Perturbed Laplacian Matrix and The Structure of a Graph

SUKANTA PATI

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Thesis supervisor : Prof. R. B. Bapat

INDIAN STATISTICAL INSTITUTE
7, SJSS Marg, New Delhi–110 016, India

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and
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SUKANTA PATI
Indian Statistical Institute
7, SJSS Marg, New Delhi, India.

email sukanta@isid1.isid.ac.in

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To

my parents

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Chapter 1

Introduction

1.1 Laplacian matrices

Let G be a connected simple graph with vertex set $V = \{1, 2, \cdots, n\}$, edge set E and let each edge be associated with a positive number, the *weight* of the edge. The above graph is called a *weighted* graph. An *unweighted* graph is just a weighted graph with each of the edges bearing weight 1. All the graphs considered are weighted and simple, unless specified otherwise; all the matrices considered are real. The *adjacency matrix* $A(G)$ related to this graph is defined as $A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \theta, & \text{if } [i,j] \in E \text{ and the weight of the edge is } \theta, \\ 0, & \text{otherwise .} \end{cases}$$

Let D be the diagonal matrix with the i-th diagonal entry equal to the sum of the weights of the edges having the vertex i as an end vertex in G. We will call such a matrix as the *degree matrix* of G or simply the degree matrix, when there is no scope of confusion. The *Laplacian matrix of* G, denoted by $L(G)$, is defined by the equation $L(G) = D - A(G)$. In case there is no scope of confusion we will write L instead of $L(G)$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of L. Let $s > 1$ and suppose that $\lambda_s > \lambda_{s-1}$. An eigenvector corresponding to the eigenvalue λ_s of L will be called a *Fiedler s-vector of* $L(G)$. The term *Fiedler vector* will mean a Fiedler 2-vector.

In the year 1847, Kirchoff proved the following result involving the Laplacian matrix which put the study of the Laplacian matrix as an interesting subject in front of many researchers. The result is popularly known as the *Kirchoff's matrix* *tree theorem.* See [26] to collect some more references on this theorem.

Theorem *Let G be an unweighted graph. Denote by $L(i|j)$ the $(n-1)\times(n-1)$ submatrix of L obtained by deleting its i-th row and j-th column. Then $(-1)^{i+j}$ det $L(i|j)$ is the number of spanning trees in G.*

In the above theorem $\det L(i|j)$ means the determinant of the matrix $L(i|j)$. It is understood that if G is disconnected then the number of spanning trees in G is zero. Since then several authors from different disciplines have enriched the subject. The fact that *two graphs G and H are isomorphic if and only if there is a permutation matrix P such that $P^T L(G)P = L(H)$*, and hence the fact that *two graphs are isomorphic only if they have unimodularly congruent Laplacian matrices,* do motivate the reader to know more about the Laplacian matrices.

Among the studies of different properties and uses of Laplacian matrices the study of Laplacian spectrum and it's relation with the structural properties of graphs has been one of the most attracting features of the subject. To begin with, we can get from the matrix-tree theorem that the rank of $L(G)$ is $n - w(G)$, where $w(G)$ is the number of connected components of G. Thus, assuming that the eigenvalues of $L(G)$ are arranged in nondecreasing order: $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, we see that $\lambda_2 = 0$ *if and only if G is connected.* Thus the graph structure is already reflected in the spectrum. This observation led M. Fiedler to define the *algebraic connectivity* of G by $\mu(G) = \lambda_2(G)$, viewing it as a quantitative measure of connectivity. Fiedler was the first to show that, for a connected graph, further information about the graph structure can be extracted from an eigenvector corresponding to the algebraic connectivity by proving some remarkable results. Subsequent observations were made by different authors. We refer the interested readers to [26] for knowing more about the Laplacian matrix and for some references.

The reader at this place might note that there exist results (though, very few) for adjacency matrices which are very similar to a few of Fiedler's results for Laplacian matrices (see for example, [10]), the only difference is that *for adjacency matrices the second largest eigenvalue and the corresponding eigenvectors are considered whereas for Laplacian matrices the second smallest eigenvalue and the corresponding eigenvectors are considered.* This gives a hope for proving some more results for adjacency matrices similar to those proved for the Laplacian matrices. Our initial approach will be in this direction. Given any diagonal matrix D the *perturbed Laplacian matrix* $\mathfrak{L}$ of a graph G is defined by $\mathfrak{L} = D - A(G)$. Thus the class of perturbed Laplacian matrices is large enough to cover the Laplacian matrices and the adjacency matrices and some others. We will prove some interesting results for this class. (Some of these results sound similar to some known results for Laplacian matrices whereas the rest are new. Not to mention, most of these results are new for the adjacency matrix itself.) The thesis is organized as follows.

In Chapter 2 we introduce the concept of a *characteristic set* and take a closer view of the relationship between the characteristic set and nonnegative matrix theory. We prove several preliminary results including some interesting results about the multiplicity of the algebraic connectivity. Other topics like Perron branches and support of a Fiedler vector are also discussed here. Many of the well-known results related to characteristic vertices and edges may be deduced as corollaries to the results presented here.

The description of a Fiedler vector of a Laplacian matrix for an arbitrary connected graph is usually hard to conceive unlike that of a tree. The primary reason to this is the number of elements in the characteristic set. In Chapter 3, Section 1, we give bounds for the cardinality of the characteristic set. It is shown that if G is a connected graph with n vertices and m edges and Y is a Fiedler vector of any perturbed Laplacian matrix of G then the characteristic set of G with respect to Y has at most $m - n + 2$ elements. We will call, sometimes, an element of the characteristic set *a characteristic element.* As a very specific corollary, follows the well-known result that *a tree possesses only one characteristic element which is either a vertex or an edge, when Fiedler vector of a Laplacian matrix is considered.* The next section is a discussion regarding the location of the characteristic set. We prove that if we take a connected graph G, any perturbed Laplacian matrix $\mathcal{L}$ of G, any two Fiedler vectors Y and Y' of $\mathcal{L}$ and assume that S, S' are the characteristic sets respectively, then *"either $S = S'$ = a singleton vertex" or "both S, S' lie in one particular block of G."*

In Chapter 4 we use a new technique to prove a result which generalizes a classical result of Fiedler to the class of perturbed Laplacian matrices. As a corollary follows the result which unifies the result of Fiedler and a striking convexity/concavity result of [22] for a tree, in the case of Laplacian matrices. Do the other eigenvectors tend to bear this increasing/decreasing and convexity/concavity nature ? The answer is "partially." One can easily see the reason when he/she goes through *the complete description of any eigenvector corresponding to the third smallest eigenvalue of a*

Laplacian matrix, when the algebraic connectivity has multiplicity one, which is also supplied in this chapter.

In Chapter 5 we obtain a formula for the Moore-Penrose inverse of the oriented vertex-edge incidence matrix of a unicyclic graph G and give a complete description of the Fiedler vectors of the Laplacian $L(G)$. A recent reference concerning formulae for the Moore-Penrose inverse of the Laplacian is [23]. A sequence of numbers $a_1, a_2, \cdots, a_n$ is called *unimodal* if there exists ℓ such that $a_1 \leq a_2 \leq \cdots \leq a_{\ell-1} \leq a_\ell \geq a_{\ell+1} \geq \cdots \geq a_n$. It is shown that the coordinates of a Fiedler vector of L are unimodal along the cycle in the unicyclic graph if we begin with a vertex corresponding to the smallest coordinate.

In Chapter 6 we consider the following graph G. Consider a tree T on n vertices $1, 2, \cdots, n$. The graph G constitutes a set of complete graphs $\{K_i : i = 1, \cdots, n\}$ such that *vertices i and j are adjacent in T if and only if $K_i \bigcap K_j \neq \emptyset$*. Label the vertices of the graph G. Such a graph will be referred to $\mathcal{T}$. The graph $\mathcal{T}$ is called an *interval graph* if T is a path and the labelling of the vertices is done in a particular fashion. In this chapter we prove some interesting results concerning these graphs. We note here that given a tree T, if each of the complete graphs is on two vertices, then $\mathcal{T} = T$, thus noting that this small class of graphs under consideration contains all trees.

In Chapter 7 we have tried to show some relationship among the two known graph invariants $\mu(G)$ (the symbol $\mu(G)$ in this chapter does not mean the algebraic connectivity) and $\lambda(G)$ and the perturbed Laplacian matrices. Much remains to be explored here.

Most of the results proved in Chapter 2 and Section 1 of Chapter 3 are proved in [3], for the case of Laplacian matrices. Results in Chapter 5 are also proved in [3].

Laplacian matrices find applications in many areas; see for example [1, 7, 16, 18, 30, 33, 34], to get more information.

1.2 Notation and terminologies

By $Y(v)$, we denote the coordinate of Y corresponding to the vertex v. By $B(i,j)$ we mean the (i,j)-th entry of a matrix B. The notation I represents the identity matrix of an appropriate order. An edge between two vertices v and w in a graph G is denoted by $[v, w]$ and if the graph is directed, the orientation of the edge will be assumed to be from v to w. By $[v_1, v_2, \cdots, v_r]$ we denote the path joining v_1 and v_r via the vertices $v_2, \cdots, v_{r-1}$. A vector means a column vector. If S is a set of vertices and edges in G, by $G - S$ we mean the graph obtained by deleting all the elements of S from G. It is understood that when a vertex is deleted, all edges incident with it are deleted as well, but when an edge is deleted, the vertices incident with it are not. If U is a set of vertices in G, then the subgraph H induced by U is defined as follows: H has vertex set U and for $u, w \in U$, the edge $[u, w] \in H$ if $[u, w] \in G$. The subgraph induced by a set of edges F in G is defined as follows: the vertex set is the set of all vertices of G which are incident with at least one edge in F and the edge set being F itself. By a nonnegative (positive) matrix we mean an entrywise nonnegative (positive) matrix. By $\tau(B)$ we denote the smallest eigenvalue of a square symmetric matrix B. The number of negative (positive) eigenvalues of a matrix B is denoted by $\lambda_-(B)$ $(\lambda_+(B))$. The vector of all ones of an appropriate order is denoted by $\hat{e}$ (it should not create any confusion with e, which sometimes, is used to denote an edge of G). Inertia of a square symmetric matrix B, denoted by $In(B)$, is the triplet (p, q, z), where p, q and z are the number of positive, negative and zero eigenvalues of B, respectively. For a matrix B, B^T denotes the transpose of B. By a nonzero (zero, negative, positive) vertex of G we mean a vertex of G such that $Y(v) \neq 0$ $(Y(v) = 0,\ Y(v) < 0,\ Y(v) > 0$, respectively), where Y is a vector (which will be clear from the context; usually Y will be an eigenvector of L). A subgraph H of G containing a nonzero vertex of G is called a nonzero subgraph of G. A subgraph H of G is called positive if each vertex of H is positive. Negative subgraph is defined similarly. Given any graph (not necessarily connected) G define a binary relation on the set of vertices as: $u \sim v$, if either $u = v$ or there is a path in G joining u and v. This is an equivalence relation. Let $\{V_i\}$ be the equivalence class defined by this relation. The graphs induced by each of the elements of $\{V_i\}$ are called *components*. In other words a component of a graph is a maximal connected subgraph.

A few words about the labels: the label of theorems, lemmas, propositions, corollaries, notes, definitions and examples are made like *c.s.n*; where c is the chapter number, s is the section number and n is the item number. Thus Lemma 2.1.2 should be found in Section 1 of Chapter 2.

Chapter 2

Fiedler s-vectors

2.1 Preliminaries

We will write ${}^s\lambda$ instead of $\lambda_s(G)$ to denote the s-th smallest eigenvalue of $\mathcal{L}$. It is assumed that $s \geq 2$. If Y is a Fiedler $s-$vector of $\mathcal{L}$ then by the *eigen condition* at a vertex v we mean the equation

$$\sum_{(i,v)\in E} \mathcal{L}(v,i)Y(i) = \left[{}^s\lambda - \mathcal{L}(v,v)\right] Y(v).$$

With respect to a vector Z, the vertex v of G is called a *characteristic vertex* of G if $Z(v) = 0$ and if there is a vertex w, adjacent to v, such that $Z(w) \neq 0$.

Note 2.1.1 *If v is a characteristic vertex of G with respect to the Fiedler $s-$vector Y then the eigen condition at v implies that there are at least two vertices u, w in G, adjacent to v such that $Y(u) > 0$ and $Y(w) < 0$.* ∎

With respect to a vector Z, an edge e with end vertices u, w is called a *characteristic edge* of G if $Z(u)Z(w) < 0$. By $C(G, Z)$ we denote the *characteristic set of G with respect to a vector Z*, which is defined as the collection of all characteristic vertices and characteristic edges of G with respect to Z.

Proposition 2.1.2 *Let P be an irreducible nonnegative matrix, D any diagonal matrix and $Q = D - P$. Then the smallest eigenvalue of Q has multiplicity one and the corresponding eigenvector is positive and unique up to a scalar multiple. An eigenvector corresponding to any other eigenvalue has a positive entry and a negative entry.*

Proof Consider the matrix $M = Q - k$, where k is a scalar matrix large enough to make M nonpositive. Since M is irreducible we know from Perron-Frobenius theory (see, for example, [20]) that the smallest eigenvalue of M (and hence the smallest eigenvalue of Q) has multiplicity one and the corresponding eigenvector say X, is positive and unique up to a scalar multiple. We know that eigenvectors corresponding to different eigenvalues of a symmetric matrix are orthogonal to each other. Thus, if Z is an eigenvector of Q corresponding to any other eigenvalue then $Z^T X = 0$. Thus Z must have a positive entry and a negative entry. ∎

The following notes are useful.

Note 2.1.3 *Let G be a connected graph. Then the smallest eigenvalue of the Laplacian matrix L is 0 and the corresponding eigenvector is $\hat{e}$.*

Proof See that $\hat{e}$ is an eigenvector corresponding to the eigenvalue 0. Since $\hat{e}$ is a positive eigenvector by Proposition 2.1.2, 0 is the smallest eigenvalue. ∎

Note 2.1.4 *Let G be a connected graph on more than one vertices, D be any diagonal matrix, Y be a Fiedler $s-$vector of L and S be the corresponding characteristic set. Then $G - S$ is disconnected with* **at least two** *nonzero components.*

Proof By Proposition 2.1.2, Y has a negative and one positive entry. If $G - S$ has exactly one nonzero component then there must be a path joining these two vertices and this path is bound to contain a characteristic element, which, according to our hypothesis, is impossible. ∎

Let us explain the emphasis the word "at least two" in the above note. One may frequently encounter the examples where $s > 2$ and the number of nonzero components is strictly greater than 2, but there are some examples where this number is exactly two. The following is one of those.

Example 2.1.5 The graph in Figure 2.1 is an unweighted tree. We consider the Laplacian matrix L. The vector Λ containing the eigenvalues (sorted) of L and the eigenvector Y corresponding to the third smallest eigenvalue are given below. One can see that the characteristic set consists of only one vertex 5 and $G - \{5\}$ has exactly two nonzero components $\{6\}$ and $\{7\}$.

Also we supply another eigenvector Z corresponding to this eigenvalue. One can see that the characteristic set corresponding to this vector has two elements.

$$\Lambda = \begin{bmatrix} 0 & 0.2254 & 1 & 1 & 2.1859 & 3.3604 & 4.2283 \end{bmatrix}^T$$

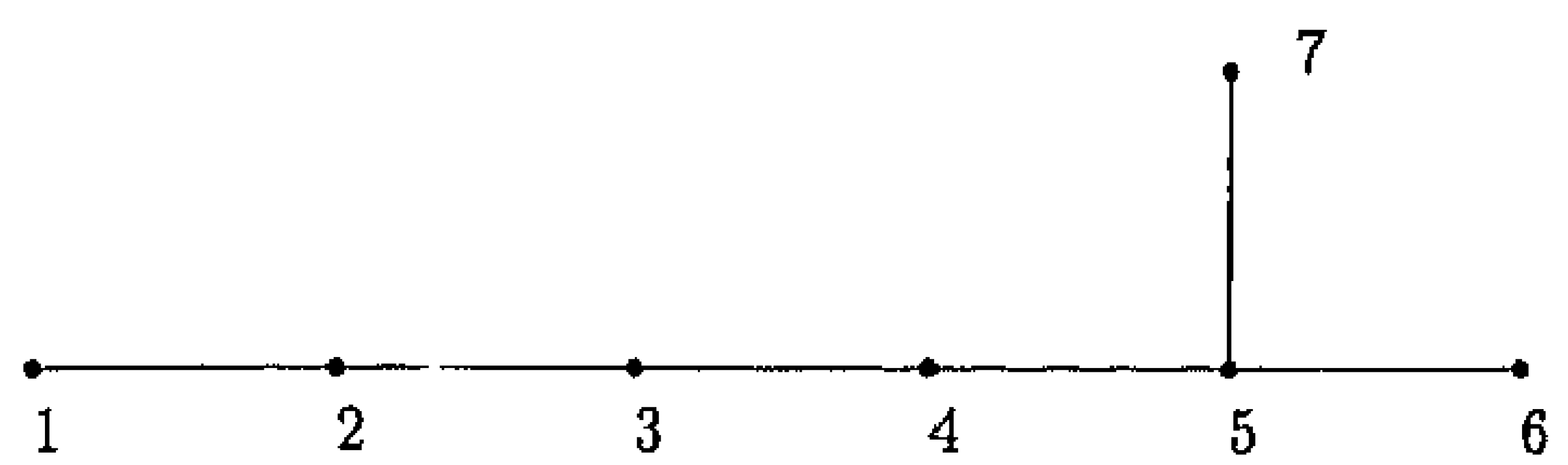


Figure 2.1: A Fiedler 3-vector for which we get exactly two nonzero components

$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}^T$
$Z = \begin{bmatrix} .1175 & 0 & -.1175 & -.1175 & 0 & .7485 & -.6310 \end{bmatrix}^T$ ∎

At times, we will use the following lemma which is a well-known result in nonnegative matrix theory.

Lemma 2.1.6 *Let C be an irreducible, symmetric, nonnegative matrix and B be a principal submatrix of C. Then the largest eigenvalue of C is strictly larger than the largest eigenvalue of B. Thus if $A = D - C$, where D is a diagonal matrix and A_1 is a principal submatrix of A then $\tau(A) < \tau(A_1)$.* ∎

Below we state a result due to Fiedler [13], which shall be used.

Lemma 2.1.7 *Let*

$$A = \begin{bmatrix} B & C \\ C^T & d \end{bmatrix}$$

be a symmetric matrix, where C is a vector and d is real. Let there exist a vector U such that $BU = 0$ and $C^TU \neq 0$. Then $In(A) = In(B) + (1, 1, -1)$. ∎

As a corollary one can prove the following. We will use this corollary to prove Lemma 2.2.13, which is one of the crucial results for further development of the thesis.

Corollary 2.1.8 *Let $A = \begin{bmatrix} B & C \\ C^T & E \end{bmatrix}$ be a symmetric matrix, where B, E are square. Let U be a vector such that $BU = 0$ and $C^TU \neq 0$. Then $\lambda_-(A) \geq \lambda_-(B) + 1$.*

Proof Since $C^TU \neq 0$, there exists a column C_1 of C such that $C_1^TU \neq 0$. Let d_1 be the diagonal entry of E corresponding to C_1. The matrix $A_1 = \begin{bmatrix} B & C_1 \\ C_1^T & d_1 \end{bmatrix}$ is a principal submatrix of A. Thus by Cauchy interlacing theorem $\lambda_-(A) \geq \lambda_-(A_1)$. By Lemma 2.1.7, we know that $In(A_1) = In(B) + (1, 1, -1)$. Hence the proof. ∎

2.2 About the graph structure

The following result reveals a nice relationship between a Fiedler vector of $\mathcal{L}$ and the graph structure.

Lemma 2.2.9 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler $s-$vector of $\mathcal{L}$. Then the subgraph induced by the vertices v in G for which $Y(v) \geq 0$ has at most $s-1$ components (similarly the subgraph induced by the vertices v in G for which $Y(v) \leq 0$ has at most $s-1$ components).*

Proof Let $L = \mathcal{L} - \tau(\mathcal{L})I$ and $\lambda = {}^s\lambda - \tau(\mathcal{L})$. Suppose that the subgraph induced by the set of vertices v for which $Y(v) \geq 0$ has more than $s-1$ components. By performing a permutation similarity transformation if necessary, we can assume that

$$LY = \lambda Y = \lambda \begin{bmatrix} Y_+ \\ Y_- \end{bmatrix}, \tag{2.2.1}$$

where Y_+ and Y_- are the subvectors of Y containing all the nonnegative and negative entries, respectively. The matrix L can be partitioned as

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 & L_{1,s+1} \\ 0 & L_{22} & \cdots & 0 & L_{2,s} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & L_{s,s} & L_{s,s+1} \\ L_{s+1,1} & L_{s+1,2} & \cdots & \cdots & L_{s+1,s+1} \end{bmatrix}, \tag{2.2.2}$$

where $L' = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ 0 & L_{22} & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & L_{s,s} \end{bmatrix}$ corresponds to Y_+ and $L_{s+1,s+1}$ corresponds to Y_-. Partitioning Y conformally, let $Y = \begin{bmatrix} Y_+^{1^T} & Y_+^{2^T} & \cdots & Y_+^{s^T} & Y_-{}^T \end{bmatrix}^T$. Since a zero vertex is either adjacent to a zero vertex or is a characteristic vertex (hence, adjacent to a positive vertex, by Note 2.1.1), none of $Y_+^{1^T}, Y_+^{2^T}, \cdots, Y_+^{s^T}$ is zero.

From Equation 2.2.1 and Equation 2.2.2, we have $L_{11}Y_+^1 + L_{1,s+1}Y_- = \lambda Y_+^1$. So we get that

$$(L_{11} - \lambda I)Y_+^1 = -L_{1,s+1}Y_- = z, \text{ (say)}.$$

Note that each entry of $L_{1,s+1}$ is nonpositive and each entry of Y_- is negative. So each entry of z is nonpositive. We know that at least one entry of $L_{1,s+1}$ is negative (since G is connected). So at least one entry of z is nonzero. Because z is nonpositive we get that

$$(Y_+^1)^T(L_{11} - \lambda I)Y_+^1 = (Y_+^1)^T z \leq 0. \tag{2.2.3}$$

So at least one eigenvalue of $L_{11} - \lambda I$ is negative (If not, let all the eigenvalues of $L_{11} - \lambda I$ be nonnegative. Then $L_{11} - \lambda I$ is positive semidefinite. Thus $(Y_+^1)^T(L_{11} - \lambda I)Y_+^1 \geq 0$. Using Equation 2.2.3, we get that $(Y_+^1)^T(L_{11} - \lambda I)Y_+^1 = 0$. Let $L_{11} - \lambda I = K^T K$, for some matrix K. So we have $(Y_+^1)^T K^T K Y_+^1 = 0$ and hence $KY_+^1 = 0$. So $z = K^T K Y_+^1 = 0$, a contradiction to the earlier statement that z has at least one entry nonzero.) Similarly, one can prove that at least one eigenvalue of $L_{ii} - \lambda I$, $i = 2, \cdots, s$ is negative. Thus, we see that at least s eigenvalues of $L' - \lambda I$ are negative. Using Cauchy interlacing theorem of eigenvalues for $L - \lambda I$ and $L' - \lambda I$, we get that at least s eigenvalues of $L - \lambda I$ are negative. But this is not possible because only $s - 1$ eigenvalues of L are less than λ. Thus the subgraph induced by the vertices v such that $Y(v) \geq 0$ has at most $s - 1$ components.

Now considering $-Y$ in place of Y as a Fiedler s-vector of $\mathcal{L}$ we conclude that the subgraph induced by the vertices v such that $Y(v) \leq 0$ has at most $s-1$ components.

■

Remark 2.2.10 *The above result is well known for the Laplacian matrix and is known for negative adjacency matrix (see for example* [10] *). Below we give an example to illustrate the above lemma.*

Corollary 2.2.11 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$. Then the subgraph induced by the positive vertices v in G is connected (similarly the subgraph induced by the negative vertices v in G is connected).* ■

Example 2.2.12 In Figure 2.2, the graph is unweighted. We consider the perturbed Laplacian matrix $\mathcal{L}$ with $D = diag(2, 3, 4, -1, 0, -1, 0, 0, 0)$. The sorted eigenvalues are

$$\begin{bmatrix} -2.8362 & -1.2434 & -0.4080 & -0.2047 & 0.5053 & 0.7856 & 1.9909 & 3.4994 & 4.9112 \end{bmatrix}.$$

Note here that the fifth smallest eigenvalue is strictly larger than the fourth smallest eigenvalue. Hence we can talk of the Fiedler 5-vector(s) of $\mathcal{L}$. (We can always talk

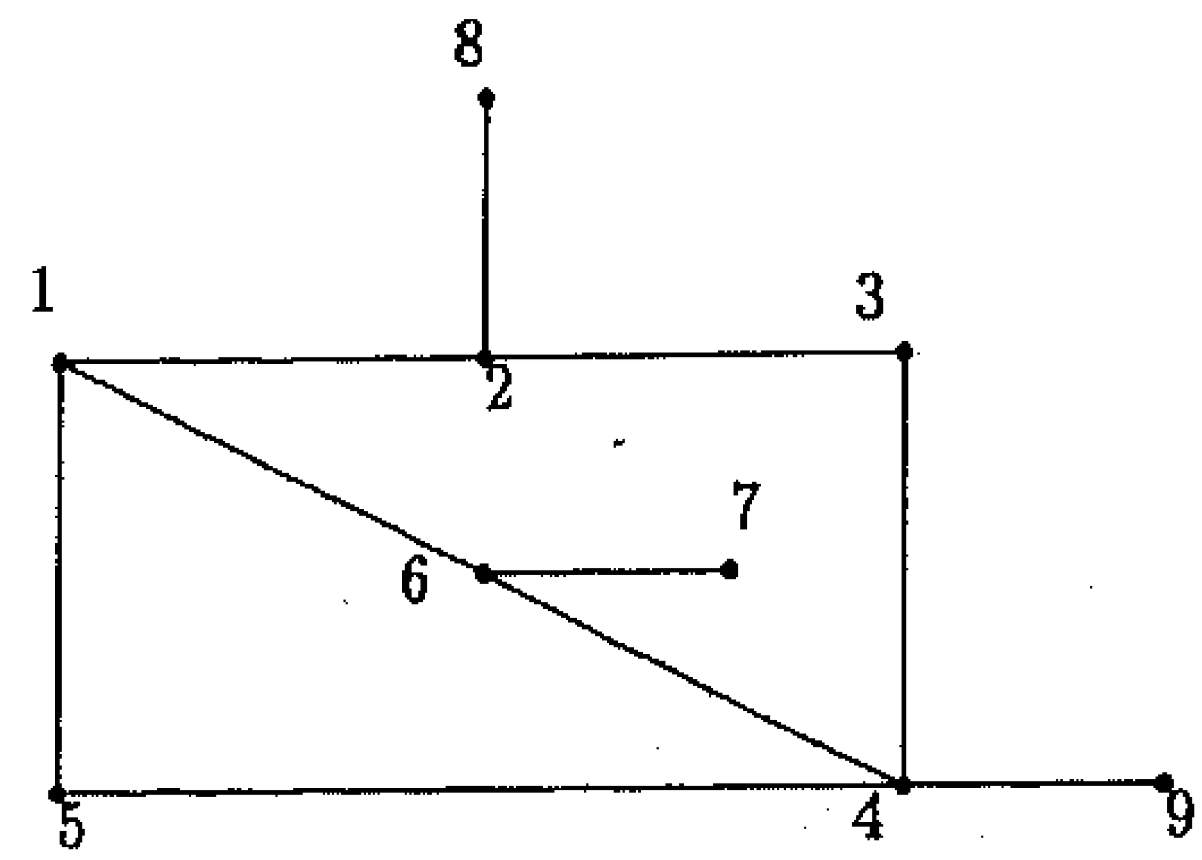


Figure 2.2: Fiedler s-vectors of perturbed Laplacian matrices.

of Fiedler 2-vectors, since by Proposition 2.1.2, the second smallest eigenvalue of $\tilde{L}$, for a connected graph, is strictly larger than the smallest eigenvalue of $\tilde{L}$.)

