}^{n} a_{ji}.$$

Adding $-a_{ii}$ on both sides, we get :

$$1 = \sum_{j=1}^{n} a_{ji} v_j \geq - \sum_{j=1}^{n} a_{ji} > 1.$$

The last inequality follows from the hypothesis. This contradiction yields the desired result. A similar proof can be given when A satisfies (2.14). □

Remark 2.3.10. It may be noted that the matrices satisfying either (2.13) or (2.14) are already covered by Mangasarian. In fact, if A satisfies (2.13), then $-A$ is a $\boldsymbol{K}$-matrix, that is, $-A$ is $\boldsymbol{P} \cap \boldsymbol{Z}$-matrix. If A satisfies (2.14), then it is a $\boldsymbol{Z}$-matrix.

We give below two more examples of matrices which satisfy the condition (2.12).

Example 2.3.11. Consider the matrices

$$A = \begin{bmatrix} 5 & -9 & -10 \\ 8 & 2 & -2 \\ -9 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -3 & 2 \\ -6 & -8 & -5 \\ -6 & -4 & 4 \end{bmatrix}.$$

Consider the matrix A. Let $v \in V_1$. Suppose (2.12) is violated for some $i \in \{1, 2, 3\}$, say $i = 1$. Then $(A^t v)_i = 1 \Rightarrow v_2 = (6 + 9v_3)/8$. Substituting this in $(A^t v)_2 \leq 1$, we get a contradiction. So $i \neq 1$. Similarly we can show that $i \neq 2$ and $i \neq 3$. Hence A satisfies (2.12) and $A \in \boldsymbol{Q}_o$. Similarly one can show that $B \in \boldsymbol{Q}_o$. Obviously A and B are not $\boldsymbol{P}_o$-matrices. Therefore, they do not satisfy Al-Khayyal's condition.

It is interesting to note that A and B are hidden $\boldsymbol{Z}$-matrices. To see this, let

$$X = \begin{bmatrix} 3 & 0 & -1 \\ -20 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 205 & -9 & -25 \\ -14 & 2 & -12 \\ -7 & -1 & 9 \end{bmatrix}.$$

Then observe that $AX = Y$, and both X and Y are K-matrices. Thus A is a hidden Z-matrix.

To see that B is hidden Z-matrix, let

$$U = \begin{bmatrix} 1 & 0 & \frac{-31}{42} \\ 0 & 1 & \frac{-1}{14} \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \quad V = \begin{bmatrix} 3 & -3 & 0 \\ -6 & -8 & 0 \\ -6 & -4 & \frac{61}{7} \end{bmatrix}.$$

Then $BU = V$, U is a K-matrix and V is Z-matrix. Hence B is a hidden Z-matrix.

2.4. LCP AND COMPLEMENTARY BASES

In linear programming problems, the existence of a solution to the problem always ensures the existence of an optimal basis. However, this is not the case with LCP. A linear complementarity problem may have a complementary solution without having a complementary basis. Consider the following example taken from Mohan (1992).

Example 2.4.1. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

Recall the definition of a complementary basis given in Definition 1.2.2. It is clear that any solution (w, z) of (q, A) cannot have a complementary basis even though it has a complementary solution, namely, $z = (1, 1, 0, 0)^t$.

Thus, in general, the existence of a complementary solution need not necessarily imply the existence of a complementary basis. However, the existence of complementary bases can be asserted in some special cases. This is established in the results that follow. These results have some interesting applications which will be dealt in chapters 3 and 5.

Definition 2.4.2. Let $A \in \boldsymbol{R}^{n \times n}$. Say that A has *property* (D), if the following implication is valid for all $\alpha \in n^*$:

$$det\, A_{\alpha\alpha} = 0 \quad \Rightarrow \quad \text{columns of } A_{.\alpha} \text{ are linearly dependent.}$$

Definition 2.4.3. Let $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{P}_o$.

(a) A is said to be *column adequate* if A has property (D),

(b) A is said to be *row adequate* if A^t is column adequate,

(c) A is said to be *adequate* if A is both column adequate and row adequate.

Remark 2.4.4. The class of matrices having property (D) is rather large. Obviously it contains column adequate matrices and all nondegenerate matrices.

Theorem 2.4.5. Suppose $A \in \boldsymbol{R}^{n \times n}$. Assume that A has property (D). If $q \in \boldsymbol{R}^n$ is such that (q, A) has a solution, then (q, A) has a solution which corresponds to a complementary basis.

Proof. Since $S(q, A) \neq \phi$, choose $z \in S(q, A)$. Let $\alpha = supp(z)$. If $\alpha = \phi$, then $z = 0$ is a solution of (q, A) and the corresponding complementary basis is I. Suppose $\alpha \neq \phi$. Without loss of generality, assume $\alpha = \{1, 2, \ldots, k\}$ where $k \leq n$. If $det\, A_{\alpha\alpha} \neq 0$, then $C_A(\alpha)$ is a complementary basis for (q, A). Suppose $det\, A_{\alpha\alpha} = 0$. Then, by property (D), there exists a $d_\alpha \in R^{|\alpha|}$ such that

$$A_{.\alpha} d_\alpha = 0, \quad d_\alpha \neq 0.$$

Since $z_\alpha > 0$, we can choose a real number λ such that $z_\alpha - \lambda d_\alpha \geq 0$ and at least one coordinate of $z_\alpha - \lambda d_\alpha$ is equal to zero. Define $\bar{z} \in \boldsymbol{R}^n_+$ by

$$\bar{z}_\alpha = z_\alpha - \lambda d_\alpha \text{ and } \bar{z}_{\bar{\alpha}} = 0.$$

Then

$$A\bar{z} + q = A_{.\alpha} z_\alpha - \lambda A_{.\alpha} d_\alpha + q = Az + q \geq 0.$$

Let $w = Az + q$. Since $z_\alpha > 0$, $w_\alpha = 0$ and hence $d^t(Az + q) = 0$, where $d^t = (d^t_\alpha, 0^t) \in R^n$. Note that

$$\begin{aligned} \bar{z}^t(A\bar{z} + q) &= \bar{z}^t(Az + q) - \lambda \bar{z}^t A d \\ &= \bar{z}^t(Az + q) \text{ (since } Ad = A_{.\alpha} d_\alpha = 0) \\ &= z^t(Az + q) - \lambda d^t(Az + q) \\ &= 0. \end{aligned}$$

Thus $\bar{z} \in S(q, A)$. Let $\beta = supp(\bar{z})$. It is clear that $|\beta| < |\alpha|$. If $det\, A_{\beta\beta} \neq 0$, then $C_A(\beta)$ is a complementary basis for (q, A). Otherwise we can repeat the above process to get a new solution whose cardinality of its support is strictly less than $|\beta|$. It is clear that in a finite number of steps (at most n) repeating the above process we end up in one of the following situations :

(a) (q, A) has a solution with a complementary basis,

(b) 0 is a solution of (q, A).

In either case (q, A) has a solution with a complementary basis. □

Corollary 2.4.6. Suppose $A \in R^{n \times n}$. Assume that A satisfies any of the following conditions :

(a) A is column adequate,

(b) A is nondegenerate.

Then for any $q \in K(A)$, (q, A) has a solution with a complementary basis.

Proof. Follows from Remark 2.4.4 and Theorem 2.4.5. □

CHAPTER 3

THE Q AND Q_0 MATRICES

3.1. INTRODUCTION

In this chapter we prove some results concerning the matrix classes $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$. The sharpness of these results is demonstrated through several applications. An elegant characterization of nonnegative $\boldsymbol{Q}_o$-matrices, which is very useful for verification, is established. Many results on $\boldsymbol{Q}$-matrices are extended to $\boldsymbol{Q}_o$-matrices. We prove some results providing sufficient conditions for principal submatrices (of order $n-1$) of a $n \times n$ $\boldsymbol{Q}_o$-matrix to be in $\boldsymbol{Q}_o$. These results are quite useful in proving several results in Chapter 5.

The study of existence of solutions to linear complementarity problems has been the heart of the subject. As mentioned in Chapter 1, there are two approaches to find out the existence of solutions to LCP - constructive and analytical. In constructive approach, one actually produces a solution to the problem, under suitable assumptions, by means of an algorithm. On the other hand, in analytical approach one ensures or asserts that a solution to the problem exists (or does not exist) by means of equivalent formulations. In this case, usually the actual solution to the problem is not known. Both the approaches have led to the study of matrix classes.

As mentioned in Chapter 1 the fundamental classes of primary importance are the $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$-matrix classes. Recall that A is a $\boldsymbol{Q}$-matrix if (q, A) has a solution for every $q \in \boldsymbol{R}^n$; and A is a $\boldsymbol{Q}_o$-matrix if (q, A) has a solution whenever q is such that $F(q, A) \neq \phi$. The class $\boldsymbol{Q}$ was introduced by Murty (1972), and $\boldsymbol{Q}_o$ was introduced by Parsons (1970). The efforts to characterize these two classes have not been fully successful. Apparently, the possibility of deriving efficient characterizations for these classes is very remote (see Murty (1988)).

However, a large number of subclasses of $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$ have been identified. The positive definite matrices, $\boldsymbol{P}$-matrices and regular matrices (see glossary for definitions) are all subclasses of $\boldsymbol{Q}$. $\boldsymbol{P}$-matrices were initially introduced by Fiedler and Ptak (1962). Samuelson, Thrall and Wesler (1958) characterized this class as : $A \in \boldsymbol{P}$ if, and only if, (q, A) has a unique solution for every $q \in \boldsymbol{R}^n$. The class of regular matrices was introduced by Karamardian (1972). Watson (1974) described some non-$\boldsymbol{Q}$ matrices through sign variations of principal minors. Using the celebrated algorithm of Lemke and Howson (1964), Lemke (1965) established that copositive-plus matrices are contained in $\boldsymbol{Q}_o$-matrices. In the same paper he obtained some remarkable results on existence of solutions to LCPs through constructive characterizations. Later Eaves (1971) enlarged this class to L and proved, using Lemke's algorithm, that $\mathrm{L} \subseteq \boldsymbol{Q}_o$. He also characterized the class $\boldsymbol{Q}_o$ and proved that $A \in \boldsymbol{Q}_o$ if, and only if, the union of complementary cones corresponding to A is a convex set. Some other subclasses of $\boldsymbol{Q}_o$-matrices are the adequate matrices introduced by Ingleton (1966), the $\boldsymbol{Z}$-matrices first studied by Fielder and Ptak (1962), and sufficient matrices introduced by Cottle, Pang and Venkateswaran (1989).

Our results in this chapter pertain to the study of the matrix classes $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$. In general, given a $A \in \boldsymbol{R}^{n \times n}$ it is difficult to check whether $A \in \boldsymbol{Q}$ or not. This is so even when $n = 3$ or 4. Finite characterizations for $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$ may be found in Aganagic and Cottle (1978) and Murty (1988). However, these are of little use for checking whether a given matrix is $\boldsymbol{Q}(\boldsymbol{Q}_o)$-matrix or not. In section 3.3, we present some elementary propositions which provide sufficient conditions for membership in $\boldsymbol{Q}$. Through a number of examples taken from the literature, we demonstrate that these elementary propositions are very useful in verification for membership in $\boldsymbol{Q}$. In section 3.4, we extend some known results on $\boldsymbol{Q}$-matrices to $\boldsymbol{Q}_o$-matrices and obtain sufficient conditions under which principal submatrices (of order $n-1$) of a $n \times n$ $\boldsymbol{Q}_o$-matrix to be in $\boldsymbol{Q}_o$. Also we present an interesting characterization of nonnegative $\boldsymbol{Q}_o$-matrices. Section 3.5 presents some results providing necessary conditions for a matrix to be in $\boldsymbol{Q}_o$. The results of sections 3.4 and 3.5 turn out to be extremely useful in settling some open problems. Before presenting our results, we present some known results on $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$-matrix classes in section 3.2.

3.2. PRELIMINARY RESULTS

Throughout this section, we shall assume that $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$.

Theorem 3.2.1. $\boldsymbol{Q} \subseteq \boldsymbol{Q}_o$.

Proof. Obvious from the definition. □

The next three theorems are on the invariance properties of $\boldsymbol{Q}$ and $\boldsymbol{Q}_o$. The first two of them may be found in Watson (1974) and the third in Parsons (1970) and Tucker (1963).

Theorem 3.2.2. If D and E are any positive diagonal matrices in $\boldsymbol{R}^n$, then $A \in \boldsymbol{Q}(\boldsymbol{Q}_o)$ if, and only if, $DAE \in \boldsymbol{Q}(\boldsymbol{Q}_o)$.

Proof. Suppose $A \in \boldsymbol{Q}_o$. We will show that $DAE \in \boldsymbol{Q}_o$. Let $q \in \boldsymbol{R}^n$ be such that $F(q, DAE) \neq \phi$. Get $w, z \in \boldsymbol{R}^n_+$ such that $w - DAEz = q$. Then we have $DD^{-1}w - DAEz = q$ or equivalently $D^{-1}w - AEz = D^{-1}q$. Since D and E are positive diagonal matrices, $(D^{-1}w, Ez)$ is a feasible solution of $(D^{-1}q, A)$. Since $A \in \boldsymbol{Q}_o$, $(D^{-1}q, A)$ has a solution, say (y, x). Let $\bar{w} = Dy$ and $\bar{z} = E^{-1}x$. Then $\bar{w} \geq 0$, $\bar{z} \geq 0$ and $\bar{w}^t\bar{z} = 0$. Also, by feasibility of (y, x), we have $y - Ax = D^{-1}q$ or equivalently $Dy - DAEE^{-1}x = q$. Thus $\bar{w} - DAE\bar{z} = q$. Hence $(\bar{w}, \bar{z})$ is a solution of (q, DAE). Therefore, $DAE \in \boldsymbol{Q}_o$. Conversely, if DAE is in $\boldsymbol{Q}_o$, then $A = D^{-1}DAEE^{-1} \in \boldsymbol{Q}_o$. A similar proof can be given in the case of $\boldsymbol{Q}$.□

The above theorem says that rows and/or columns of A can be positively scaled without disturbing its $\boldsymbol{Q}(\boldsymbol{Q}_o)$ property.

Theorem 3.2.3. Suppose $A \in \boldsymbol{Q}(\boldsymbol{Q}_o)$. Then $PAP^t \in \boldsymbol{Q}(\boldsymbol{Q}_o)$ for any permutation matrix P.

Proof. Observe that for any q, (w, z) is a solution of (q, A) if, and only if, (Pw, Pz) is a solution of (Pq, PAP^t). The theorem readily follows from the fact that $PP^t = P^tP = I$ and that both P and P^t are nonnegative. □

Theorem 3.2.4. Let M be any PPT of A. Then $A \in \boldsymbol{Q}(\boldsymbol{Q}_o)$ if, and only if,

$M \in Q(Q_o)$.

Proof. Let $\alpha \subseteq \bar{n}$ be such that $M = \wp_\alpha(A)$. Then $A = \wp_\alpha(M)$. Suffices to show that $A \in Q(Q_o)$ implies $M \in Q(Q_o)$. Suppose $A \in Q_o$. Fix any $\bar{q}$ such that $F(\bar{q}, M) \neq \phi$. By letting $q = C_A(\alpha)\bar{q}$, we note that $F(q, A) \neq \phi$. In fact, if $(\bar{w}, \bar{z})$ is a feasible solution of $(\bar{q}, M)$, then $(\begin{bmatrix} \bar{z}_\alpha \\ \bar{w}_{\bar{\alpha}} \end{bmatrix}, \begin{bmatrix} \bar{w}_\alpha \\ \bar{z}_{\bar{\alpha}} \end{bmatrix})$ is a feasible solution of (q, A). Since $A \in Q_o$, (q, A) has a solution, say (w, z). Then $(\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix}, \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix})$ is a solution of $(\bar{q}, M)$. Therefore, $M \in Q_o$. A similar proof can be given for the case of Q.□

Definition 3.2.5. A matrix $M \in R^{n \times n}$ is called an S-matrix if the value of (the matrix game) M is positive. M is called an S_o-matrix if the value is nonnegative.

Recall, from Theorem 1.4.6, that $M \in S(S_o)$ if, and only if, there exists an $x \in R^n_+$ such that $x \neq 0$ and $Mx > 0$ $(Mx \geq 0)$.

Theorem 3.2.6. Every Q-matrix is an S-matrix.

Proof. Let $q = -e$, where $e = (1, 1, \ldots, 1)^t \in R^n$. Suppose $A \in Q$. Then (q, A) has a solution. In particular, there exists $z \in F(q, A)$. Then,

$$Az + q = Az - e \geq 0 \text{ or } Az \geq e > 0.$$

Since $z \in F(q, A)$, $z \geq 0$. Obviously $z \neq 0$ as $Az > 0$. Thus $A \in S$. □

The following theorem exhibits the relationship between Q and Q_o.

Theorem 3.2.7. $Q = Q_o \cap S$.

Proof. Since $Q \subseteq Q_o$ and $Q \subseteq S$, we need to show $Q_o \cap S \subseteq Q$. Suppose $A \in Q_o \cap S$. Fix any $q \in R^n$. Since $A \in S$, there exists a $z \in R^n_+$ such that $Az > 0$. Obviously we can choose a positive real λ such that $\lambda Az + q > 0$. This means $\lambda z \in F(q, A)$. Since $A \in Q_o$, $S(q, A) \neq \phi$. Since q was arbitrary, it follows that $A \in Q$. The theorem follows. □

The following theorem due to Jeter and Pye (1985) has some interesting applications (see ref Gowda (1990), Murthy, Parthasarathy (1993)).

Theorem 3.2.8. Suppose $A \in \boldsymbol{Q}$. Let $\alpha = \bar{n} \setminus \{i\}$ for some $i \in \bar{n}$. Then either $A_{\alpha\alpha} \in \boldsymbol{Q}$ or (e_i, A) has a solution u such that $(Au)_i = -1$, where $e_i = I_{.i}$.

We shall prove an extended version of this theorem in section 3.4. By replacing $\boldsymbol{Q}_o$ by $\boldsymbol{Q}$ in Theorem 3.4.6 we get a proof of this theorem.

Definition 3.2.9. $A \in \boldsymbol{R}^{n\times n}$ is called a *regular* matrix if there exists a $d \in \boldsymbol{R}^n_{++}$ such that for all $\lambda \geq 0$, $(\lambda d, A)$ has unique solution, namely $(\lambda d, 0)$. The class is denoted by $\boldsymbol{R}$. A is called an $\boldsymbol{R}_o$-matrix if $(0, A)$ has a unique solution.

The class of regular matrices was introduced by Karamardian (1972). The class $\boldsymbol{R}_o$ was first considered by Garcia (1973) under the name $E^*(0)$. In the following theorem we list a few important subclasses of $\boldsymbol{Q}$. For definitions of various classes refer to glossary.

Theorem 3.2.10. If A is in any of the following classes, then $A \in \boldsymbol{Q}$:

(a) positive definite matrices,

(b) $\boldsymbol{P}$-matrices,

(c) $\boldsymbol{R}$-matrices,

(d) $\boldsymbol{E}_o \cap \boldsymbol{R}_o$-matrices,

(e) $\boldsymbol{E}$-matrices,

(f) $\boldsymbol{C}_o \cap \boldsymbol{R}_o$-matrices,

(g) positive matrices. □

Definition 3.2.11. Let $A \in \boldsymbol{R}^{n\times n}$. Then $K(A) = \{q \in \boldsymbol{R}^n : S(q, A) \neq \phi\}$ is called the union of all complementary cones corresponding to A.

Theorem 3.2.12.(Eaves, 1971). Let $A \in \mathbf{R}^{n \times n}$. Then the following are equivalent :

(a) $A \in Q_o$

(b) $pos[\,I\,:\,-A\,] = K(A)$

(c) $K(A)$ is a convex set

The following classical theorem is due to Lemke. He gave a constructive proof of existence of solutions to LCP with matrices of the type M mentioned in the theorem below.

Theorem 3.2.13. (Lemke, 1965). Let $A \in \mathbf{R}^{n \times n}$ and let $e = (1,1,\ldots,1)^t \in \mathbf{R}^n$. Then $M = \begin{bmatrix} A & e \\ -e^t & 0 \end{bmatrix} \in R^{(n+1)\times(n+1)}$ is a Q_o-matrix.

Later Eaves (1971a) generalized this by replacing e by (any) positive vector d. We prove this, in a more general form, using the above theorem.

Corollary 3.2.14. Let $A \in \mathbf{R}^{n \times n}$. Let $d, f \in \mathbf{R}^n_{++}$.

Then $M = \begin{bmatrix} A & d \\ -f^t & 0 \end{bmatrix}$ is a Q_o-matrix.

Proof. Let D and F be the diagonal matrices in $\mathbf{R}^{n \times n}$ with i^{th} diagonal entries as d_i and f_i respectively, $i = 1, 2, \ldots, n$. Obviously D and F are positive diagonal matrices. Let $B = D^{-1}AF^{-1}$. By Theorem 3.2.13,

$$\begin{bmatrix} B & e \\ -e^t & 0 \end{bmatrix} \in Q_o.$$

From Theorem 3.2.2, $\begin{bmatrix} D & 0 \\ 0^t & 1 \end{bmatrix} \begin{bmatrix} B & e \\ -e^t & 0 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0^t & 1 \end{bmatrix} \in Q_o$.

But $\begin{bmatrix} D & 0 \\ 0^t & 1 \end{bmatrix} \begin{bmatrix} B & e \\ -e^t & 0 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0^t & 1 \end{bmatrix} = \begin{bmatrix} A & d \\ -f^t & 0 \end{bmatrix}$. □

We close this section with the following theorem which lists some subclasses of $\boldsymbol{Q}_o$-matrices.

Theorem 3.2.15. Suppose A belongs to any of the following classes :

(a) adequate matrices

(b) copositive-plus matrices

(c) hidden $\boldsymbol{Z}$-matrices

(d) L-matrices

(e) nonpositive matrices

(f) positive semidefinite matrices

(g) sufficient matrices

(h) $\boldsymbol{Z}$-matrices

Then $A \in \boldsymbol{Q}_o$.

3.3. ELEMENTARY PROPOSITION AND ITS APPLICATIONS

In this section we first prove an elementary proposition which provides a sufficient condition for a matrix to be a $\boldsymbol{Q}$-matrix. See Murthy, Parthasarathy and Ravindran (1993b). We, then, apply this to show that several well known examples taken from Cottle, Pang and Stone (1992), Jeter and Pye (1989), Murthy, Parthasarathy and Ravindran (1993a), and Kelly and Watson (1979) are in $\boldsymbol{Q}$. We make use of the proposition in Chapter 4 as well.

Proposition 3.3.1. Suppose $A \in \boldsymbol{R}^{n \times n}$. Assume that $A_{1.} = A_{2.}$. Further, assume that $A_{\alpha\alpha}, A_{\beta\beta} \in \boldsymbol{Q}$, where $\alpha = \overline{\{1\}}$ and $\beta = \overline{\{2\}}$. Then $A \in \boldsymbol{Q}$.

Proof. Fix $q \in \boldsymbol{R}^n$. Suppose $q_1 \geq q_2$. Since $A_{\alpha\alpha} \in \boldsymbol{Q}$, there exist $w_\alpha, z_\alpha \in R^{n-1}$ such that

$$w_\alpha - A_{\alpha\alpha} z_\alpha = q_\alpha, \quad z_\alpha \geq 0, \quad w_\alpha \geq 0 \text{ and } w_\alpha^t z_\alpha = 0.$$

Let $w_1 = w_2 + q_1 - q_2$ and $z_1 = 0$. Clearly $w = (w_1, {w_\alpha}^t)^t$ and $z = (0, z_\alpha^t)^t$ are in R_+^n (as $q_1 \geq q_2$). Also $w^t z = w_\alpha^t z_\alpha = 0$ and

$$w - Az = \begin{bmatrix} w_1 \\ w_\alpha \end{bmatrix} - \begin{bmatrix} A_{1\alpha} z_\alpha \\ A_{\alpha\alpha} z_\alpha \end{bmatrix} = \begin{bmatrix} q_1 \\ q_\alpha \end{bmatrix} = q.$$

Therefore, (q, A) has a solution.

If $q_1 < q_2$, then we can get a solution to (q, A) using a solution of $(q_\beta, A_{\beta\beta})$, which exists as $A_{\beta\beta} \in \boldsymbol{Q}$, in a similar way. As q was arbitrary it follows that $A \in \boldsymbol{Q}$. □

Remark 3.3.2. In the above proposition, obviously, there is nothing special about indices 1 and 2 and the result holds good when 1 and 2 are replaced by any i and j, $i \neq j$.

We shall now give several applications of this proposition. Our first example is taken from Murthy, Parthasarathy and Ravindran (1993a).

