The Strong Law of Large Numbers for Kaplan–Meier *U*-Statistics

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We introduce a Kaplan–Meier *U*-statistic of degree two for *randomly censored* data and prove a strong law for it. We use the technique of Stute and Wang[9] by identifying appropriate reverse-time supermartingale processes. This approach avoids the stringent assumptions of Gijbels and Veraverbeke[1] who consider similar functionals.

KEY WORDS: Random censoring; Kaplan–Meier estimator; strong law of large numbers; supermartingale; Hewitt-Savage zero-one law.

1. INTRODUCTION

Let $(X_n, n \geqslant 1)$, be a sequence of nonnegative i.i.d. random variables, subject to "right censoring" by another sequence $(Y_n, n \geqslant 1)$, of nonnegative i.i.d. random variables, the two sequences being independent of each other. Thus we observe $Z_n = X_n \wedge Y_n = \min\{X_n, Y_n\}$ and $\delta_n = I(X_n \leqslant Y_n)$ only. Let $F(\cdot)$ and $G(\cdot)$ be the common distribution functions (d.f.s) of $(X_n, n \geqslant 1)$ and $(Y_n, n \geqslant 1)$ respectively. The Kaplan–Meier $(K-M)$ product-limit estimator $\hat{F}_n(\cdot)$ of $F(\cdot)$, based on $(Z_i, \delta_i, 1 \leqslant i \leqslant n)$, is given by

$$1 - \hat{F}_n(t) = \prod_{i=1}^{n} \left(1 - \frac{\delta_{[i:n]}}{n-i+1}\right)^{I(Z_{i:n} \leqslant t)} \tag{1.1}$$

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where $Z_{i:n}$, $\delta_{[i:n]}$ are respectively, the ith smallest Z_j, $1 \leqslant j \leqslant n$, and the concomitant δ_j corresponding to $Z_{i:n}$. Here ties within X-values or within Y-values may be ordered arbitrarily, but a tie between an X-value and a Y-value is treated as if the former precedes the latter.

Stute and Wang[4] have obtained a SLLN for

$$S_n(\phi) := \int \phi(x)\, \hat{F}_n(dx)$$

as a generalization of the classical SLLN for uncensored data. The main ingredient of their proof is the significant discovery that $\{S_n(\phi), n \geqslant 1\}$, for $\phi(\cdot)$ nonnegative is a reverse-time supermartingale.

We extend the Stute–Wang SLLN to Kaplan Meier U-statistics of degree two which we define later. Note that $S_n(\phi)$ can be written as

$$S_n(\phi) = \sum_{i=1}^{n} \phi(Z_{i:n})\, W_{i:n}$$

where

$$W_{i:n} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left[\frac{n-j}{n-j-1}\right]^{\delta_{[j:n]}}, \qquad 1 \leqslant i \leqslant n$$

(An empty product equals 1). Here $W_{i:n}$, are the jumps of $\hat{F}_n(\cdot)$ and equal $1/n$ for uncensored data. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be symmetric and let $E_F|\phi(X_1, X_2)| < \infty$. If one is interested in estimating $E_F\phi(X_1, X_2)$, its natural estimator is given by

$$\begin{aligned} U_{2n}(\phi) &= \sum\sum_{1 \leqslant i < j \leqslant n} \phi(Z_{i:n}, Z_{j:n})\, W_{i:n} W_{j:n} \Big/ \sum\sum_{1 \leqslant i < j \leqslant n} W_{i:n} W_{j:n} \\ &= S_{2n}(\phi)/S_{2n}(1) \end{aligned}$$

where

$$\left.\begin{aligned} S_{2n}(\phi) &= \int_{x<y} \phi(x, y)\, \hat{F}_n(dx)\, \hat{F}_n(dy) \\ S_{2n}(1) &= \int_{x<y} \hat{F}_n(dx)\, \hat{F}_n(dy) \end{aligned}\right\} \tag{1.2}$$

We call $U_{2n}(\phi)$ the $K-M$ U-statistics of degree 2. Our normalizing factor $S_{2n}(1)$ is explained by the fact that for uncensored data,

$$W_{i:n} W_{j:n} \Big/ \sum\sum_{1 \leqslant u < v \leqslant n} W_{u:n} W_{v:n} = \binom{n}{2}^{-1} \quad \forall 1 \leqslant i < j \leqslant n$$

We adopt the notation of Stute and Wang.[4] Let $H(\cdot)$ denote the common d.f. of Z_n. By independence of X and Y, $1-H=(1-F)(1-G)$. Denote also, for any d.f. P, F and G,

$$\tau_P = \inf\{x : 1-P(x)=0\}$$

$$P\{x\} = P(x) - P(x-)$$

$$A_P = \{x \in \mathbb{R} \mid P\{x\} > 0\}$$

$$D(F, G) = F(\tau_H -) + I_{\{\tau_H \in A_H\}} F\{\tau_H\}$$

$$S_2(\phi) = E_{F,G}[\phi(X_1, X_2)\, I(X_1 \leqslant Y_1)\, I(X_2 \leqslant Y_2) \times (1-G(X_1))^{-1}\, (1-G(X_2))^{-1}] \tag{1.3}$$

$$= \int_{\{x_1 < \tau_H\}} \bar{\phi}(x_1, F)\, F(dx_1) + I(\tau_H \in A_H)\, \bar{\phi}(\tau_H, F)\, F\{\tau_H\} \tag{1.4}$$

where

$$\bar{\phi}(x_1, F) := \int_{\{x_2 < \tau_H\}} \phi(x_1, x_2)\, F(dx_2) + I(\tau_H \in A_H)\, \phi(x_1, \tau_H)\, F\{\tau_H\} \tag{1.5}$$

Note that $S_2(1) = [D(F, G)]^2$.

Theorem 1. For $\phi(\cdot, \cdot)$ symmetric with $E_F|\phi| < \infty$,

$$\lim_{n \to \infty} U_{2n}(\phi) = \frac{S_2(\phi)}{S_2(1)} \qquad \text{almost surely}$$

The proof of Theorem 1 is presented in Section 2.

Remark 1. This result should hold for higher degree kernels but the notational complexity in the proof will be formidable.

