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SEQUENTIAL NONPARAMETRIC ESTIMATION OF DENSITY VIA DELTA–SEQUENCES

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SUMMARY. Sequential estimators of density using delta — sequences are studied. Large sample properties of these estimators are given. These results generalize earlier work of Deheuvels (1974) and Yamato (1972) for kernel type of sequential estimators.

1. Introduction

Estimators of the density function of a population based on a fixed sample of independent observations have been proposed by several authors. Density estimation by the kernel type method is discussed in Parzen (1962). Recently Walter and Blum (1976) proposed a method for density estimation using delta-sequences. This method has been used in Prakasa Rao (1978) to study density estimators when the observations are assumed to be sampled from a stationary Markov process. Details can be found in Basawa and Prakasa Rao.

The disadvantage of kernel type of density estimators f_n based on *fixed* sample size n is that the estimators are not *recursive* in the sense that even for a slight change in the sample size or the "window width" one cannot compute the new value of the estimator from the preceeding estimate and the new observation. It is necessary to start computation of the estimator right from the beginning. In view of this, Yamato (1972) started study of sequential estimation of density using kernel type of density estimators. Slightly more general type of sequential estimators of density of kernel type have been studied extensively by Deheuvels (1974). Here they have suggested estimators f_{n+1} of kernel type which can be computed from f_n, n and the new observation X_{n+1} and which are optimum in some sense.

Our aim in this paper is to study sequential estimation of density via delta-sequences. Results obtained here include results of Deheuvels (1974) for kernel type sequential estimators and they generalize several other methods of density estimators to sequential case.

Section 2 contains the definition of the estimator and its properties. A Gaussian process related to the estimator is studied in Section 3.

*Basawa, I. V. and Prakasa Rao, B. L. S. : *Statistical Inference for Stochastic Processes.* Academic Press, London, to appear in 1980.

2. Sequential estimation

A family $\{\delta_t, t > 0\}$ of non-negative $L_\infty(R)$ functions is called a *deltafamily of positive type* $\alpha > 0$ if there exist $A > 0$, $B > 0$ such that

$$\text{(i)} \qquad \left| 1 - \int_{-A}^{B} \delta_t(x)\, dx \right| = O(t^\alpha) \qquad \dots \quad (2.1)$$

$$\text{(ii)} \qquad \sup \{ |\delta_t(x)| : |x| \geqslant t^\alpha \} = O(t^\alpha) \qquad \dots \quad (2.2)$$

and

$$\text{(iii)} \qquad \|\delta_t\|_\infty \simeq t^{-1} \qquad \dots \quad (2.3)$$

as $t \to 0$. This definition is due to Walter and Blum (1976).

An example of a delta-family of positive type one is

$$\delta_t = \frac{1}{t} \chi_{[0,t]}, \; t > 0,$$

where χ_A is indicator function of set A.

Let $\{h_n\}$ be a sequence decreasing to zero such that $\sum_{n=1}^{\infty} h_n = \infty$. For example, $h_n = n^{-s}$, $0 < s \leqslant 1$ will be such a sequence.

Let $\gamma_n = \sum_{i=1}^{n} h_i$. Given a sample $X_1, X_2, \dots, X_n$ from a population with density function f, define

$$f_n(x) = \frac{1}{\gamma_n} \sum_{i=1}^{n} h_i \, \delta_{h_i}(x - X_i) \qquad \dots \quad (2.4)$$

and

$$\bar{f}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{h_n}(x - X_i)$$

as estimators for $f(x)$. Motivation for considering the estimator $\bar{f}_n(x)$ is given in Walter and Blum (1976) and Prakasa Rao (1978). If $\delta_h(x) = h^{-1}K(x/h)$ for some kernel $K(\cdot)$, then the above estimator reduces to that of Deheuvels (1974). It is easy to see that

$$f_{n+1}(x) = \frac{\gamma_n}{\gamma_{n+1}} f_n(x) + \frac{h_{n+1}}{\gamma_{n+1}} \delta_{h_{n+1}}(x - X_{n+1}) \qquad \dots \quad (2.5)$$

and hence the estimator $f_n(x)$ is recursive in nature.

