Moore–Penrose inverse of set inclusion matrices

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Abstract

Given integers s, k and v, let W_{sk} be the $\binom{v}{s} \times \binom{v}{k}$ 0–1 matrix, the rows and the columns of which are indexed by the s-subsets and the k-subsets of a v-set respectively, and where the entry in row S and column U is 1 if $S \subset U$ and 0 otherwise. A formula for the Moore–Penrose inverse of W_{sk} over the reals is obtained. A necessary and sufficient condition for W_{sk} to admit a Moore–Penrose inverse over the set of integers modulo a prime p is given, together with a formula for the Moore–Penrose inverse when it exists.

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1. Introduction

Let s, k, v be integers satisfying $0 \leqslant s \leqslant k \leqslant v$. The set inclusion matrix W_{sk} is a matrix of order $\binom{v}{s} \times \binom{v}{k}$ defined as follows. The rows of W_{sk} are indexed by the s-element subsets and the columns are indexed by the k-element subsets of a v-element set. The entry in row S and column U of W_{sk} is 1 if $S \subset U$ and 0 otherwise. Note that W_{0k} is a $1 \times \binom{v}{k}$ vector of all ones. These matrices arise in many combinatorial problems, particularly in design theory and extremal set theory, see [3,5,10,11] and the references contained therein.

We now introduce some definitions. If A is an $m \times n$ matrix, then an $n \times m$ matrix G is called a generalized inverse (g-inverse) of A if $AGA = A$. The *Moore–Penrose inverse* of A, denoted by A^+, is an $n \times m$ matrix G satisfying the equations $AGA = A$, $GAG = G$, $(AG)^{\mathrm{T}} = AG$ and $(GA)^{\mathrm{T}} = GA$. It is well known that any real matrix admits a unique Moore–Penrose inverse. We refer to [2,4] for basic properties of the Moore–Penrose inverse.

It is well known that when $0 \leqslant s \leqslant k \leqslant v - s$, the rows of W_{sk} are linearly independent, see, for example, [6,8,10]. In such a case, a formula for a right inverse of W_{sk} has been given in [8]. Note that when W_{sk} has full row rank, the class of right inverses of W_{sk} is the same as the class of g-inverses of W_{sk}. In Section 2, we give an explicit formula for the Moore–Penrose inverse, which is a particularly nice right inverse, of W_{sk} over the set of reals. In particular, when $k = v - s$, W_{sk} is a square nonsingular matrix and our formula gives the inverse of W_{sk}. A similar formula has been given in [8]. The class of all right inverses can be described, once we have a formula for the Moore–Penrose inverse, see [2,4].

The theory of g-inverse, and in particular of the Moore–Penrose inverse, is implicit in the work of Wilson [10,11], and one motivation for the present work is to make this connection explicit. For example, Corollary 1 of [11] gives a necessary and sufficient condition for an integer matrix to have an integer g-inverse. Similarly, Theorem 2 of [10] gives a formula (in a more general setting of t-designs) for the orthogonal projection onto the row space of W_{sk}, which is just $W_{sk}^+ W_{sk}$. The proof of Theorem 3 of [11] also employs g-inverses. It appears that there is much to be gained by incorporating the tools of the theory of g-inverses into the study of incidence matrices for set systems.

In Section 3, we determine the eigenvalues of $W_{sk} W_{sk}^{\mathrm{T}}$. This result is of independent interest and is also used in the following section.

Incidence matrices regarded as matrices over the field of integers modulo a prime p are important in design theory, see [5,9,11]. In Section 4, we give a necessary and sufficient condition for W_{sk} to admit a Moore–Penrose inverse over the set of integers modulo a prime p. We also give a formula for the Moore–Penrose inverse when it exists. As a consequence it is shown that $W_{s,v-s}$ admits a Moore–Penrose modulo p for any prime p. It is likely that these results will be applicable to extend some work in [8] to linear mappings between vector spaces over a finite field.

2. Moore–Penrose inverse over reals

Let s, k, v be integers satisfying $0 \leqslant s \leqslant k \leqslant v$. The matrix $\overline{W}_{sk}$ is a matrix of order $\binom{v}{s} \times \binom{v}{k}$ defined as follows. The rows of $\overline{W}_{sk}$ are indexed by the s-element subsets and the columns are indexed by the k-element subsets of a v-element set. The entry in row S and column U of $\overline{W}_{sk}$ is 1 if $S \cap U = \emptyset$ and 0 otherwise. As before, $\overline{W}_{0k}$ is a $1 \times \binom{v}{k}$ vector of all ones.