Example 3.3.3. Consider the matrix given by :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Note that $A_{1.} = A_{2.}$ and $A_{\alpha\alpha}$ and $A_{\beta\beta}$ are in $\boldsymbol{C}_o \cap \boldsymbol{R}_o$, where $\alpha = \{2, 3, 4\}$ and $\beta = \{1, 3, 4\}$. From Theorem 3.2.10, $A_{\alpha\alpha}, A_{\beta\beta} \in \boldsymbol{Q}$. Invoking the proposition, we conclude that $A \in \boldsymbol{Q}$.

The next example is taken from Jeter and Pye (1989).

Example 3.3.4. Let A be given by :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

Again we have $A_{1.} = A_{2.}$ and $A_{\alpha\alpha}$ and $A_{\beta\beta}$ are in $\boldsymbol{E}_\circ \cap \boldsymbol{R}_\circ$, and hence are $\boldsymbol{Q}$-matrices ($\bar{\alpha} = \{1\}, \bar{\beta} = \{2\}$). Hence the proposition implies that $A \in \boldsymbol{Q}$.

The following example is taken from Murthy, Parthasarathy and Ravindran (1993a).

Example 3.3.5. Let A be given by :

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that the matrix is a $\boldsymbol{C}_\circ$-matrix. Further, $A_{1.} = A_{2.}$ and for $\alpha = \overline{\{1\}}$ and $\beta = \overline{\{2\}}$, $A_{\alpha\alpha}, A_{\beta\beta} \in \boldsymbol{Q}$ (as they are in $\boldsymbol{C}_\circ \cap \boldsymbol{R}_\circ$). By the proposition, we conclude that $A \in \boldsymbol{Q}$. In fact, in the aforementioned reference, the proposition was repeatedly used to show that every 5×5 principal submatrix is a $\boldsymbol{Q}$-matrix.

The following example is taken from Watson(1976).

Example 3.3.6. Consider

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Let $\alpha = \{1\}$ and $M = \wp_\alpha(A)$. Then

$$\wp_\alpha(A) = \begin{bmatrix} 1 & 1 & -4 \\ 4 & 1 & -15 \\ 1 & 1 & -4 \end{bmatrix}.$$

Note that $M_{1.} = M_{3.}$. It is easy to verify that

$$\begin{bmatrix} 1 & -15 \\ 1 & -4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

are both Q-matrices. Appealing to the proposition, we conclude that M, and hence, A are Q-matrices.

The following proposition shows that the assumptions of Proposition 3.3.1 can be slightly relaxed.

Proposition 3.3.7. Let $A \in R^{nXn}$. Let α and β be such that $\bar{\alpha} = \{1\}$ and $\bar{\beta} = \{2\}$. Assume that $A_{1\theta} = A_{2\theta}$ where $\theta = \{3, 4, \ldots, n\}$. Further, assume that $a_{11} \leq a_{21}$, $a_{22} \leq a_{12}$, and $A_{\alpha\alpha}$, $A_{\beta\beta} \in Q$. Then $A \in Q$.

Proof. Let $q \in R^n$. Without loss of generality we may assume that $q_1 \geq q_2$. Since $A_{\alpha\alpha} \in Q$ there exist $w_\alpha, z_\alpha \in R^n_+$ such that (w_α, z_α) is a solution of $(q_\alpha, A_{\alpha\alpha})$. Define $\bar{w}$ and $\bar{z}$ by

$$\bar{w} = \begin{bmatrix} w_1 \\ w_\alpha \end{bmatrix} \text{ and } \bar{z} = \begin{bmatrix} 0 \\ z_\alpha \end{bmatrix}, \quad \text{where } w_1 = w_2 + q_1 - q_2 + (a_{12} - a_{22})z_2.$$

It is easy to verify that $(\bar{w}, \bar{z})$ is a solution of (q, A). Since q was arbitrary it follows that $A \in Q$. □

Remark 3.3.8. As in Remark 3.3.2, the indices 1 and 2 in the above proposition can be replaced by any i and j such that $i \neq j$.

The following examples are taken from Cottle, Pang and Stone(page 520, 1992).

Example 3.3.9. Let

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad \text{Then} \quad M = A^{-1} = \begin{bmatrix} \frac{-1}{6} & \frac{1}{6} & \frac{3}{6} \\ \frac{1}{6} & \frac{-1}{6} & \frac{3}{6} \\ \frac{3}{6} & \frac{3}{6} & \frac{-3}{6} \end{bmatrix}.$$

Note that M satisfies the assumptions of Proposition 3.3.7 and hence $M \in Q$. Therefore, $A \in Q$.

The following example is due to Howe (see Cottle, Pang and Stone (page 576, 1992)).

Example 3.3.10. Let

$$A = \begin{bmatrix} -4 & 3 & 3 & 6 \\ 3 & -4 & 3 & 6 \\ 3 & 3 & -4 & 6 \\ 6 & 6 & 6 & -4 \end{bmatrix}.$$

Scaling the last row we get :

$$M = \begin{bmatrix} -4 & 3 & 3 & 6 \\ 3 & -4 & 3 & 6 \\ 3 & 3 & -4 & 6 \\ 3 & 3 & 3 & -2 \end{bmatrix}.$$

Let $\alpha = \{1,2,3\}$ and $\beta = \{1,2,4\}$. It may be verified that $M_{\alpha\alpha}^{-1} > 0$. From Theorem 3.2.10, $M_{\alpha\alpha}$ is in $\boldsymbol{Q}$. Since

$$\begin{bmatrix} -4 & 6 \\ 3 & -2 \end{bmatrix} \in \boldsymbol{Q}$$

Proposition 3.3.7 implies $M_{\beta\beta} \in \boldsymbol{Q}$. Invoking Proposition 3.3.7 again, we conclude that $M \in \boldsymbol{Q}$. Since $\boldsymbol{Q}$ property is invariant under positive scaling, $A \in \boldsymbol{Q}$.

The following example is taken from Kelly and Watson (1979).

Example 3.3.11. Let

$$A = \begin{bmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{bmatrix}.$$

Let B be the PPT of A with respect to $\alpha = \{2,3\}$.
Let $D = diag[1/16, 1, 2/3, 1/4]$ and $E = diag[3, 48, 48, 1]$.

Then

$$C = DBE = \begin{bmatrix} 2 & -11 & 3 & 6 \\ 2 & -20 & 9 & 15 \\ 2 & -8 & 2 & 6 \\ 2 & -8 & 6 & 4 \end{bmatrix}.$$

Let $\alpha = \{1,2,3\}$ and $\beta = \{1,2,4\}$. Using Proposition 3.3.7, $C_{\alpha\alpha}$ and $C_{\beta\beta}$ are in $\boldsymbol{Q}$. Another application of Proposition 3.3.7, implies $C \in \boldsymbol{Q}$. By Theorem 3.2.2, $B \in \boldsymbol{Q}$ and by Theorem 3.2.4, $A \in \boldsymbol{Q}$.

Proposition 3.3.12. Suppose $A \in \boldsymbol{R}^{n\times n}$ is such that $a_{11} \leq \min\{a_{21}, a_{31}\}$, $a_{22} \leq \min\{a_{12}, a_{32}\}$, and $a_{33} \leq \min\{a_{13}, a_{23}\}$. Further, assume that $A_{1\theta} = A_{2\theta} = A_{3\theta}$, where $\theta = \{4, 5, \ldots, n\}$. Let α, β and γ be such that $\bar{\alpha} = \{1,2\}$, $\bar{\beta} = \{1,3\}$, and $\bar{\gamma} = \{2,3\}$. If $A_{\alpha\alpha}, A_{\beta\beta}$, and $A_{\gamma\gamma}$ are in $\boldsymbol{Q}$, then $A \in \boldsymbol{Q}$.

Proof. Fix $q \in \boldsymbol{R}^n$. Without loss of generality, we may assume $q_1 \geq q_2 \geq q_3$. Since $A_{\alpha\alpha} \in \boldsymbol{Q}$, there exists a solution (w_α, z_α) to $(q_\alpha, A_{\alpha\alpha})$. Define
$w_1 = w_3 + q_1 - q_2 + (a_{13} - a_{33})z_3$, and $w_2 = w_3 + q_2 - q_3 + (a_{23} - a_{33})z_3$.
From the hypothesis, both w_1 and w_2 are nonnegative. Then it is easy to verify that $w = (w_1, w_2, w_\alpha^t)^t$ and $z = (0, 0, z_\alpha^t)^t$ is a solution to (q, A). Since q was arbitrary, $A \in \boldsymbol{Q}$. □

The following example is taken from Cottle, Pang and Stone (page 598, 1992), where we show that (q, A) has at least two solutions for every $q \in \boldsymbol{R}^{5\times 5}$.

Example 3.3.13. Let

$$A = \begin{bmatrix} -4 & 3 & 3 & 6 & 6 \\ 3 & -4 & 3 & 6 & 6 \\ 3 & 3 & -4 & 6 & 6 \\ 6 & 6 & 6 & -4 & 6 \\ 6 & 6 & 6 & 6 & -4 \end{bmatrix}$$

Let α, β and γ be such that $\bar{\alpha} = \{1,2\}$, $\bar{\beta} = \{1,3\}$, and $\bar{\gamma} = \{2,3\}$. Note that

$$A_{\alpha\alpha} = A_{\beta\beta} = A_{\gamma\gamma} = \begin{bmatrix} -4 & 6 & 6 \\ 6 & -4 & 6 \\ 6 & 6 & -4 \end{bmatrix}.$$

It is a well known fact that $A_{\alpha\alpha}$ has 2 or 4 solutions for every $p \in \boldsymbol{R}^3$ (see Murty (page 107, 1972)). Invoking Proposition 3.3.12, we conclude that (q, A) has at least two solutions for every $q \in \boldsymbol{R}^5$.

3.4. SOME RESULTS ON Q_0-MATRICES

In this section we present a number of results concerning Q_0-matrices. Some of them are extensions of known results on Q-matrices. In particular, we give an interesting characterization of nonnegative Q_0-matrices.Though simple, it has some nice applications as will be shown in the sequel. We start with the following result on Q-matrices (see Murty (1988)).

Theorem 3.4.1. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{Q}$. Assume that $A_{i.} \geq 0$ for some $i \in \bar{n}$. Then $A_{\alpha\alpha} \in \boldsymbol{Q}$ where $\alpha = \overline{\{i\}}$.

Proof of this theorem follows from that of its extended version given below.

Theorem 3.4.2. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{Q}_0$. Assume that $A_{i.} \geq 0$ for some $i \in \bar{n}$. Then $A_{\alpha\alpha} \in \boldsymbol{Q}_0$, where $\alpha = \overline{\{i\}}$.

Proof. Without loss of generality, we may assume that $i = n$.
Suppose $A_{\alpha\alpha} \notin \boldsymbol{Q}_0$. Then there exists a $\bar{q} \in R^{n-1}$ such that $F(\bar{q}, A_{\alpha\alpha}) \neq \phi$ but $S(\bar{q}, A_{\alpha\alpha}) = \phi$. Define $q \in \boldsymbol{R}^n$ by $q_\alpha = \bar{q}$ and $q_n = 1$. Choose $\bar{z} \in F(\bar{q}, A_{\alpha\alpha})$. Define z by $z_\alpha = \bar{z}$ and $z_n = 0$. Since $A_{n.} \geq 0$, $z \in F(q, A)$. As $A \in \boldsymbol{Q}_0$, there exists an $x \in S(q, A)$. Since $A_{n.} \geq 0$, $y_n = A_{n.}x + q_n = A_{n.}x + 1 > 0$. This implies $x_n = 0$. But then $x_\alpha \in S(q_\alpha, A_{\alpha\alpha})$ which contradicts that $S(q_\alpha, A_{\alpha\alpha}) = \phi$. Hence $A_{\alpha\alpha} \in \boldsymbol{Q}_0$. □

As an application of this theorem let us consider the following example taken from Stone (1981).

Example 3.4.3. Let

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Stone shows that $A \notin \boldsymbol{Q}_o$ by producing two vectors p and q in $\boldsymbol{R}^4 \cap K(A)$ such that $\frac{1}{2}(p+q) \notin K(A)$. We shall prove that $A \notin \boldsymbol{Q}_o$ using Theorem 3.4.2. Since $A_{3.} \geq 0$, by Theorem 3.4.2,

$$B = A_{\alpha\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \boldsymbol{Q}_o, \quad \text{where } \alpha = \{1,2,4\}.$$

Since $B_{1.} \geq 0$, by Theorem 3.4.2, $B_{\beta\beta} \in \boldsymbol{Q}_o$, $\beta = \{2,3\}$. But $B_{\beta\beta}$ is not a $\boldsymbol{Q}_o$-matrix. This is a contradiction. Hence $A \notin \boldsymbol{Q}_o$.

Remark 3.4.4. We make use of Theorem 3.4.2 in Chapter 5. Arguments there will be similar to the one in the above example.

Sufficient Conditions for $(n-1)\times(n-1)$ Principal Submatrix of a $\boldsymbol{Q}_o$-Matrix to be in $\boldsymbol{Q}_o$

The following theorem is an analogy of Theorem 3.2.8 for $\boldsymbol{Q}_o$-matrices.

Theorem 3.4.5. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{Q}_o$. Let $i \in \bar{n}$ and $\alpha = \overline{\{i\}}$. Then either $A_{\alpha\alpha} \in \boldsymbol{Q}_o$ or there exists a $u \in \boldsymbol{R}^n_+$ such that $u \in S(e_i, A)$ with $(Au)_i = -1$, where $e_i = I_{.i}$.

Proof. Without loss of generality take $i = n$. Suppose $A_{\alpha\alpha} \notin \boldsymbol{Q}_o$. Then there exists a $\bar{q} \in \boldsymbol{R}^{n-1}$ such that $F(\bar{q}, A_{\alpha\alpha}) \neq \phi$ but $S(\bar{q}, A_{\alpha\alpha}) = \phi$. For each positive integer k, define $q^k_\alpha = \bar{q}/k$, $q^k_n = 1$. Observe that $F(q^k_\alpha, A_{\alpha\alpha}) \neq \phi \;\forall\; k \geq 1$. This is because, for any $z_\alpha \in F(q_\alpha, A_{\alpha\alpha})$, $\frac{1}{k} z_\alpha \in F(q^k_\alpha, A_{\alpha\alpha}) \;\forall\; k \geq 1$. Then $F(q^k, A) \neq \phi$ for all positive integers k sufficiently large. As $A \in \boldsymbol{Q}_o$, $S(q^k, A) \neq \phi$ for all k sufficiently large. So each q^k lies in a complementary cone for all

k sufficiently large. Since there are only finitely many complementary cones, there is a complementary cone containing a subsequence $q^{k_1}, q^{k_2}, q^{k_3}, \ldots$ of q^k. Since q^k converges to e_n and as the complementary cones are all closed, e_n also lies in a complementary cone containing the subsequence q^{k_i}. Also note that for each $k \geq 1$, if $z^k \in S(q^k, A)$, then $z_n^k > 0$ as otherwise it would imply that $S(\bar{q}, A_{\alpha\alpha}) \neq \phi$. Thus, for all k sufficiently large, q^k lies in a complementary cone with $A_{.n}$ as one of its generators. Therefore, e_n lies in a complementary cone with one of its generators as $A_{.n}$ and hence there exists a $u \in S(e_n, A)$ such that $(Au)_n = -1$. □

Theorem 3.4.6. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q}_o$. Let $i \in \bar{n}$ and $\alpha = \overline{\{i\}}$. Suppose A satisfies Property (D) defined in Definition 2.4.2, i.e., for all $\gamma \in n^*$, $det\, A_{\gamma\gamma} = 0$ implies columns of $A_{.\gamma}$ are linearly dependent. Then either $A_{\alpha\alpha} \in \boldsymbol{Q}_o$ or there exists a $\beta \subseteq \bar{n}$ satisfying :

- **(a)** $i \in \beta$,
- **(b)** $det\, A_{\beta\beta} \neq 0$,
- **(c)** $M_{.i} \leq 0$, where $M = \wp_\beta(A)$,
- **(d)** $u \in S(e_i, A)$, where $u_\beta = -M_{\beta i}$ and $u_{\bar{\beta}} = 0$, and
- **(e)** $(Au)_i = -1$.

Proof. Without loss of generality take $i = n$. Suppose $A_{\alpha\alpha} \notin \boldsymbol{Q}_o$. Then there exists a $\bar{q} \in R^{n-1}$ such that $F(\bar{q}, A_{\alpha\alpha}) \neq \phi$ but $S(\bar{q}, A_{\alpha\alpha}) = \phi$. For each positive integer k, define q^k by $q_\alpha^k = \bar{q}/k$, and $q_n^k = 1$. As $F(\bar{q}, A_{\alpha\alpha}) \neq \phi$, $F(q^k, A) \neq \phi$ for all k sufficiently large. Since $A \in \boldsymbol{Q}_o$, for all k sufficiently large, there exists a solution (w^k, z^k) of (q^k, A). By Theorem 2.4.5, we may assume, without loss of generality, that (w^k, z^k) corresponds to complementary basis with $\beta_k = supp(z^k)$. Then $det\, C_A(\beta_k) \neq 0$ for all k sufficiently large. Since $\bar{n}$ has only finitely many subsets, one of its subsets must repeat infinitely often in the sequence $\beta_1, \beta_2, \beta_3, \ldots$. Again, without loss of generality, we can assume $\beta_k = \beta$ for all k sufficiently large. Then $det\, C_A(\beta) \neq 0$. Note that for each k, $z_n^k > 0$, as otherwise it will imply that $S(\bar{q}, A_{\alpha\alpha}) \neq \phi$ which is a contradiction. Thus $n \in \beta$. Hence we have :

$$\begin{bmatrix} I_{|\bar{\beta}|} & -A_{\bar{\beta}\beta} \\ 0- & A_{\beta\beta} \end{bmatrix} \begin{bmatrix} w_{\bar{\beta}}^k \\ \cdot z_{\beta}^k \end{bmatrix} = \begin{bmatrix} q_{\bar{\beta}}^k \\ q_{\beta}^k \end{bmatrix} \ \forall\, k \text{ sufficiently large .}$$

Since $det\, C_A(\beta) \neq 0$,

$$\begin{bmatrix} w_{\bar{\beta}}^k \\ z_{\beta}^k \end{bmatrix} = \begin{bmatrix} I_{|\bar{\beta}|} & -A_{\bar{\beta}\beta}(A_{\beta\beta})^{-1} \\ 0 & -(A_{\beta\beta})^{-1} \end{bmatrix} \begin{bmatrix} q_{\bar{\beta}}^k \\ q_{\beta}^k \end{bmatrix} \ \forall\, k \text{ sufficiently large.}$$

Note that as $k \to \infty$, $q^k \to e_n$, and hence $\begin{bmatrix} w_{\bar{\beta}}^k \\ z_{\beta}^k \end{bmatrix} \to \begin{bmatrix} v_{\bar{\beta}} \\ u_{\beta} \end{bmatrix}$,

where

$$\begin{bmatrix} v_{\bar{\beta}} \\ u_{\beta} \end{bmatrix} = \begin{bmatrix} I_{|\bar{\beta}|} & -A_{\bar{\beta}\beta}(A_{\beta\beta})^{-1} \\ 0 & -(A_{\beta\beta})^{-1} \end{bmatrix} e_n = -M_{.n},$$

where $M = \wp_{\beta}(A)$. Since $v_{\bar{\beta}} \geq 0$, $u_{\beta} \geq 0$, $M_{.n} \leq 0$
and $u = (0^t, -M_{\beta n}^t)^t \in S(e_n, A)$.
Obviously $(Au)_n = -1$ as $w_n^k = 0$ for all k sufficiently large implies $v_n = 0$. This completes the proof of the theorem. □

Corollary 3.4.7. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q}_o$. Assume that A satisfies Property (D) defined in Definition 2.4.2. If every legitimate PPT M of A is such that $v(M^t) > 0$, then every principal submatrix of A of order $(n-1)$ is in $\boldsymbol{Q}_o$.

Proof. Suppose there exists an $\alpha \subseteq \bar{n}$ such that $|\alpha| = n-1$ and $A_{\alpha\alpha} \notin \boldsymbol{Q}_o$. By Theorem 3.4.6, there exists a PPT M of A such that $M_{.k} \leq 0$ where $\{k\} = \bar{\alpha}$. This implies $v(M^t) \leq 0$ which contradicts the hypothesis. It follows that every principal submatrix of A of order $(n-1)$ is in $\boldsymbol{Q}_o$. □

Corollary 3.4.8. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q}_o$. Assume that A satisfies Property(D) of Definition 2.4.2. Let $k \in \bar{n}$ and let $\alpha = \overline{\{k\}}$. If for every $\beta \subseteq \bar{n}$ such that $k \in \beta$ and $v((\wp_{\beta}(A))^t) > 0$ whenever $\wp_{\beta}(A)$ is well defined, then $A_{\alpha\alpha} \in \boldsymbol{Q}_o$.

Proof. Follows from Theorem 3.4.6. □

Definition 3.4.9. Let $A \in \boldsymbol{R}^{n \times n}$ and $\alpha \in n^*$. Say that A is α-nondegenerate if $det\, A_{\beta\beta} \neq 0$ for all $\beta \in n^*$ such that $\alpha \subseteq \beta$. If $|\alpha| = 1$, say $\alpha = \{k\}$, then we say A is k-nondegenerate provided A is α-nondegenerate.

Corollary 3.4.10. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q}_o$. Let $k \in \bar{n}$ be such that A is k-nondegenerate. Let $\beta = \overline{\{k\}}$. If $v((\wp_\alpha(A))^t) > 0 \; \forall \; \alpha$ such that $k \in \alpha$, then $A_{\beta\beta} \in \boldsymbol{Q}_o$.

Proof. Follows from the proof of Theorem 3.4.6. □

Characterization of Nonnegative Q_o-Matrices

Murty (1972) gave the following characterization of nonnegative $\boldsymbol{Q}$-matrices.

Theorem 3.4.11. Suppose $A \in \boldsymbol{R}^{n \times n}$ is a nonnegative matrix. Then A is a $\boldsymbol{Q}$-matrix if, and only if, $a_{ii} > 0 \; \forall \; i \in \bar{n}$.

Definition 3.4.12. $A \in \boldsymbol{R}^{n \times n}$ is said to be *completely* $\boldsymbol{Q}$ (*completely* $\boldsymbol{Q}_o$) if for all $\alpha \in n^*$, $A_{\alpha\alpha} \in \boldsymbol{Q}$ ($A_{\alpha\alpha} \in \boldsymbol{Q}_o$). These classes are denoted by $\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{Q}}_0$.

Completely $\boldsymbol{Q}$-matrices were studied by Cottle (1980) in which he gave several characterizations of the same. He remarked that characterization for $\bar{\boldsymbol{Q}}_0$-matrices must be harder to obtain. In Fredricksen, Watson and Murty (1986), characterizations for $\bar{\boldsymbol{Q}}_0$-matrices of order less than or equal to three were obtained. Characterization of $\bar{\boldsymbol{Q}}_0$ for general matrices appears to be a formidable task. However, we characterize $\bar{\boldsymbol{Q}}_0$ in some special cases.

Lemma 3.4.13. Suppose $A \in \boldsymbol{R}^{n \times n}$ is a nonnegative matrix. Then $A \in \boldsymbol{Q}_o$ if, and only if, $A \in \bar{\boldsymbol{Q}}_0$.

Proof. Suffices to show that $A \in \boldsymbol{Q}_o$ implies $A \in \bar{\boldsymbol{Q}}_0$. Let α be a proper subset of $\bar{n}$. Take any $i \in \bar{\alpha}$. Since $A_{i.} \geq 0$, by Theorem 3.4.2, $A_{\beta\beta} \in \boldsymbol{Q}_o$, where $\beta = \overline{\{i\}}$. If $\alpha = \beta$, we are done. Otherwise choose $j \in \beta \setminus \alpha$, and drop the row and column, corresponding to j, from $A_{\beta\beta}$ to get $A_{\gamma\gamma}$ where $\gamma = \beta \setminus \{j\}$. Since

$A_{\beta\beta} \in Q_o$ and $A_{j\beta} \geq 0$, we have again $A_{\gamma\gamma} \in Q_o$. We can continue this process until we reach the conclusion that $A_{\alpha\alpha} \in Q_o$. □

In the theorem below, we give a characterization of nonnegative Q_o-matrices.