A Fiedler vector is

$$\begin{bmatrix} .1033 & .0060 & -.0826 & -.4393 & -.2703 & .5993 & .4820 & .0048 & -.3534 \end{bmatrix}^T.$$

See that the nonnegative (nonpositive) vertices of G with respect to the given Fiedler vector induce a connected graph.

The Fiedler 5-vector is

$$\begin{bmatrix} -.0279 & -.0173 & -.0497 & -.1562 & .3643 & -.3886 & .7691 & .0343 & .3091 \end{bmatrix}^T.$$

Here the nonnegative vertices induce a graph with four components whereas the nonpositive vertices induce a connected graph. ∎

The following result discloses another nice relationship between some eigenvalues and some principal submatrices of $\tilde{L}$ and has various applications.

Lemma 2.2.13 *Let G be a connected graph and D be any diagonal matrix. Let W be a set of vertices of G such that $G - W$ is disconnected with at least s ($s > 1$) components. Let $G_1, G_2, \cdots, G_m$ be the components of $G - W$ and $L_1, L_2, \cdots, L_m$ be the corresponding principal submatrices of $\tilde{L}$. Suppose that $\tau(L_1) \leq \tau(L_2) \leq \cdots \leq \tau(L_m)$. Then either $\tau(L_s) > {}^s\lambda$ or $\tau(L_{s-1}) = \tau(L_s) = {}^s\lambda$, where ${}^s\lambda$ is the s-th smallest eigenvalue of $\tilde{L}$. Thus, it is always true that $\tau(L_s) \geq {}^s\lambda$.*

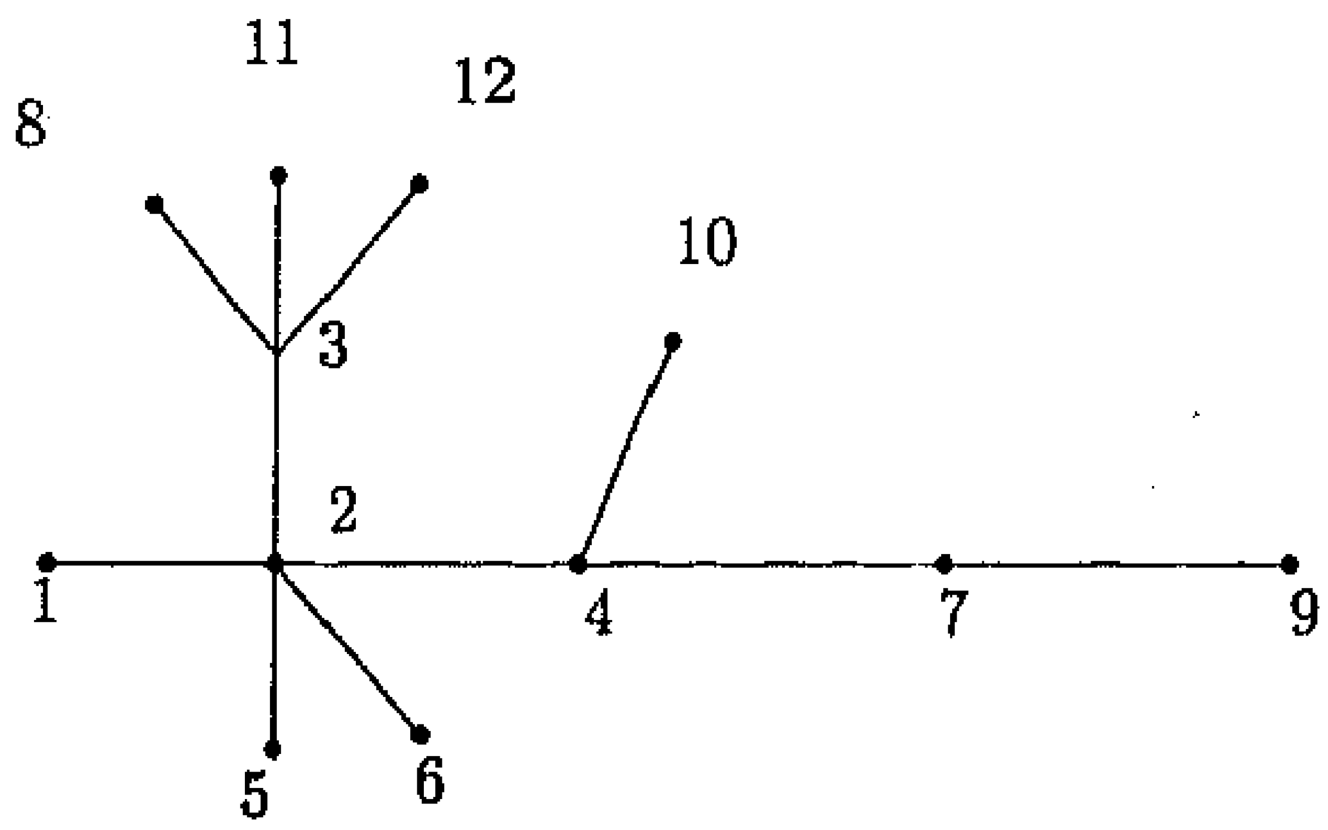


Figure 2.3:

Before proceeding for the proof let us have a look at the following example.

Example 2.2.14 In Figure 2.3, we have an unweighted graph and we consider the Laplacian matrix L. The eigenvalues (sorted) are given by the vector Λ. Let $W = \{2,3\}$. Consider the components of $G - \{2,3\}$. Let G_1 be the graph induced by $\{4,7,9,10\}$, $G_2 = \{5\}$, $G_3 = \{6\}$, $G_4 = \{8\}$, $G_5 = \{11\}$, $G_6 = \{12\}$, $G_7 = \{1\}$. Let L_i be the principal submatrix of L corresponding to the branch G_i, $i = 1, \cdots, 7$. The smallest eigenvalues of L_i's are given below.

$$\tau(L_1) = 0.1729 < \tau(L_2) = \tau(L_3) = \tau(L_4) = \tau(L_5) = \tau(L_6) = \tau(L_7) = 1.$$

Consider ${}^2\lambda$. By the above lemma, we should either have $\tau(L_2) > {}^2\lambda$ or $\tau(L_2) = \tau(L_1) = {}^2\lambda$. One can see that $1 = \tau(L_2) > {}^2\lambda = 0.1876$, for this example. Consider ${}^6\lambda$. By the above lemma, we should either have $\tau(L_6) > {}^6\lambda$ or $\tau(L_6) = \tau(L_5) = {}^6\lambda$. One can see that $1 = \tau(L_5) = \tau(L_6)$, for this example.

$$\Lambda = \begin{bmatrix} 0 & .1876 & .4146 & .6770 & 1 & 1 & 1 & 1 & 2.1755 & 3.6338 & 4.4898 & 6.4217 \end{bmatrix}^T.$$

Proof of Lemma 2.2.13 It is sufficient to show that $\tau(L_s) \leq {}^s\lambda \Rightarrow \tau(L_s) = \tau(L_{s-1}) = {}^s\lambda$. So, let $\tau(L_s) \leq {}^s\lambda$ and first suppose that $\tau(L_{s-1}) < \tau(L_s)$. Let $W = \{1,2,\cdots,k\}$ and $d_i = L(i,i)$, $i = 1,2,\cdots,k$. Let L_W be the principal submatrix of $\mathcal{L}$ corresponding to the graph $G-W$. After a permutation similarity operation we have

$$\mathcal{L} = \left[\begin{array}{cccc|ccc} L_1 & 0 & \cdots & 0 & & & \\ 0 & L_2 & \cdots & 0 & C_1 & \cdots & C_k \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & L_m & & & \\ \hline & C_1^T & & & d_1 & \cdots & * \\ & \vdots & & & * & \ddots & * \\ & C_k^T & & & * & \cdots & d_k \end{array}\right],$$

where the upper left block diagonal matrix in the above representation is L_W. Let U be the positive vector associated with $\tau(L_s)$ (refer to Proposition 2.1.2). Consider the vector $U' = \begin{bmatrix} 0 & \cdots & U^T & 0 & \cdots & 0 \end{bmatrix}^T$, where the zeros are added so that $L_W U' = \tau(L_s) U'$. Since at least one vertex in W is adjacent to one of the vertices in G_s, there exists i, $1 \le i \le k$, such that $C_i^T U' \neq 0$. Note that $\lambda_-[L_W - \tau(L_s)I] \ge s-1$, because of the hypothesis. Now, applying Corollary 2.1.8, we see that $\lambda_-[L - \tau(L_s)I] \ge s$, and hence $\tau(L_s) > {}^s\lambda$. This is a contradiction to the hypothesis that $\tau(L_s) \le {}^s\lambda$.

Next, let $\tau(L_s) = \tau(L_{s-1})$. Since L_W is a principal submatrix of $\mathcal{L}$, using Cauchy interlacing theorem we get that the s-th smallest eigenvalue of $\mathcal{L}$ is less than or equal to $\tau(L_s)$. But $\tau(L_s) \le {}^s\lambda$, by the hypothesis. So, we get that $\tau(L_s) = \tau(L_{s-1}) = {}^s\lambda$.■

As one of the applications of Lemma 2.2.13, we prove the following result.

Lemma 2.2.15 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler s-vector of $\mathcal{L}$, $s > 1$ and ${}^s\lambda$ be the corresponding eigenvalue. Let W be a nonempty set of vertices of G such that $Y(u) = 0$, for all $u \in W$ and suppose that $G - W$ is disconnected with t nonzero components, $G_1, G_2, \cdots, G_t$, $t \ge s$. Let L_i and Y_i be the principal submatrix of $\mathcal{L}$ and the subvector of Y corresponding to G_i, $i = 1, 2, \cdots, t$. Then the following occurs.*

(i) The multiplicity of ${}^s\lambda$ is at least $t-s+1$. (The multiplicity can be strictly larger than $t-s+1$, see Example 2.2.17.)

(ii) For at least $t-s+2$ indices i, $i \in \{1, 2, \cdots, t\}$, $\tau(L_i) = {}^s\lambda$; for these indices i the entries of each Y_i are nonzero and of the same sign. Thus the number of components containing both positive and negative vertices is at most $s-2$ and if F is such a component then the corresponding principal submatrix of $\mathcal{L}$ has smallest eigenvalue less than ${}^s\lambda$.

Proof Let

$$\hat{L} = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_t \end{bmatrix}.$$

For each $i = 1, \cdots, t$, $L_i Y_i = {}^s\lambda$. So we have $\tau(L_i) \leq {}^s\lambda$. So the t-th smallest eigenvalue of $\hat{L}$ is less than or equal to ${}^s\lambda$. With a permutation similarity operation, we can assume that $\hat{L}$ is a principal submatrix of $\mathcal{L}$. Thus applying Cauchy interlacing theorem we get that the t-th smallest eigenvalue of $\mathcal{L}$, that is ${}^t\lambda$, is also less than or equal to ${}^s\lambda$. But since $t \geq s$, we get that

$${}^t\lambda = {}^s\lambda.$$

Thus the multiplicity of ${}^s\lambda$ is at least $t - s + 1$.

To prove (ii), suppose that $\tau(L_1) \leq \tau(L_2) \cdots \leq \tau(L_t)$. Then using Lemma 2.2.13, we get that $\tau(L_i) = {}^s\lambda$, $i = s-1, s, \cdots, t$. By Proposition 2.1.2, the vectors Y_i, $i = s-1, s, \cdots, t$ are positive and unique up to a scalar multiple. ∎

Remark 2.2.16 *From the proof of the above lemma, it is clear that under the assumptions of the lemma, if we have a nonzero component H then $\tau(L_H) \leq {}^s\lambda$. Can we compare $\tau(L_H)$ and ${}^s\lambda$ when H is a zero component ? In general if we are given that H is a zero component then we don't have any clue about the comparison between ${}^s\lambda$ and $\tau(L_H)$. See Example 2.2.19 for this. But if the value of s is two, that is if we are considering a Fiedler vector Y then we can say that $\tau(L_H) \geq {}^2\lambda$ with the help of Lemma 2.2.9. See Example 2.2.18, where each of the comparison types occur.*

Example 2.2.17 Consider the Laplacian matrix of the unweighted cycle on six vertices and let $s = 2$. There is a Fiedler vector Y such that $C(G, Y)$ consists of exactly two vertices and no edges. Also $G - C(G, Y)$ has exactly two nonzero components. So by Lemma 2.2.15, the multiplicity of ${}^2\lambda$ is at least 1. One can see that the multiplicity of ${}^2\lambda$ is 2 which is greater than $t - s + 1 = 1$. ∎

Example 2.2.18 The graph in Figure 2.4 is unweighted. We consider the negative adjacency matrix $-A$. One Fiedler vector is

$$Y = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}^T.$$

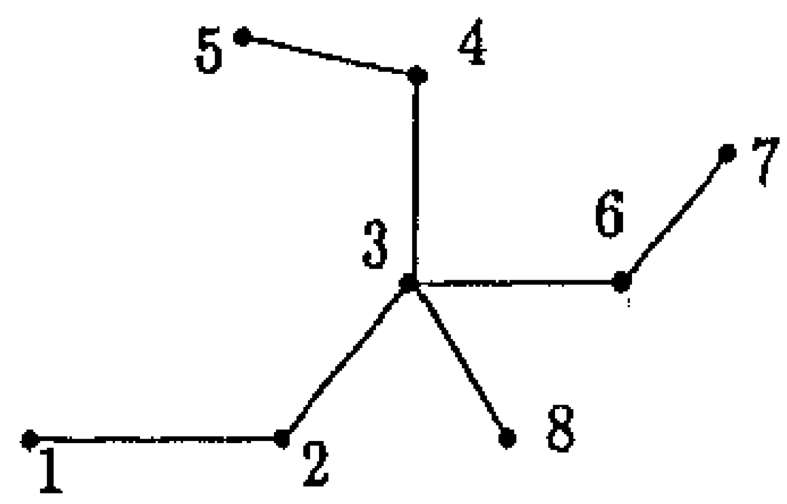


Figure 2.4:

Let $W = \{3\}$. There are two nonzero components $\{1,2\}$ and $\{4,5\}$ of $G - W$, thus meeting the assumption of Lemma 2.2.15. There are two zero components $\{6,7\}$ and $\{8\}$. The second smallest eigenvalue of $-A$ is -1; multiplicity is 2. One can check that the smallest eigenvalue of the principal submatrix of $-A$ corresponding to $\{6,7\}$ is the same as ${}^2\lambda$ and the smallest eigenvalue of the principal submatrix of $-A$ corresponding to $\{8\}$ is $0 > {}^2\lambda$. ∎

Example 2.2.19 With reference to Lemma 2.2.15 and Remark 2.2.16, the three comparison cases are shown below. In Figure 2.5, the graph G is a weighted tree. Weight of the edge $[2,7]$ is 0.7, weight of the edge $[5,9]$ is 1.1 and weight of any other edge is 1. The Laplacian matrix L is considered. The vector Λ contains the eigenvalues of L, sorted and rounded to four decimal places. The fourth smallest eigenvalue of L, ${}^4\lambda = 1$. The vector Y is a Fiedler 4-vector. Let $W = \{2,5\}$. See that $Y(2) = y(5) = 0$. Also see that $T - W$ has 4 nonzero components (thus meeting the assumption of the Lemma 2.2.15) and three zero components $H_1 = \{7\}, H_2 = \{9\}$ and $H_3 = \{10\}$. One can see that

$$\tau(L_{H_1}) = .7 < 1,\ \tau(L_{H_2}) = 1.1 > 1,\ \tau(L_{H_3}) = 1.$$

$$Y = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 1 & 0 & -2 & 0 & 0 \end{bmatrix}^T.$$

$$\Lambda = \begin{bmatrix} 0 & .1493 & .7929 & 1 & 1 & 1 & 1.075 & 2.5189 & 3.8901 & 6.1738 \end{bmatrix}^T.$$ ∎

The following result generalizes a part of Theorem 1 in [17] and a part of Theorem 1 in [21].

Theorem 2.2.20 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathfrak{L}$. Let W be a nonempty set of vertices of G such that $Y(u) = 0$, for all $u \in W$ and suppose that $G - W$ is disconnected with t nonzero*

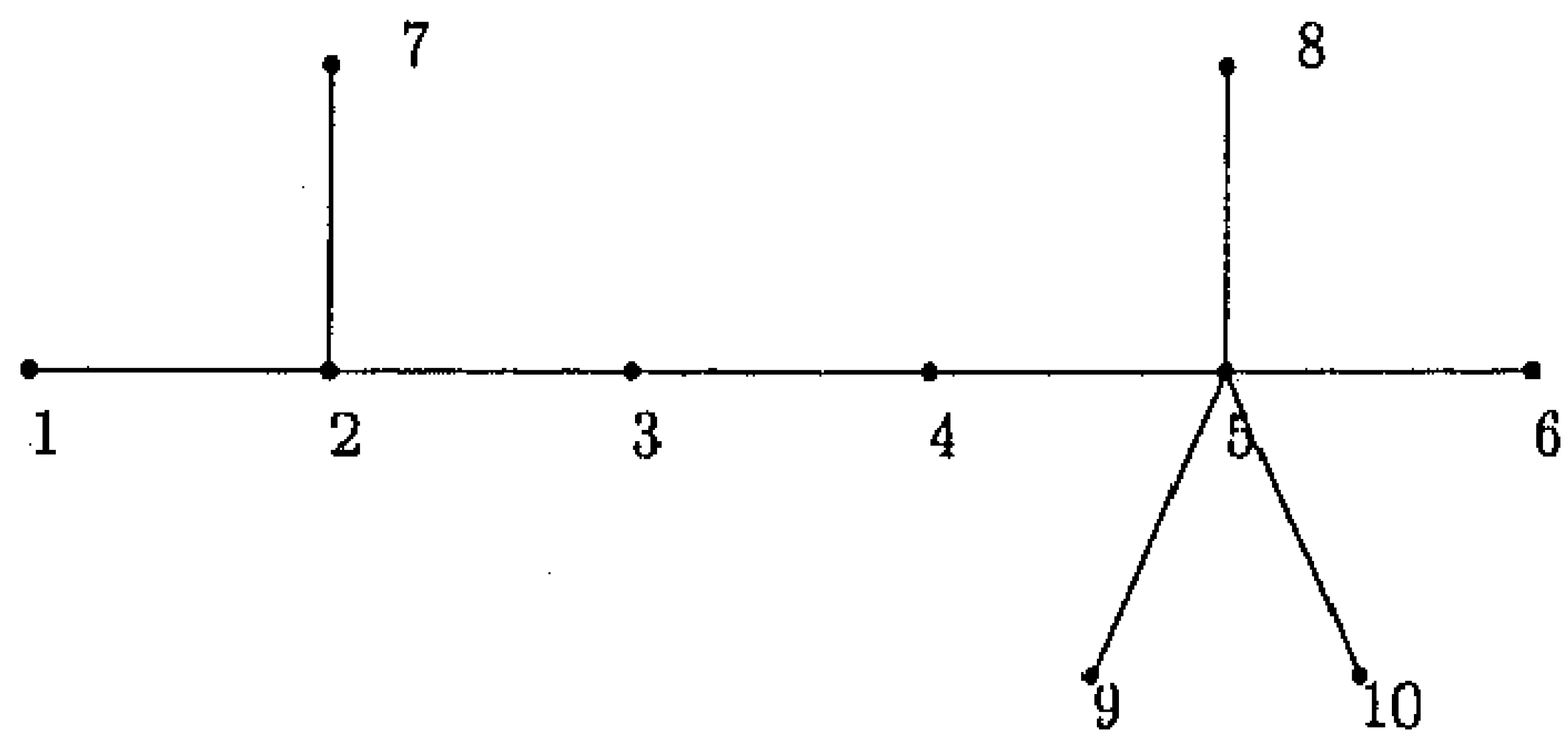


Figure 2.5:

components, $G_1, G_2, \cdots, G_t$, $t \geq 2$. *Let* L_i *and* Y_i *be the principal submatrix of* $\mathcal{L}$ *and the subvector of* Y *corresponding to* G_i, $i = 1, 2, \cdots, t$.

(i) *Then each of* G_i, $i \in \{1, \cdots, t\}$ *is either positive or negative and* $\tau(L_i) = \mu$ *and the corresponding eigenvector of* L_i *is* Y_i. *No nonzero component contains both positive and negative vertices. So* $C(G, Y)$ *does not contain an edge.*

(ii) *Further, the multiplicity of* μ *is at least* $t - 1$. *(See Example 2.2.18, where the multiplicity of* μ *is strictly more than* $t - 1$.*) We can get that* $t - 1$ *independent Fiedler vectors* $X_1, X_2, \cdots, X_{t-1}$ *of* $\mathcal{L}$ *such that for each vector* X_j, $X_j(w) = 0$, *for all* $w \in W$ *and exactly one of the components* G_i, $i \in \{1, \cdots, t\}$ *is positive and exactly one of the components* G_i, $i \in \{1, \cdots, t\}$ *is negative with respect to* X_j.

Proof Item (i) follows from Lemma 2.2.15. To prove item (ii), let

$$\hat{L} = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_t \end{bmatrix}.$$

By item (i), the t-th smallest eigenvalue of $\hat{L}$ is μ. With a permutation similarity operation, we can assume that $\hat{L}$ is a principal submatrix of $\mathcal{L}$. Thus applying Cauchy interlacing theorem we get that the t-th smallest eigenvalue of $\mathcal{L}$ is less than or equal to μ. But since $t \geq 2$ and the second smallest eigenvalue of $\mathcal{L}$ is μ, we get that the t-th smallest eigenvalue of $\mathcal{L} = \mu$. Thus the multiplicity of μ is at least $t - 1$.

Now we construct a Fiedler vector of the form described in the statement. Consider the vectors Y_1 and Y_2. Let

$$k = \frac{\sum_{v \in G_1} Z(v) Y_1(v)}{\sum_{v \in G_2} Z(v) Y_2(v)},$$

where Z is the eigenvector of $\hat{L}$ corresponding to the eigenvalue $\tau(\hat{L})$. Observe that in the above definition of k we use the fact that the entries of Y_2 agree in sign. With a permutation similarity operation we can write

$$\hat{L} = \begin{bmatrix} L_1 & 0 & L_{13} \\ 0 & L_2 & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix},$$

where L_1 corresponds to G_1, L_2 corresponds to G_2. Let

$$X_1 = \begin{bmatrix} Y_1 \\ -kY_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

One can see that

$$Z^T X_1 = 0 \text{ and}$$
$$X_1^T \hat{L} X_1 = \mu X_1^T X_1. \tag{2.2.4}$$

Recall that Z is the eigenvector of $\hat{L}$ corresponding to the smallest eigenvalue $\tau(\hat{L})$ and $\hat{L}$ is a symmetric matrix. Thus from the above equation one can easily show that

$$\hat{L} X_1 = \mu X_1.$$

Construction of other independent vectors is similar. ∎

Remark 2.2.21 *The technique used for this construction is taken from* [36].

QUESTION

We will prove later that if we have a tree and a Fiedler vector Y of $\hat{L}$ such that we have a characteristic vertex, then for every Fiedler vector the same vertex appears to be characteristic and no more vertex or edge is characteristic.

In the case of a Laplacian matrix of an even cycle one can see that the above statement does not hold except for the fact that *if Y is a Fiedler vector inducing no characteristic edge on the cycle then the number of characteristic vertices is* 2. The same example also shows that there exists Fiedler vectors Y, Y' of Ł satisfying the following:

I. $C(G, Y)$ and $C(G, Y')$ comprise of vertices only,

II. cardinality of $C(G, Y)$ is the same as the cardinality of $C(G, Y')$.

Our question is whether item I implies item II. If we consider Fiedler s-vectors, $s > 2$, then a counterexample is Example 2.1.5.

2.3 Multiplicity of the algebraic connectivity

Let G be a connected graph, D be any diagonal matrix. Let Y be a Fiedler vector of Ł. Let W be any proper subset of vertices of G. By a *branch* at W of G we mean a component of $G - W$. A branch at W is called a *Perron branch* if the principal submatrix of Ł, corresponding to the branch, has an eigenvalue less than or equal to μ. In the following few results we will discuss more about the multiplicity of μ of the perturbed Laplacian matrices.