Remark 2. Note that $S_2(\phi)/S_2(1) = E_F(\phi(X_1, X_2) \mid X_1 \leqslant \tau_H, X_2 \leqslant \tau_H)$. Let us compare this with the target value, i.e.,

$$E_F(\phi) = \int_0^{\infty} \int_0^{\infty} \phi(x_1, x_2)\, F(dx_1)\, F(dx_2)$$

Suppose, for the sake of simiplicity, that $F(\cdot)$ is continuous. If $\phi(x_1, x_2) \leqslant E_F(\phi)$ for all $x_1 \leqslant \tau_H$, $x_2 \leqslant \tau_H$, then from (1.4),

$$S_2(\phi)/S_2(1) = \int_0^{\tau_H} \int_0^{\tau_H} \phi(x_1, x_2)\, F(dx_1)\, F(dx_2)/(F(\tau_H))^2 \leqslant E_F(\phi) \tag{1.6}$$

We show that (1.6) continues to hold for nonnegative, co-ordinatewise nondecreasing $\phi(\cdot,\cdot)$, under absolute continuity of $F(\cdot)$. Let $f(\cdot)$ be the density of $F(\cdot)$. Since τ_H is unknown, we show that

$$\bar{F}_\phi(t) = \frac{\int_0^t \int_0^t \phi(x_1, x_2)\, f(x_1)\, f(x_2)\, dx_1\, dx_2}{\int_0^\infty \int_0^\infty \phi(x_1, x_2)\, f(x_1, x_2)\, dx_1\, dx_2} \leqslant \left(\int_0^t f(x)\, dx \right)^2 \;\forall t \geqslant 0 \tag{1.7}$$

Note that $\bar{F}_\phi(t) = \Pr\{\max(V_1, V_2) \leqslant t\}$ where (V_1, V_2) has joint density

$$f_\phi(x_1, x_2) = \{\phi(x_1, x_2)\, f(x_1)\, f(x_2)\}/E_F(\phi), \qquad x_1 \geqslant 0, \quad x_2 \geqslant 0$$

Now $\bar{F}_\phi(t)$ and $F^2(t)$ have densities

$$\bar{f}_\phi(t) = 2\left(\int_0^t \phi(t, x)\, f(t)\, f(x)\, dx \right) \Big/ E_F(\phi)$$

$$\bar{f}(t) = 2f(t)\, F(t)$$

Let $S = \{t > 0: \int_0^t \phi(t, x)\, f(x)\, dx > E_F(\phi).F(t)\}$. If $S = \phi$, then $\bar{f}_\phi(t) \leqslant \bar{f}(t)$ $\forall t \geqslant 0$, whence $\bar{f}_\phi = \bar{f}$ a.e. (Lebesgue measure). Thus equality holds in (1.7). If $S \neq \phi$ and $t_0 \in S$ then $\phi(t_0, t_0) > E_F(\phi)$, since otherwise by monotonicity of $\phi(\cdot,\cdot)$,

$$\phi(t_0, x) \leqslant \phi(t_0, t_0) \leqslant E_F(\phi)\; \forall 0 \leqslant x \leqslant t_0$$

$$\Rightarrow \int_0^{t_0} \pi(t_0, x)\, f(x)\, dx \leqslant E_F(\phi)\, F(t_0), \qquad \text{a contradiction}$$

We show that $t_0 + h \in S\; \forall h \geqslant 0$. Using monotonicity of ϕ,

$$\begin{aligned}
&\int_0^{t_0+h} \phi(t_0 + h, x)\, f(x)\, dx - E_F(\phi)\, F(t_0 + h) \\
&\quad = \int_0^{t_0} \phi(t_0, x)\, f(x)\, dx + \int_0^{t_0+h} (\phi(t_0 + h, x) - \phi(t_0, x))\, f(x)\, dx \\
&\qquad + \int_{t_0}^{t_0+h} \phi(t_0, x)\, f(x)\, dx - E_F(\phi)\, F(t_0 + h) \\
&\quad \geqslant E_F(\phi)\, F(t_0) - E_F(\phi)\, F(t_0 + h) + \int_{t_0}^{t_0+h} \phi(t_0, x)\, f(x)\, dx \\
&\quad = \int_{t_0}^{t_0+h} [\phi(t_0, x) - E_F(\phi)]\, f(x)\, dx \geqslant 0
\end{aligned}$$

Further $\inf S > 0$, since $\inf S = 0 \Rightarrow f_\phi(t) \geqslant \bar{f}(t)\ \forall t \geqslant 0 \Rightarrow f_\phi(t) = f(t)$ a.e. (Lebesgue measure) $\Rightarrow$ equality holds in (1.7). Thus $t^* = \inf S > 0$ and

$$f_\phi(t) \leqslant \bar{f}(t)\ \forall t \leqslant t^*$$
$$> \bar{f}(t)\ \forall t > t^*$$

whence $F_\phi(t) \leqslant F^2(t)\ \forall t \geqslant 0$, proving (1.7). Unlike our U-statistic, the one-dimensional Kaplan Meier integral of Stute and Wang[4] $S_n(\phi)$ is not normalized. Define its normalized version as

$$U_n(\phi) = \frac{S_n(\phi)}{S_n(1)} = \frac{\sum_{i=1}^n \phi(Z_{i:n})\, W_{i:n}}{\sum_{i=1}^n W_{i:n}}$$

Then the following hold:

$$\lim_{n\to\infty} S_n(\phi) = S(\phi) = \int_{\{x < \tau_H\}} \phi(x)\, F(dx) + I_{\{\tau_H \in A_H\}} \phi(\tau_H)\, F\{\tau_H\} \qquad \text{a.s.}$$

$$\lim_{n\to\infty} U_n(\phi) = S(\phi)/S(1) = S(\phi)/D(F, G) \qquad \text{a.s.}$$

In this case, it is easy to show that $S(\phi) \leqslant S(\phi)/S(1) \leqslant E_F(\phi)$ for $\phi(\cdot) \geqslant 0$ and nondecreasing. Thus normalization leads to a smaller asymptotic "bias."

Next, we present a few examples, in which continuity of $F(\cdot)$ is assumed for the sake of clarity.

Example 1. Let $\phi(x) = \phi_t(x) = I(x > t)$ for some $t > 0$, in the one-dimensional case. By Remark 2.

$$\lim_{n\to\infty} S_n(\phi_t) \overset{\text{a.s.}}{=} (F(\tau_H) - F(t))\, I(t \leqslant \tau_H)$$

$$\lim_{n\to\infty} U_n(\phi_t) \overset{\text{a.s.}}{=} \left(1 - \frac{F(t)}{F(\tau_H)}\right) I(t \leqslant \tau_H)$$

$$(F(\tau_H) - F(t))\, I(t \leqslant \tau_H) \leqslant \left(1 - \frac{F(t)}{F(\tau_H)}\right) I(t \leqslant \tau_H) \leqslant 1 - F(t)$$

Example 2. Consider the variance estimator, $\phi(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$. This ϕ is not nondecreasing, so our Remark 2 does not apply.

However for $F(x) = x^k I(0 \leqslant x \leqslant 1) + I(x > 1)$, $k \geqslant 1$ we have $E_F(\phi(X_1, X_2)) = k/[(k+1)^2 (k+2)]$. For $\tau_H \leqslant 1$,

$$\lim_{n \to \infty} U_{2n}(\phi) \overset{\text{a.s.}}{=} E_F(\phi(X_1, X_2) \mid X_1 \leqslant \tau_H, X_2 \leqslant \tau_H)$$

$$= \frac{k}{(k+1)^2 (k+2)} \tau_H^2 \leqslant E_F(\phi(X_1, X_2))$$

Thus (1.6) holds.