Now

$$\text{MSE}\,(f_n(x)) = E\,[f_n(x)-f(x)]^2$$

$$= \frac{1}{\gamma_n^2} \sum_{i=1}^{n} h_i^2 \,\text{var}\,[\delta_{h_i}(x-X_i)]+\Big[\frac{1}{\gamma_n} \sum_{i=1}^{n} h_i \{E(\delta_{h_i}(x-X_i))-f(x)\}\Big]^2$$

$$\leqslant \frac{1}{\gamma_n^2} \sum_{i=1}^{n} h_i^2\, E\,[\delta_{h_i}^2 (x-X_i)]+\frac{1}{\gamma_n^2}\Big[\sum_{i=1}^{n} h_i\, B_{h_i}(x)\Big]^2 \qquad \dots \quad (2.6)$$

where

$$B_h(x) = E[\delta_h(x-X_1)]-f(x).$$

Since $\|\delta_h\|_\infty \simeq h^{-1}$ by (2.3), it follows that

$$\text{MSE}\,[f_n(x)] = O\Big(\frac{1}{\gamma_n^2} \sum_{i=1}^{n} h_i\, E\,[\delta_{h_i}(x-X_i)]+\frac{1}{\gamma_n^2}\Big[\sum_{i=1}^{n} h_i\, B_{h_i}(x)\Big]^2\Big). \qquad \dots \quad (2.7)$$

Suppose f is Lipschitz of order λ for some $0 < \lambda \leqslant 1$, i.e., there exists $C > 0$ such that

$$|f(x)-f(y)| \leqslant C\,|x-y|^\lambda$$

for all $x, y \in R$. In particular, it follows that f is bounded.

Then

$$|B_h(x)| \leqslant c_1\, h^{\alpha\lambda}+c_2\, h^\alpha \leqslant c_3\, h^{\alpha\lambda} \qquad \dots \quad (2.8)$$

by arguments similar to those in Walter and Blum (1976, p. 6) where c_1 and c_2 are constants independent of h and x. Since $\|f\|_\infty < \infty$, it follows that

$$|E\,(\delta_h\,(x-X_1))| \leqslant c_4 \qquad \dots \quad (2.9)$$

for some constant c_4 independent of h and x. Relations (2.7) - (2.9) imply that

$$\sup_x \text{MSE}\,[f_n(x)] \leqslant c_5 \left\{\frac{1}{\gamma_n}+\left(\frac{\sum_{i=1}^{n} h_i^{1+\alpha\lambda}}{\gamma_n}\right)^2\right\} \qquad \dots \quad (2.10)$$

where c_5 is independent of $\{h_n\}$ and x.

Theorem 2.1 : *Let $\{\delta_t, t > 0\}$ be a delta-family of positive type α. Further suppose that $f \in Lip(\lambda)$ for some $0 < \lambda \leqslant 1$. Then*

$$\sup_x E[f_n(x)-f(x)]^2 = O\left\{\frac{1}{\gamma_n}+\left(\frac{\sum_{i=1}^{n} h_i^{1+\alpha\lambda}}{\gamma_n}\right)^2\right\}. \quad \dots \quad (2.11)$$

Theorem 2.1 implies that the estimator $f_n(x)$ is mean square consistent for each x. It is easy to check that $f_n(x)$ is a strongly consistent estimator of $f(x)$.

Hereafter we shall suppose that $\delta_t \in L_1(R)$ for all t. We shall now study the limiting distribution of the estimator $f_n(x)$.

