TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 1, pp. 19-34, March 2001

SOME PROPERTIES RELATED TO NESTED SEQUENCE OF BALLS IN BANACH SPACES

Pradipta Bandyopadhyay and Bor-Luh Lin

Abstract. In this survey article, we explore various natural situations where the results about strict convexity of X^* like Vlasov's Theorem or Taylor-Foguel Theorem are actually seen to be locally consequences of properties of rotund points.

1. Introduction

We work with *real* Banach spaces. Let X be a Banach space. We will denote by $B(x, r)$ (resp. $B[x, r]$) the open (resp. closed) ball of radius $r > 0$ around $x \in X$. Our notations are otherwise standard.

A Banach space X is said to be strictly convex if every point of the unit sphere $S(X)$ is an extreme point of the unit ball $B(X)$. Vlasov [18] (see also [14, Theorem 2]) showed that X^* is strictly convex if and only if the union of any unbounded nested sequence of balls in X is either the whole of X or an open affine half-space.

Definition 1.1. A sequence $\{B_n = B(x_n, r_n)\}$ of open balls in X is *nested* if for all $n \geq 1$, $B_n \subseteq B_{n+1}$.

A nested sequence $\{B_n = B(x_n, r_n)\}$ of balls in X is *unbounded* if $r_n \uparrow \infty$.

In [4], we observed that locally Vlasov's theorem is actually a consequence of the fact that if X^* is strictly convex, then every point of $S(X^*)$ is a *rotund point* of $B(X^*)$ – a notion strictly stronger than extreme points.

Definition 1.2 [8]. Let X be a Banach space. We say that $x \in S(X)$ is a rotund point of $B(X)$ (or, X is rotund at x) if $\|y\| = \|(x + y)/2\| = 1$ implies $x = y$.

Received May 4, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 46B20.
Key words and phrases: ALUR point, wALUR point, rotund point, nested sequence of balls, smooth, very smooth, Fréchet smooth, Hahn-Banach smooth, Vlasov Property, w*-ANPs.

Remark 1.3. Clearly, every rotund point of $B(X)$ is an extreme point, indeed an exposed point of $B(X)$. But the converse is not generally true. For example, no extreme point of $B(\ell_\infty)$ or $B(\ell_1)$ is a rotund point of $B(\ell_\infty)$ or $B(\ell_1)$. However, if X is strictly convex, then every point of $S(X)$ is a rotund point of $B(X)$.

Observe that x is not a rotund point of $B(X)$ if and only if there exists $z \neq 0$ such that $\|x + \lambda z\| = 1$ for all $\lambda \in [0, 1]$. We will then say that x is not rotund in the direction of z. In case x is not rotund in both the directions of z and $-z$, that is, if $\|x \pm z\| = 1$ for some $z \neq 0$, then x fails to be an extreme point as well.

A classical result of Taylor [17] and Foguel [7] showed that X^* is strictly convex if and only if every subspace Y is a U-subspace of X.

Definition 1.4. A subspace Y of a Banach space X is said to be a U-subspace of X if each $y^* \in Y^*$ has a unique Hahn-Banach (i.e., norm preserving) extension in X^*.

X is Hahn-Banach Smooth if X is an U-subspace of X^{**}.

U-subspaces are studied in [14] and [15]. They refer to them as subspaces with the Property U in X. Our terminology is borrowed from [6]. In particular, U-subspaces have been characterized in [14] in terms of unbounded nested sequence of balls.

In [3], we observed that locally the Taylor-Foguel Theorem is also a consequence of properties of rotund points.

Rotund points were introduced in [8] and, in fact, a version of Theorem 2.1 is proved there. The results are reproduced also in [9]. It is rather surprising that the notion of rotund points have not received the attention it deserves.

In this survey article, we explore various natural situations where the results about strict convexity of X^* like Vlasov's Theorem or Taylor-Foguel Theorem are actually seen to be locally consequences of properties of rotund points.

2. Local Results

We begin with a direct proof of the local version of Vlasov's Theorem in [4], which brings out the essential features and simplicity of the argument.

Theorem 2.1. *Let X be a Banach space. Then $x^* \in S(X^*)$ is a rotund point of $B(X^*)$ if and only if for every unbounded nested sequence $\{B_n\}$ of balls such that x^* is bounded below on $\cup B_n$, $\cup B_n$ is an affine half-space determined by x^*.*

Proof. Let $\{B_n = B(x_n, r_n)\}$ be an unbounded nested sequence of balls in a Banach space X, and let $B = \cup B_n$. Suppose $B \neq X$. Let

$$A = \{x^* \in S(X^*) : x^* \text{ is bounded below on } B\}.$$

Then
$$B = \bigcap_{x^* \in A} \{x \in X : x^*(x) > \inf x^*(B)\};$$
and it is easy to show that the set A is a convex subset of $S(X^*)$.

