STUDIES IN SET-THEORETIC HIERARCHIES:
FROM BOREL SETS TO R-SETS

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CONTENTS

NOTATION

ω :	set of natural numbers
ω^{ω}:	sequence of natural numbers called reals
$\omega^{<\omega}$:	finite sequence of natural number
Seq :	set of Gödel numbers of finite sequence of natural numbers
$\eta, \xi, \ldots$	usually denote subsets of ω
$\alpha, \beta, \gamma, \ldots$	usually denote reals
$s, t, u, v, \ldots$	usually vary over Seq
$\langle a_0, \ldots, a_{n-1} \rangle$	Gödel number or sequence number of $(a_0, \ldots, a_{n-1})$, sometime identified with $(a_0, \ldots, a_{n-1})$.
$s*t, s\hat{\ }t$	concatenation of two sequence numbers or concatenation of two finite sequences; otherwise taken to be 0.
sn	$s*\langle n \rangle$
$n\ s$	$\langle n \rangle * s$
$s \subseteq t$	s is an initial segment of t if $s, t \in$ Seq or $\omega^{<\omega}$
$s*\alpha$	$(a_0, a_1, \ldots, a_{n-1}, \alpha(0), \alpha(1), \ldots)$ if $s = \langle a_0, \ldots, a_{n-1} \rangle$
$lh(s)$	length of the finite sequence or sequence number s
χ_A	characteristic function of A
$\mathcal{P}(A)$	power set of A
N_s, $\Sigma(s)$	set of reals extending $s \in$ Seq
$\bar{\alpha}(n)$	$\langle \alpha(0), \ldots, \alpha(n-1) \rangle$
$E^x (E_y)$	horizontal (vertical) section of E at x.
A^c, $\neg A$, $\sim A$	complement of A
e, $\langle\rangle$	code of the empty sequence

$A \| B$	$A - B$
Dom (f)	domain of f
$f(x)\downarrow$	$x \in \text{Dom}(f)$
$c\mathfrak{F}$, $\neg\mathfrak{F}$	family of complements of sets in $\mathfrak{F}$
$\mathcal{B}_A$	Borel σ-field on A
$\sigma(\mathfrak{F})$	σ-field generated by $\mathfrak{F}$
$\leq_\alpha$	$\{(m,n) : \alpha(\langle m,n\rangle) = 0\}$
WO	set of reals α such that $\leq_\alpha$ is a wellordering on ω

For unexplained notation and terminology from descriptive set theory see Moschovakis [38]. Notation and terminology from recursion theory is from Hinman [22].

SUMMARY OF THESIS

The present thesis consists of three parts.

Part I of the thesis is devoted to the study of regularity properties of classes of sets obtained from δ-s operations applied to the class of clopen sets. In particular, the classes of sets obtained by applying the hierarchy of increasingly complex R-operations $\{R_\rho\}$, are investigated. Our points of departure in this study are the recent works on R-sets by Hinman and Burgess. In this thesis we have unified the methods of Hinman and Burgess by extensively using the game quantifier and methods of inductive definability. Using these tools we are able not only to considerably simplify the proofs of known results in the theory (e.g. the comparison of indices theorem on which so much of Lyapunov's work depends as well as the category and measure formulas for R-sets) but also to obtain a number of new results for R-sets. Among these new results we mention the measurable selection property for different levels of the hierarchy of R-sets, the scale property at all levels of the R-hierarchy (this was known only for the first two levels) and the uniformization property at all levels of the R-hierarchy (again this was known only for the first level). These last mentioned results depend on the notion of presentability introduced by Burgess in his penetrating study of the measurable selection property of various families of

sets. Our proof turns on the observation that the presentability property holds at every level of the R-hierarchy. Finally, by combining analysis of the associated games with an adaptation of the Kechris Game Formula and the existence of measurable winning strategies; we are able to extend a recent result of Srivatsa on C-sets to R-sets. Briefly stated this result ensures that an R-set in the product of two Polish spaces can be approximated in the sense of category by means of sets in appropriate product σ-fields, the importance of this result resting on the fact that setsin product σ-field have a much more tractable structure than R-sets. Following Srivatsa we use this result to obtain a selection theorem of Burgess for R-sets.

While Part I is devoted to the study of classical (bold-face) R-sets, in Part II we take up the study of the effective analogues of these classes at the finite levels. Using again the game theoretic methods of Moschovakis we are able to establish various regularity properties of these effective classes. For instance, we show that each $c\mathcal{R}_n$ is a Spector pointless. We also observe that various other results like the Measure and Category Formulas, the scale property hold in the effective case also. We also take another look at the problem of effectivizing C-sets. Hinman, by generalizing Addison's construction of the effective

Borel hierarchy, obtained a hierarchy of effective C-sets on ω which exhaust ${}_1sc(\mathbb{E}_1)$, the class of subsets of ω recursive in $\mathbb{E}_1$. We, however, obtain an effective hierarchy in ${}_1sc(\mathbb{E}_1)$ in a different manner. We first exhibit a set $D(T,\alpha,x)$, where T varies over wellfounded trees on ω, which is universal for C-sets $\upharpoonright \mathfrak{X}$. When $\mathfrak{X} = \omega$, by taking sections of D by wellfounded trees and reals of increasing complexity, we obtain a hierarchy whose scope is ${}_1sc(\mathbb{E}_1)$.

Part III of our thesis is different in spirit and content. This part is divided into two sections. In the first section we work with hereditary familes of subsets of reals satisfying a weak definability condition and think of members of these families as small sets. We show that, at the first level of the analytical hierarchy, a number of measure-theoretic and category-theoretic results of Kechris and others are instances of more general results much in the spirit of Kechris' Basis Theorem. We also obtain the Basis Theorem of Kechris for large Π^1_1 sets under a weaker definibility condition. For the basis theorem we need to consider σ-ideals instead of hereditary families. Some σ-ideals we come across in descriptive theory, like the σ-ideals of meager sets and of null sets, satisfy both the Σ^1_1 and Π^1_1 computational formulas. However, we show that for a σ-ideal which is generated by a Π_1^1-coded ideal of closed sets, small Σ_1^1 sets have the covering property and, consequently, the Σ_1^1 computational formula holds.

In Section 2, we undertake a study of the structure of hyperarithmetical sets which are of ambiguous Borel classes. There is a natural hierarchy, called the difference hierarchy, for Borel sets of ambiguous class $\xi+1$ ($\underset{\sim}{\Delta}^0_{\xi+1}$). We show that there is a analogous effective hierarchy of $\Delta^1_1 \cap \underset{\sim}{\Delta}^0_{\xi+1}$ which coincides with the difference hierarchy of $\underset{\sim}{\Delta}^0_{\xi+1}$ restricted to Δ^1_1. (Earlier, Louveau has obtained similar results in a more general setting in his paper in Cabal Seminar, 1979-81). As a consequence we have been able to obtain a complete generalization of a recent result of Debs for every level of the difference hierarchy.Debs proved that a Borel set in the product of two Polish spaces, with vertical sections the difference of two G_δ-sets, is the difference of two Borel sets with G_δ sections. The effective result mentioned above enables us to extend Debs' result. We obtain this effective result via a separation theorem for Σ_1^1 sets. From another separation theorem, we are able to establish the faithful separation property of $\underset{\sim}{\Delta}^0_\xi$ for every ξ. This extends Burgess' result to every level. Finally, we are able to establish the <u>non-existence</u> of Borel sets universal for $\underset{\sim}{\Delta}^0_{\xi+1}$, $\xi < \omega_1$ — an unpublished result of Addison and Harrington.

INTRODUCTION TO PART I

Descriptive set theory is concerned with the study of those sets of real numbers which are of interest and significance for analysis. The first large class of sets studied was the class of Borel sets which was classified, according to the degree of complexity, into a hierarchy of ω_1 levels. The discovery of operation $\mathcal{A}$ and the analytic sets (which result from application of $\mathcal{A}$ to countable sequences of open sets) led to the definition of C-sets and their classification into a hierarchy [43]. The importance of this discovery can be gauged by the fact that it led directly to Hausdorff's δ-s operations and to the general notion of a set-theoretic operation. It was then natural to search for set-theoretic operations more powerful than operation $\mathcal{A}$ which would allow a still further enlargement of the class of sets for which an "explicit construction" could be given. This led to the discovery of the operator $\mathbb{R}$ by Kolmogorov[23]. However, it was only years later that a systematic application of it was made to enlarge the class of explicitly constructible sets [33].

In [23], Lyapunov constructs for each $\rho < \omega_1$ a positive analytical set operation R_ρ and the class of sets $\mathcal{R}^\rho$ consisting of those sets obtained by the application of R_ρ to sequences of open sets. We have $R_0 = \mathcal{A}$ and hence $\mathcal{R}^0$ = class of analytic sets; while $\mathcal{R}^1 \cap c\mathcal{R}^1$ properly includes the class of C - sets. Members of $\underset{\sim}{\mathcal{R}} = \bigcup_{\rho<\omega_1} \underset{\sim}{\mathcal{R}}^{\rho}$ are classically known as the R-sets. $\underset{\sim}{\mathcal{R}}$ was also shown to consist of measurable sets and to be a proper subclass of $\underset{\sim}{\Delta}^1_2$.

Although the theory of R-sets and the $\mathbb{R}$- operator has been studied extensively by Russian mathematicians [23],[33],[34], and most of the basic properties have been deduced by them; it is only very recently that interest in the theory has been revived due mainly to the work of Hinman [20,21] who developed the effective counterpart and showed that the effective hierarchies have deep interconnections with recursion-theoretic hierarchies. The work of Burgess [12,13] has added a new dimension to the theory. He has shown that the entire hierarchy of R-sets can be obtained by applying the game-quantifier to the "difference hierarchy" of $\underset{\sim}{\Delta}^0_3$ (obtained via decreasing sequences of G_δ-sets). Burgess and Lockhart [14] have also shown that the class of sets obtained from Borel-programmable sets (of Blackwell cf.[7]) by "iteration" gives precisely the R-sets. That seemingly different definitions yield the same class of sets suggests that the R-sets form a natural class of subsets of the reals.

The first few sections of Part I will be mainly devoted to the study of R-sets and the $\mathbb{R}$-operator. In fact, we shall study these in much greater generality by conidering the classes $\underset{\sim}{\Sigma}^{\Phi^*}_1, \underset{\sim}{\Pi}^{\Phi^*}_1, \underset{\sim}{\nabla}(\Phi^*)$ etc. much in the spirit of Hinman (cf.[20]). Thus we have tried to unify the work of Burgess and Hinman; and have formulated many results appropriate for our purpose in, perhaps, a more unified manner than is available in the literature. For instance, we have obtained the prewellording (p.w.o) property of $c\mathcal{R}^{\rho}$ by showing that for most operations Φ, $\Pi_1^{\Phi*}$ has the p.w.o. property (Section 3). This is proved via a comparison of indices lemma whose proof is much along the lines of the Kunen-Aczel theorem (cf.[37], see also [34]). The comparison of indices lemma is crucial for our purpose since it helps in computing the complexity of the winning strategy for the existential player in the game associated with the operator $\mathbb{R}$.

In Section 4, we have obtained the partition-selection property of R-sets by showing, à la Burgess, that each $\nabla(R_{\rho})$ is presentable. This we exhibit by showing that the property of being presentable is "transfered" by the operator $\mathbb{R}$. The notion of presentability not only yields the partition-selection property but also the scale and uniformization properties of R-sets. We also obtain a decomposition of $E^{*} = \{x : E^{x}$ is comeager$\}$ for sets $E \in \Sigma_1^{\Phi*}$, analogous to the one obtained by Vaught for Σ_1^1 sets [51] and for $\mathcal{R}^1$-sets obtained by Burgess and Miller [8]. This immediately gives us the transfer theorem, which essentially computes E^{*} for sets $E \in \Sigma_1^{\Phi*}$, whenever the computation for F^{*} for sets $F \in \Sigma_1^{\Phi}$

is known. This is done in Section 6. An immediate application of the transfer theorem is the computation of E^* for R-sets E — a result due to Burgess and Miller [8]. These computations yield sets in product σ-fields which "approximate" section-wise (in the sense of category) R-sets in two dimensions. This is done in Section 7. Incidentally, in this section, we also prove a slightly stronger version of the Game Formula of Kechris (cf. [26]).

PART I
R-SETS

1. <u>Positive analytical operations</u> : In the section we shall introduce positive analytical operations and Hausdorff's $\delta - s$ operations and discuss some of their elementary properties which we will need in the sequel. The papers of Kantorovitch and Livenson [23] give a detailed exposition of these operations.

Let X be a non-empty set and $N \subseteq \mathcal{P}(\omega)$. (<u>As usual, we shall identify</u> $\mathcal{P}(\omega)$ <u>with</u> 2^{ω}).

1.1 <u>Definition</u> : The $\delta - s$ operation with base N is defined by

$$\Phi_N(\{E_n : n \in \omega\}) = \bigcup_{\eta \in N} \bigcap_{n \in \eta} E_n ,$$

where $\{E_n : n \in \omega\}$ is any family (sequence) of subsets of X.

To avoid trivialities we shall always assume $\varphi \notin N$ and $N \neq \varphi$.

<u>Examples</u> : If $N = \{\{n\} : n \in \omega\}$, then $\Phi_N = \cup$ (countable).

If $N = \{\eta \subseteq \omega : \eta$ is infinite$\}$, then Φ_N = limsup.

If $N = \{\{\bar{\alpha}(n) : n \in \omega\} : \alpha \in \omega^{\omega}\}$, then Φ_N = operation $\mathcal{A}$.

By an operation we shall mean a function $\Phi : \mathcal{P}(X)^{\omega} \to \mathcal{P}(X)$ for all X.

1.2 <u>Definition</u> : An operation Φ is said to be a positive analytical operation (p.a.o) if

(a) Φ is non-constant and

(b) for any families $\{E_n : n \in \omega\}$ and $\{F_n : n \in \omega\}$ of subsets of a set X,

$$x \in \Phi(\{E_n : n \in \omega\}) \& y \notin \Phi(\{F_n : n \in \omega\}) \rightarrow (\exists n)[x \in E_n \& y \notin F_n].$$

Clearly, $\cup$, $\mathcal{A}$, limsup are positive analytical operations. A p.a.o. Φ on constant sequences takes on the constant value and is <u>isotone</u> i.e. for any families $\{F_n : n \in \omega\}$ and $\{G_n : n \in \omega\}$ if $(\forall n)(F_n \subseteq G_n)$, then $\Phi(\{F_n : n \in \omega\}) \subseteq \Phi(\{G_n : n \in \omega\})$.

Notice that a $\delta-s$ operation is a p.a.o. The converse is true and follows from the following.

<u>Proposition</u> : Let Φ be a p.a.o. Then $\Phi = \Phi_N$ for some $N \subseteq \mathcal{P}(\omega)$.

<u>Proof</u> : Let $D_i = \{\eta \subseteq \omega : i \in \eta\}$. Put $N = \Phi(\{D_i : i \in \omega\})$. It is easy to check that $\Phi = \Phi_N$.

The set N obtained above is called the <u>canonical base</u> for Φ.

1.3 <u>Definition</u> : A base $N \subseteq \mathcal{P}(\omega)$ is complete if

$$\eta \in N \& \eta \subseteq \eta' \subseteq \omega \rightarrow \eta' \in N.$$

If $\tilde{N} = \Phi_N(\{D_i : i \in \omega\})$, where D_i is as above, then $\tilde{N}$ is complete (called the completion of N) and $\Phi_N = \Phi_{\tilde{N}}$. The canonical base for a p.a.o. is therefore complete.

1.4 <u>Definition</u> : For any operation Φ, the dual Φ^0 is defined by :

$$\Phi^0(\{E_n : n \in \omega\}) = [\Phi(\{E_n^c : n \in \omega\})]^c.$$

For example $\cup^0 = \cap$, $(\text{limsup})^0 = \text{liminf}$, $\mathcal{A}^0 = \mathcal{S}$ ($\mathcal{A}$ vo the Suslin operation),

where $\mathcal{S}(\{E_n : n \in \omega\}) = \{x : (\forall \alpha)(\exists n)(x \in E_{\bar{\alpha}(n)})\}$.

If Φ_N is a δ-s operation with base N, then the canonical base N^o of its dual is given by

$$N^o = \{\eta \in 2^\omega : \eta \cap \eta' \neq \varphi \text{ for every } \eta' \in N\} \qquad \ldots(1)$$

$$= \{\eta \in 2^\omega : \eta^c \notin N\}, \text{ if N is complete}$$

Plainly, N^o is always complete. Thus, for any family $\{E_n : n \in \omega\}$,

$$(\exists \eta \in N^o)(\forall n \in \eta)[x \in E_n] \leftrightarrow (\forall \eta \in N)(\exists n \in \eta)[x \in E_n]. \qquad \ldots(2)$$

If N is complete, $N^{oo} = N$ and hence

$$(\forall \eta \in N^o)(\exists n \in \eta)[x \in E_n] \leftrightarrow (\exists \eta \in N)(\forall n \in \eta)[x \in E_n]. \qquad \ldots(3)$$

2. <u>The $\mathbb{R}$-operator</u> : Although the $\mathbb{R}$-operator was first introduced by Kolmogorov, the first published account of the theory appeared in [23] and further results obtained in [33, 34]. Lyapunov also studied the hierarchies of R-sets (cf. § 3) and obtained most of their properties. The interconnection between the $\mathbb{R}$-operator and games was first noticed by Hinman [20] (and also independently by Aczel [1]). Hinman also developed the effective theory and did most of the groundwork. Much of the material in this section is adapted from these sources.

2.1 <u>Definition</u> : Let $\mathcal{N} = \{N_p \subseteq 2^\omega : p \in \omega\}$ be a sequence of nonempty bases. $\theta \subseteq \omega$ is called an $\mathcal{N}$-chain if for any s, t $\in$ Seq,

(a) $e \in \theta$,

(b) $s \in \theta$ and $t \subseteq s \rightarrow t \in \theta$,

(c) $s \in \theta \rightarrow \{n : s^\frown\langle n\rangle \in \theta\} \in N_s$.

Put $\textcircled{H} = \{\theta : \theta \text{ is an } \mathcal{N}\text{-chain}\}$.

$\mathcal{R}_{\mathcal{N}}$ is the set operation defined by

$$\mathcal{R}_{\mathcal{N}}(\{E_n : n \in \omega\}) = \bigcup_{\theta \in Ⓗ} \bigcap_{s \in \theta} E_s .$$

Clearly, $\mathcal{R}_{\mathcal{N}}$ is a $\delta-s$ operation with base Ⓗ .

If $N_p = N$ for each p, then $\mathcal{R}_{\mathcal{N}}$ is denoted by $\mathbb{R}\Phi_N$ and its base by $\mathbb{R}N$. An $\mathcal{N}$-chain will then be called an N-chain.

<u>Example</u> : Let $N = \{ \{n\} : n \in \omega\}$ so that $\Phi_N = U$ and put $\mathcal{N} = \{N\}$. Clearly, an $\mathcal{N}$-chain contains a set of the form $\{ \bar{\alpha}(n) : n \in \omega\}$. Thus $\mathbb{R}U = \mathbb{R}\Phi_N = \mathcal{A}$.

2.2 <u>Definition</u> : Let Φ_N and Φ_M be two $\delta-s$ operations with bases $N, M \subseteq 2^{\omega}$. The composed operation Ψ is given by

$$\Psi(\{F_n : n \in \omega\}) = \Phi_N(\{\Phi_M(\{F_{<p,n>} : n \in \omega\}) : p \in \omega\}).$$

Ψ is sometimes denoted by $\Phi_N \Phi_M$. By the characterization lemma, Ψ is a $\delta-s$ operation whose canonical base we shall denote by NM. Thus $\Psi = \Phi_N \Phi_M = \Phi_{NM}$ and

$$\eta \in NM \leftrightarrow (\exists \eta_1 \in N)(\forall n_1 \in \eta_1)(\exists \xi_1 \in M)(\forall m_1 \in \xi_1) [<n_1, m_1> \in \eta].$$

Henceforth, for convenience, we shall work in ω^{ω} (or, $(\omega^{\omega})^k \times \omega^{\ell}$), the space of reals, since this can be identified with the irrationals in $(0,1)$. Henceforth we shall denote such spaces by $\mathcal{X}, \mathcal{Y}$ unless otherwise stated.

2.3 <u>Definition</u> . For any operation Φ, let Σ^{Φ}_1 be the class of relations of the form $\Phi(\{F_n : n \in \omega\})$ with all F_n clopen, Π^{Φ}_1 the class of complements of such relations and $\underset{\sim}{\Delta}_1^{\Phi} = \underset{\sim}{\Sigma}_1^{\Phi} \cap \underset{\sim}{\Pi}_1^{\Phi}$. Thus $\underset{\sim}{\Sigma}_1^{U} = \underset{\sim}{\Sigma}_1^{0}$ and $\underset{\sim}{\Sigma}_1^{\mathcal{A}} = \underset{\sim}{\Sigma}_1^{1}$, the class of analytic sets. If $\Phi = \Phi_N$, then put $\Phi^* = \mathbb{R}\Phi_{NN^0}$.

By taking F_n to be recursive uniformly in n one obtains the corresponding lightface classes Σ_1^{Φ}, Π_1^{Φ} etc.

The next two lemmas show a close connection between $\mathbb{R}$-operators, inductive definability and games.

2.4 Lemma : (a) Suppose $F = \mathbb{R}\Phi_N(\{F_n : n \in \omega\})$. Then

$$x \in F \leftrightarrow (\exists \eta_0 \in N)(\forall n_0 \in \eta_0)(\exists \eta_1 \in N)(\forall n_1 \in \eta_1)\ldots \ldots(\forall k)\,[x \in F_{<n_0,n_1,\ldots,n_{k-1}>}]. \qquad (*)$$

(b) If $E = \Phi_N^*(E_n : n \in \omega)$, then

$$x \in E \leftrightarrow (\exists \eta_0 \in N)(\forall n_0 \in \eta_0)(\forall \xi_0 \in N)(\exists m_0 \in \xi_0)(\exists \eta_1 \in N)(\forall n_1 \in \eta_1)(\forall \xi_1 \in N)(\exists m_1 \in \xi_1)\ldots(\forall k)\,[x \in E_{<<n_0,m_0>,\ldots,<n_{k-1},m_{k-1}>>}].$$

(The right-hand side of each equivalence is interpreted in terms of games between two players $\forall$ and $\exists$. For instance, the right-hand expression of (*) is true just in case $\exists$ has a winning strategy in the game $\mathcal{G}_x$ played as follows : $\exists$ (player I) chooses $\eta_0 \in N$, then $\forall$ (player II) chooses $n_0 \in \eta_0$, then $\exists$ chooses $\eta_1 \in N$, etc. Thus a strategy for $\exists$ is a function from Seq into $\mathcal{P}(\omega)$, while a strategy for $\forall$ is a function from $\bigcup_k (2^{\omega})^k \to \omega$. If $\eta_0, n_0, \eta_1, \ldots$ is the sequence produced when both players have played as described above, then $\exists$ wins iff $(\forall k)\,[x \in F_{<n_0,n_1,\ldots,n_{k-1}>}]$.

<u>Proof</u> : Clearly (b) follows from (a) and the fact that

$$\eta \in NN^{o} \leftrightarrow (\exists \eta_1 \in N)(\forall n_1 \in \eta_1)(\forall \xi_1 \in N)(\exists m_1 \in \xi_1)[\langle n_1 m_1\rangle \in \eta].$$

To prove the first assertion, fix x and suppose $x \in F$. Get an N-chain $\theta \in \mathbb{R}N$ such that $(\forall s \in \theta)[x \in F_s]$. Now, consider the following strategy for $\exists$. As his first move $\exists$ plays $\eta_o = \{n : \langle n\rangle \in \theta\}$ which is clearly in N. Any response $n_o \in \eta_o$ by $\forall$ gives a set $\eta_1 = \{n : \langle n_o, n\rangle \in \theta\} \in N$ and $\exists$ should play η_1 as his next move. If $\forall$ then plays $n_1 \in \eta_1$, we still have $\langle n_o, n_1\rangle \in \theta$ and $\exists$ responds with $\eta_2 = \{n : \langle n_o, n_1, n\rangle \in \theta\}$. If $\exists$ follows this strategy, then clearly for any k, $\langle n_o, n_1, \ldots, n_{k-1}\rangle \in \theta$ and so $x \in F_{\langle n_o, \ldots, n_{k-1}\rangle}$. Hence it is a winning strategy for $\exists$ in the game (*).

For the converse implication suppose σ is a winning strategy for $\exists$. Let θ be the set of sequences $\langle n_o, \ldots, n_{k-1}\rangle$ of first k possible moves of player $\forall$ when $\exists$ follows this strategy σ. Clearly θ is an N-chain and since σ is a winning strategy for $\exists$,

$$(\forall k)[\langle n_o, \ldots, n_{k-1}\rangle \in \theta \rightarrow x \in F_{\langle n_o, n_1, \ldots, n_{k-1}\rangle}].$$

Hence $x \in \mathbb{R}\Phi_N(\{F_n : n \in \omega\}) = F$.

<u>Remark</u> : It follows from Lemma 2.4 that our definition of Φ^* is equivalent (cf. Definition 2.16) to that introduced in [22,V.4].

A set relation $\Gamma(w,x,A)$, where w varies over W and A varies over subset of W, is said to be monotone if

$$\Gamma(w,x,A) \; \& \; A \subseteq B \rightarrow \Gamma(w,x,B).$$

Such a monotone set relation induces for each x a monotone set operator given by

$$\Gamma_x(A) = \{w \in W : \Gamma(w,x,A)\}.$$

Γ_x^{μ} denotes the μ-th iterate viz.,

$$\Gamma_x^{\mu} = \Gamma_x(\bigcup_{\lambda<\mu} \Gamma_x^{\lambda}).$$

In particular, $\Gamma_x^{0} = \Gamma_x(\varphi)$. We define

$$\Gamma_x^{<\mu} = \bigcup_{\lambda<\mu} \Gamma_x^{\lambda}$$

and put $\Gamma_x^{\infty} = \bigcup_{\mu} \Gamma_x^{\mu}$.

Clearly Γ_x^{∞} is the least fixed point of the operator Γ_x. We shall use elementary facts on inductive definability as found in [38].

The following result which shows how the operator $\mathbb{R}$ gives rise to inductive operators is due to Hinman [22].

2.5 <u>Theorem</u> : For any p.a.o. Φ and any $E \subseteq \mathcal{X}$ in $\prod_1^{\Phi^*}$, there exists a monotone set relation $\Gamma(s,x,A)$ operative on ω such that for all x

$$x \in E \leftrightarrow e \in \Gamma_x^{\infty}.$$

<u>Proof</u> : Let $\{E_s : s \in \omega\}$ be a family of clopen subsets of $\mathcal{X}$ such that $E^c = \Phi^*(\{E_s : s \in \omega\})$. Without loss of generality take $E_o = \varphi$. Let $N \subseteq 2^{\omega}$ be the canonical base for Φ. Then $\Phi^* = \mathbb{R}\Phi_{NN^c}$. Define a set relation operative on ω as follows :

$$s \in \Gamma_x(A) \leftrightarrow x \notin E_s \vee (\forall \eta \in N)(\exists n \in \eta)(\exists \xi \in N)(\forall m \in \xi)[s^\frown\langle\langle n,m\rangle\rangle \in A] \quad \ldots(4)$$

By convention we take $s * t = 0$ if one of s,t is not in Seq. Clearly, $\Gamma(s,x,A)$ is a monotone set relation. Put $E^s = \Phi^*(\{E_{s*t} : t \in \omega\})$. One can easily see that $E^c = E^s, E^s \subsetneq E_s$. We claim that for all s,

$$x \notin E^s \leftrightarrow s \in \Gamma_x^\infty \quad \ldots(5)$$

and the result follows by putting $s = e$. We shall prove the implication ($\leftarrow$) by induction. If $s \in \Gamma_x^0$, then $x \notin E_s \supsetneq E^s$. Now suppose $s \in \Gamma_x^\mu$, $\mu > 0$. Then

$$x \notin E_s \vee (\forall \eta \in N)(\exists n \in \eta)(\exists \xi \in N)(\forall m \in \xi)[s * \langle\langle n,m\rangle\rangle \in \Gamma_x^{<\mu}].$$

If $s \notin$ Seq or $x \notin E_s$, we are done; otherwise by induction hypothesis

$$(\forall \eta \in N)(\exists n \in \eta)(\exists \xi \in N)(\forall m \in \xi)[x \notin E^{s*\langle\langle n,m\rangle\rangle}]$$

which by Lemma 2.4 (and determinacy of closed games) implies

$$(\forall \eta \in N)(\exists n \in \eta)(\exists \xi \in N)(\forall m \in \xi)\{(\forall \eta_1 \in N)(\exists n_1 \in \eta_1)$$

$$(\exists \xi_1 \in N)(\forall m_1 \in \xi_1)\ldots(\exists k)[x \notin E_{s*\langle\langle n,m\rangle\rangle * \langle\langle n_1,m_1\rangle \ldots \langle n_k,m_k\rangle\rangle}]\}.$$

This clearly implies

$$(\forall \eta_0 \in N)(\exists n_0 \in \eta_0)(\exists \xi_0 \in N)(\forall m_0 \in \xi_0)(\forall \eta_1 \in N)(\exists n_1 \in \eta_1)$$

$$(\exists \xi_1 \in N)(\forall m_1 \in \xi_1)\ldots(\exists k)[x \notin E_{s*\langle\langle n_0,m_0\rangle,\ldots,\langle n_{k-1},m_{k-1}\rangle\rangle}]$$

and thus $x \notin R\Phi_{NN^0}(\{E_{s*t} : t \in \omega\}) = E^s$, by Lemma 2.4 again.

Conversely, let $s \notin \Gamma_x^{\infty}$. We shall show that $x \in E^s$ i.e.,

$$(\exists \eta_0 \in N)(\forall n_0 \in \eta_0)(\forall \xi_0 \in N)(\exists m_0 \in \xi_0)(\exists \eta_1 \in N)(\forall n_1 \in \eta_1) \qquad (**)$$

$$\ldots\ldots(\forall k)\left[x \in E_{s * \langle\langle n_0, m_0\rangle, \ldots, \langle n_{k-1}, m_{k-1}\rangle\rangle}\right].$$

Since $s \notin \Gamma_x(\Gamma_x^{\infty})$, by definition of Γ, $s \in Seq$, $x \in E_s$ and moreover,

$$(\exists \eta_0 \in N)(\forall n_0 \in \eta_0)(\forall \xi_0 \in N)(\exists m_0 \in \xi_0)\left[s * \langle\langle n_0, m_0\rangle\rangle \notin \Gamma_x^{\infty}\right].$$

Now, $\exists$ can win the game (**) by adopting the following strategy. He picks $\eta_0 \in N$ such that for any choice of $n_0 \in \eta_0$ and $\xi_0 \in N$, there is an $m_0 \in \xi_0$ such that $s * \langle\langle n_0, m_0\rangle\rangle \notin \Gamma_x^{\infty} = \Gamma_x(\Gamma_x^{\infty})$. $\exists$ plays such an m_0 when n_0, ξ_0 are chosen by $\forall$. Thus $x \in E_{s * \langle\langle n_0, m_0\rangle\rangle}$ and

$$(\exists \eta_1 \in N)(\forall n_1 \in \eta_1)(\forall \xi_1 \in N)(\exists m_1 \in \xi_1)\left[s * \langle\langle n_0, m_0\rangle, \langle n_1, m_1\rangle\rangle \notin \Gamma_x^{\infty}\right].$$

$\exists$ then picks $\eta_1 \in N$ such that for any choice of $n_1 \in \eta_1$ and $\xi_1 \in N$ made by $\forall$, there is an $m_1 \in \xi_1$ (and $\exists$ plays such an m_1) such that $s * \langle\langle n_0, m_0\rangle, \langle n_1, m_1\rangle\rangle \notin \Gamma_x^{\infty}$. Proceeding this way, $\exists$ has a strategy which ensures that for all k, $x \in E_{s * \langle\langle n_0, m_0\rangle, \ldots, \langle n_{k-1}, m_{k-1}\rangle\rangle}$ and so $\exists$ wins the game (**). Consequently, $x \in E^s$.

<u>Remark</u> : If E is such that $x \notin E \leftrightarrow x \in \mathbb{R}\Phi_N(\{E_s : s \in \omega\})$, then the inductive operator takes a simpler form, viz.,

$$s \in \Gamma_x(A) \leftrightarrow x \notin E_s \vee (\forall \eta \in N)(\exists n \in \eta)\left[s * \langle n\rangle \in A\right]. \qquad \ldots(6)$$

More generally, if $\mathcal{N} = \{N_p : p \in \omega\}$ is a sequence of bases and

$$E^0 = \mathcal{R}_{\mathcal{N}}(\{E_s : s \in \omega\}),$$

then we take the following inductive operator

$$s \in \Gamma_x(A) \leftrightarrow x \notin E_s \vee (\forall \eta \in N_s)(\exists n \in \eta)[s * \langle n \rangle \in A], \quad \ldots(7)$$

and the conclusion of the above theorem still holds.

The inductive operator (4) (or (6) or (7)) is called the canonical inductive operator associated with $\{E_s : s \in \omega\}$ and N (or $\mathcal{N}$).

The above characterization theorem yields a decomposition of sets in $\prod_1^{\Phi^*}$ ($\Sigma_1^{\Phi^*}$) into simpler sets as is evident from the next theorem.

2.6 <u>Definition</u> : For any operation Φ, $\nabla(\Phi)$ is the smallest class of relations containing clopen relations and closed under Φ and Φ^d.

Thus $\nabla(\cup) = \nabla(\cap) = \Delta_1^1$.

Let $E \in \Sigma_1^{\Phi^*}$ and suppose $E = \Phi^*(\{E_s : s \in \omega\})$. Let N and Γ be as in Theorem 2.5. Then by 2.5, $E(x) \leftrightarrow s \notin \Gamma_x^\infty$. Set

$$E_s^\mu = \{x : s \notin \Gamma_x^\mu\}.$$

Then $E = \bigcap_{\mu < \omega_1} E_s^\mu$. Now define

$$T^\mu = \bigcup_{s \in \omega} (E_s^\mu - E_s^{\mu+1}).$$

It is easy to prove by induction on μ that for all $\mu < \omega_1$, $s \in \omega$, E_s^μ and T^μ are in $\nabla(\Phi)$ if it is closed under countable union. It is not hard to check that $E_s^\mu \downarrow$ while $(E_s^\mu - T^\mu)\uparrow$.

2.7 <u>Theorem</u> : $E = \bigcup_{\mu<\omega_1} (E_s^\mu - T^\mu) = \bigcap_{\mu<\omega_1} E_s^\mu$.

<u>Proof</u> : Let $x \in E$. Define

$$\beta(s) = \begin{cases} \text{least ordinal } \rho \text{ such that } x \notin E_s^\rho & \text{if } (\exists \rho)(x \notin E_s^\rho) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\omega_1 > \rho_0 > \beta(s)$, $\forall s$. Then $(\forall s)\,[x \in E_s^{\rho_0} \leftrightarrow x \in E_s^{\rho_0+1}]$. Thus, $x \notin T^{\rho_0}$ and thus $x \in E_e^{\rho_0} - T^{\rho_0}$.

Conversely, suppose for some $\rho_0 < \omega_1$, $x \in E_e^{\rho_0} - T^{\rho_0}$. Then

$$(\forall s)\,[x \in E_s^{\rho_0} \leftrightarrow x \in E_s^{\rho_0+1}].$$

One can check by induction that

$$(\forall \rho > \rho_0)(\forall s)\,[x \in E_s^{\rho_0} \leftrightarrow x \in E_s^{\rho}].$$

So in particular, $x \in \bigcap_{\rho > \rho_0} E_e^{\rho} = \bigcap_{\rho < \omega_1} E_e^{\rho} = E$.

Standard arguments using the above decomposition and the countable chain condition yield the following (cf. [22]).

2.8 <u>Theorem</u> : If Φ preserves measurability (preserves the Baire property), then so does Φ^*.

<u>Remark</u> (1) : More generally, if $\mathcal{N} = \{N_p : p \in \omega\}$ is a family of bases such that each Φ_{N_p} is measurability (Baire property) preserving, then a similar argument shows that the operation $\mathcal{R}_{\mathcal{N}}$ is also measurability (Baire property) preserving.

<u>Remark</u> (2) : The above proof uses the countable chain condition (ccc) of the Boolean algebra of sets with Baire property (measurability property) modulo meager (null) sets. Below we give a different proof which avoids the use of ccc and is an adaptation of the classical proof of Marczewski (see [28, § 11.VII]). Our result generalizes Marczewski's result to the R-operations.

2.8 (bis) <u>Theorem</u> : Let $\mathcal{B}$ be a σ-algebra of subsets of an arbitrary set X such that for every $P \subseteq X$ there is some $\overleftarrow{P} \supseteq P$ in $\mathcal{B}$ with the property that if $P \subseteq B \in \mathcal{B}$ and $A \subseteq \overleftarrow{P} - B$ then $A \in \mathcal{B}$. Then if $\mathcal{B}$ is closed under Φ then $\mathcal{B}$ is also closed under Φ^*.

<u>Proof</u> : Let N be the canonical base for the p.a.o. Φ. Let $\{E_n : n \in \omega\}$ be a family of sets in $\mathcal{B}$, and let $E = \Phi^*(\{E_n : n \in \omega\})$. Put, for each s,

$$E^s = \Phi^*(\{E_{s*t} : t \in \omega\}),$$

so that $E^\emptyset = E$ and $E^s \subseteq E_s$. For each s, find $\overleftarrow{E}^s$ in $\mathcal{B}$ such that $E^s \subseteq \overleftarrow{E}^s$ and

$$E^s \subseteq B \in \mathcal{B} \;\&\; A \subseteq \overleftarrow{E}^s - B \longrightarrow A \in \mathcal{B}.$$

We may, without loss of generality, assume that

$$\overleftarrow{E}^s \subseteq E_s.$$

We shall show that

$$\overleftarrow{E}^\emptyset - E \subseteq \bigcup_s \left[\overleftarrow{E}^s - \Phi_{NN^0}(\{\overleftarrow{E}^{s*\langle n\rangle} : n \in \omega\})\right] \qquad (*)$$

Suppose not. Then there is an $x \in \overleftarrow{E}^\emptyset - E$ such that

$$(\forall s)\left[x \in \overleftarrow{E}^s \longrightarrow x \in \Phi_{NN^0}(\{\overleftarrow{E}^{s*\langle n\rangle} : n \in \omega\})\right].$$

We claim that this implies

$$(\exists \eta_0 \in N)(\forall n_0 \in \eta_0)(\forall \xi_0 \in N)(\exists m_0 \in \xi_0) \ldots\ldots$$
$$\ldots\ldots(\forall k)\left[x \in \overleftarrow{E}^{\langle\langle n_0, m_0\rangle, \ldots, \langle n_{k-1}, m_{k-1}\rangle\rangle}\right] \qquad (**)$$

To see this first observe that $x \in \Phi_{NN^0}(\{\overset{\smile}{E}^{<n>} : n \in \omega\})$.

Hence $(\exists \eta_0 \in N)(\forall n_0 \in \eta_0)(\forall \xi_0 \in N)(\exists m_0 \in \xi_0)[x \in \overset{\smile}{E}^{<<n_0,m_0>>}]$.

Thus in the game (**) $\exists$ plays according to the following strategy. He plays $\eta_0 \in N$ such that whatever be $n_0 \in \eta_0$ and $\xi_0 \in N$ played by $\forall$ there is an $m_0 \in \xi_0$ such that $x \in \overset{\smile}{E}^{<<n_0,m_0>>}$. Then $\exists$ plays such an m_0. Since

$$x \in \overset{\smile}{E}^{<<n_0,m_0>>},\ x \in \Phi_{NN^0}(\{\overset{\smile}{E}^{<<n_0,m_0>,n>} : n \in \omega\})$$

and so $(\exists \eta_1 \in N)(\forall n_1 \in \eta_1)(\forall \xi_1 \in N)(\exists m_1 \in \xi_1)[x \in \overset{\smile}{E}^{<<n_0,m_0>,<n_1,m_1>>}]$.

Consequently $\exists$ plays as described above. Proceeding in this manner it is clear that $\exists$ wins the game (**). But since $\overset{\smile}{E}^{s} \subseteq E_s$, by Lemma 2.4 and (**) *we have* $x \in E$. *This is a contradiction.* Thus (*) holds. Now for each s,

$$F^s \overset{\text{def}}{=} \overset{\smile}{E}^{s} - \Phi_{NN^0}(\{\overset{\smile}{E}^{s*<n>} : n \in \omega\})$$

$$\subseteq \overset{\smile}{E}^{s} - \Phi_{NN^0}(\{E^{s*<n>} : n \in \omega\}) = \overset{\smile}{E}^{s} - E^{s}.$$

By hypothesis, each F^s is in $\mathcal{B}$ and since $(F^s)^c \supseteq E^s$, therefore, every subset of $F^s \in \mathcal{B}$. Hence, by (*), $\overset{\smile}{E}^{s} - E \in \mathcal{B}$. So,

$$E = \overset{\smile}{E}^{s} - (\overset{\smile}{E}^{s} - E) \in \mathcal{B}.$$

<u>Remark</u> : The family of sets with Baire property in any topological space satisfies the hypothesis of the above theorem ([20; § 11.IV]).

Consequently, in any topological space the family of sets with Baire property is closed under every R-operation (cf. § 3). (The family of universally measurable sets also has this closure property and this follows directly from Theorem 2.8).

2.9 <u>Definition</u> : Call a p.a.o. Φ a $\underset{\sim}{\Delta}^1_r$ operation if $\underset{\sim}{\Delta}^1_r$ is closed under Φ.

2.10 <u>Theorem</u> (Hinman) : For any p.a.o. Φ, and any $r \geq 1$, Φ is a $\underset{\sim}{\Delta}^1_r$ operation iff $B \in \underset{\sim}{\Delta}^1_r$, where B is the canonical base for Φ.

<u>Proof</u> : If Φ is a $\underset{\sim}{\Delta}^1_r$ operation, then clearly $B \in \underset{\sim}{\Delta}^1_r$, since $B = \Phi(\{D_i : i \in \omega\})$, where D_i is as in the Proposition of § 1. Suppose now that $B \in \underset{\sim}{\Delta}^1_r$ and let $\{E_n : n \in \omega\}$ be a family of $\underset{\sim}{\Delta}^1_r$ relations. First notice that the canonical base B^o of Φ^o is in $\underset{\sim}{\Delta}^1_r$ by (1). Then we have

$$x \in \Phi(\{E_n : n \in \omega\}) \leftrightarrow (\exists \eta \in B)(\forall n \in \eta)[x \in E_n]$$

$$\leftrightarrow (\forall \eta \in B^o)(\exists n \in \eta)[x \in E_n].$$

This shows that Φ is a $\underset{\sim}{\Delta}^1_r$ relation.

We shall omit the proof of the next theorem which can be found in [22].

2.11 <u>Theorem</u> : For any p.a.o. Φ, $\underset{\sim}{\Delta}^{1\,\Phi^*}_1$ is closed under both Φ and Φ^o. Hence $\underset{\sim}{\nabla}(\Phi) \subseteq \underset{\sim}{\Delta}^{\Phi^*}_1$. ▯

2.12 <u>Theorem</u> : Let $\mathcal{N} = \{N_p : p \in \omega\}$ be a family of complete bases such that each $N_p \in \underset{\sim}{\Delta}^1_r$, with $r \geq 2$. Then $\mathcal{R}_{\mathcal{N}}$ is a $\underset{\sim}{\Delta}^1_r$ operation.

In particular, if Φ is a $\underset{\sim}{\Delta}^1_r$ operation, then Φ^* is also $\underset{\sim}{\Delta}^1_r$.

Proof : This follows immediately from the fact that if $\{E_n : n \in \omega\}$ is a family of $\underset{\sim}{\Delta}^1_r$ sets, then the canonical inductive operator (7) is both pos $\underset{\sim}{\Pi}^1_r$ and pos $\underset{\sim}{\Sigma}^1_r$. The result then follows from Theorem 2.5 and [38:7C.2].

A standard diagonal argument gives us the following (cf [22 ; V.4.11]).

2.13 Corollary : For all $r \geq 2$ and any $\underset{\sim}{\Delta}^1_r$ p.a.o. Φ,

$$\underset{\sim}{\nabla}(\Phi)(\subseteq \underset{\sim}{\Delta}^{\Phi^*}_1) \subsetneqq \underset{\sim}{\Delta}^1_r \quad \square$$

Remark : An effective analogue of the above corollary can also be deduced (see [22 : V.5]).

A pointclass Γ is said to be closed under $\Phi = \Phi_N$ if for any relation $E \subseteq \omega \times \mathcal{X}$ in Γ the relation

$$(\Phi E)(x) \longleftrightarrow (\exists \eta \in N)(\forall n \in \eta)E(n,x)$$

is also in Γ.

The next result is due to A. Maitra (see also [1]).

2.14 Theorem : Let Γ be a Spector pointclass closed under $\forall^{\omega^\omega}$ or $\exists^{\omega^\omega}$. Let $\mathcal{N} = \{N_p : p \in \omega\}$ be a sequence of complete bases which are (uniformly) in $\neg\Gamma$. Then Γ is closed under the operation $\mathcal{R}^{o}_{\mathcal{N}}$.

Proof : We say, that a set relation $\varphi(w,x,A)$, operative on W, is Γ on Γ if for every $Q \subseteq Z \times W$, the relation

$$P(x,z) \leftrightarrow \varphi(w,z,\{w : Q(z,w)\})$$

is also in Γ.

Now suppose $E(s,x)$ is a relation in Γ and let

$$P = \mathcal{R}_{\mathcal{N}}^{o}(\{E_s : s \in \omega\}).$$

We shall show that $P \in \Gamma$. First observe that if N_p^o is the base for the dual operation, then N_p^o is (uniformly) in Γ. Now consider the following set relation :

$$s \in \varphi_x(A) \leftrightarrow x \in E_s \vee (\forall \eta \in N_s)(\exists n \in \eta)[s\hat{}\langle n\rangle \in A]$$

$$\leftrightarrow x \in E_s \vee (\exists \eta \in N_s^o)(\forall n \in \eta)[s\hat{}\langle n\rangle \in A].$$

By the remark following 2.5, we have

$$P(x) \longleftrightarrow \emptyset \in \varphi_x^{\infty}.$$

Moreover, it is easy to check that in either case φ is Γ on Γ. Hence by [38: 7C.8], $P \in \Gamma$. ▯

The proof of the above also yields the following.

