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ON THE "STRONG MEMORYLESSNESS PROPERTY" OF THE EXPONENTIAL AND GEOMETRIC PROBABILITY LAWS

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SUMMARY. If F is an exponential (resp. geometric) pr. law, relation (2) below is satisfied for all G with support on R^1_+ (resp. on Z^+). What can be said of F if (2) holds for a *specific* G? A best possible answer is obtained as Theorem A, simplifying the proof and dropping the assumption that G has m.g.f., made in an earlier version due to the author. A related result is stated as Theorem B (proof new).

Let $\mathcal{L}^+_\rho$ denote the class of all lattice probability distribution functions (d.f.'s) on the non-negative real axis *having the origin as a lattice point*, and $\rho > 0$ as a span, i.e., the class of all d.f.'s on R^1_+ whose points of increase form a subset (proper or not) of the set $\{n\rho : n = 0, 1, ...\}$ and let $\mathcal{L}^+ = \bigcup_{\rho > 0} \mathcal{L}^+_\rho$.

In Ramachandran (1977), abbreviated below as R-1977, the following result was established (under the superfluous assumption that the moment generating function (m.g.f.) of G exists) :

Theorem A : *Let X and Y be non-negative independent random variables (on some probability space) such that*

$$P[X > Y+t] = P[X > Y]\ P[X > t] \text{ for all } t > 0. \quad \dots \quad (1)$$

(1) *is equivalent to*

$$c[1-F(t)] = \int_{[0,\infty)} [1-F(t+y)]\, dG(y) \text{ for all } t > 0 \quad \dots \quad (2)$$

where F and G are the d.f.'s of X and Y respectively, and

$$c = P[X > Y] = \int_{[0,\infty)} [1-F(y)]\, dG(y). \quad \dots \quad (3)$$

The cases of interest correspond to (see Remark (ii) *below)*

$$G(0) < c < 1. \quad \dots \quad (4)$$

If then $\lambda (0 < \lambda < \infty)$ be the unique solution of the equation

$$\int_{[0,\infty)} e^{-\lambda y} dG(y) = c, \quad \dots \quad (5)$$

then, for all $x > 0$,

(a) $F(x) = 1-e^{-\lambda x}$ *(F is an exponential law) if* $G \notin \mathcal{L}^+$; *and* (b) $F(x) = 1-\xi(x)e^{-\lambda x}$, *where* ξ *is periodic with period* ρ, *if* $G \in \mathcal{L}_\rho^+$.

Remarks : (i) If X has an exponential d.f., then (1) is satisfied for any non-negative r.v. independent of X : this is a stronger property than the usual "memorylessness" of exponential laws—which corresponds to constant-valued Y. The geometric d.f. given by $1-F(n) = p^n$ for $n = 0, 1, 2, \ldots$ and constant on each interval $[n, n+1)$, for some $p \in (0, 1)$, satisfies (2) for any $G \in \mathcal{L}_1^+$, and corresponds to $\lambda = -\log p$, $\xi(x) = p^{-x}$ for $x \geqslant 0$ in Part (b) of the theorem. This remark is by way of explaining the title of this paper.

(ii) Condition (4) is rather awkward, involving as it does F as well as G, but is trivially satisfied (independently of F) if G is a continuous d.f., or, more generally, G has no atom at the origin. But, as shown by the following example due to Dr. J. S. Huang of Guelph, Canada, it cannot be improved upon; if $G(0) = c$, then our theorem need not hold : take $F(x) = x$ for $0 \leqslant x \leqslant 1$ and $G(x) = 1-e^{-1}$ for $0 \leqslant x \leqslant 1$ and $G(x) = 1-e^{-x}$ for $x > 1$. Also, the cases $c = 0$ and $c = 1$ are uninteresting : for their disposal, see R-1977.

(iii) If (2) is satisfied for $G = \delta_a$ and $G = \delta_b$ (δ_x denotes the degenerate d.f. with its sole point of increase at the point x), where $0 < a, b$ and a/b is irrational, then, appealing to either Part (a) or Part (b) of the theorem, one can deduce that F is exponential—see R-1977 for details.