Now, if x^* is a rotund point of $B(X^*)$, then the only convex subset of $S(X^*)$ that contains x^* is the singleton $\{x^*\}$. Thus, if $x^* \in A$, then $A = \{x^*\}$ and B is an affine half-space.

Conversely, suppose there exists $y^* \in S(X^*) \setminus \{x^*\}$ such that $z^* = (x^* + y^*)/2 \in S(X^*)$.

Let $\{x_n\} \subset B(X)$ be such that $(x^* + y^*)(x_n) \to 2$. Then, in fact, $x^*(x_n) \to 1$ and $y^*(x_n) \to 1$.

Choose a sequence $\{\delta_n\}$ such that $\delta_n > 0$ for all n and $\sum_{n=1}^{\infty} \delta_n < 1$. Passing to a subsequence if necessary, we may assume $x^*(x_n) > 1 - \delta_n$ and $y^*(x_n) > 1 - \delta_n$.

Let $B_n = B(\sum_{i=1}^{n} x_i; n)$. Clearly $\{B_n\}$ is an unbounded nested sequence of balls. And, for any $n \geq 1$,
$$\inf x^*(B_n) = x^*\left(\sum_{i=1}^{n} x_i\right) - n = -\sum_{i=1}^{n} [1 - x^*(x_i)] > -\sum_{i=1}^{n} \delta_i > -1$$

And similarly, $\inf y^*(B_n) > -1$. Hence, $x^* \in A$, but $A \neq \{x^*\}$. ■

Remark 2.2. Observe that our proof does not use smoothness of two-dimensional quotients as in [8] or [18]. Nor does it use the Taylor-Foguel Theorem as in [14].

Definition 2.3. Let X be a Banach space.

(a) We say that $x \in S(X)$ is

(i) an LUR (resp. wLUR) point of $B(X)$ if for any $\{x_n\} \subset B(X)$, the condition
$$\lim_n \left\| \frac{x_n + x}{2} \right\| = 1$$
implies $\lim x_n = x$ (respectively, w-lim$x_n = x$).

(ii) an almost LUR (ALUR) (resp. weakly almost LUR (wALUR)) point of $B(X)$ if for any $\{x_n\} \subset B(X)$ and $\{x_m^*\} \subset B(X^*)$, the condition
$$\lim_m \lim_n x_m^* \left(\frac{x_n + x}{2} \right) = 1$$
implies $\lim x_n = x$ (respectively, w-lim$x_n = x$).

We say that a Banach space X has one of the above properties if every point of $S(X)$ has the same property.

(b) We say that $x^* \in S(X^*)$ is *a w*-ALUR point* (respectively, w*-wALUR point, w*-nALUR point) of $B(X^*)$ if for any $\{x_n^*\} \subset B(X^*)$ and $\{x_m\} \subset B(X)$, the condition

$$\lim_m \lim_n \left(\frac{x_n^* + x^*}{2} \right) (x_m) = 1$$

implies w*-$\lim x_n^* = x^*$ (respectively, w-$\lim x_n^* = x^*$, (norm-)$\lim x_n^* = x^*$).

We actually have proved in [4]

Theorem 2.4. *Let* X *be a Banach space. For* $x^* \in S(X^*)$, *the following are equivalent*:

(a) x^* *is a rotund point of* $B(X^*)$;

(b) x^* *is a w*-ALUR point of* $B(X^*)$;

(c) *for every unbounded nested sequence* $\{B_n\}$ *of balls such that* x^* *is bounded below on* $\cup B_n$, *if for any* $\{y_n^*\} \subset S(X^*)$, *the sequence* $\{\inf y_n^*(B_n)\}$ *is bounded below*, *then* $w^*\text{-}\lim y_n^* = x^*$;

(d) *for every unbounded nested sequence* $\{B_n\}$ *of balls such that* x^* *is bounded below on* $B = \cup B_n$, *if* $y^* \in S(X^*)$ *is also bounded below on* B, *then* $y^* = x^*$;

(e) *for every unbounded nested sequence* $\{B_n\}$ *of balls*, *if* x^* *is bounded below on* $B = \cup B_n$, *then* B *is an affine half-space determined by* x^*.

And here is a direct proof of the local version of Taylor-Foguel Theorem as in [3].

Theorem 2.5. *Let* X *be a Banach space. Then* $x^* \in S(X^*)$ *is a rotund point of* $B(X^*)$ *if and only if for all subspace* $Y \subset X$ *such that* $\|x^*|_Y\| = 1$, x^* *is the unique Hahn-Banach extension of* $x^*|_Y$ *to* X.

Proof. Let $Y \subset X$ be such that $\|x^*|_Y\| = 1$. If $x^*|_Y$ has another norm preserving extension y^* to X, then clearly, $\|y^*\| = \|(x^* + y^*)/2\| = 1$.

For the converse, we follow the arguments of [7]. Suppose there exists $y^* \in S(X^*) \setminus \{x^*\}$ such that $(x^* + y^*)/2 \in S(X^*)$. Let $Y = \{x \in X : x^*(x) = y^*(x)\}$. It clearly suffices to show that $\|x^*|_Y\| = 1$.