Lemma 2.3.22 *Let G be a connected graph, D be any diagonal matrix. Let Y be a Fiedler vector of Ł. Suppose that $W = C(G, Y)$ contains vertices only. Suppose that G has $p(\geq 2)$ Perron branches $G_1, G_2, \cdots, G_p$ at W. Then the multiplicity of μ is at least $p - 1$. There exists a Fiedler vector X such that $W = C(G, X)$ and $G - W$ has p nonzero components, $G_1, G_2, \cdots, G_p$.*

Proof Let L_1 and L_2 be the principal submatrices of Ł corresponding to G_1 and G_2, respectively. Thus we have $\tau(L_1), \tau(L_2) \leq \mu$. Applying Lemma 2.2.13, we get that $\tau(L_1) = \tau(L_2) = \mu$. By Proposition 2.1.2, for each $i \in \{1, 2\}$, the eigenvector Y_i of L_i corresponding to $\tau(L_i)$ is positive. We can now proceed in a similar way as in the proof of item (ii) of Theorem 2.2.20, to construct a Fiedler vector X_{12} of Ł such that G_1 is positive and G_2 is negative with respect to X_{12} and $C(G, X_{12}) = W$. Suppose that X_{1i} be the Fiedler vector constructed, considering the branches B_1 and B_i, $i = 2, 3, \cdots, p$. It is easy to see that X_{1i}'s are independent. Thus the multiplicity of μ is at least $p-1$. The rest of the proof follows easily by taking a suitable linear combination X of X_{1i}'s. ∎

The above lemma, though looks very similar to item (ii) of Theorem 2.2.20, can extract relatively better information. Consider a very simple case, an unweighted star on 5 vertices, center is the vertex 1. Let L be the corresponding Laplacian matrix and Y be the Fiedler vector $\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix}^T$. Let $W = \{1\}$. Application of Theorem 2.2.20 will tell us that the multiplicity of the algebraic connectivity is at least 1. But an application of Lemma 2.3.22 will tell that the multiplicity is at least 3, because of the presence of four perron branches of G at W.

The following result gives another interesting information about the multiplicity of algebraic connectivity of $\mathcal{L}$.

Theorem 2.3.23 *Let G be a connected graph, D be any diagonal matrix. Let S be a set of vertices such that for every Fiedler vector X of $\mathcal{L}$, $C(G,X) = S$. Suppose that G has p Perron branches at S. Then the multiplicity of μ is exactly $p-1$.*

Proof Let $G_1, G_2, \cdots, G_p$ be the Perron branches at S. Let L_i be the principal submatrix of $\mathcal{L}$ corresponding to G_i and Y_i be the positive eigenvector of L_i corresponding to $\tau(L_i)$. By Lemma 2.3.22, the multiplicity of μ is at least $p-1$. Construct the $p-1$ Fiedler vectors X_{1i}, $i = 2, \cdots, p$ as in Theorem 2.2.20. Let Y' be any Fiedler vector. We want to show that Y' is a linear combination of X_{1i}'s. We proceed by induction on the number r of nonzero components of $G - S$ with respect to a Fiedler vector Y' We know that there exist no Fiedler vector Y' of $\mathcal{L}$ such that $G - S$ has exactly one nonzero component. (In fact, if G is connected then $G - C(G,Y)$ is always disconnected with at least two nonzero components.) Thus the statement is valid vacuously for $r = 1$. Assume that every Fiedler vector Y' with the number of nonzero components of $G - S$ less than r, is a linear combination of X_{1i}'s.

Let Y' be any Fiedler vector such that $G - S$ has r nonzero components, H_1, $H_2, \cdots, H_r$. It is easy to see that

$$\{H_1, H_2, \cdots, H_r\} \subseteq \{G_1, G_2, \cdots, G_p\}. \tag{2.3.5}$$

In fact, if H is a nonzero component then $\tau(L_H) = \mu$, by item (i) of Theorem 2.2.20, implying that H is a Perron component of G at S. In view of the above equation we can assume that

$$H_i = G_i,\ i = 1, \cdots, r;\ r \leq p.$$

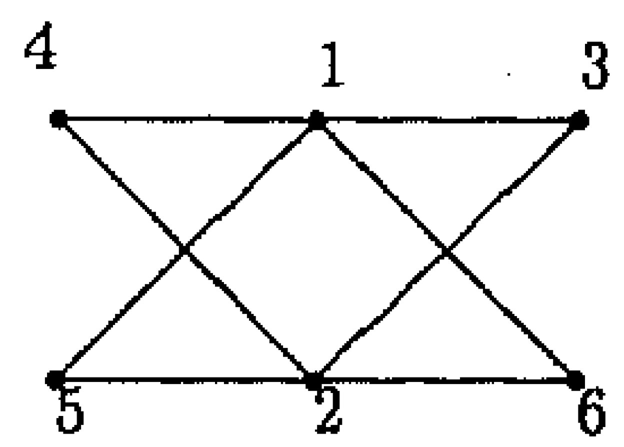


Figure 2.6: If the characteristic set consists of vertices only and does not change with Fiedler vectors then multiplicity of μ is exactly one less than the number of Perron branches of G at $C(G, Y)$.

We know, from the discussion in the proof of item (i) of Theorem 2.2.20 that

$$Y'(G_i) \propto Z_i, \; i = 1, \cdots, r.$$

Thus we can get a suitable linear combination Y'' of the vector X_{12} and Y' such that $Y''(G_2) = 0$. Clearly Y'' is a Fiedler vector and $C(G, Y'') = S$. It is evident that the number of nonzero components $G - S$, with respect to Y'' is at most $r - 1$. By the induction hypothesis Y'' is a linear combination of the vectors X_{1i}'s. Thus we get that Y' is also a linear combination of X_{1i}'s. ∎

We explain the above result by giving an example.

Example 2.3.24 The graph in Figure 2.6 is unweighted. We consider the perturbed Laplacian Matrix $\hat{\mathcal{L}}$, where $D = diag(2, 2, 1, 1, 1, 1)$. The space of Fiedler vectors is spanned by the following three vectors:

$$\begin{bmatrix} 0.0000 \\ 0.0000 \\ -0.3863 \\ -0.4206 \\ 0.8208 \\ -0.0139 \end{bmatrix}, \quad \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.5625 \\ -0.0156 \\ 0.2432 \\ -0.7901 \end{bmatrix}, \quad \begin{bmatrix} -0.0000 \\ -0.0000 \\ 0.5333 \\ -0.7569 \\ -0.1308 \\ 0.3544 \end{bmatrix}.$$

For each Fiedler vector Y, one can see that $C(G, Y) = \{1, 2\}$. The algebraic connectivity μ is 1 for $\hat{\mathcal{L}}$. Also notice that $G - C(G, Y)$ has 4 Perron components, namely $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$. Thus by Theorem 2.3.23, the multiplicity of μ must be exactly 3. This is true as the eigenvalues of $\hat{\mathcal{L}}$ are -1.3723, 1.0000, 1.0000, 1.0000, 2.0000 and 4.3723. ∎

Theorem 2.3.23 has an interesting corollary (Corollary 3.2.39) but that will be proved later.

2.4 Perron branches

In the last section we have defined a Perron branch of a graph. In this section we will try to prove some more results using this concept. The following is an application of Lemma 2.2.13 and a generalization of Corollary 1.1 of [22].

Theorem 2.4.25 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$. Suppose that Y has no zero entry. Let $F = \{e_i = [u_i, w_i], : Y(u_i) > 0, Y(w_i) < 0, i = 1, 2, \cdots, r; r \geq 1\}$ be the set of edges in $C(G, Y)$. Let $U = \{u_1, \cdots, u_r\}$ and $W = \{w_1, \cdots, w_r\}$. Let G_1 and G_2 be the two components of $G - F$, such that $u_i \in G_1$, $i = 1, 2, \cdots, r$. Then G_1 is the only Perron branch at W and G_2 is the only Perron branch at U.*

Proof Let L_F be the matrix obtained from $\mathcal{L}$ corresponding to $G - F$, that is the matrix obtained from $\mathcal{L}$ by replacing the $(u_i, w_i)-$th and $(w_i, u_i)-$th entries by 0 for each edge $e_i = [u_i, w_i] \in F$. Let L_1 and L_2 be the two submatrices of L_F corresponding to the components G_1 and G_2, respectively. By performing a permutation similarity operation, if necessary, we have $\mathcal{L} = \begin{bmatrix} L_1 & -B \\ -B^T & L_2 \end{bmatrix}$, where B is a nonnegative matrix with at least one entry nonzero. Since G_1 and G_2 are connected, L_1 and L_2 are irreducible. Partitioning Y conformally and using $\mathcal{L}Y = \mu Y$ we get that $L_1 Y_1 < \mu Y_1$ and $L_2(-Y_2) < \mu(-Y_2)$. So $[L_1 - \tau(L_1)I]Y_1 < [\mu - \tau(L_1)]Y_1$. Since $[L_1 - \tau(L_1)I]$ is positive semidefinite, it follows, by left multiplying Y_1^T, that $\mu - \tau(L_1) > 0$. Together with a similar argument applied to L_2, we obtain

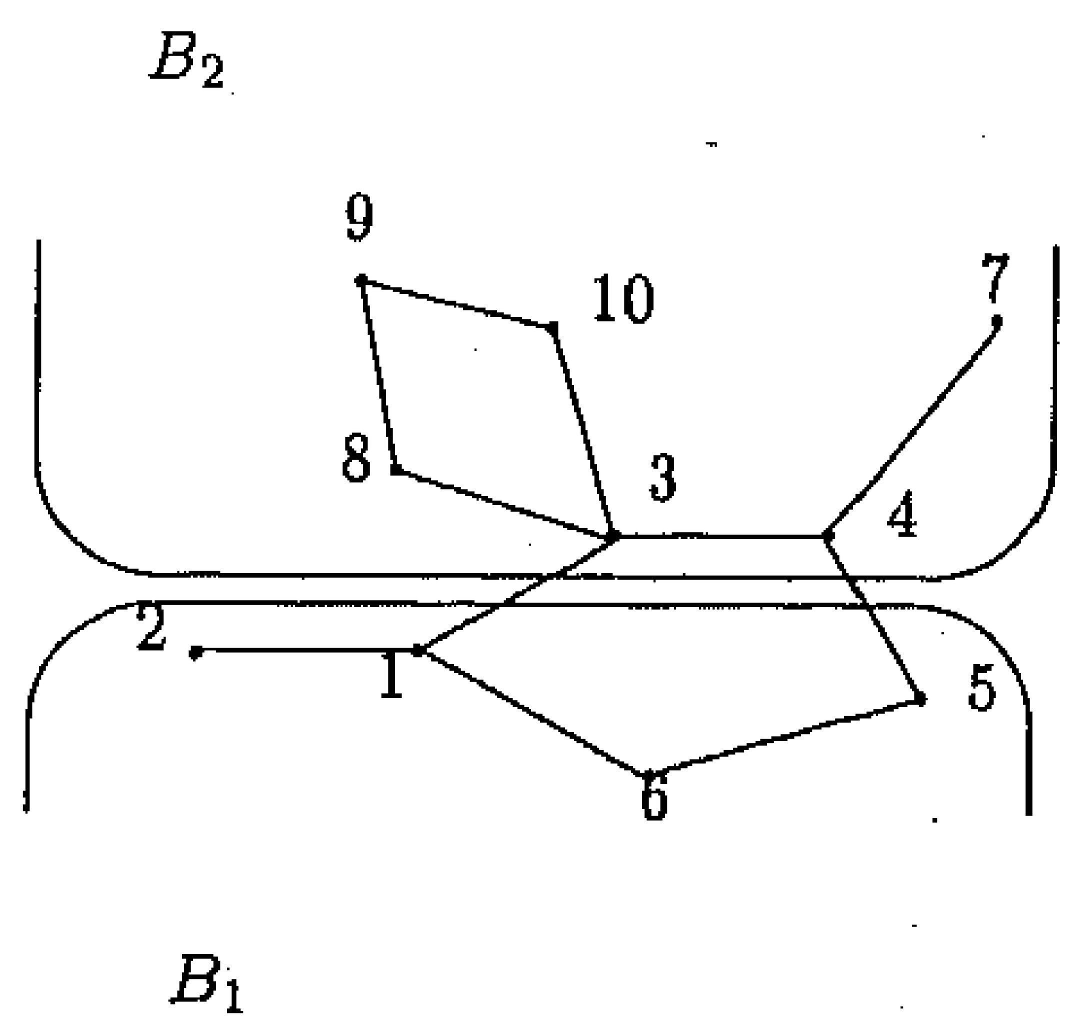
$$\tau(L_1) < \mu \text{ and } \tau(L_2) < \mu. \tag{2.4.6}$$

By Corollary 2.2.11, G_2 is a component of $G - U$. Using Lemma 2.2.13 and the fact that $\tau(L_2) < \mu$, we get that G_2 is the only Perron branch at U. Similarly G_1 is the only Perron branch at W. ∎

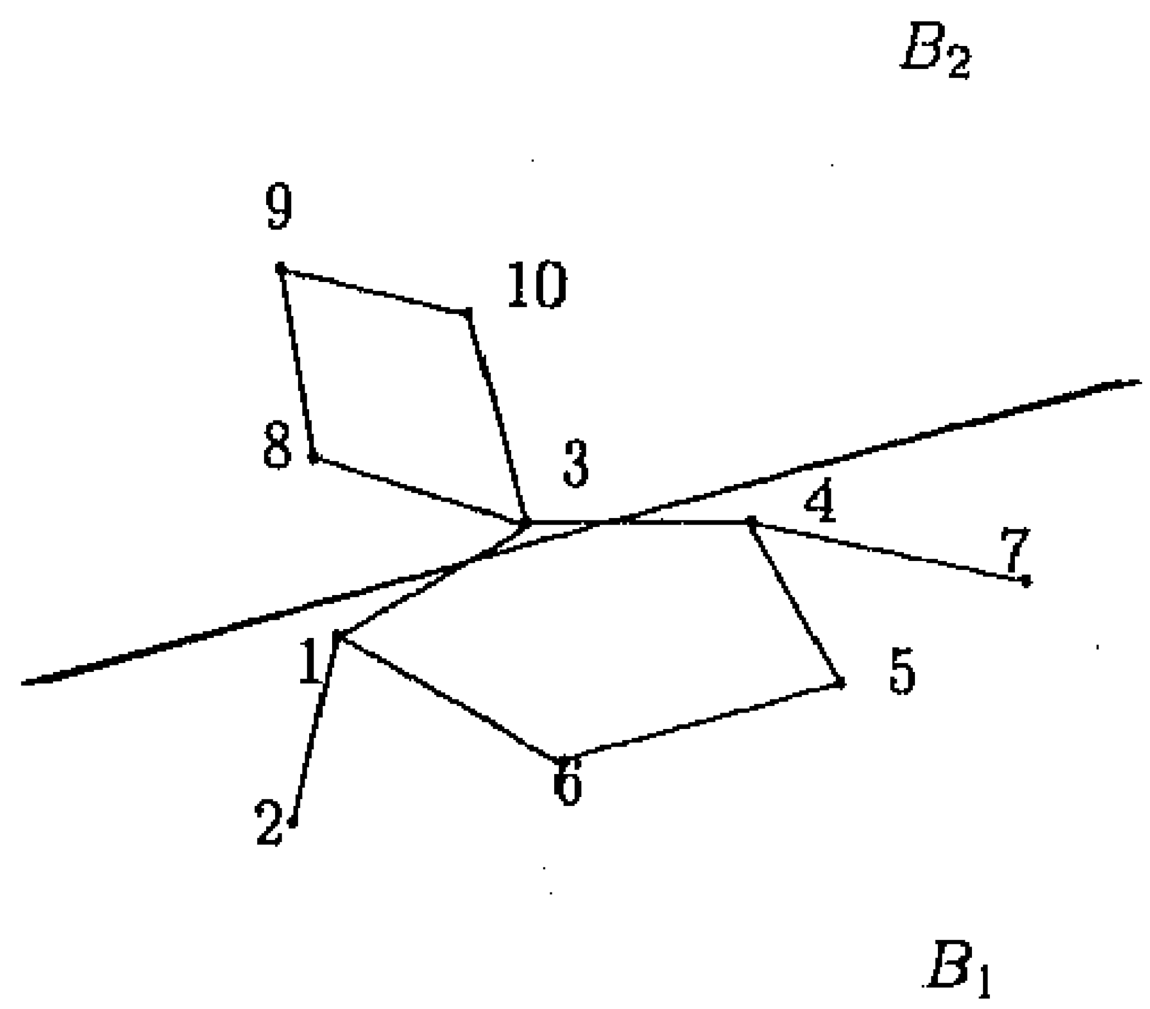
Before proceeding, let us see an example. We will prove an interesting result (Theorem 3.1.31) in the next section, related to the highlighted statement in the example.

Example 2.4.26 The graph in Figure 2.7, Case 1, is a weighted graph. Weight of an edge $[i, j]$ is $i + j$. We consider the Laplacian matrix. The Fiedler vector is

$$Y = \begin{bmatrix} .2141 & .8333 & -.1763 & -.1024 & .0123 & .1036 & -.1284 & -.2486 & -.2629 & -.2447 \end{bmatrix}^T.$$



Case 1



Case 2

Figure 2.7:

Looking at the Fiedler vector the positive and negative vertices are grouped below.
Positive: 1,2,5,6.
Negative: 3,4,7,8,9,10.
The characteristic set consists of two edges, $[1,3]$ and $[4,5]$. **Both of them lie on a simple cycle.** Let $U = \{3,4\}$ and $W = \{1,5\}$. Then B_1 is the only Perron branch at U and B_2 is the only Perron branch at W.

The graph in Figure 2.7, Case 2, is an unweighted graph. We consider the negative adjacency matrix. The Fiedler vector is

$$Y = \begin{bmatrix} -.2512 & -.1545 & .1470 & -.2512 & -.4011 & -.4011 & -.1545 & .3707 & .4559 & .3707 \end{bmatrix}^T.$$

Looking at the Fiedler vector the positive and negative vertices are grouped below.
Positive: 3,8,9,10.
Negative: 1,2,4,5,6,7.
The characteristic set consists of two edges, $[3,1]$ and $[3,4]$. **Both of them lie on a simple cycle.** Let $U = \{1,4\}$ and $W = \{3\}$. Then B_1 is the only Perron branch at U and B_2 is the only Perron branch at W. ∎

We already know from Corollary 2.2.11 that for a Fiedler vector Y of the perturbed Laplacian matrix of a connected graph G the nonnegative vertices induce a connected subgraph. Thus, if Y has no zero entry (that is, if we have characteristic edges and no characteristic vertices) then the positive vertices and the negative vertices induce connected graphs, respectively. Below we show that the presence of one characteristic edge (no condition on the presence of characteristic vertices) is enough to prove that the positive vertices and the negative vertices induce connected graphs, respectively. Before that we put the following note.

Note 2.4.27 *Let G be a connected graph and D be any diagonal matrix and Y be a Fiedler vector of $\mathfrak{L}$. Let U be the set of characteristic vertices. If $C(G,Y)$ contains an edge then by Theorem 2.2.20, $G-U$ has exactly one nonzero component.*

Lemma 2.4.28 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathfrak{L}$. Let U (may be empty) be the set of characteristic vertices of G. Suppose that there is only one component, say G', of $G-U$ is nonzero. Let L' be the principal submatrix of $\mathfrak{L}$ corresponding to G'. Then the following are valid.*

(i) The eigenvalue μ is the second smallest eigenvalue of L'.

(ii) The positive (negative) vertices of G induce a connected subgraph of G. Let the subgraph of G induced by the positive vertices be G_+ and the subgraph of G induced by the negative vertices be G_-. Let L_+ and L_- be the principal submatrices of $\mathfrak{L}$ corresponding to G_+ and G_-, respectively. Then $\tau(L_+) < \mu$ and $\tau(L_-) < \mu$.

(iii) Given a pair of distinct elements in $C(G,Y)$ there exists a simple cycle containing both of them and this cycle contains no more elements of $C(G,Y)$.

Proof If U is empty then (i) follows immediately, (ii) follows from Corollary 2.2.11 and Lemma 2.4.25. To prove (iii) let $e_i = [w_i, v_i]$, $Y(w_i) < 0$, $Y(v_i) > 0$, $i = 1, 2$ be two edges in $C(G,Y)$. By (ii), there exists a path consisting of negative vertices between w_1 and w_2. Similarly there is path consisting of positive vertices between v_1 and v_2. These two paths along with the two edges e_1 and e_2 give us the simple cycle we were looking for.

Let $U = \{u_1, u_2, \cdots, u_p\}$, $p \geq 1$. First we prove (i). Since G' is connected, L' is irreducible. The graph G' contains an edge $\bar{e}$, say, of $C(G,Y)$. This is because,

G' being the only nonzero component must contain a positive vertex and a negative vertex, by Note 2.1.1. Hence G' must contain a path P joining these two vertices. But this path must contain a characteristic edge or vertex of G. In case $C(G, Y)$ does not contain an edge, the path P (hence G') must contain a characteristic vertex. This is a contradiction to the hypothesis that U is the collection of all characteristic vertices.

Let G_1 be the only component of $G - u_1$, which is nonzero (guaranteed by the statement of the lemma). By the discussion in the above paragraph, $\bar{e} \in G_1$. Let L_1 be the principal submatrix of $\mathcal{L}$ corresponding to G_1 and Y_1 be the corresponding subvector of Y. So Y_1 contains both positive and negative entries. Since $Y(u_1) = 0$, μ is an eigenvalue of L_1 and the corresponding eigenvector is Y_1. Since the matrix L_1 is irreducible, we can use Proposition 2.1.2 to deduce that μ is not the smallest eigenvalue of L_1. Let $\lambda_1^1 \leq \lambda_2^1$ be the two smallest eigenvalues of L_1. Since L_1 is a principal submatrix of $\mathcal{L}$, using Cauchy interlacing theorem we get that $\lambda_2^1 \geq \mu$. Thus $\lambda_2^1 = \mu$.

Let G_2 be the only component of $G_1 - u_2$, which is nonzero (guaranteed by the statement of the lemma). Let L_2 be the principal submatrix of $\mathcal{L}$ corresponding to G_2 and Y_2 be the corresponding subvector of Y. Let $\lambda_1^2 \leq \lambda_2^2$ be the two smallest eigenvalues of L_2. Now one can proceed in a similar way as above to show that $\lambda_2^2 = \mu$. The rest of the proof of (i) is similar.

To prove (ii) suppose that the subgraph induced by the positive vertices is not connected. Then one can proceed in a similar way as in the proof of Lemma 2.2.9 to show that $L' - \mu I$ has at least two negative eigenvalues. But this is impossible by Proposition 2.1.2 and item (i) above. The rest of the proof is similar to the proof of Lemma 2.4.25.

Proof of (iii) is similar to the first paragraph of this proof in view of item (ii) and Note 2.1.1. ∎

Example 2.4.29 The graph G in Figure 2.8 is unweighted, $D = 0$, that is $\mathcal{L} = -A$. One can check with the help of a mathematical package that $\mu = -1.6751$ (rounded to the fourth decimal place). There is a Fiedler vector Y of $\mathcal{L}$ of the form

$$Y^T = \begin{bmatrix} 0 & 0 & 0 & - & + & - & + & + & + & + & - & - & - \end{bmatrix},$$

where + or − sign or 0 at i-th place means that the i-th entry is positive or negative or 0, respectively. Here 3 is the only characteristic vertex. Since $G - \{3\}$ has only

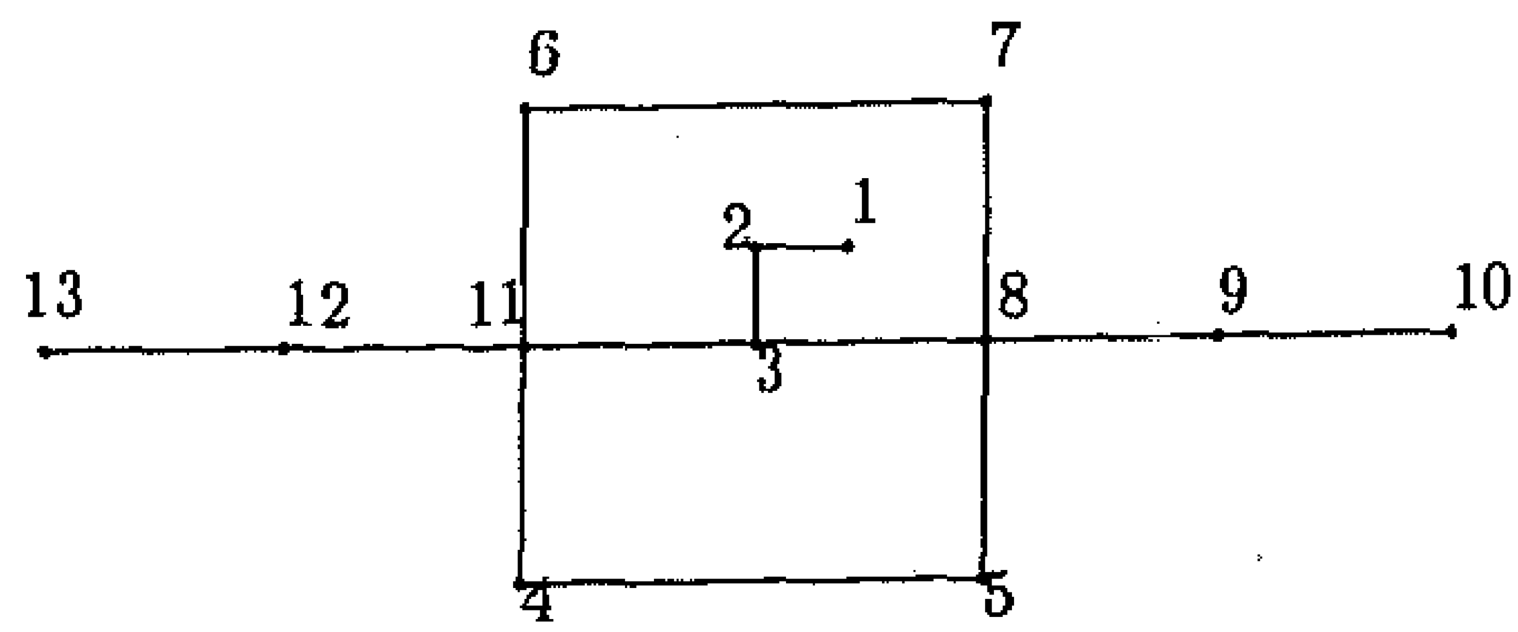


Figure 2.8:

one nonzero component, Lemma 2.4.28 can be applied. One can see that the positive vertices induce a connected subgraph, the negative vertices induce a connected subgraph, $C(G,Y) = \{3, [4,5], [6,7]\}$. Also notice that for any two elements in $C(G,Y)$ there is a simple cycle in G containing these two elements and not containing any other characteristic element. ∎

Definition: Let G be graph. For any vector X, the *support* of X, denoted by $supp(X)$, is the subgraph of G induced by the set of vertices v such that $X(v) \neq 0$. The *positive support*, denoted by $\text{supp}^+(X)$, of X is the subgraph of G induced by the vertices v such that $X(v) > 0$. The *negative support*, denoted by $\text{supp}^-(X)$, is defined similarly.

