A BOUNDARY CROSSING PROBLEM WITH APPLICATION TO SEQUENTIAL ESTIMATION

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Abstract. Let $X_1, X_2 \ldots$ be independent and identically distributed random variables with density $f(x) = \alpha x^{\alpha-1}, 0 < x < 1$ where α is a fixed positive number. Let $N_c = \inf\{j \geq m : M_j \leq (j/c)^\beta\}$ where $M_j = \max(X_1, \ldots, X_j)$ and m is a fixed positive integer. We study the properties of N_c as $c \to \infty$. As an application, we consider the problem of estimating sequentially the range of the uniform distribution and study the second order properties of an appropriate estimate.

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1. Introduction and main results. In many sequential estimation problems, the relevant stopping variable may be written in the form $N_c = \inf\{n \geq m : Z_n > c\}$ where $S_n = X_1 + \ldots + X_n$, $Z_n = S_n + \xi_n$ and $\{\xi_n\}$ is an appropriate slowly changing sequence. Probabilistic properties of N_c have been studied in the fundamental papers of Lai and Siegmund (1977, 1979). See also Woodroofe (1982). In particular, if the distribution of X_1 is nonarithmetic then as $c \to \infty$, the joint distribution of N_c and the overshoot $Z_{N_c} - c$ are asymptotically independent with the former being asymptotically normal (after appropriate centering and scaling) . This fact has been exploited by many researchers to derive second order properties of sequential estimates. See for example Woodroofe (1976, 1977, 1982) and the books by Ghosh et.al. (1997), Mukhopadhyay and Solanki (1994) and Siegmund (1985). For a version of this result for the arithmetic case see Siegmund (1985).

Even though the process of partial maximums has received much attention from probabilists, results such as above are apparently not known for maximums. In this article we deal with this in a limited framework.

Let $X_1, X_2, \ldots$ be i.i.d., with $P(0 < X_1 < 1) = 1$. Let for $n \geq 1$,

$$M_n = \max(X_1, X_2, \ldots, X_n).$$

For any positive integer m, and for positive numbers β and c, define

$$(1.1), \qquad N \equiv N_c = \inf\left\{n \geq m : M_n \leq (n/c)^\beta\right\}$$

$$[c] = \text{ integer part of } c, \quad < c >= c - [c] = \text{ fractional part of } c$$

Clearly, for $c \geq m$, $P(N_c \leq 1 + [c]) = 1$ and if c is an integer, $P(N_c \leq c) = 1$. Let

$$\tilde{N}_c = N_c/[c]$$

THEOREM 1.1. *Let $P(0 < X_1 < 1) = 1$, and $c \to \infty$. Let S be the supremum of the support of the distribution of X_1. Then*

(i) $\tilde{N}_c \to S^{1/\beta}$ almost surely.

(ii) $\tilde{N}_c^p$ is uniformly integrable for every $p > 0$.

Let

$$N_c^* = [c] - N_c \text{ and } \; M_{N_c}^* = c(1 - M_{N_c}^{1/\beta})$$

Let Z denote a random variable with density

$$f_Z(z) = \alpha\beta e^{-\alpha\beta z}, \; z > 0.$$

where α and β are positive integers. Let $\longrightarrow^{\mathcal{D}}$ denote convergence in distribution. For a restricted class of distributions of X_1, we have the following result.

THEOREM 1.2. *Suppose $f_{X_1}(x) = \alpha x^{\alpha-1}, 0 < x < 1$, and α is fixed. Suppose $c \to \infty$ such that $< c > \to \epsilon$. Then*

(i) $Z_c^* = (\; N_c^*, \; M_{N_c}^* \;) \longrightarrow^{\mathcal{D}} ([Z - \epsilon], \;\; Z)$
In particular if $c \to \infty$ through integers, then $Z_c^* \longrightarrow^{\mathcal{D}} ([Z], \;\; Z)$.

(ii) For any $p > 0$, $\{|N_c^*|^p\}$ and $\{(M_{N_c}^*)^p\}$ are uniformly integrable if $m > \frac{p}{\alpha\beta}$.