2.15 Corollary (Aczel) : If a Spector pointclass Γ is closed under both Φ and Φ^o, then it is closed under Φ^*. ▯

2.16 Definition : Let Φ and Ψ be two p.a.o.'s. Say that Φ subsumes Ψ (in smybols, $\Phi \geq \Psi$ or $\Psi \leq \Phi$) if there is a recursive function $f : \omega \longrightarrow \omega$ such that for any family $\{F_n : n \in \omega\}$,

$$\Psi(\{F_n : n \in \omega\}) = \Phi(\{F_{f(n)} : n \in \omega\}).$$

Say that Φ and Ψ are equivalent ($\Phi \leftrightarrows \Psi$) if $\Phi \geq \Psi$ and $\Psi \geq \Phi$. For example, $\mathcal{A}$ subsumes both $\bigcap$ and $\bigcup$.

A positive analytical operation Φ is said to be normal if there is recursive a function g such that for any family $\{F_n : n \in \omega\}$,

$$\Phi(\{\Phi(\{F_{<p,q>} : q \varepsilon \omega\}) : p \varepsilon \omega\}) = \Phi(\{F_{g(n)} : n \varepsilon \omega\}).$$

We shall omit the proof of the next lemma which can be found in [20,21].

2.17 Lemma : For any operations Φ and Ψ

(a) $\Phi^{oo} = \Phi$;

(b) $\Phi \geq \Psi \rightarrow \Phi^{o} \geq \Psi^{o}$;

(c) $\Phi\Phi^{o} \geq \Phi, \Phi^{o}$;

(d) $\mathbb{R}\Phi \geq \Phi$;

(e) $\Phi \geq \Psi \rightarrow \mathbb{R}\Phi \geq \mathbb{R}\Psi$;

(f) $\mathbb{R}\Phi \geq \Psi$ and $\mathbb{R}\Phi \geq \Psi^{o} \rightarrow \mathbb{R}\Phi \geq \Psi\Psi^{o}$;

(g) $\Phi\mathbb{R}\Phi \backsim \mathbb{R}\Phi$;

(h) $\mathbb{R}\Phi \backsim \mathbb{R}\mathbb{R}\Phi$;

(i) $\mathbb{R}\Phi$ is normal.

3. The R-sets : We shall first construct a sequence $\{R_\rho : \rho < \omega_1\}$ of positive analytical operations by induction as follows. Put $R_o = \mathcal{A}$ and having defined R_ρ, put

$$R_{\rho+1} = R_\rho^* .$$

If λ is a limit, choose a sequence $\rho_i \uparrow \lambda$ and set, for any family $\{E_n : n \varepsilon \omega\}$,

$$\Phi(\{E_n : n \varepsilon \omega\}) = \bigcap_{i=o}^{\infty} \Phi_{N_{\rho_i} N^o_{\rho_i}}(\{E_{<i,m>} : m \varepsilon \omega\}),$$

where N_{ρ_i} is the canonical base for R_{ρ_i}. Then define

$$R_\lambda = \Phi^*.$$

Note that any other sequence $\rho_i' \uparrow \lambda$ gives rise to an equivalent operation by Lemma 2.17. Also, it is easy to check that $R_\rho \geq R_{\rho'}$ if $\rho \geq \rho'$.

For each $\rho < \omega_1$, let $\underset{\sim}{\mathcal{R}}^\rho = \underset{\sim}{\Sigma}_1^{R_\rho}$ and $B\underset{\sim}{\mathcal{R}}^\rho = \underset{\sim}{\Delta}_1^{R_\rho}$. Let $\underset{\sim}{\mathcal{BR}}^\rho$ be the least class containing clopen relations and closed under R_ρ and complementation. Thus, for instance, $\underset{\sim}{\mathcal{R}}^0 = \underset{\sim}{\Sigma}_1^1$, $B\underset{\sim}{\mathcal{R}}^0 = \underset{\sim}{\Delta}_1^1$ and $\underset{\sim}{\mathcal{BR}}^0 = \mathcal{C}$-sets of Selivanovskii. Finally, set

$$\underset{\sim}{\mathcal{R}} = \bigcup_{\rho<\omega_1} \underset{\sim}{\mathcal{R}}^\rho.$$

Members of $\underset{\sim}{\mathcal{R}}$ are known classically as the R-sets. It is not difficult to see that for each ρ,

$$\underset{\sim}{\mathcal{R}}^\rho \subseteq \underset{\sim}{\mathcal{BR}}^\rho \subseteq B\underset{\sim}{\mathcal{R}}^{\rho+1} \subseteq \underset{\sim}{\mathcal{R}}^{\rho+1}.$$

In fact, the inclusions can be shown to be strict (cf. [34]). For instance, to show that the middle inclusion is strict one observes that $\underset{\sim}{\mathcal{BR}}^\rho$ consists precisely of those sets in $\underset{\sim}{\mathcal{R}}^{\rho+1}$ with <u>bounded constituents</u>. That the other two inclusions are also strict can be proved by the standard universal set argument.

The next result follows immediately from 2.12, 2.13, 2.8 and the following remarks and 2.17.

3.1 <u>Theorem</u> : (a) For each $\rho < \omega_1$, $\underset{\sim}{\mathcal{R}}^\rho \subsetneq \underset{\sim}{\Delta}_2^1$.

(b) R-sets are absolutely measurable and have the Baire property .

(c) For any arbitrary (T_1) topological space X, the family of sets with Baire property, $\mathcal{B}$, is closed under R_ρ, for each $\rho < \omega_1$.

(d) For every $\rho < \omega_1$, R_ρ is normal and $\mathbb{R} R_\rho \hookrightarrow R_\rho$. ||

Remark : Theorem 3.1 (b) also follows from the fact that R-sets are provably Δ^1_2, while 3.1 (c) follows from a recent result of Schilling [42] which states that absolutely Δ^1_2 operations preserve the Baire property in all topological spaces and one can show that each R_ρ is an absolute Δ^1_2-operation.

For any class of sets $\mathcal{F}$ and any operation Φ, let $\Phi[\mathcal{F}]$ denote the class of sets which result on applications of Φ to families of sets from $\mathcal{F}$; $c\mathcal{F}$ denotes the class of complements of sets in $\mathcal{F}$.

Just as in the case of the Borel class we decompose, for each ρ, the class $\mathcal{BR}^\rho$ into a hierarchy as follows. Let $R_\rho = \Phi^*$. We set $\mathcal{R}^\rho_0 = \mathcal{R}^\rho$ and define, for $\mu < \omega_1$

$$\mathcal{R}^\rho_\mu = R_\rho\left[c\left(\bigcup_{\lambda<\mu} \mathcal{R}^\rho_\lambda\right)\right],$$

$$B\mathcal{R}^\rho_\mu = \{E \mid E, E^c \in \mathcal{R}^\rho_\mu\},$$

and $\mathcal{BR}^\rho_\mu$ as the smallest class containing $\mathcal{R}^\rho_\mu$ and closed under $\cup$ and complementation. It is immediate that these classes are included in $\mathcal{BR}^\rho$ and indeed that $\mathcal{BR}^\rho = \bigcup_{\mu<\omega_1} \mathcal{R}^\rho_\mu$. As above we have, for each $\rho, \mu < \omega_1$,

$$\mathcal{R}^\rho_\mu \subsetneq \mathcal{BR}^\rho_\mu \subsetneq B\mathcal{R}^\rho_{\mu+1} \subset \mathcal{R}^\rho_{\mu+1}.$$

We shall now show that each class $\mathfrak{R}^{\rho} = \Pi_1^{R_\rho}$ has the prewellordering property. The key to this is the following lemma.

If $E = \mathcal{R}_{\mathcal{N}}(\{E_n : n \varepsilon \omega\})$, where $\mathcal{N} = \{N_p : p \varepsilon \omega\}$ and Γ the canonical inductive operator associated with $\{E_n\}$ and $\mathcal{N}$, then we have

$$x \notin E \longleftrightarrow s \varepsilon \Gamma_x^{\infty}.$$

Put

$$|(s,x)|_{\Gamma} = \text{least } \rho \text{ such that } s \varepsilon \Gamma_x^{\rho}, \text{ if such } \rho \text{ exists.}$$
$$= \omega_1, \quad \text{otherwise.}$$

3.2 <u>Lemma</u> (Comparison of Indices) : Let $\mathcal{E} = \{E_n : n \varepsilon \omega\}$ and $\mathcal{F} = \{F_n : n \varepsilon \omega\}$ be two families of subsets of $\mathcal{X}$ and further assume that $\mathcal{F}$ is regular, i.e., $F_t \subseteq F_s$ if $s \subseteq t$. Let $\mathcal{N} = \{N_p : p \varepsilon \omega\}$, $\mathcal{M} = \{M_p : p \varepsilon \omega\}$ be two sequences of bases. Define a sequence of bases $\{K_s : s \varepsilon \omega\}$ as follows. If $s = \langle\langle n_0, m_0\rangle, \ldots, \langle n_k, m_{k-1}\rangle\rangle$, then K_s is the canonical base for the positive analytical operation Φ defined by

$$\Phi(\{G_n : n \varepsilon \omega\}) = \Phi_{M^0_{\langle m_0,\ldots,m_{k-1}\rangle}}(\{\Phi_{N_{\langle n_0,\ldots,n_{k-1}\rangle}}(\{G_{\langle n,m\rangle} : n \varepsilon \omega\}) : m \varepsilon \omega\}).$$

Otherwise, $K_s = \{\omega\}$. Let $\mathcal{K} = \{K_s : s \varepsilon \omega\}$,

$$H_s = \begin{cases} E_{\langle n_0,\ldots,n_{k-1}\rangle} \cup F^c_{\langle m_0,\ldots,m_{k-1}\rangle} ; & \text{if } s = \langle\langle n_0, m_0\rangle, \ldots, \langle n_{k-1}, m_{k-1}\rangle\rangle \\ \mathcal{X}, & \text{otherwise.} \end{cases}$$

Suppose Γ is the canonical inductive operator associated with $\mathcal{E}$ and $\mathcal{N}$; Δ the inductive operator associated with $\mathcal{F}$ and $\mathcal{M}$. Then,

$$\{x : |(e,x)|_{\Gamma} < |(e,x)|_{\Delta}\}^{c} = \mathcal{R}_{\mathcal{K}}(\{H_s : s \in \omega\}).$$

Proof : First note that

$$\eta \in K_{<<n_0,m_0>,\ldots,<n_{i-1},m_{i-1}>>}$$

$$\leftrightarrow (\exists \eta' \in M^{o}_{<m_0,\ldots,m_{i-1}>})(\forall m \in \eta')(\exists \eta'' \in N_{<n_0,\ldots,n_{i-1}>})$$

$$(\forall n \in \eta'')[<n,m> \in \eta].$$

The operators Γ and Δ are as follows.

$$s \in \Gamma_x(A) \leftrightarrow x \notin E_s \vee (\forall \eta \in N_s)(\exists n \in \eta)[s * <n> \in A]$$

$$s \in \Delta_x(A) \leftrightarrow x \notin F_s \vee (\forall \eta \in M_s)(\exists n \in \eta)[s * <n> \in A].$$

To obtain the result look at the canonical inductive operator associated with $\mathcal{K}$ and $\mathcal{H} = \{H_s : s \in \omega\}$;

$$s \in \Lambda_x(A) \leftrightarrow x \notin H_s \vee (\forall \eta \in K_s)(\exists n \in \eta)[s * <n> \in A].$$

We claim that for all $x \in \mathcal{X}$ and $n_0, m_0, \ldots, n_{i-1}, m_{i-1}$

$$|(<n_0,\ldots,n_{i-1}>, x)|_{\Gamma} < |(<m_0,\ldots,m_{i-1}>, x)|_{\Delta} \qquad \ldots(8)$$

$$\leftrightarrow <<n_0,m_0>,\ldots,<n_{i-1},m_{i-1}>> \in \Lambda^{\infty}_x$$

and the result follows by taking $i = 0$.

We shall prove the implication ($\rightarrow$) by induction on $|(<n_0,\ldots,n_{i-1}>, x)|_{\Gamma}$. Suppose

$$\rho = |(<n_0,\ldots,n_{i-1}>, x)|_{\Gamma} < |(<m_0,\ldots,m_{i-1}>, x)|_{\Delta}$$

and assume, to the contrary, $<<n_0,m_0>,\ldots,<n_{i-1},m_{i-1}>> \notin \Lambda^{\infty}_x$. Then

$$x \in (E_{<n_o,\ldots,n_{i-1}>} \cup F^c_{<m_o,\ldots,m_{i-1}>}) \ \& \ (\exists \eta \in K_{<<n_o,m_o>,\ldots,<n_{i-1},m_{i-1}>>}) \quad \ldots(i)$$

$$(\forall s \in \eta)[<<n_o,m_o>,\ldots,<n_{i-1},m_{i-1}>> * <s> \notin \Lambda^{\infty}_x].$$

Now, $<m_o,\ldots,m_{i-1}> \notin \Delta^{\rho}_x$ and so

$$x \in F_{<m_o,\ldots,m_{i-1}>} \ \& \ (\exists \eta' \in M_{<m_o,\ldots,m_{i-1}>})(\forall m \in \eta')[<m_o,\ldots,m_{i-1},m> \notin \Delta^{<\rho}_s].$$

This implies

$$x \in F_{<m_o,\ldots,m_{i-1}>} \ \& \quad \ldots(ii)$$

$$(\exists \eta' \in M_{<m_o,\ldots,m_{i-1}>})(\forall m \in \eta')[\rho \leq |(<m_o,\ldots,m_{i-1},m>, x)|_{\Delta}].$$

Clearly from (i) and (ii), $x \in E_{<n_o,\ldots,n_{i-1}>}$ and since $<n_o,\ldots,n_{i-1}> \in \Gamma^{\rho}_x$,

$$(\forall \eta'' \in N_{<n_o,\ldots,n_{i-1}>})(\exists n \in \eta'')[<n_o,\ldots,n_{i-1}, n> \in \Gamma^{<\rho}_x]$$

which gives

$$(\forall \eta'' \in N_{<n_o,\ldots,n_{i-1}>})(\exists n \in \eta'')[|(<n_o,\ldots,n_{i-1},n>, x)|_{\Gamma} < \quad \ldots(iii)$$

$$\rho = |(<n_o,\ldots,n_{i-1}>, x)|_{\Gamma}].$$

Fix $\eta \in K_{<<n_o,m_o>,\ldots,<n_{i-1},m_{i-1}>>}$ to satisfy (i). Get $\eta' \in M^{o}_{<m_o,\ldots,m_{i-1}>}$ such that

$$(\forall m \in \eta')(\exists \eta'' \in N_{<n_o,\ldots,n_{i-1}>})(\forall n \in \eta'')[<n,m> \in \eta]. \quad \ldots(iv)$$

Clearly, (ii) and (1) of § 1 yield $m^* \in \eta'$ such that

$$\rho \leq |(<m_o,\ldots,m_{i-1},m^*>, x)|_{\Delta}.$$

By (iv) corresponding to m^* get $\eta'' \varepsilon N_{<n_o,\dots,n_{i-1}>}$ such that $(\forall n \varepsilon \eta'')\,[<n,m^*> \varepsilon \eta]$. By (iii) get $n^* \varepsilon \eta''$ such that $|(<n_o,\dots,n_{i-1},n^*>, x)|_\Gamma < \rho$. Clearly, $<n^*,m^*> \varepsilon \eta$ and

$$|(<n_o,\dots,n_{i-1},n^*>, x)|_\Gamma < \rho \leq |(<m_o,\dots,m_{i-1},m^*>, x)|_\Delta$$

and by the induction hypothesis this implies

$$<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>, <n^*,m^*>> \varepsilon \Lambda_x^\infty .$$

This clearly contradicts our choice of η. Hence

$$<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>> \varepsilon \Lambda_x^\infty .$$

To prove the reverse implication, set

$$A^* = \{<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>> ; \; |(<n_o,\dots,n_{i-1}>, x)|_\Gamma < |(<m_o,\dots,m_{i-1}>, x)|_\Delta\}$$
$$\cup \{t : t \text{ is not of the form } <<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>>\}.$$

We shall show that $\Lambda_x(A^*) \subseteq A^*$, from which it will follow that $\Lambda_x^\infty \subseteq A^*$. So let

$$<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>> \varepsilon \Lambda_x(A^*). \qquad \dots(v)$$

We will have to show that

$$|(<n_o,\dots,n_{i-1}>, x)|_\Gamma < |(<m_o,\dots,m_{i-1}>, x)|_\Delta .$$

Assume to the contrary that

$$|(<m_o,\dots,m_{i-1}>, x)|_\Delta \leq |(<n_o,\dots,n_{i-1}>, x)|_\Gamma .$$

From (v) we have

$$x \varepsilon E^c_{<n_o,\dots,n_{i-1}>} \cap F_{<m_o,\dots,m_{i-1}>} \vee (\forall \eta \varepsilon K_{<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>>})(\exists s \varepsilon \eta)$$

$$[<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>,s> \varepsilon A^*].$$

If $x \varepsilon E^c_{<n_o,\dots,n_{i-1}>} \cap F_{<m_o,\dots,m_{i-1}>}$, then $|(<n_o,\dots,n_{i-1}>, x)|_\Gamma = 0$ and

$$|(<m_o,\dots,m_{i-1}>, x)|_\Delta > 0,$$

and we are done. So assume

$$x \varepsilon (E_{<n_o,\dots,n_{i-1}>} \cup F^c_{<m_o,\dots,m_{i-1}>}) \;\&$$
$$\dots(vi)$$
$$(\forall \eta \varepsilon K_{<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>>})(\exists s \varepsilon \eta)[<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>,s> \varepsilon A^*].$$

If $x \varepsilon F^c_{<m_o,\dots,m_{i-1}>}$, then by regularity $x \varepsilon F^c_{<m_o,\dots,m_{i-1},m>}$ for all m, and hence $(\forall m)[|(<m_o,\dots,m_{i-1},m>, x)|_\Delta = 0]$. But this is not possible by (vi). Therefore,

$$x \varepsilon (E_{<n_o,\dots,n_{i-1}>} \cap F_{<m_o,\dots,m_{i-1}>}) \;\&$$
$$\dots(vii)$$
$$(\forall \eta \varepsilon K_{<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>>})(\exists s \varepsilon \eta)[<<n_o,m_o>,\dots,<n_{i-1},m_{i-1}>,s> \varepsilon A^*].$$

<u>Case 1</u> : $|(<n_o,\dots,n_{i-1}>,x)|_\Gamma = \omega_1$. In this case $<n_o,\dots,n_{i-1}> \not\varepsilon \Gamma^\infty_x$ and hence

$$(\exists \eta'' \varepsilon N_{<n_o,\dots,n_{i-1}>})(\forall n \varepsilon \eta'')[|(<n_o,\dots,n_{i-1},n>,x)|_\Gamma = \omega_1].$$

Fix such an $\eta'' \varepsilon N_{<n_o,\dots,n_{i-1}>}$. Pick any $\eta' \varepsilon M^o_{<m_o,\dots,m_{i-1}>}$ and put

$$\eta^* = \{<n,m> : n \varepsilon \eta'' \;\&\; m \varepsilon \eta'\}.$$

Clearly $\eta^* \in K_{<<n_0,m_0>,\ldots,<n_{i-1},m_{i-1}>>}$ and, moreover,

$$(\forall <n,m> \in \eta^*)[\ |(<m_0,\ldots,m_{i-1},m>,\ x)|_\Delta \leq \omega_1 = |(<n_0,\ldots,n_{i-1},n>,\ x)|_\Gamma].$$

This contradicts (vii).

<u>Case 2</u>: $|(<n_0,\ldots,n_{i-1}>,\ x)|_\Gamma = \rho < \omega_1$. Here we have, by our assumption, $<m_0,\ldots,m_{i-1}> \in \Delta_x^\rho$ and since $x \in F_{<m_0,\ldots,m_{i-1}>}$,

$$(\forall \eta' \in M_{<m_0,\ldots,m_{i-1}>})(\exists m \in \eta')[\ |(<m_0,\ldots,m_{i-1},m>,\ x)|_\Delta < \rho\ ]$$

which implies by (2) that

$$(\exists \eta' \in M^o_{<m_0,\ldots,m_{i-1}>})(\forall m \in \eta')[\ |(<m_0,\ldots,m_{i-1},m>,\ x)|_\Delta < \rho\ ],$$

hence

$$(\exists \eta' \in M^o_{<m_0,\ldots,m_{i-1}>})(\forall m \in \eta')[<n_0,\ldots,n_{i-1}> \notin \Gamma^{|(<m_0,\ldots,m_{i-1},m>,\ x)|_\Delta}].$$

Consequently,

$$(\exists \eta' \in M^o_{<m_0,\ldots,m_{i-1}>})(\forall m \in \eta')(\exists \eta'' \in N_{<n_0,\ldots,n_{i-1}>})(\forall n \in \eta'')$$

$$[\ |(<m_0,\ldots,m_{i-1},m>,\ x)|_\Delta \leq |(<n_0,\ldots,n_{i-1},n>,\ x)|_\Gamma].$$

This clearly contradicts (vii). Thus in either case we have a contradiction and so $\Lambda_x^\infty(A^*) \subseteq A^*$. Thus $\Lambda_x^\infty \subseteq A^*$. This proves the other implication of (8).

The following trick is due to Lyapunov.

3.3 <u>Lemma</u> (Increasing the Index by 1) : Let Γ be the canonical inductive operator associated with $\mathcal{E} = \{E_p : p \in \omega\}$ and $\mathcal{N} = \{N_k : k \in \omega\}$.

Define

$$E^*_e = \mathcal{X},$$

$$E^*_s = \begin{cases} E_{<n_0,\ldots,n_{k-1}>} & \text{if } s = <0,n_0,\ldots,n_{k-1}>, \\ \varphi & \text{otherwise.} \end{cases}$$

Put

$$N^*_e = \{\{0\}\},$$

$$N^*_s = \begin{cases} N_{<n_0,\ldots,n_{k-1}>} & \text{if } s = <0,n_0,\ldots,n_{k-1}>, \\ \{\{n\} : n \in \omega\} & \text{otherwise.} \end{cases}$$

If $\mathcal{N}^* = \{N^*_k : k \in \omega\}$, then

$$\mathcal{R}_{\mathcal{N}^*}(\{E^*_s : s \in \omega\}) = \mathcal{R}_{\mathcal{N}}(\{E_s : s \in \omega\})$$

and $|(e,x)|_{\Gamma^*} = |(e,x)|_{\Gamma} + 1$, where Γ^* is the inductive operator associated with

$$\mathcal{E}^* = \{E^*_s : s \in \omega\} \text{ and } \mathcal{N}^*$$

<u>Proof</u> : We shall prove by induction on ρ that

$$<0,n_0,\ldots,n_{k-1}> \in \Gamma^{*\rho}_x \leftrightarrow <n_0,\ldots,n_{k-1}> \in \Gamma^{\rho}_x .$$

Now,

$$<0,n_0,\ldots,n_{k-1}> \in \Gamma^{*\rho}_x \leftrightarrow x \notin E^*_{<0,n_0,\ldots,n_{k-1}>} \vee$$

$$(\forall \eta \in N^*_{<0,n_0,\ldots,n_{k-1}>})(\exists n \in \eta) [<0,n_0,\ldots,n_{k-1},n> \in \Gamma^{*<\rho}_x$$

$\leftrightarrow x \notin E_{<n_0,\ldots,n_{k-1}>} \vee$

$(\forall \eta \in N_{<n_0,\ldots,n_{k-1}>})(\exists n \in \eta)[<n_0,\ldots,n_{k-1},n> \in \Gamma_x^{<\rho}]$ by induction hypothesis

$\leftrightarrow <n_0,\ldots,n_{k-1}> \in \Gamma_x^{\rho}$.

Consequently,

$$e \in \Gamma_x^{*\rho+1} \leftrightarrow (\forall \eta \in N_e^*)(\exists n \in \eta)[<n> \in \Gamma_x^{*\rho}]$$

$$\leftrightarrow <0> \in \Gamma_x^{*\rho}$$

$$\leftrightarrow e \in \Gamma_x^{\rho}.$$

Hence $\mathcal{R}_{\mathcal{N}^*}(\{E_s^* : s \in \omega\}) = \mathcal{R}_{\mathcal{N}}(\{E_s : s \in \omega\})$ and $|(e,x)|_{\Gamma^*} = |(e,x)|_{\Gamma}+1$.

The following lemma follows from above. One has only to observe that for any positive analytical operation Φ, $\mathbb{R}\Phi$ is normal, $\mathbb{R}\Phi^0\Phi \hookrightarrow \mathbb{R}\Phi\Phi^0$ and if

$$\mathcal{N} = \{N_s : s \in \omega\}$$

is a family of bases such that $\Phi \geq \Phi_{N_s}$ for each s, then $\mathbb{R}\Phi \geq \mathcal{R}_{\mathcal{N}}$ (cf [0:§7]).

3.4 <u>Lemma</u> : Let $\Phi \geq U$ be a positive analytical operation and $\mathcal{N} = \{N_s : s \in \omega\}$, $\mathcal{M} = \{M_s : s \in \omega\}$ be two families of bases such that for each s, Φ^* subsumes Φ_{N_s} and $\Phi_{M_s^0}$. Suppose $\{E_s : s \in \omega\}$ is a family of sets in $\Sigma_1^{\Phi^*}$ and $\{F_s : s \in \omega\}$ a regular family in $\Pi_1^{\Phi^*}$. Let Γ be the canonical inductive operator associated with $\mathcal{E} = \{E_s : s \in \omega\}$ and $\mathcal{N}$, and Δ that associated with $\mathcal{F} = \{F_s : s \in \omega\}$ and $\mathcal{M}$. Put $\beta_1(x) = |(e,x)|_{\Gamma}$ and $\beta_2(x) = |(e,x)|_{\Delta}$.

Then

(a) $\{x : \beta_1(x) < \beta_2(x)\} \in \Pi_1^{\Phi^*}$,

(b) $\{x : \beta_1(x) < \omega_1 \ \& \ \beta_1(x) \leq \beta_2(x)\} \in \Pi_1^{\Phi^*}$.

Proof : The first assertion follows from the comparison of indices lemma and the observations made above. The second assertion follows from the first by increasing the index of Δ by 1.

By slightly modifying the inductive operator Λ in the proof of Lemma 3.2, one can obtain the following (see also the proof of Theorem II.1.1).

3.5 Corollary : Let Γ, Δ, Φ be as in 3.4. Then

$$\{(s,t,x) : |(s,x)|_\Gamma < |(t,x)|_\Delta\} \in \Pi_1^{\Phi^*}.$$

3.6 Theorem : For any positive analytical operation $\Phi \geq U$, $\Pi_1^{\Phi^*}$ has the prewellordering property.

Proof : Let $E \in \Pi_1^{\Phi^*}$ and suppose

$$E^c = \Phi^*(\{A_s : s \in \omega\}) \text{ with } \{A_s\} \text{ clopen and regular.}$$

Let $\beta(x)$ be the norm on E induced by the canonical inductive operator. Let N be the canonical base for Φ and put $\mathcal{N} = \{NN^o\}$, $\mathcal{M} = \{NN^o\}$. For each s, set

$$E_s = A_s \times \mathcal{X}, \quad F_s = \mathcal{X} \times A_s,$$

and let $\beta_1(x,y)$, $\beta_2(x,y)$ be the norms induced by the inductive operators associated with $\{E_s : s \in \omega\}, \mathcal{N}$ and $\{F_s : s \in \omega\}, \mathcal{M}$ respectively. Since all the hypotheses of Lemma 3.4 are satisfied, the sets

$\{(x,y) : \beta_1(x,y) < \beta_2(x,y)\}$ and $\{(x,y) : \beta_1(x,y) < \omega_1 \ \& \ \beta_1(x,y) \leq \beta_2(x,y)\}$

are in $\prod_1^{\Phi^*}$. But

$$\{(x,y) : \beta_1(x,y) < \beta_2(x,y)\} = \{(x,y) : \beta(x) < \beta(y)\}$$

and $$\{(x,y) : \beta_1(x,y) < \omega_1 \ \& \ \beta_1(x,y) \leq \beta_2(x,y)\}$$

$$= \{(x,y) : \beta(x) < \omega_1 \ \& \ \beta(x) \leq \beta(y)\}.$$

Consequently, $\prod_1^{\Phi^*}$ is normed.

3.7 <u>Corollary</u> : For each $\rho < \omega_1$, $c\mathcal{R}^\rho$ has the prewellordering property.

3.8 <u>Theorem</u> : Let Φ be a positive analytical operation which subsumes both (countable) $\bigcup$ and $\bigcap$. Let $\mathcal{F}$ be the σ-field generated by $\Sigma_1^{\Phi^*}$. Let $E \in \Sigma_1^{\Phi^*}$ such that

$$x \in E \leftrightarrow (\exists \eta_0 \in NN^0)(\forall n_0 \in \eta_0)(\exists \eta_1 \in NN^0)(\forall n_1 \in \eta_1) \ldots\ldots$$

$$\ldots\ldots (\forall k)\left[x \in E_{<n_0,\ldots,n_{k-1}>}\right], \qquad \ldots(i)$$

N being the canonical base for Φ. Then, there is a $\mathcal{F}$-measurable function $x \longrightarrow \sigma_x$ such that σ_x is a winning strategy for the player $\exists$, whenever $x \in E$.

<u>Proof</u> : Let Γ be the canonical inductive operator associated with NN^0 and $\{E_s : s \in \omega\}$ and put $NN^0 = M$. Define

$$\beta(s,x) = \begin{cases} \text{least } \rho \text{ such that } s \in \Gamma_x^\rho, & \text{if } s \in \Gamma_x^\infty \\ \omega_1, & \text{otherwise.} \end{cases}$$

Now suppose $x \in E$. So $\exists$ wins the game (i).

If $\eta_0, \eta_1, \ldots, \eta_{k-1}$ and $n_0, n_1, \ldots, n_{k-1}$ are the first k relevant moves of $\exists$ and $\forall$, notice that $\exists$ goes on to win the game (i), i.e., he is in a winning position iff $\langle n_0, \ldots, n_{k-1}\rangle \notin \Gamma_x^\infty$ i.e., iff $\beta(\langle n_0, \ldots, n_{k-1}\rangle, x) = \omega_1$. In such a case, $\exists$ has to play an $\eta \in M$ such that $(\forall n \in \eta)[\beta(\langle n_0, \ldots, n_{k-1}, n\rangle, x) = \omega_1]$. We therefore define for each x, the strategy σ_x for $\exists$ as follows.

$$p \in \sigma_x(s) \leftrightarrow \beta(s,x) \leq \beta(s * \langle p\rangle, x).$$

Clearly by 3.5, the map $x \rightarrow \sigma_x$ is $\mathcal{J}$-measurable. We shall now show that if $x \in E$, then σ_x is a winning strategy for $\exists$ in the game (i). Suppose $\eta_0, n_0, \eta_1, n_1, \ldots, \eta_{k-1}, n_{k-1}$ are the first k moves of $\exists$ and $\forall$ and assume that $\exists$ is in a winning position. Consequently, we have $\beta(\langle n_0, n_1, \ldots, n_{k-1}\rangle, x) = \omega_1$ and hence $\langle n_0, \ldots, n_{k-1}\rangle \notin \Gamma_x^\infty$. Therefore,

$$(\exists \eta \in M)(\forall n \in \eta)[\beta(\langle n_0, n_1, \ldots, n_{k-1}, n\rangle, x) = \omega_1] \qquad \ldots(ii)$$

By definition,

$$\sigma_x(\langle n_0, \ldots, n_{k-1}\rangle) = \{p : \beta(\langle n_0, \ldots, n_{k-1}\rangle, x) \leq \beta(\langle n_0, \ldots, n_{k-1}, p\rangle, x)\}$$
$$= \{p : \beta(\langle n_0, \ldots, n_{k-1}, p\rangle, x) = \omega_1\} = \eta, \text{ say}.$$

Hence, by (ii) and the completeness of M,

$$\sigma_x(\langle n_0, \ldots, n_{k-1}\rangle) = \eta \in M,$$

and $(\forall p \in \eta)(\beta(\langle n_0, \ldots, n_{k-1}, p\rangle, x) = \omega_1)$, so that $\exists$ is still in a winning position, and, moreover, $x \in E_{\langle n_0, \ldots, n_{k-1}, p\rangle}$ for all $p \in \eta$.

Remark : Notice that

$$x \notin E \leftrightarrow (\forall \eta_0 \varepsilon M)(\exists n_0 \varepsilon \eta_0)(\forall \eta_1 \varepsilon M)(\exists n \varepsilon \eta_1)\ldots\ldots$$

$$\ldots\ldots(\exists k)\left[x \notin E_{<n_0,\ldots,n_{k-1}>}\right]$$

$$\leftrightarrow (\exists \eta_0 \varepsilon M^0)(\forall n_0 \varepsilon \eta_0)\ldots(\exists k)\left[x \notin E_{<n_0,\ldots,n_{k-1}>}\right] \qquad \ldots(iii)$$

In this case also, a definable winning strategy for $\exists$ can similarly be defined. Unlike the game (i), here $\exists$ has to play such that at each stage the ordinal β is decreased. The following gives a $\mathcal{F}$-measurable winning strategy for $\exists$ in the game (iii) whenever $x \notin E$.

$$p \varepsilon \tau_x(s) \leftrightarrow x \notin E_s \vee (\beta(s * <p>, x) < \beta(s,x)).$$

4. The Ramsey property and the partition selection property of R-sets.

Let $|\omega|^\omega$ denote the set of infinite sets of natural numbers. A set $P \subseteq |\omega|^\omega$ is said B have the Ramsey property if it admits a homogeneous set H i.e. an infinite set $H \subseteq \omega$ such that every infinite subset of H belongs to P or every infinite subset of H belongs to the complement of P. Silver [45] has shown, in ZF + DC, that every Σ^1_1 set has the Ramsey property, while if measurable cardinals exist, Σ^1_2 sets have the Ramsey property. The question naturally arises whether, in ZF + DC, R-sets have the Ramsey property. A clue to the solution of this problem comes from Ellentuck's proof of Silver's theorem [17]. But first the relevant definitions which are as follow. Let s be a finite set of natural numbers and A an infinite one. The pair <s,A> is said to be a condition if every member of s is less than any member of A. The

<u>Ellentuck neighbourhood</u> (s,A) consists of all $X \varepsilon |\omega|^{\omega}$ such that $s \subseteq X \subseteq s \cup A$. Condition $\langle s,A\rangle$ extends $\langle t,B\rangle$ iff $t \subseteq s$, $s - t \subseteq B$ and $A \subseteq B$. A set $P \subseteq |\omega|^{\omega}$ is said to be <u>completely Ramsey</u> if for every condition $\langle s,A\rangle$ there is an extension $\langle s,A'\rangle$ such that $(s,A') \subseteq P$ or $(s,A') \subseteq |\omega|^{\omega} - P$. The set P, then, possesses the Ramsey property iff there is an $A \varepsilon |\omega|^{\omega}$ such that $(\varphi,A) \subseteq P$ or $(\varphi,A) \subseteq |\omega|^{\omega} - P$. In [17], Ellentuck has shown that a set $P \subseteq |\omega|^{\omega}$ is completely Ramsey iff it has the Baire property with respect to the Ellentuck topology. Thus to show that R-sets are completely Ramsey, and hence have the Ramsey property, we need only to show that R-sets have the Baire property. Since the Ellentuck topology is strictly finer than the usual one on $|\omega|^{\omega}$, $\underset{\sim}{\Pi}^{0}_{1}$ sets have the Baire property. Moreover, by Theorem 3.1 sets with Baire property are closed under the operations R_{ρ}. Thus we have

4.1 <u>Theorem</u> : R-sets are completely Ramsey and hence have the Ramsey property. ||

Next we show that an equivalence relation generated by countably many R-sets admits a $\underset{\sim}{\mathcal{R}}$-measurable selector. This is a result of Burgess [10]. No proof appears in [10], and we give a complete proof primarily because our proof enables us to obtain some very important properties of the R-sets like the scale property and the uniformization property for the class $c\underset{\sim}{\mathcal{R}}^{\rho}$ (see Theorems 5.4, 5.6). As in [10] we obtain our result via presentability, a notion introduced by Burgess.

4.2 Definition : Let X be an uncountable Polish space and $\mathcal{E}$ an equivalence relation on X. Let $[x]$ denote the $\mathcal{E}$-equivalence class containing $x \varepsilon X$. Clearly $\mathcal{E}$ gives rise to a partition $Q = \{[x] : x \varepsilon X\}$ into disjoint equivalence classes. A selector for $\mathcal{E}$ is a function $S : X \to X$ such that for each x, $S(x) \mathcal{E} x$ and $S(x) = S(x')$ whenever $x \mathcal{E} x'$. A cross-section for $\mathcal{E}$ is a set $T \subseteq X$ consisting of exactly one element from each $\mathcal{E}$-equivalence class. Plainly, if S is a selector for $\mathcal{E}$, then $T = \text{Range}(S)$ is a cross-section. A countable family $\{A_n : n \varepsilon \omega\}$ of subsets of X generates $\mathcal{E}$ if

$$x \mathcal{E} x' \leftrightarrow (\forall n)(x \varepsilon A_n \leftrightarrow x' \varepsilon A_n).$$

4.3 Definition : Let $\mathcal{B}$ be a σ-field of subsets of X. The equivalence relation $\mathcal{E}$ is said to be $\mathcal{B}$-generated if it is generated by a countable subfamily of $\mathcal{B}$. The σ-field $\mathcal{B}$ is said to have the partition-selection property if every $\mathcal{B}$-generated equivalence relation admits a $\mathcal{B}$-measurable selector.

In order to show that various families have the partition-selection property Burgess [10] introduced the following technical notion.

4.4 Definition : Let $\mathcal{B}$ and X be as above. For a set $A \subseteq X$, a $\mathcal{B}$-presentation of A is a quadruple (Y, B, P, G) such that

(i) Y is a Polish space,

(ii) $B \subseteq X \times Y$ is clopen,

(iii) $P \subseteq X \times Y$ is Π^1_1,

(iv) $G : X \to Y$ is $\mathcal{B}$-measurable with graph $\subseteq P$,

(v) $B \cap P = (A \times Y) \cap P$.

The σ-field $\mathcal{B}$ is presentable if every $A \in \mathcal{B}$ admits a $\mathcal{B}$-presentation.

The next result follows immediately from a refinement of a theorem of Kaniewski [11].

4.5 Lemma (Burgess) : Every presentable family has the partition-selection property.

In [10], Burgess proved that the $\mathcal{C}$-sets, the Borel-programmable sets, the R-sets, the absolute $\underset{\sim}{\Delta}^1_2$ sets and the Lebesgue measurable sets have the partion selection property. We shall obtain the partition-selection property of R-sets by showing that the property of being presentable is 'preserved' by the operator $\mathbb{R}$. (This is one of the several instances where a certain property is left invariant by the $\mathbb{R}$ operator. Another such 'invariance property' will be exhibited in the next ~~two sections~~). The presentability, and hence the partition-selection property, of $\underset{\sim}{\mathcal{BR}}^{\rho}$ is then obtained by a simple inductive argument.

4.6 Theorem : Let Φ be a p.a.o. which subsumes operation $(\mathcal{A})$. If $\underset{\sim}{\nabla}(\Phi)$ is presentable, then $\underset{\sim}{\nabla}(\Phi^*)$ is presentable.

Proof : We work in ω^{ω}.

Step 1. We first show that each $E \in \underset{\sim}{\Pi}^{\Phi^*}_1$ is presentable. Suppose $E^{\circ} = \Phi^*(\{E_n : n \in \omega\})$, with each E_n clopen ; and let N be the canonical base for Φ. The inductive operator associated with N and $\{E_n : n \in \omega\}$ is the following :

$$s \in \Gamma_x(A) \leftrightarrow x \notin E_s \vee (\forall \eta \in N)(\exists n \in \eta)(\exists \xi \in N)(\forall m \in \xi)[s * \langle\langle n,m\rangle\rangle \in A].$$

Then it is not hard to see that

$$s \notin \Gamma_x^{\infty} \leftrightarrow (\exists \eta_o \in N)(\forall n_o \in \eta)(\forall \xi_o \in N)(\exists m_o \in \xi_o) \ldots\ldots \qquad \ldots(i)$$
$$\ldots\ldots(\forall k)\left[x \in E_{s * \langle\langle n_o, m_o\rangle, \ldots, \langle n_{k-1}, m_{k-1}\rangle\rangle}\right].$$

In particular,

$$e \notin \Gamma_x^{\infty} \leftrightarrow x \notin E.$$

Let $Y = \omega^{\omega} \times 2^{\omega} \times \{0,1\}$. Define $C \subseteq X \times Y$ as follows :

$$C(x,\alpha,\beta,k) \overset{\text{def}}{\longleftrightarrow} WO(\alpha) \;\&$$
$$(\forall i)\left[i \in \text{Field}(\leq_\alpha) \rightarrow (\beta)_i = \Gamma_x\Big(\bigcup_{j <_\alpha i} (\beta)_j\Big)\right]$$
$$\& \; \Gamma_x\Big(\bigcup_{i \in \text{Field}(\leq_\alpha)} (\beta)_i\Big) = \bigcup_{i \in \text{Field}(\leq_\alpha)} (\beta)_i \;\&\; k = d\Big(\bigcup_{i \in \text{Field}(\leq_\alpha)} (\beta)_i\Big) ;$$

where $d : 2^{\omega} \rightarrow \{0,1\}$ is the function defined by

$$d(\eta) = \begin{cases} 0 & \text{if } e \in \eta \\ 1 & \text{otherwise.} \end{cases}$$

Put $B = X \times \omega^{\omega} \times 2^{\omega} \times \{0\}$. Plainly B is clopen and C is in $\underset{\sim}{\nabla}(\Phi)$. Moreover,

$$C \cap B = C \cap (E \times Y).$$

Let

$$\beta(n,x) = \begin{cases} \text{least ordinal } \lambda \text{ such that } n \in \Gamma_x^{\lambda}, & \text{if } n \in \Gamma_x^{\infty} \\ \omega_1 & \text{otherwise.} \end{cases}$$

Define $f : X \longrightarrow WO$ by

$$f(x)(\langle n,m\rangle) = 0 \leftrightarrow \beta(n,x) < \beta(m,x) < \omega_1$$
$$\& \ (\forall i)\left[\beta(n,x) = \beta(i,x) \rightarrow n \leq i\right]$$
$$\& \ (\forall j)\left[\beta(m,x) = \beta(j,x) \rightarrow m \leq j\right].$$

Otherwise, put $f(x)(s) = 1$.

By Corollary 3.5, f is $\underset{\sim}{\nabla}(\Phi^*)$-measurable. Now, define $g : X \rightarrow 2^\omega$ as follows :

$$(g(x))_n = \begin{cases} \ulcorner^{\beta(n,x)}_{x} & \text{if } n \in \text{Field}(f(x)) \\ \varphi & \text{otherwise.} \end{cases}$$

For s not of the form $\langle n,m\rangle$, put $g(x)(s) = 1$.

Thus

$$g(x)(\langle n,m\rangle) = 0 \leftrightarrow \beta(m,x) \leq \beta(n,x) \ \& \ n \in \text{Field}(f(x)).$$

Consequently, g is $\underset{\sim}{\nabla}(\Phi^*)$-measurable by 3.5 again. Let

$$G(x) = (f(x), g(x), \chi_E(x)).$$

Then $G : X \rightarrow Y$ is $\underset{\sim}{\nabla}(\Phi^*)$-measurable and graph $(G) \subseteq C$. By hypothesis, C is presentable. Get $(Z, \overleftarrow{B}, \overleftarrow{P}, \overleftarrow{G})$ such that

(i) $\overleftarrow{P} \subseteq X \times Y \times Z$ is $\underset{\sim}{\Pi}^1_1$

(ii) $\overleftarrow{B} \subseteq X \times Y \times Z$ is clopen

(iii) $\overleftarrow{G} : X \times Y \rightarrow Z$ is $\underset{\sim}{\nabla}(\Phi)$-measurable with graph contained in P

(iv) $\overleftarrow{P} \cap \overleftarrow{B} = \overleftarrow{P} \cap (C \times Z)$.

Put
$$P' = \overleftarrow{P} \cap \overleftarrow{B}$$
$$B' = B \times Z$$
$$G'(x) = (G(x), \overleftarrow{G}(x, G(x))).$$

It is not hard to check that $(Y\times Z, B', P', G')$ is a $\underset{\smile}{\nabla}(\Phi^*)$-presentation of E.

Step 2. Let $\mathcal{F} \subseteq \underset{\smile}{\nabla}(\Phi^*)$ be the family of presentable sets in $\underset{\smile}{\nabla}(\Phi^*)$. As shown above, $\underset{\smile}{\Pi}_1^{\Phi^*} \subseteq \mathcal{F}$ and trivially $\mathcal{F}$ is closed under complementation. We shall show that $\mathcal{F}$ if closed under Φ^*. Let $\{E_n : n \in \omega\}$ be a family of sets in $\mathcal{F}$ and let $E = \Phi^*(\{E_n : n \in \omega\})$. Since each E_n is presentable, there is $(Y', \{B_n\}, P', G)$ which is a $\underset{\smile}{\nabla}(\Phi^*)$-presentation of $\{E_n\}$ (see [11]). In other words,

(i) Y' is Polish,

(ii) each $B_n \subseteq X\times Y$ is clopen,

(iii) $P' \subseteq X\times Y$ is $\underset{\smile}{\Pi}_1^1$,

(iv) $G : X \to Y$ is $\underset{\smile}{\nabla}(\Phi^*)$-measurable with graph in P',

(v) for each n, $P' \cap B_n = P' \cap (E_n \times Y)$.

Put $B = \Phi^*(\{B_n : n \in \omega\})$. Plainly, $B \in \underset{\smile}{\Sigma}_1^{\Phi^*}$ and is therefore presentable. Let (Z, D, P'', H) be a $\underset{\smile}{\nabla}(\Phi^*)$-presentation of B. Set

$$P = P'' \cap (P' \times Z)$$

$$\tilde{G}(x) = (G(x), H(x, G(x))).$$

Then $(Y\times Z, D, P, \tilde{G})$ is a $\underset{\smile}{\nabla}(\Phi^*)$-presentation of E. ▯

4.7 Lemma : Let $\{\Phi_i : i \in \omega\}$ be a sequence of p.a.o's with canonical bases N_i and suppose for each i, $\underset{\smile}{\nabla}(\Phi_i)$ is presentable. Define a p.a.o. Ψ by

$$x \in \Psi(\{E_n : n \in \omega\}) \leftrightarrow (\forall i)(\exists \eta \in N_i)(\forall n \in \eta)[x \in E_{\langle i,n\rangle}],$$

where $\{E_n : n \in \omega\}$ is any family of subsets. Then $\underset{\smile}{\nabla}(\Psi)$ is also presentable.

