Cartesian decompositions and Schatten norms

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Abstract

Let T be a Hilbert space operator with $T = A + \mathrm{i}B$, where A and B are Hermitian. We prove sharp inequalities comparing the norms $\|T\|_p$ with $\|(A^2 + B^2)^{1/2}\|_p$ and $(\|A\|_p^2 + \|B\|_p^2)^{1/2}$

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1. Introduction

Every Hilbert space operator T can be written as $T = A + \mathrm{i}B$, where A and B are Hermitian; we have

$$A = \frac{1}{2}(T + T^*) \quad \text{and} \quad B = \frac{1}{2\mathrm{i}}(T - T^*).$$

We call this the Cartesian decomposition of T.

Let $\|T\|_2 = (\operatorname{tr} T^*T)^{1/2}$ be the Hilbert–Schmidt norm of T. This norm is one among the Schatten p-norms. If T is a compact operator with decreasingly ordered singular values $s_j(T)$, let

$$\|T\|_p = \left[\sum \left(s_j(T)\right)^p\right]^{1/p}, \quad p > 0.$$

For $1 \leqslant p < \infty$, this defines a norm (on the class of operators for which $\|T\|_p$ is a finite real number) called the Schatten p-norm. By convention, $\|T\|_\infty$ stands for the usual operator bound norm of T; when T is compact, $\|T\|_\infty = s_1(T)$.

If $T = A + \mathrm{i}B$ is the Cartesian decomposition, then

$$\|T\|_2 = \|(A^2 + B^2)^{1/2}\|_2$$

and

$$\|T\|_2^2 = \|A\|_2^2 + \|B\|_2^2.$$

These relations reflect the Euclidean character of the norm $\|\cdot\|_2$. For other Schatten p-norms $\|\cdot\|_p$, one looks for good inequalities to take the place of these relations. The purpose of this note is to provide some. We shall prove the following.

Theorem 1. *Let A, B be Hermitian operators and let $T = A + \mathrm{i}B$. Then*

$$\|(A^2 + B^2)^{1/2}\|_p \leqslant \|T\|_p \leqslant 2^{1/2-1/p}\|(A^2 + B^2)^{1/2}\|_p \tag{1}$$

for $2 \leqslant p \leqslant \infty$; and

$$2^{1/2-1/p}\|(A^2 + B^2)^{1/2}\|_p \leqslant \|T\|_p \leqslant \|(A^2 + B^2)^{1/2}\|_p \tag{2}$$

for $1 \leqslant p \leqslant 2$.

Corollary 1. *For $2 \leqslant p \leqslant \infty$, we have*

$$2^{2/p-1}(\|A\|_p^2 + \|B\|_p^2) \leqslant \|T\|_p^2 \leqslant 2^{1-2/p}(\|A\|_p^2 + \|B\|_p^2) \tag{3}$$

and for $1 \leqslant p \leqslant 2$, we have

$$2^{1-2/p}(\|A\|_p^2 + \|B\|_p^2) \leqslant \|T\|_p^2 \leqslant 2^{2/p-1}(\|A\|_p^2 + \|B\|_p^2). \tag{4}$$

All the inequalities (1)–(4) are sharp.

For several other results of this kind, and a discussion of their importance in the analysis of operators, we refer the reader to books [3,7], and papers [1,6,8].

The proofs of these inequalities are given in Section 2. In Section 3, they are reinterpreted to become comparison inequalities between different norms. For simplicity and brevity, we prove everything for finite dimensions only. Appropriate modifications are necessary in infinite dimensions.

2. Proofs

Let $|||\cdot|||$ be any unitarily invariant norm, i.e., a norm with the property $|||UTV||| = |||T|||$ for all T and unitary U, V. Such a norm is called a Q-norm, if there exists another unitarily invariant norm $|||\cdot|||^\wedge$ such that $|||T|||^2 = |||T^*T|||^\wedge$ for all T. The Schatten norms are unitarily invariant for all $1 \leqslant p \leqslant \infty$; and they are Q-norms if $2 \leqslant p \leqslant \infty$ (see [3]).