Let

$$Z_i = h_i \, \delta_{h_i}(x-X_i), \; i \geqslant 1.$$

Then

$$f_n(x) = \gamma_n^{-1} \sum_{i=1}^{n} Z_i$$

and

$$f_n(x)-E\,f_n(x) = \frac{1}{\gamma_n}\sum_{i=1}^{n}[Z_i-E(Z_i)]. \quad \dots \quad (2.12)$$

We have seen that

$$E[\delta_h(x-X)] = f(x)+O(h^{\alpha\lambda}) \quad \dots \quad (2.13)$$

uniformly in x from (2.8) and further

$$\begin{aligned} E[\delta_h^2(x-X)]-f(x)\int_R \delta_h^2(t)\,dt &= \int_R \delta_h^2(x-t)f(t)\,dt-f(x)\int_R \delta_h^2(t)\,dt \\ &= \int \delta_h^2(t)[f(x-t)-f(x)]\,dt \\ &= O(h^{-1+\alpha\lambda}) \end{aligned} \quad \dots \quad (2.14)$$

by (2.3) and (2.8) uniformly in x. It is now easily seen from (2.13) and (2.14) that

$$\begin{aligned} \text{var}(Z_i) &= h_i^2[f(x)\int_R \delta_{h_i}^2(t)\,dt+O(h_i^{1+\alpha\lambda})]-h_i^2[f(x)+O(h_i^{\alpha\lambda})]^2 \\ &= h_i^2 f(x)\int_R \delta_{h_i}^2(t)-h_i^2 f^2(x)+O(h_i^{1+\alpha\lambda}) \end{aligned} \quad \dots \quad (2.16)$$

uniformly in x.

Therefore

$$\sum_{i=1}^{n} \text{var}\,(Z_i) = f(x) \sum_{i=1}^{n} h_i^2 \int_R \delta_{h_i}(t)\,dt - f^2(x) \sum_{i=1}^{n} h_i^2 + O\Big(\sum_{i=1}^{n} h_i^{1+\alpha\lambda}\Big). \quad \ldots \quad (2.17)$$

Suppose that

$$f(x) \geqslant \Delta > 0 \text{ for all } x \,\epsilon\, [a, b] \quad \ldots \quad (2.18)$$

and

$$\frac{1}{\gamma_n} \sum_{j=1}^{n} h_j^2 \int_R \delta_{h_j}^2(t)\,dt \to \gamma > 0 \quad \ldots \quad (2.19)$$

as $n \to \infty$.

Then

$$\sum_{i=1}^{n} \text{var}\,(Z_i) = f(x)\,[\gamma\gamma_n + o(\gamma_n)] - f^2(x) \sum_{i=1}^{n} h_i^2 + O\Big(\sum_{i=1}^{n} h_i^{1+\alpha\lambda}\Big)$$

$$= \gamma_n[\gamma f(x) + o(1) - f^2(x)\,o(1) + o(1)] \quad \ldots \quad (2.20)$$

uniformly in x since

$$\frac{\sum_{i=1}^{n} h_i^2}{\gamma_n} \to 0 \quad \text{and} \quad \frac{\sum_{i=1}^{n} h_i^{1+\alpha\lambda}}{\gamma_n} \to 0$$

as $n \to \infty$. Since f is continuous, it is bounded on $[a, b]$ and we have

$$\sum_{i=1}^{n} \text{var } Z_i \geqslant \Delta^* \gamma_n \quad \ldots \quad (2.21)$$

for some $\Delta^* > 0$ uniformly for $x \,\epsilon\, [a, b]$. Further observe that

$$E\,|\,Z_j - E(Z_j)\,|^3 = h_j^3\, E(\delta_{h_j}^3\,(x - X_j))$$

$$\simeq h_j\, E(\delta_{h_j}\,(x - X_j)) \quad \text{by (2.3)}$$

$$\simeq h_j \quad \text{by (2.9)}$$

uniformly x and therefore

$$\sum_{j=1}^{n} E\,|\,Z_j - E\,Z_j\,|^3 \simeq \sum_{j=1}^{n} h_j = \gamma_n \quad \ldots \quad (2.22)$$

uniformly in x.