Let $\{x_n\} \subset S(X)$ be such that $(x^* + y^*)(x_n) \to 2$. Then, in fact, $x^*(x_n) \to 1$ and $y^*(x_n) \to 1$. Let $x_0 \in X$ be such that $(x^* - y^*)(x_0) = 1$. Then for each $n \geq 1$, $x_n = y_n + \alpha_n x_0$, where $y_n \in Y$ and $\alpha_n = (x^* - y^*)(x_n) \to 0$. It follows that $\|y_n\| \to 1$ and $x^*(y_n) \to 1$. This completes the proof. ∎

We actually have proved in [3]

Theorem 2.6. *Let* X *be a Banach space. For* $x^* \in S(X^*)$, *the following are equivalent*:

(a) x^* *is a rotund point of* $B(X^*)$;

(b) *for all subspace* $Y \subseteq X$ *such that* $\|x^*|_Y\| = 1$, *any*, *and hence all*, *of the following conditions holds*:

  (i) x^* *is the unique Hahn-Banach extension of* $x^*|_Y$ *to* X;

  (ii) *if* $x_0 \notin Y$, *then*

$$\sup\{x^*(y) - \|x_0 - y\| : y \in Y\} = \inf\{x^*(y) + \|x_0 - y\| : y \in Y\};$$

  (iii) *if* $x_0 \notin Y$ *and* $x^*(x_0) > \alpha$ (*respectively*, $x^*(x_0) < \alpha$) *for some* $\alpha \in \mathbb{R}$, *then there exists a closed ball* B *in* X *with centre in* Y *such that* $x_0 \in B$ *and* $\inf x^*(B) > \alpha$ (*respectively*, $\inf x^*(B) < \alpha$);

  (iv) *if* $\{x^*_\alpha\} \subseteq S(X^*)$ *is a net such that* $\lim_\alpha x^*_\alpha(y) = x^*(y)$ *for all* $y \in Y$, *then* w^*-$\lim x^*_\alpha = x^*$;

  (v) *if* $\{x^*_n\} \subseteq S(X^*)$ *is a sequence such that* $\lim_n x^*_n(y) = x^*(y)$ *for all* $y \in Y$, *then* $w^*\lim x^*_n = x^*$.

Proposition 2.7. *Let* X *be a Banach space. For* $x^* \in S(X^*)$, *the following are equivalent*:

(a) x^* *is a rotund point of* $B(X^*)$;

(b) *for all subspace* $Y \subseteq X$ *such that* $\|x^*|_Y\| = 1$, $x^*|_Y$ *is a rotund point of* $B(Y^*)$;

(c) *for all separable subspace* $Y \subseteq X$ *such that* $\|x^*|_Y\| = 1$, $x^*|_Y$ *is a rotund point of* $B(Y^*)$.

Corollary 2.8. *Having a strictly convex dual is a separably determined property. That is*, *for a Banach space* X, X^* *is strictly convex if and only if for all separable subspaces* $Y \subseteq X$, Y^* *is strictly convex*.

This observation appears to be new.

A recent result of [10] shows that a Banach space X is σ-fragmentable if every $x \in S(X)$ is, in our terminology, a rotund point of $B(X^{**})$. And they asked to characterize this property. As a consequence of Theorem 2.4, we also observe in [4, Corollary 8] the following

Theorem 2.9. *Let* X *be a Banach space. For* $x \in S(X)$, *the following are equivalent*:

(a) x *is a rotund point of* $B(X^{**})$;

(b) x *is a w*-ALUR point of* $B(X^{**})$;

(b$'$) x *is a wALUR point of* $B(X)$;

(c) *for every unbounded nested sequence* $\{B^*_n\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B^*_n$, *if for any* $\{y^{**}_n\} \subseteq S(X^{**})$, *the sequence* $\{\inf y^{**}_n(B^*_n)\}$ *is bounded below, then* $w^*\text{-}\lim y^{**}_n = x$;

(c$'$) *for every unbounded nested sequence* $\{B^*_n\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B^*_n$, *if for any* $\{y_n\} \subseteq S(X)$, *the sequence* $\{\inf y_n(B^*_n)\}$ *is bounded below, then* w-lim$y_n = x$;

(d) *for every unbounded nested sequence* $\{B^*_n\}$ *of balls in* X^* *such that* x *is bounded below on* $B^* = \cup B^*_n$, *if any* $x^{**} \in S(X^{**})$ *is also bounded below on* B^*, *then* $x = x^{**}$;

(e) *for every unbounded nested sequence* $\{B^*_n\}$ *of balls in* X^*, *if* x *is bounded below on* $B^* = \cup B^*_n$, *then* B^* *is an affine half-space determined by* x.

3. More on Rotund Points

We start with an elementary characterization of rotund points.