But for $F(x) = (1 - e^{-x}) I(x \geqslant 0)$, we have $E_F(\phi(X_1, X_2)) = 1$, whereas

$$E_F(\phi(X_1, X_2) \mid X_1 \leqslant \tau_H, X_2 \leqslant \tau_H)$$
$$= 1 + \tau_H e^{-\tau_H}(1 - e^{-\tau_H})^{-1} - \tau_H^2 e^{-\tau_H}(1 - e^{-\tau_H})^{-1} - \tau_H^2 e^{-2\tau_H}(1 - e^{-\tau_H})^{-2}$$
$$= 1 + g(\tau_H), \qquad \text{say}$$

Since $g(0) = 0$, $g(\cdot)$ is decreasing for $\tau_H \leqslant \frac{1}{2}$ and $g(\cdot) \geqslant 0$ for large τ_H, both over and under-estimation are possible, depending on τ_H.

Gijbels and Veraverbeke[1] consider estimation of functionals

$$\theta(F) = \int_0^T \cdots \int_0^T h(x_1, \ldots, x_m) \prod_{i=1}^{m} F(dx_i), \qquad \text{for a fixed} \quad 0 < T < \tau_H$$

the estimator being $\theta(\hat{F}_n)$. They restrict attention to "truncated" functionals due to their use of an almost-sure linear representation of $\hat{F}_n(\cdot)$, which holds only for $[0, T]$ (uniformly) for all $T < \tau_H$. They rightly remark (p. 1458), "From the practical point of view the truncation is not so desirable." As far as the SLLN is concerned, our work improve upon their results.

2. PROOF OF THEOREM 1

We show that $S_{2n}(\phi)$ converges almost surely to $L/2$, where L is the limit in (1.4), Hence $S_{2n}(1)$ converges a.s. to $S_2(1)/2$. These two facts establish Theorem 1. Our strategy is to show, following Stute and Wang,[4] that $\{S_{2n}(\phi), n \geqslant 2\}$, for $\phi \geqslant 0$, is a reverse-time supermartingale (Lemma 2.1) with respect to the filtration

$$\mathscr{F}_n = \sigma\{Z_{i:n}, \delta_{[i:n]}, 1 \leqslant i \leqslant n, Z_k, \delta_k, k \geqslant n+1\} \tag{2.1}$$

Proposition 5-3-11 of Neveu[2] implies that $\{S_{2n}(\phi), n \geqslant 2\}$ is a.s. convergent. This limit is a.s. constant, being measurable with respect to the trivial σ-field $\mathscr{F}_\infty = \bigcap_{n=1}^{\infty} \mathscr{F}_n$. The remaining lemmas identify the limit to be (1.3). The result holds for a general ϕ because $S_{2n}(\phi) = S_{2n}(\phi^+) - S_{2n}(\phi^-)$.

To carry out this program, we first consider the case when $H(\cdot)$ is continuous and show that the limit in this case is

$$\int\int_{\{x_1 < \tau_H,\, x_2 < \tau_H\}} \phi(x_1, x_2)\, F(dx_1)\, F(dx_2) \tag{2.2}$$

We then indicate how formula (1.3) and (1.4) arise in the general case.

Lemma 1. If $H(\cdot)$ is continuous, then for $\phi \geqslant 0$ $\{S_{2n}(\phi), \mathscr{F}_n, n \geqslant 2\}$ is a reverse-time supermartingale.

Proof. As in the proof of Lemma 2.2 of Stute and Wang,[4]

$$\begin{aligned} S_{2n}(\phi) &= \sum\sum_{1 \leqslant i < j \leqslant n} \phi(Z_{i:n}, Z_{j:n})\, W_{i:n} W_{j:n} \\ &= \sum\sum_{1 \leqslant i < j \leqslant n+1} \phi(Z_{i:n+1}, Z_{j:n+1})\, \hat{F}_n\{Z_{i:n+1}\}\, \hat{F}_n\{Z_{j:n+1}\} \end{aligned}$$

Next we show that, for fixed $1 \leqslant i < j \leqslant n+1$,

$$E\{\hat{F}_n\{Z_{i:n+1}\}\, \hat{F}_n\{Z_{j:n+1}\} \mid \mathscr{F}_{n+1}\} = W_{i:n+1} W_{j:n+1} Q_{ij} \tag{2.3}$$

where

$$\begin{aligned} Q_{ij} &= \bar{Q}_i, \qquad \text{if} \quad j \leqslant n \\ &= \bar{Q}_i - \pi_i \pi_n (1 - \delta_{[n:n+1]}) \frac{(n-i+2)}{(n-i+1)(n+1)}, \qquad \text{if} \quad j = n+1 \\ \bar{Q}_i &= \frac{1}{n+1} \left\{ \sum_{r=1}^{i-1} \pi_r^2 \left(\frac{n-r+2}{n-r+1}\right)^{2\delta_{[r:n+1]}} \right. \\ &\qquad \left. + \pi_i^2 (n-i+2) \left[\frac{(n-i)(n-i+2)}{(n-i+1)^2}\right]^{\delta_{[i:n+1]}} \right\} \\ \pi_r &= \prod_{k=1}^{r-1} \left[\frac{(n-k)(n-k+2)}{(n-k+1)^2}\right]^{\delta_{[k:n+1]}}, \qquad 1 \leqslant r \leqslant n \end{aligned} \tag{2.4}$$

The proof is then completed by showing that $0 < \bar{Q}_i \leqslant 1$ if $\delta_{[i:n+1]} = 1$, $1 \leqslant i \leqslant n$, since $\phi(\cdot, \cdot)$ is nonnegative. If $\delta_{[i:n+1]} = 0$ then $W_{i:n+1} = 0$, which is a trivial case.

To show (2.3), consider

$$E(\hat{F}_n\{Z_{i:n+1}\}\,\hat{F}_n\{Z_{j:n+1}\}\mid \mathscr{F}_{n+1})$$

$$= E\left(\sum_{r=1}^{n+1} \hat{F}_n\{Z_{i:n+1}\}\,\hat{F}_n\{Z_{j:n+1}\}\, I\{Z_{n+1} = Z_{r:n+1}\}\mid \mathscr{F}_{n+1}\right) \qquad (2.5)$$

Now on $\{Z_{n+1} = Z_{r:n+1}\}$ we have for $1 \leqslant l \leqslant n+1$.

$$\hat{F}_n\{Z_{l:n+1}\} = \begin{cases} W_{l-1:n} & \text{if } \; l > r \\ 0 & \text{if } \; l = r \\ W_{l:n} & \text{if } \; l < r \end{cases}$$

Also on $\{Z_{n+1} = Z_{r:n+1}\}$ for $1 \leqslant l \leqslant n$,

$$\delta_{[l:n]} = \begin{cases} \delta_{[l:n+1]} & \text{if } \; l < r \\ \delta_{[l+1:n+1]} & \text{if } \; l \geqslant r \end{cases}$$