To prove inequalities like (1)–(4), it is often helpful to use general properties of unitarily invariant norms, and the well-developed machinery of majorisation. Let x be a real n-vector, and let $x_1^{\downarrow} \geqslant \cdots \geqslant x_n^{\downarrow}$ be its coordinates rearranged in the decreasing order. We say that x is *majorised* by y if for $1 \leqslant k < n$

$$\sum_{j=1}^{k} x_j^{\downarrow} \leqslant \sum_{j=1}^{k} y_j^{\downarrow}$$

and

$$\sum_{j=1}^{n} x_j^{\downarrow} = \sum_{j=1}^{n} y_j^{\downarrow}.$$

If x is majorised by y, we write $x \prec y$.

Properties of majorisation, its relation to convex functions, and its special importance in the study of unitarily invariant norms are discussed in great detail in [3]. We will use several well-known facts from here.

For positive operators A, B, the following inequality is well known, see [4, Theorem 1]:

$$\tfrac{1}{2}|||(A+B) \oplus (A+B)||| \leqslant |||A \oplus B||| \leqslant |||(A+B) \oplus 0|||. \tag{5}$$

Using familiar majorisation and convexity arguments, one can derive from this, the inequalities

$$2^{1-p}\|A+B\|_p^p \leqslant \|A\|_p^p + \|B\|_p^p \leqslant \|A+B\|_p^p \quad \text{for } 1 \leqslant p < \infty, \tag{6}$$

$$\|A+B\|_p^p \leqslant \|A\|_p^p + \|B\|_p^p \leqslant 2^{1-p}\|A+B\|_p^p \quad \text{for } 0 < p \leqslant 1. \tag{7}$$

See [4] or [7, p. 20].

Now, consider the operator $T = A + \mathrm{i}B$, where A and B are Hermitian. Note that

$$2(A^2 + B^2) = T^*T + TT^*. \tag{8}$$

Since $|||T^*T||| = |||TT^*|||$, we get from this, $|||A^2 + B^2||| \leqslant |||T^*T|||$. Since for $p \geqslant 2$, the Schatten norms are Q-norms, the first part of (1) follows from this.

From the second inequality in (5), we have $|||T^*T \oplus TT^*||| \leqslant 2|||(A^2 + B^2) \oplus 0|||$. This is equivalent to saying

$$\|T \oplus T\|_Q \leqslant 2^{1/2}\|(A^2 + B^2)^{1/2} \oplus 0\|_Q \tag{9}$$

for all Q-norms. Now note that $\|T \oplus T\|_p = 2^{1/p}\|T\|_p$ for all p. Thus, we get the second part of (1).

Again, from (8) we have for all $0 < p < \infty$

$$2^{p/2}\|A^2 + B^2\|_{p/2}^{p/2} = \|T^*T + TT^*\|_{p/2}^{p/2}.$$

If $1 \leqslant p \leqslant 2$, using the first part of (7), we see that

$$\|T^*T + TT^*\|_{p/2}^{p/2} \leqslant 2\|T^*T\|_{p/2}^{p/2}.$$

Hence,

$$2^{1/2}\|A^2+B^2\|_{p/2}^{1/2} \leqslant 2^{1/p}\|T^*T\|_{p/2}^{1/2}.$$

This gives the first inequality in (2). A similar argument using the second part of (7) gives the second inequality in (2).

Theorem 1 has been proved, we turn to the proof of Corollary 1.

Let $p \geqslant 2$. Note that

$$\begin{aligned}(\|A\|_p^2+\|B\|_p^2)^{p/2} &= (\|A^2\|_{p/2}+\|B^2\|_{p/2})^{p/2}\\ &\leqslant 2^{p/2-1}(\|A^2\|_{p/2}^{p/2}+\|B^2\|_{p/2}^{p/2})\\ &\leqslant 2^{p/2-1}\|A^2+B^2\|_{p/2}^{p/2}.\end{aligned}$$

Here, the first inequality is a consequence of the convexity of the function $f(t) = t^{p/2}$, and the second one follows from (6). From this, we get

$$\begin{aligned}\|A\|_p^2+\|B\|_p^2 &\leqslant 2^{1-2/p}\|A^2+B^2\|_{p/2}\\ &= 2^{1-2/p}\|(A^2+B^2)^{1/2}\|_p^2.\end{aligned}$$

The first part of (3) now follows from the first part of (1).

From the second part of (1), we have

$$\begin{aligned}\|T\|_p^2 &\leqslant 2^{1-2/p}\|(A^2+B^2)^{1/2}\|_p^2\\ &= 2^{1-2/p}\|A^2+B^2\|_{p/2}\\ &\leqslant 2^{1-2/p}(\|A^2\|_{p/2}+\|B^2\|_{p/2})\\ &= 2^{1-2/p}(\|A\|_p^2+\|B\|_p^2).\end{aligned}$$

This is the second part of (3).