Theorem 3.4.14. Suppose $A \in R^{n \times n}$ is a nonnegative matrix where $n \geq 2$. Then $A \in Q_o$ if, and only if, the following implication is valid :

$$\text{for every } \; i \in \bar{n}, \;\; A_{i.} \neq 0 \Rightarrow a_{ii} > 0.$$

Proof. (Necessity) We shall prove this by induction on n. It is easy to check this when $n = 2$. Assume that the result is true for all $(n-1) \times (n-1)$ matrices, $n \geq 3$. Let $A \in R^{n \times n}$ be a nonnegative Q_o-matrix. Suppose $A_{i.} \neq 0$ for some $i \in \bar{n}$. Let j be such that $a_{ij} > 0$. If $j = i$ we are done. Suppose $j \neq i$. Choose any $k \in \overline{\{i,j\}}$ (we can do this as $n \geq 3$). Let $\alpha = \overline{\{k\}}$. Then by Lemma 3.4.13, $A_{\alpha\alpha} \in Q_o$. By choice of k, $A_{i\alpha} \neq 0$. By induction, we must have $a_{ii} > 0$.

(Sufficiency) Assume that $a_{ii} > 0 \; \forall \; i$ such that $A_{i.} \neq 0$.
Let $\alpha = \{i \in \bar{n} : a_{ii} > 0\}$. Then

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ 0 & 0 \end{bmatrix}$$

Suppose $q \in R^n$ is such that $F(q, A) \neq \phi$. Then we must have $q_{\bar{\alpha}} \geq 0$. Since $A_{\alpha\alpha}$ is nonnegative with all its diagonal entries positive, by Theorem 3.4.11, $A_{\alpha\alpha} \in Q$. Let $z_\alpha \in S(q_\alpha, A_{\alpha\alpha})$. Then $(z_\alpha^t, 0^t)^t \in S(q, A)$. As q was arbitrary, $A \in Q_o$. □

Corollary 3.4.15. Suppose $A \in R^{n \times n}$ is nonnegative nonnull matrix. Then $A \in Q_o$ if, and only if, there exists a principal rearrangement of A of the form

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ 0 & 0 \end{bmatrix}$$

such that $A_{\alpha\alpha} \in Q$. □

Corollary 3.4.16. Suppose $A \in \boldsymbol{R}^{n \times n}$. Assume that $A + A^t$ is a nonnegative $\boldsymbol{Q}_o$-matrix. Then $A \in \bar{\boldsymbol{Q}}_0$.

Proof. Let $B = A + A^t$. From Corollary 3.4.15, there exists $\alpha \subseteq \bar{n}$ such that $B_{\alpha\alpha}$ is nonnegative $\boldsymbol{Q}$-matrix (with all diagonal entries positive) and $B_{\alpha\bar{\alpha}} = 0$, $B_{\bar{\alpha}\alpha} = 0$, $B_{\bar{\alpha}\bar{\alpha}} = 0$ (because B is symmetric). Note that $x^tAx = \frac{1}{2}x^tBx$ for all $x \in \boldsymbol{R}^n_+$. Thus if $x^tAx = 0$, then $(A + A^t)x = Bx = 0$. Therefore, A is copositive-plus and hence belongs to $\bar{\boldsymbol{Q}}_0$. (see Corollary 4.2.11) □

Corollary 3.4.17. Suppose A and A^t are nonnegative $\boldsymbol{Q}_o$-matrices. Then $A + A^t$ belongs to $\bar{\boldsymbol{Q}}_0$. □.

The following examples illustrate the application of the above characterization theorem.

Example 3.4.18. Let

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} -1 & 2 & 1 & 2 \\ -3 & 1 & 0 & 4 \\ -1 & 0 & 0 & 1 \\ -1 & -2 & 1 & 0 \end{bmatrix}.$$

Note that $A_{3.} \geq 0$. If $A \in \boldsymbol{Q}_o$, then $A_{\alpha\alpha} \in \boldsymbol{Q}_o$, where $\alpha = \{1, 2, 4\}$. But by Theorem 3.4.14, $A_{\alpha\alpha} \notin \boldsymbol{Q}_o$. Hence $A \notin \boldsymbol{Q}_o$.

Next consider B. Let M $= \wp_\alpha(B)$, where $\alpha = \{3, 4\}$. Then

$$M = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 1 & 1 & 4 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Theorem 3.4.14 straightaway implies that $M \notin \boldsymbol{Q}_o$. Therefore, $B \notin \boldsymbol{Q}_o$. We generalize this feature in the following corollary.

Corollary 3.4.19. Suppose $M \in \boldsymbol{R}^{n \times n}$ is given by

$$M = \begin{bmatrix} A & B \\ C & P \end{bmatrix},$$

where $A \in \boldsymbol{R}^{k \times k}$, $B \in R^{k \times (n-k)}$, $C \in R^{(n-k) \times k}$, and $P \in \boldsymbol{R}^{(n-k)\times(n-k)}$. Assume that $B \geq 0$, $C \leq 0$, and P is a permutation matrix other than $I_{(n-k)}$. If $A \geq CP^tB$, then $M \notin \boldsymbol{Q}_o$.

Proof. Let D be the PPT of M with respect to P. As P is a permutation matrix, $PP^t = P^tP = I$. We have :

$$D = \begin{bmatrix} A - CP^tB & BP^t \\ -P^tC & P^t \end{bmatrix}$$

Note that $D \geq 0$. Further, P^t is a permutation matrix other than $I_{(n-k)}$ as P is so. Therefore, D has a nonzero row with the corresponding diagonal entry zero. Invoking Theorem 3.4.14, we conclude that D, and hence, M are not $\boldsymbol{Q}_o$-matrices. □

The following corollaries are direct consequences of Theorem 3.4.14.

Corollary 3.4.20. Suppose $A \in \boldsymbol{R}^{n \times n}$ is a nonnegative $\boldsymbol{Q}_o$-matrix. *If* $a_{ii} = 0$ *for all* $i \in \bar{n}$, *then* $A = 0$. □

Corollary 3.4.21. Suppose $A \in \boldsymbol{R}^{n \times n}$ is a nonnegative $\boldsymbol{Q}_o$-matrix. If A is *nonsingular, then* A *is completely* $\boldsymbol{Q}$. □

3.5. MORE RESULTS ON Q_o-MATRICES

In this we present some conditions under which a matrix cannot be a $\boldsymbol{Q}_o$-matrix. These results are useful in resolving some open problems in LCP (see Chapter 5).

Theorem 3.5.1. Suppose $A \in \boldsymbol{R}^{n \times n}$. Let $k \in \bar{n}$ and let $\alpha = \overline{\{k\}}$. Assume *that* A *satisfies the following conditions:*

(a) $A_{\alpha\alpha} \leq 0$,

(b) $a_{ik} > 0$ for some $i \in \alpha$, and

(c) $A_{k.} \geq 0$.

Then $A \notin \boldsymbol{Q}_o$.

Proof. Note that the assumptions of the theorem imply $i \neq k$. Since $a_{ik} > 0$, there exists a $q \in \boldsymbol{R}^n$ such that $q_i < 0,\ q_j > 0\ \forall\, j \in \bar{n},\ j \neq i$ and $F(q, A) \neq \phi$. Let $z \in F(q, A)$. Since $A_{\alpha\alpha} \leq 0$, we must have $z_k > 0$. Note that as $i \neq k$, $q_k > 0$ and $w_k = (Az)_k + q_k > 0$. This implies (q, A) cannot have a complementary solution. Therefore, $A \notin \boldsymbol{Q}_o$. □

Theorem 3.5.2. Suppose $A \in \boldsymbol{R}^{n\times n}$ where $n \geq 3$. Let $g = (a_{13}, a_{14}, \ldots, a_{1n})$ and $h = (a_{23}, a_{24}, \ldots, a_{2n})$. Assume that $a_{11} = a_{22} = 0$, $a_{12} > 0$ and $a_{21} > 0$. If $A \in \boldsymbol{Q}_o$, then A cannot satisfy any of the following conditions :

(a) $g \leq 0$ and $h \geq 0$

(b) $g \geq 0$ and $h \leq 0$.

Proof. Suppose A satisfies condition (a). Since $a_{12} > 0$, there exists a $q \in \boldsymbol{R}^n$ such that $q_1 < 0,\ q_j > 0$ for every $j \neq 1$, and $F(q, A) \neq \phi$. Hypothesis implies that for every $z \in F(q, A), z_2 > 0$. Since $A_{2.} \geq 0$, $w_2 = (Az)_2 + q_2 > 0$. Thus (q, A) cannot have a complementary solution. This contradicts that A is in $\boldsymbol{Q}_o$. Hence A does not satisfy (a). Similarly, we can show that A does not satisfy (b). □

In the above theorem the indices 1 and 2 can be replaced by any i and j, $i \neq j$.

Theorem 3.5.3. Suppose $A \in R^{3\times 3} \cap \boldsymbol{Q}_o$. Then $SP(A)$ cannot be equal to any of the following :

$$(a) \begin{bmatrix} \oplus & \oplus & \oplus \\ \star & \star & \star \\ + & \ominus & 0 \end{bmatrix} \quad (b) \begin{bmatrix} \oplus & - & \oplus \\ \oplus & \oplus & \oplus \\ + & \star & 0 \end{bmatrix}$$

Proof. Suppose $SP(A)$ is given by (a). As $a_{31} > 0$, there exists a $q \in R^3$ such that $SP(q) = (+, +, -)^t$ and $F(q, A) \neq \phi$. Let $z \in F(q, A)$. Then $z_1 > 0$.

Since $A_{1.} \geq 0$ and $q_1 > 0$, $w_1 = (Az)_1 + q_1 > 0$. This implies (q, A) cannot have a solution. This contradicts the hypothesis. Therefore, $SP(A)$ cannot be equal to the sign pattern given by (a).

Suppose $SP(A)$ is given by (b). Note that $A_{2.} \geq 0$. By Theorem 3.4.2, $A_{\alpha\alpha} \in Q_o$ where $\alpha = \{1, 3\}$. Observe that

$$SP(A_{\alpha\alpha}) = \begin{bmatrix} \oplus & \oplus \\ + & 0 \end{bmatrix}.$$

By Theorem 3.4.14, $A \notin Q_o$. From this contradiction it follows that A cannot have the sign pattern given in (b). □

CHAPTER 4

SEMIMONOTONE MATRICES IN LCP

4.1. INTRODUCTION

Our main results in this chapter are outlined as follows. We settle a conjecture, initially stated by Pang (1979b) and later modified by Jeter and Pye (1989), and Gowda (1990), by constructing a counter example. The conjecture, in its modified version, states that $C_0 \cap \boldsymbol{Q} \subseteq \boldsymbol{R}_0$. In Murthy, Parthasarathy and Ravindran (1993a), we proved that, for $n \leq 5$, if $A \in \boldsymbol{R}^{n \times n} \cap C_0 \cap \boldsymbol{Q}$ and if every 3×3 principal submatrix of A is also $\boldsymbol{Q}$, then $A \in \boldsymbol{R}_0$; and commented that it may not be possible to extend this to higher dimensions. While addressing this problem, we establish that the result is valid even when $n = 6$ and obtain sufficient conditions, for the general case, for copositive matrices to be $\boldsymbol{R}_0$. Further, we establish that for every $n \geq 5$, there exists a copositive $\boldsymbol{Q}$-matrix which is not $\boldsymbol{R}_0$ but $A_{\alpha\alpha} \in \boldsymbol{R}_0$ for all $\alpha \subseteq \bar{n}$ such that $|\alpha| \leq n-4$. Some of these results are extended to semimonotone $\boldsymbol{Q}$-matrices (see Murthy, Parthasarathy and Ravindran (1993b). These results are primarily of theoretical interest and may not have any bearing on the algorithmic aspects of LCP.

It is known that if A is symmetric semimonotone $\boldsymbol{Q}$-matrix, then A is completely $\boldsymbol{Q}$. We will establish that if A is symmetric semimonotone $\boldsymbol{Q}_0$-matrix, then A is completely $\boldsymbol{Q}_0$. Pang (1979b) showed that if A is semimonotone $\boldsymbol{Q}$-matrix, then every nontrivial solution of $(0, A)$ must have at least two nonzero coordinates. We will show that if A is semimonotone $\boldsymbol{Q}_0$-matrix and if every row of A has at least one positive entry, then every nontrivial solution of $(0, A)$ must have at least two nonzero coordinates.

The class of semimonotone matrices is one of the largest classes studied in connection with LCP. It encompasses a large number of well studied classes such as positive definite, positive semidefinite, $\boldsymbol{P}$, $\boldsymbol{P}_0$, and copositive matrices. This class was introduced by Eaves (1971) as an extension of strictly semimonotone matrices (though not with the same name) studied by Cottle and Dantzig (1968). The name "semimonotone matrices" was proposed by Karamardian (1972). The main property of this class is that if A is in this class, then (q, A) has a unique solution for every $q > 0$. A geometrical interpretation of this fact is that any complementary cone of $[\ I\ :\ -A\ ]$, other than $pos\, I$, does not intersect the nonnegative orthant in the interior. Another interesting and important result is that if A is a semimonotone matrix and if $(0, A)$ has a unique solution, then A is a Q-matrix.

4.2. PRELIMINARY RESULTS

Copositive Matrices

This class is an extension of positive semidefinite matrices. Apparently, Motzkin was the first to study this class.

Definition 4.2.1. $A \in R^{n \times n}$ is said to be

(a) *copositive* if $x^t A x \geq 0\ \forall x \in R^n_+$,

(b) *strictly copositive* if $x^t A x > 0\ \forall x \in R^n_+,\ x \neq 0$,

(c) *copositive-plus* if A is copositive and the following implication holds :

$$[x^t A x = 0,\ x \geq 0] \Rightarrow (A + A^t)x = 0$$

(d) *copositive-star* if A is copositive and the following implication holds :

$$[x \geq 0,\ Ax \geq 0,\ x^t A x = 0] \Rightarrow A^t x \leq 0.$$

These classes are denoted as follows : C_o stands for copositive matrices, C for strictly copositive matrices, C_o^+ for copositive-plus, and C_0^* for copositive-star matrices.

Definition 4.2.2. Let Y be any class of real square matrices. For any $A \in Y$, say that A is completely Y provided $A_{\alpha\alpha} \in Y$ for all $\alpha \in n^*$, where n is order of

A. The class of all completely Y matrices will be denoted by $\bar{Y}$. When $A \in \bar{Y}$, we say that 'property Y is inherited by all principal submatrices of A'.

Theorem 4.2.3. Let $A \in \boldsymbol{R}^{n \times n}$. Let $D \in \boldsymbol{R}^{n \times n}$ be any positive diagonal matrix. Then $A \in \boldsymbol{C}_o$ if, and only if, $DAD^t \in \boldsymbol{C}_o$.

Proof. Suppose $A \in \boldsymbol{C}_o$. For any $x \in \boldsymbol{R}^n_+$, let $y = D^t x$. Obviously $y \in \boldsymbol{R}^n_+$. Note that $x^t DAD^t x = y^t Ay \geq 0$ as $A \in \boldsymbol{C}_o$ and $y \geq 0$. Thus $DAD^t \in \boldsymbol{C}_o$. The converse follows from the fact that D^{-1} is a positive diagonal matrix and $A = D^{-1}(DAD^t)(D^{-1})^t$. □

Remark 4.2.4. In view of the above theorem, any copositive matrix can be converted to another (equivalent) copositive matrix with all its diagonal entries as either 0 or 1. This is because, if $A \in \boldsymbol{C}_o$, then $a_{ii} \geq 0 \; \forall \; i$.

Theorem 4.2.5. Let $A \in \boldsymbol{R}^{n \times n}$. Then $A \in \boldsymbol{C}_o$ if, and only if, $\frac{1}{2}(A + A^t) \in \boldsymbol{C}_o$.

Proof. This is a direct consequence of the fact $x^t Ax = \frac{1}{2}x^t(A + A^t)x$. □

Theorem 4.2.6. If $A \in \boldsymbol{C}_o$, then A is completely $\boldsymbol{C}_o$

Proof. Let $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{C}_o$. Let $\alpha \in n^*$. Let $x_\alpha \in R^{|\alpha|}_+$. Define $z \in \boldsymbol{R}^n_+$ by $z_\alpha = x_\alpha$, and $z_{\bar{\alpha}} = 0$. Then $x^t_\alpha A_{\alpha\alpha} x_\alpha = z^t Az \geq 0$ as $A \in \boldsymbol{C}_o$. Thus $A_{\alpha\alpha} \in \boldsymbol{C}_o$. As α was arbitrary, it follows that A is completely $\boldsymbol{C}_o$. □

Theorem 4.2.7. If $A \in \boldsymbol{C}_o{}^+$, then A is completely $\boldsymbol{C}_o{}^+$

Proof. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{C}^+_o$. Let $\alpha \in n^*$. Suppose $x_\alpha \in R^{|\alpha|}_+$ is such that $x^t_\alpha A_{\alpha\alpha} x_\alpha = 0$. Define $z \in \boldsymbol{R}^n_+$ by $z_\alpha = x_\alpha$, and $z_{\bar{\alpha}} = 0$. Then $z^t Az = x^t_\alpha A_{\alpha\alpha} x_\alpha = 0$. Since $A \in \boldsymbol{C}^+_o$, $(A + A^t)z = 0$. Therefore, $0 = (A + A^t)z = (A_{\alpha\alpha} + A^t_{\alpha\alpha})x_\alpha$. From Theorem 4.2.6, $A_{\alpha\alpha} \in \boldsymbol{C}_o$. Hence $A_{\alpha\alpha} \in \boldsymbol{C}^+_o$. As α was arbitrary, it follows that every principal submatrix of A is also a $\boldsymbol{C}^+_o$-matrix. □

Theorem 4.2.8. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{C}_o$. Then for all $\alpha \in n^*$, $v(A_{\alpha\alpha})$ is nonnegative, that is, $A \in \bar{\boldsymbol{S}}_o$. If, in addition, A is symmetric, then the following implication is valid :

$$[x \in \boldsymbol{R}^n_+,\ x^tAx = 0] \Rightarrow Ax \geq 0.$$

Proof. Fix $\alpha \in n^*$. Suppose $v(A_{\alpha\alpha})$ is negative. Then there exists a (probability) vector $y_\alpha \in R^{|\alpha|}_+$ such that $y^t_\alpha A_{\alpha\alpha} < 0$. Define $x \in \boldsymbol{R}^n_+$ by $x_\alpha = y_\alpha$ and $x_{\bar{\alpha}} = 0$. Then $x^tAx = y^t_\alpha A_{\alpha\alpha} y_\alpha < 0$, which contradicts that $A \in C_o$. Thus $v(A_{\alpha\alpha})$ is nonnegative.

We shall prove the second part by induction on n. The assertion is trivially true when $n = 1$. So assume that the result is true for all $(n-1) \times (n-1)$ matrices satisfying the hypothesis, $n > 1$. Suppose $A \in \boldsymbol{R}^{n\times n}$ is symmetric copositive matrix. Let $x \in \boldsymbol{R}^n_+$ be such that $x^tAx = 0$. We have to show $Ax \geq 0$. Suppose $(Ax)_i < 0$ for some $i \in \bar{n}$. Then, for any real $\lambda > 0$, we have :

$$(e_i + \lambda x)^t A(e_i + \lambda x) \geq 0, \text{ where } e_i = I_{.i} \text{ (because } A \in C_o)$$

$$\text{i.e., } 2\lambda(Ax)_i + a_{ii} \geq 0, \text{ where } A = (a_{ij}) \text{ (because } A = A^t\text{)}$$

But then, we can choose λ sufficiently large (positive) so that the last inequality is violated. This leads to a contradiction. Hence, we must have $Ax \geq 0$. □

The following theorem and its proof may be found in Cottle, Pang and Stone (page 179, 1992).

Theorem 4.2.9. Suppose $A \in \boldsymbol{R}^{n\times n} \cap C_o$. For any $q \in \boldsymbol{R}^n$, if the implication

$$[z \geq 0,\ Az \geq 0,\ z^tAz = 0] \Rightarrow z^tq \geq 0 \tag{4.1}$$

is valid, then (q, A) has a solution. □

Theorem 4.2.10. Let $A \in \boldsymbol{R}^{n\times n} \cap C^+_o$. *If $q \in \boldsymbol{R}^n$ is such that (q, A) has a feasible solution, then (q, A) has a solution.*

Proof. Suppose (q, A) has a feasible solution. Let $z \in \boldsymbol{R}^n_+$ be any vector satisfying $Az \geq 0$ and $z^tAz = 0$. Since $A \in C^+_o$, we have $(A + A^t)z = 0$

or $A^t z = -Az \leq 0$. Choose any $x \in F(q, A)$. Then $Ax + q \geq 0$, $x \geq 0$. Premultiplying both sides by z^t, we get $z^t Ax + z^t q \geq 0$. As $z^t A \leq 0$ and $x \geq 0$, it follows that $z^t q \geq 0$. Invoking Theorem 4.2.9, we conclude that (q, A) has a solution. □

Corollary 4.2.11. $C_o^+ \subseteq \bar{Q}_0$.

Proof. Follows from Theorem 4.2.7 and the above theorem. □

Corollary 4.2.12. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{C}$. Then $A \in \boldsymbol{Q}$.

Proof. Let $q \in \boldsymbol{R}^n$. If $z \in \boldsymbol{R}^n_+$ is such that $z^t Az = 0$, then, by strict copositivity of A, $z = 0$ and hence $z^t q = 0$. Invoking Theorem 4.2.9, we conclude that (q, A) has a solution. As q was arbitrary, we conclude that $A \in \boldsymbol{Q}$. □

Theorem 4.2.13. $\boldsymbol{C} \subseteq \bar{\boldsymbol{Q}}$.

Proof. This is a direct consequence Corollary 4.2.12 and the fact that strict copositivity of a matrix is inherited by all its principal submatrices. □

Semimonotone Matrices

Definition 4.2.14. Let $A \in \boldsymbol{R}^{n \times n}$. A is said to be a *semimonotone* matrix if for every $x \in \boldsymbol{R}^n_+$ such that $x \neq 0$, there exists an index $k \in \bar{n}$ such that $x_k > 0$ and $(Ax)_k \geq 0$. A is said to be *strictly semimonotone* if for every $x \in \boldsymbol{R}^n_+$ such that $x \neq 0$ there exists a $k \in \bar{n}$ satisfying $x_k > 0$ and $(Ax)_k > 0$. Semimonotone and strictly semimonotone matrix classes are denoted by $\boldsymbol{E}_o$ and $\boldsymbol{E}$ respectively.

Theorem 4.2.15. If $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o$, then $A_{\alpha\alpha} \in \boldsymbol{E}_o$ for all $\alpha \in n^*$.

Proof. Fix $\alpha \in n^*$. Let $z_\alpha \in \boldsymbol{R}^{|\alpha|}_+$ be such that $z_\alpha \neq 0$. Define $x \in \boldsymbol{R}^n_+$ by $x_\alpha = z_\alpha$ and $x_{\bar{\alpha}} = 0$. Obviously $0 \neq x \geq 0$. Since $A \in \boldsymbol{E}_o$, there exists a $k \in \bar{n}$ such that $x_k > 0$ and $(Ax)_k \geq 0$. As $x_{\bar{\alpha}} = 0$, $k \in \alpha$. Also $(A_{\alpha\alpha} x_\alpha)_k = (Ax)_k \geq 0$. Hence $A_{\alpha\alpha} \in \boldsymbol{E}_o$. □

Remark 4.2.16. Note that every copositive matrix is in $\boldsymbol{E}_o$. Also $\boldsymbol{P}_o \subseteq \boldsymbol{E}_o$ (see page 185, Cottle, Pang and Stone (1992)).

Theorem 4.2.17. Let $A \in \boldsymbol{R}^{n \times n}$. The following statements are equivalent :

(a) $A \in \boldsymbol{E}_o$

(b) (q, A) has a unique solution for every $q \in \boldsymbol{R}^n_{++}$

(c) $v(A^t_{\alpha\alpha}) \geq 0 \ \forall \ \alpha \in n^*$

(d) $v(A_{\alpha\alpha}) \geq 0 \ \forall \ \alpha \in n^*$

(e) $A^t \in \boldsymbol{E}_o$

Proof. **(a) $\Rightarrow$ (b).** Let $q > 0$ and assume that $z \in S(q, A)$. We will show that $z = 0$. Suppose $z_k > 0$ for some $k \in \bar{n}$. This implies $w_k = 0$, where $w = Az + q$. Then $(Az)_k + q_k = 0$ and hence $(Az)_k = -q_k < 0$. This holds for any $k \in \bar{n}$ such that $z_k > 0$, and $\{k \in \bar{n} : \ z_k > 0, \ (Az)_k \geq 0\} \neq \phi$. This contradicts that $A \in \boldsymbol{E}_o$. Hence $z = 0$ and $(q, 0)$ is the only solution of (q, A).

