ON NONPARAMETRIC FAMILIES OF LIFE DISTRIBUTIONS: SOME ISSUES AND APPLICATIONS

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Contents

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Chapter 1

Introduction

In reliability theory, one is primarily concerned with the study of the lifetime of a unit. The unit may be a mechanical device or a component of such a device; it may also be a living organism. By lifetime, we usually mean the duration for which the unit under study continues to perform certain specific functions before it passes on to what is popularly known as the 'failed state'. More generally, however, lifetime can be, and often is, interpreted as the time to occurrence of a certain event such as the elapsed time before a broken down machine starts functioning again or the time period for which a person uses a particular brand of a certain consumer product etc.

It is the practice of reliabilists to consider lifetime as a non-negative random variable. We shall generically denote it by X and assume that X is continuous. For $x \geq 0$,

$$F(x) := P[X \leq x]$$

is called the distribution function (d.f.) of X and

$$\overline{F}(x) := P[X > x]$$

denotes the corresponding *survival function.* The d.f. F is a *life d.f.* if it satisfies $F(0-) = 0$. Note that the survival function, which is also referred to as the *reliability function*, defines the probability that the unit will function at least for x units of time.

For absolutely continuous X with probability density function (p.d.f.) $f(x)$, the *failure rate function* of the unit is defined by

$$r_F(x) := \frac{f(x)}{\overline{F}(x)}, \text{ for } x \geq 0 \text{ satisfying } \overline{F}(x) > 0\,.$$

It is easy to see that $r_F(x)$ has a nice physical interpretation; heuristically, $r_F(x)dx$ can be looked upon as the probability that the unit alive at age x will fail in the interval $(x, x + dx]$ where dx is taken to be small. The above function is basic in reliability theory and is variously known as the *hazard rate function*, *intensity function* and also *force of mortality*.

Another function of fundamental importance in the study of life lengths is the *Mean Residual Life* (MRL) function (see Guess and Proschan (1985)), which is defined as $e_F(x) := E(X - x | X > x)$ where $e_F(x)$ is given by

$$e_F(x) = \begin{cases} (1/\overline{F}(x)) \int_x^\infty \overline{F}(t)dt, & \text{for } x \geq 0 \text{ satisfying } \overline{F}(x) > 0 \\ 0, \text{ when } \overline{F}(x) = 0. \end{cases}$$

As indicated above, this function gives the expected residual life of the unit given that it has survived upto age x.

The importance of the functions introduced above lies in their use in the study of the ageing pattern of units. In fact, various ageing criteria have been defined in the literature based on the behaviour of these functions. A unit is said to age *positively* (*negatively*) if its residual life tends to *decrease* (*increase*), in some probabilistic sense, with increase in age. The following definitions, which can be found in Bryson and Siddiqui (1969), Barlow and Proschan (1975), Rolski (1975), Klefsjö (1982a, 1982b) and Hollander and Proschan (1984), illustrate how the failure rate and MRL functions are used to introduce various notions of ageing.

DEFINITION 1. A life d.f. F is said to have

(i) *Increasing Failure Rate* (IFR) if $r_F(x)$ is a *non-decreasing* function of $x \geq 0$.

(ii) *Decreasing Failure Rate* (DFR) if $r_F(x)$ is a *non-increasing* function of $x \geq 0$.

More generally, (in situations where F may not have a density), we say that F is IFR (DFR) if and only if the conditional survival function $\overline{F}(x+t)/\overline{F}(t)$ is a *non-increasing* (*non-decreasing*) function of $t \geq 0$ for all $x \geq 0$. However, when the failure rate function does exist, the two definitions are equivalent.

DEFINITION 2. The life distribution F is said to have *Increasing Failure Rate Average* (IFRA) if $(1/x)\int_0^x r_F(u)du$ is *non-decreasing* in x, $x \geq 0$.

The dual class life d.f.s with *Decreasing Failure Rate Average* (DFRA) is obtained by reversing the direction of monotonicity in the above definition.

DEFINITION 3. The life d.f. F is said to have the *New Better than Used* (NBU) (*New Worse than Used* (NWU)) property if

$$\overline{F}(x+y) \leq (\geq) \overline{F}(x)\overline{F}(y) \; \forall x, y \geq 0 .$$

DEFINITION 4. The life d.f. F is said to have the *New Better than Used in Expectation* (NBUE) (*New Worse than Used in Expectation* (NWUE)) property if the mean μ of F is finite and

$$\int_x^\infty \overline{F}(t)dt \leq (\geq) \mu\overline{F}(x) \; \forall x \geq 0 ,$$

i.e. $e_F(x) \leq (\geq) e_F(0) = \mu$.

DEFINITION 5. The life distribution F has *Decreasing Mean Residual Life* (DMRL) (*Increasing Mean Residual Life* (IMRL)) if $e_F(x)$ is a *non-increasing* (*non-decreasing*) function of $x \in \{y \geq 0 : \overline{F}(y) > 0\}$.

DEFINITION 6. The life d.f. F has the *Harmonic New Better than Used in Expectation* (HNBUE) (*Harmonic New Worse than Used in Expectation* (HNWUE)) property if the mean μ of F is finite and

$$\int_x^\infty \overline{F}(t)dt \leq (\geq) \mu e^{-x/\mu} \; \forall x \geq 0 .$$

The next definition, based on Laplace ordering, is due to Klefsjö (1983).

DEFINITION 7. The life d.f. F is said to belong to the class $\mathcal{L}$ ($\overline{\mathcal{L}}$) of life distributions if the following relation holds:

$$L_F(s) := \int_0^\infty e^{-st} dF(t) \leq (\geq) \frac{1}{1+s\mu} := L_G(s) \; \forall s \geq 0,$$

μ being the finite mean of F, and where G is the exponential d.f. with mean μ.

The chain of implications among the prominent ageing classes defined above is as follows:

(i) IFR $\Longrightarrow$ IFRA $\Longrightarrow$ NBU $\Longrightarrow$ NBUE $\Longrightarrow$ HNBUE $\Longrightarrow$ $\mathcal{L}$;

(ii) IFR $\Longrightarrow$ DMRL $\Longrightarrow$ NBUE.

The identical structure holds if each of the classes in the above chain is replaced by its dual.

The classes of life distributions described in Definitions 1-7 attracted a great deal of attention during the last two decades. The theoretical properties of the corresponding families of life d.f.s have been thoroughly investigated and practical applications have been found in modelling of lifetime of devices as well as in the theory of maintenance policies.

The interesting fact to note, however, is that the ageing pattern exhibited in all the above classes happens to be *monotonic*, i.e. the *direction* of ageing, so to say, remains the same throughout the entire lifespan of the unit under consideration. But, in actual fact, it is seen that the ageing pattern in many practical situations is *non-monotonic*. Typically, there is often a 'burn-in' phase where negative ageing takes place and then there is a useful life period followed by the 'wear-out' phase characterized by positive ageing. Many physical phenomena such as the ageing of human beings (where 'infant mortality' decreases during the 'burn-in' phase) and the lifetime of cutting tools (where work hardening takes place initially) are examples of this kind of ageing. Such situations are typically modelled using bathtub-shaped failure rate functions, i.e., the failure rate is non-increasing upto a point on the time scale, then it remains constant for a while and eventually becomes non-decreasing. We shall, in future, refer to such distributions, through abuse of terminology, as bathtub failure rate or simply BFR, for which we shall give a formal (and somewhat more general) definition in Chapter 2 of the thesis.

A considerable amount of work has already been done on bathtub distributions. Bray, Crawford and Proschan (1967) derived the maximum likelihood estimate of $\overline{F}(t)$ in the BFR class of distributions. Glaser (1980) has obtained sufficient conditions to ensure that a lifetime p.d.f. has a bathtub-shaped failure rate. Gaver and Acar (1979), Hjorth (1980), Mukherjee and Islam (1983), Paranjpe, Rajarshi and Gore (1985) and Roy (1988) have proposed parametric models to represent BFR distributions. Bergman (1979) and Park (1988) have suggested procedures for testing constant versus bathtub failure rate. Aarset (1985) has obtained the null distribution of the test statistic proposed by Bergman. Deshpande and Suresh (1990) have obtained a characterization of BFR distributions in terms of the *total* *time on test* (TTT) transform. Bathtub models arising out of a variety of stochastic and reliability mechanisms have also been investigated (Canfield and Borgman (1975), Cobb (1981)). A review of the class of BFR distributions has been given by Rajarshi and Rajarshi (1988).

It would, however, be misleading to confine ourselves simply to BFR distributions in the context of modelling non-monotonic ageing situations. Attempts have been made to study such ageing through the MRL function (Guess, Hollander and Proschan (1986)) by introducing the *Increasing initially, then Decreasing Mean Residual Life* (IDMRL) family of life distributions. The *Increasing initially, then Decreasing Residual Life* (IDRL) class of Deshpande and Suresh (1990) is an effort at studying non-monotonic ageing by means of the conditional survival function.

A large part of this thesis focuses on different aspects of the theory of non-monotonic ageing classes important in reliability.

In Chapter 2, we investigate some of the basic issues concerning BFR life distributions which were, till now, unresolved. Specifically, exponential bounds have been obtained for the survival function as well as the moments of a BFR distribution. Closure properties of the BFR family under the formation of coherent structures, convolutions and mixtures have been dealt with. Closure of the BFR class under convergence in distribution and the equivalence of weak convergence and convergence of moment sequences have also been established. Much of the material of this chapter can be found in Mitra and Basu (1991).

In Chapter 3, we introduce a nonparametric family of life distributions called *New Worse then Better than Used in Expectation* (NWBUE) class. This class is shown to include (in Chapter 5) the IDMRL class of Guess, Hollander and Proschan as well as all BFR distributions. Two inequalities which are later utilized to obtain bounds for the moments of NWBUE distributions have been established. The bounds thus obtained are shown to be related to the moments of an appropriate negative exponential distribution and a characterization of the exponential law in the NWBUE class is derived as a consequence. Issues related to weak convergence have also been settled. Similar results for the dual family comprising of *New Better then Worse than Used in Expectation* (NBWUE) distributions have also been explored. This chapter is based on some of the results obtained in Mitra and Basu (1994).

Chapter 4 deals with the life distributions of devices subject to shocks occurring randomly in time according to a homogeneous Poisson process. Under appropriate conditions on the probability of surviving a given number of shocks, it is shown that the BFR and NWBUE families arise from the shock model under consideration. Though the results seem more or less natural, the proofs become quite involved and a new technique has to be applied to handle the non-monotonicity present in the model. An effort has also been made to generalize the result in the BFR context to the case of a non-homogeneous Poisson process.

In Chapter 5, we investigate the interrelationships between the different notions of non-monotonic ageing. We prove that {NWBUE} is a superclass of both {IDMRL} and {BFR}. The proof exploits characterization results involving the TTT-transform. It is also argued that the IDRL distributions introduced by Deshpande and Suresh (1990) do not form a valid non-monotonic ageing family in the class of absolutely continuous distributions. Also, a conjecture of Deshpande and Suresh is proved to be false. The material of this chapter is based on Mitra, Basu and Roy (1993) and Mitra and Basu (1994).

The next vital issue in the context of non-monotonic ageing models is to furnish estimators for the points at which the failure rate function (or the MRL function, as the case may be) changes trend. In recognition of this issue, in Chapter 6, we develop a general methodology for consistent estimation of such change points in the context of BFR, NWBUE and IDMRL distributions.

In Chapter 7, we investigate certain theoretical properties of the $\mathcal{L}$-class of life distributions introduced by Klefsjö (1983). The main focus is on the role of the coefficient of variation of an $\mathcal{L}$ distribution in providing interesting characterizations of the exponential law in specific subclasses of the $\mathcal{L}$ family. A general characterization theorem has also been obtained.

We close the thesis with Chapter 8 where the focus shifts to the applications side; here we exploit the notion of NBUE distributions in pursuing an inventory problem. Specifically, we concentrate on a one-period inventory situation of a scarce commodity. Our work has been motivated by that of Panda (1978) and Basu

(1987). The (customer) demand distribution is considered random; so is the supply from the two suppliers in the model. Under 'new better than used in expectation' assumption on the supply distributions, (an assumption particularly relevant in the context of scarce commodities), a strategy which maximizes a minimum profit has been proposed. An estimate of this maximin order quantity whenever the demand distribution is unknown, has been obtained and strong consistency of the suggested estimator established.

Before concluding, a few words about the organization of the material presented in the following chapters. Each chapter has its own introduction which motivates the investigation carried out thereafter. It also familiarizes the reader with the setup and introduces the notation and the terminology used. The results are then presented in the subsequent sections where they are arranged in the form of theorems, lemmas, propositions and examples. We shall write Result x.y.z (which can be a theorem, lemma etc.) to mean Result z of Section y of Chapter x. Result y.z will stand for Result z of Section y of the same chapter.

Finally, we provide a bibliography where we cite the relevant references.

Chapter 2

BFR Class of Life Distributions

1 Introduction

The fundamental ageing classes such as IFR and its dual DFR are defined by requiring that the failure rate function is non-decreasing or non-increasing according as the ageing pattern is positive or negative. In situations where the failure rate may not exist, the notion of an IFR (DFR) distribution can be shown to remain valid by demanding the convexity (concavity) of the cumulative hazard function. Thus it is natural to take a cue from here to introduce the following well-known (and formal) definition of a BFR distribution.

DEFINITION 1.1. A life d.f. F having support on $[0,\infty)$ is said to be a 'bathtub failure rate' (BFR) distribution if there exists a point t_0 (≥ 0) such that $R(t) := -\ln\overline{F}(t)$ is *concave* in $[0, t_0)$ and *convex* in $[t_0, \infty)$. The point t_0 is referred to as a *change point* of the d.f. F in the BFR sense, and we write F is BFR(t_0).

Clearly, a BFR distribution can be used to model non-monotonic ageing situations with an initial 'burn-in' phase followed by a useful life period and a subsequent 'wear-out' phase.

Some authors, e.g. Deshpande and Suresh (1990), took t_0 to be strictly positive in the above definition. In such a formulation, one thus excludes the IFR class ($t_0 = 0$) but the DFR distributions ($t_0 = \infty$) remain included in the BFR class. We do not see any logical justification for this apparent asymmetry which we remove by allowing the possibility of t_0 to be zero. Thus, as per Definition 1.1, $\{\text{IFR}\}\bigcup\{\text{DFR}\}\subseteq\{\text{BFR}\}$.

It is apparent that if strict concavity or convexity is not insisted upon in the above definition, then a BFR d.f. may have more than one change point. For example, consider the distribution function F having the following survival function:

$$\overline{F}(x) = \begin{cases} (1+x)^{-1}, & 0 \leq x \leq \alpha, \\ (1+\alpha)^{-1}\exp[-\lambda(x-\alpha)], & \alpha \leq x \leq \beta, \\ (1+\alpha)^{-1}\exp[-\lambda(\beta-\alpha)+\frac{1}{2}\delta\beta^2-\frac{1}{2}\delta x^2], & x \geq \beta, \end{cases}$$

where $0 < \alpha < \infty$, $\lambda > 0$, $\beta > 0$, $\delta > 0$ with $\beta\delta = \lambda = (1+\alpha)^{-1}$. The corresponding failure rate function $r_F(\cdot)$ is then given by

$$r_F(x) = \begin{cases} (1+x)^{-1}, & x \le \alpha, \\ \lambda, & \alpha \le x \le \beta, \\ \delta x, & x \ge \beta. \end{cases}$$

Here, every $t \in [\alpha, \beta]$ is a change point of F.

For a BFR d.f. F, we define

$$\mathcal{T}_F = \{t : \ R(t) \text{ is concave in } [0,t) \text{ and convex in } [t,\infty)\ \}.$$

It is simple to note that $\mathcal{T}_F$ is either a singleton set or an interval. Further, F is IFR if $0 \in \mathcal{T}_F$ and DFR if $\infty \in \mathcal{T}_F$.

We shall now introduce the notion of a *strict* BFR (BFRS) distribution in the next definition.

DEFINITION 1.2. A life d.f. $F \in \{\text{BFR}\}$ is said to be a strict BFR (BFRS) if $\mathcal{T}_F = [t_{01}, t_{02}]$ with $0 < t_{01} < t_{02} < \infty$ such that $R(t)$ is *strictly concave* in $[t_{01} - h_1, t_{01})$ and *strictly convex* in $[t_{02}, t_{02} + h_2]$ for some $0 < h_1 \le t_{01}$, $h_2 > 0$.

Interestingly, though several authors have looked into various aspects of the BFR distributions, the following issues remained to be resolved in their context:

(a) exponential bounds for the survival function and moments of a BFR distribution;

(b) characterization of the exponential distribution in the BFR family;

(c) closure properties of the BFR class under the formation of coherent systems, convolutions and mixtures.

As is well-known, the above issues have been thoroughly investigated for the IFR and DFR classes of distributions (see Barlow and Proschan (1981)). Naturally then, we look into these issues for the wider class of BFR distributions in Sections 2 and 3. In Section 4, we deal with the problem of weak convergence within the BFR family and prove an interesting theorem establishing equivalence of such convergence with convergence of moment sequences.

2 Reliability and Moment Bounds

Consider an absolutely continuous BFR life distribution F with density function $f(\cdot)$ and failure rate function $r(\cdot)$. Since F has support over $[0,\infty)$, it is trivial to note that $r(t) > 0 \ \forall\ t > 0$; of course $r(0)$ can be zero, but then F is necessarily IFR, in which case exponential bounds for moments are already known. As such, we prove the following results for F having $r(0) > 0$.

LEMMA 2.1. *Suppose F is* BFR(t_0), *$t_0 < \infty$. Then $F \overset{st}{<} G$ where G is exponential with mean $1/r(t_0)$.*

PROOF.

$$\begin{aligned} \overline{F}(x) &= \exp\{-\int_0^x r(u)du\} \\ &\leq \exp\{-\int_0^x r(t_0)du\} \\ &= \exp[-xr(t_0)] \\ &= \overline{G}(x). \quad \square \end{aligned}$$

LEMMA 2.2. *If F is* BFR(t_0), *$t_0 < \infty$, then F has finite moments of all orders $k > 0$ and*

$$\mu_k := E_F X^k \leq \frac{\Gamma(k+1)}{(r(t_0))^k}, \quad k > 0. \tag{2.1}$$

PROOF. As $F \overset{st}{<} G$ and G has finite moments of all orders, the same is true for F. Further,

$$\begin{aligned} \mu_k &= \int_0^\infty x^{k-1}\overline{F}(x)dx \\ &\leq k\int_0^\infty x^{k-1}e^{-xr(t_0)}dx \\ &= \frac{\Gamma(k+1)}{(r(t_0))^k}. \quad \square \end{aligned}$$

The next lemma yields a characterization of the exponential distribution within the BFR class.

LEMMA 2.3. *Suppose F is* BFR(t_0), *$t_0 < \infty$ and $\mu_k = \Gamma(k+1)/(r(t_0))^k$ for some $k > 0$. Then F is exponential.*

PROOF. Note that

$$\begin{aligned} & k\int_0^\infty x^{k-1}(\overline{G}(x)-\overline{F}(x))dx \\ = & \frac{\Gamma(k+1)}{(r(t_0))^k} - \mu_k \\ = & 0. \end{aligned}$$

But, by Lemma 2.1, the integrand is non-negative so that the lemma follows. □

COROLLARY 2.1. *A life distribution which is* BFR(t_0), $t_0 < \infty$ *with mean equal to* $(r(t_0))^{-1}$ *is necessarily exponential.*

3 Basic Closure Properties

We first provide an example to argue that the convolution of BFRS d.f.s is not necessarily a BFRS.

EXAMPLE 3.1. Let us consider the life distribution whose survival function is given by

$$\overline{F}(x) = \begin{cases} (1+x)^{-1}, & x \le \theta \\ (1+\theta)^{-1}\exp[-\frac{1}{2}(x^2-\theta^2)], & x > \theta \end{cases}$$

where $\theta = \frac{1}{2}(\sqrt{5}-1)$. The corresponding failure rate function is given by

$$r_F(x) = \begin{cases} (1+x)^{-1}, & x \le \theta \\ x, & x \ge \theta, \end{cases}$$

so that F is a BFR d.f. Let $H(x) = F * F(x)$ denote the convolution of F with itself. Routine but somewhat lengthy calculations show that for $x < \theta$,

$$H(x) = \frac{x}{2+x} - \frac{2}{(2+x)^2}\ln(1+x),$$

and as such, for $x < \theta$,

$$r_H(x) = \frac{2x(x+2) + 4(x+1)\ln(x+1)}{2(x+1)(x+2)(x+2+\ln(x+1))}.$$

It is clear that the numerator of $r'_H(x)$ given by

$$\begin{aligned} \varphi(x) \;:=\; & 8(x+1)(x+2)(x+2+\ln(x+1))^2 \\ & -4\{(2x+3)(x+2+\ln(x+1)) + (x+2)^2\} \\ & \times\{x(x+2) + 2(x+1)\ln(x+1)\} \end{aligned}$$

is a continuous function of x. The continuity of $\varphi(x)$, together with the fact that $\varphi(0) > 0$ implies that $\exists$ a $\delta > 0 \ni \varphi(x) > 0 \; \forall \; x \in (0, \delta)$, so that $r_H(x)$ is strictly increasing in $(0, \delta)$. Consequently, the convolution is not a BFRS d.f.

In fact, even the BFR class is *not* closed under convolution as is demostrated by the next example.