The result was established in R-1977 using complex analysis methods (including the complex inversion formula for the Laplace transform) *and* (initially) under the assumption that the m.g.f. of G exists. A "real variables proof" without any assumptions was given in Shimizu (1978); see also Huang (1978) in this connection. In the present paper, we present a simplified version of the original proof, based on first proving that the Laplace transform of F is defined for $\text{Re}\, z > -\lambda$; a preliminary establishment of this fact does away with the need for : (i) Lemma 3 in R-1977, (ii) the introduction there of an auxiliary parameter Λ, and (iii) the assumption that G has m.g.f.

We also state and prove (as Theorem B below) a slightly improved version of a result proved in Shimizu (1978), the two proofs being substantially different. This result not only is of importance for a 'real variables proof' of our characterization theorem but also leads to similar proofs of some results on characteristic functions satisfying certain functional equations, which were originally established in the years 1968-70 by R. Shimizu on the one hand and by the author and C. R. Rao on the other : for these references, see R-1977.

Proof of Theorem A : Setting $h = 1-F$, we have from (2)

$$ch(t) = \int_{[0,\infty)} h(t+y)\, dG(y). \qquad \dots \quad (6)$$

Since, by assumption (4), $c > G(0)$, $G \neq \delta_0$ in particular and (6) can be rewritten as

$$c^{*}h(t) = \int_{(0,\infty)} h(t+y)\, dG^{*}(y) \text{ for all } t > 0 \qquad \dots \quad (6)$$

where $0 < c^{*} < 1$ and G^{*} is a d.f. with $G^{*}(0) = 0$, and, further, (the same) λ satisfies the relation $c^{*} = \int_{(0,\infty)} e^{-\lambda y}\, dG^{*}(y)$. In other words, we need only consider below the case : $G(0) = 0$, and $0 < c < 1$, while λ satisfies (5).

Now choose and fix any $\alpha \in (0, \lambda)$, and let $h_{\alpha}(t) = h(t)\, e^{\alpha t}$. Then we have from (6) that

$$h_{\alpha}(t) = \int_{(0,\infty)} h_{\alpha}(t+y)\, dG_{\alpha}(y) \qquad \dots \quad (7)$$

where

$$G_{\alpha}(x) = c^{-1} \int_{(0,x]} e^{-\alpha y}\, dG(y) \text{ for all } x \geqslant 0 \qquad \dots \quad (8)$$

so that $G_{\alpha}(0) = G(0) = 0$ and $G_{\alpha}(+\infty) > 1$. We can therefore find a and b such that $0 < a < b < \infty$ and $G_{\alpha}(b) - G_{\alpha}(a) \geqslant 1$. From (7) we have

$$h_{\alpha}(t) \geqslant \int_{(a,b]} h_{\alpha}(t+y)\, dG_{\alpha}(y)$$

whence it follows by contraposition that, for some $\xi \in (a, b]$, $h_{\alpha}(t+\xi) \leqslant h_{\alpha}(t)$. Also, for $t \leqslant u \leqslant t+\xi$, $h_{\alpha}(u) \leqslant h_{\alpha}(t)\, e^{\alpha(u-t)} \leqslant h_{\alpha}(t) e^{\alpha\xi} \leqslant h_{\alpha}(t). e^{\alpha b}$. Starting therefore with an arbitrary $t_1 \geqslant 0$, we can construct a sequence $\{t_n\}$ such that (i) $t_{n+1} \geqslant t_n + a$ (so that $\{t_n\} \to \infty$), (ii) $h_{\alpha}(t_n) \leqslant h_{\alpha}(t_{n-1}) \leqslant \dots \leqslant h_{\alpha}(t_1)$ and (iii) for $t_n \leqslant u \leqslant t_{n+1}$, $h_{\alpha}(u) \leqslant h_{\alpha}(t_n)\, e^{\alpha b}$, so that, for all $u \geqslant t_1$, $h_{\alpha}(u) \leqslant h_{\alpha}(t_1).\, e^{\alpha b}$. Thus h_{α} is bounded on $(0, \infty)$.