(b) $\Rightarrow$ (c). Suppose $v(A^t_{\alpha\alpha}) < 0$ for some $\alpha \in n^*$. Then there exists a probability vector $x_\alpha \in \boldsymbol{R}^{|\alpha|}_+$ such that $0 \neq x_\alpha \geq 0$ and $x^t_\alpha A^t_{\alpha\alpha} < 0$. Define $q \in \boldsymbol{R}^n$ by $q_\alpha = -A_{\alpha\alpha}x_\alpha$ and $q_{\bar{\alpha}}$ be any positive vector in $R^{|\bar{\alpha}|}$ satisfying $q_{\bar{\alpha}} + A_{\bar{\alpha}\alpha}x_\alpha > 0$. Define $z \in \boldsymbol{R}^n_+$ by $z_\alpha = x_\alpha$ and $z_{\bar{\alpha}} = 0$. Then $z \neq 0$. Note that 0 and z are both in $S(q, A)$. This contradicts (b). Hence (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d). We shall prove this by induction on n. The result is trivially true when $n = 1$. Assume the result to be true for all real square matrices of order less than or equal to $n - 1$, $n > 1$. Suppose $A \in \boldsymbol{R}^{n \times n}$ is such that $v(A^t_{\alpha\alpha}) \geq 0 \ \forall \ \alpha \in n^*$. By induction hypothesis, $v(A_{\alpha\alpha}) \geq 0 \ \forall \ \alpha \in n^*, \ \alpha \neq \bar{n}$. If possible, let $v(A) < 0$. Then there exists a probability vector $y \in \boldsymbol{R}^n_+$ such that $y^t A < 0$. Since $v(A_{\alpha\alpha}) \geq 0 \ \forall \alpha \neq \bar{n}$, we must have $y > 0$. By Theorem 1.4.4, there exists a probability vector $z \in \boldsymbol{R}^n_+$ such that $Az = v(A)e$, where $e = (1, 1, \ldots, 1)^t \in \boldsymbol{R}^n$. But then $z^t A^t = v(A)e^t < 0$ which implies $v(A^t) < 0$, a contradiction. Hence $v(A) \geq 0$. By induction, the result follows.

(d) ⇒ (e). Suppose $A^t \notin \boldsymbol{E}_o$. Then there exists a $z \in \boldsymbol{R}^n_+$ such that $z \neq 0$ and $(A^t z)_k < 0$ for all $k \in \alpha = supp(z)$. Note that $\alpha \neq \phi$. We have $A^t_{\alpha\alpha} z_\alpha = (A^t z)_\alpha < 0$. That is, $z^t_\alpha A_{\alpha\alpha} < 0$. This contradicts (d).
Hence $A^t \in \boldsymbol{E}_o$.

(e) ⇒ (a). Since (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e), $A = (A^t)^t \in \boldsymbol{E}_o$. This completes the proof of the theorem. □

Theorem 4.2.18. Semimonotone property is invariant under principal rearrangements. That is, for any $n \geq 1$, if $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o$, and $P \in \boldsymbol{R}^{n \times n}$ is any permutation matrix, then $PAP^t \in \boldsymbol{E}_o$.

Proof. Let $P \in \boldsymbol{R}^{n \times n}$ be any permutation matrix and let $x \in \boldsymbol{R}^n_+$ be such that $x \neq 0$. Define $y = P^t x$. Then $y \in \boldsymbol{R}^n_+$ and $y \neq 0$. Since $A \in \boldsymbol{E}_o$ there exists an $i \in \bar{n}$ such that $y_i > 0$ and $(Ay)_i \geq 0$.
Since $PP^t = P^t P = I$, $(Ay)_i = (AP^t Py)_i = (AP^t x)_i \geq 0$.
Then there exists a $j \in \bar{n}$, such that

$$(Py)_j > 0 \quad \text{and} \quad (P(AP^t x))_j \geq 0.$$

Since $y = P^t x$ and $(Py)_j = (PP^t x)_j = x_j$, we have

$$x_j > 0 \quad \text{and} \quad (PAP^t x)_j \geq 0.$$

Therefore, $PAP^t \in \boldsymbol{E}_o$. □

Theorem 4.2.19. Let $A \in \boldsymbol{R}^{n \times n}$ be symmetric. Then

(a) if $A \in \boldsymbol{E}_o$ if, and only if, $A \in \boldsymbol{C}_o$

(b) if $A \in \boldsymbol{E}$ if, and only if, $A \in \boldsymbol{C}$.

Proof. (a) If $A \in \boldsymbol{C}_o$, then from the definitions of $\boldsymbol{E}_o$ and $\boldsymbol{C}_o$ it is easy to check that $A \in \boldsymbol{E}_o$.
We shall prove the converse by induction on n. If $n = 1$, the single entry of A must be nonnegative as $A \in \boldsymbol{E}_o$. Hence A is copositive. Assume that the result

is true for all real square matrices of order less than or equal to $n-1$, $n>1$. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o$ and $A = A^t$. Suppose $A \notin \boldsymbol{C}_o$. Then there exists a $y \in \boldsymbol{R}^n_+$ such that $y^t A y < 0$. Since, by induction hypothesis, every proper principal submatrix of A is in $\boldsymbol{C}_o$, $y > 0$. Since $A \in \boldsymbol{E}_o$, there exists $x \in \boldsymbol{R}^n_+$ such that $x \neq 0$ and $Ax \geq 0$. Choose a real $\lambda > 0$ such that $y - \lambda x \geq 0$ and at least one coordinate of $y - \lambda x$ is zero. Then as $Ax \geq 0$ and $y - \lambda x \geq 0$, we have $(y-\lambda x)^t Ax \geq 0$. Also, as $y > 0$ and $\lambda > 0$, $\lambda y^t Ax \geq 0$. Note

$$0 > y^t A y = (y-\lambda x)^t A (y - \lambda x) + \lambda y^t A x + \lambda (y - \lambda x)^t A x \geq 0.$$

The last inequality follows as $(y - \lambda x)$ has at least one zero coordinate and by induction hypothesis $(y-\lambda x)^t A(y-\lambda x) \geq 0$, and the other two quantities are nonnegative. The above contradiction implies that $A \in \boldsymbol{C}_o$.
Similarly we can establish (b). □

Recall the definitions of $\boldsymbol{P}_o$, $\boldsymbol{R}$, $\boldsymbol{R}_o$-matrix classes. In the rest of this section we present some results on semimonotone matrices relating to these matrix classes.

Theorem 4.2.20.(Aganagic and Cottle, 1979). Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{P}_o$. Then the following statements are equivalent :

$$(a) A \in \boldsymbol{R} \quad (b) A \in \boldsymbol{R}_o \quad (c) A \in \boldsymbol{Q}$$

Theorem 4.2.21. (Pang, 1979b). Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o$. If $A \in \boldsymbol{R}_o$, then $A \in \boldsymbol{Q}$.

Theorem 4.2.22. (Pang, 1979b). Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q} \cap \boldsymbol{E}_o$. Then the system

$$Ax = 0, \quad x > 0 \tag{4.2}$$

has no solution. Further, every nontrivial solution of $(0, A)$ has at least two nonzero coordinates. In particular, these assertions holds good for $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{C}_o \cap \boldsymbol{Q}$ as $\boldsymbol{C}_o \subseteq \boldsymbol{E}_o$.

Proof. Suppose (4.2) has a solution, say x. Then $x^t A^t = 0$. This means in the matrix game A^t, player II has a completely mixed strategy.

By Theorem 1.4.1 and Theorem 1.4.4, there exists a probability vector y such that $A^t y = 0$. Hence $y^t A = 0$. This implies that $v(A) \leq 0$. But then $A \notin \boldsymbol{Q}$, a contradiction. It follows that (4.2) has no solution. □

Theorem 4.2.23. (Pang, 1979b). Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q} \cap \boldsymbol{C}_o$. If A is symmetric, then

$$Ax = 0, \quad x \geq 0 \Rightarrow x = 0$$

Proof. Suppose there exists an x such that $Ax = 0$, $x \geq 0$, $x \neq 0$. Then $0 = x^t A^t = x^t A$. This implies that $v(A) \leq 0$ which contradicts the hypothesis $A \in \boldsymbol{Q}$. Hence the result follows. □

Theorem 4.2.24. (Gowda, 1990). Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o$. Suppose A is symmetric. Then the following statements are equivalent :

$$(a)\ A \in \boldsymbol{R}_o \qquad (b)\ A \in \boldsymbol{C} \qquad (c) A \in \boldsymbol{Q}$$

Theorem 4.2.25. (Cottle, 1980). Suppose $A \in \boldsymbol{R}^{n \times n}$. Then the following statements are equivalent :

$$(a)\ A \in \boldsymbol{E} \quad (b)\ A \in \bar{\boldsymbol{S}} \quad (c)\ A \in \bar{\boldsymbol{Q}} \quad (d)\ A \in \mathbf{V}$$

Theorem 4.2.26. (Jeter and Pye, 1985). Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{Q}$. Let $\alpha \in n^*$. Define $q \in \boldsymbol{R}^n$ by $q_\alpha = 0$ and $q_{\bar{\alpha}} = (1, 1, \ldots, 1)^t$. Then either $A_{\alpha\alpha} \in \boldsymbol{Q}$ or there exists a $u \in S(q, A)$ such that $(Au)_i = -1$ for some $i \in \bar{\alpha}$.

4.3. THE COUNTER EXAMPLE

Aganagic and Cottle (1979) characterized the class of $\boldsymbol{Q} \cap \boldsymbol{P}_o$-matrices (see Theorem 4.2.20). Pang (1979b) extended it (partially) to $\boldsymbol{E}_o$ class and established Theorem 4.2.21. He conjectured that the converse must be true, i.e., $\boldsymbol{E}_o \cap \boldsymbol{Q} \subseteq \boldsymbol{R}_o$. However, this was disproved by Jeter and Pye (1989) through their counter example given in Example 3.3.4. This is an example of a $\boldsymbol{E}_o \cap \boldsymbol{Q}$-matrix which is not $\boldsymbol{R}_o$. They conjectured that it must be true for copositive $\boldsymbol{Q}$-matrices, that is, $\boldsymbol{C}_o \cap \boldsymbol{Q} \subseteq \boldsymbol{R}_o$. Gowda (1990) proved that Pang's conjecture for symmetric copositive matrices (see Theorem 4.2.24). He too mentioned that the result is unknown for general (assymetric) copositive matrices. We settle this conjecture through the following counter example (see Murthy, Parthasarathy and Ravindran (1993a)).

Counter Example 4.3.1. Consider

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Note that

$$x^t A x = x_3^2 + x_4^2 + 2x_1x_3 + 2x_2x_4 \geq 0 \; \forall \, x \in R_+^4.$$

Thus $A \in C_o$. Let $z = (1,1,0,0)^t$. Then $z \in S(0, A)$. Hence $A \notin R_o$. In Example 3.3.3, it was shown that $A \in Q$. Thus $A \in C_o \cap Q$ but $A \notin R_o$.

Theorem 4.3.2. For every integer $n \geq 4$, there exists $M \in R^{n \times n} \cap C_o \cap Q$ such that $M \notin R_o$.

Proof. For $n = 4$, we have the above example. For any integer $n > 4$ define

$$M = \begin{bmatrix} A & 0 \\ 0 & I_{n-4} \end{bmatrix},$$

where A is given in Counter Example 4.3.1. It is easy to check that $M \in R^{n \times n} \cap C_o \cap Q$ and that $A \notin R_o$. □

Theorem 4.3.3. The set of Q-matrices in $R^{n \times n}$ is not open for $n \geq 4$.

Proof. Define, for $n \geq 4$, $M^k \in R^{n \times n}$ by

$$M^k = \begin{bmatrix} A^k & 0 \\ 0 & I_{n-4} \end{bmatrix}, \text{ where } A^k = \begin{bmatrix} \frac{1}{k} & \frac{-1}{k} & 1 & 1 \\ \frac{-1}{k} & \frac{1}{k} & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

Note that for each integer $k \geq 1$, A^k as well as M^k are in C_o.

claim : $A^k \notin \boldsymbol{Q}$ for all $k \geq 1$.
Let $q = (-4, -4, -1, -1)^t$. Then it can be verified that (q, A^k) has no solution for all $k \geq 1$. For another proof of this claim see Remark 4.4.8. It follows that (p, M^k) has no solution for all $k \geq 1$, where $p = (q^t, 0^t)^t \in \boldsymbol{R}^n$. Thus $M^k \notin \boldsymbol{Q}\ \forall\ k \geq 1$. But M^k converges to M as k tends to ∞. Here M is as defined in proof of Theorem 4.3.2. Since each M^k is copositive, it follows that the set of copositive $\boldsymbol{Q}$-matrices is not open in $\boldsymbol{R}^{n \times n}$ for $n \geq 4$. □

Remark 4.3.4. Kelly and Watson (1979) showed that the set of nondegenerate $\boldsymbol{Q}$-matrices in $\boldsymbol{R}^{n \times n}$ is open when $n \leq 3$ and produced a counter example in $R^{4 \times 4}$ to show that the set of nondegenerate $\boldsymbol{Q}$-matrices is not open.

Jeter and Pye (1985) proved that $\boldsymbol{R}^{3 \times 3} \cap \boldsymbol{C}_o \cap \boldsymbol{Q} \subseteq \boldsymbol{R}_o$. It is known that Pang's conjecture is true when $n \leq 3$.

Theorem 4.3.5. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o$, where $n \leq 3$. Then $A \in \boldsymbol{R}_o$ if, and only if, $A \in \boldsymbol{Q}$ □.

Theorem 4.3.6. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{C}_o \cap \boldsymbol{Q}$, where $n \in \{4, 5\}$. If $A_{\alpha\alpha} \in \boldsymbol{Q}\ \forall\ \alpha \subseteq \bar{n}$ such that $|\alpha| = 3$, then $A \in \boldsymbol{R}_o$.

We shall omit the proof of this theorem as it will follow from a more general theorem (see Theorem 4.4.6). The following is an example of a 6×6 $\boldsymbol{C}_o \cap \boldsymbol{Q}$-matrix which is not an $\boldsymbol{R}_o$-matrix. In this example there are 3×3 principal submatrices which are not $\boldsymbol{Q}$-matrices.

Example 4.3.7. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

It is easy to see that $A \in \boldsymbol{C}_o$. In Example 3.3.5, it was shown that $A \in \boldsymbol{Q}$.

Since $A_{.1}+A_{.2}=0$, $A \notin \boldsymbol{R}_o$. Thus $A \in C_o \cap \boldsymbol{Q}$ but $A \notin \boldsymbol{R}_o$. Note that $A_{\alpha\alpha} \notin \boldsymbol{Q}$, for $\alpha = \{1,2,3\}$. This can be seen as follows : Suppose

$$A_{\alpha\alpha} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \in \boldsymbol{Q}.$$

Since $A_{2\alpha} \geq 0$, by Theorem 3.4.1,

$$A_{\beta\beta} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in \boldsymbol{Q}, \text{ where } \beta = \{1,3\}$$

By Theorem 3.4.11, $A_{\beta\beta} \notin \boldsymbol{Q}$. This contradiction leads us to conclude that $A_{\alpha\alpha} \notin \boldsymbol{Q}$.

The above theorems and observations tempted us to raise the following question (Murthy, Parthasarathy and Ravindran (1993a)) :
Suppose $A \in \boldsymbol{R}^{n \times n} \cap C_o \cap \boldsymbol{Q}$, $n \geq 3$, and $A_{\alpha\alpha} \in \boldsymbol{Q}\ \forall\ \alpha \in n^*$ such that $|\alpha| = 3$. Can we conclude that $A \in \boldsymbol{R}_o$?

The answer is "yes" when $n = 3, 4, 5$ or 6. This is proved in Theorem 4.3.6 and Corollary 4.4.9. However, the answer is "no" when $n \geq 7$. This is established in Theorem 4.5.1. In fact, the answer provided is much more than what is desired. We shall close this section with the following observation.

Observation 4.3.8. Recently we noticed the example given by Jeter and Pye (1989) will itself serve the purpose of countering their conjecture that $C_o \cap \boldsymbol{Q} \subseteq \boldsymbol{R}_o$. Note that PPT of A, given in Example 3.3.4, with respect to $\{5\}$ is given by

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Being a PPT of a $\boldsymbol{Q}$-matrix, M is $\boldsymbol{Q}$-matrix. Obviously M is a copositive matrix.

In a way we are glad to have missed this initially because we could come up with the simple and powerful Proposition 3.3.1 and its derivatives in the process of constructing the Counter Example 4.3.1.

4.4. SUFFICIENT CONDITIONS FOR $C_o \cap Q, E_o \cap Q \subseteq R_o$

In this section we shall present some results on copositive as well as semi-monotone Q-matrices and provide some sufficient conditions under which $C_o \cap Q \subseteq R_o$ and $E_o \cap Q \subseteq R_o$.

Theorem 4.4.1. Suppose $A \in R^{n \times n} \cap C_o \cap Q$. Let α be a proper nonempty subset of $\bar{n}$. Let $B = A_{\alpha\alpha}$. Suppose there exists a vector $x \in R_+^{|\alpha|}$ such that $x > 0$ and $Bx = 0$. Let $q \in R^n$ be defined by $q_\alpha = 0$ and $q_{\bar{\alpha}} = (1, 1, \ldots, 1)^t \in R^{|\bar{\alpha}|}$. Then there exists a $u \in S(q, A)$ such that $(Au)_k + 1 = 0$ for some $k \in \bar{\alpha}$. Further, $(Au)_\alpha = 0$ and $u_{\bar{\alpha}} = 0$ for all $u \in S(q, A)$.

Proof. Since $Bx = 0$, $x > 0$ and $B \in C_o$, by Theorem 4.2.23, $B \notin Q$. By Theorem 4.2.26, there exists a $u \in S(q, A)$ such that $(Au)_k + 1 = 0$ for some $k \in \bar{\alpha}$. Next, suppose z is any solution of (q, A). Then we have

$$z^t(Az + q) = z^t Az + z_{\bar{\alpha}}^t e_{\bar{\alpha}} = 0.$$

As $A \in C_o$ and $z \geq 0$, we must have $z_{\bar{\alpha}} = 0$ and $z_\alpha^t A_{\alpha\alpha} z_\alpha = z^t Az = 0$. So $z_\alpha^t B z_\alpha = 0$. Also $0 \leq (Az + q)_\alpha = (Az)_\alpha = A_{\alpha\alpha} z_\alpha = B z_\alpha$. Let $\bar{x} = z_\alpha$. Then, for any real $\lambda \geq 0$, we have :

$$(x - \lambda\bar{x})^t B(x - \lambda\bar{x}) = x^t Bx - \lambda\bar{x}^t Bx - \lambda x^t B\bar{x} + \lambda^2 \bar{x}^t B\bar{x} \tag{4.3}$$

Since $x > 0$, for all sufficiently small $\lambda > 0$, we have $x - \lambda\bar{x} \geq 0$. As $B\bar{x} \geq 0$, $Bx = 0$ and $B \in C_o$, we have :
for all sufficiently small $\lambda > 0$

$$0 \leq (x - \lambda\bar{x})^t B(x - \lambda\bar{x}) = -\lambda x^t B\bar{x} + \lambda^2 \bar{x}^t B\bar{x}$$

Dividing by λ and taking limit as $\lambda \to 0^+$, we get $x^t B\bar{x} \leq 0$. Since $x > 0$ and $B\bar{x} \geq 0$, we must have $B\bar{x} = 0$. This completes the proof of the theorem. □

Remark 4.4.2. Theorem 4.4.1 may not hold good for semimonotone matrices in general. Especially, the last assertion of theorem may be violated. The following example demonstrates this fact.

Example 4.4.3. Consider the example of Jeter and Pye (1989) given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

We have seen, in Example 3.3.4, that $A \in \boldsymbol{Q}$. Let $\alpha = \{1,2,3\}$ and $x = (1,1,2)^t$. Let $q = (0,0,0,1,1)^t$ and $u = (2,1,0,0,0)^t$. It may be verified that $u \in S(q, A)$ with $(Au)_4 + 1 = 0$. Also $u_4 = u_5 = 0$, but $(Au)_\alpha \neq 0$ as $(Au)_3 = 1$. Thus, A does not satisfy the second assertion of Theorem 4.4.1.

Lemma 4.4.4. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C}_o \cap \boldsymbol{Q}$. Suppose there exists a $z \in (0,\ A)$ such that $|\alpha| = n-1$, where $\alpha = supp(z)$. Then there exists an $x \in \boldsymbol{R}^n_+$ such that $x \neq 0$, $|\beta| \leq n-2$, $\beta \subseteq \alpha$, $(Ax)_\alpha = 0$ and $(Ax)_{\bar{\alpha}} \geq 0$, where $\beta = supp(x)$. Further, there exists a $\gamma \subseteq \bar{n}$ such that $|\gamma| = n-2$ and $A_{\gamma\gamma} \notin \boldsymbol{R}_o$.

Proof. Assume, without loss of generality, $z_n = 0$ and $z_i > 0\ \forall\ i \neq n$. Then $\bar{\alpha} = \{n\}$. Since $z \in S(0,\ A)$ and $z_\alpha > 0$, $A_{\alpha\alpha} z_\alpha = 0$. By Theorem 4.2.22, $A_{\alpha\alpha} \notin \boldsymbol{Q}$. By Theorem 4.4.1, there exists a $u \in \boldsymbol{R}^n_+$ such that $u_n = 0$, $u_\alpha \neq 0$, $(Au)_n = -1$ and $(Au)_\alpha = 0$. Since $z_\alpha > 0$, we can choose a positive real λ such that $z_\alpha - \lambda u_\alpha \geq 0$ and $z_j - \lambda u_j = 0$ for some $j \in \alpha$. Let $x = z - \lambda u$. Since $z_n = u_n = 0$, $x \geq 0$. Note that $x \neq 0$ for otherwise $0 \leq (Az)_n = \lambda(Au)_n = -\lambda < 0$. Also $x_j = 0$, $x_n = 0$ and $(Ax)_\alpha = (Az)_\alpha - \lambda(Au)_\alpha = 0$. Further, $(Ax)_n = (Az)_n - \lambda(Au)_n > 0$ as $Az \geq 0$ and $(Au)_n = -1$. Since $x_n = 0$, $\beta = supp(x) \subseteq \alpha$. Let $\gamma = \alpha \setminus \{j\}$. Clearly x_γ is a nontrivial solution of $(0, A_{\gamma\gamma})$. Hence $A_{\gamma\gamma} \notin \boldsymbol{R}_o$. Also $|\gamma| = n-2$. This completes the proof of the lemma. □

Lemma 4.4.5. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C}_o \cap \boldsymbol{Q}$, $n \geq 3$. Assume there exists a $z \in \boldsymbol{R}^n_+$ satisfying $|\alpha| = n-2$, $z_\alpha > 0$, $(Az)_\alpha = 0$ and $(Az)_{\bar\alpha} \geq 0$, where $\alpha = supp(z)$. Then there exists an $x \in \boldsymbol{R}^n_+$ such that $|\beta| \leq n-3$, $x_\beta \neq 0$, $\beta \subseteq \alpha$ and $(Ax)_\alpha = 0$, where $\beta = supp(x)$. Further, there exists a $\gamma \subseteq \bar{n}$ such that $|\gamma| = n-3$ and $A_{\gamma\gamma} \notin \boldsymbol{R}_0$.

Proof. Since $(Az)_\alpha = A_{\alpha\alpha} z_\alpha = 0$ and $z_\alpha > 0$, by Theorem 4.2.22 $A_{\alpha\alpha} \notin \boldsymbol{Q}$. By Theorem 4.4.1, there exists a $u \in S(q, A)$, where $q \in \boldsymbol{R}^n$ is given by $q_\alpha = 0$, and $q_{\bar\alpha} = (1,1)^t$ satisfying $u_{\bar\alpha} = 0$, $(Au)_\alpha = 0$ and $(Au)_j = -1$ for some $j \in \bar\alpha$. Note that $u_\alpha \neq 0$. Since $z_\alpha > 0$, we can choose a positive real λ such that $z_\alpha - \lambda u_\alpha \geq 0$ and $z_k - \lambda u_k = 0$ for some $k \in \alpha$. Let $x = z - \lambda u$. Then, as $z_{\bar\alpha} = u_{\bar\alpha} = 0$, $x_{\bar\alpha} = 0$. Since $x_k = z_k - \lambda u_k = 0$ and $k \in \alpha$, $|\beta| \leq n-3$, where $\beta = supp(x)$. As $(Az)_\alpha = (Au)_\alpha = 0$, $(Ax)_\alpha = 0$. Lastly, note that $x = 0$ implies $Az = \lambda Au$ which is not possible as $(Az)_j \geq 0$, $(Au)_j = -1$ and $\lambda > 0$. Hence $x \neq 0$. Since $x_{\bar\alpha} = 0$, $x_\beta \neq 0$ and $\beta = supp(x) \subseteq \alpha$. Let $\gamma = \alpha \setminus \{k\}$. Then $\beta \subseteq \gamma$, $|\gamma| = n-3$ and x_γ is a nontrivial solution of $(0, A_{\gamma\gamma})$. Hence $A_{\gamma\gamma} \notin \boldsymbol{R}_0$. This completes the proof of the lemma. □

Theorem 4.4.6. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{C}_0$ where $n \geq 4$. Assume that A satisfies any of the following conditions :

- **(a)** $A_{\alpha\alpha} \in \boldsymbol{R}_0 \ \forall \ \alpha \in n^*$ such that $|\alpha| = n-1$
- **(b)** $A_{\alpha\alpha} \in \boldsymbol{R}_0 \ \forall \ \alpha \in n^*$ such that $|\alpha| = n-2$
- **(c)** $A_{\alpha\alpha} \in \boldsymbol{R}_0 \ \forall \ \alpha \in n^*$ such that $|\alpha| = n-3$

Then $A \in \boldsymbol{Q}$ if, and only if, $A \in \boldsymbol{R}_0$.