Proof : We shall first show that every $\underset{\sim}{\Sigma}^{\psi}_{1}$ set admits a $\underset{\sim}{\nabla}(\psi)$-presentation. So let $E \in \underset{\sim}{\Sigma}^{\psi}_{1}$. Find a family of clopen sets $\{E_n\}$ such that $E = \psi(\{E_n : n \in \omega\})$. Set for every i

$$E^i = \bar{\Phi}_i(\{E_{\langle i,n\rangle} : n \in \omega\}).$$

Plainly each $E^i \in \underset{\sim}{\nabla}(\bar{\Phi}_i)$ and $E = \bigcap_i E^i$. By hypothesis, each E^i admits a $\underset{\sim}{\nabla}(\bar{\Phi}_i)$ - presentation and hence a $\underset{\sim}{\nabla}(\psi)$ - presentation. It is now easy to see that $E = \bigcap_i E^i$ admits a $\underset{\sim}{\nabla}(\psi)$ - presentation. Thus the class of sets which admit a $\underset{\sim}{\nabla}(\psi)$ - presentation contains $\underset{\sim}{\Sigma}^{\psi}_{1}$ and is (trivially) closed under complementation. That it is also closed under ψ can be shown along the lines of the proof of Theorem 4.6. This shows that $\underset{\sim}{\nabla}(\psi)$ is presentable. ∎

We now have the following result of Burgess announced in [10].

4.8 Theorem : For each $\rho < \omega_1$, $\underset{\sim}{\mathcal{BR}}^{\rho}$ is presentable.

Consequently, each $\underset{\sim}{\mathcal{BR}}^{\rho}$ has the partition selection property.

Proof : By [10], the class of $\mathcal{C}$-sets $= \underset{\sim}{\mathcal{BR}}^{0}$ is presentable. Moreover, if $\underset{\sim}{\mathcal{BR}}^{\rho}$ is presentable then by Theorem 4.6 $\underset{\sim}{\mathcal{BR}}^{\rho+1}$ is also presentable. Finally if λ is limit and for each $\mu < \lambda$, $\underset{\sim}{\mathcal{BR}}^{\mu}$ is presentable, then the presentability of $\underset{\sim}{\mathcal{BR}}^{\lambda}$ can be established by Lemma 4.7 and Theorem 4.6. ∎

Remark : It is significant to note that presentability is not preserved by the game quantifier $\Game$ (at least under some determinacy hypothesis).

To see this assume $\mathrm{Det}(\underset{\sim}{\Delta}^1_2)$ and assume to the contrary that presentability is preserved by $\supset$. Since $\underset{\sim}{\Delta}^1_2$ is presentable ([10]) and $\supset \underset{\sim}{\Delta}^1_2 = \underset{\sim}{\Delta}^1_3$ ([38 ; 6E.13]) it follows that $\underset{\sim}{\Delta}^1_3$ is presentable and hence has the partition-selection property. Now fix a set $P \in \underset{\sim}{\Sigma}^1_3 - \underset{\sim}{\Delta}^1_3$. Let f be a continuous function and $A \subseteq \omega^\omega$ a $\underset{\sim}{\Pi}^1_2$ set such that $f(A) = P$. Define an equivalence relation E on ω^ω as follows :

$$E(x,y) \leftrightarrow x \notin A \;\&\; y \notin A \vee x \in A \;\&\; y \in A \;\&\; f(x) = f(y).$$

Plainly E is a $\underset{\sim}{\Delta}^1_3$-generated equivalence relation. By the partition-selection property, there is a $\underset{\sim}{\Delta}^1_3$-measurable selector S and hence a $\underset{\sim}{\Delta}^1_3$- cross-section T for E. It is easy to check that $f \restriction T \cap A$ is one-one and $f(T \cap A) = P$. By [38; 6E.14], this implies that P is $\underset{\sim}{\Delta}^1_3$ which is a contradiction.

5. The scale property and the uniformization property of R-sets : The notion of scale is of great importance in descriptive set theory and it is extremely useful to know which pointclasses have the scale property. For any p.a.o. Φ, $\Pi_1^{\Phi^*}$ behaves very much like Π_1^1 and it is not surprising that for most operations Φ, $\Pi_1^{\Phi^*}$ has the prewellordering property. It therefore seems natural to expect that the proof of the scale property should follow a similar pattern and that the key to this is the proof of the Scale Transfer Theorem of Moschovakis [38]. However, this method does not seem to work. On the other hand, by the game characterization of R-sets [13; §1], R-sets at the λth level are obtained by applying the game quantifier $\Game$ to $\lambda - \Sigma_2^0$ sets (see [48] for definition); and hence Steel's method of transfering scale applies (cf. [48]). Athough this method gives scales it does not seem to give definable scales of the right type. The failure of these methods led us to express R-sets in terms of games played on ω of length much greater than $\aleph_0$ and then using Martin's method (see [36]). Unfortunately, this method also failed to produce scales in the right definable class. By Martin's method we have, for instance, been able to show that $c\mathcal{R}^n$-sets, $n \in \omega$, admit $B\mathcal{R}^{n+1}$ scales although we believe that this can be improved. This hopeless situation was retrieved by Srivatsa by an ingenious method. By modifying our proof of the presentability of $c\mathcal{R}^\rho$, he has been able to 'transfer' the Π_1^1 scales to $c\mathcal{R}^\rho$ sets and thus obtain the scale property. We present below his proof, slightly simplified by us.

The key result for the proof is the following extremely useful and important result of Burgess which refines a theorem of Kaniewski (see [11]). The notion of presentability also plays a crucial role.

5.1 Lemma (Burgess) : Let Z be a Polish space, $P \subseteq Z$ $\underset{\sim}{\Pi}^1_1$, $Q \subseteq Z \times Z$ $\underset{\sim}{\Sigma}^1_1$, $D = Q \cap P \times P$. Suppose that D is an equivalence relation on P and that every D-equivalence class is relatively closed in P. Then D admits an analytically measurable selector whose associated cross-section is $\underset{\sim}{\Pi}^1_1$. □

We first show that $\underset{\sim}{\mathcal{BR}}^\rho$ is scaled.

5.2 Theorem : For each $\rho < \omega_1$, $\underset{\sim}{\mathcal{BR}}^\rho$ is scaled.

Proof : Let $E \in \underset{\sim}{\mathcal{BR}}^\rho$. By Theorem 4.8 E is presentable. Get a $\underset{\sim}{\mathcal{BR}}^\rho$-presentation (Y,B,P,G) as in 4.4. Define a $\underset{\sim}{\Sigma}^1_1$ (in fact, Borel) equivalence relation on $X \times Y$ such that its restriction on P has vertical lines as its equivalence classes. By Lemma 5.1 get an analytically measurable selector f such that the associated cross-section $P^* = \text{range}(f)$ is $\underset{\sim}{\Pi}^1_1$. Now fix a very good $\underset{\sim}{\Pi}^1_1$-scale ([38; 4E]) $\{\varphi_n\}$ on $P^* \cap B$ and on E define a sequence of norms $\{\psi_n\}$ as follows.

$$\psi_n(x) = \varphi_n(f(x,G(x))).$$

Plainly, $x \leq^*_{\psi_n} y \leftrightarrow f(x,G(x)) \leq^*_{\varphi_n} f(y,G(y))$

and $x <^*_{\psi_n} y \leftrightarrow f(x,G(x)) <^*_{\varphi_n} f(y,G(y))$.

To check that $\{\psi_n\}$ is a scale on E is quite routine (see the proof of the next theorem). □

To show $c\mathcal{R}^{\rho}$ has the scale property, one has to take more care. We work with partial functions and appropriately modify the proof of Theorem 4.6 to get the scale property.

5.3 <u>Theorem</u> : Let Φ be a p.a.o. which subsumes operation $\mathcal{A}$. If $\nabla(\Phi)$ is presentable, then $\prod_1^{\Phi^*}$ is scaled.

<u>Proof</u> : Let N be the canonical base for Φ and suppose $E \in \prod_1^{\Phi^*}$. Let $E^c = \Phi^*(\{E_n : n \in \omega\})$, where $\{E_n : n \in \omega\}$ is a clopen family. The canonical inductive operator determining E is the following.

$$s \in \Gamma_x(A) \leftrightarrow x \notin E_s \vee (\forall \eta \in N)(\exists n \in \eta)(\exists \xi \in N)(\forall m \in \xi)[s * \langle\langle n,m\rangle\rangle \in A].$$

Then

$$s \notin \Gamma_x^{\infty} \leftrightarrow (\exists \eta_0 \in N)(\forall n_0 \in \eta_0)(\forall \xi_0 \in N)(\exists m_0 \in \xi_0) \ldots\ldots$$

$$\ldots\ldots (\forall k)\left[x \in E_{s * \langle\langle n_0,m_0\rangle,\ldots,\langle n_1,\ldots,m_{k-1}\rangle\rangle}\right].$$

In particular, $e \in \Gamma_x^{\infty} \leftrightarrow x \in E.$

Define $C \subseteq \mathcal{X} \times \omega^{\omega} \times 2^{\omega}$ as follows.

$$C(x,\alpha,\gamma) \leftrightarrow WO(\alpha) \;\&$$

$$(\forall i)\left[i \in \mathrm{Field}(\leq_{\alpha}) \rightarrow (\gamma)_i = \Gamma_x(\bigcup_{j <_{\alpha} i} (\gamma)_j)\right]$$

$$\& \; e \in \bigcup_{i \in \mathrm{Field}(\leq_{\alpha})} (\gamma)_i$$

Plainly, C is in $\nabla(\Phi)$ and $\mathrm{proj}_{\mathcal{X}}(C) = E.$

Let $\beta(n,x)$ denote the least ordinal λ such that $n \in \Gamma_x^{\lambda}$, otherwise $\beta(n,x) = \omega_1$. Define a partial function on E into WO as follows.

For every $x \in E$,

$$f(x)(s) = 0 \leftrightarrow \beta((s)_0,x) \leq \beta(e,x) \; \& \; \beta((s)_1,x) \leq \beta(e,x)$$
$$\& \; \beta((s)_0,x) < \beta((s)_1,x)$$
$$\& \; (\forall i)\, [\beta((s)_0,x) = \beta(i,x) \rightarrow (s)_0 \leq i\,]$$
$$\& \; (\forall j)\, [\beta((s)_1,x) = \beta(j,x) \rightarrow (s)_1 \leq j\,].$$

Otherwise, put $f(x)(s) = 1$.

Observe that, since $\beta(e,x) < \omega_1$, by 3.5 it follows easily that f is a $\Pi_1^{\Phi^*}$-recursive partial function. Now define a partial function g on E into 2^ω as follows

$$g(x)(s) = 0 \leftrightarrow \beta((s)_0,x) \leq \beta(e,x)$$
$$\& \; \beta((s)_1,x) \leq \beta(e,x) \; \& \; \beta((s)_1,x) \leq \beta((s)_0,x).$$

Otherwise, $g(x)(s) = 1$.

Thus when $x \in E$ and $n \in \text{Field}(\leq_{f(x)})$,

$$(g(x))_n = \{m : \beta(m,x) \leq \beta(n,x)\}$$
$$= \Gamma_x^{\beta(n,x)}.$$

Furthermore, by 3.5 again, g is $\Pi_1^{\Phi^*}$-recursive on E.

Put, for $x \in E$,

$$G(x) = (f(x),\, g(x)).$$

Plainly, G is a partial $\Pi_1^{\Phi^*}$-recursive function whose graph is contained in C. By hypothesis, C is presentable. Get $(Z, B, P, \overleftarrow{G})$ such that

(i) Z is Polish

(ii) $P \subseteq \mathcal{X} \times \omega^\omega \times 2^\omega \times Z$ is Π^1_1

(iii) $B \subseteq \mathcal{X} \times \omega^\omega \times 2^\omega \times Z$ is clopen

(iv) $\overleftarrow{G} : \mathcal{X} \times \omega^\omega \times 2^\omega \to Z$ is $\nabla(\Phi)$-measurable with graph contained in P.

(v) $P \cap B = P \cap (C \times Z)$.

As in the proof of Theorem 5.2 get a selector F measurable with respect to the analytical σ-field which from each line $\{x\} \times (P \cap B)_x$ picks a unique point and such that the corresponding cross-section P^* is Π^1_1. Observe that $\text{proj}_{\mathcal{X}}(P^*) = E$.

Let $\{\varphi_n\}$ be a very good Π^1_1-scale on P^*. Define for any $x \in E$ and for every n,

$$\psi_n(x) = \varphi_n(F(x, G(x), \overleftarrow{G}(x, G(x))).$$

We claim that

(a) $x \leq^*_{\psi_n} y \leftrightarrow F(x,G(x), \overleftarrow{G}(x,G(x)))\downarrow$ &

$$(\forall\alpha)(\forall \gamma)(\forall z)\left[F(x,G(x), \overleftarrow{G}(x,G(x))) \leq^*_{\varphi_n} (y,\alpha,\gamma,z)\right]$$

(b) $x <^*_{\psi_n} y \leftrightarrow F(x,G(x), \overleftarrow{G}(x,G(x)))\downarrow$ &

$$(\forall\alpha)(\forall \gamma)(\forall z)\left[F(x,G(x), \overleftarrow{G}(x,G(x)) <^*_{\varphi_n} (y,\alpha,\gamma,z)\right]$$

To see (a), first assume $x \leq^*_{\psi_n} y$. Then $x \in E$ and hence

$F(x,G(x), G(x,G(x)))\downarrow$. If $y \notin E$ then clearly for any α,γ,z, $(y,\alpha,\gamma,z) \notin P^*$ and so $F(x,G(x), G(x,G(x))) \leq^*_{\varphi_n} (y,\alpha,\gamma,z)$. Now assume $y \in E$ and $\psi_n(x) \leq \psi_n(y)$. Fix any α,γ,z. If

$$(y,\alpha,\gamma,z) = F(y,G(y), \breve{G}(y,G(y))),$$

then

$$\varphi_n(F(x,G(x), \breve{G}(x,G(x)))) \leq \varphi_n(y,\alpha,\gamma,z).$$

If $(y,\alpha,\gamma,z) \neq F(y,G(y), \breve{G}(y,G(y)))$, then observe that $(y,\alpha,\gamma,z) \notin P^*$ and so

$$F(x,G(x), \breve{G}(x,G(x))) \leq^*_{\varphi_n} (y,\alpha,\gamma,z)$$

Conversely, suppose $F(x,G(x), G(x,G(x)))\downarrow$ and

$$(\forall\alpha)(\forall\gamma)(\forall z)\left[F(x,G(x), G(x,G(x))) \leq^*_{\varphi_n} (y,\alpha,\gamma,z)\right].$$

Plainly $x \in E$ and assume $y \in E$. Choose α,γ,z such that

$$(y,\alpha,\gamma,z) = F(y,G(y), \breve{G}(y,G(y))).$$

Then

$$\varphi_n(F(x,G(x), \breve{G}(x,G(x)))) \leq \varphi_n(F(y,G(y), \breve{G}(y,G(y))))$$

and so $\psi_n(x) \leq \psi_n(y)$. Thus we have (a); (b) is proved similarly. Now notice that

$$S(n,x,\alpha_1,\gamma_1,z_1,y) \overset{\text{def}}{\longleftrightarrow} (\forall\alpha)(\forall\gamma)(\forall z)\left[(x,\alpha_1,\gamma_1,z_1) \leq^*_{\varphi_n} (y,\alpha,\gamma,z)\right]$$

is Π^1_1 since $\{\varphi_n\}$ is a Π^1_1-scale. Since $\Pi^{1\,*}_1$ has the substitution property (this can be easily established),

$$x \leq^*_{\psi_n} y \leftrightarrow F(x,G(x), \overleftarrow{G}(x,G(x)))\downarrow \,\&\, S(n,F(x,G(x), \overleftarrow{G}(x,G(x))),y)$$

is $\underset{\hookrightarrow}{\Pi}{}_1^{\bar{\Phi}^*}$. Similarly, it can be shown that the relation $x <^*_{\psi_n} y$ is $\underset{\hookrightarrow}{\Pi}{}_1^{\bar{\Phi}^*}$. If now remains to show that $\{\psi_n\}$ is a scale on E. So let $\{x_n\}$ be a sequence in E and suppose $x_n \to x$ and for each i,

$$\psi_i(x_n) = \lambda_i \quad \text{ultimately.}$$

Thus, for each i,

$$\varphi_i(F(x_n,G(x_n), G(x_n,\overleftarrow{G}(x_n)))) = \lambda_i$$

for all large n. Since $\{\varphi_i\}_{i \in \omega}$ is a very good scale, there is $\vec{z} \in P^*$ such that

$$F(x_n,G(x_n), G(x_n,G(x_n))) \to \vec{z}$$

and $\quad \varphi_i(\vec{z}) \leq \lambda_i$ (c)

Since $F(x, \vec{w})$ is of the form $(x, \vec{w})$, it is clear that $\vec{z}$ is of the form $(x, \vec{w})$ for some $\vec{w}$. Thus $x \in E$ and moreover,

$$(x, \vec{w}) = F(x,G(x), \overleftarrow{G}(x,G(x))).$$

Hence $\psi_i(x) = \varphi_i(F(x,G(x), \overleftarrow{G}(x,G(x)))) \leq \lambda_i$ by (c).

This completes the proof.

As an immediate consequence of the above we have

5.4 <u>Theorem</u> : For each $\rho < \omega_1$, $\rho \geq 1$, $c\underset{\hookrightarrow}{\mathcal{R}}^\rho$ has has the scale property. ▯

The proofs of the scale property give us the following uniformization results.

5.5 <u>Theorem</u> : Let Φ be a p.a.o. which subsumes $\mathcal{A}$. If $\underset{\sim}{\nabla}(\Phi)$ is presentable, then every $E \subseteq \mathcal{X} \times \mathcal{Y}$ in $\underset{\sim}{\nabla}(\Phi)$ can be uniformized by a $\underset{\sim}{\nabla}(\Phi)$ set E^*.

In particular, for each $\rho < \omega_1$, $\underset{\sim}{\mathcal{BR}}^{\rho}$ has the uniformization property.

<u>Proof</u> : Let $E \subseteq \mathcal{X} \times \mathcal{Y}$ be in $\underset{\sim}{\nabla}(\Phi)$. By hypothesis E has a $\underset{\sim}{\nabla}(\Phi)$-presentation (Z,B,P,G). As above, let $F_3 : P \cap B \rightarrow P \cap B$ be an analytically measurable function which picks up a unique point from every vertical line $\{(x,y)\} \times (P \cap B)_{(x,y)}$ and such that the associated cross-section P^* is $\underset{\sim}{\Pi}^1_1$. Again, let $F_2 : P^* \rightarrow P^*$ be another analytically measurable function which from every vertical line $\{x_1\} \times P^*_{x_1}$ picks up a unique point and whose associated cross-section P^{**} is $\underset{\sim}{\Pi}^1_1$. Let $E^{**} = \mathrm{proj}_{\mathcal{X}\times\mathcal{Y}}(P^{**})$. Clearly, E^{**} is a graph which uniformizes E. We claim that $E^{**} \in \underset{\sim}{\nabla}(\Phi)$. To see this define $H : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y} \times Z$ by

$$H(x,y) = F_3(x,y,G(x,y)).$$

Plainly, H is $\underset{\sim}{\nabla}(\Phi)$-measurable, since $\Phi \geq \mathcal{A}$.

Moreover, $\qquad E^{**} = H^{-1}(P^{**})$,

and hence $\qquad E^{**} \in \underset{\sim}{\nabla}(\Phi)$. $\square$

<u>Remark</u> : Since Borel-programmable (absolutely $\underset{\sim}{\Delta}^1_2$; $\underset{\sim}{\Delta}^1_2$) sets are presentable (cf [10]) the above proof shows that Borel-programmable (absolutely $\underset{\sim}{\Delta}^1_2$, $\underset{\sim}{\Delta}^1_2$) sets can be uniformized by a Borel-programmable (absolutely $\underset{\sim}{\Delta}^1_2$; $\underset{\sim}{\Delta}^1_2$) graph.

5.6 Theorem : Suppose $\Phi \geq \mathcal{A}$ and $\nabla(\Phi)$ is presentable. Then every $E \subseteq \mathcal{X}_1 \times \mathcal{X}_2$ in $\Pi_1^{\Phi^*}$ can be uniformized by a set in $\Pi_1^{\Phi^*}$.

In particular, for every $\rho < \omega_1$, $c\mathcal{R}^\rho$ has the uniformization property.

Proof : Let $E \subseteq \mathcal{X}_1 \times \mathcal{X}_2$ be in $\Pi_1^{\Phi^*}$. Let C, G be as in the proof of Theorem 5.3 with $\mathcal{X}$ replaced by $\mathcal{X}_1 \times \mathcal{X}_2$. Suppose $(Z, B, P, \overleftarrow{G})$ be a $\nabla(\Phi)$- presentation of C as in the proof there. By Lemma 5.1, let $F_3 : P \cap B \to P \cap B$ be an analytically measurable function such that it picks up a unique point from every vertical line $\{(x_1,x_2)\} \times (P \cap B)_{x_1,x_2}$ and whose associated cross-section P^* is Π_1^1. By Lemma 5.1 again, let $F_2 : P^* \to P^*$ be a similar function picking up a point from each vertical line $\{x_1\} \times (P^*)_{x_1}$ such that $P^{**} = \text{range}(F_2)$ is Π_1^1. Let

$$E^{**} = \text{proj}_{\mathcal{X}_1 \times \mathcal{X}_2}(P^{**}).$$

Plainly, E^{**} is a graph which uniformizes E. We shall show that $E^{**} \in \Pi_1^{\Phi^*}$. Define a partial function

$$H : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_2$$

with domain E as follows

$$H(x_1,x_2) = F_3(x_1,x_2,G(x_1,x_2), \overleftarrow{G}(x_1,x_2,G(x_1,x_2))) \;;\; (x_1,x_2) \in E.$$

Note that H is a partial $\Pi_1^{\Phi^*}$-recursive function. We claim that

$$E^{**}(x_1,x_2) \leftrightarrow H(x_1,x_2)\downarrow \;\&\; P^{**}(H(x_1,x_2)) \qquad \text{(i)}$$

To see this assume $(x_1,x_2) \in E^{**}$. Then there exist (unique) α, γ, z

such that $(x_1,x_2,\alpha,\gamma,z) \varepsilon P^{**}$. Now since $(x_1,x_2) \varepsilon E$, $H(x_1,x_2)\downarrow$. Further,

$$H(x_1,x_2) = F_3(x_1,x_2,G(x_1,x_2), \overleftarrow{G}(x_1,x_2,G(x_1,x_2))) \varepsilon P^*.$$

So for some α',γ',z',

$$H(x_1,x_2) = (x_1,x_2,\alpha',\gamma',z') \varepsilon P^*.$$

Since $(x_1,x_2,\alpha,\gamma,z)$ and $(x_1,x_2,\alpha',\gamma',z')$ are in P^* it follows that $\alpha = \alpha', \gamma = \gamma'$ and $z = z'$. Thus

$$H(x_1,x_2) = (x_1,x_2,\alpha,\gamma,z) \varepsilon P^{**}.$$

Conversely, suppose $H(x_1,x_2) \varepsilon P^{**}$ so that for some α,γ,z, $(x_1,x_2,\alpha,\gamma,z) \varepsilon P^{**}$. But

$$E^{**} = \text{proj}_{\mathcal{X}_1 \times \mathcal{X}_2} (P^{**}),$$

and so $(x_1,x_2) \varepsilon E^{**}$. Thus we have (i). By the substitution property this shows that $E^{**} \varepsilon \underset{\hookrightarrow}{\prod}_1 \Phi^*$. ∎

6. <u>The Category and the Measure Formulas for R-sets</u> : As promised earlier we shall exibit another property which is left invariant by the operator $\mathbb{R}$. Roughly speaking, we shall show that the $*$ - transform of Vaught (see 6.1) is preserved by the operator $\mathbb{R}$. More explicitly, we shall show that if a set $E \subseteq \mathcal{X} \times \mathcal{Y}$ obtained by the application of Φ is such that $E^* = \{x : E^x$ is comeager$\}$ is computed by the application of Ψ, then for any set F computed by Φ^*, F^* is computed by Ψ^*. This invariance property is established by first obtaining a decomposition

of E^* for sets $E \in \Sigma_1^{\Phi^*}$, analogous to the one obtained by Vaught for Σ_1^1 sets [51] and for $\mathcal{R}^1$-sets obtained by Burgess and Miller [8]. Although our methods for computing E^* are implicit in Burgess' proof for the same (cf [8]), we have, by isolating the core of his proof (viz. Theorem 6.3 below), been able to compute E^* for all levels of the hierarchy of R-sets by a simple inductive argument; and thus, been able to avoid the notational complexities involved in Burgess' proof.

We first introduce the following definition due to Vaught [51].

6.1 <u>Definition</u> : A set $E \subseteq X \times Y$, the product of two Polish spaces, is said to be <u>normal</u> if for each $x \in X$, $E^x = \{y \in Y : (x,y) \in E\}$ has the Baire property. (In this section E^x denotes the vertical section at x). If E is normal and $U \subseteq Y$ is open, then define

$$E^{*U} = \{x \in X : E^x \text{ is comeager in } U\}.$$

If $U = Y$, we write E^* instead of E^{*U}.

Next we get a decomposition of E^* for sets E obtained by the application of Φ^* on a normal family. (Recall that $\Sigma(s)$ denotes the set of all reals α extending s).

6.2 <u>Lemma</u> : Let Φ be a p.a.o. which preserves the Baire property and let $E = \Phi^*(\{E_s : s \in \omega\})$, with each $E_s \subseteq \omega^\omega \times \omega^\omega$ normal. Define E_s^μ and T^μ as in 2.6. Then for any $s \in Seq$,

$$E^{*\Sigma(s)} = \bigcap_{\mu<\omega_1} [E_s^\mu]^{*\Sigma(s)} = \bigcup_{\mu<\omega_1} [E_s^\mu - T^\mu]^{*\Sigma(s)}.$$

<u>Proof</u> : As in Theorem 2.8, one can easily check that E_s^μ and T^μ are

normal for each μ. Since $E = \bigcap_{\mu<\omega_1} E_e^{\mu}$, it follows that

$$E^{*\Sigma(s)} \subseteq \bigcap_{\mu<\omega_1} [E_e^{\mu}]^{*\Sigma(s)}. \qquad \ldots(i)$$

Next, suppose $x \in [E_e^{\mu}]^{*\Sigma(s)}$ for all $\mu < \omega_1$. For each $p \in$, $\{(E_p^{\mu})^x : \mu < \omega_1\}$ is a decreasing sequence of sets with Baire property. Hence by the countable chain condition, $\exists \beta(p) < \omega_1$ such that

$$(\forall \rho > \beta(p))[(E_p^{\beta(p)} - E_p^{\rho})^x \text{ is meager}].$$

Choose ρ_o such that $\beta(p) < \rho_o < \omega_1$, for all p. Then $(\forall p)[(E_p^{\rho_o} - E_p^{\rho_o+1})^x$ is meager$]$ and hence $(T^{\rho_o})^x$ is meager. Since $x \in [E_e^{\rho_o}]^{*\Sigma(s)}$, $(E_e^{\rho_o})^x$ is comeager in $\Sigma(s)$. Therefore, $(E_e^{\rho_o})^x - (T^{\rho_o})^x$ is comeager in $\Sigma(s)$ and so $x \in [E_e^{\rho_o} - T^{\rho_o}]^{*\Sigma(s)}$.

Thus,

$$\bigcap_{\mu<\omega_1} [E_e^{\mu}]^{*\Sigma(s)} \subseteq \bigcup_{\mu<\omega_1} [E_e^{\mu} - T^{\mu}]^{*\Sigma(s)}. \qquad \ldots(ii)$$

Finally, since $(E_e^{\mu} - T^{\mu}) \subseteq E$ for each μ,

$$\bigcup_{\mu<\omega_1} [E_e^{\mu} - T^{\mu}]^{*\Sigma(s)} \subseteq E^{*\Sigma(s)}. \qquad \ldots(iii)$$

The result now follows from (i) – (iii).

6.3 <u>Transfer Theorem</u> : Let Φ and Ψ be two positive analytical operations such that Φ preserves the Baire property and Ψ is normal and subsumes both (countable) $\cup$ and $\cap$. Suppose, moreover, that there are (recursive) functions f and g such that for any normal family $\{E_p : p \in \omega\}$ of subsets of $\omega^\omega \times \omega^\omega$ with $E = \Phi(\{E_p : p \in \omega\})$,

$$E^{*\Sigma(s)} = \Psi(\{E_{f(p)}^{*\Sigma(s*g(p))} : p \in \omega\}).$$

Then for any normal family $\{F_p : p \in \omega\}$ of subsets of $\omega^\omega \times \omega^\omega$,

(a) $F = \Phi^{o}(\{F_p : p \in \omega\})$ implies that

$$F^{*\Sigma(s)} = \Psi^{o}(\{F_{\gamma(p)}^{*\Sigma(s\frown\delta(p))} : p \in \omega\}),$$

for suitable (recursive) functions γ and δ (independent of the family $\{F_p\}$).

(b) $F = \Phi\Phi^{o}(\{F_p : p \in \omega\})$ implies

$$F^{*\Sigma(s)} = \Psi\Psi^{o}(\{F_{\alpha(p)}^{*\Sigma(s\frown\beta(p))} : p \in \omega\}),$$

where

$$\alpha(s) = \begin{cases} \langle f(n), \gamma(m)\rangle & \text{if } s = \langle n,m\rangle, \\ 0 & \text{otherwise;} \end{cases}$$

$$\beta(s) = \begin{cases} g(n)\frown\delta(m) & \text{if } s = \langle n,m\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

(c) $F = \Phi^{*}(\{F_p : p \in \omega\})$ implies

$$F^{*\Sigma(s)} = \Psi^{*}(\{F_{\tilde{f}(p)}^{*\Sigma(s\frown\tilde{g}(p))} : p \in \omega\}),$$

where $$\tilde{f}(s) = \begin{cases} \langle\alpha(n_0),\ldots,\alpha(n_{k-1})\rangle & \text{if } s = \langle n_0,\ldots,n_{k-1}\rangle, \\ 0 & \text{otherwise;} \end{cases}$$

$$\tilde{g}(s) = \begin{cases} \beta(n_0)\frown\beta(n_1)\frown\ldots\frown\beta(n_{k-1}) & \text{if } s = \langle n_0,\ldots,n_{k-1}\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Proof : Let N and M be the canonical bases for Φ and Ψ, respectively.

(a) Set $G = F^c$ and $G_p = F_p^c$ for each p.

Then $G = \Phi(\{G_p : p \in \omega\})$; and since Φ preserves the Baire property, each G^x has the Baire property. Therefore,

G^x is nonmeager in $\Sigma(s)$

$\leftrightarrow (\exists u \in Seq)\,[G^x$ is comeager in $G(s\hat{\ }u)]$

$\leftrightarrow (\exists u \in Seq)(\exists \eta \in M)(\forall m \in \eta)[G^x_{f(m)}$ is comeager in $\Sigma(s\hat{\ }u\hat{\ }g(m))]$ by hypothesis

$\leftrightarrow (\exists u \in Seq)(\exists \eta \in M)(\forall m \in \eta)(\forall v \in Seq)[G^x_{f(m)}$ is nonmeager in $\Sigma(s\hat{\ }u\hat{\ }g(m)\hat{\ }v)]$

$\leftrightarrow (\exists \eta \in M)(\forall m \in \eta)[G^x_{\gamma(m)}$ is nonmeager in $\Sigma(s\hat{\ }\delta(m))]$.

for some (recursive) functions γ and δ, as $\Phi_M \geq \cup, \cap$ and is normal.

Hence,

F^x is comeager in $\Sigma(s) \leftrightarrow G^x$ is meager in $\Sigma(s)$

$\leftrightarrow (\forall \eta \in M)(\exists m \in \eta)[G^x_{\gamma(m)}$ is meager in $\Sigma(s\hat{\ }\delta(m))]$

$\leftrightarrow (\forall \eta \in M)(\exists m \in \eta)[F^x_{\gamma(m)}$ is comeager in $\Sigma(s\hat{\ }\delta(m))]$

$\leftrightarrow (\exists \eta \in M^o)(\forall m \in \eta)[F^x_{\gamma(m)}$ is comeager in $\Sigma(s\hat{\ }\delta(m))]$.

This proves (a).

To prove (b), observe that

F^x is comeager in $\Sigma(s)$

$\leftrightarrow [\Phi(\{\Phi^o(\{F_{\langle p,q\rangle} : q \in \omega\}) : p \in \omega\})]^x$ is comeager in $\Sigma(s)$

$\leftrightarrow (\exists \eta \in M)(\forall n \in \eta)[(\Phi^{o}(\{F_{<f(n),q>} : q \in \omega\}))^{x}$ is comeager in $\Sigma(s\hat{}g(n))]$ (by hypothesis),

$\leftrightarrow (\exists \eta \in M)(\forall n \in \eta)(\exists \xi \in M^{o})(\forall m \in \xi)\ [F^{x}_{<f(n),\gamma(m)>}$ is comeager in $\Sigma(s\hat{}g(n)\hat{}\delta(m))]$ (by (a)).

Setting $\alpha(<n,m>) = <f(n),\gamma(m)>$, $\beta(<n,m>) = g(n)\hat{}\delta(m)$, we get (b),

(c) Since $F = \Phi^{*}(\{F_{p} : p \in \omega\})$, the canonical inductive operator Γ is given by

$$p \in \Gamma_{z}(A) \leftrightarrow z \notin F_{p} \vee (\forall \eta \in N)(\exists n \in \eta)(\exists \xi \in N)(\forall m \in \xi) [p * <<n,m>> \in A] : z \in \omega^{\omega} \times \omega^{\omega}.$$

Define $z \in F^{\mu}_{p} \leftrightarrow p \notin \Gamma^{\mu}_{z}$. Then by Lemma 6.2,

$$F^{*\Sigma(s)} = \bigcap_{\mu<\omega_1} [F^{\mu}_{s}]^{*\Sigma(s)}. \qquad \ldots(i)$$

Define a set relation Δ operative on ω as follows :

$$t \in \Delta_{x}(A) \leftrightarrow F^{x}_{(t)_0} \text{ is not comeager in } \Sigma((t)_1) \vee (\forall \eta \in M)(\exists n \in \eta)(\exists \xi \in M)(\forall m \in \xi) [<(t)_0\hat{}<<f(n),\gamma(m)>>, (t)_1\hat{}g(n)\hat{}\delta(m)> \in A].$$

We shall show that for any $s \in Seq$

$$x \notin F^{*\Sigma(s)} \leftrightarrow <e,s> \in \Delta^{\infty}_{x}. \qquad \ldots(ii)$$

To prove (ii) we shall show by induction on μ that for any $s,t \in Seq$,

$$x \notin [F^{\mu}_{t}]^{*\Sigma(s)} \leftrightarrow <t,s> \in \Delta^{\mu}_{x}.$$

This is clearly true for $\mu = 0$, so assume $\mu > 0$. Now,

$x \notin [F_t^{\mu}]^{*\Sigma(s)} \leftrightarrow (F_t^{\mu})^x$ is not comeager in $\Sigma(s)$

$\leftrightarrow F_t^x$ is not comeager in $\Sigma(s) \vee \left[\underline{\Phi}_{NN^0}(\{\bigcap_{\lambda<\mu} F^{\lambda}_{t\hat{}<p>} : p \in \omega\})\right]^x$

is not comeager in $\Sigma(s)$

$\leftrightarrow F_t^x$ is not comeager in $\Sigma(s) \vee (\forall \eta \in M)(\exists n \in \eta)(\exists \xi \in M)(\forall m \in \xi)$

$\left[(\bigcap_{\lambda<\mu} F^{\lambda}_{t\hat{}<<f(n),\gamma(m)>>})^x \text{ is not comeager in } \Sigma(s\hat{}g(n)\hat{}\delta(m))\right]$

(by (b))

$\leftrightarrow F_t^x$ is not comeager in $\Sigma(s) \vee (\forall \eta \in M)(\exists n \in \eta)(\exists \xi \in M)(\forall m \in \xi)$

$(\exists \lambda < \mu)\left[(F^{\lambda}_{t\hat{}<<f(n),\gamma(m)>>})^x \text{ is not comeager in } \Sigma(s\hat{}g(n)\hat{}\delta(m))\right]$

$\leftrightarrow F_t^x$ is not comeager in $\Sigma(s) \vee (\forall \eta \in M)(\exists n \in \eta)(\exists \xi \in M)(\forall m \in \xi)$

$(\exists \lambda < \mu)\left[< t\hat{}<<f(n),\gamma(m)>>, s\hat{}g(n)\hat{}\delta(m) > \in \Delta_x^{\lambda}\right]$ by the

induction hypothesis

$\leftrightarrow < t,s > \in \Delta_x^{\mu}$.

Hence, putting $t = e$ and using (i), we obtain (ii). Therefore,

F^x is comeager in $\Sigma(s) \leftrightarrow x \in F^{*\Sigma(s)} \leftrightarrow < e,s > \notin \Delta_x^{\infty}$ by (ii)

$\leftrightarrow (\exists \eta_0 \in M)(\forall n_0 \in \eta_0)(\forall \xi_0 \in M)(\exists m_0 \in \xi_0)\ldots\ldots$

$\ldots(\forall k)\left[F^x_{<<f(n_0),\gamma(m_0)>,\ldots,<f(n_{k-1}),\gamma(m_{k-1})>>} \text{ is comeager in } \Sigma(s\hat{}g(n_0)\hat{}\delta(m_0)\hat{}\ldots\hat{}g(n_{k-1})\hat{}\delta(m_{k-1}))\right]$.

Define $\check{f}$ and $\check{g}$ as follows :

$$\check{f}(s) = \begin{cases} <\alpha(n_0),\dots,\alpha(n_{k-1})> & \text{if } s = <n_0,\dots,n_{k-1}>, \\ 0 & \text{otherwise} \end{cases}$$

$$g(s) = \begin{cases} \beta(n_0)^\frown\beta(n_1)^\frown\dots^\frown\beta(n_{k-1}) & \text{if } s = <n_0,\dots,n_{k-1}>, \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows immediately.

<u>Remark</u> : It is clear from the proof of 6.3(c) that, under the hypothesis of the theorem, for any normal family $\{F_p\}$ of subsets of $\omega^\omega \times \omega^\omega$ with $F = \mathbb{R}\Phi_N(\{F_p\})$,

$$F^{*\Sigma(s)} = \mathbb{R}\Phi_M(\{F^{*\Sigma(s\hat{\ }g'(p))}_{f'(p)}\ ,\ p \in \omega\}),$$

where

$$f'(<n_0,\dots,n_{k-1}>) = <f(n_0),\dots,f(n_{k-1})>,$$

$$g'(<n_0,\dots,n_{k-1}>) = g(n_0)\hat{\ }g(n_1)\hat{\ }\dots\hat{\ }g(n_{k-1}).$$

Here, *normality of* $\Psi = \Phi_M$ is not required.

We shall now apply the Transfer Theorem to deduce Vaught's Formula for E^* (cf [51] Theorem 1.6]) and the Category Formula of Burgess [13].

6.4 <u>Theorem</u> (Vaught) : Assume $E = \mathcal{A}(\{E_n : n \in \omega\})$, with each $E_n \subseteq \omega^\omega \times \omega^\omega$ normal. Then $x \in E^{*\Sigma(s)}$ *if and only if*

$$(\forall u_0 \in Seq)(\exists v_0 \in Seq)(\exists k_0)(\forall u_1 \in Seq)(\exists v_1 \in Seq)(\exists k_1)\dots\dots$$

$$\dots(\forall i)\left[E^x_{<k_0,\dots,k_{i-1}>} \text{ is comeager in } \Sigma(s\hat{\ }u_0\hat{\ }v_0\hat{\ }\dots\hat{\ }u_{i-1}\hat{\ }v_{i-1}\right].$$

Proof : Let $N = \{\{n\} : n \in \omega\}$ so that $\Phi_N = \bigcup$. Suppose $F = \Phi_N(\{F_n : n \in \omega\}) = \bigcup_{n \in \omega} F_n$, where each F_n is a normal subset $\omega^\omega \times \omega^\omega$. Then,

F^x is comeager in $\Sigma(s) \leftrightarrow (\forall u \in Seq)[F^x$ is nonmeager in $\Sigma(s^\frown u)]$

$\leftrightarrow (\forall u \in Seq)(\exists k)[F^x_k$ is nonmeager in $\Sigma(s^\frown u)]$

$\leftrightarrow (\forall u \in Seq)(\exists k)(\exists v \in Seq)[F^x_k$ is comeager in $\Sigma(s^\frown u^\frown v)]$.

Let Φ_M be a δ-s operation such that for any family $\{A_n\}$,

$$\Phi_M(\{A_n : n \in \omega\}) = \bigcap_{u \in Seq} \bigcup_{v \in Seq} \bigcup_{k \in \omega} A_{<u,<k,v>>}.$$

Hence,

F^x is comeager in $\Sigma(s)$

$\leftrightarrow (\exists \eta \in M)(\forall n \in \eta)[F^x_{(n)_{1,0}}$ is comeager in $\Sigma(s^\frown(n)_0^\frown(n)_{1,1})]$

$\leftrightarrow (\exists \eta \in M)(\forall n \in \eta)[F^x_{f(n)}$ is comeager in $\Sigma(s^\frown g(n))]$,

where $f(n) = (n)_{1,0}$ and $g(n) = (n)_0^\frown(n)_{1,1}$. Since

$$E = \mathcal{A}(\{E_n : n \in \omega\}) = \mathbb{R}\Phi_N(\{E_n\}),$$

the Transfer Theorem immediately gives

$$E^{*\Sigma(s)} = \mathbb{R}\Phi_M(\{E^{*\Sigma(s^\frown \check{g}(p))}_{\check{f}(p)} : p \in \omega\}),$$

where $\check{g}(<n_0,\ldots,n_{k-1}>) = g(n_0)^\frown g(n_1)^\frown\ldots^\frown g(n_{k-1})$

and $\check{f}(<n_0,\ldots,n_{k-1}>) = <f(n_0),\ldots,f(n_{k-1})>$. Therefore,

E^x is comeager in $\Sigma(s)$

$\leftrightarrow (\exists \eta_0 \in M)(\forall n_0 \in \eta_0)(\exists \eta_1 \in M)(\forall n_1 \in \eta_1)\ldots\ldots$

$\ldots(\forall i)[E^x_{<f(n_0),\ldots,(n_{i-1})>}$ is comeager in $\Sigma(s^\frown g(n_0)^\frown\ldots^\frown g(n_{i-1}))]$

$$\leftrightarrow (\forall u_0 \varepsilon \text{Seq})(\exists v_0 \varepsilon \text{Seq})(\exists k_0)(\forall u_1 \varepsilon \text{Seq})(\exists v_1 \varepsilon \text{Seq})(\exists k_1) \ldots\ldots$$

$$\ldots(\forall i)\left[E^x_{<k_0,\ldots,k_{i-1}>} \text{ is comeager in } \Sigma(s^\frown u_0^\frown v_0^\frown \ldots ^\frown u_{i-1}^\frown v_{i-1})\right].$$

This completes the proof.

6.5 <u>Definition</u>: Define an operation $\mathcal{V}$ as follows.

$x \varepsilon \mathcal{V}(\{E_n : n \varepsilon \omega\})$

$$\leftrightarrow (\forall u_0 \varepsilon \text{Seq})(\exists v_0 \varepsilon \text{Seq})(\exists k_0)(\forall u_1 \varepsilon \text{Seq})(\exists v_1 \varepsilon \text{Seq})(\exists k_1) \ldots\ldots$$

$$\ldots(\forall i)\left[x \varepsilon E_{<<u_0,<k_0,v_0>>,\ldots,<u_{i-1},<k_{i-1},v_{i-1}>>>}\right].$$

Clearly $\mathcal{V}$ is positive analytical and $\mathcal{V} \hookrightarrow \mathcal{A}$. Call $\mathcal{V}$ the Vaught operation.

Define a sequence of positive analytical operations $\{S_\rho : \rho < \omega_1\}$ by the induction

$$S_0 = \mathcal{V}, \quad S_{\rho+1} = S_\rho^* .$$

If λ is limit, choose $\rho_i \uparrow \lambda$ and set for any family $\{E_n\}$,

$$\Psi(\{E_n\}) = \bigcap_{i \varepsilon \omega} \Phi_{M_{\rho_i} M^0_{\rho_i}} (\{E_{<i,m>} : m \varepsilon \omega\}),$$

where M_{ρ_i} is the canonical base for S_{ρ_i}. Then define $S_\lambda = \Psi^*$. It is easy to check by induction that for each ρ, $R_\rho \hookrightarrow S_\rho$ and hence

$$\mathcal{R}^\rho = \Sigma_1^{R_\rho} = \Sigma_1^{S_\rho} .$$

We shall now deduce the Category Formula of Burgess by showing that if E is in $\mathcal{R}^\rho$, the E^* is computed by S_ρ.

6.6 <u>Theorem</u> : Let $E \subseteq \omega^\omega \times \omega^\omega$ be a set in $\mathcal{R}^\rho$. Then $E^{*\Sigma(s)} = \{x : E^x$ is comeager in $\Sigma(s)\}$ is also in $\mathcal{R}^\rho$.

<u>Proof</u> : We shall prove the theorem by induction. Let N_ρ and M_ρ denote the canonical bases for R_ρ and S_ρ, respectively.

Assume that for all $\lambda < \rho$ there are functions f_λ and g_λ such that if

$$F = R_\lambda(\{F_n : n \in \omega\}),$$

with each $F_n \subseteq \omega^\omega \times \omega^\omega$ normal, then

$$F^{*\Sigma(s)} = S_\lambda(\{F_{f_\lambda(n)}^{*\Sigma(s\hat{}g_\lambda(n))} : n \in \omega\}).$$

We shall then show that if $E = R_\rho(\{E_n\})$, then $E^{*\Sigma(s)}$ is similarly computed by S_ρ. The result then follows by observing that for each clopen E_n, E_n^* is also clopen.