Definition 3.1. The duality mapping D for a Banach space X is the set-valued map from $S(X)$ to $P(S(X^*))$ defined by

$$D(x) = \{x^* \in S(X^*) : x^*(x) = 1\}, \quad x \in S(X).$$

Lemma 3.2. *Let* X *be a Banach space.* $x \in S(X)$ *is a rotund point of* $B(X)$ *if and only if* x *is exposed by every* $x^* \in D(x)$.

Corollary 3.3. *Let* $x \in S(X)$ *be an exposed point as well as a smooth point of* $B(X)$. *Then* x *is a rotund point of* $B(X)$.

Another way to emphasize the difference of rotund points and extreme points is via the duality.

Proposition 3.4. *Let* $x \in S(X)$, $x^* \in D(x)$. *Consider the following statements*:

(a) x^* *is a rotund point of* $B(X^*)$,

(b) x *is a smooth point of* $B(X)$,

(c) x^* *is an extreme point of* $B(X^*)$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *and none of the converse is true in general.*

Proposition 3.5. *Let* X *be a Banach space. For* $x \in S(X)$, *the following are equivalent* :

(a) x *is a wALUR point of* $B(X)$;

(b) x *is w*-exposed in* $B(X^{**})$ *by every* $x^* \in D(x)$;

(c) *for every* $x^* \in D(x)$, *w*-slices of* $B(X^{**})$ *determined by* x^* *form a local base for* $(B(X^{**}), w^*)$ *at* x;

(d) *for every* $x^* \in D(x)$, *slices of* $B(X)$ *determined by* x^* *form a local base for* $(B(X), weak)$ *at* x;

(e) *for every* $x^* \in D(x)$ *and for any* $\{x_n\} \subseteq S(X)$, *if* $x^*(x_n) \to 1$, *then* w-lim $x_n = x$.

Definition 3.6. Let $K \subseteq X$ be a closed bounded convex set. A point $x \in K$ is said to be a point of continuity (PC) of K if x is a point of continuity of the identity map from (K, w) to $(K, \|\cdot\|)$.

Corollary 3.7. *Let* X *be a Banach space. For* $x \in S(X)$, *the following are equivalent* :

(a) x *is an ALUR point of* $B(X)$;

(b) x *is a wALUR point as well as a PC of* $B(X)$;

(c) *For every* $x^* \in D(x)$, *for any* $\{x_n\} \subseteq S(X)$, *if* $x^*(x_n) \to 1$, *then* $\lim x_n = x$;

(d) x *is strongly exposed in* $B(X)$ *by every* $x^* \in D(x)$;

(e) *for every unbounded nested sequence* $\{B_n^*\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B_n^*$, *if for any* $\{y_n\} \subseteq S(X)$, *the sequence* $\{\inf y_n(B_n^*)\}$ *is bounded below*, *then* $\lim y_n = x$.

Remork 3.8. Clearly, if x is an ALUR point of $B(X)$, then x is a rotund point as well as a PC of $B(X)$. Is the converse true? Notice that it would suffice to show that if x is a rotund point as well as a PC of $B(X)$, then x is a rotund point of $B(X^{**})$. Recall that if x is an extreme point as well as a PC of $B(X)$, then x is an extreme point of $B(X^{**})$ [13]. On the other hand, if x is an exposed point as well as a PC of $B(X)$, then x is not necessarily an exposed point of $B(X^{**})$ [1].

Clearly, an ALUR Banach space is strictly convex as well as Kadec and therefore has a LUR renorming. Is the same true of wALUR spaces?

Talking of renormings, it is easy to see that ℓ_1 or ℓ_∞ sums of nonzero Banach spaces cannot have any rotund points. It follows that every Banach space of dimension ≥ 2 has a renorming that lacks rotund points. Contrast this with the fact that a Banach space has the RNP if and only if the unit ball of every renorming contains a strongly exposed point. It also follows that having rotund points is not a three space property.

4. Straight Nested Sequence of Balls

Definition 4.1. An unbounded nested sequence of balls $\{B(x_n, r_n)\}$ in X is called straight if there exist $x \in S(X)$ and $\lambda_n > 0$ such that $x_n = \lambda_n x$, $n \in \mathbb{N}$. Such x is called the direction of this sequence.

Definition 4.2. $x \in S(X)$ is called a smooth (resp., very smooth, Fréchet smooth) point of $B(X)$ if for every $x^* \in D(x)$ and $\{x_n^*\} \subseteq B(X^*)$, the condition $\lim_n x_n^*(x) = 1$ implies w*-$\lim_n x_n^* = x^*$ (resp. w-$\lim_n x_n^* = x^*$, $\lim_n x_n^* = x^*$). X is said to be smooth (resp. very smooth, Fréchet smooth) if every point of $S(X)$ is a smooth (resp. very smooth, Fréchet smooth) point of $B(X)$.

Smooth, very smooth, Fréchet smooth points of $B(X)$ can be characterized in terms of straight unbounded nested sequence of balls similar to Theorem 2.4. This was obtained in [3].