Using $A_m := \{Z_{n+1} = Z_{r:n+1}\}$, the conditional expection in (2.5) is

$$\sum_{r=1}^{n+1} \hat{F}_n\{Z_{i:n+1}\}\,\hat{F}_n\{Z_{j:n+1}\}\, I_{A_m}$$

$$= \sum_{r=1}^{i-1} W_{i-1:n} W_{j-1:n} I_{A_m} + \sum_{r=i+1}^{j-1} W_{i:n} W_{j-1:n} I_{A_m} + \sum_{r=j+1}^{n+1} W_{i:n} W_{j:n} I_{A_m}$$

$$= T_1 + T_2 + T_3, \qquad \text{say}$$

Here, an empty sum is to be taken as zero. Now,

$$T_1 = \sum_{r=1}^{i-1} \frac{\delta_{[i-1:n]}}{n-i+2} \prod_{k=1}^{r-1}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k:n]}} \prod_{k=r}^{i-2}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k:n]}}$$

$$\times \frac{\delta_{[j-1:n]}}{n-j+2} \prod_{k=1}^{r-1}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k:n]}} \prod_{k=r}^{j-2}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k:n]}} I_{A_m}$$

$$= \sum_{r=1}^{i-1} \frac{\delta_{[i:n+1]}}{(n+1)-i+1} \prod_{k=1}^{r-1}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k:n+1]}} \prod_{k=r}^{i-2}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k+1:n+1]}}$$

$$\times \frac{\delta_{[j:n+1]}}{(n+1)-j+1} \prod_{k=1}^{r-1}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k:n+1]}} \prod_{k=r}^{j-2}\left[\frac{n-k}{n-k+1}\right]^{\delta_{[k+1:n+1]}} I_{A_m}$$

$$= W_{i:n+1} W_{j:n+1} \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n-r+2}{n-r+1}\right]^{2\delta_{[r:n+1]}} I_{A_m}$$

with π_r as defined in (2.4). By similar adjustments,

$$T_2 = W_{i:n+1} W_{j:n+1} \pi_i \frac{(n-i+2)}{(n-i+1)} \sum_{r=i+1}^{j-1} \pi_r \left[\frac{n-r+2}{n-r+1}\right]^{\delta_{[r:n+1]}} I_{A_{rn}}$$

$$T_3 = W_{i:n+1} W_{j:n+1} \pi_i \pi_j \frac{(n-i+2)(n-j+2)}{(n-i+1)(n-j+1)} \sum_{r=j+1}^{n+1} I_{A_{rn}}$$

Now substituting these expressions for T_1, T_2, and T_3 in (2.6), and noting that the event A_{rn} is independent of $(Z_{i:n+1}, \delta_{[i:n+1]}, 1 \leqslant i \leqslant n+1)$ by the continuity of H (cf. Stute and Wang,[4] Lemma 2.1), we have

$$\begin{aligned}
&E(\hat{F}_n\{Z_{i:n+1}\}\, \hat{F}_n\{Z_{j:n+1}\} \mid \mathscr{F}_{n+1})\\
&\quad = E(T_1 + T_2 + T_3 \mid \mathscr{F}_{n+1})\\
&\quad = W_{i:n+1} W_{j:n+1} \frac{1}{n+1} \left[\sum_{r=1}^{i-1} \pi_r^2 \left(\frac{n-r+2}{n-r+1}\right)^{2\delta_{[r:n+1]}} \right.\\
&\qquad + \sum_{r=i+1}^{j-1} \pi_i \pi_r \left(\frac{n-i+2}{n-i+1}\right)\left(\frac{n-r+2}{n-r+1}\right)^{\delta_{[r:n+1]}}\\
&\qquad \left. + \pi_i \pi_j \left(\frac{n-i+2}{n-i+1}\right)\left(\frac{n-j+2}{n-j+1}\right)(n-j+1) \right] \qquad (2.7)
\end{aligned}$$

where the last term inside $[\cdots]$ in (2.7) is zero if $j = n+1$. Using the relation,

$$\begin{aligned}
&\sum_{r=1}^{i-1} \pi_r \left(\frac{n-r+2}{n-r+1}\right)^{\delta_{[r:n+1]}} + \pi_i (n-i+2)\\
&\qquad = (n+1) - \pi_n (1 - \delta_{[n:n+1]})\, I\{i = n+1\}, \qquad 1 \leqslant i \leqslant n+1
\end{aligned}$$

given after Eq. (2.2) in Stute and Wang,[4] the last two terms inside the bracket in (2.7) simplify as

$$\begin{aligned}
&\frac{n-i+2}{n-i+1} \pi_i \left[\sum_{r=i+1}^{j-1} \pi_r \left(\frac{n-r+2}{n-r+1}\right)^{\delta_{[r:n+1]}} + \pi_j (n-j+2) \right]\\
&\quad = \frac{n-i+2}{n-i+1} \pi_i \left[\sum_{r=1}^{j-1} \pi_r (\cdots) - \sum_{r=1}^{i-1} \pi_r (\cdots) \right.\\
&\qquad \left. - \pi_i \left(\frac{n-i+2}{n-i+1}\right)^{\delta_{[i:n+1]}} + \pi_j (n-j+2) \right]
\end{aligned}$$

$$= \frac{n-i+2}{n-i+1}\pi_i \left[\left\{ (n-i+2) - \left(\frac{n-i+2}{n-i+1}\right)^{\delta_{[i:n+1]}} \right\} \times \pi_i - \pi_n (1 - \delta_{[n:n-1]})\, I\{j = n+1\} \right] \tag{2.8}$$

(Here $\pi_{n-1} = 0$). Putting (2.8) in (2.7), we get (2.3). It remains to show that $Q_i \leqslant 1$ whenever $\delta_{[i:n-1]} = 1$. We use induction on i. Note first that, for $1 \leqslant i \leqslant n-1$,

$$\begin{aligned} Q_{i+1} - \bar{Q}_i &= \frac{\pi_i^2}{n+1} \left[\left\{ \frac{(n-i)(n-i+2)}{(n-i+1)^2} \right\}^{2\delta_{[i:n+1]}} \right. \\ &\quad \times (n-i+1) \left\{ \frac{(n-i+1)(n-i+1)}{(n-i)^2} \right\}^{\delta_{[i+1:n+1]}} \\ &\quad \left. - \left(\frac{n-i+2}{n-i-1} \right)^{2\delta_{[i:n+1]}} - (n-i+2) \left\{ \frac{(n-i)(n-i+2)}{(n-i+1)^2} \right\}^{\delta_{[i:n+1]}} \right] \\ &= \begin{cases} 0, & \text{if } \delta_{[i:n+1]} = \delta_{[i+1:n+1]} = 0 \text{ or } 1 \\ -\dfrac{(n-i+1)\pi_i^2}{(n-i)^2 (n+1)}, & \text{if } \delta_{[i:n+1]} = 0, \quad \delta_{[i+1:n+1]} = 1 \\ \dfrac{(n-i+2)^2 \pi_i^2}{(n-i+1)^2 (n+1)}, & \text{if } \delta_{[i:n+1]} = 1, \quad \delta_{[i+1:n+1]} = 0 \end{cases} \end{aligned} \tag{2.9}$$

For $i = 1$ and $\delta_{[i:n+1]} = 1$, noting that $\pi_1 = 1$,

$$\bar{Q}_1 = \frac{(n-1)(n+1)}{n^2} = \frac{n^2-1}{n^2} < 1$$

Next, for $i = 2$ and $\delta_{[2:n+1]} = 1$,

$$\begin{aligned} Q_2 &= \bar{Q}_1 + (Q_2 - \bar{Q}_1) \\ &= \begin{cases} \dfrac{n^2-1}{n^2} + 0, & \text{if } \delta_{[1:n+1]} = 1 \\ 1 - \dfrac{n}{(n-1)^2 (n+1)}, & \text{if } \delta_{[1:n+1]} = 0 \end{cases} \end{aligned}$$

by (2.9). Hence $Q_2 < 1$ if $\delta_{[2:n+1]} = 1$.