COROLLARY 1.3. *If $m > \frac{1}{\alpha\beta}$ and $c \to \infty$ such that $< c > \to \epsilon$, then*

$$E([c] - N_c) = \frac{-1 + e^{-\alpha\beta} + e^{-\alpha\beta\epsilon}}{1 - e^{-\alpha\beta}} + o(1).$$

The asymptotic distribution of the *overshoot* is now easy to derive by using Theorem 1.2. First of all note that $M_N \le (\frac{N}{c})^\beta \Leftrightarrow N \ge cM_N^{1/\beta}$. Thus the overshoot may be defined as O_{1c} or O_{2c} below.

$$O_{1c} = N - cM_N^{1/\beta} \text{ or } \quad O_{2c} = (\frac{N}{c})^\beta - M_N \tag{1.2}$$

Note that asymptotically (in distribution), as $c \to \infty$,

$$\begin{aligned} O_{2c} &= \left[1 - (1 - \frac{N}{c})\right]^\beta - M_N \\ &\approx 1 - \beta(1 - \frac{N}{c}) - M_N \\ &= 1 - M_N - \beta\frac{(c-N)}{c} \end{aligned}$$

Hence asymptotically, $cO_{2c} \approx c\beta(1 - M_N^{1/\beta}) - \beta(c - N) = \beta O_{1c}$. Using the first part of Theorem 1.2, we have

COROLLARY 1.4. *If $c \to \infty$ such that $< c > \to \epsilon$, then*

$$O_{1c} \longrightarrow^{\mathcal{D}} (Z - [Z - \epsilon] - \epsilon), \tag{1.3}$$

$$cO_{2c} \longrightarrow^{\mathcal{D}} \beta(Z - [Z - \epsilon] - \epsilon) \tag{1.4}$$

Further, O_{2c}^p is uniformly integrable for all $p > 0$ and O_{1c}^p for $p > 0$ is uniformly integrable if $m > \frac{p}{\alpha\beta}$.

The behaviour for the maximums contrasts with the Lai-Seigmund results in the following ways: First, the limit depends on how $c \to \infty$ and the limiting distribution of a suitably normalised N_c is discrete. Second, the limiting marginals are not independent and are functionally related. Finally, the normalisations are completely different in this case.

In Section 2, we give the proofs of the above results. In Section 3, we give an application to a sequential estimation problem and solve a long standing problem.

2. Proofs of Theorems 1.1 and 1.2. It is easily seen that *almost surely*,

$$N_c \to \infty \text{ and hence } M_{N_c} \to S. \tag{2.1}$$

By definition (1.1) of N_c, as $c \to \infty$,

$$\frac{N_c - 1}{c} \le M_{N_c-1}^{1/\beta} \le M_{N_c}^{1/\beta} \le \frac{N_c}{c} \le 2 \tag{2.2}$$

Letting $c \to \infty$ in (2.2), (*i*) of Theorem 1.1 follows. The uniform integrability claimed in (*ii*) of Theorem 1.1 also follows from (2.2). This establishes Theorem 1.1.

PROOF OF THEOREM 1.2. Note that for fixed c, the distribution of N_c^* is discrete and its minimum value can be -1. On the other hand, the distribution of $M_{N_c}^*$ is continuous. For $j = -1, 0, 1, \ldots$ and $x \ge 0$, Let

$$G_n(j,\ x) = P[\ N_c^* = j,\ M_{N_c}^* \le x\ ]$$

and

$$G(j,\ x) = P[\ [Z - \epsilon] = j,\ Z \le\ x\ ]$$

To establish part (*i*), we need to prove that for all j and x,

$$G_n(j,\ x) \to G(j,\ x).$$

Now for $j = -1, 0, 1 \ldots$, $[Z - \epsilon] = j$ yields $j + \epsilon < Z < j + 1 + \epsilon$. With this in mind, for $j = -1, 0, 1, \ldots$, and for $x,\ y$ positive real numbers with $x + y \le 1$ let,

$$p_j(x,\ y) = P([Z - \epsilon] = j,\ j + y + \epsilon < Z < j + 1 + \epsilon - x) = e^{-\alpha\beta(j+y+\epsilon)} - e^{-\alpha\beta(j+1+\epsilon-x)}$$

The distribution G can be identified with the class of probabilities $\{p_j(x,\ y)\}$. Note that if $j = -1$, then the minimum value of y is actually $(1 - \epsilon)$.