<u>Case 1</u> : $\rho = \lambda + 1$. In this case, $E = R_\lambda^*(\{E_n : n \in \omega\})$. Hence by the Transfer Theorem,

$$E^x \text{ is comeager in } \Sigma(s) \leftrightarrow (\exists \eta_0 \in M_\lambda)(\forall n_0 \in \eta_0)(\forall \xi_0 \in M_0)(\exists m_0 \in \xi_0)\ldots$$

$$\ldots(\forall i)\left[E^x_{\overleftarrow{f}_\lambda(<<n_0,m_0>,\ldots,<n_{i-1},m_{i-1}>>)} \text{ is comeager in } \Sigma(s\hat{}\overleftarrow{g}_\lambda(<<n_0,m_0>,\ldots,<n_{i-1},m_{i-1}>>))\right]$$

where $\overleftarrow{f}_\lambda$ and $\overleftarrow{g}_\lambda$ are related to f_λ and g_λ as in 6.3. Setting $f_\rho = \overleftarrow{f}_\lambda$ and $g_\rho = \overleftarrow{g}_\lambda$ we get

$$E^{*\Sigma(s)} = S_\rho(\{E_{f_\rho(n)}^{*\Sigma(s\hat{}g_\rho(n))} : n \in \omega\}).$$

Case 2 : ρ is limit. Choose a sequence $\rho_i \uparrow \rho$ and for any normal family $\{H_n : n \in \omega\}$ set

$$H = \Phi(\{H_n\}) = \bigcap_{i \in \omega} \Phi_{N_{\rho_i} N^o_{\rho_i}} (\{H_{<i,n>} : n \in \omega\}).$$

Then,

H^x is comeager in $\Sigma(s)$

$$\leftrightarrow (\forall i) \left[(\Phi_{N_{\rho_i} N^o_{\rho_i}} (\{H_{<i,m>} : m \in \omega\}))^x \text{ is comeager in } \Sigma(s) \right]$$

$$\leftrightarrow (\forall i)(\exists \eta \in M_{\rho_i})(\forall n \in \eta)(\forall \xi \in M_{\rho_i})(\exists m \in \xi)\left[H^x_{<i,\alpha_i(<n,m>)>} \text{ is comeager in } \Sigma(s\hat{\ }\beta_i(<n,m>))\right] \text{ (by Theorem 6.3)}$$

where α_i and β_i are obtained as in 6.3(b). Therefore,

$$H^{*\Sigma(s)} = \Psi(\{H^{*\Sigma(s\hat{\ }g(m))}_{f(m)} : m \in \omega\}),$$

where Ψ is the operation in 6.5 and

$$f(<i,n>) = <i,\alpha_i(n)>, \quad g(<i,n>) = \beta_i(n).$$

Since $E = \Phi^*(\{E_n\})$, by applying the Transfer Theorem again we obtain the result as in Case 1.

As the R-sets have the Baire property the next result follows immediately.

6.7 Corollary : If $E \in c\mathcal{R}^\rho$, then $E^{*\Sigma(s)}$ is also in $c\mathcal{R}^\rho$.

By 6.6 and by repeatedly applying the Transfer Theorem to every level of the hierarchy of $\mathcal{BR}^\rho$-sets one obtains

6.8 <u>Corollary</u> : If $E \in \mathcal{BR}^{\rho}_{\mu}$ $(\mu, \rho < \omega_1)$, then $E^{*\Sigma(s)}$ is also in $\mathcal{BR}^{\rho}_{\mu}$. In particular, putting $\rho = 0$, one has for any C-set E, E^* is also a C-set.

6.9 We will now establish the measure-theoretic counterparts of Theorem 6.3 and its corollaries.

Throughout the rest of this section $G \subseteq \omega \times \omega^{\omega}$ will be a fixed good Σ^0_1 universal set which is ω-universal for Σ^0_1 subsets of ω^{ω}, and μ will denote the Lebesgue measure on ω^{ω} as defined in [24] (or any other normalized, complete, regular measure with a reasonable definability condition). We shall also fix a recursive function b such that

$$G(n,x) \ \& \ G(m,x) \leftrightarrow G(b(n,m),x).$$

(We shall also denote by E^x the vertical section of E at x and think of the rationals coded in some recursive way by elements of ω).

Call a set $A \subseteq \mathcal{X} \times \mathcal{Y}$ normal measurable if A^x is measurable for each x.

For any normal measurable set $A \subseteq \omega^{\omega} \times \omega^{\omega}$ and any $n \in \omega$, $r \geq 0$ rational, we define

$$A^{*[n,r]} = \{x : \mu(A^x \cap G^n) \geq r\}.$$

Call $s \in Seq$ <u>relevant</u> if $lh(s) = 3 \ \& \ (s)_0$ is rational $\& \ (s)_2 \in Seq$. Let s and t be two relevant sequences. Then t is an <u>extension</u> of s if

$$G^{(t)_1} \subseteq G^{(s)_1} \ \& \ (s)_2 \subseteq (t)_2 \ \& \ lh((t)_2) = lh((s)_2) + 1.$$

For any rational r, define

$$N_e^{(r)} = \{\eta \subseteq \omega : \eta \text{ is a finite set of relevant sequences} \\ \& \sum_{s \in \eta} (s)_0 > r \ \& (\forall s \in \eta)(\mathrm{lh}((s)_2) = 1) \\ \& \forall s,t \in \eta (s \neq t \to G^{(s)_1} \cap G^{(t)_1} = \varphi)\}.$$

If $s_1, s_2, \ldots, s_k$ are relevant sequences such that s_{i+1} is an extension of s_i and $\mathrm{lh}((s_i)_2) = i$, then define

$$N_{\langle s_1,\ldots,s_k\rangle} = \{\eta \subseteq \omega : \eta \text{ is a finite set of extensions of } s_k \text{ such} \\ \text{that } \sum_{t \in \eta} (t)_0 > (s_k)_0 \ \& \ \forall t,u \in \eta (t \neq u \to G^{(t)_1} \cap G^{(u)_1} = \varphi)\}.$$

We will now formulate and write down less detailed proofs of the measure analogues of the results we have proved for category.

Notice that the next result due to Burgess [13] and implicit in Kechris [24] gives us the Measure Formula at the base level.

6.10 Lemma : Let $\{E_n : n \in \omega\}$ be a family of normal measurable subsets of $\omega^\omega \times \omega^\omega$ and suppose

$$E = \mathcal{A}(\{E_n : n \in \omega\}).$$

Then for any rational $r \geq 0$,

$$\mu(E^x) > r \leftrightarrow (\exists \eta_1 \in N_e^{(r)})(\forall s_1 \in \eta_1)(\exists \eta_2 \in N_{\langle s_1\rangle})(\forall s_2 \in \eta_2)\ldots\ldots \\ \ldots(\forall k)\left[\mu(E^x_{(s_k)_2} \cap G^{(s_k)_1}) > (s_k)_0\right]. \quad \square$$

To prove Lemma 6.10 one needs the following observation due to Burgess and may be regarded as the Measure Formula for countable union.

6.11 Lemma : Let $\{E_n : n \in \omega\}$ be a family of measurable subsets of ω^ω and let $E = \bigcup_{n \geq 0} E_n$. Plainly E is measurable and moreover for any rational $r \geq 0$, $n \in \omega$

$$\mu(E \cap G^n) > r \leftrightarrow (\exists \eta \in N_{<s^*>})(\forall s \in \eta)\left[\mu(E_{(s)_2} \cap G^{(s)_1}) > (s)_0\right],$$

where $s^* = <r,n,e>$ and $N_{<s^*>}$ is defined as in 6.9.

Proof : Suppose for some $\eta \in N_{<s^*>}$ we have

$$(\forall s \in \eta)\left[\mu(E_{(s)_2} \cap G^{(s)_1}) > (s)_0\right].$$

Recall the definition of $N_{<s^*>}$. Then observe that

$$\mu(\bigcup_{i \geq 0}(E_i \cap G^n)) \geq \mu(\bigcup_{s \in \eta}(E_{(s)_2} \cap G^{(s)_1}))$$

$$= \sum_{s \in \eta} \mu(E_{(s)_2} \cap G^{(s)_1}) > \sum_{s \in \eta}(s)_0 > r.$$

To prove the reverse implication, assume without loss of generality that $G^n = \omega^\omega$. Since $\mu(\bigcup_{i \geq 0} E_i) > r$, we can find a $p \in \omega$ and $\varepsilon > 0$ such that

$$\mu(\bigcup_{i<p} E_i) > r + p\,\varepsilon.$$

Since E_i is measurable, it is easy to get a recursive (in fact, a finite union of basic open sets) B_i such that $\mu(E_i \triangle B_i) < \varepsilon$.

Set

$$B_i' = B_i - \bigcup_{j<i} B_j .$$

Clearly each B_i' is recursive and moreover,

$$\bigcup_{i<p} E_i - \bigcup_{i<p} E_i \cap B_i' \subseteq \bigcup_{i<p} (E_i \triangle B_i).$$

Hence,

$$\mu(\bigcup_{i<p} E_i) - \mu(\bigcup_{i<p} E_i \cap B_i') \le p\,\varepsilon .$$

Consequently,

$$\sum_{i<p} \mu(E_i \cap B_i') \ge \mu(\bigcup_{i<p} E_i) - p\,\varepsilon > r .$$

Let $i_1,\dots,i_k$ be an enumeration of all those $i < p$ for which $\mu(E_i \cap B_i') > 0$. Get rationals $r_1,\dots,r_k$ such that

$$\mu(E_{i_j} \cap B_{i_j}') > r_j \quad \text{and} \quad \sum_{j=1}^{k} r_j > r .$$

Get $n_1,\dots,n_k$ such that $G^{n_j} = B_{i_j}'$.

Put $\quad \eta = \{ \langle r_j, n_j, \langle i_j \rangle\rangle : 1 \le j \le k \}$.

Clearly $\eta \in N_{\langle s^* \rangle}$ and for each $s \in \eta$

$$\mu(F_{(s)_2} \cap G^{(s)_1}) > (s)_0 . \qquad \square$$

Lemma 6.11 yields Lemma 6.10 and the proof of the next theorem (6.12) will give us an idea of obtaining 6.10 from 6.11. (For details see [13]).

With notation as in 6.10, now define

$$x \in A_{<s_1,\dots,s_k>} \leftrightarrow \mu(E^x_{(s_k)_2} \cap G^{(s_k)_1}) > (s_k)_0$$

and $\quad x \in A_e \leftrightarrow \mu(E^x_e \cap G^{n^*}) > r ,$

where $G^{n^*} = \omega^\omega$. Then the above equivalence (6.10) reduces to

$$\mu(E^x) > r \leftrightarrow (\exists \eta_1 \in N_e^{(r)})(\forall s_1 \in \eta_1)(\exists \eta_2 \in N_{<s_1>})(\forall s_2 \in \eta_2)\dots\dots$$
$$\dots(\forall k)\left[x \in A_{<s_1,\dots,s_{k-1}>}\right].$$

The right side of the above equivalence clearly defines a p.a.o. whose canonical base we will denote by $M^{(r)}$. Thus we have

$$\mu(E^x) > r \leftrightarrow (\exists \eta \in M^{(r)})(\forall m \in \eta)[x \in A_m] \qquad \dots(9)$$
$$\leftrightarrow (\exists \eta \in M^{(r)})(\forall m \in \eta)\left[\mu(E^x_{f(m)} \cap G^{g(m)}) > h(r,m)\right],$$

where f,g,h are (recursive) functions such that

$$f(e) = e ,\ f(<s_1,\dots,s_k>) = (s_k)_2 \ ;$$

$$g(e) = n^*,\ g(<s_1,\dots,s_k>) = (s_k)_1 \ ;$$

$$h(r,e) = r,\ h(r, <s_1,\dots,s_k>) = (s_k)_0 .$$

This gives us a motivation to formulate the measure-theoretic counterpart of the Transfer Theorem 6.3 as follows.

6.12 <u>Theorem</u> : Let Φ be a measurability preserving p.a.o. . Suppose for each $r \geq 0$, there is an operation $\Psi^{(r)} = \Phi_{M^{(r)}} \geq \cap, \cup$ such that whenever

$$F = \Phi(\{F_n : n \in \omega\}), \text{ with each } F_n \subseteq \omega^\omega \times \omega^\omega \text{ normal measurable},$$

then

$$\mu(F^x) \geq r \leftrightarrow (\exists \eta \in M^{(r)})(\forall m \in \eta)\left[\mu(F^x_{f(m)} \cap G^{g(m)}) \geq h(r,m)\right],$$

where f,g,h are (recursive) functions from ω into ω. Then, whenever

$$E = \mathbb{R}\Phi(\{E_n : n \in \omega\}),$$

with $\{E_n\}$ a <u>regular</u> family of normal measurable sets, we have

$\mu(E^x \cap G^n) \geq r$

$$\leftrightarrow (\exists \eta_0 \in M^{(r)})(\forall m_0 \in \eta_0)(\exists \eta_1 \in M^{(\overleftarrow{h}(r, <m_0>))})(\forall m_1 \in \eta_1)\ldots\ldots$$

$$\ldots(\forall k)\left[\mu(E^x_{<f(m_0),\ldots,f(m_{k-1})>} \cap G^{\overleftarrow{b}(<m_0,\ldots,m_{k-1}>,n)}) \geq \overleftarrow{h}(r,<m_0,\ldots,m_{k-1}>)\right],$$

where $\overleftarrow{b}$ and $\overleftarrow{h}$ are defined by recursion as follows :

$$\overleftarrow{b}(e,n) = n,$$

$$\overleftarrow{b}(<m_0,\ldots,m_k>,n) = b(\overleftarrow{b}(<m_0,\ldots,m_{k-1}>,n), g(m_k)),$$

$$\overleftarrow{b}(\lambda,n) = 0 \qquad \text{otherwise}.$$

$$\overleftarrow{h}(r,e) = r,$$

$$\overleftarrow{h}(r, <m_0,\ldots,m_k>) = h(\overleftarrow{h}(r, <m_0,\ldots,m_{k-1}>), m_k),$$

$$\overleftarrow{h}(r,\lambda) = 0 \qquad \text{otherwise}.$$

<u>Proof</u> : Let N be the canonical base for Φ and let Γ be the inductive operator associated with N and $\{E_n : n \in \omega\}$. Define E^μ_s and T^μ as in 2.6. Then

$$E = \bigcap_{\rho < \omega_1} E_e^{\rho} = \bigcup_{\rho < \omega_1} (E_e^{\rho} - T^{\rho}).$$

Using this decomposition one can show as before that

$$E^{*}[n,r] = \bigcap_{\rho < \omega_1} (E_e^{\rho})^{*}[n,r] \qquad \ldots(i)$$

Now define a set relation ψ as follows

$$<s,n,r> \varepsilon\ \psi_x(A) \leftrightarrow \mu(E_s^x \cap G^n) < r$$

$$\vee\ (\forall \eta \ \varepsilon\ M^{(r)})(\exists\ m \varepsilon \eta)\ \Big[< s * <f(m)>, b(g(m),n), h(r,m)> \varepsilon\ A\Big].$$

We shall show by induction on ρ that for every $s \ \varepsilon$ Seq

$$x \notin (E_s^{\rho})^{*}[n,r] \leftrightarrow < s,n,r > \varepsilon\ \psi_x^{\rho}.$$

First observe that since $\{E_n : n \ \varepsilon\ \omega\}$ is regular,

$$E_s^{\rho} = E_s \cap \Phi_N(\{\bigcap_{\lambda < \rho} E^{\lambda}_{s * <m>} : m \ \varepsilon\ \omega\})$$

$$= \Phi_N(\{\bigcap_{\lambda < \rho} E^{\lambda}_{s * <m>} : m \ \varepsilon\ \omega\}). \qquad \ldots(ii)$$

Hence,

$$x \notin (E_s^{\rho})^{*}[n,r] \leftrightarrow \mu((E_s^{\rho})^{x} \cap G^n) < r$$

$$\leftrightarrow (\forall \eta \ \varepsilon\ M^{(r)})(\exists\ m \varepsilon \eta)\ \Big[\mu((\bigcap_{\lambda < \rho} E^{\lambda}_{s * <f(m)>})^{x} \cap G^{b(g(m),n)}) < h(r,m)\Big]$$

by (ii) and hypothesis of the theorem

$$\leftrightarrow (\forall \eta \ \varepsilon\ M^{(r)})(\exists\ m \varepsilon \eta)(\exists \lambda < \rho)\ \Big[\mu((E^{\lambda}_{s * <f(m)>})^{x} \cap G^{b(g(m),n)}) < h(r,m)\Big]$$

$$\leftrightarrow (\forall \eta \in \mathcal{M}^{(r)})(\exists m \in \eta)(\exists \lambda < \rho)\left[x \notin (E^{\lambda}_{s * <f(m)>})^{*[b(g(m),n),h(r,m)]}\right]$$

$$\leftrightarrow (\forall \eta \in \mathcal{M}^{(r)})(\exists m \in \eta)(\exists \lambda < \rho)\left[<s * <f(m)>, b(g(m),n),h(r,m)> \in \psi^{\lambda}_{x}\right]$$

by induction hypothesis

$$\leftrightarrow <s,n,r> \in \psi^{\rho}_{x}.$$

Thus by (i) we have

$$x \notin E^{*[n,r]} \leftrightarrow <s,n,r> \in \psi^{\infty}_{x}.$$

Consequently,

$$\mu(E^{x} \cap G^{n}) \geq r \leftrightarrow <s,n,r> \notin \psi^{\infty}_{x}$$

$$\leftrightarrow (\exists \eta_0 \in \mathcal{M}^{(r)})(\forall m_0 \in \eta_0)(\exists \eta_1 \in \mathcal{M}^{(\breve{h}(r,<m_0>))})(\forall m_1 \in \eta_1)\ldots$$

$$\ldots (\forall k)\left[\mu(E^{x}_{<f(m_0),\ldots,f(m_{k-1})>} \cap G^{\breve{b}(<m_0,\ldots,m_{k-1}>,n)}) \geq \breve{h}(r, <m_0,\ldots,m_{k-1}>)\right]. \quad \square$$

An immediate consequence of Theorem 6.12 is the following

6.13 <u>Theorem</u> : For every $\rho < \omega_1$ and $r \geq 0$, the sets

$$\{x : \mu(E^{x}) > r\} \text{ and } \{x : \mu(E^{x}) \geq r\}$$

are in $\mathcal{R}^{\rho}$ whenever $E \subseteq \mathcal{X} \times \mathcal{Y}$ is in $\mathcal{R}^{\rho}$.

The proof of the above theorem is exactly along the lines of the proof of Theorem 6.6. One shows by induction that for each $\rho < \omega_1$ and every rational $r \geq 0$ there are functions f_ρ, f'_ρ, g_ρ, g'_ρ, h_ρ, h'_ρ and operations $\Phi_{\mathcal{M}^r_\rho}$, $\Phi_{\mathcal{M}'^r_\rho}$ R_ρ such that whenever

$F = R_\rho(\{F_n : n \in \omega\})$, $F_n \subseteq \mathcal{X} \times \mathcal{Y}$ being normal measurable

$$\mu(F^x) > r \Leftrightarrow (\exists \eta \in M_\rho^r)(\forall n \in \eta)\{\mu(F^x_{f_\rho(n)} \cap G^{g_\rho(n)}) > h_\rho(r,n)\}$$

and $$\mu(F^x) \geq r \Leftrightarrow (\exists \eta \in M_\rho'^r)(\forall n \in \eta)\{\mu(F^x_{f_\rho'(n)} \cap G^{g_\rho'(n)}) \geq h_\rho'(r,n)\}.$$

By 6.13 and repeated applications of 6.12 we have

6.14 <u>Corollary</u> : If $E \in \mathcal{BR}_\lambda^\rho$ $(\rho, \lambda < \omega_1)$, then for every rational $r \geq 0$, the sets

$$\{x : \mu(E^x) > r\} \text{ and } \{x : \mu(E^x) \geq r\}$$

are also in $\mathcal{BR}_\lambda^\rho$. □

7. <u>Approximating R-sets in Product Spaces</u> : (In this section spaces of type 1 will be denoted by X, Y, etc. and E^x will denote the vertical section of E at x).

R-sets in $X \times Y$ are in general complicated sets and cannot be related to any reasonable product σ-field. For instance, as shown by B.V. Rao [41], the universal Σ_1^1 set U does not belong to $\mathcal{L} \otimes \mathcal{L}$, the product of σ-fields of Lebesgue measurable sets. However, V.V. Srivatsa [46] has shown that C-sets in $X \times Y$ can be approximated, in the sense of measure and category, by sets in product σ-fields. More precisely, if $A \subseteq X \times Y$, is a C-set, then there are B and C in $\mathcal{C}(X) \otimes \mathcal{B}(Y)$ ($\mathcal{C}(X)$ is the family of C-sets in X and $\mathcal{B}(Y)$ the Borel σ-field on Y), such that $B \subseteq A \subseteq C$ and $C^x - B^x$ is meager (null) for each $x \in X$. The problem of approximating sets in product spaces was first studied by V.V. Srivatsa and can be viewed as an alternative and, perhaps, easier method for proving various selection theorems (cf [46]). Srivatsa's method, however, does not seem to generalize to R-sets. In this section we shall extend Srivatsa's theorem to R-sets. We shall show that R-sets in product spaces can be approximated as above, in the sense of category, by sets in product σ-fields. (The precise statement is given in Theorem 7. 10). This incidentally will give the selection theorem of Burgess (cf [13]).

We shall prove our result through a series of lemmas.

7.1 <u>Lemma</u> : Let $\mathcal{F}$ be a σ-field closed under operation $\mathcal{A}$ and let $\mathcal{B}_{\omega^\omega}$ denote the Borel σ-field on ω^ω. Let $E \subseteq \omega^\omega \times \omega^\omega$. Suppose there is a family $\{A_n\}$ such that

(a) $A = \mathcal{A}(\{A_n\})$

(b) $A \subseteq E$

(c) Each $A_n \in \mathcal{F} \otimes \mathcal{B}_{\omega^\omega}$

(d) A^x is comeager whenever E^x is comeager.

Then there is a set $B \in \mathcal{F} \otimes \mathcal{B}_{\omega^\omega}$ such that $B \subseteq E$ and B^x is comeager whenever E^x is comeager.

<u>Proof</u> : Let $I : \omega^\omega \to \omega^\omega$ be the characteristic function of a generator for a countably generated sub-σ-field $\mathcal{F}_0$ of $\mathcal{F}$ such that each $A_n \in \mathcal{F}_0 \otimes \mathcal{B}_{\omega^\omega}$. Let $I(\omega^\omega) = D$. Then, as is well-known, I is a bimeasurable function between $\mathcal{F}_0$ and $\mathcal{B}_D$, the Borel σ-field on D.

Set

$$A_n' = \{(I(\alpha),\beta) : (\alpha,\beta) \in A_n\},$$

$$A' = \{(I(\alpha),\beta) : (\alpha,\beta) \in A\}.$$

Clearly, $A_n' \in \mathcal{B}_D \otimes \mathcal{B}_{\omega^\omega}$ and $A' = \mathcal{A}(\{A_n'\})$.

Hence A' is an analytic set in $D \times \omega^\omega$. Get an analytic set $C \subseteq \omega^\omega \times \omega^\omega$ such that

$$A' = C \cap (D \times \omega^\omega).$$

Then by 1.6 of [46], get $B' \in \mathcal{Q} \otimes \mathcal{B}_{\omega^\omega}$, where $\mathcal{Q}$ is the analytic σ-field on ω^ω, such that $B' \subseteq C$ and B'^x is comeager whenever C^x is comeager. Put

$$B = \{(\alpha,\beta) : (I(\alpha),\beta) \in B'\}.$$

Since $\mathcal{F}$ is closed under operation $\mathcal{A}$, clearly $B \in \mathcal{F} \otimes \mathcal{B}_{\omega^\omega}$ and moreover, $B \subseteq E$ and B^x is comeager whenever E^x is comeager.

We have adapted the proof of the Category Formula [13] in the next lemma.

7.2 <u>Lemma</u> : Suppose we have

$$(\forall s_0)(\exists t_0)(\forall s_1)(\exists t_1)\ldots\{(\forall a_0)(\exists b_0)(\forall a_1)(\exists b_1)\ldots P(\alpha,\beta)\}$$

$$\longleftrightarrow (\forall s_0)(\forall a_0)(\exists t_0)(\exists b_0)(\forall s_1)(\forall a_1)(\exists t_1)(\exists b_1)\ldots\ldots\ldots P(\alpha,\beta),$$

where $\alpha = (a_0,b_0,a_1,b_1,\ldots)$; $a_i,b_i \in \omega$

and $\beta = s_0 * t_0 * s_1 * t_1 \ldots$; $s_i,t_i \in \omega^{<\omega}$.

If $\exists$ wins the second game with strategy σ, then he can modify σ to a winning strategy σ^* such that, to every complete play $s_0, a_0, t_0, b_0, \ldots$ consistent with σ^*, there corresponds a complete play $s_0', t_0', s_1', t_1', \ldots$ consistent with a winning strategy for $\exists$ in the first game such that

$$s_0 * t_0 * s_1 * t_1 \ldots = s_0' * t_0' * s_1' * t_1' \ldots$$

Proof. Let $w_{-1}, w_0, w_1, w_2, \ldots$ be an enumeration of $\omega^{<\omega}$ such that w_{-1} is the empty sequence and if w_i is an initial segment of w_j, then $i < j$. For $s \in \omega^{<\omega}$, its code, denoted by $|s|$, is its position in the enumeration.

Suppose $\exists$ wins the second game with strategy σ. We shall now construct a winning strategy τ for player II in the first (Banach-Mazur) game. τ will be defined (by induction) in such a way that every partial play $s_0, t_0, \ldots, s_{n-1}, t_{n-1}$ consistent with τ corresponds to a partial play $a_0; u_0; b_0, v_0; \ldots; a_{m-1}, u_{m-1}; b_{m-1}, v_{m-1}$ consistent with σ, such that $|(a_0, \ldots, a_{m-1})| = n-1$ and $s_0 * t_0 * \ldots * t_{n-1} = u_0 * v_0 * \ldots * u_{m-1} * v_{m-1}$.

Suppose $s_0, t_0, \ldots, s_{n-1}, t_{n-1}$ have been defined consistent with τ. We now define $\tau(s_0, t_0, \ldots, s_{n-1}, t_{n-1}, s)$. Let n be the code of $(a_0, \ldots, a_{m-1}, a)$ and let $|(a_0, \ldots, a_{m-1})| = n'$. Clearly, $n' < n$. Let the partial play (consistent with σ) corresponding to $(s_0, t_0, \ldots, s_{n'}, t_{n'})$ be $(a_0, u_0; \ldots; a_{m-1}, u_{m-1}; b_{m-1}, v_{m-1})$ such that $s_0 * t_0 * \ldots * s_{n'} * t_{n'} = u_0 * v_0 * \ldots * u_{m-1} * v_{m-1}$. Put $u = s_{n'+1} * t_{n'+1} * \ldots * s_{n-1} * t_{n-1} * s$.

Let

$$\sigma(a_0, u_0; b_0, v_0; \ldots; b_{m-1}, v_{m-1}; a, u) = (b, v).$$

Then

$$\tau(s_0, t_0, \ldots, s_{n-1}, t_{n-1}, s) = v \;,$$

and the partial play associated with

$$s_0,\ t_0,\ldots,s_{n-1},\ t_{n-1},\ s,\ v$$

is

$$a_0,u_0;b_0,v_0;\ldots;b_{m-1},v_{m-1};a,u;b,v.$$

We shall now show that τ is a winning strategy for player II in the Banach-Mazur game. Let

$$s_0,\ t_0,\ s_1,\ t_1,\ s_2,\ t_2,\ \ldots.$$

be a complete play consistent with τ. We shall have to show that

$$(\forall a_0)(\exists b_0)(\forall a_1)(\exists b_1)\ldots P(\alpha,\beta),$$

where $\alpha = (a_0,b_0,a_1,b_1,\ldots.)$ and

$$\beta = s_0 * t_0 * s_1 * t_1 * \ldots. .$$

So let $\forall$ play a_0. Suppose $|(a_0)| = n_0$. By definition of τ, the partial play (consistent with σ) corresponding to $s_0,t_0,\ldots,s_{n_0},t_{n_0}$ is

$$a_0,s_0 * t_0 * \ldots\ldots * s_{n_0};\ b_0,t_{n_0}$$

for some $b_0 \varepsilon\ \omega$. $\exists$ replies with b_0. Next suppose $\forall$ plays a_1 and let $|(a_0,a_1)| = n_1$. By definition of τ, the partial play (consistent with σ) corresponding to

$$s_0,t_0,\ldots,s_{n_0},t_{n_0},\ldots,s_{n_1},t_{n_1}$$

is

$$a_0,\underbrace{s_0 * t_0 * \ldots * s_{n_0}}_{u_0};\ b_0,t_{n_0} = v_0;\ a_1,\underbrace{s_{n_0+1} * t_{n_0+1} * \ldots * s_{n_1}}_{u_1};\ b_1,t_{n_1} = v_1.$$

$\exists$ replies with b_1 and the play proceeds as described. Since the play

$$a_0, u_0;\ b_0, v_0;\ a_1, u_1;\ b_1, v_1;\ \ldots.$$

is consistent with σ, we have

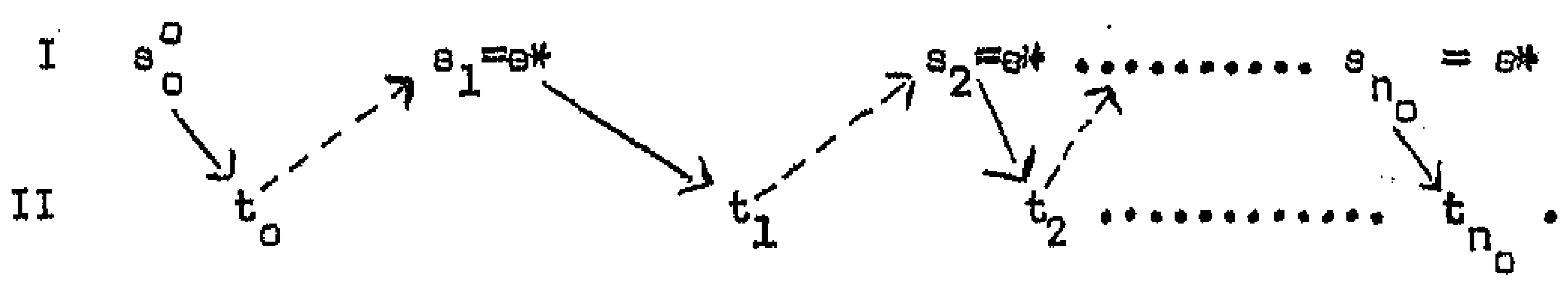
$$P(\alpha, u_0 * v_0 * u_1 * v_1 * \ldots).$$

But then $\beta = u_0 * v_0 * u_1 * v_1 * \ldots.$

Hence $P(\alpha,\beta)$. Consequently, τ is a winning strategy for II in the Banach-Mazur game.

We shall now modify σ to σ^* such that any complete play consistent with σ^* corresponds to a complete play consistent with τ.

<u>Definition of σ^*</u>: Let $\forall$ play a_0, s_0^0 and suppose $|(a_0)| = n_0$. Simulate the following partial play in the Banach-Mazur game, where II plays with strategy τ, s^* being a fixed sequence:



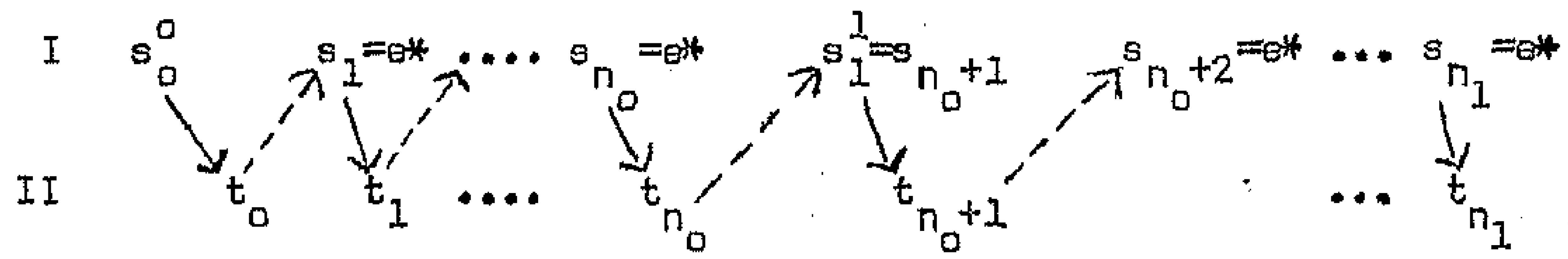
According to the definition of τ,

$$\sigma(a_0, s_0^0 * t_0 * s_1 * t_1 * \ldots * s_{n_0}) = (b_0, t_{n_0}),$$

for some $b_0 \in \omega$.

Define $\sigma^*(a_0, s_0^0) = (b_0, t_0 * s_1 * t_1 * \ldots * s_{n_0} * t_{n_0})$.

Next suppose $\forall$ plays a_1, s_1^1 and let $|(a_0, a_1)| = n_1$. Consider the following partial play consistent with τ



By definition of τ,

$\sigma(a_0, s_0^0 * t_0 * \dots * s_{n_0}; \ b_0, t_{n_0}; \ a_1, s_{n_0+1} * t_{n_0+1} * \dots * s_{n_1}) = (b_1, t_{n_1}), \ b_1 \in \omega.$

Define $\sigma^*(a_0, s_0^0; \ b_0, t_0 * s_1 * \dots * s_{n_0} * t_{n_0}; \ a_1, s_1^1)$

$$= (b_1, t_{n_0+1} * s_{n_0+2} * \dots * s_{n_1} * t_{n_1})$$

and so on. We shall show that σ^* is a winning strategy for $\exists$ in the second game. First observe that any complete play

$$a_0, u_0; \ b_0, v_0; \ a_1, u_1; \ b_1, v_1; \ \dots.$$

consistent with σ^* corresponds to a complete play

$$s_0, \ t_0, \ s_1, \ t_1, \ \dots.$$

consistent with τ such that

$$s_0 * t_0 * s_1 * t_1 * \dots = u_0 * v_0 * u_1 * v_1 * \dots = \beta, \ \text{say}.$$

To prove that σ^* is a winning strategy, we have to show $P(\alpha, \beta)$, where $\alpha = (a_0, \ b_0, \ a_1, \ b_1, \ \dots.)$. Next observe that any play consistent with σ^* is of the form

($\forall$) I $\quad a_0, s_0^0 \qquad\qquad a_1, s_1^1 = s_{n_0+1} \ \dots\dots$

($\exists$) II $\quad b_0, t_0 * s_1 * t_1 * \dots * s_{n_0} * t_{n_0} \qquad\qquad b_1, t_{n_0+1} * \dots * s_{n_1} * t_{n_1} \dots\dots$

such that

$(\forall)$ I	$a_0, s_0^0 * t_0^0 * \ldots * s_{n_0}$	$a_1, s_{n_0+1} * t_{n_0+1} * \ldots * s_{n_1}$	$\ldots\ldots\ldots$
$(\exists)$ II	b_0, t_{n_0}	b_1, t_{n_1}	$\ldots\ldots$

is consistent with σ. Consequently we have

$$P(\alpha, s_0^0 * t_0^0 * \ldots * s_{n_0} * t_{n_0} * s_{n_0+1} * t_{n_0+1} * \ldots)$$

and hence $P(\alpha,\beta)$.

<u>Remark</u>. The definition of σ^* is highly constructive and one can show that the map $\sigma \longrightarrow \sigma^*$ is Borel.

The proof of Lemma 7.2 immediately gives us the following

7.3 <u>Corollary</u>. If the second game in 7.2 is determined then

(a) $(\forall s_0)(\exists t_0)(\forall s_1)(\exists t_1)\ldots \{(\forall a_0)(\exists b_0)(\forall a_1)(\exists b_1)\ldots P(\alpha,\beta)\}$.

$\Longleftrightarrow (\forall s_0)(\forall a_0)(\exists t_0)(\exists b_0)(\forall s_1)(\forall a_1)(\exists t_1)(\exists b_1)\ldots\ldots\ldots\ldots P(\alpha,\beta)$,

(b) $(\exists s_0)(\forall t_0)(\exists s_1)(\forall t_1)\ldots \{(\exists a_0)(\forall b_0)(\exists a_1)(\forall b_1) \ldots\ldots \neg P(\alpha,\beta)\}$

$\Longleftrightarrow (\exists s_0)(\exists a_0)(\forall t_0)(\forall b_0)\ldots \ldots\ldots \ldots\ldots \ldots\ldots \neg P(\alpha,\beta)$,

and the conclusion of Lemma 7.2 also holds here.

<u>Remark</u>. (The proofs of) Lemma 7.2 and Corollary 7.3 yield a constructive proof of a particular case of the Game Formula of Kechris [26]. In fact, 7.2 and 7.3 can be formulated in the set up of Kechris.

<u>Lemma 7.4</u> For $\rho < \omega_1$ let $R_\rho = \Phi_\rho = \mathbb{R}\Phi_{NN^\rho}$. As in the proof of 6.6 there are functions $\overleftrightarrow{f}$ and $\overleftrightarrow{g}$ such that for any normal family $\{E_n\}$ with $E = R_\rho(\{E_n\})$,

E^x is comeager

$\longleftrightarrow (\exists \eta \in \overleftarrow{K})(\forall n \in \eta)\left[E^x_{\overleftarrow{f}(n)} \text{ is comeager in } \Sigma(\overleftarrow{g}(n))\right]$

$\longleftrightarrow (\exists \eta_0 \in M)(\forall n_0 \in \eta_0)\ldots(\forall k)\left[E^x_{<\alpha(<n,m>),\ldots,\alpha(<n_{k-1},m_{k-1}>)>}\right.$

$\left.\text{is comeager in } \Sigma(\beta(<n_0,m_0>)*\ldots*\beta(<n_{k-1},m_{k-1}>)\right]$,

where $\overleftarrow{f},\overleftarrow{g}$ and α,β are related is in 6.3(c) and $\Phi_M \hookrightarrow \Phi_N$ and $\overleftarrow{K} = \mathbb{R}\ M\ M^0$.

Then, one can choose $\overleftarrow{K}$ such that for any $\eta \in K$, $\bigcup_{n \in \eta} \Sigma(\overleftarrow{g}(n))$ is dense in ω^ω.

In fact, $\overleftarrow{K}$ may be taken to be the canonical base for S_ρ with the property that for any $\eta \in \overleftarrow{K}$ and $s \in Seq$ there is an $n \in \eta$ such that $\overleftarrow{g}(n)$ extends s.

Proof. We shall prove this by induction on ρ. For $\rho = 0$, $\overline{\Phi}_{\overleftarrow{K}}$ is the Vaught operation $\mathcal{V}$. Taking $\overleftarrow{K}$ to be the canonical base for $\mathcal{V}$ it is easy to check the assertion of the above theorem.

So assume $\rho > 0$ and the assertion holds for all $\lambda < \rho$.

Case 1. $\rho = \lambda + 1$.

Then $\overline{\Phi}_N = R_\lambda$. Let $\eta \in \overleftarrow{K} = R\ M\ M^0$, where M has been chosen to satisfy the assertion of the theorem for the operation R_λ.

Fix a basic clopen set $\Sigma(t)$. We have to show that there is $n \in \eta$ such that $\overleftarrow{g}(n)$ is consistent with t (in fact, extends t).

Since $\eta \in \mathbb{R}\ M\ M^0$,

$(\exists \eta_0 \in M)(\forall n_0 \in \eta_0)(\forall \xi_0 \in M)(\exists n_0 \in \xi_0)\ldots(\forall k)\,[\langle\langle n_0,m_0\rangle\ldots\langle n_{k-1},m_{k-1}\rangle\rangle \in \eta\,]$.

Fix $\eta_0 \in M$ such that for each $n_0 \in \eta_0$,

$$(\forall \xi_0 \in M)(\exists m_0 \in \xi_0)(\exists \eta_1 \in M)(\forall n_1 \in \eta_1)(\forall \xi_1 \in M)(\exists m_1 \in \xi_1)\ldots$$
$$\ldots(\forall k)\,[\langle\langle n_0,m_0\rangle,\ldots,\langle n_{k-1},m_{k-1}\rangle\rangle \in n\,].$$

By induction hypothesis, there is $n_0' \in \eta_0$ such that $g(n_0')$ extends t. Now pick a $\xi_0 \in M$ such that for some $m_0' \in \xi_0$,

$$(\exists \eta_1 \in M)\,(\forall n_1 \in \eta_1)\,(\forall \xi_1 \in M)\,(\exists m_1 \in \xi_1)\ldots$$
$$\ldots(\forall k)\,[\langle\langle n_0',m_0'\rangle,\langle n_1,m_1\rangle,\ldots,\langle n_{k-1},m_{k-1}\rangle\rangle \in \eta\,].$$

Clearly $n = \langle\langle n_0',m_0'\rangle\rangle \in \eta$ and $\overleftrightarrow{g}(n) = g(n_0')*\delta(n_0')$ (cf. 6.3).

Thus $\overleftrightarrow{g}(n)$ extends t.

Case 2. ρ is limit.

Let $\rho_i \uparrow \rho$. Then $R_\rho = \bar{\Phi}^*_N = \mathbb{R}\bar{\Phi}_{NN^0}$, where $\bar{\Phi}_N$ is the operation given by

$$\bar{\Phi}_N(\{F_n\}) = \bigcap_{i\in\omega} \bar{\Phi}_{N_{\rho_i}N^0_{\rho_i}}\,(\{F_{\langle i,m\rangle} : m \in \omega\})$$

and $\qquad \bar{\Phi}_{N_{\rho_i}} = R_{\rho_i}$.

In this case, if $E = \bar{\Phi}_N(\{E_n\})$, then E^x is comeager

$\Longleftrightarrow (\exists \eta \in \overleftrightarrow{N})\,(\forall n \in \eta)\,[E^x_{\overleftrightarrow{f}(n)}$ is comeager in $\Sigma(\overleftrightarrow{g}(n))\,]$

$\Longleftrightarrow (\forall i)(\exists \eta \in M_{\rho_i})(\forall n \in \eta)(\exists \xi \in M^0_{\rho_i})(\forall m \in \xi)\,[E^x_{\langle i,\alpha_i(\langle n,m\rangle)\rangle}$ is comeager in $\Sigma(\beta_i(\langle n,m\rangle))\,]$ (cf. 6.6, Case 2).

In view of Case 1, it suffices to show that for any $\eta \varepsilon \overleftarrow{N}$ and for any $\Sigma(t)$, there is $n \varepsilon \eta$ such that $\overleftarrow{g}(n)$ extends t.

Now, $\overleftarrow{N}$ may be chosen such that

$$\eta \varepsilon \overleftarrow{N} \longleftrightarrow (\forall i)(\exists \eta' \varepsilon M_{\rho_i})(\forall n \varepsilon \eta')(\exists \xi' \varepsilon M^o_{\rho_i})(\forall m \varepsilon \xi')[<i,<n,m>> \varepsilon \eta],$$

where each M_{ρ_i} has been chosen to satisfy the assertion of the theorem.

Let $\eta \varepsilon N$ and fix any $i' \varepsilon \omega$. Get $\eta' \varepsilon M_{\rho_i}$ such that for each $n \varepsilon \eta'$, $(\exists \xi' \varepsilon M^o_{\rho_i})(\forall m \varepsilon \xi')\ [<i',<n,m>> \varepsilon \eta]$.

Since $\eta' \varepsilon M_{\rho_i}$, by induction hypothesis, there is $n' \varepsilon \eta'$ such that $g_{i'}(n')$ extends t. Get $\xi' \varepsilon M^o_{\rho_i}$ and $m' \varepsilon \xi'$ such that $<i',<n',m'>> \varepsilon \eta$. Take $n = <i',<n',m'>>$. Then

$$\overleftarrow{g}(n) = \beta_{i'}(<n',m'>) = g_{i'}(n')*\delta_{i'}(m') \ [\text{cf. } 6.6]$$

and hence $\overleftarrow{g}(n)$ extends t.

<u>Lemma 7.5</u>: Suppose $E \subseteq \omega^\omega$ be a set in $\mathcal{R}^\rho$, $\rho < \omega_1$ and let $R_\rho = R\,\Phi_N$. Assume $E = R_\rho(\{E_n\})$, where each $\{E_n\}$ is clopen. Then, E is comeager $\longleftrightarrow (\exists \eta_o \varepsilon M)(\forall n_o \varepsilon \eta_o)(\exists \eta_1 \varepsilon M)(\forall n_1 \varepsilon \eta_1) \ldots$

$$\ldots (\forall k)[E_{<f(n_o),\ldots,f(n_{k-1})>} \text{ is comeager in } \Sigma(g(n_o)*\ldots*g(n_{k-1}))]$$

where f and g are some suitable functions and $\Phi_M \hookrightarrow \Phi_N$. Moreover, f and g can be chosen such that, with any winning strategy σ for $\exists$ one can associate a winning strategy $\sigma*$ such that for any run $\eta_o, n_o, \eta_1, n_1, \ldots$ consistent with $\sigma*$, the sequence $\beta = g(n_o)*g(n_1)*\ldots$ is in E.

<u>Proof</u>: The first assertion follows immediately from the proof of 6.6.

Moreover, by 7.4, M may be chosen such that for any $\eta \in M$ and $s \in Seq$ there is $n \in \eta$ such that $g(n)$ extends s. Therefore, we have

$$(\exists \eta_o \in M)(\forall n_o \in \eta_o)\ldots(\forall k)\left[E_{<f(n_o),\ldots,f(n_{k-1})>} \text{ is comeager in } \Sigma(g(n_o)*g(n_1)*\ldots*g(n_{k-1}))\right]$$

$$\longleftrightarrow (\exists \eta_o \in M)(\forall n_o \in \eta_o)\ldots(\forall k)\ \left[\beta \in E_{<f(n_o),\ldots,f(n_{k-1})>}\right]$$

where $\beta = g(n_o)*g(n_1)*\ldots.$

Hence,

E is comeager

$$\longleftrightarrow (\exists \eta_o \in M)(\forall n_o \in \eta_o)(\exists \eta_1 \in M)(\forall n_1 \in \eta_1)\ldots.$$

$$\ldots(\forall k)\ \left[g(n_o)*g(n_1)*\ldots \in E_{<f(n_o),\ldots,f(n_{k-1})>}\right]. \qquad (i)$$

We shall prove the second assertion only for $\rho = 1$, since for higher levels the proof involves similar ideas, although notationally cumbersome. (For a complete proof see Appendix).