Now assume $\bar{Q}_l \leqslant 1$ whenever $\delta_{[l:n+1]} = 1\ \forall 1 \leqslant l \leqslant i$. Let $\delta_{[i+1:n+1]} = 1$. Then, in view of (2.9), $\bar{Q}_{i+1} = \bar{Q}_{i+1-(k^*-1)}$, where

$$k^* = \inf\{k \geqslant 1 \mid \delta_{[i+1-k:n+1]} = 0\}$$

Hence assume $k^* = 1$, otherwise we are done by the induction hypothesis.

$$\bar{Q}_{i+1} = \bar{Q}_{i+1-l^*} + (\bar{Q}_{i+1-(l^*-1)} - \bar{Q}_{i+1-l^*}) + (\bar{Q}_{i+1} - \bar{Q}_i) \tag{2.10}$$

where

$$l^* = \begin{cases} \inf\{l \geqslant 2 \mid \delta_{[i+1-l:n+1]} = 1\} \\ i, \quad \text{if} \quad \delta_{[i+1-l:n+1]} = 0\ \forall 1 \leqslant l \leqslant i \end{cases}$$

since $\bar{Q}_{i+1-(l^*-1)} = \cdots = \bar{Q}_i$, by (2.9). Now if $\delta_{[i+1-l:n+1]} = 0\ \forall 1 \leqslant l \leqslant i$,

$$\begin{aligned} \bar{Q}_{i+1} &= \bar{Q}_1 + (\bar{Q}_{i+1} - \bar{Q}_i) \\ &= 1 - \frac{(n-i+1)}{(n-i)^2 (n+1)} < 1 \end{aligned}$$

Hence assume $\delta_{[i+1-l:n+1]} = 1$ for some $2 \leqslant l \leqslant i$. Then by the induction hypothesis, $Q_{i+1-l^*} \leqslant 1$, and since $\delta_{[i+1-l^*+1:n+1]} = 0$, we have

$$\begin{aligned} &(\bar{Q}_{i+1-l^*+1} - \bar{Q}_{i+1-l^*}) + (\bar{Q}_{i+1} - \bar{Q}_i) \\ &= \frac{\pi^2_{i+1-l^*}}{n+1} \frac{(n-i-1+l^*+2)^2}{(n-i-1+l^*+1)^3} - \frac{\pi_i^2}{n+1} \frac{(n-i+1)}{(n-i)^2} \\ &= \frac{1}{n+1} \left[\pi^2_{i+1-l^*} \frac{(n-i-1+l^*)^2 (n-i-1+l^*+2)^2}{(n-i-1+l^*+1)^4} \right. \\ &\quad \left. \times \frac{(n-i-1+l^*+1)}{(n-i-1+l^*)^2} - \pi_i^2 \frac{(n-i+1)}{(n-i)^2} \right] \\ &= \frac{1}{n+1} \left[\pi^2_{i+1-l^*+1} \frac{(n-i-1+l^*+1)}{(n-i-1+l^*)^2} - \pi_i^2 \frac{(n-i+1)}{(n-i)^2} \right] \\ &= \frac{\pi_i^2}{n+1} \left[\frac{(n-i+l^*)}{(n-i+(l^*-1))^2} - \frac{(n-i+1)}{(n-i)^2} \right] < 0 \end{aligned}$$

using the facts

$$\pi_{l+1} = \pi_l \left[\frac{(n-l)(n-l+2)}{(n-l+1)^2} \right]^{\delta_{[l:n+1]}}, \qquad 1 \leqslant l \leqslant n-1$$

and $\pi_{l+1-l^*-1} = \pi_l$, since $\delta_{[l+1-l^*+1:n+1]} = \cdots = \delta_{[l:n+1]} = 0$. Hence finally, $\bar{Q}_{l+1} < 1$ by (2.10). This completes the proof of Lemma 1. ∎

As discussed earlier, we have for $\phi \geqslant 0$,

$$\lim_{n\to\infty} S_{2n} = S = \lim_{n\to\infty} E(S_{2n}(\phi) \mid \mathscr{F}_\infty) = \lim_{n\to\infty} E(S_{2n}(\phi)) \qquad \text{a.s.}$$

We now show that $\lim_{n\to\infty} E(S_{2n}(\phi)) = L/2$ where L is (2.2) when $H(\cdot)$ is continuous and the limit in (1.3) (or (1.4)) for general $H(\cdot)$. Define

$$\begin{aligned}
m(t) &= E(\delta_1 \mid Z_1 = t)\\
C_n(t) &= \sum_{i=1}^{n+1} \left(\frac{1-m(t)}{n-i+2} \right) I(Z_{i-1:n} < t \leqslant Z_{i:n})\\
D_n(t,s) &= \prod_{k=1}^{n} \left(1 - \frac{1-m(Z_{k:n})}{n-k+2} \right)^{2I(Z_{k:n} < t)} \prod_{l=1}^{n} \left(1 - \frac{1-m(Z_{l:n})}{n-l+1} \right)^{I(t < Z_{l:n} < s)}\\
\Lambda_n(t,s) &= ED_n(t,s)\\
\Delta_n(t,s) &= ED_n(t,s)\, C_n(t) \qquad (2.11)
\end{aligned}$$

where $Z_{0:n} = -\infty$, $Z_{n+1:n} = \infty$.