EXAMPLE 3.2. We consider two life d.f.s F and G whose survival functions are given by

$$\begin{aligned} \overline{F}(x) &= \frac{1}{2}(e^{-x} + e^{-\frac{1}{2}x}), \quad x \geq 0; \\ \overline{G}(x) &= e^{-x}, \quad x \geq 0. \end{aligned}$$

Note that F, being a mixture of exponentials is DFR and hence BFR while G belongs to the BFR family trivially. Now the failure rate function of the convolution $H = F * G$ is given by

$$r_H(x) = \frac{(x-1)e^{-x} + e^{-\frac{1}{2}x}}{xe^{-x} + 2e^{-\frac{1}{2}x}};$$

accordingly, $r_H(0) = 0$, $r_H(2) = \frac{1}{2}$, $r_H(4) = 0.5533$ and $r_H(x) \to 0.5$ as $x \to \infty$. These values of $r_H(\cdot)$, obviously, reveal that it does not display the characteristic '*initially decreasing-ultimately increasing*' behaviour of the failure rate function of a BFR d.f.

The next result deals with the formation of series structure where the component lifetimes are BFR having the same change point.

THEOREM 3.1. *Suppose we have a series system where the lifetime of each of the components is* BFR (BFRS) *with a common change point* t_0. *Then the lifetime of the system again has a* BFR (BFRS) *d.f. with* t_0 *as one of its change points.*

PROOF. The conclusion of the theorem follows easily from the definition of BFR distributions and an elementary property of convex functions. □

We finally explore whether the {BFR} and {BFRS} families are closed under the formation of parallel structures. Unlike in the case of series structures, neither {BFR} nor {BFRS} is necessarily closed under the formation of parallel structures. This is evident from the examples that follow.

EXAMPLE 3.3. The failure rate function $r_P(\cdot)$ of the structure formed by connecting, in parallel, two independent components each having BFRS life distribution F as in Example 3.1, is given by

$$r_P(x) = \frac{2x}{(x+1)(2x+1)}, \quad x \leq \theta.$$

Since $r_P(x)$ is strictly increasing in $[0, \theta]$, the system lifetime cannot be BFRS.

EXAMPLE 3.4. Consider the parallel system comprising of two independent components as described in *Example* 2.1 (p. 83, Barlow and Proschan (1981)). It is clear that the system lifetime is *not* a BFR distribution; in fact, it is an *upside-down bathtub distribution*, as introduced in Glaser (1980). The distribution has a unique change point t_0 which solves

$$\lambda_1^2 e^{-\lambda_1 t} + \lambda_2^2 e^{-\lambda_2 t} = (\lambda_1 - \lambda_2)^2,$$

where λ_1 and λ_2 are as in the example mentioned.

Finally, we illustrate that the mixture of BFR life distributions need not be BFR.

EXAMPLE 3.5. Consider the life distribution given by the following survival function:

$$\overline{F}(x) = \frac{1}{2}(\overline{F}_1(x) + \overline{F}_2(x)),$$

where

$$\begin{aligned} \overline{F}_1(x) &= e^{-x}, \quad x > 0, \\ \overline{F}_2(x) &= 2e^{-2x}(x + \frac{1}{2}), \quad x > 0. \end{aligned}$$

Clearly, both F_1 and F_2 are IFR and hence BFR. Writing $\varphi(\cdot) := -\ln \overline{F}(\cdot)$, we observe that $\varphi''(x) = \psi_1(x)\psi_2(x)$, where

$$\begin{aligned} \psi_1(x) &= e^{-x}(1 + e^{-x}(2x+1))^{-2}, \\ \psi_2(x) &= -4e^{-x} + 2x - 3. \end{aligned}$$

Note that $\psi_1(x) < 0 \; \forall \, x > 0$; also, $\psi_2(0) < 0$, $\psi_2(\infty) = \infty$, and $\psi_2(\cdot)$ is a continuous increasing function. Hence it follows that $\exists$ a $t_0 > 0$ such that $\varphi(\cdot)$ is convex on $[0, t_0)$ and concave on $[t_0, \infty)$. Thus F is *not* BFR; in fact, it is an *upside-down bathtub* distribution.

We summarize the results obtained in this section in the following table together with their IFR and DFR counterparts for the purpose of comparability:

Table 1. Preservation of Life Distribution Classes under Reliability Operations.

Life distribution class	RELIABILITY Coherent systems	OPERATION Convolution	Mixture
IFR	NP	P	NP
DFR	NP	NP	P
BFR	NP	NP	NP

NP≡ Not Preserved P≡ Preserved

REMARK 3.1. As a BFR distribution has the tail behaviour of an IFR distribution, it might be intuitively appealing and rather tempting to anticipate the same kind of results to go through for BFR distributions as well. However, it is interesting to note that though this is the case in the context of mixtures, the result for convolutions is counter-intuitive, as the slightest DFR property at the initial stage upsets the natural conclusion.

4 Weak Convergence within the BFR Family

In this section, we are going to explore the connection between weak convergence and the convergence of moments within the BFR family.

We need the following lemma:

LEMMA 4.1. *Suppose F is* BFR(t_0), *$t_0 < \infty$. Then $(\overline{F}(x)/\overline{F}(t_0))^{1/x-t_0}$ is decreasing in x, $x \in (t_0, \infty)$.*

PROOF. The lemma follows easily from the fact that $R(x) = -\ln \overline{F}(x)$ is convex in (t_0, ∞). □

In Section 2 of this chapter, we derived an exponential bound for the survival function and moments of an *absolutely continuous* BFR distribution F. Even if F is *not* absolutely continuous, we can easily exploit Lemma 4.1 to claim that for any $A > t_0$, $\exists\, 0 < \theta^* = \theta^*(A, t_0) < \infty$, independent of x such that

$$\overline{F}(x) \leq \exp[-\theta^*(x - t_0)].$$

The above inequality leads to the fact that F has finite moments of all orders and for every $m > 0$,

$$E_F X^m \leq A^m + \frac{\Gamma(m+1)e^{\theta^* t_0}}{\theta^{*m}}. \tag{4.1}$$

LEMMA 4.2. *A d.f. which is* BFR(t_0), $t_0 < \infty$, *is uniquely determined by its moment sequence.*

PROOF. The lemma is an easy consequence of (4.1) which implies that the power series $\sum_{k=0}^{\infty}(u^m/m!)E_F X^m$ has a non-null radius of convergence (see e.g. Loéve [p.217,1963]). □

THEOREM 4.1. *Let* $\{F_n\}$ *be a sequence of* BFR *distributions such that* $F_n \to F$ *in law, where* F *is a continuous d.f. Then* F *is also a* BFR *distribution.*

PROOF. Let $t_{0n} \in \mathcal{T}_{F_n}$, $n = 1, 2, \ldots$. We consider the following two cases:

Case I: $\{t_{0n}\}$ **is bounded.** Consider a convergent subsequence $\{t_{0n_k}\}$ which converges to β as $k \to \infty$. Obviously, $\beta < \infty$; thus given $\epsilon > 0$, $\exists$ an integer $k_0 \geq 1$ such that

$$\beta - \epsilon < t_{0n_k} < \beta + \epsilon \;\forall\; k \geq k_0.$$

Consider $x, y > \beta + \epsilon$ which exceeds t_{0n_k} for all large k. Now, since F is continuous and R_{n_k} is convex in $[t_{0n_k}, \infty)$, for $0 \leq \alpha \leq 1$, $\overline{\alpha} = 1 - \alpha$, we have

$$\begin{aligned} R(\alpha x + \overline{\alpha} y) &= \lim_{k\to\infty} R_{n_k}(\alpha x + \overline{\alpha} y) \\ &\leq \lim_{k\to\infty} (\alpha R_{n_k}(x) + \overline{\alpha} R_{n_k}(y)) \\ &= \alpha R(x) + \overline{\alpha} R(y). \end{aligned} \tag{4.2}$$

Similarly, it follows that for all $x, y < \beta - \epsilon$, and $0 \le \alpha \le 1$,

$$R(\alpha x + \overline{\alpha} y) \ge \alpha R(x) + \overline{\alpha} R(y). \tag{4.3}$$

Since ϵ is arbitrary, (4.2) and (4.3) together imply that $\beta \in \mathcal{T}_F$. Accordingly, F is BFR.

Case II: $\{t_{0n}\}$ **is unbounded.** Consider $0 \le x \le y < \infty$. Obviously, there exists a subsequence $\{t_{0n_k}\}$ such that $y < t_{0n_k}\ \forall\, k \ge 1$. As R_{n_k} is concave in $[0, t_{0n_k})$, arguing as above and passing to the limit through this subsequence, we conclude that $R(\cdot)$ is concave in $[0, \infty)$, impling thereby that F is DFR and hence BFR. □

REMARK 4.1. Unlike in Theorem 4.1, if $F_n \in \{BFRS\}$ for each n, $F_n \to F$ in law, where F is a continuous d.f., then F may not always be a BFRS d.f. For example, consider a sequence of BFRS d.f.s $\{F_n\}$ with corresponding survival functions given by

$$\overline{F}_n(x) = \begin{cases} (1+x)^{-1}, & 0 \le x \le \alpha, \\ (1+\alpha)^{-1} \exp[-\lambda(x-\alpha)], & \alpha \le x \le \beta_n, \\ (1+\alpha)^{-1} \exp[-\lambda(\beta_n - \alpha) + \frac{1}{2}\delta_n \beta_n^2 - \frac{1}{2}\delta_n x^2], & x \ge \beta_n, \end{cases}$$

where $0 < \alpha < \infty$, $\lambda > 0$, $\beta_n > 0$, $\delta_n > 0$ with $\beta_n \delta_n = \lambda = (1+\alpha)^{-1}$ for $n \ge 1$. Then, $F_n \in \{BFRS\}$, for each n. If β_n's are such that $\beta_n \to \infty$ as $n \to \infty$, then $F_n \to F$ in law, as $n \to \infty$, where

$$\overline{F}(x) = \begin{cases} (1+x)^{-1}, & 0 < x \le \alpha, \\ (1+\alpha)^{-1} \exp[-\lambda(x-\alpha)], & x \ge \alpha. \end{cases}$$

Notice that the failure rate function $r_F(x) = (1+x)^{-1}$ or λ according as $x \le$ or $\ge \alpha$ so that $\infty \in \mathcal{T}_F$ and hence $F \notin \{BFRS\}$.

Basu and Simons (1983) proved that the weak limit of a sequence of IFR distributions is IFR provided the limit is continuous. The counterpart of their result in the context of DFR distributions was not known so far. However, as a consequence of Case II in the above theorem, we obtain the following corollary:

COROLLARY 4.1. *Let $\{F_n\}$ be a sequence of* DFR *distributions such that $F_n \to F$ in law as $n \to \infty$, where F is a continuous d.f. Then F is also a* DFR *distribution.*

Another interesting observation that results as a by-product of the proof of Theorem 4.1 above is as follows:

THEOREM 4.2. *Let* $\{F_n\}$ *be a sequence of* BFR *distributions such that* $F_n \to F$ *in law as* $n \to \infty$, *where* F *is a continuous d.f. If* F *and* F_n *have unique change points* t_0 *and* t_{0n} *respectively,* $n = 1, 2, ...,$ *then*

$$\lim_{n\to\infty} t_{0n} = t_0 \ (\leq \infty).$$

PROOF. As F has a unique change point, the proof easily follows as in Case I in the proof of Theorem 4.1 whenever $\{t_{0n}\}$ is bounded since then $\beta = t_0$ irrespective of the choice of the convergent subsequence $\{t_{0n_k}\}$. On the other hand, if $\{t_{0n}\}$ is unbounded, then necessarily F is a DFR d.f. and hence its unique change point $t_0 = \infty$, by Case II in the proof of Theorem 4.1. Now suppose that $\{t_{0n}\}$ does not converge to infinity. Then there should exist $M > 0$, such that $t_{0n} \leq M$ *infinitely often* so that it should be possible to pick a subsequence $\{t_{0n_k}\}$ which converges to a *finite* limit, say β. But, then by Case I in the proof of Theorem 4.1 again, $\beta \in \mathcal{T}_F$ and as such $\infty = \beta < \infty$ so that we arrive at a contradiction. Thus $t_{0n} \to t_0 = \infty$. □

Next we explore the connection between weak convergence of a sequence of BFR d.f.s with the convergence of the corresponding moment sequences of all orders. From Basu and Simons (1983), we note that there is a one to one correspondence between the weak convergence of a sequence of IFR distributions and the convergence of the corresponding moment sequence of any order to the moment of the same order of the limiting distribution. On the other hand, in this context, it is appropriate to also observe that the weak convergence of a sequence of DFR d.f.s $\{F_n\}$ to F (which is also then a DFR d.f. by Corollary 4.1) may not imply the convergence of, say, $E_{F_n}X^m$ to $E_F X^m$ as $n \to \infty$, even when the latter exists for some positive m. The following is an example to this effect:

EXAMPLE 4.1. For $n = 1, 2, ...,$ consider the DFR d.f.

$$\overline{F}_n(x) = \alpha_n e^{-x/\mu_n} + \overline{\alpha}_n e^{-x/\nu_n},$$

where $0 < \alpha_n < 1$ with $\alpha_n \to 1$, $0 < \mu_n \to \mu < \infty$, $\nu_n \to \infty$ and $\overline{\alpha_n}\nu_n^m \to \infty$ as $n \to \infty$, m being a positive number. For example, one could take $\alpha_n = 1 - \frac{1}{n}$, $\mu_n = \mu + \frac{1}{n}$ and $\nu_n = n$ so that all the above requirements can be satisfied for any $m > 1$. Now, obviously, as $n \to \infty$, $F_n \to F$ in law, F being the d.f. of the exponential distribution having mean μ. But $E_{F_n}X^m \to \infty$ while $E_F X^m$ is finite.

The above example illustrates that weak convergence of a sequence of DFR d.f.s need not imply the convergence of the corresponding moment sequence of any specific order to the corresponding moment of the limiting distribution. Since the $\{DFR\} \subseteq \{BFR\}$, generally, the same contention remains valid in respect of a sequence of BFR d.f.s also, unless some additional assumption is made. The following theorem is an effort in this direction.

THEOREM 4.3. *Suppose* $\{F_n\}$ *is a sequence of* BFR *distributions and let* $t_{0n} \in \mathcal{T}_{F_n}$, $n = 1, 2, \ldots$. *Assume that the sequence* $\{t_{0n}\}$ *is bounded.*

(i) *If* $F_n \to F$ *in law, where* F *is a continuous* BFR *d.f., then for every* $m > 0$,

$$E_{F_n}X^m \to E_F X^m \tag{4.4}$$

as $n \to \infty$.

(ii) *Conversely, if* (4.4) *holds for each integer* $m > 0$, *and some* BFR *d.f.* F, *then* $F_n \to F$ *in law.*

PROOF. (i) In view of finiteness of moments of F and Theorem 4.1, it is enough to show that for all $m > 0$,

$$\lim_{n\to\infty} \int_0^\infty x^{m-1}\overline{F}_n(x)dx = \int_0^\infty x^{m-1}\overline{F}(x)dx \tag{4.5}$$

Let $M > 0$ be such that $t_{0n} \leq M \ \forall\ n$. Let A be a positive number such that $A > M$ and suppose $\epsilon^* > 0$ be such that $\epsilon^* + (\overline{F}(A)/\overline{F}(M)) := e^{-\theta}$, θ being a positive number. As $F_n \to F$ in law, as $n \to \infty$, it is plain to note that there exists $n_1 = n_1(\epsilon^*)$ such that

$$\frac{\overline{F}_n(A)}{\overline{F}_n(t_{0n})} < e^{-\theta},$$

for all $n \geq n_1$. Using Lemma 4.1, we then observe that for all large n,

$$\overline{F}_n(x) \leq \begin{cases} \exp[-\theta(x - t_{0n})/(A - t_{0n})] \leq \exp[-\theta(x - M)/A], & x > A \\ 1, & x \leq A. \end{cases} \tag{4.6}$$

Since $F_n(x) \to F(x)$ at each x, F being continuous, in view of the Dominated Convergence Theorem, (4.5) follows using (4.6).

(ii) Lemma 4.2, Case I in the proof of Theorem 4.1 applied to a convergent subsequence of F_n and a standard argument based on tightness and relative compactness establish this part of the theorem. □

REMARK 4.2. It is simple to note that $t^*_{0n} := \inf \mathcal{T}_{F_n} \in \mathcal{T}_{F_n}$. As such, the above theorem will hold if $\{t^*_{0n}\}$ is bounded. In Example 4.1, each F_n is a BFR d.f. with a unique change point $t_{0n} = \infty$. It may be mentioned here that the boundedness condition regarding $\{t_{0n}\}$ is only sufficient and *not* necessary for the theorem to hold. This can be seen by taking α_n, μ_n and ν_n in Example 4.1 to be such that $\alpha_n \to 1$, $\mu_n \to \mu(< \infty)$ and $\nu_n \to \nu\ (< \infty)$ as $n \to \infty$. Then, each F_n has a unique change point $t_{0n} = \infty$, $F_n \to F$ in law, F being the exponential d.f. with mean μ, and $E_{F_n}X^m \to E_F X^m$, for every $m > 0$.

5 The Dual Class

In this section, we focus our attention on the dual family comprising of UBFR distributions and consider issues similar to those investigated for the BFR class.

We first note that Lemma 2.2. is false for UBFR distributions, i.e., an UBFR life d.f. F having a finite change point (and a finite mean) need not have finite moments of all orders. The following example demonstrates this.

EXAMPLE 5.1. Consider the life distribution whose survival function is given by

$$\overline{F}(x) = \alpha(x^2 + \alpha)^{-1}, x \geq 0$$

where $\alpha > 0$ is a constant. Its failure rate function is of the form

$$r(x) = 2(x + \frac{\alpha}{x})^{-1}$$

which is *strictly increasing* in the interval $(0, \sqrt{\alpha})$ and *strictly decreasing* in $(\sqrt{\alpha}, \infty)$; as such, F is UBFR with $t_0 = \sqrt{\alpha}$ as its finite change point. The mean of this distribution is $(\pi\sqrt{\alpha})/2$; however, for $r > 1$, $E_F X^r = \infty$.

We now discuss the UBFR analogues of the results presented in Section 4 in the folowing remarks.

REMARK 5.1. Theorem 4.1 remains valid for UBFR distributions as well; only when the sequence of change points is unbounded, the limiting distribution happens to be IFR rather than DFR. Theorem 4.2 also goes through for UBFR distributions. The arguments for proving these are exactly along the lines of the BFR versions.

REMARK 5.2. The first part of Theorem 4.3 does not hold for UBFR distributions as is evident from Example 4.1. Moreover, the fact that an UBFR distribution (with finite change point and mean) need not have moments of higher orders, makes the issue contained in the second part less relevant in this context.

Chapter 3

NWBUE Class of Life Distributions

1 Introduction

The monotonic ageing notion, characterised by the NBUE property of a life d.f. F requires that the MRL function $e_F(x)$ is dominated by the mean $\mu = e_F(0)$ of F. In this chapter, we follow up on this approach to introduce a non-monotonic analogue of the NBUE property. We shall thus develop a nonparametric class of life distributions, which we shall call the *New Worse then Better than Used in Expectation* (NWBUE). We shall show later (in Chapter 5) that this class, which we call the NWBUE class, includes the IDMRL family introduced by Guess, Hollander and Proschan (1986) as well as all BFR distributions. Consequently, the results that we prove in this chapter can also be used in the context of BFR and IDMRL distributions. We now formally define the NWBUE family of life distributions as follows:

DEFINITION 1.1. A life d.f. F having support on $[0, \infty)$ (and finite mean μ) is said to be *New Worse then Better than Used in Expectation* (NWBUE) (*New Better then Worse than Used in Expectation* (NBWUE)) if there exists a point $x_0 \geq 0$ such that

$$e_F(x) \begin{cases} \geq (\leq) e_F(0), & \text{for } x < x_0, \\ \leq (\geq) e_F(0), & \text{for } x \geq x_0. \end{cases}$$

We shall refer to such an x_0 (which need not be unique) as a change point of the d.f. F in the NWBUE sense; we shall write F is NWBUE(x_0) (NBWUE(x_0)) to indicate that the life distribution F is NWBUE (NBWUE) and x_0 is a change point of F. Let C_F be the collection of all change points of an NWBUE (or NBWUE) life d.f. F. It is easy to see that for a continuous d.f. F, C_F is either a singleton or a closed interval. Note that an NWBUE (NBWUE) life distribution F is NBUE (NWUE) if $0 \in C_F$ while it is NWUE (NBUE) if $\infty \in C_F$.

We present below a simple example of an NWBUE distribution.

EXAMPLE 1.1. Consider the life distribution characterised by the survival function

$$\overline{F}(x) = \begin{cases} 4/(2+x)^2 & \text{if } 0 \le x < 1 \\ (4x/9)\exp[-(x^2-1)/6] & \text{if } 1 \le x < \infty. \end{cases}$$

The corresponding MRL function is as follows:

$$e_F(x) = \begin{cases} 2+x & \text{if } 0 \le x < 1 \\ 3/x & \text{if } x \ge 1. \end{cases}$$

It is evident that F is NWBUE(x_0) with $x_0 = 3/2$.

In the next section, we obtain some useful bounds for the moments of NWBUE distributions; we also provide a characterization of the exponential distribution as a consequence of some of the results discussed in this section. In section 3, we prove closure under weak convergence and the equivalence of weak convergence and moment convergence in the NWBUE family of life distributions and also furnish a corollary which is an interesting by-product of our results. Our observations relating to the dual class comprising of NBWUE distributions are given in Section 4.

2 Inequalities and Moment Bounds

We would need the following basic inequality which we present in the form of a lemma.