Let the corresponding probabilities for G_n be

$$p_{n,\ j}(x,\ y) = P\left\{N_c^* = j,\ j + \epsilon + y < M_{N_c}^* < j + 1 + \epsilon - x\right\}.$$

We first show (2.3) given below.

$$\text{(2.3)} \qquad \limsup\, p_{n,\,j}(x,\ y) \leq p_j(x,\ y)$$

for all $j \geq 0$ and $x,\ y$ positive real numbers with $x + y \leq 1$.

For ease of notations in the proof, let us write $[c] = n$ and $\epsilon_n = c - [c]$. So n is the integer part of c and ϵ_n is the fractional part of c.

Using the fact that the event in (2.3) implies $N_c = n - j$ and then using the definition (1.1), it easily follows that $p_{nj}(x,\ y)$ is *bounded above* by

$$P\{M_{n-j} \leq \left(\frac{n-j}{c}\right)^{\beta},\ (1 - \frac{j+1+\epsilon-x}{c})^{\beta} < M_{n-j} < (1 - \frac{j+\epsilon+y}{c})^{\beta}\}$$

Using $\epsilon_n \to \epsilon$ and $y > 0$, it can be easily checked that for all large n,

$$(1 - \frac{j+\epsilon+y}{c}) \leq \frac{n-j}{c}.$$

Thus, the above probability for large n equals

$$P\left\{\left(1 - \tfrac{j+1+\epsilon-x}{c}\right)^{\beta} < M_{n-j} < \left(1 - \tfrac{j+\epsilon+y}{c}\right)^{\beta}\right\}$$
$$\to\ e^{-\alpha\beta(j+y+\epsilon)} - e^{-\alpha\beta(j+1+\epsilon-x)} = p_j(x,\ y).$$

It can be shown that

$$\text{(2.4)} \qquad \{N_c^*,\ M_{N_c}^*\} \text{ is a tight sequence .}$$

This is a by product of the proof of uniform integrability given later for part (*ii*). Details are given at the end of that proof.
Thus, every subsequence of it has a further subsequence which converges to say $(C_1,\ C_2)$. Let $L_j(x,\ y) = P(C_1 = j,\ j + \epsilon + y < C_2 < j + 1 + \epsilon - x)$.

Recalling the convergence in distribution criteria for *open* sets, it follows that along this subsequence,

$$\liminf\, p_{n,\,j}(x,\ y) \geq L_j(x,\ y).$$

Now using (2.3), it follows that, $L_j(x,\ y) \leq p_j(x,\ y)$ for all j, and $x,\ y$. Since both L and p define proper distributions, they must be equal. But the latter distribution is indeed the required one. This proves that every subsequence has a convergent subsequence which converges to the required limit. Thus the orginal sequence converges to the required limit, proving (i).

We now show the uniform integrability claimed in (ii). We first show that $|N_c^*|^p I_{\{N_c \geq n/2\}} = |n - N_c|^p I_{\{N_c \geq n/2\}}$ is uniformly integrable for all $p > 0$.
In the following argument, assume that J is suficiently large.

$$\begin{aligned} P\left\{N_c^* \geq J \text{ and } N_c \geq n/2\right\} &\leq \sum_{j=[n/2]}^{n-J} P\left\{M_j \leq (\frac{j}{n+\epsilon_n})^\beta\right\} \\ &= \sum_{j=[\frac{n}{2}]}^{n-J} P\left\{M_j \leq (\frac{j}{n+\epsilon_n})^\beta\right\} \\ &= T_1 \text{ say} \end{aligned}$$

For a generic constant K, using the inequality $ex \leq \exp(x)$ for $0 \leq x \leq 1$,

$$\begin{aligned} T_1 &\leq \sum_{j=[n/2]}^{n-J} (\frac{j}{n+\epsilon_n})^{\alpha j \beta} \\ &\leq K \sum_{j=[n/2]}^{n-J} e^{-\frac{(n+\epsilon_n - j)\alpha j \beta}{n+\epsilon_n}} \\ &\leq K \sum_{j=[\frac{n}{2}]}^{n-J} e^{-(n+\epsilon_n - j)\alpha\beta/4} \\ &\leq K \sum_{j=J}^{[n/2]+1} e^{-\frac{j\alpha\beta}{4}} \leq K e^{-JK} \end{aligned}$$

This estimate on the probability easily implies that $|n - N_c|^p I_{\{N_c \geq n/2\}}$ is uniformly integrable for all $p > 0$.