Lemma 2. For $H(\cdot)$ continuous,

$$E\{S_{2n}(\phi)\} = \frac{n-1}{2n} E[2\phi(Z_1, Z_2)\, m(Z_1)\, m(Z_2)\{\Lambda_{n-2}(Z_1, Z_2) + \Delta_{n-2}(Z_1, Z_2)\}\, I(Z_1 < Z_2)] \qquad (2.12)$$

Proof. To see this, following Stute and Wang,[4]

$$\begin{aligned}
E(S_{2n}(\phi)) &= E\left\{ \sum\sum_{1 \leqslant i < j \leqslant n} \phi(Z_{i:n}, Z_{j:n})\, W_{i:n} W_{j:n} \right\}\\
&= \binom{n}{2} \frac{1}{n^2} E\{\phi(Z_1, Z_2)\, m(Z_1)\, m(Z_2)\, B_n(Z_1)\, B_n(Z_2)\}
\end{aligned}$$

where

$$B_n(s) = \prod_{k=1}^{n} \left(1 + \frac{1 - m(Z_k)}{n - R_{kn}}\right)^{I(Z_k < s)}$$

$$R_{kn} = \text{rank of } Z_k \text{ among } \{Z_1, ..., Z_n\}$$

$$B_n(Z_1)\, B_n(Z_2) = \left(1 + \frac{1 - m(Z_1)}{n - R_{1n}}\right)^{I(Z_1 < Z_2)} \left(1 + \frac{1 - m(Z_2)}{n - R_{2n}}\right)^{I(Z_2 < Z_1)}$$

$$\times \prod_{k=3}^{n} \left(1 + \frac{1 - m(Z_k)}{n - R_{kn}}\right)^{I(Z_k < Z_1) + I(Z_k < Z_2)}$$

$$= I(Z_1 < Z_2) \left(1 + \frac{1 - m(Z_1)}{n - R_{1n}}\right)$$

$$\times \prod_{k=3}^{n} \left(1 + \frac{1 - m(Z_k)}{n - R_{kn}}\right)^{2I(Z_k < Z_1 < Z_2)}$$

$$\times \prod_{k=3}^{n} \left(1 + \frac{1 - m(Z_k)}{n - R_{kn}}\right)^{I(Z_1 < Z_k < Z_2)} + I(Z_1 \geqslant Z_2)\{\cdots\}$$

Hence for continuous $H(\cdot)$,

$$E\{B_n(Z_1)\, B_n(Z_2) \mid Z_1 = t, Z_2 = s\}$$

$$= I(t < s)\, E\left[\left\{\sum_{i=1}^{n-1} \left(1 + \frac{1 - m(t)}{n - i}\right) I(Z_{i-1:n-2} < t \leqslant Z_{i:n-2})\right\}\right.$$

$$\times \prod_{k=1}^{n-2} \left(1 + \frac{1 - m(Z_{k:n-2})}{n - k}\right)^{2I(Z_{k:n-2} < t)}$$

$$\left.\times \prod_{l=1}^{n-2} \left(1 + \frac{1 - m(Z_{l:n-2})}{n - l - 1}\right)^{I(t < Z_{l:n-2} < s)}\right]$$

$$+ I(t > s)\, E[\cdots] + I(t = s)\{\cdots\} \tag{2.13}$$

(here, $Z_{0:n-2} = -\infty$, $Z_{n-1:n-2} = \infty$) because, denoting by $R_{k:n-2}$ the rank of Z_k, $3 \leqslant k \leqslant n$, among themselves, we have

$$R_{kn} = \begin{cases} R_{k:n-2} & \text{on} \quad \{Z_k < Z_1 < Z_2\} \\ R_{k:n-2} + 1 & \text{on} \quad \{Z_1 < Z_k < Z_2\} \end{cases}$$

Since Z_1, Z_2 are i.i.d., by symmetry the co-efficient of $I(t > s)$ in (2.13) is the same as that of $I(t < s)$ *with t and s interchanged.* Thus while taking expectation in (2.13), the RHS can be replaced by $2I(t<s)\{\Delta_{n-2}(t,s)+\bar{\Delta}_{n-2}(t,s)\}$, as the term $I(t=s)\{\cdots\}$ has zero expectation, by continuity of $H(\cdot)$. The Lemma now follows. □

The following two lemmas are analogues of Lemmas 2.5 and 2.6 of Stute and Wang.[4]

Lemma 3. For fixed $0<t<s$ and continuous $H(\cdot)$, $\{D_n(t,s), \mathscr{G}_n = \sigma(Z_{1:n},\ldots, Z_{n:n}, Z_k, k\geqslant n+1), n\geqslant 1\}$ is a reverse-time supermartingale.

Proof. Consider

$$\begin{aligned}
&E\{D_n(t,s)\mid \mathscr{G}_{n+1}\}\\
&\quad= E\left\{\prod_{k=1}^{n}\left(1-\frac{1-m(Z_{k:n})}{n-k+2}\right)^{2I(Z_{k:n}<t)}\right.\\
&\qquad\times\left.\left(1+\frac{1-m(Z_{k:n})}{n-k+1}\right)^{I(t<Z_{k:n}<s)}\mid \mathscr{G}_{n+1}\right\}\\
&\quad=\sum_{l=1}^{n+1} E\left\{I(Z_{n+1}=Z_{l:n+1})\prod_{k=1}^{n}(\cdots)\mid \mathscr{G}_{n+1}\right\}\\
&\quad=\frac{1}{n+1}\sum_{l=1}^{n+1}\prod_{k=1}^{l-1}\left(1-\frac{1-m(Z_{k:n+1})}{n-k+2}\right)^{2I(Z_{k:n+1}<t)}\\
&\qquad\times\left(1+\frac{1-m(Z_{k:n+1})}{n-k+1}\right)^{I(t<Z_{k:n+1}<s)}\\
&\qquad\times\prod_{k=l+1}^{n+1}\left(1-\frac{1-m(Z_{k:n+1})}{n-k+3}\right)^{2I(Z_{k:n+1}<t)}\\
&\qquad\times\left(1+\frac{1-m(Z_{k:n+1})}{n-k+2}\right)^{I(t<Z_{k:n+1}<s)}
\end{aligned}$$

We now use induction on n. For $n=1$, we have

$$\begin{aligned}
&E(D_1(t,s)\mid \mathscr{G}_2)\\
&\quad=\frac{1}{2}\left[\left(1-\frac{1-m(Z_{2:2})}{2}\right)^{2I(Z_{2:2}<t)}(1+(1-m(Z_{2:2})))^{I(t<Z_{2:2}<s)}\right.\\
&\qquad+\left.\left(1-\frac{1-m(Z_{1:2})}{2}\right)^{2I(Z_{1:2}<t)}(1+(1-m(Z_{1:2})))^{I(t<Z_{1:2}<s)}\right]
\end{aligned}$$

whereas

$$D_2(t,s)=\left(1+\frac{1-m(Z_{1:2})}{3}\right)^{2I(Z_{1:2}<t)}\left(1+\frac{1-m(Z_{1:2})}{2}\right)^{I(t<Z_{1:2}<s)}$$
$$\times\left(1+\frac{1-m(Z_{2:2})}{2}\right)^{2I(Z_{2:2}<t)}(1+(1-m(Z_{2:2})))^{I(t<Z_{2:2}<s)}$$
$$=\left[1+x_2s_2+\left(\frac{x_2^2}{4}+x_2\right)t_2\right]\left[1+\frac{x_1}{2}s_1+\left(\frac{x_1^2}{9}+\frac{2x_1}{3}\right)t_1\right]$$

where

$$1-m(Z_{i:2})=x_i,\qquad I(Z_{i:2}<t)=t_i,\qquad I(t<Z_{i:2}<s)=s_i,\qquad i=1,2$$

Further

$$E(D_1(t,s)\mid\mathscr{G}_2)=\frac{1}{2}\left[1+x_2s_2+\left(\frac{x_2^2}{4}+x_2\right)t_2+1+x_1s_1+\left(\frac{x_1^2}{4}+x_1\right)t_1\right]$$

and clearly, $E(D_1(t,s)\mid\mathscr{G}_2)\leqslant D_2(t,s)$ noting, among other things, that

$$\frac{1}{2}\left(\frac{x_1^2}{4}+x_1\right)\leqslant\left(\frac{x_1^2}{9}+\frac{2}{3}x_1\right)\qquad\text{for}\quad 0\leqslant x_1\leqslant 1$$

Assume that the assertion holds for n. This is equivalent to:

(R) the inequality $E\{D_n(t,s)\mid\mathscr{G}_{n+1}\}\leqslant D_{n+1}(t,s)$ holds with $Z_{k:n+1}$ replaced by arbitrary $y_k\geqslant 0$, $1\leqslant k\leqslant n+1$, in the corresponding formula.