LEMMA 2.1. *If F is* NWBUE(x_0) *with finite mean μ, then*

$$\int_x^\infty \overline{F}(u)du \begin{cases} \ge \mu e^{-x/\mu}, & \text{for } x < x_0, \\ \le \mu e^{-(x-x_0)/\mu}, & \text{for } x \ge x_0. \end{cases} \tag{2.1}$$

Further, if $e_F(x)$ is bounded above by $M > 0$, then

$$\int_x^\infty \overline{F}(u)du \le \mu e^{-x/\mu} \exp\{-x_0(M^{-1} - \mu^{-1})\}, \; \forall\, x \ge x_0\,. \tag{2.2}$$

PROOF. The probability density function of the *first derived distribution* of F is given by

$$f_{(1)}(x) = \frac{\overline{F}(x)}{\mu}, \; x \ge 0$$

and let $r_{(1)}(x)$ denote the corresponding failure rate function. Note that

$$\begin{aligned} r_{(1)}(x) &= \frac{f_{(1)}(x)}{\int_x^\infty f_{(1)}(u)du} \\ &= \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(u)du} \\ &= 1/e_F(x). \end{aligned}$$

As F is NWBUE (x_0), it now follows that

$$r_{(1)}(x) \begin{cases} \leq 1/\mu, & \text{if } x < x_0, \\ \geq 1/\mu, & \text{if } x \geq x_0. \end{cases}$$

Writing $\overline{F}_{(1)}(x) = \exp[-\int_0^x r_{(1)}(u)du]$ and using the above inequalities, (2.1) follows easily.

If $e_F(x) \leq M$, then (2.2) follows simply from the relation between the MRL function and the failure rate of the derived distribution. □

PROPOSITION 2.1. *Suppose F is* NWBUE(x_0), $x_0 < \infty$ *with (finite) mean μ and let φ be a non-decreasing (non-increasing) function on* $[0, \infty)$. *Then*

$$\int_0^\infty \varphi(x)\overline{F}(x)dx \leq (\geq) \int_{x_0}^\infty \varphi(x)\exp[-(x - x_0)/\mu]dx \tag{2.3}$$

PROOF. We shall prove the result for non-decreasing φ; the other case can be treated similarly. Without loss of generality, we can take φ to be non-negative; the result for a general φ would then follow easily by decomposing it into its positive and negative parts. Let Z be a random variable whose d.f. is $F_{(1)}$, defined in the previous proof and U be another random variable such that $U - x_0$ is exponentially distributed with mean μ. Then, as φ is non-decreasing, in view of (2.1), we have

$$\begin{aligned} \int_0^\infty \varphi(x)\overline{F}(x)dx &= \mu\int_0^\infty P(\varphi(Z) > x)dx \\ &= \mu\int_0^\infty P(Z > \varphi^{-1}(x))dx \\ &= \mu\int_0^{\varphi(x_0)} P(Z > \varphi^{-1}(x))dx + \mu\int_{\varphi(x_0)}^\infty P(Z > \varphi^{-1}(x))dx \\ &\leq \mu\varphi(x_0) + \mu\int_{\varphi(x_0)}^\infty P(U > \varphi^{-1}(x))dx \\ &= \mu\varphi(x_0) + \mu\int_{\varphi(x_0)}^\infty P(\varphi(U) > x)dx\,. \end{aligned}$$

Now,

$$\begin{aligned}\int_{\varphi(x_0)}^{\infty} P(\varphi(U) > x)dx &= E\varphi(U) - \int_0^{\varphi(x_0)} P(\varphi(U) > x)dx \\ &= E\varphi(U) - \int_0^{\varphi(x_0)} P(\varphi(U) > x)dx \\ &= E\varphi(U) - \int_0^{\varphi(x_0)} P(U > \varphi^{-1}(x))dx \\ &= E\varphi(U) - \varphi(x_0),\end{aligned}$$

the integrand being unity, since φ is non-decreasing. Thus,

$$\begin{aligned}\int_0^{\infty} \varphi(x)\overline{F}(x)dx &\leq \mu\varphi(x_0) + \mu\{E\varphi(U) - \varphi(x_0)\} \\ &= \mu E\varphi(U),\end{aligned}$$

which completes the proof. □

COROLLARY 2.1. *If F is* NWBUE(x_0), $x_0 < \infty$, *with finite mean* μ, *then*

$$\int_0^{\infty} x^{r-1}\overline{F}(x)dx \leq (\geq) \int_{x_0}^{\infty} x^{r-1}\exp[-(x-x_0)/\mu]dx\,, \ \text{for } r \geq (<)1 \qquad (2.4)$$

PROOF. Take $\varphi(x) = x^{r-1}$ in Proposition 2.1 and note that φ is non-increasing for $r < 1$ and non-decreasing for $r \geq 1$. □

The above result leads to the following bounds for the moments of a NWBUE(x_0) distribution where x_0 is finite.

COROLLARY 2.2. *If F is* NWBUE(x_0), $x_0 < \infty$, *with finite mean* μ, *then*

$$E_F X^r \leq (\geq) r e^{x_0/\mu} \int_{x_0}^{\infty} x^{r-1} e^{-x/\mu} dx\,, \ \text{for } r \geq (<)1\,. \qquad (2.5)$$

Given a NWBUE(x_0) distribution, $x_0 < \infty$, with finite mean μ, we shall now obtain bounds for the moments of the above distribution in terms of the moments of an appropriate negative exponential distribution.

Consider the following negative exponential distribution defined by the survival function:

$$\overline{G}(x) = \begin{cases} 1, & \text{if } x < x_0\,, \\ \exp[-(x-x_0)/\mu], & \text{if } x \geq x_0. \end{cases}$$

Simple calculations show that for $r \geq 1$,

$$E_G X^r = x_0^r + re^{x_0/\mu} \int_{x_0}^{\infty} x^{r-1} e^{-x/\mu} dx\,. \tag{2.6}$$

If $r \geq 1$ is an integer,

$$E_G X^r = x_0^r + \mu^r \Gamma(r+1) \sum_{j=0}^{r-1} \frac{(x_0/\mu)^j}{j!} \tag{2.7}$$

The following corollary now gives the various moment bounds.

COROLLARY 2.3. *Let F be* NWBUE(x_0), *$x_0 < \infty$ with finite mean μ, and let G be as defined above. Then*

(i) $E_G X^r < \infty\ \forall\, r > 0$.

(ii) $E_F X^r \leq E_G X^r\ \forall\, r \geq 1$.

(iii) $E_G X^r \leq x_0^r + \mu^r \Gamma(r+1) \sum_{j=0}^{r-1} (x_0/\mu)^j / j!\ \forall$ *integers* $r \geq 1$.

(iv) $E_F X^r \leq \mu^r \Gamma(r+1) e^{x_0/\mu},\ \forall\, r \geq 1$.

PROOF. (i) This follows from (2.6).

(ii) For $r > 1$, by (2.5),

$$\begin{aligned} E_G X^r &\leq re^{x_0/\mu} \int_{x_0}^{\infty} x^{r-1} e^{-x/\mu} dx \\ &\leq x_0^r + re^{x_0/\mu} \int_{x_0}^{\infty} x^{r-1} e^{-x/\mu} dx = E_G X^r\,. \end{aligned} \tag{2.8}$$

(iii) This follows from the previous part and (2.7).

(iv) For $r \geq 1$, by (2.5),

$$\begin{aligned} E_F X^r &\leq re^{x_0/\mu} \int_{x_0}^{\infty} x^{r-1} e^{-x/\mu} dx \\ &\leq re^{x_0/\mu} \int_{0}^{\infty} x^{r-1} e^{-x/\mu} dx \\ &= \mu^r \Gamma(r+1) e^{x_0/\mu}\,. \end{aligned}$$

This completes the proof of the corollary. □

REMARK 2.1. Taking $x_0 = 0$ (i.e., if F is NBUE) in Corollary 2.1, we get the usual well-known bounds for the NBUE situation.

REMARK 2.2. Though (ii) and (iv) of the last corollary give the same bounds for the NBUE case, it is clear that the former, which has an interesting probabilistic significance is sharper than the latter, which, on the other hand, has useful applications as illustrated in the next section.

The next result gives an interesting characterization of the exponential distribution.

THEOREM 2.1. *Let F be* NWBUE(x_0), $x_0 < \infty$ *with finite mean μ and let G be as in Corollary* 2.3. *Then F is the exponential distribution if $E_F X^r = E_G X^r$ for some $r > 1$.*

PROOF. As $E_F X^r = E_G X^r$ for some $r > 1$, then equality holds in the string of inequalities (2.8) so that $x_0 = 0$ and hence F is NBUE satisfying the relation $E_F X^r = \Gamma(r+1)\mu^r$ for some $r > 1$. Consequently, F is HNBUE satisfying the above relation. Now, Lemma 2.4 of Basu and Bhattacharjee (1984) implies that F is exponential. □

3 Weak Convergence of NWBUE distributions

We start this section with the following theorem:

THEOREM 3.1. *Let F_n, $n = 1, 2, \ldots$ be a sequence of* NWBUE(x_{0n}) *life distributions with means μ_n. Suppose that*

(i) *$F_n \to F$ in law, where F is a continuous d.f.;*

(ii) *The sequences $\{\mu_n\}$ and $\{x_{0n}\}$ are bounded.*

Then F is NWBUE. *Further,*

$$\lim_{n\to\infty} \int_0^\infty x^r dF_n(x) = \int_0^\infty x^r dF(x) \text{ for every } r > 0. \tag{3.1}$$

PROOF. Let $\mu_n \leq B \ \forall\ n \geq 1$ and let μ be the mean of F. An application of Fatou's lemma together with (i) above shows that $\mu < \infty$. We first prove that $\mu_n \to \mu$ as $n \to \infty$; for $A > 0$, to be chosen suitably

$$\begin{aligned} & \int_0^\infty \{\overline{F}_n(x) - \overline{F}(x)\}dx \\ = & \int_0^A (\overline{F}_n - \overline{F})(x)dx + \int_A^\infty \overline{F}_n(x)dx - \int_A^\infty \overline{F}(x)dx \\ = & I_{1n} + I_{2n} + I_3 . \end{aligned}$$

Suppose $x_{0n} \leq M$, $\forall\, n \geq 1$. For $A > M > x_{0n}$, $\epsilon > 0$, by Lemma 2.1,

$$\begin{aligned}&\int_A^\infty \overline{F}_n(x)dx \\ \leq\ & \mu_n \exp\{-(A - x_{0n})/\mu_n\} \\ \leq\ & B\exp\{-(A-M)/B\}\end{aligned}$$

which can be made smaller than $\epsilon/3$ by choosing a sufficiently large A. Since $\mu < \infty$, $|I_3| < \epsilon/3$ for all large A. So, for $A > M$, sufficiently large, we have $|I_{2n}| < \epsilon/3\ \forall\ n \geq 1$ and $|I_3| < \epsilon/3$. As $|I_{1n}| \to 0$ by the dominated convergence theorem, $|I_{1n}| < \epsilon/3\ \forall\, n \geq n_0(A,\epsilon)$. Accordingly, as $n \to \infty$,

$$\mu_n \to \mu. \tag{3.2}$$

Since x_{0n} is bounded, $\exists$ a subsequence $\{x_{0n_k}\}$ such that $x_{0n_k} \to \beta < \infty$ as $k \to \infty$. For $x < \beta$, we can choose a sufficiently large integer $k_0 \geq 1$ such that $x_{0n_k} > x$, $\forall\, k \geq k_0$ which implies

$$\frac{1}{\overline{F}_{n_k}(x)} \int_x^\infty \overline{F}_{n_k}(u)du \geq \mu_{n_k}.$$

Taking limits as $k \to \infty$, via condition (i) of the theorem, (3.2) and the dominated convergence theorem, we get

$$\frac{1}{\overline{F}(x)} \int_x^\infty \overline{F}(u)du \geq \mu. \tag{3.3}$$

Similarly for $x > \beta$, we can show that

$$\frac{1}{\overline{F}(x)} \int_x^\infty \overline{F}(u)du \leq \mu. \tag{3.4}$$

The proof of the first part of the theorem now follows from (3.3) and (3.4).

To prove (3.1), we first note via (3.2) that it holds for $r = 1$. Let X_n, $n = 1, 2, ...$ and X be random variables having d.f.s F_n and F respectively. Then, as $\sup_n EX_n < \infty$, $\{X_n^r\}$ is uniformly integrable for $r < 1$; this together with (i) establishes (3.1) for $r < 1$.

On the other hand, for $r > 1$, by Corollary 2.3(iv),

$$EX_n^r \leq \mu_n^r \Gamma(r+1) e^{x_{0n}/\mu_n}.$$

As $\{x_{0n}\}$ is bounded and $\{\mu_n\}$ converges, the sequence $\{x_{0n}/\mu_n\}$ is bounded so that $\sup_n EX_n^r < \infty\ \forall\, r > 1$, by the above relation. So, $\{X_n^r\}$ is uniformly integrable for each $r > 1$, and this proves (3.1) for $r > 1$. □

LEMMA 3.1. *A d.f. which is* NWBUE(x_0), $x_0 < \infty$, *is uniquely determined by its moment sequence.*

PROOF. Let F be NWBUE(x_0), $x_0 < \infty$ with mean μ. By Corollary 2.3(iv), it follows easily that the power series $\sum_{r=0}^{\infty} u^r/r! E_F X^r$ has a non-null radius of convergence. The lemma now follows from a result on page 217 of Loéve (1963). □

In Theorem 3.1, we showed that under a couple of conditions, weak convergence of a sequence of NWBUE d.f.s implies convergence of moments of the sequence of d.f.s to the corresponding moments of the limiting d.f. The following theorem deals with the converse of this result.

THEOREM 3.2. *Let F_n, $n = 1, 2, \ldots$ be a sequence of* NWBUE(x_{0n}) *d.f.s with $x_{0n} < \infty \; \forall \; n \geq 1$ and supppose that F is* NWBUE(x_0), *$x_0 < \infty$ such that for all integers $r \geq 1$,*

$$\lim_{n \to \infty} \int_0^\infty x^r dF_n(x) = \int_0^\infty x^r dF(x) \tag{3.5}$$

Then $F_n \to F$ in law.

PROOF. By Lemma 3.1, F is uniquely determined by its moment sequence. In view of this and (3.5), the limiting distribution of every weakly convergent subsequence of $\{F_n\}$ happens to be necessarily F and this completes the proof. □

If condition (ii) in Theorem 3.1 is replaced by the condition

$$\lim_{n \to \infty} \int_0^\infty x dF_n(x) = \int_0^\infty x dF(x) \; (< \infty). \tag{3.6}$$

Then we will have an interesting conclusion which we present below in the form of a proposition.

PROPOSITION 3.1. *Suppose in Theorem* 3.1, *we replace condition* (ii) *by* (3.6) *given above. If the sequence $\{x_{0n}\}$ is unbounded, then F is* NWUE.

PROOF. Let μ be the mean of F; consider any $x > 0$. As $\{x_{0n}\}$ is unbounded, $\exists$ a subsequence $\{x_{0n_k}\}$ such that $x_{0n_k} > x \; \forall \; k \geq 1$. So,

$$\frac{1}{\overline{F}_{n_k}(x)} \int_x^\infty \overline{F}_{n_k}(u) du \geq \mu_{n_k}.$$

Taking limits as $k \to \infty$ and arguing as in Theorem 3.1, we get

$$\frac{1}{\overline{F}(x)} \int_x^\infty \overline{F}(u) du \geq \mu \,,$$

so that F is NWUE. □

REMARK 3.1. It is trivial to observe from Proposition 3.1 that under (3.6), the weak limit of a sequence of a NWUE d.f.s is also NWUE, provided the limiting distribution is continuous. The counterpart of this result concerning NBUE distributions was given by Basu and Bhattacharjee (1984) with more generality.

4 The Dual Class

In this section, we extend similar investigations as in Sections 2 and 3 to the dual family comprising of NBWUE distributions.

We first present an example to demonstrate that unlike NWBUE life distributions, an NBWUE distribution having a finite change point may not possess finite moments of order higher than 1.

EXAMPLE 4.1. Consider the life d.f. F described in Example 2.5.1. The mean of this distribution is $(\pi\sqrt{\alpha})/2 < \infty$; however, for $r > 1$, $E_F X^r = \infty$. As already noted, F is UBFR and hence, using the dual version of Theorem 5.4.1, we conclude that F is NBWUE. Finally, analyzing its MRL function,

$$e_F(x) = (\sqrt{\alpha})^{-1}(x^2 + \alpha)(\pi/2 - \tan^{-1}(x/\alpha))$$

we conclude that F has a unique finite change point.

In view of the above example, it is evident that a result analogous to Corollary 2.3(ii) is, in general, false for NBWUE distributions having finite change points. As such, issues contained in subsequent results in the same corollary do not seem relevant in the context of NBWUE distributions.

From the following example, we further conclude that the NBWUE version of Theorem 3.1 is false.

EXAMPLE 4.2. For $n = 1, 2, ...$, consider the sequence of NBWUE(0) life distributions defined by the survival functions

$$\overline{F}_n(x) = \alpha_n e^{-x/\mu_n} + \overline{\alpha}_n e^{-x/\nu_n},$$

where $0 < \alpha_n \leq 1$ with $\alpha_n \to 1$, $0 < \mu_n \to \mu < \infty, \nu_n \to \infty$ and $\overline{\alpha}_n \nu_n^m \to \infty$ as $n \to \infty$, m being a positive number. For example, we may take $\alpha_n = 1 - \frac{1}{n}$, $\mu_n = \mu + \frac{1}{n}$, $\nu_n = n$ so that all requirements are satisfied for any $m > 1$. Now, as $n \to \infty$, $F_n \to F$ in law, F being the d.f. of the exponential distribution having mean μ. But, $E_{F_n}X^m \to \infty$ while $E_F X^m$ is finite. Thus, for NBWUE distributions, weak convergence does not necessarily imply moment convergence even if both the sequences $\{\mu_n\}$ and $\{x_{0n}\}$ are bounded.

However, the question of closure of the NBWUE class under weak convergence still remains open.

REMARK 4.1. In view of Example 4.1, issues dealt with in Lemma 3.1 and Theorem 3.2 are, in general, *not* relevant in the context of NBWUE distributions. However, proceeding as in the proofs of Theorem 3.1 and Proposition 3.1, we can prove the following version of the latter.

THEOREM 4.1. *Let F_n, $n = 1, 2, ...$ be a sequence of* NBWUE(x_{0n}) *life distributions with means μ_n. Suppose that*

(i) *$F_n \to F$ in law, where F is a continuous d.f.;*

(ii)
$$\lim_{n\to\infty} \int_0^\infty x dF_n(x) = \int_0^\infty x dF(x) \ (< \infty) \tag{4.1}$$

If $\{x_{0n}\}$ is bounded, then F is NBWUE*; otherwise F is* NBUE.

From the above theorem, we easily observe that the NBUE class is closed under weak convergence provided (4.1) holds. This follows from the fact that $x_{0n} = \infty \ \forall n \geq 1$ when the F_n have the NBUE property.

Chapter 4

Non-monotonic Ageing through Shocks

1 Introduction

In this chapter, we consider the survival function $\overline{H}(\cdot)$ of a single device subject to shocks occurring randomly in time according to a homogeneous Poisson process.

Suppose $N(t)$ denotes the number of shocks the device experiences in the time interval $(0, t]$ and let $\overline{P}_k$ be the probability that the device survives the first k shocks, $k = 0, 1, 2, \ldots$. It is assumed that the $\overline{P}_k$' s satisfy the following natural condition:

$$1 = \overline{P}_0 \geq \overline{P}_1 \geq \overline{P}_2 \ldots . \tag{A}$$

Thus the probability $\overline{H}(t)$ of the device to survive beyond time t is given by

$$\overline{H}(t) = \sum_{k=0}^{\infty} P[N(t) = k]\, \overline{P}_k, \quad t \geq 0, \tag{1.1}$$

where $P[N(t) = k] = e^{-\lambda t}(\lambda t)^k/k!$, $k = 0, 1, 2, \ldots$, and λ (>0) is the rate of the underlying homogeneous Poisson process mentioned.

Esary, Marshall and Proschan (1973) have shown that if the sequence $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ possesses what is called the discrete IFR property, then $\overline{H}(t)$ is also an IFR survival function. It has been shown (Marshall and Proschan (1972)) that when $N(t)$ is a homogeneous Poisson process and the $\overline{P}_k$'s have a discrete NBUE (NWUE) property, then the transformation (1.1) carries this discrete property over to the corresponding continuous property, i.e., $\overline{H}(t)$ is continuous NBUE (NWUE). Analogous results for the HNBUE (HNWUE) class have been proved in Klefsjö (1981). Other interesting results in the context of shock models are given in Block and Savits (1978), Gottlieb (1980) and Ghosh and Ebrahimi (1982). In Section 2 of this chapter, we present a result showing that if the $\overline{P}_k$'s possess a discrete NWBUE property, then the continuous NWBUE property is inherited by the survival function $\overline{H}(t)$ described in (1.1). Similar results are proved for BFR distributions under appropriate conditions.

2 A Shock Model Leading to NWBUE Survival

We introduce the notion of the discrete NWBUE property as follows:

DEFINITION 2.1. Let $\overline{P}_k$, $k = 0, 1, \ldots$ as defined in the previous section satisfy the natural condition (A) with $\sum_{j=0}^{\infty} \overline{P}_j < \infty$. The sequence $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ is said to be a discrete NWBUE (NBWUE) sequence if there exists an integer $k_0 > 0$ such that

$$a_k := \overline{P}_k \sum_{j=0}^{\infty} \overline{P}_j - \sum_{j=k}^{\infty} \overline{P}_j \begin{cases} \leq (\geq) 0 & \forall\, k < k_0, \\ \geq (\leq) 0 & \forall\, k \geq k_0. \end{cases} \tag{2.1}$$

The point k_0 will be referred to as the change point of the sequence $\{\overline{P}_k, k = 0, 1, \ldots\}$ and we shall write $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ is NWBUE(k_0) (NBWUE(k_0)).

To gain an insight into the above definition, consider a device which survives k shocks with probability $\overline{P}_k$. According to the above definition, the sequence $\overline{P}_k$ has the discrete NWBUE property if the average number of shocks to failure of the device is initially smaller and subsequently larger than the expected number of additional shocks required to cause failure.