We now prove that $|n - N_c|^p I_{\{N_c \leq n/2\}}$ is also uniform integrable, but only for $m > p/\alpha\beta$.

$$\begin{aligned} P\left\{N_c^* \geq J \text{ and } N_c \leq n/2\right\} &\leq \sum_{j=m}^{[n/2]} P\left\{M_j \leq (\frac{j}{n+\epsilon_n})^\beta\right\} \\ &= T_2 \text{ say.} \end{aligned}$$

Let $0 < r < 1$ to be chosen. Then,

$$\begin{aligned} T_2 &= \sum_{j=m}^{[n^r]} (\frac{j}{n+\epsilon_n})^{\alpha\beta j} + \sum_{j=[n^r]+1}^{[n/2]} (\frac{j}{n+\epsilon_n})^{\alpha\beta j} \\ &= T_{21} + T_{22}. \end{aligned}$$

For the first term, using the crude upper bound n^r for j, we have

$$T_{21} \le \sum_{j=m}^{[n^r]} n^{\alpha\beta(r-1)j} = O(n^{\alpha\beta(r-1)m}). \tag{2.5}$$

On the other hand, we have

$$T_{22} \le \sum_{j=[n^r]+1}^{[n/2]} (\frac{1}{2})^{\alpha\beta j} = O(e^{-Kn^r}). \tag{2.6}$$

Using the bounds (2.5) and (2.6) on T_{21} and T_{22}, for $\delta > 0$,

$$E|n - N_c|^{p+\delta} I\{N_c \le n/2\} = O(n^{p+\delta+\alpha\beta(r-1)m}) \to 0 \tag{2.7}$$

provided

$$(p+\delta) + \alpha\beta(r-1)m < 0. \tag{2.8}$$

But $m > \frac{p}{\alpha\beta}$. Thus choosing δ and r sufficiently small (2.8) is satisfied.

Using (2.7) and uniform integrability of $|n - N_c|^p I_{\{N_c \ge n/2\}}$ proved earlier, establishes the uniform integrability of $|n - N_c|^p$ when $m > \frac{p}{\alpha\beta}$.

The required uniform integrability of $(M_{N_c}^{*p})$ when $m > \frac{p}{\alpha\beta}$ follows from the relation

$$N_c^* + \epsilon_n \le M_{N_c}^* \le N_c^* + \epsilon_n + 1.$$

Thus Theorem 1.2 (ii) is now proved.

We now argue the tightness of $\{N_c^*,\ M_{N_c}^*\}$ claimed in (2.4) as follows. First, the above estimates for the tail probabilities immediately yields the tightness of N_c^*. Now note that M_{N_c} cannot vary freely and is indeed controlled by the value of N_c^*, a fact heavily used so far. This shows that the joint sequence is also tight. Thus we have completed the proof of Theorem 1.2.

3. **Application to sequential estimation.** Let $Y_1, Y_2, \ldots$ be i.i.d. with density

$$f_{Y_1}(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha} I(0 < y < \theta).$$

where $\theta > 0$ is an unknown parameter and $\alpha > 0$ is known. Having observed $Y_1, Y_2, \ldots, Y_n$, the maximum likelihood estimate of θ is $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$. Suppose the loss function is

$$L_n = (\theta - Y_{(n)})^s + c_0 n \tag{3.1}$$

where $c_0 > 0$ is the known cost per unit sample. Let $R_n(c_0)$ be the expected loss. Since the density of $Y_{(n)}$ is given by

$$f_{Y_{(n)}}(y) = n\alpha y^{n\alpha-1}\theta^{-n\alpha} I(0 < y < \theta),$$

$$\begin{aligned} R_n(c_0) &= E(L_n) \\ &= \frac{\theta^s \Gamma(s+1)\Gamma(n\alpha+1)}{\Gamma(n\alpha+s+1)} + c_0 n \\ &\approx \frac{\theta^s\Gamma(s+1)}{(n\alpha)^s} + c_0 n \quad \text{as } n \to \infty. \end{aligned}$$