Now with

$$A=\prod_{k=2}^{n+2}\left(1+\frac{1-m(Z_{k:n+2})}{n-k+4}\right)^{2I(Z_{k:n+2}<t)}\left(1+\frac{1-m(Z_{k:n+2})}{n-k+3}\right)^{I(t<Z_{k:n+2}<s)}$$

$$E(D_{n+1}(t,s)\mid\mathscr{G}_{n+2})$$
$$=\frac{1}{n+2}\left\{A+\sum_{l=2}^{n+2}\prod_{k=1}^{l-1}(\cdots)\prod_{k=l+1}^{n+2}(\cdots)\right\}$$
$$=\frac{1}{n+2}\left\{A+\left(1+\frac{1-m(Z_{1:n+2})}{n+2}\right)^{2I(Z_{1:n+2}<t)}\right.$$
$$\times\left(1+\frac{1-m(Z_{1:n+2})}{n+1}\right)^{I(t<Z_{1:n+2}<s)}$$

$$
\times \sum_{l=1}^{n+1} \prod_{k=1}^{l-1} \left(1 - \frac{1 - m(Z_{k-1:n+2})}{n-k+2}\right)^{2I(Z_{k:n+2} < t)}
$$

$$
\times \left(1 + \frac{1 - m(Z_{k-1:n+2})}{n-k+1}\right)^{I(t < Z_{k:n+2} < s)}
$$

$$
\times \prod_{k=l+1}^{n+1} \left(1 - \frac{1 - m(Z_{k+1:n+2})}{n-k+3}\right)^{2I(Z_{k:n+2} < t)}
$$

$$
\times \left(1 + \frac{1 - m(Z_{k+1:n+2})}{n-k+2}\right)^{I(t < Z_{k:n+2} < s)} \Bigg\}
$$

$$
\leqslant \frac{A}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{1 - m(Z_{1:n+2})}{n+2}\right)^{2I(Z_{1:n+2} < t)}
$$

$$
\times \left(1 + \frac{1 - m(Z_{1:n+2})}{n+1}\right)^{I(t < Z_{1:n+2} < s)}
$$

$$
\times \prod_{k=1}^{n+1} \left(1 + \frac{1 - m(Z_{k+1:n+2})}{n-k+3}\right)^{2I(Z_{k+1:n+2} < t)}
$$

$$
\times \left(1 + \frac{1 - m(Z_{k+1:n+2})}{n-k+2}\right)^{I(t < Z_{k+1:n+2} < s)} \tag{2.14}
$$

where the last inequality follows using observation (R). Continuing, we have

$$
E\{D_{n+1}(t, s) \mid \mathscr{G}_{n+2}\}
$$

$$
\leqslant \frac{A}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{1 - m(Z_{1:n+2})}{n+2}\right)^{2I(Z_{1:n+2} < t)}
$$

$$
\times \left(1 + \frac{1 - m(Z_{1:n+2})}{n+1}\right)^{I(t < Z_{1:n+2} < s)} A
$$

substituting k for $(k+1)$ under the product sign in the second term in (2.14),

$$
= \left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{1 - m(Z_{1:n+2})}{n+2}\right)^{2I(Z_{1:n+2} < t)}\right]
$$

$$
\times \left(1 + \frac{1 - m(Z_{1:n+2})}{n+2}\right)^{I(t < Z_{1:n+2} < s)} A
$$

The result now follows from this last expression because for $0 \leqslant x \leqslant 1$,

$$\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x}{n+2}\right)^2 \leqslant \left(1+\frac{x}{n+3}\right)^2$$

The proof of the Lemma is now complete. □

Now refer to the quantities defined in (2.11). Further, let

$$D(t,s)=\exp\left\{\int_0^t \frac{1-m(x)}{1-H(x)}\,H(dx)+\int_0^s \frac{1-m(y)}{1-H(y)}\,H(dy)\right\}$$

Lemma 4. For continuous $H(\cdot)$ and $t<s$ such that $H(s)<1$, we have

(a) $\lim_{n\to\infty} D_n(t,s) =^{\text{a.s.}} \exp\{2\int_0^t (1-m(x))/(1-H(x))\,H(dx)+\int_t^s (1-m(y))/(1-H(y))\,H(dy)\} = D(t,s)$

(b) $\lim_{n\to\infty} C_n(t)=0$ a.s.; also, $0\leqslant C_n(t)\leqslant 1\ \forall n\geqslant 1,\ t\geqslant 0$.

Proof. (a) follows exactly as in Lemma 2.6 of Stute and Wang.[4] Details are omitted. To prove (b), note that $0\leqslant C_n(t)\leqslant 1$. Let $H_n(x)=1/n\sum_{j=1}^n I(Z_j\leqslant x)$ be the empirical d.f. of $Z_1,\dots,Z_n$.

$$\begin{aligned}
C_n(t) &= \sum_{i=1}^{n+1}\left(\frac{1-m(t)}{n-i+2}\right)[I(Z_{i-1:n}<t)-I(Z_{i:n}<t)]\\
&=(1-m(t))\left[\frac{1}{n+1}+\sum_{i=1}^{n}\left\{\frac{1}{n-i+1}-\frac{1}{n-i+2}\right\}I(Z_{i:n}<t)\right]\\
&=(1-m(t))\left[\frac{1}{n+1}+\int_0^t\left\{\frac{1}{1-H_n(x)+(1/n)}\right.\right.\\
&\qquad\left.\left.-\frac{1}{1-H_n(x)+(2/n)}\right\}H_n(dx)\right]
\end{aligned}$$

By arguments similar to those given at the end of the proof of Lemma 2.6 in Stute and Wang[4] this converges to 0 a.s. proving Lemma 4.