First observe that if $F = \mathcal{A}(\{F_n\})$, with each F_n clopen, then by the Category Formula [13], we have

F is meager

$$\longleftrightarrow(\forall k_o)(\forall s_o)(\exists t_o)(\forall k_1)(\forall s_1)(\exists t_1)\ldots.(\exists i)\left[F_{<k_o,\ldots,k_{i-1}>} \text{ is meager in } \Sigma(s_o*t_o*\ldots*s_{i-1}*t_{i-1})\right].$$

Therefore, whenever $G = \mathcal{S}(\{G_n\})$, where $\mathcal{S} = \mathcal{A}^o$,

G is comeager

$$\Longleftrightarrow (\forall k_0)(\forall s_0)(\exists\, t_0)(\forall k_1)(\forall s_1)(\exists\, t_1)\ldots(\exists\, i)\ \Big[G_{<k_0,\ldots,k_{i-1}>} \text{ is comeager in } \Sigma(s_0 * t_0 * \ldots * s_{i-1} * t_{i-1})\Big].$$

Since $R_1 \hookrightarrow \mathbb{R}\int^{\rho}$ we may assume $E = \mathbb{R}\int^{\rho}(\{E_n\})$. (This is for simplicity and does not in any way entail loss of generality). By the remark following Theorem 6.3 we have

E is comeager

$$\begin{aligned}\Longleftrightarrow\ & (\forall k_0^0)(\forall s_0^0)(\exists\, t_0^0)(\forall k_1^0)(\forall s_1^0)(\exists\, t_1^0)\ \ldots\ (\exists\, i_0)\\ & (\forall k_0^1)(\forall s_0^1)(\exists\, t_0^1)(\forall k_1^1)(\forall s_1^1)(\exists\, t_1^1)\ \ldots\ (\exists\, i_1)\\ & (\forall k_0^2)(\forall s_0^2)(\exists\, t_0^2)(\forall k_1^2)(\forall s_1^2)(\exists\, t_1^2)\ \ldots\ (\exists\, i_2)\\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad\qquad \text{(ii)}\end{aligned}$$

$$\ldots(\forall j)\ \Big[E_{<<k_0^0,k_1^0,\ldots,k_{i_0-1}^0>,\ldots,<k_0^{j-1},\ldots,k_{i_{j-1}-1}^{j-1}>>} \text{ is comeager in } \Sigma(s_0^0 * t_0^0 * \ldots * s_{i_0-1}^0 * t_{i_0-1}^0 * \ldots * s_0^{j-1} * t_0^{j-1} \ldots * s_{i_{j-1}-1}^{j-1} * t_{i_{j-1}-1}^{j-1})\Big].$$

Now, E is comeager $\Longleftrightarrow$ II wins the game $G^{**}(E^c)$

$\Longleftrightarrow (\forall s_0)(\exists\, t_0)\ (\forall s_1)(\exists\, t_1)\ \ldots(\beta \in E)$,

where $\beta = s_0 * t_0 * s_1 * t_1 * \ldots$

Hence (using (i)), the equivalence in (ii) reduces to

$$(\forall s_0)(\exists\, t_0)(\forall s_1)(\exists\, t_1)\ \ldots \Big\{(\forall \alpha_0)(\exists\, n_0)\ \ldots\ \ldots(\forall j)\ \Big[\beta \in E_{<\bar{\alpha}_0(n_0),\ldots,\bar{\alpha}_{j-1}(n_{j-1})>}\Big]\Big\}$$

$$\longleftrightarrow (\forall k_0^0)(\forall s_0^0)(\exists\, t_0^0)(\forall k_1^0)(\forall s_1^0)(\exists\, t_1^0) \ldots (\exists\, i_0)$$
$$(\forall k_0^1)(\forall s_0^1)(\exists\, t_0^1)(\forall k_1^1)(\forall s_1^1)(\exists\, t_1^1) \ldots (\exists\, i_1)$$
$$\cdots \qquad \text{(iii)}$$
$$\ldots(\forall j)\left[s_0^0 * t_0^0 \ldots * s_{i_0-1}^0 * t_{i_0-1}^0 * s_0^1 * t_0^1 * \ldots * s_{i_1-1}^1 * t_{i_1-1}^1 * \ldots \in E_{\langle\langle k_0^0,\ldots,k_{i_0-1}^0\rangle,\ldots,\langle k_0^{j-1},\ldots,k_{i_{j-1}-1}^{j-1}\rangle\rangle}\right].$$

As in [13; § 11], we shall define a game G' of length ω such that $\exists$ wins G' iff $\exists$ wins the second game in (iii). The game G' is as follows. Though its total length is ω we think of it as consisting of <u>potentially</u> infinite sequence of subgames each consisting of <u>potentially</u> infinite sequence of rounds. If in any play of G' the j-th subgame actually goes through infinitely many rounds, then the (j+1)th subgame never gets started. This keeps the total length within bounds.

In the j-th subgame the two players play the rounds of that subgame. The ℓ-th such round opens with $\exists$ signalling (by a choice of 0 or 1) either a challenge or a pass. If he challenges, the whole j-th subgame ends at once and the players proceed to the (j+1)th subgame; in this case we record $u_j = \langle k_0^j,\ldots,k_{\ell-1}^j\rangle$ and $v_j = s_0^j * t_0^j * \ldots * s_{\ell-1}^j * t_{\ell-1}^j$ formed from the moves. If $\exists$ passes, $\forall$ chooses $k_\ell^j \in \omega$ and a sequence number s_ℓ^j and $\exists$ replies with $t_\ell^j \in \mathrm{Seq.}$; then the players proceed to the $(\ell+1)$-th round.

If some j-th subgame goes on forever because $\exists$ fails to challenge on any round, $\exists$ forfiets the game. If this provision does not apply, then a sequence $(u_0,v_0; u_1,v_1; \ldots)$ will have been generated. $\exists$ wins iff for all j

$$v_0 * v_1 * v_2 * \ldots = \beta \in E_{<u_0,u_1,\ldots,u_{j-1}>}$$

Thus the game G' is essentially of the form

$$(\forall e_0)(\forall a_0)(\exists t_0)(\exists b_0)(\forall e_1)(\forall a_1)(\exists t_1)(\exists b_1) \ldots [M(\alpha,\beta)],$$

where the condition M is Borel (in fact G_δ, cf. [13; §11]). We shall now make a few observations. First observe that each subgame in the second game of (iii) is a Banach-Mazur game interlaced with operation $\mathcal{S}$ (which may be looked upon as a game!). Secondly the condition within $[\ \ ]$ is the same as the condition in the first game. Consequently, any reduction of the second game of (iii) involves the interlacing of a Banach-Mazur game with an ordinary game of length ω which is obtained by the corresponding reduction of the operation R_1. Therefore, reducing the R_1 operation within $\{\ \ \}$ in the first game of (iii) as above, we observe that the equivalence (iii) is equivalent to

$$(\forall e_0)(\exists t_0)(\forall e_1)(\exists t_1) \ldots \{(\forall a_0)(\exists b_0) \ldots M(\alpha,\beta)\}$$
$$\Longleftrightarrow (\forall e_0)(\forall a_0)(\exists t_0)(\exists b_0) \ldots M(\alpha,\beta). \qquad \text{(iv)}$$

Now, if $\exists$ wins the second game in (iii), he also wins G', the second game in (iv). By 7.2 $\exists$ wins with a strategy σ such that for any complete play $s_0, a_0, t_0, b_0, \ldots$ consistent with σ, $\beta = s_0 * t_0 * \ldots$ corresponds to a complete play of the Banach-Mazur game in (iv). Moreover, the strategy σ gives rise to a strategy σ^* in the second game of (iii) such that, if $s_0^0, t_0^0, k_0^0, \ldots, i_0; s_0^1, t_0^1, k_0^1, \ldots, i_1; \ldots$ is a complete run consistent with σ^*, then the sequence $\beta = s_0^0 * t_0^0 * \ldots * s_{i_0-1}^0 * t_{i_0-1}^0 * s_0^1 * t_0^1 * \ldots$ is also produced by a complete run of the game G', when $\exists$ plays with σ. Consequently, β satisfies the condition in $\{\ \}$ in (iv) and hence $\beta \in E$.

Remark: For sets E at the higher level we have an equivalence similarly to (iii). On the left side of the equivalence we have a Banach-Mazur game followed by the corresponding R-operation, and on the right side we have Banach-Mazur games "interlaced" with the operation (regarded as a game played on ω) with dummy moves, if necessary. As in the proof above, the second game can be reduced to a game of length ω and the proof proceeds exactly as above (see Appendix).

7.6 On the Δ-transform: We shall now look at the dual of the *-transform and obtain some analogous results. The proofs being very similar to what we have already done, we shall omit them.

For a set $E \subseteq X \times Y$ and $U \subseteq Y$, put $E^{\Delta U} = \{x \in X : E^x$ is non-meager in $U\}$.

Note that Δ is the dual of the *-transform in the sense that $E^{\Delta U} = [(E^c)^{*U}]^c$. The Δ-transform behaves very must like the *-transform and we have the counterpart of 6.2 :

With notation as in 6.2, we have

$$E^{\Delta \Sigma(s)} = \bigcap_{\mu < \omega_1} [E_s^{\mu}]^{\Delta \Sigma(s)} = \bigcup_{\mu < \omega_1} [E_0^{\mu} - T^{\mu}]^{\Delta \Sigma(s)}.$$

This decomposition of the set E^{Δ} suggests, as in the case of E^{*}, that $(E^{\Delta})^c$ may be obtained as a fixed point of an inductive operator. Indeed we have the following counterpart of 6.3.

<u>Theorem</u>: Let $\mathfrak{D}_N$ and $\mathfrak{D}_M$ be two δ-s operations such that $\mathfrak{D}_N$ preserves the Baire property and $\mathfrak{D}_M$ is normal and subsumes both (countable) $\bigcup$ and $\bigcap$. Suppose there are functions f_1 and g_1 such that for any normal family $\{E_p\}$ of subsets of $\omega^{\omega} \times \omega^{\omega}$ with $E = \mathfrak{D}_N(\{E_p\})$,

$$E^{\Delta \Sigma(s)} = \mathfrak{D}_M(\{E_{f_1(p)}^{\Sigma(s * g_1(p))} : p \in \omega\}).$$

Then for any regular normal family $\{F_p\}$, if $F = \mathfrak{D}_N(\{F_p\})$ then

$\overline{F}^{x}$ is non-meager in $\Sigma(s)$

$\Longleftrightarrow (\exists \eta_0 \in M')(\forall n_0 \in \eta_0)(\exists \eta_1 \in M')(\forall n_1 \in \eta_1) \ldots\ldots$

$\ldots\ldots (\forall k) \left[F^{x}_{\langle f'(n_0),\ldots,f'(n_{k-1})\rangle} \text{ is non-meager in } \Sigma(s * g'(n_0) * \ldots * g'(n_{k-1}))\right],$

where f' and g' are suitable function (independent of $\{F_p\}$) and $\mathfrak{D}_{M'} = \mathfrak{D}_M \bigcup \bigcap$.

<u>Proof</u>: First observe that for any regular family $\{F_p\}$, $F_s^{\mu} \subseteq F_t$ if $t \subseteq s$. As in 6.3(c) we define a set relation operative on ω as follows.

$t \in \Gamma_x(A) \Longleftrightarrow F^x_{(t)_0}$ is meager in $\Sigma((t)_1)$ v

$(\forall\eta \in M)(\exists\, n \in \eta)(\forall u)(\exists\, v)\left[<(t)_0 * <f_1(n)>, (t)_1 * g_1(n) * u * v> \in A\right].$

To obtain the result, one then shows that $x \in F^{\Delta\Sigma(s)} \Longleftrightarrow <e,s> \notin \Gamma^{\infty}_x$. □

As in 6.6, this theorem immediately implies that for a set $E \in \mathcal{R}^{\rho}$, if $E = \mathbb{R}\,\mathfrak{F}_N(\{E_n\})$, then

$$x \in E^{\Delta} \Longleftrightarrow (\exists\, \eta_0 \in M)\,(\forall n_0 \in \eta_0)\,(\exists\, \eta_1 \in M)\,(\forall n_1 \in \eta_1)\ \ldots$$
$$\ldots (\forall k)\left[E^x_{<f'(n_0),\ldots,f'(n_{k-1})>} \text{ is non-meager in } \Sigma(g'(n_0)*\ldots*g'(n_{k-1}))\right]. \qquad (9)$$

for suitable M, f', g'. Moreover, one can choose M such that

$$\eta \in M^0 \,\&\, s \in \mathrm{Seq} \Longrightarrow (\exists\, n \in \eta)\,(g'(n) \text{ extends } s).$$

Finally observe that by (9) we have for $E \subseteq \omega^{\omega}$

$$E \text{ is meager} \Longleftrightarrow (\exists\, \eta_0 \in M^0)(\forall n_0 \in \eta_0)(\exists\, \eta_1 \in M^0)(\forall n_1 \in \eta_1)\ \ldots$$
$$\ldots\ (\exists\, k)\left[g'(n_0)*g'(n_1)*\ldots \notin E_{<f'(n_0),\ldots,f'(n_{k-1})>}\right] \qquad (v)$$

Arguing as in 7.5 and invoking 7.3(b), we may show that $\exists$ can win the game (v) (if he does so) by a strategy σ^{**} such that if $\eta_0, n_0, \eta_1, n_1, \ldots$ is a complete play consistent with σ^{**}, then $\beta = g(n_0)*g(n_1)*\ldots$ is not in E.

The next theorem is the first step towards our approximation theorem.

<u>7.7 Theorem</u>: Let $E \subseteq \omega^{\omega} \times \omega^{\omega}$ be a set in $\mathcal{R}^{\rho}$ $(\rho > 0)$. Then there is a set $B \in \mathcal{BR}^{\rho}_0 \otimes \mathcal{B}_{\omega^{\omega}}$ such that $B \subseteq E$ and B^x is comeager on E^x.

Proof: Let $R_\rho = \mathbb{R}\,\underline{\Phi}_N$ and assume $E = \mathbb{R}\,\underline{\Phi}_N(\{E_n\})$, with each E_n clopen. As in 6.6, there are functions f and g such that

$$x \in E^* \longleftrightarrow (\exists\, \eta_0 \in M)(\forall n_0 \in \eta_0)(\exists\, \eta_1 \in M)(\forall n_1 \in \eta_1) \ldots$$
$$\ldots(\forall k)\left[E^x_{\langle f(n_0),\ldots,f(n_{k-1})\rangle} \text{ is comeager in } \Sigma(g(n_0)*\ldots*g(n_{k-1}))\right] \quad (i)$$

where M is such that $\underline{\Phi}_M \hookrightarrow \underline{\Phi}_N$. Further, by 3.8, there is a $\mathcal{BR}^\rho_0$-measurable function $x \longrightarrow \tau_x$ such that whenever $x \in E^*$, τ_x is a winning strategy for the player $\exists$ in the game (i). Moreover, by 7.5 and the Remark following 7.2, we may assume that the function $x \longrightarrow \tau_x$ is such that for any complete play $\eta_0, n_0, \eta_1, n_1, \ldots$ agreeing with τ_x the sequence $\beta = g(n_0)*g(n_1)*g(n_2)* \ldots$ is in E^x, and that the base M has been chosen as in 7.4.

Define for each $k \geq 1$,

$$W_k(s,x) \longleftrightarrow Seq(s) \;\&\; \ell h(s) = k \;\&\; (\forall i < \ell h(s))\left[(s)_i \in \tau_x(s \restriction i)\right] :$$

i.e., W^x_k consists of the codes of the first k possible moves of $\forall$ when the existential player plays according to the strategy τ_x. Clearly, $W_k \in \mathcal{BR}^\rho_0$. Define,

$$C(x,y) \longleftrightarrow (\exists\, \alpha)(\forall k)\left[W_k(\bar{\alpha}(k),x) \;\&\; y \in \Sigma(g(\alpha(0))*\ldots*g(\alpha(k-1)))\right] \;\&\; x \in E^*.$$

Plainly, C is the result of operation $\mathcal{A}$ on sets in $\mathcal{BR}^\rho_0 \otimes \mathcal{B}_{\omega^\omega}$. We shall show that

(a) $C \subseteq E$

and (b) C^x is comeager whenever E^x is comeager.

If $(x,y) \in C$ then $x \in E^*$ and $\exists$ wins the game (i). Further, there is a sequence $\{n_k : k \in \omega\}$ such that $n_0, n_1, n_2, \ldots$ are the moves of $\forall$ when $\exists$ plays according to τ_x and $y = g(n_0)*g(n_1)* \ldots$. Consequently, $y \in E^x$. To prove (b), fix an x such that E^x is comeager. Then

$$C^x = \{y : (\exists \alpha)(\forall k)\ [W_k(\bar{\alpha}(k), x)\ \&\ y \in \Sigma(g(\alpha(0)*\ldots*g(\bar{\alpha}(k-1))]\}$$

Define,

$$A_{<n_0,\ldots,n_{k-1}>} = \Sigma(g(n_0)*g(n_1)*\ldots*g(n_{k-1})), \text{ if } W_k(<n_0,\ldots,n_{k-1}>, x)$$

$$= \varphi, \text{ otherwise.}$$

Then, $C^x = \mathcal{A}(\{A_s\})$. Hence to show C^x is comeager, it is enough to show (by Vaught's formula) that

$$(\forall s_0)(\exists k_0)(\exists t_0)(\forall s_1)(\exists k_1)(\exists t_1)\ldots(\forall i)\ [A_{<k_0,\ldots,k_{i-1}>} \text{ is comeager in } \Sigma(s_0*t_0*\ldots*s_{i-1}*t_{i-1})].$$

So let $\forall$ play s_0. By 7.4, pick $k_0 \in \tau_x(<\ >)$ such that $g(k_0)$ extends s_0. Then $\exists$ replies with k_0 and a sequence number t_0 such that $s_0*t_0 = g(k_0)$. Next when $\forall$ plays s_1, $\exists$ plays $k_1 \in \tau_x(<k_0>)$ such that $g(k_1)$ extends s_1 and then plays t_1 such that $s_1*t_1 = g(k_1)$, and so on. By adopting this strategy, $\exists$ ensures that for each i,

$$A_{<k_0,\ldots,k_{i-1}>} \text{ is comeager in } \Sigma(s_0*t_0*\ldots*s_{i-1}*t_{i-1})$$

Thus, C^x is comeager. An application of 7.1 gives the required set B.

The next lemma is the counterpart of 7.7.

<u>Lemma 7.8</u>: Let $E \subseteq \omega^\omega \times \omega^\omega$ be a set in $\mathcal{R}^\rho$. Let $E^\# = \{x : E^x \text{ is meager}\}$. Then $E^\# \in c\mathcal{R}^\rho$ and there is a set

$C \in \underset{\hookrightarrow}{\mathcal{BR}}_0^\rho \otimes \mathcal{B}_{\omega^\omega}$ (in fact, in $\sigma(\underset{\hookrightarrow}{\mathcal{R}}^\rho) \otimes \mathcal{B}_{\omega^\omega}$) such that $E \subseteq C$ and C^x is meager on $E^\#$.

Proof: Plainly, $E^\# \in c\underset{\hookrightarrow}{\mathcal{R}}^\rho$, by 6.7.

Let f, g, N, M, E_n be as in 7.6 with $\{E_n\}$ regular. Then, by (v) of 7.6,

$$E^x \text{ is meager}$$

$$\begin{aligned} \longleftrightarrow\ & (\exists\, \eta_0 \in M^0)(\forall n_0 \in \eta_0)(\exists\, \eta_1 \in M^0)(\forall n_1 \in \eta_1) \ldots \\ & \ldots (\exists\, k)\,[g(n_0)*g(n_1)*\ldots* \notin E^x_{\langle f(n_0),\ldots,f(n_{k-1})\rangle}] \end{aligned} \tag{i}$$

Further, by the Remark following Theorem 3.8 there is a $\sigma(\underset{\hookrightarrow}{\mathcal{R}}^\rho)$-measurable function $x \longmapsto \tau_x$ such that τ_x is a winning strategy for $\exists$ in the above game whenever $x \in E$. Moreover, τ_x can be chosen as in 7.6. Call $u = \langle n_0,\ldots,n_{k-1}\rangle$ good with respect to τ_x and β if β extends $g(n_0)*\ldots*g(n_{k-1})$ and $n_0,\ldots,n_{k-1}$ are the first k possible moves of $\forall$ when $\exists$ plays with strategy τ_x. Define

$$\begin{aligned} T(u,x,\beta) \longleftrightarrow\ & Seq(u) \ \&\ (\forall i < \ell h(u))\,[(u)_i \in \tau_x(u{\restriction}i)] \\ & \&\ \beta \in \Sigma((u)_0 * g((u)_1)*\ldots*g((u)_{\ell h(u)-1}))\ \& \\ & (\forall n \in \tau_x(u))\,[\beta \notin \Sigma((u)_0 * g((u)_1 * g((u)_2)*\ldots*g((u)_{\ell h(u)-1})*g(n))]. \end{aligned}$$

In other words, for each x, β, the section $T_{x,\beta}$ consists of all maximal good sequences with respect to τ_x and β. Clearly, $T \in \sigma(\underset{\hookrightarrow}{\mathcal{R}}^\rho) \otimes \mathcal{B}_{\omega^\omega}$ and moreover $T^{u,x}$ is closed nowhere-dense for each good sequence u and x. Now define,

$$C'(x,\beta) \longleftrightarrow x \in E^\# \ \&\ (\exists\, u)\, T(u,x,\beta)$$

Set

$$C = C' \bigcup .(\omega^\omega - E^\#) \times \omega^\omega .$$

Since $E^\# \in c\mathcal{R}^\rho$, $C \in \sigma(\mathcal{R}^\rho) \otimes \mathcal{B}$; and to conclude the proof we shall show that $E \subseteq C$. So let $(x,\beta) \in E$. If $x \notin E^\#$, then we are done. So assume $x \in E^\#$. Therefore, $\exists$ wins the game (i) with strategy τ_x. Now, suppose when $\exists$ plays the game (i) according to τ_x, $\forall$ is able to play $n_0, n_1, n_2, \ldots$ such that for each k, $\langle n_0, \ldots, n_{k-1} \rangle$ is a good sequence with respect to τ_x and β. Then

$$\beta = g(n_0) * g(n_1) * \ldots$$

and hence, by 7.6, $\beta \notin E^x$. Consequently, since we have $\beta \in E^x$, $\forall$ is able to play $n_0, n_1, \ldots$ 'consistent' with β only upto a finite stage. Let u be the code of the maximal sequence. Then $T(u,x,\beta)$ and hence $(x,\beta) \in C$.

The following gives the approximation at the first level.

<u>7.9 Lemma</u>: Let $E \subseteq \omega^\omega \times \omega^\omega$ be a set in $\mathcal{R}^\rho (\rho > 0)$. Then there are sets B, C in $\mathcal{BR}^\rho_0 \otimes \mathcal{B}_{\omega^\omega}$ such that $B \subseteq E \subseteq C$ and $C^x - B^x$ is meager for each x.

<u>Proof</u>: Let $T_s = E^{*\Sigma(s)}$; $s \in Seq$.

As E^x satisfies the Baire property for each x,

$$\bigcup_s T_s = \{x : E^x \text{ is non-meager}\}.$$

Since each $\Sigma(s)$ is a (recursive) Homeomorphic copy of ω^ω , we can apply 7.7 to get B_s as in the Lemma. Put

$$B = \bigcup_s (B_s \cap (T_s \times \omega^\omega)).$$

Clearly, $B \subseteq E$, $B \in \underset{\hookrightarrow}{\mathcal{BR}}{}_0^\rho \otimes \mathcal{B}_{\omega^\omega}$ and for each x, $E^x - B^x$ is meager.

To get the set C, work with E^c and argue as above using 7.8.

<u>7.10 Approximation Theorem</u>: Let A be a $\underset{\hookrightarrow}{\mathcal{R}}{}_\mu^\rho$ subset of $\omega^\omega \times \omega^\omega$ ($\rho > 0$, $\rho, \mu < \omega_1$). Then there are sets B and C in $\underset{\hookrightarrow}{\mathcal{BR}}{}_\mu^\rho \otimes \mathcal{B}_{\omega^\omega}$ such that $B \subseteq A \subseteq C$ and $C^x - B^x$ is meager for each x.

Moreover, if $A \in \underset{\hookrightarrow}{\mathcal{BR}}{}_\mu^\rho$, then one can find B, C in $\underset{\hookrightarrow}{\mathcal{BR}}{}_\mu^\rho \otimes \mathcal{B}_{\omega^\omega}$ with the above properties.

<u>Proof</u>: We shall prove this by induction on μ. When $\mu = 0$, this is just Lemma 7.9. So suppose $\mu > 0$ and that the result is true for all $\lambda < \mu$. Let $A \in \underset{\hookrightarrow}{\mathcal{R}}{}_\mu^\rho$ and let $R_\rho = \mathbb{R}\,\Phi_N$. Then A is the result of operation R_ρ on a family $\{A_n\}$ of sets from $c[\bigcup_{\lambda<\mu} \underset{\hookrightarrow}{\mathcal{R}}{}_\lambda^\rho]$. By induction hypothesis, for each n we have B_n and C_n both in $\bigcup_{\lambda < \mu} \underset{\hookrightarrow}{\mathcal{BR}}{}_\lambda^\rho \otimes \mathcal{B}_{\omega^\omega}$ such that $B_n \subseteq A_n \subseteq C_n$ and $C_n^x - B_n^x$ is meager for each x. Let

$$B' = R_\rho(\{B_n\}) \quad \text{and} \quad C' = R_\rho(\{C_n\}).$$

Then, $B' \subseteq C'$ and for each x, $C'^x - B'^x \subseteq \bigcup_n (C_n^x - B_n^x)$, and hence meager. To complete the proof it is enough to get $B \subseteq B'$ and $C \supseteq C'$ such that B and C are in $\underset{\hookrightarrow}{\mathcal{BR}}{}_\mu^\rho \otimes \mathcal{B}_{\omega^\omega}$, $B'^x - B^x$ is meager and $C^x - C'^x$ is meager for each x. We will show how to obtain B; C can be obtained similarly.

Let $\mathfrak{F}$ be the σ-field generated by $\bigcup_{\lambda<\mu} \underset{\hookrightarrow}{\mathcal{BR}}{}_\lambda^\rho$. Then

$B' = R_\rho(\{B_n\})$ with each B_n in $\mathcal{F} \otimes \mathcal{B}_{\omega^\omega}$. Thus, one can obtain a countably generated sub-σ-field $\mathcal{J}$ of $\mathcal{F}$ such that each $B_n \in \mathcal{J} \otimes \mathcal{B}_{\omega^\omega}$. Fix a countable generator of $\mathcal{J}$ and let $f : (\omega^\omega, \mathcal{J}) \longrightarrow \omega^\omega$ be its characteristic function. Put $M = f(\omega^\omega)$. Then $(\omega^\omega, \mathcal{J})$ and $(M, \mathcal{B}_M)$ are Borel isomorphic. For each n, let $B_n' = \{(f(x),y) : (x,y) \in B_n\}$. Then $B_n' \in \mathcal{B}_{M \times \omega^\omega}$, for each n. Let $B'' = \{(f(x),y) : (x,y) \in B'\}$. Then $B'' = R_\rho(\{B_n'\})$. Hence, there is a set $E \subseteq \omega^\omega \times \omega^\omega$ in $\underset{\leftrightarrow}{\mathcal{R}}^\rho$ such that $E \cap (M \times \omega^\omega) = B''$. Apply 7.9 and get $E' \subseteq E$ such that $E' \in \underset{\leftrightarrow}{\mathcal{BR}}{}_0^\rho \otimes \mathcal{B}_{\omega^\omega}$ and $E^x - (E')^x$ is meager for each x. Let $\overset{\leftrightarrow}{f} : \omega^\omega \times \omega^\omega \longrightarrow \omega^\omega \times \omega^\omega$ be the map

$$\overset{\leftrightarrow}{f}(x,y) = (f(x),y).$$

Put $B = (\overset{\leftrightarrow}{f})^{-1}(E')$. Note that since f is a bimeasurable map of $(\omega^\omega, \mathcal{J})$ and $(M, \mathcal{B}_M)$,

$$f^{-1}(\underset{\leftrightarrow}{\mathcal{R}}^\rho \restriction \omega^\omega) \subseteq \{A \subseteq \omega^\omega : A \text{ is the result of operation } R_\rho \text{ on sets in } \mathcal{J}\} \subseteq \underset{\leftrightarrow}{\mathcal{R}}{}_\mu^\rho.$$

Consequently, $f^{-1}(\underset{\leftrightarrow}{\mathcal{BR}}{}_0^\rho \restriction \omega^\omega) \subseteq \underset{\leftrightarrow}{\mathcal{BR}}{}_\mu^\rho$. Thus $B \in \underset{\leftrightarrow}{\mathcal{BR}}{}_\mu^\rho \otimes \mathcal{B}_{\omega^\omega}$ and clearly, $B'^x - B^x$ is meager for each x.

The second assertion follows from the first by observing that the class of all sets for which the result holds is closed under Φ_N and complementation.

The next proposition is quite well-known and follows from the Von-Neumann selection theorem.

Proposition: Let $(T,\mathcal{M})$ be a measurable space, $\mathcal{M}$ being a σ-field closed under operation $\mathcal{A}$, and let Y be a polish space. Let $B \in \mathcal{M} \otimes \mathcal{B}_Y$ have non-empty vertical sections. Then B has a $\mathcal{M}$-measurable selection.

As a consequence of the approximation theorem we have the following.

7.11 Theorem: Suppose $A \subseteq \omega^\omega \times \omega^\omega$ is a $\mathcal{BR}_\mu^\rho$ set $(\rho > 0)$ such that A^x is non-meager for each x. The A has a $\mathcal{BR}_\mu^\rho$-measurable selection.

Proof: By 7.10, get $B \in \mathcal{BR}_\mu^\rho \otimes \mathcal{B}_{\omega^\omega}$ such that $B \subseteq A$ and $A^x - B^x$ is meager for each x. B^x is then non-empty for each x, and the above proposition yields the result.

The next selection theorem is due to Burgess [13].

7.12 Theorem: Let $F : \omega^\omega \longrightarrow \omega^\omega$ be a multifunction such that $F(x)$ is non-meager in its closure $cl(F(x))$. If F is $\mathcal{BR}^\rho$-measurable and its graph $Gr(F) \in \mathcal{BR}^\rho$, then F has a $\mathcal{BR}^\rho$-measurable selection.

Proof: Define G by $G(x) = cl(F(x))$. Then G is a closed-valued, $\mathcal{BR}^\rho$-measurable multifunction. Hence there is a map $g : \omega^\omega \times \omega^\omega \longrightarrow \omega^\omega$ such that g is $\mathcal{BR}^\rho \otimes \mathcal{B}_{\omega^\omega}$-measurable and $g(x,.)$ is continuous, open and onto $G(x)$, for each x(cf. [47]). Define $G' \subseteq \omega^\omega \times \omega^\omega$ by

$$G' = \{(x,y) : g(x,y) \in F(x)\}.$$

As $Gr(F) \in \mathcal{BR}^\rho$ and g is $\mathcal{BR}^\rho \otimes \mathcal{B}_{\omega^\omega}$-measurable, G' is in $\mathcal{BR}^\rho$. Also, as the inverse image of a non-meager set under a continuous open map is non-meager, G' has non-meager sections. By 7.11, G' has a $\mathcal{BR}^\rho$-measurable selection g'. Then $f(x)=g(x,g'(x))$ is a $\mathcal{BR}^\rho$-measurable selection for F.

PART II

THE FINITE LEVELS OF THE HIERARCHY OF EFFECTIVE R-SETS AND THE EFFECTIVE C-SETS

1. Until now we have only considered the boldface pointclasses $\boldsymbol{\Sigma}^0_1$, $\boldsymbol{\Pi}^0_1$ etc. and the classical R-sets and have studied their various properties. We have, however, formulated and proved many of our results so as to carry them over to the effective case without much difficulty. We first recall the definition of the effective pointclasses. For any p.a.o. Φ,

$$\Sigma^{\Phi}_1 = \left\{ \Phi(\{E_n : n \in \omega\}) : \{E_n\} \text{ is a recursive family} \right\},$$

where a family $\{E_n\}$ of subsets of a space of type 0 or 1 is recursive iff the relation $x \in E_n$ is recursive ;

$$\Pi^{\Phi}_1 = \left\{ \neg E : E \in \Sigma^{\Phi}_1 \right\}$$

and

$$\Pi^{\Phi}_1 \cap \Sigma^{\Phi}_1 = \Delta^{\Phi}_1.$$

For each n, the $n\underline{th}$ level of the hierarchy of the effective R-sets is defined to be $\Sigma^{R_n}_1$ and will be denoted by $\mathcal{R}_n$. We denote by $c\mathcal{R}_n$ the class of complements of $\mathcal{R}_n$-sets. Plainly, $\mathcal{R}_0 = \Sigma^1_1$ and by results of Aczel, Hinman, Solovay $c\mathcal{R}_1 = \Sigma^1_1\text{-ind} = \mathfrak{D}\Sigma^0_2$. The effective R-sets have been studied by Hinman in [20] and [21]. However, Hinman was more interested in the various aspects of the interplay between the operator $\mathbb{R}$ and hierarchies of recursion theory and considered various hierarchies in $\mathcal{R}_n \cap c\mathcal{R}_n$ (see [21]). He has also shown that $c\mathcal{R}_{n+1}$

is precisely those sets which are semi-recursive in $\mathbb{R}_n^{\#}$ ([21]). Here, we shall be more interested in the "set-theoretic" aspects much in the spirit of (Chapter 6 of) Moschovakis [38]. For instance, we obtain the prewellordering property of $c\mathcal{R}_n$ by adapting Blackwell's proof of the reduction property for Π_1^1 ([6]), thus showing that Blackwell's method can be extended to more general set-theoretic operations. (Blackwell's idea was further generalized to obtain a wide range of structural results by Moschovakis and others ; see e.g. [38], [36]). Of course, the prewellordering property of $c\mathcal{R}_n$ follows from Hinman's characterization. We prove it using Blackwell's idea not only for technical reasons but also to bring out structural similarities between these pointclasses and Π_1^1. (Subtle differences however exist as observed by Hinman. In his seminal paper [20], he proved that Suslin-Kleene-type theorems do not hold for operations more powerful than operation $\mathcal{A}$).

We now define the following sequence of operations.

$$Q_0 = \mathcal{A}, \quad Q_{m+1} = \mathbb{R}\, Q_n^0 .$$

Then by [21], $Q_n \mapsto R_n$ and so

$$\mathcal{R}_n = \left\{ Q_n(\{E_n\}) : \{E_n\} \text{ is recursive} \right\}.$$

Thus, for convenience, we shall work with Q_n and shall make no distinction between Q_n and R_n.

It is not very hard to see that $\left\{\mathcal{R}_n : n \in \omega\right\}$ forms a strict hierarchy. In fact, by the usual universal set argument one can show that for every n,

$$\mathcal{R}_n \subsetneq \mathcal{R}_{n+1} \cap c\mathcal{R}_{n+1}$$

It is well-known that $c\mathcal{R}_0 = \Pi^1_1$ and $c\mathcal{R}_1 = \Game\Sigma^0_2$ are Spector pointclasses (cf [38]). Our first result shows that this is true in general.

1.1 <u>Theorem</u> : For any p.a.o. $\Phi \geq \bigcup, \bigcap$; $\Pi^{\Phi^*}_1$ is a Spector pointclass.

<u>Proof</u> : It is easy to see that $\Pi^{\Phi^*}_1$ is an <u>adequate</u> pointclass and is ω-parametrized. To see that $\Pi^{\Phi^*}_1$ has the substitution property, let $P \subseteq \mathcal{Y}$ be a $\Pi^{\Phi^*}_1$-set and $f : \mathcal{X} \to \mathcal{Y}$ a partial function which is $\Pi^{\Phi^*}_1$-recursive. Let M be the canonical base for Φ^*. Let $\{F_n\}$ be a recursive family such that

$$P(y) \leftrightarrow (\forall \eta \in M)(\exists n \in \eta)\left[y \in F_n\right]. \qquad \ldots(i)$$

Since f is partial $\Pi^{\Phi^*}_1$-recursive. There is a set P' in $\Pi^{\Phi^*}_1$ such that for all s,

$$f(x)\downarrow \to \left[f(x) \in N(\mathcal{Y}; s) \leftrightarrow P'(x,s)\right] \qquad \ldots(ii)$$

Since $\{F_n\}$ is recursive (and hence Σ^0_1) there is a Σ^0_1 set $U \subseteq \omega \times \omega$ such that

$$y \in F_n \leftrightarrow (\exists p)(y \in N(\mathcal{Y}; p) \ \&\ U(p,n)). \qquad \ldots(iii)$$

Define $P^*(x) \leftrightarrow (\forall \eta \in M)(\exists n \in \eta)(\exists p)\left[P'(x,p) \ \&\ U(p,n)\right]$.

By Theorem I.2.17 it is not hard to see that $P^* \in \Pi^{\Phi^*}_1$. We claim that for every x,

$$f(x)\downarrow \rightarrow [P^*(x) \leftrightarrow P(f(x))].$$

So assume $f(x)\downarrow$. Then

$$P(f(x)) \leftrightarrow (\forall \eta \varepsilon M)(\exists n \varepsilon \eta)[f(x) \varepsilon F_n] \quad \text{by (i)}$$

$$\leftrightarrow (\forall \eta \varepsilon M)(\exists n \varepsilon \eta)(\exists p)\{f(x) \varepsilon N(\mathcal{Y}; p) \;\&\; U(p,n)\} \quad \text{by (iii)}$$

$$\leftrightarrow (\forall \eta \varepsilon M)(\exists n \varepsilon \eta)(\exists p)\{P'(x,p) \;\&\; U(p,n)\} \quad \text{by (ii)}$$

$$\leftrightarrow P^*(x).$$

Thus $\Pi_1^{\Phi^*}$ has the substitution property. It remains to show that $\Pi_1^{\Phi^*}$ is normed. To see this let N be the canonical base for $\Phi\,\Phi^o$ and $F \subseteq \mathcal{X}$ be in $\Pi_1^{\Phi^*}$. Suppose

$$\neg F = \Phi^*(\{E_n : n \varepsilon \omega\}),$$

where $\{E_n : n \varepsilon \omega\}$ is a regular recursive family, i.e.

$$E_e = \mathcal{X} \quad , \text{ and } \quad \forall s,t \varepsilon \text{Seq}[s \subseteq t \rightarrow E_t \subseteq E_s].$$

For $x,y \varepsilon \mathcal{X}$, we define a game $G(x,y)$ as follows. The game is played between $\forall$ (I) and $\exists$ (II). First $\forall$ plays ξ_0 from N, $\exists$ replies with $n_0 \varepsilon \xi_0$ and an $\eta_0 \varepsilon N$; $\forall$ then plays an $m_0 \varepsilon \eta_0$ and $\xi_1 \varepsilon N$. The game then proceeds as described above. We say that $\exists$ wins if for some k, $x \notin E_{\langle n_0,\ldots,n_k\rangle}$ and $y \varepsilon E_{\langle m_0,\ldots,m_{k-1}\rangle}$.

We define $x \leq y$ to hold iff $x,y \varepsilon F$ and $\exists$ has a winning strategy in the game $G(x,y)$. We shall show that $\leq$ is a prewellordering. (cf [36 ; § 3]).

1^o. We claim that there is no infinite sequence x_i in F such that $x_i \not\leq x_{i+1}$ for all i.

Suppose such a sequence exists. Since $x_o \in F$ we have

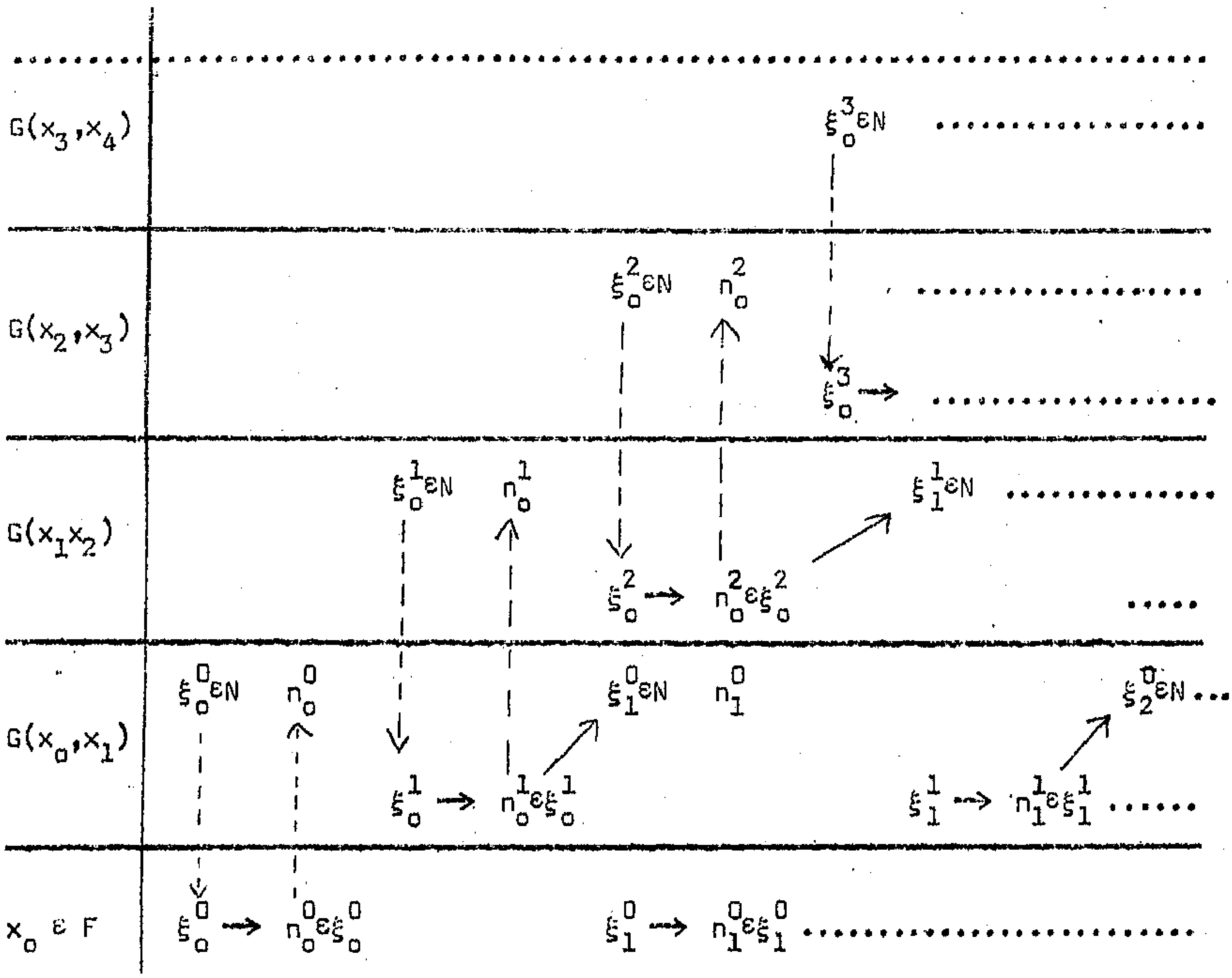
$$(\forall \xi_o \in N)(\exists n_o \in \xi_o)(\forall \xi_1 \in N)(\exists n_1 \in \xi_1)\ldots.(\exists k)\left[x_o \notin E_{<n_o,\ldots,n_{k-1}>}\right]. \qquad \ldots(iv)$$

Fix a winning strategy σ for $\exists$ in this game. Now notice that for each i, II does not win the game $G(x_i, x_{i+1})$ and since the game is determined I wins. Thus for each i,

$$(\exists \xi_o \in N)(\forall n_o \in \xi_o)(\forall \eta_o \in N)(\exists m_o \in \eta_o)\ldots.$$

$$\ldots.(\forall k)\left[x_i \in E_{<n_o,\ldots,n_k>} \vee x_{i+1} \notin E_{<m_o,\ldots,m_{k-1}>}\right] \qquad \ldots(v)$$

Let σ_i be a winning strategy for $\exists$ in (v). We now simulate the following sequence of games in which dotted lines indicate simulated moves and solid arrows show responses as dictated by the fixed winning strategies.



From the bottom game we have, for some $k' \varepsilon \omega$, $x_o \notin E_{\langle n_o^0, n_1^0, \dots, n_{k'-1}^0\rangle}$.

Since player I plays according to his winning strategy σ_o in the game $G(x_o, x_1)$ we have $(\forall k)\left[x_o \varepsilon E_{\langle n_o^0,\dots,n_k^0\rangle} \vee x_1 \notin E_{\langle n_o^1,\dots,n_{k-1}^1\rangle}\right]$.

Thus, putting $k = k'-1$, we get

$$x_1 \notin E_{\langle n_o^1,\dots,n_{k'-2}^1\rangle}.$$

At the next stage we shall obtain

$$x_2 \notin E_{\langle n_o^2,\dots,n_{k'-3}^2\rangle}.$$

Proceeding thus we see that at stage k', $x_{k'} \notin E_e$ and thus obtain a contradiction.

2^o. By 1^o we clearly have $x \preccurlyeq x$, for all x. Furthermore, we cannot have $\neg(x \preccurlyeq x' \vee x' \preccurlyeq x)$. Finally, if $x_1 \preccurlyeq x_2$ & $x_2 \preccurlyeq x_3$, then combining the two strategies gives $x_1 \preccurlyeq x_3$. Thus $\preccurlyeq$ is a prewellordering of F. Let ψ be the norm it induces on F.

3^o. Note that $\exists$ has a winning strategy in the game $G(x,y)$ iff $x \in F$ and either $y \notin F$ or $\psi(x) \leq \psi(y)$. Thus

$$x \leq^*_\psi y \Longleftrightarrow \exists \text{ wins } G(x,y)$$

$$\Longleftrightarrow (\forall \xi_o \in N)(\exists n_o \in \xi_o)(\exists \eta_o \in N)(\forall m_o \in \eta_o) \ldots\ldots$$

$$\ldots\ldots \exists k \left[x \notin E_{<n_o,\ldots,n_k>} \;\&\; y \in E_{<m_o,\ldots,m_{k-1}>}\right].$$

The last expression clearly defines a set in $\Pi^{\Phi^*}_1$.

4^o. We now claim that

$$x <^*_\psi y \Longleftrightarrow (\exists \eta_o \in N)(\forall m_o \in \eta_o)(\forall \xi_o \in N)(\exists n_o \in \xi_o) \ldots\ldots$$

$$\ldots\ldots (\exists k)\left[x \notin E_{<n_o,\ldots,n_{k-1}>} \;\&\; y \in E_{<m_o,\ldots,m_{k-1}>}\right] \quad \ldots(vi)$$

To see this suppose $\exists$ wins the above game. Then clearly $x \in F$. If $y \notin F$ then $x <^*_\psi y$. So assume $y \in F$. If $\neg(x <^*_\psi y)$ then we must have $\psi(y) \leq \psi(x)$ and hence $\exists$ wins $G(y,x)$, i.e.

$$(\forall \eta_o \in N)(\exists m_o \in \eta_o)(\exists \xi_o \in N)(\forall n_o \in \xi_o) \ldots\ldots$$

$$\ldots\ldots (\exists k)\left[y \notin E_{<m_o,\ldots,m_k>} \;\&\; x \in E_{<n_o,\ldots,n_{k-1}>}\right].$$

Combining the two games we can simulate moves such that at the end of the plays we shall obtain k_1 and k_2 satisfying

$$x \notin E_{<n_0,\dots,n_{k_1-1}>} \text{ and } y \in E_{<m_0,\dots,m_{k_1-1}>}$$

$$x \in E_{<n_0,\dots,n_{k_2-1}>} \text{ and } y \notin E_{<m_0,\dots,m_{k_2}>}.$$

By regularity of $\{E_n\}$ this gives a contradiction. We therefore have $x <^*_\psi y$.