Since $\alpha \in (0, \lambda)$ is arbitrary, it follows in particular that

$$\int_0^{\infty} e^{ru} h(u) du < \infty; \int_{[0,\infty)} e^{ry} dF(y) < \infty \text{ for all } r < \lambda. \qquad \dots \quad (9)$$

Taking the Laplace transforms (L.T.'s) of both sides in (6), and noting that g, the L.T. of h, is defined for $\operatorname{Re} z > -\lambda$ in view of (9) and that $\sigma(.)$ and $K(.)$, defined by

$$\sigma(z) = \int_{(0,\infty)} e^{zy}\, dG(y) - c;\ K(z) = \int_0^{\infty} e^{zy} \left(\int_0^{y} e^{-zu} h(u) du \right) dG(y) \qquad \dots \quad (10)$$

are both analytic in $\operatorname{Re} z < 0$, we have that

$$g(z).\sigma(z) = K(z) \text{ for } -\lambda < \operatorname{Re} z < 0 \quad \ldots \quad (11)$$

so that we may in fact write

$$g(z) = K(z)/\sigma(z) \text{ for } -\lambda < \operatorname{Re} z < 0 \quad \ldots \quad (12)$$

zeros of $K(.)$ cancelling out those of $\sigma(.)$ in that strip. We note further that

$$|\sigma(x+iy)| \geqslant c - \int_{(0,\infty)} e^{xy}\, dG(y) > 0 \text{ for } x < -\lambda,$$

so that $\sigma(.)$ has no zeros in the half-plane $\operatorname{Re} z < -\lambda$.

We then invoke the following lemmas, as in R-1977, in order to apply a complex inversion formula for g. For their proofs, we refer to R-1977.

Lemma 1 : *The number of zeros of $\sigma(.)$ in any closed rectangle of the form $a \leqslant \operatorname{Re} z \leqslant b(< 0)$, $y \leqslant \operatorname{Im} z \leqslant y+1$, is bounded by a number $n(a, b)$ which does not depend on y.*

Lemma 2 : *Given γ, ε, both positive, there exists an $m(\gamma, \varepsilon) > 0$ such that $|\sigma(z)| > m(\gamma, \varepsilon)$ for all z lying in the strip : $-\gamma+\varepsilon \leqslant Re\, z \leqslant -\varepsilon$ but outside of discs of radius ε with centres at the zeros of $\sigma(.)$.*

Lemma 3 : (i) *$\sigma(.)$ has no zeros other than $-\lambda$ on the line $Re\, z = -\lambda$, if $G \notin \mathcal{L}^+$; if, however, $G \in \mathcal{L}^+_\rho$ for some $\rho > 0$, then the zeros of $\sigma(.)$ on that line form the set $\{-\lambda+2\pi\, in/\rho;\, n = 0, \pm 1, \pm 2, \ldots\}$.*

(ii) *The zeros of $\sigma(.)$ on the line $Re\, z = -\lambda$ are all simple.*

Now we appeal to the following inversion formula for L.T.'s (see, for instance, Doetsch, 1974, p. 181, Theorem 27.2): for any x such that $-\lambda < x < 0$,

$$\text{P.V.} \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{tz} \frac{g(z)}{z}\, dz = -\int_t^\infty h(u)\, du \text{ for all } t \geqslant 0.$$

Hence, by (12),

$$-\int_t^\infty h(u)\, du = \text{P.V.} \frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} e^{tz} \frac{K(z)}{z\sigma(z)}\, dz (t \geqslant 0) \quad \ldots \quad (13)$$

for any specific $x_0 \in (-\lambda, 0)$. On the strength of Lemma 1, we can find $\delta > 0$ and a sequence $\{T_n\}$ of real numbers with $n \leqslant T_n < n+1$ for $n \geqslant 0$ integer, such that all the zeros of $\sigma(.)$ in the strip $-\lambda \leqslant \operatorname{Re} z \leqslant x_0$ lie at a distance of at least δ from the lines $|\operatorname{Im} z| = T_n$ and, as we have already seen, $\sigma(.)$ has no zeros in the half-plane $\operatorname{Re} z < -\lambda$, so that, then, by Lemma 2, $|\sigma(z)| > b$