Definition: Let G be graph and $\mathcal{V}$ be a vector space. A vector $X \in \mathcal{V}$ is said to have a *minimal support* if X is nonzero and for all nonzero vectors $X' \in \mathcal{V}$, $\text{supp}(X')$ is a subgraph of $\text{supp}(X)$ implies that $\text{supp}(X)$ is the same as $\text{supp}(X')$. A vector $X \in \mathcal{V}$ is said to have a *maximal support* if for all vectors $X' \in \mathcal{V}$, $\text{supp}(X)$ is a subgraph of $\text{supp}(X')$ implies that $\text{supp}(X)$ is the same as $\text{supp}(X')$.

Lemma 2.4.30 *Let G be any graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathfrak{L}$ with minimal support among all the Fiedler vectors. Then $supp^+(Y)$ and $supp^-(Y)$ are connected subgraphs of G.*

Proof If $C(G,Y)$ contains an edge then the result follows by Lemma 2.4.28.

Claim If $C(G,Y)$ contains vertices only then we cannot have more than two nonzero branches of G at $C(G,Y)$.

To see this let $B_1, B_2, \cdots, B_r$, $r > 2$ be the nonzero branches. Thus by Theorem 2.2.20, $\text{supp}(Y)$ is the subgraph $\bigcup_{i=1}^{r} B_i$. Let Y_1 and Y_2 be the subvectors of Y corresponding to B_1 and B_2, respectively. Then by Theorem 2.2.20, we know that we can construct a Fiedler vector X such that $\text{supp}^+(X) = B_1$ and $\text{supp}^-(X) = B_2$. Clearly $\text{supp}(X)$ is a proper subgraph of $\text{supp}(Y)$, which contradicts the fact that Y is a Fiedler vector with minimal support. Thus the claim is justified.

Now, if we have only two nonzero branches of G at $C(G, Y)$ then by Theorem 2.2.20, one branch is positive and one is negative, in which case the positive support and the negative support are connected. ∎

Remark: *The above lemma was first proved by van der Holst* [36]. *We have presented the result here because it tries to establish a relationship between the Fiedler vectors and the "minor monotone graph invariants"* [37, 35]. *We will discuss some more on this relationship in Chapter 7.*

Chapter 3

The characteristic set

3.1 Cardinality of the characteristic set

Let G be a connected graph. Define a binary relation on the set of edges of G as follows: edges e_1 and e_2 are equivalent, $e_1 \sim e_2$, if either $e_1 = e_2$ or there is a simple cycle containing both of them. It is easy to see that this defines an equivalence relation. Let $\{G_i\}$ be the collection of subgraphs of G induced by the equivalence classes. Call each of these graphs G_i a *block* of G. The following is one of our main results and reveals a nice property satisfied by the elements of a charactcristic set. Denote by $|S|$, the number of elements in the set S.

Theorem 3.1.31 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of L^D. Let $S = C(G, Y)$ and suppose that $|S| \geq 2$. Then for each pair of characteristic elemcnts there exists a simple cycle which contains them and contains no more characteristic elements.*

Proof First suppose that S contains vertices only, let $v_1, v_2 \in S$. Let us delete all characteristic vertices from G except for v_1. Applying item (i) of Theorem 2.2.20, we get that there is only one nonzero component, say H, of the resulting graph. Let u be a positive vertex adjacent to v_2 and w be a negative vertex adjacent to v_2 (guaranteed by Note 2.1.1). Since both u, w are in H, there is a path, say P, joining them in H. Since G has no characteristic edge, at least one vertex on P has to be a zero vertex. Thus P contains a characteristic vertex. Since all characteristic vertices except for v_1 have been deleted, v_1 is the only characteristic vertex on P. Thus v_1 is the only zero vertex on P. Note that the edges $[w, v_2]$ and $[v_2, u]$ along with the

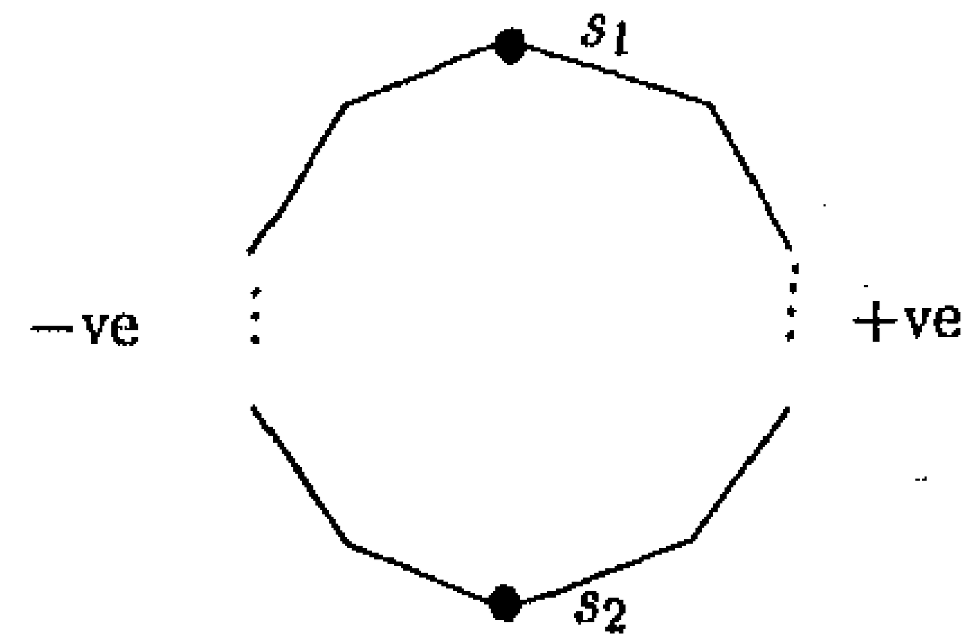


Figure 3.1:

path P form a simple cycle. This is the cycle we were looking for.

In case S contains at least one edge, the proof follows from Note 2.4.27 and item (iii) of Lemma 2.4.28. ∎

Remark 3.1.32 *It is clear from the proof of Lemma 2.4.28 and Theorem 3.1.31, that if s_1 and s_2 are two characteristic elements in $C(G, Y)$ then the simple cycle we were talking of in Theorem 3.1.31 has the structure shown in Figure 3.1.*

As an immediate corollary we have the following which generalizes a part of the Theorem (3,12), of Fiedler [14].

Corollary 3.1.33 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$ and $S = C(G,Y)$. Then either $|S| = 1$ and S contains a vertex* or *S is contained in a block of G.*

Proof Suppose that $S \neq \{v\}$, for some vertex v. If $S = \{e\}$ for some edge e, then we have nothing to prove, since e itself is a block. Suppose that S has at least two elements. Then by Theorem 3.1.31, we see that for any two elements in S, there exists a simple cycle in G containing both of them. Thus a block which contains one element of S must contain every element of S. ∎

Let B be a block of a connected graph G. If T is a spanning tree of G then a *chord of G with respect to T* is an edge of G, not in T. It is well-known that the number of chords of a connected graph is $m - n + 1$, where m and n are the number of edges and vertices in the graph. The next theorem, in which we obtain a bound on the cardinality of the characteristic set, is another main result of this chapter.

Theorem 3.1.34 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$ and $S = C(G, Y)$. Suppose that S lies in the block $\mathcal{B}$. Then $1 \leq |S| \leq \mathcal{N}_{\mathcal{B}} + 1$, where $\mathcal{N}_{\mathcal{B}}$ is the number of chords in $\mathcal{B}$ (visualizing $\mathcal{B}$ as a graph itself).*

Proof If $|S| = 1$, there is nothing to prove. Let $S = \{s_1, s_2, \cdots, s_r\}$, $r > 1$. By Theorem 3.1.31, we know that for any two elements of S there is a simple cycle in G which contains these two elements and contains no more elements of S. Denote by $\Gamma_{i,r}$ a cycle of the above type which contains s_i, s_r; $i = 1, \cdots, r-1$. From the definition of a block it is clear that these cycles are contained in $\mathcal{B}$. For $i = 1, 2 \cdots, r-1$, define

$$e_i = \begin{cases} s_i, & \text{if } s_i \text{ is an edge,} \\ \text{the edge on } \Gamma_{i,r}, \text{ joining} & \\ s_i \text{ and a positive vertex,} & \text{if } s_i \text{ is a vertex.} \end{cases}$$

Observe that e_i is well defined in view of Remark 3.1.32. Let us delete the edge e_1 from $\mathcal{B}$ to obtain $\mathcal{B}_1$. Note that none of the cycles $\Gamma_{i,r}$, $i = 2, \cdots, r-1$ contain e_1, because otherwise, they have to contain s_1, which is not possible (by Theorem 3.1.31). Let us delete the edge e_2 from $\mathcal{B}_1$ to obtain $\mathcal{B}_2$. None of the cycles $\Gamma_{i,r}$, $i = 3, \cdots, r-1$ contain e_2, because otherwise, they have to contain s_2, which is not possible (by Theorem 3.1.31). Thus repeating this process some more times, we conclude that the deletion of $e_1, \cdots, e_{r-1}$ will result in the graph, say $\mathcal{B}_{r-1}$, which is connected (because each time we are deleting an edge from a cycle only). Let T_{r-1} be a spanning tree of $\mathcal{B}_{r-1}$, thus of $\mathcal{B}$. The edges $e_1, \cdots, e_{r-1}$ are chords of $\mathcal{B}$ with respect to T_{r-1}. Hence $r - 1 \leq \mathcal{N}_{\mathcal{B}}$ and the proof is complete. ∎

One can see some interesting consequences of the above result.

Corollary 3.1.35 *Let G be a connected graph with n vertices and m edges and D be a diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$ and $S = C(G, Y)$. Then $1 \leq |S| \leq m - n + 2$. In particular if G is a tree then $|S| = 1$ and if G is unicyclic then $|S|$ is either 1 or 2.*

Proof By Corollary 3.1.33, either $|S| = 1$ or S is contained in a block $\mathcal{B}$ of G. Since $\mathcal{N}_{\mathcal{B}} \leq \mathcal{N}_G = m - n + 2$, the result follows from Theorem 3.1.34. The rest of the proof is trivial. ∎

As one can notice from Corollary 3.1.35 that, in case of a tree the bound $m-n+2$ is achieved. But this is not a typical example in the sense that the cardinality of the

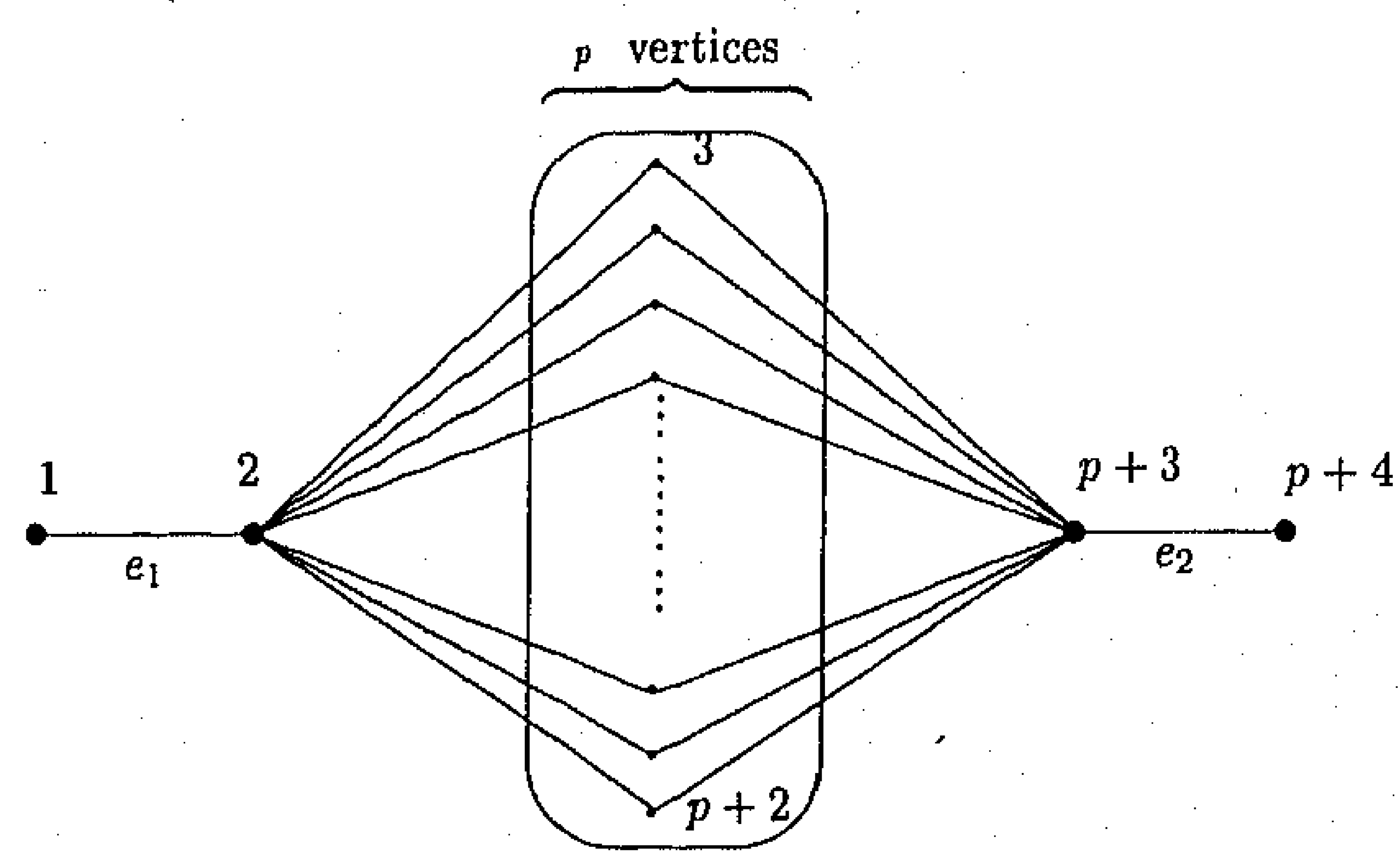


Figure 3.2:

characteristic set is 1, a small number. Let us put the following question. *Given a natural number p, is it possible to find a connected graph G and a diagonal matrix D such that the cardinality of the characteristic set is* $m - n + 2 = p$, *where m and n are the number of edges and vertices in G, respectively ?* In the following example we show that the answer is "yes".

Example 3.1.36 The graph G in Figure 3.2 is unweighted and we consider the Laplacian matrix L of G. Let $U = \{3, 4 \cdots, p+2\}$. Let Y be a Fiedler vector for G. Let $S = C(G,Y)$. By Note 2.1.1, the vertices 1 and $p+4$ cannot be characteristic vertices. We claim that $2 \notin S$. If possible, let S contain the vertex 2. Let $G_1 = \{1\}, G_2$ be the two components of $G - \{2\}$ and L_1, L_2 be the two principal submatrices of L corresponding to them, respectively. Observe that $\tau(L_1) = 1$ and L_2 has a diagonal entry 1. Thus $\tau(L_2) < 1$, by Lemma 2.1.6. Thus by Lemma 2.2.13,

$$1 = \tau(L_1) > \mu > \tau(L_2). \tag{3.1.1}$$

We have the following two cases.

Case 1: $Y(1) \neq 0$. By Note 2.1.1, the components G_1 and G_2 are nonzero. By Theorem 2.2.20, $\tau(L_1) = \tau(L_2)$, a contradiction to Equation 3.1.1.

Case 2: $Y(1) = 0$. Thus the only nonzero component of $G - \{2\}$ is G_2. Thus S contains at least one more element. By Note 2.1.4, $G-S$ has at least two nonzero components say, C_1 and C_2. Clearly these two components are two nonzero components of $G_2 - [S-\{2\}]$. Note that G_2 is nothing but $K_{p+1,1}$, the star on $p+2$ vertices. So one of C_1 and C_2 must constitute of only one vertex, say $C_1 = \{x\}$. Then applying Theorem 2.2.20 and Lemma 2.4.28 and using the fact that the degree of x is at least one, we get that $\mu \geq \tau(L(C_1)) \geq 1$, a contradiction to Equation 3.1.1. Thus the claim is justified.

Similarly, one can show that $p+3 \notin S$. Showing $S \neq \{e_i\}$, $i = 1, 2$ is trivial in view of Theorem 2.4.25 and Equation 3.1.1.

By Corollary 3.1.33, S lies in the block B, where B is the subgraph induced by the vertices $2, 3, \cdots, p+3$. Let $S' = \{[2,i], [i,p+3] \,:\, i \in U\} \bigcup U$. So $S \subset S'$. One can see that we have to delete at least p elements of S' from G to make the resulting graph disconnected. So in view of Note 2.1.4, we conclude that the cardinality of the characteristic set must be p. ∎

In the above example, we could have proved the same by computing the Fiedler vector of L. We preferred to do it in the way it is presented, because it illustrates an application of the previous results.

Let us put another question. *Let S_1 and S_2 be two characteristic sets of a graph $\mathcal{G}$ with respect to two Fiedler vectors. Is it necessary that both the sets have the same cardinality ?* The answer is **no** even if the graph is regular. This can be seen from the following example.

Example 3.1.37 Here we consider the Laplacian matrix of the complete unweighted graph on four vertices (see Figure 3.3). Two Fiedler vectors Y_1 and Y_2 are given with entries having value correct up to four decimal places. One can see that the characteristic sets are not of the same cardinality. ∎

3.2 Location of the characteristic set

It has been shown by Merris ([25], Theorem 2) that in the case of a tree, for the Laplacian case that if Fiedler vectors Y and Y' give rise to characteristic vertices u and u' respectively, then $u = u'$. The generalization of this result to connected graphs, for the Laplacian case was done in ([21], Theorem 1). Below we give a more general version of the same result which is valid for the perturbed Laplacian matrices.

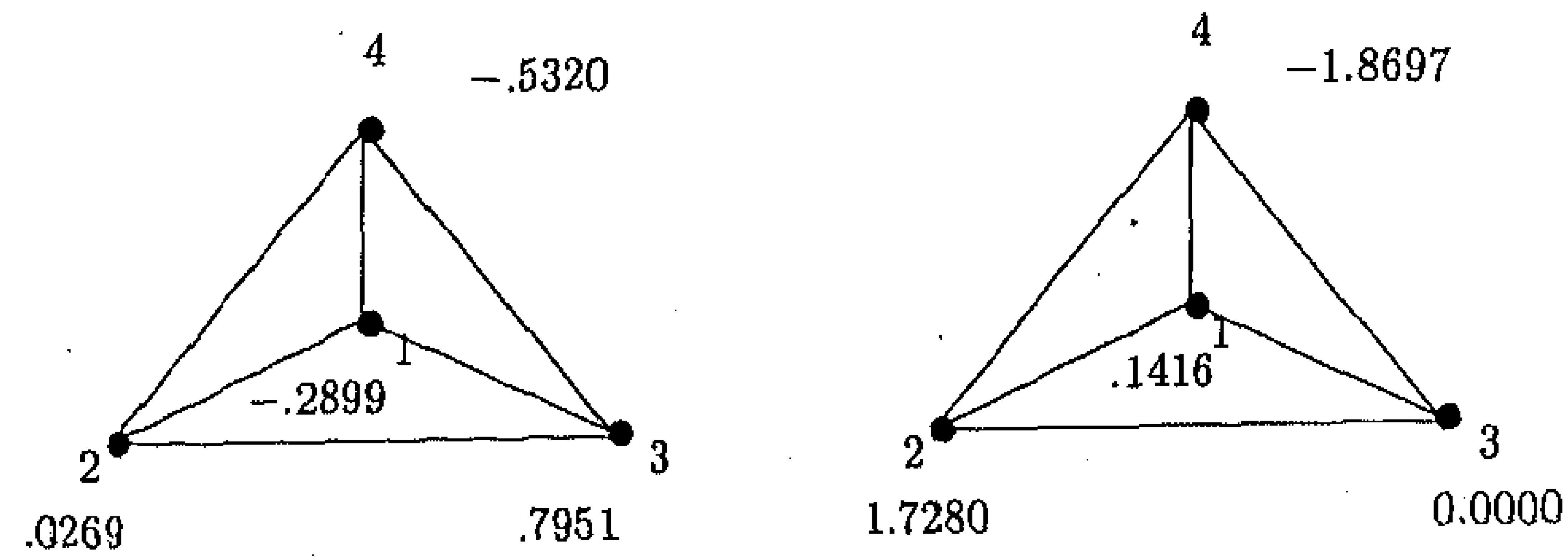


Figure 3.3:

Lemma 3.2.38 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$ and suppose that $\mathcal{C}(G,Y) = \{u\}$, for some vertex u. Then for any Fiedler vector Y' of $\mathcal{L}$, $\mathcal{C}(G,Y') = \{u\}$.*

Proof Let $S = \mathcal{C}(G,Y)$ and $S' = \mathcal{C}(G,Y')$. Consider the Fiedler vector Y. By Note 2.1.4, $G - u$ is disconnected. Let $G_1, G_2, \cdots, G_m$ be the branches at u and $L_1, L_2, \cdots, L_m$ be the respective principal submatrices of $\mathcal{L}$. We have the following cases.

Case 1. S' contains u as a characteristic vertex and $S' - \{u\} \neq \phi$.

Step 1: Without loss of generality we can assume, $S' - \{u\} \subset G_1$. If not, let $s_1 \in G_1$ and $s_2 \in G_2$, where $s_1, s_2 \in S' - \{u\}$. Then by Theorem 3.1.31, there is a simple cycle containing s_1 and s_2. But then this cycle must contain u. Thus $G - u$ cannot be disconnected, which is not true.

Step 2: Applying Theorem 2.2.20 we get that G_1 is the only nonzero component of $G - u$ with respect to Y'. By Note 2.1.4, Theorem 2.2.20, Lemma 2.4.28 there exist at least two Perron branches of G at S'. So each of these Perron branches are proper subgraphs of G_1. Let H be one of these Perron branches and L_H be the corresponding principal submatrix of $\mathcal{L}$. Then $\tau(L_H) \leq \mu$, since H is a Perron branch. Thus $\tau(L_1) < \tau(L_H) \leq \mu$, by Lemma 2.1.6. But this is a contradiction to Theorem 2.2.20, if we consider the Fiedler vector Y.

Case 2. S' contains u as the end vertex of a characteristic edge $[u, z]$.

Let $z \in G_1$. Then arguing as in the first paragraph of Case 1, we get that $S' - [u, z] \in G_1$. Thus we have a Perron component H of G at S' such that H is a subgraph of G_1. Let L_H be the principal submatrix of $\tilde{L}$ corresponding to H. So by Lemma 2.4.28 and Lemma 2.1.6, we conclude that $\tau(L_1) \leq \tau(L_H) < \mu$. But this is a contradiction to Theorem 2.2.20, if we consider the Fiedler vector Y.

Case 3. S' does not contain u at all.

This case is similar to the above cases.

So we conclude that $S' = \{u\}$. ∎

The following is an immediate corollary to Lemma 3.2.38 and Theorem 2.3.23.

Corollary 3.2.39 *Let G be a connected graph, D be any diagonal matrix. Let Y be a Fiedler vector of $\tilde{L}$. Suppose that $C(G, Y) = \{u\}$, where u is any vertex of G. Suppose that $G - u$ has p Perron branches. Then the multiplicity of the algebraic connectivity is $p - 1$.* ∎

Note 3.2.40 *Let G be a connected graph and D be a diagonal matrix. Let Y be a Fiedler vector of $\tilde{L}$. Then, except for the block containing the characteristic set, each of the other blocks are either zero blocks or positive blocks or negative blocks (otherwise some other block will contain a characteristic element, thus contradicting Corollary 3.1.33).*

The following theorem which gives us further information about the location of the characteristic set was also proved in [21], in the case of Laplacian matrices.

Theorem 3.2.41 *Let G be a connected graph and D be a diagonal matrix. Let $\tilde{L}$ be the perturbed Laplacian matrix of G and Y, Y' be two independent Fiedler vectors of $\tilde{L}$. Let $S = C(G, Y)$, $S' = C(G, Y')$ and $S \neq \{u\}$ for some vertex u. Then S and S' lie in the same block of G.*

Proof By Lemma 3.2.38, S' cannot comprise of a single vertex and no edges. Suppose that S and S' lie in two different blocks B and B', respectively. If H is a subgraph of G then Y_H will denote the subvector of Y corresponding to the vertices in H. We consider the following cases.

Case 1: $Y_{B'} = 0$ and $Y'_B > 0$.

Note that Y_B has a nonzero entry. So we can take a linear combination of Y and Y' to get a vector Z such that Z_B has a zero entry. Since $Z_{B'} = Y'_{B'}$, so by Corollary 3.1.33, we conclude that $C(G, Z)$ lies in B'. Thus $Z_B = 0$, by Note 3.2.40. Thus Y_B is a scalar multiple of $Y'_B > 0$, which is a contradiction to the fact that B contains $C(G, Y)$. So this case is not possible.

Case 2: $Y_{B'} = 0, Y'_B = 0$.

This case is clearly not possible, because if $Z = \alpha Y + \beta Y'$, $\alpha\beta \neq 0$, then $C(G, Z) = S \bigcup S'$, which obviously does not lie in one block, contradicting Corollary 3.1.33.

Case 3: $Y_{B'} > 0, Y'_B > 0$.

We have two subcases.

Subcase 1: The set S has vertices only.