Theorem 4.3. *Let* X *be a Banach space. For* $x \in S(X)$, *the following are equivalent* :

I. (a) x *is a smooth point of* $B(X)$;

(b) *for every straight unbounded nested sequence* $\{B_n\}$ *of balls in the direction of* x, *if for any* $x^*, y_n^* \in S(X^*)$, x^* *is bounded below on* $\cup B_n$ *and the sequence* $\{\inf y_n^*(B_n)\}$ *is bounded below, then* $w^*\text{-}\lim y_n^* = x^*$;

(c) *for every straight unbounded nested sequence* $\{B_n\}$ *of balls in the direction of* x, *if for any* $x^*, y^* \in S(X^*)$, *both* x^* *and* y^* *are bounded below on* $\cup B_n$, *then* $x^* = y^*$;

(d) *for every straight unbounded nested sequence* $\{B_n\}$ *of balls in the direction of* x, $B = \cup B_n$ *is either the whole of* X *or an affine half-space in* X.

II. (a) x *is a very smooth point of* $B(X)$;

(b) *for every straight unbounded nested sequence* $\{B_n\}$ *of balls in the direction of* x, *if for any* $x^*, y_n^* \in S(X^*)$, x^* *is bounded below on* $\cup B_n$ *and the sequence* $\{\inf y_n^*(B_n)\}$ *is bounded below, then w-lim*$y_n^* = x^*$;

(c) *for every straight unbounded nested sequence* $\{B_n^{**}\}$ *of balls in* X^{**} *in the direction of* x, $\cup B_n^{**}$ *is either the whole of* X^{**} *or an affine half-space in* X^{**}.

III. (a) x *is a Fréchet smooth point of* $B(X)$;

(b) for every straight unbounded nested sequence $\{B_n\}$ of balls in the direction of x, if for any $x^*, y_n^* \in S(X^*)$, x^* is bounded below on $\cup B_n$ and the sequence $\{\inf y_n^*(B_n)\}$ is bounded below, then $\lim y_n^* = x^*$.

Proposition 4.4. *Let* X *be a Banach space. For* $x \in S(X)$, *the following are equivalent* :

I. (a) x *is a wALUR point of* $B(X)$;

(b) for every straight unbounded nested sequence $\{B_n^*\}$ of balls in X^* such that x is bounded below on $\cup B_n^*$, if for any $\{y_n\} \subseteq S(X)$, the sequence $\{\inf y_n(B_n^*)\}$ is bounded below, then w-lim$y_n = x$.

II. (a) x *is an ALUR point of* $B(X)$;

(b) *for every straight unbounded nested sequence* $\{B_n^*\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B_n^*$, *if for any* $\{y_n\} \subseteq S(X)$, *the sequence* $\{\inf y_n(B_n^*)\}$ *is bounded below*, *then* $\lim y_n = x$.

Definition 4.5. A subset $B \subseteq S(X^*)$ is a boundary for X if for every $x \in S(X)$, $B \cap D(x) \neq \emptyset$.

Corollary 4.6. *Let* X *be a Banach space.*

I. (a) X *is wALUR if and only if every* $x^* \in D(S(X))$ *is a smooth point of* $B(X^*)$. *In particular*, *if* X^* *is smooth*, *then* X *is wALUR.*

(b) X *is ALUR if and only if every* $x^* \in D(S(X))$ *is a Fréchet smooth point of* $B(X^*)$. *In particular*, *if* X^* *is Fréchet smooth*, *then* X *is ALUR.*

II. (a) *If rotund points of* $B(X^*)$ *form a boundary for* X (*in particular*, *if* X^* *is rotund*); *then* X *is smooth.*

(b) *If wALUR points of* $B(X^*)$ *form a boundary for* X (*in particular*, *if* X^* *is wALUR*); *then* X *is very smooth.*

(c) *If ALUR points of* $B(X^*)$ *form a boundary for* X (*in particular, if* X^* *is ALUR*)*, then* X *is Fréchet smooth.*

Remark 4.7. If every $x^* \in D(S(X))$ is a very smooth point of $B(X^*)$, then what is the exact rotundity condition that we get in X? We will answer this at the end of the next section. Clearly that would be a notion between wALUR and ALUR. Observe that the condition X^* is very smooth already implies the reflexivity of X and therefore, we have X^* is very smooth if and only if X is rotund (wALUR) and reflexive.

5. Extending Vlasov's Theorem

Starting from Vlasov's result, Sullivan [16] introduced a stronger property, called Property (V) (called Vlasov Property in [5]). The following reformulation of the definition comes from [5, Proposition 3.1].