Proof. If $A \in \boldsymbol{R}_0$, then by Theorem 4.2.21 and the fact that $\boldsymbol{C}_0 \subseteq \boldsymbol{E}_0$, $A \in \boldsymbol{Q}$ (irrespective of the conditions (a), (b) and (c)).
Conversely, assume that $A \in \boldsymbol{Q}$.
Case (a). Suppose A satisfies (a). Assume, if possible, that $A \notin \boldsymbol{R}_0$. Then there exists a $z \in S(0, A)$ such that $z \neq 0$. Since $A \in \boldsymbol{Q}$, by Theorem 4.2.21, z must have at least one coordinate zero. Assume, without loss of generality that

$z_n = 0$. But then z_α is nontrivial solution of $(0, A_{\alpha\alpha})$, where $\alpha = \overline{\{n\}}$, which contradicts the hypothesis (a). Hence $A \in \boldsymbol{R}_0$.

Case (b). Suppose A satisfies (b). Suppose $A \notin \boldsymbol{R}_0$. Then there exists a $z \in S(0, A)$ such that $z \neq 0$. Let $\alpha = supp(z)$. Because $A \in \boldsymbol{Q}$ and A satisfies (b), we must have $|\alpha| = n - 1$. By Lemma 4.4.4, there exists a $\gamma \subseteq \bar{n}$ such that $|\gamma| = n - 2$ and $A_{\gamma\gamma} \notin \boldsymbol{R}_0$. This contradicts hypothesis (b). It follows that $A \in \boldsymbol{R}_0$.

Case (c). Suppose A satisfies (c). Suppose $A \notin \boldsymbol{R}_0$. Let $z \in S(0, A)$, $z \neq 0$. Let $\alpha = supp(z)$. Suppose $|\alpha| = n - 1$. By Lemma 4.4.4, there exists an $x \in \boldsymbol{R}^n_+$ such that $x \neq 0$, $|supp(x)| \leq n - 2$, $supp(x) \subseteq \alpha$, $(Ax)_\alpha = 0$, and $(Ax)_{\bar{\alpha}} \geq 0$. Note that $x \in S(0, A)$. If $|supp(x)| < n - 2$, then condition (c) is violated. So we must have $|supp(x)| = n - 2$. But then Lemma 4.4.5 implies that $A_{\gamma\gamma} \notin \boldsymbol{R}_0$ for some γ with $|\gamma| = n - 3$. This contradicts (c). This implies $|supp(z)| < n - 1$. By condition (c), $|supp(z)| = n - 2$. By Lemma 4.4.5, there exists a $(n - 3) \times (n - 3)$ principal submatrix of A which is not $\boldsymbol{R}_0$. This once again contradicts the hypothesis. Hence it follows that $A \in \boldsymbol{R}_0$. This completes the proof of theorem. □

Corollary 4.4.7. Suppose $A \in R^{4\times 4} \cap C_0$. Assume that all diagonal entries of A are positive. Then $A \in \boldsymbol{R}_0$ if, and only if, $A \in \boldsymbol{Q}$. □

Remark 4.4.8. In Theorem 4.3.3, it was asserted that $A^k \notin \boldsymbol{Q}\ \forall\ k \geq 1$. This can be proved using Corollary 4.4.7 as follows. Note that sum of first two columns of A^k is equal to the zero vector. Thus $A^k \notin \boldsymbol{R}_0$. Since the diagonal entries of A^k are all positive, Corollary 4.4.7 implies $A^k \notin \boldsymbol{Q}$.

Corollary 4.4.9. Suppose $A \in \boldsymbol{R}^{6\times 6} \cap C_0 \cap \boldsymbol{Q}$. If $A_{\alpha\alpha} \in \boldsymbol{Q}\ \forall\ \alpha$ such that $|\alpha| = 3$, then $A \in \boldsymbol{R}_0$.

Proof. If $A_{\alpha\alpha} \in \boldsymbol{Q}$ for all α such that $|\alpha| = 3$, then, as $\boldsymbol{C}_0 \subseteq \boldsymbol{E}_0$ Theorem 4.3.5 implies $A_{\alpha\alpha} \in \boldsymbol{R}_0$ for all α with $|\alpha| = 3$. By Theorem 4.4.6,

$A \in \boldsymbol{R}_o$. □

Remark 4.4.10. Theorem 4.3.6 also follows from Theorem 4.3.5 and Theorem 4.4.6.

The following example shows that none of the conditions (a), (b) and (c) of Theorem 4.4.6, is necessary.

Example 4.4.11. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

Clearly $A \in C_o \cap \boldsymbol{R}_o$ and hence $A \in \boldsymbol{Q}$. Let $\alpha = \{1\}$, $\beta = \{2,3\}$ and $\gamma = \{1,2,3\}$. Then none of $A_{\alpha\alpha}$, $A_{\beta\beta}$ and $A_{\gamma\gamma}$ is in $\boldsymbol{R}_o$. Thus none of the conditions (a), (b) and (c) is satisfied by A. Yet, $A \in \boldsymbol{R}_o$.

Theorem 4.4.12. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o$, where $n \geq 3$. Suppose A satisfies any of the following conditions :

(a) every principal submatrix of A of order $n-1$ is an $\boldsymbol{R}_o$-matrix,

(b) $A_{\alpha\alpha} \in \boldsymbol{R}_o \; \forall \; \alpha \in n^*$ with $|\alpha| \leq n-2$.

Then $A \in \boldsymbol{R}_o$ if, and only if, $A \in \boldsymbol{Q}$.

Proof. If $A \in \boldsymbol{R}_o$, then, by Theorem 4.2.21, $A \in \boldsymbol{Q}$. Conversely, assume that $A \in \boldsymbol{Q}$. Let us consider the case (b) first. Assume that A satisfies (b). To the contrary, suppose $A \notin \boldsymbol{R}_o$. Then there exists a $z \in S(0,\ A)$ such that $z \neq 0$. By Theorem 4.2.22, $z \notin \boldsymbol{R}^n_{++}$. Let $\alpha = supp(z)$. As every principal submatrix of A of order less than or equal to $(n-2)$ is in $\boldsymbol{R}_o$, we must have $|\alpha| = n-1$. We may assume, for simplicity, that $\bar{\alpha} = \{n\}$. Let $B = A_{\alpha\alpha}$ and let $\bar{z} = z_\alpha$. Then, observe that $B\bar{z} = 0$, $\bar{z} > 0$. By Theorem 4.2.22, $B \notin \boldsymbol{Q}$. By Theorem 3.2.8, there exists a $u \in S(e_n, A)$, where $e_n = I_{.n}$, such that $(Au)_i \geq 0 \; \forall \; i \in \alpha$ and

$(Au)_n + 1 = 0$. Note that $u \neq 0$. There are two possibilities : $u_n = 0$ or $u_n > 0$. Suppose $u_n = 0$.
Since $z_\alpha > 0$, we can choose a real $\lambda > 0$ such that $x = z - \lambda u \geq 0$ and $x_i = 0$ for some $i \in \alpha$. Without loss of generality, assume $i = n-1$. Then $x_{n-1} = x_n = 0$. Note that, if $z = ru$, where r is a real number, then as $0 \neq z \geq 0$ and $u \geq 0$, we must have $r > 0$. But then

$$0 \leq (Az)_n = (Aru)_n = r(Au)_n = -r < 0.$$

This contradiction implies that $x \neq 0$. Note that

$$(Ax)_n = (Az)_n - \lambda(Au)_n > 0 \text{ as } (Au)_n = -1.$$

Let $\beta = \{1, 2, \ldots, n-2\}$. Then $x_{\bar{\beta}} = 0$. Let $\theta = supp(u)$. Since $u \in S(e_n, A)$, we have $A_{\theta\theta}u_\theta = (Au)_\theta = 0$ and $u_\theta^t A_{\theta\theta} u_\theta. = 0$,
As $u \neq 0$, $u_\theta \neq 0$. This would imply that $A_{\theta\theta} \notin \boldsymbol{R}_o$. By hypothesis (b), we must have $|\theta| = n-1$ and $\theta = \alpha$. Hence $u_\alpha > 0$ and $A_{\alpha\alpha}u_\alpha = 0$. Then $(Ax)_\beta = (Az)_\beta - \lambda(Au)_\beta = 0$, as $(Az)_\beta = (Au)_\beta = 0$. This implies $A_{\beta\beta} \notin \boldsymbol{R}_o$ which is a contradiction to (b). Thus "$u_n = 0$" is ruled out.
Hence $u_n > 0$.
claim : $u_i = 0$ for some $i \in \bar{n}$.
Suppose $u > 0$. Then, $Au + e_n = 0$ or $u^t A^t = -e_n^t \leq 0$. This implies $v(A^t) \leq 0$. By Theorem 4.2.17, $v(A^t) = 0$. Since u is completely mixed strategy, $A^t y = 0$, y is an optimal strategy for player I (see Theorem 1.4.4). In other words, there exists a $y \in \boldsymbol{R}_+^n$ such that $y \neq 0$ and $A^t y = 0$. Then $y^t A = 0$ which would imply that $v(A) \leq 0$. But this is not possible as $A \in \boldsymbol{Q}$. Hence the claim is valid. Assume, without loss of generality, that $u_1 = 0$. Since every principal submatrix of A of order less than or equal to $n-2$ is in $\boldsymbol{R}_o$, $u_i > 0$ for $i = 2, 3, \ldots, n$. Let $\beta = \overline{\{1\}}$. Then we have

$$A_{\beta\beta}u_\beta = (Au)_\beta = \bar{q}, \; u_\beta > 0$$

where $\bar{q} = (0, 0, \ldots, 0, -1)^t \in R^{n-1}$. Repeating the argument similar to the above, we conclude that there exists a $\bar{v} \in R_+^{n-1}$ such that $\bar{v} \neq 0$ and $\bar{v}^t A_{\beta\beta} = 0$.

Let $v \in \boldsymbol{R}^n_+$ be defined by $v_1 = 0$ and $v_\beta = \bar{v}$. Note that if $v_\beta > 0$, then we get a contradiction as follows :

$$0 = (v^t_\beta A_{\beta\beta})u_\beta = v^t_\beta(A_{\beta\beta}u_\beta) = v^t_\beta(\bar{q}) = -v_n < 0.$$

Thus v_β must have at least one coordinate zero. Let $\theta = supp(v)$. Then $|\theta| \leq n-2$. Since $v^t_\beta A_{\beta\beta} = 0$ and $\theta \subseteq \beta$, we have $v^t_\theta A_{\theta\theta} = 0$. Since $0 \neq v_\theta \geq 0$, $v(A_{\theta\theta}) \leq 0$. But by hypothesis, $A_{\theta\theta} \in \boldsymbol{E}_o \cap \boldsymbol{R}_o$.
By Theorem 4.2.21, $A_{\theta\theta} \in \boldsymbol{Q}$. Thus $v(A_{\theta\theta}) \leq 0$ leads to a contradiction to hypothesis (b). It follows that $u_n > 0$ is also not possible. Therefore, $A \in \boldsymbol{R}_o$.

We shall now prove the theorem under the hypothesis (a). Suppose A satisfies (a). Suppose $A \notin \boldsymbol{R}_o$. Let z be any nontrivial solution of $(0,\ A)$. By Theorem 4.2.22, $z_i = 0$ for some i. Let $\alpha = \bar{n} \setminus \{i\}$. Note that $z_\alpha \in S(0,\ A_{\alpha\alpha})$ as $z_\alpha \neq 0$, and

$$A_{\alpha\alpha}z_\alpha = 0,\ z_\alpha > 0.$$

This contradicts the hypothesis (a). It follows that $A \in \boldsymbol{R}_o$. This completes the proof of the theorem. □

4.5. EXAMPLES OF COPOSITIVE, SEMIMONOTONE Q-MATRICES THAT ARE NOT R_o

Looking at Theorem 4.4.6 and Theorem 4.4.12, one is tempted to raise the following questions :

Question 1. Is it possible to prove Theorem 4.4.6, under the assumption that every principal submatrix of A of order $(n-4)$ is $\boldsymbol{R}_o$?

Question 2. Is it possible to prove Theorem 4.4.12, under the assumption that every principal submatrix of A of order less than or equal to $(n-3)$ is $\boldsymbol{R}_o$?

The answer to Question 1 is "No." In fact, for every $n \geq 5$, there exists a $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{C}_o \cap \boldsymbol{Q}$ such that $A_{\alpha\alpha} \in \boldsymbol{R}_o\ \forall\ \alpha \in n^*$ such that $|\alpha| \leq n-4$ but

$A \notin R_0$. Question 2 remains open as of this writing. Our belief is that the answer is "No."

Theorem 4.5.1. For every $n \geq 5$ there exists a copositive Q-matrix satisfying the following conditions

(a) $A_{\alpha\alpha} \in R_0 \ \forall \ \alpha \in n^*$ such that $|\alpha| \leq n-4$,

(b) $A \notin R_0$.

Proof. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 1 & -1 \\ 1 & 1 & 0 & \dots & 0 & -1 & 1 \\ 1 & 1 & 1 & \dots & 0 & -x & -x \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \dots & 1 & -x & -x \\ 1 & 1 & -x & \dots & -x & x & x \\ 1 & 1 & -x & \dots & -x & x & x \end{bmatrix}, \quad \text{where } x = \frac{1}{n-4}$$

First observe that $A \in C_0$. To see this, let $y \in R^n_+$. Then

$$y^tAy = y_1^2 + y_2^2 + 2y_1y_2 + (y_1+y_2)\sum_{i=3}^{n-2} y_i + y_\theta^t A_{\theta\theta} y_\theta,$$

where $\theta = \{3, 4, \dots, n\}$. Note that, to show that $y^tAy \geq 0$, it suffices to show $y_\theta^t A_{\theta\theta} y_\theta \geq 0$. Since $A_{\theta\theta}$ is symmetric and all its leading principal minors are all nonnegative, $A_{\theta\theta}$ is positive semidefinite. Therefore, $y_\theta^t A_{\theta\theta} y_\theta \geq 0$ and hence $A \in C_0$.

Clearly A is not R_0 as the vector

$$x = (0, 0, 2, 2, \dots, 2, n-4, n-4)^t \in S(0, A).$$

Let $\alpha = \overline{\{n\}}$ and $\beta = \overline{\{n-1\}}$. Note that $A_{(n-1)\cdot} = A_{n\cdot}$. In order to show A is a Q-matrix it is sufficient to show $A_{\alpha\alpha}$ and $A_{\beta\beta}$ are Q-matrices (see Proposition 3.3.1). We will in fact show $A_{\alpha\alpha}$ and $A_{\beta\beta}$ are R_0-matrices.

Suppose $A_{\alpha\alpha}$ is not $\boldsymbol{R}_o$. Then there exists a vector $y^t = (y_1, y_2, \ldots, y_{n-1}) \in R^{n-1}$ such that $A_{\alpha\alpha}y = w$ with $y \geq 0$, $w \geq 0$ and $y^t w = 0$. As the first row in $A_{\alpha\alpha}$ is nonnegative with the first entry equal to 1, by the complementarity condition $y_1 = 0$. From the last row in $A_{\alpha\alpha}$, we can infer that there exists an index $j \neq 2$ or $n-1$ such that $y_j > 0$. We may assume, without loss of generality that $j = 3$. We now have $y_2 = y_{n-1} > 0$ and $y_3 > 0$. Thus $(A_{\alpha\alpha}y)_3 = (1 - \frac{1}{n-4})y_2 + y_3 > 0$. But this contradicts the complementarity condition. Hence we must have $y_1 = 0$ and $y_2 = 0$. Since y is a nonzero nonnegative vector, we assume, without loss of generality, $y_3 > 0$. This implies $y_{n-1} > 0$. Also $y_{n-1} = (n-4)y_3$. Now it follows that $(A_{\alpha\alpha}y)_2 = -y_{n-1}$ which is negative. It leads to a contradiction. In other words, $y_i = 0$ for every $i = 1, 2, \ldots, n-1$. This proves $A_{\alpha\alpha}$ is an $\boldsymbol{R}_o$-matrix. Similar arguments show that $A_{\beta\beta}$ is also an $\boldsymbol{R}_o$-matrix. Hence A is a $\boldsymbol{Q}$-matrix which is not $\boldsymbol{R}_o$.

It is easy to check that every principal submatrix of A of order less than or equal to $(n-4)$ is an $\boldsymbol{R}_o$-matrix. This completes the proof of the theorem. □

Note that in the above matrix A there is however at least one principal submatrix of order $(n-3)$ which is not $\boldsymbol{Q}$ and hence not $\boldsymbol{R}_o$ - for example take the submatrix omitting the first two rows and the last row along with the corresponding columns.

Example 4.5.2. When $n = 5, A$ can be written explicitly as :

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

Example 4.5.3. When n = 7,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

We are not able to settle Question 2 mentioned earlier. Consider the following example with $n = 5$.

Example 4.5.4. Let

$$A = \begin{bmatrix} 1 & 2 & -\frac{3}{2} & -\frac{3}{2} & 4 \\ 2 & 1 & -\frac{3}{2} & -\frac{3}{2} & 4 \\ -\frac{1}{6} & \frac{1}{6} & 1 & -1 & 4 \\ \frac{1}{6} & -\frac{1}{6} & -1 & 1 & 4 \\ -1 & -1 & 0 & 2 & 4 \end{bmatrix}.$$

Observe A is a semimonotone matrix but not copositive. It is $\boldsymbol{Q}$ but not $\boldsymbol{R}_0$. $\boldsymbol{Q}$ property may be proved using Proposition 3.3.7, and regarding $\boldsymbol{R}_0$, note that the sum of the first four columns gives the zero vector. Here every 2×2 ($n-3 = 2$ for $n = 5$) *principal submatrix is* $\boldsymbol{R}_0$ with the exception of just one submatrix which is

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

4.6. RESULTS ON SEMIMONOTONE Q_0-MATRICES

Theorem 4.6.1. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o \cap \boldsymbol{Q}_o$. If A is symmetric, then A is in $\bar{Q}_0$.

Proof. We will show this by induction on n. If $n = 1$, there is nothing to prove. Assume that the result is true for all real square matrices of order less than or equal to $n-1$. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o \cap \boldsymbol{Q}_o$ and A is symmetric. Let α be any subset of $\bar{n}$ such that $|\alpha| = n-1$. Without loss of generality, we may assume that $\alpha = \overline{\{n\}}$. Suppose $A_{\alpha\alpha} \notin \boldsymbol{Q}_o$. Then by Theorem 3.4.5, there exists a $u \in \boldsymbol{R}^n_+$ such that $Au + e_n \geq 0$, $u^t Au + u_n = 0$, and $(Au)_n = -1$. Since A is symmetric $\boldsymbol{E}_o$-matrix, A is copositive. As $u \geq 0$ and $u^t Au + u_n = 0$, $u^t Au = 0$ and $u_n = 0$. Since A is symmetric copositive matrix, it follows that $(Au) \geq 0$ (see Theorem 4.2.8). This contradicts $(Au)_n = -1$. Hence $A_{\alpha\alpha} \in \boldsymbol{Q}_o$. As α was arbitrary, it follows that every $(n-1) \times (n-1)$ principal submatrix of A is in $\boldsymbol{Q}_o$. By induction, it follows that $A \in \bar{Q}_0$. □

Pang (1979b) proved that if A is a $\boldsymbol{E}_o \cap \boldsymbol{Q}$-matrix, then every nontrivial solution of $(0, A)$ must have at least two nonzero coordinates. Paraphrasing, if A is a $\boldsymbol{E}_o \cap \boldsymbol{Q}$-matrix, then A cannot have a diagonal entry zero and all other entries in the corresponding column nonnegative. We have the following results for $\boldsymbol{Q}_o$-matrices in this direction.

Theorem 4.6.2. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_o \cap \boldsymbol{Q}_o$. Assume that for some $i_0, j_0 \in \bar{n}$, $a_{i_0 i_0} = 0$ and $a_{i_0 j_0} > 0$. Then there exists a $k \in \bar{n}$ such that $a_{k i_0} < 0$.

Proof. Since $a_{i_0 j_0}$ is positive, we can choose a $q \in \boldsymbol{R}^n$ such that $q_{i_0} < 0$, $q_j > 0$ for all $j \neq i_0$ and $F(q, A) \neq \phi$. Since $A \in \boldsymbol{Q}_o$, $S(q, A) \neq \phi$. Let $z \in S(q, A)$ and let $\alpha = supp(z)$. Let $\beta = \alpha \setminus \{i_0\}$. Since $a_{i_0 i_0} = 0$ and $q_{i_0} < 0$, $\beta \neq \emptyset$. Since z_β is positive, we have

$$0 = A_{\beta i_0} z_{i_0} + A_{\beta\beta} z_\beta + q_\beta.$$

Note that $q_\beta > 0$. If $A_{.i_0} \geq 0$, then

$$A_{\beta\beta} z_\beta = -q_\beta - z_{i_0} A_{\beta i_0} < 0,$$

which in turn implies that $v(A^t_{\beta\beta}) < 0$. This is not possible as $A \in \boldsymbol{E}_0$. Therefore, $A_{.i_0}$ must contain a negative entry. Thus, there exists a $k \in \bar{n}$ such that $a_{ki_0} < 0$. □

Corollary 4.6.3. Suppose $A \in \boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_0 \cap \boldsymbol{Q}_0$. Assume that every row of A contains a positive entry. Then every nontrivial solution of $(0, A)$ contains at least two positive coordinates. □

CHAPTER 5

FULLY-SEMIMONOTONE MATRICES

5.1. BACKGROUND

Multiplicity of solutions to LCP has attracted several researchers. In particular, uniqueness of solutions to LCP has been of special interest. A result of Samuelson, Thrall and Wesler (1958)- which was later discovered by Ingleton (1966) independently- states that a matrix $A \in \boldsymbol{P}$ if, and only if, (q, A) has a unique solution for every $q \in \boldsymbol{R}^n$. Confining uniqueness to only those q which lie in the interior of union of complementary cones, Cottle and Stone (1983) introduced a new class of matrices called $\boldsymbol{U}$-matrices (see also Stone (1981)). A matrix A is said to be in this class if (q, A) has unique solution whenever q is in the interior of $K(A)$. Further, they enlarged this class by demanding uniqueness of solutions to (q, A) only for those q's which lie in the interior of any full complementary cone; and called it the class of fully-semimonotone matrices ($\boldsymbol{E}_0^f$).

Unfortunately, the literature on these classes, $\boldsymbol{E}_0^f$ and $\boldsymbol{U}$, is very limited. We are not aware of any nice characterizations of these classes other than those given in the aforementioned references. In Cottle and Stone (1983), it was established that $\boldsymbol{P} \subseteq \boldsymbol{U} \subseteq \boldsymbol{E}_0^f$. Further, Stone (1981) showed that $\boldsymbol{Q}_0 \cap \boldsymbol{U} \subseteq \boldsymbol{P}_0$ and raised the following conjecture.

Conjecture 5.1.1. The class of fully semimonotone matrices within the class of $\boldsymbol{Q}_0$-matrices is contained in $\boldsymbol{P}_0$. That is, $\boldsymbol{E}_0^f \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$.