The following gives an example of a discrete NWBUE sequence:

EXAMPLE 2.1. Consider a device which survives at most 3 shocks. Let $\overline{P}_0 = 1$, $\overline{P}_1 = 1/\alpha$, $\overline{P}_2 = \overline{P}_3 = 1/2\alpha$, $\overline{P}_k = 0\ \forall\ k \geq 4$, with $\alpha \geq 1$. If $\alpha > 2$, then $\overline{P}_k$ is an NWBUE(3) sequence. If $1 \leq \alpha \leq 2$, then $\overline{P}_k$ is an NBUE sequence.

We now prove a theorem which shows that the discrete NWBUE (NBWUE) property of the sequence $\overline{P}_k$ gets translated to the continuous NWBUE (NBWUE) property of $\overline{H}(t)$ under the transformation (1.1). We shall need the following lemma for this purpose:

LEMMA 2.1. *Suppose* $\{\alpha_k,\ k = 0, 1, \ldots\}$ *is a real sequence with the property that*

$$\alpha_k \begin{cases} \leq 0 & \forall k < k_0 \\ \geq 0 & \forall k \geq k_0, \end{cases} \tag{2.2}$$

for some positive integer k_0; also assume that the function defined by $\varphi(s) := \sum_{k=0}^{\infty}(\alpha_k/k!)s^k$, $s \geq 0$, converges absolutely $\forall\, s \geq 0$. Then $\exists\, s_0 \geq 0$ such that $\varphi(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0$.

PROOF. The case $k_0 = 1$ is simple. As the power series converges absolutely for all $s \geq 0$, it can be differentiated term by term as many times as we please. Note that $\varphi'(s) = \sum_{k=0}^{\infty}(\alpha_{k+1}/k!)s^k \geq 0 \;\forall\, s \geq 0$ so that $\varphi(\cdot)$ is non-decreasing; also $\varphi(0) = \alpha_0 \leq 0$ and $\varphi(\infty) = \infty$. Hence the assertion of the lemma follows easily when $k_0 = 1$. The lemma holds trivially whenever $\alpha_k = 0 \;\forall\, k \geq k_0$. Consequently, we will treat the case when $\alpha_k > 0$ for at least one $k \geq k_0 > 1$. We shall prove the result for $k_0 = 2, 3$ and indicate how our proof extends to the case of *any* positive integer k_0. Generally, for integer j, $1 \leq j \leq k_0$,

$$\varphi^{(j)}(s) = \sum_{k=0}^{\infty} \frac{\alpha_{k+j}}{k!} s^k .$$

By (2.2), $\varphi^{(k_0)}(s) > 0 \;\forall\, s > 0$.

Suppose $k_0 = 2$; then $\varphi^{(2)}(s) > 0 \;\forall\, s > 0$ so that $\varphi(s)$ is a strictly convex function of s. Additionally, $\varphi(0) \leq 0$, $\varphi(\infty) = \infty$; as such, there exists an $s_0 \geq 0$ such that $\varphi(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0$.

Now, suppose $k_0 = 3$; then $\varphi^{(3)}(s) > 0 \;\forall\, s > 0$ so that $\varphi^{(1)}(s)$ is a strictly convex function of s. Also, $\varphi^{(1)}(0) = \alpha_1 \leq 0$, $\varphi^{(1)}(\infty) = \infty$; as such, there exists an $s_0' \geq 0$ such that $\varphi^{(1)}(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0'$. Thus, $\varphi(\cdot)$ is non-increasing in $[0, s_0')$ and non-decreasing in $[s_0', \infty)$. But $\varphi(0) \leq 0$, $\varphi(\infty) = \infty$, so that $\exists\, s_0'' \geq 0$ such that $\varphi(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0''$.

Suppose, generally that the discrete change point is $k_0 \geq 4$. Then $\varphi^{(k_0-2)}(s) > 0 \;\forall\, s > 0$ and hence $\varphi^{(k_0-2)}(\cdot)$ is strictly convex. As $\varphi^{(k_0-2)}(0) = \alpha_{k_0-2} \leq 0$, $\varphi^{(k_0-2)}(\infty) = \infty$, $\exists\, s_0 \geq 0 \ni \varphi^{(k_0-2)}(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0$. Therefore, $\varphi^{(k_0-3)}(s)$ is decreasing in $[0, s_0)$ and increasing in $[s_0, \infty)$. But $\varphi^{(k_0-3)}(0) \leq 0$, $\varphi^{(k_0-3)}(\infty) = \infty$ so that $\exists\, s_0' \geq 0 \ni \varphi^{(k_0-3)}(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0'$. Thus, $\varphi^{(k_0-4)}(s)$ is decreasing in $[0, s_0')$ and increasing in $[s_0', \infty)$. Again as $\varphi^{(k_0-4)}(s) \leq 0$, $\varphi^{(k_0-4)}(\infty) = \infty$, $\exists\, s_0'' \geq 0 \ni \varphi^{(k_0-4)}(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0''$. Continuing this iterative process, we can show that $\exists\, s_0^* \geq 0$ such that $\varphi(s) <$ or $=$ or > 0 according as $s <$ or $=$ or $> s_0^*$. This completes the proof of Lemma 2.1. □

REMARK 2.1. It follows that had the inequalities in (2.2) been reversed, there would exist $s_0 \geq 0$ such that $\varphi(s) >$ or $=$ or < 0 according as $s <$ or $=$ or $>$ s_0.

THEOREM 2.1. *Consider a discrete* NWBUE (k_0) (NBWUE (k_0)) *sequence* $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ *and let* $\overline{H}(t)$ *be as in* (1.1). *Then* H *is* NWBUE (NBWUE).

PROOF. We shall prove the theorem in the NWBUE case. The corresponding assertion for the dual follows in an analogous manner by virtue of Remark 2.1.

It is easy to show (following the steps in Barlow and Proschan (1975), p.161) that

$$\begin{aligned} &\overline{H}(t)\mu - \int_t^\infty \overline{H}(x)dx \\ =\ & \frac{1}{\lambda}\sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}[\overline{P}_k \sum_{j=0}^{\infty}\overline{P}_j - \sum_{j=k}^{\infty}\overline{P}_j] \\ =\ & \frac{1}{\lambda}\sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}a_k, \end{aligned} \tag{2.3}$$

where $\mu = \int_0^\infty \overline{H}(x)dx$ is the mean of H and the a_k's are as in Definition 2.1. We want to show that $\exists$ a $t_0 \geq 0$ such that the right hand side of (2.3) is non-positive for all non-negative $t \leq t_0$ and non-negative for all $t \geq t_0$.

Since the a_k's are bounded, the power series $\sum_{k=0}^{\infty}(a_k/k!)s^k,\ s \geq 0$ converges absolutely; also, the a_k's satisfy (2.2). The theorem now follows by an application of Lemma 2.1, with $\alpha_k = a_k$. □

3 A Shock Model Leading to BFR Survival

Before going on to prove the main result, which is the analogue to Theorem 2.1 in the case of BFR distributions, we state what is meant by a discrete BFR sequence:

DEFINITION 3.1. Let $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ be as in Section 1. The sequence $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ is said to possess the discrete BFR property if there exists a positive integer k_0 such that the following holds:

$$\frac{\overline{P}_1}{\overline{P}_0} \leq \frac{\overline{P}_2}{\overline{P}_1} \leq \frac{\overline{P}_3}{\overline{P}_2} \leq \ldots \leq \frac{\overline{P}_{k_0}}{\overline{P}_{k_0-1}} \geq \frac{\overline{P}_{k_0+1}}{\overline{P}_{k_0}} \geq \ldots \tag{3.1}$$

We shall call k_0 the change point of the sequence $\overline{P}_k$ and write $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ is BFR(k_0).

The relation (3.1) implies that, in a sense, the shocks help the surviving individual/equipment to improve performance through experience/work-hardening till a certain stage, but eventually, the accumulated effect of the shocks received dominates. For example, an individual surviving the first attack of a certain disease is expected to learn to cope with the next attack better, if it occurs and so on. Initially, this learning will have an edge over the deterioration of his health caused by each attack, but the cumulative adverse effects of repeated shocks ultimately catch up with him and overtake the learning effects. Typically, the change point will be a small integer.

THEOREM 3.1. *Let* $\{\overline{P}_k,\ k = 0, 1, \ldots\}$ *be* BFR(k_0), $k_0 > 0$. *Let* $\overline{H}(\cdot)$ *be as defined in* (1.1). *Suppose that*

$$\frac{\overline{P}_{k_0+1}}{\overline{P}_{k_0}} \leq \frac{\overline{P}_1}{\overline{P}_0} \tag{3.2}$$

holds. If

(a) $k_0 \leq 3$, *then* $\overline{H}$ *is* BFR;

(b) $k_0 > 3$, *and* $b_r := \sum_{j=0}^{r} (\overline{P}_{j+2}\overline{P}_{r-j} - \overline{P}_{j+1}\overline{P}_{r-j+1})/j!(r-j)!$ *has the same sign for all* r, $k_0 - 1 \leq r \leq 2k_0 - 4$, *then* $\overline{H}(\cdot)$ *is* BFR.

PROOF. To complete the proof, it is enough to show that $\exists\ s_0 \geq 0$ such that $h(s) = \sum_{k=0}^{\infty} \overline{P}_k (s^k/k!)$ is convex on $[0, s_0)$ and concave on $[s_0, \infty)$. Note that the power series $\sum_{k=0}^{\infty} \overline{P}_k (s^k/k!)$ converges for all s, as the $\overline{P}_k$'s are bounded and hence it has derivatives of all orders. We shall show that $\exists\ s_0 \geq 0$ such that $h''(s) >$ or $=$ or < 0 according as $s <$ or $=$ or $> s_0$. Hence it is sufficient to establish the existence of an $s_0 \geq 0$ such that $\psi(s) = \sum_{r=0}^{\infty} b_r s^r >$ or $=$ or < 0 according as $s <$ or $=$ or $> s_0$. Simple calculations show that

$$\begin{aligned} b_r &= \frac{\overline{P}_2\overline{P}_r}{0!r!} - \frac{\overline{P}_1\overline{P}_{r+1}}{0!r!} \\ &\quad + \frac{\overline{P}_3\overline{P}_{r-1}}{1!(r-1)!} - \frac{\overline{P}_2\overline{P}_r}{1!(r-1)!} \\ &\quad + \frac{\overline{P}_4\overline{P}_{r-2}}{2!(r-2)!} - \frac{\overline{P}_3\overline{P}_{r-1}}{2!(r-2)!} + \ldots \end{aligned}$$

$$
\begin{aligned}
&+\frac{\overline{P}_{r+1}\overline{P}_1}{1!(r-1)!}-\frac{\overline{P}_r\overline{P}_2}{(r-1)!1!}\\
&+\frac{\overline{P}_{r+2}\overline{P}_0}{r!0!}-\frac{\overline{P}_{r+1}\overline{P}_1}{r!0!}\\
=\ &\frac{1}{r!}\{\overline{P}_{r+2}\overline{P}_0-\overline{P}_1\overline{P}_{r+1}\}+\{\frac{1}{(r-1)!}-\frac{1}{r!}\}\{\overline{P}_1\overline{P}_{r+1}-\overline{P}_2\overline{P}_r\}\\
&+\{\frac{1}{2!(r-2)!}-\frac{1}{(r-1)!}\}\{\overline{P}_r\overline{P}_2-\overline{P}_3\overline{P}_{r-1}\}+\ldots \text{ etc.} \qquad (3.3)
\end{aligned}
$$

For $k_0 = 1$, the assertion of the theorem is trivial from (3.2) of Theorem 3.1 of Esary, Marshall and Proschan (1973).

It follows from (3.1)-(3.3) that

(i) For $k_0 = 2$, $b_0 \geq 0$, $b_r \leq 0 \;\forall\; r \geq 1$; note that $|b_r| \leq 2^r/r!$ so that the power series $\sum_{r=0}^{\infty} b_r s^r$ converges absolutely and hence the conditions of Lemma 2.1 hold. Thus by Remark 2.1, there exists $s_0 \geq 0$ such that $\psi(s) >$ or $=$ or < 0 according as $s <$ or $=$ or $> s_0$.

(ii) For $k_0 = 3$, $b_0 \geq 0$, $b_1 \geq 0$, $b_r \leq 0 \;\forall\, r \geq 3$ by (3.1)-(3.3). Lemma 2.1 applies irrespective of whether $b_2 \geq 0$ or ≤ 0, thereby completing the proof.

(iii) For $k_0 > 3$, $b_r \geq 0 \;\forall\; r \leq k_0 - 2$ and $b_r \leq 0 \;\forall\; r \geq 2k_0 - 3$. From the condition in (b) of the theorem, it follows that Lemma 2.1 is once again applicable, completing the proof. □

We shall give an example of a sequence of survival probabilities satisfying (3.1) and (3.2).

EXAMPLE 3.1. For $r \geq 2$, real, let $\overline{P}_0 = 1$, $\overline{P}_k = 1/rk$, $k = 1, ..., k_0$ (positive integer), $\overline{P}_{k_0+l} = 1/r^{2^l+l}k_0$, $l = 1, 2, \ldots$.

In this case, strict inequality holds throughout in (3.1) and also in (3.2). However, if for $l = 1, 2, ...,$ the expression for $\overline{P}_{k_0+l}$ is replaced by $1/r^{l+1}k_0$, equality will hold in (3.2) as well as in (3.1) from the k_0-th stage onwards.

REMARK 3.1. Unlike in Theorem 2.1, in Theorem 3.1 we needed the additional condition (3.2). This condition signifies the severity of the shock required to reverse the improving trend thus far. This phenomenon may obtain in processes where the deterioration at a certain stage becomes too critical owing to the accumulated effect of the previous shocks.

REMARK 3.2. It is to be noted that the assertion of the above theorem would still hold if the condition in (b) is weakened to condition (b′) as follows:

Condition (b′): The first j of the $(k_0 - 2)$ numbers $b_{k_0-1}, b_{k_0}, \ldots, b_{2k_0-4}$ are *non-negative* and the remaining $(k_0 - 2 - j)$ are *non-positive* for some integer j, $0 \le j \le k_0 - 2$.

REMARK 3.3. Theorem 3.1 can be generalized to the situation where the shocks arrive according to a non-homogeneous Poisson process with mean value function $\Lambda(t)$.

Note that in this situation,

$$\overline{H}(t) = \sum_{k=0}^{\infty} e^{-\Lambda(t)} \frac{(\Lambda(t))^k}{k!} \overline{P}_k .$$

We can write $\overline{H}(t) = \overline{H}^*(\Lambda(t))$ where $\overline{H}^*(t) = \sum_{k=0}^{\infty} e^{-t}(t^k/k!)\overline{P}_k$. Under the setup and conditions of Theorem 3.1, $\overline{H}^*(t)$ is BFR, with change point t_0, say.

Suppose that the mean value function $\Lambda(t)$ is such that $\Lambda(t)$ is concave on $[0, t_0)$ and convex on $[t_0, \infty)$.

Then, noting that $-\ln \overline{H}(t) = -\ln \overline{H}^*(\Lambda(t))$ and $\overline{H}^*(t)$ is BFR with change point t_0, it follows from Lemma 2.1(a) of A-Hameed and Proschan (1975) that $\overline{H}(t)$ is BFR with change point t_0.

Chapter 5

Interrelationships amongst Non-monotonic Ageing Classes

1 Introduction

In this chapter, we shall attempt to study the interrelationships between various notions of non-monotonic ageing, namely, the IDMRL class of Guess, Hollander and Proschan (1986), the IDRL class of Deshpande and Suresh (1990), the BFR distributions and the NWBUE family introduced by Mitra and Basu (1994) and discussed in Chapter 3. A considerable part of this chapter is devoted to a discussion of the IDRL class and a conjecture of Deshpande and Suresh (1990), which we settle here. Before proceeding further, let us recapitulate various well-known non-monotonic ageing properties already existing in the literature.

DEFINITION 1.1. A life distribution F is said to be an *Increasing initially, then Decreasing Mean Residual Life* (IDMRL) distribution if $\exists$ a $t_0 \geq 0$ such that $e_F(x)$ is *non-decreasing* on $[0, t_0)$ and *non-increasing* on $[t_0, \infty)$.

We shall call t_0 a change point of the life distribution F in the IDMRL sense.

It is to be noted that unlike in the case of monotonic ageing, where it is well-known that the IFR ageing criterion is equivalent to the stochastic dominance of the residual lifetimes, in the case of non-monotonic ageing, such equivalence does not hold. This has been demonstrated by Deshpande and Suresh (1990) through an example of a BFR life distribution which does not satisfy the property that the corresponding residual life is stochastically increasing upto a certain age and then decreasing stochastically. With this observation in view, they proposed to study the property of non-monotonic ageing through stochastic dominance of residual lifetimes. With this motivation, they introduced the following definition.

DEFINITION 1.2. (Deshpande and Suresh (1990)). A life distribution F is said to be an *Increasing initially, then Decreasing Residual Life* (IDRL) distribution if $\exists$ a $t_0 > 0$ such that $\overline{F}_{t_1}(x) \leq \overline{F}_{t_2}(x)\ \forall\, x \geq 0$ whenever $0 < t_1 < t_2 < t_0$ and $\overline{F}_{t_1}(x) \geq \overline{F}_{t_2}(x)\ \forall\, x \geq 0$ whenever $t_0 \leq t_1 < t_2 < \infty$.

As before, the point t_0 will be referred to as a change point of the life distribution F in the IDRL sense.

Deshpande and Suresh (1990) have proved that
(i){IDRL}$\subseteq${BFR} and
(ii){IDRL}$\subseteq${IDMRL}.
They have put forward the following conjecture:

CONJECTURE 1.1. *If $F \in$ {BFR}$\bigcap${IDMRL}, then F is necessarily an IDRL d.f.*

Here, we note that the BFR class is not included in the IDMRL class of life distributions and conversely. We can utilize the following example appearing in Deshpande and Suresh (1990) to demonstrate the validity of the latter part of our claim.

EXAMPLE 1.1. Consider the life distribution F whose MRL function is given by

$$e_F(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ x & \text{if } 1 \leq x < 2 \\ 4/x & \text{if } 2 \leq x < \infty. \end{cases}$$

Note that $e_F(\cdot)$ is *increasing* in $[0, 2)$ and *decreasing* in $[2, \infty)$ so that F is IDMRL with $t_0 = 2$. Routine calculations yield the failure rate function as

$$r_F(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 2/x & \text{if } 1 \leq x < 2 \\ (x/4) - (1/x) & \text{if } 2 \leq x < \infty. \end{cases}$$

It is, therefore, clear that F is not BFR.

In the next section, we discuss some implications of Definition 1.2 and show that IDRL does not remain a valid non-monotonic ageing concept in a majority of regular situations. In Section 3, we put forward a counterexample to show that the above-mentioned conjecture of Deshpande and Suresh (henceforth to be referred to as D-S conjecture) is in fact, false. We end the chapter with Section 4 where the relationship of the BFR and IDMRL classes with the newly-introduced NWBUE family is investigated.

2 IDRL Class of Life Distributions

In this section, we study some interesting features of the IDRL class of life distributions. We shall state our results through a simple theorem and several remarks.

THEOREM 2.1. *Let F be an* IDRL *d.f. with a unique change point t_0. Suppose that F is absolutely continuous having a p.d.f. $f(\cdot)$ such that the failure rate function $r(\cdot)$ is left-continuous at t_0. Then, F is DFR with failure rate function $r(t)$ of the form:*

$$r(t) = \begin{cases} \varphi(t), & t < t_0 \\ \varphi(t_0), & t \geq t_0, \end{cases} \tag{2.1}$$

where $\varphi(t)$ is non-increasing.

PROOF. As F is IDRL, for all $x \geq 0$,

$$\frac{\overline{F}(x+t)}{\overline{F}(t)} \text{ is non-decreasing in } t,\ t < t_0; \tag{2.2}$$

and

$$\frac{\overline{F}(x+t)}{\overline{F}(t)} \text{ is non-increasing in } t,\ t \geq t_0. \tag{2.3}$$

From (2.2), for all $x \geq 0$, $-R(x+t) + R(t)$ is non-decreasing in t, $t < t_0$ and as such

$$r(t) \geq r(x+t),\ t < t_0. \tag{2.4}$$

Similarly, from (2.3), for all $x \geq 0$, $r(t) \leq r(x+t)$, $t \geq t_0$. This observation together with the fact that $r(t)$ is left-continuous at t_0 proves (2.1). □

REMARK 2.1. If an absolutely continuous IDRL d.f. F has more than one change point (in the IDRL sense), then using arguments as in the proof of Theorem

2.1, we observe that F is a DFR d.f. whose failure rate function has the following form:

$$r(t) = \begin{cases} \varphi(t), & t < t_0 \\ \kappa, & t \geq t_0, \end{cases}$$

where $\varphi(t)$ is non-increasing, κ is a constant, $\kappa < \varphi(t_0-)$ and $t_0 = \inf\{t : t$ is a change point of F in the IDRL sense$\}$. Thus, in this case, IDRL is *not* a valid notion of non-monotonic ageing.

REMARK 2.2. Under the conditions of Theorem 2.1, {IDRL}$\subseteq${DFR}, (with a specific form of the failure rate function) so that IDRL does not remain a valid *non-monotonic ageing* criterion in this situation as well. Accordingly, whenever the failure rate function is well-defined, a necessary condition for IDRL to be a meaningful notion of *non-monotonic* ageing is that $r(t)$ is *not* left-continuous at the (unique) change point t_0 of the respective life distribution. In case of many standard life distributions, the assumptions of Theorem 2.1 hold true and as such, the concept of IDRL there simply reduces to that of DFR with failure rate function of the form given in (2.1).

REMARK 2.3. If the assumption of left continuity of $r(\cdot)$ at the (unique) change point of the distribution is dropped in Theorem 2.1, then the notion of an IDRL distribution reduces to that of a BFR distribution for which the failure rate $r(t)$ in the interval $[t_0, \infty)$ is bounded above by $r(t_0-)$; in fact, $r(t') \geq r(t'')\ \forall\, t' < t_0,\ t'' \geq t_0$.