Definition 5.1. A Banach space X is said to have the Vlasov Property, if for every unbounded nested sequence $\{B_n\}$ of balls and x^*, $y_n^* \in S(X^*)$, if x^* is bounded below on $\cup B_n$, and the sequence $\{\inf y_n^*(B_n)\}$ is bounded below, or, specifically, if there exists $c \in \mathbb{R}$ such that

(1) $$x^*(b) \geq c \quad \text{for all } b \in \cup B_n,$$

(2) $$y_k^*(b) \geq c \quad \text{for all } b \in B_n;\ n \leq k,$$

then w-lim$y_n^* = x^*$.

Let us observe that if $\{y_k^*\}$ satisfies (2) and x^* is a cluster point of $\{y_k^*\}$ in any compatible vector topology on X^*, then x^* satisfies (1).

In [16], it is shown that X has the Vlasov Property if and only if X is Hahn-Banach Smooth and X^* is strictly convex. In [2], this characterization was used to show that the Vlasov Property is equivalent to w*-ANP-II$'$.

Definition 5.2. (a) A subset Γ of $B(X^*)$ is called a norming set for X if $\|x\| = \sup_{x^* \in \Gamma} x^*(x)$ for all $x \in X$.

(b) A sequence $\{x_n\}$ in $B(X)$ is said to be asymptotically normed by Γ if for any $\varepsilon > 0$, there exists a $x^* \in \Gamma$ and $N \in \mathbb{N}$ such that $x^*(x_n) > 1 - \varepsilon$ for all $n \geq N$.

(c) For α = I, II, II$'$ or III, a sequence $\{x_n\}$ in X is said to have the property α if

I. $\{x_n\}$ is convergent,

II. $\{x_n\}$ has a convergent subsequence,

II$'$. $\{x_n\}$ is weakly convergent,

III. $\{x_n\}$ has a weakly convergent subsequence.

(d) For κ = I, II, II$'$ or III, X is said to have the asymptotic norming property κ with respect to Φ (Φ-ANP-κ), if every sequence in $B(X)$ that is asymptotically normed by Φ has property κ.

(e) A sequence $\{x_n^*\}$ in X^* is said to have the property IV if $\{x_n^*\}$ is w*-convergent.

(f) For κ = I, II, II$'$, III or IV, X is said to have the w*-ANP-κ, if every sequence in $B(X^*)$ that is asymptotically normed by $B(X)$ has property κ.

Remark 5.3. The original definition of Φ-ANP-III was different. The equivalence with the one above was established in [11, Theorem 2.3].

For various geometric notions related to w*-ANPs, refer to [11, 12]. The Φ-ANP-II$'$ and w*-ANP-II$'$ were introduced and studied in [2]. The w*-ANP-IV is new. In particular, we recall the following result from [11, Theorem 3.1] and [2, Theorem 3.1]

Theorem 5.4. *A Banach space* X

(a) *has w*-ANP-I if and only if* X^* *is strictly convex and* $(S(X^*), w^*) = (S(X^*), \|\cdot\|)$.

(b) *has w*-ANP-II if and only if* $(S(X^*), w^*) = (S(X^*), \|\cdot\|)$.

(c) *has w*-ANP-II$'$ if and only if* X^* *is strictly convex and* $(S(X^*), w^*) = (S(X^*), w)$.

(d) *has w*-ANP-III if and only if* $(S(X^*), w^*) = (S(X^*), w)$ *if and only if* X *is Hahn-Banach smooth.*

Observe that in the definition of the Vlasov Property, if we replace "w-$\lim y_n^* = x^*$" by "w*-$\lim y_n^* = x^*$" then by Theorem 2.4, we simply get X^* is strictly convex. It was observed in [5] that if we replace it by "$\lim y_n^* = x^*$" then we get w*-ANP-I. Indeed, as observed in [5], if we replace it by $\{y_n^*\}$ has property κ, then for κ = I, II$'$ or IV, we get the corresponding w*-ANPs. But for κ = II or III, we do not get anything new. Indeed, strict convexity of X^* remains. So we need some modification, as was considered in [5].

Definition 5.5 [5]. A Banach space X has property V-κ (κ = I, II, II$'$, III or IV), if for every unbounded nested sequence $\{B_n\}$ of balls, and $\{y_n^*\} \subseteq S(X^*)$ if the sequence $\{\inf y_n^*(B_n)\}$ is bounded below, i.e., Condition (2) in Definition 5.1 is satisfied, then $\{y_n^*\}$ has property κ (κ = I, II, II$'$, III or IV).

In [5], the authors show that the above "Vlasov-like" Properties are equivalent to the w*-ANPs by observing that if for some unbounded nested sequence $\{B_n\}$ of balls, and $\{y_n^*\} \subseteq S(X^*)$, the sequence $\{\inf y_n^*(B_n)\}$ is bounded below, then $\{y_n^*\}$ is asymptotically normed by $B(X)$. In particular, they show

Theorem 5.6 [5, Theorem 3.9]. *A Banach space* X *has w*-ANP-*κ *if and only if* X *has* V-κ (κ = *I, II, II*$'$ *or III*).