Let

$$F_n(y) = P\left\{\frac{f_n(x)-E(f_n(x))}{\sqrt{\text{var}\,(f_n(x))}} \leqslant y\right\}$$

$$= P\left\{\frac{\sum\limits_{i=1}^{n} \{Z_i - EZ_i\}}{\sqrt{\sum\limits_{i=1}^{n} \text{var}\ Z_i}} \leqslant y\right\}$$

and $\Phi(y)$ be the standard normal distribution function.

Theorem 2.2 : *Under the assumptions of Theorem* 2.1 *and* (2.18) *and* (2.19) *and the fact that* $\delta_t \in L_1(R)$ *for all* $t > 0$, *there exists* $c^* > 0$ *independent of n such that*

$$\sup_{x \in (a,b)} \sup_{-\infty<y<\infty} |F_n(y)-\Phi(y)| \leqslant c\, \frac{\sum\limits_{1}^{n} h_j}{(\Delta^* \gamma_n)^{3/2}} = c^* \gamma_n^{-\frac{1}{2}} \qquad \ldots \quad (2.23)$$

where $\Phi(\cdot)$ *is the standard normal distribution function.*

Proof : This follows from Berry-Esseen bound for sums of independent random variables (cf. Loeve, 1963, p. 288) in view of inequalities (2.21) and (2.22). In particular, it follows that

$$\frac{f_n(x)-E(f_n(x))}{\sqrt{\text{var}\, f_n(x)}} \xrightarrow{\mathcal{L}} N(0, 1). \qquad \ldots \quad (2.24)$$

Note that

$$\text{var}\,(f_n(x)) = \frac{1}{\gamma_n}\,[\gamma f(x)+o(1)] \qquad \ldots \quad (2.25)$$

from (2.20) uniformly in x since f is bounded.

Remark : If $h_n \simeq n^{-1/3}$, then $\gamma_n \simeq n^{2/3}$ and the rate of convergence in Theorem 2.2 is of the order $n^{-1/3}$. A result of the above type for kernal type of density estimators when the centering is around $f(x)$ and not $E(f_n(x))$ has been obtained in Prakasa Rao (1977) for the stationary Markov case. The rate obtained is of the order $n^{-(1/3-\tau)}$, $\tau > 0$. Similar results can be obtained here. But we shall not discuss them here.

Suppose the family $\{\delta_t, t > 0\}$ satisfies the condition (cf. Deheuvels, 1974)

$$\int_{|v| \geqslant s} |v|^{\eta} \, \delta_t(v) \, dv = O\left(\frac{t^{\eta+\beta}}{s^{\beta}}\right) \qquad \ldots \quad (2.26)$$

for some $\eta \geqslant 0$, $\beta \geqslant 0$ such that $\eta+\beta > 1$. Then, for $x \neq y$,

$$\text{cov}\,(f_n(x), f_n(y)) = \frac{1}{\gamma_n} \sum_{i=1}^{n} h_i^2 \,[\text{cov}\,[\delta_{h_i}(x-X_i),\ \delta_{h_i}(y-X_i)]$$

$$\leqslant \frac{1}{\gamma_n} \sum_{i=1}^{n} h_i^2 \, O\left(\frac{h_i^{\eta+\beta-1}}{|x-y|^{\eta+\beta}}\right). \qquad \ldots \quad (2.27)$$

3. Gaussian process related to estimator $f_n(x)$

Let $\{\rho_n(x), -\infty < x < \infty\}$ be a continuous Gaussian process with $E[\rho_n(x) = 0]$ for all x and

$$\sigma_n^2(x) \equiv \text{var}\,[\rho_n(x)] = \gamma_n \,\text{var}\,[f_n(x)] \qquad \ldots \quad (3.1)$$

and

$$R_n(x, y) \equiv \text{cov}\,[\rho_n(x), \rho_n(y)] = \gamma_n \,\text{cov}\,[f_n(x), f_n(y)]. \qquad \ldots \quad (3.2)$$