The following result will be used. The proof is elementary and can be found in [10].

Lemma 1. *For $0 \leqslant i \leqslant s \leqslant k \leqslant v$, the following assertions hold*:
(i) $W_{sk}\overline{W}_{ik}^{\mathrm{T}} = \binom{v-i-s}{k-s}\overline{W}_{is}^{\mathrm{T}}$.
(ii) $\overline{W}_{ij}^{\mathrm{T}} W_{ij}$ *is symmetric for any* $j, 0 \leqslant i \leqslant j \leqslant v$.
(iii) $\sum_{i=0}^{s}(-1)^i \overline{W}_{is}^{\mathrm{T}} W_{is} = I$.
(iv) $W_{is}W_{sk} = \binom{k-i}{s-i}W_{ik}$.

A formula for the Moore–Penrose inverse of W_{sk} over the set of reals is given in the next result.

Theorem 2. *Let* $0 \leqslant s \leqslant k \leqslant v - s$. *Then*

$$W_{sk}^{+} = \sum_{i=0}^{s} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \overline{W}_{ik}^{\mathrm{T}} W_{is}.$$

Proof. Let W_{sk}^{+} be as in the statement of the theorem. We have

$$\begin{aligned}
W_{sk}W_{sk}^{+} &= \sum_{i=0}^{s} \frac{(-1)^i}{\binom{v-i-s}{k-s}} W_{sk}\overline{W}_{ik}^{\mathrm{T}} W_{is} \\
&= \sum_{i=0}^{s} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \binom{v-i-s}{k-s} \overline{W}_{is}^{\mathrm{T}} W_{is} \quad \text{(by Lemma 1(i))} \\
&= \sum_{i=0}^{s} (-1)^i \overline{W}_{is}^{\mathrm{T}} W_{is} \\
&= I \quad \text{(by Lemma 1(iii))}.
\end{aligned}$$

Thus W_{sk}^{+} is a right inverse of W_{sk}.

It follows that $W_{sk}W_{sk}^{+}W_{sk} = W_{sk}$, $W_{sk}^{+}W_{sk}W_{sk}^{+} = W_{sk}^{+}$ and that $W_{sk}W_{sk}^{+}$ is symmetric. It remains to show that $W_{sk}^{+}W_{sk}$ is symmetric. Observe that by Lemma 1,

$$\begin{aligned}
W_{sk}^{+}W_{sk} &= \sum_{i=0}^{s} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \overline{W}_{ik}^{\mathrm{T}} W_{is} W_{sk} \\
&= \sum_{i=0}^{s} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \binom{k-i}{s-i} \overline{W}_{ik}^{\mathrm{T}} W_{ik}.
\end{aligned}$$

Again by Lemma 1, $\overline{W}_{ik}^{\mathrm{T}} W_{ik}$ is symmetric and the proof is complete. □

We remark that, if $k > v - s$, then we get a formula for W_{sk}^{+} using the fact that $W_{sk}^{\mathrm{T}} = W_{v-k, v-s}$.

As seen in the proof of Theorem 2, W_{sk} has a right inverse. It follows that W_{sk} has rank $\binom{v}{s}$ when $s \leqslant k \leqslant v - s$, a result proved by several authors independently, see [5,10] and the references contained therein.

In particular, $W_{s,v-s}$ is a square, nonsingular matrix and by Theorem 2 its inverse is given by

$$W_{s,v-s}^{-1} = \sum_{i=0}^{s} \frac{(-1)^i}{\binom{v-i-s}{s-i}} \overline{W}_{i,v-s}^{\mathrm{T}} W_{is}. \tag{1}$$

In case $0 \leqslant s \leqslant k \leqslant v - s$, a formula for a right inverse of W_{sk} is given in [8], however, the formula obtained in Theorem 2 is more explicit. A formula for the inverse of $W_{s,n-s}$, different than but similar to (1), is also found in [8].

We remark that if appropriate orderings are chosen for the row and column indices, then $W_{ij} = \overline{W}_{i,v-j}$ and $\overline{W}_{ij} = W_{i,v-j}$. In view of this remark, an expression for the Moore–Penrose inverse of $\overline{W}_{sk}$ can be obtained from Theorem 2.