for all n if z lies on the lines $|\mathrm{Im}\, z| = T_n$, for some $b > 0$. $K(.)$ admits the estimate $|K(z)| \leqslant \mathrm{const.}/|\mathrm{Re}\, z|$ in the half-plane $\mathrm{Re}\, z < 0$, so that $K(z).e^{tz}$ is bounded in any half-plane of the form $\mathrm{Re}\; z \leqslant x < 0$. It then follows easily from (13) that $-\int_t^\infty h(u)\, du = \lim_{n\to\infty}$ [the sum of the residues of $\frac{e^{tz}K(z)}{z\sigma(z)}$ at those zeros of $\sigma(.)$ which lie on the line-segment: $\mathrm{Re}\, z = -\lambda$, $|\mathrm{Im}\, z| < T_n$] and, in view of Lemma 3, this leads, as in R-1977, to the stated conclusions of the theorem in the cases (a) and (b).

In fact, starting from (2), in the presence of conditions (4) and (5), we can establish as a *preliminary* fact even the stronger statement :

$$e^{\lambda x}[1-F(x)] \text{ is bounded.} \qquad \ldots \quad (14)$$

Such prior establishment is not particularly important or useful for our method of proof, but it is of independent interest as well as vital for the purposes of Shimizu (1978). As stated earlier, we give below a substantially different proof of his result concerning the boundedness of the solution of a functional inequality related in form to (6), while dropping at the same time the assumption of right-continuity of the function $k(.)$ below as well as the requirement that $C(0+) = 1$ (see also the remark at the end of this paper).

Theorem B : *Let H be a d.f. with support on $[0, \infty)$ and having m.g.f. If k is a non-negative real-valued function on $[0, \infty)$, satisfying the growth condition*

$$\sup_{0 \leqslant y \leqslant \eta} k(x+y) \leqslant k(x).C(\eta) \text{ for all } x \geqslant 0 \qquad \ldots \quad (15)$$

(*where $C(.)$ is non-decreasing on $[0, \infty)$*) *and the inequality*

$$k(x) \geqslant \int_{[0,\infty)} k(x+y)\, dH(y) \text{ for all } x \geqslant 0, \qquad \ldots \quad (16)$$

then $\inf_{[x, x+a]} k$ is uniformly bounded for all $x \geqslant 0$ for some $a > 0$ and hence k is bounded on $[0, \infty)$ (on account of (15)).

Note : If $k(x_0) = 0$ for some x_0, then (15) implies that $k(x) \equiv 0$ for $x \geqslant x_0$, so we need only consider the case where $k(x) > 0$ for every $x \geqslant 0$.

Corollary : *If F and G satisfy (2) subject to (4) and (5), then (14) holds.*

To see this, we need only take $k(x) = e^{\lambda x}[1-F(x)]$. (15) is then satisfied for $C(\eta) = \exp(\lambda\eta)$ and (16) for H given by $dH(x) = c^{-1} e^{-\lambda x}\, dG(x)$.

Remark: We may also consider k defined on $[x_0, \infty)$ where $x_0 > 0$ and satisfying (15) and (16) there: the conclusions of the theorem will hold in the relevant interval. An example is provided by $k(x) = 1/x$ in such an interval.

Proof: As before, we need only consider the cases where $H(0) = 0$, so in particular $H \neq \delta_0$. Then $k_\delta(.)$, given by $k_\delta(x) = k(x).\exp(-\delta x)$, satisfies, for all $\delta > 0$, the inequality

$$k_\delta(x) \geqslant \int_0^\infty k_\delta(x+y)\, e^{\delta y}\, dH(y).$$

We need only consider those $\delta > 0$ for which $\int_0^\infty e^{\delta y}\, dH(y) < \infty$ (H has m.g.f., so such δ exist, by assumption). Then we can conclude, as we did in the context of relation (7), that k_δ is bounded: in fact, if $0 < a < b < \infty$ be such that $\int_{(a,b]} e^{\delta y}\, dH(y) \geqslant 1 (H \neq \delta_0)$, then, for every $x \geqslant 0$, there exists a $\xi \in (a, b]$ such that $k_\delta(x+\xi) \leqslant k_\delta(x)$, and then as before it follows in view of (15) that for all $x \geqslant$ any fixed $x_1 \geqslant 0$, $k_\delta(x) \leqslant C(b) \,.\, k_\delta(x_1)$. It therefore follows that (take $\delta = u/2$)