Note that

$$E[\delta_{h_i}(x-X)-\delta_{h_i}(y-X)]^2$$

$$\simeq h_i^{-1} |E\,[\delta_{h_i}(x-X)-\delta_{h_i}(y-X)]|$$

$$\simeq h_i^{-1+\alpha\lambda} + |f(x)-f(y)| \qquad \ldots \quad (3.3)$$

when f is Lipschitz of order λ and δ_h is of positive type α. Hence it can be shown that

$$E[\rho_n(x)-\rho_n(y)]^2 \leqslant C|x-y|^{\lambda} = Ck_0(x-y) \quad \text{(say)} \qquad \ldots \quad (3.4)$$

for some constant C using (2.8). Note that if $K(\cdot)$ is any bounded density function such that $K(t) = 0(|t|^{-1-\beta})$ as $|t| \to \infty$ for some $\beta > 0$, then $\delta_h(t) = \dfrac{1}{h} K(t/h)$ is a delta-sequence of type $\beta/\beta+2$ and all the results of the previous section hold and results of this section hold under the additional conditions (2.26) and (2.19) and the fact that f is Lipschitz of order λ, $0 < \lambda \leqslant 1$.

Theorem 3.1 : *Let I_0 be an open interval in R and $M = sup \{f(x) : x \in I_0\}$. If conditions (2.19), (2.26) hold and f is Lipschitz of order λ, $0 < \lambda \leqslant 1$, then*

$$\liminf_{n \to \infty} \left(\log \frac{\gamma_n}{\sum_{i=1}^{n} h^2} \right)^{-1/2} \sup_{x \in I_0} \rho_n(x) \geqslant (2\gamma M)^{1/2} \qquad \ldots \quad (3.5)$$

and

$$\limsup_{n \to \infty} \left(\log \frac{\gamma_n}{\sum_{i=1}^{n} h_i^2} \right)^{-1/2} \inf_{x \in I_0} \rho_n(x) \leqslant -(2\gamma M)^{1/2} \qquad \ldots \quad (3.6)$$

in probability.

Proof : Since the processes ρ_n and $-\rho_n$ have the same finite dimensional distributions, it is enough to prove (3.5) and (3.6) follows from (3.5). Define ν by the relation

$$1-\nu = c(2\gamma M)^{-1/2}$$

where $0 < c < (2\gamma M)^{1/2}$. Note that f is uniformly continuous since it is Lipschitz. We can choose a subinterval I of I_0 of length l (say) such that $f(x) > (1-\nu/8)M$ for all $x \in I$ and

$$\sigma_n^2(x) = \gamma f(x)[1+o(1)]$$

as $n \to \infty$ uniformly for $x \in I$. This is possible from (2.25) and (3.1) and hence, for a large n,

$$\inf_{x \in I} \sigma_n(x) > (M\gamma)^{1/2} \left(1 - \frac{1}{2}\nu\right). \qquad \ldots \quad (3.7)$$

If we show that

$$\sup_{x \in I} \frac{\rho_n(x)}{\sigma_n(x)} \geqslant \left(2 \log \frac{\gamma_n}{\sum_{i=1}^{n} h_i^2} \right)^{1/2} \left(1 - \frac{1}{2}\nu\right) \qquad \ldots \quad (3.8)$$

in probability as $n \to \infty$, then

$$\sup_{x \in I_0} \rho_n(x) \geqslant \sup_{x \in I} \rho_n(x)$$

$$\geqslant \inf_{x \in I} \sigma_n(x) \cdot \sup_{x \in I} \frac{\rho_n(x)}{\sigma_n(x)}$$