3. Eigenvalues of $W_{sk} W_{sk}^{\mathrm{T}}$

For convenience, we will take our v-set as $\{1, 2, \ldots, v\}$. Let $0 \leqslant i \leqslant j \leqslant v$. If $T \subset \{1, 2, \ldots, v\}$ with $|T| = i$, then $W_{ij}[T]$ will denote the row of W_{ij} indexed by T. A similar notation applies to $\overline{W}_{ij}$. The row-space of the matrix A will be denoted by $\mathscr{R}(A)$.

Lemma 3. *Let* $0 < i \leqslant j \leqslant v$. *Then*

$$\mathscr{R}((-1)^i W_{ij} - \overline{W}_{ij}) \subset \mathscr{R}(W_{i-1,j}).$$

Proof. Let $T \subset \{1, 2, \ldots, v\}$, $|T| = i$. By the inclusion–exclusion principle,

$$(-1)^i W_{ij}[T] = \overline{W}_{ij}[T] - \sum_{\ell=0}^{i-1} \sum_{S \subset T, |S| = \ell} (-1)^\ell W_{\ell j}[S]. \tag{2}$$

An application of Lemma 1(iv) gives

$$W_{\ell q} W_{qj} = \binom{j-\ell}{q-\ell} W_{\ell j}$$

for $0 \leqslant \ell \leqslant q \leqslant j \leqslant v$ and hence

$$\mathscr{R}(W_{\ell j}) \subset \mathscr{R}(W_{qj}). \tag{3}$$

Therefore

$$\mathscr{R}(W_{\ell j}) \subset \mathscr{R}(W_{i-1,j}) \tag{4}$$

for $\ell = 0, 1, \ldots, i-1$. The result follows from (2) and (4). $\square$

It follows from Lemma 3 and (3) that if $0 \leqslant i \leqslant j \leqslant v$, then $\mathscr{R}(\overline{W}_{ij}) \subset \mathscr{R}(W_{ij})$; this fact will be used. Since, with appropriate orderings for the row and column indices, $W_{ij} = \overline{W}_{i,v-j}$ and $\overline{W}_{ij} = W_{i,v-j}$, it is in fact true that if $0 \leqslant i \leqslant j \leqslant v-i$, then $\mathscr{R}(\overline{W}_{ij}) = \mathscr{R}(W_{ij})$.

The following result is contained in [10] when the additional assumption that $k \geqslant 2s$ is made (see the proof of Proposition 3 in [10]). We give a different proof and do not require $k \geqslant 2s$.

Theorem 4. *Let $0 \leqslant s \leqslant k \leqslant v-s$. Then the eigenvalues of $W_{sk}W_{sk}^{\mathrm{T}}$ are given by $\binom{k-i}{s-i}\binom{v-i-s}{k-s}$ with multiplicity $\binom{v}{i} - \binom{v}{i-1}$, $i = 0, 1, \ldots, s$.*

Proof. We make the convention that $\binom{v}{-1} = 0$. Each row sum of W_{sk} is $\binom{v-s}{k-s}$ and each column sum of W_{sk} is $\binom{k}{s}$. Thus the $1 \times \binom{v}{s}$ vector of all ones is a left eigenvector of $W_{sk}W_{sk}^{\mathrm{T}}$ for the eigenvalue $\binom{k}{s}\binom{v-s}{k-s}$.

Let i be fixed, $1 \leqslant i \leqslant s$. By Lemma 3, $(-1)^i W_{is} - \overline{W}_{is} = XW_{i-1,s}$ for some matrix X. Let

$$H_i = \binom{v-i-s}{k-s} W_{is} - W_{ik}W_{sk}^{\mathrm{T}}.$$

Then

$$\begin{aligned} H_i &= (-1)^i \binom{v-i-s}{k-s} \overline{W}_{is} - W_{ik}W_{sk}^{\mathrm{T}} + (-1)^i \binom{v-i-s}{k-s} XW_{i-1,s} \\ &= (-1)^i \overline{W}_{ik}W_{sk}^{\mathrm{T}} - W_{ik}W_{sk}^{\mathrm{T}} + (-1)^i \binom{v-i-s}{k-s} XW_{i-1,s} \\ &= ((-1)^i \overline{W}_{ik} - W_{ik})W_{sk}^{\mathrm{T}} + (-1)^i \binom{v-i-s}{k-s} XW_{i-1,s}, \end{aligned} \tag{5}$$