In this case consider $G - S$. Since $Y_{B'} > 0$, the deletion of S form G will make no change to the block B'. Thus B' being connected must be contained in one of the connected components of $G - S$. By Theorem 2.2.20, there exists a component G_1 of $G - S$ containing B' such that $\tau(L_1) = \mu$, where L_1 is the principal submatrix of $\mathfrak{L}$ corresponding to G_1. Again by Theorem 2.2.20 and by Lemma 2.4.28, there exists a component G_2 of $G - S'$, which does not contain any vertex of B, such that $\tau(L_2) \leq \mu$, where L_2 is the principal submatrix of $\mathfrak{L}$ corresponding to G_2. Thus G_2 is a subgraph of G_1 with at least one vertex less (since G_2 does not contain the vertices in S' and the positive end vertices of the edges in S'). Now, by Note 2.1.4, $\tau(L_1) < \tau(L_2)$ and thus we get a contradiction to the fact that $\tau(L_1) = \mu$.

Subcase 2: Both S and S' contain edges.

Delete S from G to obtain two connected components G_1 and G_2 of negative and positive vertices, respectively (by Lemma 2.4.28). Let L_1 and L_2 be the principal submatrices of $\mathfrak{L}$ corresponding to G_1 and G_2, respectively. By Lemma 2.4.28, $\tau(L_1) < \mu$ and $\tau(L_2) < \mu$. Note that we have deleted vertices and edges from block B and $Y_{B'} > 0$. So one of G_1 and G_2 contains B', say G_2.

Now delete from G all positive end vertices of the characteristic edges of S and all characteristic vertices of S to get the graph G^*. Clearly one connected component of G^* is G_1. There must be another component which intersects with B', say G_3. Let L_3 be the principal submatrix of $\mathfrak{L}$ corresponding to G_3. Since we already know that $\tau(L_1) < \mu$, so by Lemma 2.2.13,

$$\tau(L_3) > \mu. \tag{3.2.2}$$

Now, we can proceed as in paragraph one of this Subcase, to show that there exists a connected component G_4 (the one which does not contain $\mathcal{B}$) of $G - S'$, which is a subgraph of G_3. Let L_4 be the principal submatrix of $\mathcal{L}$ corresponding to G_4. Then we have

$$\tau(L_3) \leq \tau(L_4). \tag{3.2.3}$$

But by Lemma 2.4.28,

$$\tau(L_4) < \mu. \tag{3.2.4}$$

Equations 3.2.2, 3.2.3, 3.2.4, together lead to a contradiction. That completes the proof of the theorem. ∎

As an immediate corollary we have the following.

Corollary 3.2.42 *Let G be a connected graph and D be a diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$ such that $C(G,Y) = \{[u,v]\}$, for some edge $[u,v]$ of G. Then the multiplicity of μ is one.*

Proof If not, let Y' be another Fiedler vector of G. Assume without loss that Y' has a zero entry (otherwise take a suitable linear combination of Y and Y'). By Theorem 3.2.41, there is one block containing both these characteristic sets. From the fact that $C(G,Y) = \{[u,v]\}$ we can conclude that the characteristic block is the characteristic edge itself, because, if this block contains a cycle then it must contain at least one more characteristic element, which is not the case. Thus we get that $C(G,Y') = \{[u,v]\}$. But Y' certainly contains a characteristic vertex, a contradiction. ∎

A further corollary to Corollary 3.2.42 can be the well-known fact (see for example, [13],) that if G is a tree and if Y is a Fiedler vector of the Laplacian matrix of G containing no zero entry, then the multiplicity of μ as an eigenvalue of L is one.

Chapter 4

Eigenvectors of a tree

4.1 Fiedler vectors

In this section we use a new technique for proving the monotony and convexity/concavity property satisfied by the coordinates of a Fiedler vector of a perturbed Laplacian matrix related to a tree. We illustrate the application of the technique by proving some well-known results for trees.

Let T be a directed tree on n vertices $1, 2, \cdots, n$ and $n-1$ edges $e_1, e_2, \cdots, e_{n-1}$. If $e = [u, v]$ is an edge of the tree, then the graph $T - e$, obtained by removing e from T has two components. Denote by $G_h(e, T)$, the component which contains the head (that is the terminating vertex, which is v) of e.

The next result has been observed in [22], in the case of algebraic connectivity of a Laplacian matrix. Here we use the technique and the result is applicable for any eigenvalue of a perturbed Laplacian matrix.

Lemma 4.1.43 *Let T be a directed tree, D any diagonal matrix. Let U be an eigenvector of $\mathcal{L}$ corresponding to the eigenvalue ν (not the smallest). Let Z be the eigenvector of $\mathcal{L}$ corresponding to the smallest eigenvalue. Let $e_{ij} = [i, j]$ be an edge of T with weight θ_{ij}. Then*

$$\theta_{ij}[Z(j)U(i) - Z(i)U(j)] = [\tau(\mathcal{L}) - \nu] \sum_{s \in G_h(e_{ij}, T)} Z(s)U(s). \tag{4.1.1}$$

Proof Recall that the edge $e_{ij} = [i, j]$ has j as its terminating vertex. Put $L = \mathcal{L} - \tau(\mathcal{L})I$. So Z is the vector corresponding to the eigenvalue 0 of the matrix L. By Proposition 2.1.2, Z is positive and unique up to a scalar multiple. Let P be the vector such that $P(s) = Z(s)$, if $s \in G_h(e_{ij}, T)$ and $P(s) = 0$, else. Thus we have

$$P^T LU = P^T(LU) = [\nu - \tau(\mathbb{L})]P^T U. \tag{4.1.2}$$

Let $X = P^T L$. Then

$$\begin{aligned} X(j) &= \sum_{k=1}^{n}[P(k)L(k,j)] \\ &= \sum_{k \in G_h(e_{ij},T)} [Z(k)L(k,j)] \\ &= \sum_{k \in G_h(e_{ij},T)} [Z(k)L(k,j)] + Z(i)L(i,j) - Z(i)L(i,j) \\ &= -Z(i)L(i,j), \end{aligned}$$

as $\sum_{k \in G_h(e_{ij},T)} [Z(k)L(k,j)] + Z(i)L(i,j) =$ the j-th entry of ZL which is 0.

Thus $X(j) = -L(j,i)Z(i)$. Similarly, we can show that $X(i) = L(i,j)Z(j)$ and $X(u) = 0$, if $u \notin \{i,j\}$. Note that $-L(i,j)$ is the weight of the edge $[i,j]$. Thus,

$$P^T LU = (P^T L)U = \theta_{ij}[Z(i)U(j) - Z(j)U(i)]. \tag{4.1.3}$$

Note also that

$$P^T U = \sum_{s} Z(s)U(s) = \sum_{s \in G_h(e_{ij},T)} Z(s)U(s). \tag{4.1.4}$$

Thus the proof is complete in view of Equations 4.1.2, 4.1.3 and 4.1.4. ∎

To illustrate an application of Lemma 4.1.43, we now deduce one of our main results, which generalizes a classical result of Fiedler (Theorem (3,14) of [14]) to the class of perturbed Laplacian matrices.

Theorem 4.1.44 *Let T be a tree with vertices $1, 2, \cdots, n$ and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathbb{L}$ and Z be the eigenvector of $\mathbb{L}$ corresponding to $\tau(\mathbb{L})$. Let*

$$\frac{Y}{Z} = \left[\begin{array}{cccc} \frac{Y(1)}{Z(1)} & \frac{Y(2)}{Z(2)} & \cdots & \frac{Y(n)}{Z(n)} \end{array} \right]^T.$$

Then one of the following cases occur.

(a) No entry of Y is zero. In this case, there is a unique pair of vertices i and j such that i and j are adjacent in T with $Y(i) > 0$ and $Y(j) < 0$. Further, the entries of $\frac{Y}{Z}$ increase along any path in T which starts at i and does not contain j, while the entries of $\frac{Y}{Z}$ decrease along any path in T which starts at j and does not contain i.

(b) *Some entry of Y is zero. In this case the subgraph of T induced by the set $\{v : Y(v) = 0\}$ is connected. Moreover, there is a unique vertex k such that $Y(k) = 0$ and k is adjacent to a vertex m such that $Y(m) \neq 0$. Along any path in T which starts at k, the entries of $\frac{Y}{Z}$ satisfy one of the following:*

(i) positive and increasing,

(ii) negative and decreasing,

(iii) identically zero.

Proof First we prove the case (a). In view of Corollary 3.1.35, we know that $C(T, Y)$ has cardinality 1. So there is only one characteristic edge, $e = [i, j]$ with, say, $Y(i) > 0$. Consider a path P which starts from i and does not contain j. By Lemma 2.2.9, this path lies in the positive component of $T - e$. Let $e' = [u, v]$ be any edge on P. Assume that the distance of the vertex v from i is one plus the distance of the vertex u from i. Get a directed tree by orienting each edge such that the head of the edge $[u, v]$ is v. Note here that the matrix $\hat{E}$ does not change if the orientations of the edges change. By Lemma 4.1.43, we get that

$$\theta_{uv}[Z(u)Y(v) - Z(v)Y(u)] = [\mu - \tau(\hat{E})] \sum_{s \in G_h(e',T)} Z(s)Y(s).$$

Since $G_h(e', T)$ is positive with respect to Y, we have

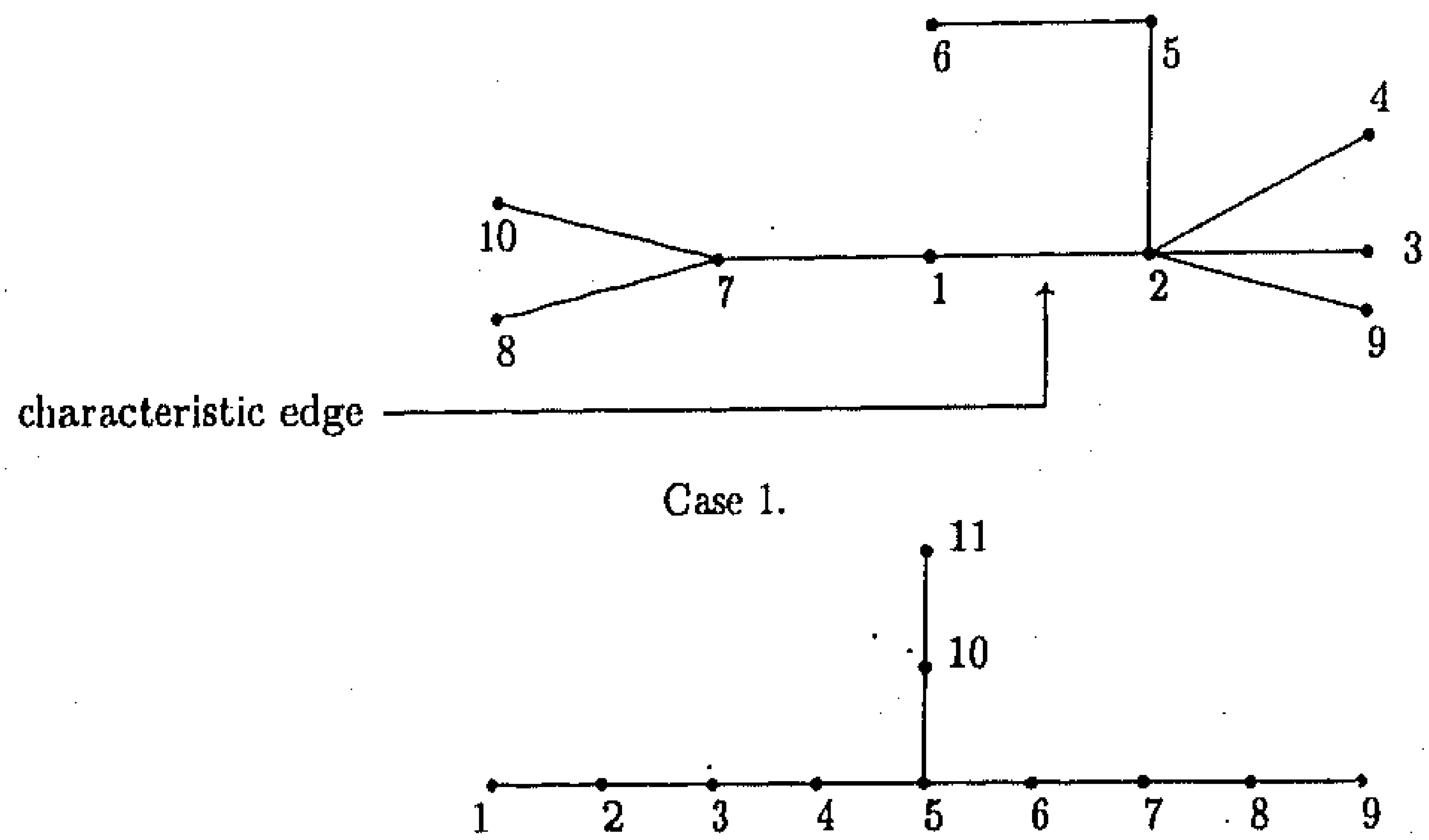
$$\sum_{s \in G_h(e',T)} Z(s)Y(s) > 0.$$

Thus

$$Z(u)Y(v) - Z(v)Y(u) > 0.$$

Rest of the proof of the case (a) is routine.

Next we prove the case (b). In view of Corollary 3.1.35, $C(T, Y)$ has cardinality 1. So there is only one characteristic vertex, say, k. The graph $T - k$, obtained by deleting k from T has at least two nonzero components. An application of Theorem 2.2.20 gives that the zero vertices induce a connected subgraph. Again in view of Theorem 2.2.20, a path starting from k is either a zero path or a positive path (except for the starting vertex) or a negative path (except for the starting vertex). Rest of the proof of the case (b) is similar to that of the case (a). ∎



Case 2

Figure 4.1:

Example 4.1.45 Here we give two illustrations of Theorem 4.1.44.
Case 1: Negative adjacency matrix and characteristic edge.

See Figure 4.1 (Case 1). Labels of vertices are written near the vertices. The weights are given according to the following description :
$\theta_{10,7} = 4$, $\theta_{8,7} = 1$, $\theta_{7,1} = 7$, $\theta_{1,2} = 6$, $\theta_{2,3} = 8$, $\theta_{2,4} = 7$, $\theta_{2,9} = 1$, $\theta_{2,5} = 9$ and $\theta_{5,6} = 5$. Here $\mathcal{L} = -A$, $\tau(-A) = -15.7994$, $\mu = -7.6442$.

$Z = [\ .3255\ \ .6765\ \ .3426\ \ .2997\ \ .4283\ \ .1355\ \ .1547\ \ .0098\ \ .0428\ \ .0392\ ]^T.$

$Y = [\ .5241\ \ -.1219\ \ -.1276\ \ -.1117\ \ -.2509\ \ -.1641\ \ .6769\ \ .0886\ \ -.0160\ \ .3542\ ]^T.$

$\frac{Y}{Z} = [\ 1.6104\ \ -.1803\ \ -.3726\ \ -.3726\ \ -.5859\ \ -1.2110\ \ 4.3744\ \ 9.0412\ \ -.3726\ \ 9.0412\ ]^T.$

Case 2: Negative adjacency matrix and characteristic vertex.

This tree (Figure 4.1, Case 2) is unweighted. The labels of vertices are written near the vertices. The vertex 5 is the characteristic vertex. Here $\mathcal{L} = -A$, $\tau(-A) = -2.0743$ and $\mu = -1.618$. The vectors Z, Y and $\frac{Y}{Z}$ are given below.

$Y = \left[\ -.2629\ \ -.4253\ \ -.4253\ \ -.2629\ \ 0\ \ .2629\ \ .4253\ \ .4253\ \ .2629\ \ 0\ \ 0\ \right]^T.$

$Z = \left[\ .0837\ \ .1735\ \ .2763\ \ .3996\ \ .5526\ \ .3996\ \ .2763\ \ .1735\ \ .0837\ \ .3470\ \ .1673\ \right]^T.$

$\frac{Y}{Z} = \left[\ -3.1423\ \ -2.4511\ \ -1.5394\ \ -.6578\ \ 0\ \ .6578\ \ 1.5394\ \ 2.4511\ \ 3.1423\ \ 0\ \ 0\ \right]^T.$ ∎

As a corollary to Theorem 4.1.44 one gets the unification of the result of Fiedler and a convexity/concavity statement obtained in ([22], Theorem 6), for the Laplacian matrix.

Corollary 4.1.46 *Let T be a tree with vertices $1, 2, \cdots, n$. Let L be the Laplacian matrix of T and Y a Fiedler vector. Then one of the following cases occur.*

(a) No entry of Y is zero. In this case, there is a unique pair of vertices i and j such that i and j are adjacent in T with $Y(i) > 0$ and $Y(j) < 0$. Further, the entries of Y increase and are concave down along any path in T which starts at i and does not contain j, while the entries decrease and are concave up along any path in T which starts at j and does not contain i.

(b) Some entry of Y is zero. In this case the subgraph of T induced by the set of vertices corresponding to the $0's$ in Y is connected. Moreover, there is a unique vertex k such that $Y(k) = 0$ and k is adjacent to a vertex m such that $Y(m) \neq 0$. The entries of Y either increase and are concave down, decrease and are concave up or are identically zero along any path in T which starts at k.

Proof In view of Theorem 4.1.44, we only prove the concavity/convexity part of the statement of the case (a). Note that the smallest eigenvalue of L is zero and the corresponding eigenvector has each entry equal to one. Let $e' = [u, v]$ be any edge on a path P which starts from i and does not contain j. Thus, assuming that the distance between vertices v and i is more than the distance between u and i, it follows from Theorem 4.1.44 that

$$[Y(v) - Y(u)] = \mu \sum_{s \in G_h(e', T)} Y(s). \tag{4.1.5}$$

Let $e'' = [v, w]$ be another edge on that path and assume that the distance between the vertices w and i is more than the distance between the vertices v and i. Then in a similar manner as above we can show that

$$Y(w) - Y(v) = \mu \sum_{s \in G_h(e'', T)} Y(s) > 0. \tag{4.1.6}$$

Since all the vertices of $G_h(e'', T)$ are vertices in $G_h(e', T)$ and since $G_h(e', T)$ has at least one extra vertex, namely v, we conclude from Equations 4.1.5 and 4.1.6, that

$$Y(v) - Y(u) > Y(w) - Y(v).$$

Proof of the case (b) is similar. ∎

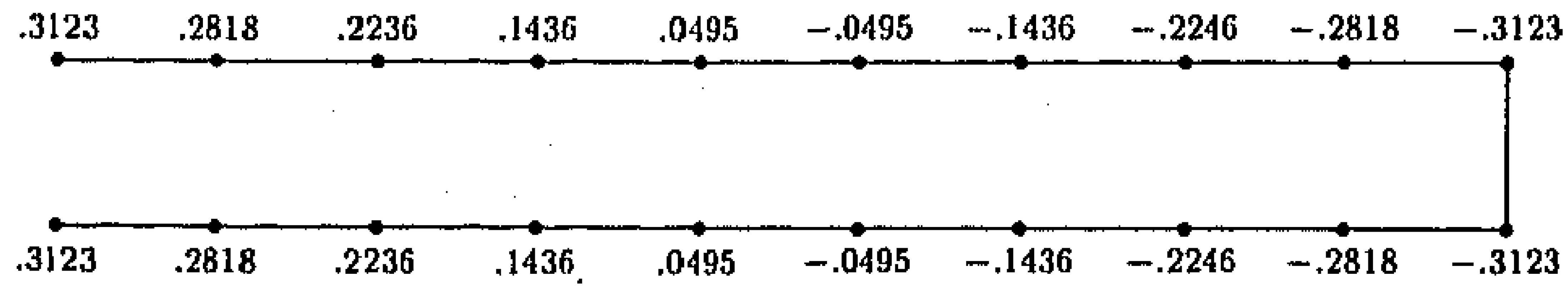


Figure 4.2:

Remark 4.1.47 *The concavity/convexity part of the statement of the above corollary is false for some of the perturbed Laplacian matrices. In Figure 4.1, Case 1, one can verify this considering the path* $[1, 7, 8]$.

Using Lemma 4.1.43, we can get the following result concerning eigenvectors corresponding to any eigenvalue (not necessarily the algebraic connectivity but not the smallest) of the Laplacian of a tree. The proof being similar to the proof of Theorem 4.1.44, is omitted.

Proposition 4.1.48 *Let T be a tree with vertices $1, 2, \cdots, n$. Let L be the Laplacian matrix of T, λ an eigenvalue (not the smallest) of L, and Z an eigenvector corresponding to λ. Let $e = [i, j]$ be an edge of T and T_i be the component of $T - e$ containing i. Suppose that $Z(k) > 0$ for every $k \in T_i$. Then along any path starting from j and containing i, the coordinates of Z increase and concave down.* ∎

We illustrate this fact by an example.

Example 4.1.49 Let T be the unweighted path on 20 vertices. Then $\lambda = .0979$ (rounded to four decimal places) is the third smallest eigenvalue of the Laplacian. The coordinates of the corresponding eigenvector are shown in Figure 4.2. ∎

4.2 Fiedler 3-vectors

In this section, we give a complete description of a Fiedler 3-vector of the Laplacian for an unweighted tree. To prove the main result we need to have some more information on the Fiedler s-vectors, $s > 1$.

An $n \times n$ matrix A will be called *acyclic* if it is symmetric and if for any mutually distinct indices $k_1, k_2, \cdots, k_s$ $(s \geq 3)$ in $\{1, 2, \cdots, n\}$ the equality

$$A(k_1, k_2)A(k_2, k_3) \cdots A(k_s, k_1) = 0$$

is fulfilled. Thus one can see that the Laplacian matrix of a tree is acyclic.

Suppose that Y is a Fiedler s-vector of $L(T)$, $s > 1$, where T is an unweighted tree. Let us put the following question. *How many characteristic elements can $C(T,Y)$ have ?* The following proposition which is due to Fiedler (a part of theorem (2,3) of [13]), helps us to get an answer to it.

Proposition 4.2.50 *Let A be a $n \times n$ acyclic matrix. Let Y be an eigenvector of A corresponding to an eigenvalue λ. Denote by ω_+ and ω_-, respectively, the number of eigenvalues of A greater than and less than λ. Let there be no "isolated" zero coordinate of Y, that is coordinate $Y(k) = 0$ such that $A(k,j)Y(j) = 0$ for all j. Then*

$$\omega_+ = a^+ + r, \qquad \omega_- = a^- + r, \tag{4.2.7}$$

where r is the number of zero coordinates of Y, a^+ is the number of those unordered pairs (i,k) for which

$$A(i,k)Y(i)Y(k) < 0$$

and a^- is the number of those unordered pairs (i,k), $i \neq k$, for which

$$A(i,k)Y(i)Y(k) > 0.$$

■

The following is an immediate corollary.

Corollary 4.2.51 *Suppose that T is an unweighted tree and L its Laplacian matrix. Let Y be a Fiedler s-vector ($s > 1$) and λ be the corresponding eigenvalue. Then the number of characteristic elements in $C(T,Y)$ is at most $s-1$.*

Proof We know that L is an acyclic matrix. In this context it is clear that an "isolated" zero coordinate of Y means a zero vertex of T which is not a characteristic vertex. Here $\omega_- = s-1$. Let $\bar{T}$ be the graph obtained from T by deleting those zero vertices of T which are not characteristic vertices. Let $\bar{L}$ and $\bar{Y}$ be the principal submatrix of L and the subvector of Y, respectively, corresponding to $\bar{T}$. It is clear that

$$C(T,Y) = C(\bar{T},\bar{Y}). \tag{4.2.8}$$

Also note that $\bar{L}$ is an acyclic matrix and λ is an eigenvalue corresponding to the eigenvector $\bar{Y}$. Since $\bar{L}$ is a principal submatrix of L,

$$\bar{\omega}_- \leq \omega_- = s-1, \tag{4.2.9}$$

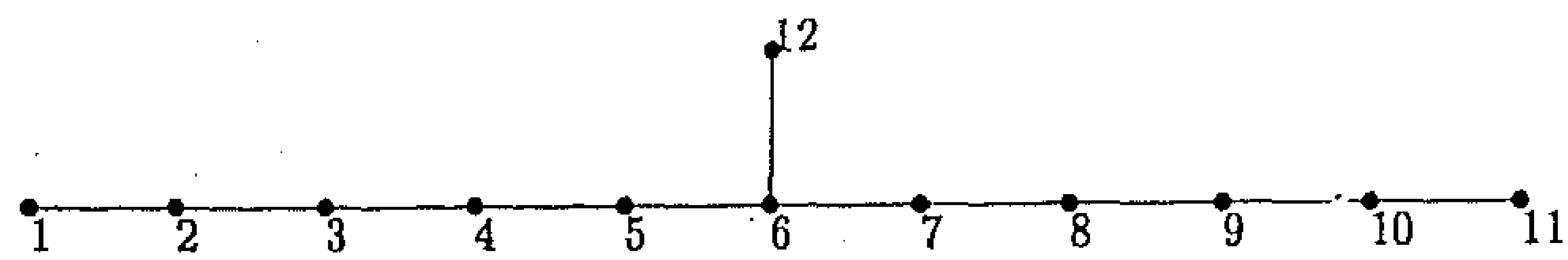


Figure 4.3:

where $\bar{\omega}_-$ is the number of eigenvalues of $\bar{L}$ less than λ. By Proposition 4.2.50,

$$\bar{\omega}_- = a^- + r, \tag{4.2.10}$$

where r is the number of zero coordinates of $\bar{Y}$, that is the number of characteristic vertices in $C(\bar{T}, \bar{Y})$ and a^- is the number of those unordered pairs (i,k), $i \neq k$, for which $\bar{L}(i,k)\bar{Y}(i)\bar{Y}(k) > 0$, that is a^- is the number of characteristic edges in $C(\bar{T}, \bar{Y})$. The result now follows in view of the above three equations. ∎

We have already seen in Example 2.1.5, a tree T and a Fiedler 3-vector Y of the Laplacian matrix such that the number of characteristic elements in $C(T,Y) = 1$, which is strictly less than $s - 1 = 2$, since $s = 3$ in that example. In that example the multiplicity of the 3rd smallest eigenvalue was 2. It has been proved by Fiedler (Corollary (2,5), [13],) that "*if T is an unweighted tree, L the Laplacian matrix and Y an eigenvector corresponding to the s-th smallest eigenvalue such that each entry of Y is different from zero, then the number of characteristic elements in $C(T,Y)$ is $s - 1$ and the corresponding eigenvalue is simple*". Thus one might get curious to know whether "*the cardinality of $C(T,Y) = s - 1$*" *is implied by* "*Y is a Fiedler s-vector corresponding to a simple eigenvalue.*" The following example shows that there exist a tree T and a Fiedler 7-vector Y corresponding to a simple eigenvalue such that $C(T,Y) = 5 < s - 1 = 6$. Obviously, Y contains a zero entry.