The proof of (4) is very similar. We use (2) instead of (1). The added ingredient is that for $1 \leqslant p \leqslant 2$, we have

$$\begin{aligned}\|A^2\|_{p/2}+\|B^2\|_{p/2} &\leqslant \|A^2+B^2\|_{p/2}\\ &\leqslant 2^{2/p-1}(\|A^2\|_{p/2}+\|B^2\|_{p/2}).\end{aligned}$$

The second of these inequalities is given in [3, Problem IV.5.6]. The first may be proved by familiar arguments. For convenience, and future reference, let us record it explicitly.

Lemma 1. *Let* $0 < p < 1$. *Then, for positive* A, B

$$\|A\|_p+\|B\|_p \leqslant \|A+B\|_p.$$

Proof. Let $\lambda^{\downarrow}(A)$ denote the n-vector, whose coordinates are the eigenvalues of A arranged in the decreasing order. It is well known [3, p. 35] that

$$\lambda^{\downarrow}(A+B) \prec \lambda^{\downarrow}(A) + \lambda^{\downarrow}(B).$$

Since the function $f(t) = t^p$ is concave, it follows that

$$\sum_{j=1}^{n} [\lambda_j(A+B)]^p \geqslant \sum_{j=1}^{n} [\lambda_j^{\downarrow}(A) + \lambda_j^{\downarrow}(B)]^p.$$

(See [3, p. 41].) Now, take the pth roots of both sides and use the Minkowski inequality. □

All the inequalities (1)–(4) have been proved.

The first inequality in (1) and the second inequality in (2) are obviously sharp – they are equalities when the operators are scalars. The 2×2 example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

shows that the other two inequalities in Theorem 1 are sharp too.

This example also shows that the second inequality in (3) and the first inequality in (4) are sharp. The 2×2 example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

shows that the other two inequalities in Corollary 1 are also sharp.

In a recent paper, Bhatia and Zhan [6] have shown that when A, B are positive the factor $2^{1-2/p}$ occurring on the right-hand side of (3) and on the left-hand side of (4) can be replaced by 1. Our example shows that this cannot be done to the factor $2^{2/p-1}$ in the other two inequalities, and that these inequalities are sharp in this restricted case.

3. Norms defined via the Cartesian decomposition

There is another interesting and illuminating way of interpreting, and proving inequalities (1)–(4). We sketch this briefly.

Given any unitarily invariant norm $|||\cdot|||$, consider the following two objects:

$$|||T|||_{\alpha} = |||(A^2 + B^2)^{1/2}|||, \tag{10}$$

$$|||T|||_{\beta} = (|||A|||^2 + |||B|||^2)^{1/2}. \tag{11}$$

For the special case of the Schatten p-norms, we will use the notation

$$\|T\|_{p,\alpha} = \|(A^2 + B^2)^{1/2}\|_p, \tag{12}$$

$$\|T\|_{p,\beta} = (\|A\|_p^2 + \|B\|_p^2)^{1/2}. \tag{13}$$

It turns out that these objects define norms on the space of matrices, with one restriction: in the case of (11) and (13), we have to restrict the scalars to real numbers.

It is easy to verify that $|||\cdot|||_\beta$ has all the properties of a norm. The restriction to real scalars is needed to show that $|||aT|||_\beta = |a|\, |||T|||_\beta$.

The case of $|||\cdot|||_\alpha$ is more interesting. Here, the triangle inequality is not obvious. One way of proving it is by using the variational expression

$$(A^2+B^2)^{1/2} = \max_{CC^*+DD^*=I} |CA+DB|. \tag{14}$$

This may be found in [2]. For the reader's convenience, we provide the simple proof.

A 2×2 block matrix

$$\begin{pmatrix} A & C^* \\ C & I \end{pmatrix}$$

is positive if and only if $A \geqslant C^*C$ (see [3, Theorem IX.5.9]). Now, given any two positive matrices A, B and any C, D satisfying $CC^* + DD^* = I$

$$\begin{bmatrix} A^2+B^2 & AC^*+BD^* \\ CA+DB & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A & C^* \\ B & D^* \end{bmatrix} \geqslant 0.$$

Hence,

$$A^2+B^2 \geqslant |CA+DB|^2. \tag{15}$$

If $A^2+B^2>0$, for the particular choice $C=(A^2+B^2)^{-1/2}A$, $D=(A^2+B^2)^{-1/2}B$, there is equality in (15). Taking square roots, we get (14). The restriction $A^2+B^2>0$ can be removed using standard arguments.