Now we are ready to identify $\lim_{n\to\infty} E(S_{2n}(\phi))$. By Proposition 5-3-11 of Neveu[2] and Lemma 3, $D_n(t,s)$, $n\geqslant 1$, is uniformly integrable for fixed $0<t<s$. Further, by the former and Lemma 4, for fixed $0<t<s$,

$$\Delta_n(t,s)=ED_n(t,s)-E(D_n(t,s)\mid \mathscr{G}_\infty)\uparrow D(t,s) \qquad \text{as} \quad n\to\infty \tag{2.15}$$

By Lemma 4, $C_n(t)\,D_n(t,s)\to 0$ a.s. as $n\to\infty$. Hence

$$\left.\begin{aligned}\bar{\Delta}_n(t,s)=EC_n(t)\,D_n(t,s)\to 0 \qquad \text{as} \quad n\to\infty\\ \bar{\Delta}_n(t,s)\leqslant ED_n(t,s)\leqslant D(t,s)\ \forall n\geqslant 1\end{aligned}\right\}\tag{2.16}$$

Under the conditions of Lemma 4, we get as in Stute and Wang,[4]

$$D(t,s)=(1-G(t))^{-1}\,(1-G(s))^{-1}$$

Thus for continuous $H(\cdot)$,

$$\begin{aligned}&E[2\phi(Z_1,Z_2)\,m(Z_1)\,m(Z_2)\,D(Z_1,Z_2)\,I(Z_1<Z_2)]\\ &\quad=E[\phi(X_1,X_2)\,I(X_1\leqslant Y_1,X_1<\tau_H)\,I(X_2\leqslant Y_2,X_2<\tau_H)\,D(X_1,X_2)]\\ &\quad=\iint_{\{x_1<\tau_H,\,x_2<\tau_H\}}\phi(x_1,x_2)\,F(dx_1)\,F(dx_2)<\infty\end{aligned}\tag{2.17}$$

Now recall from (2.12) that

$$\begin{aligned}&E(S_{2n}(\phi))\\ &\quad=(1/2-o(1))\,E[2\phi(Z_1,Z_2)\,m(Z_1)\,m(Z_2)(\Delta_{n-2}(Z_1,Z_2)\\ &\qquad+\bar{\Delta}_{n-2}(Z_1,Z_2))\,I(Z_1<Z_2)]\\ &\quad=(1/2+o(1))\,E[2\phi(X_1,X_2)\,I(X_1\leqslant Y_1,X_1<\tau_H)\\ &\qquad\times I(X_2\leqslant Y_2,X_2<\tau_H)(\Delta_{n-2}(X_1,X_2)\\ &\qquad+\bar{\Delta}_{n-2}(X_1,X_2))\,I(X_1<X_2)]\end{aligned}$$

By (2.15)–(2.17), the sequence of random variables under this expectation is uniformly integrable and converges almost surely to

$$2\phi(X_1,X_2)\,I(X_2\leqslant Y_1,X_1<\tau_H)\,I(X_2\leqslant Y_2,X_1<\tau_H)\,D(X_1,X_2)\,I(X_1<X_2)$$

Therefore $E(S_{2n}(\phi))\to\frac{1}{2}$(the limit in (2.17)) and $E\{S_{2n}(1)\}\to F^2(\tau_H)/2$ and hence for continuous $H(\cdot)$ and $\phi(\cdot,\cdot)\geqslant 0$,

$$\begin{aligned}&U_n(\phi)=S_{2n}(\phi)/S_{2n}(1)\\ &\quad\to(F(\tau_H))^{-2}\iint_{\{x_1<\tau_H,\,x_2<\tau_H\}}\phi(x_1,x_2)\,F(dx_1)\,F(dx_2)\qquad\text{a.s.}\end{aligned}$$

The argument to extend to a general $H(\cdot)$ is exactly as in Stute and Wang,[4] Lemma 2.8 onwards, combined with Stute[3] (pp. 437–438). Suppose $H(\cdot)$ is not necessarily continuous. First assume, as in Stute and Wang[4] that $A_F \cap A_G = \phi$. Define

$$U := \begin{cases} H(Z), & \text{if } Z \notin A_H \\ H(a-) + [H(a) - H(a-)]\, V, & \text{if } Z = a, \quad a \in A_H \end{cases}$$

where $V \sim \text{Uniform}[0, 1]$ independently of Z. Then $U \sim \text{Uniform}[0, 1]$, $Z = H^{-1}(U)$ a.s., $\tilde{m}(u) := E\{\delta \mid U = u\} = m(H^{-1}(u))$ for $0 < u < 1$. Also, $\{Z_{i:n}, 1 \leqslant i \leqslant n\} = \{H^{-1}(U_{i:n}), 1 \leqslant i \leqslant n\}$ a.s. Thus by no common jump assumption,

$$S_{2n}(\phi) = \sum\sum_{1 \leqslant i < j \leqslant n} \phi(H^{-1}(U_{i:n}), H^{-1}(U_{j:n}))\, W_{i:n} W_{j:n}$$

Since U_i, $i \geqslant 1$, have a continuous d.f., we get from (2.17) that

$$\lim_{n \to \infty} S_{2n}(\phi) = \frac{1}{2} \int_0^1 \int_0^1 \phi(H^{-1}(t), H^{-1}(s))\, \tilde{m}(s)\, \tilde{m}(t)\, \tilde{D}(t, s)\, dt\, ds \qquad \text{a.s.}$$

where

$$\tilde{D}(t, s) = \exp\left\{ \int_0^t \frac{1 - \tilde{m}(x)}{1 - x}\, dx + \int_0^s \frac{1 - \tilde{m}(y)}{1 - y}\, dy \right\}$$

By arguments similar to those following Eq. (2.6) in Stute and Wang,[4]

$$\tilde{D}(t, s) = [1 - G(H^{-1}(t))]^{-1} [1 - G(H^{-1}(s))]^{-1}, \qquad 0 < t, s < 1$$

whence

$$\lim_{n \to \infty} S_{2n}(\phi) = \tfrac{1}{2} E[\phi(X_1, X_2)\, I(X_1 \leqslant Y_1)\, I(1 - G(X_1))^{-1} (1 - G(X_2))^{-1}]$$

$$= \tfrac{1}{2} \qquad \text{(the limit in (1.3))}$$

Finally, if there are common jumps, say $\{x_i\}$, of F and G, replace each x_i by $[x_i, x_i + \varepsilon_i]$ where $\sum \varepsilon_i < \infty$ and move the G mass at x_i to $x_i + \varepsilon_i$. Extend the time scale for F and ϕ, by putting for example $F(x) = F(x_i)$ if $x_i \leqslant x < x_i + \varepsilon_i$. Since tied uncensored observations precede censored ties by convention, $S_{2n}(\phi) - S_2(\phi)$ remains unchanged. Now $A_F \cap A_G = \phi$. The integrability condition remains valid. This completes the proof. □

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REFERENCES

1. Gijbels, I., and Veraverbeke, N. (1991). Almost-sure asymptotic representation for a class of functionals of the Kaplan-Meier estimator. *Ann. Statist* **19**, 1457–1470.
2. Neveu, J. (1975). *Discrete-Parameter Martingales*, North-Holland, Amsterdam.
3. Stute, W. (1995). The central limit theorem under random censorship. *Ann. Statist.* **23**, 422–439.
4. Stute, W., and Wang, J. L. (1993). The strong law under random censorship. *Ann. Statist.* **21**, 1591–1607.