Example 4.2.52 Here the graph (Figure 4.3) is unweighted. Consider the Laplacian matrix. Look at the 7-th smallest eigenvalue. It is simple. But the corresponding eigenvector Y has only 5 characteristic elements. The entries of Y are rounded to four decimal places. The eigenvalues of L are given by the vector Λ.

$$\Lambda = \begin{bmatrix} 0 & .0810 & .2545 & .6903 & .7987 & 1.5089 & 1.7154 & 2.5366 & 2.8308 & 3.5200 & 3.6825 & 4.3813 \end{bmatrix}^T.$$

$$Y = \begin{bmatrix} -.3223 & .2305 & .3879 & -.1201 & -.4221 & 0 & .4221 & .1201 & -.3879 & -.2305 & .3223 & 0 \end{bmatrix}^T.$$

■

Now we are in a position to state the main result of this section.

Theorem 4.2.53 *Let T be an unweighted tree, L the Laplacian matrix of T. Suppose that the algebraic connectivity of T has multiplicity one. Let λ be the third smallest eigenvalue of L. Let Z be a Fiedler 3-vector. Then either of the following occurs.*
I. The cardinality of $C(T,Z)$ is 1. In this case $C(T,Z)$ contains a vertex say, k and the subgraph of T induced by the set of vertices corresponding to the $0'$s in Z is connected. The entries of Z either increase and are concave down or decrease and are concave up or are identically zero along any path in T which starts at k.
II. The cardinality of $C(T,Z)$ is 2. Then one of the following cases occur.

a. $C(T,Z) = \{u,v\}$.

Let P be the path joining u and v. Along any path in T which starts at u or v and does not pass through any more vertex of P, the entries of Z either increase and are concave down or decrease and are concave up or are identically zero. The entries of Z along the path P satisfy one of the following three descriptions.

1. *Entries are identically zero. Along any path in T which starts at a vertex on P (not u or v) and does not pass through any more vertex of P, the entries of Z are identically zero.*
2. *Entries are positive (except for u and v) and unimodal. Along any path in T which starts at a vertex on P (not u or v) and does not pass through any more vertex of P, the entries of Z increase and are concave down.*
3. *Entries are negative (except for u and v) and entries of $-Z$ along the path P are unimodal. Along any path in T which starts at a vertex on P (not u or v) and does not pass through any more vertex of P, the entries of Z decrease and are concave up.*

b. $C(T,Z) = \{u,[v,w]\}$.

Assume, without loss, that the distance between v and u is less than the distance between u and w and $Z(w) < 0$. Let P be the path joining u and v. Along any path in T which starts at u and does not pass through any more vertex of P, the entries of Z either increase and are concave down or decrease and are concave up or are identically zero. Along any path in T which starts at w and does not *pass through any vertex of P, the entries of Z decrease and are concave up. The entries of Z along the path P are positive (except for u) and unimodal. Along any path in T which starts at a vertex on P (not u) and does not pass through any more vertex of P, the entries of Z increase and are concave down.*

c. $C(T, Z) = \{[x, u], [v, w]\}$.

Let the distance between v and u be less than the distance between w and x by 2 units. Let $Z(x) > 0$. Let P be the path joining u and v. Along any path in T which starts at x or w and does not pass through any vertex of P, the entries of Z increase and are concave down. The entries of Z along the path P are negative and the entries of $-Z$ along the path P are unimodal. Along any path in T which starts at a vertex on P and does not pass through any more vertex of P, the entries of Z decrease and are concave up.

Proof First note that since Z is a Fiedler 3-vector, the cardinality of $C(T, Z)$ is less than 3 (by Corollary 4.2.51).

Proof of I Let the cardinality of $C(T, Z)$ be 1. We claim that Z has a characteristic vertex. It is sufficient to show that Z has a zero vertex. Suppose that Z has no zero vertex. Then by Proposition 4.2.50, we get that the number of eigenvalues of L which are less than λ is the same as the number of unordered pairs (i, k) such that $L(i, k)Z(i)Z(k) > 0$. But there are exactly two eigenvalues of L less than Λ. Thus $C(T, Z)$ must contain two edges, and this is a contradiction to the hypothesis. Since the cardinality of the characteristic set is 1, $T - \{k\}$ is disconnected with at least two nonzero components (by Note 2.1.4) and each of the components is either positive or negative or zero. Thus the subgraph of T induced by the zero vertices is connected. The rest of the proof of this item follows by Proposition 4.1.48.

Proof of II We only prove item (a). The proof of other items are similar. Note that any nonzero component of $T - \{u\}$ is either positive or negative or contains a characteristic element and the only possible characteristic element is v. Thus each of the components, except for the one which contains v, of $T - \{u\}$ is either positive or negative or zero. Also note that the component which contains v contains $P - \{u\}$ as a subgraph. Thus by Proposition 4.1.48, it follows that along any path in T which starts at u and does not pass through any more vertex of P, the entries of Z either increase and are concave down, decrease and are concave up or are identically zero. The same is true for paths starting at v and not passing through any more vertex of P.

If u and v are adjacent then we have nothing more to prove. So let $P = [u, u_1, \cdots, u_r, v]$. Observe that each of the nonzero components of $T - \{u, v\}$ has to be either positive or negative, otherwise, it will lead to the presence of another characteristic element. Suppose that H is the component of $T - \{u, v\}$ which contains the vertex u_1. If H is a zero component we have nothing to prove. So, let H be positive. If P' is any path starting at u_i, $i \in \{1, 2, \cdots, r\}$ and not passing through any more vertex of P, then by Proposition 4.1.48, entries of Z increase and are concave down along P'.

It remains to show that along the path P the entries of Z are unimodal. Towards this note that if $Z(u_i) < Z(u_{i-1})$ then the eigen condition at the vertex u_i implies that there must be a vertex x adjacent to u_i such that $Z(x) < Z(u_i)$. As $Z(u_i) < Z(u_{i-1})$ and along any path starting at u_i and not passing through any more vertex of P the entries of Z increase, we get that x is u_{i+1}. In the same way one can conclude that $Z(u_{i+2}) < Z(u_{i+1})$ and so on. Thus the proof of item (a) of (ii) is complete. ∎

Remark 4.2.54 *Though, Theorem 4.2.53 is stated for unweighted trees only, it is valid for weighted trees also. A similar result for the perturbed Laplacian matrix can be proved without the concavity/convexity property.*

Below we give some examples to show the occurrence of each of the cases described in the above theorem.

Example 4.2.55 The graph in Figure 4.4 is an unweighted tree. The third smallest eigenvalue of the Laplacian matrix is *0.1981* (rounded to four decimal places) and has multiplicity 3. The eigenvector Y_1 corresponds to the case I, eigenvector Y_2 corresponds to the case II-(a)-(1) and eigenvector Y_3 corresponds to the case II-(a)-(2). The occurrence of II-(a)-(3) can be seen by considering $-Y_2$. The occurrence of case (ii)-(b) is shown in the Example 4.2.56. One can notice the occurrence of case (ii)-(c) in Example 4.1.49. The entries of each of the vectors below are rounded to four decimal places.

$$Y_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .2638 & .4754 & .5929 & 0 & 0 & 0 & -.2638 & -.4754 & -.5929 \end{bmatrix}^T.$$

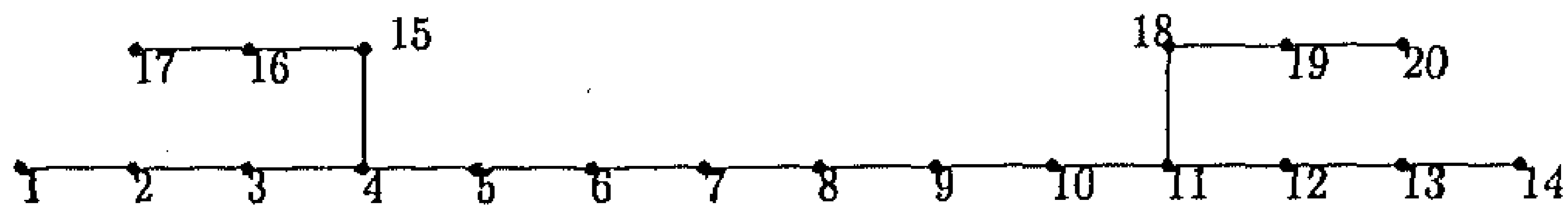


Figure 4.4:

$$Y_2 = \begin{bmatrix} .9098 & .7296 & .4049 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.0490 & -.0884 & -.1102 & -.4049 & -.7296 & -.9098 & .0490 & .0884 & .1102 \end{bmatrix}^T.$$

$$Y_3 = \begin{bmatrix} -.0434 & -.0348 & -.0193 & 0 & .0880 & .1586 & .1978 & .1978 & .1586 & .0880 \\ 0 & .1599 & .2880 & .3592 & -.0687 & -.1238 & -.1544 & -.2479 & -.4466 & -.5569 \end{bmatrix}^T.$$

Example 4.2.56 In this example we will show the occurrence of the case (ii)-(b) described in Theorem 4.2.53. Consider the weighted path on four vertices whose Laplacian matrix is given by L. The vector Y corresponding to the third smallest eigenvalue of L is given below.

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1+\frac{3+\sqrt{5}}{2} & -\frac{3+\sqrt{5}}{2} \\ 0 & 0 & -\frac{3+\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \end{bmatrix}; \; Y = \begin{bmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \\ 0 \\ \frac{1+\sqrt{5}}{3+\sqrt{5}} \end{bmatrix}.$$

The third smallest eigenvalue is $\frac{3+\sqrt{5}}{2}$.

Chapter 5

Unicyclic graphs

In this chapter, G denotes a connected unicyclic unweighted graph on vertices $1, 2, \cdots, n$ with edges $e_1, e_2, \cdots, e_n$. The unique cycle in G is assumed to be $\Gamma = [1, 2, \cdots, k, 1]$; $k > 2$. For $i = 1, \cdots, k-1$, let $e_i = [i, i+1]$, and let $e_k = [k, 1]$. Sometimes we put an orientation on G, in which case the orientation is such that for each $i = 1, \cdots, k-1$ the edge e_i has the terminating vertex $i+1$ and the edge e_k has the terminating vertex 1. Also the edges which are not on the cycle are oriented away from the cycle. We will refer to this as the *unicyclic graph with the standard orientation.*

The graph $G - \{e_1, e_2, \cdots, e_k\}$ has exactly k components $B_1, B_2, \cdots, B_k$, containing the vertices $1, 2, \cdots, k$, respectively. For $i = 1, 2, \cdots, k$, let b_i be the number of vertices in B_i. Thus $\sum_{i=1}^{k} b_i = n$. Let $e = [u, v]$ be an edge which does not lie on the cycle. Then by $G_h(e, G)$ we mean the component of the graph $G - e$, which contains v (recall that the assumed orientation of an edge $[u, v]$ is such that v is the terminating vertex). Let $\alpha_h(e, G)$ denote the number of vertices in $G_h(e, G)$. The function $\psi_G(e, j)$ is defined to be 1 or 0 according as $j \in G_h(e, G)$ or not, respectively.

Recall that the *Moore-Penrose* inverse of a real matrix A is the matrix A^+ satisfying the equations $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^T = AA^+$ and $(A^+A)^T = A^+A$. Let G be a directed graph on n vertices $1, 2, \cdots, n$ and m edges $e_1, e_2, \cdots, e_m$. The oriented vertex-edge incidence matrix of G, denoted by $\bar{A}$, is a $n \times m$ matrix defined as follows. The (i, j)-entry of $\bar{A}$ is 0 if vertex i and edge e_j are not incident and otherwise it is 1 or -1 according as e_j originates or terminates at i, respectively. Note that the Laplacian matrix $L = AA^T$ does not depend on the orientation of the edges. Let λ be an eigenvalue of L with the corresponding eigenvector Z. Thus $AA^TZ = \lambda Z$ and hence $A^+AA^TZ = \lambda A^+Z$. Since A^+A is symmetric and $(A^+)^T = (A^T)^+$, we have

$$A^+AA^T = (A^+A)^TA^T = A^T(A^+)^TA^T = (AA^+A)^T = A^T.$$

Thus

$$A^TZ = \lambda A^+Z. \qquad (5.0.1)$$

We now obtain a formula for the Moore-Penrose inverse of the vertex-edge incidence matrix of a directed unicyclic graph. It has been proved in [2] that, *if H is any directed connected graph on n vertices $1, 2, \cdots, n$, and edges $e_1, e_2, \cdots, e_n$ and A is the oriented vertex-edge incidence matrix of H, then the Moore-Penrose inverse of A is given by*

$$A^+(i,j) = \frac{1}{n\chi}\sum\{\alpha_h(e_i,T) - n\psi_T(e_i,j)\},$$

where the summation is taken over all spanning trees T containing e_i and χ is the number of spanning trees of H.

We apply this result to our unicyclic graph G. Since G has precisely k spanning trees, obtained by deleting a single edge in $[1, \cdots, k, 1]$, there will be k terms in the summation in the above formula. The following result is obtained simply by carefully writing the k terms in the summation. If a and $b\,(b \neq 0)$ are two integers then by $a \bmod b$ we mean the least nonnegative integer of the form $a - bk$, where k is an integer.

Lemma 5.0.57 *Let G be a unicyclic graph with standard orientation and A be the oriented vertex-edge incidence matrix of G. Then the Moore-Penrose inverse of A is given by the following.*

$$A^+(i,j) = \begin{cases} \frac{1}{n}\left[\alpha_h(e_i,G) - n\psi_G(e_i,j)\right] & \text{if } e_i \notin \Gamma, \\[2ex] \frac{1}{nk}\{(k-1)b_{i+1} + (k-2)b_{i+2} + \cdots + ib_k \\ \quad +(i-1)b_1 + (i-2)b_2 \cdots \\ \quad +b_{i-1} \ - n[(i-l) \bmod k]\} & \text{if } e_i \in \Gamma,\ j \in B_l, i \neq k, \\[2ex] \frac{1}{nk}\{(k-1)b_1 + (k-2)b_2 + \cdots \\ \quad +b_{k-1} \ - n(k-l)\} & \text{if } i = k,\ j \in B_l. \end{cases}$$

We now obtain the following result using Lemma 5.0.57.

Lemma 5.0.58 *Let G be a unicyclic graph. Let L be the Laplacian matrix of G, λ an eigenvalue of L and Z a corresponding eigenvector. Let $e = [u,v]$ be an edge of G. Then*

$$Z(u) - Z(v) = \begin{cases} -\frac{\lambda}{k}\sum_{j=1}^{k}\left[[(i-j) mod\ k]\sum_{s\in B_j} Z(s)\right] & \text{if } [u,v]=[i,i+1] \\ & i = 1,\cdots,k-1. \\ -\frac{\lambda}{k}\sum_{j=1}^{k}\left((k-j)\sum_{s\in B_j} Z(s)\right) & \text{if } [u,v]=[k,1], \\ -\lambda \sum_{s\in G_h(e,G)} Z(s) & \text{if } [u,v]\notin \Gamma. \end{cases}$$

Proof Without loss of generality, we assume G to be the unicyclic graph with the standard orientation. From Equation 5.0.1, which says $A^T Z = \lambda A^+ Z$ and from the fact that $\sum_{i=1}^{n} Z(i) = 0$, we get, using Lemma 5.0.57,

$$\begin{aligned} & Z(i) - Z(i+1) \\ &= \text{the } i\text{th row of } A^T Z \\ &= \text{the } i\text{th row of } \lambda A^+ Z \\ &= \lambda \sum_{j=1}^{k}\left(A^+(i,j)\sum_{s\in B_j} Z(s)\right) \\ &= -\frac{\lambda}{k}\sum_{j-1}^{k}\left[[(i-j) mod\ k]\sum_{s\in B_j} Z(s)\right], \end{aligned}$$

for $i = 1,\cdots,k-1$.

Similarly,

$$\begin{aligned} & Z(k) - Z(1) \\ &= \text{the } k\text{th row of } A^T Z \\ &= \text{the } k\text{th row of } \lambda A^+ Z \\ &= \lambda \sum_{j=1}^{k}\left(A^+(k,j)\sum_{s\in B_j} Z(s)\right) \\ &= -\frac{\lambda}{k}\sum_{j-1}^{k}\left[[(k-j) mod\ k]\sum_{s\in B_j} Z(s)\right] \\ &= -\frac{\lambda}{k}\sum_{j-1}^{k}\left[(k-j)\sum_{s\in B_j} Z(s)\right]. \end{aligned}$$

For any edge $e = [u,v] \notin \Gamma$, we have

$$Z(u) - Z(v) = \frac{\lambda}{nk}\left[\alpha_h(e,G)\sum_{s\notin G_h(e,G)} Z(s) + [\alpha_h(e,G) - n]\sum_{s\in G_h(e,G)} Z(s)\right]$$

$$= -\lambda \sum_{s \in G_h(e,G)} Z(s).$$

That completes the proof. ∎

Let G be a unicyclic graph. Let u, v and w be vertices such that $[u, v]$ and $[v, w]$ are edges on Γ. From Lemma 5.0.58, we get that

$$Z(u) + Z(w) - 2Z(v) = -\lambda \sum_{s \in B_v} Z(s). \tag{5.0.2}$$

We now prove an analog of Fiedler's result for a unicyclic graph.

Theorem 5.0.59 *Let G be a unicyclic graph. Let L be the Laplacian matrix of G, μ the algebraic connectivity of G and Y a Fiedler vector. Then one of the following cases occur.*

(a) No entry of Y is zero and there is exactly one characteristic edge $e = [v, w]$ in $C(G, Y)$. This edge does not lie on the cycle. Let $Y(v) > 0$ and $Y(w) < 0$. Then there exists an edge $e^1 \in \Gamma$, such that along any path in G which starts at v and does not contain w or e^1, the entries of Y increase. Along any path in G which starts at w and does not contain v or e^1, the entries of Y decrease.

(b) No entry of Y is zero and there are two characteristic edges $e_1 = [x_1, y_1]$ and $e_2 = [x_2, y_2]$. In this case both the characteristic edges lie on the cycle. Let $Y(x_i) > 0$ and $Y(y_i) < 0$; $i = 1, 2$. Then,

(i) along any path in G that starts at x_1 and does not contain y_1 or y_2, the entries of Y increase, if $x_1 = x_2$ and

(ii) there exists an edge e^1 on Γ such that along any path that starts at x_1 or x_2 and does not contain y_1 or y_2 or e^1, the entries of Y increase, if $x_1 \neq x_2$.

A Path starting from y_1 or y_2 have a similar property except for the fact that the entries of Y decrease along this path and in (ii) we get a different edge e^2.

(c) Some entries of Y are zero and there are two characteristic vertices u, v. In this case both u, v lie on the cycle. The graph induced by the zero vertices has exactly two connected components. There exist two edges e^1 and e^2 on the cycle Γ such that along any path that starts from u or v and does not contain e^1 or e^2 the entries of Y either increase or decrease or are identically zero.

(d) Some entries of Y are zero and there is only one characteristic vertex v and no characteristic edge. In this case the zero vertices of G induce a connected graph. There exists an edge e^1 on Γ such that along any path starting from v which does not contain e^1, the entries of Y either increase or decrease or are identically zero.

(e) Some entries of Y are zero and there are one characteristic edge $e = [i, j]$ and one characteristic vertex v. In this case both of these e and v lie on the cycle and the zero vertices of G induce a connected graph. Let $Y(i) > 0$. Then there exists an edge e^1 on Γ such that along any path that starts from i and does not contain j or e^1, the entries of Y increase. There exists an edge e^2 on Γ such that along any path that starts from j and does not contain i or e^2, the entries of Y decrease. Along any path that starts at v and does not contain e^1 or e^2, the entries of Y either increase or decrease or are identically zero.

Proof By Corollary 3.1.35, G has either 1 or 2 characteristic elements. Thus the following cases are possible. Let α, β denote the number of characteristic edges and characteristic vertices respectively. Then the pair (α, β) can take the possible values $(1,0), (0,1), (1,1), (2,0)$ and $(0,2)$. These possibilities are described in $(a), (d), (e), (b), (c)$ respectively.

We consider (a) first. Let G_+ denote the positive component of $G - e$. Without loss of generality, suppose that Γ is in G_+ and let 1 be the vertex on Γ such that $d(v, \Gamma) = d(v, 1)$.

We first claim that the sequence $Y(1), Y(2), \cdots, Y(k), Y(1)$ is unimodal. This is seen as follows. For $i = 1, \cdots, k-2$, applying Equation (5.0.2) to vertices $i, i+1$ and $i+2$, we see that

$$Y(i) + Y(i+2) - 2Y(i+1) = -\lambda \sum_{s \in B_{i+1}} Y(s).$$

Thus $Y(i)+Y(i+2) < 2Y(i+1)$, since all the vertices in B_{i+1} are positive. Similarly, applying Equation (5.0.2) to the vertices $k-1, k$ and 1, we get that $Y(k-1)+Y(1) < 2Y(k)$. Thus we see that the sequence $Y(1), \cdots, Y(k), Y(1)$ is concave. In particular, the sequence is log concave and it is well-known, (see, for example, [4], p.184), that such a sequence must be unimodal.

Let $i \in \{2, 3, \cdots, k\}$ and let $P = [i, i_1, \cdots, i_r]$ be a path in B_i starting at i. We now claim that the entries of Y increase along P. Applying Lemma 5.0.58 to the vertices i and i_1 we get that

$$Y(i) - Y(i_1) = -\lambda \sum_{s \in G_h([i,i_1],G)} Y(s). \tag{5.0.3}$$

Note that B_i contains positive vertices for all $i = 2, 3, \cdots, k$. Since $Y(s) > 0$ for all $s \in G_h([i, i_1], G)$, we get that $Y(i) < Y(i_1)$. The rest of the proof of the claim is trivial.

Similarly, we can show that along any path in B_1 starting from 1 the entries of Y decrease.

Let j be a vertex on Γ such that $Y(j)$ is the maximum coordinate of Y restricted to Γ. Let e^1 be the edge $[j, j+1]$ if $j \neq k$ or $[j, 1]$ if $j = k$. Let P' be a path which starts at v and does not contain w, e'. Write P' as a union of three paths P_1, P_2, P_3, where P_1 and P_3 do not lie on the cycle and P_2 lies on the cycle. It is clear from the above discussion that along P_1 and P_3 the entries of Y increase. From the fact that the sequence $Y(1), Y(2), \cdots, Y(k), Y(1)$ is unimodal and observing the way we selected e^1, it is also clear that along P_2 the entries of Y increase.

Next we prove (b). From Corollary 3.1.33, it is clear that both the characteristic edges will lie on the cycle. Now we claim that there exists a vertex ℓ on Γ such that the sequence $Y(\ell), Y(\ell+1), \cdots, Y(k), Y(1), Y(2), \cdots, Y(\ell)$ is unimodal. This is seen as follows. From Lemma 2.2.9, we know that the positive vertices of G are connected. Thus let $[x_1 = i, i+1, i+2, \cdots, j = x_2]$ be the positive half part of Γ. Applying Equation 5.0.2 to the above sequence in a similar way as in the proof of (a), we can see that for some vertex $s \in \{i, i+1, \cdots, j\}$

$$Y(i) \leq Y(i+1) \leq \cdots \leq Y(s) \geq Y(s+1) \geq \cdots \geq Y(j).$$

Considering $-Y$ in place of Y we can get, in a similar way, that

$$Y(j+1) \geq Y(j+2) \geq \cdots \geq Y(\ell) \leq Y(\ell+1) \leq \cdots \leq Y(i-1),$$

where ℓ is the vertex such that $Y(\ell)$ is the minimum over Y restricted to Γ. So the proof of the claim.

Let $m \in \{1, 2, \cdots, k\}$ and let $P = [m, m_1, \cdots, m_r]$ be a path in B_i starting at m. We can show, in a similar way as in the proof of (a), that the entries of Y increase along P. The rest of the proof of (b) is trivial.

To prove (c), one has to note that since we already have two characteristic elements in $\mathcal{C}(G, Y)$, a path starting from u and not passing through v can either be a zero path or a positive path (except for the starting vertex) or a negative path (except for the starting vertex). Let $P = [u, u_1, u_2, \cdots, v]$ be a path joining u and v, where $u_i \in \Gamma$.

We claim that P is not a zero path, for otherwise, since u is a characteristic vertex there must be two nonzero adjacent vertices w_1 and w_2 to u. Obviously at least one of them (say, w_1) will not lie on Γ. So the graph $G - u$ has at least two nonzero components (because, if we have only one nonzero component, then there is path P_1 joining w_1 and w_2 in $G - u$. Thus the path P_2 along with u form a cycle. Thus w_1 lies on the cycle Γ.). But now applying Corollary 2.2.15, we get that $C(G, Y) = \{u\}$, a contradiction.

We know that a zero vertex other than the two characteristic vertices is adjacent to zero vertices only (otherwise we will have more characteristic vertices). Note that the graph G is connected. Thus, if z a zero vertex then there exists either a zero path joining z and u or a zero path joining z and v. Thus by the above claim, the graph induced by the zero vertices has exactly two components.

By the above claim, both the paths on Γ joining u and v are nonzero paths. So by Corollary 2.2.15, each vertex, except for the two characteristic vertices on Γ is a nonzero vertex. In view of the proof of (b), it is now easy to prove that there exists a vertex ℓ on Γ such that the sequence $Y(\ell)$, $Y(\ell+1), \cdots, Y(k), Y(1), Y(2), \cdots, Y(\ell)$ is unimodal. The rest of the proof is similar to the proof of (a) and (b).

Proof of (d) is similar to the proof of (c).

Next we prove (e). From Corollary 3.1.33, it is clear that both the characteristic elements will lie in one block. Since the only nontrivial block is the cycle, so both of them will lie on the cycle. Also note that since we already have two elements in $C(G, Y)$, so a path starting from v and not passing through e can either be a zero path or a positive path (except for the starting vertex) or a negative path (except for the starting vertex), for otherwise we will have more elements in $C(G, Y)$. The rest of the proof is similar to the proof of (a), (b) and (c). ▪

Example 5.0.60 Here we give some examples (see Figure 5.1) to show that each of the cases described in Theorem 5.0.59 can occur. The values written near each vertex is the value of the corresponding coordinate of an eigenvector associated with the algebraic connectivity, rounded to four decimal places.