In this chapter, we establish that the conjecture is valid for matrices of order upto 4×4. Further, we establish the same for symmetric matrices of general order and identify and prove that the conjecture is valid for a number of subclasses of $\boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_0^f$ for any positive integer n. Within $\boldsymbol{E}_0^f$, $\boldsymbol{R}_0$-matrices are $\boldsymbol{Q}$-matrices. Thus, $\boldsymbol{E}_0^f \cap \boldsymbol{R}_0 \subseteq \boldsymbol{Q} \subseteq \boldsymbol{Q}_0$. It is of interest to know whether

$E_0^f \cap Q$ or $E_0^f \cap R_0$ is contained in P_0. Recently, Sridhar (1994) proved that $E_0^f \cap R_0 \subseteq P_0$. We are not aware of this result having been mentioned in the literature. The question of whether $E_0^f \cap Q$ is contained in P_0 or not still remains an open problem.

In section 5.2, we present some elementary properties and known results on fully-semimonotone matrices and some related material that will be needed in the subsequent sections. Section 5.3 presents our main results on matrices of general order. In section 5.4, we establish Conjecture 5.1.1 for matrices of order 4×4 and show that it is valid for 5×5 and 6×6 matrices with some additional assumptions.

5.2. PRELIMINARIES

Definition 5.2.1. Let $A \in \boldsymbol{R}^{n \times n}$. A is said to be a fully-semimonotone matrix if A and all its (legitimate) PPTs are semimonotone matrices. The class of fully-semimonotone matrices is denoted by E_0^f.

Remark 5.2.2. Note that the four examples given in Example 1.2.8 are all E_0^f-matrices. It may be observed, from the corresponding figures in that example, that if a q is in the interior of any complementary cone, then (q, A) has unique solution. This is a geometric characterization of the class of fully-semimonotone matrices.

Example 5.2.3. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Note that $A \in E_0$. Also $\wp_\alpha(A) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \notin E_0$, where $\alpha = \{1\}$. Observe the Figure E on next page.

Note that any q in the interior of the full cone corresponding to $\beta = \{1, 2\}$ has two solutions - one with $-A$ and the other with $[-A_{.1}\ I_{.2}]$ as complementary bases. For instance, if $q = (-2, -1)^t$, then $z = (1, 1)^t$ and $x = (2, 0)^t$ are two distinct solutions of (q, A), that is, $z, x \in S(q, A)$, and $z \neq x$.

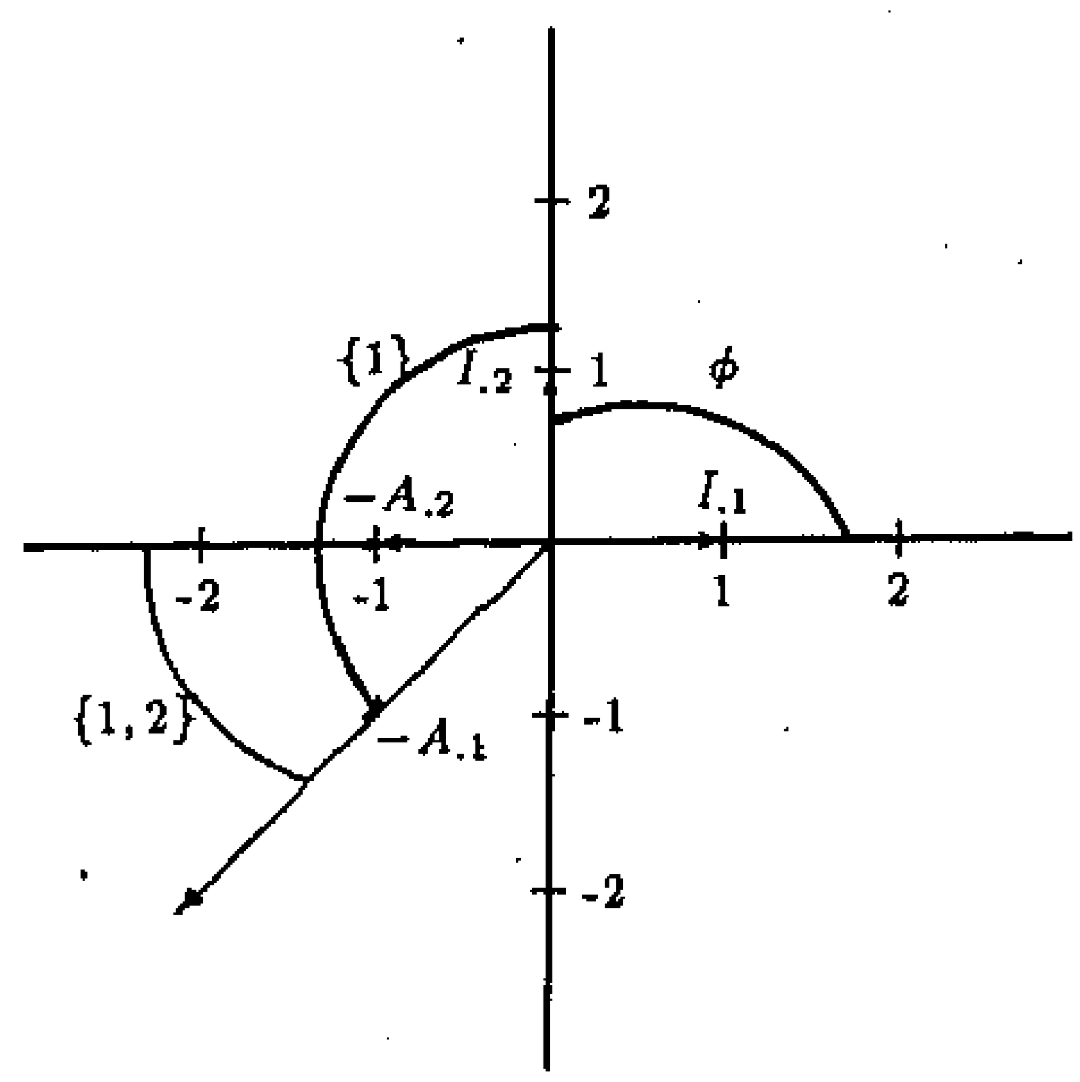

Figure E

Definition 5.2.4. Let $A \in \boldsymbol{R}^{n \times n}$ and $q \in \boldsymbol{R}^n$. A solution (w, z) of (q, A) is said to be nondegenerate if $w + z > 0$. q is said to be nondegenerate with respect to A if $S(q, A) \neq \phi$ and every solution of (q, A) is nondegenerate.

Remark 5.2.5. If q is nondegenerate with respect to A, it means that whenever $q \in pos\, C_A(\alpha)$, for any α, then it is in the interior of $pos\, C_A(\alpha)$. In other words, q is nondegenerate with respect to A if, and only if, it does not lie on the boundary of any complementary cone

Theorem 5.2.6. Suppose $A \in \boldsymbol{R}^{n \times n}$. The following statements are equivalent.

(a) $A \in E_0^f$

(b) for every q nondegenrate with respect to A, (q, A) has a unique solution.

Proof. Follows from Lemma 1.2.15 (see also Theorem 5.2.17).

Remark 5.2.7. The geometrical interpretation of Theorem 5.2.6 is that if q lies in the interior of any full cone, then through a principal pivotal transformation q can be transoformed into the interior of the nonnegative orthant in such a way that the transformed LCP has unique solution.

Observation 5.2.8. It is clear, from the definition of E_0^f and the fact that PPT is an equivalence relation, that every PPT of a E_0^f-matrix is itself a E_0^f-matrix.

Inheritance and Invariance Properties of E_0^f

Theorem 5.2.9. Suppose $A \in R^{n\times n} \cap E_0^f$. Then $A_{\alpha\alpha} \in E_0^f \ \forall \ \alpha \in n^*$. Further, if P is any permutation matrix in $R^{n\times n}$, then PAP^t is also a E_0^f-matrix.

Proof. The first assertion is obvious because any PPT of $A_{\alpha\alpha}$ is a principal submatrix of a PPT of A. The second assertion follows from the fact that every PPT of M with respect to any $\alpha \in n^*$ is also a PPT of A with respect to some $\beta \in n^*$, where $M = PAP^t$. Since E_0 is invariant under principal rearrangements, the result readily follows. □

Theorem 5.2.10. Suppose $A \in R^{n\times n} \cap E_0^f$. Let D and E be any positive diagonal matrices in $R^{n\times n}$. Then $DAE \in E_0^f$. In other words, E_0^f property is invariant under positive row and column scaling.

Proof. Suppose

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix},$$

where $\alpha \in n^*$. Let

$$M = DAE = \begin{bmatrix} D_{\alpha\alpha} & 0 \\ 0 & D_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} E_{\alpha\alpha} & 0 \\ 0 & E_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

Then

$$M = \begin{bmatrix} D_{\alpha\alpha}A_{\alpha\alpha}E_{\alpha\alpha} & D_{\alpha\alpha}A_{\alpha\bar{\alpha}}E_{\bar{\alpha}\bar{\alpha}} \\ D_{\bar{\alpha}\bar{\alpha}}A_{\bar{\alpha}\alpha}E_{\alpha\alpha} & D_{\bar{\alpha}\bar{\alpha}}A_{\bar{\alpha}\bar{\alpha}}E_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

Observe that $det\,(D_{\alpha\alpha}A_{\alpha\alpha}E_{\alpha\alpha}) \neq 0$ if, and only if $det\,A_{\alpha\alpha} \neq 0$. Therefore PPT of M with respect to α is well defined if, and only if, PPT of A with respect to α is well defined. Suppose $det\, A_{\alpha\alpha} \neq 0$. Then

$$\wp_\alpha(M) = \begin{bmatrix} E_{\alpha\alpha}^{-1}(A_{\alpha\alpha})^{-1}D_{\alpha\alpha}^{-1} & -E_{\alpha\alpha}^{-1}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}E_{\bar{\alpha}\bar{\alpha}} \\ D_{\bar{\alpha}\bar{\alpha}}A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}D_{\alpha\alpha}^{-1} & \bar{M}_{\bar{\alpha}\bar{\alpha}} \end{bmatrix},$$

where

$$\bar{M}_{\alpha\alpha} = D_{\bar{\alpha}\bar{\alpha}}A_{\bar{\alpha}\bar{\alpha}}E_{\bar{\alpha}\bar{\alpha}} - D_{\bar{\alpha}\bar{\alpha}}A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}E_{\bar{\alpha}\bar{\alpha}}$$

$$= D_{\bar{\alpha}\bar{\alpha}}(A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}})E_{\bar{\alpha}\bar{\alpha}}.$$

Simplifying $\wp_\alpha(M)$, we get :

$$\wp_\alpha(A) = \begin{bmatrix} E_{\alpha\alpha}^{-1} & 0 \\ 0 & D_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \wp_\alpha(M) \begin{bmatrix} D_{\alpha\alpha}^{-1} & 0 \\ 0 & E_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \tag{5.1}$$

Since $\wp_\alpha(A) \in \boldsymbol{E}_0$, and $\begin{bmatrix} E_{\alpha\alpha}^{-1} & 0 \\ 0 & D_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ and $\begin{bmatrix} D_{\alpha\alpha}^{-1} & 0 \\ 0 & E_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ are both positive diagonal matrices, the right hand side matrix of (5.1) must also be in $\boldsymbol{E}_0$ (it is easy to check that $\boldsymbol{E}_0$ property is invariant under row and column (positive) scaling. Thus, $\wp_\alpha(M) \in \boldsymbol{E}_0$. The theorem follows. □

Remark 5.2.11. We are not aware of an explicit mention of Theorem 5.2.10 or the formula (5.1) in the literature. However, Cottle gives a similar formula relating the transpose of PPT of A to the PPT of A^t (see page 73, Cottle, Pang and Stone (1992)).

Remark 5.2.12. From the definition, it is clear that $\boldsymbol{E}_0^f \subseteq \boldsymbol{E}_0$. Since $\boldsymbol{P}_0 \subseteq \boldsymbol{E}_0$ and every PPT of a $\boldsymbol{P}_0$-matrix is also $\boldsymbol{P}_0$, it follows that $\boldsymbol{P}_0 \subseteq \boldsymbol{E}_0^f$. Thus, $\boldsymbol{P}_0 \subseteq \boldsymbol{E}_0^f \subseteq \boldsymbol{E}_0$.

Observe that $\boldsymbol{P}_0$ property as well as $\boldsymbol{E}_0$ property are both preserved under transposition. But this is not the case with $\boldsymbol{E}_0^f$. Consider the following example.

Example 5.2.13. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Note that A has two PPTs, namely,

$$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

It easy to check that the value of any principal submatrix of any of A, B and C is nonnegative. By Theorem 4.2.17, A, B and C are $\boldsymbol{E}_o$-matrices. Hence $A \in \boldsymbol{E}_o^f$. Now look at

$$A^t = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that

$$\wp_\alpha(A^t) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \notin \boldsymbol{E}_o \text{ where } \alpha = \{2,3\}.$$

Note that $\wp_\alpha(A^t) \notin \boldsymbol{E}_o$ and hence $A^t \notin \boldsymbol{E}_o^f$.

Recall that for $A \in \boldsymbol{R}^{n \times n}$, the union of complementary cones is given by

$$K(A) = \{q \in \boldsymbol{R}^n : S(q, A) \neq \phi\}.$$

Definition 5.2.14. Let $A \in \boldsymbol{R}^{n \times n}$. A is said to be a $\boldsymbol{U}$-matrix if for all $q \in \boldsymbol{R}^n, (q, A)$ has a unique solution whenever q is in the interior of $K(A)$.

Remark 5.2.15. From Theorem 5.2.6 and the above definition, it follows that $\boldsymbol{U} \subseteq \boldsymbol{E}_o^f$. Since (q, A) has a unique solution for all q whenever $A \in \boldsymbol{P}$, $\boldsymbol{P} \subseteq \boldsymbol{U}$. Thus $\boldsymbol{P} \subseteq \boldsymbol{U} \subseteq \boldsymbol{E}_o^f$. The following example shows that these inclusions are proper.

Example 5.2.16. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Observe that A is a $\boldsymbol{U}$-matrix but is not a $\boldsymbol{P}$-matrix. Whereas

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is E_0^f-matrix but not a U-matrix.

As mentioned earlier, the classes U and E_0^f were introduced and studied by Cottle and Stone (1983) (see also Stone (1981)). We present some results from Cottle and Stone (1983) which are relevent to our discussion in the sequel.

Theorem 5.2.17. Suppose $A \in R^{n\times n} \cap E_0^f$ and $\det A_{\alpha\alpha} \neq 0$ for some $\alpha \subseteq \bar{n}$, then for all $\beta \subseteq \bar{n}$ such that

$$\beta \neq \alpha,\ pos\, C_A(\beta) \cap \text{ interior } (C_A(\alpha)) = \phi.$$

Theorem 5.2.18. Let $A \in R^{n\times n}$. Then $A \in U$ if, and only if, either $A \notin E_0^f$ or there exists $\alpha,\ \beta \subseteq \bar{n}$ and $i,\ j \in \bar{n}$ such that

(a) $\alpha \neq \beta,\ i \neq j,$

(b) $(\ det\, A_{\alpha\alpha})(\ det\, A_{\beta\beta}) \neq 0$ and there exists a nonzero vector $v \in R^n$ such that $v^t B = v^t M = 0$, where $B = (C_A(\alpha))_{.\{\bar{i}\}}$ and $M = (C_A(\beta))_{.\{\bar{j}\}}$,

(c) there exists $x \in R^{n-1}$ with $x > 0$ and $Bx \in pos\, M$.

Theorem 5.2.19. Let $A \in R^{n\times n} \cap E_0^f \cap Q_0$, then $A \notin U$ if, and only if there exists $\alpha,\ \beta \subseteq \bar{n}$ and $i,\ j \in \bar{n}$ such that

(a) $\alpha \neq \beta,\ i \neq j,$

(b) $(\ det\, A_{\alpha\alpha})(\ det\, A_{\beta\beta}) \neq 0$ and there exists a nonzero vector $v \in R^n$ such that $v^t B = v^t M = 0$, where $B = (C_A(\alpha))_{.\{\bar{i}\}}$ and $M = (C_A(\beta))_{.\{\bar{j}\}}$,

(c) $(C_A(\alpha))_{.i}$ and $(C_A(\beta))_{.j}$ are on the opposite sides of span $B =$ span M, where "span B " stands for subspace generated by columns of B.

Theorem 5.2.20. $Q_0 \cap U \subseteq P_0$.

Even though Conjecture 5.1.1 is true in a number of special cases, it remains an open problem. In this context two questions that can be raised are : (i) Is $E_0^f \cap Q$ a subset of P_0? (ii) Is $E_0^f \cap R_0$ a subset of P_0? The questions are relevant because Q is a subset of Q_0 and within E_0^f, R_0 is contained in Q. Recently Sridhar (1994) settled the latter question using degree theory. We present the proof, briefly introducing the relevant material.

Definition 5.2.21. Let $A \in R^{n \times n} \cap R_0$. Let $q \in R^n$ be such that $|S(q, A)| < \infty$. For any $z \in S(q, A)$, define the index of z, denoted as $i_z(A)$, as $det\, A_{\alpha\alpha}/|det\, A_{\alpha\alpha}|$, where $\alpha = supp(z)$. Define degree of A at q as :

$$\deg(A, q) = \sum_{z \in S(q,A)} i_z(A).$$

The degree as defined above is well defined. Further, the value of $\deg(A, q)$ is the same for all $q \in R^n$ for which $|S(q, A)| < \infty$. This common value is called degree of A and will be denoted by $\deg(A)$ (see Cottle, Pang, and Stone (1992)). The following theorem relates degree of an R_0-matrix with that of its PPTs (see page 595, Cottle, Pang, and Stone (1992)).

Theorem 5.2.22. Let $A \in R^{n \times n} \cap R_0$. Suppose $det\, A_{\alpha\alpha} \neq 0$ for some $\alpha \in n^*$. Then

$$\deg(\wp_\alpha(A)) = (det\, A_{\alpha\alpha}/|det\, A_{\alpha\alpha}|).\ \deg(A).$$

Theorem 5.2.23. Let $A \in R^{n \times n} \cap E_0^f \cap R_0$. Then $\deg(A) = 1$.

Proof. Follows from the fact that for any $q > 0$, $|S(q, A)| = 1$ and $det\, A_{\emptyset\emptyset} = 1$. □

Theorem 5.2.24. (Sridhar (1994)). Let $A \in R^{n \times n} \cap E_0^f \cap R_0$. Then A belongs to P_0.

Proof. Suppose, to the contrary, assume that $det\, A_{\alpha\alpha} < 0$, for some $\alpha \in n^*$. Let $M = \wp_\alpha(A)$. Then from Theorem 5.2.22, it follows that $\deg(M) = -1$. This contradicts Theorem 5.2.23. Hence, it follows that $A \in P_0$. □

In the rest of this section, we present some results on N_0, $E_0^f \cap N_0$ and almost P_0-matrices. These are needed for our results in the next section.

Definition 5.2.25. Let $A \in \boldsymbol{R}^{n \times n}$. Say that A is an almost $\boldsymbol{P}_o$-matrix (almost $\boldsymbol{P}$-matrix) if $det\, A_{\alpha\alpha} \geq 0$ ($det\, A_{\alpha\alpha} > 0$) for all $\alpha \in n^*$, $\alpha \neq \bar{n}$ and $det\, A < 0$.

Definition 5.2.26. Let $A \in \boldsymbol{R}^{n \times n}$. Say that A is an $\boldsymbol{N}_o$-matrix ($\boldsymbol{N}$-matrix) if for every $\alpha \in n^*$, $det\, A_{\alpha\alpha} \leq 0$ ($det\, A_{\alpha\alpha} < 0$).

Remark 5.2.27. Obviously the $\boldsymbol{N}_o$ and $\boldsymbol{N}$ properties are inheritance properties. That is if $A \in \boldsymbol{N}_o$ ($A \in \boldsymbol{N}$), then $A_{\alpha\alpha} \in \boldsymbol{N}_o$ ($A_{\alpha\alpha} \in \boldsymbol{N}$) $\forall\ \alpha \in n^*$. Further, $\boldsymbol{N}_o$ and $\boldsymbol{N}$ properties are invariant under principal rearrangements.

It is a well known fact (see Pye (1992)) that a matrix A is an almost $\boldsymbol{P}_o$-matrix ($\boldsymbol{P}$-matrix) if, and only if A is nonsingular and A^{-1} is an $\boldsymbol{N}_o$-matrix ($\boldsymbol{N}$-matrix). A matrix A is said to be an $\bar{\boldsymbol{N}}$-matrix if it can be obtained as a limit of a sequence of $\boldsymbol{N}$-matrices. If $A \in \boldsymbol{R}^{n \times n}$ is an $\bar{\boldsymbol{N}}$-matrix, then there exists a nonempty subset α of $\bar{n}$ such that $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\bar{\alpha}}$ are nonpositive, and $A_{\alpha\bar{\alpha}}$ and $A_{\bar{\alpha}\alpha}$ are nonnegative (see Mohan, Parthasarathy and Sridhar (1990), Mohan and Sridhar (1992), and Parthasarathy and Ravindran (1990)).

Theorem 5.2.28. Suppose $A \in \boldsymbol{R}^{n \times n}$ is an almost $\boldsymbol{P}_o$-matrix. Let $B = A^{-1}$. Then there exists a nonempty subset α of $\bar{n}$ satisfying :

$$B_{\alpha\alpha} \leq 0, \quad B_{\bar{\alpha}\bar{\alpha}} \leq 0, \quad B_{\alpha\bar{\alpha}} \geq 0 \text{ and } B_{\bar{\alpha}\alpha} \geq 0.$$

Proof. Suffices to show that B is an $\bar{\boldsymbol{N}}$-matrix. It is easy to show that for all positive ε sufficiently small, $A + \varepsilon I$ is an almost $\boldsymbol{P}$-matrix. Therefore, $(A + \varepsilon I)^{-1}$ is an $\boldsymbol{N}$-matrix for all positive ε sufficiently small. Note that $(A + \varepsilon I)^{-1}$ converges to B as ε converges to 0 and hence B is an $\bar{\boldsymbol{N}}$-matrix. □

5.3. PROPERTIES OF $E_0^f \cap Q_o$-MATRICES

In this section our main goal is to establish the validity of the long standing Conjecture 5.1.1 for a number of subclasses of $\boldsymbol{R}^{n \times n} \cap \boldsymbol{E}_0^f \cap \boldsymbol{Q}_o$-matrix class for any general n. We shall start with the following lemma.

Lemma 5.3.1. Let $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{E}_o \cap \boldsymbol{N}_o$. Suppose $A \leq 0$. Then there exists a principal rearrangement M of A such that M is a strict upper triangular matrix, that is, $m_{ij} = 0$ for all $i, j \in \bar{n}$ such that $i \geq j$. In other words, there exists a permutation matrix $P \in \boldsymbol{R}^{n\times n}$ such that PAP^t is a strict upper triangular matrix.

Proof. We shall prove this by induction on n. If $n = 1$, the result is trivially true. So assume that the lemma is valid for all square matrices of order upto $(n-1)$, $n > 1$. Now assume $A \in \boldsymbol{R}^{n\times n}$ satisfies the hypothesis of the lemma.

If every column of A has a negative entry, then, as $A \leq 0$, we have

$$e^t A < 0, \text{ where } e = (1, 1, \ldots, 1)^t \in \boldsymbol{R}^n.$$

This implies $v(A)$ is negative. This contradicts the hypothesis that $A \in \boldsymbol{E}_o$. Hence A must have a zero column. Suppose $A_{\cdot j} = 0$. Then interchange the first column and j^{th} column and then the first row and j^{th} row. In the resulting matrix the first column will be zero. Since both $\boldsymbol{E}_o$ and $\boldsymbol{N}_o$ properties are invariant under principal rearrangements, the new matrix is also in $\boldsymbol{E}_o \cap \boldsymbol{N}_o$. Hence assume, without loss of generality, that $A_{\cdot 1} = 0$. Let $\alpha = \{2, 3, \ldots, n\}$. Then $A_{\alpha\alpha} \in R^{(n-1)\times(n-1)} \cap \boldsymbol{E}_o \cap \boldsymbol{N}_o$. Also $A_{\alpha\alpha} \leq 0$. By induction hypothesis, there exists a permutation matrix $\bar{P} \in R^{(n-1)\times(n-1)}$ such that $\bar{P} A_{\alpha\alpha} \bar{P}^t$ is a strict upper triangular matrix. Let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & \bar{P} \end{bmatrix}.$$

Then

$$PAP^t = \begin{bmatrix} 0 & A_{1\alpha}\bar{P}^t \\ 0 & \bar{P} A_{\alpha\alpha} \bar{P}^t \end{bmatrix}.$$

Since $\bar{P} A_{\alpha\alpha} \bar{P}^t$ is a strict upper triangular matrix so is PAP^t. □

Theorem 5.3.2. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{E}_o \cap \boldsymbol{N}_o$. Assume that A is nonsingular.