We observe in [3] that V-IV and w*-ANP-IV are also equivalent and, as expected, equivalent to the strict convexity of X^*. That is,

Proposition 5.7. *For a Banach space* X, *the following are equivalent*:

(a) X *has w*-ANP-IV*;

(b) X *has* V-*IV*;

(c) X^* *is strictly convex*.

In attempting to localize these properties, we observe that the formulation of the Vlasov Property is readily localized as : $x^* \in S(X^*)$ is a V-κ (κ = I, II$'$ or IV) point of $B(X^*)$, if for every unbounded nested sequence $\{B_n\}$ of balls such that x^* satisfies (1), if for any $\{y_n^*\} \subseteq S(X^*)$, (2) is satisfied, then $y_n^* \to x^*$ in w*, weak or norm topology, respectively. From Theorem 2.4 again, a V-IV point is simply a rotund point of $B(X^*)$. Later we will identify V-I and II$'$ points as respectively w*-nALUR and w*-wALUR points of $B(X^*)$. But again similar localization for II or III does not work. We can get an alternative localization for III via w*-w PCs. But localization for II appears to be much more difficult.

We now give a reformulation of rotund points which makes the role of strict convexity of X^* in the discussion on w*-ANP more transparent.

Theorem 5.8. *Let* X *be a Banach space. For* $x^* \in S(X^*)$, *the following are equivalent*:

(a) x^* *is a rotund point of* $B(X^*)$;

(b) *for any* $\{x_n^*\} \subseteq B(X^*)$, *if* $\{(x_n^* + x^*)/2\}$ *is asymptotically normed by* $B(X)$, *then w*-lim* $x_n^* = x^*$.

It follows the following

Proposition 5.9. *Let* X *be a Banach space. For* $x \in S(X)$, *the following are equivalent*:

I. (a) x *is a wALUR point of* $B(X)$;

(b) *for any* $\{x_n\} \subseteq B(X)$, *if* $\{(x_n + x)/2)\}$ *is asymptotically normed by* $B(X^*)$, *then w-lim*$x_n = x$.

II. (a) x *is an ALUR point of* $B(X)$;

(b) *for any* $\{x_n\} \subseteq B(X)$, *if* $\{(x_n + x)/2)\}$ *is asymptotically normed by* $B(X^*)$, *then* $\lim x_n = x$.

Definition 5.10. Let $K \subseteq X^*$ be a closed bounded convex set.

(a) A point $x^* \in K$ is said to be a weak* point of continuity (w* PC) (respectively, weak*-weak point of continuity (w*-w PC)) of K if x^* is a point of continuity of the identity map from (K, w^*) to $(K, \|\cdot\|)$ (respectively, (K, w)).

(b) A point $x^* \in K$ is said to be a weak* point of sequential continuity (w* seq PC) (respectively, weak*-weak point of sequential continuity (w*-w seq PC)) of K if $\{x_n^*\} \subseteq K$ and w*-lim$x_n^* = x^*$ implies $\lim x_n^* = x^*$ (respectively, w-lim$x_n^* = x^*$).

The Taylor-Foguel Theorem says that X^* is strictly convex if and only if every subspace Y of X is a U-subspace of X, while X is Hahn-Banach Smooth if and only if X is a U-subspace of X^{**}. It follows that X^* is strictly convex and X is Hahn-Banach Smooth if and only if every subspace Y of X is a U-subspace of X^{**}. The following local version of this phenomenon was obtained in [3].

Theorem 5.11. *Let* X *be a Banach space. For* $x^* \in S(X^*)$, *the following are equivalent*:

(a) x^* *is a rotund point of* $B(X^*)$ *as well as a w*-w PC of* $B(X^*)$;

(b) x^* *is a rotund point of* $B(X^*)$ *as well as a w*-w seq PC of* $B(X^*)$;

(c) *for every unbounded nested sequence* $\{B_n\}$ *of balls in* X *such that* x^* *is bounded below on* $\cup B_n$, *if for any* $\{y_n^*\} \subseteq S(X^*)$, *the sequence* $\{\inf y_n^*(B_n)\}$ *is bounded below, then w-lim*$y_n^* = x^*$;

(d) *for every unbounded nested sequence* $\{B_n^{**}\}$ *of balls in* X^{**} *with centres in* X *such that* x^* *is bounded below on* $\cup B_n^{**}$, *if any* $x^{***} \in S(X^{***})$ *is also bounded below on* $\cup B_n^{**}$, *then* $x^{***} = x^*$;

(e) *for every unbounded nested sequence* $\{B_n^{**}\}$ *of balls in* X^{**} *with centres in* X *such that* x^* *is bounded below on* $\cup B_n^{**}$, $\cup B_n^{**}$ *is an affine half-space in* X^{**} *determined by* x^*;

(f) x^* *is a w*-wALUR point of* $B(X^*)$;

(g) *for all subspace* $Y \subseteq X$ *such that* $\|x^*|_Y\| = 1$; *any of the following conditions holds.*

(i) x^* *is the unique Hahn-Banach extension of* $x^*|_Y$ *to* X^{**};

(ii) *if* $\{x^*_\alpha\} \subseteq S(X^*)$ *is a net such that* $\lim_\alpha x^*_\alpha(y) = x^*(y)$ *for all* $y \in Y$; *then w-lim*$x^*_\alpha = x^*$;

(iii) *if* $\{x^*_n\} \subseteq S(X^*)$ *is a sequence such that* $\lim_n x^*_n(y) = x^*(y)$ *for all* $y \in Y$; *then w-lim*$x^*_n = x^*$.