3 A Counterexample to D-S Conjecture

Here, we present an example to show that the D-S conjecture mentioned in Section 1 is false, i.e., a distribution which is both IDMRL and BFR need not be an IDRL distribution.

EXAMPLE 3.1. Consider the life distribution determined by the following failure rate function:

$$r(t) = \begin{cases} (1+t)^{-1}, & 0 \leq t \leq \alpha,\ \alpha > 0 \\ \delta t, & t \geq \alpha \end{cases} \tag{3.1}$$

where $\delta^{-1} = \alpha(1+\alpha)$. It is clear that for each $\alpha > 0$, the above life distribution is BFR (with change point α) but *not* IDRL, since (2.4) is violated. So it is enough to check that the distribution is IDMRL; in fact, we will make it an IDMRL distribution by choosing a suitable α.

Simple computations show that the corresponding survival function $\overline{F}(\cdot)$ and the MRL function $e_F(\cdot)$ are given by:

$$\overline{F}(x) = \begin{cases} (1+x)^{-1}, & x \le \alpha \\ (1+\alpha)^{-1}\exp[\frac{1}{2}\delta\alpha^2 - \frac{1}{2}\delta x^2], & x \ge \alpha; \end{cases}$$

$$e_F(x) = \begin{cases} (1+x)\ln[(1+\alpha)/(1+x)] + (1+x)(1+B_\alpha), & x \le \alpha, \\ \exp[\frac{1}{2}\delta x^2]\int_x^\infty \exp[-\frac{1}{2}\delta u^2]du, & x \ge \alpha, \end{cases}$$

where $B_\alpha = \sqrt{\alpha/2(1+\alpha)}\exp\{\alpha/2(1+\alpha)\}\int_{\alpha/2(1+\alpha)}^\infty u^{-1/2}e^{-u}du - 1$. First we note that $-1 < B_\alpha < 0$ for all positive α. To prove this, it is enough to show that for $\beta > 0$, $0 < \sqrt{\beta}e^\beta\int_\beta^\infty e^{-u}u^{-1/2}du < 1$. The first inequality is trivial while the second is an easy consequence of the following observation:

$$\begin{aligned} & \int_\beta^\infty \sqrt{\beta}e^{-(u-\beta)}u^{-1/2}du \\ = & \int_0^\infty e^{-v}\frac{\sqrt{\beta}}{\sqrt{\beta+v}}dv \\ < & \int_0^\infty e^{-v}dv < 1. \end{aligned}$$

Now, the derivative of $e_F(x)$ is given by

$$e_F'(x) = \ln[(1+\alpha)/(1+x)] + B_\alpha$$

so that $e_F'(x) > 0$ or < 0 according as $x <$ or $> (1+\alpha)e^{B_\alpha} - 1 = \alpha^*$, say. To make $\alpha^* > 0$, it is enough to choose $\alpha > 0$ such that $\ln(1+\alpha) > -B_\alpha$. As $0 < -B_\alpha < 1$, take $\alpha > 0$ satisfying $\ln(1+\alpha) > 1$, e.g., take $\alpha = e^2 - 1$. Also, $\alpha^* < \alpha$ as $B_\alpha < 0$. Accordingly, $e_F(x)$ is increasing in the interval $[0, \alpha^*)$, and it is decreasing in $[\alpha^*, \alpha)$; moreover, as $\overline{F}(x+t)/\overline{F}(x)$ is decreasing in the interval $[\alpha, \infty)$ for every $t \ge 0$, so is $e_F(x)$ for $x \in [\alpha, \infty)$. As such, F is IDMRL whenever $\alpha = e^2 - 1$. We thus have a life distribution which is both BFR and IDMRL, but *not* IDRL. Thus the D-S conjecture is false.

4 Interrelations amongst NWBUE, BFR and IDMRL Classes

In this section, we explore the interrelationships amongst NWBUE, BFR and IDMRL classes of life distributions.

THEOREM 4.1. *A continuous and strictly increasing* BFR *life d.f.* F *is necessarily* NWBUE.

We need a theorem due to Deshpande and Suresh (1990) for proving the above proposition. Since our definition of the BFR class is slightly more general than theirs, (for symmetry, we have included the IFR class), Theorem 3.1 of Deshpande and Suresh can be modified to yield the following result.

RESULT 4.1. *A life distribution* F *is* BFR *if and only if* $\exists$ *a* t_0, $0 \leq t_0 \leq 1$ *such that* $\psi_F(t)$ *is convex in* $[0, t_0)$ *and concave in* $[t_0, 1]$.

We shall now use the above result to prove the theorem.

PROOF OF THEOREM 4.1. Since F is continuous and strictly increasing, we can write the *total time on test* (TTT)-transform $\psi_F(\cdot)$ of F as

$$e_F(x) = \mu(1 - \psi_F(t))/(1 - t),$$

where $F(x) = t$. It then follows that F is NWBUE if and only if $\exists$ a t'_0, $0 \leq t'_0 \leq 1$, such that

$$\psi_F(t) = \begin{cases} \leq t & \text{for } t < t'_0 \\ \geq t & \text{for } t \geq t'_0. \end{cases} \tag{4.2}$$

We now discuss the proof in 4 different cases as follows:

Case I. $\psi_F(t)$ *does not intersect the line* $g(t) = t$, $0 \leq t \leq 1$.
In view of the result, F is then either NBUE or NWUE according as $\psi_F(t) \geq$ or $\leq t$, $t \in [0, 1]$ and the conclusion of the proposition follows.

Case II. *First crossing of* $g(t) = t$ *by the curve* $\psi_F(t)$ *is from above.*
Let the point of interswection be t^*; then, $\psi_F(t) = t$, $t \in [t^*, 1]$, for, otherwise, either

the result or the fact that $\psi_F(1) = 1$ will be violated. As such, F is NBUE and the proposition follows.

Case III. *First crossing of $g(t) = t$ by the curve $\psi_F(t)$ is from below and there are no further crossings.*
Here also $\psi_F(t) = t \; \forall \, t \in [t^*, 1]$, and the proposition follows.

Case IV. *First crossing of $g(t) = t$ by the curve $\psi_F(t)$ is from below and there is a second crossing.*
In this case, in view of the result, the second crosssing must be from above, and the proof will follow as in Case II. □

REMARK 4.1. The converse of the above theorem is false. This can be seen readily by considering the life distribution exhibited in Example 1.1. It is easy to observe that the distribution is NWBUE(x_0) with $x_0 = 4$, but its failure rate function does not possess the BFR property as has already been noted in the example quoted above.

The following theorem reveals the relationship between the IDMRL and NWBUE families. The proof is trivial and hence is omitted here.

THEOREM 4.2. *If F is* IDMRL(t_0), *then F is* NWBUE(t_0') *with $t_0' \geq t_0$.*

We now provide an example to convey that a life distribution can be NWBUE without being IDMRL.

EXAMPLE 4.1. Consider the life distribution having the following survival function:

$$\overline{F}(x) = \begin{cases} \{\theta/(\theta+x)\}^2 & \text{if } 0 \leq x < \alpha \\ \{\theta/(\theta+\alpha)\}^2 & \text{if } \alpha \leq x < \beta \\ \{\theta/(\theta+\alpha)\}^2[(\theta+2\alpha-\beta)/(\theta+2\alpha-2\beta+x)]^2 & \text{if } \beta \leq x < \gamma \\ \frac{\theta^2}{(\theta+\alpha)^2}\frac{(\theta+2\alpha-\beta)^2}{(\theta+2\alpha-2\beta+\gamma)^2}(x/\gamma)\exp[-\frac{1}{2}\frac{(x^2-\gamma^2)}{\gamma(\theta+2\alpha-2\beta+\gamma)}] & \text{if } x \geq \gamma \end{cases}$$

where the parameters α, β, γ and θ are such that $0 < \alpha \leq \beta \leq \gamma$, $2\alpha > \beta$ and $\theta > 0$. The corresponding MRL function then works out as

$$e_F(x) = \begin{cases} \theta + x & \text{if } 0 \leq x < \alpha \\ \theta + 2\alpha - x & \text{if } \alpha \leq x < \beta \\ \theta + 2\alpha - 2\beta + x & \text{if } \beta \leq x < \gamma \\ \gamma(\theta + 2\alpha - 2\beta + \gamma)/x & \text{if } x \geq \gamma. \end{cases}$$

It is easy to observe that the life distribution F is NWBUE with change point $x_0 = \gamma(\theta + 2\alpha - 2\beta + \gamma)/\theta$; however, F is *not* IDMRL whenever $\alpha < \beta < \gamma$. As such, the converse of Theorem 4.2 is false, in general.

Typically, for a life distribution to be NWBUE, its MRL function has to lie above the mean life μ till x_0 and below it then onwards. This property seems to imply a natural non-monotonic ageing phenomenon which cannot be captured, as is illustrated in the above-mentioned examples, through either BFR or IDMRL notions.

Chapter 6

Change Point Estimation

1 Introduction

In reliability theory, it is well-known that in most practical situations, the ageing pattern is non-monotonic and is typically characterized by a trend change in the failure rate or mean residual life functions. In this chapter, we shall concentrate on such situations and outline a procedure for estimating the 'change points' which indicate precisely where the ageing process is reversed. We shall demonstrate that our method applies in the standard non-monotonic ageing classes discussed earlier, namely, the NWBUE, BFR and IDMRL families.

Earlier efforts in estimating change points were limited to specific life distributions only. For example, Nguyen, Rogers and Walker (1984) and Yao (1986) considered the life distribution having failure rate function

$$r(t) = \alpha\chi(0 \le t \le \tau) + \beta\chi(t > \tau)$$

where $\chi(A)$ is the indicator function of the set A and α, β, $\tau \in \mathbb{R}$. The following general model characterized by the hazard function

$$r(t) = \sum_{k=1}^{n} \alpha_k \chi(\tau_{k-1} \le t < \tau_k)$$

with $0 = \tau_0 < \tau_1 < \ldots < \tau_m = \infty$ and $\alpha_k \ge 0 \; \forall \; k$, was subsequently considered by Pham and Nguyen (1990). They used techniques of maximum likelihood estimation (the 'pseudo-maximum likelihood' approach) and established strong consistency of the suggested estimators. The application of bootstrap methods in the case $m = 2$ has been studied by Pham and Nguyen (1993). Basu, Ghosh and Joshi (1988) treated the 'truncated bathtub model' specified by the rate function

$$r(t) = \lambda(t)\chi(0 \le t \le \tau) + \lambda_0\chi(t > \tau)\,,$$

where $\lambda(t)$ is a decreasing positive function and λ_0 is a positive constant. As opposed to the above parametric and semiparametric models, here we propose to consider the problem of estimating the change point in purely nonparametric setups. An earlier effort in this direction has been by Kulasekera and Lal Saxena (1991) who considered life distributions displaying bathtub failure rates having a unique change point and proposed a consistent estimator of the same. The methodology described in the next section when applied to the BFR case, seems to work under less restrictive assumptions.

2 Change Point Estimation

In this section, we consider BFR, IDMRL and NWBUE life distributions having *unique* change points and suggest a unified methodology for estimating these. Estimation of change points is relevant particularly in the context of maintenance policies, since, as is natural, one would hardly think of preventively replacing a component having such a life distribution before the 'threshold' (unknown) age of x_0 is achieved.

In a general setup, our problem can be formulated as follows:

Given a random sample, $X_1, X_2, ..., X_n$ of size n from an unknown life d.f. F, where F is NM(x_0), $x_0 < \infty$, (NM is the abbreviation for non-monotonic and it would stand for either BFR or IDMRL or NWBUE in subsequent discussions), how to estimate the unknown change point of the said life distribution F?

We shall make the following two assumptions:

(A1) The finite change point x_0 of the d.f. F is the *unique minimizer/maximizer* of a suitable non-negative transform $h_F(\cdot)$ of the life d.f. F.

(A2) We have an upper bound for the unknown change point – call it B, i.e., $x_0 < B < \infty$, $F(B) < 1$.

Note that (A2) is quite a weak assumption because, in most practical cases, an idea about B can be formed on the basis of some prior knowledge concerning the phenomenon under consideration.

While estimating the change point of a BFR distribution F, we shall take $h_F(x) \equiv r_F(x)$; likewise, in the IDMRL case, $h_F(x)$ is taken to be $e_F(x)$. Consequently, in the context of both these cases, the assumption (A1) is equivalent to

(A1*) The finite change point x_0 of the d.f. F is unique.

However, the above equivalence fails in the NWBUE case where we choose $h_F(x) \equiv |e_F(x) - \mu_F|$ with μ_F being the mean of F. This is so because here, under (A1*), $h_F(x) = 0$ may have multiple solutions, exactly one of which is the change point.

Our method consists of identifying x_0 as the *unique* minimizer/maximizer of a specific non-negative transform $h_F(x)$ of F. We then estimate $h_F(x)$ by $h_{F_n^*}(x)$, where F_n^* is a *suitable* estimator of F and is based on a random sample of size n from it.

Let Λ_n be the set of minimizers/maximizers of $h_{F_n^*}(x)$. We propose to estimate x_0 by x_{0n}, where $x_{0n} := \inf \Lambda_n$ whenever Λ_n is non-empty; x_{0n} is defined appropriately otherwise. The choice of F_n^* would depend on the specific problem being tackled as will be evident from the three cases dealing with the estimation of the change point when F is (i) NWBUE(x_0), (ii) IDMRL(x_0) and (iii) BFR(x_0).

(i) The NWBUE(x_0) Case.

Suppose F is NWBUE(x_0) with finite mean μ_F. Condition (A1) implies that x_0 is the *unique* minimizer of the non-negative transform defined by

$$h_F(x) := |e_F(x) - \mu_F| \tag{2.1}$$

whenever $x_0 = 0$; otherwise, it is the unique *positive* minimizer. Here $e_F(\cdot)$ is the mean residual life function of F. Let F_n be the empirical c.d.f. based on the random sample $X_1, X_2, ..., X_n$ and $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be the corresponding order statistics. In this case, we estimate $h_F(x)$ by

$$h_{F_n}(x) := |e_{F_n}(x) - \mu_{F_n}|, \tag{2.2}$$

where $e_{F_n}(x)$ is given by

$$e_{F_n}(x) = \begin{cases} \frac{1}{\overline{F}_n(x)} \int_x^\infty \overline{F}_n(v)dv & \text{if } X_{(n)} > x, \\ 0 & \text{otherwise,} \end{cases}$$

and μ_{F_n} is the sample mean $\overline{X}_n$. Note that $e_{F_n}(x)$ is nothing but the usual life table estimate of the life expectancy at age x. Also, it is easy to see that

$$e_{F_n}(x) = \begin{cases} \frac{\sum_{j=1}^{n}(X_j - x)I(X_j - x)}{\sum_{j=1}^{n} I(X_j - x)} & \text{if } X_{(n)} > x, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

where $I(a) = 1$ or 0 according as $a > 0$ or $a \le 0$. It follows from Yang (1978) that as $n \to \infty$,

$$\sup_{0 \le x \le b} |e_{F_n}(x) - e_F(x)| \to 0 \text{ a.s.} \tag{2.4}$$

for every $b > 0$ satisfying $F(b) < 1$. Further, it is clear from (2.3) that $e_{F_n}(x)$ is a right continuous function having finite left limits and is piecewise linearly decreasing in the intervals $[0, X_{(1)})$, $[X_{(1)}, X_{(2)})$, ..., $[X_{(n-1)}, X_{(n)})$. Also, note that $e_{F_n}(x)$ has jumps at $x = X_{(i)}$, $i = 1, 2, ..., n-1$. These jumps are of *positive* magnitude, since the averages $(n-k+1)^{-1}\sum_{j=k}^{n} X_{(j)}$ increase with k. Now, define

$$\Lambda_n := \{0 < x \le B : |e_{F_n}(x) - \overline{X}_n| \text{ is minimum}\}. \tag{2.5}$$

Observe that $\lim_{x \downarrow 0} |e_{F_n}(x) - \overline{X}_n| = 0$; hence, if Λ_n is non-empty, the minimum value of $|e_{F_n}(x) - \overline{X}_n|$ has to be zero; we thus have,

$$\Lambda_n = \{0 < x \le B : e_{F_n}(x) = \overline{X}_n\}.$$

Then we have the following useful lemma:

LEMMA 2.1. *If $x_0 > 0$ (=0), there exists an integer $n_0 \ge 1$ such that for all $n \ge n_0$, Λ_n is non-empty (empty) with probability* 1.

PROOF. Consider the case $x_0 > 0$. Let $0 < x_1 < x_0 < x_2 < B$. As F is NWBUE(x_0), and x_0 is the unique change point of F,

$$\begin{aligned} e_F(x_1) - \mu_F &> 0, \\ e_F(x_2) - \mu_F &< 0. \end{aligned} \tag{2.6}$$

Using (2.4) and Kolmogorov's SLLN, we note that

$$e_{F_n}(x) - \overline{X}_n \to e_F(x) - \mu_F \text{ a.s. as } n \to \infty.$$

It then follows from (2.6) that there exists $n_0 \geq 1$, sufficiently large, such that for all $n \geq n_0$, $e_{F_n}(x_1) - \overline{X}_n > 0$ and $e_{F_n}(x_2) - \overline{X}_n < 0$ hold with probability 1.

The above inequalities, together with the fact that $e_{F_n}(\cdot)$ has jumps that can only be *positive* in magnitude, guarantee the existence of a solution to the equation $e_{F_n}(x) = \overline{X}_n$ with probability 1, for all sufficiently large n.

If $x_0 = 0$, then via (2.4) and SLLN, we observe that for all $x > 0$, $e_{F_n}(x) - \overline{X}_n > 0$ a.s. $\forall$ large n, since F is then NWBUE with unique change point zero. □

It seems natural to estimate x_0 by $x_{0n} := 0$ or $\inf \Lambda_n$ according as $\Lambda_n = \emptyset$ or $\Lambda_n \neq \emptyset$.

REMARK 2.1. Lemma 2.1 implies that $x_{0n} = 0$ for all large n with probability 1 whenever $x_0 = 0$.

The following theorem justifies the use of the estimator proposed above.

THEOREM 2.1. *The estimator x_{0n} is strongly consistent for x_0.*

PROOF. In case $x_0 = 0$, a much stronger conclusion holds in view of Remark 2.1; in fact, beyond a certain stage, x_{0n} becomes zero identically. Fix any $\omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is the probability space on which the X_i's are defined. By definition, $\{x_{0n}\}$ is a bounded sequence and as such, it has a convergent subsequence. Let $\{x_{0n_k}\}$ be any convergent subsequence of $\{x_{0n}\}$ and suppose that

$$x_{0n_k} \to x_0^* \text{ as } n \to \infty. \tag{2.7}$$

At this stage, we note that the graph of $e_{F_n}(x)$ consists of n linear segments, each having negative slope. Each such segment extends between two successive order statistics. Moreover, $e_{F_n}(0) = \overline{X}_n$ and $e_{F_n}(x)$ is linearly decreasing between 0 and $X_{(1)}$. Thus it is clear that Λ_n can have at most $(n-1)$ elements so that $\inf \Lambda_n = \min \Lambda_n$ and as such $x_{0n_k} \in \Lambda_{n_k}$. Therefore,

$$0 \leq |e_{F_{0n_k}}(x_{0n_k}) - \overline{X}_{n_k}| \leq |e_{F_{0n_k}}(x_0) - \overline{X}_{n_k}|. \tag{2.8}$$

Taking limits as $k \to \infty$ and using (2.4) and the SLLN, we have, with probability 1,

$$0 \leq |e_F(x_0^*) - \mu_F| \leq |e_F(x_0) - \mu_F| = 0,$$

by definition of x_0. Thus, with probability 1, $x_0^* = x_0$, by the uniqueness of the change point. Hence, any convergent subsequence of $\{x_{0n}\}$ converges a.s. to x_0. This completes the proof. □

(ii) The IDMRL(x_0) Case.

We take $h_F(x) \equiv e_F(x)$, identify x_0 as the unique maximizer of $e_F(x)$, estimate $e_F(x)$ by $e_{F_n}(x)$ as in Case (i), and define

$$\Lambda_n := \{0 < x \leq B : e_{F_n}(x) \text{ is maximum}\}.$$

It follows from the graph of $e_{F_n}(\cdot)$ that Λ_n is empty if and only if $e_{F_n}(x) - \overline{X}_n < 0$ a.s. $\forall\ x > 0$; likewise if $e_{F_n}(x) - \overline{X}_n \geq 0$ a.s. for some $x > 0$, then Λ_n will comprise exclusively of one or more of the order statistics. Now (2.4) and the IDMRL property of F imply that for all sufficiently large n, Λ_n is non-empty with probability 1. Define

$$x_{0n} = \begin{cases} 0 & \text{if } \Lambda_n = \emptyset \\ \inf \Lambda_n = \min \Lambda_n & \text{if } \Lambda_n \neq \emptyset. \end{cases}$$

It is to be noted that the estimate of the change point would either be zero (if Λ_n is empty) or one of the order statistics (if Λ_n is non-empty).

The consistency of x_{0n} can be proved very easily along lines similar to those in Case (i).

(iii) The BFR(x_0) Case.

We now discuss the estimation of the change point x_0 of a BFR distribution F for which the failure rate function $r_F(x)$ is well-defined. For this purpose, we take $h_F(x) \equiv r_F(x)$; by (A1), we note that x_0 is the *unique* minimizer of $h_F(x)$.