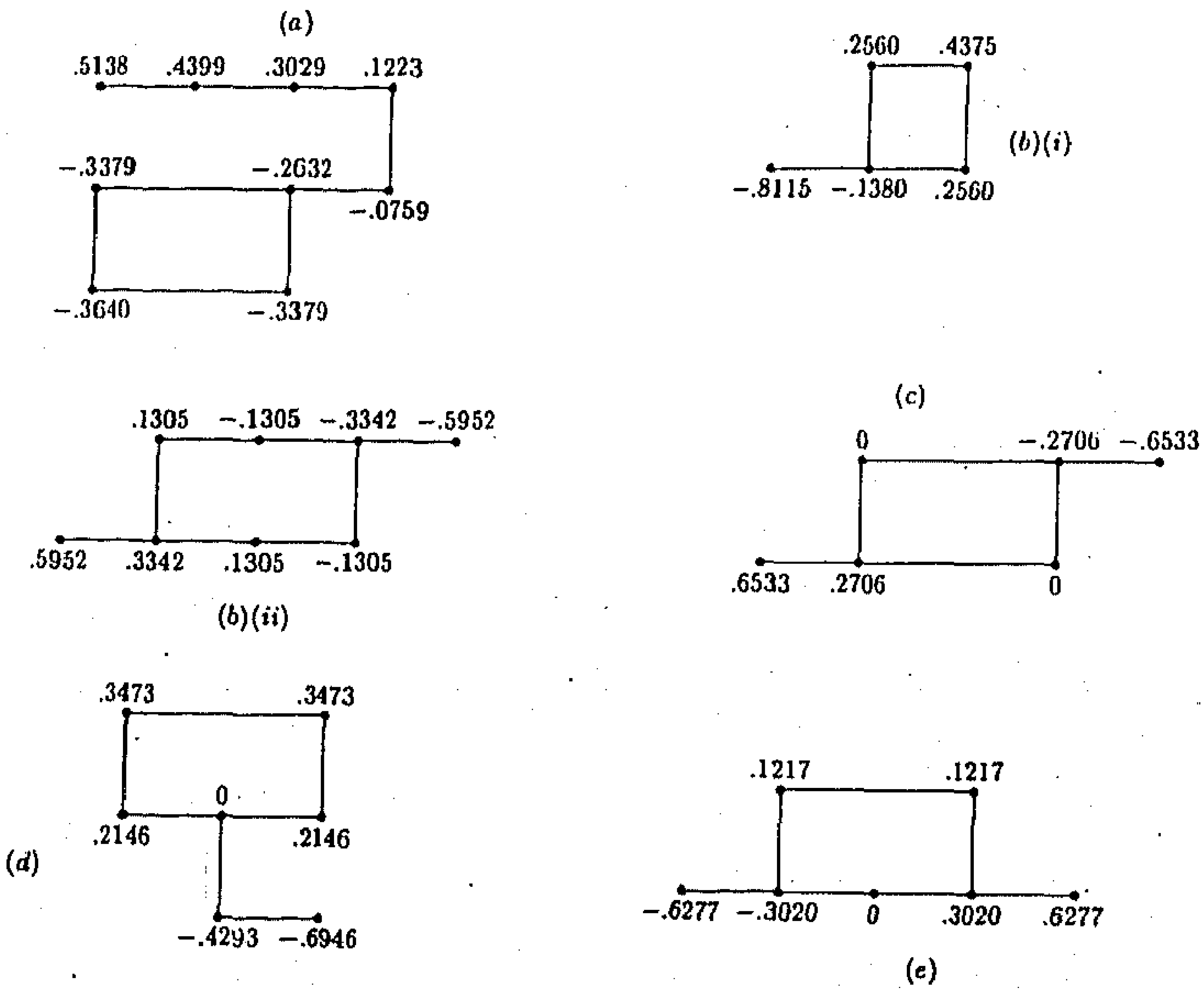

(a)
.5138 .4399 .3029 .1223
−.3379 −.2632
−.0759
−.3640 −.3379
.2560 .4375
(b)(i)
−.8115 −.1380 .2560
(c)
.1305 −.1305 −.3342 −.5952
.5952 .3342 .1305 −.1305
(b)(ii)
0 −.2706 −.6533
.6533 .2706 0
.3473 .3473
0
.2146 .2146
(d)
−.4293 −.6946
.1217 .1217
−.6277 −.3020 0 .3020 .6277
(e)

Figure 5.1:

Chapter 6

A tree type graph

Consider a tree T on n vertices $1, 2, \cdots, n$. Consider a set of complete graphs $\{K_i : i = 1, \cdots, n\}$ such that *vertices i and j are adjacent in T if and only if* $K_i \bigcap K_j \neq \emptyset$, $K_i \bigcap K_j \neq K_i$, $K_i \bigcap K_j \neq K_j$. The graph obtained by the union of all these complete graphs will be referred to as $\mathcal{T}$. It may be noted here that given a tree T, if each of the complete graphs is on two vertices then $\mathcal{T} = T$. If $\mathcal{T}$ has r vertices then we assume that the vertices are labelled $1, 2, \cdots, r$. The letters i, j, k are reserved for vertices of T. We assume the graph $\mathcal{T}$ is unweighted.

Definitions: Consider a tree T and consider $\mathcal{T}$. Suppose that $[i, j]$ is an edge in T. Then the *overlapping* O_{ij} is defined as $K_i \bigcap K_j$ in $\mathcal{T}$. If $i \in T$ is a vertex then the *mid-part* M_i is defined as $K_i - \{u : u \in K_j, \ [j, i] \in T\}$.

Example 6.0.61 See Figure 6.1. The tree T is on 5 vertices. The complete graphs are the following:

K_1 is on $\{1, 2, 3, 4\}$, K_2 is on $\{2, 3, 5\}$, K_3 is on $\{5, 6, 7, 8\}$, K_4 is on $\{7, 10\}$ and K_5 is on $\{6, 9\}$.

The overlappings are the following:

$$O_{1,2} = K_1 \bigcap K_2 = \text{the complete graph induced by } \{2, 3\}.$$
$$O_{2,3} = K_2 \bigcap K_3 = \text{the complete graph induced by } \{5\}.$$
$$O_{3,4} = K_3 \bigcap K_4 = \text{the complete graph induced by } \{7\}.$$
$$O_{3,5} = K_3 \bigcap K_5 = \text{the complete graph induced by } \{6\}.$$

The mid-parts are the following:

$$M_1 = \{1, 4\}.$$
$$M_2 = \emptyset.$$

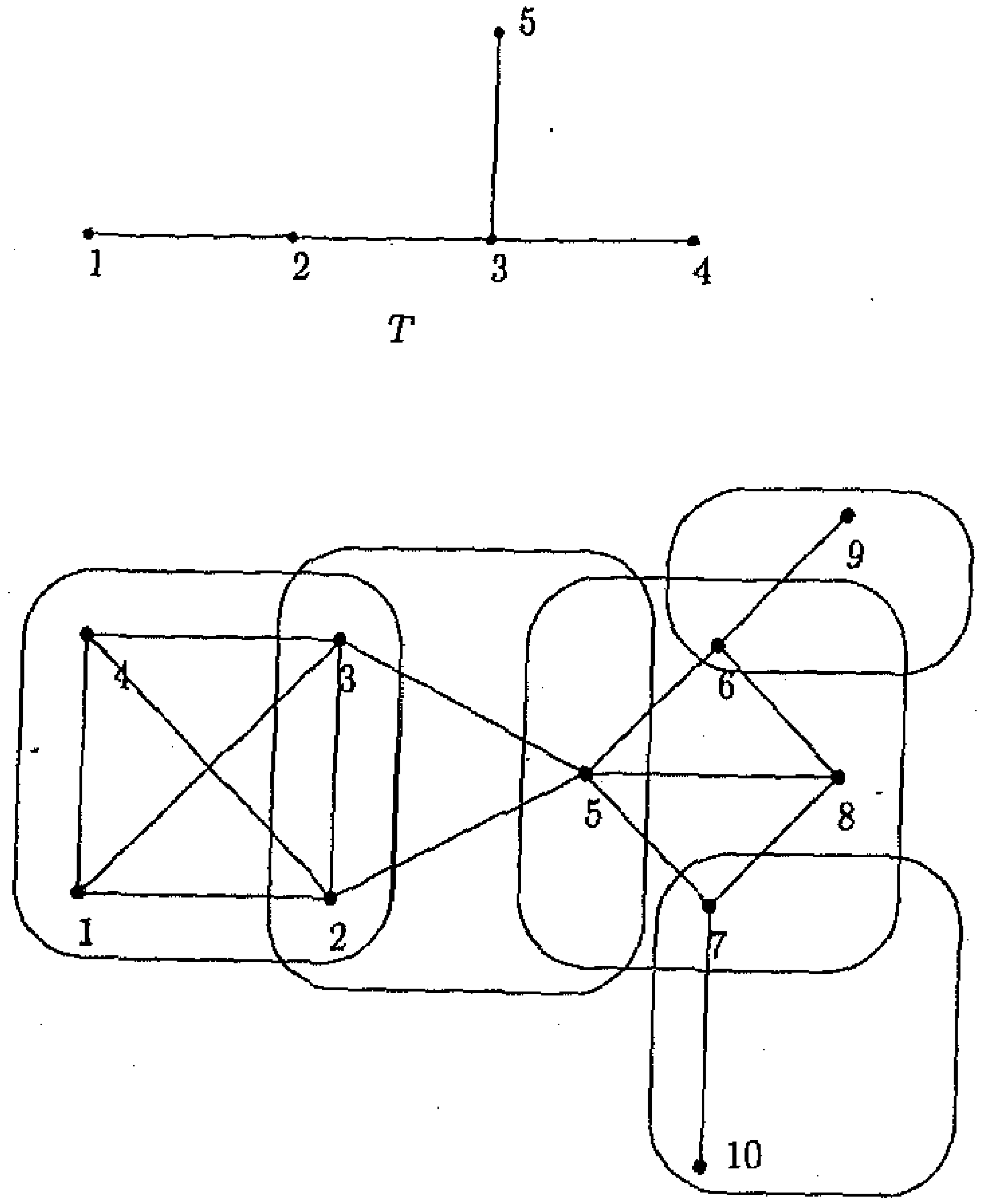


Figure 6.1: T and $\mathcal{T}$

$$
\begin{aligned}
M_3 &= \{8\}. \\
M_4 &= \{10\}. \\
M_5 &= \{9\}.
\end{aligned}
$$

∎

Henceforth, we follow the following notation. If S is a set or a graph or a subgraph, by $|S|$ we mean the number of points or vertices contained in S. By N_v we mean the neighbours of the vertex v in the graph under consideration, that is the set of vertices adjacent to v. Given a graph $\mathcal{T}$, by the *degree matrix* D, we mean the diagonal matrix such that the u-th diagonal entry of D is the degree of the vertex u in $\mathcal{T}$.

Note 6.0.62 *Let $\mathcal{T}$ be given and $0 \leq \alpha \leq 1$. Let D be the degree matrix and $\mathcal{L} = \alpha D - A$. Let $u, v \in O_{ij}$ and $x, y \in M_i$, for some i, j. Thus $N_u - \{v\} = N_v - \{u\}$ and so $\mathcal{L} = P\mathcal{L}P^T$, where P is the permutation matrix which interchanges the u-th row and the v-th row. Similarly $N_x - \{y\} = N_y - \{x\}$ and so $\mathcal{L} = Q\mathcal{L}Q^T$, where Q is the permutation matrix which interchanges the x-th row and the y-th row.* ∎

Lemma 6.0.63 *Let $\mathcal{T}$ be given and $0 \leq \alpha \leq 1$. Let D be the degree matrix and $\mathcal{L} = \alpha D - A$. Let Z be the eigenvector corresponding to the smallest eigenvalue of $\mathcal{L}$. Let $u, v \in O_{ij}$. Then $Z(u) = Z(v)$. If $x, y \in M_i$ then $Z(x) = Z(y)$.*

Proof If $Z(u) \neq Z(v)$ then interchange the values $Z(u)$ and $Z(v)$, to obtain Z'. By Note 6.0.62, we see that $\mathcal{L}Z' = \tau(\mathcal{L})Z'$. Thus $\tau(L)$ has multiplicity more than one; a contradiction to Proposition 2.1.2. Thus $Z(u) = Z(v)$. A similar argument shows that $Z(x) = Z(y)$. ∎

Lemma 6.0.64 *Let $\mathcal{T}$ be given and $0 \leq \alpha \leq 1$. Let D be the degree matrix and $\mathcal{L} = \alpha D - A$. Let Y be a Fiedler vector of $\mathcal{L}$. Let $u, v \in O_{ij}$. Then $Y(u) = Y(v)$.*

Proof If $Y(u) \neq Y(v)$ then by Note 6.0.62, we can get another Fiedler vector Y' of $\mathcal{L}$, independent of Y, by interchanging the values $Y(u)$ and $Y(v)$. Then

$$
Y' - Y = \begin{cases} Y(v) - Y(u) & \text{at} \quad u, \\ Y(u) - Y(v) & \text{at} \quad v, \\ 0 & \text{at other vertices.} \end{cases}
$$

Let $Y_1 = Y' - Y$. The u-th entry of $\mathcal{L}Y_1$ is

$$
\begin{aligned}
&\sum_x \mathcal{L}(u,x)Y_1(x) \\
&= \mathcal{L}(u,u)Y_1(u) + \mathcal{L}(u,v)Y_1(v) \\
&= \mathcal{L}(u,u)Y_1(u) - Y_1(v) \quad (\text{since } \mathcal{L}(u,v) = -1)
\end{aligned}
$$

Since $\mathfrak{L}Y_1 = \mu Y_1$, it follows that μ, the algebraic connectivity is equal to $\alpha d(u) + 1$, where $d(u)$ is the degree of the vertex u. We note down this by the following equation.

$$\mu = \alpha d(u) + 1. \tag{6.0.1}$$

Consider the graphs G_1 and G_2 obtained from $\mathfrak{T}$ by deleting O_{ij}; G_1 is the part which has nonempty intersection with K_i (say). Let L_1 and L_2 be the principal submatrices of $\mathfrak{L}$ corresponding to G_1 and G_2, respectively. Let D_i be the diagonal matrix defined as the following:

$$D_i(x) = \begin{cases} |O_{ij}| & \text{if} \quad x \in K_i - O_{ij}, \\ 0 & \text{if} \quad x \in G_1 - K_i \end{cases}$$

Let $U = [L_1 + (1-\alpha)(D_1 - D_i)]\hat{e}$, where D_1 is the principal submatrix of D corresponding to G_1. Let A_1 be the adjacency matrix of G_1. Thus U

$$\begin{aligned} &= [\alpha D_1 - A_1 + D_1 - D_i - \alpha D_1 + \alpha D_i]\hat{e} \\ &= [(D_1 - D_i - A_1) + \alpha D_i]\hat{e} \\ &= \alpha D_i \hat{e} \quad \text{(since } D_1 - D_i - A_1 \text{ is the Laplacian matrix of } G_1\text{)} \\ &\leq \alpha |O_{ij}| \hat{e} \\ &\Rightarrow \tau([L_1 + (1-\alpha)(D_1 - D_i)]) \leq \alpha |O_{ij}| \end{aligned}$$

Since $(1-\alpha)(D_1 - D_i)$ is a nonnegative diagonal matrix with at least one entry positive, we have

$$\tau(L_1) < \tau[L_1 + (1-\alpha)(D - D_i)] \leq \alpha |O_{ij}| \tag{6.0.2}$$

Similarly, one can show that

$$\tau(L_2) < \alpha d(u).$$

It is easy to see from the graph that $|O_{ij}| \leq d(u)$. Now applying Lemma 2.2.13, we get that $\mu \leq \alpha d(u)$, which is a contradiction to Equation 6.0.1. Thus the proof. ∎

Lemma 6.0.65 *Let $\mathfrak{T}$ be given and $0 \leq \alpha \leq 1$. Let D be the degree matrix and $\mathfrak{L} = \alpha D - A$. Let Y be a Fiedler vector of $\mathfrak{L}$. Let $u, v \in M_i$. Then $Y(u) = Y(v)$.*

Proof Suppose that $Y(u) \neq Y(v)$. Then arguing in a similar way as in the proof of the above lemma, one can get a Fiedler vector Y' of the following form :

$$Y' = \begin{cases} Y(v) - Y(u) & \text{at} \quad u, \\ Y(u) - Y(v) & \text{at} \quad v, \\ 0 & \text{at} \quad \text{other vertices.} \end{cases}$$

Thus $\mu = \alpha d(u) + 1$. An application of Lemma 2.2.13 tells that if H is any zero component of $\mathcal{T}$ obtained by deleting some zero vertices then $\tau(L_H) \geq \mu$. Here in this case, we consider Y' and delete the vertices in O_{ij}, for some j such that $[i,j] \in T$ and take the component having nonempty intersection with K_j, as the subgraph H. Thus one gets the following equation.

$$\tau(L_H) \geq \mu = \alpha d(u) + 1, \tag{6.0.3}$$

where $d(u)$ is the degree of the vertex u in $\mathcal{T}$. But we already know from the discussion in the proof of the previous lemma (Equation 6.0.2) that in this case $\tau(L_H) < \alpha|O_{ij}|$. It is easy to see from the graph structure that $|O_{ij}| \leq d(u)$, but then we have got a contradiction to Equation 6.0.3. ∎

The following is the main result of this chapter.

Theorem 6.0.66 *Consider the graph $\mathcal{T}$. Let D be the degree matrix of $\mathcal{T}$. Let $0 \leq \alpha \leq 1$ and $\mathcal{L} = \alpha D - A$. Let Y be Fiedler vector of $\mathcal{L}$. Let Z be the eigenvector corresponding to the smallest eigenvalue of $\mathcal{L}$. With respect to Y, suppose that O_{ij} is positive and the component containing $K_j - O_{ij}$ of $G - O_{ij}$ is positive. Then the following hold.*

(i) Let $u \in O_{ij}$ and $v \in M_j$. Then $\frac{Y(u)}{Z(u)} \leq \frac{Y(v)}{Z(v)}$.

(ii) Let $u \in M_j$ and $v \in O_{jk}$, $k \neq i$, $[j,k] \in T$. Then $\frac{Y(u)}{Z(u)} \leq \frac{Y(v)}{Z(v)}$.

Proof Let $\tau = \tau(\mathcal{L})$ and $M = [\mathcal{L} - \tau]diag(Z)$, where $diag(Z)$ is the diagonal matrix with $Z(i)$ as the i-th diagonal entry. Then $M\hat{e} = 0$ and the second smallest eigenvalue of M is $\mu' = \mu - \tau$ and the corresponding eigenvector is $Y' = \begin{bmatrix} \frac{Y(1)}{Z(1)} & \frac{Y(2)}{Z(2)} & \cdots & \frac{Y(n)}{Z(n)} \end{bmatrix}^T$. We may recall at this point that Z is a positive eigenvector.

First we prove item (i). Suppose that $\frac{Y(u)}{Z(u)} > \frac{Y(v)}{Z(v)}$. From $MY' = \mu'Y'$, we get that

$$\begin{aligned}
\mu' Y'(v) &= M(v,v)Y'(v) + \sum_{x \in N_v} M(v,x)Y'(x). \\
&= \sum_{x \in N_v} -M(v,x)Y'(v) + \sum_{x \in N_v} M(v,x)Y'(x) \qquad \text{Since } M\hat{e} = 0 \\
&= \sum_{x \in O_{ij}} -M(v,x)[Y'(v) - Y'(x)]
\end{aligned}$$

$$+ \sum_{[j,l]\in T\ l\neq i} \sum_{x\in O_{jl}} -M(v,x)[Y'(v)-Y'(x)]$$
$$+ \sum_{x\in[M_j-v]} -M(v,x)[Y'(v)-Y'(x)]. \qquad (6.0.4)$$

By Lemma 6.0.63, 6.0.65, we know that $Y'(v) = Y'(x)$ for each $x \in M_j$ and thus the third summand in the right hand side of the above equation is zero. Also, by the hypothesis for each $x \in O_{ij}$ the difference $[Y'(v) - Y'(x)] < 0$. Thus the first summand in the right hand side of the above equation is negative. Since $\mu'Y'(v) > 0$, we conclude that there must exist a k, $(k \neq i,\ k \neq j)$ such that

$$\sum_{x\in O_{jk}} -M(v,x)[Y'(v)-Y'(x)] > 0. \qquad (6.0.5)$$

By Lemma 6.0.63, 6.0.65, we know that for any pair $s,t \in O_{jk}$, $Y'(s) = Y'(t)$. Also it is easy to see that $M(v,s) = M(v,t)$. Let $w \in O_{jk}$. Thus Equation 6.0.5 reduces to

$$-M(v,w)|O_{jk}|[Y'(v)-Y'(w)] > 0.$$

Thus $Y'(v) > Y'(w)$, since $M(v,w) < 0$. So we have got a k such that $Y'(v) > Y'(x)$ holds for all $x \in O_{jk}$.

Now we have $v \in M_j, w \in O_{jk}$ and $Y'(v) > Y'(w)$. From $MY' = \mu'Y'$ we get that

$$\begin{aligned}
\mu'Y'(w) &= M(w,w)Y'(w) + \sum_{x\in N_w} M(w,x)Y'(x). \\
&= \sum_{x\in N_w} -M(w,x)Y'(w) + \sum_{x\in N_w} M(w,x)Y'(x) \qquad \text{Since } M\hat{e}=0 \\
&= \sum_{x\in M_j} -M(w,x)[Y'(w)-Y'(x)] \\
&+ \sum_{x\in O_{ji}} -M(w,x)[Y'(w)-Y'(x)] \\
&+ \sum_{[k,l]\in T} \sum_{x\in O_{kl}} -M(w,x)[Y'(w)-Y'(x)] \\
&+ \sum_{[j,l]\in T,\ l\neq i} \sum_{x\in O_{jl}} -M(w,x)[Y'(w)-Y'(x)] \\
&+ \sum_{x\in M_k} -M(w,x)[Y'(w)-Y'(x)]. \qquad (6.0.6)
\end{aligned}$$

By Lemma 6.0.63, 6.0.64, 6.0.65, we know that

(1) For each pair of vertices $x, z \in O_{kl}$ or O_{jl}, $Y'(x) = Y'(z)$.

(2) For each l, for each pair of vertices $x, z \in M_l$, $Y'(x) = Y'(z)$.

Since $Y'(w) < Y'(v) < Y'(u)$, for each $x \in M_j$ the difference $[Y'(w) - Y'(x)] < 0$ and $\sum_{x \in O_{ki}} -M(w,x)[Y'(w) - Y'(x)] < 0$.

Thus the first two summands in the right hand side of the above equation are negative. Since $\mu' Y'(w) > 0$, we conclude, in view of Lemma 6.0.63, 6.0.64 and 6.0.65 that either

(1) there exists $[k, m] \in T$ such that $Y'(w) > Y'(x)$ for all $x \in O_{k,m}$ or

(2) there exists $[j, m] \in T$ such that $Y'(w) > Y'(x)$ for all $x \in O_{jm}$ or

(3) $Y'(w) > Y'(k)$ for all $x \in M_k$.

Repeated application of the above argument will then result in a sequence of nonempty sets of vertices (S_r), S_r is either M_α or $O_{\alpha\beta}$, such that if $x_1 \in S_r$ and $x_2 \in S_{r+1}$ then $Y'(x_1) > Y'(x_2)$. One can see that, this is an infinite sequence of pairwise disjoint sets. *This is not possible, because we have a finite graph.*

The proof of item (ii) is similar to the proof of item (i). ∎

Remark 6.0.67 *Since $\mathcal{T} = T$, if all the complete graphs under consideration are complete graphs on two vertices, Theorem 6.0.66 also proves the increasing/decreasing property of the Fiedler vector of a perturbed Laplacian matrix for a tree, as a special case.*

Definition: Let $T = [1, 2, \cdots, n]$ be a path. Consider the graph $\mathcal{T}$ with labelling of the vertices satisfying the following property (in addition to those in the definition of $\mathcal{T}$): the vertices in M_i have labels less than the vertices in $O_{i,i+1}, M_{i+1}$ and the vertices in $O_{i,i+1}$ have labels less than the vertices in $M_{i+1}, O_{i+1,i+2}$. The graph thus described will be referred to an *interval graph.*

The following is an interesting application of Theorem 6.0.66.

Corollary 6.0.68 *Let $\mathcal{T}$ be an interval graph on vertices $1, 2, \cdots, m$. Let D be any diagonal matrix, $\mathcal{L} = \alpha D - A$ and Y be a Fiedler vector with $Y(1) \leq 0$ and Z be the positive eigenvector corresponding to the smallest eigenvalue of $\mathcal{L}$. Then*

$$\frac{Y(1)}{Z(1)} \leq \frac{Y(2)}{Z(2)} \leq \cdots \leq \frac{Y(m)}{Z(m)}.$$

Proof Follows immediately from Theorem 6.0.66. ∎

Remark 6.0.69 *A more general version of the above corollary and some more results about interval graphs can be found in* [1].

We conclude the chapter by illustrating the above fact by some examples.

Example 6.0.70 In this example we consider an unweighted interval graph. In Case 1, we consider the Laplacian matrix of the graph and in Case 2, we consider the adjacency matrix of the graph. For our graph, T is the path on 5 vertices. The graph $\mathcal{T}$ is constructed with K^1, K^2, K^3, K^4, K^5; where each of them are complete graph on vertices $\{1,2,3,4,5\}$, $\{3,4,5,6,7,8\}$, $\{7,8,9\}$, $\{9,10,11,12\}$, $\{11,12,13,14\}$, respectively. Drawing so many lines would not look nice. That is the reason we have not supplied the figure.