Conversely suppose $x <^*_\psi y$. If $x \in F$ and $y \notin F$ then plainly $\exists$ wins (vi). So assume $x \in F$ and $y \in F$ and $\psi(x) < \psi(y)$. Then we have $\neg(y \preceq x)$. Thus

$$(\exists \eta_0 \in N)(\forall m_0 \in \eta_0)(\forall \xi_0 \in N)(\exists n_0 \in \xi) \dots\dots$$

$$\dots\dots (\forall k)\left[y \in E_{<m_0,\dots,m_k>} \vee x \notin E_{<n_0,\dots,n_{k-1}>}\right]. \qquad \dots(\text{vii})$$

If $\exists$ does not win (vi) then we have

$$(\forall \eta_0 \in N)(\exists m_0 \in \eta_0)(\exists \xi_0 \in N)(\forall n_0 \in \xi_0) \dots\dots$$

$$\dots\dots (\forall k)\left[x \in E_{<n_0,\dots,n_{k-1}>} \vee y \notin E_{<m_0,\dots,m_{k-1}>}\right] \qquad \dots(\text{viii})$$

From (vii), (viii) and the game witnessing $y \in F$ we can play a series of games as in 1^0 and arrive at a contradiction. This shows that $\exists$ must win the game (vi). Thus $<^*_\psi \in \Pi_1^{\Phi^*}$. This completes the proof.

1.2 <u>Corollary</u> : For each n, $o\mathcal{R}_n$ is a Spector pointclass. □

The effective versions of I.6.6, I.6.7 and I.6.12 give us the following. The proofs follow the same pattern and are therefore omitted.

1.3 <u>Theorem</u> : Let $\Gamma = \mathcal{R}_n$ or $c\mathcal{R}_n$. Then for any $A \in \Gamma$ and any s,n,r the following sets are also in Γ (uniformly).

(a) $A^{*\Sigma(s)} = \{ x : A^x \text{ is comeager in } \Sigma(s) \}$

(b) $A^{*[n,r]} = \{ x : \mu(A^x \cap G^n) \geq r \}$

(c) $A^{*(n,r)} \underline{\underline{def}} \{ x : \mu(A^x \cap G^n) > r \}$. ▯

The next result is an effective analogue of Theorem I.3.8 and can be viewed as the counterpart of the third periodicity theorem of Moschovakis [38].

1.4 <u>Theorem</u> : Suppose $F \in \Pi_1^{\mathbb{R}\Phi_N}$ and assume that

$$F(x) \leftrightarrow (\forall \eta_0 \in N)(\exists n_0 \in \eta_0)(\forall \eta_1 \in N)(\exists n_1 \in \eta_1) \ldots\ldots$$

$$\ldots\ldots (\exists k) \left[\neg E(x, \langle n_0, \ldots, n_{k-1} \rangle) \right] \qquad \ldots(i)$$

where E is a regular recursive relation. Assume, moreover, that Φ_N is normal.

Then, there is a $\Pi_1^{\mathbb{R}\Phi_N}$-recursive partial function τ such that $\tau_x = \tau(x,\cdot)$ is a winning strategy for $\exists$ in the above game, whenever $x \in F$.

<u>Proof</u> : Let Γ be the canonical inductive operator associated with N and E so that

$$s \in \Gamma_x(A) \leftrightarrow \neg E(x,s) \vee (\forall \eta \in N)(\exists n \in \eta)[s * \langle n \rangle \in A].$$

Plainly, $F(x) \leftrightarrow e \in \Gamma_x^{\infty}$.

Define
$$\beta(s,x) = \begin{cases} \text{least } \rho \text{ such that } s \in \Gamma_x^{\rho}, \text{ if such a } \rho \text{ exists} \\ \omega_1 \quad \text{otherwise.} \end{cases}$$

Clearly, $F(x) \leftrightarrow \beta(e,x) < \omega_1$.

Define $F^s(x) \leftrightarrow s \in \Gamma_x^{\infty}$.

This relation in x and s is in $\Pi_1^{\mathbb{R}\Phi_N}$. Now let $x \in F$, so $\exists$ wins the above game. If $\eta_0, \eta_1, \ldots, \eta_{i-1}$; $n_0, n_1, \ldots, n_{i-1}$ are the first 2i relevant moves of $\forall$ and $\exists$, observe that $\exists$ is in a winning position i.e. he goes on to win the game (if he has not yet done so) iff $\langle n_0, \ldots, n_{i-1}\rangle \in \Gamma_x^{\infty}$ i.e. iff $\beta(\langle n_0, \ldots, n_{i-1}\rangle, x) < \omega_1$. Hence, corresponding to any subsequent move $\eta \in N$ made by $\forall$, $\exists$ has to play an $n \in \eta$ such that $\beta(\langle n_0, \ldots, n_{i-1}, n\rangle, x) < \beta(\langle n_0, \ldots, n_{i-1}\rangle, x)$ holds. Playing thus $\exists$ ensures that he is in a winning position at each stage and ultimately arrives at a stage k when $\beta(\langle n_0, \ldots, n_{k-1}\rangle, x) = 0$ i.e. $\neg E(x, \langle n_0, \ldots, n_{k-1}\rangle)$ and, therefore, wins the game (i). We shall show that $\exists$ can ensure this in a $\Delta_1^{\mathbb{R}\Phi_N}(x)$ - recursive way. Put

$$\begin{aligned} &\mathrm{Min}(x, \langle \eta_0, \ldots, \eta_{k-1}\rangle, \langle n_0, \ldots, n_{k-1}\rangle) \\ \longleftrightarrow\ &(\forall i < k)(\eta_i \in N \ \& \ n_i \in \eta_i) \ \& \ x \in F^{\langle n_0, \ldots, n_{k-1}\rangle} \ \& \\ &(\forall v)\{(\mathrm{Seq}(v) \ \& \ \mathrm{lh}(v) = k \ \& \ (\forall i < k-1)((v)_i = n_i) \ \& \ (v)_{k-1} \in \eta_{k-1}) \\ &\qquad \longrightarrow \beta(x, \langle n_0, \ldots, n_{k-1}\rangle) \le \beta(x,v)\}. \end{aligned}$$

The proof of Theorem 1.1 shows that the relation

$$\beta(x,u) < \omega_1 \ \& \ \beta(x,u) \leq \beta(x,v)$$

is in $\Pi_1^{\mathbb{R}\Phi_N}$. It follows that the relation Min is in $\Pi_1^{\mathbb{R}\Phi_N}$.

Call $\langle a_0,\ldots,a_k\rangle$ best with respect to x and $\eta_0,\ldots,\eta_k$ if it is "minimal" with respect to x and $\eta_0,\ldots,\eta_k$ and if, moreover, there is no $b < a_k$ in η_k such that $\langle a_0,\ldots,a_{k-1},b\rangle$ is minimal. Thus

$$\begin{aligned}&\mathrm{Best}(x, \langle\eta_0,\ldots,\eta_k\rangle, \langle a_0,\ldots,a_k\rangle)\\ \longleftrightarrow\ &\mathrm{Min}(x, \langle\eta_0,\ldots,\eta_k\rangle, \langle a_0,\ldots,a_k\rangle) \ \&\\ &(\forall b < a_k)\big[b \in \eta_k \rightarrow \beta(x, \langle a_0,\ldots,a_k\rangle) < \beta(x, \langle a_0,\ldots,a_{k-1}, b\rangle)\big]\end{aligned}$$

The relation Best is also in $\Pi_1^{\mathbb{R}\Phi_N}$. Finally, τ is defined as follows

$$\begin{aligned}&\tau(x\ ;\ \eta_0, n_0, \eta_1, n_1,\ldots,\eta_{k-1}, n_{k-1}, \eta_k)\downarrow\\ \longleftrightarrow\ &F(x) \ \&\ x \in F^s \ \&\ \beta(x,s) > 0.\end{aligned}$$

In this case, put

$$\begin{aligned}&\tau(x\ ;\ \eta_0, n_0,\ldots,\eta_{k-1}, n_{k-1}, \eta_k) = a_k\\ \longleftrightarrow\ &(\exists a_0)\ldots(\exists a_{k-1})\big[(\forall j < k)\ \mathrm{Best}(x, \langle\eta_0,\ldots,\eta_j\rangle, \langle a_0,\ldots,a_j\rangle)\\ &\qquad \& \ \mathrm{Best}(x, \langle\eta_0,\ldots,\eta_k\rangle, \langle a_0,\ldots,a_k\rangle)\big].\end{aligned}$$

It is quite routine to check that if $x \in F$, τ_x is a winning strategy for $\exists$ in the game (i). ▯

The natural question that arises now is whether the Scale-Transfer Theorem of Moschovakis [38] can be generalized to the type of games that we have dealing with so far. We do not know whether this can be done. However, we can establish the scale property for $c\mathcal{R}_n$ by observing the following facts. First note that by extending the notion of presentability to any pointclass, one can obtain an effective version of Theorem I.4.6. Then observe that the result of Burgess and Kaniewski (Lemma I.5.1) can be effectivized using the canonical scales on Π^1_1 sets and the effective analogue of the theorem of Von-Neumann-Yankov. Using these facts and the effective version of Corollary I:3.5 (which follows easily from the proof of the norm property in Theorem 1.1) one can effectivize Theorem I:5.3. Without going into any further details we announce

1.5 <u>Theorem</u> : For every n, $c\mathcal{R}_n$ has the scale property. $\square$

The above results show that for each n, $c\mathcal{R}_n$ is a " nice " pointclass with pleasent structural properties and is therefore an interesting class to study.

2. <u>The effective C-sets</u> : In this section we shall consider the problem of effectivizing the $\underset{\sim}{\mathcal{BR}}^0$ = C-sets of Selivanovskij in a rather unorthodox fashion. The effective C-sets have already been constructed by Hinman in [20] where he has given a fairly general and satisfactory method of effectivizing $\underset{\sim}{\nabla}(\Phi)$ on ω for operations Φ more powerful than $\cup$. Hinman gets the effective hierarchy by generalizing Addison's construction of " effective " Borel hierarchy which, roughly, consists

of alternating applications of r.e. union and complementation. Let us briefly discuss Hinman's method. We shall confine ourselves to operation $\mathcal{A}$ for simplicity. Briefly, Hinman's method consists of assigning to each set C as it is generated an index or code $i(C)$ and at each stage of the inductive definition applying operation $\mathcal{A}$ to those families $\{F_n\}$ of subsets of ω for which the function $n \longrightarrow i(F_n)$ is recursive in some set previously generated. The class of sets thus obtained is exactly the class of sets recursive in $\mathbb{E}_1$, the type-2 object associated with operation $\mathcal{A}$. This generalizes the result for the " effective " Borel sets or HYP, since HYP is precisely the class of sets recursive in ${}^2\mathbb{E}$, the Kleene's type-2 object associated with $\bigcup$. More specifically, let I be the smallest subset of ω such that for all $a \in I$, there exist $P_a \subseteq \omega$ which satisfy the following conditions : for all a,b,c

(i) if c is an index of a partial recursive function on ω, then $<7,c> \in I$ and $P_{<7,c>} = Dm\{c\}$;

(ii) if $b \in I$ and for all k, $\{a\}(k,P_b) \in I$, then $<a,b> \in I$ and $P_{<a,b>} = \mathcal{A}(\{\neg P_{\{a\}(k,P_b)} : k \in \omega\})$.

This is an inductive definition whose closure ordinal is $\omega_1[\mathbb{E}_1]$, the least ordinal not the order-type of a well-ordering of ω recursive in $\mathbb{E}_1$ (cf [22]). Put

$$\nabla(\mathcal{A}) = \{P_a : a \in I\}.$$

Then by [20; Theorem 2.10],

$$\nabla(\mathcal{A}) = {}_1sc(\mathbb{E}_1), \text{ the class of subsets of } \omega \text{ recursive in } \mathbb{E}_1.$$

This, in short, is the effective C-sets obtained by Hinman by appropriately generalizing Addison's construction.

Now observe that the C-sets can be obtained by different methods. For example, as shown in [13], the C-sets are obtained by applying the game quantifier $\mathfrak{D}$ to $\underset{\sim}{\Delta}^0_2$ sets. In fact, by applying $\mathfrak{D}$ to the λth level of the " difference hierarchy " of $\underset{\sim}{\Delta}^0_2$ given by alternated series of decreasing closed sets (of [28, § 34 VI]) one obtains the λth level of the hierarchy of C-sets of Selivanovskij. This naturally raises the following question. Can one obtain the effective C-sets by applying $\mathfrak{D}$ to " effective " alternated series of Π^0_1 sets of appropriate lengths ? We feel that this should be possible. In this section we shall confine ourselves to another, albeit unorthodox, method of obtaining the heirarchy of the classical C-sets and then obtain an effective analogue. But first we need the following definitions.

2.1 <u>Definition</u> : $S_1(\alpha) \overset{\text{def}}{\longleftrightarrow} \alpha$ codes some tree on ω

$$\longleftrightarrow (\forall s)\left[\alpha(s) = 0 \rightarrow (Seq(s)\ \& \right.$$

$$\left.(\forall t)(Seq(t)\ \&\ t \subseteq s \rightarrow \alpha(t) = 0))\right].$$

$S_2(\alpha,s) \overset{\text{def}}{\longleftrightarrow} s$ is terminal for the tree coded by α

$$\overset{\text{def}}{\longleftrightarrow} S_1(\alpha)\ \&\ \alpha(s) = 0\ \&\ (\forall n)(\alpha(s * \langle n\rangle) \neq 0)).$$

$WF(\alpha) \overset{\text{def}}{\longleftrightarrow} \alpha$ codes a wellfounded (wff) tree on ω

$$\longleftrightarrow S_1(\alpha)\ \&\ (\forall\beta)(\exists i)\left[\alpha(\bar\beta(i)) \neq 0\right]$$

$$\longleftrightarrow S_1(\alpha)\ \&\ (\forall\beta)(\exists i) S_2(\alpha, \bar\beta(i)).$$

Plainly S_1 and S_2 are Π^0_1 while WF is Π^1_1.

If T is a wff tree on ω then $|T|$ denotes its length. If $T \neq \varphi$, then $|T| = \rho(e)$, where ρ is the rank function (cf [38 ; 2 D]). If α codes a wff tree T then put $\|\alpha\| = |T|$.

Let $\mathcal{X}$ be a space of type 0 or 1 and let $G \subseteq \omega^\omega \times \mathcal{X}$ be a good universal for $\Sigma^0_1 \restriction \mathcal{X}$. Define

$$H(\alpha,x) \leftrightarrow G(\alpha^*,x) \ \&\ \alpha(0) = 0 \ \vee\ \neg G(\alpha^*,x) \ \&\ \alpha(0) \neq 0,$$

where $\alpha^* = \lambda n.\alpha(n+1)$.

Now, define $D^{\mathcal{X}} \subseteq \omega^\omega \times \omega^\omega \times \mathcal{X}$ (or D when $\mathcal{X}$ is clear from the context) as follows.

$$D(\gamma,\delta,x) \leftrightarrow WF(\gamma) \ \&\ (\exists\alpha_0)(\forall n_0)(\forall\alpha_1)(\exists n_1)(\exists\alpha_2)(\forall n_2)\ldots$$

$$\ldots(\exists k)\Big\{S_2(\gamma,\underbrace{<\bar{\alpha}_0(n_0),\ldots,\bar{\alpha}_{k-1}(n_{k-1})>}_{t}) \ \&\ H((\delta)_t,x)\Big\}$$

Notice that the above equivalence can also be written as

$$D(\gamma,\delta,x) \leftrightarrow WF(\gamma) \ \&\ (\exists\alpha_0)(\forall n_0)(\forall\alpha_1)(\exists n_1)\ldots$$

$$\ldots(\forall k)\Big\{S_2(\gamma,\underbrace{<\bar{\alpha}_0(n_0),\ldots,\bar{\alpha}_{k-1}(n_{k-1})>}_{t}) \rightarrow H((\delta)_t,x)\Big\} \quad (*)$$

This is because for any wff tree T and any sequence $\{m_i\}$ there is a unique i such that $<m_0,\ldots,m_{i-1}>$ is terminal for T. From the above two equivalences it follows that D is in $\Delta_1^{\mathbb{R}_1} = c\mathcal{R}_1 \cap \mathcal{R}_1$.

The set D above (for spaces $\mathcal{X}$ of type 1) was constructed by Burgess and Lockhart in [14] where they show that D is universal for the C-sets. (If fact they prove that D is a Borel-programmable set and so the class of C-sets is strictly included in the class of Borel-programmable sets of Blackwell (see [14 ; § 14])).

A set $A \subseteq \mathfrak{X}$, a space of type-1, is at the λth level ($\lambda < \omega_1$) of the hierarchy of C-sets iff there exist $\gamma \in WF$, $\delta \in \omega^\omega$, $\|\gamma\| \leq \lambda$ and $A = D_{\gamma,\delta}$.

The effectivization in achieved by taking sections of D by wff trees and reals of increasing complexity (obtained à la Hinman). We first show that D is recursive in $\mathbb{E}_1$.

2.2 Lemma : For any space $\mathfrak{X}$ (of type 0 or 1), $D^{\mathfrak{X}}$ is recursive in $\mathbb{E}_1$.

Proof : We shall omit the superscript. We first fix a recursive function $(\delta,s) \longrightarrow \delta^s$ from $\omega^\omega \times \omega \longrightarrow \omega^\omega$ such that $(\delta^s)_t = (\delta)_{s*t}$. If γ codes a tree T then γ_s is a code for $T_s = \{t : s*t \in T\}$ such that the function $(\gamma,s) \longrightarrow \gamma_s$ is recursive. Now define a partial function φ as follows.

$$\varphi(c,\gamma,\delta,x) \simeq 0 \qquad \text{if } WF(\gamma) \;\&\; \|\gamma\| = 0 \;\&\; G((\delta)_e,x)$$

$$\simeq \mathbb{E}_1(\lambda k.\, \mathbb{E}_1^0(\lambda k'.\, \{c\}(\gamma_{<k,k'>}, \delta^{<k,k'>}, x)))$$

$$\text{if } WF(\gamma) \;\&\; \|\gamma\| > 0$$

$$\simeq 1 \qquad \text{otherwise ;}$$

$$\text{where} \quad \mathbb{E}_1^0(\alpha) = \begin{cases} 0 & \text{if } (\forall \beta)(\exists m)(\alpha(\bar{\beta}(m)) = 0) \\ 1 & \text{if } (\exists \beta)(\forall m)(\alpha(\bar{\beta}(m)) \neq 0). \end{cases}$$

Clearly φ is partial recursive in $\mathbb{E}_1$. By the recursion theorem there is an index c^* such that

$$\varphi(c^*,\gamma,\delta,x) \simeq \{c^*\}^{\mathbb{E}_1}(\gamma,\delta,x).$$

It is easy to check by induction on $\|\gamma\|$ that $\{c^*\}^{\mathbb{E}_1}$ is the characteristic function of D. Thus D is recursive in $\mathbb{E}_1$. ||

Since D^{ω^ω} is recursive in $\mathbb{E}_1$, any effective hierarchy we wish to construct over ω^ω will not exhaust $_2sc(\mathbb{E}_1)$ as can be shown by diagonalizing. We therefore confine ourselves to ω and from now on denote D^ω by D.

2.3 <u>Definition</u>: Let $\overleftarrow{I}$ be the smallest subset of ω such that for every $a \in \overleftarrow{I}$ there exist $tr(a) \in \omega^\omega$ coding a wff tree and a real $r(a)$ satisfying the following conditions.

For all a,b,b',c,k,s,n,m

(i) if c is the index of a total recursive function on ω, then $\langle 17,c\rangle \in \overleftarrow{I}$ and $tr(\langle 17,c\rangle)$ is the canonical code (i.e. the characteristic function) for the tree $\{e\}$ and

$$r(\langle 17,c\rangle)(\langle e,n\rangle) = \{c\}(n)$$
$$r(\langle 17,c\rangle)(m) \quad = 0 \qquad \text{otherwise};$$

(ii) if $b,b' \in \overleftarrow{I}$ and for all k, $\{a\}(k, D_{tr(b),r(b')}) \in \overleftarrow{I}$, then $\langle a,b,b'\rangle \in \overleftarrow{I}$ and

$tr(\langle a,b,b'\rangle)(e) = 0$

$$tr(\langle a,b,b'\rangle)(ks) = \begin{cases} 0 & \text{if } s = e \vee \left[Seq(s) \; \& \; tr(\{a\}(k, D_{tr(b),r(b')})(s) = 0 \right] \\ 1 & \text{otherwise} \end{cases}$$

$$r(\langle a,b,b'\rangle)(\langle ks,n\rangle) = r(\{a\}(k, D_{tr(b),r(b')}))(\langle s,n\rangle),$$

otherwise $r(\langle a,b,b'\rangle)(m) = 1$.

Now put

$$\mathcal{K} = \left\{ D_{tr(a),r(a')} : a,a' \in \overleftarrow{I} \;\&\; |a| = |a'| \right\}.$$

Obviously, there is a natural way of assigning ordinals to each $a \in \overleftarrow{I}$ (viz $|a|$) so that one can define a natural hierarchy whose scope is $\mathcal{K}$. We shall first show that it is a hierarchy in ${}_1sc(\mathbb{E}_1)$.

2.4 <u>Lemma</u> : There are primitive recursive functions σ_1 and σ_2 such that for all $a \in \overleftarrow{I}$,

$$tr(a) = \lambda n. \{\sigma_1(a)\}^{\mathbb{E}_1}(n)$$

and $$r(a) = \lambda n. \{\sigma_2(a)\}^{\mathbb{E}_1}(n).$$

<u>Proof</u> : Define two partial functions ψ_1, ψ_2 as follows. For any $e_1, e_2, a, b, b', c, n, k, s, u$

(i) $$\psi_1(e_1, e_2, \langle 17, c\rangle, n) = \begin{cases} 0 & \text{if } n = e \\ 1 & \text{otherwise.} \end{cases}$$

(ii) $\psi_1(e_1,e_2,\langle a,b,b'\rangle, t) = 0$ if $t = e \vee Seq(t) \;\&\; lh(t) = 1$

$$\psi_1(e_1,e_2,\langle a,b,b'\rangle,ks) \simeq \{e_1\}^{\mathbb{E}_1}(e_2,\{a\}(k,D_{\lambda n.\{e_1\}^{\mathbb{E}_1}(e_2,b,n),\lambda n.\{e_2\}^{\mathbb{E}_1}(e_1,b',n)}),s)$$

$\psi_1(e_1,e_2,\langle a,b,b'\rangle, n) = 1$ if $n \notin Seq$

(iii) $\psi_1(e_1,e_2,u,n)$ is undefined otherwise.

(i') $\psi_2(e_2,e_1,\langle 17,c\rangle, \langle e,n\rangle) \simeq \{c\}(n)$

(ii') $\psi_2(e_2,e_1, \langle a,b,b'\rangle, \langle ks,n\rangle)$

$$\simeq \{e_2\}^{\mathbb{E}_1}(e_1,\{a\}(k,D_{\lambda n.\{e_1\}^{\mathbb{E}_1}(e_2,b,n),\lambda n.\{e_2\}^{\mathbb{E}_1}(e_1,b',n)}),\langle s,n\rangle)$$

$\psi_2(e_2,e_1, \langle a,b,b'\rangle, m) = 1$ if $m \notin Seq \vee (m)_0 \notin Seq$

(iii') $\psi_2(e_2,e_1,u,n)$ is undefined otherwise.

Clearly, ψ_1,ψ_2 are partial recursive in $\mathbb{E}_1$ and so by the recursion theorem there are indices $\tilde{e}_1$, $\tilde{e}_2$ such that

$$\psi_1(\tilde{e}_1,e_2,a,n) \simeq \{\tilde{e}_1\}^{\mathbb{E}_1}(e_2,a,n)$$

and
$$\psi_2(\tilde{e}_2,e_1,a,n) \simeq \{\tilde{e}_2\}^{\mathbb{E}_1}(e_1,a,n).$$

Let σ_1 and σ_2 be primitive recursive functions such that

$$\{\sigma_1(a)\}^{\mathbb{E}_1}(n) \simeq \{\tilde{e}_1\}^{\mathbb{E}_1}(\tilde{e}_2,a,n)$$
$$\{\sigma_2(a)\}^{\mathbb{E}_1}(n) \simeq \{\tilde{e}_2\}^{\mathbb{E}_1}(\tilde{e}_1,a,n).$$

It is routine to check by induction on $\overleftarrow{I}$ that σ_1 and σ_2 have the required properties. ||

2.5 Theorem : $\mathcal{K} = {}_1\mathrm{sc}(\mathbb{E}_1)$.

Proof : One part of the inclusion follows from Lemmas 2.2 and 2.4. For the other inclusion we shall show

$$\nabla(\mathcal{A}) = \{ p_a : a \in I \} \subseteq \mathcal{K} .$$

In fact, we shall define partial recursive functions ρ_1 and ρ_2 such that for every $a \in I$, $\rho_1(a), \rho_2(a) \in \overleftarrow{I}$; $|\rho_1(a)| = |\rho_2(a)|$ and

$$p_a = D_{\mathrm{tr}(\rho_1(a)),\ \mathrm{r}(\rho_2(a))} . \qquad \ldots(1)$$

By the usual argument via the recursion theorem one can get a primitive recursive function π such that for every $a \in \overleftarrow{I}$ and $t \in \mathrm{Seq}$, $\pi(a) \in \overleftarrow{I}$,

$$\neg H_{(\mathrm{r}(a))_t} = H_{(\mathrm{r}(\pi(a)))_t} \qquad \ldots(2)$$

and $\quad |a| = |\pi(a)|$.

We now define two partial functions ψ_1 and ψ_2 by cases as follows.

(i) $\quad \psi_1(e_1, e_2, \langle 7, c\rangle) = \langle 17, c\rangle$

$\quad \psi_2(e_2, e_1, \langle 7, c\rangle) = \langle 17, c'\rangle$,

where $\mathrm{Dm}\{c\} = G_{\lambda n.\{c''\}(n)}$

and $\{c'\}(n) = \begin{cases} 0 & \text{if } n = 0 \\ \{c''\}(n) & \text{if } n > 0 . \end{cases}$

(ii) If $u = \langle a,b\rangle$, then let τ_1 and τ_2 be primitive recursive functions such that for any v,t,A,

$$\{e_1\}(e_2,\{v\}(t,A)) \simeq \{\tau_1(e_1,e_2,\ v)\}(t,A) \qquad \ldots(3)$$

and

$$\pi(\{e_2\}(e_1,\{v\}(t,A))) \simeq \{\tau_2(e_1,e_2,v)\}(t,A) \qquad \ldots(4)$$

Set

$$\psi_1(e_1,e_2,u) \simeq \langle\tau_1(e_1,e_2,a),\{e_1\}(e_2,b),\{e_2\}(e_1,b)\rangle$$

$$\psi_2(e_2,e_1,u) \simeq \langle\tau_2(e_1,e_2,a),\{e_1\}(e_2,b),\{e_2\}(e_1,b)\rangle.$$

By the recursion theorem there are indices $\tilde{e}_1$ and $\tilde{e}_2$ such that

$$\psi_1(\tilde{e}_1,e_2,u) \simeq \{\tilde{e}_1\}(e_2,u)$$

and

$$\psi_2(\tilde{e}_2,e_1,u) \simeq \{\tilde{e}_2\}(e_1,u).$$

Set

$$\rho_1(u) \simeq \{\tilde{e}_1\}(\tilde{e}_2,u)$$

$$\rho_2(u) \simeq \{\tilde{e}_2\}(\tilde{e}_1,u).$$

We shall show by induction on I that ρ_1 and ρ_2 have the desired properties. So let $u \in I$. If u is of the form $\langle 7,c\rangle$, then (1) holds easily. So let $u = \langle a,b\rangle$, $a \neq 7$. Then

$$x \in P_u \leftrightarrow (\exists\alpha)(\forall n)\left[x \notin P_{\{a\}(\bar{\alpha}(n),P_b)}\right]$$

$$\leftrightarrow (\exists\alpha)(\forall n)\left[x \notin D_{tr(\rho_1(\{a\}(\bar{\alpha}(n),D_{tr(\rho_1(b)),r(\rho_2(b))}))),r(\rho_2(\{a\}(\bar{\alpha}(n),D_{tr(\rho_1(b)),r(\rho_2(b))})))}\right]$$

by induction hypothesis.

$$\leftrightarrow (\exists \alpha)(\forall n)(\forall \alpha_o)\overset{(\exists n_o)}{\wedge}(\exists \alpha_1)(\forall n_1) \;\ldots\ldots$$

$$\ldots(\exists k)\Big[S_2(tr(\rho_1(\{a\}(\tilde{\alpha}(n), D_{tr(\rho_1(b)), r(\rho_2(b))}))), \underbrace{<\bar{\alpha}_o(n_o), \ldots, \bar{\alpha}_{k-1}(n_{k-1})>}_{t})$$

$$\& \; \neg H((r(\rho_2(\{a\}(\tilde{\alpha}(n), D_{tr(\rho_1(b)), r(\rho_2(b))}))))_t, x) \Big],$$

by (*) of 2.1 and determinacy (of closed games).

$$\leftrightarrow (\exists \alpha_o)(\forall n_o)(\forall \alpha_1)(\exists n_1)(\exists \alpha_2)(\forall n_2) \;\ldots\ldots$$

$$\ldots(\exists k)\Big[S_2(tr(\rho_1(\{a\}(\tilde{\alpha}_o(n_o), D_{tr(\rho_1(b)), r(\rho_2(b))}))), \underbrace{<\tilde{\alpha}_1(n_1), \ldots, \bar{\alpha}_k(n_k)>}_{t})$$

$$\& \; H((r(\pi(\rho_2(\{a\}(\tilde{\alpha}_o(n_o), D_{tr(\rho_1(b)), r(\rho_2(b))})))))_t, x) \Big] \quad \text{by (2)}$$

$$\leftrightarrow (\exists \alpha_o)(\forall n_o)(\forall \alpha_1)(\exists n_1)(\exists \alpha_2)(\forall n_2) \;\ldots\ldots$$

$$\ldots(\exists k)\Big[S_2(tr(\{\tau_1(\bar{e}_1, \tilde{e}_2, a)\}(\tilde{\alpha}_o(n_o), D_{tr(\rho_1(b)), r(\rho_2(b))})), t) \;\&$$

$$H((r(\{\tau_2(\bar{e}_1, \tilde{e}_2, a)\}(\tilde{\alpha}_o(n_o), D_{tr(\rho_1(b)), r(\rho_2(b))}))_t, x) \Big], \quad \text{by (3) and (4)}$$

$$\leftrightarrow D(tr(<\tau_1(\bar{e}_1, \tilde{e}_2, a), \rho_1(b), \rho_2(b)>), r(<\tau_2(\bar{e}_1, \tilde{e}_2, a), \rho_1(b), \rho_2(b)>), x),$$

$$\leftrightarrow D(tr(\rho_1(u)), r(\rho_2(u)), x), \quad \text{by definition.}$$

This completes the proof.

<u>Remark</u> : It is clear that the above construction is perfectly general and can be generalized to obtain a hierarchy over ${}_1sc(F_{\Phi})$, for any p.a.o Φ subsuming U. ∎

PART III

SOME PROPERTIES OF LARGE Π^1_1 SETS AND THE STRUCTURE OF HYPERARITHMETICAL SETS OF AMBIGUOUS BOREL CLASSES

1. <u>Large Π^1_1 sets</u> : In his paper 'Measure and category in effective descriptive set theory' [24], Kechris proved a number of measure-theoretic and category-theoretic results for the analytical classes under Projective Determinacy (PD). Although separate proofs were given for measure and category, Kechris, however, unified (and extended to the higher levels of the analytical hierarchy under PD) the basis theorem of Tanaka-Sacks ([49], [40]) and Hinman-Thomason ([19], [50]) by formulating the problem in terms of σ-ideals of subsets of reals satisfying certain definability conditions. In this section we shall show that, at the first level of the analytical hierarchy, some of these results (see (1) - (5) of Theorem 1.4) are instances of more general results much in the spirit of Kechris' Basis Theorem. Except for the basis theorem, we need only to consider hereditary families $\mathfrak{J}$ (i.e. closed under subsets) satisfying a weak definability condition viz., the Π^1_1 computational formula (see Defination 1.3 below). Elements of $\mathfrak{J}$ are thought of being small sets and we shall obtain the basis theorem for large Π^1_1 sets when $\mathfrak{J}$ is a σ-ideal satisfying the Π^1_1 computational formula. The corresponding results for measure and category are, therefore, particular instances of our results. Furthermore, the basis theorem of Kechris for large Π^1_1 sets is strengthened since we prove it under a much weaker definability assumption.

Our original results and proofs (see [3]) were strengthened and modified by V.V. Srivatsa and we present below mainly the modified version.

We work in ZF + DC. Our notation and terminology is as in Moschovakis [38]. Standard results in effective theory used here will be found in [38]. The following definitions would be needed in the sequel.

1.1 <u>Definition</u> : Let $A \subseteq \omega^\omega$ be a Π^1_1 set. The core of A, denoted by c(A), is the union of all Δ^1_1 subsets (possibly empty) of A.

1.2 <u>Definition</u> : For reals α, β we write $\alpha \leq_1 \beta$ iff $\alpha \in \Delta^1_1(\beta)$ and $\alpha =_1 \beta$ iff $\alpha \leq_1 \beta$ & $\beta \leq_1 \alpha$. Clearly $=_1$ is an equivalence relation and both $\leq_1$ and $=_1$ are Π^1_1 relations.

Let $\mathfrak{I}$ be a family of subsets of ω^ω and let Γ be a pointclass.

1.3 <u>Definition</u> : The family $\mathfrak{I}$ is said to satisfy the Γ-computational formula if every $B \in \Gamma$, $B \subseteq \mathcal{X} \times \omega^\omega$, $\mathcal{X} = (\omega^\omega)^k \times \omega^\ell$, the set

$$B^\Delta \stackrel{\text{def}}{=} \{x \in \mathcal{X} : B_x \notin \mathfrak{I}\}$$

is also in Γ, where $B_x = \{\beta : (x,\beta) \in B\}$.

<u>Examples</u> : The σ-ideals of meager sets and Lebesgue measure zero sets satisfy both Σ^1_1 and Π^1_1 computational formulas ([24]); the σ-ideal of countable sets satisfy the Σ^1_1 but not the Π^1_1 computational formula, while the σ-ideal of Ramsey null sets (i.e. meager with respect to the Ellentuck topology) satisfies neither the Σ^1_1 nor the Π^1_1 computational formula.

We shall also need the following result whose proof can be found in [25].

<u>Gandy's basis theorem</u> : If $A \subseteq \omega^\omega$ is a non-empty Σ^1_1 set, then A contains a real α with $\omega_1^\alpha = \omega_1^{ck}$. $\square$

1.4 <u>Theorem</u> : Let $\mathfrak{J}$ be a hereditary family of subsets of reals not containing ω^ω which satisfies the Π^1_1 computational formula. Then we have the following:

(1) Every Π^1_1 set of reals $A \notin \mathfrak{J}$ contains a Δ^1_1 subset $B \notin \mathfrak{J}$

(2) $\{\alpha \in \omega^\omega : \omega_1^{ck} < \omega_1^\alpha\} \in \mathfrak{J}$.

(3) $\alpha \in \Delta^1_1 \cap \omega^\omega \leftrightarrow \{\beta \in \omega^\omega : \alpha \leq_1 \beta\} \notin \mathfrak{J}$

where $\Delta^1_1 \cap \omega^\omega$ is the set of all Δ^1_1 reals.

(4) For every Π^1_1 set $A \subseteq \omega^\omega$, $A - c(A) \in \mathfrak{J}$.

If, moreover, $\mathfrak{J}$ is a σ-ideal, then

(5) every Π^1_1 set of reals $A \notin \mathfrak{J}$ contains a Δ^1_1 real.

<u>Proof</u> (1) This is implicitly (rather dualization of) an effective analogue of Piętkiewicz's result (cf [39]).

Fix a Π^1_1 norm φ on A. Suppose A is not Δ^1_1. Let

$$C = \{\alpha : \{\beta : \beta <^*_\varphi \alpha\} \in \mathfrak{J}\}.$$

Since $\mathfrak{J}$ satisfies the Π^1_1 computational formula, C is Σ^1_1 and since $A \notin \mathfrak{J}$, $C \subseteq A$. Moreover

$$C(\alpha) \,\&\, (\varphi(\beta) \leq \varphi(\alpha)) \rightarrow C(\beta).$$

Hence there is $\alpha_0 \varepsilon A$ such that $C = \{\alpha : \alpha <^*_\varphi \alpha_0\}$. Since $\alpha_0 \notin C$, it follow that

$$C = \{\beta : \beta <^*_\varphi \alpha_0\} \notin \mathfrak{H}.$$

Let B be a Δ^1_1 set such that $C \subseteq B \subseteq A$. Plainly B is the required set.

(2) Let $A = \{\alpha \varepsilon \omega^\omega : \omega_1^{ck} < \omega_1^\alpha\}$. Plainly A is Π^1_1 ([38 ; 4F]) and if $A \notin \mathfrak{H}$ then by (1) A includes a non-empty Δ^1_1 set. But this is not possible by Gandy's basis theorem. So $A \varepsilon \mathfrak{H}$.

(3) If α is a Δ^1_1 point then clearly $\{\beta \varepsilon \omega^\omega : \alpha \leq_1 \beta\} = \omega^\omega \notin \mathfrak{H}$. For the converse, first observe that if α is a $\Pi^1_1(\gamma)$ singleton i.e. $\{\alpha\}$ is $\Pi^1_1(\gamma)$, then the set $E[\alpha] = \{\beta \varepsilon \omega^\omega : \alpha \leq_1 \beta\}$ is $\Pi^1_1(\gamma)$. If furthermore $E[\alpha] \notin \mathfrak{H}$, then by the relativized version of (1), there is a $\Delta^1_1(\gamma)$ set $B \subseteq E[\alpha]$ such that $B \notin \mathfrak{H}$. So $\alpha \leq_1 \beta$ for all $\beta \varepsilon B$. But then

$$\alpha(n) = m \leftrightarrow (\forall \beta)[B(\beta) \rightarrow (\exists \delta \varepsilon \Delta^1_1(\beta)(\delta \varepsilon \{\alpha\} \;\&\; \delta(n) = m)],$$

so that α is a $\Delta^1_1(\gamma)$ real. Hence the set

$$G = \{\alpha \varepsilon \omega^\omega : \{\beta : \alpha \leq_1 \beta\} \notin \mathfrak{H}\}$$

contains no non-trivial $\Pi^1_1(\gamma)$ singleton for any γ. Also by hypothesis G is Π^1_1. We now claim that G is thin i.e. contains no perfect set. If not, then there is a $\Delta^1_1(\gamma)$, one-one function $g : 2^\omega \rightarrow G$ for some γ. But 2^ω contains a non-trivial $\Pi^1_1(\gamma)$ singleton and so G contains a non-trivial $\Pi^1_1(\gamma)$ singleton. But this is not possible. So G is thin and consequently, $G \subseteq C_1$, the largest thin Π^1_1 set (cf [25]). Since G

is closed under $\leq_1$ and G contains no non-trivial Π^1_1 singleton G must be the set of Δ^1_1 reals (see [25; §18]). This completes the proof of (3).

(4) Let $A \subseteq \omega^\omega$ be a Π^1_1 set. Let $A' = \{\alpha \in \omega^\omega : \omega_1^{ck} = \omega_1^\alpha\}$. Fix a recursive function $f : \omega^\omega \longrightarrow \omega^\omega$ such that $A(\alpha) \leftrightarrow WO(f(\alpha))$. Suppose $\alpha \in A'$. Then $f(\alpha) \in WO$ and $\omega_1^\alpha = \omega_1^{ck}$. But $f(\alpha)$ is $\Delta^1_1(\alpha)$ and hence $|f(\alpha)| < \omega_1^\alpha = \omega_1^{ck}$. For each $\xi < \omega_1^{ck}$, let A $A^*_\xi = \{\alpha \in A : |f(\alpha)| \leq \xi\}$. Then $A' \subseteq \bigcup_{\xi < \omega_1^{ck}} A^*_\xi$. By boundedness, $\bigcup_{\xi < \omega_1^{ck}} A^*_\xi = c(A)$, the union of all Δ^1_1 subsets of A. Thus

$A - c(A) \subseteq \{\alpha : \omega_1^{ck} < \omega_1^\alpha\}$ and by (2), $A - c(A) \in \mathfrak{J}$.

Assume now that $\mathfrak{J}$ is a σ-ideal.

(5) Let $A \subseteq \omega^\omega$ be a Π^1_1 set, $A \notin \mathfrak{J}$. By (1) get a Δ^1_1 set $B \subseteq A$ such that $B \notin \mathfrak{J}$. Fix a recursive function $f : \omega^\omega \longrightarrow \omega^\omega$ and a Π^0_1 set C such that $f \restriction C$ is one-one and $f(C) = B$. Consider the predicate $Q(s) \leftrightarrow f(C \cap N_s) \notin \mathfrak{J}$. Clearly, Q is Π^1_1. Fix a Π^1_1 norm on Q and define

$$P(s,t) \leftrightarrow Q(s) \;\&\; Q(t) \;\&\; s \subseteq t \;\&\; lh(t) = lh(s) + 1 \;\&$$
$$(\forall t')\{s \subseteq t' \;\&\; lh(t') = lh(s) + 1$$
$$\rightarrow (t \leq^*_\varphi t' \;\&\; [t <^*_\varphi t' \vee (t)_{lh(t)-1} \leq (t')_{lh(t')-1}])\}.$$

Plainly P is Π^1_1 and since $\mathfrak{J}$ is a σ-ideal for each $s \in Q$ there is a (unique) $t \in Q$ such that $P(s,t)$. By the Recursion Theorem ([38; 7A.2]) the unique α that P defines is a Δ^1_1 point in C. Hence $f(\alpha)$ is a Δ^1_1 point in A.

Remark 1 : It is clear from the proof of (1) that if Γ is a Spector pointclass and $\mathfrak{F}$ a hereditary family of subsets of ω^{ω} which satisfies the Γ computational formula, then for any Γ set $A \notin \mathfrak{F}$ there is a $\Delta (= \Gamma \cap \neg\Gamma)$ set $B \subseteq A$ such that $B \notin \mathfrak{F}$.

Remark 2 : An appropriate generalization of (3) to higher levels of the analytical hierarchy has been obtained by Srivatsa.

Remark 3 : The following example of Srivatsa shows that one cannot hope to get the basis theorem for hereditary families.

Example : Let $A \subseteq \omega^{\omega}$ be a Δ^1_1 set not containing any Δ^1_1 real. Define $I(A) = \{B \subseteq \omega^{\omega} : A \not\subseteq B\}$. Clearly $I(A)$ is a hereditary family and it is easy to see that it satisfies the Π^1_1 computational formula. Further $A \notin I(A)$ and A contains no Δ^1_1 real.

Remark 4 : A boldface analogue of (1) above yields the following (cf [35]).

Let $\mathfrak{F}$ be a σ-ideal which satisfies the $\boldsymbol{\Pi}^1_1$ computational formula. Let $C \subseteq \omega^{\omega} \times \omega^{\omega}$ be a $\boldsymbol{\Pi}^1_1$ set such that for each α, $C_{\alpha} = \{\beta : C(\alpha,\beta)\} \notin \mathfrak{F}$. Then there is a Borel measurable function f such that $(\alpha, f(\alpha)) \in C$ for each $\alpha \in \omega^{\omega}$.

To see this first obtain a Borel set $B \subseteq C$ such that $B_{\alpha} \notin \mathfrak{F}$ for each α, by (dualizing) Piątkiewicz's theorem. Then proceed as in the proof of (5).

We now obtain an analogue of a result of Kechris on the existence of the largest Π^1_1 set in $\mathfrak{F}$ where $\mathfrak{F}$ is a σ-ideal. Predictably, our proof relies on the techniques developed in [25].

1.5 <u>Theorem</u> : Let the σ-ideal $\mathfrak{I}$ satisfy both the Π^1_1 and Σ^1_1 computational formulas. Then there is a largest Π^1_1 set in $\mathfrak{I}$.

<u>Proof</u> : Let $W \subseteq \omega \times \omega^\omega$ be a Π^1_1 set which is ω-universal for Π^1_1 sets of reals. Fix a regular Π^1_1 norm ψ on W. Define

$$Q(n,\alpha) \leftrightarrow W(n,\alpha) \;\&\; \{\beta : (n,\beta) \leq^*_\psi (n,\alpha)\} \in \mathfrak{I}.$$

Put $R(\alpha) \leftrightarrow (\exists n)Q(n,\alpha)$. By hypothesis, Q is Π^1_1 and so R is Π^1_1. It is easy to check that if $W_n = \{\alpha : W(n,\alpha)\} \in \mathfrak{I}$ then $W_n \subseteq R$. To complete the proof it remains to show that $R \in \mathfrak{I}$. For this it suffices to show $Q_n \in \mathfrak{I}$ for each n. Suppose there is an $n \in \omega$ such that $Q_n \notin \mathfrak{I}$. Put

$$\alpha \in Q'_n \leftrightarrow \alpha \in Q_n \;\&\; \omega^\alpha_1 = \omega^{ck}_1 .$$

By Theorem 1, $Q_n \setminus Q'_n \in \mathfrak{I}$ and hence $Q'_n \notin \mathfrak{I}$. For each ξ, let $Q_{n,\xi} = \{\alpha \in Q_n : \psi(n,\alpha) = \xi\}$. Suppose $\alpha \in Q'_n$ and let $\psi(n,\alpha) = \mu$. Then the intial segment of $\leq^*_\psi$ determined by (n,α) is a $\Delta^1_1(\alpha)$ prewellordering of $\omega \times \omega^\omega$ of length μ. Hence μ is the length of a $\Delta^1_1(\alpha)$ prewellordering of ω^ω and so $\mu < \omega^\alpha_1 = \omega^{ck}_1$ (cf [24]). Thus

$$Q'_n \subseteq \bigcup_{\xi < \omega^{ck}_1} Q_{n,\xi} ,$$

and, consequently, there is a $\xi < \omega^{ck}_1$ such that $Q_{n,\xi} \notin \mathfrak{I}$. But this is clearly a contradiction, since for each ξ, $Q_{n,\xi}$ is easily seen to be in $\mathfrak{I}$. This completes the proof.

Obviously the problem that is of interest how is to find necessary conditions for σ-ideals $\mathfrak{I}$ to satisfy the Π^1_1 computational formula.

No satisfactory answer has been obtained. However, we shall show that if $\mathfrak{I}$ is generated by an ideal of closed sets which is Π^1_1 coded, then every Σ^1_1 set in $\mathfrak{I}$ has the covering property and hence, the Σ^1_1 computational formula holds. The covering theorem incidentally gives alternative proofs of some well-known results like the covering property of Σ^1_1 meager sets, of Σ^1_1 σ-bounded sets and Harrison's theorem on countable Σ^1_1 sets. Louveau ([30], [31]) has already obtained these earlier but our methods are different. Results in the rest of this section were obtained jointly with A. Maitra and V.V. Srivatsa.