Now, given positive matrices A_1, A_2, B_1, B_2, choose C, D, such that

$$[(A_1+A_2)^2+(B_1+B_2)^2]^{1/2} = |C(A_1+A_2)+D(B_1+B_2)|.$$

Regroup terms, use the matrix triangle inequality [3, Theorem III.5.6] and then use (14) to see that

$$\begin{aligned}[(A_1+A_2)^2+(B_1+B_2)^2]^{1/2} &= |(CA_1+DB_1)+(CA_2+DB_2)| \\ &\leqslant U|CA_1+DB_1|U^* + V|CA_2+DB_2|V^* \\ &\leqslant U(A^2+B^2)^{1/2}U^* + V(A^2+B^2)^{1/2}V^*\end{aligned}$$

for some unitaries U, V. The triangle inequality for $|||\cdot|||_\alpha$ is a special consequence of this inequality.

Inequalities (1) and (2) can be rewritten as

$$\|T\|_{p,\alpha} \leqslant \|T\|_p \leqslant 2^{1/2-1/p}\|T\|_{p,\alpha} \quad \text{for } 2 \leqslant p \leqslant \infty \tag{16}$$

and

$$2^{1/2-1/p}\|T\|_{p,\alpha} \leqslant \|T\|_p \leqslant \|T\|_{p,\alpha} \quad \text{for } 1 \leqslant p \leqslant 2. \tag{17}$$

One can see that the norm $\|T\|_{p,\alpha}$ is dual to the norm $\|T\|_{q,\alpha}$ if

$$\frac{1}{p}+\frac{1}{q}=1.$$

This brings out even more clearly the fact that inequalities (16) and (17) are dual relations.

It is worth remarking that for the operator norm the right-hand side inequality in (16) can be proved by another simple argument:

$$\begin{aligned}\|T\|^2 &= \|T^*T\| = \|A^2+B^2+\mathrm{i}(AB-BA)\|\\ &\leqslant \|A^2+B^2\| + 2\|AB\| \leqslant 2\|A^2+B^2\|\\ &= 2\|(A^2+B^2)^{1/2}\|^2.\end{aligned}$$

Here, we have used the arithmetic–geometric mean inequality $2\|AB\| \leqslant \|A^2+B^2\|$, [3, p. 263] and [5].

Since $\|T\|_2 = \|T\|_{2,\alpha}$, the other inequalities on the right-hand side in (16) could be obtained by an interpolation argument from the ones for $p=2$ and ∞.

We should point out that in Section 2, we have proved that for all unitarily invariant norms

$$|||(A^2+B^2)\oplus(A^2+B^2)||| \leqslant |||T^*T\oplus T^*T||| \leqslant 2|||(A^2+B^2)\oplus 0|||.$$

Equivalently, for all Q-norms

$$\|T\oplus T\|_{Q,\alpha} \leqslant \|T\oplus T\|_Q \leqslant 2^{1/2}\|T\oplus 0\|_{Q,\alpha}.$$

The inequalities are reversed for Q^*-norms (the duals of Q norms). These include (16) and (17) as special cases.

Similar remarks can be made about (3) and (4). They can be rewritten as

$$2^{1/p-1/2}\|T\|_{p,\beta} \leqslant \|T\|_p \leqslant 2^{1/2-1/p}\|T\|_{p,\beta} \quad \text{for } 2 \leqslant p \leqslant \infty \tag{18}$$

and

$$2^{1/2-1/p}\|T\|_{p,\beta} \leqslant \|T\|_p \leqslant 2^{1/p-1/2}\|T\|_{p,\beta} \quad \text{for } 1 \leqslant p \leqslant 2. \tag{19}$$

Once again the triangle inequality and the ordinary arithmetic–geometric mean inequality show that $\|T\|_p \leqslant 2^{1/2}\|T\|_{p,\beta}$ for all p. For the case $p=\infty$, this is best possible. For $p=2$, we can replace $2^{1/2}$ by 1. For other p, we could appeal to duality and to interpolation.

Finally, we remark that the norms defined in (10) and (11) are weakly unitarily invariant [3]. More such norms may be constructed using these ideas.

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