$$\geqslant (M\gamma)^{1/2} \left(1-\frac{1}{2}\nu\right)^2 \left(2 \log \frac{\gamma_n}{\Sigma h_i^2}\right)^{1/2}$$

$$\geqslant c \left(\log \frac{\gamma_n}{\Sigma h_i^2}\right)^{1/2} \quad \ldots \quad (3.9)$$

and since this is true for every $0 < c < (2\gamma M)^{1/2}$, the result follows. Hence it is sufficient to prove that

$$P\left[\sup_{x \in I} \frac{\rho_n(x)}{\sigma_n(x)} \geqslant \left(1-\frac{1}{2}\nu\right)\left(2 \log \frac{\gamma_n}{\Sigma h_i^2}\right)^{1/2}\right] \to 1 \text{ as } n \to \infty. \quad \ldots \quad (3.10)$$

Define

$$\eta(x) = \begin{cases} 1 & \text{if } \dfrac{\rho_n(x)}{\sigma_n(x)} \geqslant \left(1-\dfrac{1}{2}\nu\right)\left(2 \log \dfrac{\gamma_n}{\Sigma h_i^2}\right)^{1/2} \\ 0 & \text{otherwise} \end{cases} \quad \ldots \quad (3.11)$$

and

$$Z_n = \frac{1}{l} \int_I \eta_n(x)\, dx. \quad \ldots \quad (3.12)$$

Then

$$P\left[\sup_{x \in I} \frac{\rho_n(x)}{\sigma_n(x)} \leqslant \left(1-\frac{1}{2}\nu\right)\left(2 \log \frac{\gamma_n}{\Sigma h_i^2}\right)^{1/2}\right]$$

$$= P[Z_0 = 0]$$

$$\leqslant \frac{\text{var } Z_0}{EZ_0^2}$$

$$\leqslant K \log \frac{\gamma_n}{\Sigma h_i^2} \cdot L_n \quad \ldots \quad (3.13)$$

where

$$L_n \equiv \iint_{I \times I} |\chi_n'| \ \exp\left\{2 \log \frac{\gamma_n}{\Sigma h_i^2} \left(1-\frac{1}{2}\nu\right)^2 |\chi_n|/1+|\chi_n|\right\} dx\, dy, \quad \ldots \quad (3.14)$$

K is constant and $\chi_n(x, y) = R_n(x, y)/\sigma_n(x)\sigma_n(y)$. This follows from Cramér and Leadbetter (1967, Sec. 13.5). Since $|\chi_n|/1+|\chi_n| \leqslant \frac{1}{2}$ and $|\chi_n| \leqslant 1$, it follows that

$$L_n \leqslant [\inf_{I\times I} \sigma_n(x)\,\sigma_n(y)]^{-1} \left(\log \frac{\gamma_n}{\Sigma h_i^2}\right)\left(1-\frac{1}{2}\nu\right)^2 \int\int_{I\times I} |R_n(x, y)|\,dx\,dy. \quad \ldots \quad (3.15)$$

But

$$\int\int_I |R_n(x, y)|\,dx\,dy \leqslant \frac{1}{\gamma_n}\sum_{i=1}^{n} h_i^2 \int\int_{I\times I} \{E[\delta_{h_i}(x-X_i)\delta_{h_i}(y-X_i)]\}dx\,dy$$

$$= \frac{1}{\gamma_n}\sum_{i=1}^{n} h_i^2 \int\int_{I\times I} \frac{O(h_i^{\eta+\beta-1})}{|x-y|^{\eta+\beta}}\,dx\,dy \text{ (by (2.27))}$$

$$= \frac{1}{\gamma_n}\sum_{i=1}^{n} h_i^{1+\eta+\beta} \int\int_{I\times J_i} \frac{1}{h_i^{\eta+\beta}|\xi|^{\eta+\beta}}\,h_i\,dx\,d\xi$$

by applying the transformation $(x, y) \to (x, \xi)$ with $y-x = h_i\,\xi$.