If we decide to choose the sample size to minimize the risk, the approximate optimal sample size n_0 is the smallest integer greater than or equal to

$$(\frac{s^2\Gamma(s)\theta^s}{c_0\alpha^s})^{1/(s+1)} = c, \text{ say .}$$

Note that

$$n_0 = [c] + I(0 << c >).$$

Since c is unknown, sequential procedures are called for. The following purely sequential stopping rule was proposed in Mukhopadhyay et al. (1983). Fix an initial sample size m. Define

$$N = N_{c_0} = \inf\left\{ n \geq m : n \geq \left(\frac{dY_{(n)}^s}{c_0}\right)^{(1/(s+1)}\right\}$$

where

$$d = s^2\Gamma(s)/\alpha^s.$$

For the uniform distribution ($\alpha = 1$), this rule reduces to that given by Ghosh and Mukhopadhyay (1975) if $s = 1$. Mukhopadhyay et al. (1983) verified various first order results including, as $c_0 \to 0$ (or $c \to \infty$),

(3.2) $$\frac{N}{c} \to 1 \text{ almost surely.}$$

(3.3) $$E(N/c) \to 1.$$

(3.4) $$E(L_N)/E(L_c)) \to 1.$$

(3.4) holds if $m > s^2/(\alpha s + \alpha)$. Bose and Mukhopadhyay (1997) showed that if $c_0 \to 0$ so that *c remains an integer* and if $m > s/(\alpha(s+1))$, then

(3.5)
$$-\frac{s}{\alpha(s+1)} \leq \liminf E(N-c) \leq \limsup E(N-c) \leq -\left[\exp\{\alpha(s+1)/s\} - 1\right]^{-1}$$

However, the problem of obtaining the so called second order properties of the estimate were unsolved. This includes obtaining an expansion of the *regret* and obtaining the exact limit in the above result. Below, we give a complete solution to these problems.

Define $X_i = Y_i/\theta, i > 1$. Note that the stopping time N_{c_0} may be written as

$$N_c = \inf\left\{n \geq m : M_n \leq (n/c)^\beta\right\}$$

where

$$\beta = (s+1)/s.$$

The stopping time is thus exactly of the form (1.1). We immediately have, if $c \to \infty$ such that $< c > \to \epsilon$ then

$$\lim E(N - [c]) = 1 - \frac{e^{-\alpha\beta\epsilon}}{1 - e^{-\alpha\beta}}.$$

Note that this implies that if $c \to \infty$ through integers, then the upper bound in (3.5) is exact.

The second order analysis of the regret is more delicate. The regret function is given by (here n_0 is the optimal sample size if θ were known),

$$\begin{aligned} R_c &= E(L_N) - E(L_{n_0}) \\ &= E\left[(\theta - Y_{(N)})^s + c_0 N\right] - E\left[(\theta - Y_{(n_0)})^s + c_0 n_0\right] . \\ &= \theta^s E\left[(1-M_N)^s - (1-M_{n_0})^s\right] + c_0 E(N - n_0) \end{aligned}$$

Note that

$$c_0 = d\theta^s / c^{s+1}.$$

Hence

$$\begin{aligned} \tfrac{R_c}{c_0} &= \frac{c^{s+1}}{d} E\{(1-M_N)^s - (1-M_{n_0})^s\} + E(N - n_0) \\ &= T_1 + T_2 \ \text{ say.} \end{aligned}$$

To compute the limit of T_1 as $c_0 \to 0$ (or $c \to \infty$), note that given N and M_N, M_{n_0} has a mixed distribution with $P\{M_{n_0} = M_N\} = M_N^{\alpha(n_0 - N)}$ and with conditional density (given $N = n$ and M_N), $f_{M_{n_0}}(x) = \alpha(n_0 - N)x^{\alpha(n-N)-1)}$, $M_N < x < 1$. Thus

$$\begin{aligned} T_1 &= \frac{c^{s+1}}{d}\left[E\left\{(1-M_N)^s(1 - M_N^{\alpha(n_0-N)}) - \alpha\int_{M_N}^1 (n_0 - N)(1-x)^s x^{\alpha(n-N)-1} dx\right\}\right] \\ &= E(T_{11}) - E(T_{12}) \ \text{ say.} \end{aligned}$$