$$\int_0^\infty e^{-ux} k(x)\, dx \text{ exists finitely for all } u > 0.$$

Then, δ being any number > 0 such that $\int_0^\infty e^{\delta y}\, dH(y) < \infty$, we have from (16) that for any u such that $0 < u < \delta$,

$$\left(\int_0^\infty e^{-ux} k(x)\, dx\right)\left(\int_0^\infty e^{uy}\, dH(y) - 1\right) \leqslant \int_0^\infty e^{uy} \left(\int_0^y k(t)\, e^{-ut}\, dt\right) dH(y). \quad \ldots \quad (17)$$

If C_δ is any bound for the function $k_\delta(.)$, so that $k(t) \leqslant C_\delta\, e^{t\delta}$, the R.H.S. of (17) is easily verified to admit the estimate

$$C_\delta \int_0^\infty (e^{\delta y} - e^{uy})\, dH(y)/(\delta - u)$$

$$\leqslant \text{const.} \int_0^\infty e^{\delta y}\, dH(y) \text{ for } 0 < u \leqslant \delta/2.$$

It then follows from (17) that

$$u \int_0^\infty e^{-ux}\, k(x)\, dx \leqslant A \text{ for } 0 < u \leqslant 1$$

where A is some absolute constant. Hence, for every $x > 0$ and all such u,

$$\int_0^\infty e^{-uy}\, k(y)\, dy \leqslant A/u$$

so that, for $x \geqslant 1$, we may conclude (take $u = 1/x$) that

$$\xi(x) = \int_0^x k(y)\, dy \leqslant Ae\, x (x \geqslant 1). \qquad \ldots \quad (18)$$

Since H has m.g.f., $\int_0^\infty y\, dH(y)$ exists finitely, and hence so do

$$\int_0^\infty \xi(x+y)\, dH(y) \quad \text{and} \quad \int_0^\infty \xi(y)\, dH(y)$$

by (18). Denoting the value of the last integral by B, we have from (16) that

$$\xi(x) \geqslant \int_0^\infty \xi(x+y)\, dH(y) - B \text{ for all } x \geqslant 0$$

whence, for any x, $a \geqslant 0$, we have

$$\int_a^\infty \left(\int_x^{x+a} k(t)\, dt \right) dH(y) \leqslant \int_a^\infty \left(\int_x^{x+y} k(t)\, dt \right) dH(y) \leqslant B$$

so that, choosing any $a > 0$ such that $H(a) < 1$, we see that

$$\inf_{[x,\, x+a]} k \leqslant \text{some } K > 0, \text{ uniformly for all } x \geqslant 0. \qquad \ldots \quad (19)$$

It then follows from (15) that for any $x > a$ (k is obviously bounded on $[0, a]$),

$$k(x) \leqslant k(x-y)\, .\, C(a) \text{ for all } y \in [0, a]$$

so that

$$k(x) \leqslant \inf_{0 \leqslant y \leqslant a} k(x-y)\, .\, C(a) \leqslant K.C(a) \text{ for } x > a$$

by (19). Hence the theorem.

Remark : Theorem 1 of Shimizu (1978), of which the result above is a modification, assumes in addition that k is right-continuous and that $C(0+) = 1$, these requirements being automatically satisfied in both the applications of the theorem (including a proof of our characterizaion theorem) considered there : further, the proof of the above theorem given there also depends essentially on k being right-continuous as well as on $C(0+)$ being $= 1$, through the fact that k, subject to (15), is also lower semi-continuous then and in particular attains its infimum on every compact interval.

For details of how the above theorem is used to prove our characterization theorem, we refer to Shimizu (1978). In this connection we may also refer to related results due to Choquet and Deny, statements and (martingale-theoretic) proofs of which may be found in P. A. Meyer, *Probability and Potentials*, Blaisdell (1966).

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