where we used Lemma 1(i) to get the second equality. By Lemma 3,

$$(-1)^i \overline{W}_{ik} - W_{ik} = Y\overline{W}_{i-1,k} \tag{6}$$

for some matrix Y. It follows from (5), (6) and Lemma 1(i) that

$$\begin{aligned} H_i &= Y\overline{W}_{i-1,k}W_{sk}^{\mathrm{T}} + (-1)^i \binom{v-i-s}{k-s} XW_{i-1,s} \\ &= \binom{v-i+1-s}{k-s} Y\overline{W}_{i-1,s} + (-1)^i \binom{v-i-s}{k-s} XW_{i-1,s}. \end{aligned} \tag{7}$$

In view of the remark following Lemma 3 it follows from (7) that any vector in the row space of H_i is contained in the row space of $W_{i-1,s}$ and hence rank $(H_i) \leqslant \binom{v}{i-1}$. Consider the vector space

$$\mathscr{S} = \{x : x^{\mathrm{T}} = y^{\mathrm{T}}W_{is} \text{ for some } y \text{ such that } y^{\mathrm{T}}H_i = 0\}.$$

Since W_{is} has full row rank, the dimension of $\mathscr{S}$ equals the dimension of the left null space of H_i. As observed earlier, rank $(H_i) \leqslant \binom{v}{i-1}$ and hence dim $(\mathscr{S})$ is at least $\binom{v}{i} - \binom{v}{i-1}$.

By Lemma 1(iv),

$$W_{is} W_{sk} W_{sk}^{\mathrm{T}} = \binom{k-i}{s-i} W_{ik} W_{sk}^{\mathrm{T}}$$
$$= \binom{k-i}{s-i}\binom{v-i-s}{k-s} \frac{1}{\binom{v-i-s}{k-s}} W_{ik} W_{sk}^{\mathrm{T}}. \quad (8)$$

Let $x \in \mathscr{S}$ and suppose $x^{\mathrm{T}} = y^{\mathrm{T}} W_{is}$ where $y^{\mathrm{T}} H_i = 0$. Premultiply (8) to conclude that x is a left eigenvector of $W_{sk} W_{sk}^{\mathrm{T}}$ with eigenvalue $\binom{k-i}{s-i}\binom{v-i-s}{k-s}$. The multiplicity of the eigenvalue equals dim $(\mathscr{S})$ which is at least $\binom{v}{i} - \binom{v}{i-1}$. Since this is true for $i = 0, 1, \ldots s$ (the case $i = 0$ was covered in the beginning of this proof) adding up the multiplicities over $i = 0, 1, \ldots, s$ we see that the multiplicity of $\binom{k-i}{s-i}\binom{v-i-s}{k-s}$ must in fact be equal to $\binom{v}{i} - \binom{v}{i-1}$. That completes the proof. □

4. Moore–Penrose inverse over a finite field

In this section, we view W_{sk} as a matrix over the field of integers modulo p, a prime. The following result has been proved in [11].

Theorem 5. *Let $0 \leqslant s \leqslant k \leqslant v - s$. Let p be a prime and let*

$$\mathscr{N} = \left\{ i : 0 \leqslant i \leqslant s, p \nmid \binom{k-i}{s-i} \right\}.$$

Then, the rank of W_{sk} over the integers modulo p equals

$$\sum_{i \in \mathscr{N}} \binom{v}{i} - \binom{v}{i-1}.$$

The next result, proved in [1], gives a necessary and sufficient condition for a matrix to admit a Moore–Penrose inverse over an arbitrary field. An equivalent condition has been obtained in [7].

Theorem 6. *An $m \times n$ matrix A of rank r over an arbitrary field admits a Moore–Penrose inverse over the field if and only if the sum of squares of the $r \times r$ minors of A is nonzero.*

We now prove the main result of this paper.