1.6 <u>Definition</u> : A family $\mathcal{J}$ of closed subsets of ω^ω will be called an <u>ideal</u> of closed sets if whenever $A \varepsilon \mathcal{J}$ and $B \subseteq A$ is closed, then $B \varepsilon \mathcal{J}$. $\mathfrak{I}$ will be called the σ-ideal generated by $\mathcal{J}$ provided $B \varepsilon \mathfrak{I}$ iff there is a sequence $\{B_n\}$ in $\mathcal{J}$ such that $B \subseteq \bigcup_n B_n$.

An ideal of closed sets $\mathcal{J}$ is said to be Π^1_1-coded if the set of codes

$$C = \{\alpha \varepsilon \omega^\omega : \alpha \text{ codes a tree } T \text{ on } \omega \text{ such that } [T] \varepsilon \mathcal{J}\}$$

if Π^1_1. If $\mathfrak{I}$ is the σ-ideal generated by a Π^1_1-coded ideal, then $\mathfrak{I}$ is also said to be Π^1_1-coded.

1.7 <u>Lemma</u> : Let $\mathcal{J}$ be a Π^1_1 coded ideal of closed sets. If $A \varepsilon \mathcal{J}$, A is Σ^1_1, then there is a Δ^1_1 tree T on ω such that $[T] \varepsilon \mathcal{J}$ and $A \subseteq [T]$.

<u>Proof</u> : Suppose not, i.e. whenever T is a Δ^1_1 tree such that $A \subseteq [T]$, $[T] \notin \mathcal{J}$. In particular, T' is not Δ^1_1, where $T' = \{s \varepsilon \text{Seq} : N_s \cap A \neq \varphi\}$

so that T' is a Σ^1_1 tree and $[T'] \in \mathcal{G}$. Let φ be a Π^1_1 norm on $Seq \setminus T'$. Define $\psi(t) = \min\{\varphi(u) : Seq(u)\ \&\ u \subseteq t\ \&\ \neg T'(u)\}$, $t \in Seq \setminus T'$. Clearly $u,t \in Seq \setminus T'\ \&\ u \subseteq t \rightarrow \psi(t) \leq \psi(u)$. It is not hard to check that ψ is a Π^1_1 norm on $Seq \setminus T'$. Then

$$s \in T' \leftrightarrow Seq(s)\ \&\ (\forall\alpha)\left[(\alpha \text{ codes a tree } \& \right. \qquad \ldots(i)$$
$$\left.(\forall t)(\alpha(t) = 0 \rightarrow \neg(t <^*_\psi s))) \rightarrow C(\alpha)\right].$$

To see this, let $s \in T'$. Now if α codes a tree T'' such that $(\forall t)(t \in T'' \rightarrow \neg(t <^*_\psi s))$, then $T'' \subseteq T'$ so that $[T''] \in \mathcal{G}$ and so $\alpha \in C$. Conversely, suppose $s \notin T'\ \&\ Seq(s)$. Let

$$S = \{t \in Seq : \neg(t <^*_\psi s)\}.$$

Clearly S is Δ^1_1. We define

$$S^* = \{u \in Seq : (\exists t)(S(t)\ \&\ u \subseteq t)\}.$$

Then S^* is a Δ^1_1 tree and $T' \subseteq S \subseteq S^*$. Choose $\alpha \in \omega^\omega$ such that $\alpha = 0$ on S^* and $\alpha \neq 0$ off S^*. Then $(\forall t)[\alpha(t) = 0 \rightarrow \neg(t <^*_\psi s)]$. Since α codes a Δ^1_1 tree S^* and $A \subseteq [S^*]$, so by assumption $\neg C(\alpha)$. This proves (i). But (i) is a Π^1_1 definition of T', so T' is Δ^1_1, which is a contradiction.

The above lemma can also be formulated in terms of a Spector pointclass closed under $\forall^{\omega^\omega}$ and is of independent interest.

We now prove the covering theorem.

1.8 <u>Theorem</u>: Let $\mathcal{G}$ be as above and $\mathcal{H}$ the σ-ideal generated by $\mathcal{G}$. Suppose $A \subseteq \omega^\omega$ is Σ^1_1 and $A \in \mathcal{H}$. Then there is a Δ^1_1 set $B \subseteq \omega \times \omega^\omega$

such that for each n, $B_n = \{\alpha : B(n,\alpha)\} \varepsilon \mathcal{I}$ and $(\forall\alpha)[A(\alpha) \to (\exists n)B(n,\alpha)]$.

Proof : Let $M = \cup\{[T] : T$ is a Δ^1_1 tree & $[T] \varepsilon \mathcal{I}\}$. It is easy to check that M is Π^1_1.

We claim that if $L \neq \varphi$ is Σ^1_1 and $L \varepsilon \mathcal{H}$, then $L \cap M \neq \varphi$. To see this, let $f : \omega^\omega \to \omega^\omega$ be a recursive function and $D \subseteq \omega^\omega$ be a Π^0_1 set such that $f(D) = L$. Now $L \subseteq \bigcup_{n \geq 0} L_n$ where each $L_n \varepsilon \mathcal{I}$ By Baire Category theorem, there is $s_o \varepsilon$ Seq and $n_o \varepsilon \omega$ such that $\varphi \neq f(D \cap N_{s_o}) \subseteq L_{n_o}$. Now $f(D \cap N_{s_o})$ is Σ^1_1 and so $cl(f(D \cap N_{s_o}))$ is Σ^1_1 and in $\mathcal{I}$. Hence by Lemma 1 there is a Δ^1_1 tree T such that $[T] \varepsilon \mathcal{I}$ and $f(D \cap N_{s_o}) \subseteq [T]$. It follows that $L \cap M \supseteq L \cap [T] \supseteq f(D \cap N_{s_o}) \neq \varphi$.

Now if $A \not\subseteq M$, then $L' = A \setminus M$ is Σ^1_1, nonempty and in $\mathcal{H}$. Hence $L' \cap M \neq \varphi$ which is impossible. So $A \subseteq M$. Let

$$R(n,\alpha) \leftrightarrow d(n)\downarrow \;\&\; C(d(n)) \;\&\; (\forall m)(d(n)(\bar{\alpha}(m)) = 0),$$

where $d : \omega \to \omega^\omega$ is a partial Π^1_1 recursive function which parametrizes $\Delta^1_1 \cap \omega^\omega$. R is Π^1_1 and $A(\alpha) \to (\exists n)R(n,\alpha)$. Get a Δ^1_1 function $g : \omega^\omega \to \omega$ such that $(\forall\alpha \varepsilon A)R(g(\alpha), \alpha)$.

Let $Q_1(n) \leftrightarrow (\exists\alpha \varepsilon A)(g(\alpha) = n)$

$$Q_2(n) \leftrightarrow d(n)\downarrow \;\&\; C(d(n)).$$

So Q_1 is Σ^1_1, Q_2 is Π^1_1 and $Q_1 \subseteq Q_2$. Get a Δ^1_1 Q such that $Q_1 \subseteq Q \subseteq Q_2$. Define

$$B(n,\alpha) \leftrightarrow Q(n) \;\&\; (\forall m)(d(n)(\bar{\alpha}(m)) = 0).$$

Clearly, B is Δ^1_1 and does the job.

The Σ^1_1 computational formula now follows immediately.

1.9 <u>Theorem</u> : Let $\mathcal{J}$ be as above. Then $\mathcal{J}$ satisfies the Σ^1_1 computational formula.

<u>Proof</u> : Let $A \subseteq \mathcal{X} \times \omega^\omega$ be a Σ^1_1 set, where $\mathcal{X} = (\omega^\omega)^k \times \omega^l$. Let

$$B = \{x : A_x \in \mathcal{J}\}.$$

It suffices to show that B is Π^1_1. By a relativization of Theorem 1.8 we have

$$B(x) \leftrightarrow (\forall \beta)\, [A(x,\beta) \rightarrow (\exists n)(d(n,x)\downarrow \;\&\; c(d(n,x))$$
$$\&(\forall m)\,(d(n,x)(\bar{\beta}(m)) = 0))],$$

where d is as in [38; 4D.2]. Thus B is Π^1_1. ‖

As an illustration, we shall we shall adopt the above methods to obtain the following separation theorem (see [16] and the next section).

Suppose A and B are two Σ^1_1 sets and A can be separated from B by the difference of two $\mathbf{\Sigma}^0_2$ sets. Then A can be separated from B by the difference of two $\Delta^1_1 \cap \mathbf{\Sigma}^0_2$ sets.

To see this consider the following ideal of closed sets

$\mathcal{J} = \{[T] : T$ is a tree & $[T] \cap B$ is contained in a $\mathbf{\Sigma}^0_2$ set disjoint from $A\}$.

$= \{[T] : T$ is a tree & $[T] \cap B$ is contained in a $\Delta^1_1(T)$, $\mathbf{\Sigma}^0_2$ set disjoint from $A\}$, by the separation theorem of Louveau [29].

Then $\mathcal{Y}$ is Π^1_1-coded and if $\mathcal{J}$ is the σ-ideal generated by $\mathcal{Y}$, then $A \in \mathcal{J}$. So by the covering theorem, there is a Δ^1_1-sequence of sets $F_n - \bigcup_m G_{nm}$ such that $A \subseteq \bigcup_n (F_n - \bigcup_m G_{nm})$ and each F_n, G_{nm} is closed and (uniformly) Δ^1_1 and $G_{nm} \cap A = \varphi$, $F_n \cap B \subseteq \bigcup_m G_{nm}$. Then

$$A \subseteq \bigcup_n F_n - \bigcup_n \bigcup_m G_{nm} \quad \text{and} \quad (\bigcup_n F_n - \bigcup_{n,m} G_{nm}) \cap B = \varphi .$$

2. Hyperarithmetical Sets of Ambiguous Borel Classes with Applications to Borel Sets in Product Spaces.

2.0 Introduction : In this part of our dissertation we shall digress a bit and undertake a study of the structure of hyperarithmetical sets which are of ambiguous Borel classes. Thus our object of study is mainly $\Delta^1_1 \cap \Delta^0_{\xi+1}$ ($\xi < \omega^{ck}_1$). Borel sets of ambiguous class $\xi+1$ (i.e. belonging to $\Delta^0_{\xi+1}$) are precisely those sets which can be written as the union of alternating differences of a transfinite decreasing sequence of Π^0_ξ sets. Such a representation gives rise to a natural hierarchy (the so-called difference hierarchy) of $\Delta^0_{\xi+1}$. (cf [28; § 37 IV]). Here we shall show that there is an analogous effective hierarchy of $\Delta^1_1 \cap \Delta^0_{\xi+1}$ which coincides with the difference hierarchy of $\Delta^0_{\xi+1}$ restricted to Δ^1_1.

Although we believe that the study of such hierarchies is not entirely out of place in the context of the effective R-sets, the content of this section stems mainly from a result of G. Debs [16] on the structure of Borel sets in product spaces which states the following.

Suppose X and Y be two Polish (analytic) spaces and $B \subseteq X \times Y$ a Borel set such that each vertical section is the difference of two G_δ sets. Then $B = C - D$, where C, D are Borel subsets of $X \times Y$ with G_δ sections. In the previous section, we have obtained an effective analogue by showing that a Δ_1^1 set of reals which is the difference of two Π_2^0 sets is of the form $C - D$, where C and D are both $\Delta_1^1 \cap \Pi_2^0$. This immediately gives Debs' result in product spaces. We shall generalize Debs' result by showing the following[1]. Suppose A is a Δ_1^1 set of reals and λ, ξ are fixed recursive ordinals. Suppose there is a decreasing transfinite sequence of Π_ξ^0 sets $\{G_\mu : \mu < \lambda\}$ such that

$$A = \bigcup_{\substack{\mu < \lambda \\ \mu \text{ even}}} (G_\mu - G_{\mu+1}).$$

Then there is a decreasing "Δ_1^1-sequence" of $\Delta_1^1 \cap \Pi_\xi^0$ sets $\{G_\mu^* : \mu < \lambda\}$ such that

$$A = \bigcup_{\substack{\mu < \lambda \\ \mu \text{ even}}} (G_\mu^* - G_{\mu+1}^*).$$

This completely generalizes Debs' result on Borel sets in product spaces and answers a question raised by us in [3]. (Note that when λ is odd G_λ and G_λ^* above are taken to be the empty set). Now observe that

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1. After we have obtained this result we were informed by Prof. Louveau that he has obtained this and much more in his paper " Some properties of the Wadge hierarchy of Borel sets " in [32].

by a result of Burgess and Louveau [29; § 2 Theorem 6] (see Theorem 2.2.8 below) there is an effective analogue of the difference hierarchy in $\Delta^1_1 \cap \underset{\sim}{\Delta}^0_{\xi+1}$. Our result quoted above shows that this hierarchy coincides with the hierarchy in $\Delta^1_1 \cap \underset{\sim}{\Delta}^0_{\xi+1}$ obtained by restricting the difference hierarchy in $\underset{\sim}{\Delta}^0_{\xi+1}$ to Δ^1_1 sets. We shall also show (via a separation theorem for Σ^1_1 sets) that if P and Q are two disjoint $\underset{\sim}{\Sigma}^1_1$ sets in $\omega^\omega \times \omega^\omega$ such that each vertical section P_α is separated from Q_α by a $\underset{\sim}{\Delta}^0_\xi$ set, then there is a Borel set B separating P and Q such that each vertical section B_α is $\underset{\sim}{\Delta}^0_\xi$. This is a result of Burgess for $\xi = 2$; our result extends Burgess' result to every level. Analogues of these results for Δ^1_1, clopen sets are also obtained — results due to J.W. Addison, but never published. We include them for the sake of completeness.

As applications we shall, as is usually the case, obtain structural results for Borel sets in product spaces. These include the generalization of Debs' result mentioned earlier and an unpublished result of Addison and Harrington on the non-existence of Borel sets universal for $\underset{\sim}{\Delta}^0_{\xi+1}$.

2.1 <u>Coding pair ; the ξ - and ∞ - topologies</u> : Most of the material in this subsection is taken from Louveau's article [29]. Here, we fix a recursively presented (r.p.) Polish space X (for definition see [38]; otherwise one may think of X as the space 2^ω or ω^ω or $(\omega^\omega)^k \times \omega^\ell$). Given such a space X, by a coding pair we shall mean a pair $(W(X), C(X))$ (or (W, C), when X is clear from the context) such that

(i) W, the set of codes, is a Π^1_1 set of integers,

(ii) C is a Π^1_1 subset of $\omega \times X$ which is universal for non-empty Δ^1_1 subsets of X

(iii) $\pi(C) = W$, and

(iv) the relation $n \in W \;\&\; x \notin C_n$ is Π^1_1.

Observe that such a coding pair exists (cf [38]).

A set $A \subseteq X$ is the Δ^1_1-union of a sequence $\{B_n : n \in \omega\}$, written $A = U^1_1 B_n$, if $A = \bigcup_n B_n$ and the set B defined by

$$B(n,x) \leftrightarrow x \in B_n$$

is Δ^1_1. If Γ is a family of sets, we denote by $U^1_1\Gamma$ the family of sets obtained by Δ^1_1-union performed on sequence of sets from Γ.

The Δ^1_1-recursive hierarchy $(\Sigma^*_\xi, \pi^*_\xi)_{\xi \in On}$ is defined by the induction

$$\Sigma^*_0 = \pi^*_0 = \{N_s : s \in Seq\}$$

$$\Sigma^*_\xi = U^1_1(\bigcup_{\eta < \xi} \pi^*_\eta)$$

and $\quad \pi^*_\xi = \neg\, \Sigma^*_\xi = \{A \subseteq X : X - A \in \Sigma^*_\xi\}$.

Clearly, $\Sigma^*_\xi \subseteq \underset{\sim}{\Sigma}^0_\xi \cap \Delta^1_1$; in fact Louveau proves, for $\xi < \omega^{ck}_1$,

$$\Sigma^*_\xi = \underset{\sim}{\Sigma}^0_\xi \cap \Delta^1_1 \text{ and } \Delta^1_1 = \bigcup_{\xi < \omega^{ck}_1} \Sigma^*_\xi .$$

For details regarding this hierarchy we refer to [29].

For each $\xi < \omega^{ck}_1$, the set W_ξ of ξ-codes is defined by

$$n \in W_\xi \leftrightarrow n \in W \ \& \ c_n \in \pi^*_\xi .$$

One can show that W_ξ and $\bigcup_{\eta < \xi} W_\eta$ are Π^1_1 ([29]).

We now consider a family of topologies as follows.

For each ξ, $1 \le \xi < \omega_1^{ck}$, the topology T_ξ on X is the topology on X generated by $\Sigma^1_1 \cap (\bigcup_{\eta<\xi} \underset{\sim}{\pi}^0_\eta)$. T_∞ is the topology on X generated by all Σ^1_1 subsets of X. The topology T_∞ will be called the <u>∞-topology</u> or the <u>Harrington topology</u>. Clearly T_1 is the usual topology on X and the topologies $\{T_\xi : \xi < \omega_1^{ck}\}$ are increasing, each T_ξ being coarser than T_∞. Terms and notation like "ξ-open", "ξ-closed", "$\overline{A}^\xi$" ... refer to the corresponding topological notion in the ξ-topology.

We shall need the following results in the sequal. The proofs can be found in [29].

2.1.1 <u>Lemma</u> (Harrington, Louveau) : The space (X, T_∞) is a Baire space i.e. the countable intersection of ∞-open-dense sets is ∞-dense.

Consequently, any open subspace of (X, T_∞) is also a Baire space. ▯

2.1.2 <u>Lemma</u> (Louveau) : Let $A \subseteq X$ be a Σ^1_1 set. Then $\overline{A}^\xi$ is $\underset{\sim}{\pi}^0_\xi$ and Σ^1_1. ▯

For each set $H \subseteq X$, let G_H denote the largest ξ-open set such that $H \cap G_H$ is ∞-meager. We set $\widetilde{H}^\xi = X - G_H$. Clearly $\widetilde{H}^\xi$ is ξ-closed and $H - \widetilde{H}^\xi$ is ∞-meager. In fact we have

2.1.3 <u>Lemma</u> (Louveau) : If H is $\underset{\sim}{\pi}^0_\xi$, then $H \triangle \widetilde{H}^\xi$ is ∞-meager.

<u>Proof</u> : See [29].

For any two subsets A, B of X we write $A \underset{a.e}{\subseteq} B$ if $A - B$ is ω-meager. Clearly, if $A \subseteq B$ then $A \underset{a.e}{\subseteq} B$; $\underset{a.e}{\subseteq}$ is transitive and satisfies many of the properties of $\subseteq$.

2.1.4 <u>Lemma</u> : If $A \subseteq X$ is ω-open, and $A \underset{a.e}{\subseteq} H$, then $A \subseteq \widetilde{H}^{\xi}$.

<u>Proof</u> : Clearly, $A \underset{a.e}{\subseteq} \widetilde{H}^{\xi}$ and hence $A - \widetilde{H}^{\xi}$ is ω-meager. But $A - \widetilde{H}^{\xi}$ is ω-open. So by Lemma 2.1.1, $A - \widetilde{H}^{\xi} = \varphi$.

2.2 <u>The main results</u> : For any Polish space X, a property $\mathbb{P}$ of subsets of X is called a <u>faithful separation</u> (FS) property if for any disjoint Σ_1^1 $A, E \subseteq X^2$, if there exist a set $B \subseteq X^2$ separating A, E such that every vertical section B_x of B has $\mathbb{P}$, then there exists a Borel set with the same properties. Burgess [9] has shown that the property of being Δ_2^0 is FS. In this section we shall extend this result to every level. We first need the following separation principle.

2.2.1 <u>Lemma</u> : Let $1 \leq \xi < \omega_1^{ck}$. Let P and Q be two Σ_1^1 sets of reals which can be separate by a Δ_ξ^0 set. Then they can be separated by a $\Delta_1^1 \cap \Delta_\xi^0$ set.

<u>Proof</u> : Suppose H is a Δ_ξ^0 set such that $P \subseteq H$ and $H \cap Q = \varphi$. Since P is Σ_1^1, by Lemma 2.1.4, $P \subseteq \widetilde{H}^{\xi}$. Hence $\overline{P}^{\xi} \subseteq \widetilde{H}^{\xi}$. Similarly, $\overline{Q}^{\xi} \subseteq \widetilde{H^c}^{\xi}$. So,

$$\overline{P}^{\xi} \cap \overline{Q}^{\xi} \subseteq \widetilde{H}^{\xi} \cap \widetilde{H^c}^{\xi} \subseteq (\widetilde{H}^{\xi} - H) \cup (\widetilde{H^c}^{\xi} - H^c).$$

Since both H and H^c are Π_ξ^0, by Lemma 2.1.3, both $\widetilde{H}^{\xi} - H$ and $\widetilde{H^c}^{\xi} - H^c$ are ω-meager. Hence $\overline{P}^{\xi} \cap \overline{Q}^{\xi}$ is ω-meager. By Lemma 2.1.2, $\overline{P}^{\xi} \cap \overline{Q}^{\xi}$

is Σ_1^1 and hence by Lemma 2.1.1, $\bar{P}^{\xi} \cap \bar{Q}^{\xi} = \varphi$. Applying the separation theorem of Louveau twice one obtains two disjoint Δ_1^1, Π_{ξ}^0 sets P^* and Q^* such that $\bar{P}^{\xi} \subseteq P^*$ and $\bar{Q}^{\xi} \subseteq Q^*$. Now by [29 ; § 2A, Corollary 3] or by effectivizing [28 ; § 30 VII Theorem 2] we get a $\Delta_1^1 \cap \Delta_{\xi}^0$ set separating P^* from Q^*. This completes the proof.

The relativized version of the above immediately gives the FS property of Δ_{ξ}^0.

2.2.2 <u>Theorem</u> : Let P and Q be two disjoint Σ_1^1 subsets of $\omega^{\omega} \times \omega^{\omega}$ such that each vertical section $P_{\alpha} = \{\beta : P(\alpha,\beta)\}$ of P is separated from Q_{α} by a Δ_{ξ}^0 set, $1 \leq \xi < \omega_1$. Then there is a Borel set B separating P and Q such that each vertical section B_{α} is Δ_{ξ}^0.

<u>Proof</u> : Fix z such that P and Q are $\Sigma_1^1(z)$ and $\xi < \omega_1^z$. Then for each α, P_{α} and Q_{α} are $\Sigma_1^1(<\alpha,z>)$ sets which can be separated by a Δ_{ξ}^0 set. Hence by Lemma 2.2.1 (relativized) there is a $\Delta_1^1(<\alpha,z>) \cap \Delta_{\xi}^0$ set separating P_{α} and Q_{α}. As in [29 , § 2C] fix a pair (W,C) of Π_1^1 relations, $W \subseteq X \times \omega^{\omega}$, $C \subseteq X \times \omega \times \omega^{\omega}$ such that for each x, $<W_x, C_x>$ is a coding pair for $\Delta_1^1(x)$ subsets of ω^{ω}. Consider the following (cf. [29] for notation)

$$R(\alpha,m,n) \leftrightarrow m \in W_{\xi}^{<\alpha,z>} \;\&\; n \in W_{\xi}^{<\alpha,z>}$$

$$\& \; C_{<\alpha,z>,m} = \neg C_{<\alpha,z>,n} \;\&\; P_{\alpha} \subseteq C_{<\alpha,z>,m}$$

$$\& \; C_{<\alpha,z>,m} \cap Q_{\alpha} = \varphi .$$

One can check that R is $\Pi_1^1(z)$ (cf [29]) and $(\forall\alpha)(\exists m)(\exists n) R(\alpha,m,n)$.

Hence by Easy Uniformization [38] there exist $\Delta^1_1(z)$ functions f and g such that $(\forall\alpha) R(\alpha, f(\alpha), g(\alpha))$. Now define B as follows.

$$B(\alpha,\beta) \leftrightarrow C(<\alpha,z>, f(\alpha),\beta)$$

Clearly, B is $\Delta^1_1(z)$ and hence Borel, B separates P from Q and for each α, B_α is $\underset{\sim}{\Delta}^0_\xi$. This completes the proof.

We shall now prove the structural result mentioned in the introduction. <u>We fix for the rest of this section a recursive function</u> φ <u>such that if $\alpha \in WO$ then $\varphi(\alpha,n)$ codes the strict initial segment of $\leq_\alpha$ with top n, whenever</u> $n \in \text{Field}(\leq_\alpha)$; <u>otherwise it codes the empty relation</u>.

2.2.3 <u>Theorem</u>: Let L and M be two disjoint Σ^1_1 sets of reals, $P \subseteq \omega^\omega$ any Σ^1_1 set and $1 \leq \lambda < \omega_1^{ck}$, $1 \leq \xi < \omega_1^{ck}$. Suppose there is a decreasing transfinite sequence $\{H_\mu : \mu < \lambda\}$ of $\underset{\sim}{\Pi}^0_\xi$ sets such that

(a) $L \subseteq H_o$

(b) $H_\mu = \bigcap_{\mu' < \mu} H_{\mu'}$, if $\mu < \lambda$ is limit

(c) $H_\mu \cap M \subseteq H_{\mu+1}$, if $\mu < \lambda$ is even

(d) $H_\mu \cap L \subseteq H_{\mu+1}$, if $\mu < \lambda$ is odd

(e) $(\bigcap_{\mu<\lambda} H_\mu) \cap P = \varphi$

Then, for each recursive $\alpha \in WO$ with $|\alpha| = \lambda$, there is a π^*_ξ set $H^* \subseteq \omega \times \omega^\omega$ such that

(o) $H^*_i \supseteq H^*_j$ if $i \leq_\alpha j$

(1) $L \subseteq H^*_n$ if $n \in \mathrm{Field}(\leq_\alpha)$ and $|\varphi(\alpha,n)| = 0$

(2) $H^*_i = \bigcap_{j <_\alpha i} H^*_j$ if $|\varphi(\alpha,i)|$ is limit

(3) $H^*_i \cap M \subseteq H^*_j$ if $|\varphi(\alpha,i)|$ is even and $|\varphi(\alpha,i)| + 1 = |\varphi(\alpha,j)|$

(4) $H^*_i \cap L \subseteq H^*_j$ if $|\varphi(\alpha,i)|$ is odd and $|\varphi(\alpha,i)| + 1 = |\varphi(\alpha,j)|$

(5) $(\bigcap_{i \in \mathrm{Field}(\leq_\alpha)} H^*_i) \cap P = \varphi$.

Proof : We shall prove this by induction on λ. So assume that the result is true for all $\lambda' < \lambda$.

Case 1. $\lambda = \bar{\lambda} + 1$.

If $\bar{\lambda}$ is limit, then observe that the sequence $\{H_\mu : \mu < \bar{\lambda}\}$ satisfies the hypotheses (a) – (e) of the theorem. Hence induction hypothesis applies. So assume $\lambda = \lambda_1 + 2$. Fix a recursive function f and π^0_1 set $A \subseteq \omega^\omega$ such that $f(A) = P$. Let (W,C) and (W',C') be coding pairs which code Δ^1_1 subsets of ω^ω and $\omega \times \omega^\omega$ respectively.

Subcase (i). λ_1 is even.

In this case we define A^*_1 as follows.

$$x \in A_1^* \leftrightarrow (\exists i)\Big\{ i \in \bigcup_{\eta<\xi} W_\eta \ \& \ x \in C_i \ \& \ (\exists m \in \bigcup_{\eta<\xi} W_\eta)(\exists n \in W_\xi')(\forall \beta)$$

$$\Big[\beta \in A \cap C_i \longrightarrow (f(\beta) \in C_m \text{ and}$$

$$(0): (\forall i)(\forall j)(i \in \text{Field}(\leq_\alpha) \ \& \ i <_\alpha j \rightarrow C'_{n,i} \supseteq C'_{n,j})$$

$$\& \ (1): (\forall i)(i \in \text{Field}(\leq_\alpha) \ \& \ |\varphi(\alpha,i)| = 0 \rightarrow L \subseteq C'_{n,i})$$

$$\& (2): (\forall i)(|\varphi(\alpha,i)| \text{ is limit} \rightarrow C'_{n,i} = \bigcap_{j<_\alpha i} C'_{n,j})$$

$$\& (3): (\forall i)(\forall j)(i \in \text{Field}(\leq_\alpha) \ \& \ |\varphi(\alpha,i)| \text{ is even}$$

$$\& \ |\varphi(\alpha,i)|+1 = |\varphi(\alpha,j)| \rightarrow C'_{n,i} \cap M \subseteq C'_{n,j})$$

$$\& (4): (\forall i)(\forall j)(|\varphi(\alpha,i)| \text{ is odd} \ \& \ |\varphi(\alpha,i)|+1 = |\varphi(\alpha,j)|$$

$$\rightarrow C'_{n,i} \cap L \subseteq C'_{n,j})$$

$$\& (5'): C'_{n,i*} \cap C_m \cap M = \varphi)\Big]\Big\},$$

where $|\varphi(\alpha,i^*)| = \lambda_1$. Clearly, A_1^* is ξ-open and hence ω-open. Moreover, it is not hard to check that A_1^* is also π_1^1. We now claim that $A \subseteq A_1^*$. If not, then $B = A - A_1^* \neq \varphi$ and so $f(B) \subseteq f(A) = P$. Hence by (e), $f(B) \subseteq F_{\lambda_1+1}$, where $F_{\lambda_1+1} = \omega^\omega - H_{\lambda_1+1}$ is $\underset{\sim}{\Sigma}{}_\xi^0$. So we may write $F_{\lambda_1+1} = \bigcup_{k \in \omega} F^k_{\lambda_1+1}$, where each $F^k_{\lambda_1+1}$ is in $\underset{\sim}{\pi}{}^0_{\eta(k)}$ for some $\eta(k) < \xi$. Consequently, $f(B) \underset{a.e}{\subseteq} \bigcup_k \widetilde{F^k_{\lambda_1+1}}^{\eta(k)}$. Now observe that B is ω-clopen and hence (B, T_ω) is a Baire space. Further $f: (B,T_\omega) \rightarrow (\omega^\omega, T_\omega)$ is continuous and open. Hence $B \underset{a.e}{\subseteq} \bigcup_k f^{-1}(\widetilde{F^k_{\lambda_1+1}}^{\eta(k)})$. So there is a $k \in \omega$ such that $f^{-1}(\widetilde{F^k_{\lambda_1+1}}^{\eta(k)})$ is not - ω-nowheredense.

Hence there is a non-empty Σ^1_1 set $B' \subseteq B$ such that $f(B') \subseteq \widetilde{F}^{k\,\eta(k)}_{\lambda_1+1}$. But $\widetilde{F}^{k\,\eta(k)}_{\lambda_1+1} \underset{a.e}{\subseteq} F^k_{\lambda_1+1} \subseteq F_{\lambda_1+1}$. Hence $\overline{f(B')}^{\eta(k)} \underset{a.e}{\subseteq} F_{\lambda_1+1}$. This shows that $\overline{f(B')}^{\eta(k)} \cap H_{\lambda_1+1}$ is ω-meager. Since $\overline{f(B')}^{\eta(k)}$ is ξ-open, we have

$$\overline{f(B')}^{\eta(k)} \cap \widetilde{H}^{\xi}_{\lambda_1+1} = \varphi. \qquad \ldots(I)$$

We now define a decreasing sequence of ξ-closed sets $\{J_\mu : \mu \leq \lambda_1 + 1\}$ satisfying the following

(i) $L \subseteq J_0$

(ii) $J_\mu \subseteq \widetilde{H}^{\xi}_{\mu}$ for each $\mu \leq \lambda_1 + 1$

(iii) each J_μ is Σ^1_1

(iv) $J_\mu \cap M \subseteq J_{\mu+1}$ if $\mu < \lambda_1 + 1$ is even

(v) $J_\mu \cap L \subseteq J_{\mu+1}$ if $\mu < \lambda_1 + 1$ is odd

(vi) $J_\mu = \bigcap_{\mu' < \mu} J_{\mu'}$ if $\mu < \lambda_1 + 1$ is limit.

Such a sequence is defined by recursion as follow. Fix once and for all $m^*, n^* \in \text{Field}(\leq_\alpha)$ such that $|\varphi(\alpha, m^*)| = 0$ and $|\varphi(\alpha, n^*)| = \lambda_1 + 1$. Let G be a good ω-universal set for $\Sigma^1_1 \restriction \omega \times \omega^\omega$. Define

$$R(p,n,\alpha) \longleftrightarrow (n = m^* \ \& \ \alpha \in \overline{L}^{\xi})$$

$$\vee \ (|\varphi(\alpha,n)| \text{ is limit } \& \ (\forall j <_\alpha n)G(p,j,\alpha))$$

$$\vee \ (\exists m \in \text{Field}(\leq_\alpha))(|\varphi(\alpha,m)| + 1 = |\varphi(\alpha,n)| \ \& \ n <_\alpha n^*$$

$$\& \ |\varphi(\alpha,n)| \text{ is even } \& \ \alpha \in \overline{G_{p,m} \cap L^{\xi}})$$

$$\vee \ (\exists m \in \text{Field}(\leq_\alpha)(|\varphi(\alpha,m)| + 1 = |\varphi(\alpha,n)| \ \& \ n \leq_\alpha n^*$$

$$\& \ |\varphi(\alpha,n)| \text{ is odd } \& \ \alpha \in \overline{G_{p,m} \cap M^{\xi}}).$$

As in the proof of Lemma 2.2.1 (cf [29]) one can show that R is Σ^1_1. Hence by the Recursion Theorem of Kleene ([3B ; 3H]) there is a $p^* \in \omega$ such that

$$R(p^*,n,\alpha) \longleftrightarrow G(p^*,n,\alpha).$$

Then for $\eta \leq \lambda_1 + 1$, we define J_η by

$$\alpha \in J_\eta \longleftrightarrow R(p^*,n,\alpha),$$

where $n \in \text{Field}(\leq_\alpha) \ \& \ |\varphi(\alpha,n)| = \eta$.

Clearly J_η's satisfy (i), (iii), (vi) above; (ii), (iv) and (v) are proved by induction. To see (ii), for instance, observe that

$$J_{\eta+1} = \overline{J_\eta \cap L^{\xi}} \quad \text{when } \eta + 1 \text{ is even.}$$

But $J_\eta \cap L \subseteq \tilde{H}^{\xi}_\eta \cap L$ by induction hypothesis

$$\underset{\text{a.e.}}{\subseteq} H_\eta \cap L \subseteq H_{\eta+1}.$$

Hence by Lemma 2.1.4, $J_\eta \cap L \subseteq \tilde{H}^{\xi}_{\eta+1}$.

Consequently, $J_{\eta+1} = \overline{J_\eta \cap L}^{\xi} \subsetneq \tilde{H}^{\xi}_{\eta+1}$. To see (iv), observe that, when η is even,

$$J_\eta \cap M \subsetneq \overline{J_\eta \cap M}^{\xi} = J_{\eta+1}, \text{ by definition.}$$

The other cases can be proved similarly. Having defined such a sequence we have by (I),

$$\overline{f(B')}^{\eta(k)} \cap J_{\lambda_1+1} = \varphi$$

So by the separation theorem of Louveau [29], there is a $\pi^*_{\eta(k)}$ set E such that

$$f(B') \subsetneq E \quad \text{and} \quad E \cap J_{\lambda_1+1} = \varphi .$$

Now $\quad E \cap M \subsetneq M - J_{\lambda_1+1} \subsetneq \omega^\omega - J_{\lambda_1}$ by (iv).

Thus the decreasing sequence of $\underset{\sim}{\pi}^0_\xi$ sets $\{ J_\mu : \mu \leq \lambda_1 \}$ satisfies the hypotheses (a) – (e) of the theorem with P replaced by $E \cap M$. Hence by induction hypothesis, there is an $n \in W'_\xi$ such that C'_n satisfies (0) – (5) in the statement of the theorem with P replaced by $E \cap M$ and α replaced by $\varphi(\alpha, n^*)$.

Now fix $x \in B'$. Get $m \in \bigcup_{\eta<\xi} W_\eta$ such that $E = C_m$. Since f is recursive, there is an $i \in \bigcup_{\eta<\xi} W_\eta$ such that $C_i = f^{-1}(C_m)$. Clearly $x \in C_i$. Suppose $\beta \in A \cap C_i$. Then $f(\beta) \in C_m$ and moreover C'_n satisfies conditions (0) – (5') in the definition of A^*_1. This establishes $x \in A^*_1$, which is a contradiction. Thus $A \subsetneq A^*_1$. This means that for each $y \in P$ there exist $m \in \bigcup_{\eta<\xi} W_\eta$, $n \in W'_\xi$ such that $y \in C_m$ and C'_n

satisfies conditions (0) - (5') in the definition of A_1^*. Define

$$Q(y,m) \longleftrightarrow m \in \bigcup_{\eta<\xi} W_\eta \ \&\ y \in C_m$$

$$\&\ (\exists n)\left[n \in W'_\xi \ \&\ C'_n \text{ satisfies } (0) - (5')\right].$$

Q is Π_1^1 and for each $y \in P$ there is an m such that $Q(y,m)$ holds. Hence there is a Δ_1^1-function f such that $(\forall y \in P)R(y,f(y))$.

Let $U = f[P]$ and $V = \{m \in \bigcup_{\eta<\xi} W_\eta : (\exists n)[n \in W'_\xi \ \&\ C'_n$

satisfies $(0) - (5')]\}$.

U is Σ_1^1, V is Π_1^1 and $U \subseteq V$. Let $U^* \subseteq \omega$ be Δ_1^1 such that $U \subseteq U^* \subseteq V$. Clearly, $P \subseteq \bigcup_{m \in U^*} C_m$ and, moreover, for each $m \in U^*$, $(\exists n)[n \in W'_\xi \ \&\ C'_n$ satisfies $(0) - (5')$ in the definition of $A_1^*]$. By Δ-selection again, there is a Δ_1^1 function g which for each $m \in U^*$ picks an integer $n \in W'_\xi$ such that C'_n satisfies $(0) - (5')$ in the definition of A_1^*. Now define

$$H^*(i,x) \longleftrightarrow i \notin \text{Field}(\leq_\alpha) \vee ((i <_\alpha n^*) \ \&\ (\forall m \in U^*)C'(g(m),i,x))$$

$$\vee ((i = n^*) \ \&\ (\forall m \in U^*)[\neg C(m,x) \ \&\ C'(g(m),i^*,x)]),$$

where $|\varphi(\alpha,i^*)| = \lambda_1$. A tedious but straightforward computation shows that H^* satisfies the conclusion of the theorem.

<u>Subcase (ii)</u> λ_1 is odd.

In this case we define A_2^* exactly like A_1^* except that we replace in the definition of A_1^* condition (5') by

(5'') $\qquad C'_{n,i^*} \cap C_m \cap L = \varphi$;

and the proof proceeds exactly as above.

Case 2 λ is limit.

In this case we define A_3^* as follows.

$$x \in A_3^* \leftrightarrow (\exists i)\Big\{ i \in \bigcup_{\eta<\xi} W_\eta \;\&\; x \in C_i \;\&$$

$$(\exists p \in \text{Field}(\leq_\alpha))(\exists m \in \bigcup_{\eta<\xi} W_\eta)(\exists n \in W'_\xi)(\forall \beta)\Big[\beta \in A \cap C_i \rightarrow$$

$$f(\beta) \in C_m \;\&\; (C'_n \text{ satisfies } (0) - (4) \text{ in the definition of } A_1^* \text{ with } \varphi(\alpha,p) \text{ replacing } \alpha) \;\&$$

$$(5''')((\bigcap_{i<_\alpha p} C'_{n,i}) \cap C_m = \varphi)\Big]\Big\}.$$

As in subcase (i) we can show that $A \subseteq A_3^*$, and further U^* is defined similarly such that for each $m \in U^*$, there exist $p \in \text{Field}(\leq_\alpha)$ and $n \in W'_\xi$ such that C'_n satisfies (0) – (5''') above with α replaced by $\varphi(\alpha,p)$. Get two Δ_1^1 functions f and g which for each $m \in U^*$ select a p and an n as above. Then define

$$x \in H'_i \leftrightarrow i \notin \text{Field}(\leq_\alpha) \vee (\forall m \in U^*)[i \leq_\alpha f(m) \rightarrow C'(g(m), i, x)].$$

Finally define

$$x \in H_i^* \leftrightarrow (\forall j \leq_\alpha i)H'(j,x) \vee [(|\varphi(\alpha,i)| \text{ is limit}) \;\&\; (\forall j <_\alpha i)H'(j,x)].$$

One can now check that H^* is our required set. This completes the proof.

As an immediate consequence of Theorem 2.2.3 we have the following separation result.

2.2.4 <u>Theorem</u> : Let L and M be two disjoint Σ^1_1 sets of reals. Let λ and ξ be two recursive ordinals. *Suppose there is a decreasing* sequence of $\underset{\sim}{\pi}^0_\xi$ sets $\{H_\mu : \mu < \lambda\}$ such that, at limit ordinals μ, $H_\mu = \bigcap_{\mu' < \mu} H_{\mu'}$ and the set $B = \bigcup_{\mu < \lambda, \mu \text{ even}} (H_\mu - H_{\mu+1})$ separates L from M. Then, whenever α is a recursive code of λ, there is a π^*_ξ set $H^* \subseteq \omega \times \omega^\omega$ such that

(a) $H^*_n \subseteq H^*_m$ if $m \leq_\alpha n$

(b) $H^*_n = \bigcap_{n' <_\alpha n} H^*_{n'}$, if $|\varphi(\alpha,n)|$ is limit

and (c) $B^* = \bigcup \{H^*_n - H^*_m : |\varphi(\alpha,n)|$ is even, $n \in \text{Field}(\leq_\alpha)$ and $|\varphi(\alpha,n)| + 1 = |\varphi(\alpha,m)|\}$

separates L from M (when λ is odd we take $H_\lambda = \varphi$. Similarly for H^*).

<u>Proof</u> : This follows from Theorem 2.2.3 by taking $P = L$ when λ is even and $P = M$ when λ is odd. ▯

We now state the main result of this section.

2.2.5 <u>Theorem</u> : Let $A \subseteq \omega^\omega$ be a Δ^1_1 set and $\{H_\mu : \mu < \lambda\}$ a decreasing sequence of $\underset{\sim}{\pi}^0_\xi$ sets as above such that

$$A = \bigcup_{\mu < \lambda, \mu \text{ even}} (H_\mu - H_{\mu+1}).$$

Then for each recursive α coding λ, there is a π^*_ξ set $H^* \subseteq \omega \times \omega^\omega$ as in Theorem 2.2.4 such that

$$A = \bigcup \{H^*_n - H^*_m : |\varphi(\alpha,n)| \text{ is even}, \ n \leq_\alpha m \ \& \ |\varphi(\alpha,n)| + 1 = |\varphi(\alpha,m)|\},$$

Proof : Follows from Theorem 2.2.4 by taking $L = A$ and $M = A^c$. $\square$

To obtain structural properties of Δ^1_1 sets in the ambiguous Borel class $\xi+1$, $\xi < \omega_1^{ck}$, we shall first obtain a structural result for $\Delta^1_1 \cap \underset{\sim}{\Delta}^0_2$ sets. This result is due to Burgess and since no published account exists we shall give a proof for the sake of completeness. We need the following result from inductive definability due to Cenzer [15]. (For basic facts about inductive operators we refer to [38]).

2.2.6 Lemma : Let Γ be a monotone Π^1_1 operator on ω^ω and E a Σ^1_1 set of reals such that $E \subseteq \Gamma^\infty$. Then there is a recursive ordinal λ such that $E \subseteq \Gamma^\lambda$.

The above lemma enables us to effectivize [20; §34 VI].

2.2.7 Theorem : Let $E \subseteq \omega^\omega$ be a $\Delta^1_1 \cap \underset{\sim}{\Delta}^0_2$ set. Then there is a recursive ordinal λ such that for every recursive code α of λ there is Δ^1_1, closed set $F \subseteq \omega \times \omega^\omega$ with the following properties

(a) $F_n \subseteq F_m$ if $m <_\alpha n$

(b) $F_n = \bigcap_{n' <_\alpha n} F_{n'}$ if $|\varphi(\alpha,n)|$ is limit

and (c) $E = \bigcup\{F_n - F_m : |\varphi(\alpha,n)|$ is even $\&\ n <_\alpha m$ $\&\ |\varphi(\alpha,n)| + 1 = |\varphi(\alpha,m)|\}$.

Proof : Define a transfinite decreasing sequence of closed sets by the induction

$$X_0 = \omega^\omega, \quad X_\mu = \bigcap_{\mu' < \mu} X_{\mu'} \text{ if } \mu \text{ is limit,}$$

$$X_{\mu+1} = \overline{X_\mu \cap E} \cap \overline{(X_\mu - E)}.$$

Since X_μ is a decreasing sequence of closed sets there is an ordinal $\mu^* < \omega_1$ such that $X_{\mu^*+1} = X_{\mu^*}$. Thus we have

$$X_{\mu^*} = \overline{X_{\mu^*} \cap E} \cap \overline{X_{\mu^*} - E}$$

This implies

$$\overline{X_{\mu^*} \cap E} = X_{\mu^*} = \overline{X_{\mu^*} - E}.$$

Now, X_{μ^*} is closed and hence a Baire space. Moreover, both E and E^c are dense G_δ sets in X_{μ^*}. This is not possible unless $X_{\mu^*} = \varphi$. Set $Y_\mu = \omega^\omega - X_\mu$. Then $Y_\infty = \bigcup_\mu Y_\mu = \omega^\omega$. We shall now define a Π^1_1 monotone operator Γ on ω^ω whose μth iterate $\Gamma^\mu = Y_\mu$ if $\mu < \omega$ and $\Gamma^\mu = Y_{\mu+1}$ if $\mu \geq \omega$. The operator Γ is defined as follows, where A varies over subsets of ω^ω.

$$\alpha \in \Gamma(A) \leftrightarrow (\exists s)\{ (\alpha \in N_s) \;\&\; [(\forall\beta)(\beta \in N_s \cap E \rightarrow \beta \in A) \vee (\forall\beta)(\beta \in N_s \cap E^c \rightarrow \beta \in A)]\}.$$

This is a monotone Π^1_1 operator such that $\Gamma^\infty = Y_\infty = \omega^\omega$. Hence by Lemma 2.2.6, there is a recursive ordinal λ such that $\Gamma^\lambda = \omega^\omega$. This implies that, for a recusive ordinal λ', $X_{\lambda'} = \varphi$. Now let $x \in E$ and let $\mu < \lambda'$ be the least ordinal such that $x \notin X_{\mu+1}$. Then $x \in X_\mu - \overline{X_\mu - E}$. Thus

$$E \subseteq \bigcup_{\mu<\lambda'} (X_\mu - \overline{X_\mu - E})$$

Further $\bigcup_{\mu<\lambda'} (X_\mu - \overline{X_\mu - E}) \subseteq E$. Hence

$$E = \bigcup_{\mu<\lambda'} (X_\mu - \overline{X_\mu - E}).$$

An application of Theorem 2.2.5 (with $\xi = 1$) gives us the result. ∎

To get the general result we need a transfer theorem (see [29]).