We assume the p.d.f. $f(\cdot)$ of F to be uniformly continuous and let $f_n(x)$ be a continuous kernel estimate of $f(\cdot)$. Accordingly, under suitable assumptions on the kernel function, we have, by Theorem A of Silverman (1978),

$$\sup\{|f_n(x) - f(x)| : x \geq 0\} \to 0 \text{ a.s. as } n \to \infty. \tag{2.9}$$

We propose to estimate $r_F(x)$ by

$$r_n(x) := \frac{f_n(x)}{\overline{F}_n(x)}, \tag{2.10}$$

where $\overline{F}_n(x) = 1 - n/(n+1)\int_0^x f_n(y)dy$, $f_n(\cdot)$ being the above kernel estimate of $f(\cdot)$. The factor $n/(n+1)$ is introduced in the definition of $\overline{F}_n(\cdot)$ simply to ensure that $r_n(\cdot)$ in (2.9) is well-defined, as $\int_0^\infty f_n(x)dx \leq 1$. It is simple to deduce that

$$|r_n(x) - r(x)| \rightarrow \text{ a.s. as } n \rightarrow \infty \tag{2.11}$$

uniformly, over any bounded interval.

Define

$$\Lambda_n := \{0 \leq x \leq B : r_n(x) \text{ is minimum}\}. \tag{2.12}$$

Since, for each fixed sample point, $r_n(x)$ is a continuous function of x, it attains its bounds over the compact set $[0, B]$. Thus Λ_n is non-empty, and we set

$$x_{0n} := \inf \Lambda_n = \min \Lambda_n. \tag{2.13}$$

The last equality holds in view of the continuity of $r_n(\cdot)$.

The estimator proposed in (2.12) seems more natural and intuitively appealing compared to the one given by Kulasekera and Lal Saxena (1991), which looks complicated, besides being computationally rather involved. Moreover, they needed a number of assumptions on the failure rate as well as the kernel function to establish the strong consistency of their estimator. But, the estimator in (2.12) is strongly consistent under much less restrictive conditions, as can be seen in Theorem 2.2 below.

For the purpose of Theorem 2.2 below, we assume the conditions of Theorem A of Silverman (1978), so that (2.8) follows.

THEOREM 2.2. *The estimator x_{0n} defined in* (2.12) *is strongly consistent for x_0.*

PROOF. Let $\{x_{0n_k}\}$ be a convergent subsequence of the bounded sequence $\{x_{0n}\}$ and let

$$x_{0n_k} \rightarrow x_0^*.$$

By definition of x_{0n_k},

$$r_{n_k}(x_{0n_k}) \leq r_{n_k}(x_0).$$

Taking limits as $k \to \infty$ and using (2.10), we get,

$$r(x_0^*) \leq r(x_0),$$

which completes the proof because of the uniqueness of x_0. □

Chapter 7

Some New Properties of the $\mathcal{L}$ Class

1 Introduction

The notion of ageing is central to the statistical theory of reliability and maintenance. Consequently, an increasing number of life distribution classes have been proposed and used to model various aspects of ageing. Among the ageing classes (and their duals) which are considered as benchmark standards in reliability research, the largest such class is known as $\mathcal{L}$ ($\overline{\mathcal{L}}$ resp.) which was introduced by Klefsjö (1983). The immediately smaller sub-class of $\mathcal{L}$ ($\overline{\mathcal{L}}$) in the hierarchy of standard and nested sub-classes is {HNBUE} ({HNWUE} resp.). Definitions and properties of these stronger ageing notions can be found in Barlow and Proschan (1975), Rolski (1975) and Klefsjö (1982, 1983).In this chapter, we examine some additional ramifications of the $\mathcal{L}$ ($\overline{\mathcal{L}}$)-class property.

In what follows, the Laplace transform of a life distribution F of a random variable (non-negative) X will be denoted by

$$L_F(s) := E(e^{-sX}) = \int_0^\infty e^{-st}dF(t)\,,\ s \geq 0.$$

By $\mu_{r,F}$ we shall denote the r-th moment of F whenever it exists and η_F will denote its coefficient of variation (c.v.). Let F_1 denote the *first derived distribution* corresponding to F defined by

$$F_1(x) = \frac{1}{\mu_{1,F}} \int_0^x \overline{F}(t)dt,$$

whose interpretation in the context of renewal theory is well-known. The Laplace transform of F and F_1 are related as

$$L_{F_1}(s) = \int_0^\infty e^{-st}\frac{\overline{F}(t)}{\mu_{1,F}}dt = \frac{1 - L_F(s)}{s\mu_{1,F}},\ s > 0. \tag{1.1}$$

DEFINITION 1.1. (Klefsjö (1983)). A life distribution F with finite mean $\mu_{1,F} = \mu$ belongs to the $\mathcal{L}$ ($\overline{\mathcal{L}}$)-class if

$$\int_0^\infty e^{-st}\overline{F}(t)dt \geq (\leq) \frac{\mu}{1+s\mu}\,,\ \forall\, s \geq 0. \tag{1.2}$$

By virtue of (1.1), we see that $F \in \mathcal{L}$ if and only if $L_F(s) \leq (1+s\mu)^{-1}, \ \forall\, s > 0$; i.e. the Laplace transform of F is dominated pointwise by the Laplace transform of the exponential distribution having the same mean as F. Trivially, the exponential distributions are members of the above class. For various interesting interpretations of the above definition, the reader is referred to Klefsjö (1983).

In Section 2, we prove two basic results involving a distribution belonging to the $\mathcal{L}$-class; the first is a useful moment inequality while the second deals with the weak convergence of $\mathcal{L}$ distributions. However, our main objective in this chapter is to present two interesting characterizations (Theorems 3.1 and 3.2) of the exponential distribution within the $\mathcal{L}$ ($\overline{\mathcal{L}}$) class of life distributions. Our results show that the c.v. has a basic role in identifying $\mathcal{L}$ ($\overline{\mathcal{L}}$) distributions in some situations, except when the c.v. has the extremal value 1. To examine the relationship of $\mathcal{L}$ with the standard ageing classes, we introduce a sub-class $\mathcal{L}_D$ of $\mathcal{L}$ and explore its relationship with {DMRL}. We also mention a result (Theorem 3.3) relating convergence in distribution of a sequence of $\mathcal{L}_D$-random variables scaled by their means to the unit exponential distribution with the convergence of their c.v.s to 1.

2 Two Basic Results

The existence of the second moment of a distribution belonging to the $\mathcal{L}$-class is guaranteed by Remark 1 of Bhattacharjee and Sengupta (1994). They also proved the following lemma which shows that the coefficient of variation η_F of a distribution belonging to the $\mathcal{L}$ ($\overline{\mathcal{L}}$) class satisfies $\eta_F \leq (\geq)\ 1$.

LEMMA 2.1. *If $F \in \mathcal{L}$ ($\overline{\mathcal{L}}$), then $\mu_2 \leq (\geq) 2\mu^2$.*

The above result can be proved using an alternative approach as follows. Define

$$h_F(s) = L_F(s) - \frac{1}{1+s\mu_{1,F}}, \ s \geq 0. \tag{2.1}$$

Note that $h_F(0) = h_F'(0) = 0$. A d.f. $F \in \mathcal{L}$ if and only if $h_F(s) \leq 0, \ \forall\, s \geq 0$. If $F \in \mathcal{L}$, then $h_F(s)$ has a maximum at $s = 0$, since, then $h_F(s) \leq 0 = h_F(0), \ \forall\, s \geq 0$. This implies that $\mu_{2,F} - 2\mu_{1,F}^2 = h_F''(0) \leq 0$; or, $\eta_F \leq 1$. If $F \in \overline{\mathcal{L}}$ instead, assume $\mu_{2,F} < \infty$ since the conclusion holds trivially whenever $\mu_{2,F} = \infty$. Since $e^{-x} \geq$

$1 - x \; \forall \, x \geq 0$, we have

$$\begin{aligned} \mu_{2,F} &= 2\int_0^\infty t\overline{F}(t)dt \\ &\geq 2\int_0^\infty \frac{1-e^{-st}}{s}\overline{F}(t)dt \\ &\geq \frac{2\mu_{1,F}^2}{1+s\mu_{1,F}}, \end{aligned}$$

for all $s > 0$, by the $\overline{\mathcal{L}}$-property in (1.2). This implies

$$\mu_{2,F} \geq \sup_{s\geq 0} \frac{2\mu_{1,F}^2}{1+s\mu_{1,F}},$$

which is equivalent to $\eta_F \geq 1$.

The following results illustrate some simple applications of Lemma 2.1.

LEMMA 2.2. *If $F \in \overline{\mathcal{L}}$, then*
(i) $\mu_r := \int_0^\infty x^r dF(x) \geq 2^{r/4}\mu^{r/2}$ *for* $r > 2$,
(ii) $\mu_r \geq r(r-1)^{r-1}(2-r)^{2-r}\mu^r$ *for* $1 < r < 2$.

PROOF. (i) follows from Lemma 2.1 and Liapounov's inequality while (ii) can easily be proved by an argument analogous to that used to establish the second part of Lemma 2.1. □

REMARK 2.1. If $F \in \mathcal{L}$, then $\mu_r \leq 2^{r/4}\mu^{r/2}$ for $0 < r < 2$.

We shall now utilize Lemma 2.1 to prove the closure of the $\mathcal{L}$-class under weak convergence. First we make the following definition:

DEFINITION 2.1. A non-negative random variable is said to be $\mathcal{L}$ if and only if its distribution function belongs to the $\mathcal{L}$-class.

LEMMA 2.3. *Let $\{X_n\}$ be a sequence of $\mathcal{L}$ random variables such that $X_n \to X$ in law. Then X is also an $\mathcal{L}$ random variable.*

PROOF. As $X_n \to X$ in law, by the Helly-Bray theorem,

$$\frac{1}{1+EX_n} \geq L_{F_n}(1) \to L_F(1) \text{ as } n \to \infty.$$

So, $\exists\, B > 0$ such that $EX_n \leq B < \infty\ \forall\, n \geq 1$. By Lemma 2.1, $EX_n{}^2 \leq 2(EX_n)^2 \leq 2B^2$ so that $\sup_{n\geq 1}\ EX_n{}^2 < \infty$, which is a sufficient condition for the $\{X_n\}$ to be uniformly integrable. As such, (see e.g. Billingsley (1985), Theorem 25.12), $EX_n \to EX$ as $n \to \infty$.

The lemma now follows from the Helly-Bray theorem. □

COROLLARY 2.1. *$\{\mathcal{L}\}^{LD} = \{\mathcal{L}\}$, where $\{\mathcal{L}\}^{LD}$ denotes the class obtained by taking limits in distribution of members in $\mathcal{L}$.*

3 The Characterization Theorems

It may be recalled that Bhattacharjee and Sengupta (1994) gave an example where $F \in \mathcal{L}$, $\eta_F = 1$, but F is *not* exponential. We now discuss two results (Theorems 3.1 and 3.2 below) concerning characterization of the exponential distribution within two specific subclasses of the $\mathcal{L}$-class. Basu and Bhattacharjee (1984) proved that for an HNBUE distribution to be exponential, it is necessary and sufficient that its c.v. is 1. However, the counterexample of Bhattacharjee and Sengupta (1994) shows that such a characterization does not hold in the wider class of $\mathcal{L}$-distributions. This, therefore, raises the question: is there a nonparametric property stronger than $\mathcal{L}$ but *not* HNBUE, under which $\eta_F = 1$ is both necessary and sufficient for F to be exponential? The answer is affirmative, as we show in Theorems 3.1 and 3.2 below.

DEFINITION 3.1. A life d.f. F having finite mean belongs to $\mathcal{L}_D$-class if both F and F_1 belong to $\mathcal{L}$.

Obviously, $\mathcal{L}_D \subseteq \mathcal{L}$. Further, $\{DMRL\} \subseteq \mathcal{L}_D$. This is so because, if F is DMRL, then $F \in \{HNBUE\}$ and hence is a member of $\mathcal{L}$; on the other hand, $F \in \{DMRL\}$ implies that F_1 is IFR and consequently belongs to $\mathcal{L}$. It is well-known that {HNBUE}, the largest among the standard ageing classes, is a strict subset of $\mathcal{L}$. However, as of now, we are unable to comment as to whether any hierarchical relationship holds between the classes {HNBUE} and $\mathcal{L}_D$.

We now present the first characterization theorem.

THEOREM 3.1. *Suppose $F \in \mathcal{L}_D$. Then F is exponential if and only if its coefficient of variation is* 1.

PROOF. The 'only if' part is trivial. To prove the converse, suppose F, with mean μ and second moment $\mu_{2,F} = \mu_2$ is in $\mathcal{L}_D$. Straightforward calculations and standard arguments involving the interchange of the order of integration together with the $\mathcal{L}$-class property (1.2) show that

$$\begin{aligned} \frac{(\mu_2/2\mu)}{1+s(\mu_2/2\mu)} &\leq \int_0^\infty e^{-st}\overline{F}_{(1)}(t)dt \\ &= \frac{1}{s} - \frac{1}{s\mu}\int_0^\infty e^{-st}\overline{F}(t)dt \\ &\leq \frac{\mu}{1+s\mu}. \end{aligned} \tag{3.1}$$

The first inequality in (3.1) uses the fact that F_1 has mean $\mu_2/2\mu$ and $F_1 \in \mathcal{L}$. The second inequality uses the $\mathcal{L}$-class property of F. If F has c.v. $\eta_F = 1$ (i.e., $\mu_2 = 2\mu^2$), then the bounds on both sides of (3.1) collapse to yield

$$\int_0^\infty e^{-st}\overline{F}_{(1)}(t)dt = \frac{\mu}{1+s\mu} = \int_0^\infty e^{-st}\overline{G}(t)dt,$$

where G is exponential with mean μ. From the uniqueness of the Laplace transform, we conclude that $\overline{F}(x) = \overline{F_1}(x) = e^{-x/\mu}$, $\forall\, x \geq 0$. □

REMARK 3.1. By virtue of Theorem 3.1, the counter-example in Bhattacharjee and Sengupta (1994) implies that $\mathcal{L}$ is strictly larger than $\mathcal{L}_D$.

REMARK 3.2. The dual version of Theorem 3.1 provides a similar characterization of the exponentials within the $\overline{\mathcal{L}}$-class.

We would need the following definition to present a general characterization theorem which will later be exploited in the context of $\mathcal{L}$-distributions.

DEFINITION 3.2. For arbitrary life distributions F and G, we say that $\overline{F}$ crosses $\overline{G}$ from above (below) if for some $x_0 \in [0, \infty)$,

$$\begin{aligned} \overline{F}(x) &\geq (\leq)\overline{G}(x) \quad \text{if } x \leq x_0, \\ \overline{F}(x) &\leq (\geq)\overline{G}(x) \quad \text{if } x > x_0. \end{aligned}$$

Note that the above definition admits the possibility of F being identically equal to G. Taking a cue from this definition, we shall say that $\overline{F}$ *crosses* $\overline{G}$ if $\overline{F}$ crosses $\overline{G}$ either from above or from below.

Interestingly, the crossing properties defined above introduce a class of life distributions (and its dual) with ageing property stronger than $\mathcal{L}$ ($\overline{\mathcal{L}}$). Specifically, if $\overline{F}$ crosses $\overline{G}$ from above (below), then $F \in \mathcal{L}$ ($\overline{\mathcal{L}}$). This can be seen as follows. Since F and G have the same mean, we can write

$$\begin{aligned}
&\int_0^\infty e^{-sx}(\overline{F}(x) - \overline{G}(x))dx \\
= &\int_0^\infty (e^{-sx} - e^{-sx_0})(\overline{F}(x) - \overline{G}(x))dx \\
= &(\int_0^{x_0} + \int_{x_0}^\infty)(e^{-sx} - e^{-sx_0})(\overline{F}(x) - \overline{G}(x))dx \\
\geq &\ 0,
\end{aligned}$$

since, the two factors in the integrand have the same (opposite) signs according as $\overline{F}$ crosses $\overline{G}$ from above (below). In the spirit of the above discussion, the following theorem provides yet another characterization of the exponential distribution within the $\mathcal{L}$ ($\overline{\mathcal{L}}$)-class.

THEOREM 3.2. *Suppose F is a life distribution with mean μ, such that $\overline{F}(x)$ crosses $\overline{G}(x) = e^{-x/\mu}$. Then F is exponential if and only if $\eta_F = 1$.*

PROOF. It is enough to prove sufficiency as necessity is trivial. Consider first the case when $\overline{F}$ crosses $\overline{G}$ from above and suppose that the crossing takes place at x_0. Notice that the function $(1 - e^{-sx})/s\mu$ decreases in s for each $x > 0$. Accordingly, $\eta_F = 1$ implies that

$$g(s) := \int_0^\infty (\frac{1 - e^{-sx}}{s\mu})(e^{-x/\mu} - \overline{F}(x))dx \to 0 \tag{3.2}$$

as $s \to 0+$. Elementary computations yield

$$g'(s) = (s^2\mu)^{-1}\int_0^\infty \psi(s,x)(e^{-x/\mu} - \overline{F}(x))dx$$

for all $s > 0$, where $\psi(s,x) := (1 + sx)e^{-sx}$. Clearly, $\psi(s,x)$ is strictly decreasing in x on $(0,\infty)$. Hence, as F has mean μ, we have

$$(s^2\mu)g'(s) = \int_0^\infty (\psi(s,x) - \psi(s,x_0))(e^{-x/\mu} - \overline{F}(x))dx$$

$$
\begin{aligned}
&= \left(\int_0^{x_0} + \int_{x_0}^{\infty}\right)(\psi(s,x) - \psi(s,x_0))(e^{-x/\mu} - \overline{F}(x))dx \\
&\leq 0,
\end{aligned}
$$

since the two factors in the integrand have opposite signs for both the integrals. Thus we conclude that $g(s)$ is *non-increasing* and $g(s) \leq g(0+) = 0$. These, together with the fact that $g(s) \to 0$ as $s \to \infty$ implies that $g(s) = 0 \; \forall \; s > 0$. From the definition of $g(s)$, it now follows that $L_F(s) = (1 + s\mu)^{-1}$, and as such, $F(x) = 1 - e^{-x/\mu}, x \geq 0$ by the uniqueness of the Laplace transform. For the case when $\overline{F}$ crosses $\overline{G}$ from below, the factors in the integrand would have the same sign and consequently $g(s)$ would this time be *non-decreasing* so that $g(s) \geq g(0+) = 0$ would hold. Now, we argue just as before to complete the proof. □

Our final theorem reveals the importance of the c.v. of a distribution belonging to the $\mathcal{L}$-class as a measure of its distance from the exponential distribution.

THEOREM 3.3. *Suppose $\{X_n\}$ is a sequence of non-negative random variables with d.f. $F_n \in \mathcal{L}_D$, $n = 1, 2, \ldots$; $EX_n = \mu_n < \infty$ and c.v. η_n. Then X_n/μ_n converges in distribution to the unit exponential distribution if and only if $\eta_n \to 1$ as $n \to \infty$.*

Obretenov (1977) proved the above result under the much more restrictive IFR assumption by exploiting the upper bound on the Laplace transforms of IFR distributions. We omit the proof of Theorem 3.3, since exactly similar arguments are easily seen to remain valid for life distributions in $\mathcal{L}_D$.

Chapter 8

An Optimal Ordering Policy involving NBUE Supplies

1 Introduction

In a one-period inventory situation, suppose an order of amounts $Q_1(\geq 0)$ and $Q_2(\geq 0)$ are placed with two suppliers available at the commencement of the period, depending on the initial stock $y(\geq 0)$. The supplies U_1 and U_2 which are made instantaneously by supplier 1 and supplier 2 respectively are, however, random variables having distribution functions (d.f.) $G_1(\cdot|Q_1)$ and $G_2(\cdot|Q_2)$ respectively with $EU_i = Q_i$, $i = 1, 2$. We assume that the customer demand $X(\geq 0)$ has d.f. $F(\cdot)$ with mean strictly positive and finite and U_1, U_2, X are mutually independent. Let p, s and L denote the sale price, shortage and salvage costs per unit, respectively of the commodity/item under consideration and let c_1 and c_2 be the purchase (production) costs corresponding to the two suppliers and let c be the cost at which the initial stock was procured. We assume, as in most typical cases that

$$\begin{aligned} p > c_1 > L \geq 0 \\ p > c_2 > L \geq 0 \end{aligned} \tag{1.1}$$

and

$$s > \max(p - c_1, p - c_2). \tag{1.2}$$

In this chapter, we suggest a somewhat conservative policy which maximizes a minimum profit (in a sense to be made explicit later) under a broad nonparametric assumption on the structures of G_1 and G_2. More precisely, we take G_1, G_2 to be NBUE (Barlow and Proschan (1981)), i.e., for $i = 1, 2$,

$$E(U_i - u|U_i > u) \leq EU_i. \tag{1.3}$$

This assumption seems to make sense, especially in the context of scarce commodities, since it merely means that on the average, the excess supply over any arbitrary amount u does not exceed the average amount supplied. As is well-known from standard literature on reliability, the NBUE class is fairly large and includes in particular, the exponential distributions. Under this assumption, we show that at all order levels $Q = (Q_1, Q_2)$, the expected profit will never fall short of the corresponding expected profit had the supplies been distributed exponentially with the same means. Consequently, in absence of knowledge about G_1 and G_2, it seems sensible to place an order Q_o where Q_o maximizes the latter average profit since the strategy Q_o ensures the best minimal return on the average. We shall refer to the maximin strategy Q_o as the optimal exponential order quantity. However, Q_o depends on F, which is typically unknown. So, we propose an estimator Q_n of Q_o and establish its desirable large sample properties.

The corresponding inventory problem in the single supplier case was treated by Panda (1978) and Basu (1987). In the context of the two-supplier problem, *apriori* it would be tempting to select the supplier whose product is less expensive and then proceed exactly as in the one-supplier problem to maximize the expected profit. However, the following example demonstrates that this strategy may not always be as beneficial as the one that allocates orders between the suppliers in an optimum manner.