Theorem 7. *Let $0 \leqslant s \leqslant k \leqslant v - s$, let p be a prime and let*

$$\mathcal{N} = \left\{ i : 0 \leqslant i \leqslant s, p \nmid \binom{k-i}{s-i} \right\}.$$

Then W_{sk} has a Moore–Penrose inverse modulo p if and only if $p \nmid \binom{v-i-s}{k-s}$ for all $i \in \mathcal{N}$. Furthermore, the Moore–Penrose inverse, when it exists, is given by

$$W_{sk}^{+} = \sum_{i,j \in \mathcal{N}} (-1)^{i+j} \frac{\binom{v-i-j}{s-i}}{\binom{v-i-s}{k-s}} \overline{W}_{ik}^{\mathrm{T}} \overline{W}_{ji}^{\mathrm{T}} W_{js}.$$

Proof. Let

$$\mathcal{N}' = \left\{ i : 0 \leqslant i \leqslant s, p \nmid \binom{k-i}{s-i}, p \nmid \binom{v-i-s}{k-s} \right\}.$$

Set

$$r = \sum_{i \in N} \binom{v}{i} - \binom{v}{i-1} \quad \text{and} \quad r' = \sum_{i \in \mathcal{N}'} \binom{v}{i} - \binom{v}{i-1}.$$

Since $s \leqslant 2v$, then $\binom{v}{i} - \binom{v}{i-1} > 0, i = 0, 1, \ldots, s$, and therefore $r \geqslant r'$. By Theorem 5, the rank of W_{sk} equals r. By Theorem 4, the eigenvalues of $W_{sk} W_{sk}^{\mathrm{T}}$ over the reals are given by $\binom{k-i}{s-i}\binom{v-i-s}{k-s}$, with multiplicity $\binom{v}{i} - \binom{v}{i-1}$, $i = 0, 1, \ldots, s$. Thus the number of eigenvalues of $W_{sk} W_{sk}^{\mathrm{T}}$ which are nonzero modulo p is r'. The sum of squares of the $r \times r$ minors of W_{sk} equals the sum of the principal $r \times r$ minors of $W_{sk} W_{sk}^{\mathrm{T}}$, which in turn is the rth elementary symmetric function in the eigenvalues of $W_{sk} W_{sk}^{\mathrm{T}}$. Thus, if $r > r'$, then the sum of squares of the $r \times r$ minors of W_{sk} must be zero modulo p. It follows by Theorem 6 that, if W_{sk} admits a Moore–Penrose inverse modulo p, then $r = r'$. Clearly, if $r = r'$, then $i \in \mathcal{N}'$ whenever $i \in \mathcal{N}$ and this proves the necessity part of the theorem.

Now suppose that $p \nmid \binom{v-i-s}{k-s}$ for all $i \in \mathcal{N}$. We work over the integers modulo p in what follows.

Let

$$\tilde{W}_{sk} = \sum_{i \in \mathcal{N}} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \overline{W}_{ik}^{\mathrm{T}} W_{is}. \tag{9}$$

We have

$$\begin{aligned} W_{sk} \tilde{W}_{sk} &= \sum_{i \in \mathcal{N}} \frac{(-1)^i}{\binom{v-i-s}{k-s}} W_{sk} \overline{W}_{ik}^{\mathrm{T}} W_{is} \\ &= \sum_{i \in \mathcal{N}} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \binom{v-i-s}{k-s} \overline{W}_{is}^{\mathrm{T}} W_{is} \\ &= \sum_{i \in \mathcal{N}} (-1)^i \overline{W}_{is}^{\mathrm{T}} W_{is}. \end{aligned} \tag{10}$$

It follows from (10) and Lemma 1(ii) that $W_{sk}\tilde{W}_{sk}$ is symmetric.

Now

$$\tilde{W}_{sk}W_{sk} = \sum_{i\in\mathcal{N}} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \overline{W}_{ik}^{\mathrm{T}} W_{is} W_{sk}$$
$$= \sum_{i\in\mathcal{N}} \frac{(-1)^i}{\binom{v-i-s}{k-s}} \binom{k-i}{s-i} \overline{W}_{ik}^{\mathrm{T}} W_{ik}, \tag{11}$$

using Lemma 1(iv). It follows from (11) and Lemma 1(ii) that $\tilde{W}_{sk}W_{sk}$ is symmetric.