By Theorem 1.2 (i),

$$(c(1 - M_N^{1/\beta}),\ ([c] - N)) \xrightarrow{\mathcal{D}} (Z,\ \ [Z - \epsilon])$$

and hence asymptotically, in distribution (if $< c > \to \epsilon$)

$$\begin{aligned} T_{11} &\approx \alpha\frac{(\beta X_{N_c}^*)^s}{d} c(1 - M_N^{n_0 - N}) \\ &\approx \alpha\frac{(\beta M_{N_c}^*)^s}{d} c(1 - M_{N_c}) \sum_{j=0}^{n_0 - N - 1} M_{N_c}^j \qquad (3.6) \\ &\approx \alpha\frac{(\beta M_{N_c}^*)^{s+1}}{d}(n_0 - N) \\ &\xrightarrow{\mathcal{D}} \alpha\tfrac{\beta^{s+1} Z^{s+1}}{d}([Z - \epsilon] + I(0 < \epsilon)). \end{aligned}$$

By Theorem 1.2 (*ii*), $\{T_{11}\}$ is uniformly integrable if $m > \frac{1}{\alpha\beta}$. Hence under this condition, the limit of $E(T_{11})$ is given by the expectation of the limit in (3.6).

To compute $\lim E(T_{12})$, note that

$$\begin{aligned} T_{12} &\le \frac{\alpha}{d} c^{s+1}(n_0 - N)(1 - M_N)^s \int_{M_N}^1 M_N^{\alpha(n_0-N)-1} dx \\ &= \tfrac{c^{s+1}}{d}(1 - M_N)^s(1 - M_{(N)}^{\alpha(n_0-N)}) = T_{11}. \end{aligned}$$

Thus T_{12} is also uniformly integrable if $m > \frac{1}{\alpha\beta}$. Further adding and subtracting an appropriate term,

$$\begin{aligned} T_{12} &= \frac{\alpha(n_0 - N)c^{s+1}}{d(s+1)} M_N^{\alpha(n_0-N)-1}(1 - M_N)^{s+1} \\ &\quad + \frac{\alpha(n_0 - N)c^{s+1}}{d} \int_{M_{(N)}}^1 (1-x)^s (x^{\alpha(n_0-N)-1} - M_N^{\alpha(n-N)-1}) dx \end{aligned}$$

The second term of T_{12} in absolute value is bounded by

$$\frac{\alpha(n_0 - N)c^{s+1}}{d(s+1)} (1 - M_N)^{s+2} \max\{1, M_N^{\alpha(n_0-N)-2}\}$$

which converges to zero in distribution since by Theorem 1.2 (*i*), $(n_0 - N)$ and $c^{s+2}(1 - M_n)^{s+2}$ converge in distribution as $c \to \infty$.

On the other hand, the first term of T_{12} is asymptotically equivalent in distribution to

$$\text{(3.7)} \qquad \frac{\alpha(n_0 - N)}{d(s+1)} (\beta M_{N_c}^*)^{s+1} \xrightarrow{\mathcal{D}} \frac{\alpha\beta^{s+1} Z^{s+1}}{d(s+1)} ([Z - \epsilon] + I(0 < \epsilon)).$$

Thus the limit of $E(T_{12})$ is given by the expectation of the limit given above. Note that the indicator in the above expression comes from the fact that $n_0 = [c] + I(< c > > 0)$. Now combining the results from (3.6) and (3.7), and using the facts that $d = s^2\Gamma(s)/\alpha^s$ and $\beta = (s+1)/s$, we have the following theorem:

THEOREM 3.1. If $m > \frac{1}{\alpha\beta}$ such that $< c > \to \epsilon$, then

$$E(N - n_0) = -I(0 < \epsilon) + 1 - \frac{e^{-\alpha\beta\epsilon}}{1 - e^{-\alpha\beta}} + o(1)$$

$$\frac{R_c}{c_0} \to \frac{\alpha\beta^s}{s^2\Gamma(s)}[E(Z^{s+1}\{[Z - \epsilon]) + I(0 < \epsilon)\}] + 1 - I(0 < \epsilon) - \frac{\exp(-\alpha\beta\epsilon)}{1 - \exp(-\alpha\beta)}$$

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