Then there exists a principal rearrangement

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A such that $\alpha \neq \phi$, $\alpha \neq \bar{n}$, $A_{\alpha\alpha}$, and $A_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices, and $A_{\alpha\bar{\alpha}}$, and $A_{\bar{\alpha}\alpha}$ are nonnegative matrices.

Proof. Since A is nonsingular $\boldsymbol{N}_o$-matrix, A^{-1} is an almost $\boldsymbol{P}_o$-matrix. By Theorem 5.2.28, *there exists a nonempty subset α of $\bar{n}$ such that* $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\bar{\alpha}}$ are nonpositive, and $A_{\alpha\bar{\alpha}}$ and $A_{\bar{\alpha}\alpha}$ are nonnegative matrices. Since A is nonsingular, $\alpha \neq \bar{n}$. By Lemma 5.3.1, there exist permutation matrices $M \in R^{|\alpha|\times|\alpha|}$ and $L \in R^{|\bar{\alpha}|\times|\bar{\alpha}|}$ such that $MA_{\alpha\alpha}M^t$ and $LA_{\bar{\alpha}\bar{\alpha}}L^t$ are strict upper triangular matrices. Let

$$P = \begin{bmatrix} M & 0 \\ 0 & L \end{bmatrix}.$$

Then

$$PAP^t = \begin{bmatrix} MA_{\alpha\alpha}M^t & MA_{\alpha\bar{\alpha}}L^t \\ LA_{\bar{\alpha}\alpha}M^t & LA_{\bar{\alpha}\bar{\alpha}}L^t \end{bmatrix}.$$

Since $A_{\bar{\alpha}\alpha}$, $A_{\alpha\bar{\alpha}}$, M, and L are all nonnegative, we have $LA_{\bar{\alpha}\alpha}M^t \geq 0$ and $MA_{\alpha\bar{\alpha}}L^t \geq 0$. This completes the proof. □

Theorem 5.3.3. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{E}_o^f \cap \boldsymbol{Q}_o$. Assume that every proper principal minor of A is nonnegative. Then A belongs to $\boldsymbol{P}_o$.

Proof. Suffices to show that $\det A \geq 0$. Suppose $\det A < 0$. Then A is an almost $\boldsymbol{P}_o$-matrix and hence $A^{-1} \in \boldsymbol{N}_o$. Since A^{-1} is a PPT of A, $A^{-1} \in \boldsymbol{E}_o^f \cap \boldsymbol{N}_o \cap \boldsymbol{Q}_o$. Let $B = A^{-1}$. Then by Theorem 5.3.2, there exists a principal rearrangement of B such that $B_{\alpha\alpha}$ and $B_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices, and $B_{\alpha\bar{\alpha}}$ and $B_{\bar{\alpha}\alpha}$ are nonnegative matrices for some $\alpha \subseteq \bar{n}$ with $\alpha \neq \phi$ and $\alpha \neq \bar{n}$. For simplicity, we assume $\alpha = \{1, 2, \ldots, k\}$, $k < n$. Observe that

$$b_{ij} \geq 0 \;\forall\, i,\; j \in \bar{n} \text{ such that } i \geq j \tag{5.2}$$

In particular, $B_{n.} \geq 0$. By Theorem 3.4.2, $B_{\beta\beta} \in \boldsymbol{Q}_o$, where $\beta = \overline{\{n\}}$. Note that, from the above observation (5.2), the last row of $B_{\beta\beta}$ is nonnegative. By

Theorem 3.4.2, $B_{\gamma\gamma} \in Q_o$, where $\gamma = \{1, 2, \ldots, n-2\}$. Thus it can be seen that all the leading principal submatrices of B are in Q_o.

We will now show that $B_{\alpha(k+1)} = 0$ which will in turn imply that $B_{.(k+1)} = 0$ leading to the contradiction that B is singular.

Let

$$M = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha(k+1)} \\ B_{(k+1)\alpha} & 0 \end{bmatrix}.$$

Being a leading principal submatrix of B, M belongs to Q_o. If $B_{\alpha(k+1)}$ has a positive entry, then by Theorem 3.5.1, $M \notin Q_o$. Hence $B_{\alpha(k+1)} = 0$. It follows that A belongs to P_o. □

We shall now identify a number of subclasses of E_o^f for which Conjecture 5.1.1 is valid.

Corollary 5.3.4. Suppose $A \in R^{n \times n} \cap E_o^f \cap \bar{Q}_0$. Then A belongs to P_o.

Proof. We prove this by induction on n. If $n = 1$, the result is obviously true. Hence assume that the result is true for all real square matrices of order less than or equal to $n-1$, $n > 1$. Suppose $A \in R^{n \times n} \cap E_o^f \cap \bar{Q}_0$. Then $A_{\alpha\alpha} \in R^{(n-1)\times(n-1)} \cap E_o^f \cap \bar{Q}_0$ for all $\alpha \in n^*$. By induction hypothesis, $A_{\alpha\alpha} \in P_o$ for all $\alpha \in n^*$ with $|\alpha| < n$. By Theorem 5.3.3, A belongs to P_o. □

Corollary 5.3.5. Suppose $A \in R^{n \times n} \cap E_o^f$. Assume that A satisfies any one of the following conditions :

(a) A is nonnegative Q_o-matrix

(b) $A + A^t$ is a nonnegative Q_o-matrix

(c) A is symmetric Q_o-matrix

(d) A is a Z-matrix

(e) A is a E-matrix

(f) A is a copositive-plus matrix.

Then A belongs to P_0.

Proof. This is a direct consequence of the above corollary and the fact that if A *satisfies any of the conditions* (a) to (f), then A is in $\bar{Q}_0$ (see Lemma 3.4.13, Corollary 3.4.16, Theorem 4.6.1, and pages 181, 196, 201 of Cottle, Pang and Stone (1992)). □

A Note on a Topological Aspect of E_0^f

It is a well known fact that the set of P-matrices is an open set. Intuitively one feels that the interior of the set of E_0^f-matrices in $R^{n\times n}$ should coincide with P-matrices of $R^{n\times n}$. Our aim, here, is to show that this is indeed the case. We are not aware of a specific mention of this result in the literature. Our main interest here is to establish this as an application of our Theorem 5.3.3. The following results are fairly well known (see Cottle (1980) and Cottle, Pang and Stone (1992)).

Theorem 5.3.6. Suppose $A \in R^{n\times n} \cap E_0$. Then for every positive ε, $A+\varepsilon I$ is in E and hence in $\bar{Q}$.

Proof. One can easily check that if $A \in E_0$, then $A+\varepsilon I$ is in E. The second assertion follows from Cottle's (1980) result, $E = \bar{Q}$. □

Lemma 5.3.7. Closure ($R^{n\times n} \cap P) = R^{n\times n} \cap P_0$.

Proof. Follows from the fact that if $A \in P_0$, then $A+\varepsilon I \in P$ for all $\varepsilon > 0$. □

Theorem 5.3.8. Let $T = \{A \in R^{n\times n} : A \in E_0^f\}$. Then

$$\text{interior}(T) = R^{n\times n} \cap P.$$

Proof. In the light of Lemma 5.3.7, it is sufficient to show that if M is in the interior(T), then M belongs to the interior of $R^{n\times n} \cap P_0$.

Since $M \in$ interior(T), there exists a $\delta > 0$ such that

$$B_\delta(M) = \{A \in \boldsymbol{R}^{n\times n} : ||M - A|| < \delta\} \subseteq T,$$

where $||.||$ is any norm on $\boldsymbol{R}^{n\times n}$. We will show that if $A \in B_\delta(M)$, then $A \in \boldsymbol{P}_0$. This will then imply that M is an interior point of $\{A \in \boldsymbol{R}^{n\times n} : A \in \boldsymbol{P}_0\}$. Observe that $A \in B_\delta(M)$ implies $A + \varepsilon I \in B_\delta(M)$ for all positive ε sufficiently small. Also $A + \varepsilon I \in \boldsymbol{E}_0^f$ for all positive ε sufficiently small. By Theorem 5.3.6, $A + \varepsilon I \in \bar{\boldsymbol{Q}}$ for all positive ε sufficiently small. By Corollary 5.3.4, $A + \varepsilon I \in \boldsymbol{P}_0$ for all positive ε sufficiently small. It follows that $A \in \boldsymbol{P}_0$ as $\boldsymbol{R}^{n\times n} \cap \boldsymbol{P}_0$ is a closed set. Therefore, M is an interior point of $\boldsymbol{R}^{n\times n} \cap \boldsymbol{P}_0$. Since interior of $\boldsymbol{R}^{n\times n} \cap \boldsymbol{P}_0$ is $\boldsymbol{P}$, the theorem follows. □

Theorem 5.3.9. Suppose $A \in \boldsymbol{R}^{n\times n} \cap \boldsymbol{E}_0^f$, where $n \leq 3$. If all the diagonal entries of A are positive, then A is a $\boldsymbol{P}_0$-matrix.

Proof. If A is a 2×2 $\boldsymbol{E}_0^f$-matrix, then it is easy to check that even with one diagonal entry positive, A has to be a $\boldsymbol{P}_0$-matrix. Now suppose A is a 3×3 matrix satisfying hypothesis of the theorem. Assume, to the contrary, that A is not a $\boldsymbol{P}_0$-matrix. Then A is an almost $\boldsymbol{P}_0$-matrix. Let B be the inverse of A. Then B is an $\boldsymbol{N}_0 \cap \boldsymbol{E}_0^f$-matrix and must have all diagonal entries zero. Since all diagonal entries of A are positive, b_{ij} and b_{ji} must have the same sign for all $i \neq j$. Since B is in $\boldsymbol{E}_0^f$, this would imply that B is nonnegative. But then $\det B$ must be positive which is a contradiction. Therefore, A is $\boldsymbol{P}_0$-matrix. □

From the above theorem a logical question that can arise is that : If A is in $\boldsymbol{R}^{n\times n} \cap \boldsymbol{E}_0^f$ and $a_{ii} > 0$ for all i, then is true that A belongs to $\boldsymbol{P}_0$? Our investigation for $n = 4$ proved that this is not true for $n \geq 4$. Consider the following example.

Example 5.3.10. Let

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ -2 & 1 & -1 & 1 \\ -1 & 2 & 1 & -1 \\ 2 & -1 & -2 & 2 \end{bmatrix}.$$

It can be checked that A is an almost P_o-matrix and that all the PPTs of A are E_o-matrices. Thus, A is a E_o^f-matrix with all diagonal entries positive but $A \notin P_o$ as $det\, A < 0$.

Fully Copositive Matrices

Definition 5.3.11. Let $A \in R^{n\times n}$. Say that A is a *fully copositive* matrix if A and all its PPTs are all copositive matrices. This class will be denoted by C_o^f.

Remark 5.3.12. As $C_o \subseteq E_o$, it is obvious from the definition that $C_o^f \subseteq E_o^f$. Observe that positive semidefinite matrices and permutation matrices are C_o^f-matrices. Further, if $A \in R^{n\times n} \cap C_o^f$, then $A_{\alpha\alpha} \in C_o^f$ for every $\alpha \in n^*$.

Example 5.3.13. Let

$$A = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Note that A is P-matrix (and hence E_o^f) but not a fully copositive matrix (A^{-1} does not belong to C_o). B is fully copositive matrix but not a positive semidefinite matrix. Lastly C is E_o^f-matrix but not a fully copositive matrix.

The following theorem establishes that within the class of symmetric matrices there is no difference between C_o^f and E_o^f.

Theorem 5.3.14. Suppose $A \in R^{n\times n}$. Assume that A is symmetric. Then A belongs to C_o^f if, and only if A belongs to E_o^f.

Proof. Since $C_o^f \subseteq E_o^f$, we need to show that $E_o^f \subseteq C_o^f$. Suppose $A \in E_o^f$. Since A is symmetric, $A \in C_o$. Let $\alpha \in n^*$ be such that $det\, A_{\alpha\alpha} \neq 0$. We will show that $\wp_\alpha(A) \in C_o$. Let $B = \wp_\alpha(A)$. Since A is symmetric $(A_{\bar{\alpha}\alpha})^t = A_{\alpha\bar{\alpha}}$ and $(A_{\alpha\bar{\alpha}})^t = A_{\bar{\alpha}\alpha}$. Therefore,

$$\frac{1}{2}(B + B^t) = \begin{bmatrix} (A_{\alpha\alpha})^{-1} & 0 \\ 0 & (\, A/A_{\alpha\alpha}\,) \end{bmatrix}.$$

Since $A \in E_o^f$, $(A_{\alpha\alpha})^{-1}$, and $(\, A/A_{\alpha\alpha}\,)$ are both E_o-matrices. Observe, also, that both $(A_{\alpha\alpha})^{-1}$, and $(\, A/A_{\alpha\alpha}\,)$ are symmetric. Therefore, $(A_{\alpha\alpha})^{-1}$, and

($A/A_{\alpha\alpha}$) are copositive matrices. Hence $\frac{1}{2}$ $(B + B^t)$ is a copositive. Since $x^tBx = \frac{1}{2}x^t(B + B^t)x$, it follows that $B \in C_o$. Since α was arbitrary, it follows that $A \in C_o^f$. □

Recall that for $n \leq 3$, if $A \in R^{n\times n} \cap E_o^f$ and if $a_{ii} > 0$ for all i, then $A \in P_o$. In Example 5.3.10, it was shown that this assertion does not hold good for $n = 4$. However, for A in C_o^f we have the following results.

Theorem 5.3.15. Suppose $A \in R^{n\times n} \cap C_o^f$. Assume that $a_{ii} > 0$ for all $i \in \bar{n}$. Then $A \in P_o$.

Proof. We prove this by induction on n. If $n = 1$, then the result is trivially true. Assume that the result is true for all $(n-1) \times (n-1)$ real matrices, $n > 1$. Suppose $A \in R^{n\times n} \cap C_o^f$ and $a_{ii} > 0$ for every $i \in \bar{n}$. Observe that for all $\alpha \subseteq \bar{n}$ with $|\alpha| = n - 1$, $A_{\alpha\alpha}$ satisfies the assumptions of the theorem. Suppose $A \notin P_o$. By induction hypothesis, A is an almost P_o-matrix. Then $det\, A < 0$ and $A^{-1} \in C_o^f \cap N_o$. Let $B = A^{-1}$. By Theorem 5.3.2, there exists a subset α of $\bar{n}$ such that

$$\phi \neq \alpha \neq \bar{n},\ B_{\alpha\alpha} \leq 0,\ B_{\bar{\alpha}\bar{\alpha}} \leq 0,\ B_{\alpha\bar{\alpha}} \geq 0 \text{ and } B_{\bar{\alpha}\alpha} \geq 0.$$

Since $B \in C_o$, we must have $B_{\alpha\alpha} = 0$ and $B_{\bar{\alpha}\bar{\alpha}} = 0$. Without loss of generality, we may assume

$$B = \begin{bmatrix} 0 & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & 0 \end{bmatrix}$$

Let $k = |\alpha|$. Since B is nonsingular, we must have $|\alpha| = |\bar{\alpha}| = k$. It is easy to see, from the structure of B, that if we drop any row and the corresponding column from B, then the resulting $(n-1) \times (n-1)$ principal submatrix of B must be singular. Let $\beta = \overline{\{1\}}$. Then

$$a_{11} = \frac{det\, B_{\beta\beta}}{det\, B} = 0$$

which contradicts the hypothesis. It follows that A belongs to P_o. □

Corollary 5.3.16. Suppose $A \in R^{n\times n} \cap C_o^f$. Assume that A has at most one zero diagonal entry. Then A belongs to P_o.

Proof. The proof is exactly similar to the proof of the above theorem. Note that induction hypothesis works because if A has the property that it has at most one zero diagonal entry, then every principal submatrix of A also has this property. □

Our next goal is to establish that $C_0^f \cap Q_o \subseteq P_o$.

Definition 5.3.17. Let $A \in R^{n \times n}$ and let $\alpha \subseteq \bar{n}$ be such that $pos\, C_A(\alpha)$ is full. Let $B = C_A(\alpha)$. Then $pos\, B_{.\beta}$ is called a facet of $pos\, C_A(\alpha)$ provided $|\beta| = n - 1$.

Definition 5.3.18. Let $A \in R^{n \times n}$ and let $\alpha, \beta \subseteq \bar{n}$ be such that $pos\, C_A(\alpha)$ and $pos\, C_A(\beta)$ are full cones. Say that the cones $pos\, C_A(\alpha)$ and $pos\, C_A(\beta)$ are *incident* to each other on a hyperplane H if the relative interior (with respect to H) S of $H \cap pos\, C_A(\alpha) \cap pos\, C_A(\beta)$ is nonempty.

Lemma 5.3.19. Suppose $A \in R^{n \times n} \cap C_0^f$. Suppose α is a nonempty subset of $\bar{n}$ such that $pos\, C_A(\alpha)$ is full and is incident to $R_+^n (= pos\, C_A(\emptyset))$. Then $det\, A_{\alpha\alpha} > 0$.

Proof. We shall prove this by induction on n. When $n = 1$ the lemma is obvious. Assume that the lemma is valid for all matrices of order $n-1$, $n > 1$. Let $A \in R^{n \times n}$ satisfy hypothesis of the lemma along with a subset α of $\bar{n}$. Let $B = C_A(\alpha)$. Since $A \in C_0^f$, $pos\, C_A(\alpha)$ and R_+^n cannot intersect in the interior. For simplicity, we assume $S = pos\, [I_{.2}, I_{.3}, \ldots, I_{.n}] \cap pos\, C_A(\alpha)$. Note that the common hyperplane containing the facets of $pos\, I$ and $pos\, C_A(\alpha)$ is given by $H = \{x \in R^n : x_1 = 0\}$. Choose $(n-1)$ linearly independent vectors $q^1, q^2, \ldots, q^{(n-1)}$ from the relative interior of S. Let $B_{i_1}, B_{i_2}, \ldots, B_{i_{(n-1)}}$ be the generators of the facet (of $pos\, C_A(\alpha)$) containing S. Then there exists a nonsingular matrix X (strictly positive) of order $(n-1)$ such that

$$[q^1, q^2, \ldots, q^{(n-1)}] = [B_{i_1}, B_{i_2}, \ldots, B_{i_{(n-1)}}]X.$$

From this it follows that the first coordinates of $B_{i_1}, B_{i_2}, \ldots, B_{i_{(n-1)}}$ are equal to zero. Note that as $A \in C_0^f$, $I_{.1}$ cannot be a generator of $pos\, C_A(\alpha)$. Hence

$1 \in \alpha$.

Case (i). $-A_{.1} \notin H$.
In this case we must have $\alpha = \{1\}$ as otherwise the interior points of the line segment joining $-A_{.1}$ and q^1 will belong to interiors of two cones- $pos\, C_A(\alpha)$ and $pos\,[-A_{.1},\ I_{.2},\ I_{.3},\ \ldots,\ I_{.n}]$. This is not possible as $A \in C_{\circ}^f$ (follows from Theorem 5.2.17 and the fact that $C_{\circ}^f \subseteq E_{\circ}^f$). But if $\alpha = \{1\}$, as $pos\, C_A(\alpha)$ is full and $A \in C_{\circ}^f$, $det\, A_{\alpha\alpha} = a_{11} > 0$.

Case (ii). $-A_{.1} \in H$.
Since $pos\, C_A(\alpha)$ is full, we must have a $k \in \bar{n}$ such that $-A_{.k} \notin H$. Without loss of generality assume $k = n$.
Suppose $|\alpha| < n$, say $(n-1) \notin \alpha$. Let $\beta = \bar{n} \setminus \{n-1\}$ and let $M = A_{\beta\beta}$. It can be verified that M together with α satisfies the assumptions of the lemma. That is, $pos\, C_M(\alpha)$ is full and is incident to R_+^{n-1} on the hyperplane $\bar{H} = \{(x_1, \ldots, x_{(n-2)}, x_n)^t \in R^{n-1} : x_1 = 0\}$. By induction hypothesis, $det\, M_{\alpha\alpha} > 0$. But $M_{\alpha\alpha} = A_{\alpha\alpha}$ and hence $det\, A_{\alpha\alpha} > 0$.

Suppose $|\alpha| = n$. Since $S \subsetneq pos\,[-A_{.1}, \ldots, -A_{.(n-1)}]$, there exists a positive vector $(x_1, \ldots, x_{(n-1)})^t$ such that

$$-\begin{bmatrix} a_{21} & a_{22} & \ldots & a_{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \ldots & a_{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{(n-1)} \end{bmatrix} > \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If $a_{i1} \geq 0$ for all $i \in \gamma = \{2, \ldots, (n-1)\}$, then it follows that $v(A_{\gamma\gamma}{}^t)$ is negative which is a contradiction. Hence there must exist an index $k \in \gamma$ such that $a_{k1} < 0$. But then for $\theta = \{1, k\}$, $A_{\theta\theta} = \begin{bmatrix} 0 & 0 \\ a_{k1} & a_{kk} \end{bmatrix} \notin C_{\circ}$, which contradicts that $A \in C_{\circ}^f$. It follows that $|\alpha|$ cannot be equal to n. This completes the proof of the lemma. □

Lemma 5.3.20. Suppose $A \in R^{n \times n} \cap C_{\circ}^f$. Assume that $\alpha, \beta \subseteq \bar{n}$ are such that $pos\, C_A(\alpha)$ and $pos\, C_A(\beta)$ are full. If $pos\, C_A(\alpha)$ and $pos\, C_A(\beta)$ are incident to each other (with respect to a common hyperplane containing the facets), then *det* $A_{\alpha\alpha}$ and *det* $A_{\beta\beta}$ have the same sign.

Proof. Let $M = \wp_\alpha(A)$. Note that the principal pivotal transformation merely transforms the cones of $K(A)$ to the cones of $K(M)$ through the nonsingular linear transformation q going to $C_A(\alpha)^{-1}q$. In particular, $pos\, C_A(\alpha)$ gets transformed to $\boldsymbol{R}^n_+$ and $pos\, C_A(\beta)$ to $pos\, C_M(\gamma)$ where $\gamma = \alpha \triangle \beta$. As $pos\, C_A(\alpha)$ and $pos\, C_A(\beta)$ are incident to each other, it follows that $\boldsymbol{R}^n_+$ and $pos\, C_M(\gamma)$ are incident to each other. By Lemma 5.3.19, it follows that $det\, M_{\gamma\gamma}$ is positive. From Theorem 1.2.19, it follows that $det\, A_{\alpha\alpha}$ and $A_{\beta\beta}$ have the same sign. □

Theorem 5.3.21. Suppose $A \in \boldsymbol{R}^{n\times n} \cap C_0^f \cap \boldsymbol{Q}_0$. Then $A \in \boldsymbol{P}_0$.

Proof. Let $\alpha \in n^*$ be such that $pos\, C_A(\alpha)$ is full. Let $q^o \in$ interior $pos\, C_A(\alpha)$. Let $r > 0$ be such that $B_r(q^o) \subseteq pos\, C_A(\alpha)$. Since $A \in \boldsymbol{Q}_0$, $K(A)$ is convex. Define the set

$$P = \{q \in \boldsymbol{R}^n : q = \lambda p + (1-\lambda)e \text{ for some } \lambda \in [0,1] \text{ and some } p \in B_r(q^o)\},$$

where $e = (1, 1, \ldots, 1)^t \in \boldsymbol{R}^n$. Clearly P is an open set and is contained in the interior of $K(A)$. Note that for any β and γ, if $P \cap pos\, C_A(\beta) \cap pos\, C_A(\gamma)$ is nonempty, then $pos\, C_A(\beta)$ and $pos\, C_A(\gamma)$ are incident to each other and by Lemma 5.3.20, *det* $A_{\beta\beta}$ and *det* $A_{\gamma\gamma}$ have the same sign. From this it is clear that there exist $\emptyset = \alpha_0, \alpha_1, \ldots, \alpha_m = \alpha \subseteq \bar{n}$, $m \geq 1$, such that $pos\, C_A(\alpha_i)$ is full for each $i \in \{1, 2, \ldots, m\}$, and $pos\, C_A(\alpha_i)$ and $pos\, C_A(\alpha_{i+1})$ are incident to each other for $i = 0, 1, \ldots, m-1$. From Lemma 5.3.19 and Lemma 5.3.20, it follows that *det* $A_{\alpha\alpha}$ is positive. As α was arbitrary, this completes the proof of the theorem. □

Remark 5.3.22. It may be observed that Lemma 5.3.19 is valid when C_0^f is replaced by $\boldsymbol{U}$. This gives an alternative proof of Stone's result that $\boldsymbol{U} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$. Unfortunately the lemma is valid for $\boldsymbol{E}_0^f$-matrices when $n \leq 3$ and fails for $n \geq 4$. The following serves as a counter example.