Also

$$\begin{aligned}
W_{sk}\tilde{W}_{sk}W_{sk} &= \sum_{i\in\mathcal{N}} (-1)^i \overline{W}_{is}^{\mathrm{T}} W_{is} W_{sk} \quad \text{(by (10))} \\
&= \sum_{i\in\mathcal{N}} (-1)^i \binom{k-i}{s-i} \overline{W}_{is}^{\mathrm{T}} W_{ik} \quad \text{(by Lemma 1(iv))} \\
&= \sum_{i=0}^{s} (-1)^i \binom{k-i}{s-i} \overline{W}_{is}^{\mathrm{T}} W_{ik} \\
&= \sum_{i=0}^{s} (-1)^i \overline{W}_{is}^{\mathrm{T}} W_{is} W_{sk} \quad \text{(by Lemma 1(iv))} \\
&= W_{sk} \quad \text{(by Lemma 1(iii))}.
\end{aligned}$$

Thus $\tilde{W}_{sk}$ is a g-inverse of W_{sk} and $W_{sk}\tilde{W}_{sk}$, $\tilde{W}_{sk}W_{sk}$ are both symmetric. It follows that

$$W_{sk}^{+} = \tilde{W}_{sk}W_{sk}\tilde{W}_{sk}$$

is the Moore–Penrose inverse of W_{sk}. Now

$$\begin{aligned}
W_{sk}^{+} &= \tilde{W}_{sk} \sum_{j\in\mathcal{N}} (-1)^j \overline{W}_{js}^{\mathrm{T}} W_{js} \quad \text{(by (10))} \\
&= \sum_{i,j\in\mathcal{N}} \frac{(-1)^{i+j}}{\binom{v-i-s}{k-s}} \overline{W}_{ik}^{\mathrm{T}} W_{is} \overline{W}_{js}^{\mathrm{T}} W_{js} \quad \text{(by (9))} \\
&= \sum_{i,j\in\mathcal{N}} (-1)^{i+j} \frac{\binom{v-i-j}{s-i}}{\binom{v-i-s}{k-s}} \overline{W}_{ik}^{\mathrm{T}} \overline{W}_{ji}^{\mathrm{T}} W_{js} \quad \text{(by Lemma 1(i))},
\end{aligned}$$

and the proof is complete. □

We remark that if $p \not| \binom{k-i}{s-i}$, $p \not| \binom{v-i-s}{k-s}$, $i = 0, 1, \ldots, s$, then W_{sk}^+ exists modulo p and is given by (9). The proof is similar to that of Theorem 7, keeping in mind that $W_{sk}\tilde{W}_{sk} = I$ holds in this case. In particular, for all sufficiently large primes p, W_{sk}^+ exists modulo p and is given by (9).

Example. Let $s = 2$, $k = 4$, $v = 6$ and let $p = 3$. Using the notation of Theorem 7, $\mathscr{N}$ consists of the single element 2. Then $\tilde{W}_{24} = \overline{W}_{24}^{\mathrm{T}} W_{22}$. Note that $W_{22} = I$, the identity matrix. Also, if we agree to list the 2-subsets and the 4-subsets in such a way that an element in the first list has its complement as the corresponding element in the second list, then $\overline{W}_{24} = I$ as well. Thus $\tilde{W}_{24} = I$. Now $W_{24}^+ = \tilde{W}_{24} W_{24} \tilde{W}_{24} = W_{24}$. Note that W_{24} is symmetric and it is easily verified that $W_{24}^2 = W_{24}$. We indicate the argument briefly: Let X, Y be subsets of $\{1, 2, \ldots, 6\}$ with $|X| = 2, |Y| = 4$. Then $|X \cap Y|$ equals 0, 1 or 2. Suppose $|X \cap Y| = 0$. Then the (X, Y)-element of W_{24} is zero. Also, there are precisely $\binom{4}{2} = 6$ sets $Z \subset \{1, 2, \ldots, 6\}$ such that the (X, Z)-element and the (Z, Y)-element of W_{24} are both 1. Since we are working over integers modulo 3, it follows that the (X, Y)-element of W_{24}^2 is zero. The cases that $|X \cap Y| = 1, 2$ are treated similarly and hence $W_{24}^2 = W_{24}$. We thus have another verification of the fact that $W_{24}^+ = W_{24}$. This example also shows that $W_{sk}^+ \neq \tilde{W}_{sk}$ in general.

We conclude with the following easy consequence of Theorem 7.

Corollary 8. *Let $0 \leqslant s \leqslant v - s$ and let p be a prime. Then $W_{s,v-s}$ has Moore–Penrose inverse over the integers modulo p.*

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