Case 1 The Laplacian matrix. The algebraic connectivity is 0.3279 and has multiplicity one. The Fiedler vector is the following.

$$Y = \begin{bmatrix} -.2636 & -.2636 & -.2348 & -.2348 & -.2348 & -.2246 & -.1725 & -.1725 & .1228 & .2836 \\ .3175 & .3175 & .3798 & .3798 \end{bmatrix}^T.$$

It is well known that the eigenvector corresponding to the smallest eigenvalue of the Laplacian matrix of a connected graph is $\hat{e}$. One can see the reflection of the behavior of the Fiedler vector noticed in Corollary 6.0.68.

Case 2 The negative adjacency matrix. The algebraic connectivity is -4.8509 with multiplicity one. The Fiedler vector and the eigenvector for the smallest eigenvalue are given below.

$$Y = \begin{bmatrix} -.0802 & -.0802 & -.0762 & -.0762 & -.0762 & -.0431 & .0312 & .0312 & .3607 & .3479 \\ .4894 & .4894 & .3433 & .3433 \end{bmatrix}^T.$$

$$Z = \begin{bmatrix} .2524 & .2524 & .4041 & .4041 & .4041 & .3299 & .3510 & .3510 & .1441 & .0409 \\ .0466 & .0466 & .0194 & .0194 \end{bmatrix}^T.$$

$$\frac{Y}{Z} = \begin{bmatrix} -.3177 & -.3177 & -.1886 & -.1886 & -.1886 & -.1308 & .0890 & .0890 & 2.5035 & 8.5080 \\ 10.5049 & 10.5049 & 17.6986 & 17.6986 \end{bmatrix}^T.$$

Chapter 7

On two minor-monotone graph invariants

In this chapter, our main aim is to establish a relationship between two minor-monotone graph invariants and the Laplacian matrices or the perturbed Laplacian matrices.

Definition: Let G be an undirected graph. A minor of G is a graph obtained from G by a series of deletions and contractions of edges and deletions of isolated vertices, suppressing any multiple edges and loops that may arise.

Definition: Let G be an undirected graph. A function $\phi(G)$ is called minor-monotone if for any minor H of G the inequality

$$\phi(H) \leq \phi(G)$$

holds.

A few reasons which raise the interest of the reader to study the minor-monotone graph parameters, are given in [35]. The following graph parameter $\mu(G)$ was introduced by Colin de Verdiere [9].

Definition: The *corank* of a matrix M is the dimension of it's null space. It is denoted by corank(M).

Definition 7.0.71 Let G be an undirected graph with the vertex set $\{1, 2, \cdots, n\}$. *Then* $\mu(G)$ *is the largest corank of any symmetric real-valued* $n \times n$ matrix $M = (m_{i,j})$ such that:

(i) M has exactly one negative eigenvalue,

(ii) for all i, j with $i \neq j$, $m_{i,j} < 0$ if i and j are adjacent and $m_{i,j} = 0$ if i and j are nonadjacent,

(iii) there is no nonzero symmetric matrix $X = (x_{i,j})$ such that $MX = 0$ and such that $x_{i,j} = 0$, whenever $i = j$ or $m_{i,j} \neq 0$.

There is no condition on the diagonal entries $m_{i,i}$.

Throughout this chapter $\mu(G)$ will denote the above graph invariant. It is known (see for example, [35]) that $\mu(G)$ is minor-monotone.

One immediately recognizes that the matrix M defined by (i) and (ii) of the above definition is nothing but a perturbed Laplacian matrix of G with some suitable weights on the edges. On the other hand if L is a perturbed Laplacian matrix of G and ${}^2\lambda$ is the second smallest eigenvalue of it, then $L - {}^2\lambda I$ behaves like a matrix satisfying the first two conditions of the above definition. Thus an equivalent definition of $\mu(G)$ is the following.

Definition 7.0.72 The parameter $\mu(G)$ is the largest multiplicity of the second smallest eigenvalue of a perturbed Laplacian matrix M satisfying the following condition: there is no nonzero symmetric matrix $X = (x_{i,j})$ such that $MX = {}^2\lambda X$ and such that $x_{i,j} = 0$ whenever $i = j$ or $m_{i,j} \neq 0$.

Let G be any graph and M be any matrix satisfying the above definition. Let Y be an eigenvector in the null space of M. Since M is a perturbed Laplacian matrix and Y is a Fiedler vector, many relevant results form the previous chapters remain valid.

It is well-known that

(i) $\mu(G) \leq 1$ if and only if G is a disjoint union of paths,

(ii) $\mu(G) \leq 2$ if and only if G is outerplanar,

(iii) $\mu(G) \leq 3$ if and only if G is planar and

(iv) $\mu(G) \leq 4$ if and only if G is linklessly embeddable.

van der Holst [36] gave a proof of item (iii) of the above listed properties of $\mu(G)$ using the following lemma.

Lemma 7.0.73 (van der Holst): *Let M be any matrix satisfying the conditions given in Definition 7.0.71 and let X be a vector in the null space of M with minimal support. Then $supp^+(X)$ and $supp^-(X)$ are connected.*

The proof of this lemma directly follows from the Lemma 2.4.30.

Motivated by the above lemma van der Holst, Laurent and Schrijver [37] introduced a graph parameter $\lambda(G)$.

Definition 7.0.74 Let G be a graph on n vertices. Call a subspace $\mathcal{X}$ of $\mathbb{R}^n$ *representative* for G if for each nonzero vector $X \in \mathcal{X}$, $\text{supp}^+(X)$ is nonempty and connected. (Thus $\text{supp}^+(-X)$, that is $\text{supp}^-(X)$, is nonempty and connected.) Then $\lambda(G)$ is defined as the maximum dimension of a representative subspace of $\mathbb{R}^n$.

It is known that $\lambda(G)$ is minor-monotone. (see [35].) Henceforth, by saying $\mathcal{X}$ is a *max-representative space* for G, we will mean the following:

(i) $\mathcal{X}$ is a representative subspace of $\mathbb{R}^n$, where n is the number of vertices of G and

(ii) the dimension of $\mathcal{X}$ is $\lambda(G)$.

Suppose that G is a connected graph. Let X be a nonzero vector in the max-representative space for G. Then we know that $\text{supp}^+(X)$ and $\text{supp}^-(X)$ are nonempty and connected. Thus $\text{supp}(X)$ has exactly two nonzero components. Let M be a matrix satisfying the conditions in Definition 7.0.71 such that $\mu(G) =$ corank(M). Let Y be a nonzero Fiedler vector of M. At this point we put the following fundamental question: *how many nonzero components can supp(Y) have ?* The following result supplies an answer to the above question.

Lemma 7.0.75 *Let G be a connected graph and M a matrix satisfying all conditions of Definition 7.0.71. Let Y be a nonzero Fiedler vector of M. Then supp(Y) has at most three nonzero components.*

Proof Suppose that $\text{supp}(Y)$ has $r(\geq 4)$ components. Let W be the set of vertices u in G such that $Y(u) = 0$. Since $\text{supp}(Y)$ is disconnected, W is nonempty. So, by Theorem 2.2.20, we see that there is no edge in the characteristic set $S = \mathcal{C}(G, Y)$. Thus with a permutation similarity operation we can write

$$M = \begin{bmatrix} M_{11} & 0 & 0 & 0 & \cdots & 0 & M_{1S} \\ 0 & M_{22} & 0 & 0 & \cdots & 0 & M_{2S} \\ 0 & 0 & M_{33} & 0 & \cdots & 0 & M_{3S} \\ 0 & 0 & 0 & M_{44} & \cdots & 0 & M_{4s} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M_{rr} & M_{rS} \\ M_{S1} & M_{S2} & M_{S3} & M_{S4} & \cdots & M_{Sr} & M_{SS} \end{bmatrix}.$$

Let Z be the eigenvector of M corresponding to the smallest eigenvalue. Let Y_i, Z_i be the subvector of Y and Z, respectively, corresponding to M_{ii}, $i = 1, \cdots, r$. We know that Z is positive and Y_i's are either negative or positive. Thus, there is no chance of $Z_i^T Y_i$ being zero. Let

$$\alpha = \frac{-Z_3^T Y_3}{Z_2^T Y_2}, \ \beta = \frac{-Z_4^T Y_4}{Z_1^T Y_1}.$$

Consider the vectors

$$U_1 = \begin{bmatrix} 0 \\ \alpha Y_2 \\ Y_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ U_2 = \begin{bmatrix} \beta Y_1 \\ 0 \\ 0 \\ Y_4 \\ \vdots \\ 0 \end{bmatrix},$$

where the i-th zero entry stands for the zero vector of the order as that of M_{ii}, $i = 1, \cdots, 4$ and more zeros are added so that the order of the vectors are the same as the order of M. It is evident from the proof of item (ii) of Theorem 2.2.20 that the above two vectors are Fiedler vectors of M. Thus $MU_1 = MU_2 = 0$. Consider the matrix

$$X = \left[\begin{array}{cccc|c} 0 & \alpha Y_1 Y_2^T & Y_1 Y_3^T & 0 & \\ \alpha Y_2 Y_1^T & 0 & 0 & \frac{\alpha}{\beta} Y_2 Y_4^T & 0 \\ Y_3 Y_1^T & 0 & 0 & \frac{1}{\beta} Y_3 Y_4^T & \\ 0 & \frac{\alpha}{\beta} Y_1 Y_2^T & \frac{1}{\beta} Y_1 Y_3^T & 0 & \\ \hline & & 0 & & 0 \end{array}\right],$$

where the order of X is the same as that of M. One can easily check that $MX = 0$ and X is of the form described in Definition 7.0.71. The previous statement contradicts the hypothesis that M satisfies all conditions given in Definition 7.0.71. ∎

Below, we give an example of a very simple graph G, a matrix M satisfying all conditions of Definition 7.0.71 and a nonzero Fiedler vector Y of M such that supp(Y) has exactly three nonzero components.

Example 7.0.76 The graph in Figure 7.1 is the unweighted star on 4 vertices, that is $K_{1,3}$. Let L be the Laplacian matrix of G. We know that the second smallest eigenvalue of L is 1. Let $M = L - I$. The matrix M and a Fiedler vector Y are given

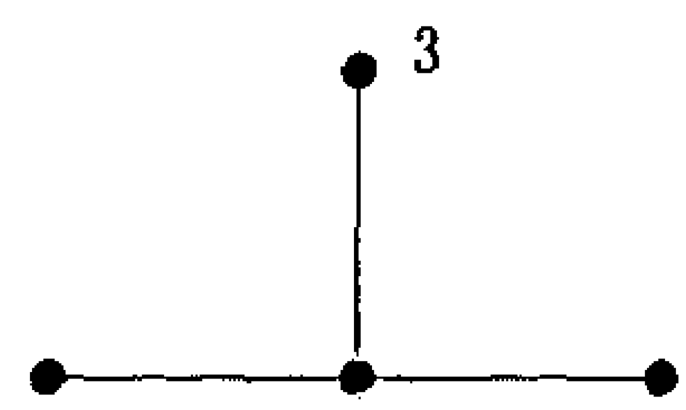


Figure 7.1:

below.

$$M = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

One can see that M satisfies the first two conditions given in Definition 7.0.71. Suppose that there is a *nonzero symmetric matrix* $X = (x_{i,j})$ such that $MX = 0$ and such that $x_{i,j} = 0$, whenever $i = j$ or $m_{i,j} \neq 0$. Thus

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & a & 0 & c \\ 0 & b & c & 0 \end{bmatrix}, \quad a, b, c \in I\!R.$$

From the condition $MX = 0$ we get that

$$a = -b, \; a = -c \text{ and } b = -c \tag{7.0.1}$$

and one can notice that the first two equalities contradict the third, unless $a = b = c = 0$. Thus the matrix M satisfies all conditions given in Definition 7.0.71. One can see that supp(Y) has three nonzero components. ■

Definition 7.0.77 Let G be a graph and $\mathcal{X}$ be a max-representative space for G. We say $\mathcal{X}$ has **full support** if there exists a vector $Y \in \mathcal{X}$ such that supp$(Y) = G$.

We will show that if G is a connected graph then there exists a max-representative space for G with full support. For this purpose we need the following lemma.

Lemma 7.0.78 *Let G be a connected graph and $\mathcal{X}$ be a max-representative space for G. Let u and v be two vertices in G. Consider the space $\mathcal{X}'$ obtained from $\mathcal{X}$ by* *replacing the value $Y(v)$ by $Y(u)$, (denoted by, $Y(v) \leftarrow Y(u)$) for all $Y \in \mathcal{X}$. Then the dimension of $\mathcal{X}'$ is $\lambda(G)$.*

Proof Let $\{Z_i : i = 1, \cdots, \lambda(G)\}$ be a basis for $\mathcal{X}$. Let $\{Z_i' : i = 1, \cdots, \lambda(G)\}$ be the set obtained from the above set by performing the operation $Z_i(v) \leftarrow Z_i(u)$, for all i. If the dimension of $\mathcal{X}'$ is less than $\lambda(G)$ then the set $\{Z_i'\}$ will be a linearly dependent set. Thus we can get a natural number k (> 1) and $k-1$ constants α_i, $i = 1, \cdots, k-1$ such that

$$Z_k' = \sum_{i=1}^{k-1} \alpha_i Z_i'.$$

Recall that Z_i's are linearly independent. Thus we conclude that the vector

$$Z = Z_k - \sum_{i=1}^{k-1} \alpha_i Z_i$$

has only one nonzero entry that is the v-th entry. But this is a contradiction to the fact that $\mathcal{X}$ is a representative space for G (since $Z \in \mathcal{X}$, $\text{supp}^+(Z)$ and $\text{supp}^-(Z)$ should be nonempty). Thus the dimension of $\mathcal{X}' \geq \lambda(G)$. On the other hand, since $\mathcal{X}'$ is isomorphic to a subspace of $\mathcal{X}$, we get that the dimension of $\mathcal{X}' \leq \lambda(G)$. Thus the proof is complete. ∎

Theorem 7.0.79 *Let G be a connected graph. Then there exists a max-representative space $\mathcal{X}$ of G with full support.*

Proof Let $\mathcal{X}$ be a max-representative space for G. If $\mathcal{X}$ is of full support then we have nothing to do. Otherwise, let S be the maximal (with respect to set inclusion) set of vertices of G such that $Y(S) = 0$, for all $Y \in \mathcal{X}$. Choose a vertex $u \in S$ which is adjacent to a vertex $v \notin S$. Consider the vector space $\mathcal{X}'$ obtained from $\mathcal{X}$ by performing the operation $Y(u) \leftarrow Y(v)$, for all $Y \in \mathcal{X}$. By Lemma 7.0.78, the dimension of $\mathcal{X}'$ is $\lambda(G)$. To show that $\mathcal{X}'$ is a max-representative space, let $X' \in \mathcal{X}'$ and let X be the vector obtained from X' by putting $X(u) = 0$. We know that $X \in \mathcal{X}$. Since $\text{supp}^+(X) \neq \emptyset$, $\text{supp}^+(X') \neq \emptyset$. Note that if $v \notin \text{supp}^+(X)$ then $\text{supp}^+(X')$ is the same as $\text{supp}^+(X)$, thus, in this case $\text{supp}^+(X')$ is connected. If $v \in \text{supp}^+(X)$ then $u, v \in \text{supp}^+(X')$ and since the vertices u and v are adjacent, $\text{supp}^+(X')$ is connected. A similar argument shows that $\text{supp}^-(X')$ is connected. Thus $\mathcal{X}'$ is also a max-representative space for G. The proof is complete by induction on the cardinality of S. ∎

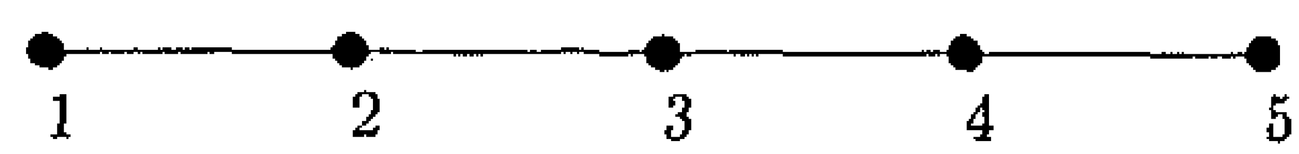


Figure 7.2:

The above theorem shows a way to get a max-representative space for G of full support, starting with any max-representative space. Let us see an example.

Example 7.0.80 *Here the graph G is the path on 5 vertices; Figure 7.2. It is known* that $\lambda(G) = 1$. We start with the max-representative space

$$\mathcal{X} = \left\{ \alpha \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \alpha \in I\!R \right\}.$$

With reference to the above theorem we see that here $S = \{1, 5\}$. Let $u = 1$ and $v = 2$. Then proceeding in the way as described in the proof of the above theorem, we get the new max-representative space

$$\mathcal{X} = \left\{ \alpha \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \alpha \in I\!R \right\}.$$

Here $S = \{1, 2, 5\}$. Let $u = 5$ and $v = 4$. Then we get the new max-representative space

$$\mathcal{X} = \left\{ \alpha \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \alpha \in I\!R \right\}.$$

Here $S = \{1, 2, 4, 5\}$. Let $u = 2$ and $v = 3$. Then we get the new max-representative space

$$\mathcal{X} = \left\{ \alpha \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \alpha \in I\!R \right\},$$

which has full support. ∎

Recall the definition of a block from Chapter 3, Section 2. A vertex is called a *point of articulation* if it is common to more than one blocks. Two points of articulation are called *neighboring* if both of them belong to the same block. The following result is well-known. (See, for example, [19].)

Proposition 7.0.81 *Let G be a connected graph and B_1, B_2 be two different blocks of G. Then there is a unique sequence $v_1, v_2, \cdots, v_r$ $(r \geq 1)$ of points of articulation such that $v_1 \in B_1$ $v_r \in B_2$ and v_i, v_{i+1} are neighboring for $i = 1, \cdots, r-1$.*

With reference to the above proposition, call a block B *an intermediate block* of B_1 and B_2 if B contains the points v_i, v_{i+1}, for some $i \in \{1, \cdots, r-1\}$. The following lemma will be used in the sequel.

Lemma 7.0.82 *Let G be a connected graph and $\mathcal{X}$ be a representative space. Let $Y \in \mathcal{X}$ be a vector with full support. Then there is exactly one block B of G with both positive and negative vertices with respect to Y and any other block is either positive or negative.*

Proof Since $\mathcal{X}$ is a representative space, $\text{supp}^+(Y)$ and $\text{supp}^-(Y)$ are nonempty and connected. Since Y is of full support, $C(G, Y)$ contains edges only. If $C(G, Y)$ is a singleton set then we have nothing to prove. So, assume $e_i = [u_i, v_i]$, $i = 1, 2$ are two edges in $C(G, Y)$ such that $Y(u_i) > 0$ and $Y(v_i) < 0$. Since $\text{supp}^+(Y)$ and $\text{supp}^-(Y)$ are nonempty and connected there are paths P_1 and P_2 joining u_1, u_2 and v_1, v_2 respectively. These two paths along with the two edges form a simple cycle. Thus, for any pair of distinct element in $C(G, Y)$, there is a simple cycle containing them. This shows that all of the edges in $C(G, Y)$ are contained in one block, say B. If there is any other block which contains both positive and negative vertices then that block must contain an edge of $\mathcal{C}(G, Y)$, which is a contradiction to the previous statement. ■

The following is a crucial result in establishing the relationship between the Fiedler vectors of the perturbed Laplacian matrices and the vectors in some max-representative space.

Lemma 7.0.83 *Let G be a connected graph and $\mathcal{X}$ be a representative space (not necessarily a max-representative space). Let $Y, Y' \in \mathcal{X}$ be vectors with full support. Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are the blocks of G with both positive and negative vertices with respect to Y and Y', respectively. Then $\mathcal{B} = \mathcal{B}'$.*

Proof Assume $\mathcal{B} \neq \mathcal{B}'$. Let u be a point of articulation, in the unique sequence of points articulation between the two blocks. (Refer to the Proposition 7.0.81.) By considering scalar multiples, we can assume $Y'(u) = Y(u) < 0$.

Let G_1 be the component of $G - u$ containing $\mathcal{B} - u$. It follows from Lemma 7.0.82 that $Y'(G_1) < 0$. Let

$$\alpha = \min\left\{\frac{|Y'(v)|}{Y(v)} : v \in G_1,\ Y(v) > 0\right\}$$

and suppose that the minimum occurs at a vertex w that is

$$\alpha Y(w) + Y'(w) = 0 \qquad (7.0.2)$$

The number α is well defined, because we know that $Y(\mathcal{B})$ contains one positive entry. Also, let $\alpha' = \alpha + \epsilon$, ϵ very small, positive. Let $X = \alpha' Y + Y'$. Then $X(G_1)$ has a positive entry. In fact it follows from Equation 7.0.2 that

$$X(w) = \alpha' Y(w) + Y'(w) = \alpha Y(w) + Y'(w) + \epsilon Y(w) = \epsilon Y(w) > 0.$$

Note that $X(u) < 0$ and X is a vector in the representative space $\mathcal{X}$. Since $\mathrm{supp}^+(X)$ is connected, we see that **each component of $G-u$, except for G_1, is nonpositive with respect to X.** The above statement is true for all positive ϵ. Thus we conclude, by continuity that if $X = \alpha Y + Y'$, then each component of $G - u$, except for G_1, is nonpositive with respect to X. But note here that G_1 is also nonpositive with respect to X and $X(u) < 0$. In fact, if $v \in G_1$ then

$$\alpha \leq \frac{|Y'(v)|}{Y(v)} \Rightarrow \alpha Y(v) - |Y'(v)| \leq 0 \Rightarrow \alpha Y(v) + Y'(v) \leq 0,$$

since $Y'(G_1) < 0$. Thus we see that the whole graph G is nonpositive with respect to X. This is impossible for $\mathcal{X}$ is a representative space and every nonzero vector in $\mathcal{X}$ must have a nonempty positive support. ∎

Let G be a connected graph and Y be any nonzero vector. Call a block of G, a *c-block* of Y if the block contains both positive and negative vertices with respect to Y. The following theorem which is similar to the Corollary 3.1.33 and Theorem 3.2.41 is the main result of this chapter. The result shows that, given a connected graph G, we can get a *max-representative space* $\mathcal{X}$ for G, with full support where we can talk of c-blocks of vectors in $\mathcal{X}$ and then we can see the similarity in the behavior between the vectors in $\mathcal{X}$ and the *Fiedler vectors* of perturbed Laplacian matrices of G.

Theorem 7.0.84 *Let G be a connected graph and $\mathcal{X}$ be a representative space. Let $Y \in \mathcal{X}$ and suppose that Y has full support. Let B be the c-block for Y. Let $Y' \in \mathcal{X}$ be any nonzero vector. Then there exists a c-block for Y' and it is B.*

Proof Let B' be a block of G satisfying the following conditions:

(i) B' is neither positive nor negative with respect to Y'.

(ii) B' is not zero with respect to Y'.

For example, B' might be a block with some zero vertices and some negative vertices only. Choose (one can easily do so) a very small real number, not necessarily positive, α such that the vector $X = Y' + \alpha Y$ is of full support and $X(B')$ contains both positive and negative entry. Now applying Lemma 7.0.83 to Y and X we get that

$$B = B'. \tag{7.0.3}$$

It follows from the above equation that *given any nonzero vector $Y' \in \mathcal{X}$, there is only one block B of G which satisfies the conditions (i) and (ii) given above.* Thus each other block of G is either positive or negative or zero with respect to Y'. (Otherwise, there are two blocks, one containing some positive and some zero vertices and the other containing some negative and some zero vertices. But this is not possible by the above discussion.) And hence it follows that the block B contains both positive and negative vertices with respect to Y'. Thus B is the c-block for Y'.∎

The reader might, at this stage, wonder about the difference between a c-block of a vector in a max-representative space with full support and the characteristic block

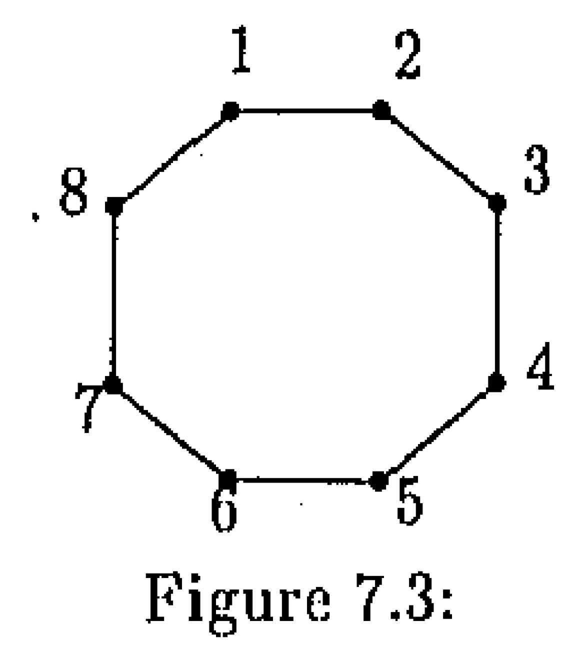


Figure 7.3:

of a Fiedler vector of a perturbed Laplacian matrix. There is only one difference. For a Fiedler vector we have the notion of a characteristic vertex. For example we know that a characteristic vertex with respect to a Fiedler vector, has at least two nonzero adjacent vertices, one positive and one negative. But for a vector in a max-representative space with full support we cannot say the same thing. *If a vertex in the characteristic block is adjacent to a nonzero vertex then it must be adjacent to at least two nonzero vertices, one negative and one positive* whereas *the same is not necessarily true for a c-block.* Below we give an example to show this.

Example 7.0.85 Here the graph G is a cycle on 8 vertices; Figure 7.3. It is well-known that $\lambda(G)$ is two. A max-representative space with full support $\mathcal{X}$ is given by the basis. Basis of $\mathcal{X} = \{Z, Y\}$, where

$$Z = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}, \text{ and } Y = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix}.$$

It is not difficult to check that $\mathcal{X}$ is actually a representative space. Due to symmetry we only have to check the linear combination $X = Z + \alpha Y$, $\alpha \geq 0$.

	$\text{supp}^+(X)$	$\text{supp}^-(X)$
$\alpha > 1$	$\{1, 2, 3, 4\}$	$\{5, 6, 7, 8\}$
$0 < \alpha < 1$	$\{2, 3, 4, 5\}$	$\{6, 7, 8, 1\}$

The cycle it self is the c-block for each nonzero vector in $\mathcal{X}$. One can see that with respect to the vector Y, we have a zero vertex adjacent to only one nonzero vertex. ∎

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