Since $\eta+\beta > 1$, the double integral

$$\int\int_{I\times J_i} \frac{1}{|\xi|^{\eta+\beta}}\,dx\,d\xi$$

is uniformly bounded in i by $l \int_{-\infty}^{\infty} \frac{1}{|\xi|^{\eta+\beta}}\,d\xi$ which is convergent. Hence

$$\int\int_{I\times I} |R_n(x, y)|\,dx\,dy = O\left(\frac{\Sigma h_i^2}{\gamma_n}\right). \quad \ldots \quad (3.16)$$

(3.7) and (3.13)-(3.16) prove that

$$P\left[\sup_{x\in I} \frac{\rho_n(x)}{\sigma_n(x)} \geqslant \left(1-\frac{1}{2}\nu\right)\left(2\log\frac{\gamma_n}{\Sigma h_i^2}\right)^{1/2}\right]$$

$$= O\left(\log\frac{\gamma_n}{\Sigma h_i^2}\cdot\left(\frac{\gamma_n}{\Sigma h_i^2}\right)^{\left(1-\frac{1}{2}\nu\right)^2}\frac{\Sigma h_i^2}{\gamma_n}\right)$$

$$= O\left(\log\frac{\gamma_n}{\Sigma h_i^2}\cdot\left(\frac{\gamma_n}{\Sigma h_i^2}\right)^{-(\nu-\nu^2/4)}\right) \quad \ldots \quad (3.17)$$

and

$$\log\frac{\gamma_n}{\Sigma h_i^2}\cdot\left(\frac{\gamma_n}{\Sigma h_i^2}\right)^{-(\nu-\nu^2/4)} \to 0 \text{ as } n\to\infty$$

since $0 < \nu < 1$ and $\gamma_n / \Sigma h_i^2 \to \infty$ as $n \to \infty$. This completes the proof of Theorem 3.1. The proof given above is an adaptation of the proof of proposition 1 in Silverman (1979) to the sequential case.

Let

$$\zeta_n(x) = (-\log \Delta_n)^{-\frac{1}{2}} \rho_n(x), \; x \epsilon I \qquad \dots \quad (3.18)$$

where

$$\Delta_n = \frac{\sum_{i=1}^{n} h_i^2}{\gamma_n}. \qquad \dots \quad (3.19)$$

Note that $\Delta_n \to 0$ as $n \to \infty$. (3.4) implies that

$$E[\zeta_n(x) - \zeta_n(y)]^2 \leqslant c\,(-\log \Delta_n)^{-1} k_0(y-x) \qquad \dots \quad (3.20)$$

and hence

$$\sup_{|x-y|<u} \{E\,[\zeta_n(x) - \zeta_n(y)]^2\}^{1/2}$$

$$\leqslant c^{1/2}(-\log \Delta_n)^{-1/2} \sup_{|x-y|<u} [k_0(y-x)]^{1/2}$$

$$= c^{1/2}(-\log \Delta_n)^{-1/2} k(u) \equiv p(u) \quad \text{(say)} \qquad \dots \quad (3.21)$$

where

$$k(u) = \sup_{|x-y|<u} [k_0(x-y)]^{1/2}. \qquad \dots \quad (3.22)$$

Assume that

$$\int_0^1 (-\log u)^{1/2}\, k(du) < \infty. \qquad \dots \quad (3.23)$$

In particular, it follows that $k(u) \to 0$ as $u \to 0$.

Let

$$w_n(u) = \sup_{|x-y|<u} |\zeta_n(x) - \zeta_n(y)|. \qquad \dots \quad (3.24)$$

Lemma 3.2 : *Under the assumptions* (3.3) *and* (3.23)

$$\lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}}\; w_n(\Delta_n \epsilon) = 0 \quad \textit{in probability}. \qquad \dots \quad (3.25)$$

Proof of Lemma 3.2 is similar to that of Lemma 4 in Silverman (1970). We omit it.