<u>Transfer Lemma</u> (Kuratowski–Louveau) : For each A in $\underset{\sim}{\Delta}^0_{\xi+1} \cap \Delta^1_1$, $2 \leq \xi < \omega_1^{ck}$, there is a Δ^1_1, closed set $F \subseteq \omega^\omega$, and a Δ^1_1-recursive function f from F onto ω^ω which is injective and continuous, such that for each open subset G of ω^ω, $f(G) \in \underset{\sim}{\Sigma}^0_\xi$, and such that $f^{-1}(A)$ is both $\underset{\sim}{\Delta}^0_2$ and Δ^1_1 in ω^ω.

2.2.8 <u>Theorem</u>(Louveau) : For each $E \subseteq \omega^\omega$ in $\Delta^1_1 \cap \underset{\sim}{\Delta}^0_{\xi+1}$, $1 \leq \xi < \omega_1^{ck}$, there is a recursive ordinal λ and a $\Delta^1_1 \cap \underset{\sim}{\pi}^0_\xi$ set $H \subseteq \omega \times \omega^\omega$ (corresponding to each recursive code α for λ) such that

(a) $H_n \subseteq H_m$ if $m \leq_\alpha n$

(b) $H_n = \bigcap_{n' <_\alpha n} H_{n'}$, if $|\varphi(\alpha,n)|$ is limit, and

(c) $E = \bigcup \{H_n - H_m : |\varphi(\alpha,n)|$ is even & $n <_\alpha m$ & $|\varphi(\alpha,n)| + 1 = |\varphi(\alpha,m)|\}$.

<u>Proof</u> : This follows from Theorem 2.2.7 and the Transfer Lemma.

<u>Remark</u> : Theorem 2.2.8 enables us to define an effective hierarchy in $\Delta^1_1 \cap \underset{\sim}{\Delta}^0_{\xi+1}$ analogous to the difference hierarchy in $\underset{\sim}{\Delta}^0_{\xi+1}$.

2.2.9 <u>Corollary</u> : Suppose $E, E' \subseteq \omega^\omega$ be two Σ^1_1 sets which can be separated by a $\underset{\sim}{\Delta}^0_{\xi+1}$ set, $1 \leq \xi < \omega_1^{ck}$. Then there is a recursive ordinal λ and a π^*_ξ set $H \subseteq \omega \times \omega^\omega$ (corresponding to each recursive α for λ), as in Theorem 2.2.8, such that the set

$B = \bigcup\{H_n - H_m : |\varphi(\alpha,n)|$ is even $\& \ n <_\alpha m \ \& \ |\varphi(\alpha,n)| + 1 = |\varphi(\alpha,m)|\}$

separates E from E'.

Proof : By Lemma 2.2.1 there is a $\Delta^1_1 \cap \Delta^0_{\xi+1}$ set B separating E from E'. The result now follows from Theorem 2.2.8.

2.3 The Kalmar hierarchy : In this sub-section we shall study the Kalmar hierarchy for clopen subsets of ω^ω which are Δ^1_1.

For any set $K \subseteq \omega^\omega$ and any $s \in \omega^{<\omega}$ let

$$K_{(s)} = \{\alpha \in \omega^\omega : s^\frown\alpha \in K\}.$$

The Kalmar hierarchy $\{\mathcal{K}_\xi : \xi < \omega_1\}$ is defined by induction as follows :

$$\mathcal{K}_0 = \{\varphi, \omega^\omega\}.$$

Having defined $\mathcal{K}_\eta$ for every $\eta < \xi$, put

$$K \in \mathcal{K}_\xi \leftrightarrow (\forall n)\left[K_{((n))} \in \bigcup_{\eta<\xi}\mathcal{K}_\eta\right].$$

It is not very difficult to see that

$$\bigcup_{\xi<\omega_1}\mathcal{K}_\xi = \Delta^0_1 \restriction \omega^\omega.$$

Given a clopen set $\varphi \neq K \subseteq \omega^\omega$ we shall associate a well-founded tree T_K as follows.

$$T_K = \{\langle s\rangle \in \text{Seq} \mid K_{(s)} \neq \varphi, \omega^\omega \ \vee \ K_{(s)} = \omega^\omega \ \& \ K_{(s')} \neq \omega^\omega\},$$

where $s' = (n_0,\ldots,n_{\ell-1})$ if $s = (n_0,\ldots,n_{\ell-1},n_\ell)$. If $K = \varphi$, T_K is taken to be the empty tree. Conversely, given a well-founded tree T, there is a clopen set $K \subseteq \omega^{\omega}$ such that $T = T_K$. If T is a well-founded tree, then its length, denoted by $|T|$, is defined as in [38].

2.3.1 Proposition : Let $K \subseteq \omega^{\omega}$ be a clopen set. Then

$$|T_K| \leq \xi < \omega_1 \text{ iff } K \in \mathcal{K}_{\xi}.$$

Proof : This is proved by induction on ξ and is straightforward.

2.3.2 Proposition : If $K \subseteq \omega^{\omega}$ is Δ^1_1 and clopen, then there is a recursive $\xi < \omega_1^{ck}$ such that $K \in \mathcal{K}_{\xi}$.

Proof : Let K be Δ^1_1 and clopen. We shall first show that T_K is a Δ^1_1 well-founded tree. Consider the set

$$T'_K = \{ \langle s \rangle \in \text{Seq} : K_{(s)} = \omega^{\omega} \,\&\, K_{(s')} \neq \omega^{\omega} \}.$$

Then
$$s \in T'_K \leftrightarrow N_s \subseteq K \,\&\, N_{s'} \not\subseteq K$$
$$\leftrightarrow N_s \subseteq K \,\&\, (\exists t)(s' \subseteq t \,\&\, N_t \subseteq K^c)$$

This shows that T'_K is Π^1_1. Similarly, T'_{K^c} is also Π^1_1. Further

$$s \in T'_K \leftrightarrow N_s \subseteq K \,\&\, N_{s'} \not\subseteq K$$
$$\leftrightarrow (\forall t)(s \subseteq t \rightarrow t \notin T'_{K^c}) \,\&\, N_{s'} \not\subseteq K$$

Thus, T'_K is also Σ^1_1. Since

$$s \in T_K \leftrightarrow (\exists t)(s \subseteq t \,\&\, t \in T'_K)$$

it follows that T_K is Δ^1_1. Hence T_K is a Δ^1_1 well-founded tree and so $\xi = |T_K| < \omega_1^{ck}$. By Proposition 2.3.1, $K \in \mathcal{K}_{\xi}$. This completes the proof.

We shall now obtain a seperation result analogous to Theorem 2.2.4 If α codes a well-founded tree T, we shall denote by $\|\alpha\|$ the length of the tree T. As in [44 ; 7.10] we can define a Π^1_1 relation $S_<$ such that if β codes a well-founded tree, then

$$S_<(\alpha,\beta) \leftrightarrow \alpha \text{ codes a well-founded tree } \& \ \|\alpha\| < \|\beta\|.$$

2.3.3 <u>Theorem</u> : Let P,Q be two disjoint Σ^1_1 sets of reals and $\lambda < \omega_1^{ck}$. Suppose there is a set $K \in \mathcal{K}_\lambda$ which separates P from Q. Then there is a Δ^1_1 set $K^* \in \mathcal{K}_\lambda$ which separates P from Q.

<u>Proof</u> : We shall prove this by induction on λ. Now for each n, $P_{((n))}$ and $Q_{((n))}$ are Σ^1_1 sets and $K_{((n))} \in \bigcup_{\xi<\lambda} \mathcal{K}_\xi$ separates $P_{((n))}$ from $Q_{((n))}$. Hence by induction hypothesis, there is a Δ^1_1 real α, $\|\alpha\| < \lambda$, which codes a wellfounded tree with the associated clopen set separating $P_{((n))}$ and $Q_{((n))}$. Fix a Δ^1_1 γ such that γ codes a Δ^1_1 wellfounded tree and $\|\gamma\| = \lambda$ and define

$$R(n,\alpha) \leftrightarrow S_<(\alpha,\gamma) \ \& \ (\forall\beta)\ [\beta \in P_{((n))} \rightarrow (\exists s)(\forall k)\ (\alpha(s) = 0$$
$$\& \ \alpha(s^\frown\langle k\rangle) \neq 0 \ \& \ \beta \in N_s)]$$
$$\& \ (\forall\beta)\ [(\forall s)(\exists k)\ (\alpha(s) = 0 \ \& \ \alpha(s^\frown\langle k\rangle) \neq 0 \ \& \ \beta \in N_s)$$
$$\rightarrow \beta \notin Q_{((n))}].$$

R is Π^1_1 and $(\forall n)(\exists\alpha \in \Delta^1_1) R(n,\alpha)$. Hence by the Uniformization Lemma there is a Δ^1_1-function f such that $\forall n R(n,f(n))$. Now define a tree T as follows :

$$T = \{\langle n\rangle^\frown s \mid f(n)(s) = 0\}.$$

T is wellfounded and $|T| \leq \lambda$. If K^* be the clopen set such that $T = T_{K^*}$, then it is not hard to see that K^* is Δ^1_1 and separates P from Q.

An immediate consequence of Lemma 2.2.1 and Proposition 2.3.2 is the following

2.3.4 Theorem : Let P and Q be two Σ^1_1 sets of reals and suppose there is a $\underset{\sim}{\Delta}^0_1$ set which separates P and Q. Then there is a $\lambda < \omega_1^{ck}$ and a Δ^1_1 set $K \in \mathcal{H}_\lambda$ such that K separates P from Q. ∎

2.4 Applications to Borel sets in product spaces : In this subsection we shall obtain consequences of the results obtained in the previous subsections to the boldface theory. As usual for convenience we shall work in ω^ω. Our first result is the generalization of Debs' result mentioned in the introduction.

2.4.1 Theorem : Let B be a Borel subset of $\omega^\omega \times \omega^\omega$ and λ, ξ two countable ordinals. Suppose for each $x \in \omega^\omega$ there is a decreasing transfinite sequence of $\underset{\sim}{\pi}^0_\xi$ sets $\{H_\mu : \mu < \lambda\}$, with $H_\mu = \bigcap_{\mu' < \mu} H_{\mu'}$ at limits, such that the vertical section $B_x = \bigcup_{\mu < \lambda, \mu \text{ even}} (H_\mu - H_{\mu+1})$. Then there is a decreasing transfinite sequence of Borel sets $\{B^\mu : \mu < \lambda\}$ with sections in $\underset{\sim}{\pi}^0_\xi$ such that

$$B^\mu = \bigcap_{\mu' < \mu} B^{\mu'} \text{ if } \mu \text{ is limit}$$

and
$$B = \bigcup_{\mu < \lambda, \mu \text{ even}} (B^\mu - B^{\mu+1}).$$

<u>Proof</u> : Fix z such that B is $\Delta^1_1(z)$ and $\xi, \lambda < \omega_1^z$. Then each vertical section B_x is $\Delta^1_1(<x,z>)$ and is also the union of alternating differences of a decreasing λ-sequence of π^0_ξ sets. Hence by Theorem 2.2.5 (relativized), there is a $\pi^*_\xi(<x,z>)$ set $H^* \subseteq \omega \times \omega^\omega$ such that

(a) $H^*_n \subseteq H^*_m$ if $m \leq_\alpha n$

(b) $H^*_n = \bigcap_{n' <_\alpha n} H^*_{n'}$ if $|\varphi(\alpha,n)|$ is limit

and (c) $B_x = \bigcup \{ H^*_n - H^*_m : |\varphi(\alpha,n)|$ is even & $n <_\alpha m$ & $|\varphi(\alpha,n)| + 1 = |\varphi(\alpha,m)| \}$,

where α is a recursive in z code for λ. Just as in the proof of Theorem 2.2.2, get a $\Delta^1_1(z)$-recursive function f such that for each x, $f(x) \in W_\xi^{<x,z>}$ and $C_{<x,z>, f(x)} \overset{\text{def}}{=} H^*$ satisfies (a) - (c) above. (Here for each y, $<W_y, C_y>$ is a coding pair for $\Delta^1_1(y) \upharpoonright \omega \times \omega^\omega$). For each $\mu < \lambda$ fix $n^* \in \text{Field}(\leq_\alpha)$ such that $|\varphi(\alpha,n^*)| = \mu$ and define

$$B^\mu(x,y) \leftrightarrow C(<x,z>, f(x), n^*, y).$$

It can be checked that $\{ B^\mu ; \mu < \lambda \}$ satisfies the conclusion of the theorem.

2.4.2 <u>Theorem</u> : Let $P, Q \subseteq \omega^\omega \times \omega^\omega$ be two Σ^1_1 sets such that for each x, P_x is separated from Q_x by a $\Delta^0_{\xi+1}$ set, $1 \leq \xi < \omega_1$. Then there is a countable ordinal λ and a decreasing transfinite sequence of Borel sets $\{ B^\mu ; \mu < \lambda \}$ as above such that the set

$$B = \bigcup_{\mu < \lambda, \mu \text{ even}} (B_\mu - B_{\mu+1})$$

separates P from Q.

Proof : By Theorem 2.2.2 there is a Borel set $B \subseteq \omega^\omega \times \omega^\omega$ with sections in $\underset{\sim}{\Delta}^0_{\xi+1}$ which separates P from Q. Fix z such that B is $\Delta^1_1(z)$ and $\xi < \omega^z_1$. Thus for each x, B_x is $\Delta^1_1(\langle x,z\rangle)$ and $\underset{\sim}{\Delta}^0_{\xi+1}$. Hence by Theorem 2.2.8, relativized, there is an $\alpha \in WO$ recursive in $\langle x,z\rangle$ and a $\pi^*_\xi(\langle x,z\rangle)$ set $H^* \subseteq \omega \times \omega^\omega$ as in the proof of Theorem 2.4.1 above. By the Uniformization Lemma and the boundedness theorem there is a $\lambda < \omega_1$ such that each $\alpha \in WO$ above has length less than λ. An application of Theorem 2.4.1 gives a sequence $\{B^\mu : \mu < \lambda\}$ of Borel sets with required properties such that $B = \bigcup_{\mu<\lambda,\mu \text{ even}} (B^\mu - B^{\mu+1})$.

Remark : The above result for $\xi = 1$ is due to Burgess [9].

The following is now immediate.

2.4.3 Corollary : Let B be a Borel subset of $\omega^\omega \times \omega^\omega$ with each vertical section in $\underset{\sim}{\Delta}^0_{\xi+1}$, $1 \leq \xi < \omega_1$. Then there is a countable ordinal λ and a decreasing sequence of Borel sets $\{B^\mu : \mu < \lambda\}$ as above such that $B = \bigcup_{\mu<\lambda,\mu \text{ even}} (B^\mu - B^{\mu+1})$.

Finally we establish the following unpublished result of Addison and Harrington.

2.4.4 Corollary : There is no Borel set $B \subseteq \omega^\omega \times \omega^\omega$ which is universal for $\underset{\sim}{\Delta}^0_{\xi+1} \restriction \omega^\omega$, $\xi < \omega_1$.

Proof : Suppose there is such a Borel set B. Fix z such that B is $\Delta^1_1(z)$ and assume $\xi \geq 1$. Then for each x, B_x is $\Delta^1_1(\langle x,z\rangle)$ and $\underset{\sim}{\Delta}^0_{\xi+1}$. Hence by Theorem 2.2.8, relativized, there is an $\alpha \in WO$ recursive in $\langle x,z\rangle$ and a $\pi^*_\xi(\langle x,z\rangle)$ set $H \subseteq \omega \times \omega^\omega$ such that

(i) $H_n \subseteq H_m$ if $m \leq_\alpha n$

(ii) $H_n = \bigcap_{n' <_\alpha n} H_{n'}$ if $|\varphi(\alpha,n)|$ is limit

and (iii) $B_x = \bigcup\{H_n - H_m : |\varphi(\alpha,n)|$ is even & $n <_\alpha m$

& $|\varphi(\alpha,n)| + 1 = |\varphi(\alpha,m)|\}$.

By the usual selection and boundedness argument there is $\lambda < \omega_1$ such that each $\alpha \in WO$ as above has length less than λ. This would imply that each set in $\Delta^0_{\xi+1}$ is available at the λth level of the difference hierarchy of $\Delta^0_{\xi+1}$. This is clearly a contradiction (cf [28; §37 IV]). The proof for $\xi = 0$ is similar. ∎

APPENDIX

Proof of Lemma 7.5 of Part I:

With each operation $R_\rho, \rho < \omega_1$, every family $\{E_n\}$ of subsets of X and every $x \in X$ we will associate a (closed) game on the integers, $G_\rho(x)$, played between $\forall$ and $\exists$ such that

$$x \in R_\rho(\{E_n\}) \leftrightarrow \exists \text{ wins } G_\rho(x).$$

(Observe that $G_\rho(x)$ depends on $\{E_n\}$ though our notation does not indicate this).

The game $G_\rho(x)$ will be defined by induction on ρ; it will be of length $\lambda + 1$ for some limit ordinal $\lambda < \omega_1$ with $\forall$ making a move at position λ. (This move will be called the stopping move of the game $G_\rho(x)$).

Corresponding to each limit ordinal $\lambda < \omega_1$ fix, for the rest of this section, a sequence $\varphi_\lambda(n) \uparrow \lambda$.

For $\rho = 0$, the game $G_\rho(x)$ is played with $\exists$ playing successively $a_0, a_1, \ldots$ while at position ω $\forall$ plays a k (a stopping move) thereby producing, what we shall call a relevant sequence, $s = \langle a_0, a_1, \ldots, a_{k-1}\rangle$. We say that $\exists$ wins iff $x \in E_s$.

Now suppose $\rho = \mu + 1$. Then $R_\rho = \mathbb{R}\, R_\mu R_\mu^o$. Set $R_\mu = \Phi$ and define

$$E = \Phi^*_n(\{E_n\})$$
$$E^u = \Phi^*_n(\{E_{u*n}\})$$
$$E'^{<u_o,\dots,u_{k-1},s>} = \Phi^o_m(\{\Phi^*_n(\{E_{<u_o,\dots,u_{k-1}>*<<s,m>>*n}\})\}). \tag{i}$$

The game $G_\rho(x)$ is played as follows. First $\forall$ and $\exists$ play $G_\mu(x)$ corresponding to the family $\{E'^{<k>}\}$ with $\forall$ making a (stopping) move at the last position thereby producing a relevant sequence s_o. Then $\forall$ and $\exists$ play the dual of the game $G_\mu(x)$ (denoted by $G^o_\mu(x)$ which is played as in $G_\mu(x)$ by interchanging the moves of $\forall$ and $\exists$) corresponding to the family $\{\sim E^{<<s_o,m>>}\}_m$ with $\exists$ making a (stopping) move at the last position to produce a relevant sequence t_o. Let $u_o = <s_o,t_o>$. Next $\forall$ and $\exists$ play $G_\mu(x)$ corresponding to the family $\{E'^{<u_o,n>}\}_n$ with $\forall$ making a (stopping) move at the last position to produce a relevant sequence s_1. Then $\forall$ and $\exists$ play the dual game $G^o_\mu(x)$ associated with the family $\{\sim E^{<u_o,<s_1,m>>}\}_m$ where $\exists$ makes the last (stopping) move to produce a relevant sequence t_1. Set $u_1 = <s_1,t_1>$; $\exists$ and $\forall$ continue to play alternately $G_\mu(x)$ and $G^o_\mu(x)$ as described above. At the end $\forall$ plays some k so that $<u_o,\dots,u_{k-1}>$ is the relevant sequence produced. Then $\exists$ wins iff $x \in E_{<u_o,\dots,u_{k-1}>}$.

Now suppose ρ is limit. Then $R_\rho = \mathbb{R}\Phi\Phi^0$, where Φ is defined by

$$\Phi(\{F_n\}) = \bigcap_i R_{\varphi_\rho(i)}(\{F_{<i,n>}\}), \tag{ii}$$

for any family $\{F_n\}$. The game $G_\rho(x)$ is defined as follows. First $\forall$ makes a move i_o (called a <u>choice</u> move) which specifies the game to be played. Then $\forall$ and $\exists$ play the game $G_{\varphi_\rho(i_o)}(x)$ associated with the family $\{E^{,<<i_o,n>>}\}_n$ (see (i)) at the end of which $\forall$ makes a (stopping) move producing a relevant sequence s_o. Next $\exists$ plays j_o (a choice move) and then $\forall$, $\exists$ start playing the dual game $G^o_{\varphi_\rho(j_o)}(x)$ associated with $\{\sim E^{<<<i_o,s_o>,<j_o,m>>>}\}_m$ with $\exists$ moving at the last position producing a relevant sequence t_o. Let $u_o = <<i_o,s_o>, <j_o,t_o>>$. Then $\forall$ makes a choice move i_1 signaling the start of the game $G_{\varphi_\rho(i_1)}(x)$ associated with $\{E^{,<u_o,<i_1,n>>}\}_n$ and the above procedure is repeated. At the end a stopping move k is made by $\forall$ so that $<u_o,\ldots,u_{k-1}>$ is the relevant sequence produced. Then $\exists$ wins iff $x \in E_{<u_o,\ldots,u_{k-1}>}$.

It is quite straightforward, though tedious, to show that for every $\rho < \omega_1$, if $x \in R_\rho(\{E_n\})$, then $\exists$ wins $G_\rho(x)$.

For the converse, recall the definition of E^μ_s from I:2.6. Define for every $s \in Seq$, the game $G_{\rho,s}(x)$ exactly as above with the following winning condition for $\exists$: if $u_o,u_1,\ldots$

are the relevant sequences produced and if $\forall$ plays k at the last position (stopping move), then $\exists$ wins iff $x \in E_{s*\langle u_0, u_1, \ldots, u_{k-1}\rangle}$. Then one can show by induction on μ that if $x \notin E^{\mu}_{s}$ then $\exists$ does not win $G_{\rho,s}(x)$. It then follows from Theorem I.2.7 that if $x \notin E$ then $\exists$ does not win $G_{\rho,e}(x) = G_{\rho}(x)$. Thus we have

$$x \in R_{\rho}(\{E_n\}) \leftrightarrow \exists \text{ wins } G_{\rho}(x).$$

Note that if $\rho < \omega$, then $G_{\rho}(x)$ is of length $\omega^{\rho+1}+1$.

We now define by induction on ρ a game $G'_{\rho}(x)$ of length $\omega+1$ (and can thus be regarded as an ordinary Gale-Stewart game with the last move contributing to the pay-off set) associated with $\{E_n\}$, $x \in X$ such that

$$\exists \text{ wins } G'_{\rho}(x) \leftrightarrow \exists \text{ wins } G_{\rho}(x).$$

In this game also the $\forall$ player makes a choice move at the last position ω to produce a relevant sequence.

At the base step, $G'_{\rho}(x)$ is the same as $G_{\rho}(x)$. We shall now describe $G'_{\rho}(x)$ for successor ordinals ρ only, the limit case being much similar. So suppose $\rho = \mu + 1$. The game $G'_{\rho}(x)$ will be such that if both players play according to some specified rules, then at the end $\forall$ plays a k producing a relevant sequence $\langle u_0, \ldots, u_{k-1}\rangle$ and $\exists$ wins iff $x \in E_{\langle u_0, \ldots, u_{k-1}\rangle}$. To see how $G'_{\rho}(x)$ is played recall how the game $G_\rho(x)$ is described. We shall first define an intermediate game $G''_\rho(x)$ of length ω^2+1 and then obtain $G'_\rho(x)$. In the game $G''_\rho(x)$, first $\forall$ and $\exists$ play the game $G'_\mu(x)$ (associated with $\{E'^{<n>}\}$) with $\forall$ making a move at position ω (thereby producing some relevant sequence s_o). Next $\forall$ and $\exists$ play $(G'_\mu(x))^o$, the dual of the game $G'_\mu(x)$ (associated with $\{\sim E^{<<s_o,m>>}\}_m$), with $\exists$ making a stopping move at the last position so as to produce some relevant sequence t_o. Let $u_o = <s_o,t_o>$. The game then proceeds with the two players alternately playing $G'_\mu(x)$ and $(G'_\mu(x))^o$. At position ω^2+1, $\forall$ plays k and produces the relevant sequence $<u_o,\ldots,u_{k-1}>$. If none of the players violate the prescribed rules in each of the above subgames, then $\exists$ wins iff $x \in E_{<u_o,\ldots,u_{k-1}>}$.

The game $G'_\rho(x)$ is now easy to describe. Though its total length is $\omega+1$ we think of it as consisting of <u>potentially</u> infinite sequence of subgames each consisting of <u>potentially</u> infinite sequence of rounds. Each of these subgames "corresponds" to the corresponding subgame in $G''_\rho(x)$. If in any play of $G'_\rho(x)$ the j-th subgame actually goes through infinitely many rounds, then the (j+1)th subgame never gets started. This keeps the total length within bounds.

In the j-th subgame of $G'_\rho(x)$, $\forall$ and $\exists$ make moves maintaining the rules of the j-th subgame of $G''_\rho(x)$. A move of $\forall$ followed by a move of $\exists$ (or vice versa, depending on the order of the moves in the j-th subgame of $G''_\rho(x)$) will be called a round.

The i-th round of the 2j-th subgame opens with $\forall$ signalling (by a choice of 0 or 1) either a <u>challenge</u> or a <u>pass</u>. If he challenges, the whole 2j-th subgame ends at once and the players proceed to the (2j+1)th subgame; in this case we record $u_j \in Seq$ which codes the moves of the players in this subgame. If $\forall$ passes, then the two players play the i-th round (according to the rules of 2j-th subgame in $G''_\rho(x)$); then the players proceed to the (i+1)th round.

The (2j+1)th subgame is played exactly like the 2j-th subgame with $\exists$ (instead of $\forall$) signalling a challenge or a pass at the start of each round.

If some subgame goes on forever because $\exists$ or $\forall$ (as the case may be) fails to challenge on any round, then he forfeits the game. If this provision does not apply, a sequence $u_o, v_o;\ u_1, v_1; \ldots$ (of relevant sequences) is generated. Then at position ω, $\forall$ plays a j and $\exists$ wins iff $x \in E_{<<u_o,v_o>,\ldots,<u_{j-1},v_{j-1}>>}$.

One can check that $\exists$ wins $G'_\rho(x)$ iff $\exists$ wins $G_\rho(x)$. In fact any winning strategy σ for $\exists$ in $G'_\rho(x)$ gives rise in a canonical way to a winning strategy σ^*

for $\exists$ in the game $G_\rho(x)$; and vice versa. Moreover, $G'_\rho(x)$ is a $\underset{\sim}{\Delta}^0_3$ game. (For details at the finite levels, see [13, § 11]).

<u>The game $J_\rho(x)$</u>: We shall now define a game $J_\rho(x)$ associated with normal families $\{E_n\}$, analogous to $G_\rho(x)$, such that

$$\exists \text{ wins } J_\rho(x) \text{ iff } E^x \text{ is comeager.}$$

Now observe that in each ω-block in the game $G_\rho(x)$ (apart from the choice moves and the stopping moves) either $\exists$ plays successively or $\forall$ plays successively. Each move by $\exists$, which is not a choice move or a stopping move, in $G_\rho(x)$ will "correspond" (in the game $J_\rho(x)$) to a sequence move (i.e., a choice from $\omega^{<\omega}$) by $\forall$ followed by a sequence move and a integer move by $\exists$. Similarly, each move by $\forall$ in $G_\rho(x)$ which is not a choice move or a stopping move, will correspond to a sequence move and an integer move by $\forall$ followed by a sequence move by $\exists$ in $J_\rho(x)$. A choice move by $\forall$ ($\exists$) in $G_\rho(x)$ corresponds to a choice move by $\forall$($\exists$) in $J_\rho(x)$ and a stopping move by $\forall$ ($\exists$) in $G_\rho(x)$ corresponds to a stopping move by $\forall$($\exists$) in $J_\rho(x)$. Thus each λ-block, λ limit, in $J_\rho(x)$ will give rise to <u>two</u> relevant sequences, one from the integer moves and the other from the sequence moves. At the end of the game $J_\rho(x)$ $\forall$ plays some k thus producing two relevant sequences of the form $u = \langle u_0,\dots,u_{k-1}\rangle$ and

$v = v_0 * v_1 * \ldots * v_{k-1} \ldots$. We say that $\exists$ wins iff E_u^x is comeager in $\Sigma(v)$.

More formally we define $J_\rho(x)$ as follows. First recall the definition of the Vaught operation $\mathcal{V}$ and the operations S_ρ from §6.5. For convenience we replace Seq by $\omega^{<\omega}$. Let $\{E_n\}$ be a normal family of subsets of $\omega^\omega \times \omega^\omega$, s a finite sequence and $x \in \omega^\omega$. When $\rho = 0$, $J_{\rho,s}(x)$ is the Vaught game and is played as follows: First $\forall$ chooses a finite sequence s_0, then $\exists$ replies with a finite sequence t_0 and a natural number k_0; then $\forall$ chooses a finite sequence s_1 while $\exists$ replies with t_1, k_1 and so on. In the end $\forall$ plays an integer i (a stopping move) to produce two relevant sequences $u = \langle k_0, \ldots, k_{i-1} \rangle$ and $v = s_0 * t_0 * \ldots * s_{i-1} * t_{i-1}$. $\exists$ wins iff E_u^x is comeager in $\Sigma(s*v)$.

Now suppose $\rho = \mu + 1$. Then $S_\rho = \mathbb{R} S_\mu S_\mu^o$. Now $\forall$ and $\exists$ first play the game $J_{\mu,s}(x)$ associated with the normal family $\{E'^{\langle n \rangle}\}$ (defined as in (i) above) with $\forall$ making a (stopping) move at the end so as to produce two relevant sequences $s_0 \in$ Seq and $t_0 \in \omega^{<\omega}$. Then $\forall$ plays a finite sequence t and the two players play the dual game $J_{\mu, s*t_0*t}(x)$ (associated with $\{\sim E^{\langle\langle s_0, m \rangle\rangle}\}_m$) with $\exists$ playing at the end to produce two relevant sequences $s_0' \in$ Seq, $t_0' \in \omega^{<\omega}$ with $t_0' = t*t_0''$ for some t_0''. Let

$u_o = \langle s_o, s_o' \rangle$, $v_o = t_o * t_o'$. Next $\exists$ plays a $t' \in \omega^{<\omega}$ and two players play $J_{\mu, s*v_o*t'}(x)$ (associated with $\{E^{, \langle u_o, n \rangle}\}_n$) with $\forall$ making the last move to produce two relevant sequences s_1, t_1 and then $\forall$ plays $t'' \in \omega^{<\omega}$ and they play the dual game $J^o_{\mu, s*v_o*t_1*t''}(x)$. Thus the two players alternately play $J_\mu(x)$ and $J^o_\mu(x)$ as described above producing two sequences $\{u_i\}$ and $\{v_i\}$ of relevant sequences. At the end of the play $\forall$ plays a k to produce the relevant sequences $u = \langle u_o, \ldots, u_{k-1} \rangle \in$ Seq and $v = v_o * v_1 * \ldots * v_{k-1}$. Then $\exists$ wins iff E^x_u is comeager in $\Sigma(s*v*v')$, where v' is the next sequence move (played by $\forall$).

As in the case of $G_\rho(x)$ we can now define exactly as above, a game $J'_{\rho,s}(x)$ of length $\omega+1$ with $\forall$ playing in the end to produce two relevant sequences such that

$$\exists \text{ wins } J'_{\rho,s}(x) \leftrightarrow \exists \text{ wins } J_{\rho,s}(x).$$

We now show that E^x is comeager in $\Sigma(s)$ iff $\exists$ wins $J_{\rho,s}(x)$. This is really Theorem I:6.6 in disguise.

Let $\mathfrak{J}_{\rho,s}(x)$ be a variant of the game $J_{\rho,s}(x)$ which is played exactly as in $J_{\rho,s}(x)$ such that if $u_o, u_1, \ldots; v_o, v_1, \ldots$ are the two sequences of relevant sequences produced before $\forall$ plays an integer k in the last position, then $\exists$ wins $\mathfrak{J}_{\rho,s}(x)$ iff $s*v_o*v_1*\ldots \in E^x_{\langle u_o, \ldots, u_{k-1} \rangle}$.

Lemma 1: With notation as above, we have E^x is comeager in $\Sigma(s)$ iff $\exists$ wins $J_{\rho,s}(x)$ iff $\exists$ wins $\tilde{J}_{\rho,s}(x)$.

Proof. We shall prove this by induction on ρ. We prove it only for the successor case, the limit case being similar. Without loss of generality assume that $\{E_n\}$ is regular. Suppose $\rho = \mu + 1$. Then $R_\rho = \mathbb{R} R_\mu R_\mu^o$. Put $\Phi = R_\mu$.

1^o. Since $E = \Phi_n(\{E^{'<n>}\})$, E^x is comeager in $\Sigma(s)$ implies $\exists$ wins $J_{\mu,s}(x)$ associated with $\{E^{'<n>}\}$, by induction hypothesis. So to win the game $J_{\rho,s}(x)$ $\exists$ first plays with a winning strategy σ_o, say, in $J_{\mu,s}(x)$. If in a complete run of this subgame s_o, t_o are the relevant sequences produced, then $(E^{'<s_o>})^x$ is comeager in $\Sigma(s*t_o)$. Now $E^{'<s_o>} = \Phi^o_m(\{E^{<<s_o,m>>}\})$. Thus $(E^{'<s_o>})^x$ is comeager in $\Sigma(S*t_o)$

$\to$ $(\Phi_m(\{\sim E^{<<s_o,m>>}\}))^x$ is meager in $\Sigma(s*t_o)$

$\to$ $(\forall u) \sim [(\Phi_m(\{\sim E^{<<s_o,m>>}\})^x$ is comeager in $\Sigma(s*t_o*u)]$

$\to$ $(\forall u)$ [$\exists$ does not win the game $J_{\mu,s*t_o*u}(x)$ associated with $\{\sim E^{<<s_o,m>>}\}$]

$\to$ $(\forall u)$[$\exists$ wins the dual game $J^o_{\mu,s*t_o*u}(x)$ associated with $\{\sim E^{<<s_o,m>>}\}$].

Thus $\exists$ wins the second subgame of $J_{\rho,s}(x)$ with strategy τ_0, say, when the first subgame has been played with s_0, t_0 as the generated relevant sequences. Thus $\exists$ plays with τ_0 and if s_0', t_0' are the relevant sequences produced at the end of a run, then we have (since $\exists$ wins the dual game)

$$(E^{<<s_0,s_0'>>})^x \text{ is nonmeager in } \Sigma(s*t_0*t_0')$$

and so for some $v \in \omega^{<\omega}$,

$$(E^{<<s_0,s_0'>>})^x \text{ is comeager in } \Sigma(s*t_0*t_0'*v).$$

By regularity $E^x_{<<s_0,s_0'>>}$ is comeager in $\Sigma(s*t_0*t_0'*v)$. In the next subgame $\exists$ plays this v and then continue as described above. This clearly defines a winning strategy for $\exists$ in the game $J_{\rho,s}(x)$.

2^o. It is clear that if $\exists$ wins $\mathcal{J}_{\rho,s}(x)$ then $\exists$ wins $J_{\rho,s}(x)$. For, without loss of generality, $\{E^x_n\}$ may be taken to be a family of clopen sets and so E^x_u is comeager in $\Sigma(v)$ iff $\Sigma(u) \subseteq E^x_u$.

3^o. Now suppose $\exists$ wins $J_{\rho,s}(x)$ with strategy σ. Now suppose the two players play the first two subgames with $\exists$ playing according to σ. Suppose u_0, v_0 are the relevant sequences produced. In the next subgame $\exists$ plays some v as dictated by σ. If for some $t \in \omega^{<\omega}$, $E^x_{u_0} \cap \Sigma(s*v_0*v*t) = \varphi$, then $\forall$ can beat σ by playing this t as his next move. (Apart from the choice move, if there is one, it is $\forall$'s turn to play a finite sequenc). Thus for all $t \in \omega^{<\omega}$, $E^x_{u_0} \cap \Sigma(s*v_0*v*t) \neq \varphi$; and this is true at every stage. This shows that if $\{u_i\}$ and $\{v_i\}$ are the two sequences produced before $\forall$ plays at the last position, then $s*v_0*v_1* \ldots \in E^x_{<u_0,\ldots,u_{k-1}>}$ for every k and so $\exists$ wins $\overleftarrow{J}_{\rho,s}(x)$ with the same strategy σ.

4^o. It now remains to show that if $\exists$ wins $J_{\rho,s}(x)$ then E^x is comeager in $\Sigma(s)$. To this end we show that if E^x is not comeager in $\Sigma(s)$, then $\exists$ wins the dual game $J^o_{\rho,s}(x)$. This is shown as in 1^o using the decomposition in Lemma I:6.2. We omit the proof. ||

With notation as above we now have the following

<u>Lemma 2</u>: E^x is comeager iff $\exists$ wins $J_{\rho,e}(x)$. Moreover, with any winning strategy σ for $\exists$ in $J_{\rho,e}(x)$ one can associate a winning strategy σ^* such that if $\{u_i\}$ and $\{v_i\}$ are two relevant sequences produced (before $\forall$ plays a k in the last position) in a complete run of $J_{\rho,e}(x)$ when $\exists$ plays with σ^*, the sequence $\delta = v_0*v_1*v_2* \ldots$ is in E^x.

<u>Proof</u>. The first part of the lemma is simply Lemma 1. Now E^x is comeager iff player II wins the Banach-Mazur game $\sim E^x$ i.e., iff

$$(\forall s_0)\ (\exists\, t_0)\ (\forall s_1)\ (\exists\, t_1)\ \ldots\ \{\delta \in E^x\}$$

$$\leftrightarrow (\forall s_0)\ (\exists\, t_0)\ (\forall s_1)\ (\exists\, t_1)\ \ldots\ \{\exists \text{ wins } G_\rho(x,\delta)\}$$

$$\leftrightarrow (\forall s_0)\ (\exists\, t_0)\ (\forall s_1)\ (\exists\, t_1)\ \ldots\ \{\exists \text{ wins } G'_\rho(x,\delta)\}$$

where $\delta = s_0 * t_0 * s_1 * t_1 * \ldots$

Now, $\exists$ wins $J_{\rho,e}(x) = J_\rho(x)$ iff $\exists$ wins $\mathfrak{J}_\rho(x)$ (with same strategy). Further, the game $\tilde{\mathfrak{J}}_\rho(x)$ is obtained from $G_\rho(x,\delta)$ by interlacing the integer moves of $G_\rho(x,\delta)$ with sequence moves such that the sequence moves of a complete run in $\tilde{\mathfrak{J}}_\rho(x)$ generate δ. Thus if $\tilde{\mathfrak{J}}'_\rho(x)$ is obtained from $\tilde{\mathfrak{J}}_\rho(x)$ in the same way as $J'_\rho(x)$ is obtained from $J_\rho(x)$ or $G'_\rho(x)$ is obtained from $G_\rho(x)$; then observe that $\tilde{\mathfrak{J}}'_\rho(x)$ is simply $G'_\rho(x,\delta)$ interlaced with a Banach-Mazur game. Thus we have

$$(\forall s_0)(\exists\, t_0)(\forall s_1)(\exists\, t_1)\ \ldots\ \ \{\exists \text{ wins } G'_\rho(x,\delta)\} \leftrightarrow \exists \text{ wins } \mathfrak{J}'_\rho(x),$$

which is of the form of the equivalence in Lemma I:7.2. Now suppose $\exists$ wins $J_\rho(x)$. Then he wins $\tilde{\mathfrak{J}}'_\rho(x)$ with some strategy σ. Now modify σ to σ^* so that any complete play in $\tilde{\mathfrak{J}}'_\rho(x)$ corresponds to a complete play in the (left-side) Banach-Mazur game as in Lemma I:7.2. The strategy σ^* gives rise to a strategy σ^{**} for $\exists$ in $J_\rho(x)$ such that if $\{u_i\}$ and $\{v_i\}$ are two sequences of relevant sequences formed when $\exists$ plays σ^{**}, then the same two relevant sequences can be obtained, by simulation, in a play of $\mathfrak{J}'_\rho(x)$ when $\exists$ plays σ^*. This can be easily seen from the way $\mathfrak{J}'_\rho(x)$ is obtained from $\mathfrak{J}_\rho(x)$. Consequently, $\delta = v_o*v_1*v_2*v_3*\ \ldots$ is generated by a complete play of the (left-side) Banach-Mazur game when player II plays with a winning strategy. Thus $\delta \in E^x$. ||

As we have mentioned earlier the above lemma is really Theorem I:6.6, stated in a different form. Thus it is not suprizing that for every ρ and every normal family $\{E_n\}$, one can show that

$$\exists \text{ wins } J_{\rho,s}(x) \leftrightarrow (\exists\ \eta \in M_\rho)(\forall n \in \eta)[E^x_{f_\rho}(x) \text{ is comeager in } \Sigma(s*g_\rho(n))],$$

where $\Phi_{M_\rho} = S_\rho$ and f_ρ and g_ρ are suitable functions independent of $\{E_n\}$. We shall now shat that if $\exists$ wins $J_{\rho,s}(x)$ $(J^o_{\rho,s}(x))$ with some strategy σ_x, then there is some $\eta \in M_\rho(M^o_\rho)$ such that for each $n \in \eta$, there is a complete play in $J_{\rho,s}(x)$ $(J^o_{\rho,s}(x))$ with $\exists$ playing with σ_x such that $f_\rho(n)$ and $g_\rho(n)$ are relevant sequences produced. We will show this by induction on ρ. Suppose $\rho = 0$ and $\exists$ wins $J_{\rho,s}(x)$ with strategy σ_x. Then $\exists$ wins the following game with strategy σ_x :

$$(\forall s_o)(\exists\ t_o)(\exists\ k_o)(\forall s_1)(\exists\ t_1)(\exists\ k_1)\ \ldots$$
$$\ldots\ (\forall i)[E^x_{k_o,\ldots,k_{i-1}} \text{ in comeager in } \Sigma(s*s_o*t_o*\ldots*s_{i-1}*t_{i-1})]$$

Let $\eta = \{<<s_o,<k_o,t_o>>,\ldots,<s_{i-1},<k_{i-1},t_{i-1}>>>:$

$s_o,t_o,k_o;s_1,t_1,k_1;\ldots;\ s_{i-1},t_{i-1},k_{i-1}$ is an initial segment of a play in the above game when $\exists$ follows $\sigma_x\}$.

It is easy to see that η is in the canonical base for S_o. Define

$$f_o(<<s_o,<k_o,t_o>>,\ldots,<s_{i-1},<k_{i-1},t_{i-1}>>>) = <k_o,\ldots,k_{i-1}>$$

$$f_o(n) = 1 \text{ otherwise;}$$

$$g_o(<<s_o,<k_o,t_o>>,\ldots,<s_{i-1},<k_{i-1},t_{i-1}>>>) = s_o*t_o*\ldots*s_{i-1}*t_{i-1}.$$

$$g_o(n) = 1 \text{ otherwise.}$$

The dual case is shown similarly.

We now prove our assertion for successor ordinals ρ, the limit case being similar. So let $\rho = \mu+1$ and suppose $\exists$ wins $J_{\rho,s}(x)$ with strategy σ_x. We show that $\exists$ wins the game (ii) below with a strategy σ'_x with the following property: If $\eta_o,n_o,u_o,\xi_o,m_o,v_o,\ldots$ is a complete play consistent with σ'_x then for each k there is a complete play in $J_{\rho,s}(x)$ consistent with σ_x such that

$<<f_\mu(n_o),\ f_\mu(m_o)>,\ldots,<f_\mu(n_{k-1}),\ f_\mu(m_{k-1})>>$ and

$g_\mu(n_o)*u_o*g_\mu(m_o)*v_o*\ldots*g_\mu(m_{k-1})*v_{k-1}$ are the corresponding relevant sequences produced.

$(\exists \eta_o \in M_\rho)(\forall n_o \in \eta_o)(\forall u_o)(\forall \xi_o \in M_\mu)(\exists m_o \in \xi_o)(\exists v_o)\ldots.$

$\ldots. (\forall k)[E^x_{<<f_\mu(n_o), f_\mu(m_o)>, \ldots, <f_\mu(n_{k-1}), f_\mu(m_{k-1})>>}$

is comeager in $\Sigma(s * g_\mu(n_o) * u_o * g_\mu(m_o) * v_o * \ldots * g_\mu(m_{k-1}) * v_{k-1})]$

... (ii)

Since $\exists$ wins $J_{\rho,s}(x)$, he wins $J_{\mu,s}(x)$ (associated with the normal family $\{E^{'<n>}\}$). So for some $\eta_o \in M_\mu$ and for every $n \in \eta_o$ there is a complete play in the subgame $J_{\mu,s}(x)$ (consistent with σ_x) such that $f_\mu(n)$ and $g_\mu(n)$ are the relevant sequences produced. So in the game (ii) $\exists$ plays such an η_o. Suppose $\forall$ plays $n_o \in \eta_o$, a finite sequence u_o and $\xi_o \in M_\mu$. Now $\exists$ wins the dual game $J^o_{\mu, s * g_\mu(n_o) * u_o}(x)$ associated with $\{\sim E^{<<f_\mu(n_o), m>>}\}_m$. Hence for every $\xi_o \in M_\mu$ there is a $m_o \in \xi_o$ such that $<f_\mu(n_o), f_\mu(m_o)>$ and $g_\mu(n_o) * u_o * g_\mu(m_o)$ are the relevant sequences (consistent with σ_x) produced in the dual game and consequently

$E^{<<f_\mu(n_o), f_\mu(m_o)>>}$ is comeager in $\Sigma(s * g_\mu(n_o) * u_o * g_\mu(m_o) * v_o)$

for some $v_o \in \omega^{<\omega}$. Thus $\exists$ plays such m_o, v_o. By regularity,

$E_{<<f_\mu(n_o), f_\mu(m_o)>>}$ is comeager in $\Sigma(s * g_\mu(n_o) * u_o * g_\mu(m_o) * v_o)$.

Proceeding thus it is clear that $\exists$ can win the game (ii). Finally observe that the function $x \longrightarrow \sigma_x$ can be chosen to be $\mathcal{BR}_0^\rho$-measurable (see the proof of Theorem II:1.1 and Theorem I:3.8). Consequently, $x \longrightarrow \sigma'_x$ can also be chosen to be $\mathcal{BR}_0^\rho$-measurable, since to describe σ'_x one needs to describe relevant sequences with respect to σ_x. Finally, for each $x \in E^*$ we choose σ_x (in a $\mathcal{BR}_0^\rho$-measurable way) to satisfy the conclusion of Lemma 2 above. Thus for each $x \in E^*$ we can choose (in a $\mathcal{BR}_0^\rho$-measurable way) σ'_x such that σ'_x is a winning strategy in the game (ii) above and satisfies the conclusion of Lemma I:7.5. $\|$

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