By Theorems 2.4 and 5.11, we have the following :

Corollary 5.12. *Let* X *be a Banach space. If* $x^* \in S(X^*)$ *is a rotund point of* $B(X^{***})$; *then* x^* *is a rotund point of* $B(X^*)$ *as well as a w*-w PC of* $B(X^*)$. *In particular; if* X^{***} *is strictly convex; then* X^* *is strictly convex and* X *is Hahn-Banach Smooth.*

Is the converse of any of the above results true?

Remark 5.13. It follows that the well-known result that X^{***} strictly convex implies X is Hahn-Banach Smooth [16] is again a consequence of properties of rotund points of $B(X^{***})$.

Replacing the weak topology by the norm topology in the above Theorem, we immediately obtain

Corollary 5.14. *Let* X *be a normed linear space. For* $x^* \in S(X^*)$; *the following are equivalent* :

(a) x^* *is a rotund point of* $B(X^*)$ *as well as a w* PC of* $B(X^*)$;

(b) x^* *is a rotund point of* $B(X^*)$ *as well as a w* seq PC of* $B(X^*)$;

(c) *for every unbounded nested sequence* $\{B_n\}$ *of balls such that* x^* *is bounded below on* $\cup B_n$; *if for any* $\{y^*_n\} \subseteq S(X^*)$; *the sequence* $\{\inf y^*_n(B_n)\}$ *is bounded below, then* $\lim y^*_n = x^*$;

(d) x^* *is a w*-nALUR point of* $B(X^*)$;

(e) *for all subspace* $Y \subseteq X$ *such that* $\|x^*|_Y\| = 1$; *any of the following conditions holds:*

(i) *if* $\{x^*_\alpha\} \subseteq S(X^*)$ *is a net such that* $\lim_\alpha x^*_\alpha(y) = x^*(y)$ *for all* $y \in Y$; *then* $\lim x^*_\alpha = x^*$;

(ii) *if* $\{x_n^*\} \subseteq S(X^*)$ *is a sequence such that* $\lim_n x_n^*(y) = x^*(y)$ *for all* $y \in Y$; *then* $\lim x_n^* = x^*$.

We now answer the question raised at the end of Section 4.

Corollary 5.15. *Let* X *be a Banach space. For* $x \in S(X)$; *the following are equivalent*:

(a) x *is a wALUR point of* $B(X)$ *as well as a w*-w PC of* $B(X^{**})$;

(b) x *is a wALUR point of* $B(X)$ *as well as a w*-w seq PC of* $B(X^{**})$;

(c) *every* $x^* \in D(x)$ *is a very smooth point of* $B(X^*)$;

(d) *for every* $x^* \in D(x)$; *w*-slices of* $B(X^{**})$ *determined by* x^* *form a local base for* $(B(X^{**}); w)$ *at* x;

(e) *for every* $x^* \in D(x)$ *and for any* $\{x_n^{**}\} \subseteq S(X^{**})$; *if* $x_n^{**}(x^*) \to 1$; *then* *w-lim*$x_n^{**} = x$;

(f) *for every unbounded nested sequence* $\{B_n^*\}$ *of balls in* X^* *such that* x *is bounded below on* $\cup B_n^*$; *if for any* $\{y_n^{**}\} \subseteq S(X^{**})$; *the sequence* $\{\inf y_n^{**}(B_n^*)\}$ *is bounded below*; *then w-lim*$y_n^{**} = x$;

(g) *for every unbounded nested sequence* $\{B_n^{***}\}$ *of balls in* X^{***} *with centres in* X^* *such that* x *is bounded below on* $\cup B_n^{***}$; $\cup B_n^{***}$ *is an affine half-space in* X^{***} *determined by* x;

(h) x *is a w*-wALUR point of* $B(X^{**})$;

(i) *for any* $\{x_n^{**}\} \subseteq B(X^{**})$; *if* $\{(x_n^{**} + x)/2)\}$ *is asymptotically normed by* $B(X^*)$; *then w-lim*$x_n^{**} = x^*$.

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Pradipta Bandyopadhyay
Stat-Math Division Indian Statistical Institute
203, B. T. Road, Calcutta 700 035, India,
E-mail: pradipta@isical.ac.in

Bor-Luh Lin
Department of Mathematics, The University of Iowa
Iowa City, IA 52242 USA
E-mail: bllin@math.uiowa.edu