Theorem 3.3 : *Let* $\varepsilon_0 = \inf \{\varepsilon > 0 : l_n \Delta_n^{-\varepsilon} \to 0 \text{ as } n \to \infty\}$ *where* l_n *is length of interval* I_n. *Under the assumptions stated above,*

$$\overline{\lim_{n\to\infty}} \sup_{x \in I_n} \{(-\log \Delta_n)^{-1/2} |\rho_n(x)|\} \leqslant \{2(1+\varepsilon_0)M_1\gamma\}^{1/2} \quad \ldots \quad (3.26)$$

in probability where $M_1 = \sup_{\cup_n I_n} f(x)$.

The above theorem follows by the method similar to that of proposition 2 of Silverman (1976) using Lemma 3.2.

Theorem 3.4 : *For any finite interval I, under the conditions stated above,*

$$\sup_{x \in I} (-\log \Delta_n)^{-1/2} \rho_n(x) \xrightarrow{p} (2\gamma \sup_{x \in I} f(x))^{1/2} \quad \ldots \quad (3.27)$$

and

$$\inf_{x \in I} (-\log \Delta_n)^{-1/2} \rho_n(x) \xrightarrow{p} -(2\gamma \sup_{x \in I} f(x))^{1/2} \quad \ldots \quad (3.28)$$

where $\Delta_n = (\Sigma h_i^2)/\gamma_n$ *and* $\gamma = \lim_{n \to \infty} \frac{1}{\gamma_n} \Sigma h_i^2 \int_R \delta_{h_i}^2 (u)\, du$.

Theorem 3.4 follows from Theorems 3.1 and 3.3 by taking $I_n = I$ for all n.

Example : Let $K(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, $t \in R$.

It is clear that $K(t) = O(|t|^{-\tau})$ for every $\tau > 0$ as $|t| \to \infty$ and $K(t)$ is a bounded density function. Therefore $\delta_h(t) = \frac{1}{h} K\left(\frac{t}{h}\right)$ is a delta-sequence of positive type. It is trivial to see that (2.19) holds since, for all $n \geqslant 1$,

$$\frac{1}{\gamma_n} \sum_{j=1}^{n} h_j^2 \int_R \delta_{h_j}^2 (t)\, dt = \int_R K^2(u)\, du > 0$$

and (2.26) holds with $\eta = 2$ and $\beta = 0$ since normal distribution has second moment. Hence, if $f \in \text{Lip}(\lambda)$ for some $0 < \lambda \leqslant 1$, then Theorems 2.1 and 3.1 holds for the sequential estimator $f_n(x)$ defined by the delta-family given above.

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REFERENCES

CRAMER, H. C. and LEADBETTER, M. R. (1967): *Stationary and Related Stochastic Processes*, Wiley, New York.

DEHEUVELS, P. (1974): Estimation sequentielle de la densite. Doctoral thesis, Universite de Paris VI.

LOEVE, M. (1963): *Probability Theory*, Van Nostrand, Princeton.

PARZEN, M. (1962): On the estimation of a probability density function and the mode. *Ann. Math. Statist.*, **33**, 1065-1076.

PRAKASA RAO, B. L. S. (1977): Berry-Essen type bound for density estimators of stationary Markov processes. *Bull. Math. Statist.*, **17**, 15-21.

——— (1978): Density estimation for Markov processes using delta-sequences. *Ann. Inst. Statist. Math.*, **30**, 321-328.

SILVERMAN, B. (1976): On a Gaussian process related to multivariate probability density estimation. *Math. Proc. Camb. Phil. Soc.*, **80**, 135-144.

WALTER, G. and BLUM, J. R. (1976): Probability density estimation using delta-sequences. (Preprint). The University of Wisconsin, Milwaukee.

YAMATO, H. (1972): Sequential estimation of a continuous probability density function and mode. *Bull. Math. Statist.*, **14**, 1-18.

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