Example 5.3.23. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

It can be checked that $A \in \boldsymbol{E}_o^f$ and $pos - A$ is incident to $\boldsymbol{R}_+^{4\times4}$ (on the hyperplane $H = \{x \in \boldsymbol{R}^{4\times4} : x_1 = 0\}$). However, $det\, A < 0$. It may be worth noting that A is not a $\boldsymbol{Q}_o$-matrix. This can be seen as follows. Since $A_{1.} \geq 0$, if A is in $\boldsymbol{Q}_o$, then $A_{\alpha\alpha}$, $\alpha = \{2, 3, 4\}$, must also be in $\boldsymbol{Q}_o$. But from Theorem 3.4.14 we can see that $A_{\alpha\alpha}$ is not in $\boldsymbol{Q}_o$.

Sign Structures of $\boldsymbol{E}_o^f$-Matrices

The following results on sign structures of $\boldsymbol{E}_o^f$-matrices are crucial in establishing the results of next section.

Theorem 5.3.24. Suppose $A \in R^{2\times2} \cap \boldsymbol{E}_o^f$. Then $SP(A)$ cannot be equal to any of the following :

$$(a)\begin{bmatrix} \oplus & - \\ - & 0 \end{bmatrix} \quad (b)\begin{bmatrix} 0 & - \\ - & \oplus \end{bmatrix} (c)\begin{bmatrix} + & + \\ + & 0 \end{bmatrix} \quad (d)\begin{bmatrix} 0 & + \\ + & + \end{bmatrix}$$

Proof. It is easy to check that the value of any matrix having the above sign pattern given by (a) or (b) is negative and hence cannot be a $\boldsymbol{E}_o$-matrix.
If A has the sign pattern as in (c), then the second diagonal entry of $\wp_\alpha(A)$ is negative, where $\alpha = \{1\}$. Hence A cannot have the sign pattern given by (c). Similarly we can show that A cannot have the sign pattern given by (d). □

Theorem 5.3.25. Suppose $A \in \boldsymbol{R}^{3\times3} \cap \boldsymbol{E}_o^f$. Then $SP(A)$ cannot be equal to any of the following :

$$(a)\begin{bmatrix} \star & - & \star \\ + & 0 & + \\ \star & + & 0 \end{bmatrix} \quad (b)\begin{bmatrix} \oplus & \star & - \\ \star & 0 & + \\ + & + & 0 \end{bmatrix} \quad (c)\begin{bmatrix} \star & - & \star \\ \star & 0 & - \\ - & 0 & 0 \end{bmatrix} \quad (d)\begin{bmatrix} \star & \star & - \\ - & \star & 0 \\ 0 & - & 0 \end{bmatrix}$$

Proof. Suppose $SP(A) = \begin{bmatrix} \oplus & - & \star \\ + & 0 & + \\ \star & + & 0 \end{bmatrix}$

Let M be a PPT of A with respect to $\alpha = \{2,3\}$. Then note that $SP(M_{\alpha\alpha}) = SP((A_{\alpha\alpha})^{-1}) = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$. Further,

$$SP(M_{\alpha\bar{\alpha}}) = SP(-(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}})$$

$$= SP(-(A_{\alpha\alpha})^{-1})SP(A_{\alpha\bar{\alpha}})$$

$$= \begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix} \begin{bmatrix} + \\ \star \end{bmatrix} = \begin{bmatrix} \star \\ - \end{bmatrix}$$

Similarly, $SP(M_{\bar{\alpha}\alpha}) = (\star, -)$. Since M is a PPT of a E_0^f-matrix, $m_{11} \geq 0$. Thus,

$$SP(M) = \begin{bmatrix} \oplus & \star & - \\ \star & 0 & + \\ - & + & 0 \end{bmatrix}$$

Note that $SP(M_{\beta\beta}) = \begin{bmatrix} \oplus & - \\ - & 0 \end{bmatrix}$, where $\beta = \{1,3\}$. This implies $M \notin E_0^f$. This shows that $SP(A)$ cannot be equal to the one given by (a). Similarly we can show that $SP(A)$ cannot be equal to the one given by (b).

Suppose $SP(A)$ is given by (c). Choose $x_2 > 0$ such that $(a_{13} + a_{23}x_2)$ is negative (we can do this as $SP(A)$ implies that $a_{23} < 0$). Now choose $x_3 > 0$ such that $(a_{11} + a_{21}x_2 + a_{31}x_3)$ is negative (we can do this also as $a_{31} < 0$). Let $x = (1, x_2, x_3)^t$. Then $x^tA < 0$. This contradicts the hypothesis that $A \in E_0^f$. Hence $SP(A)$ can not be equal to the one given by (c). A similar argument will show that $SP(A)$ cannot be equal to the one given by (d). □

Corollary 5.3.26. Suppose $A \in R^{n \times n} \cap E_0^f$. Then no principal submatrix of A or any of its principal rearrangements can have any of the sign patterns listed in Theorem 5.3.24 and Theorem 5.3.25.

Proof. Follows from the fact that every principal submatrix of E_0^f-matrix is also in E_0^f. □

Theorem 5.3.27. Suppose $A \in R^{3\times3}$. Assume that $SP(A)$ is equal to one of the following sign patterns:

$$(a)\begin{bmatrix} \oplus & \oplus & \oplus \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \quad (b)\begin{bmatrix} \oplus & \oplus & \oplus \\ - & 0 & + \\ - & + & 0 \end{bmatrix}$$

If $a_{12} + a_{13} > 0$, then $A \notin E_0^f$.

Proof. Suppose $A \in E_0^f$. Since $a_{12} + a_{13} > 0$, we may assume, without loss of generality, that $a_{12} > 0$. If $a_{11} > 0$, then, by Corollary 5.3.26, $A \notin E_0^f$. So $a_{11} = 0$. But then the first diagonal entry of $\wp_\alpha(A)$ is negative, where $\alpha = \{2,3\}$. This contradicts our supposition that $A \in E_0^f$. It follows that $A \notin E_0^f$.
Suppose $SP(A)$ is given by (b). Let $B = \wp_\alpha(A)$, where $\alpha = \{2,3\}$. Then

$$SP(B) = \begin{bmatrix} \oplus & \oplus & \oplus \\ + & 0 & + \\ + & + & 0 \end{bmatrix}.$$

Since $a_{12}+a_{13} > 0$, it can be seen that $b_{12}+b_{13} > 0$. From the earlier argument $B \notin E_0^f$. Therefore, $A \notin E_0^f$.□

5.4. $E_0^f \cap Q_0$-Matrices of Order Less Than 7

Theorem 5.4.1. Suppose $A \in R^{3\times3} \cap E_0^f \cap Q_0$. Then $A \in P_0$.

Proof. In view of Theorem 5.3.3, it is sufficient to show that $A_{\alpha\alpha} \in P_0$ for all $\alpha \subseteq \{1,2,3\}$ such that $|\alpha| = 2$. Since $A \in E_0^f$, $a_{ii} \geq 0$ for all i. Suppose there exists an $\alpha \subseteq \{1,2,3\}$ such that $|\alpha| = 2$ and $A_{\alpha\alpha} \notin P_0$. Since $E_0^f \cap Q_0$ property is invariant under principal rearrangements, we may assume, without loss of generality, that $\alpha = \{2,3\}$. Since $A_{\alpha\alpha} \in \boldsymbol{R}^{2\times 2} \cap \boldsymbol{E}_{\circ}^{f}$ and $A_{\alpha\alpha} \notin \boldsymbol{P}_{\circ}$,

$$SP(A_{\alpha\alpha}) = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}.$$

By Theorem 4.6.2, $a_{12} < 0$ and $a_{13} < 0$. By Theorem 3.5.2, we must have either $(a_{21}, a_{31}) < 0$ or $(a_{21}, a_{31}) > 0$. Since $(a_{12}, a_{13}) < 0$, $(a_{21}, a_{31}) > 0$ (Corollary 5.3.26). But this will contradict Corollary 5.3.26. Hence, it follows that A belongs to $\boldsymbol{P}_{\circ}$. □

We now establish Conjecture 5.1.1 for 4×4 matrices. The outline of the proof is as follows. We first show that every 2×2 principal submatrix of a $\boldsymbol{R}^{4\times 4} \cap \boldsymbol{E}_{\circ}^{f} \cap \boldsymbol{Q}_{\circ}$-matrix is in $\boldsymbol{P}_{\circ}$ (see Lemma 5.4.3) and then show that every 3×3 principal submatrix of a $\boldsymbol{R}^{4\times 4} \cap \boldsymbol{E}_{\circ}^{f} \cap \boldsymbol{Q}_{\circ}$-matrix is in $\boldsymbol{P}_{\circ}$. Then invoking Theorem 5.3.3, we conclude the result.

Lemma 5.4.2. Suppose $A \in \boldsymbol{R}^{4\times 4} \cap \boldsymbol{Q}_{\circ}$. Assume that $a_{33} = a_{44} = 0$, and a_{34} and a_{43} are positive. Then there exists a PPT B of A such that, subject to principal rearrangement,

$$b_{33} = b_{44} = 0, \; b_{31} > 0, \; b_{34} > 0 \; \text{ and } b_{43} > 0. \tag{5.3}$$

Proof. Let $\alpha = \{3,4\}$. From Theorem 3.5.2, it follows that $A_{\alpha\bar{\alpha}} \neq 0$. If $A_{\alpha\bar{\alpha}}$ contains a positive entry, then it is easy to see that A or a principal rearrangement B of it will satisfy (5.3). If $A_{\alpha\bar{\alpha}}$ has no positive entry, then it must have a negative entry. Let $M = \wp_{\alpha}(A)$. Then $m_{33} = m_{44} = 0$, and m_{34} and m_{43} are positive. Also $M_{\alpha\bar{\alpha}}$ will have a positive entry. Then a principal rearrangement B of M will satisfy (4.1). □

Lemma 5.4.3. Suppose $A \in \boldsymbol{R}^{4\times 4} \cap \boldsymbol{E}_{\circ}^{f} \cap \boldsymbol{Q}_{\circ}$. Assume that $a_{33} = a_{44} = 0$, $a_{34} > 0$ and $a_{43} > 0$. Then $A_{1.}$ and $A_{2.}$ both must have negative entries.

Proof. Let $\alpha = \{2,3,4\}$ and $\beta = \{1,3,4\}$. From the hypothesis, $A_{\alpha\alpha}$ and $A_{\beta\beta}$ are not in $\boldsymbol{P}_{\circ}$. If $A_{1.}$ is nonnegative, then by Theorem 3.4.2, $A_{\alpha\alpha} \in \boldsymbol{R}^{3\times 3} \cap \boldsymbol{E}_{\circ}^{f} \cap \boldsymbol{Q}_{\circ}$, and by Theorem 5.4.1, $A_{\alpha\alpha} \in \boldsymbol{P}_{\circ}$. This contradiction implies that $A_{1.}$

must have a negative entry. Similar argument shows that $A_{2.}$ must contain a negative entry. □

Lemma 5.4.4. Suppose $A \in \boldsymbol{R}^{4\times 4} \cap \boldsymbol{E}_{o}^{f} \cap \boldsymbol{Q}_{o}$. Then every 2×2 principal submatrix of A is in $\boldsymbol{P}_{o}$.

Proof. Suppose A has a 2×2 principal submatrix which is not in $\boldsymbol{P}_{o}$. Let $\alpha = \{3, 4\}$. Without loss of generality, assume $A_{\alpha\alpha} \notin \boldsymbol{P}_{o}$. Then, we must have $a_{33} = a_{44} = 0$, and a_{34} and a_{43} positive. In view of Lemma 5.4.2, we may assume, without loss of generality, that a_{31} is positive. By Theorem 5.3.25 and Corollary 5.3.26, we must have $a_{13} \geq 0$. Since $A \in \boldsymbol{E}_{o}^{f}$, a_{11} and a_{22} are nonnegative. By Theorem 4.6.2, $a_{23} < 0$ and this in turn implies $a_{32} = 0$ (Corollary 5.3.26). By Theorem 3.5.2, we must have :

$$\text{either } a_{41} > 0 \text{ or } a_{42} > 0.$$

Suppose $a_{41} > 0$.
Then by Corollary 5.3.26, $a_{14} \geq 0$ and by Theorem 4.6.2, $a_{24} < 0$.
Since $a_{24} < 0$, $a_{42} = 0$. Thus,

$$SP(A) = \begin{bmatrix} \oplus & - & \oplus & \oplus \\ \star & \oplus & - & - \\ + & 0 & 0 & + \\ + & 0 & + & 0 \end{bmatrix}.$$

Note that $a_{12} < 0$, as otherwise it would contradict Lemma 5.4.3.
By Theorem 5.3.27, $a_{13} = a_{14} = 0$. Then

$$SP(\wp_{\alpha}(A)) = \begin{bmatrix} \oplus & - & 0 & 0 \\ \star & \oplus & - & - \\ - & 0 & 0 & + \\ - & 0 & + & 0 \end{bmatrix}.$$

This contradicts Corollary 5.3.26. So we must have $a_{41} \leq 0$ and $a_{42} > 0$. This in turn implies $a_{24} \geq 0$ (Corollary 5.3.26), $a_{14} < 0$ (Theorem 4.6.2) and $a_{41} = 0$ (Corollary 5.3.26). Let $\beta = \{1, 2, 3\}$ and $\gamma = \{1, 2, 4\}$. Observe that $A_{3.}$ and

$A_{4.}$ are nonnegative. This implies that $A_{\beta\beta}$ and $A_{\gamma\gamma}$ are in $\boldsymbol{P}_{\text{o}}$, which in turn implies that $a_{13} = a_{24} = 0$. Thus,

$$SP(A) = \begin{bmatrix} \oplus & \star & 0 & - \\ \star & \oplus & - & 0 \\ + & 0 & 0 & + \\ 0 & + & + & 0 \end{bmatrix}.$$

As $A_{\beta\beta} \in \boldsymbol{Q}_{\text{o}}$, by Theorem 3.5.3, we must have $a_{12} < 0$. Observe that

$$SP(\wp_\alpha(A)) = \begin{bmatrix} + & - & - & 0 \\ \star & + & 0 & - \\ 0 & - & 0 & + \\ - & 0 & + & 0 \end{bmatrix}.$$

This contradicts Corollary 5.3.26. Hence every 2×2 principal submatrix of a $\boldsymbol{R}^{4\times 4} \cap \boldsymbol{E}_{\text{o}}^{f} \cap \boldsymbol{Q}_{\text{o}}$-matrix must be in $\boldsymbol{P}_{\text{o}}$. □

Theorem 5.4.5. Suppose $A \in \boldsymbol{R}^{4\times 4} \cap \boldsymbol{E}_{\text{o}}^{f} \cap \boldsymbol{Q}_{\text{o}}$. Then A belongs to $\boldsymbol{P}_{\text{o}}$.

Proof. By Lemma 5.4.4, every 2×2 principal submatrix of A is in $\boldsymbol{P}_{\text{o}}$. If every 3×3 principal submatrix of A is also in $\boldsymbol{P}_{\text{o}}$, then by Theorem 5.3.3, $A \in \boldsymbol{P}_{\text{o}}$. Suppose there exists an $\alpha \subseteq \{1,2,3,4\}$ such that $|\alpha| = 3$ and $A_{\alpha\alpha} \notin \boldsymbol{P}_{\text{o}}$. Since $\boldsymbol{E}_{\text{o}}^{f} \cap \boldsymbol{Q}_{\text{o}}$ property is invariant under principal rearrangements, we may assume, without loss of generality, that $\alpha = \{2,3,4\}$. Since every 2×2 principal submatrix of A is in $\boldsymbol{P}_{\text{o}}$, we must have $det\, A_{\alpha\alpha} < 0$ and $(A_{\alpha\alpha})^{-1} \in \boldsymbol{E}_{\text{o}}^{f} \cap \boldsymbol{N}_{\text{o}}$. Let $B = \wp_\alpha(A)$. Then $B \in \boldsymbol{E}_{\text{o}}^{f} \cap \boldsymbol{Q}_{\text{o}}$. Note that $B_{\alpha\alpha} = (A_{\alpha\alpha})^{-1} \in \boldsymbol{E}_{\text{o}} \cap \boldsymbol{N}_{\text{o}}$. By Theorem 5.3.2, we can assume, without loss of generality, that

$$SP(B_{\alpha\alpha}) = \begin{bmatrix} 0 & \ominus & \oplus \\ 0 & 0 & \oplus \\ \oplus & \oplus & 0 \end{bmatrix}.$$

Since $B_{\alpha\alpha}$ is nonsingular, $b_{42} > 0$, $b_{34} > 0$, and $b_{23} < 0$. Since $B \in \boldsymbol{E}_{\text{o}}^{f} \cap \boldsymbol{Q}_{\text{o}}$, all its 2×2 principal submatrices are in $\boldsymbol{P}_0$. This implies $b_{24} = b_{43} = 0$. Thus

$$SP(B) = \begin{bmatrix} \oplus & \star & \star & \star \\ \star & 0 & - & 0 \\ \star & 0 & 0 & + \\ \star & + & 0 & 0 \end{bmatrix}.$$

From Theorem 4.6.2, b_{12} and b_{14} must both be negative. This in turn implies that b_{21} and b_{41} are both nonnegative. If $b_{31} \leq 0$, then we can choose a $q \in \boldsymbol{R}^{4\times4}$ with $SP(q) = (+,+,-,+)^t$ satisfying $F(q, A) \neq \phi$ and $S(q, A) = \phi$. So b_{31} must be positive. But then

$$SP(A) = \begin{bmatrix} \oplus & \star & \star & - \\ \star & 0 & 0 & + \\ \star & - & 0 & 0 \\ - & 0 & + & 0 \end{bmatrix}.$$

This contradicts Corollary 5.3.26. It follows that every 3×3 principal submatrix of A is in $\boldsymbol{P}_0$. Invoking Theorem 5.3.3, we conclude that A belongs to $\boldsymbol{P}_0$. □

The following result is due to Jeter and Pye (1989). We give an alternative proof of this using Theorem 5.4.5 and Theorem 4.2.20.

Theorem 5.4.6. Suppose $A \in \boldsymbol{R}^{4\times4} \cap \boldsymbol{E}_0^f \cap \boldsymbol{Q}$. Then A belongs to $\boldsymbol{R}_0$.

Proof. Since $\boldsymbol{Q} \subseteq \boldsymbol{Q}_0$, by Theorem 5.4.5, $A \in \boldsymbol{P}_0$. From Theorem 4.2.20, it follows that A belongs to $\boldsymbol{R}_0$. □

In Murthy, Parthasarathy and Ravindran (1993b), it was shown that if $A \in \boldsymbol{R}^{4\times4} \cap \boldsymbol{E}_0^f \cap \boldsymbol{Q}$ and $a_{ii} > 0$ for all i, then $A \in \boldsymbol{P}_0$. In this direction we have the following result for $A \in \boldsymbol{R}^{5\times5}$.

Theorem 5.4.7. Suppose $A \in \boldsymbol{R}^{5\times5} \cap \boldsymbol{E}_0^f \cap \boldsymbol{Q}_0$. If $a_{ii} > 0$ for all $i \in \{1, 2, \ldots, 5\}$, then A belongs to $\boldsymbol{P}_0$.

Proof. Since $A \in \boldsymbol{E}_0^f$, and $a_{ii} > 0$ for all i, by Theorem 5.3.9, every 3×3 principal submatrix of A is a $\boldsymbol{P}_0$-matrix. If every 4×4 principal submatrix of A is in

$\boldsymbol{P}_o$, then by Theorem 5.3.3, $A \in \boldsymbol{P}_o$. Suppose A has a 4×4 principal submatrix which is not a $\boldsymbol{P}_o$-matrix. Without loss of generality, assume $A_{\alpha\alpha} \notin \boldsymbol{P}_o$, where $\alpha = \{1, 2, 3, 4\}$. Then, by the above observation, $det\, A_{\alpha\alpha}$ is negative and $(A_{\alpha\alpha})^{-1} \in \boldsymbol{E}_o^f \cap \boldsymbol{N}_o$. Let $B = (A_{\alpha\alpha})^{-1}$. By Theorem 5.3.2, there exists a principal rearrangement of B whose sign pattern is :

$$\text{either (a)} \begin{bmatrix} 0 & \ominus & \oplus & \oplus \\ 0 & 0 & \oplus & \oplus \\ \oplus & \oplus & 0 & \ominus \\ \oplus & \oplus & 0 & 0 \end{bmatrix} \quad \text{or (b)} \begin{bmatrix} 0 & \ominus & \ominus & \oplus \\ 0 & 0 & \ominus & \oplus \\ 0 & 0 & 0 & \oplus \\ \oplus & \oplus & \oplus & 0 \end{bmatrix}.$$

We may assume, without loss of generality, that $SP(B)$ itself is given by either (a) or (b). Suppose $SP(B)$ is as in (a). Note that $det\, B < 0$.

Since $a_{11} = \frac{det\, B_{\beta\beta}}{det\, B}$, where $\beta = \{2, 3, 4\}$,

$$a_{11} > 0 \Rightarrow b_{42} > 0,\ b_{34} < 0 \text{ and } b_{23} > 0.$$

Similarly,

$$a_{22} > 0 \Rightarrow b_{41} > 0,\ b_{13} > 0.$$

$$a_{33} > 0 \Rightarrow b_{24} > 0.$$

$$a_{44} > 0 \Rightarrow b_{31} > 0,\ b_{23} > 0,\ b_{12} < 0.$$

Then

$$SP(B) = \begin{bmatrix} 0 & - & + & \oplus \\ 0 & 0 & + & + \\ + & \oplus & 0 & - \\ + & + & 0 & 0 \end{bmatrix}$$

Let $D = \wp_\alpha(A)$. Then

$$SP(D) = \begin{bmatrix} 0 & - & + & \oplus & \star \\ 0 & 0 & + & + & \star \\ + & \oplus & 0 & - & \star \\ + & + & 0 & 0 & \star \\ \star & \star & \star & \star & \oplus \end{bmatrix}$$

Write $D = (d_{ij})$. If $d_{25} \geq 0$, then $D_{2.} \geq 0$, and $D_{\beta\beta} \in E_0^f \cap Q_0$, where $\beta = \{1, 3, 4, 5\}$. But $D_{\beta\beta}$ has a principal submatrix with determinant negative. This contradicts Theorem 5.4.5. Hence $d_{25} < 0$. By Theorem 4.6.2, $d_{51} < 0$. Let $\gamma = \{1, 2, 5\}$. Then $v(D_{\gamma\gamma})$ is negative (observe $SP(D_{\gamma\gamma})$), which contradicts the hypothesis that A is in E_0^f. Thus, B cannot have the sign pattern given by (a). So B must be given by (b). But this sign pattern is ruled out because it implies that $a_{44} = 0$ which contradicts the hypothesis. It follows that A is in P_0. □

Corollary 5.4.8. Suppose $A \in R^{5\times5} \cap E_0^f \cap Q$. Assume that all the diagonal entries of A (or any of its PPTs) are positive. Then A belongs to R_0.

Proof. If A (or any of its PPTs) has all diagonal entries positive, then A belongs to P_0. Since A is in Q, A belongs to R_0. □

Theorem 5.4.9. Suppose $A \in R^{6\times6} \cap E_0^f \cap Q_0$. Suppose A satisfies the *following conditions :*

(a) $a_{ii} > 0$ for every $i \in \{1, 2, \ldots, 6\}$,

(b) A has property (D),

(c) for every PPT M of A, $v(M^t) > 0$.

Then A belongs to P_0.

Proof. Note that Corollary 3.4.7 implies $A_{\alpha\alpha} \in Q_0$ for all $\alpha \subseteq \{1, 2, \ldots, 6\}$ with $|\alpha| = 5$. By Theorem 5.4.7, $A_{\alpha\alpha} \in P_0$ for every $\alpha \subseteq \{1, 2, \ldots, 6\}$ such that $|\alpha| = 5$. By Theorem 5.3.3, A belongs to P_0. □

Concluding Remarks

Aganagic and Cottle (1987) gave a constructive characterization of $P_0 \cap Q_0$ and showed that Lemke's algorithm processes (q, A) when A is in this class. Hence for all the cases for which we have established Conjecture 5.1.1 this result will apply. We believe that Conjecture 5.1.1 can be established even in the case of 5×5 matrices using sign patterns. It can be shown, using sign patterns, that if $A \in \boldsymbol{E}_o^f \cap \boldsymbol{Q}_o$ and every principal submatrix of A of order $(n-2)$ is in $\boldsymbol{P}$, then A belongs to $\boldsymbol{P}_o$ (this also follows from Theorem 5.3.3).

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