EXAMPLE 1.1. Suppose that the customer demand distribution F is exponential with mean μ and let $c_2 \geq c_1$. Without loss of generality, take the initial stock y to be zero as it adds only a constant term to the expected profit function. Considering the supplier with cost c_1, the expected profit function (Basu (1987)) given by

$$R_1(Q) = (L - c_1)Q - s\mu + (p - s + L)\mu \frac{Q}{\mu + Q}$$

is maximized at $Q^* = (D^{1/2} - 1)\mu$, where $D = (p + s - L)/(c_1 - L)$. Considering both suppliers, using (2.6)-(2.9) below, the expected profit function turns out to be

$$R_2(Q_1, Q_2) = (L - c_1)Q_1 + (L - c_2)Q_2 - s\mu + (p + s - L)\frac{Q_1Q_2 + \mu(Q_1 + Q_2)}{(\mu + Q_1)(\mu + Q_2)}$$

which is maximized at (Q_1^o, Q_2^o) where for $i = 1, 2$, $Q_i^o = (D_i^{1/3} - 1)\mu$, $D_i = (p + s - L)(c_{i+1} - L)/(c_i - L)^2$, with c_3 equal to c_1. Now routine calculations yield that $R_2(Q_1^o, Q_2^o) > R_1(Q^*)$ since $D_2 > 0$, in view of (1.1) and (1.2). This example, therefore, confirms that we would be better off utilizing both the suppliers optimally rather than using the one whose price is less and in this sense, the example summarizes our reasons for following up the present problem.

2 Optimal Exponential Order Quantity

Let G_1, G_2, F, y and Q_1, Q_2 be as in the previous section; also let $R_{F,G_1,G_2}(y,Q_1,Q_2)$ denote the expected profit when amounts Q_1, Q_2 are ordered (to suppliers 1 and 2 respectively) at the commencement of the period.

It is clear that for a specific demand x and supplies u_1 and u_2 from supplier 1 and 2 respectively, the profit is

$$p(y+u_1+u_2)-s(x-y-u_1-u_2)-(cy+c_1u_1+c_2u_2)$$

or

$$px+L(y+u_1+u_2-x)-(cy+c_1u_1+c_2u_2),$$

according as the total inventory $y+u_1+u_2 \leq x$ or $\geq x$. After a certain amount of routine algebra involving integration by parts and interchange of order of integration (which are permitted since EX is finite), we obtain

$$\begin{aligned}
& R_{F,G_1,G_2}(y,Q_1,Q_2) \\
= \; & (L-c_1)Q_1+(L-c_2)Q_2+(L-c)y+(p-L)EX \\
& -(p+s-L)\int_0^\infty \int_{u+y}^\infty \int_0^{x-y-u} G_2(v)dv\, dF(x)\, dG_1(u) \\
= \; & (L-c_1)Q_1+(L-c_2)Q_2+(L-c)y-sEX+(p+s-L)[\int_0^y \overline{F}(x)dx \\
& +\int_y^\infty \int_0^{x-y} \{\overline{G}_1(v)+\overline{G}_2(v)-\overline{G}_1(x-y-v)\overline{G}_2(v)\}dv\, dF(x)]. \qquad (2.1)
\end{aligned}$$

In what follows, we assume that G_1, G_2 are NBUE in the sense of (1.3). Let H_i denote the d.f. of an exponential random variable having the same expectation as U_i, $(i=1,2)$.

LEMMA 2.1. *For all supply distributions F, under* (1.1), (1.2),

$$R_{F,G_1,G_2}(y,Q_1,Q_2) \geq R_{F,H_1,H_2}(y,Q_1,Q_2) \qquad (2.2)$$

for all $Q_1 \geq 0$, $Q_2 \geq 0$.

PROOF. In view of (2.1), it is enough to show that for each $x > y$,

$$\psi_{F,G_1,G_2}(x,y) \geq \psi_{F,H_1,H_2}(x,y),$$

where $\psi_{F,G_1,G_2}(x,y) = \int_0^{x-y} \{\overline{G}_1(v) + \overline{G}_2(v) - \overline{G}_1(x-y-v)\overline{G}_2(v)\}dv.$

For $x > y$,

$$\psi_{F,G_1,G_2}(x,y) = \int_0^{x-y} \overline{G}_1(v)dv + \int_0^{\infty} \varphi_{x-y}(v)\overline{G}_2(v)dv \tag{2.3}$$

where

$$\varphi_{x-y}(v) = \begin{cases} G_1(x-y-v) & \text{if } 0 < v < x-y \\ 0 & \text{if } v \geq x-y \end{cases}$$

is a non-increasing function of v. Hence, it follows, in view of a lemma in Bhattacharjee (1981), that

$$\begin{aligned} \int_0^{\infty} \varphi_{x-y}(v)\overline{G}_2(v)dv &\geq \int_0^{\infty} \varphi_{x-y}(v)\overline{H}_2(v)dv \\ &= \int_0^{x-y} \overline{H}_2(v)dv \\ &\quad + \int_0^{x-y} \{-\overline{H}_2(x-y-v)\}\overline{G}_1(v)dv \end{aligned} \tag{2.4}$$

Noting that the function

$$\gamma_{x-y}(v) = \begin{cases} -\overline{H}_2(x-y-v) & \text{if } 0 < v < x-y \\ -1 & \text{if } v \geq x-y \end{cases}$$

is non-increasing and using the same lemma, we get,

$$\begin{aligned} &\int_0^{x-y} \{-\overline{H}_2(x-y-v)\}\overline{G}_1(v)dv \\ \geq\ &\int_0^{x-y} \{-\overline{H}_2(x-y-v)\}\overline{H}_1(v)dv \\ &+ \int_{x-y}^{\infty} \overline{G}_1(v)dv - \int_{x-y}^{\infty} \overline{H}_1(v)dv \end{aligned} \tag{2.5}$$

By (2.3)-(2.5), we have,

$$\begin{aligned} \psi_{F,G_1,G_2}(x,y) \geq\ &\int_0^{x-y} \overline{G}_1(v)dv + \int_0^{x-y} \overline{H}_2(v)dv \\ &+ \int_0^{x-y} \{-\overline{H}_2(x-y-v)\}\overline{H}_1(v)dv \end{aligned}$$

$$
\begin{aligned}
&+\int_{x-y}^{\infty}\overline{G}_1(v)dv - \int_{x-y}^{\infty}\overline{H}_1(v)dv \\
=& \int_0^{x-y}\{\overline{H}_1(v)+\overline{H}_2(v)-\overline{H}_1(x-y-v)\overline{H}_2(v)\}dv \\
=& \psi_{F,H_1,H_2}(x,y),
\end{aligned}
$$

which completes the proof. □

We shall, for the sake of brevity, denote the right hand member of (2.2) by $R(Q_1, Q_2)$.

Write $B = p + s - L$ (>0); now simple calculations yield, for $Q_1, Q_2 > 0$,

$$
R(Q_1,Q_2) = \sum_{i=1}^{2}(L-c_i)Q_i + (L-c)y - sEX + B[\int_0^y \overline{F}(x)dx + \Lambda(Q_1,Q_2)] \tag{2.6}
$$

where $\Lambda(Q_1, Q_2)$ is given by

$$
\Lambda(Q_1,Q_2) = \begin{cases} \int_y^{\infty} \frac{1}{Q_1-Q_2}\{Q_1 e^{-\frac{(x-y)}{Q_1}} - Q_2 e^{-\frac{(x-y)}{Q_2}}\}\overline{F}(x)dx & \text{if } Q_1 \neq Q_2 \\ \int_y^{\infty}(1+\frac{x-y}{Q})e^{-\frac{(x-y)}{Q}}\overline{F}(x)dx & \text{if } Q_1 = Q_2 = Q. \end{cases}
$$

Similarly, we have,

$$
\begin{aligned}
R(Q_1,0) &= (L-c_1)Q_1 + (L-c)y - sEX \\
&\quad + B[\int_0^y \overline{F}(x)dx + \int_y^{\infty}\exp[-(x-y)/Q_1]\overline{F}(x)dx]
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
R(0,Q_2) &= (L-c_2)Q_2 + (L-c)y - sEX \\
&\quad + B[\int_0^y \overline{F}(x)dx + \int_y^{\infty}\exp[-(x-y)/Q_2]\overline{F}(x)dx]
\end{aligned} \tag{2.8}
$$

$$
R(0,0) = (L-c)y - sEX + B\int_0^y \overline{F}(x)dx \tag{2.9}
$$

It is clear that for all $Q_1, Q_2 \geq 0$, $(Q_1, Q_2) \neq (0,0)$, the expression for $R(Q_1, Q_2)$ is given by (2.6)-(2.8) while (2.9) gives $R(0,0)$.

LEMMA 2.2. *For all* $Q_1 > 0$, $Q_2 > 0$,

$$
R_{F,G_1,G_2}(y,Q_1,Q_2) < R_{F,G_1,G_2}(y,0,0)
$$

whenever y *is such that*

$$
\overline{F}(y) < \min\{\frac{c_1-L}{p+s-L}, \frac{c_2-L}{p+s-L}\}. \tag{2.10}
$$

PROOF. Simple calculations yield

$$R_{F,G_1,G_2}(y,0,0) = (L-c)y - sEX + B\int_0^y \overline{F}(x)dx$$

Thus, by (2.1),

$$\begin{aligned}
& R_{F,G_1,G_2}(y,Q_1,Q_2) - R_{F,G_1,G_2}(y,0,0) \\
= & \sum_{i=1}^{2}(L-c_i)Q_i + B\int_y^\infty \int_0^{x-y} \{\overline{G}_1(v) + \overline{G}_2(v) - \overline{G}_1(x-y-v)\overline{G}_2(v)\}dvdF(x) \\
\leq & \sum_{i=1}^{2}(L-c_i)Q_i + B\int_y^\infty \int_0^{x-y} \{\overline{G}_1(v) + \overline{G}_2(v)\}dvdF(x) \\
\leq & \sum_{i=1}^{2}(L-c_i)Q_i + B(Q_1+Q_2)\overline{F}(y) \\
= & \sum_{i=1}^{2} Q_i\{(L-c_i) + B\overline{F}(y)\} < 0.
\end{aligned}$$

Hence the lemma follows. □

LEMMA 2.3. *Under the conditions (1.1), (1.2), the function $R(Q_1,Q_2)$ is bounded above for $Q_1, Q_2 \geq 0$.*

PROOF. Using the inequalities

$$\frac{Q_1 e^{-u/Q_1} - Q_2 e^{-u/Q_2}}{Q_1 - Q_2} \leq 1 \text{ for } u > 0, Q_1, Q_2 > 0, Q_1 \neq Q_2, \tag{2.11}$$

$$\text{and } e^x > 1 + x \text{ for } x > 0, \tag{2.12}$$

we have, from (2.6)-(2.9),

$$R(Q_1,Q_2) \leq B.EX, \ \forall\, Q_1, Q_2 \geq 0.$$

This completes the proof of the lemma. □

It is clear that under (2.10), the ordering level $(0,0)$ maximizes the expected profit. Otherwise, i.e., whenever (2.10) is violated, it would, in view of Lemma 2.3 make sense to locate $\boldsymbol{Q}_o = (Q_{1o}, Q_{2o}) \in \mathbb{R}_+^2$ such that the expected profit function under exponentiality of the supply distributions is maximized at $\boldsymbol{Q}_o$.

Accordingly, in what follows, we suppose that the initial stock y is such that (2.10) fails.

Let $\xi = \sup\{R(Q_1, Q_2) : Q_1, Q_2 \geq 0\}$. We shall prove that if the demand distribution F is non-degenerate at 0, then there exists a *unique* $\boldsymbol{Q}_o = (Q_{1o}, Q_{2o}) \neq (0,0)$ such that $R(Q_{1o}, Q_{2o}) = \xi$. Consequently, $\boldsymbol{Q}_o$ is the optimal exponential order quantity which, as in Theorem 2.4 in Basu (1987), is maximin by virtue of (2.2). Thus it seems reasonable to adopt the optimum ordering policy $\boldsymbol{Q}_o$ whenever the supply distributions G_1, G_2 are not completely known, but are known to be NBUE. We now present the main theorem of this section.

THEOREM 2.1. *If F is non-degenerate at 0, then under* (1.1), (1.2), *$R(Q_1, Q_2)$ has a unique maximum at $\boldsymbol{Q}_o = (Q_{1o}, Q_{2o}) \neq (0,0)$, $0 \leq Q_{1o}, Q_{2o} < \infty$. In fact, $\boldsymbol{Q}_o$ solves $\partial R(Q_1, Q_2)/\partial Q_1 = 0$ and $\partial R(Q_1, Q_2)/\partial Q_2 = 0$ uniquely.*

We shall prove the above theorem at the end of this section. First, we need the following lemmas necessary to prove the theorem.

LEMMA 2.4. *Under conditions* (1.1), (1.2), *there exists a positive real number θ such that $R(Q_1, Q_2) < R(0,0)$ whenever $Q_1 > \theta$ and/or $Q_2 > \theta$.*

PROOF. Let $Q_1, Q_2 > 0$, $Q_1 \neq Q_2$. From (2.6) and (2.9), it follows using (2.11), that

$$\begin{aligned} & R(Q_1, Q_2) - R(0,0) \\ \leq\ & (L - c_1)Q_1 + (L - c_2)Q_2 + B.EX \\ \leq\ & (L - \min(c_1, c_2))(Q_1 + Q_2) + B.EX < 0 \\ & \text{if } Q_1 + Q_2 > \frac{B.EX}{\min(c_1, c_2) - L} > 0 \end{aligned} \tag{2.13}$$

Again, for, $Q_1, Q_2 > 0$, $Q_1 = Q_2$, we have, using (2.11),

$$\begin{aligned} & R(Q_1, Q_2) - R(0,0) \\ \leq\ & (2L - c_1 - c_2) + B.EX < 0 \\ & \text{if } Q_1 > \frac{B.EX}{c_1 + c_2 - 2L} > 0. \end{aligned} \tag{2.14}$$

Also, from (2.7)-(2.9), we get,

$$R(Q_1, 0) - R(0,0) < 0 \text{ if } Q_1 > \frac{B.EX}{c_1 - L} \tag{2.15}$$

$$R(0, Q_2) - R(0,0) < 0 \text{ if } Q_2 > \frac{B.EX}{c_2 - L} \tag{2.16}$$

Taking $\theta = B.EX/(\min(c_1, c_2) - L)$, the proof follows from (2.13)-(2.16). □

LEMMA 2.5. *Under* (1.1), (1.2), *there exists*

(a) $Q_1^* > 0$ *such that* $R(0,0) < R(Q_1^*, 0)$ *whenever* y *satisfies* $\overline{F}(y) > (c_1 - L)/B$.

(b) $Q_2^* > 0$ *such that* $R(0,0) < R(0, Q_2^*)$ *whenever* y *satisfies* $\overline{F}(y) > (c_2 - L)/B$.

PROOF. (a) For $Q_1 > 0$,

$$R(Q_1, 0) - R(0,0) = Q_1[(L - c_1) + B\int_0^\infty e^{-u}\overline{F}(y + uQ_1)du]. \tag{2.17}$$

As $Q_1 \to 0+$, the bracketed term tends to $(L - c_1) + B\overline{F}(y)$ which is positive by condition (a) of the lemma. The assertion of the lemma now follows from (2.17).

(b) The proof is similar to that of part (a). □

In what follows, we shall, for the sake of algebraic simplicity, assume that the initial inventory $y = 0$. The expression for the expected profit function now reduces to

$$R(Q_1, Q_2) = \sum_{i=1}^{2}(L - c_i)Q_i - s.EX + B\psi(Q_1, Q_2) \tag{2.18}$$

for $Q_1, Q_2 \geq 0$, $(Q_1, Q_2) \neq (0,0)$, where

$$\psi(Q_1, Q_2) = \begin{cases} \int_0^\infty \frac{Q_1 e^{-x/Q_1} - Q_2 e^{-x/Q_2}}{Q_1 - Q_2}\overline{F}(x)dx, & Q_1 \neq Q_2, \\ \int_0^\infty (1 + \frac{x}{Q_1})e^{-x/Q_1}\overline{F}(x)dx, & Q_1 = Q_2. \end{cases}$$

We shall prove that the function $\psi(Q_1, Q_2)$ is *strictly concave*. To this end, we notice that

$$Q_1 e^{-x/Q_1} - Q_2 e^{-x/Q_2} = \int_{Q_2}^{Q_1} h(Q; x)dQ,$$

where $h(Q, x) = (1 + x/Q)e^{-x/Q}$, $x > 0, Q > 0$. So, we can write, for $Q_1 \neq Q_2$,

$$\psi(Q_1, Q_2) = \frac{1}{Q_1 - Q_2}\int_{Q_2}^{Q_1} \xi(Q)dQ,$$

where

$$\begin{aligned}\xi(Q) &= \int_0^\infty h(Q,x)\overline{F}(x)dx \\ &= \int_0^\infty (1+\frac{x}{Q})e^{-x/Q}\overline{F}(x)dx.\end{aligned}$$

Note that the interchange of integrals is permissible by Fubini's theorem. Thus,

$$\psi(Q_1,Q_2) = \begin{cases} \dfrac{1}{Q_1-Q_2}\displaystyle\int_{Q_2}^{Q_1}\xi(Q)dQ, & Q_1 \neq Q_2, \\ \xi(Q), & Q_1 = Q_2 = Q. \end{cases} \tag{2.19}$$

Straightforward calculations yield, for $Q > 0$,

$$\xi''(Q) = \frac{1}{Q}\int_0^\infty e^{-u}(u^3 - 3u^2)\overline{F}(uQ)du. \tag{2.20}$$

If F is absolutely continuous, integrating the righthand member of (2.20) by parts, we get,

$$\xi''(Q) = -\int_0^\infty u^3 e^{-u} f(uQ)du < 0,$$

where $f(\cdot)$ is the probability density function corresponding to F. This shows that $\xi(Q)$ is a *strictly concave* function, in the case where F has a density.

We need a slightly different argument as follows to deal with the case where F has jumps. Suppose $0 \leq x_1 \leq x_2 \ldots$ are the jump points of F with jumps $p_1, p_2, \ldots$, respectively. For symmetry, let $x_0 = 0$; also for $j = 0, 1, 2, \ldots$ let $k_j = x_j/Q$, $Q > 0$. Supposing the number of jumps to be infinite (the modification required whenever this is finite is obvious), we note that

$$\begin{aligned}\xi''(Q) &= \frac{1}{Q}\sum_{j=0}^{\infty}\int_{k_j}^{k_{j+1}}\overline{F}(uQ)(u^3 - 3u^2)e^{-u}du \\ &= \frac{1}{Q}\sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty}p_i\int_{k_j}^{k_{j+1}}(u^3 - 3u^2)e^{-u}du \\ &= -\frac{1}{Q}\sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty}p_i\{k_{j+1}^3\exp(-k_{j+1}) - k_j^3\exp(-k_j)\} \\ &= -\frac{1}{Q}\sum_{i=1}^{\infty}p_i\sum_{j=0}^{i-1}\{k_{j+1}^3\exp(-k_{j+1}) - k_j^3\exp(-k_j)\} \\ &= -\frac{1}{Q}\sum_{i=1}^{\infty}p_i\{k_i^3\exp(-k_i) - k_0^3\exp(-k_0)\} \\ &= -\frac{1}{Q}\sum_{i=1}^{\infty}p_i k_i^3\exp(-k_i), \text{ as } k_0 = 0.\end{aligned}$$

Since F is non-degenerate at zero, $\xi''(Q) < 0$.

We shall now use the strict concavity of $\xi(Q)$ to establish the strict concavity of $\psi(Q_1, Q_2)$ via (2.19). We want to show that for $0 \leq \alpha \leq 1$,

$$\psi(\alpha Q_1 + \overline{\alpha} Q_1^*, \alpha Q_2 + \overline{\alpha} Q_2^*) > \alpha\psi(Q_1, Q_2) + \overline{\alpha}\psi(Q_1^*, Q_2^*) \tag{2.21}$$

for all (Q_1, Q_2), $(Q_1^*, Q_2^*) \in S$, where $S = \{(x, y) : x \geq 0, y \geq 0, (x, y) \neq (0, 0)\}$ and $\overline{\alpha} = 1 - \alpha$. We split the proof into several cases. All tuples (Q_1, Q_2), (Q_1^*, Q_2^*) considered in what follows belong to S.

Case I. $Q_1 = Q_2 = Q$, $Q_1^* = Q_2^* = Q^*$.

$$\begin{aligned}
\text{RHS of (2.21)} &= \alpha\xi(Q) + \overline{\alpha}\xi(Q^*) \\
&< \xi(\alpha Q + \overline{\alpha} Q^*) \\
&= \text{LHS of (2.21)}.
\end{aligned}$$

Case II. $Q_1 = Q_2 = Q$, $Q_1^* \neq Q_2^*$.

$$\begin{aligned}
\text{LHS of (2.21)} &= \psi(\alpha Q + \overline{\alpha} Q_1^*, \alpha Q + \overline{\alpha} Q_2^*) \\
&= \frac{1}{\overline{\alpha}(Q_1^* - Q_2^*)} \int_{\alpha Q + \overline{\alpha} Q_2^*}^{\alpha Q + \overline{\alpha} Q_1^*} \xi(t) dt \\
&= \frac{1}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \xi(\alpha Q + \overline{\alpha} t) dt \\
&> \frac{1}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \{\alpha\xi(Q) + \overline{\alpha}\xi(t)\} dt \\
&= \alpha\xi(Q) + \overline{\alpha} \frac{1}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \xi(t) dt \\
&= \text{RHS of (2.21)}.
\end{aligned}$$

Case III(a). $Q_1 \neq Q_2$, $Q_1^* \neq Q_2^*$, $\alpha Q_1 + \overline{\alpha} Q_1^* \neq \alpha Q_2 + \overline{\alpha} Q_2^*$.

LHS of (2.21)

$$= \frac{1}{\alpha(Q_1 - Q_2) + \overline{\alpha}(Q_1^* - Q_2^*)} \int_{\alpha Q_2 + \overline{\alpha} Q_2^*}^{\alpha Q_1 + \overline{\alpha} Q_1^*} \xi(t) dt$$

$$
\begin{aligned}
&= \frac{1}{Q_1 - Q_2} \int_{Q_2}^{Q_1} \xi(\alpha u + \overline{\alpha}\{\frac{(Q_1^* - Q_2^*)u - (Q_1^* Q_2 - Q_1 Q_2^*)}{Q_1 - Q_2}\})du \\
&> \frac{\alpha}{Q_1 - Q_2} \int_{Q_2}^{Q_1} \xi(u)du + \frac{\overline{\alpha}}{Q_1 - Q_2} \int_{Q_2}^{Q_1} \xi(\frac{(Q_1^* - Q_2^*)u - (Q_1^* Q_2 - Q_1 Q_2^*)}{Q_1 - Q_2})du \\
&= \frac{\alpha}{Q_1 - Q_2} \int_{Q_2}^{Q_1} \xi(u)du + \frac{\overline{\alpha}}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \xi(v)dv \\
&= \text{RHS of (2.21)}.
\end{aligned}
$$

Case III(b). $Q_1 \neq Q_2$, $Q_1^* \neq Q_2^*$, $\alpha Q_1 + \overline{\alpha} Q_1^* = \alpha Q_2 + \overline{\alpha} Q_2^*$.

$$
\begin{aligned}
& \text{RHS of (2.21)} \\
&= \frac{\alpha}{Q_1 - Q_2} \int_{Q_2}^{Q_1} \xi(t)dt + \frac{\overline{\alpha}}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \xi(t)dt \\
&= \frac{\alpha}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \xi(\frac{\alpha Q_1 + \overline{\alpha} Q_1^*}{\alpha} - \frac{\overline{\alpha} u}{\alpha})du + \frac{\overline{\alpha}}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \xi(u)du \\
&= \frac{1}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \{\alpha \xi(\frac{\alpha Q_1 + \overline{\alpha} Q_1^*}{\alpha} - \frac{\overline{\alpha} u}{\alpha}) + \overline{\alpha}\xi(u)\}du \\
&< \frac{1}{Q_1^* - Q_2^*} \int_{Q_2^*}^{Q_1^*} \xi(\alpha\{\frac{\alpha Q_1 + \overline{\alpha} Q_1^*}{\alpha} - \frac{\overline{\alpha} u}{\alpha}\} + \overline{\alpha} u)du \\
&= \xi(\alpha Q_1 + \overline{\alpha} Q_1^*) \\
&= \text{LHS of (2.21)}.
\end{aligned}
$$

This completes the proof of strict concavity of the function $\psi(Q_1, Q_2)$ on the set S. Now the strict concavity of $R(Q_1, Q_2)$ follows from (2.18). Thus we have proved the following:

LEMMA 2.6. *If F is non-degenerate at 0 and (1.1), (1.2) hold, then the expected profit function $R(Q_1, Q_2)$ is strictly concave on the set S.*

We are now in a position to provide a proof of the main theorem.

PROOF OF THEOREM 2.1. Write $\zeta = \sup\{R(Q_1, Q_2) : Q_1, Q_2 \geq 0\}$ and $\zeta^* = \sup\{R(Q_1, Q_2) : (Q_1, Q_2) \in \Gamma\}$ where $\Gamma = \{(Q_1, Q_2) : 0 \leq Q_1, Q_2 \leq \theta\}$ and θ is as in Lemma 2.4. Clearly, $\zeta^* \leq \zeta$, by definition. But if $\zeta^* < \zeta$, then for some (Q_1, Q_2) with $Q_1 > \theta$ and/or $Q_2 > \theta$,

$$
R(Q_1, Q_2) > \zeta^* \geq R(0, 0),
$$

which contradicts Lemma 2.4. Thus $\zeta^* = \zeta$. Also, because of continuity, $R(Q_1, Q_2)$ must attain its maximum over the compact set Γ, say at (Q_{1o}, Q_{2o}). Then, clearly, $R(Q_{1o}, Q_{2o}) = \zeta^* = \zeta$, and by Lemma 2.5 both Q_{1o} and Q_{2o} cannot be zero simultaneously. Evidently, (Q_{1o}, Q_{2o}) solves $\partial R(Q_1, Q_2)/\partial Q_1 = 0$, $\partial R(Q_1, Q_2)/\partial Q_2 = 0$. The uniqueness of $Q_o = (Q_{1o}, Q_{2o})$ is a consequence of Lemma 2.6. □

3 Estimation of the Optimal Exponential Order Quantity

Typically, the demand distribution F is unknown so that it is not possible to determine exactly the optimal exponential order quantity $Q_o = (Q_{1o}, Q_{2o})$. In such a situation, the statistical estimation of Q_o is relevant. Suppose F_n is the empirical c.d.f. based on a random sample of size n. Then, the Glivenko-Cantelli theorem (see, e.g. Loéve (1963)) ensures that

$$\Delta_n := \sup_x |F_n(x) - F(x)| \to 0 \text{ w.p. } 1 \text{ as } n \to \infty. \tag{3.1}$$

Thus a natural estimate of Q_o is $Q_n = (Q_{1n}, Q_{2n})$ where Q_n maximizes

$$R^{(n)}(Q_1, Q_2) = R_{F_n, H_1, H_2}(Q_1, Q_2). \tag{3.2}$$

The uniqueness of Q_n follows as in Theorem 2.1 noting that if F is non-degenerate at 0, so is F_n with probability 1 because of (3.1). (Note that the strict concavity of $R(Q_1, Q_2)$ in the case when F has jumps becomes relevant in this context.)

Before going on to prove the main theorem of this section, we require the following lemma.

LEMMA 3.1. *For* $x, Q_1, Q_2 > 0$, $Q_1 > Q_2$,

$$(1 + \frac{x}{Q_2})e^{-x/Q_2} < \frac{Q_1 e^{-x/Q_1} - Q_2 1 e^{-x/Q_2}}{Q_1 - Q_2} < (1 + \frac{x}{Q_1})e^{-x/Q_1}.$$

PROOF. It is easy to see that

$$(1 + \frac{x}{Q_1})e^{-x/Q_1} > \frac{Q_1 e^{-x/Q_1} - Q_2 1 e^{-x/Q_2}}{Q_1 - Q_2}$$

is equivalent to

$$\exp[-x(1/Q_2 - 1/Q_1)] > 1 - x(1/Q_2 - 1/Q_1),$$

which is true since $e^{-y} > 1 - y \,\forall\, y > 0$; here, $y = x(1/Q_2 - 1/Q_1) > 0$ as $Q_1 > Q_2$. The other inequality follows similarly using the relation $e^y > 1 + y \,\forall\, y > 0$. □

We next present our main result concerning the estimation of $\boldsymbol{Q}_o$.

THEOREM 3.1. *If F is non-degenerate at 0, and (1.1), (1.2) hold, then $Q_n \to Q_o$ a.s. as $n \to \infty$.*

PROOF. For $Q_1, Q_2 \geq 0$, $(Q_1, Q_2) \neq (0,0)$, let

$$R_{ij}^{(n)}(Q_1, Q_2) := \partial^{i+j} R^{(n)}(Q_1, Q_2)/\partial Q_1^i \partial Q_2^j,$$

$i, j = 0, 1$, and let $\psi_{ij}^{(n)}(Q_1, Q_2)$ be defined similarly, where $R^{(n)}$ is as in (3.2) and $\psi^{(n)}$ is analogous to ψ in (2.18) with F replaced by F_n. From (2.18),

$$R_{10}(Q_1, Q_2) = (L - c_1) + B\psi_{10}(Q_1, Q_2),$$

where

$$\begin{aligned}
&\psi_{10}(Q_1, Q_2) \\
=\; & \int_0^\infty \frac{1}{Q_1 - Q_2}(1 + x/Q_1)e^{-x/Q_1}\overline{F}(x)dx - \int_0^\infty \frac{Q_1 e^{-x/Q_1} - Q_2 e^{-x/Q_2}}{(Q_1 - Q_2)^2}\overline{F}(x)dx.
\end{aligned}$$

$$\begin{aligned}
&|R_{10}(Q_{1n}, Q_{2n}) - R_{10}^{(n)}(Q_{1n}, Q_{2n})| \\
=\; & B|\psi_{10}(Q_{1n}, Q_{2n}) - \psi_{10}^{(n)}(Q_{1n}, Q_{2n})| \\
=\; & B|\{\int_0^\infty \frac{1}{Q_{1n} - Q_{2n}}(1 + x/Q_{1n})e^{-x/Q_{1n}} - \frac{Q_{1n}e^{-x/Q_{1n}} - Q_{2n}e^{-x/Q_{2n}}}{(Q_{1n} - Q_{2n})^2}\}\{F_n(x) - F(x)\}dx| \\
\leq\; & B\int_0^\infty |\frac{1}{Q_{1n} - Q_{2n}}(1 + x/Q_{1n})e^{-x/Q_{1n}} - \frac{Q_{1n}e^{-x/Q_{1n}} - Q_{2n}e^{-x/Q_{2n}}}{(Q_{1n} - Q_{2n})^2}|\Delta_n dx \\
=\; & B\Delta_n\{\int_0^\infty |\frac{1}{Q_{1n} - Q_{2n}}(1 + x/Q_{1n})e^{-x/Q_{1n}} - \frac{Q_{1n}e^{-x/Q_{1n}} - Q_{2n}e^{-x/Q_{2n}}}{(Q_{1n} - Q_{2n})^2}\}dx,
\end{aligned}$$

since, by Lemma 3.1, the quantity within bracket is positive with probability 1. Then, after completing the integration, we note that

$$\begin{aligned}
&|R_{10}(Q_{1n}, Q_{2n}) - R_{10}^{(n)}(Q_{1n}, Q_{2n})| \\
\leq\; & B\Delta_n[\frac{2Q_{1n}}{Q_{1n} - Q_{2n}} - \frac{Q_{1n}^2 - Q_{2n}^2}{(Q_{1n} - Q_{2n})^2}] \\
=\; & B\Delta_n \to 0, \text{ a.s. as } n \to \infty,
\end{aligned}$$

by virtue of (3.1). As $R_{10}^{(n)}(Q_{1n}, Q_{2n}) = 0$, by definition, it now follows that

$$R_{10}(Q_{1n}, Q_{2n}) \to 0 \text{ a.s. as } n \to \infty. \tag{3.3}$$

Similarly,

$$R_{01}(Q_{1n}, Q_{2n}) \to 0 \text{ a.s.} \tag{3.4}$$

We shall prove that

$$\|Q_n - Q_o\| := \{(Q_{1n} - Q_{1o})^2 + (Q_{2n} - Q_{2o})^2\}^{\frac{1}{2}} \to 0 \text{ a.s.} \tag{3.5}$$

We divide the proof into two cases.

Case I. $Q_{1o}, Q_{2o} > 0$. Define for $0 \leq r \leq \min(Q_{1o}, Q_{2o})$, $0 \leq \varphi \leq 2\pi$,

$$\begin{aligned} h(r, \varphi) \quad := \quad & (\cos\varphi) R_{10}(Q_{1o} + r\cos\varphi, Q_{2o} + r\sin\varphi) \\ & + (\sin\varphi) R_{01}(Q_{1o} + r\cos\varphi, Q_{2o} + r\sin\varphi) \end{aligned} \tag{3.6}$$

Consider the function

$$g_\varphi(r) := R(Q_{1o} + r\cos\varphi, Q_{2o} + r\sin\varphi)$$

as a function of r only for fixed φ: call it $g(r)$. It is easy to show, using Lemma 2.6, that $g(r)$ is a strictly concave function of r. So $g'(r)$ is a strictly decreasing function of r. Noting that $g'(r) = h(r, \varphi)$, we conclude that for each fixed φ, $h(r, \varphi)$ is a strictly decreasing function of r. Thus $h(r, \varphi) < h(0, \varphi) = 0$ for $r > 0$, the equality holding in view of Theorem 2.1.

Writing

$$\begin{aligned} Q_{1n} &= Q_{1o} + r_n \cos\varphi_n, \\ \text{and } Q_{2n} &= Q_{2o} + r_n \sin\varphi_n, \end{aligned}$$

by (3.3) and (3.4), we have,

$$|h(r_n, \varphi_n)| \to 0 \text{ a.s. as } n \to \infty. \tag{3.7}$$

Now, consider an arbitrary $\epsilon > 0$. The function $h(\epsilon, \varphi)$, being continuous in φ over the compact set $[0, 2\pi]$, $\exists$ a δ (> 0) depending on ϵ, such that

$$\sup\{h(\epsilon, \varphi) : 0 \leq \varphi \leq 2\pi\} = \max\{h(\epsilon, \varphi) : 0 \leq \varphi \leq 2\pi\} = -\delta,$$

since $h(r,\varphi) < 0$, $\forall r$ in the range $0 < r < \min(Q_{1o}, Q_{2o})$. Since for fixed φ, $h(r,\varphi)$ is decreasing in r, it follows that for all $\varphi \in [0, 2\pi]$, $h(r,\varphi) \leq h(\epsilon,\varphi) \leq -\delta$ whenever $r \geq \epsilon$. As ϵ is arbitrary and δ is strictly positive, we conclude from (3.7) that $r_n \to 0$ as $n \to \infty$. This proves (3.5) in Case I.

Case II. Suppose one of the co-ordinates of Q_o is zero. (Note that both cannot be zero in view of Lemma 2.5). Without loss of generality, suppose $Q_o = (Q_{1o}, 0)$. Then, we shall define $h(r,\varphi)$ as before with $\varphi \in [0,\pi]$ and $0 < r < Q_{1o}$. As Q_{1n}, Q_{1n} are positive with probability 1, we can use the representation

$$\begin{aligned} Q_{1n} &= Q_{1o} + r_n \cos\varphi_n \\ Q_{2n} &= r_n \sin\varphi_n \end{aligned}$$

and arguing as in Case I, complete the proof of the theorem. □

References

Aarset, M.V. (1985). The null distribution for a test of constant versus "bathtub" failure rate. *Scand. J. Statist.* **12**, 55-61.

A-Hameed, M.S. and Proschan, F. (1975). Shock models with underlying birth processes. *J. Appl. Probab.* **12**, 18-28.

Barlow, R.E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing.* To Begin With, Silver Spring, M.D.

Basu, A.P., Ghosh, J.K. and Joshi, S.N. (1988). On estimating change point in a failure rate. *Statistical Decision Theory and Related Topics IV, No. 2.* Springer-Verlag, New York, 239-252.

Basu, S.K. (1987). An optimum ordering policy for situations with uncertainty in supply. *Naval Res. Logist. Quart.* **34**, 151-159.

Basu, S.K. and Bhattacharjee, M.C. (1984). On weak convergence within the HNBUE family of life distributions. *J. Appl. Probab.* **21**, 654-660.

Basu, S.K. and Simons, G. (1983). Moment spaces of IFR distributions, applications and related materials. In *Contributions to Statistics: Essays in Honour of Norman L. Johnson*, ed. P.K. Sen, North Holland, Amsterdam.

Bergman, B. (1979). On age replacement and the total time on test concept. *Scand. J. Statist.* **6**, 161-168.

Bhattacharjee, A. and Sengupta, D. (1994). On the coefficient of variation for the $\mathcal{L}$ and $\overline{\mathcal{L}}$-classes of life distributions. *Submitted.*

Bhattacharjee, M. C. (1981). A note on maximum disposal policy under NWUE pricing. *Naval Res. Logist. Quart.* **28**, 341-345.

Billingsley, P. (1985). *Probability and Measure.* John Wiley and Sons, New York.

Block, H.W. and Savits, T.H. (1978). Shock models with NBUE survival. *J. Appl. Probab.* **15**, 621-628.

Bray, T., Crawford, G. and Proschan, F. (1967). Maximum Likelihood estimation of a U-shaped failure rate function. *Math Note 534*, Boeing Scientific Research Laboratories, Seattle, W.A.

Bryson, M.C. and Siddiqui, M.M. (1969). Some criteria for ageing. *J. Amer. Stat. Soc.* **64**, 472-483.

Canfield, R.V. and Borgman, L.E. (1975). Some distributions of time to failure for reliability applications. *Technometrics* **17**, 263-268.

Cobb, L. (1981). The multimodal exponential families of statistical catastrophe theory in *Statistical distributions in scientific work*, Ed. C. Taillie, G.P. Patil and B. Baldessari, Dordrecht, Holland: Reidel Press.

Deshpande, J.V. and Suresh, R.P. (1990). Non-monotonic ageing. *Scand. J. Statist.* **17**, 257-262.

Esary, J.D., Marshall, A.W. and Proschan, F. (1973). Shock models and wear processes. *Ann. Probab.* **1**, 627-649.

Gaver, O.P. and Acar, M. (1979). Analytical hazard representations for use in reliability, mortality and simulation studies. *Comm. Statist.- Simula Computa* **B8**(2), 91-111.

Ghosh, M. and Ebrahimi, N. (1982). Shock models leading to increasing failure rate and decreasing mean residual life survival. *J. Appl. Probab.* **19**, 158-166.

Glaser, R.E. (1980). Bathtub and related failure rate characterizations. *J. Amer. Statist. Assoc.* **75**, 667-672.

Gottlieb, G. (1980). Failure distributions of shock models. *J. Appl. Probab.* **17**, 745-752.

Guess, F. and Proschan, F. (1985). Mean residual life: theory and applications. Florida State University Statistics Report M702. (Air Force Office of Scientific Research Technical Report No. 85-178).

Guess, F., Hollander, M. and Proschan, F. (1986). Testing exponentiality versus a trend change in mean residual life. *Ann. Statist.* **14**, 1388-1398.

Haines, A. (1973). Some contributions to the theory of restricted classes of distributions with applications to reliability. *Ph.D. Dissertation.* The George Washington University.

Hjorth, U. (1980). A reliability distribution with increasing, decreasing, constant bathtub failure rates. *Technometrics* **22**, 99-107.

Hollander, M. and Proschan, F. (1984). Nonparametric concepts and methods in reliability. *Handbook of Statistics*, **Vol. 4** (edited by P.R. Krishnaiah and P.K. Sen), 613-655.

Klefsjö, B. (1981). HNBUE survival under some shock models. *Scand. J. Statist.* **8**, 39-47.

Klefsjö, B. (1982a). The HNBUE and HNWUE classes of life distributions. *Naval Res. Logist. Quart.* **29**, 331-344.

Klefsjö, B. (1982b). On ageing properties and total time on test transforms. *Scand. J. Statist.* **9**, 37-41.

Klefsjö, B. (1983). A useful ageing property based on the Laplace transform. *J. Appl. Probab.* **20**, 615-626.

Kulasekera, K.B. and Lal Saxena, K.M. (1991). Estimation of change point in failure rate models. *J. Statist. Planning and Inference* **29**, 111-124.

Loéve, M. (1963). *Probability Theory, 3rd ed.* Van Nostrand, New York.

Marshall, A.W. and Proschan, F. (1972). Classes of distributions applicable in replacement, with renewal theory implications. *Proceedings of the VIth Berkeley Symposium on Mathematical Statistics and Probability*, **Vol. I.** Eds. L. LeCam, J. Neyman, and E.L. Scott, 395-415. Berkeley, Calif.: University of California Press.

Mitra, M. and Basu, S.K. (1991). On some properties of the BFR family of life distributions. *IIMC Working Paper Series No. WPS-169(91).*

Mitra, M., Basu, S.K. and Roy, D. (1993). Interrelationships between certain notions of non-monotonic ageing. *Probability and Statistics: Proceedings of the First International Triennial Calcutta Symposium*, 14-19.

Mitra, M. and Basu, S.K. (1994). On a nonparametric family of life distributions and its dual. *J. Statist. Planning and Inference* **39**, 385-397.

Mukherjee, S.P. and Islam, A. (1983). A finite range distribution of failure rates. *Naval Res. Logist. Quart.* **30**, 487-491.

Nguyen, H.T., Rogers, G.S. and Walker, E.A. (1984). Estimation in change point hazard rate models. *Biometrika* **71**, 299-304.

Obretenov, A. (1977). Convergence of IFR-distribution to the exponential one. *Comptes Rendus de l'Academie des Sciences*, **30**, 1385-1387.

Panda, R. M. (1978). Static inventory model with uncertainty in supply. *Calcutta Statistical Association Bulletin* **27**, 141-148.

Paranjpe, S.A., Rajarshi, M.B. and Gore, A.P. (1985). On a model for failure rates. *Biometrical J.* **27**, 913-917.

Park, D.H. (1988). Testing whether failure rate changes its trend. *IEEE Transactions on Reliability* Vol. **37**, No. 4, 375-378.

Pham, T.D. and Nguyen, H.T. (1990). Strong consistency of the maximum likelihood estimators in the change-point hazard rate model. *Statistics* **21**, No. 2, 203-216.

Pham, T.D. and Nguyen, H.T. (1993). Bootstrapping the change-point of a hazard rate. *Ann. Inst. Statist. Math.* **45**, No. 2, 331-340.

Rajarshi, M.B. and Rajarshi, S.M. (1988). Bathtub distributions–a review. *Comm. Statist.* **17**, 2597-2622.

Rolski, T. (1975). Mean residual life. *Proc. 40th Session Internat. Statist. Inst.* **4**, 266-270.

Roy, D. (1988). *Life Distributions- Characterizations and Classifications.* Ph.D. dissertation. The University of Calcutta.

Silverman, B.W. (1978). Strong and weak uniform consistency of the kernel estimate of a density and its derivatives. *Ann. Statist.* **6**, 177-184.

Yang, G.L. (1978). Estimation of a biometric function. *Ann. Statist.* **6**, 112-116.

Yao, Y.C. (1986). Maximum likelihood estimation in hazard rate models with a change point. *Comm. Statist. Theor. Meth.* **15**, 2455-2466.